Correlation Functions of One-Dimensional Impenetrable Anyons

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Abstract of the Dissertation

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In the last years we have witnessed the experimental realization of many one-dimensional physical systems. This has renewed the interest of theoretical physicists in computing relevant quantities for such models, which are experimental accessible.

We have investigated the field-field correlation functions of a model of one-dimensional impenetrable anyons which is relevant for systems that can be realized with edges of the electron liquids in the Fractional Quantum Hall Effect (FQHE) regime. Varying the statistics parameter, the correlation functions of this model, interpolate between the ones for impenetrable bosons and free fermions.

We have computed the large distance asymptotic behavior of the field-field correlator at finite temperatures solving a Riemann-Hilbert problem associated with the integrable system of nonlinear partial differential equations characterizing the correlator. As a preliminary step we have obtained a Fredholm determinant representation using two methods: the anyonic generalization of Lenard's formula and the summation of form factors. We show that the leading term of the asymptotics is oscillatory with the period of oscillation proportional with the statistics parameter. Also as the statistics parameter approaches the free fermionic point the second leading term becomes comparable in magnitude with the leading term but with opposite phase producing fermionic beats.

Contents

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Chapter 1

Introduction

For hard-core particles moving in two spatial dimensions, one can unambiguously define the notion of braiding of the particle trajectories by introducing the winding number n that gives the number of times the trajectory of one particle encircles another particle. This fact makes it possible to consider "anyonic" particles with fractional exchange statistics [36, 57], for which the wavefunction acquires the non-trivial phase factor $e^{\pm i2\pi\kappa}$, where κ is the "statistical parameter", whenever n changes by ± 1. This situation can be contrasted with the case of three spatial dimensions where one can define only permutations (no braiding) of point-like particles leading to only integer statistics, i.e. $\kappa = 0, 1$ for bosons and fermions, respectively. In physical terms, the anyons in two dimensions can be viewed as the charge-flux composites for which the statistical phase arises as the result of the Aharonov-Bohm interaction between the charge of one particle and the flux of the other [76]. Experimentally, anyons can be realized as quasiparticles of the two-dimensional (2D) electron liquids in the Fractional Quantum Hall Effect (FQHE) [4]. Individual quasiparticles are localized and controlled by quantum antidots in the FQHE regime [37], and the transport properties of multi-antidot systems should provide direct manifestations of their fractional exchange statistics [5]. Dynamics of individual FQHE quasiparticles attracted considerable attention (see, e.g., [6, 25]) as a possible basis for realization of the topological quantum computation [52].

Both conceptually and in practice (e.g., in FQHE systems), the 2D anyons can be confined to move in one dimension. There are, however, the aspects of fractional statistics in one dimension that make its introduction more complicated than in two dimensions. One is that for strictly 1D particles, a trajectory of one particle can not wind around another, making the sign of the exchange phase $e^{\pm i\pi\kappa sgn(x_i - x_j)/2}$ that the wavefunction should acquire when the particle with coordinate x_i moves past the one with x_j, undetermined. The sign of this phase depends on whether x_i rotates clockwise or counter-clockwise around x_j

in the underlying 2D geometry, which also explains why the signs of the phase change at $x_i = x_j$ are opposite for the two particles in the pair: rotation of one sense for increasing coordinate x_i implies the opposite rotation for increasing x_j. This fact hindered the early attempts at direct introduction of the 1D anyons as charge-flux composites [1, 72]. It implies that any description of the 1D anyons requires an additional convention on the choice of the sign of the statistical phase for each pair of particles. As discussed in more details below, this choice can be arbitrary and affects the appropriate boundary conditions of the quantum-mechanical wavefunctions of the system of anyons.

One-dimensional impenetrable anyons in a "box" of length L are described by the following second-quantized hamiltonian (we consider $\hbar = 2m = 1$)

$$H = \int_0^L dx \; [\partial_x \Psi^\dagger(x)][\partial_x \Psi(x)] \, ,$$

supplemented with the impenetrability condition on the energy eigenstates

$$\left(\int dx' \; \Psi^\dagger(x)\Psi^\dagger(x')\delta(x-x')\Psi(x')\Psi(x) \right) |\Psi\rangle_N = 0 \, ,$$

or

$$(\Psi(x))^2 |\Psi\rangle_N = 0 \, .$$

The anyonic fields obey the following commutation relations

$$\Psi(x_1)\Psi^\dagger(x_2) = e^{-i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2)\Psi(x_1) + \delta(x_1-x_2) \, ,$$

$$\Psi^\dagger(x_1)\Psi^\dagger(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2)\Psi^\dagger(x_1) \, ,$$

$$\Psi(x_1)\Psi(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi(x_2)\Psi(x_1) \, ,$$

where $\epsilon(x) = x/|x|, \epsilon(0) = 0$ and $\kappa \in [0,1]$ is the statistics parameter. For $\kappa = 0$ the previous relations are bosonic and for $\kappa = 1$ they are fermionic. Anyonic fields can be constructed in terms of bosonic fields in the following manner

$$\Psi^\dagger(x) = \Psi_B^\dagger(x) e^{i\pi\kappa \int_0^x dx' \rho(x')}, \qquad \Psi(x) = e^{-i\pi\kappa \int_0^x dx' \rho(x')} \Psi_B(x) \, ,$$

with

$$\rho(x) \equiv \Psi^\dagger(x)\Psi(x) = \Psi_B^\dagger(x)\Psi_B(x) \, .$$

This type of construction is the generalization of the Fermi-Bose correspondence

$$\Psi_B^\dagger(x) = \Psi_F^\dagger(x) e^{i\pi \int_0^x dx' \rho(x')}, \qquad \Psi_B(x) = e^{-i\pi \int_0^x dx' \rho(x')} \Psi_F(x) \, ,$$

which gives impenetrable bosonic fields in terms of canonical anticommuting fermionic ones. Due to the fact that

$$\rho(x) \equiv \Psi^\dagger(x)\Psi(x) = \Psi_B^\dagger(x)\Psi_B(x) = \Psi_F^\dagger(x)\Psi_F(x)\,,$$

and the wavefunctions differ only by a phase, the density-density correlation functions of impenetrable anyons, bosons and free fermions are the same. However, the field-field correlation function

$$\langle \Psi^\dagger(x_2)\Psi(x_1)\rangle_T$$

of impenetrable anyons is different from the similar correlation of free fermions and impenetrable bosons and will be the main object of study of this thesis.

Investigating the correlation functions of physical models is in general an extremely difficult task. In our case it is possibly to perform this analysis rigorously due to the fact that the wavefunctions of the system can be obtained in an exact form. In the case of a N-particles eigenstate of the Hamiltonian

$$|\Psi\rangle_N = \frac{1}{\sqrt{N!}}\int d^N z\ \chi_N(z_1, \cdots, z_N)\Psi^\dagger(z_N)\cdots\Psi^\dagger(z_1)|0\rangle\,,$$

the quantum mechanical wavefunction obeys

$$\chi_N(z_1, \cdots, z_i, z_{i+1}, \cdots, z_N) = e^{i\pi\kappa\epsilon(z_i - z_{i+1})}\chi_N(z_1, \cdots, z_{i+1}, z_i, \cdots, z_N) \quad (1.1)$$

and has the form

$$\chi_N = \frac{e^{+i\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j-z_k)}}{\sqrt{N!}}\prod_{j>k}\epsilon(z_j - z_k)\sum_{\pi\in S_N}(-1)^\pi e^{i\sum_{n=1}^N z_n\lambda_{\pi(n)}}\,.$$

where $\lambda' s$ are the quasimomenta of the particles. The individual quasimomenta λ_j depend of the boundary conditions imposed on the wavefunctions. In contrast to particles of integer statistics, wavefunctions of the anyons satisfy different quasi-periodic boundary conditions in their different arguments. This fact was first noticed by Averin and Nesteroff in [5]. For example, in the case of two particles, if we impose periodic boundary conditions on the first variable

$$\chi_2(0, z_2) = \chi_2(L, z_2)\,, \quad (1.2)$$

then from the anyonic property of the wavefunction (1.1) we obtain

$$\chi_2(z_1, 0) = e^{i2\pi\kappa}\chi_2(z_1, L)\,. \quad (1.3)$$

and the Bethe equations are given by

$$e^{i\lambda_j L} = (-1)e^{-i\pi\kappa} \,.$$

Analogously imposing periodic boundary conditions on the second variable

$$\chi_2(z_1, 0) = \chi_2(z_1, L) \,,$$

implies

$$\chi_2(0, z_2) = e^{-i2\pi\kappa}\chi_2(L, z_2) \,,$$

and the Bethe equations

$$e^{i\lambda_j L} = (-1)e^{i\pi\kappa} \,.$$

The origin of this difference can be traced back to the fact that the fractional statistics requires braiding of particles, something that strictly speaking can not be done in one dimension. To define the braiding of 1D particles one needs to first adopt a convention on how the particles pass each other at coinciding points, something that is done by choosing a specific sign of the exchange phase $e^{i\pi\kappa\epsilon(z_1-z_2)/2}$. After that, one more choice that needs to be made is how the 1D loop with anyons is imbedded into the underlying 2D anyonic system. In the case of two particles, this choice is reflected in the possibility of choosing different boundary conditions for two different anyonic coordinates and determines how the particle trajectories enclose each other as the particles move along the loop [5]. As reflected in Eqs. (1.2) and (1.3), periodicity in z_1 means that the trajectory of z_1 does not enclose the particle z_2. This implies that z_1 is itself enclosed by the trajectory of z_2, producing the twist in the boundary condition for z_2 variable. The different choice of the boundary condition would mean that the 1D loop in imbedded into the 2D system in such a way that the trajectory of z_1 encloses z_2. This means that the wavefunction periodicity in both variables correspond to different but valid physical situations.

As we have said, our main object of interest is the field-field correlation function of impenetrable anyons at finite temperature, more specifically the determination of the large distance asymptotic behavior. We will present two methods of calculating the asymptotic behavior. The first method will make use of the methods of conformal field theory and is valid only at zero and low temperatures. The second method is more rigorous and gives the results for any temperature and is more technically involved.

Obtaining the large distance asymptotics of correlation functions using CFT requires the computation of the spectrum of low-lying excitations of our model. Impenetrable anyons can be considered as the limiting case of infinite repulsive coupling constant of the more general model called the Lieb-Liniger anyonic gas introduced by Kundu [56]. The model introduced by Kundu is the generalization for arbitrary statistics κ of the Bose gas with δ-function interaction studied by Lieb and Liniger [62]. In Chapter 2, following [66] we study the Bethe Ansatz solution, the properties of the ground state for quasi-periodic boundary conditions and the spectrum of low-lying excitations. The results for impenetrable anyons are obtained by taking the limit $c \to \infty$.

The conformal field theory predictions are presented in Chapter 3. They provide a generalization of similar results obtained in [19] using the harmonic fluid approach.

The rigorous investigation of the field-field correlation functions of impenetrable anyons was performed along the lines of the original investigations of Korepin, Slavnov, Its and Izergin [46–49, 54, 55] on impenetrable bosons. As a first step we have derived a representation of the correlator in terms of a Fredholm determinant of an integral operator. We will present two different methods which produce different but equivalent results. The first method presented in Chapter 4 and based on the results of [67] is the anyonic generalization of Lenard's formula [59]. In the original paper, Lenard, using the Bose-Fermi mapping introduced by Girardeau [34], was able to obtain a representation of the correlation functions of impenetrable bosons (at zero and finite temperature) in terms of correlation functions of free fermions. This representation can be shown to be equivalent with the Fredholm minor of an integral operator with the kernel being given by the Fourier transform of the Fermi distribution function. In [67] it was shown that Lenard's formula can be generalized in the anyonic case making use of the Anyon-Fermi mapping. The kernel of the integral operator is also the Fourier transform of the Fermi distribution function but the constant appearing in front of the operator is no longer $2/\pi$ as in the bosonic case but $(1 + e^{\pm i\pi\kappa})/\pi$. It is interesting to note that, unlike the bosonic case, the anyonic correlator $\langle\Psi^\dagger(x_2)\Psi(x_1)\rangle_T$ depends on the sign of $x_2 - x_1$. The advantage of this method is the fact that also produces results for other 2n-point correlation functions but it cannot be extended in the case of dynamical correlators.

In order to obtain the determinant representation for dynamical correlators we have used the method of summation of form factors which is presented in Chapter 5 and is based on the results obtained in [68]. A fundamental step is constituted by the computation of the finite volume form factors in which extreme care has to be taken in selecting wavefunctions for N and N+1 particles with similar periodicity. The summation of finite volume form factors is performed first and then the thermodynamic limit is taken. The equivalence of the determinant representation for static correlators obtained via this method with the results from the anyonic generalization of Lenard's formula is presented at the end of Chapter 5.

The integral operators that appears in the determinant representation of the field correlator are of a special type called "integrable" [54]. This means that the kernel of these operators can be factorized in a specific way and the same property is shared by the resolvent kernel. As a result the determinant representation can be used to obtain a system of integrable partial nonlinear differential equations characterizing completely the correlation functions. This system is the same as the one obtained by Its, Izergin, Korepin and Slavnov in [49] which characterizes the correlation functions of impenetrable bosons but with different boundary conditions. At zero temperature we obtain the same ordinary differential equation of Painlevé V type obtained by Jimbo, Miwa, Môri and Sato in their celebrated work on the one-particle reduced density matrix (field-field correlator) of impenetrable bosons [51] but of course with statistics dependent boundary conditions. These results were obtained in [69] and are presented in Chapter 6.

The most important result of this dissertation, the large distance asymptotic behavior (including the amplitude) of the correlation functions at any temperature, is presented in Chapter 7 and was announced in [69]. In order to obtain the asymptotics we have solved in the appropriate limit a matrix Riemann-Hilbert problem associated with the integrable system of differential equations. The main feature of the asymptotics of the field correlator of impenetrable anyons is the fact that the leading term is oscillatory with the period of oscillation proportional with the statistics parameter κ. Also as the statistics parameter approaches the free fermionic point ($\kappa = 1$) the second leading term becomes comparable in magnitude with the leading term but with opposite phase producing fermionic beats.

Except some parts of Chap. 2 this dissertation is based on the papers [66–70] written together with Vladimir Korepin and Dmitri Averin.

Chapter 2

The Lieb-Liniger Gas of Anyons

From the historic point of view, the first continuous model of one-dimensional anyons was introduced by Kundu in [56]. This model, which is referred in the literature as the Lieb-Liniger gas of anyons, can be considered as an anyonic generalization of the Bose gas with δ-function interaction and in the limit of infinite coupling constant it reduces to impenetrable anyons. In the last years there has been an increasing number of papers investigating the properties of the model starting with the work of Batchelor *et al.*,[7–9] investigating the properties of the ground-state and thermodynamics. The spectrum of low-lying excitations was computed in [66] and other relevant works are [21, 39]. The correlation functions were studied using the harmonic fluid approach in [19] and conformal field theory in [66]. Similarly to the case of the Bose gas with δ-function interaction in the case of infinitely repulsive coupling constant the quantity of information on the correlation functions is much more substantive and will be presented in the next chapters. In this case the relevant papers are [20, 66–69, 73, 74].

Some of the features of the Lieb-Liniger gas of anyons will bring tears in the eyes of the mathematically inclined. If the reader feels uncomfortable it should be reminded that we are interested in the impenetrable case, $c \to \infty$, in which all the conceptual difficulties disappear.

It should be mentioned that other models of one-dimensional anyons have been proposed [3, 10, 13, 29, 40, 65] and recently Batchelor, Foerster, Guan, Links and Zhou have introduced the quantum inverse scattering method with anyonic grading for models on the lattice. On the more fundamental level regarding the construction of the Fock space and anyonic fields rigorous work was done by Liguori and Mintchev in [60] and Ilieva and Thirring in [45].

2.1 The model

The Lieb-Liniger gas of anyons is characterized by the second quantized Hamiltonian

$$H = \int_0^L dx \, \{[\partial_x \Psi^\dagger(x)][\partial_x \Psi(x)] + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\} \,, \tag{2.1}$$

where $c > 0$ is the coupling constant and L the length of the system. The anyonic fields obey the equal-time commutation relations

$$\Psi(x_1)\Psi^\dagger(x_2) = e^{-i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2)\Psi(x_1) + \delta(x_1 - x_2) \,, \tag{2.2}$$

$$\Psi^\dagger(x_1)\Psi^\dagger(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2)\Psi^\dagger(x_1) \,, \tag{2.3}$$

$$\Psi(x_1)\Psi(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi(x_2)\Psi(x_1) \,, \tag{2.4}$$

where

$$\epsilon(x_1 - x_2) = \begin{cases} 1 & \text{when} \quad x_1 > x_2 \,, \\ -1 & \text{when} \quad x_1 < x_2 \,, \\ 0 & \text{when} \quad x_1 = x_2 \,. \end{cases} \tag{2.5}$$

In the original work [56] introducing this model, the anyonic fields were realized in terms of the bosonic fields

$$\Psi^\dagger(x) = \Psi_B^\dagger(x) e^{i\pi\kappa \int_0^x dx' \rho(x')}, \qquad \Psi(x) = e^{-i\pi\kappa \int_0^x dx' \rho(x')}\Psi_B(x) \,, \tag{2.6}$$

where

$$\rho(x) \equiv \Psi^\dagger(x)\Psi(x) = \Psi_B^\dagger(x)\Psi_B(x) \,.$$

Due to the fact that at coinciding points $\epsilon(0) = 0$, the commutation relations (2.2), (2.3), (2.4) are indeed bosonic. An alternative realization in terms of the fermionic fields was proposed in [35]. However, in this case, the interaction term in the Hamiltonian (2.1) vanishes, since $\Psi^2(x) = [\Psi^\dagger(x)]^2 = 0$ in coinciding points (see also the discussion in [9]). One implication of this difference is that in comparison to the bosonic representation (2.6), similar fermionic representation with appropriate modification of the statistical parameter, effectively makes it possible to describe only the infinite repulsion limit $c \to \infty$.

The corresponding equation of motion $-i\partial_t \Psi(x, t) = [H, \Psi(x, t)]$ is the nonlinear Schrödinger equation

$$i\partial_t \Psi(x, t) = \partial_x \Psi(x, t) + 2c\Psi^\dagger(x, t)\Psi^2(x, t) \,.$$

The number of particle operator Q and the momentum operator P are defined as

$$Q = \int_0^L dx \ \Psi^\dagger(x)\Psi(x)\,,$$

$$P = -\frac{i}{2}\int_0^L dx \ \left(\Psi^\dagger(x)\partial_x\Psi(x) - [\partial_x\Psi^\dagger(x)]\Psi(x)\right)\,.$$

Both of them are hermitian operators which commute with the Hamiltonian

$$[H,P] = [H,Q] = 0\,.$$

If we define the Fock vacuum as

$$\Psi(x)|0\rangle \ \ , \ \ x\in[0,L]\,,$$

the N-particle eigenstate of the Hamiltonian (and also of P and Q) can be then written as

$$|\Psi\rangle_N = \frac{1}{\sqrt{N!}}\int d^N z \ \chi_N(z_1,\cdots,z_N)\Psi^\dagger(z_N)\cdots\Psi^\dagger(z_1)|0\rangle\,,$$

where the many-body wavefunction obeys

$$\chi_N(z_1,\cdots,z_i,z_{i+1},\cdots,z_N) = e^{i\pi\kappa\epsilon(z_i-z_{i+1})}\chi_N(z_1,\cdots,z_{i+1},z_i,\cdots,z_N)\,. \tag{2.7}$$

This can be seen directly by using the exchange relation of the field operators $\Psi^\dagger(z_{i+1})\Psi^\dagger(z_i) = e^{i\pi\kappa\epsilon(z_{i+1}-z_i)}\Psi^\dagger(z_i)\Psi^\dagger(z_{i+1})$ and interchanging the name of the integration variables z_i, z_{i+1}. Iterating the exchanges several times we obtain

$$\chi_N(z_1,\cdots,z_i,\cdots,z_j,\cdots,z_N) = e^{i\pi\kappa[\sum_{k=i+1}^{j}\epsilon(z_i-z_k)-\sum_{k=i+1}^{j-1}\epsilon(z_j-z_k)]} \times \chi_N(z_1,\cdots,z_j,\cdots,z_i,\cdots,z_N)\,. \tag{2.8}$$

2.2 The equivalent quantum mechanical problem

In [9, 56], it was shown that the eigenvalue problem

$$H|\Psi\rangle_N = E_N|\Psi\rangle_N, \qquad P|\Psi\rangle_N = p_N|\Psi\rangle_N,$$

can be reduced to the quantum-mechanical problem

$$\mathcal{H}\chi_N(z_1,\cdots,z_N) = E_N\chi_N(z_1,\cdots,z_N)\,, \tag{2.9}$$

$$\mathcal{P}\chi_N(z_1,\cdots,z_N) = p_N\chi_N(z_1,\cdots,z_N)\,,$$

where

$$\mathcal{H}_N = \sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial z_j^2}\right) + 2c\sum_{1\leq j\leq k\leq N}\delta(z_j - z_k)\,, \tag{2.10}$$

and

$$\mathcal{P} = \sum_{j=1}^{N}\left(-\frac{\partial}{\partial_{z_j}}\right)\,.$$

The quantum mechanical hamiltonian is the same as the one considered by Lieb and Liniger [62] in their study of the Bose gas with δ-function interaction. In our case the wavefunction is no longer symmetrical under the exchange of two coordinates it obeys (2.7). Due to the fact that the interaction potential is given by a delta function the eigenvalue problem (2.9) is equivalent with

$$-\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial z_i^2}\right)\chi_N(z_1,\cdots,z_N) = E_N\chi_N(z_1,\cdots,z_N)\,, \tag{2.11}$$

supplemented by the boundary condition

$$\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)\chi_N(z_1,\cdots,z_N)\Big|_{z_i=z_j+\epsilon} - \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)\chi_N(z_1,\cdots,z_N)\,|_{z_i=z_j-\epsilon}$$
$$= 2c\chi_N(z_1,\cdots,z_N)|_{z_i=z_j}\,. \tag{2.12}$$

2.3 Bethe Ansatz solution

In this section we are going to construct the Bethe Ansatz solution for the eigenvalue problem (2.11), (2.12) following the papers [9, 56]. It is easy to see that we can construct a wavefunction with the required symmetry (2.7) in the following manner

$$\chi_N(z_1,\cdots,z_N) = e^{i\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j-z_k)}\chi_N^b(z_1,\cdots,z_N)\,,$$

where χ_N^b is a symmetric wavefunction. This means that we are going to consider the Bethe Ansatz wavefunction of the form

$$\chi_N(z_1,\cdots,z_N) = e^{i\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j-z_k)}\chi_N^b(z_1,\cdots,z_N)\,, \tag{2.13}$$

with

$$\chi_N^b(z_1,\cdots,z_N) = \sum_{\pi\in S_N} A(\lambda_{\pi(1)},\cdots,\lambda_{\pi(N)}) e^{i(\lambda_{\pi(1)}z_1+\cdots+\lambda_{\pi(N)}z_N)}\,, \qquad (2.14)$$

where S_N is the group of permutations with N elements. It is instructive to consider first the case of two particles. In this case $\chi_2(z_1,z_2)$ is given by

$$\chi_2(z_1,z_2) = e^{-i\pi\kappa/2}\left[A(12)e^{i(\lambda_1 z_1+\lambda_2 z_2)} + A(21)e^{i(\lambda_2 z_1+\lambda_1 z_2)}\right]\,,$$

in the region $z_1 < z_2$ and by

$$\chi_2(z_1,z_2) = e^{+i\pi\kappa/2}\left[A(12)e^{i(\lambda_2 z_1+\lambda_1 z_2)} + A(21)e^{i(\lambda_1 z_1+\lambda_2 z_2)}\right]\,,$$

in the region $z_1 > z_2$. In the previous expressions we have denoted $A(ij) = A(\lambda_i,\lambda_j)$. At the coinciding points $z_1 = z_2$ we have

$$\chi_2(z_1,z_2) = A(12)e^{i(\lambda_1+\lambda_2)z_1} + A(21)e^{i(\lambda_1+\lambda_2)z_1}\,.$$

Application of the boundary condition (2.12) gives

$$(e^{i\pi\kappa/2}+e^{-i\pi\kappa/2})i(\lambda_2-\lambda_1)[A(12)-A(21)] = 2c[A(12)+A(21)]\,,$$

which means that

$$A(21) = A(12)\frac{\lambda_2-\lambda_1+ic'}{\lambda_2-\lambda_1-ic'}\,,$$

where we have introduced

$$c' = \frac{c}{\cos(\pi\kappa/2)}\,. \qquad (2.15)$$

In the general case of N particles the wavefunction $\chi_N(z_1,\cdots,z_N)$ in the wedge $z_1 < z_2 < \cdots < z_i < z_j < \cdots < z_N$ is given by

$$\chi_N(z_1,\cdots,z_N) = e^{-i\frac{\pi\kappa}{2}+i\frac{\pi\kappa}{2}\sum_{l<m;[i,j]}\epsilon(z_l-z_m)}[\cdots + A(\cdots ij\cdots)e^{i(\cdots+\lambda_i z_i+\lambda_j z_j+\cdots)}$$
$$+A(\cdots ji\cdots)e^{i(\cdots+\lambda_j z_i+\lambda_i z_j+\cdots)} + \cdots]\,,$$

and by

$$\chi_N(z_1,\cdots,z_N) = e^{i\frac{\pi\kappa}{2}+i\frac{\pi\kappa}{2}\sum_{l<m;[i,j]}\epsilon(z_l-z_m)}[\cdots + A(\cdots ij\cdots)e^{i(\cdots+\lambda_j z_i+\lambda_i z_j+\cdots)}$$
$$+A(\cdots ji\cdots)e^{i(\cdots+\lambda_i z_i+\lambda_j z_j+\cdots)} + \cdots]\,,$$

in the wedge $z_1 < z_2 < \cdots < z_j < z_i < \cdots < z_N$. At the coinciding points $z_i = z_j$ the wavefunction is given by

$$\chi_N(z_1, \cdots, z_N) = e^{i\frac{\pi\kappa}{2}\sum_{l<m;[i,j]}\epsilon(z_l - z_m)}[\cdots + A(\cdots ij \cdots)e^{i(\cdots+(\lambda_i+\lambda_j)z_i+\cdots)}$$
$$+A(\cdots ji \cdots)e^{i(\cdots+(\lambda_j+\lambda_i)z_i+\cdots)} + \cdots].$$

In the previous expressions the pair $[i, j]$ is excepted in the sum of the statistics phase of of the wavefunction. Again the application of the discontinuity condition (2.12) gives

$$(e^{i\pi\kappa/2}+e^{-i\pi\kappa/2})i(\lambda_j-\lambda_i)[A(\cdots ij\cdots)-A(\cdots ji\cdots)] = 2c[A(\cdots ij\cdots)+A(\cdots ji\cdots)],$$

which results in

$$A(\cdots ji \cdots) = A(\cdots ij \cdots)\frac{\lambda_j - \lambda_i + ic'}{\lambda_j - \lambda_i - ic'}.$$

Therefore the eigenfunctions of the Hamiltonian (2.10) are given by

$$\chi_N = \frac{e^{i\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j-z_k)}}{\sqrt{N!\prod_{j>k}[(\lambda_j-\lambda_k)^2+c'^2]}}\sum_{\pi\in S_N}(-1)^{[\pi]}e^{i\sum_{n=1}^{N} z_n\lambda_{\pi(n)}}$$
$$\times\prod_{j>k}[\lambda_{\pi(j)} - \lambda_{\pi(k)} - ic'\epsilon(z_j - z_k)], \qquad (2.16)$$

where we have used the same normalization factor as for impenetrable bosons [54].

2.4 Boundary conditions

The characteristics of the anyonic gas (2.1) depend on the boundary conditions imposed on the system at $x = 0 = L$. In this work, we use two different quasiperiodic boundary conditions which impose periodicity either directly on the anyonic or on the bosonic fields. Equations (2.6) imply that the periodic boundary condition for anyons correspond to twisted boundary conditions for bosons and viceversa. In terms of the anyonic fields, the boundary condition we use are,

$$\text{periodic BC:} \quad \Psi^\dagger(0) = \Psi^\dagger(L), \qquad (2.17)$$

and

$$\text{twisted BC:} \quad \Psi^\dagger(0) = \Psi^\dagger(L)e^{-i\pi\kappa(N-1)}, \qquad (2.18)$$

where N is the number of particle in the system. One can see directly from Eq. (2.6) that the external phase shift $\pi\kappa(N-1)$ introduced into the conditions (2.18), ensures the periodicity of the bosonic fields. As will be shown in more details below, this means that this phase removes the anyonic shift of the quasiparticle momenta. Below, we use the common notation for the two types of boundary conditions:

$$\Psi^\dagger(0) = \Psi^\dagger(L)e^{-i\pi\beta\kappa(N-1)}\,, \quad \beta = 0, 1\,. \tag{2.19}$$

An important difference of the anyons with fractional exchange statistics from the integer-statistics particles is that the boundary conditions (2.19) for the fields do not translate directly into the same boundary conditions for the quantum-mechanical wavefunctions of the N-anyon system [5], which have a more complicated structure (2.20).

2.5 Bethe Ansatz equations

In this section we are going to show that the boundary conditions for the many-anyon wavefunctions are given by

$$\begin{array}{rcll} \chi_N(0, z_2, \cdots, z_N) &=& e^{-i\pi\beta\kappa(N-1)} & \chi_N(L, x_2, \cdots, x_N)\,, \\ \chi_N(z_1, 0, \cdots, z_N) &=& e^{i2\pi\kappa}e^{-i\pi\beta\kappa(N-1)} & \chi_N(z_1, L, \cdots, z_N)\,, \\ && \vdots & \\ \chi_N(z_1, z_2, \cdots, 0) &=& e^{i2(N-1)\pi\kappa}e^{-i\pi\beta\kappa(N-1)} & \chi_N(z_1, z_2 \cdots, L)\,, \end{array} \tag{2.20}$$

with the associated Bethe equations

$$e^{i\lambda_j L} = e^{i\pi(\beta-1)\kappa(N-1)} \prod_{k=1,k\neq j}^{N} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'}\right). \tag{2.21}$$

The Bethe equations (2.21) are similar to those obtained by Lieb and Liniger for the Bose gas with repulsive δ-function interaction. In our case, however, the effective coupling constant c' (2.15) can take negative values. While it can be shown (see, e.g., [54]) that the Bethe roots λ_j are real for $c' > 0$, the roots can become complex for $c' < 0$, and one gets bound states [64]. From now on we will consider only the case $c' > 0$.

The treatment in this section generalizes the approach of [5] to the case of several penetrable particles. In physical terms, the situation we consider corresponds to anyons confined to move along a loop with, in general, an external phase shift ϕ created, e.g., by a magnetic filed threading the loop. We start with the case of *two particles* and no external phase shift, $\phi = 0$. The Bethe-Anzatz wavefunction (2.16) reduces in this case to the following form: In the region I ($z_1 < z_2$) one has

$$\chi_I(z_1, z_2) = \frac{e^{-i\pi\kappa/2}}{\sqrt{2[(\lambda_2 - \lambda_1)^2 + c'^2]}}[e^{i(z_1\lambda_1 + z_2\lambda_2)}(\lambda_2 - \lambda_1 - ic')$$

$$+e^{i(z_1\lambda_2 + z_2\lambda_1)}(\lambda_2 - \lambda_1 + ic')] \, ,$$

and in the region II ($z_1 > z_2$):

$$\chi_{II}(z_1, z_2) = \frac{e^{i\pi\kappa/2}}{\sqrt{2[(\lambda_2 - \lambda_1)^2 + c'^2]}}[e^{i(z_1\lambda_1 + z_2\lambda_2)}(\lambda_2 - \lambda_1 + ic')$$

$$+e^{i(z_1\lambda_2 + z_2\lambda_1)}(\lambda_2 - \lambda_1 - ic')] \, .$$

The general exchange symmetry of this wavefunction given by Eq. (2.7) imply that for fractional κ it can not satisfy the same boundary conditions in the two coordinates. As one can see by exchanging the coordinates, if the wavefunction is periodic in the first one, the boundary conditions in second one should have a twist,

$$\chi(0, z_2) = \chi(L, z_2) \quad \rightarrow \quad \chi(z_1, 0) = \chi(z_1, L)e^{2i\pi\kappa} \, , \tag{2.22}$$

and viceversa. One consequence of this is that the exact form of the Bethe equations (2.21) depends on whether we impose periodic boundary conditions on one or the other coordinate. Indeed, if one requires periodicity in z_1, $\chi(0, z_2) = \chi(L, z_2)$, the Bethe equations are:

$$e^{iL\lambda_j} = e^{-i\pi\kappa} \prod_{k=1, k\neq j}^{2} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'} \right) \, ,$$

whereas the periodicity in z_2, $\chi(z_1, 0) = \chi(z_1, L)$, results in the equations that differ by the sign of the statistics parameter κ:

$$e^{iL\lambda_j} = e^{i\pi\kappa} \prod_{k=1, k\neq j}^{2} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'} \right) \, .$$

Since the Bethe equations determine the spectrum of the quasiparticle momenta λ_j the κ shifts of different signs produce two physically different situations.

As we have said and explained in Introduction this difference is due to the fact that in one-dimension we cannot define properly the braiding of particles. From a more mathematical point of view this is explained by the fact that our anyons are on a circle and the exchange phase $e^{i\pi\kappa\epsilon(z_1-z_2)/2}$ depends on the positions of the particles. This means that we have to chose first an "origin" on the circle and then the direction in which the coordinate on the circle "increases". Depending on the way we choose this "origin" we will have different but valid physical situations.

The situation is somewhat more complicated for larger number of particles, as can be seen in the case of *three particles*. In the wavefunction (2.16), one needs to distinguish then six regions corresponding to the six permutation of the particles. The wavefunction (2.16) in these regions is:

Region I ($z_1 < z_2 < z_3$)

$$\begin{aligned}
&\chi_I(z_1, z_2, z_3)\\
&= Ae^{\frac{-i3\pi\kappa}{2}} \Big[e^{i(z_1\lambda_1+z_2\lambda_2+z_3\lambda_3)}(\lambda_3-\lambda_2-ic')(\lambda_3-\lambda_1-ic')(\lambda_2-\lambda_1-ic')\\
&\quad -e^{i(z_1\lambda_1+z_2\lambda_3+z_3\lambda_2)}(\lambda_2-\lambda_3-ic')(\lambda_2-\lambda_1-ic')(\lambda_3-\lambda_1-ic')\\
&\quad +e^{i(z_1\lambda_3+z_2\lambda_1+z_3\lambda_2)}(\lambda_2-\lambda_1-ic')(\lambda_2-\lambda_3-ic')(\lambda_1-\lambda_3-ic')\\
&\quad -e^{i(z_1\lambda_3+z_2\lambda_2+z_3\lambda_1)}(\lambda_1-\lambda_2-ic')(\lambda_1-\lambda_3-ic')(\lambda_2-\lambda_3-ic')\\
&\quad +e^{i(z_1\lambda_2+z_2\lambda_3+z_3\lambda_1)}(\lambda_1-\lambda_3-ic')(\lambda_1-\lambda_2-ic')(\lambda_3-\lambda_2-ic')\\
&\quad -e^{i(z_1\lambda_2+z_2\lambda_1+z_3\lambda_3)}(\lambda_3-\lambda_1-ic')(\lambda_3-\lambda_2-ic')(\lambda_1-\lambda_2-ic')\Big],
\end{aligned}$$

Region II ($z_1 < z_3 < z_2$)

$$\begin{aligned}
&\chi_{II}(z_1, z_2, z_3)\\
&= Ae^{\frac{-i\pi\kappa}{2}} \Big[e^{i(z_1\lambda_1+z_2\lambda_2+z_3\lambda_3)}(\lambda_3-\lambda_2+ic')(\lambda_3-\lambda_1-ic')(\lambda_2-\lambda_1-ic')\\
&\quad -e^{i(z_1\lambda_1+z_2\lambda_3+z_3\lambda_2)}(\lambda_2-\lambda_3+ic')(\lambda_2-\lambda_1-ic')(\lambda_3-\lambda_1-ic')\\
&\quad +e^{i(z_1\lambda_3+z_2\lambda_1+z_3\lambda_2)}(\lambda_2-\lambda_1+ic')(\lambda_2-\lambda_3-ic')(\lambda_1-\lambda_3-ic')\\
&\quad -e^{i(z_1\lambda_3+z_2\lambda_2+z_3\lambda_1)}(\lambda_1-\lambda_2+ic')(\lambda_1-\lambda_3-ic')(\lambda_2-\lambda_3-ic')\\
&\quad +e^{i(z_1\lambda_2+z_2\lambda_3+z_3\lambda_1)}(\lambda_1-\lambda_3+ic')(\lambda_1-\lambda_2-ic')(\lambda_3-\lambda_2-ic')\\
&\quad -e^{i(z_1\lambda_2+z_2\lambda_1+z_3\lambda_3)}(\lambda_3-\lambda_1+ic')(\lambda_3-\lambda_2-ic')(\lambda_1-\lambda_2-ic')\Big],
\end{aligned}$$

Region III ($z_3 < z_1 < z_2$)

$$
\begin{aligned}
&\chi_{III}(z_1, z_2, z_3) \\
&= \; Ae^{\frac{i\pi\kappa}{2}} \left[e^{i(z_1\lambda_1+z_2\lambda_2+z_3\lambda_3)}(\lambda_3-\lambda_2+ic')(\lambda_3-\lambda_1+ic')(\lambda_2-\lambda_1-ic') \right. \\
&\qquad -e^{i(z_1\lambda_1+z_2\lambda_3+z_3\lambda_2)}(\lambda_2-\lambda_3+ic')(\lambda_2-\lambda_1+ic')(\lambda_3-\lambda_1-ic') \\
&\qquad +e^{i(z_1\lambda_3+z_2\lambda_1+z_3\lambda_2)}(\lambda_2-\lambda_1+ic')(\lambda_2-\lambda_3+ic')(\lambda_1-\lambda_3-ic') \\
&\qquad -e^{i(z_1\lambda_3+z_2\lambda_2+z_3\lambda_1)}(\lambda_1-\lambda_2+ic')(\lambda_1-\lambda_3+ic')(\lambda_2-\lambda_3-ic') \\
&\qquad +e^{i(z_1\lambda_2+z_2\lambda_3+z_3\lambda_1)}(\lambda_1-\lambda_3+ic')(\lambda_1-\lambda_2+ic')(\lambda_3-\lambda_2-ic') \\
&\qquad \left. -e^{i(z_1\lambda_2+z_2\lambda_1+z_3\lambda_3)}(\lambda_3-\lambda_1+ic')(\lambda_3-\lambda_2+ic')(\lambda_1-\lambda_2-ic') \right],
\end{aligned}
$$

Region IV ($z_3 < z_2 < z_1$)

$$
\begin{aligned}
&\chi_{IV}(z_1, z_2, z_3) \\
&= \; Ae^{\frac{i3\pi\kappa}{2}} \left[e^{i(z_1\lambda_1+z_2\lambda_2+z_3\lambda_3)}(\lambda_3-\lambda_2+ic')(\lambda_3-\lambda_1+ic')(\lambda_2-\lambda_1+ic') \right. \\
&\qquad -e^{i(z_1\lambda_1+z_2\lambda_3+z_3\lambda_2)}(\lambda_2-\lambda_3+ic')(\lambda_2-\lambda_1+ic')(\lambda_3-\lambda_1+ic') \\
&\qquad +e^{i(z_1\lambda_3+z_2\lambda_1+z_3\lambda_2)}(\lambda_2-\lambda_1+ic')(\lambda_2-\lambda_3+ic')(\lambda_1-\lambda_3+ic') \\
&\qquad -e^{i(z_1\lambda_3+z_2\lambda_2+z_3\lambda_1)}(\lambda_1-\lambda_2+ic')(\lambda_1-\lambda_3+ic')(\lambda_2-\lambda_3+ic') \\
&\qquad +e^{i(z_1\lambda_2+z_2\lambda_3+z_3\lambda_1)}(\lambda_1-\lambda_3+ic')(\lambda_1-\lambda_2+ic')(\lambda_3-\lambda_2+ic') \\
&\qquad \left. -e^{i(z_1\lambda_2+z_2\lambda_1+z_3\lambda_3)}(\lambda_3-\lambda_1+ic')(\lambda_3-\lambda_2+ic')(\lambda_1-\lambda_2+ic') \right],
\end{aligned}
$$

Region V ($z_2 < z_1 < z_3$)

$$
\begin{aligned}
&\chi_{V}(z_1, z_2, z_3) \\
&= \; Ae^{\frac{-i\pi\kappa}{2}} \left[e^{i(z_1\lambda_1+z_2\lambda_2+z_3\lambda_3)}(\lambda_3-\lambda_2-ic')(\lambda_3-\lambda_1-ic')(\lambda_2-\lambda_1+ic') \right. \\
&\qquad -e^{i(z_1\lambda_1+z_2\lambda_3+z_3\lambda_2)}(\lambda_2-\lambda_3-ic')(\lambda_2-\lambda_1-ic')(\lambda_3-\lambda_1+ic') \\
&\qquad +e^{i(z_1\lambda_3+z_2\lambda_1+z_3\lambda_2)}(\lambda_2-\lambda_1-ic')(\lambda_2-\lambda_3-ic')(\lambda_1-\lambda_3+ic') \\
&\qquad -e^{i(z_1\lambda_3+z_2\lambda_2+z_3\lambda_1)}(\lambda_1-\lambda_2-ic')(\lambda_1-\lambda_3-ic')(\lambda_2-\lambda_3+ic') \\
&\qquad +e^{i(z_1\lambda_2+z_2\lambda_3+z_3\lambda_1)}(\lambda_1-\lambda_3-ic')(\lambda_1-\lambda_2-ic')(\lambda_3-\lambda_2+ic') \\
&\qquad \left. -e^{i(z_1\lambda_2+z_2\lambda_1+z_3\lambda_3)}(\lambda_3-\lambda_1-ic')(\lambda_3-\lambda_2-ic')(\lambda_1-\lambda_2+ic') \right],
\end{aligned}
$$

Region VI ($z_2 < z_3 < z_1$)

$$\begin{aligned}
&\chi_{VI}(z_1, z_2, z_3) \\
&= \; Ae^{\frac{i\pi\kappa}{2}} \big[e^{i(z_1\lambda_1+z_2\lambda_2+z_3\lambda_3)}(\lambda_3-\lambda_2-ic')(\lambda_3-\lambda_1+ic')(\lambda_2-\lambda_1+ic') \\
&\qquad -e^{i(z_1\lambda_1+z_2\lambda_3+z_3\lambda_2)}(\lambda_2-\lambda_3-ic')(\lambda_2-\lambda_1+ic')(\lambda_3-\lambda_1+ic') \\
&\qquad +e^{i(z_1\lambda_3+z_2\lambda_1+z_3\lambda_2)}(\lambda_2-\lambda_1-ic')(\lambda_2-\lambda_3+ic')(\lambda_1-\lambda_3+ic') \\
&\qquad -e^{i(z_1\lambda_3+z_2\lambda_2+z_3\lambda_1)}(\lambda_1-\lambda_2-ic')(\lambda_1-\lambda_3+ic')(\lambda_2-\lambda_3+ic') \\
&\qquad +e^{i(z_1\lambda_2+z_2\lambda_3+z_3\lambda_1)}(\lambda_1-\lambda_3-ic')(\lambda_1-\lambda_2+ic')(\lambda_3-\lambda_2+ic') \\
&\qquad -e^{i(z_1\lambda_2+z_2\lambda_1+z_3\lambda_3)}(\lambda_3-\lambda_1-ic')(\lambda_3-\lambda_2+ic')(\lambda_1-\lambda_2+ic')\big] \,,
\end{aligned}$$

where

$$A = \frac{1}{\sqrt{6\prod_{j>k}[(\lambda_j-\lambda_k)^2+c'^2]}} \,.$$

As discussed above for the two particles, the periodic boundary conditions can be imposed in principle on any of the wavefunction arguments. Requiring z_1 to be periodic, $\chi(0, z_2, z_3) = \chi(L, z_2, z_3)$, gives

$$\chi_I(0, z_2, z_3) \; = \; \chi_{VI}(L, z_2, z_3) \,, \quad \text{for } z_2 < z_3 \,, \tag{2.23}$$
$$\chi_{II}(0, z_2, z_3) \; = \; \chi_{IV}(L, z_2, z_3) \,, \quad \text{for } z_3 < z_2 \,. \tag{2.24}$$

Except for the exchange-statistics phase factors, the wavefunctions in the six regions coincide with the wavefunctions of the Bose gas with the δ-function interaction of strength c' (2.15). Therefore, the Bethe equations we obtain are the same as in the bosonic case with the only difference coming from the statistical phase factors. Conditions (2.23) and (2.24) produce six equations each, with only three of them being independent

$$\begin{aligned}
e^{iL\lambda_1} &= e^{-2i\pi\kappa}\left(\frac{\lambda_1-\lambda_2+ic'}{\lambda_1-\lambda_2-ic'}\right)\left(\frac{\lambda_1-\lambda_3+ic'}{\lambda_1-\lambda_3-ic'}\right), \\
e^{iL\lambda_2} &= e^{-2i\pi\kappa}\left(\frac{\lambda_2-\lambda_1+ic'}{\lambda_2-\lambda_1-ic'}\right)\left(\frac{\lambda_2-\lambda_3+ic'}{\lambda_2-\lambda_3-ic'}\right), \\
e^{iL\lambda_3} &= e^{-2i\pi\kappa}\left(\frac{\lambda_3-\lambda_1+ic'}{\lambda_3-\lambda_1-ic'}\right)\left(\frac{\lambda_3-\lambda_2+ic'}{\lambda_3-\lambda_2-ic'}\right).
\end{aligned}$$

These equations can be written in the compact form similar to Eq. (2.21):

$$e^{iL\lambda_j} = e^{-2i\pi\kappa}\prod_{k=1,k\neq j}^{3}\left(\frac{\lambda_j-\lambda_k+ic'}{\lambda_j-\lambda_k-ic'}\right). \tag{2.25}$$

If the periodic boundary conditions are imposed on the second variable, $\chi(z_1, 0, z_3) = \chi(z_1, L, z_3)$, i.e.,

$$\chi_V(0, z_2, z_3) = \chi_{II}(L, z_2, z_3)\,, \quad \text{for } z_1 < z_3\,, \tag{2.26}$$

$$\chi_{VI}(z_1, 0, z_3) = \chi_{III}(z_1, L, z_3)\,, \quad \text{for } z_1 > z_3\,, \tag{2.27}$$

we obtain either from (2.26) or (2.27) the following Bethe equations

$$e^{iL\lambda_j} = \prod_{k=1, k\neq j}^{3} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'} \right). \tag{2.28}$$

Finally, if we impose periodic boundary conditions on the third variable, $\chi(z_1, z_2, 0) = \chi(z_1, z_2, L)$, i.e.,

$$\chi_{III}(z_1, z_2, 0) = \chi_I(z_1, z_2, L)\,, \quad \text{for } z_1 < z_2\,, \tag{2.29}$$

$$\chi_{IV}(z_1, z_2, 0) = \chi_V(z_1, z_2, L)\,, \quad \text{for } z_2 < z_1\,, \tag{2.30}$$

the resulting Bethe equations are

$$e^{iL\lambda_j} = e^{2i\pi\kappa} \prod_{k=1, k\neq j}^{3} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'} \right). \tag{2.31}$$

The difference between the three forms of the Bethe equations (2.25), (2.28), (2.31) means that the periodic boundary conditions imposed on one variable automatically require the twisted boundary conditions on the other variables if one wants to keep the same Bethe equations. Similarly to the case of two particles, this can also be seen directly from the anyonic exchange symmetry (2.7) of the wavefunction. Suppose we set the periodic boundary conditions on the first variable:

$$\chi(0, z_2, z_3) = \chi(L, z_2, z_3)\,. \tag{2.32}$$

Exchanging then the first two variables on both sides of Eq. (2.32) with the help of Eq. (2.7), we get the twisted boundary conditions for the second variable:

$$\chi(z_2, 0, z_3) = \chi(z_2, L, z_3) e^{2i\pi\kappa}\,. \tag{2.33}$$

From (2.33), using again (2.7) we have

$$\chi(z_2, z_3, 0) = \chi(z_2, z_3, L) e^{4i\pi\kappa}\,, \tag{2.34}$$

which are the twisted boundary conditions for the third variable which follow from the periodic conditions on the first. From any of the boundary conditions (2.32), (2.33), (2.34) we obtain the Bethe equations (2.25).

Similarly, periodic boundary conditions on the second variable give the following boundary conditions for the three-anyon wavefunction:

$$\begin{aligned} \chi(0,z_2,z_3) &= \chi(L,z_2,z_3)e^{-2i\pi\kappa}\,, \\ \chi(z_1,0,z_3) &= \chi(z_1,L,z_2)\,, \\ \chi(z_2,z_3,0) &= \chi(z_2,z_3,L)e^{2i\pi\kappa}\,, \end{aligned} \tag{2.35}$$

and the Bethe equations (2.28). The same can be done starting with periodicity in the third variable. As in the case of two particles, we see that imposing periodic boundary conditions on the first and the last variables produces the Bethe equations, (2.25) and (2.31), which differ only by the sign of the statistical parameter κ. As discussed in detail for the two particles, this difference corresponds physically to different imbedding of the 1D loop of anyons into the underlying 2D system. In the two situations, the number of particles enclosed by the trajectories of successive particles z_j, $j = 1, 2, ..., N$, either increases from 0 to $N-1$ or decreases from $N-1$ to 0, as reflected in the corresponding boundary conditions of the multi-anyon wavefunction. In contrast to this, the requirement of periodicity of one of the "internal" variables (e.g., z_2 in the case of three particles) produces the Bethe equations and boundary conditions, e.g. (2.28) and (2.35), that do not have this interpretation. They describe the situations with appropriate non-vanishing external phase shift $\phi \neq 0$, which twists uniformly the boundary conditions of all the variables. In what follows we use the periodic boundary conditions with respect to the first variable of the anyonic wavefunction or introduce the external twist $\phi = -\pi\kappa(N-1)$ which removes the anyonic shift of the quasiparticle momenta. As follows from this discussion discussion, the boundary conditions for the wavefunction of N anyons are given in these two situations by Eqs. (2.20) and the Bethe equations by (2.21).

2.6 Properties of the ground state

Bethe equations (2.21) can also be written as

$$\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j + \pi\kappa(\beta-1)(N-1)\,, \qquad j = 1, \cdots, N\,, \tag{2.36}$$

where

$$\theta(\lambda) = i \ln \left(\frac{ic' + \lambda}{ic' - \lambda} \right) ,$$

and n_j are integers when N is odd and half-integers when N is even.

2.6.1 Twisted boundary conditions

In this case ($\beta = 1$), the Bethe equations are similar to those for the Bose gas with periodic boundary conditions [54, 62] with c' as a coupling constant. The ground state is characterized by the set of integers (half-integers) $n_j = j - (N+1)/2$, so the Bethe equations take the form

$$\lambda_j^B L + \sum_{k=1}^{N} \theta(\lambda_j^B - \lambda_k^B) = 2\pi \left(j - \frac{N+1}{2} \right) , \quad j = 1, \cdots, N . \tag{2.37}$$

From now on the superscript B will mean that the variables and physical quantities are the same as the ones for the Bose gas with periodic boundary conditions and coupling constant c'. In the thermodynamic limit $N, L \to \infty$, $D = N/L =$const, the Bethe roots become dense and fill the symmetric interval $[-q, q]$. The density of roots in this interval obeys the Lieb-Liniger integral equation

$$\rho(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) \rho(\mu) \, d\mu = \frac{1}{2\pi} ,$$

where $K(\lambda, \mu) = \theta'(\lambda - \mu) = 2c'/(c'^2 + (\lambda - \mu)^2)$. The Fermi momentum q can be obtained from the Lieb-Liniger integral equation and the particle density is

$$D = \frac{N}{L} = \int_{-q}^{q} \rho(\lambda) \, d\lambda .$$

Finally, the energy and the momentum of the ground state are

$$E_0^B = L \int_{-q}^{q} \lambda^2 \rho(\lambda) \, d\lambda , \qquad P_0^B = 0 . \tag{2.38}$$

2.6.2 Periodic boundary conditions

This is the case treated in [7–9]. The Bethe equations (2.36) in this case ($\beta = 0$) are similar to those for the Bose gas with twisted boundary conditions:

$$\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j - \pi\kappa(N-1)\,, \qquad j = 1, \cdots, N\,.$$

Introducing the notation $\{[...]\}$ such that

$$\{[x]\} = \gamma\,, \qquad \text{if } x = 2\pi \times \text{ integer } + 2\pi\gamma\,, \quad \gamma \in (-1,1)\,, \tag{2.39}$$

we can describe the ground state by the following set of the Bethe equations:

$$\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi\left(j - \frac{N+1}{2}\right) + 2\pi\delta\,, \qquad j = 1, \cdots, N\,, \tag{2.40}$$

where $\delta = \{[-\pi\kappa(N-1)]\}$. Comparison of Eqs. (2.40) and (2.37) shows that we have the following connection between the Bethe roots for periodic and twisted boundary conditions:

$$\lambda_j = \lambda_j^B + 2\pi\delta/L\,. \tag{2.41}$$

This relation is exact and holds also for the excited states if the (half)integers in the Bethe equations are the same. In the periodic case, the ground state is shifted by $2\pi\delta/L$, so that the Bethe roots are now distributed in the interval $[-q + 2\pi\delta/L, q + 2\pi\delta/L]$, and momentum of the ground state P_0 in general does not vanish:

$$P_0 = \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} (\lambda_i^B + 2\pi\delta/L) = 2\pi D\delta\,. \tag{2.42}$$

The ground-state energy is:

$$E_0 = \sum_{i=1}^{N} \lambda_i^2 = \sum_{i=1}^{N} \left((\lambda_i^B)^2 + \frac{4\pi\delta\lambda_i^B}{L} + \frac{(2\pi\delta)^2}{L^2} \right) = E_0^B + \frac{D(2\pi\delta)^2}{L}\,, \tag{2.43}$$

where we have used that the total momentum in the case of twisted boundary conditions is zero and E_0^B in the thermodynamic limit is given by Eq. (2.38).

2.7 Finite size corrections

In this section, we are going to calculate the finite size corrections for the energy of the ground state and characteristics of the low-lying excitations. Based on the results of this section, we will be able to find the large-distance asymptotics of the correlations functions using conformal field theory. A chemical potential h is added to the Hamiltonian (2.1) throughout this section, so that the total Hamiltonian is

$$H_h = \int_0^L dx \; \{[\partial_x \Psi^\dagger(x)][\partial_x \Psi(x)] + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) - h\Psi^\dagger(x)\Psi(x)\} \, .$$

2.7.1 Finite size corrections for the ground state energy

As we have seen in the previous section, the ground state of the gas of anyons with twisted boundary conditions ($\beta = 1$) is characterized by the same set of Bethe equations as the Bose gas with coupling constant c' and periodic boundary conditions. So in this case we can use the results for the Bose gas [15–18, 54, 77]:

$$E_0^B = L \int_{-q}^{q} \varepsilon_0(\lambda)\rho(\lambda) \; d\lambda - \frac{\pi v_F}{6L} + \mathcal{O}\left(\frac{1}{L^2}\right) , \tag{2.44}$$

where $\varepsilon_0(\lambda) = \lambda^2 - h$ and v_F is the Fermi velocity for the Bose gas with coupling constant c'. In the case of periodic boundary conditions ($\beta = 0$), Eq. (2.43) then gives:

$$E_0 = L \int_{-q}^{q} \varepsilon_0(\lambda)\rho(\lambda) \; d\lambda - \frac{\pi v_F}{6L} + \frac{D(2\pi\delta)^2}{L} + \mathcal{O}\left(\frac{1}{L^2}\right) .$$

2.7.2 Finite size corrections for the low-lying excitations

In our discussion of the low-lying excitations, we consider several different types of excitation processes:

- Addition of a finite number ΔN of particles into the ground state of the system.
- Backscattering: all integers n_j in the set $\{n_j\}$ characterizing the ground-state distribution are shifted by an integer d.

- Particle-hole excitations: the integer n_j that characterizes the particle at the Fermi surface is modified from its value in the ground state distribution by N^+ for the particle with momentum q, (or $q+2\pi\delta/L$, depending on the boundary conditions) or by N^- at the opposite point of the Fermi surface with momentum $-q, (-q+2\pi\delta/L)$.

The central feature of the gas of anyons is that the boundary conditions for the field operators and the wavefunctions depend on the number of particles in the system. This means that any modification of the number of particles in the system changes the Bethe equations and, as a result, the quasiparticle momenta given by the Bethe roots. If we add one particle to the system of N particles, the boundary conditions are:

$$\begin{aligned}
\chi_{N+1}(0, z_2, \cdots, z_N, z_{N+1}) &= e^{-i\pi\beta\kappa(N-1)}\chi_{N+1}(L, z_2, \cdots, z_N, z_{N+1}),\\
\chi_{N+1}(z_1, 0, \cdots, z_N, z_{N+1}) &= e^{i2\pi\kappa}e^{-i\pi\beta\kappa(N-1)}\chi_{N+1}(z_1, L, \cdots, z_N, z_{N+1}),\\
&\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.45)\\
\chi_{N+1}(z_1, z_2, \cdots, 0, z_{N+1}) &= e^{i2(N-1)\pi\kappa}e^{-i\pi\beta\kappa(N-1)}\chi_{N+1}(z_1, z_2 \cdots, L, z_{N+1}),\\
\chi_{N+1}(z_1, z_2, \cdots, z_N, 0) &= e^{i2N\pi\kappa}e^{-i\pi\beta\kappa(N-1)}\chi_{N+1}(z_1, z_2 \cdots, z_N, L),
\end{aligned}$$

and the Bethe equations become

$$e^{i\lambda_j L} = e^{-i\pi\kappa N}e^{i\pi\beta\kappa(N-1)}\prod_{k=1,k\neq j}^{N+1}\left(\frac{\lambda_j-\lambda_k+ic'}{\lambda_j-\lambda_k-ic'}\right).$$

The ground states for N and $N+1$ particles are characterized by the Bethe roots satisfying different equations:

$$\begin{aligned}
\lambda_j L + \sum_{k=1}^{N}\theta(\lambda_j-\lambda_k) &= 2\pi\left(j-\frac{N+1}{2}\right)+2\pi\omega, \quad j=1,\cdots,N, \quad (2.46)\\
\tilde{\lambda}_j L + \sum_{k=1}^{N+1}\theta(\tilde{\lambda}_j-\tilde{\lambda}_k) &= 2\pi\left(j-\frac{N+2}{2}\right)+2\pi\omega', \quad j=1,\cdots,N+1,
\end{aligned}$$

where

$$\omega=0,\ \ \omega'=\kappa/2, \quad \text{and} \quad \omega=\{[-\pi\kappa(N-1)]\},\ \ \omega'=\{[-\pi\kappa N]\},$$

for the twisted ($\beta=1$) and periodic ($\beta=0$) boundary conditions, respectively, and {[...]} is defined by Eq. (2.39). Comparing Eq. (2.46) with Eq. (2.37) we see that

$$\lambda_j = \lambda^B_{jN}+2\pi\omega/L, \qquad \tilde{\lambda}_j=\lambda^B_{j,N+1}+2\pi\omega'/L, \tag{2.47}$$

where λ_{jN}^B are the Bethe roots characterizing the ground state of a gas of N bosons with periodic boundary conditions and coupling constant c'.

Addition of one particle to the system

For excitations of this type we assume that both before and after the addition of a particle, the system is in the ground state. In order to calculate the energy and momentum of this excitation, we use Eq. (2.47) which enables one to express energy and momentum through corrections to the same characteristics of excitations of the Bose gas.

For the energy we get from Eq. (2.47):

$$\begin{aligned}\Delta E(\Delta N = 1) &= \sum_{j=1}^{N+1} \varepsilon_0(\tilde{\lambda}_j) - \sum_{j=1}^{N} \varepsilon_0(\lambda_j) \\ &= \Delta E^B(\Delta N = 1) + (N+1)\left(\frac{2\pi\omega'}{L}\right)^2 - N\left(\frac{2\pi\omega}{L}\right)^2, \end{aligned} \tag{2.48}$$

where $\Delta E^B(\Delta N = 1)$ is the energy of the corresponding bosonic excitation. As known in the literature (see, e.g., [15, 16, 18, 54, 77]) it is convenient to express this energy in terms of the "dressed charge" $Z(\lambda)$:

$$\Delta E^B(\Delta N = 1) = \frac{2\pi v_F}{L}\left(\frac{1}{2\mathcal{Z}}\right)^2, \tag{2.49}$$

where $\mathcal{Z} = Z(q) = Z(-q)$, and $Z(\lambda)$ is defined as solution of the equation

$$Z(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu) Z(\mu)\, d\mu = 1\,. \tag{2.50}$$

From (2.48) and (2.49) we obtain

$$\Delta E(\Delta N = 1) = \frac{2\pi v_F}{L}\left(\frac{1}{2\mathcal{Z}}\right)^2 + (N+1)\left(\frac{2\pi\omega'}{L}\right)^2 - N\left(\frac{2\pi\omega}{L}\right)^2. \tag{2.51}$$

The momentum of the excitation is:

$$\Delta P(\Delta N = 1) = \sum_{j=1}^{N+1} \tilde{\lambda}_j - \sum_{j=1}^{N} \lambda_j = (N+1)\frac{2\pi\omega'}{L} - N\frac{2\pi\omega}{L}\,, \tag{2.52}$$

where we again used the fact that for the ground state of bosons with periodic boundary conditions and any number of particles the total momentum is vanishing.

Backscattering

The uniform shift of the ground-state distribution in a backscattering process can be understood as a jump of some number d of particles between the opposite boundaries of the Fermi surface. The Bethe equations relevant for this process (in the case of N and $N+1$ particles in the ground state) take the form:

$$\begin{aligned}
\lambda_j^d L + \sum_{k=1}^{N} \theta(\lambda_j^d - \lambda_k^d) &= 2\pi \left(j - \frac{N+1}{2} \right) + 2\pi d + 2\pi\omega\,,\;\; j = 1, \cdots, N\,, \\
\tilde{\lambda}_j^d L + \sum_{k=1}^{N+1} \theta(\tilde{\lambda}_j^d - \tilde{\lambda}_k^d) &= 2\pi \left(j - \frac{N+2}{2} \right) + 2\pi d \\
&\qquad + 2\pi\omega', \quad j = 1, \cdots, N+1\,. \qquad (2.53)
\end{aligned}$$

Again, comparison with Eq. (2.37) shows that

$$\lambda_j^d = \lambda_{jN}^B + 2\pi(\omega + d)/L\,, \qquad \tilde{\lambda}_j^d = \lambda_{j,N+1}^B + 2\pi(\omega' + d)/L\,, \tag{2.54}$$

and the ground states are characterized by Eq. (2.47). Using Eqs. (2.47) and (2.54) we get the excitation energy:

$$\begin{aligned}
N \text{ particles:} \qquad \Delta E(d) &= \sum_{j=1}^{N} (\varepsilon_0(\lambda_j^d) - \varepsilon_0(\lambda_j)) \\
&= N \frac{(2\pi\omega + 2\pi d)^2}{L^2} - N \frac{(2\pi\omega)^2}{L^2}\,, \\
N+1 \text{ particles:} \qquad \Delta E(d) &= \sum_{j=1}^{N+1} (\varepsilon_0(\tilde{\lambda}_j^d) - \varepsilon_0(\tilde{\lambda}_j)) \\
&= (N+1) \frac{(2\pi\omega' + 2\pi d)^2}{L^2} - (N+1) \frac{(2\pi\omega')^2}{L^2}\,.
\end{aligned}$$

This result can be rewritten using the relation $\mathcal{Z}^2 = 2\pi D/v_F$ (see [54], Chap. I.9) obtaining

$$\begin{aligned} N \text{ particles:} \quad \Delta E(d) &= \frac{2\pi v_F}{L}\mathcal{Z}^2(d+\omega)^2 - \frac{2\pi v_F}{L}\mathcal{Z}^2\omega^2, \\ N+1 \text{ particles:} \quad \Delta E(d) &= \frac{2\pi v_F}{L}\mathcal{Z}^2(d+\omega')^2 - \frac{2\pi v_F}{L}\mathcal{Z}^2\omega'^2 \\ &\quad + \frac{(2\pi\omega' + 2\pi d)^2}{L^2} - \frac{(2\pi\omega')^2}{L^2}\,. \end{aligned} \tag{2.55}$$

The momentum of the backscattering excitation is simply

$$\Delta P(d) = N(2\pi d/L)\,, \tag{2.56}$$

expression that is valid for any number of particles N.

Particle-Hole Excitations at the Fermi Surface

In this case, the excitations we consider consist in changing the maximal (minimal) n_j in the ground state by $N^{\pm}$. For N particles and "excitation magnitude" N^+ the Bethe equations are

$$\begin{aligned} \lambda_j L + \sum_{k=1}^{N}\theta(\lambda_j - \lambda_k) &= 2\pi\left(j - \frac{N+1}{2}\right) + 2\pi\omega\,, \qquad j = 1, \cdots, N-1\,, \\ \lambda_N L + \sum_{k=1}^{N}\theta(\lambda_N - \lambda_k) &= 2\pi\left(N - \frac{N+1}{2}\right) + 2\pi\omega + 2\pi N^+. \end{aligned} \tag{2.57}$$

From (2.57) we see that the momentum of the excitation N^+ is $\Delta P(N^+) = 2\pi N^+/L$ and, similarly, for the excitation N^- the momentum is $\Delta P(N^-) = -2\pi N^-/L$. These excitation can be considered as a special case of the general particle-hole excitations, and we can use the results of Appendix A for them. Using (A.4) we see that the excitation energy and momentum

$$\Delta E(N^{\pm}) = \frac{2\pi v_F}{L}N^{\pm} + \mathcal{O}\left(\frac{1}{L^2}\right), \qquad \Delta P(N^{\pm}) = \pm\frac{2\pi}{L}N^{\pm}, \tag{2.58}$$

coincide with those for the similar excitations of the Bose gas (see Appendix I.4 of [54]):

$$\Delta E^B(N^{\pm}) = \frac{2\pi v_F}{L}N^{\pm}, \qquad \Delta P^B(N^{\pm}) = \pm\frac{2\pi}{L}N^{\pm}.$$

For $N+1$ particles, the energy and momentum of the excitations are given by the same expressions as in (2.58).

Chapter 3

Asymptotic Behavior of Correlation Functions from Conformal Field Theory

The results of the previous chapter allows us to compute the asymptotics of the correlation functions using the methods of conformal field theory. We will consider the case of twisted boundary conditions ($\beta = 1$), or the periodic boundary conditions ($\beta = 0$) when κ is a integer multiple of $2/(N-1)$, so that the shift in (2.41) vanishes, $\delta = 0$, and the two boundary conditions are equivalent – see (2.19). The main feature of this case that is important for the direct applicability of the conformal field theory approach is that the momentum of the ground state (2.42) of the gas of anyons is zero for these boundary conditions. For general gapless 1+1-dimensional systems, $T = 0$ is a critical point making the correlation functions decay as a power of distance at $T = 0$ but exponentially at $T > 0$. As we have seen in the previous section, the Lieb-Liniger anyonic gas is gapless and the excitation spectrum has a linear dispersion law in the vicinity of the Fermi level. These features support the expectation that the critical behavior of the anyon system is described by conformal field theory (CFT).

CFT is a vast subject and we refer the reader to [12, 23, 33, 50] and Chap. XVIII of [54] for more information. A conformal theory is characterized by the central charge c (not to be confused with the coupling constant in (2.1)) of the underlying Virasoro algebra, and conformal invariance constrains the critical behavior of the systems under consideration. The critical exponents (the powers that characterizes the algebraic decay at $T = 0$) are related to the conformal dimensions of the operators within the CFT, so to obtain the complete information about the critical behavior of the system we need to calculate the central charge and the conformal dimensions of the primary fields.

3.1 Central charge

In order to find the central charge we use the fact that for unitary conformal theories it can be found from the finite-size corrections, specifically the coefficient of the $1/L$ term in the expansion of the ground state energy for $L \to \infty$ [2, 14]:

$$E = L\epsilon_\infty - \frac{\pi v_F}{6L} c + \mathcal{O}\left(\frac{1}{L}\right) . \tag{3.1}$$

Comparing this relation to Eq. (2.44) valid for the boundary conditions we are assuming in this Section, we see that the central charge $c = 1$. The fact that the central charge $c = 1$ means that the critical exponents can depend continuously on the parameters of the model [12, 27, 30].

3.2 Conformal dimensions from finite size effects

Following the original idea of Cardy [22] subsequently developed in [15, 16, 18], we obtain below the conformal dimensions of the conformal fields in the theory from the spectrum of the low-lying excitations described in the previous Chapter. The local fields of the model can be represented as a combination of conformal fields

$$\phi(x,t) = \sum_Q \tilde{\tilde{A}}(Q) \phi_Q(z,\bar{z}) , \tag{3.2}$$

where $\tilde{\tilde{A}}(Q)$ are some coefficients and $z = ix + v_F\tau$, with v_F the Fermi velocity and τ the Euclidean time. The conformal fields are related to excitations with quantum numbers $Q = \{\Delta N, N^\pm, d\}$, where ΔN represents the number of particles created by the field ϕ, and all the fields in the expansion (3.2) should have the same ΔN. The quantum number d gives the number of particles backscattered across the Fermi "sphere", and $N^\pm$ characterizes the change of the maximal or minimal n_j in the Bethe equations from its values in the ground state. While ΔN has to be the same for all the terms in the expansion, d and $N^\pm$ can be different.

For two conformal fields, ϕ_Q and $\phi_{Q'}$, with the same conformal dimensions denoted $\Delta^{\pm}$, their correlation function is given by

$$\langle \phi_Q(z_1,\bar z_1)\phi_{Q'}(z_2,\bar z_2)\rangle = \frac{1}{(z_1-z_2)^{2\Delta^+}(\bar z_1-\bar z_2)^{2\Delta^-}}. \qquad (3.3)$$

Under a conformal transformation $z = z(w), \bar z = \bar z(\bar w)$, it transforms like

$$\langle \phi_Q(w_1,\bar w_1)\phi_{Q'}(w_2,\bar w_2)\rangle = \prod_{j=1}^{2}\left(\frac{\partial z_j}{\partial w_j}\right)^{\Delta^+}\left(\frac{\partial \bar z_j}{\partial \bar w_j}\right)^{\Delta^-}$$
$$\times \langle \phi_Q(z_1(w_1),\bar z_1(\bar w_1))\phi_{Q'}(z_2(w_2),\bar z_2(\bar w_2))\rangle\,. \qquad (3.4)$$

Using the expansion (3.2), the fact that the two conformal fields with different conformal dimensions are orthogonal (their correlation function is zero), and (3.3) we then have:

$$\langle \phi(z_1,\bar z_1)\phi(z_2,\bar z_2)\rangle = \sum_Q \frac{\tilde A(Q)}{(z_1-z_2)^{2\Delta_Q^+}(\bar z_1-\bar z_2)^{2\Delta_Q^-}}\,, \qquad (3.5)$$

which is valid in the whole complex plane without the origin ($z_1 \neq z_2$). Conformal mapping of this plane to a cylinder (periodic strip) with the help of transformation

$$z = e^{2\pi w/L}\,, \quad w = ix + v_F\tau \quad \text{with} \quad 0 < x \le L\,, \qquad (3.6)$$

applied to (3.4) gives

$$\langle \phi(w_1,\bar w_1)\phi(w_2,\bar w_2)\rangle = \sum_Q \tilde A(Q)\left(\frac{\pi/L}{\sinh[\pi(w_1-w_2)/L]}\right)^{2\Delta_Q^+}$$
$$\times\left(\frac{\pi/L}{\sinh[\pi(\bar w_1-\bar w_2)/L]}\right)^{2\Delta_Q^-},$$

with the asymptotics

$$\langle \phi(w_1,\bar w_1)\phi(w_2,\bar w_2)\rangle \sim \sum_Q e^{-\frac{2\pi v_F}{L}(\Delta_Q^+ + \Delta_Q^-)(\tau_1-\tau_2) - i\frac{2\pi}{L}(\Delta_Q^+ - \Delta_Q^-)(x_1-x_2)}\,.$$

Comparison with the spectral decomposition of the correlation function in the periodic strip ($\tau_1 > \tau_2$)

$$\langle\phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle_L = \sum_Q |\langle 0|\phi(0,0)|Q\rangle|^2 e^{-(E_Q-E_0)(\tau_1-\tau_2)-i(P_Q-P_0)(x_1-x_2)}\,,$$

where $|0\rangle$ is the ground state and $E_0\,, P_0$ are the energy and momentum of the ground state, leads to

$$E_Q - E_0 = \frac{2\pi v_F}{L}(\Delta_Q^+ + \Delta_Q^-)\,,\quad P_Q - P_0 = \frac{2\pi}{L}(\Delta_Q^+ - \Delta_Q^-)\,, \tag{3.7}$$

assuming that both the energy and momentum gaps are of order $\mathcal{O}(1/L)$. However, as we have seen in Sect. 2.7, for some of the excitations considered (addition of a particle in the system, $\Delta N = 1$, backscattering processes characterized by d, and particle-hole excitations at the Fermi surface characterized by $N^{\pm}$), the momentum gap is macroscopic. For example, if $Q = \{\Delta N = 0, d \neq 0, N^{\pm} = 0\}$, the momentum gap is $2k_F d$, $k_F \equiv \pi D$, and for $Q = \{\Delta N = 1, d = 0, N^{\pm} = 0\}$ the momentum gap is $-\pi k_F \kappa + \pi\kappa/L$. For these excitations, following [15, 16, 18], the coefficients $\tilde{A}(Q)$ will depend on x as

$$\tilde{A}(Q) = A(Q)e^{ip_Q x}\,, \tag{3.8}$$

where p_Q is the macroscopic part of the momentum gap $P_Q - P_0$. From (3.5) and (3.8) we obtain the generic formula for the asymptotics of correlations functions at $T = 0$

$$\langle\phi(x,t)\phi(0,0)\rangle = \sum_Q \frac{A(Q)e^{ip_Q x}}{(ix + v_F\tau)^{2\Delta_Q^+}(-ix + v_F\tau)^{2\Delta_Q^-}}\,, \tag{3.9}$$

where $\Delta_Q^{\pm}$ can be found from (3.7) and the leading term corresponds to the smallest $\Delta_Q^{\pm}$.

We also can find the low-temperature asymptotics of the correlation functions if we use instead of the conformal mapping (3.6), the mapping

$$z = e^{2\pi T w/v_F}\,,\quad z = x - iv_F\tau\,, \tag{3.10}$$

which differ from (3.6) by interchanging the space and time variables. The computations are similar those described above for the correlation functions in a finite box, and the final result is

$$\langle\phi(x,t)\phi(0,0)\rangle_T = \sum_Q B(Q)e^{ip_Q x}\left(\frac{\pi T/v_F}{\sinh[\pi T(x - iv_F\tau)/v_F]}\right)^{2\Delta_Q^+}$$

$$\times \left(\frac{\pi T/v_F}{\sinh[\pi T(x+iv_F\tau)/v_F]} \right)^{2\Delta_Q^-} . \quad (3.11)$$

This result is valid only at temperatures close to zero.

3.3 Density-density correlation function

In the case of the density correlation function, $\langle j(x,t)j(0,0)\rangle$, where $j(x) = \Psi^\dagger(x)\Psi(x)$, we have $\Delta N = 0$ so the most general excitation is constructed by backscattering d particles and creating a particle-hole pair at the Fermi surface characterized by $N^\pm$. Making use of (2.55,2.56,2.58), we obtain for the energy and momentum gap of the excitation characterized by $Q = \{\Delta N = 0, d, N^\pm\}$:

$$P_{N^\pm,d} - P_0 = 2k_F d + \frac{2\pi}{L}(N^+ - N^-) ,$$

$$E_{N^\pm,d} - E_0 = \frac{2\pi v_F}{L}[(\mathcal{Z}d)^2 + N^+ + N^-] .$$

Here we have taken into account only the terms of order 1 and $\mathcal{O}(1/L)$. Equation (3.7) gives the conformal dimensions

$$2\Delta_Q^\pm = 2N^\pm + (\mathcal{Z}d)^2 , \quad (3.12)$$

and from the general formula (3.9)

$$\langle j(x,t)j(0,0)\rangle - \langle j(0,0)\rangle^2 = \sum_{Q=\{N^\pm,d\}} A(Q) \frac{e^{2ixk_F d}}{(ix+v_F\tau)^{2\Delta_Q^+}(-ix+v_F\tau)^{2\Delta_Q^-}} . \quad (3.13)$$

Defining $\theta \equiv 2\mathcal{Z}^2 = 4\pi D/v_F$, where $\mathcal{Z} = Z(-q) = Z(q)$, and $Z(\lambda)$ given by the integral equation (2.50), the leading terms are

$$\langle j(x,t)j(0,0)\rangle - \langle j(0,0)\rangle^2 = \frac{a}{(ix+v_F\tau)^2} + \frac{a}{(-ix+v_F\tau)^2} + b\frac{\cos(2k_F x)}{|ix+v_F\tau|^\theta} .$$

For equal times, Eq. (3.13) takes the form

$$\langle j(x,0)j(0,0)\rangle - \langle j(0,0)\rangle^2 = \sum_{Q=\{N^\pm,d\}} \hat{A}(Q) \frac{e^{2ixk_F d}}{|x|^{d^2\theta+2N^++2N^-}} . \quad (3.14)$$

The presence of the oscillatory terms in this expression can be explained by the following simple computation [18]:

$$\begin{aligned}\langle j(x,0)j(0,0)\rangle &= \sum_Q \langle 0|j(x,0)|Q\rangle\langle Q|j(0,0)|0\rangle \\ &= \sum_Q |\langle 0|j(0,0)|Q\rangle|^2 e^{i(P_Q-P_0)x} \\ &= \sum_{d=-\infty}^{\infty} e^{i2k_F dx} \sum_{N^\pm} |\langle 0|j(0,0)|d,N^\pm\rangle|^2 e^{\frac{i2\pi x}{L}(N^+-N^-)}\,,\end{aligned}$$

where in the second line, we broke the sum over Q into disjoint sums characterized by different macroscopic momenta. The second part of the sum gives the power-law decay for $k_F^{-1} \ll x \ll L$. The formulae (3.13) and (3.14) are the same as in the case of a Bose gas with coupling constant $c' = c/\cos(\pi\kappa/2)$ and periodic boundary conditions [18] – see Chap. XVII of [54]. This situation is expected, since

$$j(x) = \Psi^\dagger(x)\Psi(x) = \Psi_B^\dagger(x)\Psi_B(x)\,.$$

Finally, from (3.11), the finite temperature density correlation function is

$$\langle j(x,t)j(0,0)\rangle_T = \sum_{Q=\{d,N^\pm\}} B(Q)e^{i2k_F dx}\left(\frac{\pi T/v_F}{\sinh[\pi T(x-iv_F\tau)/v_F]}\right)^{2\Delta_Q^+} \times\left(\frac{\pi T/v_F}{\sinh[\pi T(x+iv_F\tau)/v_F]}\right)^{2\Delta_Q^-}\,, \quad (3.15)$$

with $\Delta_Q^\pm$ given by (3.12).

3.4 Field-field correlation function

In contrast to the density correlators, for the field correlator $\langle\Psi(x,t)\Psi^\dagger(0,0)\rangle$, one has $\Delta N = 1$. For the ground states with N and $N+1$ particles and the boundary conditions considered in this Section the Bethe equations are:

$$\begin{aligned}\lambda_j L + \sum_{k=1}^{N}\theta(\lambda_j-\lambda_k) &= 2\pi\left(j-\frac{N+1}{2}\right)\,, \qquad j=1,\cdots,N\,,\\ \tilde\lambda_j L + \sum_{k=1}^{N+1}\theta(\tilde\lambda_j-\tilde\lambda_k) &= 2\pi\left(j-\frac{N+2}{2}\right)-\pi\kappa\,, \qquad j=1,\cdots,N+1\,.\end{aligned}$$

The shift $-\pi\kappa$ in the second equation implies that the anyonic wavefunctions for N and $N+1$ particles live in two orthogonal sectors of the Hilbert space. The addition of one particle produces in this case a macroscopic change in the momentum, $-\pi k_F \kappa - \pi\kappa/L$, which gives rise to oscillations even in the dominant term of the field correlator.

The most general excitation is obtained by an addition of one particle to the system, followed by the backscattering of d particles and creation of a particle-hole pair at the Fermi surface. Using the results (2.51,2.52,2.55,2.56,2.58) with $\omega = 0, \omega' = -\kappa/2$, we obtain the following expressions for the energy and momentum gaps of an excitation with $Q = \{\Delta N = 1, d, N^\pm\}$ (retaining, as before, the terms of order 1 and $\mathcal{O}(1/L)$):

$$P^{\Delta N=1}_{N^\pm,d} - P_0 = 2k_F(d-\kappa/2) + \frac{2\pi}{L}\left[(d-\kappa/2) + N^+ - N^-\right] ,$$

$$E^{\Delta N=1}_{N^\pm,d} - E_0 = \frac{2\pi v_F}{L}\left[\left(\frac{1}{2\mathcal{Z}}\right)^2 + \mathcal{Z}^2(d-\kappa/2)^2 + N^+ + N^-\right] ,$$

so the conformal dimensions are

$$2\Delta^\pm_Q = 2N^\pm + \left(\frac{1}{2\mathcal{Z}} \pm \mathcal{Z}(d-\kappa/2)\right)^2 . \tag{3.16}$$

From Eq. (3.9), the field correlator is

$$\langle \Psi(x,t)\Psi^\dagger(0,0)\rangle = \sum_{Q=\{N^\pm,d\}} A(Q) \frac{e^{2ik_F(d-\frac{\kappa}{2})x}}{(ix+v_F\tau)^{-2\Delta^+_Q}(-ix+v_F\tau)^{-2\Delta^-_Q}} , \tag{3.17}$$

or in the equal-time case

$$\langle \Psi(x,0)\Psi^\dagger(0,0)\rangle = \sum_{Q=\{N^\pm,d\}} \hat{A}(Q) \frac{e^{2ik_F(d-\frac{\kappa}{2})x}}{|x|^{(d+\frac{\kappa}{2})^2\theta + \frac{1}{\theta} + 2N^+ + 2N^-}} ,$$

where $\theta = 2\mathcal{Z}^2$. Again, we can heuristically justify the presence of the oscillatory terms in the correlation function in the same way as for the density correlator, but for the field correlator, the complete set of states that is inserted between Ψ and $\Psi^\dagger$ is from the sector with $N+1$ particles

$$\begin{aligned}\langle\Psi(x,0)\Psi^\dagger(0,0)\rangle &= \sum_Q \langle 0|\Psi(x,0)|Q\rangle\langle Q|\Psi^\dagger(0,0)|0\rangle \\ &= \sum_Q |\langle 0|\Psi(0,0)|Q\rangle|^2 e^{i(P_Q-P_0)x} \\ &= \sum_{d=-\infty}^{\infty} e^{i2k_F(d-\frac{\kappa}{2})x} \sum_{N^\pm} |\langle 0|\Psi(0,0)|d,N^\pm\rangle|^2 e^{\frac{i2\pi x}{L}(N^+-N^-)}\,.\end{aligned}$$

In this case, the terms of the correlation function containing $e^{i2k_F(d-\kappa/2)x}$ that are responsible for the oscillatory behavior at $x \ll L$, exhibit dependence on the statistical parameter.

Equation (3.17) can be compared to the result of Calabrese and Mintchev [19], who calculated the field correlation function for anyonic gapless systems in the low-momentum regime using the harmonic fluid approach [24, 42], obtaining

$$\langle\Psi^\dagger(x,0)\Psi(0,0)\rangle = D\sum_{d=-\infty}^{\infty} b_d \frac{e^{-2i(d+\frac{\kappa}{2})k_F x} e^{-2i(m+\frac{\kappa}{2})\pi\epsilon(x)/2}}{(Dc(x))^{(d+\frac{\kappa}{2})^2 2K+\frac{1}{2K}}}\,,$$

where D is the density, b_d unknown non-universal amplitudes, $c(x) = L\sin(\pi x/L)$, and K is a universal parameter that can be expressed in terms of the phenomenological velocity parameters v_N, v_J as $K = \sqrt{v_J/v_N}$. For the Lieb-Liniger anyons,

$$K = \frac{2\pi D}{v_F} = \frac{\theta}{2}\,. \tag{3.18}$$

They have checked their results in the limit $c\to\infty, K=1$ against the exact results of Santachiara *et al.* [73], who calculated the asymptotic behavior of the field correlator as a Toeplitz determinant, which is a generalization for anyonic statistics of a result obtained by Lenard in [58]. We see that our conformal field theory approach agrees with the leading asymptotics produced by the harmonic liquid approximation but also gives the higher-order terms in the large-distance expansion.

Using the conformal mapping (3.10) that leads to general Eq. (3.11), we find also the finite-temperature field correlator:

$$\langle\Psi(x,t)\Psi^\dagger(0,0)\rangle_T = \sum_{Q=\{d,N^\pm\}} B(Q)e^{i2k_F(d+\frac{\kappa}{2})x}\left(\frac{\pi T/v_F}{\sinh[\pi T(x-iv_F\tau)/v_F]}\right)^{2\Delta_Q^+}$$

$$\times \left(\frac{\pi T/v_F}{\sinh[\pi T(x + iv_F\tau)/v_F]} \right)^{2\Delta_Q^-} , \quad (3.19)$$

where $\Delta_Q^\pm$ is given by (3.16).

Chapter 4

Anyonic Generalization of Lenard's Formula

In this chapter we are starting the rigorous investigation of the field-field correlation function for impenetrable anyons. As a first step we are going to compute the anyonic generalization of Lenard's formula. In [59], Lenard was able to show, using the Fermi-Bose mapping for wavefunctions, that the reduced density matrices (correlation functions) of bosons interacting with hard-core potentials can be expressed in terms of reduced density matrices of free-fermions and vice-versa. In the particular case of zero-point interactions, this result can be summarized in the following form: n-particle RDMs of impenetrable bosons are given by the n-th Fredholm minor of an integral operator whose kernel is the Fourier transform of the Fermi distribution. In [67], making use of the Anyon-Fermi and Anyon-Bose mappings for wavefunctions, we were able to generalize this result for arbitrary statistics. In this case, again, the n-particle RDMs of impenetrable anyons are given by the n-th Fredholm minor of an integral operator whose kernel is the Fourier transform of the Fermi distribution, however, now the constant in front of the integral operator depends on the statistics parameter κ and is given by $(1+e^{i\pi\kappa})/\pi$ which can be contrasted with the constant in the impenetrable bosons case which is $2/\pi$. Another interesting feature of the anyonic RDM is the fact that in this case they depend on the sign of the difference of the arguments.

Compared with the technique presented in the next chapter, the anyonic generalization of Lenard's formula, has the advantage of giving results also for n-particle correlation functions. The disadvantage is that it does not address time-dependent correlation functions. The results presented in this chapter are based on [67].

4.1 From correlation functions to reduced density matrices

The one-dimensional anyons considered in this chapter are characterized by anyonic fields $\Psi^\dagger(x), \Psi(x)$ which obey the following commutation relations

$$\Psi(x_1)\Psi^\dagger(x_2) = e^{-i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2)\Psi(x_1) + \delta(x_1 - x_2)\,, \tag{4.1}$$

$$\Psi^\dagger(x_1)\Psi^\dagger(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2)\Psi^\dagger(x_1)\,, \tag{4.2}$$

$$\Psi(x_1)\Psi(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi(x_2)\Psi(x_1)\,. \tag{4.3}$$

Here κ is the statistics parameter, and $\epsilon(x) = x/|x|,\ \epsilon(0) = 0$. The commutation relations become bosonic for $\kappa = 0$ and fermionic for $\kappa = 1$. For an arbitrary Hamiltonian of the anyons confined to the interval $V = [-L/2, L/2]$, the N-particle eigenstates are defined as

$$|\Psi_N(\{\lambda\})\rangle = \frac{1}{\sqrt{N!}}\int_V dz_1\cdots\int_V dz_N\ \chi_N^a(z_1,\cdots,z_N|\{\lambda\})\Psi^\dagger(z_N)\cdots\Psi^\dagger(z_1)|0\rangle\,, \tag{4.4}$$

$$\langle\Psi_N(\{\lambda\})| = \frac{1}{\sqrt{N!}}\int_V dz_1\cdots\int_V dz_N\ \langle 0|\Psi(z_1)\cdots\Psi(z_N)\chi_N^{*a}(z_1,\cdots,z_N|\{\lambda\})\,, \tag{4.5}$$

where χ_N^a are the (norm one) quantum-mechanical wavefunctions of N anyons, and $\{\lambda\}$ is a set of quantum numbers specifying the state. The wavefunctions χ have the anyonic symmetry

$$\chi_N^a(z_1,\cdots,z_i,z_{i+1},\cdots,z_N) = e^{i\pi\kappa\epsilon(z_i-z_{i+1})}\chi_N^a(z_1,\cdots,z_{i+1},z_i,\cdots,z_N)\,, \tag{4.6}$$

that reflects the field commutation relations. We are interested in computing the finite-temperature correlation functions of anyonic fields. The simplest example of these correlators is

$$\langle\Psi^\dagger(x')\Psi(x)\rangle\,.$$

In the grand canonical ensemble characterized by temperature T and chemical potential h, the field correlation function is given by following relation

$$\langle\Psi^\dagger(x')\Psi(x)\rangle_{T,h} = \sum_{N=1}^{\infty}\sum_{\{\lambda\}} e^{hN/T}\frac{e^{-E(\{\lambda\})/T}}{Z(h,V,T)}\langle\Psi_N(\{\lambda\})|\Psi^\dagger(x')\Psi(x)|\Psi_N(\{\lambda\})\rangle\,, \tag{4.7}$$

where $E(\{\lambda\})$ is the energy of the eigenstate with quantum numbers $\{\lambda\}$, and $Z(h, V, T)$ is the grand-canonical partition function

$$Z(h, V, T) = \sum_{N=0}^{\infty} \sum_{\{\lambda\}} e^{hN/T} e^{-E(\{\lambda\})/T} . \quad (4.8)$$

As shown in the Appendix B, the correlator at a fixed number of particles N is given by an overlap integral of the corresponding wavefunction

$$\langle \Psi_N(\{\lambda\}) | \Psi^\dagger(x') \Psi(x) | \Psi_N(\{\lambda\}) \rangle = N \int_V dz_1 \cdots \int_V dz_{N-1}$$
$$\times \chi_N^{*a}(z_1, \cdots, z_{N-1}, x' | \{\lambda\}) \chi_N^a(z_1, \cdots, z_{N-1}, x | \{\lambda\}) , \quad (4.9)$$

so that the correlation function (4.7) can be written as

$$\langle \Psi^\dagger(x') \Psi(x) \rangle_{T,h} = \sum_{N=1}^{\infty} \sum_{\{\lambda\}} e^{hN/T} \frac{e^{-E(\{\lambda\})/T}}{Z(h, V, T)} N \int_V dz_1 \cdots \int_V dz_{N-1}$$
$$\times \chi_N^{*a}(z_1, \cdots, z_{N-1}, x' | \{\lambda\}) \chi_N^a(z_1, \cdots, z_{N-1}, x | \{\lambda\}) . \quad (4.10)$$

We will also be interested in a class of $2n$-point field correlation functions at finite temperature:

$$\langle \Psi^\dagger(x'_n) \cdots \Psi^\dagger(x'_1) \Psi(x_1) \cdots \Psi(x_n) \rangle_{T,h} . \quad (4.11)$$

These correlators can be expressed similarly to Eq. (4.10)

$$\langle \Psi^\dagger(x'_n) \cdots \Psi^\dagger(x'_1) \Psi(x_1) \cdots \Psi(x_n) \rangle_{T,h} =$$
$$\sum_{N=n}^{\infty} \sum_{\{\lambda\}} e^{hN/T} \frac{e^{-E(\{\lambda\})/T}}{Z(h, V, T)} \frac{N!}{(N-n)!} \int_V dz_1 \cdots \int_V dz_{N-n}$$
$$\times \chi_N^{*a}(z_1, \cdots, z_{N-n}, x'_1, \cdots, x'_n | \{\lambda\}) \chi_N^a(z_1, \cdots, z_{N-n}, x_1, \cdots, x_n | \{\lambda\}) . \quad (4.12)$$

As one can see from (4.10) and (4.12), the correlation functions are obtained as a combination of the wavefunctions and ensemble probabilities. In this context, it is useful, similarly to the case of fermionic or bosonic particles, to introduce the reduced density matrices of anyons:

Definition 1. *For a statistical ensemble characterized by the probabilities* $p^N_{\{\lambda\}}$, *the anyonic n-particle reduced density matrix is defined as*

$$(x_1, \cdots, x_n|\rho_n^a|x_1', \cdots, x_n') = \sum_{N=n}^{\infty}\sum_{\{\lambda\}} p_{\{\lambda\}}^N \frac{N!}{(N-n)!}\int_V dz_1 \cdots \int_V dz_{N-n}$$

$$\times\, \chi_N^{*a}(z_1, \cdots, z_{N-n}, x_1', \cdots, x_n'|\{\lambda\})\chi_N^a(z_1, \cdots, z_{N-n}, x_1, \cdots, x_n|\{\lambda\})\,, \quad (4.13)$$

where the wavefunctions $\chi_{N,\{\lambda\}}^a$ *are normalized to one.*

If the probabilities $p_{\{\lambda\}}^N$ coincide with those in the grand canonical ensemble, $p_{\{\lambda\}}^N = e^{Nh/T}e^{-E(\{\lambda\})/T}/Z(h,V,T)$, the one-particle reduced density matrix is just the 2-point correlator (4.10), and the n-particle reduced density matrix is the particular $2n$-point correlator (4.12). These relations are exactly the same as in the case of bosonic and fermionic statistics. A particular "anyonic" feature of the Definition 1, is the fact that we integrate over the first $N-n$ arguments of the wavefunctions. In the case of bosonic and fermionic reduced density matrices, integration over any subset of the $N-n$ out of N arguments produces the same result due to the parity of the wavefunctions. This is not the case for the reduced density matrices of anyons due to the anyonic symmetry (4.6) which in general, e.g., in the periodic or quasi-periodic situation ("anyons on a ring") makes different arguments of the wavefunctions inequivalent – see discussion in the next Section.

Under a certain set of conditions, which is also made precise in the next Section, there is a correspondence between the anyonic and the fermionic or bosonic wavefunctions. This correspondence will be used later to express the anyonic reduced density matrices as expansions in terms of fermionic or bosonic ones.

4.2 Anyon-Fermi and Anyon-Bose mapping

To establish the correspondence between the wavefunctions of anyons and fermions or bosons we define the two functions which essentially incorporate statistical properties of the wavefunctions of different statistics in one dimension:

$$A_\kappa(z_1, \cdots, z_N) = e^{i\pi\kappa\sum_{j<k}\epsilon(z_j-z_k)/2} \qquad (4.14)$$

and

$$B(z_1, \cdots, z_N) = \prod_{j>k}\epsilon(z_j - z_k)\,, \qquad (4.15)$$

where the notations are the same as in Eq. (2.2).

The mapping between anyons and fermions or bosons is analogous to the Bose-Fermi mapping discovered in [34], where it was noticed that any wavefunction of N fermions has a bosonic counterpart given by

$$\chi^b(z_1, \cdots, z_N) = B(z_1, \cdots, z_N)\chi^f(z_1, \cdots, z_N)\,. \tag{4.16}$$

This correspondence is valid under very general conditions, with no restrictions on the external or particle-particle interaction potential, except for the requirement of the hard-core condition which should make the bosons impenetrable. For particles confined to a box with "hard wall" boundary conditions (BC), bosonic and fermionic wavefunctions satisfy the same BC. In this case, if χ^f is an eigenfunction of the Hamiltonian, then χ^b is also an eigenfunction with the same eigenvalue. However, in the case of a ring of length L with periodic BC for bosons, the BC for the fermions are in general different and given by

$$\chi^f(0, \cdots, z_N) = (-1)^{N-1}\chi^f(L, \cdots, z_N)\,. \tag{4.17}$$

For even N, when Eq. (4.17) means that the BC for fermions and bosons are different by a phase shift π, the relation between the eigenenergies of the fermionic and bosonic systems is less direct. Since for non-coincident coordinates $B^2 \equiv 1$, the Bose-Fermi mapping (4.16) is symmetric and remains true if the superscripts b and f are interchanged.

4.2.1 Anyon-Fermi mapping

It is straightforward to see that similarly to the Bose-Fermi mapping (4.16), the wavefunction with anyonic symmetry (4.6) can be obtained by multiplication of a fermionic wavefunction with the statistics factors (4.14) and (4.15) [5, 35]:

$$\chi^a(z_1, \cdots, z_N) = A_\kappa(z_1, \cdots, z_N)B(z_1, \cdots, z_N)\chi^f(z_1, \cdots, z_N)\,, \tag{4.18}$$

Besides the anyonic symmetry (4.6), the wavefunction χ^a (4.18) satisfies the condition

$$\chi^a(z_1, \cdots, z_N)|_{z_i=z_j} = 0 \quad \text{for all } \{i, j\} \in \{1, \cdots, N\}\,. \tag{4.19}$$

This means that the correspondence (4.18) is valid as long as potential energy contains a hard-core part which ensures that the anyons are impenetrable and condition (4.19) is indeed satisfied. Other properties of the Anyon-Fermi mapping (4.18) are similar to those of the Bose-Fermi mapping. It is valid for an arbitrary form of the potential energy in the particle Hamiltonian. When the particles are confined to an interval with *"hard wall" boundary* *conditions*, the fermionic and anyonic systems both satisfy the same BC of the wavefunctions vanishing at the ends of the interval. In this case, if χ^f is an eigenfunction of the particle Hamiltonian, then χ^a is also an eigenfunction of this Hamiltonian with the same eigenvalue. This follows from the fact that the statistics factors (4.14) and (4.15) are constant everywhere except for the points of coincident coordinates, where the wavefunctions vanish.

When particles are confined to a ring with *periodic or quasi-periodic BC*, the properties of the Anyon-Fermi mapping are more complicated. In this case, the anyonic wavefunction will have different boundary conditions for each of its coordinate (see Chap. 2), the difference being given by an extra phase shift that depends on the statistics parameter κ. Specifically, if the fermion wave function obeys some generic quasi-periodic BC (the same in all coordinates) which can be written as

$$\chi^f(0, \cdots, z_N) = (-1)^{N-1} e^{-i\phi} \chi^f(L, \cdots, z_N)\,, \tag{4.20}$$

then the anyonic wavefunction obeys the following BC in its different arguments

$$\begin{aligned} \chi^a(0, z_2, \cdots, z_N) &= e^{-i\overline{\phi}}\ \chi^a(L, z_2, \cdots, z_N)\,, \\ \chi^a(z_1, 0, \cdots, z_N) &= e^{i(2\pi\kappa - \overline{\phi})} \chi^a(z_1, L, \cdots, z_N)\,, \\ &\vdots \\ \chi^a(z_1, z_2, \cdots, 0) &= e^{i(2(N-1)\pi\kappa - \overline{\phi})} \chi^a(z_1, z_2 \cdots, L)\,, \end{aligned} \tag{4.21}$$

where $\overline{\phi} = \phi + \pi\kappa(N-1)$. As for the Bose-Fermi mapping (4.16) with even N, the anyonic and fermionic eigenenergies are not related directly in the situation of a ring with quasiperiodic BC. In physics terms, this difference between anyons and fermions corresponds to the statistical magnetic flux $\pi\kappa(N-1)$ through the ring produced by N one-dimensional anyons of statistics κ.

The Anyon-Fermi mapping (4.18) is not symmetric. The inverse relation can be written as

$$\chi^f(z_1, \cdots, z_N) = A_{-\kappa}(z_1, \cdots, z_N) B(z_1, \cdots, z_N) \chi^a(z_1, \cdots, z_N)\,. \tag{4.22}$$

4.2.2 Anyon-Bose mappping

The Anyon-Bose mapping was historically the first mapping of this kind introduced for one-dimensional anyons in [56]:

$$\chi^a(z_1, \cdots, z_N) = A_\kappa(z_1, \cdots, z_N) \chi^b(z_1, \cdots, z_N)\,. \tag{4.23}$$

The wavefunction χ^a in (4.23) has the correct anyonic symmetry (4.6), and in contrast to Anyon-Fermi mapping (4.18), need not vanish when a pair of coordinates coincide. It should be noted, however, that without this hard-core condition, the discontinuity of the statistics factor A_κ (4.14) at coinciding coordinates translates into discontinuity of the wavefunctions (4.23). In this case one needs an additional condition regularizing the wavefunctions. Also, without the hard-core condition, the statistics factor A changes substantially the behavior of the wavefunctions at the points of coincident coordinates (for instance, if the particle-particle interaction is δ-functional, statistics renormalizes the interaction strength) and the energy eigenvalues of the bosonic and anyonic problems are different regardless of the boundary conditions.

Other properties of the Anyon-Bose mapping (4.23) are very similar to those of the Anyon-Fermi mapping. It is valid for arbitrary potential energy. For particles in a box with “hard wall” BC, both wavefunctions (4.23) satisfy the same condition $\chi^a = 0$ and $\chi^b = 0$ at the boundary. For particles on a ring with generic quasi-periodic BC for bosons that can be written as

$$\chi^b(0, \cdots, z_N) = e^{-i\phi}\chi^b(L, \cdots, z_N) \tag{4.24}$$

(and have the same form for all other arguments of χ^b), the anyonic wavefunction obeys the following BC

$$\begin{aligned} \chi^a(0, z_2, \cdots, z_N) &= e^{-i\overline{\phi}} \quad \chi^a(L, z_2, \cdots, z_N)\,, \\ \chi^a(z_1, 0, \cdots, z_N) &= e^{i(2\pi\kappa-\overline{\phi})}\chi^a(z_1, L, \cdots, z_N)\,, \\ &\vdots \\ \chi^a(z_1, z_2, \cdots, 0) &= e^{i(2(N-1)\pi\kappa-\overline{\phi})}\chi^a(z_1, z_2 \cdots, L)\,, \end{aligned} \tag{4.25}$$

where $\overline{\phi} = \phi + \pi\kappa(N-1)$. The Anyon-Bose mapping is also not symmetric. The inverse of (4.23) is

$$\chi^b(z_1, \cdots, z_N) = A_{-\kappa}(z_1, \cdots, z_N)\chi^a(z_1, \cdots, z_N)\,. \tag{4.26}$$

4.3 An important theorem

In this section, we consider an arbitrary statistical ensemble in which the states $\chi^a_{N,\{\lambda\}}$ occur with probabilities $p^N_{\{\lambda\}}$. The anyons are assumed to be confined to an interval $V = [-L/2, L/2]$, and wavefunctions are normalized to 1: $\|\chi^a_{N,\{\lambda\}}\| = 1$. Our goal is to establish a relation between the reduced density matrices ρ^a_n of anyons (4.13) and similarly defined reduced density matrices of bosons ρ^b_m and fermions ρ^f_p. The fermionic and bosonic states that correspond to the anyonic states $\chi^a_{N,\{\lambda\}}$ in the Anyon-Fermi (4.18) or Anyon-Bose (4.23) mappings have similarly normalized wavefunctions, and we assume that they have the same probabilities $p^N_{\{\lambda\}}$. This assumption is natural under the conditions (discussed in the previous Section) for which the energies of the states of different statistics are the same, as they are, for instance, when the wavefunctions satisfy the "hard wall" boundary conditions and hard-core condition on the particle-particle interaction. The relation between the reduced density matrices is established by the following theorem:

Theorem 1. *Let $x_1, \cdots, x_n, x'_1, \cdots, x'_n$ be $2n$ coordinates in the interval V, and $O_\pm$ are the parts of the space of these coordinates in which they are ordered, respectively, as $x_1 < x'_1 < \cdots < x_n < x'_n$ and $x'_1 < x_1 < \cdots < x'_n < x_n$. For the O_+ ordering, one can define the subset of V: $I_+ = [x_1, x'_1] \cup \cdots \cup [x_n, x'_n] \subset V$, and the subset $I_- = [x'_1, x_1] \cup \cdots \cup [x'_n, x_n] \subset V$ for ordering as in O_-. If the conditions of validity of the Anyon-Fermi (4.18) or the Anyon-Bose mapping (4.23) are fulfilled, the reduced density matrices of anyons can be expressed then in terms of the reduced density matrices of fermions as*

$$(x_1, \cdots, x_n|\rho^a_n|x'_1, \cdots, x'_n)_\pm = A_{-\kappa}(x'_1, \cdots, x'_n)B(x'_1, \cdots, x'_n)$$
$$\times A_\kappa(x_1, \cdots, x_n)B(x_1, \cdots, x_n)\sum_{j=0}^{\infty}(-1)^j\frac{(1+e^{\pm i\pi\kappa})^j}{j!}$$
$$\int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_j \; (x_1, \cdots, x_n, z_1, \cdots, z_j|\rho^f_{n+j}|x'_1, \cdots, x'_n, z_1, \cdots, z_j) \quad (4.27)$$

or bosons as

$$(x_1, \cdots, x_n|\rho^a_n|x'_1, \cdots, x'_n)_\pm = A_{-\kappa}(x'_1, \cdots, x'_n)A_\kappa(x_1, \cdots, x_n)$$
$$\times\sum_{j=0}^{\infty}(-1)^j\frac{(1-e^{\pm i\pi\kappa})^j}{j!}\int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_j$$
$$(x_1, \cdots, x_n, z_1, \cdots, z_j|\rho^b_{n+j}|x'_1, \cdots, x'_n, z_1, \cdots, z_j)\,. \quad (4.28)$$

The subscript $\pm$ in these expressions specifies whether $x_1, \cdots, x_n, x'_1, \cdots, x'_n$ are ordered as in O_+ or O_-.

Proof. The proof follows that of Lenard [59], generalizing it to the anyonic statistics. As a first step, we need a preliminary result.

Lemma 1. *For any symmetric function $f(z_1, \cdots, z_n)$ and a constant α,*

$$\int_V dz_1 \cdots \int_V dz_n \; \alpha^{\sigma(I_\pm)} f(z_1, \cdots, z_n) = \sum_{j=0}^{n} C_j^n (-1+\alpha)^j$$

$$\times \int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_j \int_V dz_{j+1} \cdots \int_V dz_n f(z_1, \cdots, z_n), \quad (4.29)$$

where $C_j^n = \frac{n!}{(n-j)!j!}$ and $\sigma(I_\pm)$ is the number of variables $z_1, \cdots, z_n$ contained in $I_\pm$.

Proof. The L.H.S. of (4.29) can be written explicitly as

$$Q = \sum_{m=0}^{n} C_m^n \alpha^m \int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_m \int_{V\setminus I_\pm} dz_{m+1} \cdots \int_{V\setminus I_\pm} dz_n f(z_1, \cdots, z_n)\,,$$

and combined with an obvious relation $\int_{V\setminus I_\pm} dz_i = \int_V dz_i - \int_{I_\pm} dz_i\,,\ (i = m+1, \cdots, n)$ can be further transformed into

$$Q = \sum_{m=0}^{n} C_m^n \alpha^m \sum_{k=0}^{n-m} C_k^{n-m} (-1)^k \int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_{m+k} \int_V dz_{m+k+1} \cdots \int_V dz_n$$

$$\times f(z_1, \cdots, z_n)\,.$$

Collecting the terms in this expression with the same $j = m + k$, we obtain the desired result

$$Q = \sum_{j=0}^{n} C_j^n (-1+\alpha)^j \int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_j \int_V dz_{j+1} \cdots \int_V dz_n f(z_1, \cdots, z_n)\,.$$

□

Now we can prove the Theorem 1 starting with (4.27). Using the Anyon-Fermi mapping (4.18) we have

$$(x_1, \cdots, x_n|\rho_n^a|x_1', \cdots, x_n')_\pm =$$

$$\sum_{N=n}^{\infty} \sum_{\{\lambda\}} p_{\{\lambda\}}^N \frac{N!}{(N-n)!} \int_V dz_1 \cdots \int_V dz_{N-n} C(x_1, \cdots, x_n, x_1', \ldots, x_n')_\pm$$

$$\times \chi_N^{*f}(z_1, \cdots, z_{N-n}, x_1', \cdots, x_n'|\{\lambda\}) \chi_N^f(z_1, \cdots, z_{N-n}, x_1, \cdots, x_n|\{\lambda\})\,,$$

where

$$C(x_1, \cdots, x_n, x'_1, \cdots, x'_n)_\pm =$$

$$A_{-\kappa}(x'_1, \cdots, x'_n) B(x'_1, \cdots, x'_n) A_\kappa(x_1, \cdots, x_n) B(x_1, \cdots, x_n)$$

$$\times \prod_{j=1}^{n} \prod_{i=1}^{N-n} e^{-i\pi\kappa\epsilon(z_i - x'_j)/2} e^{i\pi\kappa\epsilon(z_i - x_j)/2} \epsilon(x'_j - z_i) \epsilon(x_j - z_i) .$$

One can see directly that

$$\prod_{j=1}^{n} e^{-i\pi\kappa\epsilon(z - x'_j)/2} e^{i\pi\kappa\epsilon(z - x_j)/2} \epsilon(x'_j - z) \epsilon(x_j - z) = \begin{cases} -e^{\pm i\pi\kappa}, & z \text{ in } I_\pm , \\ 1, & z \text{ not in } I_\pm . \end{cases}$$

This means that

$$C(x_1, \cdots, x_n, x'_1, \cdots, x'_n)_\pm = A_{-\kappa}(x'_1, \cdots, x'_n) B(x'_1, \cdots, x'_n)$$

$$\times A_\kappa(x_1, \cdots, x_n) B(x_1, \cdots, x_n) (-e^{\pm i\pi\kappa})^{\sigma'(I_\pm)} ,$$

where $\sigma'(I_\pm)$ is the number of variables $z_1, \cdots, z_{N-n}$ in $I_\pm$. Applying now Lemma 1 with $\alpha = -e^{\pm i\pi\kappa}$, we obtain for the anyonic reduced density matrices

$$(x_1, \cdots, x_n | \rho^a_n | x'_1, \cdots, x'_n)_\pm = A_{-\kappa}(x'_1, \cdots, x'_n) B(x'_1, \cdots, x'_n) A_\kappa(x_1, \cdots, x_n)$$

$$B(x_1, \cdots, x_n) \sum_{N=n}^{\infty} \sum_{\{\lambda\}} p^N_{\{\lambda\}} \frac{N!}{(N-n)!} \sum_{j=0}^{N-n} C^{N-n}_j (-1)^j (1 + e^{\pm i\pi\kappa})^j$$

$$\times \int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_j \int_V dz_{j+1} \cdots \int_V dz_{N-n} \; \chi^{*f}_{N,\{\lambda\}} \chi^f_{N,\{\lambda\}} .$$

Interchanging the order of summations, one can notice that the sum over N and $\{\lambda\}$ is precisely ρ^f_{n+j}. Therefore finally

$$(x_1, \cdots, x_n | \rho^a_n | x'_1, \cdots, x_n)_\pm = A_{-\kappa}(x'_1, \cdots, x'_n) B(x'_1, \cdots, x'_n) A_\kappa(x_1, \cdots, x_n)$$

$$B(x_1, \cdots, x_n) \sum_{j=0}^{\infty} (-1)^j \frac{(1 + e^{\pm i\pi\kappa})^j}{j!}$$

$$\times \int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_j \; (x_1, \cdots, x_n, z_1, \cdots, z_j | \rho^f_{n+j} | x'_1, \cdots, x'_n, z_1, \cdots, z_j) .$$

The proof of (4.28) is similar. In this case, we use the Anyon-Bose mapping

(4.23), and the $C_\pm$ function is

$$C(x_1,\cdots,x_n,x'_1,\cdots,x'_n)_\pm = A_{-\kappa}(x'_1,\cdots,x'_n)A_\kappa(x_1,\cdots,x_n) \prod_{j=1}^{n}\prod_{i=1}^{N-n} e^{-i\pi\kappa\epsilon(z_i-x'_j)/2}e^{i\pi\kappa\epsilon(z_i-x_j)/2}\,.$$

This means that we can use Lemma 1 with $\alpha = e^{\pm i\pi\kappa}$. Interchanging the order of summation and identifying the bosonic reduced density matrices ρ^b_{n+j} we obtain (4.28). □

The results of Theorem 1 do not depend on the statistical ensemble used in the computation of the reduced density matrices as long as the particles are subject to the hard-wall boundary conditions making the state energies independent of the statistics. They also do not depend on the form of the interparticle potential beyond the need for the hard-core part which ensures that the wavefunctions satisfy the hard-core condition. If both of these conditions are satisfied, we can see from (4.27) and (4.28) that there is also no explicit dependence on the length L of the confining box V, and the results remain valid in the thermodynamic limit $L \to \infty$.

4.4 Lenard's formula for impenetrable anyons

The Anyon-Fermi relation derived above for the reduced density matrices is particularly useful in the situation when the radius of the hard-core interaction is vanishingly small, and no other interactions are present. In this case, the fermionic problem is identical to free fermions, since the hard-core potential of zero radius effectively vanished due to antisymmetry of the wavefunctions. The reduced density matrices ρ^f_n coincide then with those of free fermions [59] (see Appendix D):

$$(x_1,\cdots,x_n|\rho^f_n|x'_1,\cdots,x'_n) = \frac{1}{\pi^n}\theta_T\begin{pmatrix} x_1,\cdots,x_n \\ x'_1,\cdots,x'_n \end{pmatrix}, \tag{4.30}$$

where $\theta_T(x,y)/\pi$ is the Fourier transform of the Fermi distribution function:

$$\theta_T(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} dk \frac{e^{ik(x-y)}}{1+e^{(k^2-h)/T}}\,. \tag{4.31}$$

At $T = 0$ we have

$$\theta_0(x,y) = \frac{\sin q(x-y)}{x-y}\,, \tag{4.32}$$

where $q = \sqrt{h}$ is the Fermi momentum.

Applying Theorem 1 and (D.4) we have

$$(x_1, \cdots, x_n|\rho_n^a|x_1', \cdots, x_n')_\pm = C \sum_{j=0}^{\infty} (-1)^j \frac{(1 + e^{\pm i\pi\kappa})^j}{j!}$$

$$\times \int_{I_\pm} dz_1 \cdots \int_{I_\pm} dz_j \, \frac{1}{\pi^{n+j}} \theta_T \begin{pmatrix} x_1, \cdots, x_n, & z_1, \cdots, z_j \\ x_1', \cdots, x_n', & z_1, \cdots, z_j \end{pmatrix},$$

where

$$C(x_1', \cdots, x_n) \equiv A_{-\kappa}(x_1', \cdots, x_n') B(x_1', \cdots, x_n') A_\kappa(x_1, \cdots, x_n) B(x_1, \cdots, x_n) \tag{4.33}$$

and the subscript $\pm$ specifies particular ordering of $x_1, \cdots, x_n, x_1', \cdots, x_n'$ as in Theorem 1. This result can be rewritten in terms of Fredholm minors using (C.5)

$$(x_1, \cdots, x_n|\rho_n^a|x_1', \cdots, x_n')_\pm = \frac{1}{\pi^n} C(x_1', \cdots, x_n)$$

$$\times \det \left(1 - \gamma \hat{\theta}_T^\pm \left| \begin{array}{c} x_1, \cdots, x_n \\ x_1', \cdots, x_n' \end{array} \right. \right) \Bigg|_{\gamma = (1 + e^{\pm i\pi\kappa})/\pi},$$

where the integral operator $\hat{\theta}_T^\pm$ with kernel $\theta_T(x, y)$ is defined by its action on an arbitrary function f:

$$(\hat{\theta}_T^\pm f)(x) = \int_{I_\pm} \theta_T(x, y) f(y) dy$$

Finally, introducing the resolvent kernel $\varrho_T^\pm(x, y)$ associated with the the kernel $\theta_T(x, y)$, which satisfies

$$\varrho_T^\pm(x, y) - \frac{(1 + e^{\pm i\pi\kappa})}{\pi} \int_{I_\pm} \theta_T(x - z) \varrho_T^\pm(z, y) dz = \theta_T(x - y)\,,$$

and making use of (C.6), (4.12) and (4.10), we obtain

$$\langle \Psi^\dagger(x_n') \cdots \Psi^\dagger(x_1') \Psi(x_1) \cdots \Psi(x_n) \rangle_{T,h,\pm} = \frac{C(x_1', \cdots, x_n)}{\pi^n} \varrho_T^\pm \begin{pmatrix} x_1, \cdots, x_n \\ x_1', \cdots, x_n' \end{pmatrix}$$

$$\times \det \left(1 - \gamma \hat{\theta}_T^\pm \right) \Big|_{\gamma = (1 + e^{\pm i\pi\kappa})/\pi}.$$

In the particular case of the simplest two-point correlator, this expression reduces to

$$\langle\Psi^\dagger(x')\Psi(x)\rangle_{T,h,\pm} = \frac{1}{\pi}\varrho_T^\pm(x',x)\,\det\left(1-\gamma\hat{\theta}_T^\pm\right)\Big|_{\gamma=(1+e^{\pm i\pi\kappa})/\pi}\,, \tag{4.34}$$

and gives the correlator of two anyonic fields in terms of the Fredholm determinant of the integral operator $\hat{\theta}_T$ and its resolvent kernel.

Chapter 5

Determinant Representation for the Correlation Functions of Impenetrable Anyons

In this chapter we are going to obtain the determinant representation of the time and temperature dependent field-field correlator using the summation of form factors. The method is similar with the one used by Korepin and Slavnov [55] in obtaining the same representation for impenetrable bosons. However, in the case of anyons the computation of form-factors has to be done with extra care due to the fact that the wavefunctions for systems with different number of particles obey different quasi-periodic boundary conditions. At the end of the chapter, in the particular case of static correlators, we show the equivalence with Lenard's formula. The results presented are based on [68].

5.1 The gas of impenetrable anyons

The Lieb-Liniger gas of 1D anyons with the the second-quantized Hamiltonian given by

$$H=\int_{-L/2}^{L/2} dx\ \left([\partial_x\Psi^\dagger(x)][\partial_x\Psi(x)]+c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)-h\Psi^\dagger(x)\Psi(x)\right)\ , \tag{5.1}$$

was studied in Chap. 2. In the previous formula $c>0$ is the coupling constant, L is the length of normalization interval, and h is the chemical potential. In this chapter we will be interested only in the limiting case $c\rightarrow\infty$.

The canonical Heisenberg fields

$$\Psi^\dagger(x,t) = e^{iHt}\Psi^\dagger(x)e^{-iHt}\,, \quad \Psi(x,t) = e^{iHt}\Psi(x)e^{-iHt}\,, \tag{5.2}$$

obey the anyonic equal-time commutation relations

$$\Psi(x_1,t)\Psi^\dagger(x_2,t) = e^{-i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2,t)\Psi(x_1,t) + \delta(x_1-x_2)\,, \tag{5.3}$$

$$\Psi^\dagger(x_1,t)\Psi^\dagger(x_2,t) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi^\dagger(x_2,t)\Psi^\dagger(x_1,t)\,, \tag{5.4}$$

$$\Psi(x_1,t)\Psi(x_2,t) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi(x_2,t)\Psi(x_1,t)\,, \tag{5.5}$$

where κ is the statistics parameter, which we assume to be rational, and $\epsilon(x) = x/|x|,\ \epsilon(0) = 0$. The Fock vacuum is defined as usual by

$$\Psi(x)|0\rangle = 0 = \langle 0|\Psi^\dagger(x)\,, \quad \langle 0|0\rangle = 1\,, \tag{5.6}$$

and the eigenstates $|\Psi_N\rangle$ of the Hamiltonian are

$$|\Psi_N\rangle = \frac{1}{\sqrt{N!}}\int_{-L/2}^{L/2} dz_1 \cdots \int_{-L/2}^{L/2} dz_N$$

$$\times\, \chi_N(z_1,\cdots,z_N|\lambda_1,\cdots,\lambda_N)\Psi^\dagger(z_N)\cdots\Psi^\dagger(z_1)|0\rangle\,, \tag{5.7}$$

where quantum-mechanical wavefunctions have the property of anyonic exchange statistics:

$$\chi_N(\cdots,z_i,z_{i+1},\cdots) = e^{i\pi\kappa\epsilon(z_i-z_{i+1})}\chi_N(\cdots,z_{i+1},z_i,\cdots)\,. \tag{5.8}$$

Note that the sign in front of the statistical phase in this expression ($+i\pi\kappa$ or $-i\pi\kappa$) depends on the choice of ordering of the creation operators in the definition of the eigenstates (5.7). The order of these operators adopted in Eq. (5.7) (leading to the phase $+i\pi\kappa$): the particle with the first coordinate z_1 created first, then z_2, etc., is convenient [5] for the subsequent calculation of the form factors.

As we have said, we limit our discussion to the case of infinitely strong interaction, $c \to \infty$, which corresponds to impenetrable anyons. In general, the eigenfunctions χ_N are

$$\chi_N = \frac{e^{+i\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j-z_k)}}{\sqrt{N!\prod_{j>k}[(\lambda_j-\lambda_k)^2+c'^2]}}\sum_{\pi\in S_N}(-1)^\pi e^{i\sum_{n=1}^N z_n\lambda_{\pi(n)}}$$

$$\times \prod_{j>k} [\lambda_{\pi(j)} - \lambda_{\pi(k)} - ic'\epsilon(z_j - z_k)] \,, \qquad (5.9)$$

where $c' \equiv c/\cos(\pi\kappa/2)$, and reduce for impenetrable anyons to a simpler form:

$$\chi_N = \frac{e^{+i\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j-z_k)}}{\sqrt{N!}} \prod_{j>k} \epsilon(z_j - z_k) \sum_{\pi \in S_N} (-1)^{\pi} e^{i\sum_{n=1}^{N} z_n \lambda_{\pi(n)}} \,. \qquad (5.10)$$

Here S_N is the group of permutations of N elements, and $(-1)^{\pi}$ is the sign of the permutation. The energy eigenvalues

$$H|\Psi_N\rangle = E|\Psi_N\rangle$$

are given by the sum of effectively single-particle contributions:

$$E = \sum_{i=1}^{N} \varepsilon(\lambda_j), \quad \text{with} \quad \varepsilon(\lambda) = \lambda^2 - h \,.$$

The individual momenta λ_j depend of the boundary conditions imposed on the wavefunctions. In contrast to particles of integer statistics, wavefunctions of the anyons satisfy different quasi-periodic boundary conditions in their different arguments, the difference resulting from the statistical phase shift $2\pi\kappa$ [5, 66]. In general, the quasi-periodic boundary conditions also include the external phase shift η (we will consider η=$2\pi\times$ rational), so that the boundary conditions on the wavefunctions (5.10) are:

$$\begin{aligned}
\chi_N(-L/2, z_2, \cdots, z_N) &= e^{-i\eta} \quad \chi_N(L/2, z_2, \cdots, z_N) \,, \\
\chi_N(z_1, -L/2, \cdots, z_N) &= e^{i(2\pi\kappa-\eta)} \chi_N(z_1, L/2, \cdots, z_N) \,, \\
&\vdots \\
\chi_N(z_1, z_2, \cdots, -L/2) &= e^{i(2\pi(N-1)\kappa-\eta)} \chi_N(z_1, z_2 \cdots, L/2) \,.
\end{aligned} \qquad (5.11)$$

The difference in the boundary conditions for different arguments of χ_N makes it possible, in general, to impose the condition without the statistical phase shift on any of the arguments z_j. The precise form of the Bethe equations for the momenta λ_j in the wavefunction (5.9) depends on specific choice of the boundary conditions. The choice (5.11), in which the first coordinate z_1 does not have the statistical shift in its boundary condition, gives rise to the Bethe equations which include the full statistical contribution $\pi\kappa(N-1)$ to the momentum shift of each of the anyons produced by the $N-1$ other anyons in the system [66]:

$$e^{i\lambda_j L} = e^{i\overline{\eta}} \prod_{k=1,k\neq j}^{N} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'}\right), \tag{5.12}$$

where $\overline{\eta} = \eta - \pi\kappa(N-1)$. Similarly to the wavefunctions, the general Bethe equations (5.12) are simplified in the impenetrable limit $c \to \infty$:

$$e^{i\lambda_j L} = (-1)^{N-1} e^{i\overline{\eta}}. \tag{5.13}$$

5.1.1 Structure of the ground state

We assume that the ground state of the gas contains N anyons, and take, for convenience, N to be even, although this does not affect our final results. We denote the momenta of the particles in the ground state as μ_j, where $j = 1, \cdots, N$, and introduce the notation $\{[...]\}$ such that

$$\{[x]\} = \gamma\,, \qquad \text{if } x = 2\pi \times \text{ integer } + 2\pi\gamma\,, \quad \gamma \in (-1, 1)\,. \tag{5.14}$$

The Bethe equations (5.13) give then the momenta μ_j:

$$\mu_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2\pi\delta}{L}, \quad j = 1, \cdots, N_0\,, \tag{5.15}$$

where $\delta = \{[\overline{\eta}]\}$. In the thermodynamic limit $L \to \infty,\ N \to \infty,\ N/L = D$, momenta of the particles fill densely the Fermi sea $[-q, q]$, where $q = \sqrt{h}$ is the Fermi momentum and the gas density is $D = q/\pi$.

5.1.2 Thermodynamics

The thermodynamics of the Lieb-Liniger anyonic gas was considered in [8, 9]. Similarly to the structure of the ground state, all local thermodynamic characteristics in the case of impenetrable anyons are equivalent to those of the free fermions. At non-vanishing temperature T, the quasiparticle distribution is given by the Fermi weight

$$\vartheta(\lambda, T, h) = \frac{1}{1 + e^{(\lambda^2 - h)/T}}\,. \tag{5.16}$$

and the density and energy are

$$D = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vartheta(\lambda, h, T)\ d\lambda\,, \quad E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda^2\vartheta(\lambda, h, T)\ d\lambda\,. \tag{5.17}$$

The density increases monotonically as a function of the chemical potential h. At $T = 0$,we have $D = 0$ for $h \leq 0$, and $0 < D < \infty$ if $0 < h < \infty$. At non-vanishing temperature, the density is zero for $h = -\infty$ and monotonically increases with h for $-\infty < h < \infty$.

5.2 Time dependent field-field correlator

In Chap. 4, we have derived the anyonic generalization of the Lenard formula, which for impenetrable free anyons, is an expansion of the anyonic reduced density matrices in terms of the reduced density matrices of free fermions. In the simplest case, the correlator

$$(x_1|\rho_1^a|x_2) = \langle \Psi^\dagger(x_2)\Psi(x_1)\rangle_T \tag{5.18}$$

is the first Fredholm minor of an integral operator, whose kernel is the Fourier transform of the Fermi weight (5.16). In this chapter, we obtain the time dependent generalization of this result. Our approach will be based on the following considerations. We start with the zero temperature field correlator

$$\langle \Psi(x_2, t_2)\Psi^\dagger(x_1, t_1)\rangle_N = \frac{\langle \Psi(\mu_1, \cdots, \mu_N)|\Psi(x_2, t_2)\Psi^\dagger(x_1, t_1)|\Psi(\mu_1, \cdots, \mu_N)\rangle}{\langle \Psi(\mu_1, \cdots, \mu_N)|\Psi(\mu_1, \cdots, \mu_N)\rangle}\,, \tag{5.19}$$

where the wavefunctions are taken to be normalized as

$$\langle \Psi(\mu_1, \cdots, \mu_N)|\Psi(\mu_1, \cdots, \mu_N)\rangle = L^N, \tag{5.20}$$

and $\mu_1, \cdots, \mu_N$ are the momenta in the ground state (5.15). Using the resolution of identity for the Hilbert space of $N + 1$ particles

$$\mathbf{1} = \sum_{\text{all } \{\lambda\}_{N+1}} \frac{|\Psi(\lambda_1, \cdots, \lambda_{N+1})\rangle\langle\Psi(\lambda_1, \cdots, \lambda_{N+1})|}{\langle \Psi(\lambda_1, \cdots, \lambda_{N+1})|\Psi(\lambda_1, \cdots, \lambda_{N+1})\rangle}\,, \tag{5.21}$$

where, according to (5.20)

$$\langle \Psi(\lambda_1, \cdots, \lambda_{N+1})|\Psi(\lambda_1, \cdots, \lambda_{N+1})\rangle = L^{N+1}\,,$$

and the sum is over all possible solutions of the Bethe equations with $N+1$ particles, we have

$$\langle \Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\rangle_N = \frac{1}{L^{2N+1}} \sum_{\text{all}\,\{\lambda\}_{N+1}} \langle \Psi_N(\{\mu\})|\Psi(x_2,t_2)|\Psi_{N+1}(\{\lambda\})\rangle$$

$$\times \langle \Psi_{N+1}(\{\lambda\})|\Psi^\dagger(x_1,t_1)|\Psi_N(\{\mu\})\rangle\,. \qquad (5.22)$$

Defining the form factors

$$\begin{aligned} F_{N+1,N}(x,t) &= \langle \Psi_{N+1}(\{\lambda\})|\Psi^\dagger(x,t)|\Psi_N(\{\mu\})\rangle\,, \\ F^*_{N+1,N}(x,t) &= \langle \Psi_N(\{\mu\})|\Psi(x,t)|\Psi_{N+1}(\{\lambda\})\rangle\,, \end{aligned} \qquad (5.23)$$

we can rewrite Eq. (5.22) as

$$\langle \Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\rangle_N = \frac{1}{L^{2N+1}} \sum_{\text{all}\,\{\lambda\}_{N+1}} F^*_{N+1,N}(x_2,t_2)F_{N+1,N}(x_1,t_1)\,. \quad (5.24)$$

Equation (5.24) means that in order to find the dynamic field correlator, we need to compute the form factors and sum over all of them. After the summation, one can take the thermodynamic limit. In general, such a summation of form factors is extremely difficult. The main simplification which makes it possible to perform this summation in the model of anyons we consider here, is the fact that, similarly to the problem of impenetrable bosons [54, 55], the local thermodynamic properties of particles with δ-function interaction are identical with those of free fermions regardless of the actual exchange statistics. Finally, the finite-temperature correlator can be obtained from the zero-temperature result using the standard argument developed for the Bose gas (see, e.g., Appendix XIII.1 of [54]), which is also applicable in the case of anyons.

5.2.1 Form factors

As a first step in carrying out the program outlined above, we compute the form factors. In the definition (5.23) of the form factors, the eigenstates $|\Psi_N(\{\mu\})\rangle$, $|\Psi_{N+1}(\{\lambda\})\rangle$ have, respectively, N and $N+1$ particles. Although the set $\{\mu\}$ represents in (5.23) momenta in the ground state of N particles, our calculation below is valid also when $|\Psi_N(\{\mu\})\rangle$ is not the ground state. As before, we assume for convenience that N is even. We denote by $\{\mu_j\}$ the momenta of the anyons in the N-particle eigenstate, and by $\{\lambda_j\}$ the momenta in the $N+1$ eigenstate.

Using the definition (5.7) for the eigenstates with N and $N+1$ anyons

$$|\Psi_N(\{\mu\})\rangle = \frac{1}{\sqrt{N!}}\int d^N z\ \chi_N(z_1,\cdots,z_N|\{\mu\})\Psi^\dagger(z_N)\cdots\Psi^\dagger(z_1)|0\rangle\,,$$

$$\langle\Psi_{N+1}(\{\lambda\})| = \frac{1}{\sqrt{N+1!}}\int d^{N+1}y\ \langle 0|\Psi(y_1)\cdots\Psi(y_{N+1})\chi^*_{N+1}(y_1,\cdots,y_{N+1}|\{\lambda\})$$

one can write the form factor as

$$F_{N+1,N}(x,0) = \frac{1}{\sqrt{(N+1)!N!}}\int d^{N+1}y\ d^N z\ \chi^*_{N+1}(y_1,\cdots,y_{N+1}|\{\lambda\})$$

$$\times\,\chi_N(z_1,\cdots,z_N|\{\mu\})\langle 0|\Psi(y_1)\cdots\Psi(y_{N+1})\Psi^\dagger(x)\Psi^\dagger(z_N)\cdots\Psi^\dagger(z_1)|0\rangle\,. \quad (5.25)$$

A direct application of the anyonic commutation relation (2.2) and Eq. (5.6) described in more details in Appendix E, reduces this expression to

$$\begin{aligned} F_{N+1,N}(x,0) &= \langle\Psi_{N+1}|\Psi^\dagger(x)|\Psi_N\rangle \quad (5.26)\\ &= \sqrt{N+1}\int d^N z\ \chi^*_{N+1}(z_1,\cdots,z_N,x|\{\lambda\})\chi_N(z_1,\cdots,z_N|\{\mu\})\,. \end{aligned}$$

An important feature of Eq. (5.26) is that the order of the creation operators chosen in Eq. (5.7) makes the "free" coordinate x in (5.26) the last argument of the wavefunction χ_{N+1}. This ensures that both wavefunctions, χ_N and χ_{N+1}, have the same phase shifts (2.20) at the boundary of the normalization interval in all other variables z_j. Since these phase shifts are canceled in Eq. (5.26), the expression under the integrals over z_j is periodic in each of the variable [5]. This feature is the necessary consistency condition for the Hilbert spaces of anyon wavefunctions with different numbers of particles, and is important in what follows for the appropriate calculation of the form factors (5.26).

The sets of momenta $\{\mu_j\}$ and $\{\lambda_j\}$ in the wavefunctions χ_N and χ_{N+1} in (5.26) are determined by the Bethe equations (5.13) as

$$\mu_j = \frac{2\pi}{L}\left(m_j+\frac{1}{2}\right)+\frac{2\pi\delta}{L},\quad \delta=\{[\eta-\pi\kappa(N-1)]\},\ j=1,\cdots,N,\ m_j\in\mathbb{Z}\,, \quad (5.27)$$

$$\lambda_j = \frac{2\pi}{L}n_j+\frac{2\pi\delta'}{L},\quad \delta'=\{[\eta-\pi\kappa N]\},\ j=1,\cdots,N+1,\ n_j\in\mathbb{Z}\,. \quad (5.28)$$

These equations show that

$$\lambda_j-\mu_k = \frac{2\pi}{L}\left(l-\frac{\kappa+1}{2}\right),\quad l\in\mathbb{Z}\,, \quad (5.29)$$

which means that λ_j and μ_k never coincide except in the trivial case $\kappa = 1$, when we have a gas of non-interacting fermions. In all other situations, λ_j and μ_k are different. This difference between them comes from the phase shift due to the hard-core condition on the added particle described by the factor 1/2 in (5.29), and the extra anyonic statistical phase added to the anyon system together with the particle [5]. This difference between λ_j and μ_k plays an important role in the following calculations. Using the identity

$$e^{+i\frac{\pi\kappa}{2}\epsilon(x-y)}\epsilon(y-x) = \cos(\pi\kappa/2)\,\epsilon(y-x) - i\sin(\pi\kappa/2)\ , \tag{5.30}$$

we can rewrite the anyonic wavefunction (5.10) as

$$\chi_N(z_1, \cdots, z_N|\{\mu\}) = \frac{\prod_{j>k}[\cos(\pi\kappa/2)\,\epsilon(z_j - z_k) - i\sin(\pi\kappa/2)]}{\sqrt{N!}} \times \sum_{\pi\in S_N}(-1)^\pi e^{i\sum_{n=1}^N z_n\mu_{\pi(n)}}\,.$$

Using this expression for both of the wavefunctions in (5.26) we obtain

$$F_{N+1,N}(x,0) = \frac{1}{N!}\sum_{\pi\in S_{N+1}}\sum_{\sigma\in S_N}(-1)^{\pi+\sigma}e^{-ix\lambda_{\pi(n+1)}}$$

$$\times\int_{-L/2}^{L/2}\prod_{n=1}^N dz_n\,[\cos(\pi\kappa/2)\,\epsilon(x-z_n) + i\sin(\pi\kappa/2)]\,e^{-i\sum_{n=1}^N z_n(\lambda_{\pi(n)}-\mu_{\sigma(n)})}\,.$$

Integration by parts in this equation produces the boundary terms in the following form

$$\frac{e^{-iz_n(\lambda_{\pi(n)}-\mu_{\sigma(n)})}}{-i(\lambda_{\pi(n)}-\mu_{\sigma(n)})}\left(\cos(\pi\kappa/2)\,\epsilon(x-z_n) + i\sin(\pi\kappa/2)\right)\Big|_{z_n=-L/2}^{z_n=L/2} =$$

$$\frac{e^{-i\frac{\pi\kappa}{2}}e^{-i\frac{L}{2}(\lambda_{\pi(n)}-\mu_{\sigma(n)})}}{i(\lambda_{\pi(n)}-\mu_{\sigma(n)})}\left(1 + e^{+i\pi\kappa}e^{iL(\lambda_{\pi(n)}-\mu_{\sigma(n)})}\right).$$

All these terms vanish due to Eq. (5.29). Then, using the relation

$$\frac{d\epsilon(x-z_n)}{dz_n} = -2\delta(x-z_n)\,,$$

we obtain the following expression for the form factors

$$
\begin{aligned}
F_{N+1,N}(x,0) &= \frac{[2i\cos(\pi\kappa/2)]^N}{N!}\exp\left\{ix\left[\sum_{j=1}^{N}\mu_j-\sum_{j=1}^{N+1}\lambda_j\right]\right\} \\
&\quad\times\sum_{\pi\in S_{N+1}}\sum_{\sigma\in S_N}(-1)^{\pi+\sigma}\prod_{j=1}^{N}\frac{1}{\lambda_{\pi(j)}-\mu_{\sigma(j)}}\,. \qquad (5.31)
\end{aligned}
$$

This expression differs from the corresponding result for impenetrable bosons [54, 55] by the spectrum of the momenta which now include the statistical shift, and by the overall $[\cos(\pi\kappa/2)]^N$ factor. For $\kappa=0$, both differences disappear, and Eq. (5.31) reproduces, as should be, the case of the Bose gas. We transform this equation following the corresponding steps for bosons [54, 55]. One can see directly that the sums over permutations in (5.31) can be written in the form of a determinant:

$$
\frac{1}{N!}\sum_{\pi\in S_{N+1}}\sum_{\sigma\in S_N}(-1)^{\pi+\sigma}\prod_{j=1}^{N}\frac{1}{\lambda_{\pi(j)}-\mu_{\sigma(j)}}=\left(1+\frac{\partial}{\partial\alpha}\right)\det\nolimits_N\left(M^{\alpha}_{jk}\right)\Big|_{\alpha=0}\,, \qquad (5.32)
$$

with

$$
M^{\alpha}_{jk}=\frac{1}{\lambda_j-\mu_k}-\frac{\alpha}{\lambda_{N+1}-\mu_k},\quad j,k=1,\cdots,N\,, \qquad (5.33)
$$

reducing Eq. (5.31) to

$$
\begin{aligned}
F_{N+1,N}(x,0) &= (2i\cos(\pi\kappa/2))^N\exp\left\{ix\left[\sum_{j=1}^{N}\mu_j-\sum_{j=1}^{N+1}\lambda_j\right]\right\} \\
&\quad\left(1+\frac{\partial}{\partial\alpha}\right)\det\nolimits_N\left(M^{\alpha}_{jk}\right)\Big|_{\alpha=0}\,. \qquad (5.34)
\end{aligned}
$$

The determinant part of this equation can also be written as

$$
\left(1+\frac{\partial}{\partial\alpha}\right)\det\nolimits_N\left(M^{\alpha}_{jk}\right)\Big|_{\alpha=0}=\sum_{\pi\in S_{N+1}}(-1)^{\pi}\prod_{j=1}^{N}\frac{1}{\lambda_{\pi(j)}-\mu_j}\,, \qquad (5.35)
$$

as one can see directly from the L.H.S. of (5.32) by noticing that due to the permutations π of λ_j, all permutations of μ_j give identical contributions to the sum over $\pi\in S_{N+1}$.

Alternatively, one can introduce a fictitious momentum μ_{N+1}, and obtain the following representation [63] of the form factor in terms of this momentum:

$$F_{N+1,N}(x,0) = (2i\cos(\pi\kappa/2))^N \exp\left\{ix\left[\sum_{j=1}^{N}\mu_j - \sum_{j=1}^{N+1}\lambda_j\right]\right\}$$

$$\times \lim_{\mu_{N+1}\to\infty}\left[-\mu_{N+1}\ \det{}_{N+1}\left(\frac{1}{\lambda_j-\mu_k}\right)\right],$$

where $\det_{N+1}(a_{jk})$ is the determinant of the $(N+1)\times(N+1)$ matrix with elements a_{jk}. We will not be using this representation explicitly below.

The time-dependent form factors can be obtained from the timeless form (5.34) using the following simple relations:

$$e^{-iHt}|\Psi_N(\{\mu\})\rangle = e^{-it\sum_{j=1}^{N}(\mu_j^2-h)}|\Psi_N(\{\mu\})\rangle\,,$$

and

$$\langle\Psi_N(\{\lambda\})|e^{iHt} = e^{it\sum_{j=1}^{N+1}(\lambda_j^2-h)}\langle\Psi_N(\{\lambda\})|\,.$$

Combining the exponential factors in these expressions with those in Eq. (5.34), we arrive at the final result for the time-dependent form factor:

$$F_{N+1,N}(x,t) = (2i\cos(\pi\kappa/2))^N e^{-iht}\left(\prod_{i=1}^{N+1} e(\lambda_i|t,x)\right)\left(\prod_{j=1}^{N} e^*(\mu_j|t,x)\right)$$

$$\times\left(1+\frac{\partial}{\partial\alpha}\right)\det{}_N\left(M^\alpha_{jk}\right)\Big|_{\alpha=0}\,, \qquad (5.36)$$

where we have introduced the function

$$e(\lambda|t,x) = e^{it\lambda^2-ix\lambda}\,, \qquad (5.37)$$

$e^*(\lambda|t,x)$ is its complex conjugate, and M^α_{jk} is defined in (5.33). The form factor of the annihilation operator $\Psi(x,t)$ is obtained through complex conjugation

$$\langle\Psi_N(\{\mu\})|\Psi(x,t)|\Psi_{N+1}(\{\lambda\})\rangle = F^*_{N+1,N}(x,t)\,. \qquad (5.38)$$

5.2.2 Summation of the form factors

Using Eqs. (5.36) and (5.38), we write the field correlator (5.24) as a sum over intermediate momenta $\{\lambda\}$:

$$\langle\Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\rangle_N = \sum_{\text{all }\{\lambda\}_{N+1}} \frac{(2\cos(\pi\kappa/2))^{2N}}{L^{2N+1}} e^{iht_{21}} \left(\prod_{i=1}^{N+1} e^*(\lambda_i|t_{21},x_{21})\right)$$

$$\times\left(\prod_{j=1}^{N} e(\mu_j|t_{21},x_{21})\right)\left(1+\frac{\partial}{\partial\alpha}\right)\det{}_N\left(M^\alpha_{jk}\right)\Big|_{\alpha=0}\left(1+\frac{\partial}{\partial\beta}\right)\det{}_N\left(M^\beta_{jk}\right)\Big|_{\beta=0}, \tag{5.39}$$

with the notations $x_{ab}=x_a-x_b,\ t_{ab}=t_a-t_b,\ a,b=1,2$. The matrix M^β_{jk} here is the same as (5.33) with α replaced by β. As was mentioned above, modulo the $[\cos(\pi\kappa/2)]^{2N}$ factors and the spectrum of momenta, Eq. (5.39) is identical with the expression for the bosonic field correlators [54, 55]. This means that the summation process over $\{\lambda\}$ is very similar, and we just sketch the derivation here. Since we sum over all momenta $\{\lambda\}$, individual momenta λ_j are equivalent up to permutation. This means that one of the two permutations of $\{\lambda_j\}$ involved in the definition of the two determinants in (5.39) produces coinciding terms, so that under the sum over $\{\lambda_j\}$, one can replace one of the determinants, e.g., the second one, with

$$(N+1)!\prod_{j=1}^{N}\frac{1}{\lambda_j-\mu_j}, \tag{5.40}$$

obtaining

$$\begin{aligned}&\langle\Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\rangle_N\\ &= e^{iht_{21}}\left(\prod_{j=1}^{N} e(\mu_j|t_{21},x_{21})\right)\frac{1}{L}\left(\frac{2\cos(\pi\kappa/2)}{L}\right)^{2N}(N+1)!\\ &\times \sum_{\text{all }\{\lambda\}_{N+1}}\left(e^*(\lambda_{N+1}|t_{21},x_{21})+\frac{\partial}{\partial\alpha}\right)\det{}_N\left(\frac{e^*(\lambda_j|t_{21},x_{21})}{(\lambda_j-\mu_k)(\lambda_j-\mu_j)}\right.\\ &\qquad\left.-\alpha\frac{e^*(\lambda_j|t_{21},x_{21})}{(\lambda_j-\mu_j)}\frac{e^*(\lambda_{N+1}|t_{21},x_{21})}{(\lambda_{N+1}-\mu_j)}\right)\Big|_{\alpha=0}.\end{aligned} \tag{5.41}$$

The summation over the momenta $\{\lambda_j\}$ can be done then independently over each λ_j inside the determinant. Also, we transfer the factors $e(\mu_j|t_{21},x_{21})$ in (5.41) into the determinant splitting them between the rows and columns, and use the formula

$$\frac{1}{(\lambda_j-\mu_k)}\frac{1}{(\lambda_j-\mu_j)}=\left(\frac{1}{\lambda_j-\mu_j}-\frac{1}{\lambda_j-\mu_k}\right)\frac{1}{\mu_j-\mu_k}\,. \tag{5.42}$$

This gives the correlator as

$$\begin{aligned}\langle\Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\rangle_N &= e^{iht_{21}}\left(\frac{1}{2\pi}G_L(t_{12},x_{12})+\frac{\partial}{\partial\alpha}\right)\\ &\times \det_N\Big[\delta_{jk}\tilde{E}_L(\mu_k|t_{12},x_{12})e(\mu_j|t_{21},x_{21})+e(\mu_j|t_2,x_2)e^*(\mu_k|t_1,x_1)\\ &\times\cos^2(\pi\kappa/2)\left(\frac{2(1-\delta_{jk})}{\pi L(\mu_j-\mu_k)}(E_L(\mu_j|t_{12},x_{12})-E_L(\mu_k,t_{12},x_{12}))\right.\\ &\left.\left.\left.-\frac{\alpha}{L\pi^2}E_L(\mu_j|t_{12},x_{12})E_L(\mu_k|t_{12},x_{12})\right)\right]\right|_{\alpha=0},\end{aligned} \tag{5.43}$$

where we have defined the functions

$$\frac{1}{2\pi}G_L(t,x)=\frac{1}{L}\sum_\lambda e(\lambda|t,x)\,, \tag{5.44}$$

$$\frac{1}{2\pi}E_L(\mu_k|t,x)=\frac{1}{L}\sum_\lambda\frac{e(\lambda|t,x)}{\lambda-\mu_k}\,, \tag{5.45}$$

$$\tilde{E}_L(\mu_k|t,x)=\frac{4\cos(\pi\kappa/2)^2}{L^2}\sum_\lambda\frac{e(\lambda|t,x)}{(\lambda-\mu_k)^2}\,, \tag{5.46}$$

and $\lambda=\frac{2\pi}{L}(\mathbb{Z}+\delta')$ – see (5.28). Formula (5.43) is the final expression for the field correlator in the ground state of N anyons on a finite interval with quasi-periodic boundary conditions.

5.2.3 Thermodynamic limit

In order to obtain the correlator in the thermodynamic limit, we need to compute the large-L limit of the functions (5.44), (5.45), and (5.46). This is done in Appendix F with the results

$$G(t,x)\equiv\lim_{L\to\infty}G_L(t,x)=\int_{-\infty}^{\infty}e(\lambda|t,x)\;d\lambda\,, \tag{5.47}$$

$$E(\mu_k|t,x)\equiv\lim_{L\to\infty}E_L(\mu_k|t,x)=\text{P.V.}\int_{-\infty}^{\infty}d\lambda\,\frac{e(\lambda|t,x)}{\lambda-\mu_k}+e(\mu_k|t,x)\pi\tan\left(\frac{\pi\kappa}{2}\right)\,, \tag{5.48}$$

$$\tilde{E}(\mu_k|t,x) \equiv \lim_{L\to\infty} \tilde{E}_L(\mu_k|t,x) = e(\mu_k|t,x) + \frac{2\cos^2(\pi\kappa/2)}{\pi L}\frac{\partial}{\partial\mu_k}E(\mu_k|t,x)\,. \tag{5.49}$$

In the thermodynamic limit $L, N \to \infty$ with $D = N/L$ constant, the anyon momenta fill densely the Fermi interval $[-q, q]$, where $q = \sqrt{h}$ and $D = q/\pi$. In this case, the determinant in the correlator (5.43) can be understood as the Fredholm determinant of an integral operator. Indeed, for an arbitrary integral operator $\hat{V}$, whose action on a function $f(\lambda)$ is defined by

$$(\hat{V}f)(\lambda) = \int_a^b V(\lambda,\mu)f(\mu)\;d\mu\,,$$

the associated Fredholm determinant is (see, e.g., [75])

$$\det(1+\hat{V}) = \lim_{n\to\infty}\begin{vmatrix} 1+\xi V(\lambda_1,\lambda_1) & \xi V(\lambda_1,\lambda_2) & \cdots & \xi V(\lambda_1,\lambda_n) \\ \xi V(\lambda_2,\lambda_1) & 1+\xi V(\lambda_2,\lambda_2) & \cdots & \xi V(\lambda_2,\lambda_n) \\ \vdots & \vdots & \ddots & \vdots \\ \xi V(\lambda_n,\lambda_1) & \xi V(\lambda_n,\lambda_2) & \cdots & 1+\xi V(\lambda_n,\lambda_n) \end{vmatrix}\,,$$

where $\xi = (b-a)/n,\ \lambda_p - \lambda_{p-1} = \xi$ and $\lambda_0 = a,\ \lambda_n = b$. One can see directly that, in the thermodynamic limit, the determinant part of Eq. (5.43) has the same structure with N momenta μ_j separated by $\xi = 2\pi/L$ filling the Fermi interval $[-q, q]$. This means that the correlator can be expressed as

$$\langle\Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\rangle = e^{iht_{21}}\left(\frac{1}{2\pi}G(t_{12},x_{12}) + \frac{\partial}{\partial\alpha}\right)\det(1+\hat{\tilde{V}}_0)\bigg|_{\alpha=0}\,,$$

where $\hat{\tilde{V}}_0$ acts on an arbitrary function $f(\lambda)$ as

$$(\hat{\tilde{V}}_0 f)(\lambda) = \int_{-q}^{q}\tilde{V}_0(\lambda,\mu)f(\mu)\;d\mu\,,$$

and

$$\tilde{V}_0(\lambda,\mu) = \cos^2(\pi\kappa/2)e(\lambda|t_2,x_2)e^*(\mu|t_1,x_1)$$
$$\times\left[\frac{E(\lambda|t_{12},x_{12}) - E(\mu|t_{12},x_{12})}{\pi^2(\lambda-\mu)} - \frac{\alpha}{2\pi^3}E(\lambda|t_{12},x_{12})E(\mu|t_{12},x_{12})\right]\,. \tag{5.50}$$

Performing the unitary transformation

$$V_0(\lambda,\mu) = \exp\left\{-i\frac{(t_1+t_2)}{2}(\lambda^2-\mu^2) + i\frac{(x_1+x_2)}{2}(\lambda-\mu)\right\}\tilde{V}_0(\lambda,\mu)\,,$$

with the property

$$\det(1 + \hat{\tilde{V}}_0) = \det(1 + \hat{V}_0)\,,$$

we transform the kernel $\tilde{V}_0(\lambda, \mu)$ (5.50) into the symmetric form:

$$V_0(\lambda,\mu) = \cos^2(\pi\kappa/2)\exp\left\{-\frac{i}{2}t_{12}(\lambda^2+\mu^2) + \frac{i}{2}x_{12}(\lambda+\mu)\right\}$$
$$\times\left[\frac{E(\lambda|t_{12},x_{12}) - E(\mu|t_{12},x_{12})}{\pi^2(\lambda-\mu)} - \frac{\alpha}{2\pi^3}E(\lambda|t_{12},x_{12})E(\mu|t_{12},x_{12})\right]. \tag{5.51}$$

Two observations are in order. First, one can check that the second term in (5.49) is obtained from the first term in the square bracket of (5.51) in the limit $\lambda \to \mu$. Second, in the limit $\kappa \to 0$, Eq. (5.51) reproduces the known result [54, 55] for impenetrable bosons.

In the static case ($t_1 = t_2$), which is discussed in the next Section, the kernel (5.51) can be simplified further. One needs to distinguish two cases.

- $x_1 > x_2$. In this case,

$$E(\lambda|0,x_{12}) = -i\pi e^{-ix_{12}\lambda}[1 + i\tan(\pi\kappa/2)]\,, \tag{5.52}$$

and the kernel (5.51) becomes

$$V_0^+(\lambda,\mu) = -\frac{(1+e^{+i\pi\kappa})}{\pi}\left(\frac{\sin(x_{12}(\lambda-\mu)/2)}{\lambda-\mu}\right.$$
$$\left.+\frac{\alpha}{2\pi}e^{+i\pi\kappa}\exp\left\{-i\frac{x_{12}}{2}(\lambda+\mu)\right\}\right). \tag{5.53}$$

- $x_1 < x_2$. In this case,

$$E(\lambda|0,x_{12}) = i\pi e^{-ix_{12}\lambda}[1 - i\tan(\pi\kappa/2)]\,, \tag{5.54}$$

and

$$V_0^-(\lambda,\mu) = \frac{(1+e^{-i\pi\kappa})}{\pi}\left(\frac{\sin(x_{12}(\lambda-\mu)/2)}{\lambda-\mu}\right.$$
$$\left.+\frac{\alpha}{2\pi}e^{-i\pi\kappa}\exp\left\{-i\frac{x_{12}}{2}(\lambda+\mu)\right\}\right). \tag{5.55}$$

We now extend the discussion to the situation of non-vanishing temperature T. The temperature-dependent field correlator is defined as

$$\langle\Psi(x_2,t_2)\Psi^\dagger(x_1,t_2)\rangle_T = \frac{\mathrm{Tr}\left(e^{-H/T}\Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\right)}{\mathrm{Tr}\ e^{-H/T}}\,.$$

According to the well-known argument developed for the Bose gas [54], this correlator can be found as the mean value over any one of the "typical" eigenfunctions Ω_T of the Hamiltonian which characterizes the given state of thermal equilibrium:

$$\frac{\langle\Omega_T|\Psi(x_2,t_2)\Psi^\dagger(x_1,t_2)|\Omega_T\rangle}{\langle\Omega_T|\Omega_T\rangle}\,. \tag{5.56}$$

This argument depends only on the general saddle-point approximation in the description of the state of equilibrium, and also holds in the case of anyons. The further computation of the field correlator based on Eq. (5.56) is similar to the zero-temperature case, the main difference being the change of the measure of integration:

$$\int_{-q}^{q} d\lambda \quad\rightarrow\quad \int_{-\infty}^{\infty} d\lambda\;\vartheta(\lambda,T,h) \quad \text{with}\;\; \vartheta(\lambda,T,h)=\frac{1}{1+e^{(\lambda^2-h)/T}}\,.$$

The final result for the temperature-dependent correlator is then

$$\langle\Psi(x_2,t_2)\Psi^\dagger(x_1,t_1)\rangle_T = e^{iht_{21}}\left(\frac{1}{2\pi}G(t_{12},x_{12})+\frac{\partial}{\partial\alpha}\right)\det(1+\hat{V}_T)\bigg|_{\alpha=0}\,,$$

where the kernel of the integral operator $\hat{V}_T$ is

$$\begin{aligned} V_T(\lambda,\mu) &= \sqrt{\vartheta(\lambda)}V_0(\lambda,\mu)\sqrt{\vartheta(\mu)}\,, \qquad (5.57)\\ &= \cos^2(\pi\kappa/2)\exp\left\{-\frac{i}{2}t_{12}(\lambda^2+\mu^2)+\frac{i}{2}x_{12}(\lambda+\mu)\right\}\sqrt{\vartheta(\lambda)\vartheta(\mu)}\\ &\quad\left[\frac{E(\lambda|t_{12},x_{12})-E(\mu|t_{12},x_{12})}{\pi^2(\lambda-\mu)}-\frac{\alpha}{2\pi^3}E(\lambda|t_{12},x_{12})E(\mu|t_{12},x_{12})\right]\,, \end{aligned}$$

and the operator acts on an arbitrary function $f(\mu)$ as

$$\left(V_T f\right)(\lambda)=\int_{-\infty}^{\infty}V_T(\lambda,\mu)f(\mu)\;d\mu\,.$$

5.3 Equivalence with Lenard's formula

In the previous chapter we obtained the anyonic generalization of the Lenard formula for the equal-time field correlator or, equivalently, reduced density matrices of anyons. In the case of the first reduced density matrix, the anyonic

Lenard formula reads

$$(x|\rho_1^a|x')_{\pm} = \frac{1}{\pi}\det\left(1-\gamma\hat{\theta}_T^{\pm}\,\middle|\,\begin{matrix} x \\ x' \end{matrix}\right)\Bigg|_{\gamma=(1+e^{\pm i\pi\kappa})/\pi}, \qquad (5.58)$$

where the kernel of the integral operators $\hat{\theta}_T^{\pm}$ is

$$\theta_T(\xi-\eta) = \frac{1}{2}\int_{-\infty}^{\infty} d\lambda\, \frac{e^{i(\xi-\eta)\lambda}}{1+e^{(\lambda^2-h)/T}}, \qquad (5.59)$$

and their action on an arbitrary function is defined as

$$(\hat{\theta}_T^{\pm}f)(\xi) = \int_{I_\pm} \theta_T(\xi-\eta)f(\eta)\, d\eta\,. \qquad (5.60)$$

In these expressions, the plus sign refers to the situation when $x' > x$ and $I_+ = [x, x']$, and the minus sign – to the situation when $x' < x$ and $I_- = [x', x]$. The resolvent kernels associated with the kernel $\theta_T(x, y)$ acting on the intervals $I_\pm$ are denoted by $\varrho_T^{\pm}(\xi, \eta)$ and satisfy the equations:

$$\varrho_T^{\pm}(\xi,\eta) - \frac{(1+e^{\pm i\pi\kappa})}{\pi}\int_{I_\pm}\theta_T(\xi-\xi')\varrho_T^{\pm}(\xi',\eta)d\xi' = \theta_T(\xi-\eta)\,.$$

One can rewrite Eq. (5.58) in terms of the resolvent kernel ϱ_T and the field correlator as

$$\langle\Psi^\dagger(x')\Psi(x)\rangle_{T,\pm} = \frac{1}{\pi}\varrho_T^{\pm}(x',x)\,\det\left(1-\gamma\hat{\theta}_T^{\pm}\right)\Big|_{\gamma=(1+e^{\pm i\pi\kappa})/\pi}, \qquad (5.61)$$

where again, the plus sign refers to the case $x' > x$ and the minus sign – to $x < x'$. Next, we show that Eq. (5.61) is reproduced by the results obtained in the previous section when they are specialized to the equal-time correlators. We treat the two cases, $x' > x$ and $x' < x$, separately.

5.3.1 The static correlator $\langle\Psi(-x)\Psi^\dagger(x)\rangle_T$

Equations (5.37) and (5.47) show that in the static case

$$\frac{1}{2\pi}G(0,x) = \delta(x)\,.$$

Using this relation and Eqs. (5.53) and (5.57), we see that the equal-time field correlator can be written as

$$\langle\Psi(-x)\Psi^\dagger(x)\rangle_T = \left(\delta(2x)+\frac{\partial}{\partial\alpha}\right)\det\left(1-\frac{(1+e^{i\pi\kappa})}{\pi}\hat{K}_T+\alpha\frac{e^{i\pi\kappa}}{2\pi}\hat{A}_T^+\right)\Bigg|_{\alpha=0}\,, \tag{5.62}$$

where $\hat{K}_T$ and $\hat{A}_T^+$ are the integral operators acting on the real axis and defined by kernels

$$K_T(\lambda,\mu)=\sqrt{\vartheta(\lambda)}\frac{\sin x(\lambda-\mu)}{\lambda-\mu}\sqrt{\vartheta(\mu)}\,, \tag{5.63}$$

and

$$A_T^+(\lambda,\mu)=\sqrt{\vartheta(\lambda)}e^{-ix(\lambda+\mu)}\sqrt{\vartheta(\mu)}\,.$$

At zero temperature, both operators act on the interval $[-q,q]$ and their kernels are

$$K(\lambda,\mu)=\frac{\sin x(\lambda-\mu)}{\lambda-\mu}\,,\qquad A^+(\lambda,\mu)=e^{-ix(\lambda+\mu)}\,.$$

The commutation relation (2.2) shows that

$$\langle\Psi(-x)\Psi^\dagger(x)\rangle_T = e^{i\pi\kappa}\langle\Psi^\dagger(x)\Psi(-x)\rangle_T+\delta(2x)\,.$$

This means that in order to prove the equivalence with Lenard formula, we have to show that

$$G^+(\kappa,x,T)\equiv\frac{\partial}{\partial\alpha}\det\left(1-\frac{(1+e^{i\pi\kappa})}{\pi}\hat{K}_T+\alpha\frac{e^{i\pi\kappa}}{2\pi}\hat{A}_T^+\right)\Bigg|_{\alpha=0}=$$

$$=e^{i\pi\kappa}\langle\Psi^\dagger(x)\Psi(-x)\rangle_T\,,$$

where $\langle\Psi^\dagger(x)\Psi(-x)\rangle_T$ is given by (5.61). For a general integral operator with kernel V, one of the useful expressions for the Fredholm determinant is

$$\ln\,\det(1-\gamma\hat{V})=-\sum_{n=1}^{\infty}\frac{\gamma^n}{n}\mathrm{Tr}\;V^n\,.$$

Making use of this formula, we obtain

$$G^+(\kappa,x,T)=\frac{e^{i\pi\kappa}}{2\pi}\mathrm{Tr}\;\left[(1-\gamma\hat{K}_T)^{-1}\hat{A}_T^+\right]\det(1-\gamma\hat{K}_T)|_{\gamma=(1+e^{i\pi\kappa})/\pi}\,. \tag{5.64}$$

Denoting as $f_-^+(\lambda)$ the solution of the integral equation

$$f_-^+(\lambda) - \frac{(1+e^{i\pi\kappa})}{\pi} \int_{-\infty}^{\infty} K_T(\lambda,\mu) f_-^+(\mu)\, d\mu = \sqrt{\vartheta(\lambda)} e^{-ix\lambda}\,, \tag{5.65}$$

we can rewrite (5.64) as

$$G^+(\kappa, x, T) = \frac{e^{i\pi\kappa}}{2\pi} \int_{-\infty}^{\infty} e^{-ix\lambda} f_-^+(\lambda) \sqrt{\vartheta(\lambda)} d\lambda \quad \det(1-\gamma \hat{K}_T)|_{\gamma=(1+e^{i\pi\kappa})/\pi}\,.$$

We will show now that

$$\det(1-\gamma\hat{K}_T) = \det\left(1-\gamma\hat{\theta}_T^+\right)\,, \tag{5.66}$$

where the operator $\hat{\theta}_T$ is described by Eqs. (5.59) and (5.60), and $\gamma = (1+e^{i\pi\kappa})/\pi$. Direct and inverse Fourier transforms of a function g can be defined to include as integration measure $\sqrt{\vartheta(\lambda)}$:

$$\tilde{g}(\lambda) = \frac{1}{2\pi\sqrt{\vartheta(\lambda)}} \int_{-\infty}^{\infty} d\xi\; e^{i\lambda\xi} g(\xi)\,, \qquad g(\xi) = \int_{-\infty}^{\infty} d\lambda\; \sqrt{\vartheta(\lambda)} e^{-i\lambda\xi} \tilde{g}(\lambda)\,.$$

With this definition, taking the Fourier transform of the integral equation

$$g(\xi) - \gamma \int_{-x}^{x} \theta_T(\xi-\xi') g(\xi')\; d\xi' = G(\xi)\,,$$

we obtain

$$\tilde{g}(\lambda) - \gamma \int_{-\infty}^{\infty} K_T(\lambda-\mu)\tilde{g}(\mu)\; d\mu = \tilde{G}(\lambda)\,.$$

Coincidence of the two equations implies the equality (5.66) of the determinants.

The final step in proving the equivalence of Eqs. (5.61) and (5.62) is to show that

$$\varrho_T^+(x,-x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ix\lambda} f_-^+(\lambda)\sqrt{\vartheta(\lambda)} d\lambda\,. \tag{5.67}$$

The Fourier transform of the equation defining the resolvent kernel ϱ_T

$$\varrho_T^+(\xi,-x) - \frac{(1+e^{i\pi\kappa})}{\pi}\int_{-x}^{x} \theta_T(\xi-\xi')\varrho_T^+(\xi',-x)\; d\xi' = \theta_T(\xi+x)\,,$$

gives

$$\tilde{\varrho}_T^+(\lambda,-x) - \frac{(1+e^{i\pi\kappa})}{\pi}\int_{-\infty}^{\infty} K_T(\lambda-\mu)\tilde{\varrho}_T^+(\mu,-x)\ d\mu = \frac{1}{2}e^{-ix\lambda}\sqrt{\vartheta(\lambda)}\,.$$

Comparison of this equation with the definition of $f_-^+(\lambda)$ (5.65) shows that

$$\tilde{\varrho}_T^+(\lambda,-x) = \frac{1}{2}f_-^+(\lambda)\,. \tag{5.68}$$

Taking the inverse Fourier transform of (5.68) proves (5.67). Thus, we have shown that for $x' > x$, the Lenard formula (5.61) is equivalent with the result (5.62) for the static field correlator that follows from the direct summation of the form factors.

5.3.2 The static correlator $\langle\Psi(x)\Psi^\dagger(-x)\rangle_T$

In this case, the proof of the equivalence of the two approaches is very similar to what was just discussed for $x' > x$. Equations (5.55) and (5.57) show that the static field correlator is

$$\langle\Psi(x)\Psi^\dagger(-x)\rangle_T = \left(\delta(2x) + \frac{\partial}{\partial\alpha}\right)\det\left(1 - \frac{(1+e^{-i\pi\kappa})}{\pi}\hat{K}_T + \alpha\frac{e^{-i\pi\kappa}}{2\pi}\hat{A}_T^-\right)\Bigg|_{\alpha=0}\,,$$

where $\hat{K}_T$ is given by (5.63) and

$$A_T^-(\lambda,\mu) = \sqrt{\vartheta(\lambda)}e^{ix(\lambda+\mu)}\sqrt{\vartheta(\mu)}\,.$$

From the commutation relation (2.2) we see that

$$\langle\Psi(x)\Psi^\dagger(-x)\rangle_T = e^{-i\pi\kappa}\langle\Psi^\dagger(-x)\Psi(x)\rangle_T + \delta(2x)\,,$$

so we have to show that

$$G^-(\kappa,x,T) \equiv \frac{\partial}{\partial\alpha}\det\left(1 - \frac{(1+e^{-i\pi\kappa})}{\pi}\hat{K}_T + \alpha\frac{e^{-i\pi\kappa}}{2\pi}\hat{A}_T^-\right)\Bigg|_{\alpha=0} =$$

$$= e^{-i\pi\kappa}\langle\Psi^\dagger(-x)\Psi(x)\rangle_T\,.$$

where $\langle\Psi^\dagger(-x)\Psi(x)\rangle_T$ is given by Eq. (5.61). Similarly to the discussion in the previous section, we can rewrite G^- as

$$G^-(\kappa,x,T) = \frac{e^{-i\pi\kappa}}{2\pi}\int_{-\infty}^{\infty} e^{+ix\lambda}f_+^-(\lambda)\sqrt{\vartheta(\lambda)}d\lambda\quad \det(1-\gamma\hat{K}_T)|_{\gamma=(1+e^{-i\pi\kappa})/\pi}\,,$$

where $f_+^-(\lambda)$ is the solution of the integral equation

$$f_+^-(\lambda) - \frac{(1+e^{-i\pi\kappa})}{\pi} \int_{-\infty}^{\infty} K_T(\lambda,\mu) f_+^-(\mu)\ d\mu = \sqrt{\vartheta(\lambda)} e^{+ix\lambda}\,.$$

The equality of the Fredholm determinants of the operators $\hat{K}_T$ and $\hat{\theta}_T$ was shown in the previous Section, so it remains to prove that

$$\varrho_T^-(-x,x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{+ix\lambda} f_+^-(\lambda) \sqrt{\vartheta(\lambda)} d\lambda\,. \tag{5.69}$$

Again, taking the Fourier transform of

$$\varrho_T^-(\xi,x) - \frac{(1+e^{-i\pi\kappa})}{\pi} \int_{-x}^{x} \theta_T(\xi-\xi') \varrho_T^+(\xi',x)\ d\xi' = \theta_T(\xi - x)\,,$$

we obtain

$$\tilde{\varrho}_T^-(\lambda,x) - \frac{(1+e^{-i\pi\kappa})}{\pi} \int_{-\infty}^{\infty} K_T(\lambda-\mu) \tilde{\varrho}_T^-(\mu,x)\ d\mu = \frac{1}{2} e^{+ix\lambda} \sqrt{\vartheta(\lambda)}\,,$$

which shows that

$$\tilde{\varrho}_T^-(\lambda,x) = \frac{1}{2} f_+^-(\lambda)\,. \tag{5.70}$$

The inverse Fourier transform of (5.70) gives the correct result (5.69).

Chapter 6

Differential Equations for the Field-Field Correlator

In this chapter we are going to obtain an integrable system of nonlinear partial differential equations and the appropriate boundary conditions, that provides a complete characterization of the field-field correlation functions at finite temperature. This system is the same as the one obtained by Its, Izergin, Korepin and Slavnov [49] for impenetrable bosons but with different boundary conditions. At $T = 0$ the system reduces to the Painlevé V equation obtained by Jimbo, Miwa, Môri and Sato in their study of the one-particle reduced density matrix of impenetrable bosons [51], but again with different boundary conditions.

6.1 Auxiliary potentials

In the previous chapter we have obtained the following expressions for the the static, i.e. equal-time, correlators at finite temperature:

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T = \frac{1}{2\pi}\mathrm{Tr}\ \left[(1-\gamma\hat{K}_T)^{-1}\hat{A}_T^+\right]\det(1-\gamma\hat{K}_T)|_{\gamma=(1+e^{+i\pi\kappa})/\pi}\ , \quad (6.1)$$

when $x_1 > x_2$, and

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T = \frac{1}{2\pi}\mathrm{Tr}\ \left[(1-\gamma\hat{K}_T)^{-1}\hat{A}_T^-\right]\det(1-\gamma\hat{K}_T)|_{\gamma=(1+e^{-i\pi\kappa})/\pi}\ , \quad (6.2)$$

when $x_1 < x_2$. In (6.1) and (6.2), $\hat{K}_T$ and $\hat{A}_T^{\pm}$ are integral operators acting on the entire real axis and have kernels ($x_{12} = (x_1 - x_2)$)

$$K_T(\lambda, \mu) = \sqrt{\vartheta(\lambda)} \frac{\sin x_{12}(\lambda - \mu)}{\lambda - \mu} \sqrt{\vartheta(\mu)}\,,$$

$$A_T^{\pm}(\lambda, \mu) = \sqrt{\vartheta(\lambda)} e^{\mp i x_{12}(\lambda+\mu)} \sqrt{\vartheta(\mu)}\,,$$

where

$$\vartheta(\lambda) \equiv \vartheta(\lambda, T, h) = \frac{1}{1 + e^{(\lambda^2 - h)/T}}\,, \tag{6.3}$$

is the Fermi distribution function at temperature T and chemical potential h and $\text{Tr}\,[f(x,y)] \equiv \int f(x,x)\,dx$. At zero temperature (6.1) and (6.2) remain valid but the integral operators act on the interval $[-q, q]$ with $q = \sqrt{h}$ and have kernels

$$K(\lambda, \mu) = \frac{\sin x_{12}(\lambda - \mu)}{\lambda - \mu}\,, \qquad A^{\pm}(\lambda, \mu) = e^{\mp i x_{12}(\lambda+\mu)}\,.$$

In the anyonic case the corrrelator $\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T$ depends on the sign of $x_1 - x_2$. However, it is easy to see that (6.1) and (6.2) are related via complex conjugation. This means that we can focus our attention only on the correlator (6.1). This correlator depends on four variables, the coordinate difference $x_1 - x_2 > 0$, temperature T, chemical potential h and statistics parameter κ. We will show bellow that by introducing new variables, the scaled distance x and scaled chemical potential β defined by

$$x = \frac{1}{2}(x_1 - x_2)\sqrt{T}\,, \qquad \beta = \frac{h}{T}$$

and by changing the spectral parameter $\lambda \to \lambda/\sqrt{T}$ the dependence on temperature will become extremely simple. More specifically

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T = \frac{\sqrt{T}}{2\pi\gamma} g(x, \beta, \gamma)|_{\gamma = (1+e^{i\pi\kappa})/\pi}\,, \tag{6.4}$$

where $g(x, \beta, \gamma)$ will be defined below.

The Fredholm integral operator appearing in the expression of the field correlators belongs to a specific class called "integrable integral operators" [43, 49, 54]. This means that introducing "plane waves"

$$e_{\pm}(\lambda) = \sqrt{\vartheta(\lambda)} e^{\pm i\lambda x} \tag{6.5}$$

then the kernel of $\hat{K}_T$ can be written as

$$K_T(\lambda,\mu) = \frac{e_+(\lambda)e_-(\lambda) - e_-(\lambda)e_+(\mu)}{2i(\lambda-\mu)} \tag{6.6}$$

which is a particular case of the more general case studied in [43, 49] (see also Chap XIV of [54]). One important feature of this particular class of integrable operators is the fact that that the resolvent kernel $R_T(\lambda,\mu)$ of the resolvent operator $\hat{R}$ has a similar form [49, 54]. The resolvent operator is defined by

$$\hat{R}_T = (1-\gamma\hat{K}_T)^{-1}\hat{K}_T\,, \quad (1-\gamma\hat{K}_T)(1+\gamma\hat{R}_T) = 1\,, \tag{6.7}$$

and the resolvent kernel solves the integral equation

$$R_T(\lambda,\mu) - \gamma\int_{-\infty}^{+\infty} K_T(\lambda,\nu)R_T(\nu,\mu)\ d\nu = K_T(\lambda,\mu)\,.$$

Then introducing functions $f_\pm(\lambda)$ which are solutions of the integral equations

$$f_\pm(\lambda) - \gamma\int_{-\infty}^{+\infty} K_T(\lambda,\mu)f_\pm(\mu)d\mu = e_\pm(\lambda)\,, \tag{6.8}$$

the resolvent kernel can be written as

$$R_T(\lambda,\mu) = \frac{f_+(\lambda)f_-(\mu) - f_-(\lambda)f_+(\mu)}{2i(\lambda-\mu)}\,. \tag{6.9}$$

For a proof of this statement see Chap XIV of [54]. The auxiliary potentials B_{lm} are defined defined by

$$B_{lm}(x,\beta,\kappa) \equiv \gamma\int_{-\infty}^{+\infty} e_l(\lambda)f_m(\lambda)\ d\lambda\,, \qquad l=\pm,\ m=\pm\,. \tag{6.10}$$

where $\gamma = (1+e^{i\pi\kappa})/\pi$. Compared with the bosonic case ($\gamma = 2/\pi$) the auxiliary potentials are now complex. However, as in the bosonic case we have $B_{+-}(x,\beta,\kappa) = B_{-+}(x,\beta,\kappa)$ and $B_{++}(x,\beta,\kappa) = B_{--}(x,\beta,\kappa)$. First we have

$$\begin{aligned} B_{+-} &\equiv \gamma\int_{-\infty}^{+\infty} e_+(\lambda)f_-(\lambda)\ d\lambda \\ &= \gamma\int_{-\infty}^{+\infty} e_+(\lambda)\int_{-\infty}^{+\infty}(1-\gamma\hat{K}_T)^{-1}(\lambda,\mu)e_-(\mu)\ d\mu\ d\lambda \\ &= \gamma\int_{-\infty}^{+\infty} e_-(\lambda)f_+(\lambda)\ d\lambda \equiv B_{-+}\,, \end{aligned}$$

where in the last line we have used the fact that the kernel is symmetric $K_T(\lambda,\mu) = K_T(\mu,\lambda)$. In order to prove the second assertion we start with the integral equation (6.8) for $f_+(-\lambda)$

$$f_+(-\lambda) - \gamma \int_{-\infty}^{+\infty} K_T(-\lambda,\mu) f_+(\mu) d\mu = \sqrt{\vartheta(\lambda)} e^{-i\lambda x},$$

which can be rewritten as

$$f_+(-\lambda) - \gamma \int_{-\infty}^{+\infty} K_T(-\lambda,-\mu) f_+(-\mu) d\mu = \sqrt{\vartheta(\lambda)} e^{-i\lambda x}.$$

Using $K_T(-\lambda,-\mu) = K_T(\lambda,\mu)$ we obtain that $f_+(-\lambda) = f_-(\lambda)$. Therefore we have

$$\begin{aligned} B_{++} &\equiv \gamma \int_{-\infty}^{+\infty} e_+(\lambda) f_+(\lambda)\ d\lambda = \gamma \int_{-\infty}^{+\infty} e_+(-\lambda) f_+(-\lambda)\ d\lambda \\ &= \gamma \int_{-\infty}^{+\infty} e_-(\lambda) f_-(\lambda)\ d\lambda \equiv B_{--}\,. \end{aligned}$$

Finally it is easy to see that (6.1) can be rewritten as

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T = \frac{\sqrt{T}}{2\pi} \det(1-\gamma \hat{K}_T)|_{\gamma=(1+e^{+i\pi\kappa})/\pi} \int_{-\infty}^{+\infty} f_-(\lambda) e_-(\lambda)\ d\lambda\,,$$

which shows that the function $g(x,\beta,\gamma)$ appearing in (6.4) is given by

$$g(x,\beta,\gamma) = B_{++}(x,\beta,\gamma) \det(1-\gamma\hat{K}_T)|_{\gamma=(1+e^{+i\pi\kappa})/\pi}\,. \tag{6.11}$$

6.2 Correlators as a completely integrable system

In general is very difficult to obtain differential equations for the entire correlator (6.1). In our case first, we will obtain nonlinear partial differential equations for the potentials B_{++}, B_{+-} and then we will show that $\sigma(x,\beta,\gamma) = \ln\det(1-\gamma\hat{K}_T)$ can be expressed in terms of B_{++} and B_{--}. The strategy will be the following. We are looking for two operators L and M depending on B_{++} and B_{--} such that we have

$$\begin{aligned} \partial_x F(\lambda) &= \mathsf{L}\ F(\lambda)\,, \\ (2\lambda\partial_\beta + \partial_\lambda) F(\lambda) &= \mathsf{M}\ F(\lambda)\,, \end{aligned} \tag{6.12}$$

where $F(\lambda)$ is the two-component vector function

$$F(\lambda) = \begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix} .$$

Then from the compatibility condition for (6.12)

$$(2\lambda\partial_\beta + \partial_\lambda)\mathsf{L} - \partial_x \mathsf{M} + [\mathsf{L}, \mathsf{M}] = 0 ,$$

we will obtain a system of partial differential equations in x and β. It should be mentioned that the Fredholm determinant appearing in (6.1) differs from the one appearing in the similar representation for impenetrable bosons only in the value of γ ($\gamma = (1 + e^{i\pi\kappa})/\pi$ for anyons and $\gamma = 2/\pi$ for bosons). This means that the computations are similar with the ones performed in the case of impenetrable bosons [46, 54]. The only difference is given by the fact that the auxiliary potentials B_{++}, B_{+-} are now complex and not real.

6.2.1 The **L** operator

We will start from the integral equations for the functions $f_\pm(\lambda)$

$$f_\pm(\lambda) - \gamma \int_{-\infty}^{+\infty} K_T(\lambda, \mu) f_\pm(\mu) d\mu = e_\pm(\lambda) , \quad \gamma = (1 + e^{i\pi\kappa})/\pi.$$

Taking the derivative with respect to x we obtain

$$\left[(1 - \gamma\hat{K}_T)\partial_x f_\pm\right](\lambda) - \gamma \left[\partial_x \hat{K}_T f_\pm\right](\lambda) = \pm i\lambda e_\pm(\lambda) .$$

Multiplying both sides with $(1 - \gamma\hat{K}_T)^{-1}$ we have

$$\partial_x f_\pm(\lambda) = \gamma \left[(1 - \gamma\hat{K}_T)^{-1}(\partial_x \hat{K}_T) f_\pm\right](\lambda)$$
$$\pm i \int_{-\infty}^{+\infty} \left[\delta(\lambda - \mu) + \gamma R_T(\lambda, \mu)\right] \mu e_\pm(\mu)\, d\mu , \tag{6.13}$$

where in the second term on the R.H.S. of (6.13) of the previous equation we have used $(1 - \gamma\hat{K}_T)^{-1} = (1 + \gamma\hat{R}_T)$. The first term is computed easily using $\left[(1 - \gamma\hat{K}_T)^{-1} e_\pm\right](\lambda) = f_\pm(\lambda)$ and $\partial_x K_T(\lambda, \mu) = (e_+(\lambda)e_-(\mu) + e_-(\lambda)e_+(\mu))/2$ with the result

$$\gamma \left[(1 - \gamma\hat{K}_T)^{-1}(\partial_x \hat{K}_T) f_\pm\right](\lambda) = \frac{1}{2} f_+(\lambda) B_{-,\pm} + \frac{1}{2} f_-(\lambda) B_{+,\pm} . \tag{6.14}$$

For the second term we have

$$
\begin{aligned}
&\pm i\int_{-\infty}^{+\infty}\left[\delta(\lambda-\mu)+\gamma R_T(\lambda,\mu)\right]\mu e_\pm(\mu)\ d\mu \\
&= \pm i\lambda f_\pm \mp i\int_{-\infty}^{+\infty}[\delta(\lambda-\mu)+\gamma R_T(\lambda,\mu)]((\lambda-\mu)e_\pm(\mu))\ d\mu \\
&= \pm i\lambda f_\pm \mp \left(\frac{1}{2}f_+(\lambda)B_{\pm,-}-\frac{1}{2}f_-(\lambda)B_{\pm,+}\right), \qquad (6.15)
\end{aligned}
$$

where we have used (6.9). From (6.14) and (6.15) we obtain

$$
\begin{aligned}
\partial_x f_+(\lambda) &= i\lambda f_+(\lambda)+B_{++}f_-(\lambda)\,, \\
\partial_x f_-(\lambda) &= -i\lambda f_+(\lambda)+B_{--}f_-(\lambda)\,,
\end{aligned}
$$

which implies that our L operator has the form

$$
\mathsf{L} = i\lambda\sigma_3 + B_{++}\sigma_1\,, \qquad (6.16)
$$

where we have used the fact that $B_{++}=B_{--}$ and σ_i are the Pauli matrices

$$
\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.
$$

6.2.2 The M operator

The derivation of the M operator is more complicated. In this case we will rely heavily on the following property of the Fermi distribution function

$$
(2\lambda\partial_\beta+\partial_\lambda)\vartheta(\lambda)=0\,.
$$

This property is essential because in this case we will have terms which contains the resolvent $R_T(\lambda,\mu)$ but for which we cannot apply the trick used in the previous section when we were able to reduce it to terms of the form $(\lambda-\mu)R(\lambda,\mu)$ which are "disentangled" (are products of one dimensional projectors). This is also why the differential operator associated with M is $(2\lambda\partial_\beta+\partial_\lambda)$ and not ∂_β as we would expect.

First we will compute the derivative with respect to β. Using

$$
\partial_\beta K_T(\lambda,\mu)=\frac{\partial_\beta\vartheta(\lambda)}{2\vartheta(\lambda)}K_T(\lambda,\mu)+\frac{\partial_\beta\vartheta(\mu)}{2\vartheta(\mu)}K_T(\lambda,\mu) \qquad (6.17)
$$

then differentiating (6.8) we obtain

$$\left[(1-\gamma\hat{K}_T)\partial_\beta f_\pm\right](\lambda)=\left[(1+\gamma\hat{K}_T)\frac{\partial_\beta\vartheta}{2\vartheta}f_\pm\right](\lambda)\,. \tag{6.18}$$

Again, multiplying both sides with $(1-\gamma\hat{K}_T)^{-1}$ and using the relation $(1-\gamma\hat{K}_T)^{-1}\hat{K}_T=\hat{R}_T$ we can rewrite the previous equation as

$$\partial_\beta f_\pm(\lambda)=\frac{\partial_\beta\vartheta(\lambda)}{2\vartheta(\lambda)}f_\pm(\lambda)+2\gamma\int_{-\infty}^{+\infty}R_T(\lambda,\mu)\frac{\partial_\beta\vartheta(\mu)}{2\vartheta(\mu)}f_\pm(\mu)\;d\mu\,.$$

Furthermore, using (6.9) we can obtain

$$\begin{aligned}2\lambda\partial_\beta f_\pm(\lambda) &= 2\lambda\frac{\partial_\beta\vartheta(\lambda)}{2\vartheta(\lambda)}f_\pm(\lambda)\\ &+\frac{2\gamma}{i}\int_{-\infty}^{+\infty}[f_+(\lambda)f_-(\mu)-f_-(\lambda)f_+(\mu)]\frac{\partial_\beta\vartheta(\mu)}{2\vartheta(\mu)}f_\pm(\mu)\;d\mu\\ &+2\gamma\int_{-\infty}^{+\infty}R_T(\lambda,\mu)\mu\frac{\partial_\beta\vartheta(\mu)}{\vartheta(\mu)}f_\pm(\mu)\;d\mu\,.\end{aligned} \tag{6.19}$$

The derivative with respect to λ is more laborious. First we note that

$$\partial_\lambda e_\pm(\lambda)=\left(\frac{\partial_\lambda\vartheta(\lambda)}{2\vartheta(\lambda)}\pm ix\right)e_\pm(\lambda)\,,$$

and

$$\begin{aligned}\partial_\lambda K_T(\lambda,\mu)=\frac{\partial_\lambda\vartheta(\lambda)}{2\vartheta(\lambda)}K(\lambda,\mu)+ix\frac{e_+(\lambda)e_-(\mu)+e_+(\mu)e_-(\lambda)}{2i(\lambda-\mu)}\\ -\frac{e_+(\lambda)e_-(\mu)-e_+(\mu)e_-(\lambda)}{2i(\lambda-\mu)^2}\end{aligned} \tag{6.20}$$

Differentiating (6.8) and integrating by parts the third term in (6.20) we obtain

$$\begin{aligned}\partial_\lambda f_\pm(\lambda)-\gamma\int_{-\infty}^{+\infty}\left[\frac{\partial_\lambda\vartheta(\lambda)}{2\vartheta(\lambda)}K(\lambda,\mu)+\frac{\partial_\mu\vartheta(\mu)}{2\vartheta(\mu)}K(\lambda,\mu)\right]f_\pm(\lambda)\\ +K(\lambda,\mu)\partial_\mu f_\pm(\mu)d\mu=\left(\frac{\partial_\lambda\vartheta(\lambda)}{2\vartheta(\lambda)}\pm ix\right)e_\pm(\lambda)\end{aligned}$$

which can be rewritten as

$$\left[(1-\gamma\hat{K}_T)\partial_\lambda f_\pm\right](\lambda)=\left[(1+\gamma\hat{K}_T)\frac{\partial_\lambda\vartheta}{2\vartheta}f_\pm\right](\lambda)\pm ixe_\pm(\lambda)\,.$$

Multiplying both sides with $(1-\gamma\hat{K}_T)^{-1}$ and using the relation $(1-\gamma\hat{K}_T)^{-1}\hat{K}_T = \hat{R}_T$ produces the final result

$$\partial_\lambda f_\pm(\lambda) = \pm ix f_\pm(\lambda) + \frac{\partial_\lambda\vartheta(\lambda)}{2\vartheta(\lambda)} f_\pm(\lambda) + 2\gamma\int_{-\infty}^{+\infty} R_T(\lambda,\mu)\frac{\partial_\mu\vartheta(\mu)}{2\vartheta(\mu)} f_\pm(\mu)d\mu\,. \tag{6.21}$$

Adding the results given by (6.19) and (6.21) we obtain

$$(2\lambda\partial_\beta + \partial_\lambda)f_\pm(\lambda) = \pm ix f_\pm(\lambda) - if_+ U_{-,\pm} + if_-(\lambda)U_{+,\pm}\,,$$

where

$$U_{lm} \equiv \gamma\int_{-\infty}^{+\infty} f_l(\mu)f_m(\mu)\frac{\partial_\beta\vartheta(\mu)}{\vartheta(\mu)}\ d\mu\,, \quad l=\pm,\ m=\pm\,. \tag{6.22}$$

Now we will prove that

$$\partial_\beta B_{lm} = U_{lm}\,. \tag{6.23}$$

Indeed we have

$$\begin{aligned}
\partial_\beta B_{lm} &= \gamma\int_{-\infty}^{+\infty}\left(\frac{\partial_\beta\vartheta(\mu)}{2\vartheta(\mu)}e_l(\mu)f_m(\mu) + e_l(\mu)\partial_\beta f_m(\mu)\right)d\mu\,,\\
&= \gamma\int_{-\infty}^{+\infty}\frac{\partial_\beta\vartheta(\mu)}{2\vartheta(\mu)}e_l(\mu)f_m(\mu)d\mu\\
&\quad +\gamma\int_{-\infty}^{+\infty}[1-\gamma\hat{K}_T](\mu,\lambda)f_l(\lambda)\partial_\beta f_m(\mu)\ d\mu d\lambda\,,\\
&= \gamma\int_{-\infty}^{+\infty}[1-\gamma\hat{K}_T](\mu,\lambda)f_l(\lambda)\frac{\partial_\beta\vartheta(\mu)}{2\vartheta(\mu)}f_m(\mu)d\lambda d\mu\\
&\quad +\gamma\int_{-\infty}^{+\infty}[1+\gamma\hat{K}_T](\mu,\lambda)f_l(\lambda)\frac{\partial_\beta\vartheta(\mu)}{2\vartheta(\mu)}f_m(\mu)d\lambda d\mu\,,\\
&= \gamma\int_{-\infty}^{+\infty} f_l(\mu)f_m(\mu)\frac{\partial_\beta\vartheta(\mu)}{\vartheta(\mu)} \equiv U_{lm}\,,
\end{aligned} \tag{6.24}$$

where we have used (6.18) in the second line of (6.24). This results in

$$\begin{aligned}
(2\lambda\partial_\beta + \partial_\lambda)f_+(\lambda) &= ixf_+(\lambda) - if_+(\lambda)\partial_\beta(B_{-+}) + if_-(\lambda)\partial_\beta(B_{++})\,,\\
(2\lambda\partial_\beta + \partial_\lambda)f_-(\lambda) &= -ixf_+(\lambda) - if_+(\lambda)\partial_\beta(B_{--}) + if_-(\lambda)\partial_\beta(B_{+-})\,,
\end{aligned}$$

therefore the M operator is given by $(B_{-+} = B_{+-},\ B_{++} = B_{--})$

$$\mathsf{M} = ix\sigma_3 - i(\partial_\beta B_{+-})\sigma_3 - (\partial_\beta B_{++})\sigma_2\,. \tag{6.25}$$

6.2.3 Differential equations for the potentials

The results obtained in the previous sections allow us to state the following theorem:

Theorem 2. *For all* $\gamma = (1+e^{i\pi\kappa})/\pi$ *with* $\kappa \in [0,1)$*, the potentials* $B_{++}(x,\beta,\gamma)$, $B_{++}(x,\beta,\gamma)$ *satisfy the following system of partial differential equations*

$$\partial_\beta B_{+-} = x + \frac{1}{2}\frac{\partial_x\partial_\beta B_{++}}{B_{++}}\,, \tag{6.26}$$

$$\partial_x B_{+-} = B_{++}^2\,, \tag{6.27}$$

with the initial conditions (at β *fixed)*

$$\begin{aligned} B_{++}(x,\beta,\gamma) &= \gamma d(\beta) + [\gamma d(\beta)]^2 x + O(x^2)\,, \quad x \to 0\,, \\ B_{+-}(x,\beta,\gamma) &= \gamma d(\beta) + [\gamma d(\beta)]^2 x + O(x^2)\,, \quad x \to 0\,, \end{aligned} \tag{6.28}$$

where $d(\beta) = \int_{-\infty}^{+\infty} \vartheta(\lambda) d\lambda$ *and*

$$B_{++}(x,\beta,\gamma) = B_{+-}(x,\beta,\gamma) = 0\,, \quad \beta \to -\infty\,. \tag{6.29}$$

The potential $B_{++}(x,\beta,\gamma)$ *satisfies the nonlinear equation*

$$\partial_\beta B_{++}^2 = 1 + \frac{1}{2}\frac{\partial}{\partial_x}\left(\frac{\partial_x\partial_\beta B_{++}}{B_{++}}\right)\,, \tag{6.30}$$

with the same initial conditions.

Proof. In the previous sections we have shown that the two-component vector function $F(\lambda)$ satisfies the following differential equations

$$\begin{aligned} \partial_x F(\lambda) &= \mathsf{L}\ F(\lambda)\,, \\ (2\lambda\partial_\beta + \partial_\lambda)F(\lambda) &= \mathsf{M}\ F(\lambda)\,. \end{aligned}$$

where L,M are given by (6.16) and (6.25). The compatibility condition for these equation is given by

$$[\partial_x - \mathsf{L}, (2\lambda\partial_\beta + \partial_\lambda) - \mathsf{M}] = (2\lambda\partial_\beta + \partial_\lambda)\mathsf{L} - \partial_x\mathsf{M} + [\mathsf{L},\mathsf{M}] = 0\,. \tag{6.31}$$

Using

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\,,$$

(6.16) and (6.25) we obtain

$$(2\lambda\partial_\beta + \partial_\lambda)\mathsf{L} = 2\lambda(\partial_\beta B_{++})\sigma_1 + i\sigma_3\,,$$

$$\partial_x \mathsf{M} = i\sigma_3 - i(\partial_x\partial_\beta B_{+-})\sigma_3 - (\partial_x\partial_\beta B_{++})\sigma_2\,,$$

and

$$[\mathsf{L},\mathsf{M}] = -2\lambda(\partial_\beta B_{++})\sigma_1 + [2xB_{++} - 2B_{++}(\partial_\beta B_{+-})]\,\sigma_2 - 2iB_{++}(\partial_\beta B_{++})\sigma_3\,.$$

Plugging these results in (6.31) we obtain a matrix which should vanish with only two distinct elements

$$2iB_{++}(\partial_\beta B_{+-}) - 2ixB_{++} + i\partial_x\partial_\beta B_{++} = 0\,,$$

and

$$\partial_x\partial_\beta B_{+-} - \partial_\beta B_{++}^2 = 0\,.$$

The first equation is just (6.26) while the second is equivalent with (6.27) due to (6.29). The equation (6.30) is obtained by differentiating (6.26) with respect to x and (6.27) with respect to β and then equating the right sides. The asymptotic behavior at short distances (6.28) and at low density 6.29, which give the initial conditions, are obtained in Appendices G and (H). □

6.2.4 Differential equations for $\sigma(x,\beta,\kappa)$

Theorem 3. *For any $\gamma = (1+e^{i\pi\kappa})/\pi$ with $\kappa \in [0,1)$ the partial derivatives of $\sigma(x,\beta,\gamma)$ with respect to x and β are given by*

$$\partial_x\sigma = -B_{+-}\,,\quad \partial_x^2\sigma = -B_{++}^2\,, \tag{6.32}$$

$$\partial_\beta\sigma = -x\partial_\beta B_{+-} + \frac{1}{2}(\partial_\beta B_{+-})^2 - \frac{1}{2}(\partial_\beta B_{++})^2\,. \tag{6.33}$$

Furthermore the function $\sigma(x,\beta,\gamma)$ satisfies for all γ the following nonlinear partial differential equation

$$(\partial_\beta\partial_x^2\sigma)^2 + 4(\partial_x^2\sigma)[2x\partial_\beta\partial_x\sigma + (\partial_\beta\partial_x\sigma)^2 - 2\partial_\beta\sigma] = 0 \tag{6.34}$$

with initial conditions

$$\sigma = -\gamma d(\beta)x - [\gamma d(\beta)]^2\frac{x^2}{2} + O(x^3)\,,\quad x\to 0\,;\quad \sigma = 0\,,\quad \beta\to-\infty\,, \tag{6.35}$$

where $d(\beta) = \int_{-\infty}^{+\infty}\vartheta(\lambda)d\lambda$.

Proof. We will start with the derivative with respect to x which is simpler.

Using the following representation for the Fredholm determinant of an operator

$$\det(1-\gamma\hat{K}_T) = \exp\left(-\sum_{n=1}^{\infty}\frac{\gamma^n}{n}\ \mathrm{Tr}\ K_T^n\right)\,, \tag{6.36}$$

where $\mathrm{Tr}\ K_T = \int K_T(\lambda,\lambda)$, $\ \mathrm{Tr}\ K_T^2 = \int K_T(\lambda,\mu)K_T(\mu,\lambda)\ d\lambda d\mu$ and so on we obtain

$$\partial_x\sigma = -\gamma\ \mathrm{Tr}\ \left[(1-\gamma\hat{K}_T)^{-1}\partial_x\hat{K}_T\right]\,.$$

Using $\left[(1-\gamma\hat{K}_T)^{-1}e_\pm\right](\lambda) = f_\pm(\lambda)$ and $\partial_x K_T(\lambda,\mu) = (e_+(\lambda)e_-(\mu)+e_-(\lambda)e_+(\mu))/2$ we have

$$\begin{aligned} &-\gamma\ \mathrm{Tr}\ \left[(1-\gamma\hat{K}_T)^{-1}\partial_x\hat{K}_T\right] \\ &= \ -\frac{\gamma}{2}\int_{-\infty}^{+\infty} f+(\lambda)e_(\lambda)d\lambda - \frac{\gamma}{2}\int_{-\infty}^{+\infty} f_-(\lambda)e_+(\lambda)d\lambda\,, \\ &= \ -\frac{1}{2}(B_{-+}+B_{+-})\,, \end{aligned}$$

which due to $B_{-+}=B_{+-}$ means that

$$\partial_x\sigma = -B_{+-}\,.$$

proving the first part of (6.32). The second part is obtained differentiating once more with respect to x and using (6.27).

The derivative with respect to β is more involved. Starting again with the representation (6.36) we obtain

$$\partial_\beta\sigma = -\gamma\ \mathrm{Tr}\ \left[(1-\gamma\hat{K}_T)^{-1}\partial_\beta\hat{K}_T\right]\,.$$

The trace in the previous formula can be expressed as

$$\partial_\beta\sigma = -\gamma\int_{-\infty}^{+\infty} R_T(\lambda,\lambda)\frac{\partial_\beta\vartheta(\lambda)}{\vartheta(\lambda)}d\lambda\,,$$

where we have used (6.17) for the derivative of the resolvent kernel and $(1-\gamma\hat{K}_T)^{-1}\hat{K}_T = \hat{R}_T$. Using L'Hopital rule in (6.7), $R_T(\lambda,\lambda) = [f_-(\lambda)\partial_\lambda f_+(\lambda) - f_+(\lambda)\partial_\lambda f_-(\lambda)]/2i$ therefore

$$\partial_\beta\sigma = \frac{i\gamma}{2}\int_{-\infty}^{+\infty}\frac{\partial_\beta\vartheta(\lambda)}{\vartheta(\lambda)}[f_-(\lambda)\partial_\lambda f_+(\lambda) - f_+(\lambda)\partial_\lambda f_-(\lambda)]\ d\lambda\,.$$

Employing (6.21) we obtain the cumbersome expression

$$\begin{aligned}\partial_\beta \sigma &= -\gamma x \int_{-\infty}^{+\infty} f_+(\lambda) f_-(\lambda) \frac{\partial_\beta \vartheta(\lambda)}{\vartheta(\lambda)} d\lambda \\ &+ \frac{i\gamma^2}{2} \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\mu \left(f_-(\lambda) f_+(\mu) R_T(\lambda,\mu) \frac{\partial_\mu \vartheta(\mu)}{\vartheta(\mu)} \frac{\partial_\beta \vartheta(\lambda)}{\vartheta(\lambda)} \right. \\ &\qquad \left. - f_+(\lambda) f_-(\mu) R_T(\lambda,\mu) \frac{\partial_\mu \vartheta(\mu)}{\vartheta(\mu)} \frac{\partial_\beta \vartheta(\lambda)}{\vartheta(\lambda)} \right) .\end{aligned}$$

The first term is just $-x\partial_\beta B_{+-}$ (see (6.22) and (6.23)) so making the change of variables $\mu \to \lambda$ in the second term of the double integral and using the symmetry of the resolvent kernel we have

$$\begin{aligned}\partial_\beta \sigma &= -x\partial_\beta B_{+-} + \frac{i\gamma^2}{2} \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\mu f_-(\lambda) f_+(\mu) R_T(\lambda,\mu) \\ &\quad \times \left(\frac{\partial_\mu \vartheta(\mu)}{\vartheta(\mu)} \frac{\partial_\beta \vartheta(\lambda)}{\vartheta(\lambda)} - \frac{\partial_\lambda \vartheta(\lambda)}{\vartheta(\lambda)} \frac{\partial_\beta \vartheta(\lambda)}{\vartheta(\lambda)} \right) .\end{aligned}$$

Making use of the relation $2\lambda\partial_\beta\vartheta(\lambda) = -\partial_\lambda\vartheta(\lambda)$ we can replace the terms which do not contain derivatives of the Fermi distribution function with respect to β arriving at

$$\begin{aligned}\partial_\beta \sigma &= -x\partial_\beta B_{+-} \\ &+ i\gamma^2 \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\mu f_-(\lambda) f_+(\mu) R_T(\lambda,\mu)(\lambda-\mu) \frac{\partial_\beta \vartheta(\mu)}{\vartheta(\mu)} \frac{\partial_\beta \vartheta(\lambda)}{\vartheta(\lambda)} .\end{aligned}$$

The $(\lambda - \mu)$ factor in the integral allows us to cancel the denominator of the resolvent kernel (6.7) obtaining an expression in which the two dimensional integrals factorizes in products of one dimensional ones

$$\begin{aligned}\partial_\beta \sigma &= -x\partial_\beta B_{+-} \\ &+ \frac{\gamma^2}{2} \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\mu f_-(\lambda) f_+(\mu) [f_+(\lambda) f_-(\mu) - f_-(\lambda) f_+(\mu)] \\ &\qquad \times \frac{\partial_\beta \vartheta(\mu)}{\vartheta(\mu)} \frac{\partial_\beta \vartheta(\lambda)}{\vartheta(\lambda)} .\end{aligned}$$

Using (6.22) and (6.19) it easy to see now that

$$\partial_\beta \sigma = -x\partial_\beta B_{+-} + \frac{1}{2}(\partial_\beta B_{+-})^2 - \frac{1}{2}(\partial_\beta B_{++})^2$$

proving (6.33).

The last differential equation (6.34) is obtained by replacing $B_{+-} = -\partial_x \sigma$ and $B_{++} = (-\partial_x^2 \sigma)^{1/2}$ in the R.H.S. of (6.33). It is easy to see that $K_T(\lambda, \mu) \to 0$ when $x \to 0$ or $\beta \to -\infty$ which means that $\sigma = \ln\det(1 - \gamma \hat{K}_T) = 0$ in the same limits. The first part of (6.35) then results from (6.28). This concludes the proof. □

We conclude this section with a simple observation. Integrating $\partial_x \sigma = -B_{+-}$ and using (6.35) we can obtain an expression for $g(x, \beta, \gamma)$ (or equivalently the field correlator) only in terms of the potentials B_{++}, B_{+-}

$$g(x, \beta, \gamma) = B_{++}(x, \beta, \gamma) e^{-\int_0^x B_{+-}(y,\beta,\gamma)dy} . \tag{6.37}$$

6.2.5 The zero temperature limit

In the zero temperature limit the field-field correlator (6.4) depends only on three variables, the distance $(x_1 - x_2) > 0$, the Fermi momentum (or chemical potential) $q = \sqrt{h}$ and the statistics parameter κ. This dependence can be encoded in the new variable

$$\xi = \frac{(x_1 - x_2)}{2} q = x\beta^{1/2} ,$$

where now the rescaled variables are $x = (x_1 - x_2)/2$ and $\beta = h$. The integral operator $\hat{K}$ now acts on the interval $[-q, q]$ and has the kernel $K(\lambda, \mu) = \sin x(\lambda - \mu)/(\lambda - \mu)$. The logarithm of the determinant is given by

$$\tilde{\sigma}_0(\xi, \gamma) = \ln\det(1 - \gamma \hat{K})|_{\gamma = (1+e^{i\pi\kappa})/\pi} ,$$

The partial differential equation (6.34) characterizing $\sigma(x, \beta, \gamma)$ becomes an ordinary differential equation in ξ. Introducing the new function

$$\sigma_0 = \xi(\tilde{\sigma}_0)' \tag{6.38}$$

where prime denotes the derivative with respect to ξ then (6.34) becomes

$$(\xi\sigma_0'')^2 + 4(\xi\sigma_0' - \sigma_0)[4\xi\sigma_0 - 4\sigma_0 + (\sigma_0')^2] = 0 \tag{6.39}$$

with boundary conditions

$$\sigma_0 = -2\gamma\xi - 4\gamma^2\xi^2 + O(\xi^3) , \quad \gamma = (1 + e^{i\pi\kappa})/\pi . \tag{6.40}$$

The ordinary differential equation (6.39) is the same Painlevé V equation obtained by Jimbo, Miwa, Môri and Sato in their celebrated work on the one-particle reduced density matrix (field-field correlator) of impenetrable bosons [51]. The only difference is in the initial conditions (6.40) which unlike the differential equation depend on the statistics parameter. In a certain way we can say that this was to be expected taking into account the fact that it was already noted by Jimbo et *al.*, that the same equation, but with different boundary conditions, characterizes the density matrix of impenetrable bosons and free fermions. The same situation was noted by Forrester, Frankel, Garoni and Witte [31] in their study of systems with finite number of particles with periodic boundary conditions. In this case the reduced density matrix satisfies a Painlevé VI differential equation. Their work was recently extended in the case of impenetrable anyons by Santachiara and Calabrese [74]. From (G.5) the first terms of the short distance expansion of the field-field correlator are

$$\langle\Psi^\dagger(x_1)\Psi(x_2)\rangle_T = D_0\left(1-\frac{\pi^2}{6}D_0^2(x_1-x_2)^2+\gamma\frac{\pi^3}{18}D_0^3(x_1-x_2)^3\right)+O\left((x_1-x_2)^4\right)\,, \tag{6.41}$$

where $D_0 = q/\pi$ is the density at $T=0$. This result agrees with the expansion obtained by Santachiara and Calabrese (see Eq. (47) of [74] and note that their statistical parameter κ differ in sign from ours).

Chapter 7

Large Distance Asymptotics for the Field-Field Correlator

The most difficult part in the analysis of the the field-field correlation functions is obtaining the large distance asymptotics. We have seen in the previous chapter the correlation function are characterized by a completely integrable system of nonlinear partial differential equations (see Thm. 2, (6.4) and (6.37)). A powerful method of investigating these differential equations is the matrix Riemann-Hilbert problem formalism [28]. The solution of the associated matrix RHP will allow us to obtain large distance asymptotics for the potentials B_{+-}, B_{++} and therefore the large distance asymptotics for the correlator. This method was first used in [47, 48] by Its, Izergin and Korepin to obtain the asymptotic behavior of correlation functions of impenetrable bosons.

In order to solve the RHP we are going to make use of the fact that the matrix RHP can be shown to be equivalent with a system of singular integral equations. In the large x limit we are going to use contour integration in order to extract the leading asymptotic terms. It should be mentioned that the same results can be obtained using the nonlinear steepest descent method of Deift and Zhou [26]. The constant (amplitude) in front of the leading term was obtained using a method of Kitanine, Kozlowski, Maillet, Slavnov and Terras [53] used by them in their investigation of the generalized sine-kernel. The presentation in this chapter follows [69, 70].

7.1 The matrix Riemann-Hilbert problem for the field-field corelator

Let us present the matrix Riemann-Hilbert problem relevant for our problem. In this chapter $\kappa \in (0, 1]$. We are looking for a 2×2 matrix valued function $\chi(\lambda)$, nonsingular for all $\lambda \in \mathbb{C}$, analytic in the upper and lower half-plane, it is equal with the unit matrix at $\lambda = \infty$

$$\chi(\infty) = I\,,$$

and the boundary values on the real axis satisfy the condition

$$\chi_-(\lambda) \;=\; \chi_+(\lambda)G(\lambda)\,, \qquad \chi_\pm(\lambda) = \lim_{\epsilon\to 0^+} \chi(\lambda \pm i\epsilon) \;\; \lambda \in \mathbb{R}\,. \tag{7.1}$$

The matrix $G(\lambda)$ is called the conjugation matrix associated with the RHP and is defined only for λ real. In our case it has the form

$$G(\lambda) = \begin{pmatrix} 1 + \pi\gamma e_+(\lambda)e_-(\lambda) & -\pi\gamma e_+^2(\lambda) \\ \pi\gamma e_-^2(\lambda) & 1 - \pi\gamma e_+(\lambda)e_-(\lambda) \end{pmatrix} \tag{7.2}$$

where $e_\pm(\lambda) = \sqrt{\vartheta(\lambda)}e^{\pm i\lambda x}$, $\theta(\lambda) = (e^{\lambda^2-\beta} + 1)^{-1}$ is the rescaled Fermi weight and $\gamma = (1 + e^{i\pi\kappa})/\pi$. $G(\lambda)$ and $\chi(\lambda)$ depend also on x, β and κ but this dependence will be suppressed in our notation.

7.1.1 Connection with the auxiliary potentials

In this section we are going to show how we can obtain the auxiliary potentials from the solution of the RHP (7.1). The normalization condition $\chi(\infty) = I$ means that the solution of the RHP has the following expansion

$$\chi(\lambda) = I + \frac{\Psi_1}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\,, \tag{7.3}$$

where $\Psi_1(x, \beta)$ is a 2×2 matrix depending only on β and x. Then the following symmetry of the conjugation matrix

$$G(\lambda) = \sigma_1 G^{-1}(-\lambda)\sigma_1\,, \tag{7.4}$$

determines the structure of Ψ_1 which can be written as

$$\Psi_1 = \frac{1}{2i}\begin{pmatrix} B_{+-} & -B_{++} \\ B_{++} & -B_{+-} \end{pmatrix}\,, \tag{7.5}$$

where the constant factor $1/2i$ was introduced for convenience. At the moment we do not know that B_{++} and B_{+-} in (7.5) are the auxiliary potentials (6.10). In order to prove this assertion we are going to use the singular integral formulation of the RHP. It easy to prove (see Chap. XV of [54]) that the matrix RHP is equivalent with the following system of singular integral equations

$$\chi_+(\lambda) = I + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\chi_+(\mu)[I - G(\mu)]}{\mu - \lambda - i0} d\mu\,, \quad \lambda \in \mathbb{R}\,. \tag{7.6}$$

Then, if we define

$$\tilde{\chi}(\lambda) = \chi_+(\lambda)E(\lambda)\,, \tag{7.7}$$

where $E(\lambda)$ is the triangular matrix

$$E(\lambda) = \begin{pmatrix} 1 & e_+(\lambda) \\ 0 & e_-(\lambda) \end{pmatrix}$$

the system of integral equations (7.6) can be rewritten as

$$\tilde{\chi}(\lambda) = E(\lambda) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\tilde{\chi}(\mu)\tilde{G}(\mu,\lambda)}{\mu - \lambda - i0} d\mu\,, \quad \lambda \in \mathbb{R}\,, \tag{7.8}$$

with

$$\begin{aligned} \tilde{G}(\lambda,\mu) &= E^{-1}(\mu)[I - G(\mu)]E(\lambda) \\ &= \begin{pmatrix} 0 & 0 \\ -\pi\gamma e_-(\mu) & \pi\gamma(e_+(\mu)e_-(\lambda) - e_+(\lambda)e_-(\mu)) \end{pmatrix}. \end{aligned}$$

Explicitly, from (7.8) we obtain

$$\begin{aligned} \tilde{\chi}_{12} &= e_+(\lambda) + \gamma\int_{-\infty}^{+\infty} K_T(\lambda,\mu)\tilde{\chi}_{12}(\mu)d\mu\,, \\ \tilde{\chi}_{22} &= e_-(\lambda) + \gamma\int_{-\infty}^{+\infty} K_T(\lambda,\mu)\tilde{\chi}_{22}(\mu)d\mu\,, \end{aligned}$$

with the kernel

$$K_T(\lambda,\mu) = \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{2i(\lambda - \mu)}\,, \tag{7.9}$$

and

$$
\begin{aligned}
\tilde{\chi}_{11}(\lambda) &= 1 - \frac{\gamma}{2i} \int_{-\infty}^{+\infty} \frac{\tilde{\chi}_{12}(\mu) e_-(\mu)}{\mu - \lambda - i0} d\mu \,, \\
\tilde{\chi}_{21}(\lambda) &= -\frac{\gamma}{2i} \int_{-\infty}^{+\infty} \frac{\tilde{\chi}_{22}(\mu) e_+(\mu)}{\mu - \lambda - i0} d\mu \,.
\end{aligned}
$$

The first two equations are just the integral equations (6.8) defining $f_{\pm}(\lambda)$, therefore we have

$$
\tilde{\chi}_{12}(\lambda) = f_+(\lambda) \,, \quad \tilde{\chi}_{22}(\lambda) = f_-(\lambda) \,, \tag{7.10}
$$

and noting from (7.7) that $\tilde{\chi}_{11}(\lambda) = \chi_{11,+}(\lambda)$ and $\tilde{\chi}_{21}(\lambda) = \chi_{22,+}(\lambda)$ then we obtain

$$
\begin{aligned}
\chi_{11}(\lambda) &= 1 - \frac{\gamma}{2i} \int_{-\infty}^{+\infty} \frac{f_+(\mu) e_-(\mu)}{\mu - \lambda} d\mu \,, \quad \Im\lambda > 0 \,, \\
\chi_{21}(\lambda) &= -\frac{\gamma}{2i} \int_{-\infty}^{+\infty} \frac{f_-(\mu) e_-(\mu)}{\mu - \lambda} d\mu \,, \quad \Im\lambda > 0 \,.
\end{aligned} \tag{7.11}
$$

Taking the limit $\lambda \to \infty$ in (7.11) we obtain

$$
\begin{aligned}
\chi_{11}(\lambda) &= 1 + \frac{\gamma}{2i\lambda} \int_{-\infty}^{+\infty} f_+(\mu) e_-(\mu) d\mu + O\left(\frac{1}{\lambda^2}\right) \,, \\
\chi_{21}(\lambda) &= \frac{\gamma}{2i\lambda} \int_{-\infty}^{+\infty} f_-(\mu) e_-(\mu) d\mu + O\left(\frac{1}{\lambda^2}\right) \,.
\end{aligned}
$$

which shows that the components of the matrix Ψ_1 are given by (6.10). Also it can be shown using (7.7) and (7.10) that

$$
\begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix} = \chi_+(\lambda) \begin{pmatrix} e_+(\lambda) \\ e_-(\lambda) \end{pmatrix} \,. \tag{7.12}
$$

7.1.2 Lax representation

Before we go any further we should investigate for what values of our parameters the RHP (7.1) has a unique solution. In Appendix I it is shown that our RHP is uniquely solvable except for when x is in a countable set denoted by X. In the following section we are going to consider that x is not in X which means that (7.1) has a unique solution. Now we are going to show that the function $\Psi(\lambda)$ defined by

$$\Psi(\lambda) = \chi(\lambda) e^{i\lambda x\sigma_3}\,, \tag{7.13}$$

is a matrix solution of the linear system (6.12) which is called the Lax representation. $\Psi(\lambda)$ is the solution of the matrix RHP

$$\begin{aligned} \Psi_-(\lambda) &= \Psi_+(\lambda) G_0(\lambda)\,, \quad \lambda \in \mathbb{R}\,, \\ \Psi(\infty) &= e^{i\lambda x\sigma_3}\,, \end{aligned}$$

with the conjugation matrix G_0 given by

$$G_0(\lambda) \equiv e^{-i\lambda x\sigma_3} G(\lambda) e^{i\lambda x\sigma_3} = \begin{pmatrix} 1+\pi\gamma\vartheta(\lambda) & -\pi\gamma\vartheta(\lambda) \\ \pi\gamma\vartheta(\lambda) & 1-\pi\gamma\vartheta(\lambda) \end{pmatrix}.$$

The main advantage of the matrix G_0 is the fact that does not depend on x and the dependence on β is through the Fermi distribution function $\vartheta(\lambda)$ which means that

$$\partial_x G_0(\lambda) = 0\,, \qquad (2\lambda\partial_\beta + \partial_\lambda) G_0(\lambda) = 0\,.$$

Consequently, the derivatives of $\Psi(\lambda)$ will satisfy the same RHP as $\Psi(\lambda)$

$$\begin{aligned} [\partial_x \Psi(\lambda)]_- &= [\partial_x \Psi(\lambda)]_+ G_0(\lambda)\,, \\ [(2\lambda\partial_\beta + \partial_\lambda)\Psi(\lambda)]_- &= [(2\lambda\partial_\beta + \partial_\lambda)\Psi(\lambda)]_+ G_0(\lambda)\,, \end{aligned}$$

but with different behavior for large λ, and the logarithmic derivatives are entire functions of λ

$$\begin{aligned} F_1(\lambda) &\equiv [\partial_x \Psi(\lambda)]\, \Psi^{-1}(\lambda)\,, \\ F_2(\lambda) &\equiv [(2\lambda\partial_\beta + \partial_\lambda)\Psi(\lambda)]\, \Psi^{-1}(\lambda)\,, \end{aligned} \tag{7.14}$$

denoted by $F_1(\lambda)$ and $F_2(\lambda)$. The leading terms of these functions when λ is large can be obtain easily from (7.3) and (7.13)

$$F_1(\lambda) = i\lambda\sigma_3 + O(1)\,, \quad F_2(\lambda) = O(1)\,, \;\; \lambda \to \infty\,,$$

so from Liouville theorem we conclude that F_1 and F_2 are polynomials of the first and zeroth degree respectively

$$F_1(\lambda) = i\lambda\sigma_3 + U_0\,, \quad F_2(\lambda) = V_0\,.$$

The matrices $U_0(x,\beta,\gamma)$ and $V_0(x,\beta,\gamma)$ can be expressed in terms of Ψ_1 if we use the fact that $\chi^{-1}(\lambda) = 1 - \Psi_1/\lambda$ when $\lambda \to \infty$, with the result

$$U_0 = i[\Psi_1, \sigma_3]\,, \quad V_0 = i\sigma_3 + 2\partial_\beta \Psi_1\,.$$

Using (7.5) we can obtain explicit expressions of U_0 and V_0 in terms of B_{+-}, B_{++}, and we can rewrite (7.14) as

$$\begin{aligned} \partial_x \Psi(\lambda) &= [i\lambda\sigma_3 + B_{++}\sigma_1]\,\Psi(\lambda)\,, \\ (2\lambda\partial_\beta + \partial_\lambda)\Psi(\lambda) &= [ix\sigma_3 - i(\partial_\beta B_{+-})\sigma_3 - (\partial_\beta B_{++})\sigma_2]\,\Psi(\lambda)\,. \end{aligned} \tag{7.15}$$

We recognize in the RHS of (7.15) the L and M operators (6.16),(6.25), but now they act on a 2×2 matrix. As an immediate consequence of the Lax representation (7.15) the potentials B_{+-}, B_{++} satisfy the differential equations (6.26),(6.27) and (6.30) (see Thm 2 for a proof).

7.1.3 An useful transformation of the RHP

In the previous section we showed that the potentials B_{++}, B_{+-} that characterize the field-field correlator (see (6.4),(6.11) and (6.37)) can be obtained from the expansion of the solution of the matrix RHP (7.1). This means that the large distance asymptotics of the potentials, and hence of the field corrrelator, can be obtained from the solution of the RHP. Our analysis will use the the equivalent formulation of the RHP in terms of the singular integral equation (7.6) so as a first step we will try to use a matrix RHP whose conjugation matrix has 1 on the diagonal. In order to achieve this goal we will introduce a new matrix Φ defined by

$$\Phi(\lambda) = \chi(\lambda) \begin{pmatrix} \beta^{-1}(\lambda) & 0 \\ 0 & \alpha^{-1}(\lambda) \end{pmatrix}\,,$$

where the functions $\alpha(\lambda), \beta(\lambda)$ are defined by

$$\alpha(\lambda) = \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\mu}{\mu - \lambda} \ln\left(\frac{e^{\mu^2-\beta} - e^{i\pi\kappa}}{e^{\mu^2-\beta} + 1} \right) \right\}\,, \tag{7.16}$$

with the branch of the logarithm specified by

$$\ln\left(\frac{e^{\mu^2-\beta} - e^{i\pi\kappa}}{e^{\mu^2-\beta} + 1} \right) \to 0, \quad \mu \to \infty\,,$$

and

$$\beta(\lambda) = \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\mu}{\mu - \lambda} \ln\left(\frac{e^{\mu^2-\beta} + e^{i\pi\kappa} + 2}{e^{\mu^2-\beta} + 1} \right) \right\},$$

with the branch of the logarithm specified by

$$\ln\left(\frac{e^{\mu^2-\beta} + e^{i\pi\kappa} + 2}{e^{\mu^2-\beta} + 1} \right) \to 0, \quad \mu \to \infty\,.$$

The functions $\alpha(\lambda)$ and $\beta(\lambda)$ are analytic in the upper and lower half-plane, remember $\kappa \in (0,1]$, and they are the solutions of the following scalar Riemann-Hilbert problems (see Appendix J)

$$\begin{aligned} \alpha_-(\lambda) &= \alpha_+(\lambda) g_\alpha(\lambda)\,, \;\; \lambda \in \mathbb{R}\,, \\ \alpha(\infty) &= 1\,, \\ g_\alpha(\lambda) &= \frac{e^{\mu^2-\beta} - e^{i\pi\kappa}}{e^{\mu^2-\beta} + 1}\,, \end{aligned} \tag{7.17}$$

and

$$\begin{aligned} \beta_-(\lambda) &= \beta_+(\lambda) g_\beta(\lambda)\,, \;\; \lambda \in \mathbb{R}\,, \\ \beta(\infty) &= 1\,, \\ g_\beta(\lambda) &= \frac{e^{\mu^2-\beta} + e^{i\pi\kappa} + 2}{e^{\mu^2-\beta} + 1}\,, \end{aligned} \tag{7.18}$$

and have the properties

$$\alpha^{-1}(\lambda) = \alpha(-\lambda)\,, \quad \beta^{-1}(\lambda) = \beta(-\lambda)\,.$$

The matrix Φ satisfies the RHP (see(7.1))

$$\begin{aligned} \Phi_-(\lambda) &= \Phi_+(\lambda) G_\Phi(\lambda)\,, \quad \lambda \in \mathbb{R}\,, \\ \Phi(\lambda) &= 1\,, \end{aligned} \tag{7.19}$$

with the conjugation matrix

$$G_\Phi(\lambda) = \begin{pmatrix} 1 & -\pi\gamma\vartheta(\lambda)\beta_+(\lambda)\alpha_-^{-1}(\lambda)e^{2i\lambda x} \\ \pi\gamma\vartheta(\lambda)\alpha_+(\lambda)\beta_-^{-1}e^{-2i\lambda x} & 1 \end{pmatrix}. \tag{7.20}$$

In terms of the components of the matrix Φ that solves the RHP (7.19) the potentials B_{++}, B_{-+} are expressed as

$$\begin{aligned} B_{+-} &= 2i \lim_{\lambda\to\infty} \lambda \left[\Phi_{11}(\lambda) - \beta^{-1}(\lambda)\right] \,, \\ B_{++} &= -2i \lim_{\lambda\to\infty} \lambda \Phi_{12}(\lambda) \,. \end{aligned} \tag{7.21}$$

The RHP (7.19) and formulae (7.21) will constitute the basis of our analysis of the large distance asymptotics of the field-field correlator of impenetrable anyons.

7.2 Large distance asymptotic analysis

In this section we are going to perform the large distance asymptotic analysis of the RHP (7.1). Using (7.21) we will obtain the auxiliary potentials $B_{++}(x,\beta,\kappa), B_{+-}(x,\beta,\kappa)$ from the large λ expansion of $\Phi_{11,+}(\lambda), \Phi_{12,+}(\lambda)$. Then $\sigma(x,\beta,\kappa)$ can be calculated using the differential equations (6.32), (6.33).

7.2.1 The auxiliary potentials in the large x limit

From the integral equation formulation (7.6) of the RH problem (7.19), and using the properties of α and β we obtain the following expressions for $\Phi_{11,+}(\lambda)$, $\Phi_{12,+}(\lambda)$

$$\Phi_{11,+}(\lambda) = 1 - \frac{(1+e^{i\pi\kappa})}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi_{12,+}(\mu)}{(\mu-\lambda-i0)} \frac{\alpha_-(\mu)}{\beta_-(\mu)} \frac{e^{-2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})}\,, \quad \Im\lambda = 0\,, \tag{7.22}$$

and

$$\Phi_{12,+}(\lambda) = \frac{(1+e^{i\pi\kappa})}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi_{11,+}(\mu)}{(\mu-\lambda-i0)} \frac{\beta_+(\mu)}{\alpha_+(\mu)} \frac{e^{2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})}\,, \quad \Im\lambda = 0\,. \tag{7.23}$$

Using the analyticity of α and β and $\Phi_{11}(\infty) = \alpha(\infty) = \beta(\infty) = 1$, we are trying to obtain an estimate of $\Phi_{12,+}$ by shifting the contour in the upper half-plane and the replacing the integral by the sum of the residues. Rigorously speaking this will require an estimate of the type $|\Phi_{12}(\lambda,x,\beta)| \leq C/\lambda$ and as we will see further $|\Phi_{11}(\lambda,\beta,x) - 1| \leq D/\lambda$ for $\Im\lambda \geq 0\,, \beta < \beta_0\,, x < x_0$ where C, D depends on β_0, x_0 only. In what follows we will assume that these estimates hold, noting that these assumptions can be justified self-consistently as in Sect. 4 of [48]. The poles of the integrand in (7.23) are given by $\lambda + i0$

and the zeros of the function

$$e^{\lambda^2-\beta} - e^{i\pi\kappa}$$

situated in the upper half-plane. Explicitly these zeros are given by the formulae

$$(\Re\lambda_k)^2 = \frac{1}{2}\left(\beta + \sqrt{\beta^2 + \pi^2[\kappa+2k]^2}\right)\,,$$

$$(\Im\lambda_k)^2 = \frac{1}{2}\left(-\beta + \sqrt{\beta^2 + \pi^2[\kappa+2k]^2}\right)\,,\quad k = 0, \pm 1, \pm 2, \cdots \tag{7.24}$$

and for us an important role will be played by λ_0^+ and λ_{-1}^+ given by

$$\lambda_0^+ = \left(\beta + \sqrt{\beta^2 + \pi^2\kappa^2}\right)^{1/2}/\sqrt{2} + i\left(-\beta + \sqrt{\beta^2 + \pi^2\kappa^2}\right)^{1/2}/\sqrt{2}\,, \tag{7.25}$$

and

$$\lambda_{-1}^+ = -\left(\beta + \sqrt{\beta^2 + \pi^2[\kappa-2]^2}\right)^{1/2}/\sqrt{2} + i\left(-\beta + \sqrt{\beta^2 + \pi^2[\kappa-2]^2}\right)^{1/2}/\sqrt{2}\,, \tag{7.26}$$

where the superscript + distinguishes the solutions from the upper half-plane. Closing the contour in the upper half-plane we obtain

$$\Phi_{12,+}(\lambda) = (1 + e^{i\pi\kappa})\Phi_{11,+}(\lambda)\frac{\beta_+(\lambda)}{\alpha_+(\lambda)}\frac{e^{2i\lambda x}}{(e^{\lambda^2-\beta} - e^{i\pi\kappa})} + S^+(\lambda)\,, \tag{7.27}$$

with

$$S^+(\lambda) = \sum_{k=-\infty}^{+\infty} \frac{(1+e^{i\pi\kappa})}{2e^{i\pi\kappa}}\frac{\beta(\lambda_k^+)}{\alpha(\lambda_k^+)}\frac{\Phi_{11}(\lambda_k^+)e^{2i\lambda_k^+ x}}{(\lambda_k^+ - \lambda)\lambda_k^+}\,.$$

The series $S^+(\lambda)$ is uniformly convergent for $\lambda \in \mathbb{R},\ x_0 \le x,\ \beta_1 \le \beta \le \beta_0$ and $0 < \kappa \le 1$. Plugging (7.27) in (7.22) and using the scalar RH problems (7.17) and (7.18) we obtain the following representation

$$\begin{aligned}\Phi_{11,+}(\lambda) &= 1 - \frac{(1+e^{i\pi\kappa})^2}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Phi_{11,+}(\mu)}{\mu-\lambda-i0}\frac{d\mu}{(e^{\mu^2-\beta}-e^{i\pi\kappa})(e^{\mu^2-\beta}+e^{i\pi\kappa}+2)}\\ &\quad -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{R^+(\mu)}{\mu-\lambda-i0}d\mu\,,\end{aligned} \tag{7.28}$$

with

$$R^+(\mu) = (1+e^{i\pi\kappa})\frac{\alpha_-(\mu)}{\beta_-(\mu)}\frac{e^{-2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})}S^+(\mu)\,.$$

The particular form of the integral equation(7.28) (see J.5) indicates that it is useful to consider the following inhomogeneous scalar RH problem (Appendix J)

$$\begin{aligned}\tilde{\Phi}_-(\lambda) &= \tilde{\Phi}_+(\lambda)g(\lambda) + R^+(\lambda)\,, \quad \lambda \in \mathbb{R}\,, \\ \tilde{\Phi}_+(\lambda) &= \Phi_{11,+}(\lambda)\,, \quad \lambda \in \mathbb{R}\,, \\ \tilde{\Phi}(\infty) &= 1\,, \end{aligned} \tag{7.29}$$

with

$$1 - g(\lambda) = -\frac{(1+e^{i\pi\kappa})^2}{(e^{\lambda^2-\beta} - e^{i\pi\kappa})(e^{\lambda^2-\beta} + e^{i\pi\kappa} + 2)}\,,$$

Simple computations corroborated with (7.17) and (7.18) allow us to obtain the following expression for g

$$g(\lambda) = \frac{(e^{\lambda^2-\beta}+1)^2}{(e^{\lambda^2-\beta} - e^{i\pi\kappa})(e^{\lambda^2-\beta} + e^{i\pi\kappa} + 2)} = \frac{\alpha_-^{-1}(\lambda)\beta_-^{-1}(\lambda)}{\alpha_+^{-1}(\lambda)\beta_+^{-1}(\lambda)}\,.$$

Using (J.4) the solution of the RH problem (7.29) is given by

$$\tilde{\Phi}(\lambda) = \alpha^{-1}(\lambda)\beta^{-1}(\lambda) - \frac{\alpha^{-1}(\lambda)\beta^{-1}(\lambda)}{2\pi i}\int_{-\infty}^{+\infty} \frac{\alpha_-(\mu)\beta_-(\mu)R(\mu)}{\mu-\lambda}d\mu\,, \quad \lambda \in \mathbb{C}/\mathbb{R}\,. \tag{7.30}$$

The functions $\tilde{\Phi}(\lambda)$ and $\Phi_{11}(\lambda)$ are analytic in the upper half-plane have the same behavior at infinity $\tilde{\Phi}(\infty) = \Phi_{11}(\infty) = 1$ and their boundary values close to the real axis $\tilde{\Phi}_+(\lambda), \Phi_{11,+}(\lambda)\,, \lambda \in \mathbb{R}$ are equal which means that $\Phi_{11}(\lambda) = \tilde{\Phi}(\lambda)$ for $\Im\lambda > 0$. Therefore from (7.30) and the explicit expression for $R^+(\lambda)$ we have

$$\Phi_{11}(\lambda) = \alpha^{-1}(\lambda)\beta^{-1}(\lambda) - \frac{\alpha^{-1}(\lambda)\beta^{-1}(\lambda)}{2\pi i}\sum_{k=-\infty}^{+\infty} A(\lambda_k^+,\lambda)\Phi_{11}(\lambda_k^+)\,, \quad \Im\lambda \geq 0\,,$$

where

$$A(\lambda_k^+,\lambda) = \frac{(1+e^{i\pi\kappa})^2}{2\lambda_k^+ e^{i\pi\kappa}}\frac{\beta(\lambda_k^+)}{\alpha(\lambda_k^+)}e^{2i\lambda_k^+ x}\int_{-\infty}^{+\infty}\frac{\alpha_-^2(\mu)e^{-2i\mu x}}{(e^{\mu^2-\beta} - e^{i\pi\kappa})(\lambda_k^+ - \mu)(\mu-\lambda)}d\mu\,.$$

The integral in the last term can be estimated as $C|e^{-2i\lambda_0^- x}|$, where λ_0^- is the solution of (7.24) for $k=0$ in the lower half-plane, so the leading term of Φ_{11} when $x \to \infty$ is given by

$$\Phi_{11}(\lambda) = \alpha^{-1}(\lambda)\beta^{-1}(\lambda) + O(e^{-4\Im\lambda_0^+ x})\,, \quad \Im\lambda > 0\,. \tag{7.31}$$

Using (7.23) we obtain the leading term of Φ_{12} when x is large

$$\Phi_{12}(\lambda)=\frac{(1+e^{i\pi\kappa})}{2\pi i}\int_{-\infty}^{+\infty}\frac{\alpha_+^{-2}(\mu)}{(\mu-\lambda-i0)}\frac{e^{2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})}+O(e^{-4\Im\lambda_0^+x})\,,\quad \Im\lambda>0\,. \tag{7.32}$$

The large distance asymptotics of the potentials B_{++}, B_{+-} are obtained from (7.31) and (7.32) using (7.21) with the results

$$B_{+-}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\ln\left(\frac{e^{\mu^2-\beta}+1}{e^{\mu^2-\beta}-e^{i\pi\kappa}}\right)d\mu+O(e^{-4\Im\lambda_0^+x})\,,\ x\to\infty \tag{7.33}$$

where the branch of the logarithm is fixed as

$$\ln\left(\frac{e^{\mu^2-\beta}+1}{e^{\mu^2-\beta}-e^{i\pi\kappa}}\right)\to 0\,,\quad \mu\to\infty\,, \tag{7.34}$$

and

$$B_{++}=\frac{(1+e^{i\pi\kappa})}{\pi}\int_{-\infty}^{+\infty}\frac{\alpha_+^{-2}(\mu)e^{2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})}+O(e^{-4\Im\lambda_0^+x})\,,\ x\to\infty\,.$$

The expression for B_{++} can be made more precise if we close the contour in the upper half-plane obtaining

$$B_{++}=i(1+e^{-i\pi\kappa})\sum_{k=min}^{max}\frac{\alpha^{-2}(\lambda_k^+)}{\lambda_k^+}e^{2i\lambda_k^+x}+O(e^{-4\Im\lambda_0^+x})\,,\ x\to\infty\,, \tag{7.35}$$

where $min(\beta,\kappa)<0$ and $max(\beta,\kappa)>0$ are defined by the relations

$$\Im\lambda_{min}^+<2\Im\lambda_0^+\,,\quad \Im\lambda_{min-1}^+>2\Im\lambda_0^+\,, \tag{7.36}$$

and

$$\Im\lambda_{max}^+<2\Im\lambda_0^+\,,\quad \Im\lambda_{max+1}^+>2\Im\lambda_0^+\,. \tag{7.37}$$

7.2.2 Determination of $\sigma(x,\beta,\kappa)$ in the large x limit

Define

$$\nu(\lambda,\beta)=\frac{1}{2\pi i}\log\left(\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta}-e^{i\pi\kappa}}\right)\,, \tag{7.38}$$

where, $\kappa \in [0,1)$ is the statistics parameter and the branch of the logarithm is specified by

$$\lim_{\lambda\to\infty} \log\left(\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta}-e^{i\pi\kappa}}\right) \to 0\,,$$

and no branch cut intersects the real axis. The function $C(\beta,\kappa)$ and is introduced as

$$C(\beta,\kappa) \equiv 2i\int_{-\infty}^{+\infty} \nu(\lambda,\beta)d\lambda = \frac{1}{\pi}\int_{-\infty}^{+\infty} \log\left(\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta}-e^{i\pi\kappa}}\right) d\lambda\,. \tag{7.39}$$

Then using the the large distance asymptotics for the auxiliary potentials obtained in the previous section and the differential equations (6.32) and (6.33) we obtain

$$\sigma(x,\beta,\kappa) = -xC(\beta,\kappa) + c(\beta,\kappa) + O(e^{-4\Im\lambda_0^+ x})\,,\ x\to\infty\,, \tag{7.40}$$

where $c(\beta,\kappa)$ is a constant that that depends on β and κ but not on x and needs to be determined. In order to obtain this constant which appears in the expression for $\sigma(x,\beta,\kappa)$ we are going to integrate the relation

$$\partial_\gamma \sigma = -\int_{-\infty}^{+\infty} R_T(\lambda,\lambda)\; d\lambda\,, \tag{7.41}$$

which can be obtained from the differentiation of (6.36). In the previous expression $R_T(\lambda,\lambda)$ is given by

$$R_T(\lambda,\lambda) = \frac{1}{2i}(\partial_\lambda f_+(\lambda) f_-(\lambda) - f_+(\lambda)\partial_\lambda f_-(\lambda))\,. \tag{7.42}$$

The technique used is similar with the one employed in [53]. In the cited paper the authors considered a generalized sine-kernel at zero temperature and studied the large distance asymptotic behavior of the Fredholm determinant using the nonlinear steepest descent method of Deift and Zhou [26]. At zero temperature $\det(1-\gamma\hat{K})$ becomes a particular case of the Fredholm determinant considered in [53].

The first step will be the computation of $f_\pm(\lambda)$ using the relation

$$\begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix} = \chi_+(\lambda) \begin{pmatrix} e_+(\lambda) \\ e_-(\lambda) \end{pmatrix}\,. \tag{7.43}$$

Remember that

$$\chi(\lambda) = \Phi(\lambda) \begin{pmatrix} \beta(\lambda) & 0 \\ 0 & \alpha(\lambda) \end{pmatrix}, \tag{7.44}$$

but and in the previous sections we have obtained only $\Phi_{11}(\lambda)$ and $\Phi_{12}(\lambda)$ in the large x limit. We have

$$\Phi_{11}(\lambda) = \alpha^{-1}(\lambda)\beta^{-1}(\lambda) + O(e^{-4\Im\lambda_0^+ x}), \quad \Im\lambda > 0, \tag{7.45}$$

and

$$\Phi_{12}(\lambda) = \frac{(1+e^{i\pi\kappa})}{2\pi i} \int_{-\infty}^{+\infty} \frac{\alpha_+^{-2}(\mu)}{(\mu-\lambda-i0)} \frac{e^{2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})} + O(e^{-4\Im\lambda_0^+ x}), \quad \Im\lambda > 0. \tag{7.46}$$

If in the expression for Φ_{12} we close the contour in the upper half plane we obtain

$$\Phi_{12}(\lambda) = \pi\gamma \frac{\alpha_+^{-2}(\lambda)}{(e^{\lambda^2-\beta}-e^{i\pi\kappa})} e^{2i\lambda x} + O(e^{-2\Im\lambda_0^+ x}). \tag{7.47}$$

We need to obtain similar relations for Φ_{21} and Φ_{22}. From the integral formulation of the RHP (7.6) and the properties of $\alpha(\lambda)$ we have

$$\Phi_{21,+}(\lambda) = -\frac{\gamma}{2i} \int_{-\infty}^{+\infty} \frac{\Phi_{22,+}(\mu)}{\mu-\lambda-i0} \frac{\alpha_-(\mu)}{\beta_-(\mu)} \frac{e^{-2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})} d\mu, \quad \lambda \in \mathbb{R}, \tag{7.48}$$

and

$$\Phi_{22,+}(\lambda) = 1 + \frac{\gamma}{2i} \int_{-\infty}^{+\infty} \frac{\Phi_{21,+}(\mu)}{\mu-\lambda-i0} \frac{\beta_+(\mu)}{\alpha_+(\mu)} \frac{e^{2i\mu x}}{(e^{\mu^2-\beta}-e^{i\pi\kappa})} d\mu, \quad \lambda \in \mathbb{R}. \tag{7.49}$$

In the integral equation (7.48) we can close the contour in the lower half plane obtaining

$$\Phi_{21,+}(\lambda) = -\pi\gamma \sum_{k=-\infty}^{\infty} \frac{\Phi_{22,+}(\lambda_k^-)}{\lambda_k^- - \lambda - i0} \frac{\alpha_-(\lambda_k^-)}{\beta_-(\lambda_k^-)} \frac{e^{-2|\Im\lambda_k^-|x}}{2\lambda_k^- e^{i\pi\kappa}},$$

where λ_k^- are the zeroes of $e^{\mu^2-\beta} - e^{i\pi\kappa}$ in the lower half plane. Plugging this result in the integral equation for $\Phi_{22,+}$ and closing the contour in the upper half plane we obtain

$$\Phi_{22,+}(\lambda) = 1 + O(e^{-2(|\Im\lambda_0^-| + \Im\lambda_0^+)x}). \tag{7.50}$$

Using (7.50) in (7.48) we have

$$\Phi_{21,+}(\lambda) = O(e^{-2|\Im\lambda_0^-|x}) \,. \tag{7.51}$$

Now we are able to compute $f_\pm(\lambda)$ up to exponentially small corrections in x. Explicitly from (7.44) and (7.43) we have

$$\begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix} = \begin{pmatrix} \Phi_{11,+}(\lambda)\beta_+(\lambda)e_+(\lambda) + \Phi_{12,+}(\lambda)\alpha_+(\lambda)e_-(\lambda) \\ \Phi_{21,+}(\lambda)\beta_+(\lambda)e_+(\lambda) + \Phi_{22,+}(\lambda)\alpha_+(\lambda)e_-(\lambda) \end{pmatrix} ,$$

therefore using (7.45), (7.47), (7.50) and (7.51) we obtain

$$\begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix} = \begin{pmatrix} \alpha_+^{-1}(\lambda)e_+(\lambda)e^{2\pi i\nu(\lambda)} \\ \alpha_+(\lambda)e_-(\lambda) \end{pmatrix} , \tag{7.52}$$

where $\nu(\lambda)$ (we have suppressed the dependence on β) is defined in (7.38). It is easy to see that $\nu(\lambda)$ can be written in terms of the rescaled Fermi weight function $\theta(\lambda) = (1 + e^{\lambda^2-\beta})^{-1}$ as

$$\begin{aligned} \nu(\lambda) &= -\frac{1}{2\pi i}\log(1 - \pi\gamma\theta(\lambda)) \,, \\ &= \frac{1}{2\pi i}\log\left(\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta} - e^{i\pi\kappa}}\right) . \end{aligned}$$

Also we have

$$\alpha_+(\lambda) = \exp\left\{ i\pi\nu(\lambda) + \text{P.V.}\int_{-\infty}^{+\infty} \frac{\nu(\mu)}{\mu-\lambda}d\mu \right\} .$$

Using (7.52) and (7.42) in (7.41) we obtain

$$\begin{aligned} \partial_\gamma\sigma &= -\int_{-\infty}^{+\infty} R_T(\lambda,\lambda)d\lambda \,, \\ &= -\int_{-\infty}^{+\infty} \frac{1}{2\pi i}\frac{\pi\theta(\lambda)}{(1-\pi\gamma\theta(\lambda))}(2ix - 2\partial_\lambda\log\alpha_+(\lambda) + 2i\pi\nu(\lambda))\, d\lambda \,, \\ &= -\int_{-\infty}^{+\infty} \partial_\gamma\nu(\lambda)(2ix - 2\partial_\lambda\log\alpha_+(\lambda) + 2i\pi\nu(\lambda))\, d\lambda \,. \end{aligned} \tag{7.53}$$

Our goal is to write the RHS of (7.53) as a derivative with respect to γ. The term depending on x is in this form but not the constant one. We have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \partial_\gamma \nu(\lambda)(2\partial_\lambda \log \alpha_+(\lambda) - 2i\pi\nu(\lambda)) \\
&= \int_{-\infty}^{+\infty} \partial_\gamma \nu(\lambda)\partial_\lambda \int_{-\infty}^{+\infty} \left(\frac{\nu(\mu)}{\mu - \lambda - i0} + \frac{\nu(\mu)}{\mu - \lambda + i0} \right) d\mu d\lambda \,, \\
&= \int_{-\infty}^{+\infty} \partial_\gamma \nu(\lambda) \left(\frac{\nu(\mu)}{(\lambda - \mu - i0)^2} + \frac{\nu(\mu)}{(\lambda - \mu + i0)^2} \right) d\mu d\lambda \,.
\end{aligned}$$

We will show that this expression is equal with

$$\partial_\gamma \int_{-\infty}^{+\infty} \frac{\partial_\lambda \nu(\lambda)\nu(\mu) - \nu(\lambda)\partial_\mu(\mu)}{2(\lambda - \mu)} \; d\lambda d\mu \,.$$

Indeed we have

$$\begin{aligned}
&\partial_\gamma \int_{-\infty}^{+\infty} \frac{\partial_\lambda \nu(\lambda)\nu(\mu) - \nu(\lambda)\partial_\mu(\mu)}{2(\lambda - \mu)} \; d\lambda d\mu \\
&= \int_{-\infty}^{+\infty} \frac{[(\partial_\gamma \partial_\lambda \nu(\lambda))\nu(\mu) + \partial_\lambda(\lambda)\partial_\gamma(\mu)]}{\lambda - \mu} d\lambda d\mu \,, \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} [(\partial_\gamma \partial_\lambda \nu(\lambda))\nu(\mu) + \partial_\lambda(\lambda)\partial_\gamma(\mu)] \left(\frac{1}{\lambda - \mu + i0} + \frac{1}{\lambda - \mu - i0} \right) d\lambda d\mu \,, \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} [\partial_\gamma \nu(\lambda)\nu(\mu) + \nu(\lambda)\partial_\gamma \nu(\mu)] \left(\frac{1}{(\lambda - \mu + i0)^2} + \frac{1}{(\lambda - \mu - i0)^2} \right) d\lambda d\mu \,, \\
&= \int_{-\infty}^{+\infty} \partial_\gamma \nu(\lambda) \left(\frac{\nu(\mu)}{(\lambda - \mu - i0)^2} + \frac{\nu(\mu)}{(\lambda - \mu + i0)^2} \right) d\mu d\lambda \,.
\end{aligned}$$

Therefore

$$\partial_\gamma \sigma = -2ix\partial_\gamma \int_{-\infty}^{+\infty} \nu(\lambda) d\lambda + \partial_\gamma \int_{-\infty}^{+\infty} \frac{\partial_\lambda \nu(\lambda)\nu(\mu) - \nu(\lambda)\partial_\mu(\mu)}{2(\lambda - \mu)} \; d\lambda d\mu \,,$$

and integrating with respect to γ and taking into account that $\sigma(\gamma = 0) = 0$ we obtain

$$\sigma(x, \beta, \kappa) = -xC(\beta, \kappa) + \int_{-\infty}^{+\infty} \frac{\partial_\lambda \nu(\lambda)\nu(\mu) - \nu(\lambda)\partial_\mu(\mu)}{2(\lambda - \mu)} \; d\lambda d\mu \,. \qquad (7.54)$$

Comparison with (7.40) shows that

$$c(\beta,\kappa)=\int_{-\infty}^{+\infty}\frac{\partial_\lambda\nu(\lambda)\nu(\mu)-\nu(\lambda)\partial_\mu(\mu)}{2(\lambda-\mu)}\,d\lambda d\mu\,. \tag{7.55}$$

7.2.3 Large distance asymptotic behavior for the field-field correlator

We remind the reader that the field-field correlation function is given by

$$\langle\Psi^\dagger(x_1)\Psi(x_2)\rangle_T=\frac{\sqrt{T}}{2\pi\gamma}B_{++}(x,\beta,\kappa)e^{\sigma(x,\beta,\kappa)}\,,\quad \gamma=(1+e^{i\pi\kappa})/\pi\,.$$

Using the results obtained in the previous sections for the potential B_{++} and σ, and going back to the original variables $x=x_{12}\sqrt{T}/2$ and $\beta=h/T$, the large distance behavior of the field-field correlator is given by

$$\langle\Psi^\dagger(x_1)\Psi(x_2)\rangle_T=e^{-x_{12}\frac{\sqrt{T}}{2}C(h/T,\kappa)}e^{c(h/T,\kappa)}\sum_{k=min}^{max}c_k e^{ix_{12}\sqrt{T}\lambda_k^+}+O(e^{-4\Im\lambda_0^+ x}) \tag{7.56}$$

where

$$c_k=i\frac{e^{i\pi\kappa}\sqrt{T}}{2}\frac{\alpha^{-2}(\lambda_k^+)}{\lambda_k^+}\,. \tag{7.57}$$

and λ_k^+ are the solutions of (7.24) in the upper half-plane. In (7.56) $C(\beta,\kappa)$ and $\alpha(\lambda)$ are defined in (7.39) and (7.17)and the summation limits are given by (7.36) and (7.37). The leading term is $k=0$ but as we approach the free fermionic point, $\kappa\to 1$, the $k=-1$ term also becomes relevant.

7.3 Analysis of the results

We will check the validity of our results in three appropriate limits. When the statistics parameter $\kappa\to 0$ we should reproduce the results obtained for impenetrable bosons in [46, 48, 54] and when $\kappa=1$ we should obtain the results for free fermions. Even though in the large distance analysis performed in the previous section we have not made any distinction between the cases of negative and positive chemical potential (or equivalently β) we will see that in the bosonic limit the asymptotic behavior of the corrrelation function is fundamentally different in the two regions as we would have expected from [46, 48, 54]. In the limit of low temperatures ($\beta\to\infty$) at positive chemical potential our system becomes critical and we can verify our result with the predictions of conformal field theory [19, 66].

7.3.1 The bosonic limit

The bosonic limit is defined by $\kappa \to 0$. Then from (7.39) in the case of positive chemical potential we obtain

$$\lim_{\kappa\to 0} C(\beta,\kappa) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln\left|\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta}-1}\right| d\lambda + 2i\sqrt{\beta}\,,\ \ \beta > 0\,, \tag{7.58}$$

and in the case of negative chemical potential

$$\lim_{\kappa\to 0} C(\beta,\kappa) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln\left(\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta}-1}\right) d\lambda\,,\quad \beta < 0\,. \tag{7.59}$$

Also from (7.25) we have

$$\lambda_0^+ = \sqrt{\beta}\,,\quad \beta > 0\,, \kappa \to 0\,, \tag{7.60}$$

$$\lambda_0^+ = i\sqrt{|\beta|}\,,\quad \beta < 0\,, \kappa \to 0\,. \tag{7.61}$$

Introducing the function $C(\beta)$ given by

$$C(\beta) \equiv \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln\left|\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta}-1}\right| d\lambda\,,\quad \beta = h/T\,, \tag{7.62}$$

using (7.56), (7.59) and (7.61) we obtain the following asymptotic behavior of the correlation function at negative chemical potential in the original variables ($x = (x_1 - x_2)\sqrt{T}/2, \beta = h/T$)

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T \simeq e^{-x_{12}\left[\frac{\sqrt{T}}{2}C(h/T)+\sqrt{|h|}\right]}\,,\qquad h < 0\,,\ \ x_{12} \equiv (x_1 - x_2) \to \infty\,. \tag{7.63}$$

In the case of positive chemical potential from (7.56), (7.58) and (7.60) we obtain

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T \simeq e^{-x_{12}\frac{\sqrt{T}}{2}C(h/T)}\,,\quad h > 0\,,\ \ x_{12} \equiv (x_1 - x_2) \to \infty\,. \tag{7.64}$$

Both asymptotics (7.63) and (7.64) agree with the result obtained for impenetrable bosons in [47, 48, 54].

7.3.2 Free fermionic limit

The field-field correlator for free fermions is given by

$$\langle \Psi_F^\dagger(x_1)\Psi_F(x_2)\rangle = \frac{\sqrt{T}}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ix_{12}\sqrt{T}\lambda}}{e^{\lambda^2-\beta}+1}d\lambda\,. \tag{7.65}$$

In the large x_{12} limit we can close the contour in the upper half-plane and take the contributions of the leading residues which are given by the zeros of the denominator. We obtain

$$\langle \Psi_F^\dagger(x_1)\Psi_F(x_2)\rangle \sim -i\frac{\sqrt{T}}{2}\sum_{k=0,-1}\frac{e^{ix_{12}\sqrt{T}\lambda_k^+}}{\lambda_k^+} \tag{7.66}$$

where λ_k^+ are given by (7.25) and (7.26) with $\kappa = 1$. This is exactly the same result that we obtain from (7.56) if we notice that $C(\beta,\kappa=1) = c(\beta,\kappa=1) = 0$ and $\alpha(\lambda,\kappa=1)=1$.

7.3.3 The conformal limit

Before we embark on the analysis of our result in the conformal limit it will be useful to investigate more thoroughly the function $C(\beta,\kappa)$ defined by (7.39). We will start with case of negative β which is simpler. Using the expansions $\ln(1+z) = \sum_{n=1}^\infty (-1)^{n+1}z^n/n\,,\ |z|<1$ and $\ln(1+z) = -\sum_{n=1}^\infty z^n/n\,,\ |z|<1$ we obtain

$$\begin{aligned} C(\beta,\kappa) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\sum_{n=1}^\infty\left(\frac{(-1)^{n+1}e^{-n(\lambda^2+|\beta|)}}{n}+\frac{e^{in\pi\kappa}e^{-n(\lambda^2+|\beta|)}}{n}\right)d\lambda\,,\\ &= \frac{1}{\sqrt{\pi}}\sum_{n=1}^\infty\left(\frac{(-1)^{n+1}e^{-n|\beta|}}{n^{3/2}}+\frac{e^{in\pi\kappa}e^{-n|\beta|}}{n^{3/2}}\right)\,. \end{aligned}$$

Therefore, in the large and small $|\beta|$ limits the leading terms are given by

$$C(\beta,\kappa) = \frac{e^{-|\beta|}}{\sqrt{\pi}}(1+\cos\pi\kappa) + i\frac{e^{-|\beta|}}{\sqrt{\pi}}\sin\pi\kappa\,,\quad (\beta\to-\infty)\,, \tag{7.67}$$

and

$$C(\beta,\kappa) = \frac{1}{\sqrt{\pi}}\sum_{n=1}^\infty\left(\frac{(-1)^{n+1}+\cos n\pi\kappa}{n^{3/2}}\right)+i\frac{1}{\sqrt{\pi}}\sum_{n=1}^\infty\frac{\sin n\pi\kappa}{n^{3/2}}\,,\ \ (\beta\to0,\ \beta<0)\,. \tag{7.68}$$

Let us now investigate the case of positive β. Using the same expansions for the logarithms we obtain

$$\begin{aligned} C(\beta,\kappa) &= \frac{2}{\pi}\int_0^{\sqrt{\beta}}\left[-i\pi(\kappa-1)+\sum_{n=1}^{\infty}e^{n(\lambda^2-\beta)}\frac{(-1)^{n+1}+e^{-in\pi\kappa}}{n}\right]d\lambda \\ &\quad +\frac{2}{\pi}\int_{\sqrt{\beta}}^{\infty}\sum_{n=1}^{\infty}e^{n(\lambda^2-\beta)}\frac{(-1)^{n+1}+e^{-in\pi\kappa}}{n}d\lambda\,. \end{aligned}$$

For β large we can use the formulae

$$e^{-\beta n}\int_0^{\sqrt{\beta}}e^{\lambda^2 n}d\lambda=\frac{1}{2n\sqrt{\beta}}+O\left(\frac{1}{\beta^{3/2}}\right)\,,\ e^{\beta n}\int_{\sqrt{\beta}}^{\infty}e^{-\lambda^2 n}d\lambda=\frac{1}{2n\sqrt{\beta}}+O\left(\frac{1}{\beta^{3/2}}\right)\,,$$

obtaining

$$C(\beta,\kappa)=\frac{2}{\pi\sqrt{\beta}}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^2}+\frac{\cos n\pi\kappa}{n^2}\right)-2i\sqrt{\beta}(\kappa-1)+O\left(\frac{1}{\beta^{3/2}}\right)\,.$$

Using the formulae (0.234) and (1.443) of [38] we have $\sum_{k=1}^{\infty}(-1)^{n+1}/n^2=\pi^2/12$ and $\sum_{k=1}^{\infty}\cos n\pi\kappa/n^2=\pi^2 B_2(\kappa/2)$ where $B_2(x)=x^2-x+1/6$ is the second Bernoulli polynomial which allow us to rewrite the previous result as

$$C(\beta,\kappa)=\frac{\pi}{\sqrt{\beta}}\left(\frac{\kappa^2}{2}-\kappa+\frac{1}{2}\right)-2i\sqrt{\beta}(\kappa-1)\,,\quad(\beta\to\infty)\,. \tag{7.69}$$

For small β from (7.69) we obtain

$$C(\beta,\kappa)=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}+\cos n\pi\kappa}{n^{3/2}}\right)+i\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\sin n\pi\kappa}{n^{3/2}}\,,\ (\beta\to 0,\ \beta>0)\,. \tag{7.70}$$

In the case of positive chemical potential at low temperatures the system is conformal. In the limit $\beta\to\infty$ we have

$$\begin{aligned} \lambda_0^+ &= \left[\left(\beta+\sqrt{\beta^2+\pi^2\kappa^2}\right)^{1/2}+i\left(-\beta+\sqrt{\beta^2+\pi^2\kappa^2}\right)^{1/2}\right]/\sqrt{2} \\ &\to \sqrt{\beta}+i\frac{\pi\kappa}{2\sqrt{\beta}}\,, \end{aligned} \tag{7.71}$$

$$\lambda_{-1}^{+} = \left[-\left(\beta+\sqrt{\beta^2+\pi^2[\kappa-2]^2}\right)^{1/2} + i\left(-\beta+\sqrt{\beta^2+\pi^2[\kappa-2]^2}\right)^{1/2}\right]/\sqrt{2}$$

$$\to -\sqrt{\beta} + i\frac{\pi|\kappa-2|}{2\sqrt{\beta}} . \tag{7.72}$$

and the behavior of $C(\beta,\kappa)$ in the same limit was studied above. In order to make the connection with the results obtained in [19, 66] we are going to use the original variables $x = (x_1 - x_2)\sqrt{T}/2, \beta = h/T$ and the fact that the Fermi momentum $k_F = \sqrt{h}$ and the Fermi velocity $v_F = 2\sqrt{h}$. Then from (7.56) (7.71) and (7.72) we obtain the following asymptotics for the field-field correlator at low temperatures

- For $0 < \kappa < 2/3$

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T \simeq c_0 e^{-x_{12}\frac{\pi T}{v_F}\left(\frac{\kappa^2}{2}+\frac{1}{2}\right)} e^{ix_{12}k_F\kappa} , \tag{7.73}$$

- For $2/3 < \kappa < 1$

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T \simeq c_0 e^{-x_{12}\frac{\pi T}{v_F}\left(\frac{\kappa^2}{2}+\frac{1}{2}\right)} e^{ix_{12}k_F\kappa} + c_{-1} e^{-x_{12}\frac{\pi T}{v_F}\left[2\left(\frac{\kappa}{2}-1\right)^2+\frac{1}{2}\right]} e^{ix_{12}k_F(\kappa-2)} . \tag{7.74}$$

The presence of the second term in (7.74) is a consequence of (7.36) and (7.37). The conformal result obtained in ([19, 66]) also presented in Chap. 3 is

$$\langle \Psi^\dagger(x_1)\Psi(x_2)\rangle_T \simeq \sum_{Q=\{N^\pm,d\}} B(Q) e^{-x_{12}\frac{\pi T}{v_F}\left[2N^+ + 2N^- + \frac{1}{2} + 2\left(d+\frac{\kappa}{2}\right)^2\right]} \times e^{ix_{12}k_F(2d+\kappa)} . \tag{7.75}$$

It is easy to see that the leading terms in the expansion (7.75), which correspond to $Q = 0, 0, 0$ and $Q = 0, 0, -1$, are identical (modulo the constants) with (7.73) and (7.74). The presence of the second term in (7.74) (and the term with $Q = 0, 0, -1$ in the conformal expansion) explains why close to the fermionic limit the correlation function exhibit beats. This phenomenon was noticed and explained by Calabrese and Mintchev [19] at $T = 0$.

Chapter 8

Conclusions

In this dissertation we have investigated the properties of the field-field correlation function of impenetrable anyons at finite temperature. The main result presented is the large distance asymptotic behavior of this correlator. We have also obtained the determinant representation, the short distance and low density asymptotics and the system of partial differential equations that characterize the field-field correlator. Even though we do not present the results here, together with Vladimir Korepin and Dmitri Averin we have obtained the large time and distance asymptotics of the same correlation function. In this case it seems that the asymptotic behavior predicted by conformal field theory is not the same with the one obtained solving the associated Riemann-Hilbert problem. We will present this computations elsewhere.

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Appendix A: Particle-hole excitation

In this appendix we find the energy and momentum of particle-hole excitations of the gas of anyons. As discussed in the main text, for twisted boundary conditions ($\beta = 1$), the ground state of anyons is equivalent to that of the Bose gas with periodic boundary conditions and coupling constant c', so the excitation energy and momentum coincide in this case with those known for the Bose gas (see Chap. I.4 of [54]). For periodic boundary conditions ($\beta = 0$), the Bethe equations are the same as for the Bose gas with the boundary conditions twisted by the phase shift $2\pi\delta$, where $\delta = \{[-\pi\kappa(N-1)]\}$. In the case of one hole with momentum λ_h and one particle with momentum λ_p the equations for the ground state and the excited state are:

$$\text{Ground State, PBC}: \ \lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi\left(j - \frac{N+1}{2}\right) + 2\pi\delta\,, \quad j = 1, \cdots, N\,, \tag{A.1}$$

$$\text{Excited State, PBC}: \ \tilde{\lambda}_j L + \sum_{k=1}^{N} \theta(\tilde{\lambda}_j - \tilde{\lambda}_k) + \theta(\tilde{\lambda}_j - \lambda_p) - \theta(\tilde{\lambda}_j - \lambda_h) = 2\pi\left(j - \frac{N+1}{2}\right) + 2\pi\delta\,, \quad j = 1, \cdots, N\,. \tag{A.2}$$

Comparing the equations for a particle-hole excitation in the case of twisted boundary conditions

$$\text{Ground State, TBC}: \ \lambda_j^B L + \sum_{k=1}^{N} \theta(\lambda_j^B - \lambda_k^B) = 2\pi\left(j - \frac{N+1}{2}\right), \ j = 1, \cdots, N\,,$$

$$\text{Excited State, TBC}: \ \tilde{\lambda}_j^B L + \sum_{k=1}^{N} \theta(\tilde{\lambda}_j^B - \tilde{\lambda}_k^B) + \theta(\tilde{\lambda}_j^B - \lambda_p^B) - \theta(\tilde{\lambda}_j^B - \lambda_h^B)$$
$$= 2\pi \left(j - \frac{N+1}{2} \right), \quad j = 1, \cdots, N,$$

with (A.1) and (A.2), we find the following relations

$$\lambda_j = \lambda_j^B + 2\pi\delta/L, \quad \tilde{\lambda}_j = \tilde{\lambda}_j^B + 2\pi\delta/L, \quad (j = 1, \cdots, N)$$
$$\lambda_p = \lambda_p^B + 2\pi\delta/L, \quad \lambda_h = \lambda_h^B + 2\pi\delta/L. \tag{A.3}$$

The energy and momentum of this excited state with respect to the ground state is ($\varepsilon_0(\lambda) = \lambda^2 - h$):

$$\begin{aligned}
\Delta E(\lambda_p, \lambda_h) &= \varepsilon_0(\lambda_p) - \varepsilon_0(\lambda_h) + \sum_{j=1}^{N} (\varepsilon_0(\tilde{\lambda}_j) - \varepsilon_0(\lambda_j)) \\
&= \varepsilon_0(\lambda_p^B) - \varepsilon_0(\lambda_h^B) + \sum_{j=1}^{N} (\varepsilon_0(\tilde{\lambda}_j^B) - \varepsilon_0(\lambda_j^B)) \\
&\quad + 2\frac{2\pi\delta}{L} \left(\lambda_p^B - \lambda_h^B + \sum_{j=1}^{N} (\tilde{\lambda}_j^B - \lambda_j^B) \right) \\
&= \Delta E^B(\lambda_p^B, \lambda_h^B) + 2\frac{2\pi\delta}{L} \Delta P^P(\lambda_h^B, \lambda_p^B), & \text{(A.4)} \\
\Delta P(\lambda_p, \lambda_h) &= \Delta P^B(\lambda_p^B, \lambda_h^B), & \text{(A.5)}
\end{aligned}$$

where $\Delta E^B(\lambda_p^B, \lambda_h^B)$ and $\Delta P^B(\lambda_p^B, \lambda_h^B)$ are the energy and momentum of a particle-hole excitation in the Bose gas with periodic boundary conditions, and λ_h^B and λ_p^B are given by (A.3).

From (A.4) we see that in the case of twisted boundary conditions, the Fermi velocity v_F^{TBC} will be the same as in the Bose gas, whereas for the periodic boundary conditions the Fermi velocity will be modified as

$$v_F^{PBC} = v_F^{TBC} + \frac{4\pi\delta}{L}. \tag{A.6}$$

Appendix B: Anyonic correlators

In this Appendix, we prove Eq. (4.9) following the approach used in [5] for calculation of the anyonic matrix elements. We start with the simple case of the correlator

$$\langle\Psi_2|\Psi^\dagger(x')\Psi(x)|\Psi_2\rangle\,, \tag{B.1}$$

where we have omitted the quantum numbers $\{\lambda\}$ unimportant for the present computation. From (4.4) and (4.5), we have

$$\langle\Psi_2|\Psi^\dagger(x')\Psi(x)|\Psi_2\rangle = \frac{1}{2}\int dz^2dy^2\ \chi_2^{*a}(y_1,y_2)\chi_2^a(z_1,z_2) \\ \times\langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x')\Psi(x)\Psi^\dagger(z_2)\Psi^\dagger(z_1)|0\rangle\,. \tag{B.2}$$

Defining for the moment

$$A = \langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x')\Psi(x)\Psi^\dagger(z_2)\Psi^\dagger(z_1)|0\rangle\,, \tag{B.3}$$

and using the commutation relation (2.2), $\Psi(x)|0\rangle = 0$, and $\langle 0|0\rangle = 1$ we obtain

$$\begin{aligned} A &= \langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x')\left[\Psi^\dagger(z_2)\Psi(x)e^{-i\pi\kappa\epsilon(x-z_2)} + \delta(x-z_2)\right]\Psi^\dagger(z_1)|0\rangle\,, \\ &= \langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x')\Psi^\dagger(z_2)\Psi(x)\Psi^\dagger(z_1)|0\rangle e^{-i\pi\kappa\epsilon(x-z_2)} \\ &\qquad + \langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x')\Psi^\dagger(z_1)|0\rangle\delta(x-z_2)\,, \\ &= \underbrace{\langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x')\Psi^\dagger(z_2)|0\rangle}_{(\mathbf{a})}\delta(x-z_1)e^{-i\pi\kappa\epsilon(x-z_2)} \\ &\qquad + \underbrace{\langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x')\Psi^\dagger(z_1)|0\rangle}_{(\mathbf{b})}\delta(x-z_2)\,. \end{aligned} \tag{B.4}$$

Performing similar transformations we find that

$$\begin{aligned} \mathbf{a} &= \delta(y_1-x')\delta(y_2-z_2)e^{-i\pi\kappa\epsilon(y_2-x')}+\delta(y_2-x')\delta(y_1-z_2)\,, \\ \mathbf{b} &= \delta(y_1-x')\delta(y_2-z_1)e^{-i\pi\kappa\epsilon(y_2-x')}+\delta(y_1-z_1)\delta(y_2-x')\,. \end{aligned} \tag{B.5}$$

Substitution of (B.4) and (B.5) into (B.2) gives

$$\begin{aligned} \langle\Psi_2|\Psi^\dagger(x')\Psi(x)|\Psi_2\rangle = \frac{1}{2}\int dz_1 \Big\{\chi_2^{*a}(x',z_1)\chi_2^a(x,z_1)e^{-i\pi\kappa[\epsilon(z_1-x')+\epsilon(x-z_1)]} \\ +\chi_2^{*a}(z_1,x')\chi_2^a(x,z_1)e^{-i\pi\kappa\epsilon(x-z_1)}+\chi_2^{*a}(x',z_1)\chi_2^a(z_1,x)e^{-i\pi\kappa\epsilon(z_1-x')} \\ +\chi_2^{*a}(z_1,x')\chi_2^a(z_1,x)\Big\}\;. \end{aligned} \tag{B.6}$$

Anyonic property of the wavefunctions (4.6) together with its complex conjugate

$$\chi_N^{*a}(z_1,\cdots,z_i,z_{i+1},\cdots,z_N)=e^{-i\pi\kappa\epsilon(z_i-z_{i+1})}\chi_N^{*a}(z_1,\cdots,z_{i+1},z_i,\cdots,z_N)\,, \tag{B.7}$$

means that (B.6) reduces to a simple form

$$\langle\Psi_2|\Psi^\dagger(x')\Psi(x)|\Psi_2\rangle = 2\int dz_1\;\chi_2^{*a}(z_1,x')\chi_2^a(z_1,x)\,. \tag{B.8}$$

The generalization to the N-particle eigenstate is straightforward and gives

$$\langle\Psi_N|\Psi^\dagger(x')\Psi(x)|\Psi_N\rangle = N\int dz^{N-1}\;\chi_N^{*a}(z_1,\cdots,z_{N-1},x')\chi_N^a(z_1,\cdots,z_{N-1},x)\,. \tag{B.9}$$

Appendix C: Fredholm determinants

In this Appendix, we give a brief summary of results of Fredholm theory of integral equations. For more details, see, e.g., [75]. Consider the Fredholm equation of the second kind

$$f(x) - \gamma \int_a^b K(x,y) f(y) dy = g(x) \,,$$

where the kernel $K(x,y)$ is a symmetric, bounded and continuous function.

Defining operations with kernel $K(x,y)$ similarly to the usual matrix operations:

$$K^n(x,y) = \int_a^b K(x,z) K^{n-1}(z,y)\; dz \,, \quad \text{with } K^1(x,y) = K(x,y) \,,$$

and

$$\mathrm{Tr}K = \int_a^b K(x,x)\; dx \,, \;\; \mathrm{Tr}K^2 = \int_a^b \int_a^b K(x,y) K(y,x)\; dx dy \,, \quad \text{and so on,}$$

we have the formulae that are useful for calculation of the Fredholm determinant of the integral operator $1 - \gamma \hat{K}$:

$$(1 - \gamma \hat{K})^{-1} = 1 + \gamma K^1 + \gamma^2 K^2 + \cdots \,,$$

and

$$\ln \det(1 - \gamma \hat{K}) = - \sum_{n=1}^{\infty} \frac{\gamma^n}{n} \mathrm{Tr} K^n \,. \tag{C.1}$$

Indeed, writing (C.1) as

$$\det(1-\gamma\hat{K}) = \prod_{n=1}^{\infty} \exp\{-\frac{\gamma^n}{n}\mathrm{Tr}\ K^n\}\,, \tag{C.2}$$

and collecting terms of the same order in γ one can see that the determinant can be written conveniently as

$$\det\left(1-\gamma\hat{K}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\gamma^n}{n!}\int_a^b dx_1\cdots\int_a^b dx_n K_n\begin{pmatrix} x_1,\cdots,x_n \\ x_1,\cdots,x_n\end{pmatrix}, \tag{C.3}$$

where

$$K_n\begin{pmatrix} x_1,\cdots,x_n \\ y_1,\cdots,y_n\end{pmatrix} \equiv \det_{1\le j,k\le n}\left[K(x_j,y_k)\right]. \tag{C.4}$$

The resolvent kernel $R(x,y)$ associated with the kernel $K(x,y)$ is defined as $\hat{R} = (1-\gamma\hat{K})^{-1}\hat{K}$, i.e.,

$$R(x,y) - \gamma\int_a^b\ K(x,z)R(z,y)dz = K(x,y)\,.$$

If one introduces the determinants R_n of kernels R similarly to (C.4), an important relation can be proven to exist between R_n and the r-th Fredholm minor defined as a natural generalization of Eq. (C.3):

$$\det\left(1-\gamma\hat{K}\,\middle|\,\begin{matrix} y_1, & \cdots, & y_r \\ y_1', & \cdots, & y_r'\end{matrix}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\gamma^n}{n!}\int_a^b dx_1\cdots\int_a^b dx_n$$

$$\times\, K_{n+r}\begin{pmatrix} y_1, & \cdots, & y_r, & x_1, & \cdots, & x_n \\ y_1', & \cdots, & y_r', & x_1, & \cdots, & x_n\end{pmatrix}. \tag{C.5}$$

The relation is [44]

$$\det\left(1-\gamma\hat{K}\,\middle|\,\begin{matrix} y_1, & \cdots, & y_r \\ y_1', & \cdots, & y_r'\end{matrix}\right) = \det\left(1-\gamma\hat{K}\right) R_n\begin{pmatrix} y_1,\cdots,y_r \\ y_1',\cdots,y_r'\end{pmatrix}. \tag{C.6}$$

Appendix D: Reduced density matrices of free fermions

The reduced density matrices for free 1D fermions were calculated in the original paper of Lenard [59]. To make our discussion self-contained, we provide here a sketch of the proof and the main results in the notations that in general allow for an external potential $U(z)$ acting on the particles.

We assume that the fermions are confined to the domain $V = [-L/2, L/2]$, and have a complete set $\{u_\lambda(z)\}$ of normalized single-particle wavefunctions with energies ϵ_λ in the potential $U(z)$. For instance, for $U(z) \equiv 0$, and the "hard wall" boundary conditions at the boundaries of the domain V,

$$u_\lambda(z) = \begin{cases} \sqrt{\frac{2}{L}}\sin(\lambda z), & \lambda = \frac{2\pi}{L}, \frac{4\pi}{L}, \cdots, \\ \sqrt{\frac{2}{L}}\cos(\lambda z), & \lambda = \frac{\pi}{L}, \frac{3\pi}{L}, \cdots, \end{cases}$$

and, with appropriate conventions, $\epsilon_\lambda = \lambda^2$. The N-body wavefunction of a stationary state is given by the Slater determinant

$$\chi_N^f(z_1, \cdots, z_N|\{\lambda\}) = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (-1)^\pi \prod_{i=1}^{N} u_{\lambda_i}(z_{\pi(i)}),$$

where the set $\{\lambda\}$ consists of non-coincident single-particle states λ_i, and the energy eigenvalue is $E(\{\lambda\}) = \sum_{i=1}^N \epsilon_{\lambda_i}$. In the grand canonical ensemble, the Gibbs measure is

$$p_{\{\lambda\}}^N = e^{hN/T}\frac{e^{-E(\{\lambda\})/T}}{Z(h,L,T)}, \quad \text{with} \quad Z(h,L,T) = \sum_{N=0}^{\infty}\sum_{\{\lambda\}} e^{hN/T}e^{-E(\{\lambda\})/T}, \tag{D.1}$$

where h is the chemical potential. Using the fact that, with an extra factor $1/N!$ included to compensate for overcounting, summation over $\{\lambda\}$ can be replaced with summation over independent individual λ_i's, one obtains the following fundamental formula

$$\sum_{\{\lambda\}} e^{hN/T} e^{-E(\{\lambda\})/T} \chi_N^{*f}(z_1, \cdots, z_N|\{\lambda\}) \chi_N^f(z_1', \cdots, z_N'|\{\lambda\}) =$$

$$\frac{1}{N!} \sum_{\pi \in S_N} (-1)^\pi \prod_{i=1}^N F(z_i, z_{\pi(i)}') = \frac{1}{N!} F_N \left(\begin{array}{c} z_1, \cdots, z_N \\ z_1', \cdots, z_N' \end{array} \right). \tag{D.2}$$

Here we have used (C.4) and

$$F(x,y) \equiv e^{h/T} \sum_\lambda e^{-\epsilon_\lambda/T} u_\lambda^*(x) u_\lambda(y)\,. \tag{D.3}$$

Equation (D.2) and proper normalization of the wavefunctions u_λ show that the fermionic statistical sum (D.1) can be expressed as the determinant (C.3) of the integral operator with kernel (D.3):

$$Z(h,L,T) = \sum_{N=0}^{\infty} \frac{1}{N!} \int_V dz_1 \cdots \int_V dz_N \; F_N \left(\begin{array}{c} z_1, \cdots, z_N \\ z_1, \cdots, z_N \end{array} \right) = \det(1+\hat{F}).$$

Similarly, using the definition (C.5) of Fredholm minor of the same operator we see that

$$\sum_{N=n}^{\infty} e^{hN/T} \sum_{\{\lambda\}} e^{-E(\{\lambda\})/T} \frac{N!}{(N-n)!} \int_V dz_1 \cdots \int_V dz_{N-n}$$

$$\times \chi_N^{*f}(z_1, \cdots, z_{N-n}, x_1, \cdots, x_n|\{\lambda\}) \chi_N^f(z_1, \cdots, z_{N-n}, x_1', \cdots, x_n'|\{\lambda\})$$

$$= \det \left(1 + \hat{F} \left| \begin{array}{c} x_1, \cdots, x_n \\ x_1', \cdots, x_n' \end{array} \right. \right), \tag{D.4}$$

so that the reduced density matrix of the fermions can be expressed as:

$$(x_1, \cdots, x_n|\rho_n^f|x_1', \cdots, x_n') = \det \left(1 + \hat{F} \left| \begin{array}{c} x_1, \cdots, x_n \\ x_1', \cdots, x_n' \end{array} \right. \right) / \det(1+\hat{F}).$$

The relation (C.6) for Fredholm minors means that this result can be expressed simply in terms of the resolvent kernel $\theta_T(x,y)/\pi$ associated with kernel $F(x,y)$ (D.3) (factors of π are chosen so that the notations are the same as in the main text):

$$(x_1, \cdots, x_n|\rho_n^f|x_1', \cdots, x_n') = \frac{1}{\pi^n} \theta_T \left(\begin{array}{c} x_1, \cdots, x_n \\ x_1', \cdots, x_n' \end{array} \right).$$

In the thermodynamic limit with no external potential, $U(z) \equiv 0$, the resolvent kernel θ_T is given by

$$\lim_{L\to\infty} \theta_T(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} dk \frac{e^{ik(x-y)}}{1+e^{(k^2-h)/T}} \, .$$

Appendix E: Anyonic form factors

In this appendix, we prove Eq. (5.26). Consider first the simple example of the form factor $F_{3,2}$:

$$F_{3,2}(x) = \frac{1}{2\sqrt{3}} \int d^3y \; d^2z \; \chi_3^*(y_1, y_2, y_3)\chi_2(z_1, z_2)$$
$$\times \langle 0|\Psi(y_1)\Psi(y_2)\Psi(y_3)\Psi^\dagger(x)\Psi^\dagger(z_2)\Psi^\dagger(z_1)|0\rangle \, . \tag{E.1}$$

If one defines

$$A = \langle 0|\Psi(y_1)\Psi(y_2)\Psi(y_3)\Psi^\dagger(x)\Psi^\dagger(z_2)\Psi^\dagger(z_1)|0\rangle \, ,$$

then successive applications of the commutation relation (2.2) followed by the Eq. (5.6) give

$$\begin{aligned} A &= \langle 0|\Psi(y_1)\Psi(y_2)\left[\Psi^\dagger(x)\Psi(y_3)e^{-i\pi\kappa\epsilon(y_3-x)} + \delta(y_3-x)\right]\Psi^\dagger(z_2)\Psi^\dagger(z_1)|0\rangle \\ &= \langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x)\left[\Psi^\dagger(z_2)\Psi(y_3)e^{-i\pi\kappa\epsilon(y_3-z_2)} + \delta(y_3-z_2)\right] \\ &\quad \times\Psi^\dagger(z_1)|0\rangle e^{-i\pi\kappa\epsilon(y_3-x)} + \langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(z_2)\Psi^\dagger(z_1)|0\rangle\delta(y_3-x) \\ &= \underbrace{\langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x)\Psi^\dagger(z_2)|0\rangle\delta(y_3-z_1)e^{-i\pi\kappa[\epsilon(y_3-z_2)+\epsilon(y_3-x)]}}_{(\mathbf{a})} \\ &\quad + \underbrace{\langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(x)\Psi^\dagger(z_1)|0\rangle\delta(y_3-z_2)e^{-i\pi\kappa\epsilon(y_3-x)}}_{(\mathbf{b})} \\ &\quad + \underbrace{\langle 0|\Psi(y_1)\Psi(y_2)\Psi^\dagger(z_2)\Psi^\dagger(z_1)|0\rangle\delta(y_3-x)}_{(\mathbf{c})} \, . \end{aligned} \tag{E.2}$$

Performing similar transformations, we obtain

$$
\begin{aligned}
\mathbf{a} &= \delta(y_1 - x)\delta(y_2 - z_2)\delta(y_3 - z_1)e^{-i\pi\kappa[\epsilon(y_2-x)+\epsilon(y_3-z_2)+\epsilon(y_3-x)]} \\
&\quad +\delta(y_1 - z_2)\delta(y_2 - x)\delta(y_3 - z_1)e^{-i\pi\kappa[\epsilon(y_3-z_2)+\epsilon(y_3-x)]}\,, \\
\mathbf{b} &= \delta(y_1 - x)\delta(y_2 - z_1)\delta(y_3 - z_2)e^{-i\pi\kappa[\epsilon(y_2-x)+\epsilon(y_3-x)]} \\
&\quad +\delta(y_1 - z_1)\delta(y_2 - x)\delta(y_3 - z_2)e^{-i\pi\kappa\epsilon(y_3-x)}\,, \\
\mathbf{c} &= \delta(y_1 - z_2)\delta(y_2 - z_1)\delta(y_3 - x)e^{-i\pi\kappa\epsilon(y_2-z_2)} \\
&\qquad\qquad\qquad +\delta(y_1 - z_1)\delta(y_2 - z_2)\delta(y_3 - z_3)\,.
\end{aligned}
$$

Substituting $A = \mathbf{a} + \mathbf{b} + \mathbf{c}$ into (E.1), we have for the form factor

$$
\begin{aligned}
F_{3,2}(x) = \frac{1}{2\sqrt{3}} \int d^2z \ \ &\left\{\chi_3^*(x, z_2, z_1)\chi_2(z_1, z_2)e^{-i\pi\kappa[\epsilon(z_2-x)+\epsilon(z_1-z_2)+\epsilon(z_1-x)]}\right. \\
&+\chi_3^*(z_2, x, z_1)\chi_2(z_1, z_2)e^{-i\pi\kappa[\epsilon(z_1-z_2)+\epsilon(z_1-x)]} + \chi_3^*(z_1, z_2, x)\chi_2(z_1, z_2) \\
&+\chi_3^*(z_1, x, z_2)\chi_2(z_1, z_2)e^{-i\pi\kappa(z_2-x)} + \chi_3^*(z_2, z_1, x)\chi_2(z_1, z_2)e^{-i\pi\kappa\epsilon(z_1-z_2)} \\
&\left.+\chi_3^*(x, z_1, z_2)\chi_2(z_1, z_2)e^{-i\pi\kappa[\epsilon(z_1-x)+\epsilon(z_2-x)]}\right\}. \qquad \text{(E.3)}
\end{aligned}
$$

Using the anyonic property (5.8) of the wavefunctions, and its complex conjugate:

$$
\chi^*(\cdots, z_i, z_{i+1}, \cdots) = e^{-i\pi\kappa\epsilon(z_i-z_{i+1})}\chi^*(\cdots, z_{i+1}, z_i, \cdots)\,,
$$

we reduce Eq. E.3 to the final expression for the form factor

$$
F_{3,2}(x) = \sqrt{3} \int d^2z \ \chi_3^*(z_1, z_2, x)\chi_2(z_1, z_2)\,. \qquad \text{(E.4)}
$$

The calculations leading to Eq. (E.4) can be generalized to arbitrary N:

$$
\begin{aligned}
F_{N+1,N}(x) &= \langle\Psi_{N+1}|\Psi^\dagger(x)|\Psi_N\rangle \\
&= \sqrt{N+1} \int d^Nz \ \chi_{N+1}^*(z_1, \cdots, z_N, x)\chi_N(z_1, \cdots, z_N)\,. \qquad \text{(E.5)}
\end{aligned}
$$

This result follows from Eq. (5.25) by noticing that the statistical phase factors in the commutation relations (2.2)–(2.4) of the field operators are compensated by the exchange property (5.8) of the wavefunctions. This means that the pairing of the $\Psi^\dagger(x)$ operator with any of the $\Psi(y_j)$ operators produces $N+1$ identical terms in which the coordinate x is made the last coordinate of the wavefunction χ_{N+1}. After that, the integrals over z's and remaining y's can be limited to the ordered regions $z_1 > z_2 > ... > z_N$ and $y_1 > y_2 > ... > y_N$ giving directly (E.5).

Appendix F: Thermodynamic limit of singular sums

In this appendix, we study the behavior of the functions defined by Eqs. (5.44), (5.45), and (5.46) in the thermodynamic limit of large length L of normalization interval. We start with (5.44). In this case, the function summed over the momenta λ is sufficiently smooth, so that the anyonic shift $2\pi\delta'/L$ of the momenta becomes negligible when $L \to \infty$, and one can pass directly from the sum to the integral over λ:

$$G(t,x) \equiv \lim_{L\to\infty} G_L(t,x) = \frac{2\pi}{L} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} e(\lambda_j|t,x) = \int_{-\infty}^{\infty} e(\lambda|t,x)\, d\lambda\,. \quad \text{(F.1)}$$

The regularization $t \to t + i0$ for $e(\lambda|t,x) = \exp(it\lambda^2 - ix\lambda)$ is implied in these expressions.

Next, we turn to Eq. (5.45). In this case, the function under the sum is no longer smooth in the thermodynamic limit. We transform it by separating the singular part that can be summed explicitly:

$$\begin{aligned} E(\mu_k|t,x) \equiv \lim_{L\to\infty} E_L(\mu_k|t,x) &= \frac{2\pi}{L} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x)}{\lambda_j - \mu_k} \\ &= \frac{2\pi}{L} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{\lambda_j - \mu_k} \\ &\quad + e(\mu_k|t,x) \sum_{n=-\infty}^{\infty} \left(n - \frac{\kappa+1}{2}\right)^{-1} . \end{aligned} \quad \text{(F.2)}$$

In the last line here we have used Eq. (5.29). The first term in (F.2) is now a smooth function, so as before, we can directly replace the sum with the integral, since the anyonic shift of the momenta does not affect the value of the integral. The integral can then be transformed as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\lambda \; \frac{e(\lambda|t,x) - e(\mu_k|t,x)}{\lambda - \mu_k} \\
&= \; \text{P.V.} \int_{-\infty}^{\infty} d\lambda \; \frac{e(\lambda|t,x)}{\lambda - \mu_k} - e(\mu_k|t,x) \; \text{P.V.} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda - \mu_k} \\
&= \; \text{P.V.} \int_{-\infty}^{\infty} d\lambda \; \frac{e(\lambda|t,x)}{\lambda - \mu_k} \, . \qquad \text{(F.3)}
\end{aligned}$$

Under the natural interpretation of the sum in the second term in (F.2), it can be simplified using formula 1.421.(3) of [38], $\pi\cot(\pi x) = (1/x) + 2x\sum_{n=1}^{\infty}(x^2 - n^2)^{-1}$:

$$\sum_{n=-\infty}^{\infty} \left(n - \frac{\kappa+1}{2}\right)^{-1} = \pi \tan\left(\frac{\pi\kappa}{2}\right) \, . \qquad \text{(F.4)}$$

Collecting the two terms we finally get

$$E(\mu_k|t,x) = \text{P.V.} \int_{-\infty}^{\infty} d\lambda \; \frac{e(\lambda|t,x)}{\lambda - \mu_k} + e(\mu_k|t,x)\pi\tan\left(\frac{\pi\kappa}{2}\right) \, .$$

The function defined by Eq. (5.46) is more singular than $E(\mu_k|t,x)$. To transform it, we use the same strategy of separating the most divergent terms that can be summed explicitly:

$$\begin{aligned}
\tilde{E}(\mu_k|t,x) \;&\equiv\; \lim_{L\to\infty} \tilde{E}_L(\mu_k|t,x) = \frac{4}{L^2}\cos^2(\pi\kappa/2) \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x)}{(\lambda_j - \mu_k)^2} \, , \\
&= \; \frac{4}{L^2}\cos^2(\pi\kappa/2) \left(\sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{(\lambda_j - \mu_k)^2} \right. \\
&\qquad \left. + e(\mu_k|t,x) \frac{L^2}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{\left(n - \frac{\kappa+1}{2}\right)^2} \right) . \qquad \text{(F.5)}
\end{aligned}$$

Defining

$$f(\mu_k) = \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{\lambda_j - \mu_k} \, ,$$

one has

$$\sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{(\lambda_j - \mu_k)^2} = \frac{\partial f(\mu_k)}{\partial \mu_k} + \frac{\partial e(\mu_k|t,x)}{\partial \mu_k} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{1}{\lambda_j - \mu_k}$$

Taking the limit $L \to \infty$ and using (F.3) and (F.4) in this equation we obtain

$$\lim_{L\to\infty} \frac{L}{2\pi} \sum_{\lambda_j \in \frac{2\pi}{L}(\mathbb{Z}+\delta')} \frac{e(\lambda_j|t,x) - e(\mu_k|t,x)}{(\lambda_j - \mu_k)^2} =$$

$$= \frac{\partial}{\partial \mu_k} \left(\text{P.V.} \int_{-\infty}^{\infty} d\lambda \; \frac{e(\lambda|t,x)}{\lambda - \mu_k} \right) + \frac{\partial e(\mu_k|t,x)}{\partial \mu_k} \pi \tan\left(\frac{\pi\kappa}{2}\right) .$$

For the second term in the R.H.S. of (F.5) we use the formula 1.422.(4) of [38] $\pi^2/\sin^2(\pi x) = \sum_{n=-\infty}^{\infty}(n-x)^{-2}$ to get

$$\sum_{n=-\infty}^{\infty} \left(n - \frac{\kappa+1}{2} \right)^{-2} = \frac{\pi^2}{\cos^2(\pi\kappa/2)} .$$

Collecting all the terms we have the final result

$$\tilde{E}(\mu_k|t,x) = e(\mu_k|t,x) + \frac{2\cos^2(\pi\kappa/2)}{\pi L} \frac{\partial e(\mu_k|t,x)}{\partial \mu_k} \pi \tan\left(\frac{\pi\kappa}{2}\right)$$

$$+ \frac{2\cos^2(\pi\kappa/2)}{\pi L} \frac{\partial}{\partial \mu_k} \left(\text{P.V.} \int_{-\infty}^{\infty} d\lambda \; \frac{e(\lambda|t,x)}{\lambda - \mu_k} \right) .$$

Appendix G: Short distance asymptotics

In this appendix we will obtain the short distance asymptotics of the potentials B_{+-} and B_{++} and as a byproduct we will compute the short distance asymptotics of the field correlator. First we need to express the potentials into a form more amenable for short distance computations. We will start with B_{+-}. Using the integral equations (6.8) it is easy to see that

$$\begin{aligned}
&\gamma f_+(\lambda)\sqrt{\vartheta(\lambda)}e^{-i\lambda x} \\
&= \gamma\vartheta(\lambda) + \gamma^2\sqrt{\vartheta(\lambda)}e^{-i\lambda x}\int_{-\infty}^{+\infty}\sqrt{\vartheta(\lambda)}\frac{\sin x(\lambda-\mu)}{2i(\lambda-\mu)}\sqrt{\vartheta(\mu)}f_+(\mu)\,d\mu\,, \\
&= \gamma\vartheta(\lambda) + \gamma\vartheta(\lambda)\int_{-\infty}^{+\infty}\frac{1-e^{-2i(\lambda-\mu)x}}{2i(\lambda-\mu)}\gamma f_+(\mu)\vartheta(\mu)e^{-i\mu x}d\mu\,,
\end{aligned}$$

which shows that B_{+-} can be written in the form

$$B_{+-}(x,\beta,\gamma) = \int_{-\infty}^{+\infty} s(\lambda)d\lambda \tag{G.1}$$

where $s(\lambda)$ solves the following integral equation

$$s(\lambda) - \gamma\vartheta(\lambda)\int_{-\infty}^{+\infty}\frac{1-e^{-2i(\lambda-\mu)x}}{2i(\lambda-\mu)}s(\mu)d\mu = \gamma\vartheta(\lambda)\,. \tag{G.2}$$

In a similar fashion we obtain

$$B_{++}(x,\beta,\gamma) = \int_{-\infty}^{+\infty} e^{2i\lambda x}s(\lambda)d\lambda\,, \tag{G.3}$$

where $s(\lambda)$ is the solution of the same integral equation (G.2).

For x small the solution of (G.2) can be expanded as

$$s(\lambda) \equiv s(\lambda, \beta, \gamma) = \sum_{k=0}^{\infty} s_k(\lambda, \beta, \gamma) x^k ,$$

where s_k are defined by the following recursion relations

$$\begin{aligned} s_0(\lambda) &= \gamma \vartheta(\lambda) , \\ s_k(\lambda) &= s_0(\gamma) \sum_{k=0}^{m-1} \frac{(2i)^{m-k-1}}{(m-k)!} \int_{-\infty}^{+\infty} (\mu - \lambda)^{m-k-1} s_k(\mu) d\mu . \end{aligned} \qquad \text{(G.4)}$$

Defining

$$\beta_l(\beta, \gamma) = \gamma \int_{-\infty}^{+\infty} \lambda^l \vartheta(\lambda) d\lambda , \qquad \beta_{2n+1} = 0 ,$$

and using (G.4) in (G.1) and (G.3) we obtain the short distance asymptotics for the potentials

$$\begin{aligned} B_{++}(x, \beta, \gamma) &= \beta_0 + \beta_0^2 x + \left(\beta_0^3 - 2\beta_2\right) x^2 + \left(\beta_0^4 - \frac{4}{3}\beta_0\beta_2\right) x^3 + O(x^4) , \\ B_{+-}(x, \beta, \gamma) &= \beta_0 + \beta_0^2 x + \beta_0^3 x^2 + \left(\beta_0^4 - \frac{4}{3}\beta_0\beta_2\right) x^3 + O(x^4) . \end{aligned}$$

Now we can obtain the short distance asymptotics of the field correlator. Using

$$g(x, \beta, \gamma) = B_{++}(x, \beta, \gamma) e^{\sigma(x,\beta,\gamma)}|_{\gamma=(1+e^{i\pi\kappa})/\pi} ,$$

$$\sigma(x, \beta, \gamma) = -\int_0^x B_{+-}(y, \beta, \gamma) dy ,$$

we successively obtain

$$\sigma(x, \beta, \gamma) = -\beta_0 x - \frac{1}{2}\beta_0^2 x^2 - \frac{1}{3}\beta_0^3 x^3 + O(x^4) ,$$

and

$$g(x, \beta, \gamma) = \beta_0 \left(1 - 2\frac{\beta_2}{\beta_0} x^2 + \frac{2}{3}\beta_2 x^3\right) + O(x^4) .$$

In the original variables $x = (x_1 - x_2)\sqrt{T}/2 > 0$, $\beta = h/T$ this result is rewritten as (see (6.4))

$$\langle\Psi^\dagger(x_1)\Psi(x_2)\rangle_T = D\left(1-\frac{E}{2D}(x_1-x_2)^2+\gamma\frac{\pi E}{6}(x_1-x_2)^3\right)+O\left((x_1-x_2)^4\right)\ , \tag{G.5}$$

where

$$D=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d\lambda}{1+e^{(\lambda^2-h)/T}}\ ,\quad E=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\lambda^2 d\lambda}{1+e^{(\lambda^2-h)/T}}\ ,$$

are the density and the kinetic energy density.

Appendix H: Low density expansions

The low density limit is obtained when $\beta \to -\infty$. In terms of our rescaled variables the density of impenetrable anyons is given by

$$D = \frac{\sqrt{T}}{2\pi} \int_{-\infty}^{+\infty} \frac{d\lambda}{1 + e^{\lambda^2 - \beta}}$$

so $D \to 0$ when $\beta \to -\infty$. In what will follow it will be useful to use the variable

$$\zeta = -e^{\beta}, \quad \zeta \to 0 \quad \text{when } \beta \to -\infty\,.$$

In order to obtain low density expansions for the potentials

$$B_{++} = \sum_{k=1}^{\infty} b_k(x)\zeta^k\,, \quad B_{+-} = \sum_{k=1}^{\infty} c_k(x)\zeta^k\,,$$

we will use again (G.1), (G.3) and the integral equation (G.2). Representing the Fermi weight as

$$\vartheta(\lambda) = -\sum_{k=1}^{\infty} \zeta^k e^{-k\lambda^2}\,,$$

and $s(\lambda)$ as

$$s(\lambda) = \sum_{k=1}^{\infty} \zeta^k s_k(\lambda, x)\,,$$

we obtain the following recursion relations for s_k

$$\begin{aligned} s_1(\lambda) &= -\gamma e^{-\lambda^2}\,, \\ s_k(\lambda, x) &= e^{-\lambda^2} s_{k-1}(\lambda, x) - \gamma e^{-\lambda^2} \int_{-\infty}^{+\infty} \frac{1 - e^{-2i(\lambda-\mu)x}}{2i(\lambda - \mu)} s_{k-1}(\mu, x) d\mu\,, \quad k \geq 2\,. \end{aligned}$$

Let's give the first terms of the expansions of the potentials

$$
\begin{aligned}
B_{+-}(x,\zeta,\kappa) &= -\gamma\sqrt{\pi}\zeta + \left(-\gamma\sqrt{\frac{\pi}{2}} + \gamma^2\pi\int_0^x e^{-x_1^2}dx_1\right)\zeta^2 + O(\zeta^3)\,, \\
B_{++}(x,\zeta,\kappa) &= -\gamma\sqrt{\pi}e^{-x^2}\zeta + \left(-\gamma\sqrt{\frac{\pi}{2}}e^{-x^2}\right. \\
&\qquad \left. +\gamma^2\pi e^{-x^2}\int_0^x e^{-2x_1^2+2x_1x}dx_1\right)\zeta^2 + O(\zeta^3)\,.
\end{aligned}
$$

Using (H.1) we obtain

$$
\sigma(x,\beta,\gamma) = -\gamma\sqrt{\pi}xe^{\beta} + O(e^{2\beta})\,,
$$

and

$$
\frac{\sqrt{T}}{2\pi\gamma}g(x,\beta,\gamma) = \frac{\sqrt{T}}{2\pi^{1/2}}e^{-x^2}e^{\beta} + O(e^{2\beta})\,.
$$

In the original variables the result for the correlator is rewritten as

$$
\langle\Psi^\dagger(x_1)\Psi(x_2)\rangle_T = De^{-T(x_1-x_2)^2/4} \qquad h \ll -(x_1-x_2)^2T^2. \tag{H.1}
$$

Appendix I: Solvability of the matrix Riemann-Hilbert problem

We have shown in Section 7.1.1 that the matrix RH problem (7.1) is equivalent with the system of nonsingular integral equations

$$f_{\pm}(\lambda) - \gamma \int_{-\infty}^{+\infty} K_T(\lambda,\mu) f_{\pm}(\mu) d\mu = e_{\pm}(\lambda)\,,$$

with kernel (7.9). This means that the RH problem will have a unique solution whenever the system of integral equations of Fredholm type have a unique solution. Fix β and κ, and let D be an open connected subset of the complex plane, $\mathcal{L}(\mathcal{H})$ the space of operators acting on a separable Hilbert space $\mathcal{H}$ and consider the function

$$f(x) : D \to \mathcal{L}(\mathcal{H})\,,$$

which gives for each x in D an integral operator with kernel $K_T(\lambda,\mu)$. Then for each x in the finite strip

$$0 < a < \Re x < b\,, \quad \Im x < \epsilon\,.$$

$f(x)$ is an analytical operator valued function. The kernel $K_T(\lambda,\mu)$ also satisfies the estimate

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_T(\lambda,\mu) d\lambda\; d\mu < Cb^2$$

where C is a constant, which means that $f(x)$ is compact for each $x \in D$ (see Thm. VI. 23 of [71]). Now we can apply the analytic Fredholm theorem.

Theorem 4 (Thm VI. 14 of [71]). *Let D be an open connected subset of $\mathbb{C}$. Let $f : D \to \mathcal{L}(\mathcal{H})$ be an analytic operator-valued function such that for each $z \in D$, $f(z)$ is compact. Then, either* *a)* $(I - f(z))^{-1}$ *exists for no* $z \in D$

or

b) $(I - f(z))^{-1}$ *exists for all* $z \in D\backslash S$ *where* S *is a discrete subset of* D *(i.e. a set which has no limit points in* D*). In this case* $(I - f(z))^{-1}$ *is meromorphic in* D*, analytic in* $D\backslash S$*, the residues at the poles are finite rank operators, and if* $z \in S$ *then* $f(z)\psi = \psi$ *has a nonzero solution in* $\mathcal{H}$*.*

As a consequence of the theorem we have to prove that for at least one point in the strip D the integral equations have a unique solution. But this is definitely true for small x where the Liouville-Neumann series is convergent. Thus we have shown that the matrix RH problem has a unique solution except for a countable set of values of x_n which we will denote by $X = \{x_n\}$.

Appendix J: Scalar Riemann-Hilbert problem

In the following we will consider the scalar Riemann-Hilbert problem for the semi-plane which is defined as follows. Consider two functions $g(\lambda)$ and $r(\lambda)$ defined on the real axis which satisfy the Hölder condition ($|g(\lambda_1) - g(\lambda_2)| < C|\lambda_1 - \lambda_2|^k$, $0 < k \leq 1$, similarly for $r(\lambda)$) and $g(\lambda)$ does not vanish. We need to find functions $\alpha(\lambda), \tilde{\alpha}(\lambda)$ which are analytic in the upper and lower half-plane with the boundary values on the real axis satisfying the conditions

$$\alpha_-(\lambda) = \alpha_+(\lambda) g(\lambda)\,, \quad \lambda \in \mathbb{R} \quad \text{homogeneous problem} \tag{J.1}$$

or

$$\tilde{\alpha}_-(\lambda) = \tilde{\alpha}_+(\lambda) g(\lambda) + r(\lambda)\,, \quad \lambda \in \mathbb{R} \quad \text{inhomogeneous problem.} \tag{J.2}$$

For our purposes we will also consider the normalization condition $\alpha(\infty) = \tilde{\alpha}(\infty) = 1$. The considerations below also hold in the more general case of a simply-connected closed contour in the complex plane. For a complete treatment of this problems the interested reader should consult the excellent textbook [32].

J.0.4 The homogeneous case

We distinguish three cases depending on the index $\chi(g) = (1/2\pi)\mathrm{Var}_{[-\infty,+\infty]}\arg g(\lambda)$ of the function $g(\lambda)$. When $\chi = 0$ the RH problem with the normalization condition is uniquely solvable, if the index $\chi > 0$ the problem has $\chi + 1$ linearly independent solutions and if $\chi < 0$ the problem has no solution. In the $\chi = 0$ case which is the most important for us the solution of the RH problem (J.1) is given by

$$\alpha(\lambda) = \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln g(\mu)}{\mu - \lambda} d\mu \right\}\,, \quad \lambda \in \mathbb{C}/\mathbb{R}\,. \tag{J.3}$$

As in the matrix case it can be shown that the scalar RH problem (J.1) is equivalent with the singular integral equation

$$\alpha_+(\lambda) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\alpha_+(\mu)(1-g(\mu))}{\mu-\lambda-i0}d\mu\,, \quad \lambda \in \mathbb{R}\,.$$

J.0.5 The inhomogeneous case

Again in the most interesting case for us $\chi = 0$ the solution of the RH problem (J.2) with the normalization condition is unique. The solution can be obtained from the solution of the homogeneous problem with the same $g(\lambda)$. If $\alpha(\lambda)$ solves (J.1) then (J.2) can be written as

$$\frac{\tilde{\alpha}_+(\lambda)}{\alpha_+(\lambda)} - \frac{\tilde{\alpha}_-(\lambda)}{\alpha_-(\lambda)} = -\frac{r(\lambda)}{\alpha_-(\lambda)}\,,$$

where we have used $g(\lambda) = \alpha_-(\lambda)/\alpha_+(\lambda)\,, \lambda \in \mathbb{R}$. The functions $\tilde{\alpha}_+(\lambda)/\alpha_+(\lambda)$ and $\tilde{\alpha}_-(\lambda)/\alpha_-(\lambda)$ are the boundary values of the function $\tilde{\alpha}(\lambda)/\alpha(\lambda)$ which is analytic in the complex plane minus the real axis and is 1 at infinity (because of the normalization $\tilde{\alpha}(\infty) = \alpha(\infty) = 1$). Using the properties of the Cauchy integral we obtain

$$\frac{\tilde{\alpha}(\lambda)}{\alpha(\lambda)} = 1 - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{r(\mu)}{\alpha_-(\mu)(\lambda-\mu)}d\mu\,, \quad \lambda \in \mathbb{C}/\mathbb{R}\,,$$

which shows that the solution of the inhomogeneous scalar RH problem (J.2) is given by

$$\tilde{\alpha}(\lambda) = \alpha(\lambda)\left(1 - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{r(\mu)}{\alpha_-(\mu)(\mu-\lambda)}d\mu\right)\,, \quad \lambda \in \mathbb{C}/\mathbb{R}\,. \tag{J.4}$$

where $\alpha(\lambda)$ is the solution of the homogeneous problem (J.1). The inhomogeneous RH problem is equivalent with the singular integral equation

$$\tilde{\alpha}_+(\lambda) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\tilde{\alpha}_+(\mu)(1-g(\mu))}{\mu-\lambda-i0}d\mu - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{r(\mu)}{\mu-\lambda-i0}d\mu\,, \quad \lambda \in \mathbb{R}\,. \tag{J.5}$$