Structure of N=4 SYM

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Abstract of the Dissertation

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In the first part of this thesis I present my research into the issue of on-shell structures of N=4 SYM amplitudes at both tree and loop level. In both cases the presence of supersymmetry is incorporated through on-shell superspace. At tree level the recent perturbative expansion inspired from twistor string theory will be given a field theory explanation: it corresponds to a perturbation expansion around the self-dual sector of the action, which is free classically. At loop level, in the absence of off-shell superspace, one can only anticipate a superspace representation for the kinematic invariants in front of the loop integrals. I will present such a description for the non-trivial 6-point one loop NMHV amplitude. Since a large part of this research utilizes spinor helicity formalism, I'll summarize some useful result in the appendix.

In the second part, I will discuss an approach for off-shell superspace such that one can compute amplitudes either using first or second quantization methods. Since first quantization may present a closer relation with the string formulation of this theory, I will first introduce first quantization approach for ordinary YM theory as a toy model. This entails the construction of constraints, BRST charge and vertex operators. A useful new result from this study is a recipe to define Green function on spaces that are not a simple line or circle, thus paving the way for multi-loop calculations. Finally a superspace in which such a approach can be used for N=4 SYM will be introduced, it is based on a coset of super-anti de Sitter space, the free constraints will be given along with the ghost structure for BRST quantization. In the end I will give a brief discussion of the second quantized theory.

To my family.

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Acknowledgements

I remember the first day I arrived on Stony Brook's campus, not knowing how close I was to the sea, I was surprised to find a seagull picking through the empty parking lot. I thought at the time that it was lost, for it had no partners and there was not a scent of salt in the air to guide it back. In the back of my mind I wondered that perhaps in a way so was I. Six years after that what I've learned is that not only would that first impression be true, I would spend the next six years learning how to be comfortable with being clueless. For this I would like to thank Prof. Warren Siegel's guidance, both for his generosity of time and his always refreshing point of view. It was always a pleasure to be confused along with my advisor.

I've been fortunate to take part in a long series of excellent lectures on various topics[1] in theoretical physics taught by Prof. Peter van Nieuwenhuizen who has spent tremendous effort in researching these lectures and was always energetic in delivering them in class. Taking his courses has always been a baffling experience, on one hand one feels like a novice whose ignorance is displayed on the blackboard over and over in precise definition and beautiful handwriting, on the other hand one feels like the old guy who can barely keep up with the energy and enthusiasm of the lecturer. As a student I thank Peter for his energy and persistence. I would also like to take this opportunity to express my gratitude for his generous assistance during my personal hardship in the last years of my study, I could not have finished this thesis without his help, thank you Peter.

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[1]This really is a long list, General relativity(1), Supersymmetry and Supergravity(1), Group Theory(1), Advanced quantum field theory (n) , and the periodic String theory course co-taught with my advisor, Pro. Martin Roček and Prof. Leonardo Rastelli. The parenthesis notes the number of times I participated.

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Chapter 1

Introduction: N=4 SYM

N=4 super Yang-Mills theory is a quantum gauge theory in 4-dimensions with 16 supersymmetries. The on-shell field content consists of a vector gauge field A_μ, 4 complex Weyl spinors λ^A_α and 6 real scalars ϕ^{AB} all in the adjoint representation of the gauge group.

The action for N=4 SYM theory was first written down more than three decades ago[1], yet new structures and symmetries are still being discovered for this theory thirty years later (The latest along this line would be the dual superconformal symmetry proposed in the summer of 2008.) Much of these new information does not manifest itself in the action, instead they were discovered, in a sense, through reformulation of the theory.

There are two major reformulations, both in terms of string theory. First is the AdS/CFT [2]correspondence which relates the large N_c(planar) limit of this theory at strong coupling to a string theory in $AdS_5\times S_5$ background at weak coupling. This duality led to a wide range of discoveries for properties of N=4 SYM; from its integrability in the planar limit and using it to determine the spectrum of scaling dimensions[3], to the dual superconformal invariance [4] of the scattering amplitudes which is related to the T-duality transformations of the string sigma-model in the $AdS_5\times S_5$ background[5].

The second is the Wittens twistor [6] topological B model in twistor space background($CP^{3|4}$). This is a weak weak duality, namely perturbative amplitude in the string theory correspond to perturbative amplitude in the field theory. Since the string theory construction was topological, this led to new "topological" expansion in field theory such as the expansion in chiral amplitudes (CSW approach) or a recursive relationship between higher and lower point amplitudes (BCFW relation).

The supersymmetric action is uniquely determined by requirement of invariance under global super Poincare group, the principle of local gauge symmetry and locality. In terms of these properties N=4 SYM is special only in the sense that it has the largest possible supersymmetry for a quantum field theory in 4 dimensions (quantum gravity not included). However, from the traditional field theory point of view, this large number of supersymmetries is actually a headache in the form of complicated helicity states for the S matrix elements and the lack of an off-shell formulation.

On the other hand, in suitable on-shell variables the amplitudes for this theory are extremely simple. For example the Maximal Helicity Violating (MHV) (precise definition given in the next chapter) amplitude is given as [7][1]

$$A(\mathrm{MHV})_{tree} = \frac{\delta^8(\sum_{i=1}^{n} \lambda_i \theta_i^A)}{\Pi_{i=1}^{n} < ii+1 >} \tag{1.1}$$

This form for the amplitude cannot be derived from first principle using the action and Feynman rules (This amplitude is verified by expanding in components and compared with the field theory computation.) Instead this amplitude was only "derived" as the expression for specific amplitudes in Witten's twistor string. Another example would be the recent investigation of tree level amplitude in ambi-twistor space [8][2] The four point amplitude is simply a product of 4 sign functions:

$$M_4 = sgn(W_1 \cdot Z_2)sgn(Z_2 \cdot W_3)sgn(W_3 \cdot Z_4)sgn(Z_4 \cdot W_1) \tag{1.2}$$

the subscripts label the external lines, Zs and Ws are conjugate ambi-twistor variables. This does not even look like an amplitude derived from any first or second quantized action.

Since most of these results are not based on an action, this would imply

[1] A similar example is the Parke-Taylor form for MHV amplitudes [9]. Its simplicity perhaps can also be credited to N=4 SYM since both give the same tree level gluon amplitude.

[2] In ambi-twistor space one enlarges the original twistor space construction by keeping all the twistor coordinate and their conjugate momenta, while the original twistor approach keeps only half of them. For detail see (A). In a sense the usual twistor approach is based on holomorphicity (chiral basis).

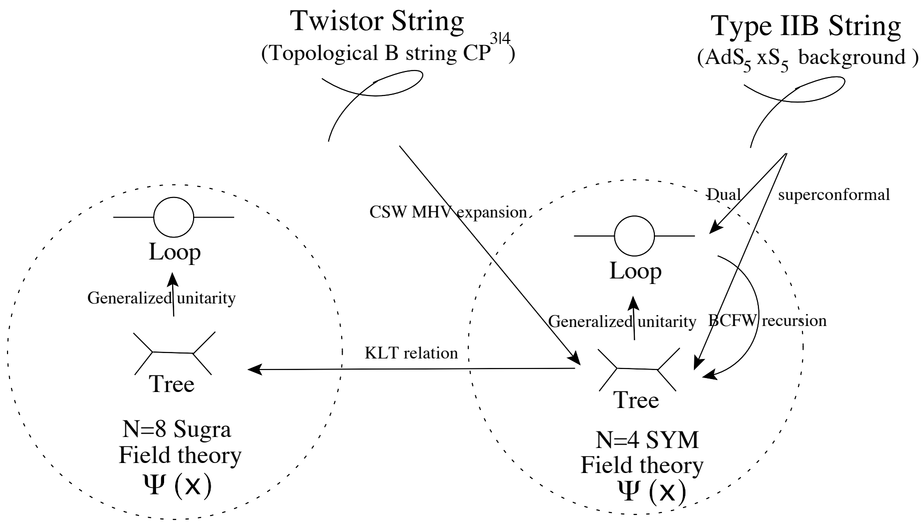


Figure 1.1: Interplay of field theory and string theory at the level of scattering amplitudes.

a deeper reason for this structure other then supersymmetry, locality and gauge symmetry. A similar situation occurs for N=8 super gravity theory (Sugra), which exhibits finiteness beyond that predicted from supersymmetry. In fact the situation for N=4 SYM and N=8 Sugra are closely related. The modern approach to N=8 computations is to use generalized unitarity methods (multi-particle cuts) to relate loop level amplitudes to tree level amplitudes. Since tree-level gravity amplitudes can be rewritten in terms of gauge-theory amplitudes, a fact first uncovered from string theory by Kawai, Lewellen and Tye (KLT)[10], the finiteness of N=8 Sugra is then tied to simple properties of tree level amplitudes of N=4 SYM. These on-shell relations are displayed schematically as:

The issue of off-shell supersymmetry is an even more glaring mystery. By the field content of the theory and its global symmetry properties, it is well known that this theory possesses even at the quantum level a large space time symmetry: the super conformal symmetry. However, there is no known formulation, even for its amplitudes, such that this symmetry is manifest. In comparison, ordinary(N=0) YM theory has Lorentz symmetry and its amplitude can be written in terms of Lorentz invariants.

All these mysteries seem to imply that there is a deeper principle at work here, that is beyond the structure of standard field theory which is based on action principle determined by global(and local)symmetries, renormalizability and locality. Uncovering this principle should shed light on other 4-dimensional theories that encounter difficulties when constructed as a field theory, such as quantum gravity.

We first introduce the action for N=4 SYM.

1.1 N=4 SYM from dimension reduction

We derive D=4 N=4 SYM from D=10 N=1 SYM. This was originally derived in [1], here we follow [11]. The d=10 theory is written as

$$\mathcal{L} = \frac{1}{g_{10}^2} tr(\frac{1}{2} F^a_{MN} F^{aMN} + \bar{\psi}\Gamma^M D_M \psi) \;\; M, N = 0, 1, \cdot\cdot 9 \tag{1.3}$$

The spinors ψ satisfy Majorana-Weyl condition, that is $\bar{\psi} = \psi^T C_{10}$ and $\Gamma_{10}\psi = \psi$, where C_{10} and Γ_{10} are the 10 dimensional charge conjugation and chirality matrices respectively. This gives the correct on-shell counting since there are 8 bosonic degrees of freedom in A_M and $\frac{32}{2\times 2} = 8$ degrees of freedom for ψ. It is invariant under the susy transformation

$$\delta A_M = \bar{\epsilon}\Gamma_M \psi \;\; ; \quad \delta\psi = -\frac{1}{2} F_{MN}\Gamma^{MN}\epsilon \tag{1.4}$$

where $\Gamma^{MN} = \frac{1}{2}\Gamma^{[M}\Gamma^{N]}$. Since ψ is a Majorana-Weyl spinor, so is ϵ. To see that it is invariant under this transformation, note that the variation for the F^2 term gives $-2\bar{\epsilon}\Gamma_N \psi D_M F^{MN}$ while varying ψ gives $-\bar{\psi}\Gamma^M \Gamma^{PQ}\epsilon D_M F_{PQ}$. Using $\Gamma^M\Gamma^{PQ} = \Gamma^{MPQ} + 2\eta^{M[P}\Gamma^{Q]}$ and $\bar{\psi}\Gamma^M\epsilon = -\bar{\epsilon}\Gamma^M\psi$, since both spinors are Majorana, these two terms cancel after using the Bianchi identity $D_{[M}F_{PQ]}$. Then one is left with a term coming from varying the gauge field in the Dirac Lagrangian, $(\bar{\psi}^a\Gamma^M\psi^b)(\bar{\epsilon}\Gamma_M\psi^c)f_{acb}$ where f_{acb} is the structure constant for the gauge group. To see that it vanishes one uses Feirz recoupling

$$(\bar{\lambda}M\xi)(\bar{\psi}N\eta) = -\frac{1}{32}\Sigma_I(\bar{\psi}N\mathcal{O}_I M\xi)(\bar{\lambda}\mathcal{O}_I\eta) \tag{1.5}$$

where one $\mathcal{O}_I$ is a complete set of matrices in the spinor space, and can be taken to be $\mathcal{O}_I = \{I, \Gamma^M, i\Gamma^{MN}, \Gamma^{MNP}..., i\Gamma\Gamma^P, \Gamma\}$ with the normalization

$tr(\mathcal{O}_I\mathcal{O}_J) = 32\delta_{IJ}$. One can now rewrite

$$(\bar{\psi}^a\Gamma^M\psi^b)(\bar{\epsilon}\Gamma_M\psi^c)f_{acb} = -\frac{1}{32}\Sigma_I(\bar{\epsilon}\Gamma_M\mathcal{O}_I\Gamma^M\psi^b)(\bar{\psi}^a\mathcal{O}_I\psi^c)f_{acb} \tag{1.6}$$

For $(\bar{\psi}^a\mathcal{O}_I\psi^c)$ to be non-vanishing it must be antisymmetric in ac, and since ψ satisfies Weyl condition, $\mathcal{O}_I$ must have odd number of gamma matrices. These two requirement leads to two possibilities, $\mathcal{O}_I = \Gamma^M, \Gamma^{MNPQR}$. [3] Using $\Gamma^M\Gamma^N\Gamma_M = -8\Gamma^N$ and $\Gamma^M\Gamma^{PN}\Gamma_M = 0$, we see that $(\bar{\psi}^a\Gamma^M\psi^b)(\bar{\epsilon}\Gamma_M\psi^c)f_{acb} = 0$, the action is indeed invariant under these transformations.

To reduce to four dimensions we choose a specific representation for the gamma matrices.

$$\Gamma^M = \{\gamma^\mu \otimes I_8, \gamma_{(4)} \otimes \tilde{\gamma}^I\} \quad ; \mu = 1, 2 \cdot\cdot 4,\ I = 1, 2 \cdot\cdot 6 \tag{1.7}$$

This breaks the 10 d representation into a product of SO(3,1) and SO(6) representations, $\gamma_{(4)}$ is the chirality matrice in 4-d, and $\tilde{\gamma}^I$ is defined as

$$\tilde{\gamma}^I = \begin{pmatrix} 0 & (\Sigma^I)^{\mu\nu} \\ (\bar{\Sigma}^I)_{\mu\nu} & 0 \end{pmatrix} \quad ; \quad \begin{array}{l} (\Sigma^I)^{\mu\nu} = (\eta^{i\mu\nu}, i\bar{\eta}^{2i\mu\nu}) \\ (\bar{\Sigma}^I)_{\mu\nu} = (-\eta^i_{\mu\nu}, i\bar{\eta}^i_{\mu\nu}) \end{array} \quad i = 1, 2, 3$$
$$\eta^{ijk} = \epsilon^{ijk},\ \eta^{i\mu 4} = \eta^{i4\mu} = \delta^{i\mu}$$
$$\bar{\eta}_{ijk} = \epsilon_{ijk},\ \bar{\eta}_{i\mu 4} = \bar{\eta}_{i\mu 4} = -\delta_{i\mu}$$

$\eta^{i\mu\nu}$ and $\bar{\eta}_{i\mu\nu}$ are 't Hooft symbols[12][4]. In this basis the chirality and the charge conjugation matrix can also be written into a product of 4 and 6 dimension[5]. Now we arrive at the 4-d theory by separating the gauge field into a four

[3] Using $\Gamma^T = \Gamma$ and $\Gamma\psi = \psi$ one can show that for Weyl spinors, $(\bar{\psi}\Gamma^{M_1M_2\ldots M_k}\psi^c) = (-1)^{k+1}(\bar{\psi}\Gamma^{M_1M_2\ldots M_k}\psi^c)$. Thus for it to not vanish $k = odd$. Taking the transpose of $(\bar{\psi}\Gamma^{M_1M_2\ldots M_k}\psi^c)$ and using $C^T = -C$ in d=10, one has $(\bar{\psi}\Gamma^{M_1M_2\ldots M_k}\psi^c) = (\bar{\psi}\Gamma^{M_1M_2\ldots M_k}\psi^c)^T = (-1)^{\frac{k(k+1)}{2}}(\bar{\psi}\Gamma^{M_1M_2\ldots M_k}\psi^c)$. Thus one is left with $k = 1, 5$, since for $k = 9$ it is equivalent to $\Gamma^M\Gamma$ which is equivalent to Γ^M on Weyl spinors.

[4] They form a basis for 4×4 anti-symetric tensors. They satisfy (anti)self-dual relationship, $\eta^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\eta^a_{\rho\sigma}$ and $\bar{\eta}^{a\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\eta}^a_{\rho\sigma}$.

[5] As discussed in the appendix, for 10-d and 4-d $C^T = -C$, while $C^T = C$ for d=6. Specifically $C_{10} = C_6 \times C_4 = \begin{pmatrix} 0 & \delta_{AB} \\ \delta^{AB} & 0 \end{pmatrix} \otimes \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$

dimension vector and six scalars, while the spinor splits into 4 Weyl spinors

$$\begin{aligned} A_M &= \{A_\mu, P^i, S^i\} \;\; i=1,2,3 \\ \psi &= \begin{pmatrix} \delta^{AB} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \lambda^\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta_{AB} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar\lambda_{\dot\alpha} \end{pmatrix} \quad A=1,2,3,4 \end{aligned} \tag{1.8}$$

Note that the way ψ is written satisfies the Majorana-Weyl condition for 10-d for our specific representation of gamma matrix chosen. Since they are complex, they transform as the 4 and $\bar{4}$ of SU(4), the covering group for SO(6). Due to the gamma matrices in the Dirac action, when separated into 4-d the 6 scalars couple to the spinors in the form of

$$f_{abc}\bar\lambda^a_{\dot\alpha A}(\Sigma^{iAB}P^b_i + \Sigma^{2i\,AB}S^b_{2i})\bar\lambda^{c\dot\alpha}_B - f_{abc}\lambda^{a\alpha A}(\bar\Sigma^i_{AB}P^b_i + \bar\Sigma^{2i}{}_{AB}S^b_{2i})\lambda^{cB}_\alpha \tag{1.9}$$

One can then simplify things by the following redefinition of the scalars

$$\begin{aligned} \phi^{aAB} &= \frac{1}{\sqrt{2}}\Sigma^{I\,AB}A^a_I = \frac{1}{\sqrt{2}}(P^{ai}\eta^{iAB} + iS^{ai}\bar\eta^{iAB}) \\ \bar\phi^a_{AB} &= \frac{-1}{\sqrt{2}}\bar\Sigma^I_{AB}A^a_I = \frac{-1}{\sqrt{2}}(P^{ai}\eta^i_{AB} - iS^{ai}\bar\eta^i_{AB}) \end{aligned} \tag{1.10}$$

Due to the self-duality relation for the 't Hooft symbols, we have $-\bar\Sigma^a_{AB} = \frac{1}{2}\epsilon_{ABCD}\Sigma^{aCD}$ and hence the scalars satisfy the following self-dual relationship[6].

$$\bar\phi_{AB} = \frac{1}{2}\epsilon_{ABCD}\phi^{CD} \tag{1.11}$$

Finally we have the following 4-d action for N=4 SYM

$$\begin{aligned} S = \frac{1}{g^2}\int d^4x tr\{&\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar\lambda^{\dot\alpha}_A \not{D}_{\dot\alpha\beta}\lambda^{\beta A} - i\lambda^A_\alpha \not{D}^{\alpha\dot\beta}\bar\lambda_{A\dot\beta} + \frac{1}{2}(D_\mu\bar\phi_{AB})(D^\mu\phi^{AB}) \\ &-\sqrt{2}\bar\phi_{AB}\{\lambda^{\alpha A}, \lambda^B_\alpha\} - \sqrt{2}\phi^{AB}\{\bar\lambda^{\dot\alpha}_A, \bar\lambda_{\dot\alpha B}\} + \frac{1}{8}[\phi^{AB}, \phi^{CD}][\bar\phi_{AB}, \bar\phi_{CD}]\} \end{aligned} \tag{1.12}$$

[6] In superspace approach, this self-duality relationship is a result from modified Bianchi identity as we will discuss later

1.2 Light-cone superspace

The above susy transformations in d=10 close up to field equations. This can be seen by two susy transformation on the spinor

$$
\begin{aligned}
(\delta_{\epsilon_2}\delta_{\epsilon_1}-\delta_{\epsilon_2}\delta_{\epsilon_1})\psi &= -\frac{1}{2}(2\bar{\epsilon}_2\Gamma_N D_M\psi)\Gamma^{MN}\epsilon_1-(1\leftrightarrow 2)\\
&= -\frac{1}{32}\Sigma_I\Gamma^{MN}\mathcal{O}_I\Gamma_N D_M\psi(\bar{\epsilon}_2\mathcal{O}_I\epsilon_1)-(1\leftrightarrow 2)\\
&= -\frac{1}{32}\Gamma^{MN}\Gamma^P\Gamma_N D_M\psi(\bar{\epsilon}_2\Gamma_P\epsilon_1)-(1\leftrightarrow 2)\\
&= (\bar{\epsilon}_2\Gamma^P\epsilon_1)D_P\psi-\frac{1}{2}(\bar{\epsilon}_2\Gamma^P\epsilon_1)\Gamma_P\not{D}\psi
\end{aligned}
\tag{1.13}
$$

This it closes up to the field equation $\not{D}\psi=0$. For off-shell susy, one needs auxiliary fields which we post-pone to later chapter. At this point one can still manifest half of the susy on-shell. In a frame where only p^+ is nonvanishing, the Dirac equation is solved if $\Gamma^-p_-\psi=\Gamma^-p^+\psi=0$ where $\Gamma^{\pm}=\frac{1}{\sqrt{2}}(\Gamma^0\pm\Gamma^1)$. This means that if one splits the spinor ψ into

$$
\psi=-\frac{1}{2}(\Gamma^+\Gamma^-+\Gamma^-\Gamma^+)\psi\equiv\psi^++\psi^- \tag{1.14}
$$

an on-shell spinor means that one has only ψ^-, or $\Gamma^+\psi$ the "+" projected spinor. Looking back at (1.13) indeed the susy algebra with ϵ^+ closes on ψ^-. From the transformation of A_M one sees that only the transverse direction transforms under this reduced susy [7]. This is the basis for light-cone superfield formalism [13], where half of the susy is manifest with the on-shell degrees of freedom, $A_\perp$ and ψ^-. The susy algebra one is left with is

$$
\{Q_{\alpha+},\bar{Q}^{\beta}_{+}\}=(\gamma_+)_\alpha{}^\beta p^+ \tag{1.15}
$$

Preserving half of the susy means that only the SO(8) subgroup of the original Lorentz group is manifest. Dimensionally reducing to four dimensions breaks the SO(8) into SO(6)×SO(2)∼SU(4)× U(1). The four dimensional algebra is then

$$
\{q^m,\bar{q}_n\}=-\sqrt{2}\delta^m_n p^+ \tag{1.16}
$$

[7] $\delta A_{\pm}=(\bar{\epsilon}_+\Gamma_{\pm}\psi_-)=0$ since $\Gamma^-\psi^-=\Gamma^+\epsilon^+=0$.

where m,n are SU(4) indices, there are 4 complex supercharges. One can then define covariant derivatives with anti-commuting grassman variables, θ_m such that the susy generators and covariant derivatives are given by

$$q^m = -\frac{\partial}{\partial\bar{\theta}_m} + \frac{i}{\sqrt{2}}\theta\frac{\partial}{\partial x^-} \quad ; \quad d^m = -\frac{\partial}{\partial\bar{\theta}_m} - \frac{i}{\sqrt{2}}\theta\frac{\partial}{\partial x^-}$$
$$\bar{q}_n = \frac{\partial}{\partial\theta^n} - \frac{i}{\sqrt{2}}\bar{\theta}\frac{\partial}{\partial x^-} \quad ; \quad \bar{d}_n = \frac{\partial}{\partial\theta^n} + \frac{i}{\sqrt{2}}\bar{\theta}\frac{\partial}{\partial x^-}$$

The four dimensional physical fields $\{A, \lambda^m, \phi^{mn}, \bar{\lambda}_n, \bar{A}\}$ transform as the $\{1, 4, 6, \bar{4}, 1\}$ of SU(4). It is then natural to incorporate them in a scalar superfield, a chiral superfield

$$\bar{d}^m\Phi = 0 \tag{1.17}$$

For N=4 SYM its multiplet is TCP self-conjugate, therefore there is a further constraint on the chiral fields.

$$\bar{\Phi} = \frac{1}{(\partial^+)^2}\epsilon^{mnpq}d_m d_n d_p d_q\Phi \tag{1.18}$$

which reflects the self-duality relationship of the scalar fields. Expanding in components

$$\Phi(x,\theta) = \frac{1}{\partial^+}A(y) + \frac{i}{\partial^+}\theta^m\Lambda(y) + i\frac{1}{2}\theta^m\theta^n\bar{C}_{mn}(y) \tag{1.19}$$
$$+\frac{1}{3!}\theta^m\theta^n\theta^p\epsilon_{mnpq}\bar{\Lambda}^q(y) + \frac{1}{4!}\theta^m\theta^n\theta^p\theta^q\epsilon_{mnpq}\partial^+\bar{A}(y)$$

where $y = (x^+, x^- + \frac{1}{2}i\theta^m\bar{\theta}_m, x, \bar{x})$ and p^+ appears such that each term is dimensionless. The 4 d action can then be written as

$$S = tr\int d^4xd^4\theta d^4\bar{\theta}\{\bar{\Phi}\frac{\partial^+\partial^- - \bar{\partial}\tilde{\partial}}{2\partial^{+2}}\Phi - \frac{2}{3}gf^{abc}[\frac{1}{\partial^+}\bar{\Phi}^a\Phi^b\bar{\partial}\Phi^c + \text{complex conjugate}]$$
$$-g^2f^{abc}f^{ade}[\frac{1}{\partial^+}(\Phi^b\partial^+\Phi^c)\frac{1}{\partial^+}(\bar{\Phi}^d\partial^+\bar{\Phi}^e) + \frac{1}{2}\Phi^b\bar{\Phi}^c\Phi^d\bar{\Phi}^e]\} \tag{1.20}$$

Chapter 2

On-shell amplitudes (tree)

2.1 Introduction

In recent years the attention has turned to on-shell methods for the S-matrix of the theory, see [14, 15] for review. These methods were built upon either Cachazo, Svrcek and Witten's (CSW)[16]'s MHV vertex expansion or Britto, Cachazo, Feng, and Witten's [17, 18] recursion relations (sometimes its a combination of the two). Though these two methods was preliminarily developed for N=0 YM theory, extension to N=4 has been straight forward [19]. The power of these methods is that higher point amplitudes which are complicated in traditional Feynman rules can now be constructed from simpler amplitudes which are in compact form. The utility of these approach has been demonstrated in the computation of previous inaccessible N=8 Supergravity loop amplitudes as mentioned in the general introduction.

Various efforts has been made on providing a proof for the CSW program. Risager [20] showed that the CSW program is just a result of certain recursion relationship similar to that developed by BCFW, which uses the fact that one can use unitarity to relate one loop amplitudes to tree amplitudes, while infrared consistency conditions relate different tree amplitudes to satisfy a recursion relationship. However, in the proof for the BCFW recursion relationship [18] one actually uses the CSW program to prove the behavior of tree amplitudes in certain limits.[1] In the following we give a brief description of the

[1] Recently, one has been able to prove that BCFW eventually leads to the CSW expansion [21].

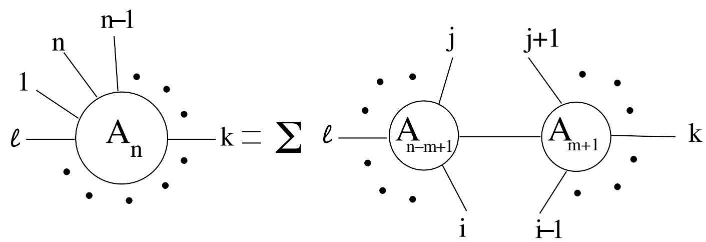


Figure 2.1: The BCFW recursion relation.

BCFW and CSW approach, the main point will be to prove that CSW has its origin hidden in the Lagrangian of the theory: it corresponds to a perturbative expansion around the self-dual part of the action.

2.1.1 BCFW recursion relations

The BCFW recursion relation is an algebraic relation between higher point amplitudes and lower point amplitudes. An n-point on-shell tree amplitude can be expressed in terms of two lower point on-shell amplitudes with $n-l+1$ and $l+1$ external legs:

$$A_n = \sum_{\{ij\}} \hat{A}_{n-l+1,(i\cdot\cdot j)} \frac{1}{P_{ij}^2} \hat{A}_{l+1,(j+1\cdot\cdot i-1)} \tag{2.1}$$

where the sum is over different sets of external momenta that sits on one side of the propagator. I used hat for the lower point amplitude $\hat{A}$, indicating that the two of the external lines (reference lines) are slightly modified. This will become clear shortly. This relationship can be expressed pictorially as: where l and k label the reference lines which we now discuss. Consider a general tree amplitude with n external lines, choose two external lines (the reference lines) and shift them by a null momentum q,

$$p_l \to p_l + zq \ , \ p_k \to p_k - zq \tag{2.2}$$

This shift preserves momentum conservation and q is chosen such that $q \cdot p_l = q \cdot p_k = 0$. One can choose $q = \lambda_l \tilde{\lambda}_k$, then the above shift corresponds to the following shift for the spinors of the reference lines

$$\tilde{\lambda}_l \to \tilde{\lambda}_l + z\tilde{\lambda}_k \ ; \ \lambda_k \to \lambda_k - z\lambda_l \tag{2.3}$$

Note that this shift violates the relationship $\tilde{\lambda} = \pm\bar{\lambda}$ required for Minkowski signature, therefore we are really looking at amplitudes in split signature $(+, +, -, -)$ which does not make a difference for tree amplitudes. After the shift the only singularity in z for an arbitrary tree graph comes from propagators: the shifted amplitude $A_n(z)$ is then a rational function in z and has simple poles in the propagators that have the two reference lines on opposite sides[2]

$$\frac{1}{(P_{ij})^2} \rightarrow \frac{1}{(P_{ij})^2 - 2z\langle\lambda_l|P_{ij}|\tilde{\lambda}_k]} \tag{2.4}$$

This is true for general amplitudes. The crucial point is that if the amplitudes vanish for z taken to infinity, then the function $A(z)$ is uniquely determined by the residues of the simple poles. That is, $A(z)$ has a unique expansion as

$$A(z) = \sum_{\{i,j\}} \frac{c_{i,j}}{z - z_{i,j}} = -\sum_{\{i,j\}} A_L \frac{1}{2(\langle\lambda_l|P_{ij}|\tilde{\lambda}_k])(z - \frac{P_{i,j}^2}{2\langle\lambda_l|P_{ij}|\tilde{\lambda}_k]})} A_R \tag{2.5}$$

again the sum runs over all sets of external line configuration i, j in the figure such that the reference lines sit on opposite side of the propagator, and $z_{i,j} = \frac{P_{i,j}^2}{2\langle\lambda_l|P_{ij}|\tilde{\lambda}_k]}$. At this point $A_L(A_R)$ are just functions depending on the polarization and momentum of the external lines on the left(right) of the propagators, they are not amplitudes yet.

Whether or not the tree amplitudes vanish for large z is discussed in [22]. We will use the fact that indeed they do for N=4 SYM.

The real amplitude corresponds to the (2.5) evaluated at $z = 0$: $A(0) = -\sum_{\{i,j\}} \frac{c_{i,j}}{z_{i,j}}$. The residues $c_{i,j}$ take the form

$$c_{i,j} = -\hat{A}_L \times \hat{A}_R \frac{1}{2\langle\lambda_l|P_{ij}|\tilde{\lambda}_k]}\Bigg|_{z=\frac{P_{i,j}^2}{2\langle\lambda_l|P_{ij}|\tilde{\lambda}_k]}} \tag{2.6}$$

where $\hat{A}_L$ and $\hat{A}_R$ contains the shifted reference momentum (2.2,2.3) with $z = \frac{P_{i,j}^2}{2\langle\lambda_l|P_{ij}|\tilde{\lambda}_k]}$. Now the line in $\hat{A}_R$ and $\hat{A}_L$ that was connected to the propagator has momenta $\pm(P_{ij} - z_{i,j}q)$, which is now massless $(P_{ij} - z_{i,j}q)^2 = P_{ij}^2 - 2z_{i,j}q \cdot P_{ij} = 0$. Thus all lines for $\hat{A}_R$ and $\hat{A}_L$ are now on-shell: these are just lower point tree amplitudes with the reference spinors redefined as 2.3. Thus we've finally

[2] Note that it has only simple poles due to $q^2 = 0$. Since q is constructed from the momenta of the external lines, it'll will be difficult to construct an off-shell version of BCFW.

arrived at the BCFW recursion relation:

$$A_n = \sum_{\{ij\}} \hat{A}_{n-m+1,(i\cdot\cdot j),l} \frac{1}{P_{ij}^2} \hat{A}_{m+1,(j+1\cdot\cdot i-1),k} \tag{2.7}$$

In this final form we've labelled the reference lines l, k.

2.1.2 CSW from Twistor string

The origin of CSW expansion is quite different from BCFW (except the authors). BCFW relations were originally realized from analysing loop amplitudes of N=4 SYM which are UV finite but have infrared singularity. Using unitarity based methods, the computation of these amplitudes reduces to evaluating coefficients in front of a set of scalar box integrals (discussed in next chapter). These coefficients can be expressed in terms of tree amplitudes across the cuts, thus these tree amplitudes must combine in a way such that the infrared singularities coming from the scalar box combine nicely into known results. This gives a recursive relationship for the tree amplitudes.

CSW expansion came from Witten's twistor string formulation of tree level N=4 SYM amplitudes[6]. Perturbative expansion of amplitudes correspond to perturbation in instanton number (target space D-instantons for Witten's topological string, world sheet instantons for Berkovits and Siegels construction[23, 24]). An instanton number 1 amplitude is the MHV amplitude while instanton number k amplitude gives N^{k-1}MHV amplitude. For $k > 1$ it is not clear whether one should consider only a k instanton or multiples of lower instanton number. So far there is evidence for both case. If completely disconnected instantons can give the correct field theory amplitude, then this implies that there must be a perturbative formulation in field theory which uses MHV amplitudes as vertices, this is the CSW approach[16].

The definition of MHV is as follows. For the usual Yang-Mills theory, amplitudes are labelled by their external momenta and helicity($\pm$). Amplitudes with all plus(minus) or just one plus(minus) helicity vanish. Thus the first non-vanishing amplitude that has mostly plus(minus) helicity must have at least two minus(plus) helicities, these are the Maximal Helicity Violating (MHV)amplitudes. For example 5 point amplitude takes the form $(-,-,+,+,+)$, $(-,-,-,+,+)\cdot\cdot$e.t.c. while $(-,-,-,-,+) = 0$. In N=4

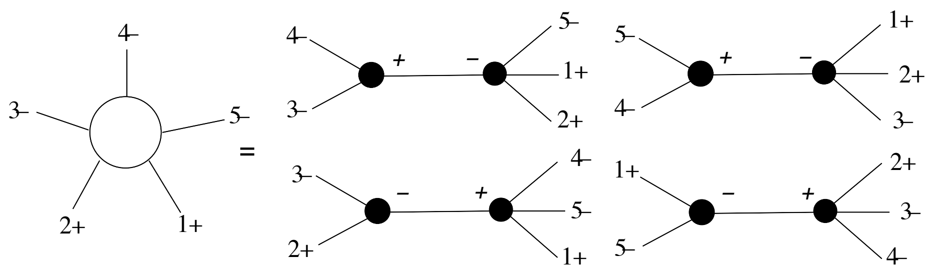


Figure 2.2: A 5-pt $\overline{\text{MHV}}$ amplitude constructed in CSW method.

SYM this assignment is extended to fermions and scalars. Thus (anti)chiral fermions has helicity $(-\frac{1}{2})+\frac{1}{2}$ while the scalars has helicity 0. Then MHV amplitudes can be defined as those that have total helicity -4 with respect to an all plus helicity amplitude. For example for 4 point we have $(-1,-1,+1,+1)$, $(0,0,0,0)$ or $(-\frac{1}{2},-\frac{1}{2},0,+1)$.

In the CSW approach one constructs an arbitrary tree amplitude by using MHV vertices as its only building block. For example a 5 point $(++---)$ amplitude can be written as four different combinations of a 3 point and a 4 point MHV amplitude.

Each MHV vertex takes an on-shell form while the leg that connects the propagator is continued off shell by the following prescription

$$\lambda_P = P_{\alpha\dot{\alpha}}\tilde{\eta}^{\dot{\alpha}} \tag{2.8}$$

where η corresponds to an arbitrary null vector. Note that only the holomorphic spinor is redefined, this is because the MHV amplitude only depends on the holomorphic spinor. This is why the CSW expansion is in a sense expanding on a chiral basis. In practice one usually picks an arbitrary external line for this null vector, then one anticipates that the final amplitude does not depend on the choice of reference spinor. In the next section we begin to derive a field theory explanation [25] for such a perturbation base on the light-cone superspace action, then the reference spinor is identified as the frame that defines the light-cone gauge. Hence reference spinor independence is equivalent to gauge choice independence.

2.2 MHV Lagraingian

Even though the relation between various on-shell methods has become clear, one would still like to see its relationship to the action approach of QFT, since originally the theory was defined by its Lagrangian. Making the connection may well shed light on what properties of the Lagrangian lead to such simple structures for its scatering amplitudes. Effort along this line of thought began by Gorsky and Rosly [26] where they proposed a non-local field redefinition to transform the self-dual part of the YM action into a free action, while the remaining vertices would transform into an infinite series of MHV vertices. In this sense the MHV lagrangian can be viewed as a perturbation around the self-dual sector of ordinary Yang-Mills. This seems natural since self-dual Yang-Mills is essentially a free theory classically. Yang-Mills lagrangian in light-cone (or space-cone[27]) gauge is a natural framework for such a field redefinition since the positive and negative helicity component of the gauge field are connected by a scalar propagator. Work on the light-cone action began by Mansfield[28] emphasizing the canonical nature of the field redefinition. The formulation was also extended to massless fermions. The explicit redefinition for Yang-Mills was worked out by Ettle and Morris [29]. The canonical condition in [28][29] ensures that using the field redefinition, complications will not arise when taking into account currents in computing scattering amplitude. This will not be true for more general field redefinitions as we show in this letter.

The progress above was mostly done in the framework of ordinary Yang-Mills. However, the CSW program has also achieved various successes in N=4 SYM as priorly mentioned. It is also interesting in [29] the redefinition for positive and negative helicity have very similar form which begs for a formulation putting them on equal footing. This formulation is present in N=4 light-cone superspace [13] where both the positive and negative helicity gauge field sit on opposite end of the multiplet contained in a single chiral superfield. Thus a field redefinition for one superfield contains the redefinition for the entire multiplet, which would be very difficult if one tried the CSW program for the component fields separately. Moreover, N=4 Self-dual YM is free at quantum level, implying the CSW program should work better at loop level for SYM compared to YM.

In this section we formulate such a field redefinition using the N=4 SYM light-cone Lagrangian. We proceed in two ways, first we try to formulate a general redefinition by simply requiring the self-dual part of the SYM lagrangian becomes free in the new Lagrangian. Subtleties arise when using it to compute scattering amplitudes that require one to take into account the contribution of currents under field redefinition. Latter, we will impose the redefinition to be canonical. In both cases only the redefinition of the chiral field is needed, thus giving the transformations for components in a compact manner. However, it is the second redefinition that corresponds to CSW program, and we will see that once stripped away of the superpartners, it gives the result for YM derived in [29]. We calculate the on-shell amplitude in the new lagrangian for 4-pt MHV amplitude and show that it matches the simple form derived in [7]. In the end we briefly discuss the relation between the off-shell MHV vertices here and the on-shell form, with off-shell continuation for propagators, used in CSW.

2.2.1 The Field Redefinition

Transforming (1.20) to the chiral basis using (1.18), one arrives at a quadratic term, a 3-pt vertex with 4 covariant derivatives, a three pt and 4-pt vertex with 8 covariant derivatives. As shown by Chalmers and Siegel [30], the quadratic term and the three point vertex which contains only 4 covariant derivatives describes self-dual SYM. Since self-dual SYM is free classically, at tree level one should be able to consider the self-dual sector to be simply a free action in the full SYM, i.e. one considers the full SYM as a perturbative expansion around the self-dual sector. Therefore the aim is to redefine the chiral field so that the self-dual sector transforms into a free action: one then tries to find $\Phi(\chi)$ such that

$$\begin{aligned} S_{SD} = tr \int d^4x d^4\theta \; \{\Phi\partial^+\partial^-\Phi - \Phi\partial\bar{\partial}\Phi + \frac{2}{3}\partial^+\Phi[\Phi, \bar{\partial}\Phi]\} \\ = tr \int d^4x d^4\theta \; \{\chi\partial^+\partial^-\chi - \chi\partial\bar{\partial}\chi\} \end{aligned} \tag{2.9}$$

Note that if the field redefinition does not contain covariant derivatives, the remaining interaction terms will becomes MHV vertices, the infinite series generated by the field redefinition from the remaining 3 and 4-pt vertex will all have 8 covariant derivatives. This result is implied by the known MHV amplitude [7]

$$A(...j^-.....i^-...)_{tree} = \frac{\delta^8(\sum_{i=1}^n \lambda_i \theta_i^A)}{\Pi_{i=1}^n < i\, i+1 >} \tag{2.10}$$

where

$$\delta^8(\sum_{i=1}^n \lambda_i \theta_i^A) = \frac{1}{2}\prod_{A=1}^4 (\sum_{i=1}^n \lambda_i^\alpha \theta_i^A)(\sum_{i=1}^n \lambda_{i\alpha}\theta_i^A) \tag{2.11}$$

The amplitude contains various combination of 8 θ's and thus imply 8 covariant derivatives to extract the amplitude.

In the Yang-Mills MHV lagrangian [28][29], the positive helicity gauge field A transforms into a function of only the new positive helicity field B, while the negative helicity $\bar{A}$ transforms linearly with respect to $\bar{B}$, $\bar{A}(\bar{B}, B)$. One can see this result by noting that in order to preserve the equal time commutation relationship,

$$[\partial^+ \bar{A}, A] = [\partial^+ \bar{B}, B] \tag{2.12}$$

that is, the field redefinition is canonical. This implies $\partial^+ \bar{A} = \partial^+ \bar{B} \frac{\delta B}{\delta A}$, therefore $\bar{A}$ transform into one $\bar{B}$ and multiple B fields. This result for the gauge fields becomes natural in the N=4 framework since now the chiral field Φ is redefined in terms of series of new chiral field χ. The positive helicity gauge field A which can be defined in the superfield as $\frac{1}{\partial^+}A = \Phi|_{\theta=0} = \Phi(\chi|_{\theta=0})$ resulting in a function that depends only on B. For the negative helicity $\partial^+ \bar{A} = D^4 \Phi|_{\theta=0} = ...\chi(D^4\chi)\chi|_{\theta=0}...$, dropping contributions from the super partners we see that $\bar{A}(\bar{B}, B)$ depends on $\bar{B}$ linearly.

Another advantage of working with superfields is that as long as the field redefinition does not contain covariant derivatives, the super determinant arising from the field redefinition will always be unity due to cancellation between bosonic and fermionic contributions. Therefore there will be no jacobian factor arising.

The requirement that the field redefinition must be canonical is necessary for the equivalence between MHV lagrangian and the original lagrangian in the framework of the LSZ reduction formula for scattering amplitudes. Indeed we will illustrate this fact by solving the field redefinition for (2.10) disregarding the canonical constraint. We will show that this gives a solution that by itself does not give the correct form of MHV amplitude on-shell, one needs to incorporate the change induced on the external currents. After imposing the canonical constraint we derive the correct on-shell result.

Field redefinition I $\Phi(\chi)$

We proceed by expanding Φ in terms of χ. Since the light-cone action in the component language corresponds to choosing a light-cone gauge, the redefinition should be performed on the equal light-cone time surface to preserve the gauge condition. We thus Fourier transform the remaining three coordinates into momentum space, leaving the time direction alone, understanding that all fields are defined on the same time surface.

$$\Phi(\vec{p}_1) = \chi(1) + \sum_{n=2}^{\infty} \int_{\vec{p}_2\vec{p}_3..\vec{p}_{n+1}} C(\vec{p}_2, \vec{p}_3 \cdots \vec{p}_{n+1})\chi(2)\chi(3)..\chi(n+1)\delta(\vec{p}_1 + \sum_{i=2}^{n+1} \vec{p}_i) \tag{2.13}$$

Here we follow the simplified notation in [29], the light-cone momenta are labeled $p = \{p^-, p^+, p, \bar{p}\}$, the later spatial momenta are collected as a three vector $\vec{p}$, and introduce abbreviation for the momentum carried by the fields, $\chi(i) = \chi(-\vec{p}_i)$. Plugging into (2.10), the coefficient in front of the first term is determined by equating terms quadratic in χ on the left hand side with the right. Similarly for cubic terms we have :

$$\begin{aligned} \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) tr \int d^4\theta \int_{\vec{p}_2\vec{p}_3\vec{p}_1} [-2C(\vec{p}_2, \vec{p}_3) P^2_{2,3} \\ + \frac{2}{3}(p^+_3 \bar{p}_2 - p^+_2 \bar{p}_3)]\chi(1)\chi(2)\chi(3) = 0 \end{aligned} \tag{2.14}$$

Thus we have

$$C(\vec{p}_2, \vec{p}_3) = -\frac{1\{23\}}{3P^2_{2,3}} \tag{2.15}$$

where $P^2_{i..j} = (p_i + ...p_j)^2$, $\{i, j\} = p^+_i \bar{p}_j - p^+_j \bar{p}_i$, and for later $(i, j) = p^+_i p_j - p^+_j p_i$.

For four field terms :

$$\delta(\Sigma_{i=2}^{5}\vec{p}_i)tr\int d^4\theta\int_{\vec{p}_2\cdot\cdot\vec{p}_5}[-C(\vec{p}_2,\vec{p}_3)C(\vec{p}_4,\vec{p}_5)P^2_{2,3}-2C(\vec{p}_2,\vec{p}_3,\vec{p}_4)P^2_{2,3,4}$$
$$-\frac{2}{3}C(\vec{p}_2,\vec{p}_3)\{4,5\}-\frac{2}{3}C(\vec{p}_3,\vec{p}_4)\{(3,4),5\}$$
$$-\frac{2}{3}C(\vec{p}_4,\vec{p}_5)\{3,(4,5)\}]\chi(2)\chi(3)\chi(4)\chi(5)=0 \tag{2.16}$$

Using our solution for $C(\vec{p}_2,\vec{p}_3)$ from (2.15), cyclic identity within trace and relabeling the momenta for the last three terms we have:

$$C(\vec{p}_2,\vec{p}_3,\vec{p}_4)=\frac{5}{18}\frac{\{2,3\}\{4,5\}}{P^2_{2,3,4}P^2_{2,3}} \tag{2.17}$$

One can again use this result to obtain higher terms iteratively. The field redefinition does not contain covariant derivatives, thus guarantees the remaining vertex after field redefinition will be only of MHV vertex. However if we directly use the new vertices to calculate on-shell amplitude we find that it will differ from the original amplitude computed using the old action. In the next subsection we use YM to illustrate the discrepancy and its remedy.

Field redefinition I for YM

One can easily follow the above procedure to solve YM field redefinition[3]. Again we have :

$$tr\int d^4x\quad \bar{A}\partial^+\partial^-A-\bar{A}\bar{\partial}\partial A-\frac{\bar{\partial}}{\partial^+}A[A,\partial^+\bar{A}] \tag{2.18}$$
$$=tr\int d^4x\quad \bar{B}\partial^+\partial^-B-\bar{B}\bar{\partial}\partial B$$

We can choose to leave $\bar{A}$ alone, $\bar{A}=\bar{B}$. Following steps similar to the above, for the next to linear term one have:

$$A(1)=B(1)+\int_{\vec{p}_2\vec{p}_3}C(\vec{p}_2,\vec{p}_3)B(2)B(3)\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)..... \tag{2.19}$$

[3]This redefinition was also investigated in [31].

with

$$C(\vec{p}_2, \vec{p}_3) = \frac{ip_1^+\{2,3\}}{p_2^+ p_3^+ P_{2,3}^2} \tag{2.20}$$

One can then use this result to compute a four point MHV amplitude. With the momentum being on shell now one has

$$C(\vec{p}_2, \vec{p}_3) = \frac{ip_1^+}{(2,3)} \tag{2.21}$$

To see that this does not give the correct result, note that (2.21) is exactly the required redefinition, $\Upsilon(123)$, for A field derived [29]. However, in [29] there is also a field redefinition for $\bar{A}$ while in our approach we left it alone, thus it is obvious that our redefinition will not give the correct on-shell MHV amplitude. The difference between our approach and [29] is the lacking of canonical constraint of the field redefinition. One might guess the discrepancy comes from the jacobian factor in the measure generated by our redefinition (which will be present for YM). However these only contribute at loop level. It is peculiar that field redefinition in the lagrangian formalism should be submitted to constraints in the canonical formalism. From direct comparison for the four pt MHV (- -++) we see that we reproduce the last two terms in eq.(3.13) [29] while the first two terms are missing, the two terms coming from the result of redefining the the $\bar{A}$ field.

The resolution to the missing terms comes from new contribution arising from the currents. In a beautiful discussion of field redefinitions in lagrangian formalism [32], it was pointed out that since scattering amplitudes are really computed in the lagrangian formalism with currents, one should also take into account the effect of the field redefinition for the currents. In the LSZ reduction formula for amplitude, one connects the source to the Feynman diagrams being computed through propagators and then amputate the propagator by multiplying p^2 and taking it on-shell. For YM the currents are $J\bar{A}$ and $\bar{J}A$ where J carries the A external field and $\bar{J}$ carries the $\bar{A}$ field, as can be seen by connecting them to $\langle A\bar{A}\rangle$ propagator. When performing a field redefinition the coupling of the current with the new fields now takes a very different form

$$\bar{J}A(B) \to \bar{J}B + C_2\bar{J}BB + \cdot\cdot \tag{2.22}$$

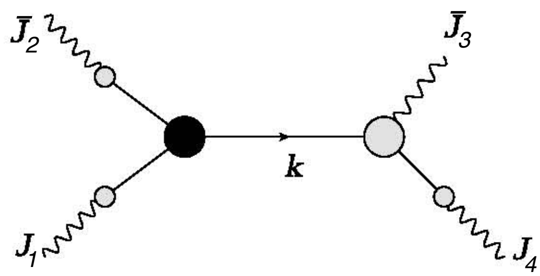


Figure 2.3: In this figure we show how the field redefinition may contribute to tree graphs from the modification of coupling to the source current. The solid circle indicate the $(--+)$ vertex while the empty circle indicates contraction with the currents. Due to new terms in coupling, the $C\bar{J}_3BB$ term, one can actually construct contribution to the $(--++)$ amplitude by using this term, denoted by the larger empty circle, as a vertex.

Due to these higher order terms, the currents themselves behave as interaction terms. In [29] these higher order contributions vanish after multiplying p^2 and taking them on-shell in the LSZ procedure. In our approach these higher terms will not vanish because of the $\frac{1}{p^2}$ always sitting in front of each field redefinition coefficient as in (2.15)(2.17). Remember the scattering amplitudes are always computed by taking $\frac{\delta}{\delta J}$ (or $\frac{\delta}{\delta \bar{J}}$)of the path integral and multiplying each J (or $\bar{J}$) by p^2 and external wave function, taking everything on-shell in the end. The non-vanishing of the additional terms means we have new contributions to the amplitude.

Adding the contribution of these terms we shall see that one gets the correct amplitude. Consider the 4pt MHV(- -++) or $(\bar{J}\bar{J}JJ)$ amplitude. Now there are four new terms present, two for two different ways of connecting the $\bar{J}BB$ term to the original three pt.vertex, and there are two three point vertices available. A typical graph would be that shown in fig.2.3,

Consider the 3-pt vertex $\frac{-ip_2}{p_2^+}p_1^+\bar{B}(k)\bar{B}(2)B(1)$ in the original lagrangian. The $\bar{B}(k)$ leg is now connected to the $\bar{J}BB$ vertex, thus contributing a $\frac{1}{P_{1,2}^2}$. From the LSZ procedure there are p^2's multiplying each current. These cancel the remaining propagators except the $\bar{J}$ for the empty circle, the p^2 of that current cancels the $\frac{1}{p^2}$ in front of the field redefinition in (2.20). Putting everything together we have.

$$-\frac{p_2}{p_2^+}p_1^+\delta(\vec{p}_k+\vec{p}_2+\vec{p}_1)\times\frac{1}{P_{1,2}^2}\times\frac{p_3^+\{-k,4\}}{-p_k^+p_4^+}\delta(\vec{p}_3+\vec{p}_4-\vec{p}_k) \qquad (2.23)$$

Using the delta function and putting all external momenta on-shell we arrive at

$$-\frac{p_2 p_1^+ p_3^{+2}}{p_2^+(p_3^+ + p_4^+)(3,4)} \tag{2.24}$$

One can proceed the same way to generate other terms by connecting the $\bar{B}(2)$ leg to the $\bar{J}BB$ vertex, and also doing the same thing to the other MHV 3-pt vertex $-i\frac{p_k p_2^+}{p_k^+}\bar{B}(k)B(2)\bar{B}(3)$. Collecting everything we reproduce the missing terms. Thus our field redefinition does provide the same on-shell amplitude if we take into account contributions coming from the currents.

Field redefinition II (canonical redefinition)

Due to the extra terms coming from the currents, the field redefinition from the previous sections does not relate to the CSW program, since for CSW the only ingredients are the MHV vertices while above one needs current contribution. In order to avoid complication arising from the currents we impose canonical constraint as in [29]. This implies the following relationship

$$tr\int d^4x d^4\theta \quad \Phi(\chi)\partial^+\partial^-\Phi(\chi) = tr\int d^4x d^4\theta \quad \chi\partial^+\partial^-\chi \tag{2.25}$$

This is true because the canonical constraint (2.12) implies that the new field depends on the time coordinate through the old field, there cannot be inverse derivative of time in the coefficients that define the redefinition. Thus our field redefinition should satisfy (2.25) and

$$tr\int d^4x d^4\theta - \Phi\partial\bar{\partial}\Phi + \frac{2}{3}\partial^+\Phi[\Phi,\bar{\partial}\Phi] = tr\int d^4x d^4\theta \ -\chi\partial\bar{\partial}\chi \tag{2.26}$$

separately. To find a solution to both (2.25) and (2.26) one notes that the component fields are defined in the same way for both chiral superfields, we see that the A field under redefinition will not mix with other super partners in the supersymmetric theory. Thus we can basically read off the redefinition coefficient from the A field redefinition derived in [29].

$$A(1) = B(1) + \sum_{n=2}^{\infty} -(i)^{n-1}\int_{\vec{p}_2 \cdot\cdot \vec{p}_{n+1}} \frac{p_1^+ p_3^+ \cdot\cdot p_n^+}{(23)(34).(n,n+1)} B(2)...B(n+1)\delta(\sum_{i=1}^{n}\vec{p}_i) \tag{2.27}$$

The A field redefinition coming from the superfield redefinition in (2.13) would read

$$\frac{A(1)}{ip_1^+} = \frac{B(1)}{ip_1^+} + \sum_{n=2}^{\infty} \int_{\vec{p}_2..\vec{p}_{n+1}} C(2,..n+1)(i)^n \frac{B(2)...B(n+1)}{p_2^+..p_{n+1}^+} \delta(\sum_{i=1}^{n} \vec{p}_i) \quad (2.28)$$

Comparing (2.27) and (2.28) implies the field redefinitions for the superfields are

$$\Phi(1) = \chi(1) + \sum_{n=2}^{\infty} \int_{\vec{p}_2..\vec{p}_{n+1}} \frac{p_2^+ p_3^{+2}..p_n^{+2} p_{n+1}^+}{(2,3)(3,4)..(n,n+1)} \chi(2)\chi(3)..\chi(n+1)\delta(\sum_{i=1}^{n} \vec{p}_i) \quad (2.29)$$

One can check this straight forwardly by computing the redefinition for the $\bar{A}$. Stripping away the superpartner contributions gives

$$\bar{A}(1) = \bar{B}(1) + \sum_{n=2}^{\infty} \int_{\vec{p}_2..\vec{p}_{n+1}} \sum_{s=2}^{n} \frac{(i)^{n+1} p_s^{+2} p_3^+ p_4^+ ..p_n^+}{p_1^+(2,3)..(n,n+1)} B(2)..\bar{B}(s)..B(n+1)\delta(\sum_{i=1}^{n} \vec{p}_i) \quad (2.30)$$

This agrees with the result in [29]. It remains to see that the solution in (2.29) satisfy the constraint (2.25) and eq.(2.26). However the fact that the pure YM sector resulting from the super field redefinition satisfies the constraint implies that this is indeed the correct answer. In the appendix we use this solution to prove (2.25) and eq.(2.26) is satisfied. In the next section we use our new field redefinition to reproduce supersymmetric MHV amplitude $\bar{\Lambda}\bar{\text{A}}\Lambda\text{A}$.

Explicit Calculation for MHV amplitude $\bar{\Lambda}\bar{\text{A}}\Lambda\text{A}$

Here we calculate the MHV amplitude in our new lagrangian and compare to known results. For the amplitude $\bar{\Lambda}(1)\bar{\text{A}}(2)\Lambda(3)\text{A}(4)$ we know that the result is

$$\frac{\langle 12\rangle^2}{\langle 34\rangle\langle 41\rangle} \quad (2.31)$$

To transform this into momentum space we follow [29] conventions (we'll discuss more in the next section). For a massless on-shell momentum we write the spinor variables to be :

$$\lambda_\alpha = \begin{pmatrix} \frac{-\tilde{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \ \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} \frac{-\bar{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \tag{2.32}$$

Then we have

$$\langle 12 \rangle = \frac{(1,2)}{\sqrt{p_1^+ p_2^+}} \quad [12] = \frac{\{1,2\}}{\sqrt{p_1^+ p_2^+}} \tag{2.33}$$

Thus (2.31) becomes

$$\frac{(1,2)^2 p_4^+ \sqrt{p_1^+ p_3^+}}{(3,4)(4,1) p_1^+ p_2^+} \tag{2.34}$$

To compute this amplitude from our MHV Lagrangian, we use the relevant field redefinition in components, and then substitute them in the following three and four point vertex of the original Lagrangain.

$$-i\frac{\partial \bar{\mathrm{A}}}{\partial^+} \Lambda \bar{\Lambda} + i \bar{\mathrm{A}} \Lambda \frac{\partial \bar{\Lambda}}{\partial^+} - i \bar{\Lambda} \frac{(\bar{\mathrm{A}} \Lambda)}{\partial^+} \mathrm{A} \tag{2.35}$$

From our field redefinition we can extract the relevant redefinition for $\Lambda \bar{\Lambda}$

$$\Lambda(1) \rightarrow \int_{\vec{p}_2 \vec{p}_3} i \frac{(p_2^+ + p_3^+)}{(2,3)} \Lambda'(2) \mathrm{A}'(3) \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \tag{2.36}$$

$$\bar{\Lambda}(1) \rightarrow \int_{p_2 p_3} -i \frac{p_3^+}{(2,3)} \mathrm{A}'(2) \bar{\Lambda}'(3) \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$$

Plugging into (2.35) we have five terms. Cyclically rotating the fields to the desired order and relabeling the momenta we arrive at

$$-\frac{1}{p_2^+ + p_3^+} - \frac{p_2(p_4^+ + p_3^+)}{p_2^+(3,4)} + \frac{p_1(p_4^+ + p_3^+)}{p_1^+(3,4)} - \frac{p_1^+ p_2}{(4,1) p_2^+} + \frac{p_1^+(p_2 + p_3)}{(4,1)(p_2^+ + p_3^+)}$$

$$= \ -\frac{(1,2)}{(4,1) p_2^+} - \frac{(1,2)(p_4^+ + p_3^+)}{(3,4) p_1^+ p_2^+} = \frac{(1,2)^2 p_4^+}{(3,4)(4,1) p_1^+ p_2^+} \tag{2.37}$$

Using that the on shell external line factor in light cone for the gauge fields is 1 and for the fermion pair is $\sqrt{p_1^+ p_3^+}$, one reproduces the MHV amplitude in (2.31).

2.2.2 CSW-off-shell continuation

An on-shell four momentum can be written in the bispinor form

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} p\bar{p}/p^+ & -p \\ -\bar{p} & p^+ \end{pmatrix} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} \;\; ; \;\; \lambda_\alpha = \begin{pmatrix} \frac{-p}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix}, \; \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} \frac{-\bar{p}}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \quad (2.38)$$

For an off-shell momentum the relationship is modified

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} + z\eta_\alpha \bar{\eta}_{\dot{\alpha}} \;\; ; \;\; z = p^- - \frac{p\bar{p}}{p^+}, \; \eta_\alpha = \bar{\eta}_{\dot{\alpha}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.39)$$

imposing $p^2 = 0$ we see that $z = 0$ and we are back at (2.38). The spinors λ_α and $\bar{\lambda}_{\dot{\alpha}}$ are written in terms of $p^+, p, \bar{p}$, so that it can be directly related to amplitudes computed by the light-cone action which only contains these momenta in the interaction vertices. One can then use these spinors for the off-shell lines by keeping in mind that they relate to the momentum through (2.39). To see this one can compute the three point MHV amplitude by looking directly at the 3 point $--+$ vertex from the light-cone action (even though these vanish by kinematic constraint, but it is sufficient to demonstrate the equivalence since the three point MHV vertex is part of the ingredient of CSW). The 3pt vertex for light-cone YM reads $i[\bar{A}, p^+ A]\frac{p}{p^+}\bar{A}$, then the amplitude is

$$(1^-2^-3^+) = i(\frac{p_1}{p_1^+}p_3^+ - p_3^+\frac{p_2}{p_2^+}) = -i\frac{p_3^+}{p_2^+ p_1^+}(1,2) = -i\frac{p_3^+}{p_2^+ p_1^+}\frac{(1,2)^3}{(2,3)(3,1)} \quad (2.40)$$

where in the last equivalence we used $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$. In our definition for the spinors, we have the identity $\langle 1, 2\rangle = \frac{(1,2)}{\sqrt{p_1^+ p_2^+}}$. We see that

$$-i\frac{p_3^+}{p_2^+ p_1^+}\frac{(1,2)^3}{(2,3)(3,1)} = -i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \quad (2.41)$$

Thus using this relation between the spinors and the momenta, one can relate the "on-shell" form (in terms of $\langle ij\rangle$) to its off-shell value (in terms of momentum).

Now in the CSW approach the spinor for an off-shell momentum is written as $\lambda_\alpha = p_{\alpha\dot{\alpha}}\bar{X}^{\dot{\alpha}}$, where $\bar{X}^{\dot{\alpha}}$ is the complex conjugate spinor from an arbitrary null external line. Since in the previous analysis, one should take the identification in (2.38) to make the connection between the MHV on-shell form and its off-shell value, for this to work the CSW offshell continuation must be equivalent to our map, that is

$$\begin{pmatrix} p^- & -p \\ -\bar{p} & p^+ \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix} = \lambda_\alpha = \begin{pmatrix} \frac{-p}{\sqrt{p^+}} \\ \sqrt{p^+} \end{pmatrix} \tag{2.42}$$

this leads to the requirement that $\bar{X}^{\dot{\alpha}} = \frac{1}{\sqrt{p^+}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For an arbitrary null momentum one can always find a frame such that $k_{\alpha\dot{\alpha}} = k^+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, this leads to $\bar{X}^{\dot{\alpha}} = \sqrt{k^+} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which differs with the desired result by an overall factor $\frac{1}{\sqrt{k^+ p^+}}$. This overall factor cancels in the CSW calculation since the propagator always connect two MHV graphs with one side + helicity and the other − helicity, the + helicity side has a factor $(\sqrt{k^+ p^+})^2$ while the negative helicity side $(\sqrt{k^+ p^+})^{-2}$.

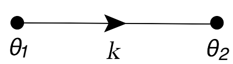
To see that one of the vertices generated by the redefinition can be written in terms of the holomorphic off-shell spinors (2.38), one needs to prove that these vertices will not depend on $\bar{p}$. This was shown in [28] to be true.

Therefore in the MHV lagrangian, all vertices are MHV vertices and this indicates that one should be able to do perturbative calculation simply by computing Feynman graphs with only MHV vertices. Defining the map between momentum and spinor according to (2.38), one can compute arbitrary off-shell amplitude in light-cone gauge in terms of momentum, and then map to their spinor form. Their spinor form will then take the well known holomorphic form via Nair. The difference between off-shell and on-shell is then encoded in how these spinors relate to their momentum. In a suitable basis, we see that the CSW definition for the spinor is equivalent to our on-shell off-shell map up to an overall factor that cancels in the calcualtion.

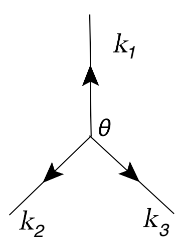
2.2.3 Equivalence Theorem at one-loop

Again for this to be a proof of the CSW approach, one needs to show that the field redefinition does not introduce new terms that will survive the LSZ procedure and contribute to amplitude calculations. As discussed previously, at tree level all terms generated from the field redefinition of the coupling to source current will cancel through the LSZ procedure except the linear term. The only other possibility will be the self-energy diagram where multiplying by p^2 cancels the propagator that connects this diagram to other parts of the amplitude, and thus survives. The argument that it vanishes follows closely along the line of [29], one should be able to prove with the requirement of Lorentz invariance that all the loop integrals will be dependent only on the external momentum p^2 which we take to zero in the LSZ procedure. This implies that the self-energy diagrams are scaleless integrals and thus vanish.[4]

We would like to compute the self-energy diagram in light-cone superspace. The Feynman rules for light-cone superspace are defined for the chiral superfield Φ, thus one uses (1.18) to convert all the $\bar{\Phi}$ into Φ. The rules have been derived in [35], and here we simply use the result. [5]



$$\sim \frac{\bar{d}_1^4(k)}{k^2}\delta^8(\theta_1 - \theta_2)$$



$$\sim \int d^4\theta d^4\bar{\theta}\frac{d^4(p_1)}{p_1^{+2}}[\frac{1}{p_2^+}\frac{p_3 d^4(p_3)}{p_3^{+2}} - \frac{p_2 d^4(p_2)}{p_2^{+2}}\frac{1}{p_3^+}] \tag{2.43}$$

Here $d(k) = \frac{\partial}{\partial\theta^A} - \frac{k^+}{\sqrt{2}}\bar{\theta}_A$. The relevant graphs is now shown in fig.2.2.3.

[4] There is of course the question of whether dimensional regularization is the correct scheme for this approach. However since in [33] dimensional regularization was used to give the correct one loop amplitudes from Yang-Mills MHV Lagrangian, the analysis here should hold. However, in [34] a different scheme was used, and it would be interesting to see if there will be any equivalence theorem violation within this scheme.

[5] Note that the propagators given here have already included the factor of $\bar{d}^4$ from the functional derivative $\frac{\delta\Phi(x_1,\theta_1,\bar{\theta}_1)}{\delta\Phi(x_2,\theta_2,\bar{\theta}_2)} = \frac{\bar{d}_1^4}{(4!)^2}\delta^4(x_1 - x_2)\delta^4(\theta_1 - \theta_2)\delta^4(\bar{\theta}_1 - \bar{\theta}_2)$.

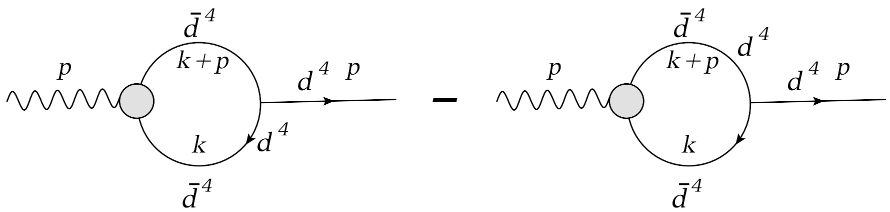


Figure 2.4: These are the two relevant contributions to the one-loop self-energy diagram. For simplicity we only denote the positions of d^4 and $\bar{d}^4$ to indicate which legs of the vertex were used for the loop contraction.

Note that other graphs can be manipulated into the same form by partially integrating the fermionic derivatives. Using (2.29) with $n = 2$, the two terms give

$$\begin{aligned} &\int d^4\theta d^4\bar{\theta} J[\frac{k^{+2}(k+p)}{(k,p)k^2(k+p)^2(k^+ + p^+)} - \frac{k^+ k}{(k,p)k^2(p+k)^2}]\Phi \\ = &\int d^4\theta d^4\bar{\theta} J[\frac{k^+}{k^2(k+p)^2(k^+ + p^+)}]\Phi \end{aligned} \tag{2.44}$$

Writing in Lorentz invariant fashion we introduce a light-like reference vector μ in the + direction. The result is rewritten as

$$\int d^4\theta d^4\bar{\theta} J[\frac{(k\cdot\mu)}{k^2(k+p)^2(k+p)\cdot\mu}]\Phi \tag{2.45}$$

Again following [29], since by rescaling $\mu \to r\mu$ the factor cancels, thus the resulting integral can only depend on p^2. Since we take $p^2 \to 0$ in LSZ reduction this means that the integral becomes a scaleless integral, and vanishes in dimensional regularization.

2.3 Conclusion

We've shown that by redefining the chiral superfield such that the self-dual part of N=4 SYM becomes free, one generates a new lagrangian with infinite interaction terms which are all MHV vertex. When restricting to equal time field redefinitions the solution gives the suitable off-shell lagrangian that corresponds to the CSW off-shell continuation. The redefinition is preformed by requiring the self-dual part of the action becomes free since the self-dual sector is essentially free classically. It does not, however, give a derivation of Nair's holomorphic form of n-point super MHV amplitude. For this purpose it is more useful to start from an action that was directly written in twistor space. Indeed such an action has been constructed in[36] and its relation to CSW has been discussed. The extremely non-local form of the redefined action makes understanding CSW in terms of field theory very difficult. This non-locality can be again traced back to the on-shell light-cone action that we began with. Presumably an off-shell formalism well aid this discussion inmensely.

Chapter 3

On-shell amplitudes (loop)

3.1 Introduction

In the study of N=4 SuperYang-Mills one loop amplitudes in component fields [37][38], it has been show that they have the special property of being cut constructible, that is they are uniquely determined by their unitary cuts. It was shown in [37] that $N = 4$ SYM one loop amplitudes can be decomposed on the basis of scalar box integrals with rational coefficients:

$$A = \sum(c_{4m}I_{4m} + c_{3m}I_{3m} + c_{2mh}I_{2mh} + c_{2me}I_{2me} + c_{1m}I_{1m}) \tag{3.1}$$

Each integral is defined as

$$I(K_1, K_2, K_3, K_4) = -i4(\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2(l+K_1)^2(l+K_1+K_2)^2(l-K_4)^2}$$

The external lines are organized into four corners of the box graph, with K_i representing the sum of their momenta. Depending on how the external lines are organized they are separated into the above five different scalar integrals. Four-mass integrals I_{4m} have all four momentum sums massive: $K_i^2 \neq 0$. Three-mass integrals I_{3m} have one massless $K_i^2 = 0$, while for two-mass integrals depending on whether the massless K's are adjacent or not we have I_{2mh} for $K_i^2 = K_{i+1}^2 = 0$ and I_{2me} for $K_i^2 = K_{i+2}^2 = 0$. We illustrate these integrals in the following fig.**??**.

Therefore the calculation of one loop amplitude is reduced to determining

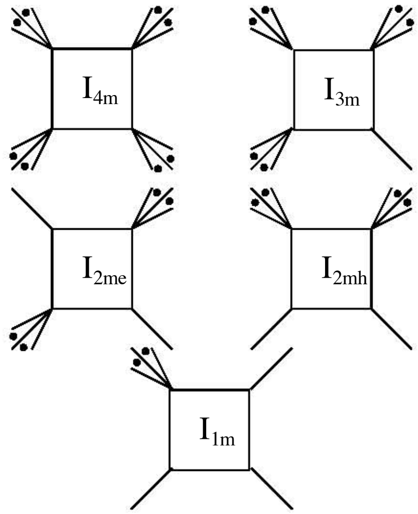


Figure 3.1: The scalar box integrals.

the coefficients in front of these box integrals. These coefficients are rational functions and therefore they are not affected by branch cut singularities. Thus in principle one can extract the coefficients by cutting two Feynman propagators in a given channel on the right hand side of (3.1), and the same for the scalar box integrals on the left. Then (3.1) yields (for a given channel)

$$\begin{aligned} &\int d\mu A^{tree}(l_1, i \cdots j, l_2) A^{tree}(-l_2, j+1 \cdots i-1, -l_1) \\ = &\sum (\Delta c_{4m} I_{4m} + \Delta c_{3m} I_{3m} + \Delta c_{2mh} I_{2mh} + \Delta c_{2me} I_{2me} + \Delta c_{1m} I_{1m}) \end{aligned} \tag{3.2}$$

with Δ denoting the discontinuity across the branch cut of the box integrals, and μ the Lorentz invariant measure

$$d\mu = \delta^+(l_1^2)\delta^+(l_2^2)\delta^{(4)}(l_1 + l_2 - P_{ij}) \tag{3.3}$$

Unfortunately complication arises from the fact that some of the cuts are shared by more than one box integral. Therefore their coefficients come in this equation at the same time. This problem was solved in [39] by using generalized unitarity (quadruple) cuts [40] to analyze the leading singularities which turn out to be unique in the box integrals. The construction is to cut four propagators on both sides of (3.1), therefore analysing the leading singularity of the box integrals

$$\int d^4l\delta^+(l^2)\delta^+((l+K_1)^2)\delta^+((l+K_1+K_2)^2)\delta^+((l-K_4)^2)$$
$$A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} = \sum(\Delta_{LS}c_{4m}I_{4m}) \tag{3.4}$$

where the tree amplitudes correspond to amplitudes with the corresponding external lines K_i. Note that for other box integrals, one of the tree amplitudes on the left hand side of (3.4) will be a three point amplitude which is zero for Minkowski signature. This issue is resolved by going to split signature with the corresponding modification for the cut measure [39].

At this point it is natural to ask if one may reconstruct the above results in a superspace language. The most natural approach would then be the CSW construction discussed in the previous chapter, which uses MHV vertices, which is already in superspace form, as the basic building block of the scattering amplitudes. At loop level the valediction of CSW approach was proven to give the same result as that in field theory in [41] for MHV loop amplitudes, and [42] reproduces the relationship between the color leading amplitudes and sub-leading amplitudes.

To compute NMHV one loop amplitude using CSW construction would require three MHV vertices connected by three propagators. At this point it is not clear how the correct scalar box functions should arise in this formalism. One of the complications is for more than two fermionic delta functions (there is one for each MHV vertex), after the expansion in superspace there will be multiple spinor products that contain the off shell continuation spinor of the propagator, η. Since the external line factor for different species is different, the integration over these spinor products requires separation: The integrand for the gluonic amplitudes will be dramatically different from the ones with gluinos, implying one can only derive the box functions from the superspace expansion one term at a time and not in the original superspace full form.

This is not surprising since one would anticipate the scalar box decomposition to naturally arise only for an off-shell superspace formalism. At the current stage one can at most anticipate a superspace representation for the coefficients in front of these scalar box integrals, since in principle they are products of on-shell tree amplitudes. The aim of this section is to formulate such results using different techniques. Here we take the NMHV 6 point one loop amplitude as an example. NMHV tree and loop amplitudes with gluinos or scalars were previously [43] derived from their pure gluonic partners by solving the supersymmetric Ward identities (SWI)[44]. Since the the Ward identities act linearly on the amplitudes, one would anticipate the simple superspace representation of this amplitude.

In [43] the SWI were not used directly upon the coefficients in front of the box integrals for the gluonic amplitude, but rather the coefficients in front of a particular combination of box integrals, which originated from the three different two particle cuts [38]. Here we show one can construct the superspace amplitude by noting that for the six point one loop amplitude, the tree graphs on either side of the cuts always come in MHV and $\overline{MHV}$ pair. Since MHV and $\overline{MHV}$ trees can be written straight forwardly in superspace form, one naturally derives the six point one loop NMHV amplitude for all helicity configurations and external species as one superspace amplitude by fusing the two tree amplitudes. Note that there is already progress for deriving the coefficients in (3.1) directly in superspace[45], though at least for six point NMHV it is not simpler then our result [46].

In the following we present the amplitude in its full superspace form and confirm our result by explicitly expanding out the terms that give the correct amplitudes with two gluinos obtained in [43]. We will also give a brief demonstration of how one could obtain the field theory result for the loop amplitude from the MHV vertex approach (CSW).

3.2 The Construction

The n point MHV and $\overline{MHV}$ tree level amplitudes have a remarkably simple form. For MHV tree [7]:

$$A(...j^-.....i^-...)_{tree}=\frac{\delta^8(\sum_{i=1}^n\lambda_i\eta_i^A)}{\Pi_{i=1}^n<ii+1>} \tag{3.5}$$

where

$$\delta^8(\sum_{i=1}^n\lambda_i\eta_i^A)=\frac{1}{2}\prod_{A=1}^4(\sum_{i=1}^n\lambda_i^\alpha\eta_i^A)(\sum_{i=1}^n\lambda_{i\alpha}\eta_i^A) \tag{3.6}$$

as for $\overline{MHV}$ tree:

$$A(...j^+.....i^+...) = \frac{\delta^8(\sum_{i=1}^{n}\widetilde{\lambda}_i\widetilde{\eta_i^A})}{\Pi_{i=1}^{6}[ii+1]} \tag{3.7}$$

Here we've omitted the energy momentum conserving delta function and the group theory factor. After expansion in the fermionic parameters η_i^A, one can obtain MHV amplitudes with different helicity ordering $(++---, +-+--...$etc) and different particle content.

We proceed to construct the full N=4 SYM NMHV 1-loop six point amplitudes by following the original gluonic calculation [38], where the amplitude was computed from the cuts of the three channels t_{123} t_{234} t_{345} $(t_{ijk} = (k_i + k_j + k_l)^2)$, except now the tree amplitudes across the cuts are written in supersymmetric form. We find that the propagator momentum integrals from which the various scalar box functions arise are the same for different external particles. Thus with the gluon amplitude already computed all we need to do is extract away the part of the gluon coefficient that came from the expansion of the two fermionic delta function, the remaining pre-factor will be universal and has its origin from the denominator of eq.(1) and (3). The N=4 SYM 6 point NMHV loop amplitude for the gluonic case was given [38] as

$$A(...j^-.....i^-...)_{loop} = c_\Gamma[B_1W_6^{(1)} + B_2W_6^{(2)} + B_3W_6^{(3)}] \tag{3.8}$$

where $W_6^{(i)}$ contains particular combination of the two-mass-hard and one-mass box functions [37]. The full 6 point NMHV loop amplitude for any given set of external particle and helicity ordering are then given with the following coefficients :

$$\begin{aligned} B_1 = &\frac{\delta^8(\sum_{i=1}^{3}\widetilde{\lambda}_i\widetilde{\eta}_i - \widetilde{l}_1\widetilde{\eta}_1 + \widetilde{l}_2\widetilde{\eta}_2)\delta^8(\sum_{i=4}^{6}\lambda_i\eta_i - l_2\eta_2 + l_1\eta_1)}{t_{123}}B_0 \\ &+\frac{\delta^8(\sum_{i=1}^{3}\lambda_i\eta_i - l_1\eta_1 + l_2\eta_2)\delta^8(\sum_{i=4}^{6}\widetilde{\lambda}_i\widetilde{\eta}_i - \widetilde{l}_2\widetilde{\eta}_2 + \widetilde{l}_1\widetilde{\eta}_1)}{t_{123}}B_0^\dagger \end{aligned} \tag{3.9}$$

$$B_2 = \frac{\delta^8(\sum_{i=2}^4 \widetilde{\lambda}_i \widetilde{\eta}_i - \widetilde{l}_1 \widetilde{\eta}_1 + \widetilde{l}_2 \widetilde{\eta}_2)\delta^8(\sum_{i=5}^1 \lambda_i \eta_i - l_2 \eta_2 + l_1 \eta_1)}{t_{234}} B_+ + \frac{\delta^8(\sum_{i=2}^4 \lambda_i \eta_i - l_1 \eta_1 + l_2 \eta_2)\delta^8(\sum_{i=5}^1 \widetilde{\lambda}_i \widetilde{\eta}_i - \widetilde{l}_2 \widetilde{\eta}_2 + \widetilde{l}_1 \widetilde{\eta}_1)}{t_{234}} B_+^\dagger \quad (3.10)$$

$$B_3 = \frac{\delta^8(\sum_{i=3}^5 \widetilde{\lambda}_i \widetilde{\eta}_i - \widetilde{l}_1 \widetilde{\eta}_1 + \widetilde{l}_2 \widetilde{\eta}_2)\delta^8(\sum_{i=6}^2 \lambda_i \eta_i - l_2 \eta_2 + l_1 \eta_1)}{t_{345}} B_- + \frac{\delta^8(\sum_{i=3}^5 \lambda_i \eta_i - l_1 \eta_1 + l_2 \eta_2)\delta^8(\sum_{i=6}^2 \widetilde{\lambda}_i \widetilde{\eta}_i - \widetilde{l}_2 \widetilde{\eta}_2 + \widetilde{l}_1 \widetilde{\eta}_1)}{t_{345}} B_-^\dagger \quad (3.11)$$

where we define :

$$B_0 = i \frac{1}{[12][23] < 45 >< 56 >< 1|K_{123}|4 >< 3|K_{123}|6 >} \quad (3.12)$$

and

$$B_+ = B_0|_{j \to j+1} \; B_- = B_0|_{j \to j-1} \quad (3.13)$$

with $< A|K_{ijk}|B >= [Ai]\langle iB\rangle + [Aj]\langle jB\rangle + [Ak]\langle kB\rangle$. Each coefficient is expressed in two terms, this corresponds to the assignment of helicity for the propagators l_1 and l_2 which for specific assignments will reverse the MHV and $\overline{MHV}$ nature of the two tree amplitude across the cut fig3.2. The presence of the loop momenta seems perplexing at this point since all loop momenta should have been integrated out to give the box functions. As we will see on a case by case basis this comes as a blessing. The actual expansion for a particular set of helicity ordering and external particles contains multiple terms. The presence of loop momentum forces one to regroup the terms such that the loop momentum forms kinematic invariants. It is after this regrouping that one obtains previous known results.

The amplitudes for different external particles are computed as an expansion in the $SU(4)_R$ anti-commuting fermionic variables η. Choosing particular combinations following [19]

$$g_i^- = \eta_i^1 \eta_i^2 \eta_i^3 \eta_i^4, \; \phi_i^{AB} = \eta_i^A \eta_i^B, \; \Lambda_i^{1-} = -\eta_i^2 \eta_i^3 \eta_i^4, \; \Lambda_i^{2-} = -\eta_i^1 \eta_i^3 \eta_i^4 \quad (3.14)$$
$$\Lambda_i^{3-} = -\eta_i^1 \eta_i^2 \eta_i^4, \qquad \Lambda_i^{4-} = -\eta_i^1 \eta_i^2 \eta_i^3, \qquad \Lambda_i^{A+} = \eta_i^A, \qquad g_i^+ = 1$$

The superscript represents which flavor the particle carries, in the N=4 multi-

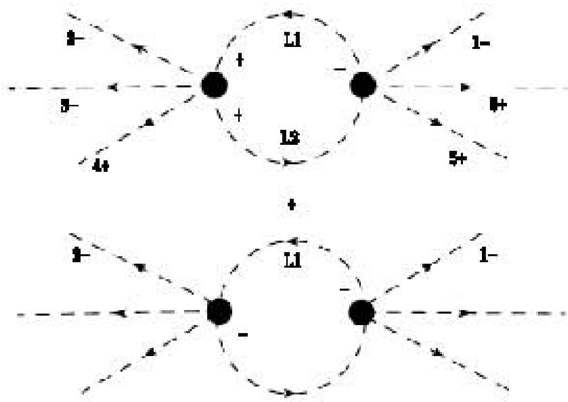


Figure 3.2: Here we show for a particular case of the gluonic NMHV loop amplitude, different assignment of helicity for the propagators will change the MHV or $\overline{MHV}$ nature of each vertex which is the reason we have two terms in eq.(5)-(7).

plet there are four gluinos and six scalars. Corresponding combination in the $\widetilde{\eta}$ follows:

$$g_i^+ = \widetilde{\eta_i^1}\widetilde{\eta_i^2}\widetilde{\eta_i^3}\widetilde{\eta_i^4},\ \phi_i^{AB} = \widetilde{\eta_i^C}\widetilde{\eta_i^C},\ \Lambda_i^{1+} = -\widetilde{\eta_i^2}\widetilde{\eta_i^3}\widetilde{\eta_i^4},\ \Lambda_i^{2+} = -\widetilde{\eta_i^1}\widetilde{\eta_i^3}\widetilde{\eta_i^4} \quad (3.15)$$
$$\Lambda_i^{3+} = -\widetilde{\eta_i^1}\widetilde{\eta_i^2}\widetilde{\eta_i^4}, \qquad \Lambda_i^{4+} = -\widetilde{\eta_i^1}\widetilde{\eta_i^2}\widetilde{\eta_i^3}, \qquad \Lambda_i^{A-} = \widetilde{\eta_i^A}, \qquad g_i^- = 1$$

Thus a particular term in the expansion corresponds to a particular assignment of the fermionic variables to the external particle and results in an amplitude with a particular set of external particle species and helicity ordering. In the next two sections we show by expanding eq.(5),(6),(7) and following the above dictionary one can recover the amplitudes containing two same color gluinos with different helicity ordering computed in [43].

3.2.1 B_1 Coefficient $\Rightarrow t_{123}$ cut

First we look at the t_{123} cut which corresponds to the B_1 coefficient. For the purely gluonic amplitude $A(g_1^- g_2^- g_3^- | g_4^+ g_5^+ g_6^+)$ (we use a bar to indicate the cut), we have only one particle assignment for the loop propagators:

$$l_1 = g^+ \,, l_2 = g^+ \quad (3.16)$$

Here the assignment of helicity is labeled with respect to the $\overline{MHV}$ vertex. Therefore we get only contribution from the first term in eq.(5), the expansion from the delta function gives $\langle l_1 l_2\rangle^4[l_1 l_2]^4 = (l_1 - l_2)^8 = t_{123}^4$ and therefore $B_1 = t_{123}^3 B_0$ which matches eq.(5.4) in [43].

For the two gluino amplitudes first we look at $A(\Lambda_1^- g_2^- g_3^- |\Lambda_4^+ g_5^+ g_6^+)$ from the delta function expansion we have helicity assignments :

$$l_1 = \Lambda^+ \; l_2 = g^+ \;, \; +(\; exchage \; between \; l_1 \; and \; l_2 \;) \tag{3.17}$$

Again only the first term in eq.(5) gives contribution :

$$\langle l_1 l_2\rangle^3[l_1 l_2]^3([1l_1]\langle l_1 4\rangle - [1l_2]\langle l_2 4\rangle) = t_{123}^3\langle 1|K_{123}|4\rangle \tag{3.18}$$

Note that only when the external gluino carry the same flavor will this term contribute. Since in [43] the two gluino amplitude was derived using N=1 SWI, the two gluinos carry the same flavor. Thus we have

$$B_1(\Lambda_1^- g_2^- g_3^- |\Lambda_4^+ g_5^+ g_6^+) = i\frac{t_{123}^2\langle 1|K_{123}|4\rangle}{[12][23] < 45 >< 56 >< 1|K_{123}|4 >< 3|K_{123}|6 >} \tag{3.19}$$

This is exactly the result of [43]. Other non-cyclic permutations of two gluino amplitude calculated in [43] at this cut do not change the assignment of the propagators, thus the amplitude remains the same form apart from the labelling of the position of the two gluinos.

3.2.2 B_2 Coefficient $\Rightarrow t_{234}$ cut

For this cut with different helicity assignment of the propagators, contribution can arise from both terms. Propagators with the same helicities (here we mean they are both plus or minus regardless of the species) get their contribution from one term while the rest from the other, this is why B_2 was split in two terms in the original computation of the gluon amplitude [38]. We deal with the same helicity first since there is only one way of assigning propagators.

$B_2(\Lambda_1^- |g_2^- g_3^- \Lambda_4^+ |g_5^+ g_6^+)_{same\ helicity} = 0$ since there is no way of assigning same helicity particles to the propagators.

For $B_2(g_1^-|\Lambda_2^- g_3^- \Lambda_4^+|g_5^+ g_6^+)_{same\ helicity}$ we have

$$l_1 = g^-\ l_2 = g^- \tag{3.20}$$

This receives contribution from the second term in eq.(3.10) which is $\langle 23\rangle^3\langle 43\rangle[56]^4$, thus giving

$$B_2(g_1^-|\Lambda_2^- g_3^- \Lambda_4^+|g_5^+ g_6^+)_{samehelicity} = (\frac{\langle 23\rangle^3\langle 43\rangle[56]^4}{t_{234}})B_+^\dagger \tag{3.21}$$

For $B_2(g_1^-|\Lambda_2^- g_3^- g_4^+|\Lambda_5^+ g_6^+)_{same\ helicity}$ we have

$$l_1 = g^-\ l_2 = \Lambda^-\ , + \ (exchange\ between\ l_1, l_2) \tag{3.22}$$

This gives contribution $\langle 23\rangle^3[56]^3(\langle 3l_1\rangle[l_1 6] - \langle 3l_2\rangle[l_2 6]) = \langle 23\rangle^3[56]^3\langle 3|K_{234}|6\rangle$ giving

$$B_2(g_1^-|\Lambda_2^- g_3^- g_4^+|\Lambda_5^+ g_6^+)_{same\ helicity} = (\frac{\langle 23\rangle^3[56]^3\langle 3|K_{234}|6\rangle}{t_{234}})B_+^\dagger \tag{3.23}$$

Now we move to configurations with different helicity. For

$$B_2(\Lambda_1^-|g_2^- g_3^- \Lambda_4^+|g_5^+ g_6^+)_{Diff\ helicity}$$

we have :

$$l_1 = \Lambda^+\ l_2 = g^-\ , l_1 = \Lambda^-\ l_2 = \phi\ , + \ (exchange\ between\ l_1, l_2) \tag{3.24}$$

For fixed external states Λ_4^+ and Λ_1^- we have to sum up all possible flavors for the internal gluino. This gives a contribution of

$$\begin{aligned}[1l_1]^3[l_1 l_2]\langle 4l_2\rangle^3\langle l_1 l_2\rangle - 3[l_1 l_2][l_2 4][l_1 4]^2\langle l_1 l_2\rangle\langle l_2 1\rangle\langle l_1 1\rangle^2 \\ +3[l_1 l_2][l_1 4][l_2 4]^2\langle l_1 l_2\rangle\langle l_1 1\rangle\langle l_2 1\rangle^2 \\ -[1l_2]^3[l_1 l_2]\langle 4l_2\rangle^3\langle l_1 l_2\rangle = t_{123}(\langle 1|l_1 - l_2|4\rangle)^3 = t_{123}\langle 1|K_{123}|4\rangle^3\end{aligned} \tag{3.25}$$

Therefore

$$B_2(\Lambda_1^-|g_2^- g_3^- \Lambda_4^+|g_5^+ g_6^+)_{Diff\ Helicity} = \frac{\langle 1|K_{123}|4\rangle^3}{t_{123}^3}B_+ \tag{3.26}$$

For $B_2(g_1^-|\Lambda_2^- g_3^- \Lambda_4^+|g_5^+ g_6^+)_{Diff\ Helicity}$ we have:

$$l_1 = g^-\ l_2 = g^+\ , l_1 = \Lambda^+\ l_2 = \Lambda^-\ , l_1 = \phi\ l_2 = \phi\ ,\ + (exchange\ between\ l_1, l_2) \tag{3.27}$$

Here whether or not Λ_4^+ and Λ_1^- carry the same flavor will effect the number of ways one can assign flavor to the internal gluino and scalar. For the same flavor we have

$$-([4l_1]\langle l_1 1\rangle - [4l_2]\langle l_2 1\rangle)^3([2l_1]\langle l_1 1\rangle - [2l_2]\langle l_2 1\rangle) = -(\langle 4|K_{234}|1\rangle)^3(\langle 2|K_{234}|1\rangle) \tag{3.28}$$

Thus

$$B_2(g_1^-|\Lambda_2^- g_3^- \Lambda_4^+|g_5^+ g_6^+)_{Diff\ Helicity} = (\frac{-(\langle 4|K_{234}|1\rangle)^3(\langle 2|K_{234}|1\rangle)}{t_{234}})B_+ \tag{3.29}$$

For $B_2(g_1^-|\Lambda_2^- g_3^- g_4^+|\Lambda_4^+ g_6^+)_{Diff\ helicity}$ we have :

$$l_1 = g^-\ l_2 = \Lambda^+\ , l_1 = \Lambda^-\ l_2 = \phi\ ,\ + (exchange\ between\ l_1, l_2) \tag{3.30}$$

This gives contribution :

$$\begin{aligned} & -\langle 1l_1\rangle^3\langle 15\rangle[l_1 4]^3[42] + 3\langle 1l_1\rangle^2[l_1 4]^2\langle 1l_2\rangle[l_2 4][42]\langle 15\rangle \\ & -3\langle 1l_2\rangle^2[l_2 4]^2\langle 1l_1\rangle[l_1 4][42]\langle 15\rangle + \langle 1l_1\rangle^3\langle 15\rangle[l_1 4]^3[42] \\ & = -(\langle 4|K_{234}|1\rangle)^3[42]\langle 15\rangle \end{aligned} \tag{3.31}$$

Thus

$$B_2(g_1^-|\Lambda_2^- g_3^- g_4^+|\Lambda_4^+ g_6^+)_{Diffhelicity} = \frac{-(\langle 4|K_{234}|1\rangle)^3[42]\langle 15\rangle}{t_{234}}B_+ \tag{3.32}$$

Adding eq.(17),(19),(22),(25) and (28) together gives the B_2 coefficient of the gluino anti-gluino pair amplitudes computed in [43]. Coefficients for the next cut can be calculated in similar way, we've checked it gives the same result as that derived in [43].

It is straight forward to compute amplitudes that involve more than one pair of gluino or scalars. The new amplitudes are :

$$A(g^-g^+\Lambda^+\Lambda^-\Lambda^-\Lambda^+)\ ,A(g^-g^+\phi\phi\phi\phi)\ ,A(\phi\phi\phi\phi\phi\phi)\ A(\Lambda^-\Lambda^-\Lambda^-\Lambda^+\Lambda^+\Lambda^+) \tag{3.33}$$
$$A(\Lambda^-\Lambda^+\phi\phi\phi\phi)\ ,A(\Lambda^-\Lambda^-\Lambda^+\Lambda^+\phi\phi)\ ,A(\Lambda^-\Lambda^+\phi g^-g^+g^+)$$

Complication arises for these amplitudes because non-gluon particles carry less superspace variables and increase the amount of spinor combination. Luckily with the specification of the flavor for the external particles, the propagators are restricted to take certain species. This is discussed in detail in the next section where we calculate the all gluino and all scalar amplitude.

3.3 Amplitudes with all gluinos and all scalars

Here we present N=4 SYM NMHV loop amplitudes with all gluino and all scalars. These amplitudes were derived from explicit expansion of eq.(5)-(7). Since scalars and gluinos carry less fermionic parameters as seen in eq.(10)(11), the spinor product that arises from the fermionic delta function becomes complicated. The final coefficient should not contain the off shell propagator spinor, thus one can use this as a guideline to group the spinor products to form kinematic invariant terms. With specific flavors this also restricts the possible species for propagators.

3.3.1 $A(\Lambda_1^{1+}\Lambda_2^{2+}\Lambda_3^{3+}\Lambda_4^{1-}\Lambda_5^{2-}\Lambda_6^{3-})$

For the six gluino amplitude we look at amplitudes with all three positive helicity gluinos carrying different flavor. The negative helicities also carry different flavor and is the same set as the positive. For t_{123} the flavors of the internal particles are uniquely determined.

$$l_1=\Lambda^-\ l_2=g^+\ ,\ l_1=\Lambda^+\ l_2=\phi\ ,\ +exchange \tag{3.34}$$

This gives

$$B_1(\Lambda_1^{1+}\Lambda_2^{2+}\Lambda_3^{3+}|\Lambda_4^{1-}\Lambda_5^{2-}\Lambda_6^{3-})=(\langle 1|K_{123}|5\rangle\langle 2|K_{123}|6\rangle\langle 3|K_{123}|4\rangle \tag{3.35}$$
$$+\langle 1|K_{123}|4\rangle\langle 2|K_{123}|5\rangle\langle 3|K_{123}|6\rangle+\langle 1|K_{123}|6\rangle\langle 2|K_{123}|4\rangle\langle 3|K_{123}|5\rangle)B_0^{\dagger}$$

Next we look at t_{345} cut. The propagator assignment with same helicity (the definition of same or different helicity again follows that of the previous paragraph) would be :

$$l_1 = g^- \; l_2 = \Lambda^- \; , \; +exchange \; propagator \tag{3.36}$$

this gives

$$\begin{aligned} B_3(\Lambda_1^{1+}\Lambda_2^{2+}|\Lambda_3^{3+}\Lambda_4^{1-}\Lambda_5^{2-}|\Lambda_6^{3-})_{Same\ helicity} &= \langle 45\rangle^2[12]^2\{\langle 34\rangle[61]\langle 5|K_{345}|2\rangle \\ &+\langle 34\rangle[62]\langle 5|K_{345}|1\rangle + \langle 35\rangle[61]\langle 4|K_{345}|2\rangle + \langle 35\rangle[62]\langle 4|K_{345}|1\rangle\}B_-^\dagger \end{aligned} \tag{3.37}$$

There are two ways of assigning different helicity propagators

$$l_1 = g^- \; l_2 = \Lambda^+ \; , \; or \; l_1 = \Lambda^- \; l_2 = \phi \; , \; + \; exchange \tag{3.38}$$

Note however for the present set of flavors, there is no consistent way of assigning flavors when the propagators are a gluon and a gluino. Thus we are left with the gluino scalar possibility with its flavor uniquely determined.

$$B_3(\Lambda_1^{1+}\Lambda_2^{2+}|\Lambda_3^{3+}\Lambda_4^{1-}\Lambda_5^{2-}|\Lambda_6^{3-})_{Diff\ helicity} = \langle 16\rangle\langle 62\rangle[43][35]\langle 6|K_{345}|3\rangle t_{345}B_- \tag{3.39}$$

Luckily there is no need to compute B_2 coefficients since it is related to B_3 by symmetry.

3.3.2 $A(\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6)$

The power of deriving amplitudes from a superspace expansion is that one can rule out certain amplitudes just by inspection. Amplitudes with more than two scalars carrying the same color vanish since there is no way of assigning the correct fermionic variables. Here we look at six-scalar amplitude all carrying different flavor. This should be the simplest amplitude since the flavor carried by the internal particle is uniquely determined. We give the result for cut t_{123}

while the other cuts are related by symmetry.

$$\begin{aligned}B_1(\phi_1\cdots\phi_6) = \{(&\langle 12\rangle[56]\langle 3|K_{123}|4\rangle + \langle 12\rangle[64]\langle 3|K_{123}|5\rangle + \langle 12\rangle[45]\langle 3|K_{123}|6\rangle \\ &+\langle 31\rangle[56]\langle 2|K_{123}|4\rangle + \langle 31\rangle[64]\langle 2|K_{123}|5\rangle + \langle 31\rangle[45]\langle 2|K_{123}|6\rangle \\ &+\langle 23\rangle[56]\langle 1|K_{123}|4\rangle + \langle 23\rangle[64]\langle 1|K_{123}|5\rangle + \langle 23\rangle[45]\langle 1|K_{123}|6\rangle)^2\}B_0 \\ &+\mathit{complex\ conjugate}\end{aligned} \tag{3.40}$$

3.4 A brief discussion on the MHV vertex approach

As discussed in the introduction, the straight forward way to compute amplitudes in superspace is the generalization of the MHV vertex approach. It is also of conceptual interest to see if this approach actually works for the NMHV loop amplitude. Here we give a brief discussion of the extension.

The MHV vertex approach was shown to be successful [41] in constructing the n point MHV loop amplitude. This is partly due to the similarity between the cut diagrams [37] originally used to compute the amplitude and the MHV vertex diagram, so that one can use a dispersion type integral to reconstruct the box functions from its discontinuity across the branch cut. For the NMHV loop amplitude, one requires three propagator for the three MHV vertex one-particle-irreducible(1PI) diagram and two propagators for the one-particle-reducible(1PR) diagram (fig-2)[42]. We would then encounter the following

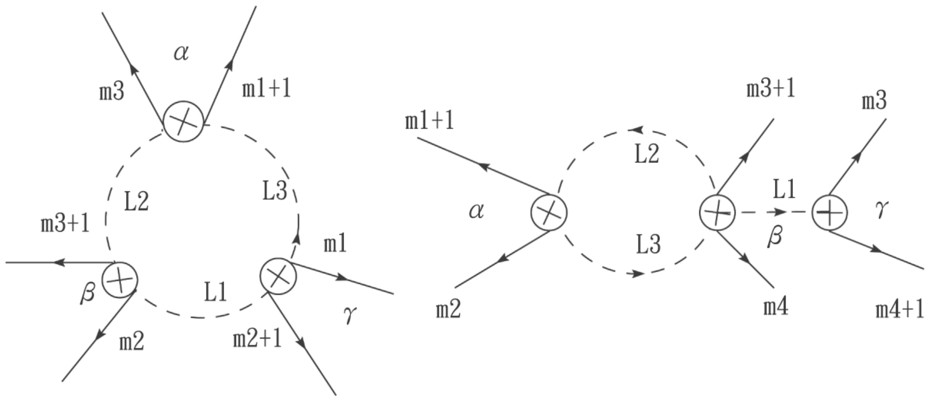


Figure 3.3: MHV diagrams for NMHV loop amplitude, includes the one-particle-irreducible and one-particle-reducible graph.

integration:

$$\begin{aligned}
&\frac{1}{\prod_{i=1}^{n}\langle i i+1\rangle}\int \frac{d^4L_1}{L_1^2}\frac{d^4L_2}{L_2^2}\frac{d^4L_3}{L_3^2}\delta(P_\alpha+L_2-L_3)\delta(P_\beta+L_3-L_1)\\
&\qquad\delta(P_\gamma+L_1-L_2)\int d^8\eta_{l_1}d^8\eta_{l_2}d^8\eta_{l_3}\\
&\qquad\frac{\delta^8(\Theta_1)\delta^8(\Theta_2)\delta^8(\Theta_3)\langle m_2m_2+1\rangle\langle m_1m_1+1\rangle\langle m_3m_3+1\rangle}{\langle l_2l_1\rangle\langle l_3l_2\rangle\langle l_1l_3\rangle\langle l_1m_2+1\rangle\langle m_2l_1\rangle\langle l_2m_3+1\rangle\langle m_3l_2\rangle\langle l_3m_1+1\rangle\langle m_1l_3\rangle}\\
&+\frac{\delta(L_1-P_\gamma)}{\prod_{i=1}^{n}\langle i i+1\rangle}\int\frac{d^4L_2}{L_2^2}\frac{d^4L_3}{L_3^2}\delta(P_\alpha+L_3-L_2)\delta(P_\beta+L_2+L_1-L_3)\int d^8\eta_{l_1}d^8\eta_{l_2}d^8\eta_{l_3}\\
&\quad\times\frac{\delta^8(\Theta_1)\delta^8(\Theta_2)\delta^8(\Theta_3)\langle m_2m_2+1\rangle\langle m_1m_1+1\rangle\langle m_3m_3+1\rangle\langle m_4m_4+1\rangle}{L_1^2\langle l_3l_2\rangle^2\langle m_1l_2\rangle\langle l_2m_{1+1}\rangle\langle l_3m_2+1\rangle\langle m_2l_3\rangle\langle l_1m_3+1\rangle\langle m_3l_1\rangle\langle l_1m_4+1\rangle\langle m_4l_1\rangle}
\end{aligned}\tag{3.41}$$

where for the first term

$$\begin{aligned}
\Theta_1 &= \sum_{i=\alpha}\eta_i\lambda_i + l_2\eta_{l_2} - l_3\eta_{l_3} \qquad (3.42)\\
\Theta_2 &= \sum_{i=\beta}\eta_i\lambda_i + l_3\eta_{l_3} - l_1\eta_{l_1}\\
\Theta_3 &= \sum_{i=\gamma}\eta_i\lambda_i + l_1\eta_{l_1} - l_2\eta_{l_2}
\end{aligned}$$

for the second term

$$\Theta_1 = \sum_{i=\alpha} \eta_i \lambda_i - l_2 \eta_{l_2} + l_3 \eta_{l_3} \tag{3.43}$$
$$\Theta_2 = \sum_{i=\beta} \eta_i \lambda_i + l_1 \eta_{l_1} + l_2 \eta_{l_2} - l_3 \eta_{l_3}$$
$$\Theta_3 = \sum_{i=\gamma} \eta_i \lambda_i - l_1 \eta_{l_1}$$

$\alpha\beta\gamma$ labels the external momenta assigned to the three MHV vertex and the l_is are the off shell continuation spinor following the CSW prescription. We can reorganize the delta functions to reproduce the overall momentum conservation. For the first term in eq.(37) we have

$$\delta(P_{\alpha+\beta+\gamma})\delta(P_{\beta+\gamma} + L_3 - L_2)\delta(P_\gamma + L_1 - L_2) \tag{3.44}$$

For the second term

$$\delta(P_{\alpha+\beta+\gamma})\delta(P_\alpha + L_2 - L_3)$$

If we integrate the last delta function away in the first term and combine with the 1PR graphs, it is equivalent to using two MHV vertices to construct NMHV tree amplitude on one side of the two remaining propagators, namely this combines vertex γ and β through propagator L_1. To see this note that the momentum conserving delta function forces L_1 propagator to carry the correct momentum as it would for the CSW method and the $\frac{1}{P^2_{L_1}}$ is present in the integral measure in the first place. This would obviously affect the off shell spinor in the following way.

$$l_1 = L_1 \widetilde{\eta} \rightarrow (L_2 - P_\gamma)\widetilde{\eta} \tag{3.45}$$

This simply fixes the off shell spinor to be computed from the correct momentum as the CSW method. Thus we have come to a two propagator integral with two tree level amplitudes on both side constructed from the CSW method. This is exactly the picture one would have if one applied the standard cut, except the propagators are off shell instead of on shell. For higher number of MHV vertices this can be applied straight forwardly, by integrating the momentum conserving propagators one at a time one can reduce the number of propagators until one arrives at the standard cut picture. As shown in [41] one can then proceed to recast the two propagator integral into a dispersion integral which computes the discontinuity across the cut of the integrand, by using the cut constructibility of N=4 SYM loop amplitudes, one can reconstruct the box function and it's coefficient. However there is one subtlety. In the original standard cut one has to analyze every cut channel, and then disentangle the information since more than one box integral shares the same cuts. If we follow the CSW prescription we can always reduce the loop diagrams down to two propagator one loop diagrams with an MHV vertex on one side of the two propagators. Thus this implies if the CSW approach is valid at one loop, then the full loop amplitude should be able to be reconstructed from the cuts of a subgroup of two propagator diagrams which always have an MHV vertex on one side of the cut.

This construction makes the connection between MHV vertex and $\overline{MHV}$ loop amplitude more transparent. $\overline{MHV}$ loop are just the parity transformation of the MHV loop, where one simply takes the complex conjugate of the MHV loop:

$$A(\overline{MHV})_{loop} = A(\overline{MHV})_{tree}\sum_{i=1}^{n}\sum_{r=1}^{[n/2]-1}(1-\frac{1}{2}\delta_{\frac{n}{2}},r)F^{2me}_{n:r;i} \tag{3.46}$$

It's derivation from MHV vertex is as follows. In [47] it was shown that using MHV vertices one can reconstruct the $\overline{MHV}$ tree amplitude in its complex conjugate spinor form. Since by integrating out one loop propagator corresponds to using MHV vertex to construct NMHV tree amplitude, one can proceed in a specific manner to reduce the number of loop propagators down to two with two $\overline{MHV}$ trees on both side. Since from [47] the two $\overline{MHV}$ tree amplitudes on both side are expressed in complex conjugate form, following exactly the same lines in [41] one can reproduce eq.(43).

3.5 Conclusion

In this chapter we constructed the 6 point NMHV loop amplitude for N=4 SuperYang-Mills in a compact form using its cut constructible nature. The expansion with respect to the fermionic parameter gives amplitudes with different particle content and helicity ordering. To extend further to higher point NMHV loops one may have to resolve to the MHV vertex approach since the tree level amplitudes on both side of the cut in general will not be in simple MHV and $\overline{MHV}$ combination. We also give a general discussion on how to proceed with the MHV vertex construction for higher than MHV loop(more than two negative helicities). The fact that it reproduces the two propagator picture for any one loop diagram combined with earlier results that have reproduced the MHV loop[41] and the relationship between the leading order and sub leading order amplitudes[42], gives a strong support for the CSW approach beyond tree level.

Chapter 4

First quantized Yang-Mills

4.1 Introduction

Here we discuss first quantization for ordinary YM which is based on quantizing the N=2 spinning particle(worldline supersymmetry). This is a simpler model then superparticles(spacetime supersymmetry) since the constraints are easier (the situation is similar to the quantization of the spinning string instead of the Green-Schwarz string), however we will be able to introduce how one construct a BRST charge and by extending it to background fields, generate vertex operators for the external states which are the standard procedure for any first quantized model and thus relevant for the discussion in the next section.

First-quantization has provided an efficient way of calculating Yang-Mills amplitudes. A set of rules for writing down 1-loop Yang-Mills amplitudes was first derived by Bern and Kosower from evaluating heterotic string amplitudes in the infinite string tension limit [48]. Later an alternative derivation of the same rules (but only for the 1-loop effective action) from first-quantization of particles was given by Strassler [49]. However, the generalization of these first-quantized rules to multi-loop amplitudes has not been clear. In fact, such rules have not yet been given even for Yang-Mills tree amplitudes. This is partially because the vacuum, ghost measure and Green function needed for the calculation of trees and multi-loops have not been clarified. Although there are already many ways to compute Yang-Mills tree amplitudes, it is important to clarify how first-quantization works at tree level first for the purpose of generalizing this method to multi-loop level. This is the main purpose of this paper.

To derive the first-quantized rules for trees, we start from theories of free relativistic spinning particles, which were first developed by Brink et al. [50] and many others [51]. In these theories the spin degree of freedom is encoded in the worldline supersymmetry. More precisely, the BRST quantization of the particle action with N-extended worldline supersymmetry shows that the cohomology is of a spin-$\frac{N}{2}$ particle.

In this chapter we study the $N = 2$ theory, which describes a spin-1 particle. We derive the vertex operator for background gauge field via the usual BRST quantization method, thus ensuring background gauge invariance. (The coupling of background vector fields to spin 1/2 was formulated in [50]. It was used to calculate effective actions in [52].) We proceed to show how the correct amplitudes can be derived. In the usual worldline approach, all interactions are derived by coupling external fields to the 1-dimensional worldline or loop. This is insufficient for $n \geq 6$-point tree and multi-loop amplitudes because there is no consistent way to draw a line through these graphs such that all lines attached are background fields. Here we propose an alternative ("worldgraph") approach that includes spaces that are not strictly 1D manifolds: They are not always locally R^1, but only fail to be so at a finite number of points. Taking these spaces into account we derive a set of rules for computing amplitudes that can be extended to all possible graphs[56].

We organize this chapter as follows: First we give a brief review of a general formalism to describe free spinning particles with arbitrary spin. We then focus on the spin-1 particle: introducing background Yang-Mills interaction to the theory and deriving the vertex operator for the external Yang-Mills fields. Then we define the vacuum, ghost measure and Green functions for Yang-Mills tree amplitudes. For examples we present the calculation of 3 and 4-point trees, and one-loop amplitudes, using the worldline approach, since it is sufficient for these amplitudes. Finally we discuss the worldgraph approach that follows string calculations more closely, and show how it can reproduce the tree results derived from the worldline approach.

4.2 Free spinning particles

We begin with the free BRST charge for arbitrary spin. A useful method for deriving gauge invariant actions is the OSp(1,1|2) formalism [53], where one starts with the light-cone SO(D−2) linearly realized by the physical states, and adds two bosonic coordinates to restore Lorentz covariance and two fermionic coordinates to cancel the additional degrees of freedom. Thus the $\mathrm{SO(D-2)}$ representation is extended to $\mathrm{OSp(D-1,1|2)}$, and the non-linearly realized $\mathrm{SO(D-1,1)}$ of the physical states is extended to $\mathrm{OSp(D,2|2)}$. The action then uses only the subgroup $\mathrm{SO(D-1,1)}\otimes\mathrm{OSp(1,1|2)}$, where the $\mathrm{OSp(1,1|2)}$ is a symmetry of the unphysical (orthogonal) directions under which the physical states should be singlets (in the cohomology). We use $(A,B...)$ for $\mathrm{OSp(D,2|2)}$ indices, $(a,b...)$ for the $\mathrm{SO(D-1,1)}$ part and $(+,-)$, $(\oplus,\ominus)$ for the bosonic and fermionic indices of $\mathrm{OSp(1,1|2)}$ respectively. The easiest way is then to begin with linear generators J^{AB} of $\mathrm{OSp(D,2|2)}$, use the gauge symmetry to gauge away the $+$ direction of $\mathrm{OSp(1,1|2)}$ and use equations of motion to fix the $-$ direction. Then the kinetic operator of the action is simply the delta function of the $\mathrm{OSp(1,1|2)}$ generators (now non-trivial due to solving the equation of motion).

One can further simplify things by utilizing only a subset of the generators of $\mathrm{OSp(1,1|2)}$. (This is analogous to the method of finding SU(2) singlets by looking at states annihilated by J_3 and J_-.) In the end one is left with the group IGL(1) with generators $J^{\oplus\ominus}$ and $J^{\oplus-}$. Relabeling $c=x^\oplus$ and $b=\partial_\oplus$,

$$J=iJ^{\oplus\ominus}+1=cb+iS^{\oplus\ominus},\quad Q=J^{\oplus-}=\tfrac{1}{2}c\partial^2+S^{\oplus a}\partial_a+S^{\oplus\oplus}b \tag{4.1}$$

J will be the ghost number and Q the BRST charge. One is then left with the task of finding different representation for S^{AB} satisfying the algebra

$$\left[S_{AB},S^{CD}\right\}=-\delta_{[A}^{[C}S_{B\}}{}^{C\}}$$

There may be more than one representation corresponding to the same spin. It is easy to build massless spin-$\frac{1}{2}$ representations using gamma matrices

$$\text{spin}-\frac{1}{2}:\qquad S_{AB}=-\frac{1}{2}[\gamma_A.\gamma_B\},\quad\{\gamma_A,\gamma_B]=-\eta_{AB} \tag{4.2}$$

and spin-1 using ket-bra

$$\text{spin}{-}1: \qquad S_{AB} = |_{[A}\rangle\langle_{B\}}|, \quad \langle_A|_B\rangle = \eta_{AB}$$

All higher spins can be built out of these two. For a review of the OSp$(1,1|2)$ formalism see [54].

For our purpose we use first-quantized fields (i.e. fields on a worldline) to form representations. It is known that the free relativistic spin-$\frac{N}{2}$ particle can be described by a first-quantized action with N-extended worldline supersymmetry [50]. For example, for spin $\frac{1}{2}$ we use $N=1$ worldline fields ψ^A where ψ^a are fermionic fields and $\psi^\oplus = i\gamma$, $\psi^\ominus = i\beta$ are the bosonic ghosts for SUSY. We summarize this representation as follows

$$\begin{aligned} S^{ab} &= \tfrac{1}{2}\left[\psi^a, \psi^b\right] = \psi^a\psi^b \\ S^{\oplus a} &= \tfrac{i}{2}\left\{\gamma, \psi^a\right\} = i\gamma\psi^a \\ S^{\oplus\oplus} &= \tfrac{1}{2}\left\{\gamma, \gamma\right\} = \gamma^2 \end{aligned} \tag{4.3}$$

and

$$\begin{gathered} \{\psi^a, \psi^b\} = \eta^{ab} \\ [\gamma, \psi^a] = 0 \\ [\gamma, \gamma] = 0 \end{gathered} \tag{4.4}$$

In this letter we focus on the $N=2$ spinning particle representation for massless vector states. Now, due to $N=2$ there are a pair of worldline spinors $\left(\psi^a, \bar{\psi}^b\right)$ and similarly bosonic ghosts $\left(\gamma, \bar{\gamma}, \beta, \bar{\beta}\right)$. The spin operators are then:

$$\begin{aligned} S^{ab} &= \bar{\psi}^a\psi^b - \bar{\psi}^b\psi^a \\ S^{\oplus a} &= i\gamma\bar{\psi}^a + i\bar{\gamma}\psi^a \\ S^{\oplus\oplus} &= 2\gamma\bar{\gamma} \end{aligned} \tag{4.5}$$

with the following (anti-)commutation relations for the fields:

$$\{\bar{\psi}^a, \psi^b\} = \eta^{ab} \tag{4.6}$$

$$\{\bar{\psi}^a, \bar{\psi}^b\} = \{\psi^a, \psi^b\} = [\gamma, \beta] = [\bar{\gamma}, \bar{\beta}] = 0 \tag{4.7}$$

$$[\gamma, \bar{\beta}] = [\bar{\gamma}, \beta] = \{b, c\} = 1 \tag{4.8}$$

4.3 Interacting spinning particles

Interaction with external fields is introduced by covariantizing all the derivatives in the free BRST charge and adding a term proportional to $iF_{ab}S^{ab}$, which is the only term allowed by dimension analysis and Lorentz symmetry. The relative coefficient can be fixed by requiring the new interacting BRST charge Q_{I} to be nilpotent. In general the result is:

$$Q_{\mathrm{I}} = \tfrac{1}{2}c\left(\nabla^2 + iF_{ab}S^{ab}\right) + S^{\oplus a}\nabla_a + S^{\oplus\oplus}b \tag{4.9}$$

where we use the following convention for the covariant derivative and the field strength:

$$\nabla_a \equiv \partial_a + iA_a \tag{4.10}$$

$$iF_{ab} \equiv [\nabla_a, \nabla_b] \tag{4.11}$$

The nilpotency of Q_{I} can be used to derive vertex operators that are Q closed. If we define the vertex operator as

$$V = Q_{\mathrm{I}} - Q \tag{4.12}$$

Then

$$Q_{\mathrm{I}}^2 = 0 \Rightarrow \{Q, V\} + V^2 = 0 \tag{4.13}$$

In the linearized limit, which is relevant for asymptotic states, we take only the part of V that is linear in background fields (denoted by V_0). Then one has

$$\{Q, V_0\} = 0 \tag{4.14}$$

There will be an additional U(1) symmetry in the $N = 2$ model. The vector states should be U(1) singlets and can be picked out by multiplying the original Q_{I} in eq. (4.9) with an additional δ function (a U(1) projector).

$$Q_{\mathrm{I}}' = \delta\left(J_{\mathrm{U}(1)}\right) Q_{\mathrm{I}} \tag{4.15}$$

$J_{\mathrm{U}(1)}$ is the U(1) current:

$$J_{\mathrm{U}(1)} = \tfrac{1}{2}\left(\psi \cdot \bar{\psi} - \bar{\psi} \cdot \psi\right) - \gamma\bar{\beta} + \bar{\gamma}\beta = -\bar{\psi} \cdot \psi + \tfrac{D}{2} - \gamma\bar{\beta} + \bar{\gamma}\beta \qquad (4.16)$$

where D is the spacetime dimension and $\bar{\psi}^a$, $\bar{\gamma}$, $\bar{\beta}$ have U(1) charge -1, and their complex conjugates have $+1$. This U(1) constraint is important in that it ensures that Q_{I} for the $N = 2$ model is indeed nilpotent. We will show this is the case.

Before choosing any specific representation, we have

$$\begin{aligned} Q_{\mathrm{I}}'^2 &= \delta\left(J_{\mathrm{U}(1)}\right) Q_{\mathrm{I}}^2 = \delta\left(J_{\mathrm{U}(1)}\right) \tfrac{1}{2}\{Q_{\mathrm{I}}, Q_{\mathrm{I}}\} \\ &= \delta\left(J_{\mathrm{U}(1)}\right) \tfrac{1}{2}\Big\{-icS^{\oplus a}[\nabla^b, F_{ab}] - icS^{\oplus c}S^{ab}\left[\nabla_c, F_{ab}\right] \\ &\quad +iS^{\oplus\oplus}S^{ab}F_{ab} + iS^{\oplus a}S^{\oplus b}F_{ab}\Big\} \end{aligned} \qquad (4.17)$$

To understand how the projector works for the $N = 2$ model, consider normal ordering with respect to the following scalar vacuum:

$$(\gamma,\, \beta,\, \psi,\, b)\,|0\rangle = 0 \qquad (4.18)$$

This vacuum has U(1) charge $+1$. A general normal-ordered operator with ≥ 2 barred fields on the left (unbarred fields are on the right), acting on any state built from the above vacuum, will either vanish or have negative U(1) charge. Therefore normal-ordered operators with ≥ 2 barred fields will be projected out by $\delta(J_{\mathrm{U}(1)})$. Actually this property can be made true for any vacuum: One just needs to shift the current by a constant in the projection operator.

With this in mind we have the following:

$$\begin{aligned} \delta\left(J_{\mathrm{U}(1)}\right) S^{\oplus a}S^{\oplus b} &= \delta\left(J_{\mathrm{U}(1)}\right)\left(i\gamma\bar{\psi}^a + i\bar{\gamma}\psi^a\right)\left(i\gamma\bar{\psi}^b + i\bar{\gamma}\psi^b\right) \\ &= -\delta\left(J_{\mathrm{U}(1)}\right)\bar{\gamma}\gamma\eta^{ab} = -\delta\left(J_{\mathrm{U}(1)}\right)\frac{1}{2}S^{\oplus\oplus}\eta^{ab} \\ \delta\left(J_{\mathrm{U}(1)}\right) S^{\oplus\oplus}S^{ab} &= \delta\left(J_{\mathrm{U}(1)}\right) 2\gamma\bar{\gamma}\left(\bar{\psi}^a\psi^b - \bar{\psi}^b\psi^a\right) = 0 \\ \delta\left(J_{\mathrm{U}(1)}\right) S^{\oplus c}S^{ab} &= \delta\left(J_{\mathrm{U}(1)}\right)\left(i\gamma\bar{\psi}^c + i\bar{\gamma}\psi^c\right)\left(\bar{\psi}^a\psi^b - \bar{\psi}^b\psi^a\right) \\ &= \delta\left(J_{\mathrm{U}(1)}\right)\left(i\bar{\gamma}\psi^b\eta^{ac} - i\bar{\gamma}\psi^a\eta^{bc}\right) \end{aligned} \qquad (4.19)$$

Note that one could arrive at the same algebra for the spin operators if one were to use the spin-1 ket-bra representation introduced in the previous section; thus one again sees that the U(1) projector acts as projecting out vector states. In fact the nilpotency of the BRST charge can be checked more easily using the ket-bra representation; however, for completeness we plug the above result into our previous calculation for $Q_I^{2\prime}$. We have

$$\delta\left(J_{\mathrm{U}(1)}\right) Q_{\mathrm{I}}^2 = c\delta\left(J_{\mathrm{U}(1)}\right)\left(\bar{\psi}^a\gamma - \bar{\gamma}\psi^a\right)\left[\nabla^b, F_{ab}\right] \tag{4.20}$$

which is proportional to the equation of motion satisfied by the asymptotic states. So we have proved that $\delta\left(J_{\mathrm{U}(1)}\right) Q_{\mathrm{I}}^2 = 0$.

The vertex operator is then easily obtained by considering Q_{I} as an expansion of Q,

$$\begin{aligned} V_0 &= \left[Q_{\mathrm{I}} - Q\right]_{\mathrm{linearin}A} \\ &\equiv cW_{\mathrm{I}} + W_{\mathrm{II}} \\ &= \tfrac{1}{2}c\left[2iA\cdot\partial + i\left(\partial_a A_b - \partial_b A_a\right)S^{ab}\right] + iA_a S^{\oplus a} \\ &= -\epsilon_a\left[c\left(i\dot{X}^a + \bar{\psi}^b\psi^a k_b - \bar{\psi}^a\psi^b k_b\right) + \left(\gamma\bar{\psi}^a + \bar{\gamma}\psi^a\right)\right]\exp\left[ik\cdot X\left(\tau\right)\right] \end{aligned} \tag{4.21}$$

This vertex operator satisfies

$$\{Q, V_0\} = 0 \tag{4.22}$$

The integrated vertex can be derived by noting:

$$[Q, W_{\mathrm{I}}] = \partial V_0 \;\rightarrow\; \left[Q, \int W_{\mathrm{I}}\right] = 0 \tag{4.23}$$

More complicated vertex operators are needed for the usual worldline formalism. We will discuss in detail how these operators arise in section V. In the world graph formalism linearized vertex operators derived above will be sufficient.

4.4 Vacuum, Ghost measure and Green Functions

When calculating amplitudes, the vacuum with which one chooses to work dictates the form of vertex operator and insertions one needs. In string theory, different choices of vacuum are called different pictures. The scalar vacuum discussed above is defined by the expectation value

$$\langle 0|c|0\rangle \sim 1 \tag{4.24}$$

The conformal vacuum of string theory

$$\langle 0|ccc|0\rangle \sim 1 \tag{4.25}$$

does not exist in particle theory since there aren't that many zero modes to saturate at tree level. On the other hand one could also treat the worldline SUSY ghosts' zero modes, which would require additional insertions. These are defined by the vacuum

$$(\bar{\beta},\, \beta,\, \psi,\, b)\left|\hat{0}\right\rangle = 0 \Rightarrow \left\langle \hat{0}|\mathrm{c}\delta(\gamma)\delta(\bar{\gamma})|\hat{0}\right\rangle \sim 1 \tag{4.26}$$

which has U(1) charge 2 and is thus not a physical vacuum.

To use the vertex operator we found above, we need to find a U(1) neutral vacuum $\left|\tilde{0}\right\rangle$ that is in the cohomology of the free BRST charge Q. It is related to the previous vacuum through the following relation:

$$\left|\tilde{0}\right\rangle = \bar{\beta}\left|0\right\rangle = \delta\left(\frac{1}{2}\gamma^2\right)\left|\hat{0}\right\rangle, \tag{4.27}$$

which leads to

$$\left\langle\tilde{0}\right|\gamma c\bar{\gamma}\left|\tilde{0}\right\rangle \sim 1 \tag{4.28}$$

This vacuum can be understood as the Yang-Mills ghost. It has ghost number -1 and lies in the cohomology only at zero momentum, indicating a constant field. Therefore it corresponds to the global part of the gauge symmetry: Gauge parameters satisfying $Q\Lambda = 0$ have no effect on the gauge transformations in the free theory, $\delta\phi = Q\Lambda$. In principal one could proceed to compute amplitudes in the available vacua mentioned above; however, due to its U(1) neutral property, the Yang-Mills ghost vacuum should be the easiest to extend to higher loops, since it would be easier to enforce U(1) neutrality.

With the above definition of the vacuum and the ghost measure, we can easily obtain the tree-level Green function. For the worldline formalism the Green function for the X fields at tree level is as usual,

$$\eta^{ab}G_{\rm B}\left(\tau,\tau'\right)\equiv\left\langle X^{a}\left(\tau\right)X^{b}\left(\tau'\right)\right\rangle=-\frac{1}{2}\eta^{ab}\left|\tau-\tau'\right| \tag{4.29}$$

For the fermions:

$$\eta^{ab}G_{F}(\tau,\tau')\equiv\left\langle\psi^{a}\left(\tau\right)\bar{\psi}^{b}\left(\tau'\right)\right\rangle=\eta^{ab}\Theta\left(\tau-\tau'\right) \tag{4.30}$$

where Θ is a step function which is zero if the argument is negative. Note that the fermionic Green function does not have the naive relation with the bosonic Green function

$$G_{F}\neq-\dot{G}_{\rm B}=\frac{1}{2}sign(\tau-\tau') \tag{4.31}$$

It differs by a constant $\frac{1}{2}$. This is due to different boundary conditions: The vacuum we choose, which is at $t=-\infty$, is defined to be annihilated by ψ^{a}; therefore on a time ordered line the expectation can be non-vanishing only if ψ is at later time then $\bar{\psi}$.

4.5 Scattering Amplitudes (worldline approach)

In the worldline approach, one starts by choosing a specific worldline, and then inserts relevant vertex operators for external states. For YM theory, where the worldline state is the same as external states, namely a vector, the choice for worldline is less obvious. Previous work on the worldline formalism was geared toward the calculation of one-loop amplitudes, where the loop itself provides a natural candidate for the worldline. This advantage is not present for tree or higher-loop amplitudes. Furthermore, for higher-point tree graphs, calculating the amplitude from the worldline requires sewing lower-point tree amplitudes to the worldline. This is unsatisfactory from the viewpoint of first-quantized

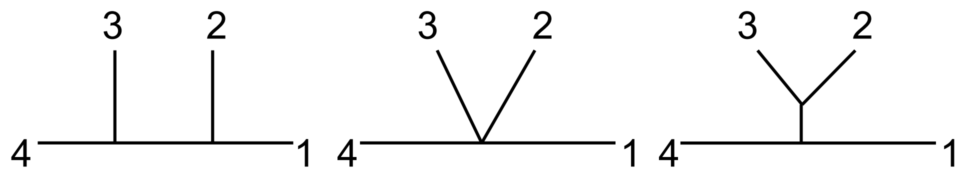


Figure 4.1: Three diagrams to be calculated if one chooses to connect lines 1 and 4 as the worldline. The second diagram needs a pinch operator, and the third diagram needs a vertex operator representing the tree attached to the worldline.

perturbation theory.

In general, to calculate an n-point tree-level partial amplitude in the worldline approach:

1. Choose a specific color ordering (e.g., $12...n$). Label external lines counter-clockwise.

2. Draw a worldline between any two of the external lines (e.g., line 1 and line n) and connect all other external lines to this worldline in the following three ways: (a) Use the linearized vertex operator V_0 defined in section III. (b) Use a vertex operator that is quadratic in background fields ("pinching"). This quadratic vertex operator ("pinch operator") can be derived from eq.(4.21) by extending the field strength to contain the non-abelian terms and takes the form

$$v^{(ij)} = \epsilon_{ia}\epsilon_{jb}c\left(\bar{\psi}^b\psi^a - \bar{\psi}^a\psi^b\right)e^{i(k_i+k_j)\cdot X} \tag{4.32}$$

(c) Have the external lines first form a lower-point tree graph and then connect to the worldline through either of the two vertex operators mentioned previously. This corresponds to replacing $A^a = \epsilon^a e^{ik\cdot X}$ with the non-linear part of the solution to the field equations that the background field satisfies. For example, for a four-point tree amplitude there are the three graphs shown in fig. (4.1), representing the three different ways external fields can attach to the worldline.

For lower-point graphs it is possible to choose the worldline in such a way that only linear vertex operators are required. We will show this in our actual computation for the four-point amplitude.

3. For each of the diagrams from above, insert three fixed vertex operators (respectively fixed at $\tau = \infty, 0, -\infty$). Two of them represent the initial and

final external states that were connected to form the worldline, while the remaining one can be any of the operators described above. For example, one has:

$$V^{(n)}(\infty)V^{(2)}(0)V^{(1)}(-\infty) \text{ or } V^{(n)}(\infty)v^{(32)}(0)V^{(1)}(-\infty) \tag{4.33}$$

where the superscript (i) represents the momentum and polarization vector of the external line i.

4. Insert the remaining vertex operators as the integrated ones, e.g.,

$$\int W_{\mathrm{I}}^{(i)} \text{ or } \int v^{(ij)} \tag{4.34}$$

with the integration regions so chosen that the diagram is kept planar.

5. Evaluate the expectation value with respect to the Yang-Mills ghost vacuum.

The fact that one needs to calculate lower-point tree graphs for a general tree graph is unsatisfactory, since one should be able to calculate an arbitrary-point amplitude without the knowledge of its lower-point counterparts. This was less a problem in the previous one-loop calculations, since one can claim that the method was really for one-particle-irreducible (1PI) graphs, and therefore sewing is necessary to calculate graphs that are not 1PI. It is more desirable to be able to calculate any amplitude with the knowledge of just the vertex operators and Green functions. This will be the aim of the "worldgraph" approach, which we leave to section VI. We first proceed to show how to calculate 3- and 4-point trees, and one-loop amplitudes, by the worldline approach.

4.5.1 3-Point Tree

In the 3-point case, we connect line 1 and line 3 as the worldline. The three vertex operators are respectively fixed at $\tau_C \to \infty$, $\tau_B = 0$ and $\tau_A \to -\infty$. Note that we need one c ghost to saturate the zero-mode and give a nonvanishing expectation value:

$$
\begin{aligned}
A_3 &= \left\langle V^{(3)}(\tau_C)\, V^{(2)}(\tau_B)\, V^{(1)}(\tau_A)\right\rangle \qquad (4.35)\\
&= \left\langle \left[cW_{\mathrm{I}}^{(3)}(\tau_C)\right]\left[W_{\mathrm{II}}^{(2)}(\tau_B)\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle\\
&+ \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_C)\right]\left[cW_{\mathrm{I}}^{(2)}(\tau_B)\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle\\
&+ \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_C)\right]\left[W_{\mathrm{II}}^{(2)}(\tau_B)\right]\left[cW_{\mathrm{I}}^{(1)}(\tau_A)\right]\right\rangle
\end{aligned}
$$

The first term and the third term vanish due to $\epsilon\cdot\dot{X}$ in W_{I} contracting with the $e^{ik\cdot X}$ in the other two W_{II}'s, which are proportional to $\epsilon_3\cdot k_3$ and $\epsilon_1\cdot k_1$ respectively, and vanish in the Lorenz gauge. The remaining term becomes

$$
\begin{aligned}
A_3 &= \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_C)\right]\left[cW_{\mathrm{I}}^{(2)}(\tau_B)\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle \qquad (4.36)\\
&= -\epsilon_{3a}\epsilon_{2c}\epsilon_{1d}\Big\langle \left[\left(\gamma\bar{\psi}^a+\bar{\gamma}\psi^a\right)e^{ik_3\cdot X}\right]_{\tau_C} c\left[\left(k_1^c+(\bar{\psi}^b\psi^c-\bar{\psi}^c\psi^b)k_{2b}\right)e^{ik_2\cdot X}\right]_{\tau_B}\\
&\quad \left[\left(\gamma\bar{\psi}^d+\bar{\gamma}\psi^d\right)e^{ik_1\cdot X}\right]_{\tau_A}\Big\rangle\\
&= -\left[(\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot k_3)+(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot k_1)+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot k_2)\right]
\end{aligned}
$$

As usual (see, e.g., [61]), the contractions among the exponentials give an overall factor of $e^{-\sum_{A\le i<j\le C} k_i\cdot k_j G_B(\tau_i-\tau_j)}$ in the final result, but this factor equals 1 if we go on-shell.

4.5.2 4-Point Tree

For the 4-point amplitude (with color-ordering 1234), one can calculate the three diagrams in fig. (4.1), but as we have mentioned, one can simplify the calculation by choosing a worldline between line 1 and line 3. In this case, there is only one diagram to be calculated (fig. (4.2)), and there is only one integrated vertex operator — line 4. We fix the other three as $\tau_{\mathrm{D}}\to\infty$, $\tau_{\mathrm{C}}=0$ and $\tau_{\mathrm{A}}\to-\infty$, and the integrated vertex has integration region $\tau_{\mathrm{D}}\ge\tau_{\mathrm{B}}\ge\tau_{\mathrm{A}}$. We then have:

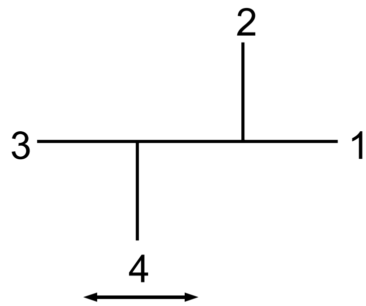


Figure 4.2: If one chooses to connect line 1 and line 3 as the worldline, there is only one diagram to be calculated. There is no need for pinch or more complicated operators. Note that line 4 is the integrated vertex and the integration region can be from $-\infty$ to $+\infty$, still keeping the graph planar.

$$\begin{aligned} A_4 &= \left\langle \left[V^{(3)}(\tau_D)\right]\left[V^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[V^{(1)}(\tau_A)\right]\right\rangle \\ &= \left\langle \left[cW_{\mathrm{I}}^{(3)}(\tau_D)\right]\left[W_{\mathrm{II}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle \\ &\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[cW_{\mathrm{I}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle \\ &\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[W_{\mathrm{II}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[cW_{\mathrm{I}}^{(1)}(\tau_A)\right]\right\rangle \end{aligned} \tag{4.37}$$

The first and third term again vanish, for the same reason as in the three-point case. The remaining term can be written in two parts by separating the integration region:

$$\begin{aligned} A_4 &= A_{4s} + A_{4t} \\ &= \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[cW_{\mathrm{I}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_C} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle \\ &\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[\int_{\tau_C}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[cW_{\mathrm{I}}^{(4)}(\tau_C)\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle \end{aligned} \tag{4.38}$$

Actually one can see these two terms as representing the s-channel and t-channel graphs from the second-quantized approach (see fig. (4.3)). The τ's

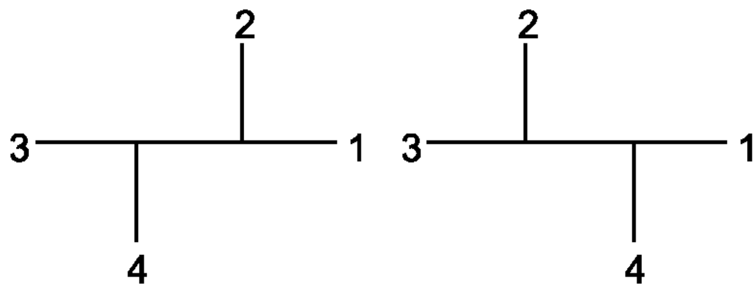


Figure 4.3: Two integration regions. The integrated vertex sits at 2.

are time ordered according to the order they appear on the worldline. The results are:

$$A_{4s} = -\frac{2}{s}\left[\begin{array}{l} -\frac{s}{4}\left(\epsilon_1 \cdot \epsilon_3\right)\left(\epsilon_2 \cdot \epsilon_4\right) - \frac{u}{2}\left(\epsilon_1 \cdot \epsilon_2\right)\left(\epsilon_4 \cdot \epsilon_3\right) \\ +\left(\epsilon_2 \cdot k_1\right)\left(\epsilon_4 \cdot k_3\right)\left(\epsilon_1 \cdot \epsilon_3\right) + \left(\epsilon_1 \cdot k_2\right)\left(\epsilon_3 \cdot k_4\right)\left(\epsilon_2 \cdot \epsilon_4\right) \\ +\left(\epsilon_1 \cdot k_3\right)\left(\epsilon_2 \cdot k_4\right)\left(\epsilon_3 \cdot \epsilon_4\right) + \left(\epsilon_4 \cdot k_2\right)\left(\epsilon_3 \cdot k_1\right)\left(\epsilon_1 \cdot \epsilon_2\right) \\ -\left(\epsilon_1 \cdot k_2\right)\left(\epsilon_4 \cdot k_3\right)\left(\epsilon_2 \cdot \epsilon_3\right) - \left(\epsilon_3 \cdot k_4\right)\left(\epsilon_2 \cdot k_1\right)\left(\epsilon_1 \cdot \epsilon_4\right) \\ -\left(\epsilon_1 \cdot k_4\right)\left(\epsilon_2 \cdot k_3\right)\left(\epsilon_3 \cdot \epsilon_4\right) - \left(\epsilon_3 \cdot k_2\right)\left(\epsilon_4 \cdot k_1\right)\left(\epsilon_1 \cdot \epsilon_2\right) \end{array}\right] \tag{4.39}$$

$$A_{4t} = -\frac{2}{t}\left[\begin{array}{l} -\frac{t}{4}\left(\epsilon_1 \cdot \epsilon_3\right)\left(\epsilon_2 \cdot \epsilon_4\right) - \frac{u}{2}\left(\epsilon_1 \cdot \epsilon_4\right)\left(\epsilon_2 \cdot \epsilon_3\right) \\ +\left(\epsilon_1 \cdot k_4\right)\left(\epsilon_3 \cdot k_2\right)\left(\epsilon_2 \cdot \epsilon_4\right) + \left(\epsilon_2 \cdot k_3\right)\left(\epsilon_4 \cdot k_1\right)\left(\epsilon_1 \cdot \epsilon_3\right) \\ +\left(\epsilon_1 \cdot k_3\right)\left(\epsilon_4 \cdot k_2\right)\left(\epsilon_2 \cdot \epsilon_3\right) + \left(\epsilon_2 \cdot k_4\right)\left(\epsilon_3 \cdot k_1\right)\left(\epsilon_1 \cdot \epsilon_4\right) \\ -\left(\epsilon_1 \cdot k_2\right)\left(\epsilon_4 \cdot k_3\right)\left(\epsilon_2 \cdot \epsilon_3\right) - \left(\epsilon_2 \cdot k_1\right)\left(\epsilon_3 \cdot k_4\right)\left(\epsilon_1 \cdot \epsilon_4\right) \\ -\left(\epsilon_1 \cdot k_4\right)\left(\epsilon_2 \cdot k_3\right)\left(\epsilon_3 \cdot \epsilon_4\right) - \left(\epsilon_3 \cdot k_2\right)\left(\epsilon_4 \cdot k_1\right)\left(\epsilon_1 \cdot \epsilon_2\right) \end{array}\right] \tag{4.40}$$

The sum of the above two parts is exactly the 4-point Yang-Mills tree amplitude. Note that we don't need the pinch operator in this calculation. This is because there cannot be a pinch operator representing line 2 and line 4, since they are not adjacent in the color ordering.

4.5.3 One-Loop Amplitude

It is straightforward to generalize this method to the calculation of 1-loop 1PI diagrams. The new feature in this case is that one must ensure U(1) neutrality inside the loop. One can think of the diagram as connecting both ends of a tree diagram, and only sum over U(1) neutral states. The U(1) neutral states are written as:

$$|A,p\rangle = \begin{cases} |a,p\rangle = \gamma\bar{\psi}^a \left|\tilde{0}\right\rangle \otimes |p\rangle \\ |\text{ghost},p\rangle = \left|\tilde{0}\right\rangle \otimes |p\rangle \\ |\text{antighost},p\rangle = \gamma\bar{\gamma} \left|\tilde{0}\right\rangle \otimes |p\rangle \end{cases} \tag{4.41}$$

where p is the momentum of the state, and the last two states are the Faddeev-Popov ghosts for background gauge fixing. The general expression for the amplitude of n-point 1-loop 1PI diagrams is then

$$\begin{aligned} A_n^{1-\text{loop}} &= \sum_{A,p} \int_0^\infty dT \, \langle A,p| \, V^{(n)}(\tau_n) \prod_{i=1}^{n-1} \int_{\tau_{i-1}\le\tau_i\le\tau_{i+1}} d\tau_i W_{\text{I}}^{(i)}(\tau_i) \, |A,p\rangle \\ &+ \text{ diagrams with pinch operators} \end{aligned} \tag{4.42}$$

where we define $\tau_0 = 0$ and fix $\tau_n = T$. Note that at one-loop we don't have the freedom to choose worldline (it should always be the loop), so one cannot avoid using the pinch operators.

Another approach is to insert a U(1) projector in the loop to pick out all the U(1) neutral states. That is, one inserts:

$$\begin{aligned} \delta\left[J_{\text{U}(1)}\right] &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp\left[i\frac{\theta}{T}\int_0^T d\tau J_{\text{U}(1)}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp\left[i\frac{\theta}{T}\int_0^T d\tau(-\bar{\psi}\cdot\psi + \tfrac{D}{2} - \gamma\bar{\beta} + \bar{\gamma}\beta)\right] \end{aligned} \tag{4.43}$$

Similar approaches have been taken in [49] and [55]. In [49], $i\theta$ is interpreted as a mass to be taken to infinity at the end, and together with GSO-like projection kills all U(1) non-neutral states. For us the U(1) projector naturally gets rid of all unwanted states. Furthermore the worldline ghosts were not taken into account in [49]; therefore they need to include the effect of Faddeev-Popov ghosts by adding covariant scalars to the action. This is sufficient for one loop, since they couple in the same way, yet will no longer be true for higher loops. Here we've (and also [55]) included all the worldline ghosts; thus the Faddeev-Popov ghosts are naturally included. In [55] gauge fixing the U(1) gauge field on a loop leads to a modulus, which is equivalent to θ in our U(1) projector insertion. The two views are analogous.

The inclusion of a U(1) projector amounts to additional quadratic terms in the action which will modify the Green function and introduce an additional θ-dependent term to the measure. Here we give a brief discussion of its effect. The kinetic operator for the SUSY partners and SUSY ghosts is now:

$$\partial_\tau + i\frac{\theta}{T} \tag{4.44}$$

The θ term can be absorbed by redefining the U(1) charged fields,

$$\Psi' = e^{i\theta\tau/T}\Psi \quad \bar{\Psi}' = e^{-i\theta\tau/T}\bar{\Psi} \tag{4.45}$$

where $\Psi = (\psi^a, \gamma, \beta)$. Then the integration over θ is really integrating over all possible boundary conditions since:

$$\Psi'(T) = e^{i\theta}\Psi'(0) \tag{4.46}$$

Without loss of generality, we choose the periodic boundary condition for the original fields Ψ.

The 1-loop vacuum bubble is then computed through mode expansion on a circle with periodic boundary condition:

$$\mathrm{Det}\left(\partial_\tau + i\frac{\theta}{T}\right)^{D-2} = \left[2i\sin\left(\tfrac{\theta}{2}\right)\right]^{D-2} \tag{4.47}$$

where D comes from the $\psi\bar{\psi}$ integration and -2 comes from SUSY ghosts. The fermionic Green function will be modified to

$$G_F(\tau,\tau') = \frac{e^{-\frac{i\theta(\tau-\tau')}{T}}}{2i\sin\frac{\theta}{2}}\left[e^{i\frac{\theta}{2}}\Theta(\tau-\tau') + e^{-i\frac{\theta}{2}}\Theta(\tau'-\tau)\right] \tag{4.48}$$

which satisfies the periodic boundary condition and differential equation

$$\left(\partial_\tau + i\frac{\theta}{T}\right)G_F(\tau,\tau') = \delta(\tau-\tau') \tag{4.49}$$

Also, at one loop there are two zero-modes, one modulus (the circumference of the loop) and one Killing vector. The proper insertions for the vacuum are:

$$\left\langle \tilde{0}|bc|\tilde{0}\right\rangle \sim 1 \tag{4.50}$$

In general, the n-point 1-loop 1PI amplitude can thus be written as

$$\begin{aligned} A_n^{1-\text{loop}} &= g^n \int_0^\infty \frac{dT}{T^{D/2}} \left\langle \delta\left[J_{\mathrm{U}(1)}\right] bV^{(n)}\left(\tau_n\right) \prod_{i=1}^{n-1} \int_{\tau_{i-1}\leq\tau_i\leq\tau_{i+1}} d\tau_i W_{\mathrm{I}}^{(i)}\left(\tau_i\right)\right\rangle \\ &+ \text{ diagrams with pinch operators} \\ &= \frac{g^n}{2\pi} \int_0^\infty \frac{dT}{T^{D/2}} \int_0^{2\pi} d\theta \left[2i\sin\left(\tfrac{\theta}{2}\right)\right]^{D-2} \left\langle W_{\mathrm{I}}^{(n)} \prod_{i=1}^{n-1} \int d\tau_i W_{\mathrm{I}}^{(i)}\right\rangle \\ &+ \text{ diagrams with pinch operators} \end{aligned} \tag{4.51}$$

We've added the coupling constant g, but omitted group theory factors, such as a trace and a factor N_c of the number of colors for the planar contribution. The XX contraction should be calculated by the 1-loop bosonic Green function:

$$\left\langle X^a\left(\tau\right) X^b\left(\tau'\right)\right\rangle = \eta^{ab} G_B\left(\tau-\tau'\right) = \eta^{ab}\left[-\frac{1}{2}\left|\tau-\tau'\right| + \frac{\left(\tau-\tau'\right)^2}{2T}\right] \tag{4.52}$$

For example, the two-point contribution to the effective action is (including the usual $-$ sign for the action, $\frac{1}{2}$ for permutations, and group theory factor for this case)

$$
\begin{aligned}
\Gamma_2^{1-\text{loop}} \quad &= \frac{-g^2 N_c}{4\pi} \int_0^\infty \frac{dT}{T^{D/2}} \int_0^{2\pi} d\theta \left[2i \sin\left(\tfrac{\theta}{2}\right)\right]^{D-2} \\
&\times \int_0^T d\tau \left\langle W_{\rm I}^{(2)}(T)\, W_{\rm I}^{(1)}(\tau) \right\rangle \\
&= -g^2 N_c \int_0^\infty \frac{dT}{T^{D/2}} \int_0^T d\tau \left[\begin{array}{l} \left(\delta(T-\tau) - \frac{1}{T}\right)(\epsilon_1 \cdot \epsilon_2) \\ + \left(\frac{1}{2} - \frac{\tau}{T}\right)^2 (\epsilon_2 \cdot k_1)(\epsilon_1 \cdot k_2) \\ - (\epsilon_2 \cdot k_1)(\epsilon_1 \cdot k_2) + (k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_2) \end{array} \right] \\
&\times e^{\frac{1}{2} k_1 \cdot k_2 \left(T - \tau - \frac{(T-\tau)^2}{T}\right)} \\
&= g^2 N_c \left(\frac{k_1^2}{2}\right)^{-\epsilon} \left(-1 + \frac{1}{12}\right) \Gamma(\epsilon) \left[(\epsilon_1 \cdot \epsilon_2)(k_1 \cdot k_2) - (\epsilon_1 \cdot k_2)(\epsilon_2 \cdot k_1)\right] \\
&= -\frac{11}{24} \text{tr} \left\{ F_1^{ab} \left[\frac{1}{\epsilon} - \log\left(\frac{1}{2} k_1^2\right)\right] F_{2ab} \right\}
\end{aligned}
\tag{4.53}
$$

In the final line we have used dimensional regularization $D = 4 - 2\epsilon$, and dropped the term with the δ function, which gives the tadpole contribution. Modified minimal subtraction was used, with the conventions of ref.[57]. Note that the $-\frac{1}{12}$ piece comes from the scalar graph while the 1 comes from terms with the fermion Green function. The diagram with pinch operator does not contribute in this case.

4.6 Worldgraph Approach

As mentioned previously, it is desirable even for tree graphs to develop a formalism that does not require an identification of a worldline to which external states are attached. Intuitively such a formalism would require one to simply identify 1D topological spaces that connect the external lines. This idea is very similar to string theory calculations and goes back as far as 1974 [58]. The main challenge for this "worldgraph approach" (following [59]) is the definition of Green functions on these non-differentiable topological spaces (non-differentiable because at interacting points it is not locally R^1). Previously, for multi-loops such Green functions have been derived by a combination of one-loop Green functions and insertions: See [60] for review. Recently in [61] a

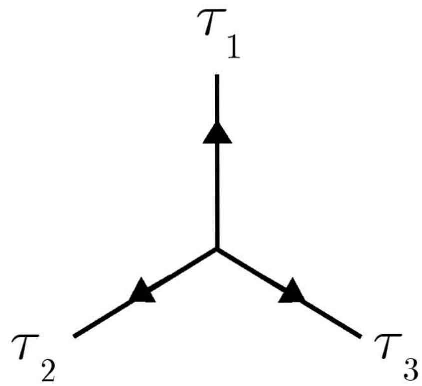


Figure 4.4: The topological space for a three-point interaction

more straightforward way to derive multi-loop Green functions was developed for scalar particles using the electric circuit analog. (A similar approach was used in [59].) Since fermion Green functions are related to bosons through a derivative (up to additional terms due to choice of vacuum or boundary conditions), what remains is to consistently define derivatives on these 1D topological spaces. We will use tree graphs as our testing ground.

Consider the three-point amplitude: One has only one graph, fig. (4.4). The arrows indicate the direction in which each τ_i is increasing. For scalar fields it was shown [61] that the appropriate Green function is proportional to the distance between two insertions; for the 3-point graph this is taken to be $-\frac{1}{2}(\tau_i + \tau_j)$.

To define derivatives, one notes that they are worldline vectors and therefore must be conserved at each interaction point. This leads to the conclusion that if we denote the worldgraph derivative on each line as $D(\tau_i)$, for the three-point graph they must satisfy:

$$D_{\tau_1} + D_{\tau_2} + D_{\tau_3} = 0 \tag{4.54}$$

This can be solved by defining the worldgraph derivatives as follows:

$$\begin{aligned} D_{\tau_1} &= \partial_{\tau_2} - \partial_{\tau_3} \\ D_{\tau_2} &= \partial_{\tau_3} - \partial_{\tau_1} \\ D_{\tau_3} &= \partial_{\tau_1} - \partial_{\tau_2} \end{aligned} \tag{4.55}$$

There is another solution which corresponds to (counter-)clockwise orientation. The choice of orientation can be fixed by matching it with the color ordering. Since the derivative is a local operator, its definition will not be altered if the three-point graph is connected to other pieces to form larger graphs. The fermionic Green function then follows from the bosonic by taking ψ as a worldline scalar and $\bar{\psi}$ as a worldline vector:

$$G_F(\tau_i, \tau_j) \equiv \langle \bar{\psi}(\tau_i)\psi(\tau_j)\rangle = 2\langle D_{\tau_i} X(\tau_i) X(\tau_j)\rangle \tag{4.56}$$

Armed with these two Green functions we can show how the three-point amplitude works.

3-Point Tree

For the three-point tree graph fig. (4.4) we start with:

$$\begin{aligned} A_3 &= \left\langle V^{(3)}(\tau_3)\, V^{(2)}(\tau_2)\, V^{(1)}(\tau_1)\right\rangle \\ &= \left\langle \left[cW_{\mathrm{I}}^{(3)}(\tau_3)\right]\left[W_{\mathrm{II}}^{(2)}(\tau_2)\right]\left[W_{\mathrm{II}}^{(1)}(\tau_1)\right]\right\rangle \\ &\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_3)\right]\left[cW_{\mathrm{I}}^{(2)}(\tau_2)\right]\left[W_{\mathrm{II}}^{(1)}(\tau_1)\right]\right\rangle \\ &\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_3)\right]\left[W_{\mathrm{II}}^{(2)}(\tau_2)\right]\left[cW_{\mathrm{I}}^{(1)}(\tau_1)\right]\right\rangle \end{aligned} \tag{4.57}$$

Now the worldline derivatives in W_{I} are replaced by worldgraph derivatives defined in eq. (4.55) and they give:

$$\left\langle i\epsilon_1 \cdot D_{\tau_1} X(\tau_1) e^{i\left[\sum_{i=1}^{3} k_i \cdot X(\tau_i)\right]}\right\rangle = -(\epsilon_1 \cdot k_3) \tag{4.58}$$

$$\left\langle i\epsilon_2 \cdot D_{\tau_2} X(\tau_2) e^{i\left[\sum_{i=1}^{3} k_i \cdot X(\tau_i)\right]}\right\rangle = -(\epsilon_2 \cdot k_1) \tag{4.59}$$

$$\left\langle i\epsilon_3 \cdot D_{\tau_3} X(\tau_3) e^{i\left[\sum_{i=1}^{3} k_i \cdot X(\tau_i)\right]}\right\rangle = -(\epsilon_3 \cdot k_2) \tag{4.60}$$

The fermionic Green functions are (with $F_{ij} \equiv \langle \bar{\psi}(\tau_i)\psi(\tau_j)\rangle$):

$$\begin{aligned} F_{12} = -1, \quad F_{23} = -1, \quad F_{31} = -1 \\ F_{21} = +1, \quad F_{32} = +1, \quad F_{13} = +1 \end{aligned} \tag{4.61}$$

Using the above one can compute eq. (4.57). The first term becomes:

$$\begin{aligned}
A_{3-1} &= \langle [cW_{\rm I}\,(\tau_3)]\,[W_{\rm II}\,(\tau_2)]\,[W_{\rm II}\,(\tau_1)]\rangle \\
&= -\epsilon_{3a}\epsilon_{2c}\epsilon_{1d}\langle c[iDX^a + (\bar\psi^b\psi^a - \bar\psi^a\psi^b)k_{3b}]_{\tau_3}[\gamma\bar\psi^c + \bar\gamma\psi^c]_{\tau_2} \\
&\quad \times[\gamma\bar\psi^d + \bar\gamma\psi^d]_{\tau_1} e^{ik_1\cdot X_{\tau_1}} e^{ik_2\cdot X_{\tau_2}} e^{ik_3\cdot X_{\tau_3}}\rangle \\
&= -\epsilon_{2c}\epsilon_{1d}\langle[-(\epsilon_3\cdot k_2) + (\bar\psi^b\psi^a - \bar\psi^a\psi^b)\epsilon_{3a}k_{3b}]_{\tau_3} \\
&\quad \times[\bar\psi^c(\tau_2)\psi^d(\tau_1) + \psi^c(\tau_2)\bar\psi^d(\tau_1)]\rangle \\
&= 2(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_1) + 2(\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_1) + 2(\epsilon_1\cdot k_3)(\epsilon_2\cdot\epsilon_3)
\end{aligned} \tag{4.62}$$

A similar derivation gives the second and third terms:

$$\begin{aligned}
A_{3-2} &= \langle [W_{\rm II}\,(\tau_3)]\,[cW_{\rm I}\,(\tau_2)]\,[W_{\rm II}\,(\tau_1)]\rangle \\
&= 2(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_1) + 2(\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_1) + 2(\epsilon_1\cdot k_3)(\epsilon_2\cdot\epsilon_3) \\
A_{3-3} &= \langle [W_{\rm II}\,(\tau_3)]\,[W_{\rm II}\,(\tau_2)]\,[cW_{\rm I}\,(\tau_1)]\rangle \\
&= 2(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_1) + 2(\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_1) + 2(\epsilon_1\cdot k_3)(\epsilon_2\cdot\epsilon_3)
\end{aligned} \tag{4.63}$$

Note that the three terms are the same, which respects the symmetry of the graph.

4.6.1 4-point Tree

For the 4-point amplitude we have two graphs (s channel and t channel, see fig. (4.5)) constructed by connecting two three-point worldgraphs on a worldline. The worldline in the middle is actually a modulus of the theory, and one must insert a b ghost. We focus on the s-channel graph; the t-channel graph can later be derived by exchanging the external momenta and polarizations in the

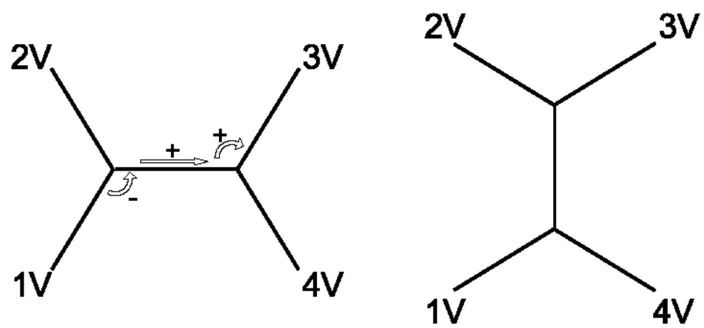


Figure 4.5: The two graphs for the four-point interaction

s-channel amplitude. We wish to derive

$$\begin{aligned} A_{4s} &= \int_0^\infty dT \left\langle V^{(4)}(\tau_4)V^{(3)}(\tau_3)b(T)V^{(2)}(\tau_2)V^{(1)}(\tau_1)\right\rangle \qquad (4.64) \\ &= \int_0^\infty dT \left[\begin{array}{l} \left\langle W_{\rm II}^{(4)}(\tau_4)cW_{\rm I}^{(3)}(\tau_3)b(T)cW_{\rm I}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ +\left\langle cW_{\rm I}^{(4)}(\tau_4)cW_{\rm I}^{(3)}(\tau_3)b(T)W_{\rm II}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ +\left\langle cW_{\rm I}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)b(T)cW_{\rm I}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ +\left\langle cW_{\rm I}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)b(T)W_{\rm II}^{(2)}(\tau_2)cW_{\rm I}^{(1)}(\tau_1)\right\rangle \\ +\left\langle W_{\rm II}^{(4)}(\tau_4)cW_{\rm I}^{(3)}(\tau_3)b(T)W_{\rm II}^{(2)}(\tau_2)cW_{\rm I}^{(1)}(\tau_1)\right\rangle \\ +\left\langle W_{\rm II}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)b(T)cW_{\rm I}^{(2)}(\tau_2)cW_{\rm I}^{(1)}(\tau_1)\right\rangle \end{array}\right] \end{aligned}$$

First we address the Green functions. As in [61] the bosonic Green function is still $-\frac{1}{2}L$, where L is the length between two fields. Thus it is the same as in the three-point case, except that when the two fields sit on opposite ends of the modulus, one needs to add the value of the modulus T. The worldgraph derivatives still act the same way, since the definition is local, irrespective of other parts of the graph. This gives the following result for the s-channel graph:

$$\left\langle i\epsilon_1 \cdot D_{\tau_1} X(\tau_1) e^{i\left[\sum_{i=1}^4 k_i\cdot X(\tau_i)\right]}\right\rangle = -(\epsilon_1 \cdot k_2) \qquad (4.65)$$

$$\left\langle i\epsilon_2 \cdot D_{\tau_2} X(\tau_2) e^{i\left[\sum_{i=1}^4 k_i\cdot X(\tau_i)\right]}\right\rangle = +(\epsilon_2 \cdot k_1) \qquad (4.66)$$

$$\left\langle i\epsilon_3 \cdot D_{\tau_3} X(\tau_3) e^{i\left[\sum_{i=1}^4 k_i\cdot X(\tau_i)\right]}\right\rangle = -(\epsilon_3 \cdot k_4) \qquad (4.67)$$

$$\left\langle i\epsilon_4 \cdot D_{\tau_4} X(\tau_4) e^{i\left[\sum_{i=1}^4 k_i\cdot X(\tau_i)\right]}\right\rangle = +(\epsilon_4 \cdot k_3) \qquad (4.68)$$

The fermionic Green functions are again more subtle. There are two types, that for bc ghosts and that for the $\bar{\psi}\psi$. First one notes that on the modulus, which is a worldline, both Green functions should be a step function, as explained in section IV. This is sufficient for the b, c ghosts. For $\bar{\psi}\psi$, since they can contract with each other on the same three-point graph or contract across the modulus, one must take the combined result: For contractions on the same three-point graph the rules are just as eq. (4.61), while for contraction across the modulus one multiplies the two Green functions on the two vertices with one from the modulus. For example, in the s-channel graph fig. (4.5):

$$\left\langle \bar{\psi}(\tau_1)\psi(\tau_3)\right\rangle = \left\langle \bar{\psi}(\tau_1)\psi(\tau_T)\right\rangle \left\langle \bar{\psi}(\tau_T)\psi(\tau_3)\right\rangle \Theta(T) = -1 \tag{4.69}$$

As one can see, the contraction across the modulus is broken down as if there were a pair $\bar{\psi}\psi$ on each end of the modulus, contracting with the vertices separately, and a final step function due to the fact that the modulus is a worldline. (We choose the left time to be earlier.) We now list all the relevant Green functions for the s-channel graph. The Green functions for the bc ghosts are

$$\langle c(\tau_1)b(T)\rangle = 1, \quad \langle c(\tau_2)b(T)\rangle = 1, \quad \langle c(\tau_3)b(T)\rangle = 0, \quad \langle c(\tau_4)b(T)\rangle = 0 \tag{4.70}$$

and the Green functions for the $\bar{\psi}\psi$ are (recall that we have defined $F_{ij} \equiv \langle\bar{\psi}(\tau_i)\psi(\tau_j)\rangle$)

$$\begin{aligned} &F_{12} = +1, \quad F_{21} = -1, \quad F_{34} = +1, \quad F_{43} = -1 \\ &F_{23} = +1, \quad F_{32} = 0, \quad\ \ F_{14} = +1, \quad F_{41} = -1 \\ &F_{13} = -1, \quad F_{31} = 0, \quad\ \ F_{24} = -1, \quad F_{42} = 0 \end{aligned} \tag{4.71}$$

Equipped with the Green functions one can compute eq. (4.64). We do the bc contractions first. Each term has two such contractions; using the above

Green functions we see that the second and last terms cancel. We then have:

$$A_{4s} = \int_0^\infty dT \left[\begin{array}{c} \left\langle cW_{\rm II}^{(4)}(\tau_4)W_{\rm I}^{(3)}(\tau_3)W_{\rm I}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ -\left\langle cW_{\rm I}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)W_{\rm I}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ +\left\langle cW_{\rm I}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)W_{\rm II}^{(2)}(\tau_2)W_{\rm I}^{(1)}(\tau_1)\right\rangle \\ -\left\langle cW_{\rm II}^{(4)}(\tau_4)W_{\rm I}^{(3)}(\tau_3)W_{\rm II}^{(2)}(\tau_2)W_{\rm I}^{(1)}(\tau_1)\right\rangle \end{array} \right] \tag{4.72}$$

Expanding out all possible contractions and implementing the Green functions and noting that

$$\begin{array}{ll} \langle DX(\tau_1)DX(\tau_3)\rangle = -2\delta(T), & \langle DX(\tau_2)DX(\tau_4)\rangle = -2\delta(T) \\ \langle DX(\tau_1)DX(\tau_4)\rangle = +2\delta(T), & \langle DX(\tau_2)DX(\tau_3)\rangle = +2\delta(T) \end{array} \tag{4.73}$$

With these Green functions in hand we arrive at the following s-channel amplitude:

$$A_{4s} = \frac{8}{s} \left[\begin{array}{l} +\frac{s}{4}\left(\epsilon_1\cdot\epsilon_4\right)\left(\epsilon_2\cdot\epsilon_3\right) - \frac{s}{4}\left(\epsilon_2\cdot\epsilon_4\right)\left(\epsilon_1\cdot\epsilon_3\right) - \left(\frac{s}{4}+\frac{u}{2}\right)\left(\epsilon_1\cdot\epsilon_2\right)\left(\epsilon_4\cdot\epsilon_3\right) \\ +\left(\epsilon_2\cdot k_1\right)\left(\epsilon_4\cdot k_3\right)\left(\epsilon_1\cdot\epsilon_3\right) + \left(\epsilon_1\cdot k_2\right)\left(\epsilon_3\cdot k_4\right)\left(\epsilon_2\cdot\epsilon_4\right) \\ +\left(\epsilon_1\cdot k_3\right)\left(\epsilon_2\cdot k_4\right)\left(\epsilon_3\cdot\epsilon_4\right) + \left(\epsilon_4\cdot k_2\right)\left(\epsilon_3\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_2\right) \\ -\left(\epsilon_1\cdot k_2\right)\left(\epsilon_4\cdot k_3\right)\left(\epsilon_2\cdot\epsilon_3\right) - \left(\epsilon_3\cdot k_4\right)\left(\epsilon_2\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_4\right) \\ -\left(\epsilon_1\cdot k_4\right)\left(\epsilon_2\cdot k_3\right)\left(\epsilon_3\cdot\epsilon_4\right) - \left(\epsilon_3\cdot k_2\right)\left(\epsilon_4\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_2\right) \end{array} \right] \tag{4.74}$$

A similar calculation can be done for the t-channel graph, and the result is simply changing the labeling of all momenta and polarizations in the s-channel result according to:

$$\begin{array}{ccc} s & \rightarrow & t \\ 1 & \rightarrow & 4 \\ 2 & \rightarrow & 1 \\ 3 & \rightarrow & 2 \\ 4 & \rightarrow & 3 \end{array} \tag{4.75}$$

We arrive at:

$$A_{4t} = \frac{8}{t} \left[\begin{array}{l} +\frac{t}{4}\,(\epsilon_4 \cdot \epsilon_3)\,(\epsilon_2 \cdot \epsilon_1) - \frac{t}{4}\,(\epsilon_2 \cdot \epsilon_4)\,(\epsilon_1 \cdot \epsilon_3) - (\frac{t}{4} + \frac{u}{2})\,(\epsilon_1 \cdot \epsilon_4)\,(\epsilon_2 \cdot \epsilon_3) \\ +\,(\epsilon_1 \cdot k_4)\,(\epsilon_3 \cdot k_2)\,(\epsilon_2 \cdot \epsilon_4) + (\epsilon_2 \cdot k_3)\,(\epsilon_4 \cdot k_1)\,(\epsilon_1 \cdot \epsilon_3) \\ +\,(\epsilon_1 \cdot k_3)\,(\epsilon_4 \cdot k_2)\,(\epsilon_2 \cdot \epsilon_3) + (\epsilon_2 \cdot k_4)\,(\epsilon_3 \cdot k_1)\,(\epsilon_1 \cdot \epsilon_4) \\ -\,(\epsilon_1 \cdot k_2)\,(\epsilon_4 \cdot k_3)\,(\epsilon_2 \cdot \epsilon_3) - (\epsilon_2 \cdot k_1)\,(\epsilon_3 \cdot k_4)\,(\epsilon_1 \cdot \epsilon_4) \\ -\,(\epsilon_1 \cdot k_4)\,(\epsilon_2 \cdot k_3)\,(\epsilon_3 \cdot \epsilon_4) - (\epsilon_3 \cdot k_2)\,(\epsilon_4 \cdot k_1)\,(\epsilon_1 \cdot \epsilon_2) \end{array} \right] \tag{4.76}$$

Adding the two channels again gives the complete 4-point amplitude.

Chapter 5

Off-shell superspace

5.1 Introduction

N=4 super Yang-Mills is the simplest four-dimensional quantum field theory in terms of properties relating to symmetry, finiteness, vanishing of amplitudes, resummation, etc. However, there is still no tractable formalism for calculating its amplitudes that directly incorporates these features.

Up to now we have explored the structure of N=4 SYM on-shell amplitudes. In order to efficiently explore the quantum properties of this theory, it is desirable to have an off-shell formulation. From the light-cone superspace formulation studied in previous chapters, one sees that the physical degree of freedom only requires a quater of the spinor variables of the full superspace(one only needs the chiral superfield to contain all the on-shell degrees of freedom). This is done non-covariantly in light-cone superspace by going to the light-cone gauge. Thus the crux of obtaining an off-shell formulation is to find a covariant way of truncating the full superspace to a subset in which it contains only half of the fermionic coordinates.

Approaches (for maximal supersymmetry) that incorporate the full off-shell supersymmetry manifestly prefer the ten-dimensional theory (the d=10 N=1 introduced earlier), showing no advantages unique to four dimensions: (1) Pure spinors [62] have complicated loop insertions (related to picture changing) that resemble BRST operators. (There is also the related problem of the lack of a gauge-invariant classical mechanics action, and thus of the usual b and c ghosts.) (2) The use of a ghost pyramid of spinor coordinates [63] has a BRST operator (following from an infinite set of constraints) that becomes complicated in the presence of a background (although it can be simply truncated in applications so far), and its viability at higher loops is still being investigated.

Although a complete formalism exists for describing 4D N=2 renormalizable quantum field theories in N=2 projective superspace [64] (which, however, could stand some further elucidation), little has been done for the N=4 analog at the interacting level. (There is an N=3 harmonic formulation [65], but no amplitudes have been calculated with it. Recently, a modified N=4 harmonic superspace has been proposed [66]; however, it failed to obtain the correct propagating degrees of freedom). In the harmonic construction, the harmonics are elements of the coset (G/H) of the R symmetry group SU(N) and are used to project out a subset of fermionic derivatives (d_ϑ) that closes

$$\{d_\vartheta, d_\vartheta\} \sim d_\vartheta \text{ or } 0$$

such that the prepotential depends on half the superspace $d_\vartheta V = 0$(this is sometimes called Grassmann analyticity condition). Since the local subgroup H usually has U(1)s, the measure acquires specific U(1) charge. In the case for N=4,3, one has so far failed to construct an action with the correct charge that cancels the measure(either the action has the wrong charge or an action with the right charge but wrong degrees of freedom).

Here we present the ingredients for a new formulation of this theory based on N=4 projective superspace [67, 68]. The basic idea is instead of using harmonics to project out the analytic fermionic derivatives, one takes fermionic coset. (That is, one includes some of the fermionic generators in H, as in chiral superspace). The coset is based on super anti-de Sitter coset OSp(4|4)/OSp(2|2)^2. The global group is (anti)symmetrized subgroup of the original superconformal group SU(4|2,2), thus the superconfomal transformation is obscured, however it is due to curved nature of the global group the makes it possible to define this coset in a SYM background.

In this chapter [69] we set up the ingredients for either first or second approach. We introduce projective superspace based on both the superconformal and the super anti-de Sitter group. We discuss the construction of constraints using suitable group generators, and proceed to solve them for the simple

N=0 case. A simple set of first-class superconformal (super AdS) constraints for first-quantization of the superparticle (based on an earlier description for $AdS_5 \otimes S^5$ [70]) will be given and results in a simple BRST operator. There is a supersymmetric ghost "tower" (but not a "pyramid") for *all* the coordinates. Projective superspace is found as a first-quantized (partial) gauge choice. It is both a (partial) unitary gauge, in that it eliminates constraints and their corresponding ghosts, and a covariant gauge. The projective superspace formalism for N=4 Yang-Mills is derived by the corresponding truncation from the full superspace, which is possible only with projective superspace. We show how four-point amplitudes are simpler in projective superspace than chiral. Finally we will discuss the second quantization in this space, by introducing new field strengths to this theory, one can show that the Bianchi identity no longer puts the theory on-shell, and one arrives at an action possible for off-shell quantization. The new fields strengths basically breaks the self-duality relationship among the original scalar field strengths.

Note that the simplest expression for the 4-point amplitude kinematic factor (and thus presumably the amplitude to all orders, after including the usual scalar loop factors) in normal (super) spacetime (or its conjugate momentum superspace), as opposed to supertwistor space, is in projective superspace [71]. (We will present a new derivation of this result from supertwistor space below.) This is due to the fact that the projective superfield strength is a scalar, while the chiral superfield strength, as follows from chiral supertwistor space (which is geared for MHV amplitudes), is a tensor, whose chirality holds only at the linearized level. This suggests that a projective formulation, at the (interacting) first- or second-quantized level, would provide the simplest derivation of this result. Also, being in spacetime as opposed to twistor space, it would directly allow an off-shell extension.

5.2 Superspace for superconformal symmetry

Assuming one successfully construct a first or second quantized theory that describes N=4 SYM, the large spacetime symmetry of this theory(superconformal) implies that one should be able to write the resulting amplitudes in a space where these symmetries are manifest. Thus one can instead first look for such a space and then try to construct a first or second quantized theory in it. Since the superconformal group in 4 dimensions is (P)SU(2,2|N), one would anticipate that a suitable superspace would arise as some coset or coset of a subgroup of this group. We would also like to have a natural truncation of the fermionic degrees of freedom to half of the full superspace. Combining these reasons leads us to a "Projective Superspace" [67, 68, 72]. Most of its simplifications follow from the fact that its coordinates are conveniently arranged in a single matrix.

There are various ways to arrive at this space, here we begin with a half-coset description in which the coordinates appear in a square matrix and therefore superconformal transformations are straightforward. Later we will introduce an alternative derivation, more useful in constructing first class constraints for first quantization, which is based on the super anti-de Sitter group.

We start with the group U(2,2|N) which is a square matrix with extra U(1)'s in comparison with the usual superconformal group. Then one defines a half-coset of this group by (i) first taking the coset $\frac{U(2,2|N)}{U(1,1|N-n)U(1,1|n)}$, thus the extra U(1)s cancel, (ii) choose a U(1) generator in the isotropy group and divide the U(2,2|N) generators into those with positive, minus or vanishing eigenvalue (G_+, G_-, G_0.) (iii) restrict to only the G_+ generators. In a sense we've mod out G_0 and G_-. We label this half-coset by $\frac{U(2,2|N)}{U(1,1|N-n)U(1,1|n)+}$. We write the U(2,2|N) coordinate as follows

$$\mathcal{Z}_{\mathcal{A}}{}^{\mathcal{M}} \quad \begin{cases} \mathcal{M} = M,\ M' \begin{cases} M = \mu,\ m \\ M' = \dot{\mu},\ m' \end{cases} \\ \mathcal{A} = A,\ A' \begin{cases} A = \alpha,\ a \\ A' = \dot{\alpha}, a' \end{cases} \end{cases}$$

The global superconformal group acts on $\mathcal{M}$ while the isotropy group acts on

$\mathcal{A}$.[1] One can now decompose a general U(2,2|N) element as

$$\mathcal{Z}_{\mathcal{A}}{}^{\mathcal{M}} = \begin{pmatrix} Z_A{}^M & Z_A{}^{M'} \\ Z_{A'}{}^M & Z_{A'}{}^{M'} \end{pmatrix} = \begin{pmatrix} \mathrm{I} & v \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ w & \mathrm{I} \end{pmatrix}$$
$$= \begin{pmatrix} u + vu'w & vu' \\ u'w & u' \end{pmatrix}$$

Then the half coset correspond to using G_- and G_0 to gauge away the v and u, u' coordinates respectively. One is then left with the w coordinates which can be represented by taking a rectangular part of the original matrix $\mathcal{Z}_{A'}{}^{\mathcal{M}} = (Z_{A'}{}^M, \, Z_{A'}{}^{M'})$ and define it as a fraction of the original coordinates:

$$w_{M'}{}^N = (Z_{A'}{}^{M'})^{-1} Z_{A'}{}^N = \begin{pmatrix} y_{m'}{}^m & \theta_{m'}{}^\mu \\ \bar{\theta}_{\dot{\mu}}{}^m & x_{\dot{\mu}}{}^\mu \end{pmatrix}$$

w is an $(n|2)$ by $(N - n|2)$ matrix for the case of N supersymmetries, where "n" indicates their "twisting": $n = 0$ (or N) describes (anti)chiral superspace. Since $\mu, \dot{\mu} = 1, 2$, one always result in 2N spinor coordinates corresponding to half of the fermionic coordinate of the full superspace. Thus one naturally achieves the goal of truncating the fermionic degrees of freedom in half. The case $n = N/2$ describes the preferred superspace, which allows a type of reality condition because this matrix is then square, satisfying a form of hermiticity. In this case for N=4 one has $m, m' = 1, 2$, then besides the usual 4 space time coordinate x, there is an equal number of θ, $\bar{\theta}$ and 4 internal coordinates y. Unfortunately using this type of coset gives second class constraints and quantization is difficult. This will be remedied by introducing another coset at the expense of manifest superconformal symmetry, however the resulting coordinates are the same thus one can still hope for manifest symmetry for the amplitudes.

Alternatively one can start with elements $\mathcal{Z}_{A'}{}^{\mathcal{M}}$ and $\mathcal{Z}_{\mathcal{M}}{}^A$ supplemented by the constraint $\mathcal{Z}_{A'}{}^{\mathcal{M}} \mathcal{Z}_{\mathcal{M}}{}^A = 0$. The general solution can be written as

$$\mathcal{Z}_{A'}{}^{\mathcal{M}} = u'(w, \, I), \quad \mathcal{Z}_{\mathcal{M}}{}^A = \begin{pmatrix} I \\ -w \end{pmatrix} u \tag{5.1}$$

[1] $\mathcal{M}$ is also separated into M and M' so that one can easily read where the coordinate sits in the matrix just by reading the indices.

one arrives at the same result.

Since $\mathcal{Z}$ transforms linearly under the (P)SU(2,2|N) superconformal group ${g_{\mathcal{N}}}^{\mathcal{M}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, transformations for w is then represented as a fractional linear transformations:

$$w' = (wc + d)^{-1}(wa + b)$$

or the equivalent

$$w' = (\tilde{a}w + \tilde{b})(\tilde{c}w + \tilde{d})^{-1}$$

in terms of the (P)SU(2,2|N) group element

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \tilde{d} & -\tilde{c} \\ -\tilde{b} & \tilde{a} \end{pmatrix}^{-1}$$

or in linearized form as

$$\delta w = \alpha + \beta w + w\gamma + w\epsilon w$$

Superconformal invariants can then be constructed by taking superdeterminants of w (or multiple supertraces since $sdet(e^M) = e^{strM}$).

Charge conjugation for the w coordinates can be easily defined once it is defined for the $\mathcal{Z}$ coordinates[68]. The goal is to find a conjugation operation $\mathcal{C}\mathcal{Z}$ such that the conjugated coordinate transforms the same way as $\mathcal{Z}$ under the superconformal group. Using (pseudo)unitarity of the group element one has $\mathcal{C}\mathcal{Z} \equiv \mathcal{Z}^{-1\dagger}\Upsilon$ since

$$\mathcal{Z}' = \mathcal{Z}g \rightarrow (\mathcal{C}\mathcal{Z})' = (\mathcal{Z}')^{-1\dagger}\Upsilon = (\mathcal{Z})^{-1\dagger}g^{-1\dagger}\Upsilon = \mathcal{C}\mathcal{Z}g \tag{5.2}$$

Where Υ is the (P)SU(2,2|4) metric. Then $\mathcal{C}w$ transforms

$$(\mathcal{C}w)^\dagger \equiv \begin{pmatrix} -y_a^{-1\,a'} & iy_a^{-1\,b'}\theta_{b'}{}^{\beta}C_{\beta\alpha} \\ -iC^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}}{}^{b}y_b^{-1\,a,} & C^{\dot{\alpha}\dot{\beta}}(x_{\dot{\beta}}{}^{\beta} - \bar{\theta}_{\dot{\beta}}{}^{b}y_b^{-1\,b'}\theta_{b'}{}^{\beta})C_{\beta\alpha} \end{pmatrix}$$

The superconformally invariant (4D extension of the Hilbert space) inner product is then

$$\langle A|B\rangle \equiv \int dw(\mathcal{C}A)(w)B(w) = \langle B|A\rangle^*$$

for any A and B that transform as half-densities

$$dw'[A'(w')]^2 = dw[A(w)]^2$$

where $\mathcal{C}A$, which transforms in the same way as A, is defined by the relation of complex conjugation to charge conjugation in the above inner product,

$$(\mathcal{C}A)(w)[det(y)]^{-str(I)}[A(\mathcal{C}w)]^\dagger,\ str(I) = N - 2$$

This superspace has various descriptions in terms of cosets [67] or related projections [68, 72]. A manifestly superconformal description is most natural in a projective lightcone formalism [68, 72]: In that approach, one would start with the coset of the superconformal group with respect to a classical (isotropy) group, and take a contraction of the latter ("projective lightcone limit"), which makes some of the original coordinates (including one from spacetime) nondynamical. Unfortunately, the interpretation of the resulting action remains unclear.

5.3 Super anti-de Sitter

Here we present a new approach: We first formulate the first quantized theory in the full group space(the isometry group), that is we define the constraints using covariant derivatives of the full group. As the isometry group we choose the super-anti de Sitter group (in four dimensions). We then choose first-quantized gauge conditions, which corresponds to the isotropy group. The isotropy group is the super-anti de Sitter group in one lower dimension (three), up to questions of signature. (Some Wick rotation is involved, since we really want 3D de Sitter symmetry, not 3D anti-de Sitter, to get anti-de Sitter space SO(3,2)/SO(3,1), but only the anti-de Sitter symmetry can be supersymmetrized. This Wick rotation leads to modified reality conditions, which always occur in projective space [64, 68].) Although the manifest symmetry is only super-anti de Sitter, the superconformal invariance of super Yang-Mills in D=4 guarantees the result is directly applicable to flat space. (Pure spinors have also been used to describe 4D N=4 Yang-Mills in super AdS, but using the maximal bosonic isotropy group [73])

Thus the "full" superspace of the isometry group is reduced to a projective coset which is represented by the w coordinates we discussed previously. This approach has the advantage that before gauge fixing the super Yang-Mills background can be constructed with covariant derivatives, which require the full superspace in order to incorporate all the physical field strengths, while after gauge fixing the theory can be quantized using just projective superspace, which is all the super Yang-Mills prepotential should require. The fact that our SYM theory is defined in a space that arises from a partial gauge fixing of the larger space mirrors the fact that the action for the N=2 vector multiplet in harmonic superspace is nonlocal in the internal coordinates, this is analogous to Coulomb-like interactions suggesting such spaces are the result of partial gauge fixing from larger superspaces with local actions. As discussed previously, reduction of the number of fermionic coordinates is useful for quantization because only one quarter of those of the full superspace are physical; any unphysical coordinates must be canceled by ghosts. However, in such spaces nonrenormalization theorems are not obvious.

The relevant cosets are then

$$\frac{\mathrm{OSp(N|4)}}{\mathrm{OSp(n|2)OSp(N-n|2)}}$$

which can readily be seen to yield the rectangle of coordinates given above. (This isotropy group was also found in the projective lightcone approach. The case n=0 of this coset, namely $\frac{\mathrm{OSp(N|4)}}{\mathrm{OSp(N|2)Sp(2)}}$, was used in [74] to describe self-dual supergravity).

We can also take a contraction of the isometry, a graded generalization of the contraction used to obtain the Poincare group from the anti-de Sitter group: In terms of the algebra

$$[H,H\} = H,\ [H,G/H\} = G/H,\ [G/H,G/H\} = H$$

we simply drop the right-hand side of the last equation[2]. The result is

$$\frac{\mathrm{I[OSp(n|2)OSp(N-n|2)]}}{\mathrm{OSp(n|2)OSp(N-n|2)}}$$

where the "I(nhomogeneous)" part is just translations with respect to the coordinates of the rectangle described above. The coset is now Abelian, and consists of the usual "supertranslations" of the projective group: spacetime translations($p_{\alpha\dot{\alpha}}$), half of the supersymmetries($\pi_{a'\alpha}, \bar{\pi}_{a\dot{\alpha}}$), and part of the R-symmetry($t_{aa'}$). However, although the part of the isometry group acting on Lorentz (spinor) indices is just the Poincare group, the full group is not the super Poincare group, because the isotropy group is unchanged: The last consists of the Lorentz group, a subgroup of R-symmetry, and the sum of the other half of the supersymmetries and the corresponding S-supersymmetries. So we have the usual flat spacetime, but not the usual flat full superspace. Since the coset space is our projective superspace on which our superfields depend, while the isotropy group is the tangent space[3], this means in the contracted case we have a flat (and torsion-free) coordinate space with a curved tangent space, the opposite of the usual.

We will use both these cosets below (more or less simultaneously, since it's easy to see how to contract the former to the latter). Both isometry groups are subgroups of the superconformal group. Before going to the coset we first show how constraints can be constructed in the full group space. This is a general discussion, for readers only interested in the construction for N=4 SYM, one can skip directly to the next section.

5.4 Groups without cosets

5.4.1 Constraints

All the constraints we consider in the following sections are first-class, thus the first quantized actions are of the form

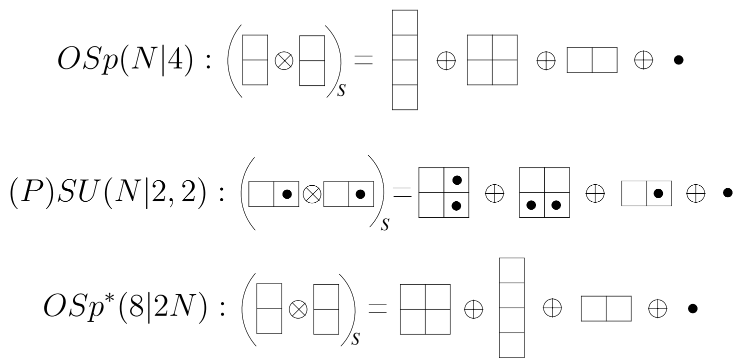
$$S = \int d\tau\ L, \quad L = -\dot{z}^a p_a + \lambda^i \mathcal{G}_i$$

[2]This can be done by multiplying G/H by R and taking R to ∞

[3]For the usual superspace, the isotropy group is the Lorentz group

for superspace coordinates z and their conjugate momenta p, and constraints $\mathcal{G}$ and their Lagrange multipliers λ^i, all as functions of the worldline parameter τ.

For reasons mentioned above, we define our "full" superspace as the entire supergroup space, rather than just a coset. The free field equations of the theory are expressed at the first-quantized level as constraints quadratic in the group generators. (This is "dual" to writing them in the same form in terms of covariant derivatives.) We begin by reducing the (symmetrized) square of the generators: The three general (finite-dimensional) cases are for superconformal in D=3,4,6 (or 1 more dimension for super-AdS),



(OSp has a real defining representation, OSp* has pseudoreal; thus the former has bosonic subgroup SO(N)Sp(4), while the latter has SO*(8)USp(2N).) In each case we have listed the 4 irreducible representations in the following order:

- (1) nonvanishing on shell, most symmetric in spacetime spinor indices;
- (2) vanishing for superconformal only, most antisymmetric in spacetime spinor indices, includes flat Klein-Gordon;
- (3) vanishing for both superconformal and AdS, single supertrace, includes Pauli-Lubanski;
- (4) vanishing for both superconformal and AdS, double supertrace (Casimir, with a dot for its singlet tableau), AdS Klein-Gordon.

We use graded symmetrization, so "symmetric" in the tableaux means symmetric in the former label of the group, since in the first and last cases that has the symmetric metric. For the cases of interest in D=4, this means that

R-indices are considered as bosonic, and Lorentz spinor indices as fermionic, for purposes of sign factors from reordering indices. So, e.g., $str(M_{\mathcal{A}}{}^{\mathcal{B}}) = +M^{\mathcal{A}}{}_{\mathcal{A}}$. For the unitary case, dots in boxes refer to the complex conjugate representation; ordering of the dotted block with respect to the undotted block is arbitrary. In the cases where the ranges of the bosonic and fermionic indices are the same (N=4), supertracelessness is undefined ($str(I) = 0$), so the 3rd constraint implies the 4th, and the 3rd is implied by the 1st (for the latter 2 cases) or the 2nd (for the former 2 cases). By "vanishing", we mean up to constants, for the case of vanishing superhelicity. (Nonvanishing superhelicity can be described by introducing "spin" operators, in addition to these "orbital" ones defined in terms of just the supergroup coordinates.)

To see that this classification of (quadratic) constraints is consistent with the usual identification of the superconformal mass shell, we evaluate them in the supertwistor representation, which exists for D=3,4,6: The generators $\hat{G}$ in terms of the supertwistors ζ are

$$\begin{aligned}
D=3\ &:\ \hat{G}_{\mathcal{AB}} = \frac{1}{2}[\zeta_{\mathcal{A}}, \zeta_{\mathcal{B}}\},\ \ \{\zeta_{\mathcal{A}}, \zeta_{\mathcal{B}}] = \eta_{\mathcal{AB}} \\
D=4\ &:\ \hat{G}^{\mathcal{A}}{}_{\mathcal{B}} = \frac{1}{2}[\zeta^{\mathcal{A}}, \bar{\zeta}_{\mathcal{B}}\},\ \ \{\bar{\zeta}^{\mathcal{A}}, \zeta_{\mathcal{B}}] = \delta^{\mathcal{A}}{}_{\mathcal{B}} \qquad\quad h = \frac{1}{2}[\zeta^{\mathcal{A}}, \bar{\zeta}_{\mathcal{A}}\} \\
D=6\ &:\ \hat{G}_{\mathcal{AB}} = \frac{1}{2}[\zeta^{a}{}_{\mathcal{A}}, \zeta_{a\mathcal{B}}\},\ \ \{\zeta_{a\mathcal{A}}, \zeta_{b\mathcal{B}}] = C_{ab}\eta_{\mathcal{AB}},\ \ h_{ab} = \frac{1}{2}[\zeta_{a}{}^{\mathcal{A}}, \zeta_{b\mathcal{A}}\}
\end{aligned} \tag{5.3}$$

with indices $\mathcal{A}, \mathcal{B}$ in the defining representation (and defining SU(2) indices a, b for D=6), where h is the superhelicity (generating the little group U(1) for D=4, or SU(2) for chiral D=6), which is set to vanish in our case. (For D=4 we have given the U(N|2,2) generators; in coordinate representations, only (P)SU(N|2,2) need be defined. Note that twistors are essentially γ-matrices for OSp, or creation/annihilation operators for U, satisfying graded anticommutation relations; thus the bosonic ones anticommute with the fermionic ones.)

Substitution of these representations into the corresponding constraints numbers 2,3,4 above shows they vanish up to constants for vanishing superhelicity, and do not vanish for number 1.

Once the correct constraints have been identified, it's more convenient (for purposes of applying isotropy conditions or coupling background fields) to express the constraints in their dual form in terms of (free) covariant derivatives ($\hat{G} \to d$).

5.4.2 N=0

For the case N=0, we can examine arbitrary dimensions, with the generators carrying vector indices; then we have

$$SO(D,2): \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)_S = \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix} \oplus \square\square \oplus \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} \oplus \bullet$$

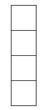
(The ordering is as above, but the roles of symmetry and traces have changed.)

We now outline the solution of the constraints. In the conformal case, the constraints are, in terms of SO(D−1,1) indices,

dim	$\square\square$	$\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}$	$\bullet$
2	P^2		
1	$S_m{}^n P_n + wP_m$	$S_{[mn}P_{p]}$	
0	$S_m{}^p S_{pn} + K_{(m}P_{n)} - tr$	$wS_{mn} + K_{[m}P_{n]}$	
0	$S^2 + (D+1)w^2 + (D-2)K \cdot P$	$S_{[mn}S_{pq]}$	$\frac{1}{2}S^2 - w^2 - 2K \cdot P$
−1	$S_m{}^n K_n - wK_m$	$S_{[mn}K_{p]}$	
−2	K^2		

for momentum P, spin S, scale weight w, and covariant derivative for conformal boosts K. The dimension-2 constraint allows us to choose a lightcone frame to make the analysis simpler. The dimension-1 constraints then imply $S = w = 0$ (assuming the momentum is not identically 0). The surviving dimension-0 constraints then imply $K = 0$. So we are left with a massless scalar.

In the AdS case, in terms of SO(D,1) indices, we have



$\bullet$

$P^2 - \frac{1}{2}S^2$

$S_{[mn}P_{p]}$

$S_{[mn}P_{p]}$

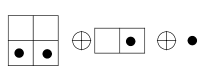
We again find $S = 0$, describing a scalar.

5.5 AdS$_4$ supergroup space

5.5.1 Constraints

We now examine the constraints in more detail for the cases of interest for D=4: (P)SU(N|2,2) for superconformal, and OSp(N|4) for super-AdS. Conveniently, D=4 is the only case for which there are classical supergroups for all N that can be applied for both superconformal and super-AdS. As a result, a supertwistor analysis can be applied for super-AdS as well as for superconformal (using the same supertwistors).

For the superconformal group we find from the analysis of the previous section, substituting the supertwistor expression,

superconformal  :

$$\hat{\mathcal{G}}_{\mathcal{AB}}{}^{\mathcal{CD}} \equiv \hat{G}_{(\mathcal{A}}{}^{(\mathcal{C}}\hat{G}_{\mathcal{B}]}{}^{\mathcal{D}]} - \frac{1}{2}\delta_{\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B}]}{}^{\mathcal{D}} = 0$$

The Kronecker δ term can be considered as arising from "normal ordering". Note that these constraints, with all indices uncontracted, have large amount of redundancy and thus implementing quantization leads to complicated ghosts structure that was the main reason previous attempts of first quantization failed.

The constraints for the AdS case are a subset of (a linear combination of) the superconformal ones, because the OSp(N|4) generators themselves can be derived from (graded) antisymmetrization of the (P)SU(N|2,2) ones:

$$\hat{G}_{\mathcal{A}}{}^{\mathcal{B}} \;\rightarrow\; G_{\mathcal{AB}} \equiv \hat{G}_{[\mathcal{A}}{}^{\mathcal{C}}\eta_{\mathcal{C}|\mathcal{B})}$$

where η is the graded-symmetric OSp(N|4) metric. Consequently the maximal subset of the above constraints that can be expressed in terms of OSp(N|4) generators requires a supertrace on the indices of the above constraints, in agreement with our previous analysis:

$$super - AdS \ \square\!\square \oplus \bullet \ : \qquad \mathcal{G}_{\mathcal{A}\mathcal{B}} \equiv G_{(\mathcal{A}|}{}^{\mathcal{C}} G_{\mathcal{C}|\mathcal{B})} + str(I)\eta\mathcal{A}\mathcal{B} = 0$$

The superhelicity is again required to vanish. (Note the "anomalous" η term vanishes for N=4.) Similar constraints were proposed previously for the (10D) superparticle on AdS$_5$⊗S^5 [71], using the superconformal isometry group.

5.5.2 Lightcone gauge

The constraints are most easily solved in a lightcone-type decomposition. First, it's useful to identify how the constraints relate to the more reducible superconformal constraints. It will only be necessary to look at those constraints that do not include terms with covariant derivatives corresponding to conformal boosts and S-supersymmetry, since those constraints can be applied to arbitrary massless representations of supersymmetry [75], of which the minimal representation appearing in first-quantization is a special case. (So the rest are redundant, at least after choosing conformal boosts and S-supersymmetry as the isotropy group.) Separating out the PSU(N|2,2) indices as $\mathcal{A} = (\mathrm{a}, \alpha, \dot\alpha)$ (where $\mathrm{a} = (a, a'))$, those constraints are

$$\begin{array}{cccc}
\mathcal{A} & \equiv & \frac{1}{4}\hat{\mathcal{G}}_{\alpha\beta}{}^{\dot\gamma\dot\delta} C^{\beta\alpha}\bar{C}_{\dot\delta\dot\gamma} & = & p^{\alpha\dot\alpha}p_{\alpha\dot\alpha} \\
\mathcal{B}_{\mathrm{a}\alpha} & \equiv & \frac{1}{2}\hat{\mathcal{G}}_{\mathrm{a}\alpha}{}^{\dot\gamma\dot\delta}\bar{C}_{\dot\delta\dot\gamma} & = & p_\alpha{}^{\dot\alpha}\bar\pi_{a\dot\alpha} \\
\bar{\mathcal{B}}^{\mathrm{a}\dot\alpha} & \equiv & \frac{1}{2}\hat{\mathcal{G}}_{\alpha\beta}{}^{\mathrm{a}\dot\alpha}C^{\beta\alpha} & = & p^{\alpha\dot\alpha}\pi^a{}_\alpha \\
\mathcal{C}_\alpha{}^{\dot\alpha} & \equiv & \hat{\mathcal{G}}_\alpha{}^{\mathcal{B}}{}_{\mathcal{B}}{}^{\dot\alpha} & = & S_\alpha{}^\beta p_\beta{}^{\dot\alpha} - \bar{S}^{\dot\alpha\dot\beta}p_{\alpha\dot\beta} - \pi^{\mathrm{a}}_\alpha\bar\pi_{\mathrm{a}}{}^{\dot\alpha} \\
\mathcal{C}^{b\dot\alpha}_{int,\mathrm{a}\alpha} & \equiv & \hat{\mathcal{G}}_{\mathrm{a}\alpha}{}^{\mathrm{b}\dot\alpha} & = & t_{\mathrm{a}}{}^{\mathrm{b}}p^{\alpha\dot\alpha} + \pi^{\mathrm{b}}{}_\alpha\bar\pi_{\mathrm{a}}{}^{\dot\alpha} \\
\mathcal{C}_\chi{}^{\mathrm{ab}}{}_{\alpha\beta} & \equiv & \hat{\mathcal{G}}_{\alpha\beta}{}^{\mathrm{ab}} & = & \pi^{\mathrm{a}}{}_\alpha\pi^{\mathrm{b}}{}_\beta \\
\mathcal{C}_{\bar\chi,\mathrm{ab}}{}^{\dot\alpha\dot\beta} & \equiv & \hat{\mathcal{G}}_{\mathrm{ab}}{}^{\dot\alpha\dot\beta} & = & \bar\pi_{\mathrm{a}}{}^{\dot\alpha}\bar\pi_{\mathrm{b}}{}^{\dot\beta} \\
\mathcal{D}_\alpha{}^{\dot\alpha} & \equiv & \frac{1}{2}(\hat{\mathcal{G}}_{\alpha\beta}{}^{\beta\dot\alpha} + \hat{\mathcal{G}}_{\alpha\dot\beta}{}^{\dot\alpha\dot\beta}) & = & S_\alpha{}^\beta p_\beta{}^{\dot\alpha} + \bar{S}^{\dot\alpha\dot\beta}p_{\alpha\dot\beta} + wp_\alpha{}^{\dot\alpha}
\end{array}$$

where we have labeled the constraints as in [75]. Note that in D=4 the Pauli-Lubanski equation $\mathcal{C}$ is equivalent to the $\mathcal{D}$ constraint in the presence of the

$\mathcal{B}$ constraint and the Klein-Gordon equation $\mathcal{A}$.

We now compare these to the OSp(N|4) constraints: In terms of covariant derivatives of the isometry group

$$d^{\mathcal{A}\mathcal{B}} = \begin{pmatrix} t^{\mathrm{ab}} & \pi^{\mathrm{a}\beta} & \bar{\pi}^{\mathrm{a}\dot{\beta}} \\ -\pi^{\mathrm{b}\alpha} & S^{\alpha\beta} & p^{\alpha\dot{\beta}} \\ -\bar{\pi}^{\mathrm{b}\dot{\alpha}} & p^{\beta\alpha} & \bar{S}^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$

we have

$$0 = \mathcal{G}^{\mathcal{A}\mathcal{B}} \equiv d^{\mathcal{A}\mathcal{C}} d^{\mathcal{B}\mathcal{D}} \eta_{\mathcal{D}\mathcal{C}} =$$
$$\begin{pmatrix} t^{\mathrm{ac}} t^{\mathrm{b}}{}_{\mathrm{c}} + \pi^{\mathrm{a}\gamma}\pi^{\mathrm{b}}{}_{\gamma} + \bar{\pi}^{\mathrm{a}\dot{\gamma}}\bar{\pi}^{\mathrm{b}}{}_{\dot{\gamma}} & t^{\mathrm{ac}}\pi^{\beta}{}_{\mathrm{c}} - \pi^{\mathrm{a}\gamma} S^{\beta}{}_{\gamma} - \bar{\pi}^{\mathrm{a}\dot{\gamma}} p^{\beta}{}_{\dot{\gamma}} & t^{\mathrm{ac}}\bar{\pi}^{\dot{\beta}}{}_{\mathrm{c}} - \pi^{\mathrm{a}\gamma} p^{\dot{\beta}}{}_{\gamma} - \bar{\pi}^{\mathrm{a}\dot{\gamma}} \bar{S}^{\dot{\beta}}{}_{\dot{\gamma}} \\ \cdots & \pi^{\alpha\mathrm{c}}\pi^{\beta}{}_{\mathrm{c}} - S^{\alpha\gamma} S^{\beta}{}_{\gamma} - p^{\alpha\dot{\gamma}} p^{\beta}{}_{\dot{\gamma}} & \pi^{\alpha\mathrm{c}}\bar{\pi}^{\dot{\beta}}{}_{\mathrm{c}} - S^{\alpha\gamma} p^{\dot{\beta}}{}_{\gamma} - p^{\alpha\dot{\gamma}} \bar{S}^{\dot{\beta}}{}_{\dot{\gamma}} \\ \cdots & \cdots & \bar{\pi}^{\dot{\alpha}\mathrm{c}}\bar{\pi}^{\dot{\beta}}{}_{\mathrm{c}} - p^{\dot{\alpha}\gamma} p^{\dot{\beta}}{}_{\gamma} - \bar{S}^{\dot{\alpha}\dot{\gamma}} \bar{S}^{\dot{\beta}}{}_{\dot{\gamma}} \end{pmatrix}$$
$$\approx \begin{pmatrix} ? & \mathcal{B}^{\alpha\beta} & \bar{\mathcal{B}}^{\dot{\alpha}\dot{\beta}} \\ \cdots & C^{\alpha\beta}\mathcal{A} & \mathcal{C}^{\alpha\dot{\beta}} \\ \cdots & \cdots & \bar{C}^{\dot{\alpha}\dot{\beta}}\mathcal{A} \end{pmatrix} \tag{5.4}$$

where "$\cdots$" is proportional to the transposed element, "$\approx$" refers to extra terms, and we have ignored symmetrization of indices, which produces terms linear in generators. Note that extra signs from reordering of super-indices are implicit: For example, in the supertrace of indices in the definition of the OSp constraints, there is a factor of $(-1)^{\mathcal{B}\mathcal{D}}$, which is -1 if both indices are Sp(4) and $+1$ otherwise, because supertraced indices belong next to each other. (We could also just use the graded symmetry of the second d factor, but we want to use notation that applies to the general case, where d has no symmetry.) The constraint "?" will be found to be the square of the lightcone part of the $\mathcal{C}_{int}$ constraint.

To analyze these constraints it's instructive to look first at the case N=0: In the superconformal case, we can solve the $\mathcal{A}$ constraint as usual. The $\mathcal{C}$ constraint is then the usual Pauli-Lubanski equation for vanishing helicity: We can thus set the spin operators $S^{\alpha\beta}$ and $\bar{S}^{\dot{\alpha}\dot{\beta}}$ to vanish. (The components of the spin not explicitly set to vanish by this equation do not appear, and so can be eliminated from the theory by unitary transformation, or equivalently by a gauge condition for the gauge transformation generated by this equation.) The $\mathcal{D}$ constraint does the same if we set the conformal weight $w = 0$: It's the same as the Pauli-Lubanski equation except for a (Hodge) duality transformation on the spin (and in general also switching helicity with conformal weight). The

AdS case is similar except for the extra terms in $\mathcal{A}$; but these drop out after solving the Pauli-Lubanski equation.

We then choose a lightcone Lorentz frame

$$p^{\alpha\dot{\alpha}} = \begin{pmatrix} p^+ & 0 \\ 0 & p^- \end{pmatrix}$$

For general N, the $\mathcal{A}$ and $\mathcal{B}$ constraints are used to solve for $p^{-\dot{-}}$, and $\pi_{\rm a}^-$ and $\bar{\pi}_{\rm a}^{\dot{-}}$, as usual. The $\mathcal{C}$ constraint then determines $S^{\alpha\beta}$ and $\bar{S}^{\dot{\alpha}\dot{\beta}}$. (Again a $\mathcal{D}$ constraint is unnecessary.) Now the ? constraint will perform a similar function for $t_{\rm ab}$: After plugging in the solution for $\pi_{\rm a}^-$ and $\bar{\pi}_{\rm a}^{\dot{-}}$, it becomes

$$\tilde{t}^{\rm ac}\tilde{t}^{\rm b}{}_{\rm c} = 0, \qquad \tilde{t}^{\rm ab} = t^{\rm ab} - \frac{1}{p^+}\bar{\pi}^{\rm a\dot{+}}\pi^{\rm b+}$$

We recognize $t^{\rm ab}$ as proportional to the superconformal $\mathcal{C}_{int}^{\rm ab+\dot{+}}$. Since the internal space is compact, the vanishing of the square of this operator implies its own vanishing. (In particular, we see all the Casimirs of these modified group generators vanish.) Thus the AdS constraints are equivalent to the larger superconformal set: They yield the supertwistor representation.

5.5.3 Constructing the BRST Charge

Isotropy constraints (really gauge conditions) are expressed in terms of covariant derivatives (since they preserve the global symmetry), so from now on we also represent the quadratic constraints (field equations) in terms of them, also. (The supertwistor representation of the previous subsection applies only to the group generators.) The covariant derivatives are again a subset of those for the superconformal group. The explicit form of the latter has been given previously; we won't need them here (only their algebra).

In matrix notation, these constraints are

$$d\eta d^T = d^T\eta d = 0$$

(for graded transpose "^{T}"). The most interesting things about these constraints are that: (1) Their index structure is that of a matrix, as for the covariant derivatives d themselves, except perhaps for the symmetry on their two indices: for example, the covariant derivative d^{ab} is anti-symmetric in its two R indices while the constraint $t^{\mathrm{ac}}t^{\mathrm{b}}{}_{\mathrm{c}} + \pi^{\mathrm{a}\gamma}\pi^{\mathrm{b}}{}_{\gamma} + \bar{\pi}^{\mathrm{a}\dot{\gamma}}\bar{\pi}^{\mathrm{b}}{}_{\dot{\gamma}}$ is symmetric in the two R indices. (2) These constraints are reducible, the index structure for their reducibility condition, and the reducibility of the reducibility condition etc, are also just a matrix. Therefore ghost for the constraints and their reducibility condition have the same structure.

In the present case d is graded antisymmetric on its 2 indices. We then have

$$d^T = -d$$

$$(d\eta d)^T = +(d\eta d)$$

The constraints are written as $\mathcal{G}_1 \equiv d\eta d = 0$. Then the reducibility conditions are

$$\mathcal{G}_2 \equiv d\eta\mathcal{G}_1 - \mathcal{G}_1\eta d = +\mathcal{G}_2^T = 0$$

$$\mathcal{G}_3 \equiv d\eta\mathcal{G}_2 + \mathcal{G}_2\eta d = -\mathcal{G}_3^T = 0$$

etc., where the sign for the symmetry of $\mathcal{G}_n$ alternates as $-++--++--...$. Explicitly, with $\mathcal{G}_0 \equiv d$,

$$\mathcal{G}_{n+1} \equiv d\eta\mathcal{G}_n + (-1)^n\mathcal{G}_n\eta d = (-1)^{n(n-1)/2}\mathcal{G}_{n+1}^T = 0$$

Using this construction for the BRST operator, and including terms for closure ($Q^2 = 0$) leads to the result that the complete minimal BRST operator can be written in the simple form (matrix multiplication with metric, and trace, implied)

$$Q = \sum_{m,n=0}^{\infty} c_{m+n+1}b_m b_n + f...$$

where the indices label the ghost generation,

$$b_0 = d$$

while higher generation of b represents the reducible constraint, etc. We have

$$c_n = -(-1)^{n(n+1)/2}c_n^T$$

and similarly for b while "f..." denotes structure-constant terms. (We won't need those for the contracted projective case since the group is abelian.)

No nonminimal degrees of freedom are needed; we can choose a "temporal" gauge for the first-quantized gauge fields, as is standard for D=1 and 2 (because it doesn't break worldsheet or spacetime Lorentz invariance). Thus, there is only a ("1D") tower of ghosts [72] (for all of x, θ and y), as opposed to the ("2D") pyramid of ghosts (for just θ) for the approach of that name.

5.6 N=4 projective superspace

5.6.1 Projective gauge

In the previous section we analyzed the first-quantized theory on shell by simultaneously solving all the constraints explicitly and choosing a lightcone gauge for the symmetry generated by the constraints. We can instead solve a subset of the constraints and choose their corresponding gauges in such a way as to manifestly preserve Lorentz covariance. This can be achieved in a way that is equivalent to completely eliminating some of the coordinates (a subset of those eliminated in the lightcone gauge). Since the algebra of gauge conditions must close, this is the same as choosing an isotropy subgroup. Then the isotropy group can be used to reduce the original constraints, eliminating constraints, or terms in constraints, that vanish off shell as a consequence of the vanishing of the isotropy covariant derivatives themselves. This leads to the coset construction discussed in (5.3) which correspond to projective superspace.

To treat these cosets, we again divide the range of OSp(N|4) indices in half as

$$\mathcal{A} = (A, A')$$

for the two OSp($\frac{N}{2}$|2)'s. In the original matrix (or rectangular) notation the isotropy coordinates correspond to the us, thus the coset simply means that we are dropping the u dependence. The constraints resulting from dropping the isotropy covariant derivatives d_u, leaving just the projective ones d_w, will have a similar form as before, but the indices will be reduced from OSp(N|4) to (one of the) OSp($\frac{N}{2}$|2): d_w has the index structure $d_A{}^{A'}$ and indices are then contracted with one of the $\mathrm{OSp}(\frac{N}{2}|2)$ metrics (or its inverse). Thus the constraints for this coset are

$$\eta^{AB} = (\delta^{ab}, C^{\alpha\beta}),\ \ \eta_{A'B'} = (\delta_{a'b'}, \bar{C}_{\dot{\alpha}\dot{\beta}});$$
$$C_{\alpha\beta} = \bar{C}_{\dot{\alpha}\dot{\beta}} = -C^{\alpha\beta} = -\bar{C}^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{5.5}$$

(If we want to keep track of dimensional analysis, we can include a factor of the anti-de Sitter radius $1/R$ with the Kronecker δ's, but being careful to distinguish the inverse metrics, where R appears instead of $1/R$.)

In the explicit form for the BRST operator (which takes the same form as above, but with different symmetry for the matrices, as discussed below), the algebra for d_w closes only on d_u, so $Q^2 = 0$ modulo such terms. A separate term to enforce $d_u = 0$ can easily be added, along with the corresponding terms for closure of the $d_w d_w$ algebra on d_u. (Similar remarks apply to adding a Lagrange multiplier term for d_u to the Hamiltonian.) Alternatively, we can work just in the contracted coset space, and d_u can be ignored altogether. If we use the contracted coset, the d_w are simply partial derivatives with respect to w (up to factors of isotropy coordinates u, which can be ignored upon restriction to the coset space). To see that this gives the desired description for N=4 SYM we solve it in the light-cone gauge.

5.6.2 Lightcone gauge again

First we write out the different Lorentz pieces of the constraints: In terms of

$$d_A{}^{A'} = \begin{pmatrix} t_a{}^{a'} & \bar{\pi}_a{}^{\dot{\alpha}} \\ \pi_\alpha{}^{a'} & p_\alpha{}^{\dot{\alpha}} \end{pmatrix}$$

we have

$$\mathcal{G}_{AB} = \begin{pmatrix} G_{(ab)} & G_{a\beta} \\ G_{\alpha b} & G_{[\alpha\beta]} \end{pmatrix} = \begin{pmatrix} t_a{}^{a'} t_{ba'} + \bar{\pi}_a{}^{\dot{\alpha}} \bar{\pi}_{b\dot{\alpha}} & t_a{}^{a'} \pi_{\beta a'} - \bar{\pi}_a{}^{\dot{\alpha}} p_{\beta\dot{\alpha}} \\ \cdots & \pi_\alpha{}^{a'} \pi_{\beta a'} - p_\alpha{}^{\dot{\alpha}} p_{\beta\dot{\alpha}} \end{pmatrix}$$

(with the usual signs for a fermionic index interrupting the contraction of two fermionic indices) and similarly for the complex-conjugate constraints $\bar{\mathcal{G}}^{A'B'}$.

In particular we have, from $\mathcal{G}_{\alpha\beta}$, $\mathcal{G}^a{}_a$, and their complex conjugates,

$$p^2 = \pi^2 = \bar{\pi}^2 = -t^2$$

(plus one redundant equation).

From $\mathcal{G}^{+a}$ and its complex conjugate we find

$$\pi^{-a'} = \frac{1}{p^+} i t^{aa'} \bar{\pi}_a{}^{\dot{+}}$$

and its complex conjugate. ($\mathcal{G}^{-a}$ and its complex conjugate are redundant.) This tells us

$$\pi^2 = \frac{1}{p^+} t^{aa'} \pi_a{}^{\dot{+}} \pi^+{}_{a'}$$

We then have, from the remaining constraint Γ_{ab} and complex conjugate,

$$\hat{t}_a{}^{a'} \hat{t}_{ba'} = \hat{t}^a{}_{a'} \hat{t}_{ab'} = 0$$

defining

$$\hat{t}_{aa'} \equiv t_{aa'} - \frac{1}{p^+} \bar{\pi}_a{}^{\dot{+}} \pi^+{}_{a'}$$

This expression is the independent piece of the constraint from the bigger superconformal set,

$$p_{\alpha\dot{\alpha}} t_{aa'} - \bar{\pi}_{a\dot{\alpha}} \pi_{\alpha a'} = 0$$

Since the vanishing of the square of a Hermitian operator implies the vanishing of the operator, we find

$$\hat{t}_{aa'} = 0$$

(This is clear on the original coset, since the internal space is compact, so there is no ambiguity in normalization of states. However, things might be more subtle on the contracted coset.) The hermiticity of this operator follows from the fact that it is a piece of the superconformal field equations, which can be expressed in terms of group generators (instead of covariant derivatives), which are by definition Hermitian.

It then follows that

$$p^2 = \pi^2 = \bar{\pi}^2 = -t^2 = 0$$

($t^2 = 0$ does not imply $t_{aa'} = 0$, since t is not Hermitian with respect to the charge conjugation that defines the inner product.) So x dependence is determined by the Klein-Gordon equation (as usual in the lightcone formalism), y dependence is completely determined, and θ dependence is determined in terms of half the original θ's, i.e., 1/4 of those of the full superspace, as usual.

5.6.3 Counting degrees of freedom

We can count the degrees, subtracting the constraint, adding back due to reducibility, subtracting reducibility of reducibility $\cdots$. This is equivalent to counting ghosts. The results of (5.5.3) for counting ghosts can be applied directly to the 4D case by using OSp(N|4) indices, dividing their ranges in half, and dropping irrelevant blocks. The ghosts for odd n both indices are primed or both unprimed, while for even n we have mixed indices:[4]

$$\begin{cases} c_{2n+1,AB},\ c_{2n+1,A'B'} & \text{where } c_{2n+1} = (-1)^n c^T_{2n+1} \\ & c_{2n,AB'} \end{cases}$$

Thus the symmetry has a cycle of 4, going as asymmetric, (twice) graded symmetric, asymmetric, (twice) graded antisymmetric.

We can now count the naive effective number of modes for any of x, θ, y . Infinite sums can be defined, e.g., by regularization:

$$(1+z)^{-1} = 1 - z + z^2 - ... \to 1 - 1 + 1 - ... = \frac{1}{2}$$

We then have for each variable (remember w carries OSp($\frac{N}{2}$|2) indices)

$$\begin{aligned} x \text{ (two Weyl indices)} \quad &: \quad 4 - 2\cdot 1 + 4 - 2\cdot 3 + \cdots \\ &= \quad (1 - 1 + 1 - \cdot\cdot)\cdot 4 + (1 - 1 + 1 - \cdot\cdot)\cdot 2 \\ &= \quad 3 = D - 1 \end{aligned}$$

[4]For example the zeroeth level are the original constraints which have both primed or unprimed indices. The first level correspond to the first reducibility condition, which are d_w on constraints, thus have mixed indices.

$$
\begin{aligned}
\theta \text{ (one Weyl indices)} \quad : \quad & 2N - 2N + 2N - 2N + \cdots \\
= \quad & (1 - 1 + 1 - \cdot\cdot) \cdot 2N \\
= \quad & N \rightarrow \ \frac{1}{4} \text{of full superspace}
\end{aligned}
$$

$$
\begin{aligned}
y \text{ (no Weyl indices)} \quad : \quad & (\frac{N}{2})^2 - 2 \cdot \frac{\frac{N}{2}(\frac{N}{2}+1)}{2} + (\frac{N}{2})^2 - 2 \cdot \frac{\frac{N}{2}(\frac{N}{2}-1)}{2} \cdots \\
= \quad & (1 - 1 + 1 - \cdot\cdot) \cdot \frac{1}{4} N^2 - (1 - 1 + 1 - \cdot\cdot) \cdot \frac{N}{2} \\
= \quad & \frac{N(N-2)}{8}
\end{aligned}
$$

where for x and y we have separated the sum into averages over symmetry/antisymmetry plus the deviations due to either. The x's (and p's) have just the "transverse" degrees of freedom $D-1$, which in the equivalent ghost-free lightcone analysis arise from the gauge choice $x^+ = \tau$ (and $p^2 = 0$ eliminating p^-). This agrees with the usual scalar particle, which has just x and 1 c; but here there are 2 (identical) constraints for $N = 0$, resulting in reducibility to cancel 1. For θ we also find the number of physical degrees, which is just 1/4 that of the full superspace. However, though the y's have no physical degrees of freedom, they do not cancel by this counting because they are eliminated by quadratic constraints, not linear. (But note that net bosons and fermions cancel for $N = 4$, as they do at each ghost level. Also, because of the grading the x counting is just the $N = -4$ case of the y counting.) Interestingly, for the case of OSp(n|2)OSp(N−n|2) with $n \neq \frac{1}{2}N$ the sum diverges for y, even with the above regularization, giving an extra term $-(n - \frac{1}{2}N)^2(1 + 1 + ...)$.

5.7 N=4 super Yang-Mills

5.7.1 Covariant derivatives

We now consider the formulation of N=4 super Yang-Mills in this projective superspace. We first note that it is difficult to define the projective superspace in the half-coset approach starting from the SU(4|2,2) group. For a (half)coset to be consistently defined in a SYM background, it must be consistent to define the vanishing of the isotropy group once the covariant derivative is extended to include SYM connections $d_u \to \nabla_u$, that is:

$$[\nabla_u, \nabla_u\} \sim \nabla_u \text{ or } 0$$

For the half coset approach based on SU(4|2,2), the problem is $[\nabla_u, \nabla_u\} = F_{uu}$, $[\nabla_u, \nabla_v\} = \nabla_v + F_{uv}$ e.t.c. The presence of field strengths in the SYM background renders the definition of the half-coset inconsistent. We will show that in the anti de Sitter coset, the coset can be defined in SYM background.

We begin by defining the algebra of the (gauge)covariant derivatives in the "full" OSp(N|4) space. The algebra is simply the original OSp(N|4) algebra plus the usual flat SYM field strengths. We need to start with the full superspace in order to incorporate all the physical field strengths. Then we will see that when separating out the isotropy sector of the algebra(the two OSp(2|2)s, the algebra of ∇_us), a redefinition can be performed such that the isotropy constraint can be defined in the SYM background.

As is well known the usual flat algebra for the covariant derivatives of SYM implies field equations for the field strength for N=3 and 4. This will be also true for our case, and we will derive the field equations shortly. In the first quantization point of view this is not a problem since from our previous analysis in the YM case 4.1, the nilpotency of the interacting BRST charge, which is necessary for the construction of vertex operators, implies the field equation for the background field. In fact in any linearized quantum gauge theory in a background of the same gauge theory, linearized gauge invariance of the quantum theory requires the background to be on shell [76], we will here restrict ourselves to an on-shell background. However, this background is on shell with respect to the full nonlinear field equations, which is sufficient to construct the Feynman rules: For example, tree graphs can be derived from perturbative solutions to the classical equations of motion. Thus, the existence of this construction, combined with the off-shell formulation of the linearized theory, should be sufficient to prove the existence of the nonlinear off-shell theory.

In addition to the usual 4 spacetime and 4N anticommuting coordinates, this full superspace contains also internal (bosonic) coordinates for not only the AdS R-symmetry group SO(4) but also the Lorentz group. Of course, as for (N=0) gravity in curved space, we treat the spacetime derivatives and

Lorentz (spin) generators as separate, even though the Killing vectors of AdS that generate SO(3,2) do not distinguish "translations" from "Lorentz transformations". This is fully consistent with the distinction between symmetry generators and covariant derivatives, and thus the usual coset construction for Sp(4)/Sp(2)^2 ("left" and "right" action on the group elements). What is unusual here is that we introduce coordinates for the Lorentz spin, as well as corresponding components for the Yang-Mills gauge fields. (The Yang-Mills gauge group is still the same; it is only that it is defined over a bigger manifold.) This is already done for R-symmetry, in projective and harmonic superspace. The Yang-Mills field strengths in these directions vanish, and thus gauges can be chosen where their gauge fields do also. However, in some cases it may prove convenient to choose gauges where they do not, as in the usual N=2 harmonic construction [77].[5] An interesting example is the case of selfdual Yang-Mills ($f_{\alpha\beta} = 0$), even for N=0, which is known to be analogous to N=2 projective and harmonic superspaces [64, 77]. In the lightcone gauge for this theory, we separate the + and − components of the undotted spinor index (but not the dotted one), then one can solve some of the selfduality conditions as

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] = C_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}} \begin{cases} \left[\nabla_{+\dot{\alpha}}, \nabla_{+\dot{\beta}}\right] = 0 \rightarrow\ \nabla_{+\dot{\alpha}} = \partial_{+\dot{\alpha}},\ A_{+\dot{\alpha}} = 0 \\ \left[\nabla_{+[\dot{\alpha}}, \nabla_{-\dot{\beta}]}\right] = \partial_{+[\dot{\alpha}}A_{-\dot{\beta}]} = 0 \rightarrow\ A_{-\dot{\alpha}} = \partial_{+\dot{\alpha}}A_{--} \end{cases} \tag{5.6}$$

(where $\nabla = d + iA$) in terms of some "prepotential" A_{--}. But the solution of the second equation automatically follows from the first when we gauge the Lorentz invariance; the prepotential A_{--} appears as the connection for the Lorentz connection ∇_{--}, then we derive the same result from the algebra

$$[\nabla_{+\dot{\alpha}}, \nabla_{--}] = \nabla_{-\dot{\alpha}}\ \rightarrow\ A_{-\dot{\alpha}} = \partial_{+\dot{\alpha}}A_{--}$$

Pure spinors are also related to (coset) Lorentz coordinates.

[5] In N=2 Harmonic superspace [77] the prepotential is the connection introduced for one of the y coordinates in the coset $\frac{SU(2)}{U(1)}$

5.7.2 SYM Background

We first express the covariant derivatives in manifestly SO(N) covariant form; their algebra is the obvious combination of the OSp(N|4) algebra with that of the flat-space super Yang-Mills covariant derivatives (and hence their field strength):

$$
\begin{aligned}
[\nabla_{\mathrm{ab}}, \nabla_{\mathrm{cd}}] &= -\eta_{[\mathrm{a}[\mathrm{c}|}\nabla_{\mathrm{b}]|\mathrm{d}]} \\
[\nabla_{\mathrm{ab}}, \nabla_{\mathrm{c}\gamma}] &= -\eta_{[\mathrm{ac}}\nabla_{\mathrm{b}]\gamma} \\
\{\nabla_{\mathrm{a}\alpha}, \nabla_{\mathrm{b}\beta}\} &= -C_{\alpha\beta}(\nabla_{\mathrm{ab}} + \phi_{\mathrm{ab}}) - \eta_{\mathrm{ab}}\nabla_{\alpha\beta} \\
\{\nabla_{\mathrm{a}\dot{\alpha}}, \nabla_{\mathrm{b}\dot{\beta}}\} &= -C_{\dot{\alpha}\dot{\beta}}(\nabla_{\mathrm{ab}} + \bar{\phi}_{\mathrm{ab}}) - \eta_{\mathrm{ab}}\nabla_{\dot{\alpha}\dot{\beta}} \\
\{\nabla_{\mathrm{a}\alpha}, \nabla_{\mathrm{b}\dot{\beta}}\} &= -\eta_{\mathrm{ab}}\nabla_{\alpha\dot{\beta}} \\
[\nabla_{\alpha\beta}, \nabla_{\mathrm{b}\gamma}] &= C_{(\alpha\gamma}\nabla_{b\beta)} \\
[\nabla_{\mathrm{a}\alpha}, \nabla_{\gamma\dot{\beta}}] &= C_{\alpha\gamma}(\nabla_{\mathrm{a}\dot{\beta}} + \overline{W}_{\mathrm{a}\dot{\beta}}) \\
[\nabla_{\mathrm{a}\dot{\alpha}}, \nabla_{\gamma\dot{\beta}}] &= C_{\dot{\alpha}\dot{\beta}}(\nabla_{\mathrm{a}\gamma} + W_{\mathrm{a}\gamma}) \\
[\nabla_{\alpha\beta}, \nabla_{\gamma\dot{\delta}}] &= C_{(\alpha\gamma}\nabla_{\beta)\dot{\delta}} \\
[\nabla_{\alpha\dot{\beta}}, \nabla_{\gamma\dot{\rho}}] &= C_{\alpha\gamma}(\nabla_{\dot{\beta}\dot{\rho}} + f_{\dot{\beta}\dot{\rho}}) + C_{\dot{\beta}\dot{\rho}}(\nabla_{\alpha\gamma} + f_{\alpha\gamma}) \\
[\nabla_{\dot{\alpha}\dot{\beta}}, \nabla_{\gamma\dot{\delta}}] &= C_{(\dot{\alpha}\dot{\delta}}\nabla_{\gamma\dot{\beta})}
\end{aligned}
$$

Using Bianchi identity on this algebra relates different field strengths

$$
\begin{aligned}
[\nabla_{\mathrm{ab}}, \phi_{\mathrm{cd}}] &= -\eta_{[\mathrm{a}[\mathrm{c}|}\phi_{\mathrm{b}]|\mathrm{d}]} \\
[\nabla^{\mathrm{c}}_{\dot{\beta}}, \phi_{\mathrm{cd}}] &= 3\overline{W}_{\mathrm{d}\dot{\beta}} \\
\{\nabla^{\mathrm{a}}_{(\alpha}, W_{\mathrm{a}\beta)}\} &= -8f_{\alpha\beta}
\end{aligned}
\tag{5.7}
$$

For N=4 they imply self-duality for the scalar field strength as well as field equations for the other field strength

$$
\begin{aligned}
&\mathrm{Self-duality}\ \bar{\phi}^{\mathrm{ab}} = \frac{1}{2}\epsilon^{\mathrm{abcd}}\phi_{\mathrm{cd}} \\
&[\nabla_{\alpha\dot{\beta}}, \overline{W}_a{}^{\dot{\beta}}] - [\phi_{ab}, W^b{}_\alpha] = 0 \\
&[\nabla_\alpha{}^{\dot{\beta}}, f_{\dot{\gamma}\dot{\beta}}] + \frac{1}{4}[\phi_{\mathrm{ab}}, [\nabla_{\alpha\dot{\gamma}}, \bar{\phi}^{\mathrm{ab}}]] - \{W^{\mathrm{b}}{}_\alpha, \overline{W}_{\mathrm{b}\dot{\gamma}}\} = 0 \\
&[\nabla^{\gamma\dot{\beta}}, [\nabla_{\gamma\dot{\beta}}, \phi_{\mathrm{ab}}]] - 2\{\bar{W}_{\mathrm{a}}{}^{\dot{\beta}}, \bar{W}_{\mathrm{b}\dot{\beta}}\} - \epsilon_{abcd}\{W^{c\gamma}, W_\gamma{}^d\} \\
&-4\phi_{\mathrm{ab}} - [\phi_{\mathrm{bc}}, [\phi_{\mathrm{ad}}, \bar{\phi}^{\mathrm{cd}}]] = 0
\end{aligned}
\tag{5.8}
$$

where the self-duality relationship is determined only up to a phase. Later when we discuss second quantization the goal is then to modify this algebra such that the Bianchi identity no longer implies field equations, therefore it is instructive to see how self-duality and field equations arise. We give a brief sketch of the derivation in the Appendix.

5.7.3 Projective gauge

We now separate the OSp(N|4) algebra into the subsets by labelling the covariant derivatives as either ∇_u or ∇_w. Then the above algebra can be represented as

$$[\nabla_u, \nabla_u\} = \nabla_u + f_{uu},\ [\nabla_u, \nabla_w\} = \nabla_w + f_{uw},\ [\nabla_w, \nabla_w\} = \nabla_u + f_{ww}$$

The field strengths f_{uu}, f_{uw}, f_{ww} are denoted by their position in the algebra. Note that N=4 is the only projective case where the ∇_u algebra has field strengths, they appear as scalar field strengths $\check{\phi}$

$$
\begin{aligned}
[\nabla_{a'b'}, \nabla_{c'\dot{\alpha}}] &= \eta_{a'c'}\nabla_{b'\dot{\alpha}} - \eta_{b'c'}\nabla_{a'\dot{\alpha}} \\
[\nabla_{bc}, \nabla_{a\alpha}] &= \eta_{ab}\nabla_{c\alpha} - \eta_{ac}\nabla_{b\alpha} \\
\{\nabla_{a\alpha}, \nabla_{b\beta}\} &= -C_{\alpha\beta}(\nabla_{ab} + \check{\phi}_{ab}) - \eta_{ab}\nabla_{\alpha\beta} \\
\{\nabla_{a'\dot{\alpha}}, \nabla_{b'\dot{\beta}}\} &= -C_{\dot{\alpha}\dot{\beta}}(\nabla_{a'b'} + \check{\phi}_{a'b'}) - \eta_{a'b'}\nabla_{\dot{\alpha}\dot{\beta}} \\
[\nabla_{\alpha\beta}, \nabla_{b\gamma}] &= -C_{\gamma\alpha}\nabla_{b\beta} - C_{\gamma\beta}\nabla_{b\alpha} \\
[\nabla_{\dot{\alpha}\dot{\beta}}, \nabla_{b'\dot{\gamma}}] &= -C_{\dot{\gamma}\dot{\alpha}}\nabla_{b'\dot{\beta}} - C_{\dot{\gamma}\dot{\beta}}\nabla_{b'\dot{\alpha}} \\
[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] &= -C_{\gamma\alpha}\nabla_{\beta\delta} - C_{\delta\beta}\nabla_{\alpha\gamma} - C_{\delta\alpha}\nabla_{\beta\gamma} - C_{\gamma\beta}\nabla_{\alpha\delta}
\end{aligned}
$$

From the self-duality relation derived previously one has

$$\check{\phi}_{ab} = \frac{1}{2}\epsilon_{ab}\epsilon_{c'd'}\check{\phi}^{c'd'} \rightarrow \check{\phi}_{ab} = C_{ab}\varphi, \;\; \bar{\phi}_{a'b'} = C_{a'b'}\varphi$$

Thus there is only one f_{uu}, furthermore it is projective:

$$
\begin{aligned}
[\nabla_u, \varphi] = 0 \\
\nabla_u = (\nabla_{ab}, \nabla_{a'b'}, \nabla_{a\alpha}, \nabla_{a'\dot{\alpha}}, \nabla_{\alpha\beta}, \nabla_{\dot{\alpha}\dot{\beta}})
\end{aligned}
\tag{5.9}
$$

which just arise from the Bianchi identity. φ can then be absorbed by (the gauge fields of) the SO(2) derivatives :$\nabla'_{a'b'} = \nabla_{a'b'} + \varphi C_{a'b'}$ and $\nabla'_{ab} = \nabla_{ab} + \varphi C_{ab}$. (A similar procedure works for the N=2 chiral case, but not for N=4 chiral.) The new set of isotropy covariant derivatives closes without field strength due to 5.9, and thus it is now consistent to impose them as a constraint

$$\nabla_{\alpha\beta} = \nabla_{\dot{\alpha}\dot{\beta}} = \nabla_{a\alpha} = \nabla_{a'\dot{\alpha}} = \nabla'_{ab} = \nabla'_{a'b'} = 0$$

In particular, we can choose the gauge $d_u = 0$ (i.e., the above minus $d_u = 0$). In this gauge, there is a residual gauge invariance with $d_u\lambda = 0$; i.e., the gauge parameter is projective. At that point we can work exclusively in terms of ∇_w.

Some interesting features of this required modification are: (1) It involves the SO(n)SO(N−n) isotropy derivatives, this is needed to absorb the projective field strength, and hence requires the super anti-de Sitter construction. (The analogous derivatives in flat superspace would be central charges, which would break superconformal invariance. However, we can still use our contracted coset, since the isotropy group is unchanged.) (2) The modifications must involve only a single field strength to avoid generation of field-strength commutator terms (and hence nonclosure) in the algebra of isotropy constraints, and hence both n and N−n $\leqslant 2$. This shows that chiral superspace does not exist for N=4 Yang-Mills.

5.8 Projective amplitude

5.8.1 Duality

The four-point amplitude in this theory has been shown to have a simple form in projective superspace, where coordinate/momentum duality is almost manifest [71]. This duality is the one that results from (super) Fourier transformation, whereupon coordinates (of vertices) are replaced with loop momenta, after applying momentum conservation. (External line momenta are also expressed as differences, by interpreting paths connecting adjacent external lines as "half-loops", with their own momenta.) Thus graphs are replaced with (geometrically) dual graphs [78]. In string theory, introducing a (random) lattice for the worldsheet, this is recognized as T-duality [79]. The $AdS_5 \otimes S^5$ string has been shown to have invariance under such a T-duality [5], implying that N=4 super Yang-Mills has another PSU(4|2,2) symmetry that includes the usual Lorentz and R-symmetry, but also "translation" invariance in the loop supermomenta (of a projective or chiral superspace), and their completion to a full dual superconformal group.

A proposal for this dual superconformal invariance of the theory had already been made directly on the N=4 Yang-Mills amplitudes [4]; however, it requires the inclusion of twistor coordinates with both the coordinate and momentum spaces, and is thus not a complete duality. The reason why the twistors were found necessary is that this formulation is based on chiral superspace, which is simplest for MHV amplitudes. In that space the chiral field strengths are the selfdual parts of the (superfield which at $\theta = 0$ is the) Yang-Mills field strength, which carries Lorentz indices. They thus use twistors to carry these indices. An alternative would be to introduce spin coordinates; but these do not naturally appear in chiral superspace (at least according to our projective construction). The scalar factor of the amplitude (the purely spacetime-momentum factor) would then acquire additional denominator factors of momenta to cancel those introduced into the chiral external line factors, since the Yang-Mills field strength has higher dimension than the scalars. At least effective actions would be expected to be more complicated in this approach, since the chirality of this field strength holds only at the linearized level.

5.8.2 From chiral to projective

Although we do not yet give the Feynman rules, we present an alternate derivation of this amplitude in projective superspace from chiral supertwistor space, which could be generalized to known higher-point amplitudes. We do this not to illustrate the method, which can be complicated in general (especially if we include the effort required to derive the chiral supertwistor expressions with which we begin), but to show the simplicity of the projective superspace result. The method is to transform the chiral supertwistor into projective supertwistor space by Fourier transforming half the fermionic twistor coordinates; the result can then be put into projective supercoordinate space by the usual (projective super) Penrose transform. The result can already be guessed by noting that the four-point amplitude is both MHV and anti-MHV: For the tree case, the chiral and antichiral supertwistor expressions are

$$\mathcal{A}_{4\chi} = \frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^8(\sum \pi_{\mathrm{a}\alpha})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \quad \mathcal{A}_{4\bar{\chi}} = \frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^8(\sum \bar{\pi}^{\mathrm{a}}{}_{\dot{\alpha}})}{[12][23][34][41]}$$

(The sums are over external lines.) Thus we'll find that the ubiquitous twistor denominator of MHV, and its complex conjugate of anti-MHV, are replaced in projective supertwistor space by their magnitude, which is directly expressible in terms of momenta (e.g., st for the tree case).

We use the notation $ijkl$ to label the 4 distinct external lines. Then the only twistor identity we need is the equality of the MHV and anti-MHV expressions for the pure-gluon amplitude:

$$\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \frac{[kl]^4}{[12][23][34][41]}$$

This allows us to evaluate the fermionic Fourier transform with respect to any *one* of the 4 twistor fermions (with respect to N=4, but all *four* of the external lines):

$$\int d^4\bar{\zeta} e^{i\bar{\zeta}_i\zeta_i}\delta^2(\sum \lambda_i\zeta_i) = \sum \langle ij\rangle \zeta_k\zeta_l = \delta^2(\sum \bar{\lambda}_i\zeta_i)\left(\frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}{[12][23][34][41]}\right)^{\frac{1}{4}}$$

with Einstein summation understood on identical indices. Thus this Fourier transformation replaces the conservation δ-function for total $\pi_\alpha = \lambda_\alpha\bar{\zeta}$ with one for the corresponding $\bar{\pi}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}}\zeta$, and throws in a phase factor. In addition to reproducing the correct relation between the above forms of the amplitude in chiral and antichiral supertwistor space, it gives the intermediate result for projective supertwistor space:

$$\mathcal{A}_{4\Pi} = \frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^4(\sum \pi_{\dot{\alpha}\mathrm{a}'})\delta^4(\sum \bar{\pi}^{\mathrm{a}}{}_{\dot{\alpha}})}{\frac{1}{4}st}$$

Note that this amplitude is missing an explicit δ-function for conservation of $t_{aa'}$ (which would actually be a Kronecker δ, because of the compactness of the R-space): This conservation is implied by the other δ-functions (in twistor superspace, or on shell).

In this form, the amplitude is already expressed directly in momentum superspace; we need only attach external line factors, which are just the (linearized) projective superfield strengths φ:

$$\hat{\mathcal{A}}_{4\Pi} = \int d^{16}p_i d^{32}\pi_i d^{16}t_i \tilde{\varphi}(1)\tilde{\varphi}(2)\tilde{\varphi}(3)\tilde{\varphi}(4)\frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^4(\sum \pi_{\dot{\alpha}\mathrm{a}'})\delta^4(\sum \bar{\pi}^{\mathrm{a}}{}_{\dot{\alpha}})}{\frac{1}{4}st}$$

(The $\int d^{16}t_i$ should really be a sum. Of course, the t_i conjugate to y_i should not be confused with the Mandelstam variable t.) For comparison, in the chiral case, we need to multiply numerator and denominator by [12][23][34][41] to put the amplitude into momentum space: The denominator becomes $(st)^2$, while for the numerator factor, including external line factors, we have

$$\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \frac{[12][23][34][41]}{(\frac{1}{4}st)^2} \to \frac{\bar{f}_{\dot{\alpha}}{}^{\dot{\beta}}\bar{f}_{\dot{\beta}}{}^{\dot{\gamma}}\bar{f}_{\dot{\gamma}}{}^{\dot{\delta}}\bar{f}_{\dot{\delta}}{}^{\dot{\alpha}}}{(\frac{1}{4}st)^2}$$

$$\to \mathcal{A}_{4\chi} = \int d^{16}p_i d^{32}\pi_i \bar{f}_{\dot{\alpha}}{}^{\dot{\beta}}\bar{f}_{\dot{\beta}}{}^{\dot{\gamma}}\bar{f}_{\dot{\gamma}}{}^{\dot{\delta}}\bar{f}_{\dot{\delta}}{}^{\dot{\alpha}}\frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^8(\sum \pi_{\mathrm{a}\alpha})}{(\frac{1}{4}st)^2}$$

Note that there is no direct analog for the chiral supertwistor scalar wave function in momentum (or coordinate) superspace, unlike the projective case: It is a (super)helicity amplitude, and does not directly covariantize (except by mutliplying by twistors to get tensors, as above). This is a consequence of the fact that the only scalar superfield strength is projective, not chiral (even in the linear approximation).

We can easily Fourier transform the projective amplitude back to coordinate superspace: The x dependence is as usual, the dependence is the local product, and the y dependence evaluates at $y = 0$:

$$\hat{\mathcal{A}}_{4\Pi} = \int d^{16}x_i d^8\theta \varphi(x_1, \theta, 0)\varphi(x_2, \theta, 0)\varphi(x_3, \theta, 0)\varphi(x_4, \theta, 0)\frac{\delta^4(x_1 - x_2 + x_3 - x_4)}{x_{12}^2 x_{23}^2}$$

5.8.3 From 6 dimensions to 4 dimensions

Alternatively the same amplitude can be derived from the on-shell N=2 SYM amplitudes in 6-dimensions since on-shell 6 dimensions is similar to off-shell in 4 dimensions. The relevant spinor-helicity formalism for six dimensions is given in the appendix. It is based on representing an on-shell momentum $p^2 = 0$ in terms of SU(4) twistors

$$p^2 = 0 \rightarrow p^\mu = p^{AB} = Z^{Aa} Z^B{}_a \ , \ p_{AB} = \frac{1}{2}\epsilon_{ABCD} p^{CD} = \bar{Z}_A{}^{\dot{a}} \bar{Z}_{B\dot{a}}$$

where $A \in SU(4)$ and $a, \dot{a} \in SU(2)$. One then includes the on-shell N=2 spinor coordinates as q_{1a} and $q_{2\dot{a}}$ and $1, 2$ labels the N=2. One can then map them to the projective ones as:

$$\begin{aligned} \check{q}^A = \check{q}_1^A &= Z^{Aa} q_{1a} \\ \check{\bar{q}}_A = \check{q}_{2A} &= \bar{Z}_A{}^{\dot{a}} q_{2\dot{a}} \end{aligned} \tag{5.10}$$

The 4 point amplitude in terms of these spinor-helicity in six dimension reads[80]

$$\frac{\epsilon^{ABCD}\epsilon_{EFGH} F^E{}_A(1) F^F{}_B(2) F^G{}_C(3) F^H{}_D(4)}{st}\delta(\sum p) \tag{5.11}$$

this can be seen as derived from (one term in the expansion)

$$\frac{\delta(\sum p)\delta^4(\sum \check{q})\delta^4(\sum \check{\check{q}})}{st}\phi(1)\phi(2)\phi(3)\phi(4) \tag{5.12}$$

which is exactly the result we derived previously. One can then look at all the higher point amplitudes and find their corresponding projective ones. The higher point amplitude was derived in [80] using BCFW recursion relation [18] for 6 dimension tree amplitude. For example for 5 point

$$\begin{aligned} &\frac{\delta^6(p)\delta^4(\check{q})\delta^4(\check{\check{q}})}{s_{12}s_{23}s_{34}s_{45}s_{51}}[\check{q}^M\check{\check{q}}_N(p_2\cdot p_3\cdot p_4\cdot p_5)_M{}^N + \text{ cyclic} \\ &+((p_2\cdot p_3\cdot p_4\cdot p_5)-(p_5\cdot p_4\cdot p_3\cdot p_2))\,\check{q}^M\check{\check{q}}_M + \text{ cyclic}] \end{aligned} \tag{5.13}$$

5.9 Second-quantization

Here we give some tentative discussion for possible second quantized construction. Since the algebra of the covariant derivatives leads to field equations for the field strength, an off-shell formulation would require a deformed version of this algebra. Similar situation was discussed for N=3 [65] where one introduces new few strengths for the internal symmetry direction, the action then simply gives vanishing field strength in these directions which gives back the on-shell theory.

For the N=3 case, extra field field strengths were introduced to the covariant derivatives of some of the coset coordinates. Using the coset $SU(3)/U(1)\times U(1)$ one introduces nonvanishing field strength

$$[\nabla_{-1,2},\nabla_{1,1}]=F_{0,3},\ [\nabla_{2,-1},\nabla_{1,1}]=F_{3,0},\ [\nabla_{-1,2},\nabla_{2,-1}]=F_{1,1}$$

The subscripts label the charge of the harmonic variables (R coordinates) with respect to the two U(1)s. The vanishing of these three field strengths then leads to the original on-shell algebra, thus the field equation can now be translated to $F_{0,3}=F_{3,0}=F_{1,1}=0$. This leads to the following action

$$S\sim\int A_{-1,2}F_{3,0}+A_{2,-1}F_{0,3}+A_{1,1}F_{1,1}+\cdot\cdot$$

So the connection now becomes Lagrange multipliers and gives the desired on-shell degree of freedom.

Note that one should look for deformation which can be removed by setting the R field strengths(such as the Fs above) to zero. This is because the projective(analytic for harmonic approach) measure has engineering dimension zero, and thus the action can only be built out of dimensionless field strengths.

Here we try to find the minimum set of new field strengths that puts the theory off-shell. Since one would still like to be able to consistently impose analytic condition $\nabla_\vartheta f = 0$,[6] we do not introduce field strengths in the ∇_u (isotropy group) algebra. We list the relevant part of the algebra under discussion.[7]

$$1.[\nabla_u, \nabla_w\} = \nabla_w + F_{wu}$$

- $[\nabla_{ab}, \nabla_{cd'}] = -\eta_{ac}(\nabla_{bd'} + K_{bd'}) + \eta_{bc}(\nabla_{ad'} + K_{ad'})$
- $[\nabla_{c\dot\beta}, \nabla_{ab}] = \eta_{ac}(\nabla_{b\dot\beta} + W_{b\dot\beta})$
- $[\nabla_{a\alpha}, \nabla_{cb'}] = \eta_{ac}(\nabla_{b'\alpha} + M_{b'\alpha})$
- $\{\nabla_{a\alpha}, \nabla_{b\dot\beta}\} = -\eta_{ab}\nabla_{\alpha\dot\beta}$
- $\{\nabla_{a\alpha}, \nabla_{b'\beta}\} = -C_{\alpha\beta}(\nabla_{ab'} + \phi_{ab'})$
- $\{\nabla_{a'\dot\alpha}, \nabla_{b\dot\beta}\} = -C_{\dot\alpha\dot\beta}(\nabla_{a'b} + \bar\phi_{a'b})$
- $[\nabla_{\alpha\beta}, \nabla_{b'\gamma}] = -C_{\gamma\alpha}\nabla_{b'\beta} - C_{\gamma\beta}\nabla_{b'\alpha}$
- $[\nabla_{a\alpha}, \nabla_{\gamma\dot\beta}] = C_{\alpha\gamma}(\nabla_{a\dot\beta} + \overline{W}_{a\dot\beta})$
- $[\nabla_{a'\dot\alpha}, \nabla_{\beta\dot\gamma}] = C_{\dot\alpha\dot\gamma}(\nabla_{a'\beta} + W_{a'\beta})$
- $[\nabla_{\alpha\beta}, \nabla_{\gamma\dot\delta}] = -C_{\gamma\alpha}\nabla_{\beta\dot\delta} - C_{\gamma\beta}\nabla_{\alpha\dot\delta}$

$$2.[\nabla_w, \nabla_w\} = \nabla_u + F_{ww}$$

- $[\nabla_{ab'}, \nabla_{cd'}] = -\eta_{ac}(\nabla_{b'd'} - \check\phi_{b'd'}) - \eta_{b'd'}(\nabla_{ac} - \check\phi_{ac})$

[6] We label the fermionic derivatives in the isotropy group as $\nabla_\vartheta = (\nabla_{a\alpha}, \nabla_{b'\dot\beta})$.

[7] This algebra is the shifted one thus there will be no field strength for $[\nabla_u, \nabla_u\}$, the original projective field strength now appears in other places.

- $[\nabla_{a\dot{\alpha}}, \nabla_{bc'}] = \eta_{ab}(\nabla_{c'\dot{\alpha}} + \bar{G}_{c'\dot{\alpha}})$
- $\{\nabla_{\gamma d'}, \nabla_{\alpha c'}\} = -C_{\gamma\alpha}(\nabla_{d'c'} + \varphi_{d'c'}) - \eta_{c'd'}\nabla_{\gamma\alpha}$
- $\{\nabla_{a\dot{\alpha}}, \nabla_{b\dot{\beta}}\} = -C_{\dot{\alpha}\dot{\beta}}(\nabla_{ab} + \varphi_{ab}) - \eta_{ab}\nabla_{\dot{\alpha}\dot{\beta}}$
- $[\nabla_{c\dot{\alpha}}, \nabla_{\gamma\dot{\beta}}] = C_{\dot{\alpha}\dot{\beta}}(\nabla_{c\gamma} + W_{c\gamma})$
- $[\nabla_{a'\alpha}, \nabla_{\gamma\dot{\beta}}] = C_{\alpha\gamma}(\nabla_{a'\dot{\beta}} + \overline{W}_{a'\dot{\beta}})$
- $[\nabla_{\alpha\dot{\beta}}, \nabla_{\gamma\dot{\delta}}] = C_{\alpha\gamma}(\nabla_{\dot{\delta}\dot{\beta}} + f_{\dot{\delta}\dot{\beta}}) + C_{\dot{\beta}\dot{\delta}}(\nabla_{\alpha\gamma} + f_{\alpha\gamma})$

where φ_{ab} is the linear combination $\phi_{ab} - \check{\phi}_{ab}$. Since the shifted algebra already contains field strengths in the R direction, the only modify the relationship of the R field strengths. In the new algebra $K_{ab'}$ would be identified with $\phi_{ab'}$ for the on-shell algebra. This disassociation leads to the additional spinor field strengths $G_{a\alpha}$ and $M_{a'\alpha}$ since the Bianchi identity relates them:

$$\begin{aligned} K_{ab'} - \phi_{ab'} &= \frac{1}{2}\{\nabla_a{}^{\alpha}, M_{b'\alpha}\} \\ G_{a'\dot{\alpha}} &= [\nabla^{b'}{}_{\dot{\alpha}}, \check{\phi}_{b'a'}] - \frac{1}{2}[\nabla_{a'}{}^{a}, M_{a\dot{\alpha}}] \end{aligned}$$

In this language the on-shell equation becomes

$$K_{ab'} = \phi_{ab'}$$

and one can anticipate an action

$$S \sim \int A^{ab'}(K_{ab'} - \phi_{ab'}) + \cdot\cdot$$

However due to the curved nature of the coset, the projective prepotential can not be the connections since $[\nabla_u, \nabla_w\} \neq 0$.

5.10 Conclusion

These results can be generalized to other N: For example, the simpler case of N=2 would be useful to compare with the known harmonic and projective formalisms. Since in general such first-quantization describes superspin 0, the "smallest" supermultiplet (unless additional spin variables are included), N=2 would describe a scalar multiplet, which could also be coupled to external Yang-Mills. The R-space in that case is SO(2), corresponding to the identification of the usual projective R-space with the unit circle. Similarly, N=8 would describe (gauged) supergravity. All cases N8 could be coupled to external (gauged) supergravity; the formalism suggests that the tangent space for this supergravity would be OSp(n|2)OSp(N−n|2), rather than the purely bosonic (Lorentz and R-symmetry) tangent spaces that have been used so far.

These results can also be generalized to some other dimensions; we have at least

$$\begin{aligned}
D = 5 \quad &: \quad \frac{\mathrm{(P)SU(N|2,2)}}{\mathrm{OSp(N|4)}} \\
D = 4 \quad &: \quad \frac{\mathrm{OSp(N|4)}}{\mathrm{OSp(n|2)OSp(N-n|2)}} \\
D = 3 \quad &: \quad \frac{\mathrm{OSp(\tfrac{1}{2}N|2)^2}}{\mathrm{OSp(\tfrac{1}{2}N|2)}} \\
D = 2 \quad &: \quad \frac{\mathrm{OSp(N|2)}}{\mathrm{U(\tfrac{1}{2}N|1)}}
\end{aligned} \tag{5.14}$$

The method for solving the constraints is similar, and correctly produces (at least the free) supersymmetric theories in those dimensions.

Another problem is whether superconformal invariance can be made manifest. A better understanding of projective lightcone limits might do that. This might also shed some light on the relationship between the harmonic and projective approaches: For example, for the N=2 case, we see the R-symmetry part of the coset space in D=5 is SU(2)/SO(2), as in the usual N=2 harmonic superspace, while the coset space in D=4 is just SO(2), so the sphere reduces to the circle.

In principle, superconformal invariance could be made manifest by using an action with the full superconformal set of constraints:

$$d_{(A}{}^{(A'} d_{B]}{}^{B']} = 0$$

However, this set is highly reducible: In particular, it reduces to our anti-de

Sitter ones

$$d_A{}^{A'} d_B{}^{B'} \eta_{B'A'}, \quad d_A{}^{A'} d_B{}^{B'} \eta^{BA}$$

(with or without contraction), and the ghosts are much messier [81]. Alternatively, one could consider the flat-space limit (R→ ∞) of the anti-de Sitter constraints,

$$d_A{}^{\dot{\alpha}} d_B{}^{\dot{\beta}} \bar{C}_{\dot{\beta}\dot{\alpha}}, \quad d_\alpha{}^{A'} d_\beta{}^{B'} C^{\beta\alpha}$$

but this loses some necessary constraints: $t_{aa'}$ drops out altogether. These different sets of constraints can be considered as related to partial gauge fixing of the corresponding Lagrange multipliers.

It may be possible to find at least some of the superconformal invariance through transformations of the Lagrange multipliers. For example, the usual action for a scalar particle, $\int g\dot{x}^2$, is conformal through transformation of g, and an (A)dS metric can be obtained by redefining g by the appropriate Weyl scale factor.

An obvious topic is the gauge-invariant field theory action for N=4 Yang-Mills, and its second-quantization. It should be noted that the supergraph rules will not be manifestly superconformal: The second-quantized gauge-fixing term for Yang-Mills breaks conformal invariance, and first-quantization requires gauge-fixing the worldline metric, which also breaks conformal invariance. However, it should be possible to preserve some useful affine subalgebra.

Alternatively, by finishing the treatment of first-quantization in an external N=4 super Yang-Mills background, it should be possible to define vertex operators that allow supergraph calculations directly in a first-quantized approach, in analogy to string theory. It may then be possible to reproduce many of the results of the gauge/string correspondence without requiring the full string machinery. For example, properties such as N=4 superconformal symmetry, or its "dual", may be sufficient.

Unique to the case N=4, the numbers of commuting and anticommuting coordinates cancel (at each ghost level). This suggests that potential zero-mode problems (and their resulting picture-changing or equivalent vacuum problems) could be directly canceled, after an appropriate (worldline-infrared) regularization.

Even if these first-quantized methods prove useful for deriving expressions for S-matrices, a more important question is whether it can be helpful in calculating anything relevant to confinement. In this regard, a random lattice approach to the string would suggest that this first-quantized action for the N=4 superparticle might lead to a first-quantized action for a 4D N=4 superstring.

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Appendix A

Appendix: Spinor-helicity formalism

A.1 4-dimensions

In 4-dimensions a four vector can be written in two component spinors as a 2×2 matrix $p_{\alpha\dot{\beta}} = p^{\mu}\sigma_{\alpha\dot{\beta}}$. Then an on-shell massless momentum

$$p^{\mu}p_{\mu} = det(p_{\alpha\dot{\alpha}}) = 0 \rightarrow p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}$$

where λ is a bosonic spinor. For SO(3,1) $\tilde{\lambda}_{\dot{\alpha}} = \pm\bar{\lambda}_{\dot{\alpha}}$ and for SO(2,2) λ_{α} and $\tilde{\lambda}_{\dot{\alpha}}$ are real and independent. These bosonic spinors are solutions to the Dirac equation

$$p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}} = \lambda_{\alpha}\langle\tilde{\lambda}\tilde{\lambda}\rangle = 0$$

where we've used the notation

$$\langle\lambda_i\lambda_j\rangle = \lambda_i{}^{\alpha}\lambda_{j\alpha} = -\langle\lambda_j\lambda_i\rangle$$

$$[\tilde{\lambda}_i\tilde{\lambda}_j] = \tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_{j\dot{\alpha}} = -[\tilde{\lambda}_j\tilde{\lambda}_i]$$

$$p_{\alpha\dot{\alpha}} = {}_{\alpha}|p\rangle[p|_{\dot{\alpha}}$$

The indices are raised and lowered by SL(2,C) (the covering group of SO(3,1)) metric $C_{\alpha\beta} = \bar{C}_{\dot{\alpha}\dot{\beta}} = -C^{\alpha\beta} = -\bar{C}^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The final minus sign for both lines comes from when i, j spinors are exchanged[1] we need to raise and lower the SL(2,C) indices so that they are contracted northwest-sountheast direction.

The polarization vector for a gauge vector can be represented by an arbitrary reference null vector $v_{\alpha\dot{\alpha}} = \mu_\alpha \tilde{\mu}_{\dot{\alpha}}$:

$$\begin{aligned} \epsilon_\mu = \epsilon_{\alpha\dot{\alpha}} &= \frac{\lambda_\alpha \tilde{\mu}_{\dot{\alpha}}}{[\tilde{\lambda}\tilde{\mu}]} \\ \bar{\epsilon}_\mu = \bar{\epsilon}_{\alpha\dot{\alpha}} &= \frac{\mu_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle \mu\lambda \rangle} \end{aligned} \tag{A.1}$$

These polarization vectors obviously satisfy $p \cdot \epsilon = p \cdot \bar{\epsilon} = \epsilon \cdot \bar{\epsilon} = 0$. Changing the choice of reference vector corresponds to a gauge transformation: An arbitrary change of v can be written as [2]

$$\tilde{\mu} \to \tilde{\mu} + \alpha\tilde{\mu} + \beta\tilde{\lambda}$$

One can see that α corresponds to a rescaling of ϵ_μ, while under β

$$\epsilon_\mu \to \epsilon_\mu + \beta \frac{p_{\alpha\dot{\alpha}}}{[\tilde{\lambda}\tilde{\mu}]}$$

which is just a gauge transformation.

The little group which in 4 dimensions is called the helicity is a U(1) phase for SO(3,1), while for SO(2,2) it is a rescaling by R. Then $(\tilde{\lambda}_{\dot{\alpha}})\lambda_\alpha$ has helicity $-\frac{1}{2}(+\frac{1}{2})$. Thus we see that $\epsilon_\mu(\bar{\epsilon}_\mu)$ has helicity $-1(+1)$. Plugging ϵ_μ into $F_{\mu\nu}$

$$F_{\mu\nu} = C_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}} + C_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} \begin{cases} p_m\epsilon_n - p_n\epsilon_m = C_{\dot{\alpha}\dot{\beta}}\lambda_\alpha\lambda_\beta \to f_{\alpha\beta} \\ p_m\bar{\epsilon}_n - p_n\bar{\epsilon}_m = C_{\alpha\beta}\tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}} \to \bar{f}_{\dot{\alpha}\dot{\beta}} \end{cases} \tag{A.2}$$

Thus $f_{\alpha\beta}(\bar{f}_{\dot{\alpha}\dot{\beta}})$ corresponds to helicity $-1(+1)$. We summarize the Feynman rules in spinor helicity

[1] These are bosonic spinors, so no extra minus signs when one simply exchange them. The minus sign comes from maintaining the northwest-southeast contraction in the definition of $\langle i, j\rangle$ and $[i, j]$.

[2] The change in v cannot be arbitrary, however since the new v' still has to satisfy $v'^2 = 0$, this change can be understood as $v' = \mu\tilde{\mu}'$ so that v' remains a null vector. The space of $\tilde{\mu}$ is two dimensional, so the new $\tilde{\mu}'$ can be written on two bases, $\tilde{\mu}$ and $\tilde{\lambda}$.

A.1.1 External line factors

In defining the helicities, we consider all partical to be outgoing.

$$\begin{aligned} \text{Scalar } \phi &: \quad 1 \\ \text{Spinors } \psi^+(\bar{\psi}^-) &: \quad \langle p|_\alpha = \lambda_\alpha \\ \text{Spinors } \psi^-(\bar{\psi}^+) &: \quad [p|_{\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \\ \text{Vector } A_\mu^+ &= \frac{\mu_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle \mu\lambda\rangle} \\ \text{Vector } A_\mu^- &= \frac{\lambda_\alpha \tilde{\mu}_{\dot{\alpha}}}{[\tilde{\lambda}\tilde{\mu}]} \end{aligned} \tag{A.3}$$

A.1.2 Propagators

$$\begin{aligned} \text{Scalar } \phi &: \quad \frac{1}{p^2} \\ \text{Spinors } \psi &: \quad \frac{\not{p}}{p^2} = \frac{\sum_I |I]\langle I|}{p^2} \\ \text{Vector } A_\mu &: \quad \frac{C_{\alpha\beta} C_{\dot{\alpha}\dot{\beta}}}{p^2} \end{aligned} \tag{A.4}$$

where the sum $\sum_I$ sums over momentum of the external lines on ones side of the propagator. In general $\not{V} \equiv \begin{pmatrix} 0 & V_\alpha{}^{\dot{\beta}} \\ V^\beta{}_{\dot{\alpha}} & 0 \end{pmatrix}$. One can check easily $\{\not{V}, \not{W}\} = -V \cdot W$.

A.2 ambi-twistor

Extending the usual twistor coordinates to include their conjugate gives the ambi-twistor approach which transforms nicely under conformal group SU(2,2)

$$[Z_A, W^B] = \delta_A{}^B$$

with

$$Z_A = (\mu_\alpha, \tilde{\lambda}_{\dot{\alpha}})\,,\ W^A = (\lambda^\alpha, \tilde{\mu}^{\dot{\alpha}})$$

Using these variables one can represent the conformal generators as

$$G_A{}^B = W^B Z_A - \frac{1}{4}\delta_A^B W^C Z_C$$

Extending to N=4 one includes Grassmann variables η_I and $\bar{\eta}^I$, where $I \in SU(4)$

$$super: \quad Z_A = (\mu_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \eta_I)\,, \; W^A = (\lambda^\alpha, \tilde{\mu}^{\dot{\alpha}}, \bar{\eta}^I)$$

A.3 6-dimensions

Massless momentum in 6 dimensions can also be written in terms of bosonic spinors of the covering group of SO(6), namely SU*(4)

$$\begin{aligned} &p^2 = 0 \to p^\mu = p^{AB} = Z^{Aa} Z^B{}_a\,,\; p_{AB} = \frac{1}{2}\epsilon_{ABCD} p^{CD} = \bar{Z}_A{}^{\dot{a}} \bar{Z}_{B\dot{a}} \\ &\text{then } p^2 = \frac{1}{4}\epsilon_{ABCD} p^{AB} p^{CD} = \frac{1}{4}\epsilon_{ABCD} Z^{Aa} Z^B{}_a Z^{Cb} Z^D{}_b = 0 \end{aligned} \tag{A.5}$$

where $A \in$SU*(4) and $a, \dot{a} \in$ SU(2) which corresponds to the little group for 6 dimension, SO*(4)=SU(2)$\otimes$SU(2). For polarization vector one again find a null reference vector $\eta^{AB} = W^{Aa} W^B{}_a$

$$\epsilon^\mu_{a\dot{a}} = \epsilon^{AB}_{a\dot{a}} \equiv \frac{Z^A{}_a W^B{}_c}{W^{Cc} \bar{Z}_C{}^{\dot{a}}} \tag{A.6}$$

Note that $\epsilon^\mu{}_{a\dot{a}} \epsilon_{\mu b\dot{b}} = C_{ab} C_{\dot{a}\dot{b}}$.

$F_{\mu\nu} = p_\mu \epsilon_\nu - p_\nu \epsilon_\mu$ can now be written in terms of spinors by substituting ϵ given above:

$$\begin{aligned} F_{\mu\nu} &= (F_{AB,CD})_{a\dot{a}} \\ &\sim (\epsilon_{ABCE} Z^E{}_a \bar{Z}_{D\dot{a}} + \epsilon_{DBCE} Z^E{}_a \bar{Z}_{A\dot{a}} - \epsilon_{ABDE} Z^E{}_a \bar{Z}_{C\dot{a}} - \epsilon_{DACE} Z^E{}_a \bar{Z}_{B\dot{a}}) \end{aligned} \tag{A.7}$$

In terms of spinor space the field strength has the following irreducible pieces

$$\begin{aligned} F^{AB}{}_{CD} = \epsilon^{ABMN} F_{MNCD} = 0 \\ F^M{}_D = \epsilon^{MABC} F_{ABCD} = Z^M{}_a \bar{Z}_{D\dot{a}} \end{aligned} \tag{A.8}$$

Appendix B

Appendix: Spinors in d dimensions

Here we discuss spinors for various dimensions by building up the representation for the Clifford algebra. We follow closely the discussion in Polchinski [82]. We begin with the Clifford algebra,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{B.1}$$

For our discussion we are interested in Minkowski space, and the Clifford algebra gives the spinor representation of SO$(d-1,1)$. We can build up the representation by using creation and annihilation operators, consider for even dimensions[1]

$$\begin{aligned} \alpha^{0\dagger} &= \frac{1}{2}(\Gamma^0 + \Gamma^1)\,, \qquad \alpha^0 = \frac{1}{2}(-\Gamma^0 + \Gamma^1) \\ \alpha^{i\dagger} &= \frac{1}{2}(\Gamma^{2i} + i\Gamma^{2i+1})\,, \alpha^i = \frac{1}{2}(\Gamma^{2i} - i\Gamma^{2i+1}) \;\; i = 1, 2 \cdot \cdot k \end{aligned} \tag{B.2}$$

where $d = 2k+2$. Note that in this construction Γ^0 will be anti-hermitian while all others will be hermitian. One can easily see they satisfy $\{\alpha^I, \alpha^{K\dagger}\} = \delta^{IK}$ and $\{\alpha^I, \alpha^J\} = \{\alpha^{I\dagger}, \alpha^{J\dagger}\} = 0$, where $J = \{0, j\}$. We can define a spinor such that $\Gamma^I\psi = 0$. Since $\Gamma^{j2} = 0$ the dimension of states is $2^{d/2}$. On this spinor, creation and annihilation operators have simple form and we can use them

[1] We only consider even dimensions, odd dimensions can be incorporated by simply identifying $\Gamma^{d-1} = \Gamma$ where $\Gamma \equiv (i)^{\frac{d-3}{2}}\Gamma^0\Gamma^1 \cdot \cdot \Gamma^{d-2}$, the chirality matrix for the lower even dimension.

to construct the gamma matrices. For example for d=2 k=1, we have on the spinor $\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have

$$\alpha^{0\dagger} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ \alpha^0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow \Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Iteratively one can construct higher dimension gamma matrices with lower dimension ones using

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mu = 0, 1 \cdots d-3 \tag{B.3}$$
$$\Gamma^{d-2} = I \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \Gamma^{d-1} = I \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

This is the Dirac representation. Note that in this basis, Γ^3, Γ^5, Γ^{2n+1} with $n = 1, 2..$ are purely imaginary while all other are real.

The chirality matrix is defined as $\Gamma \equiv (i)^{\frac{d-3}{2}} \Gamma^0 \Gamma^1 \cdot \cdot \Gamma^{d-1}$, and has the property that $\{\Gamma, \Gamma^I\} = 0$, $(\Gamma)^2 = 1$ and most importantly

$$[\Gamma, \Sigma^{\mu\nu}] = 0 \ \ \Sigma^{\mu\nu} = -\frac{1}{4}[\Gamma^\mu, \Gamma^\nu] \tag{B.4}$$

where $\Sigma^{\mu\nu}$ is the (Lorentz) generators for SO$(D-1, 1)$. This implies that in suitable basis where Γ is diagonalized, the spinors are eigenstates of Γ half of them with eigenvalue $+1$ the other half -1, and each sign transform within itself under Lorentz transformation. Thus the Dirac representation is reducible with respect to the Lorentz group. To see that the spinors splits exactly in half with respect to Γ, note that each state built from ψ can be labeled by whether it is an excited state with respect α^I. Consider the operator:

$$\hat{s}^I = \alpha^{I\dagger}\alpha^I - \frac{1}{2} \ , \ \begin{cases} |+\rangle = \alpha^{I\dagger}\psi \ , & \hat{s}^I|+\rangle = \frac{1}{2}|+\rangle \\ |-\rangle = \psi \ , & \hat{s}^I|-\rangle = -\frac{1}{2}|-\rangle \end{cases} \tag{B.5}$$

one can then label each state by a vector $\vec{S} = \{s^1, s^2, \cdots s^{\frac{d}{2}}\}$, where s^I are the eigenvalue of $\hat{s}^I$. The chirality matrix can be written as $\Gamma = 2^{d/2}\hat{s}^0\hat{s}^1\hat{s}^2 \cdots \hat{s}^{\frac{d}{2}}$, thus one can see half of the states have even number of $-\frac{1}{2}$ eigenvalue s^I are $+1$

when acted on by Γ, while the other half are -1. Thus the Dirac representation can be written as two inequivalent **Weyl** representation. One can project the spinor into two independent pieces by the projection operator $P_{\pm} = \frac{1}{2}(1 \pm \Gamma)$, a Weyl spinor satisfies $\lambda_{\pm} = P_{\pm}\lambda_{\pm}$, so there are $2^{d/2-1}$ complex components for a Weyl spinor, half of a Dirac spinor. Note that in odd dimensions Γ^{d-1} is the chirality matrix of the lower even dimension, thus one cannot define chirality matrix in odd dimensions.

Reality conditions: Another way of reducing the number of components of a spinor is to impose reality conditions. One has to be careful because the condition must be imposed in such a way that it is compatible with the Lorentz transformations. Since the Dirac representation is an irreducible representation, the fact that Γ^{μ} and $\Gamma^{\mu *}$ satisfy the same Clifford algebra implies that they must be related through a similarity transformation. We state that

$$B_1 \Gamma^{\mu} B_1^{-1} = (-1)^k \Gamma^{\mu *} \quad B_1 = \Gamma^3 \Gamma^5 \cdot \cdot \Gamma^{2n+1} \;\; n = 1, 2 \cdot \cdot k \tag{B.6}$$

This is true since when $\Gamma^{\mu} \neq \Gamma^{2n+1}$ they are real, $\Gamma^{\mu} = \Gamma^{\mu *}$, and B_1^{-1} just passes through Γ^{μ} to cancel B_1 with a factor of $(-1)^k$. On the other hand when $\Gamma^{\mu} = \Gamma^{2n+1}$ it is imaginary, $\Gamma^{\mu} = -\Gamma^{\mu *}$, and B_1^{-1} just passes through Γ^{μ} to cancel B_1 with a factor of $(-1)^{d/2}$. from B_1 one can also construct $B_2 = \Gamma B_1$, with

$$B_2 \Gamma^{\mu} B_2^{-1} = (-1)^{k+1} \Gamma^{\mu *} \tag{B.7}$$

Under these similarity transformations $B \Sigma^{\mu\nu} B^{-1} = \Sigma^{\mu\nu *}$. Since under Lorentz transformation, a spinor transform as $\lambda' = \Sigma^{\mu\nu}\lambda$, then

$$(B^{-1}\lambda^*)' \to B^{-1}\Sigma^{\mu\nu *}\lambda^* = \Sigma^{\mu\nu} B^{-1}\lambda^* \tag{B.8}$$

Both λ and $B^{-1}\lambda^*$ transform the same way under Lorentz. Thus we can consistently impose the **Majorana condition** .

$$\lambda^* = B\lambda \tag{B.9}$$

Note that this implies $\lambda^* = B\lambda = BB^*\lambda^* \to BB^* = 1$. For B_1, each gamma matrix inside is imaginary, hence $B_1^* = (-1)^k B_1$, and $B_1^2 = (-1)^{\frac{k(k-1)}{2}}$. This leads to the requirement for $B_1 B_1^* = (-1)^{\frac{k(k+1)}{2}} = 1$ which is only true when

$k = 0, 3 (\text{mod } 4)$. For B_2, each gamma matrix inside is real, so $B_2 B_2^* = B_2^2 = (-1)^{\frac{(k+1)(k+2)}{2}}(-1)$ (the extra minus sign comes from the Γ^0 inside B_2 and $(\Gamma^0)^2 = -1$). Thus $B_2 B_2^* = (-1)^{\frac{k(k+3)}{2}} = 1$ which is only true for $k = 0, 1 (\text{mod } 4)$. this restricts the kind of B one can use for Majorana condition in various dimensions.

To impose **both** Majorana and Weyl conditions one must insure that the helicity for λ^* is the same as $B\lambda$. Using $B\Gamma B^{-1} = (-1)^k \Gamma^*$

$$\Gamma B^{-1}\lambda^* = B^{-1}B\Gamma B^{-1}\lambda^* = (-1)^k B^{-1}\Gamma^*\lambda^* \tag{B.10}$$

one sees that in order for $B^{-1}\lambda^*$ to have the same helicity as λ, k must be even. Thus combined with previous constraint **Majorana-Weyl conditions are only possible when $k = 0 (\text{mod } 4)$ or $d = 2 (\text{mod } 8)$.**

Charge conjugation: Here we define charge conjugation matrix for arbitrary dimensions. The charge conjugation matrix is defined as [2]

$$C\Gamma^\mu C^{-1} = -\Gamma^{\mu T} \tag{B.11}$$

Note that C is defined up to an overall factor which will not be important. The reason this is charge conjugation is because for a spinor ψ that satisfies the positive energy Dirac equation $(\not{p} - m)\psi = 0$ then $C(\not{p} - m)\psi = C(\not{p} - m)C^{-1}C\psi = (-\not{p}^T - m)C\psi = 0$. Taking the transpose we have $(C\psi)^T(-\not{p} - m)$ thus $\bar{\psi}^c = (C\psi)^T$ describes a negative energy solution or an anti-particle. To obtain C one uses the the fact that in our representation,

$$\Gamma^0\Gamma^\mu(\Gamma^0)^{-1} = -\Gamma^{\mu\dagger} \tag{B.12}$$

This is true since only Γ^0 is anti-hermitian while all other gammas are hermitian, and $(\Gamma^0)^{-1} = -\Gamma^0$. Then we have

$$-\Gamma^{\mu\dagger} = -(\Gamma^{\mu *})^T = C\Gamma^{\mu *}C^{-1} = \begin{cases} (-1)^k C B_1 \Gamma^\mu B_1^{-1} C^{-1} \\ (-1)^{k-1} C B_2 \Gamma^\mu B_2^{-1} C^{-1} \end{cases} \tag{B.13}$$

[2] Note that we can also define C_+ which satisfies $C_+\Gamma^\mu C_+^{-1} = +\Gamma^{\mu T}$. This is achieved by $C_+ = C\Gamma$. Then all properties of C_+ can be derived from C. Note that since one needs to use Γ, it is only possible to have both C and C_+ in even dimensions.

Thus for even k, $\Gamma^0 = CB_1 \rightarrow C = \Gamma^0 B_1^{-1}$, for odd k, $\Gamma^0 = CB_2 \rightarrow C = \Gamma^0 B_2^{-1}$. Then we have

$$\begin{array}{lrcl} \text{even } k & C^T & = & B_1^* \Gamma^{0T} = (-1)^{k+1} B_1 \Gamma^0 = (-1)^{\frac{k(k-1)}{2}+1} \Gamma^0 B_1^{-1} = (-1)^{\frac{k(k-1)}{2}+1} C \\ \text{odd } k & C^T & = & \Gamma^T B_1^{-1T} \Gamma^{0T} = (-1)^{k+1} \Gamma B_1 \Gamma^0 = \Gamma^0 B_2 = (-1)^{\frac{k(k+1)}{2}} \Gamma^0 B_2^{-1} \\ & & = & (-1)^{\frac{k(k+1)}{2}} C \end{array} \tag{B.14}$$

where we've used $B_1^{-1} = B_1^\dagger = (-1)^{\frac{k(k-1)}{2}} B_1$, $B_2^{-1} = (-1)^{\frac{k(k+1)}{2}} B_2$, and $\Gamma^T = \Gamma$. Thus for d=2,4,10 $C^T = -C$ while for d=6 and 8 $C^T = C$.

Equipped with C we can now rewrite the Majorana condition in more standard form

$$\lambda^* = B\lambda \rightarrow \lambda^\dagger = \lambda^T B^T \rightarrow \lambda^\dagger \Gamma^0 = \lambda^T B^T \Gamma^0 = \lambda^T C \tag{B.15}$$

This is the usual way of stating the Majorana condition, the Dirac conjugate ($\lambda^\dagger \Gamma^0$) is equal to the Majorana conjugate ($\lambda^T C$).

Appendix C

Appendix: Proof of field redefinition

Here we will prove that our field redefinition introduced in sec.(2.2.1) satisfies both (2.25) and (2.26). We first produce the proof at leading order, terms with three new fields on the LHS of both equations vanish. From this experience we will then show that the same holds for all higher order terms, namely, written in terms of new fields, terms that are more than quadratic in χ on LHS of these equations vanish.

For (2.25) terms with three field comes from the second order term in the field redefinition, namely $\phi(1) \to C(2,3)\chi(2)\chi(3)$ with $C(2,3) = \frac{p_2^+ p_3^+}{(2,3)}$, they give

$$tr \int_{\vec{p}_1\vec{p}_2\vec{p}_3} p^- [\frac{p_1^+ p_2^+ p_3^+}{(1,2)} - \frac{p_1^+ p_2^+ p_3^+}{(2,3)}] \Phi(1)\Phi(2)\Phi(3)\delta(\Sigma_i \vec{p}_i) \tag{C.1}$$

Using momentum conservation, $(1,2) = -(3,2) = (2,3)$, these two terms indeed cancel each other. The 3 field term that is generated on the LHS for (2.26)

$$tr \int_{\vec{p}_1\vec{p}_2\vec{p}_3} -\frac{p_2^+ p_3^+ p_1 \bar{p}_1}{(2,3)} \chi(1)\chi(2)\chi(3) + \frac{(\bar{p}_2 p_3^+ - \bar{p}_3 p_2^+)}{3} \chi(1)\chi(2)\chi(3) \tag{C.2}$$

Using cyclic identity and relabelling the momentum for the first term we have

$$
\begin{aligned}
& tr\int_{\vec{p}_1\vec{p}_2\vec{p}_3} -\chi(1)\chi(2)\chi(3)\frac{1}{3}[\frac{p_2^+p_3^+\tilde{p}_1\bar{p}_1}{(2,3)}+\frac{p_1^+p_2^+\tilde{p}_3\bar{p}_3}{(1,2)}+\frac{p_3^+p_1^+\tilde{p}_2\bar{p}_2}{(3,1)}] \\
= \quad & tr\int_{\vec{p}_1\vec{p}_2\vec{p}_3} -\chi(1)\chi(2)\chi(3)[\frac{p_2^+p_3^+\tilde{p}_2\bar{p}_3+p_2^+p_3^+\tilde{p}_3\bar{p}_2-p_2^{+2}\tilde{p}_3\bar{p}_3-p_3^{+2}\tilde{p}_2\bar{p}_2}{3(2,3)}] \\
= \quad & tr\int_{\vec{p}_1\vec{p}_2\vec{p}_3} \chi(1)\chi(2)\chi(3)\frac{\{2,3\}}{3} \qquad \text{(C.3)}
\end{aligned}
$$

where in the last two lines we used momentum conservation. This gives the same term as the second term in (C.2) with a minus sign.

To prove that higher field terms also cancel in (2.25) for our field redefinition, note that for n-fields the coefficients combine into

$$
\begin{aligned}
& \sum_{j=3}^{n-1} C(2,\cdot\cdot,j)p^+_{(j+1,n)}C(j+1,\cdot\cdot,n) \\
= \quad & -\frac{(\prod_{i=2}^n p_i^+)(\sum_{j=3}^{n-2} S_j)}{(2,3)(3,4)\cdot\cdot(n,n-1)} \qquad \text{(C.4)}
\end{aligned}
$$

where we've used the notation that $p^+_{(1,n)} \equiv \sum_{i=1}^n p_i^+$ and

$$
S_j \equiv p^+_{n-j}\cdot\cdot p_4^+p_3^+[p^+_{n-1}\cdot\cdot p^+_{n+3-j}p^+_{n+2-j}(n+1-j,n-j)+\text{cyclic rotations}] \qquad \text{(C.5)}
$$

For example for $n = 7$

$$
\begin{aligned}
S_3 &= p_4^+p_3^+[p_6^+(5,4)+p_5^+(4,6)+p_4^+(6,5)] \\
S_4 &= p_3^+[p_6^+p_5^+(4,3)+p_5^+p_4^+(3,6)+p_4^+p_3^+(6,5)+p_3^+p_6^+(5,4)] \\
S_5 &= [p_6^+p_5^+p_4^+(3,2)+p_5^+p_4^+p_3^+(2,6)+p_4^+p_3^+p_2^+(6,5) \\
&\quad +p_3^+p_2^+p_6^+(5,4)+p_2^+p_6^+p_5^+(4,3)]
\end{aligned}
\qquad \text{(C.6)}
$$

The important point is since these S_j are cyclic sums over terms that are partially anti-symmetric, $S_j = 0$. Hence we've proven that (2.25) is indeed satisfied.

Moving on to (2.26), we use the fact that since (2.25) is satisfied, this implies that[1]

$$\partial^+\Phi = \frac{\delta\chi}{\delta\Phi}\partial^+\chi. \tag{C.7}$$

From the discussion above we see that this is indeed true. Plugging back into 2.26 we have

$$\frac{1}{\partial^+}[\partial^+\Phi, \bar{\partial}\Phi] = -\frac{\bar{\partial}\tilde{\partial}}{\partial^+}\Phi + \frac{\delta\Phi}{\delta\chi}\frac{\bar{\partial}\tilde{\partial}}{\partial^+}\chi \tag{C.8}$$

Fourier transforming into momentum space and plugging in (2.29) we have

$$\begin{aligned}
&(-\frac{\tilde{p}_1\bar{p}_1}{p_1^+} + \sum_{i=2}^{n}\frac{\tilde{p}_i\bar{p}_i}{p_i^+})C(2,3,\cdots,n) \\
=&\ \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,\cdots,j)C(j+1,\cdots,n)\{p_{(j+1,n)},p_{(2,j)}\} \\
=&\ \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)\frac{(j,j+1)}{p_j^+p_{j+1}^+}\{p_{(j+1,n)},p_{(2,j)}\}
\end{aligned} \tag{C.9}$$

Again $\{p_{(j+1,n)},p_{(2,j)}\} = p^+_{(j+1,n)}\bar{p}_{(2,j)} - \bar{p}_{(j+1,n)}p^+_{(2,j)}$. Since $\frac{(j,j+1)}{p_j^+p_{j+1}^+} = \frac{\tilde{p}_{j+1}}{p_{j+1}^+} - \frac{\tilde{p}_j}{p_j^+}$ the RHS becomes

$$\begin{aligned}
&\frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)[\frac{\tilde{p}_{j+1}}{p_{j+1}^+} - \frac{\tilde{p}_j}{p_j^+}]\{p_{(j+1,n)},p_{(2,j)}\} \\
=&\ \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)\frac{\tilde{p}_j}{p_j^+}[\{p_{(j,n)},p_{(2,j-1)}\} - \{p_{(j+1,n)},p_{(2,j)}\}] \\
=&\ \frac{1}{p_1^+}\sum_{j=2}^{n}C(2,3,\cdots,n)\frac{\tilde{p}_j}{p_j^+}\{1,j\}
\end{aligned} \tag{C.10}$$

Momentum conversation then gives the LHS of (C.9).

[1]Written in this form we neglect the superspace delta functions and spinor derivatives that usually arise, since we know that the chiral superfield Φ is now already written in terms of chiral superfield χ.

Appendix D

Appendix: Field equations from Bianchi identity

D.1 Self-duality

The self-duality relations among the scalar field strengths arise from a particular kind of Bianchi identity[83]. Here we demonstrate on the ones in the isotropy group, which is simpler:

$$\begin{aligned}
[\check{\phi}_{ab}, \check{\phi}_{a'b'}] &= [\{\nabla_a{}^{\alpha}, \nabla_{b\alpha}\}, \{\nabla_{a'}{}^{\dot{\beta}}, \nabla_{b'\dot{\beta}}\}] \\
&= \{\nabla_a{}^{\alpha}, [\nabla_{b\alpha}, \{\nabla_{a'}{}^{\dot{\beta}}, \nabla_{b'\dot{\beta}}\}]\} + \{\nabla_{b\alpha}, [\nabla_a{}^{\alpha}, \{\nabla_{a'}{}^{\dot{\beta}}, \nabla_{b'\dot{\beta}}\}]\} \\
&= 0 \qquad \text{(D.1)}
\end{aligned}$$

thus one arrives at $\check{\phi}_{ab} = \frac{1}{2} C_{ab} C_{a'b'} \check{\phi}^{a'b'}$. Similar procedure gives us $\phi_{ab} = \frac{1}{2} C_{ab} C_{a'b'} \phi^{a'b'}$ and $\phi_{ab'} = C_{ac} C_{b'd'} \bar{\phi}^{cd'}$.

D.2 Field-equations

We start from the field equation for the spinor field strength, others can be derived from the spinor field strength by acting on it with spinor covariant derivatives. The relevant results from Bianchi identities that will be useful are as follows:

$$
\begin{aligned}
(\nabla_{a'\dot{\alpha}}, \nabla_{b'\beta}, \nabla_{c'\sigma}) &\to [\nabla_{a'\dot{\alpha}}, \phi_{b'c'}] = \eta_{a'b'} W_{c'\dot{\alpha}} - \eta_{a'c'} W_{b'\dot{\alpha}} \\
(\nabla_{\gamma\dot{\rho}}, \nabla_{a\alpha}, \nabla_{b'\beta}) &\to \{\nabla_{b'\gamma}, W_{a\dot{\rho}}\} = -\{\nabla_{a\gamma}, W_{b'\dot{\rho}}\} = [\nabla_{\gamma\dot{\rho}}, \phi_{ab'}]
\end{aligned} \tag{D.2}
$$

We begin with

$$
\begin{aligned}
& [\{\nabla_{a\alpha}, \nabla^{b'\dot{\beta}}\}, \overline{W}_{b'\dot{\beta}}] = 0 \\
= \quad & -[\{\nabla^{b'\dot{\beta}}, \overline{W}_{b'\dot{\beta}}\}, \nabla_{a\alpha}] - [\{\nabla_{a\alpha}, \overline{W}_{b'\dot{\beta}}\}, \nabla^{b'\dot{\beta}}]
\end{aligned} \tag{D.3}
$$

From the first result in (D.2)

$$
\begin{aligned}
\{\overline{\nabla}^{c'\dot{\sigma}}, \overline{W}_{c'\dot{\sigma}}\} &= \{\overline{\nabla}^{c'\dot{\sigma}}, [\overline{\nabla}^{a'}{}_{\dot{\sigma}}, \phi_{a'c'}]\} \\
&= \{\overline{\nabla}^{a'}{}_{\dot{\sigma}}, [\phi_{a'c'}, \overline{\nabla}^{c'\dot{\sigma}}]\} - [\phi_{a'c'}, \{\overline{\nabla}^{c'\dot{\sigma}}, \overline{\nabla}^{a'}{}_{\dot{\sigma}}\}] \\
&\to \{\overline{\nabla}^{c'\dot{\sigma}}, \overline{W}_{c'\dot{\sigma}}\} = [\phi_{a'c'}, \check{\phi}^{a'c'}]
\end{aligned} \tag{D.4}
$$

Then the first term in (D.3)

$$
\begin{aligned}
-[\{\nabla^{b'\dot{\beta}}, \overline{W}_{b'\dot{\beta}}\}, \nabla_{a\alpha}] &= [\nabla_{a\alpha}, [\phi_{a'c'}, \check{\phi}^{a'c'}]] \\
&= -[\check{\phi}^{a'c'}, [\nabla_{a\alpha}, \phi_{a'c'}]
\end{aligned} \tag{D.5}
$$

Using the fact that $(\check{\phi}_{ab}, \phi_{a'b'}, \phi_{ab'})$ are dual to $(\check{\phi}_{a'b'}, \phi_{ab}, \overline{\phi}_{a'b})$. More precisely

$$
\phi_{a'c'} = \frac{1}{2}\epsilon_{a'c'}\epsilon^{bd}\phi_{bd}
$$

$$
\check{\phi}^{a'c'} = \frac{1}{2}\epsilon^{a'c'}\epsilon_{bd}\check{\phi}^{bd}
$$

$$
\begin{aligned}
[\check{\phi}^{a'c'}, [\nabla_{a\alpha}, \phi_{a'c'}]] &= [\check{\phi}^{bd}, [\nabla_{a\alpha}, \phi_{bd}]] \\
&= 2[\check{\phi}_a{}^d, W_{d\alpha}]
\end{aligned} \tag{D.6}
$$

Using the second result in (D.2) we find the second term in (D.3) becomes

$$
\begin{aligned}
[\nabla^{b'\dot{\beta}}, \{\nabla_{a\alpha}, \overline{W}_{b'\dot{\beta}}\}] &= -[\nabla^{b'\dot{\beta}}, [\nabla_{\alpha\dot{\beta}}, \phi_{ab'}]] \\
&= 2[\nabla_{\alpha\dot{\beta}}, \overline{W}_a{}^{\dot{\beta}}] + 2[\phi_{ab'}, W^{b'}{}_{\alpha}]
\end{aligned} \tag{D.7}
$$

Putting everything together we finally arrive at the field equation

$$[\nabla_{\alpha\dot{\beta}}, \overline{W}_a{}^{\dot{\beta}}] + [\phi_{ab'}, W^{b'}{}_{\alpha}] + [\check{\phi}_a{}^d, W_{d\alpha}] = 0$$

Note that self-duality of the scalar field strengths is crucial in getting the field equation. Violating this relationship will then lead to an off-shell construction.