Marginal Deformations of Gauge Theories and their Dual Description

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Abstract of the Dissertation

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Holography and its realization in string theory as the AdS/CFT correspondence, offers an equivalence between gauge theories and gravity that provides a means to explore the otherwise inaccessible large N and strong coupling region of SU(N) gauge theories. While considerable progress has been made in this area, a concrete method for specifying the gravitational background dual to a given gauge theory is still lacking. This is the question addressed in this thesis in the context of exactly marginal deformations of $\mathcal{N} = 4$ SYM. First, a precise relation between the deformation of the superpotential and transverse space noncommutativity is established. In particular, the appropriate noncommutativity matrix Θ is determined, relying solely on data from the gauge theory lagrangian and basic notions of the AdS/CFT correspondence. The set $(\mathcal{G}, \Theta)$ of open string parameters, with $\mathcal{G}$ the metric of the transverse space, is then understood as a way to encode information pertaining to the moduli space of the gauge theory. It seems thus natural to expect that it may be possible to obtain the corresponding gravitational solution by mapping the open string fields $(\mathcal{G}, \Theta)$ to the closed string ones (g, B). This hints at a purely algebraic method for constructing gravity duals to given conformal gauge theories. The idea is tested within the context of the β–deformed theory where the dual gravity description is known and then used to construct the background for the ρ–deformed theory up to third order in the deformation parameter ρ. Discrepancy of the higher order in ρ terms in the latter case is traced to the nonassociativity of the noncommutative matrix Θ.

Contents

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Chapter 1

Introduction

The principle of holography [1, 2] states that a gravitational theory defined within a spacetime region can be equivalently described by another non–gravitational theory living on the boundary of this region. Though an astonishing proposal, holography has a sound origin in the properties of black holes. Since the work of Bekenstein and Hawking, it is known that black holes have entropy proportional to the area of their event horizon: $S = \frac{1}{2}\mathcal{A}$. However, it can be reasonably argued [2] that black holes are objects of maximal entropy, providing a bound for the entropy of any physical system. This naturally suggests that the number of degrees of freedom should scale with the area and not with the volume of a given region — as it usually expected for non–gravity theories — thus leading to the thought that all information pertinent to a gravitational theory defined in a domain of spacetime could be captured by another theory residing on the boundary of that same domain.

While the idea of holography as a potential path in further exploring gravity and perhaps succesfully combining it with quantum mechanics was proposed early on, its conrete realization became possible only within the context of string theory, in what is now known as the AdS/CFT correspondence [3, 4, 5]. String theory offered a new perspective on the relation between gauge theories and gravity through the double nature of Dp–branes [6]. Gauge theories emerged naturally as low energy effective descriptions of the dynamics of Dp–branes, following their definition within perturbative string theory as hypersurfaces where open strings are confined to end. At the same time, Dp–branes can be regarded as sources for closed string fields. In this picture, they are non–perturbative classical solitons of string theory with a low energy effective description as particular solutions of the supergravity equations of motion.

In its original formulation, the AdS/CFT correspondence originated from the dual nature of D3–branes, relating $\mathcal{N} = 4$ U(N) Super–Yang–Mills in 3+1 dimensions, with Type IIB closed string theory on $AdS_5 \times S^5$ with N units of RR–flux.

Consider N parallel, coincident, D3–branes immersed in flat ten–dimensional space, extended along 3+1 directions. String perturbation theory in such a background contains both open and closed strings with the former attached to the D3–branes (therefore describing the excitations of the branes) and the latter being understood as excitations of empty space. In the limit of low — compared to the string scale $1/l_s$ — but fixed energy, open and closed strings decouple from each other while being effectively described through $\mathcal{N} = 4$ U(N) SYM in 3+1 dimensions and free bulk Type II B supegravity i.e. ten dimensional flat space.

As discussed above, the stack of D3–branes admits an alternative low energy

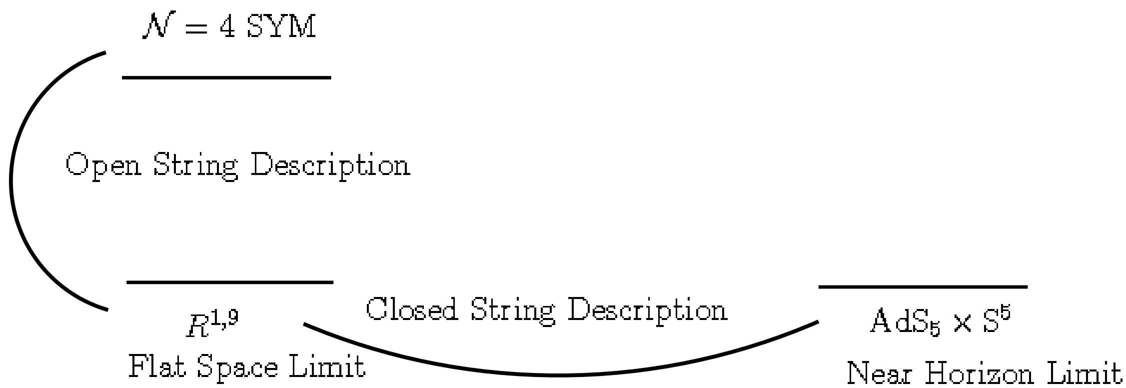

Figure 1.1: It is the double nature of D3–branes that lead to the AdS/CFT proposal.

description as a a collection of 3-branes, particular solutions of type II B supergravity:

$$\begin{aligned} \mathrm{d}s^2 &= f^{-1/2}\left(-\mathrm{d}t^2 + \sum_{i=1}^{i=3}\mathrm{d}x_i^2\right) + f^{1/2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right) \\ F_5 &= (1+*)\,\mathrm{d}t\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}f^{-1} \\ f &= 1 + \frac{R^4}{r^4}, \quad \text{with} \quad R^4 = 4\pi g_s \alpha'^2 \mathrm{N} \end{aligned} \tag{1.1}$$

In this description, the limit of fixed energies corresponds to $\frac{r}{\alpha'}$ = fixed while $\alpha' \to 0$ and string theory in this background reduces to two decoupled systems: one of free supergravity and the other of the "near horizon region" of the original geometry, namely type II B closed string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{r^2}{R^2}\left(-\mathrm{d}t^2 + \sum_{i=1}^{i=3}\mathrm{d}x_i^2\right) + R^2\frac{\mathrm{d}r^2}{r^2} + R^2\mathrm{d}\Omega_5^2 \\ F_5 &= (1+*)\,\mathrm{d}t\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}\left(\frac{r^4}{R^4}\right) \end{aligned} \tag{1.2}$$

Since taking the same limit either in terms of open or closed strings, results in two decoupled systems one of which is common in both descriptions (free supergravity that is), it is natural to identify the second system that appears in the two independent pictures. We are thus lead to the conjecture, that $\mathcal{N} = 4$ U(N) SYM theory in 3+1 dimensions is equivalent to type II B superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. As an additional confirmation comes the fact that the full symmetry algebra of $\mathrm{AdS}_5 \times \mathrm{S}^5$, being the product of $SO(4,2) \times SO(6)$ coincides with the conformal symmetry of the SYM theory in 3+1 dimensions along with the $SU(4)_R$ R–symmetry group of the field theory.

In this form, referred to as the *strong* form of the conjecture, AdS/CFT is of little practical use since the problem of quantizing string theory on curved spacetimes with RR—fluxes resists solution [1]. Fortunately, there are limiting cases where the proposal is nontrivial yet tractable. The 't Hooft limit on the SYM side for instance, where $\lambda = g_{YM}^2 N$ is held fixed but N is taken to infinity, $N \to \infty$, corresponds to classical string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. An even more useful limit, is the $\lambda \to \infty$ limit which corresponds to classical type II B supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. It implies that the strong coupling and large N region of the gauge theory, which is not accessible at the moment by other means, can be studied through classical supergravity. In this sense, the AdS/CFT duality provides a practical tool in analysing gauge theories — an equally significant task though different from the original objective of holography.

Since the date of its birth, AdS/CFT has passed several tests that confirm its validity although a consistent proof is still lacking. As a result, considerable effort has been directed into extending its original form, mainly aiming at describing gauge theories closer to real–life QCD. This naturally includes examples of theories with less or no supersymmetry and/or broken conformal invariance. Despite however the interest initiated in the subject and the progress in this direction, a precise method

[1] For recent progress in this direction see [7].

for identifying and constructing the supergravity, if not string theory, dual to a given gauge theory, remains to be proposed. Most of the cases examined in the literature, depend on the high amount of symmetry and are mainly focused on determining methods for solving the supergravity equations of motion [8, 9, 10, 11, 12, 13] based on symmetry requirements. No other *direct input* from the gauge theory is used. Thus the question/issue which motivated this work:

Is there a *concrete* way to use information pertinent to a gauge theory, to construct its geometric dual configuration?

We will take the point of view that this is indeed possible and systematically explore ways for doing so. This is however a notoriously difficult question to be answered in all generality, so we will concentrate on the specific examples of Leigh—Strassler deformations of $\mathcal{N} = 4$ SYM. These are $\mathcal{N} = 1$ superconformal gauge theories labelled by two parameters β and ρ, and described by the superpotential:

$$\mathcal{W} = ih\mathrm{Tr}\left[\left(e^{i\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\beta}\Phi_1\Phi_3\Phi_2\right) + \rho\left(\Phi_1^3 + \Phi_2^3 + \Phi_3^3\right)\right] \tag{1.3}$$

Despite being conformal and superymmetric thus quite dissimilar to QCD, these theories provide a natural starting point for this study due to the following reasons:

- Since they are continuous deformations of the $\mathcal{N} = 4$ SYM theory preserving conformal invariance, the setup of the original AdS/CFT proposal is not dramatically altered. In particular, we can still hope to describe them by placing D3–branes in some background continuously connected to flat space and recover their gravity dual in the near horizon limit.

- In addition, although it would have been natural to expect that their gravity

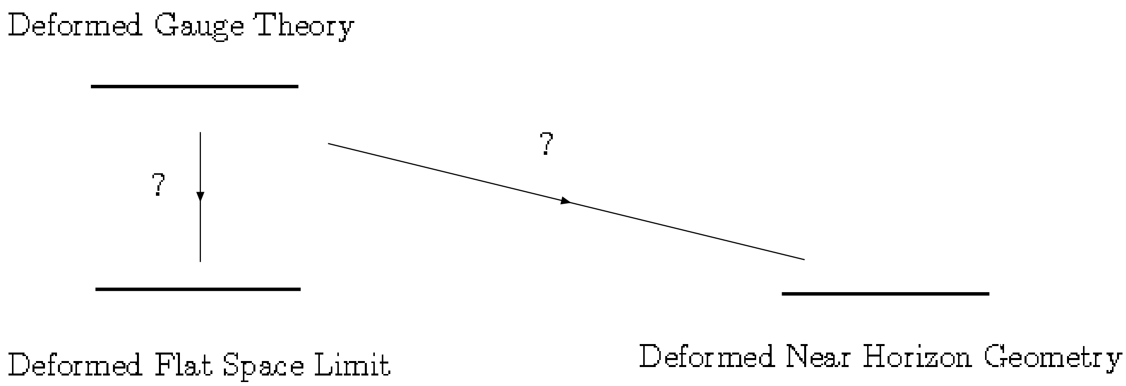

Figure 1.2: The problem of finding the dual geometry is twofold.

duals would be known by now, this is actually not the case. In fact, it was not until fairly recently that definite progress in this direction was made, when Lunin and Maldacena [14] constructed the dual gravity background of the β–deformed theory. The gravitational description for the ρ– deformation is still unknown. This implies that any proposal can be explicitly checked in the former case before being applied to the latter.

Exactly marginal deformations thus offer a sound testing ground for exploring new ideas along with the possibility of obtaining new results.

Following the arguments that lead to the Maldacena conjecture, the problem of finding the gravity dual to a given gauge theory appears to be twofold. One must first specify the analogue of the flat space geometry into which D3–branes are embedded, then construct the supergravity description of these branes and take the near horizon limit. In other words, one has to move vertically and then to the right in Fig. 1.2. Let us focus on the first part; given a gauge theory arising as a deformation of $\mathcal{N} = 4$ SYM, is there a way to identify the deformed flat space geometry where D3–branes should be immersed?

It is helpful to first consider the $\mathcal{N}=4$ theory and examine how the ten dimensional flat space geometry is encoded in the Lagrangian. The embedding space is decomposed into a 3+1 dimensional space parallel to the brane worldvolume and a 6 dimensional transverse part. The former is characterized by the flat Minkowski metric since it is the space where the theory lives. The latter on the other hand, is characterized by the Káhler metric for the scalars, since they can be identified with the embedding transverse space coordinates. The potential term for the scalars provides additional information on the moduli space. Equation (1.4) below,

$$\mathcal{V} = \sum_{i=1}^{i=6} [X^i, X^j][X_i, X_j] = 0 \tag{1.4}$$

shows that the transverse space can be parametrized by *commutative* coordinates. The embedding space is then represented in the $\mathcal{N}=4$ SYM Lagrangian through $(\mathcal{G}_{\mu\nu} = \eta_{\mu\nu}, \mathcal{G}_{IJ} = \delta_{IJ})$ where $\mu, \nu = 0, 1, 2, 3$ and $I, J = 4, ..9$.

Now consider deforming the superpotential of the theory. This will obviously have an impact on the transverse space geometry. The Káhler metric remains unchanged but the potential for the scalars is modified. It is natural to expect based on (1.4), that the deformation in the superpotential will manifest itself as a noncommutative deformation of the transverse space. Making this precise ,*i.e.*, constructing the appropriate noncommutativity parameter Θ^{IJ} to describe the deformation, is however a quite nontrivial task that will be one of the main goals of this work.

Suppose now, that the matrix Θ^{IJ} encoding the deformation is determined. Then, the background into which D3–branes are placed will be specified by the following set of fields $(\mathcal{G}_{\mu\nu}, \mathcal{G}_{IJ}, \Theta^{IJ})$. Recall however, that spacetime noncommutativity arises entirely in the context of open strings. It is the way open strings perceive the presence of nonvanishing antisymmetric tensor fields in the background [15, 16]. To be precise, turning on a nontrivial B–field from the viewpoint of closed strings results in making the space noncommutative from the point of view of the open strings. It is then natural that any information on the background extracted from the gauge theory, will be carried over to the set of *open* string parameters: $(\mathcal{G}_{\mu\nu}, \mathcal{G}_{IJ}, \Theta^{IJ})$. To determine the geometry for the purposes of the AdS/CFT however, one needs to specify the appropriate set of *closed* and not *open* string fields. This is not difficult though, once the latter are known, since there exists a precise map relating the two sets of fields. The above reasoning suggests that a concrete method for constructing the deformed flat space geometry indeed exists; it consists into specifying the open string fields from the gauge theory Lagrangian and mapping them to the corresponding closed string ones.

Most of this work will actually focus on realizing these ideas in a precise manner for the exactly marginal deformations of $\mathcal{N} = 4$. We will first construct a noncommutativity matrix to encode the deformation of the transverse space geometry. Having obtained the set $(\mathcal{G}, \Theta)$ we will then use the known map relating open and closed string fields in order to specify the deformed flat space geometry, the set (g, B) that is. The map in question was constructed for c–number noncommutative but associative deformations, and we will see that it will be sufficient for the β–deformed theory. In the case of the ρ–deformation, the noncommutativity parameter will not obey the Jacobi identity and as a result, the closed string fields will satisfy the supergravity equations of motion only up to third order in the deformation parameter ρ.

Let us now move on to the main objective, which is to find the gravity dual description of the deformed gauge theory. Is there any way, we can use the parametrization of its moduli space in terms of open string fields, to attack this problem? Put differently,

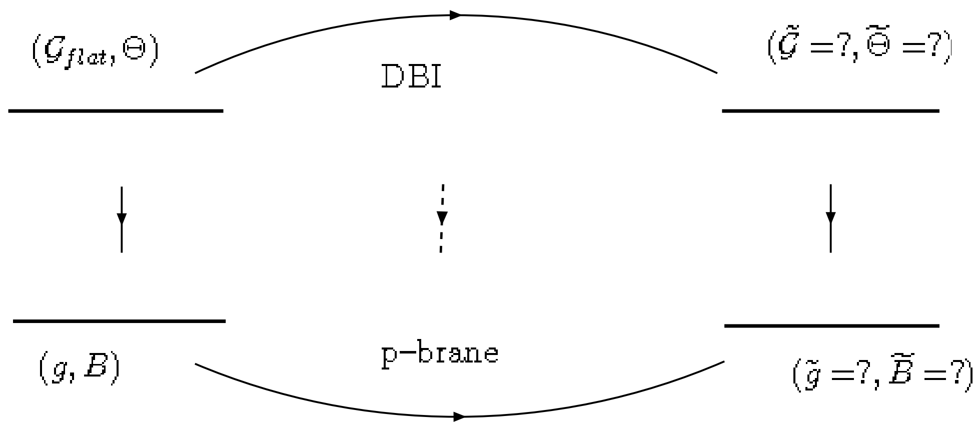

Figure 1.3: Is there a way to deduce the open string parameters for the gauge theory living on a stack of branes placed in the *dual* gravity background?

how do we move diagonally in Fig. 1.2?

Unfortunately a concrete setup for studying this problem is not clear to us yet. Preliminary ideas and a natural conjecture however will lead us into the dual gravity solution for the ρ–deformed theory up to third order in ρ. The notions that we will be dealing with in this case, are the $\mathcal{N} = 4$ IR quantum effective action — obtained by giving a vacuum expectation value to one of the scalars and integrating over the massive fields, its connection to the Dirac–Born–Infeld (DBI) action describing the fluctuating degrees of freedom of D–branes and the dual geometry. The main question here, can be better expressed in Fig. 1.3. Is there a way to move in this diagram from left to right first and then vertically? What is the reasoning that will help us obtain the corresponding open string parameters for a stack of D–branes immersed in the near horizon geometry?

This report is divided into two main parts. In the next chapter, based on [17], we focus on the β–deformed theory, identify the noncommutativity matrix describing the deformation and its role in the solution generating transformation employed by

Lunin and Maldacena. In chapter 2, based on [18], we construct the appropriate noncommutativity matrix for the case of the ρ–deformation. Following the logic outlined above, specifying the deformed flat space geometry to third order in the deformation parameter — as well as the gravity dual background to the same order — reduces to a simple algebraic procedure. Finally, we present an extensive discussion on the approach proposed in this note, illuminating difficulties and obscure points as well as bringing forth interesting directions for future study. These, we hope, could prove fruitful in completing the program of determining a precise method for constructing gravity descriptions of given gauge theories.

Chapter 2

β–deformations and Noncommutativity

2.1 Introduction

As discussed in the introduction, the Maldacena conjecture [3, 4, 5] relates four–dimensional theories at strong t'Hooft coupling with weakly coupled gravitational ones. In [14] Lunin and Maldacena presented a further development in this direction by constructing the gravity duals of gauge theories deformed in a particular manner that maintains a global $U(1) \times U(1)$ symmetry present in the original undeformed theory. The prototype of these deformations is a Leigh–Strassler [19] exactly marginal deformation of $\mathcal{N} = 4$ SYM theory, characterized by a complex parameter β which preserves $\mathcal{N} = 1$ supersymmetry. The method of Lunin and Maldacena is not however restricted to conformal field theories. It can be applied to any field theory as long as its dual gravity background contains a two torus geometrically realizing the global $U(1)$ symmetries in question. When $\beta \in \mathbb{R}$ — usually denoted as γ in the literature — the prescription presented in [14] amounts to performing an $SL(2,\mathbb{R})$ transformation on the complexified Kähler modulus τ of this two torus. The specific element of $SL(2,\mathbb{R})$ under consideration has only one free parameter which is then identified with the real deformation parameter γ of the gauge theory. Subsequent work on the subject of the β–deformed gauge theories has provided further checks of the AdS/CFT correspondence [20, 21, 22, 23, 24, 25] whereas the possibility of an underlying integrable structure in this context was explored in [26, 27, 28]. Several aspects of these deformations were analysed from the gauge theory viewpoint in [29, 30, 31, 32, 33, 34, 35]. Furthermore, generalizations as well as applications of the solution generating technique introduced in [14] were considered in [36, 28, 25, 37, 38, 39].

Meanwhile, it became clear [40] that embedding $SL(2,\mathbb{R})$ into the T–duality group $O(2,2,\mathbb{R})$ may be a significantly easier way to obtain the deformed backgrounds since it suffices then to consider the action of the appropriate $O(2,2,\mathbb{R})$ group element on the background matrix $E = g + B$. In this framework, an extraordinary similarity between the proposal of [14] and the method for constructing gravity duals of noncommutative gauge theories becomes evident [1]. From the gauge theory point of view this analogy is not surprising since the deformation amounts to modifying the commutator of the matter fields in the Lagrangian or equivalently, their product. A natural proposal for the product rule was set forth in [14] and subsequently verified in the dual field theory context in [29, 23].

The central aim of this chapter is to clarify the relation between noncommutativity and β–deformations. We will consider the deformations in their original context as marginal deformations of $\mathcal{N} = 4$ SYM and show how to obtain a noncommutativity matrix Θ describing them. The main point will be to think of the matter fields

[1] Actually, this connection was already noted in [14].

in the dual theory as coordinates parametrizing the space transverse to the D3–brane where the gauge theory lives. Then, reality properties, global symmetries and marginality will severely constrain the form of the noncommutativity matrix leaving one possible choice, the one which leads to the correct gravity dual description. In other words, Θ^{ij} along with the metric of the transverse space can be thought of as another way to encode the moduli space of the gauge theory. This suggests an alternative way in which to investigate deformations of the original AdS/CFT proposal [3] by determining the *open* string parameters pertaining to them. Related ideas will be explored in the next chapter in order to study another Leigh–Strassler marginal deformation of $\mathcal{N} = 4$ SYM the gravity dual of which is yet unkown.

The plan of this chapter is as follows. In the next section, we review the solution generating technique proposed in [14] as well as its formulation through T–duality [40]. In section 2.3, we present some basic facts about noncommutative geometry. Then we describe the methods employed in finding the gravity duals of these theories in a fashion that makes evident the similarity with the approach of [14]. In particular, it is shown that the T–duality group elements used in both cases can be identified if the deformation submatrix referred to as $\mathbf{\Gamma}$ in [40] is interpreted as a noncommutativity matrix. In section 2.4, we explain how one can determine a suitable noncommutativity matrix for the β–deformed gauge theory. This construction is purely based on gauge theory data and basic notions of AdS/CFT. We then show that Θ^{ij} is precisely the submatrix $\mathbf{\Gamma}$ appearing in section 1. We conclude with 2.6 after a short discussion on possible applications of our techniques within the context of gauge/gravity duality. Extension of our results to noncommutative gauge theories is left to A since it lies somewhat outside the main scope of this report.

2.2 The Lunin–Maldacena solution generating technique.

As it was shown in [19] $\mathcal{N}=4$ Super Yang Mills admits a complex three parameter family of exactly marginal deformations [2] preserving $\mathcal{N}=1$ supersymmetry which is described by the following superpotential:

$$\mathcal{W} = \kappa\epsilon_{IJK}\mathrm{Tr}\left([\Phi^I,\Phi^J]_\beta\Phi^K\right) + \rho\mathrm{Tr}\left(\sum_{I=1}^{3}(\Phi^I)^3\right) \tag{2.1}$$

Here Φ^I are three chiral superfields and $[\Phi^I,\Phi^J]_\beta \equiv e^{i\beta}\Phi^I\Phi^J - e^{-i\beta}\Phi^J\Phi^I$. Together with the gauge coupling g_{YM}, the complex parameters (κ,β,ρ) constitute the four couplings of the theory. Conformal invariance imposes one condition on these couplings thus (2.1) describes a three parameter family of deformations. When $\rho=0$ the theory is often referred to as the β–deformed gauge theory and preserves an additional global $U(1)\times U(1)$ symmetry (apart from the $U(1)_R$ R–symmetry) which acts on the superfields as follows:

$$\begin{aligned} U(1)_1: &\quad (\Phi_1,\Phi_2,\Phi_3)\rightarrow(\Phi_1,e^{i\alpha_1}\Phi_2,e^{-i\alpha_1}\Phi_3) \\ U(1)_2: &\quad (\Phi_1,\Phi_2,\Phi_3)\rightarrow(e^{-i\alpha_2}\Phi_1,e^{i\alpha_2}\Phi_2,\Phi_3) \end{aligned} \tag{2.2}$$

In this thesis we will be mainly considering the β–deformed theory for $\beta\in\mathbb{R}$. It is then customary to denote the deformation parameter by γ and we will adhere to this notation in this section. Lunin and Maldacena in [14] succeeded in finding the gravity dual of this theory by implemeting a generating solution technique which can be applied to any field theory with $U(1)\times U(1)$ global symmetry realized geometrically.

[2] More on the Leigh–Strassler deformations can be found in chapter 3 and section 3.2.

Their method essentially consists in performing an $SL(2,\mathbb{R})$ transformation on the complexified Kähler modulus of the two torus associated with the U(1) symmetries in question. Suppose for instance that one knows the gravity dual of the undeformed theory and furthermore that the two global U(1)'s of the parent theory also preserved by the deformation are indeed realized geometrically. Then the supergravity dual of the deformed theory is given by the following substitution:

$$\tau = (B_{12} + \sqrt{g}) \rightarrow \frac{\tau}{1+\gamma\tau} \tag{2.3}$$

where τ is the complexified Kähler modulus of the two torus (associated to the U(1) symmetries of the original solution)with B_{12} the B–field along the torus and $\sqrt{g}$ its volume. In other words, one considers the theory compactified on the two torus and subsequently acts on its Kähler modulus with the particular element of $SL(2,\mathbb{R})$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)$ with γ the parameter of the theory. This element of $SL(2,\mathbb{R})$ is chosen because it ensures that the new solution will present no singularities as long as the original metric is non–singular. An alternative way of thinking about this solution generating transformation is in terms of applying a series of T–dualities. More precisely, the method illustrated above is equivalent to doing a T–duality on a circle, a coordinate transformation and then another T–duality (TsT).

Subsequently it was shown [40] that one can embed the $SL(2,\mathbb{R})$ that acts on the Kähler modulus into the T–duality group $O(2,2,\mathbb{R})$ and thus consider the action of the latter on the background matrix $E = g + B$. This provides a considerably simpler way of obtaining the new solutions. For the generic element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $SL(2,\mathbb{R})$ the appropriate embedding is the following:

$$\mathcal{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ c & 0 & 0 & d \end{pmatrix} \tag{2.4}$$

It is then easy to see [41] that $\mathcal{T}$ transforms the original background matrix E_0 as:

$$E_0 \to E = (\mathbf{A}E_0 + \mathbf{B})(\mathbf{C}E_0 + \mathbf{D})^{-1} \equiv \frac{\mathbf{A}E_0 + \mathbf{B}}{\mathbf{C}E_0 + \mathbf{D}} \tag{2.5}$$

where the 2×2 matrices A,B,C,D are defined through (2.4). According to [14] we should not consider any $SL(2,\mathbb{R})$ element but the precise one with $a = d = 1$, $b = 0$ and $c = \gamma$. In this case (2.4) reads:

$$\mathcal{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{\Gamma} & \mathbf{1} \end{pmatrix} \quad \text{with} \quad \mathbf{\Gamma} = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix} \tag{2.6}$$

where $\mathbf{1}$ and $\mathbf{0}$ represent the 2×2 identity and zero matrices respectively. Following now the T–duality rules in [41] we can write the NS–NS fields of the new solution in terms of E_0 and Γ as follows:

$$\begin{aligned} E &= \frac{1}{E_0^{-1} + \mathbf{\Gamma}} \\ e^{2\Phi} &= \det(1 + E_0\mathbf{\Gamma})e^{2\Phi_0} \end{aligned} \tag{2.7}$$

The RR-fields of the new background can be obtained in a similar fashion using the transformation rules of [42, 43, 44, 45, 46]. Nevertheless, it suffices for us to know that appropriate rules exist and can be applied.

There are however cases where one needs to slightly modify the method illustrated above. This happens when non–trivial fibrations mix the isometry diretions of the two torus with other directions in the metric. It is then necessary to embed $SL(2,\mathbb{R})$ into $O(n+2,n+2,\mathbb{R})$ with n the number of non–trivial coordinate fibrations. A particular example of this is the $\mathrm{AdS}_5\times\mathrm{T}^{1,1}$ solution of [47]. If we want to apply the deformation to this background instead of (2.6) we should employ:

$$\mathcal{T}=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{\Gamma} & \mathbf{1}\end{pmatrix}\quad \text{where}\quad \mathbf{\Gamma}=\begin{pmatrix}0 & -\gamma & 0\\ \gamma & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.8}$$

Furthermore, as it was again pointed out in [40], the appropriate T–duality matrix one should use for the deformation of $\mathrm{AdS}_5\times\mathrm{S}^5$which gives rise to the gravity dual of the β–deformed gauge theory is:

$$\mathcal{T}=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{\Gamma} & \mathbf{1}\end{pmatrix}\quad \text{where now}\quad \mathbf{\Gamma}=\begin{pmatrix}0 & -\gamma & \gamma\\ \gamma & 0 & -\gamma\\ -\gamma & \gamma & 0\end{pmatrix} \tag{2.9}$$

This particular choice of $\mathbf{\Gamma}$ with the necessary embedding of $SL(2,\mathbb{R})$ into $O(3,3,\mathbb{R})$ can be understood in this case as the result of performing a change of coordinates and a T–duality transformation of the form (2.8) followed by another coordinate transformation [40]. For future reference and as a concrete illustration of the above we would like to give an explicit construction of the background in this case. What we have to do is to simply act with (2.9) on the background matrix E_0 which in this example is none other but the metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$. Since we are interested in obtaining the gravity dual of a conformal gauge theory we expect that only the S^5 part of $\mathrm{AdS}_5 \times \mathrm{S}^5$ will be affected by the deformation. We can write the metric on S^5 in the following way:

$$ds^2 = R^2\left(\sum_{i=1}^{3} d\mu_i^2 + \mu_i^2 d\phi_i^2\right) \quad \text{where} \quad \sum_{i=1}^{3}\mu_i^2 = 1 \tag{2.10}$$

Note here that we want to deform the geometry along the U(1) isometry directions of S^5, therefore the relevant part of the background matrix is:

$$E_0 = R^2\begin{pmatrix} \mu_1^2 & 0 & 0 \\ 0 & \mu_2^2 & 0 \\ 0 & 0 & \mu_3^2 \end{pmatrix} \tag{2.11}$$

Using now equation (2.7) and its generalization for RR–fields, we find [40]:

$$\begin{aligned}
\mathrm{d}s^2 =& \mathrm{R}^2(\mathrm{d}s^2_{\mathrm{AdS}_5} + \mathrm{d}s^2_5), \quad \text{where}: \quad \mathrm{d}s^2_5 = \sum_i(\mathrm{d}\mu_i^2 + G\mu_i^2\mathrm{d}\phi_i^2) + \hat{\gamma}G\mu_1^2\mu_2^2\mu_3^2(\sum_i \mathrm{d}\phi_i)^2 \\
G^{-1} =& 1 + \hat{\gamma}^2(\sum_{i\neq j}\mu_i^2\mu_j^2), \qquad \hat{\gamma} = \mathrm{R}^2\gamma, \qquad \mathrm{R}^4 = 4\pi e^{\Phi_0}\mathrm{N} \\
e^{2\varphi} =& e^{2\varphi_0}G, \quad B = \hat{\gamma}R^2G\left(\sum_{i\neq j}\mu_i^2\mu_j^2\mathrm{d}\phi_i\mathrm{d}\phi_j\right) \\
C_2 =& -\gamma(16\pi N)\omega_1(\sum_i \mathrm{d}\phi_i), \quad C_4 = (16\pi N)(\omega_4 + G\omega_1\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\phi_3) \\
F_5 =& (16\pi N)(\omega_{\mathrm{AdS}_5} + G\omega_{\mathrm{S}^5}), \quad \omega_{\mathrm{S}^5} = \mathrm{d}\omega_1\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\phi_3, \quad \omega_{\mathrm{AdS}_5} = \mathrm{d}\omega_4
\end{aligned} \tag{2.12}$$

which is precisely the gravity solution given in [14].

2.3 The gravity duals of noncommutative gauge theories.

In this section we would like to focus on yet another class of supergravity duals which can be obtained in manner analogous to the one described earlier. These are the gravity duals of noncommutative gauge theories [3] and in fact the methodology used in both cases is almost identical.

Noncommutative — as opposed to ordinary — gauge theories, live in a space of noncommuting coordinates [4]. Such a deformation of space is encoded in what is referred to as the noncommutativity parameter Θ^{ij} defined as:

$$[x^i, x^j] = i\Theta^{ij} \tag{2.13}$$

where $\{x^i\}$ is a set of coordinates parametrizing the space and Θ^{ij} a real antisymmetric matrix. In general, the easiest way to deal with functions on these spaces is to replace noncommuting variables with commuting ones by simply defining a new product rule between them, usually called a star product. The star product will then contain all the information on the noncommutative structure of the space.

Out of all the possible forms of Θ^{ij} the case most well understood is by far the one in which the commutators of (2.13) are c–numbers and therefore the noncommutativity parameter is essentially a constant. In this case, associativity is preserved

[3] For an introduction to noncommutative geometry see for example [48] and references therein.

[4] We limit the discussion in this section to Euclidean spaces or to noncommutativity which does not affect the time–like coordinate.

and the appropriate star product has the form:

$$f(x) * g(x) = f(x+\xi) e^{\frac{i}{2} \overleftarrow{\frac{\partial}{\partial \xi^i}} \Theta^{ij} \overrightarrow{\frac{\partial}{\partial \zeta^j}}} g(x+\zeta) = f \left(1 + \overleftarrow{\partial}_i \Theta^{ij} \overrightarrow{\partial}_j + \mathcal{O}(\Theta^2)\right) g \qquad (2.14)$$

Gravity duals of theories living on noncommutative spaces with constant noncommutativity parameter were first found in [49][50]. The basic technique for constructing these solutions is to combine diagonal T–dualities, constant shifts of the NS–NS two form and $SO(p,1)$ transformations, where p is the number of spatial dimensions. One first T–dualizes in the directions where one wants to turn on fluxes, shifts the B field by a constant in these directions and then T–dualizes back. Equivalently, one can T–dualize along one of the directions of the fluxes, use a boost/rotation between a non compact and a compact direction and the T–dualize back. Both methods give the same result. It was later on realized that [51] these solutions can be generated from the action of the $O(p,p,\mathbb{R})$ T–duality group element

$$\mathcal{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{\Theta} & \mathbf{1} \end{pmatrix} \qquad (2.15)$$

on the original undeformed solution where now $\mathbf{0}, \mathbf{1}, \mathbf{\Theta}$ are p dimensional square matrices with p denoting the number of spatial directions along which noncommutativity is turned on. Suppose for instance that one wants to describe a gauge theory living in four dimensional Euclidean space employed with cartesian coordinates x^μ where: $[x^0, x^1] = ib_1$ and $[x^2, x^3] = ib_2$. It is then clear that one should consider the embedding of the noncommutativity parameter into the T–duality group $O(4,4,\mathbb{R})$ as follows:

$$\mathcal{T} = \begin{pmatrix} \mathbf{1}_4 & \mathbf{0}_4 \\ \mathbf{\Theta} & \mathbf{1}_4 \end{pmatrix} \quad \text{with} \quad \mathbf{\Theta} = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix} \tag{2.16}$$

The original solution to be deformed in this context is again $AdS_5 \times S^5$, however now Θ lies along the non–compact, AdS_5 piece of the geometry. Writting the metric on AdS_5 as:

$$ds^2_{AdS} = R^2 u^2 (dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{du^2}{u^2} \tag{2.17}$$

we see that the relevant part of the background matrix E_0 in this case is:

$$E_0 = R^2 u^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

and acting now on E_0 with the T–duality matrix $\mathcal{T}$ of equation (2.16) we obtain [49]:

$$\begin{aligned} ds^2_{str} &= u^2 R^2 (G_1(dx_0^2 + dx_1^2) + G_2(dx_2^2 + dx_3^2)) + \frac{R^2}{u^2}(du^2 + u^2 d\Omega_5^2) \\ B &= \hat{b}_1 R^2 G_1 u^4 dx_0 \wedge dx_1 + \hat{b}_2 R^2 G_2 u^4 dx_2 \wedge dx_3 \\ e^{2\Phi} &= G_1 G_2 e^{2\Phi_0}, \quad G_1 = \frac{1}{1+\hat{b}_1^2 u^4}, \quad G_2 = \frac{1}{1+\hat{b}_2^2 u^4} \\ \hat{b}_1 &= R^2 b_1, \quad \hat{b}_2 = R^2 b_2 \end{aligned} \tag{2.19}$$

which is the gravity dual [5] of a noncommutative gauge theory defined in Euclidean

[5]Note the resemblance between (2.12) and (2.19).

space with $[x^0, x^1] = ib_1$ and $[x^2, x^3] = ib_2$. The Langrangian description of this theory can be easily derived from the $\mathcal{N} = 4$ SYM Langrangian by replacing the ordinary product of functions with the Moyal star designated in (2.14).

Although we have so far considered applying this method directly to the near horizon geometry one can, perhaps even more appropriately, perform it on the p–brane solutions as well [52, 51, 53, 54]. The near horizon limit that needs to be taken in this case, requires a relative scaling between the B–field and the metric g which actually corresponds to the Seiberg–Witten limit proposed in [15].

It should now be evident that the solution generating transform employed by Lunin and Maldacena in order to find the gravity duals of β–deformed gauge theories is almost identical to the one used for the same purpose within the context of noncommutative gauge theories. The only difference is that in the former case it is the transverse space to the brane, or rather the compact part of the near horizon geometry that is being deformed. This naturally suggests interpreting the matrix $\mathbf{\Gamma}$ appearing in equation (2.9) as some kind of noncommutativity parameter. Since noncommutativity in this case is a property of the transverse space it manifests itself as a deformation of the matter content of the theory.

Before we proceed to the next section where we will further clarify this point, we would like to make some final remarks about the applicability of the solution generating transformations illustrated above. Despite the fact that this method has had a rather remarkable set of applications so far its utility is unfortunately restricted to the following conditions. First of all, the directions one wants to introduce fluxes — or equivalently noncommutativity — should be isometry directions realized geometrically, meaning as shift symmetries of the metric [55]. In addition, the noncommutativity matrix should have constant entries. Expressed in a more precise manner this means that there should exist a coordinate system where the noncommutativity is reduced to a constant along isometry directions of the metric.

As an example of this, let us consider the Melvin Twist gauge theory. This has been studied in [54, 56, 57]. The relevant noncommutativity parameter can be written in cartesian coordinates as [6]:

$$[x_2, x_3] = ibx_1, \quad [x_3, x_1] = ibx_2 \quad \text{and} \quad [x_1, x_2] = 0 \tag{2.20}$$

but in polar coordinates on the (x_1, x_2)–plane it becomes:

$$[\rho, \theta] = 0, \quad [\rho, x_3] = 0, \quad \text{and} \quad [\theta, x_3] = ib \tag{2.21}$$

In these coordinates $(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x_3})$ are indeed Killing vectors of the flat space metric and therefore the solution generating technique is applicable.

In general it seems reasonable to expect that given a noncommutativity parameter, the following two conditions should hold for a coordinate system to exist in which Θ^{ij} is reduced to a constant matrix:

$$\left.\begin{array}{r} \partial_i \Theta^{ij} = 0 \\ \Theta^{il}\partial_l\Theta^{jk} + \Theta^{kl}\partial_l\Theta^{ij} + \Theta^{jl}\partial_l\Theta^{ki} = 0 \end{array}\right\} \quad \Rightarrow \quad T^{[ijk]} = \partial_l(\Theta^{l[i}\Theta^{jk]}) = 0 \tag{2.22}$$

Although neither have we been able to find a proof of this nor have we come accross a proof of it in the literature, we find that it is natural to think of the second (associativity) condition in analogy with the vanishing of the Nijenhus tensor condition

[6] Here we consider the case of a non compact direction x_3 in contrast to the most widely used case.

for an almost complex structure [7]. We thus understand (2.22) as ensuring that one can always find a local coordinate system in order to put Θ^{ij} in a constant form. Then, the first condition in (2.22) can be read as the possibility of extending the local coordinates to global ones.[8]

We would like to conclude this section by stressing once more that (2.22) cannot be seen as the requirement for the solution generating transformation to work since there is no way to make sure that the coordinate transformation employed to bring Θ^{ij} into a constant form will not spoil the shift symmetries present in the metric. One example of this is the nongeometric background also referred to as the Q–space in the literature [58, 59, 60]. The relevant noncommutativity parameter in this case is:

$$[x_1, x_2] = ibx_3, \quad [x_1, x_3] = [x_2, x_3] = 0 \tag{2.23}$$

While it is obvious from the discussion above that Θ^{ij} can be reduced to a constant, the coordinate transformation that makes this possible is [60]: $x_1 \to y_1 y_3, x_2 \to y_2, x_3 \to y_3$ and in these coordinates the metric looks like:

$$ds^2 = -dt^2 + (y_1 dy_3 + y_3 dy_1)^2 + dy_2^2 + dy_3^2 \tag{2.24}$$

Indeed it has not been possible to embed this noncommutative deformation of flat space directly into string theory. It nevertheless naturally emerges when a D3–brane probe is immersed in the background of smeared NS5–branes.

[7] It may thus be interesting to formulate generalized complex geometry from the point of view of open strings.

[8] This is actually not true for the two–dimensional case, which is particularly simple. For instance, all noncommutative deformations are also associative ones.

2.4 β–deformations and noncommutativity

The aim of this section is to establish a precise relation between transverse space noncommutativity and β–deformations of $\mathcal{N} = 4$ SYM. In general the connection between marginal deformations and noncommutativity is not new. A study of the moduli space clearly points into this direction — a thorough analysis can be found in [61, 62, 63, 64]. The F–term constraints for instance read:

$$\Phi^I \Phi^J = q \Phi^J \Phi^I, \quad \overline{\Phi}^{\bar{I}} \overline{\Phi}^{\bar{J}} = q \overline{\Phi}^{\bar{J}} \overline{\Phi}^{\bar{I}} \quad \text{where } q = e^{2i\beta} \text{ and I,J are cyclically ordered.} \tag{2.25}$$

and Φ^I here indicate the first components of the corresponding superfields. These equations are usually understood to represent the space where the D–branes can move. For small enough deformations we can interpret the eigenvalues of these matrices as coordinates parametrizing the transverse space to the worldvolume of the D3–brane. The eignevalues should however now be thought of as noncommuting numbers according to equation (2.25). If we denote the coordinates of the moduli space as $(z^I, \overline{z}^{\bar{I}})$ with $I, \bar{I} = 1, 2, 3$ we have that:

$$z^I z^J = q z^J z^I, \quad \overline{z}^{\bar{I}} \overline{z}^{\bar{J}} = q \overline{z}^{\bar{J}} \overline{z}^{\bar{I}} \quad \text{with I,J cyclically ordered.} \tag{2.26}$$

Later on, it will become clear that a noncommutative interpretation is meaningful only when $\beta \in \mathbb{R}$. Henceforth we replace β with γ in order to avoid confusion and to be consistent with existing notations in the literature.

As it was mentioned in the previous section we can identify the prescription of [14] with the one used within the context of noncommutative gauge theories so long as matrix $\mathbf{\Gamma}$ appearing in equation (2.9) is the noncommutativity matrix associated to the deformation of the transverse space. Therefore, our main objective here is to construct a noncommutativity matrix, or rather a contravariant antisymmetric tensor field Θ^{IJ} to describe the deformed space. A natural way to define it is through the commutation relations implied by (2.26). That is:

$$[z^I, z^J] = i2e^{i\gamma}\sin\gamma z^I z^J \quad [\overline{z}^I, \overline{z}^J] = i2e^{i\gamma}\sin\gamma \overline{z}^{\bar{I}}\overline{z}^{\bar{J}} \tag{2.27}$$

Clearly such a definition would require a whole different notion of differential geometry since the noncommutativity parameter is position dependent and the coordinates themselves are now nonocommuting objects. We circumvent this by implementing an alternative procedure. As mentioned in the previous section one can replace noncommuting coordinates with commuting ones by defining a star product between them. In general, constructing an appropriate star product can be an equally formidable task as dealing with noncommuting variables. In this case however a natural proposal was set forth in [14]. Specifically, the authors of [14] suggested:

$$f * g = f e^{i\pi\beta\left(\overleftarrow{Q_1}\overrightarrow{Q_2} - \overleftarrow{Q_2}\overrightarrow{Q_2}\right)} g \tag{2.28}$$

where f, g belong to the set of chiral/antichiral multiplets of the theory and $Q_{1,2}$ are the global U(1) charges associated with these fields (see equation (2.2)). This proposal was subsequently used [29] in order to rewrite the component Lagrangian of the β–deformed gauge theory as the $\mathcal{N} = 4$ SYM Lagrangian with the product of matter fields now replaced by the above star product. This enabled the author of [29] to show that all the amplitudes in the planar limit of the deformed theory with $\beta \in \mathbb{R}$ are proportional to their $\mathcal{N} = 4$ counterparts. Note that the star here is not explicitly written in terms of derivatives acting on the fields (f, g). Knowledge of the product in this form however will be sufficient for the purposes of this letter.

In what follows we will use equation (2.28) in order to write down a noncommutativity matrix and compare it with (2.9). Then we will discuss ways to derive the appropriate Θ^{ij} without prior knowledge of the star product. We therefore define the noncommutativity parameter through the following relations:

$$\left.\begin{aligned} [z^I, z^J]_* &= \left(z^I * z^J - z^J * z^I\right) = i\Theta^{IJ} \\ [\overline{z}^{\bar{I}}, \overline{z}^{\bar{J}}]_* &= \left(\overline{z}^{\bar{I}} * \overline{z}^{\bar{J}} - \overline{z}^{\bar{J}} * \overline{z}^{\bar{I}}\right) = i\Theta^{\bar{I}\bar{J}} \\ [z^I, \overline{z}^{\bar{J}}]_* &= \left(z^I * \overline{z}^{\bar{J}} - \overline{z}^{\bar{J}} * z^I\right) = i\Theta^{I\bar{J}} \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} \Theta^{IJ} &= 2\sin\gamma z^I z^J \\ \Theta^{\bar{I}\bar{J}} &= 2\sin\gamma \overline{z}^{\bar{I}}\overline{z}^{\bar{J}} \\ \Theta^{I\bar{J}} &= -2\sin\gamma z^I \overline{z}^{\bar{J}} \end{aligned} \tag{2.29}$$

with (I, J) cyclically ordered. Setting $a \equiv 2\sin\gamma$ and writting this in matrix notation, we obtain:

$$\Theta = a \begin{pmatrix} 0 & z_1 z_2 & -z_1 z_3 & 0 & -z_1\overline{z}_2 & z_1\overline{z}_3 \\ -z_1 z_2 & 0 & z_2 z_3 & \overline{z}_1 z_2 & 0 & -z_2\overline{z}_3 \\ z_3 z_1 & -z_2 z_3 & 0 & -\overline{z}_1 z_3 & \overline{z}_2 z_3 & 0 \\ 0 & -\overline{z}_1 z_2 & \overline{z}_1 z_3 & 0 & \overline{z}_1\overline{z}_2 & -\overline{z}_1\overline{z}_3 \\ z_1\overline{z}_2 & 0 & -\overline{z}_2 z_3 & -\overline{z}_1\overline{z}_2 & 0 & \overline{z}_2\overline{z}_3 \\ -\overline{z}_3 z_1 & \overline{z}_3 z_2 & 0 & \overline{z}_1\overline{z}_3 & -\overline{z}_2\overline{z}_3 & 0 \end{pmatrix} \tag{2.30}$$

Clearly, the result obtained above is not exactly a satisfactory one. Despite the fact that we managed to describe the deformation of the transverse space in a noncommutative way, the associated noncommutativity matrix Θ is both position dependent and six dimensional. It does not therefore in any sense resemble to matrix $\mathbf{\Gamma}$ of equation (2.9). An additional interesting but perhaps perplexing feature of Θ is that it is not a purely holomorphic/antiholomorphic matrix as we might have expected from the F–term constraints. We will return to this point later in this section after we outline a more general prescription of identifying the appropriate Θ^{ij}.

Let us however proceed to make a coordinate transformation on (2.30). Since Θ^{IJ} thus defined is a contravariant tensor we have no trouble doing so. In other words we know that when changing coordinates from $\{x^i\}$ to $\{x'^{i'}\}$, the noncommutativity parameter transforms as:

$$\Theta^{i'j'} = \frac{\partial x'^{i'}}{\partial x^i}\frac{\partial x'^{j'}}{\partial x^j}\Theta^{ij} \tag{2.31}$$

Here, we chose to rewrite Θ^{IJ} in spherical coordinates $(r, \alpha, \theta, \phi_1, \phi_2, \phi_3)$ defined through:

$$\begin{aligned} z_1 &= r\cos\alpha \mathrm{e}^{i\phi_1}, \quad z_2 = r\sin\alpha\sin\theta\mathrm{e}^{i\phi_2}, \quad z_3 = r\sin\alpha\cos\theta\mathrm{e}^{i\phi_3} \\ \overline{z}_1 &= r\cos\alpha\mathrm{e}^{-i\phi_1}, \quad \overline{z}_2 = r\sin\alpha\sin\theta\mathrm{e}^{-i\phi_2}, \quad \overline{z}_3 = r\sin\alpha\cos\theta\mathrm{e}^{-i\phi_3} \end{aligned} \tag{2.32}$$

Note that in these coordinates we should be careful to define if possible the parameter γ of our matrix so as to have $\Theta \in \mathbb{R}$. Only then can Θ be interpreted as a noncommutativity parameter in the usual sense. Applying (2.31) to (2.30) we obtain in matrix notation:

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & a \\ 0 & 0 & 0 & a & 0 & -a \\ 0 & 0 & 0 & -a & a & 0 \end{pmatrix} \tag{2.33}$$

and we immediately see that we can indeed think of Θ as a noncommutativity matrix only when $a \in \mathbb{R}$. More importantly, from equation (2.33) it is clear that we can reduce Θ to the 3×3 matrix denoted as $\mathbf{\Gamma}$ in section 2.2. [9]. The only difference is that now the deformation parameter γ of the gauge theory is replaced by $a = 2\sin\gamma$. Recall however, that the Lunin–Maldacena solution (2.12) has small curvature only when: $\gamma R \ll 1$ and $R \gg 1$. Then $a \simeq 2\gamma$ and the solutions generated by using either Γ or Θ are basically equivalent. Yet we find it interesting that the periodicity of the parameter γ is manifest in this description. Nonetheless, note that this is not quite the correct periodicity condition. Our result is periodic when $\gamma \to \gamma + 2\pi$ whereas from (2.25) we expect: $\gamma \to \gamma + \pi$. The reason for this discrepancy lies in equation (2.29). Indeed, the two ways of defining deformed commutators, one in terms of commuting variables multiplied with a star product and the other in terms of noncommuting ones, are only strictly equivalent when the commutation relations are c–numbers. "Comparing" equations (2.29) and (2.27) in this case we see that there is a a phase difference between the parameters entering the two definitions. The absence of this phase in (2.29) is responsible for the discrepancy in periodicity. Nevertheless, the star product gives a more natural way to think of Θ^{ij} as a contravariant antisymmetric tensor thus having well defined transformation properties under a change of coordinates.

Suppose now that no precise definition of a star product between the superfields of the theory was known. Would we be able to construct the noncommutativity matrix and therefore find the gravity dual of the β–deformed gauge theory? A glance at the

[9]We can actually reduce Θ^{ij} even further using coordinates: $\psi = \frac{1}{3}\sum_{i=1}^{3}\phi_i, \sigma_1 = \frac{1}{3}(\phi_2 + \phi_3 - 2\phi_1), \sigma_2 = \frac{1}{3}(\phi_1 + \phi_3 - 2\phi_2)$. In this parametrization ψ denotes the U(1) circle associated with the R–symmetry of the original background and Θ^{ij} reads: $\Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{pmatrix}$. It is then obvious that the solution generating transformation does not act on the $U(1)_R$ therefore preserving $\mathcal{N} = 1$ supersymmetry.

superpotential of the theory would naturally lead us to define:

$$\Theta^{IJ} = 2\sin\gamma z^I z^J \quad \text{and} \quad \Theta^{\bar{I}\bar{J}} = 2\sin\gamma \bar{z}^{\bar{I}} \bar{z}^{\bar{J}} \tag{2.34}$$

and therefore correctly guess the purely holomorphic and purely antiholomorphic parts of Θ^{ij}. What about the other parts though? We can actually constrain the form of $\Theta^{I\bar{J}}$ by the following requirements:

- Definite Reality Properties.

  In order to be able to describe the deformation in noncommutative terms we should define the parameters appearing in Θ^{ij} so as to have a matrix with real entries after going to real coordinates.

- Symmetries.

  Since we expect the global symmetries of the Langrangian to be preserved in the strong coupling limit as well, we should ensure that the noncommutativity matrix respects those symmetries. This is true as long as [65]:

  $$[z^I, z^J] = i\Theta^{IJ}(z) \xrightarrow{z\to z'} [z'^I, z'^J] = i\Theta^{IJ}(z') \tag{2.35}$$

  Note that this is precisely analogous to the condition for a certain symmetry to be an isometry of the metric. Assuming that $\Theta^{I\bar{J}}$ is quadratic (2.35) implies that up to a sign there exist only two possibilities: $\Theta^{I\bar{J}} = 0$ or $\Theta^{I\bar{J}} = z^I \bar{z}^{\bar{J}}$.

- Marginality condition.

  According to the usual reasoning of AdS/CFT, marginal defromations should be described by AdS geometries with different compact pieces. This suggests that when the noncommutativity parameter is transformed in spherical coordinates, it should be independent of and have no components along the radial direction of AdS. In other words, $\frac{\partial \Theta^{a_i a_j}}{\partial r} = 0$ where a_i are angular variables parametrizing the five sphere and $\Theta^{r a_i} = 0$. This last requirement completely determines the form of $\Theta^{I\bar{J}}$ to be the one appearing in (2.29).

We see as remarkable as it may seem that there exists a *unique* noncommutativity matrix which respects the above conditions. Stated differently, simple gauge theory data and elementary notions from the AdS/CFT correspondence, made it possible to fully determine the form of Θ^{ij}. We thus want to understand this matrix as a way of encoding the deformation of the transverse space or in other words, the moduli space of the gauge theory — at least insofar as information relevant to the gauge/gravity duality in the large N limit is concerned. Indeed given the F–term constraints we seem to have extracted information coming from the D–terms. We can convince ourselves of this with the following observation. Recall that the β–deformation of $\mathcal{N} = 4$ SYM is exactly marginal and that the deformation enters only in the superpotential of the theory. This means that we wish not to deform the D–terms in the Lagrangian. Note however that we can write the D–terms of the $\mathcal{N} = 4$ theory as:

$$\mathrm{Tr}[\Phi_I, \tilde{\Phi}^I][\Phi_J, \tilde{\Phi}^J] = \mathrm{Tr}[\Phi_I, \Phi_J][\tilde{\Phi}^I, \tilde{\Phi}^J] + \mathrm{Tr}[\Phi_I, \tilde{\Phi}^J][\Phi_J, \tilde{\Phi}^I] \tag{2.36}$$

The first term on the right hand side of equation (2.36) is precisely the contribution to the potential coming from the F–terms. We then deduce that if we wish to retain the D–terms unaffected by the deformation of the F–term commutator we must induce an appropriate deformation on the commutator between holomorphic and antiholomorphic fields as well. Surprisingly enough, the reasoning outlined above seems to have granted us this exact piece of information.

It is now evident that we can identify the Lunin–Maldacena generating solution technique with the method employed in the case of noncommutative gauge theories [51]. The noncommutative data in this context are basically given to us from the gauge theory Lagrangian. This is quite natural since the deformations we are dealing with are exactly marginal. It is worth pointing out here that combined with the knowledge of the gravity dual of the parent $\mathcal{N} = 4$ theory, these data made it possible to find the gravity solution dual to the deformed theory. Unfortunately, this is not as general a statement as it may seem since the particular method employed was applicable only because there existed a coordinate system in which Θ^{IJ} was reduced to a constant and along isometry directions of the metric. In a forthcoming letter [66] we will nevertheless be able to extract some information on the gravity duals of the marginally deformed $\mathcal{N} = 4$ theory when the parameter ρ in (2.1) is different than zero.

2.5 Applications and New Backgrounds

In the previous section we were able to associate a specific noncommutativity matrix to the β–deformed gauge theory. We found that indeed there exists a coordinate system for which Θ^{ij} is position independent and lies along U(1) isometries of the transverse space metric as well as of the S^5. Identifying the solution generating transforms of [14] and [49] was then a straightforward task. This result naturally opens up two main directions for further study — the first one pertaining to noncommutative gauge theories and the second to deformations of $\mathcal{N} = 4$ SYM. In what follows we will try to touch upon several questions arising in the latter case. An extensive discussion and applications to noncommutative gauge theories for the interested reader is presented in A.

2.5.1 Matter–content deformations of $\mathcal{N} = 4$ SYM

A natural question to ask in this context is whether we can now borrow results pertaining to noncommutative gauge theories in order to explore different kinds of (super)potential deformations of $\mathcal{N} = 4$ SYM. A few cases where the solution generating technique was applicable were already mentioned in section 2.3. Consider for instance the original situation where a constant noncommutativity parameter is turned on. Here, we would like to translate this deformation to some kind of transverse space noncommutativity. If we parametrize our six dimensional space with complex coordinates $(z^I, \overline{z}^{\bar{I}})$, we can write:

$$[z^I, z^J] = ib, \quad [\overline{z}^{\bar{I}}, \overline{z}^{\bar{J}}] = ib, \quad [z^I, \overline{z}^{\bar{J}}] = -ib \quad \text{with I,J cyclically ordered} \tag{2.37}$$

We may then associate these commutation relations to a deformation of the gauge theory potential $\mathcal{V}$. Since the type of deformations considered in this section may generically break supersymmetry we prefer to state the deformation in terms of the potential which of course may when appropriate be promoted to the superpotential. Identifying the coordinates $(z^I, \overline{z}^{\bar{I}})$ with the scalar fields of the theory would naturally lead to[10]:

$$\mathcal{V}_{\mathcal{N}=4} = \text{Tr}[\Phi^I, \Phi^J][\Phi^{\bar{I}}, \Phi^{\bar{J}}] \to \text{Tr}[\Phi^I, \Phi^J][\Phi^{\bar{I}}, \Phi^{\bar{J}}]_*. \tag{2.38}$$

[10]This deformation is only meaningful for gauge groups other than SU(N).

Here the star product is defined according to (2.37) as: $\Phi^I * \Phi^J = \Phi^I\Phi^J + ib$. In a similar fashion we could relate the noncommutative deformation of the Melvin Universe which in complex coordinates looks like [11]:

$$[z_1, z_2] = -bz_1, \quad [\overline{z}_1, \overline{z}_2] = b\overline{z}_1, \quad [z_1, \overline{z}_2] = -bz_1 \quad [\overline{z}_1, z_2] = b\overline{z}_1 \\ \text{with all other commutators vanishing} \tag{2.39}$$

to a potential deformation of the same form as in (2.38) but with a different star product as indicated from (2.39).

Yet the true story is not as simple as this. These deformations are not marginal and the theory will generically flow from some UV point to an IR one. This means that we cannot solely rely on the data given to us from the Lagrangian of the theory which we can only take to be a valid description near the UV (small b). Moreover, the precise arguments that helped us construct the noncommutativity matrix encoding the moduli space in the β–deformed case are not applicable anymore. We do not therefore have a means of understanding the commutation relations between holomorphic and antiholomorphic fields/coordinates despite the fact that we believe such a cosntruction may be possible in the future. In addition we do not even know whether a noncommutative description of the transverse space will be valid throughout the flow [12].

Nevertheless, we could still expect to find the relevant supergravity solutions and use that as a means of understanding the precise gauge theory duals. Unfortunately this is again a difficult task to pursue because the solution generating technique

[11] Here we defined $z_1 \equiv x_1 + ix_2$ and $z_2 \equiv x_3 + ix_4$ with x_i as appearing in equation (2.20).

[12] Note however that it is possible to further examine this in certain cases, especially when some of the fields can be integrated out by considering the theory at appropriate energy scales.

discussed in this paper in not applicable anymore. The reason for this lies in the fact that the directions where the noncommutativity parameter is constant are not isometry directions of the transverse part of the D3–brane geometry. One could of course apply the T–duality transform on flat space. This would give rise to a deformed flat space geometry with non–trivial B–field and dilaton turned on, in which once D3–branes are immersed and the near horizon limit is taken, would result in the appropriate gravity dual. We think that it will be very interesting to explore this point further as well as to study the corresponding gauge theories which we schemmatically described above.

2.6 Discussion

In this chapter, we established a precise relation between noncommutativity and β–deformations of $\mathcal{N} = 4$ SYM theory. We first identified a specific matrix within the solution generating transform of [14, 40] which plays the role of noncommutativity parameter Θ^{ij} and then showed how it arises from the gauge theory point of view. Moreover, we explained that it is possible to fully specify Θ^{ij} by imposing requirements on its particular form naturally deduced from the gauge theory and AdS/CFT. We further argued that Θ^{ij} thus constructed encompasses all the relevant information on the moduli space of the gauge theory.

This hints at an alternative path in exploring deformations of the original AdS/CFT proposal [3] which consists in first specifying the associated *open* string parameters and then mapping them to the *closed* string ones. Here we investigated the former issue for the particular case of a Leigh–Strassler marginal deformation of the $\mathcal{N} = 4$ SYM theory. The mapping to the closed degrees of freedom in this case was granted to us in the form of T–duality transformation rules. In the next chapter, we will combine the basic reasoning set forth in this note with an attempt to address the latter issue in a situation where U(1) symmetries are absent and the T–duality prescription is not applicable. Such is the case for the superpotential deformation of equation (2.1) with $\rho \neq 0$.

Chapter 3

Open vs. Closed string parameters

3.1 Introduction

The AdS/CFT correspondence [67, 4, 5] offers an equivalence between gauge theory and gravity. In its original form, relates superconformal $\mathcal{N} = 4$ SU(N) Super Yang–Mills to closed string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$with N units of RR–flux. While closed strings on nontrivial backgrounds with RR–fluxes are still in many ways intractable, their low energy description in terms of supergravity is not. From the gauge theory point of view, this limit corresponds to large N and strong t'Hooft coupling λ. This makes the correspondence extremely useful in that it provides a window into understanding the physics of gauge theories in a region that is otherwise difficult to explore. By now the original proposal has been greatly extended so as to cover gauge theories with less amount of supersymmetry and/or a running coupling constant [47, 68, 69, 70, 71, 72]. Despite however the progress made thus far the natural question of how to specify the gravitational background corresponding to a given gauge theory, is still far from being understood [1]. In this chapter, we will attempt to give an alternative point of view on this matter while investigating exactly marginal deformations of the $\mathcal{N} = 4$ theory.

Supersymmetric deformations of the original AdS/CFT proposal have been extensively explored with the main cases of interest being, exactly marginal and relevant deformations. Whereas it would have been natural to think that the gravity dual backgrounds of the former would be more accessible than those of the latter it actually turned out the opposite. In fact the gravity duals of a class of supersymmetric mass deformations, all along their renormalization group flow, were discovered quite early on — see for example [73, 74, 75] and references therein — even though in most cases an analytic solution was provided only for the associated conformal fixed points. The main reason is that these backgrounds can be analyzed using the truncation to five–dimensional supergravity, something which is not possible for marginal deformations of the $\mathcal{N} = 4$ theory. Actualy it was only fairly recently that the authors of [14] succeeded in constructing the corresponding backgrounds for a subclass of these latter theories.

Marginal deformations of $\mathcal{N} = 4$ SYM preserve $\mathcal{N} = 1$ supersymmetry and are mainly described by two parameters, denoted as β and ρ, in addition to the gauge coupling g_{YM}. In [14] Lunin and Maldacena discovered the geometry dual to the β–deformed theory i.e. when $\rho = 0$. In this case, extra U(1) global symmetries are present which played a significant part in the construction of the new solution. When $\rho \neq 0$ however, the theory does not preserve any continuous symmetries other than the U(1) R–symmetry and the problem has resisted solution thus far. In this note, we

[1] For an existing approach to related issues see for instance [8, 9, 10, 11, 12, 13] and references therein.

construct the supergravity solution dual to the ρ–deformed theory up to third order in the deformation parameter. We propose a method which — although rudimentary at this stage — may provide additional means in exploring deformations of the original AdS/CFT proposal. The main idea, is to encode information from the gauge theory moduli space into the set of *open* string parameters $(\mathcal{G}, \Theta)$ and then map them to the *closed* string ones (g, B).

In the previous chapter, we achieved the first step in this direction by establishing a precise relation between transverse space noncommutativity and β–deformations of $\mathcal{N} = 4$ SYM. We identified the role of the noncommutativity matrix Θ in the solution generating technique of Lunin and Maldacena and showed how to explicitly construct it, relying on gauge theory data and basic notions of AdS/CFT. The resulting noncommutativity matrix Θ, along with the flat metric of the transverse space $\mathcal{G}_{flat}$, provided an alternative way of encoding the moduli space. They constitute the *open* string parameters $(\mathcal{G}, \Theta)$. Following here the method of [17] we acquire the corresponding set of open string data for the ρ–deformed gauge theory. In this case, it turns out that Θ defines a *nonassociative*, as well as noncommutative, deformation of the transverse space.

Having obtained the open string parameters, we move on to explore possible mappings to the closed string ones. It is natural to expect by T–duality, that the results of Seiberg and Witten [15] — see also [16] — relating noncommutativity to the presence of a nonvanishing B–field, will be valid independently of whether the noncommutative deformation is along or transverse to the D–brane. It follows that the equations which establish a connection between the open $(\mathcal{G}, \Theta)$ and closed (g, B) string fields in that setting, present a potential mapping of the associated parameters in this case as well. We therefore employ them in the context of the ρ–deformation and determine the deformed flat space background into where D–branes are immersed up to third order in the deformation parameter.

We then consider the effective action of the ρ–deformed gauge theory, obtained by giving a vacuum expectation value to one of the scalars and integrating out the massive fields. According to [76, 77, 78, 79, 80, 81], the leading IR large N part of this action should coincide with the DBI action for a D3–brane immersed in the dual background. We observe that in the case of the β–deformed gauge theory, the corresponding DBI action is characterized by the open string data $(\mathcal{G}_{AdS_5\times S^5}, \Theta)$ and that the associated NS–NS closed string fields (g, B) are part of the *exact* Lunin–Maldacena solution. This is not surprising. Indeed, the Lagrangian description of this theory can be given in terms of the $\mathcal{N} = 4$ Lagrangian with the product of matter fields replaced by a star product of the Moyal type. Subsequently, all amplitudes in the planar limit can be shown [29] to be proportional up to a phase to their $\mathcal{N} = 4$ counterparts. Then the open string data $(\mathcal{G}_{AdS_5\times S^5}, \Theta = 0)$ of the $\mathcal{N} = 4$ SYM theory are naturally promoted to the set $(\mathcal{G}_{AdS_5\times S^5}, \Theta)$. Can something similar occur for the ρ–deformation?

Since the noncommutativity parameter now defines a nonassociative deformation, it is hard to imagine how the planar equivalence with the parent $\mathcal{N} = 4$ theory could be achieved. Yet nonassociativity is a second–order in ρ effect and it turns out that up to third order in ρ this is indeed the case. In other words, mapping the open string fields to the closed ones provides again a supergravity solution to this order — the gravity dual of the ρ–deformed gauge theory.

The structure of this chapter is the following: In section 2 we review some known facts about marginal deformations of the $\mathcal{N} = 4$ theory and their gravity duals. In addition, we explore some special points in the deformation space for which the general theory with $\beta \neq 0$ and $\rho \neq 0$ is equivalent to an exactly marginal deformation with either $\widetilde{\beta} = 0$ or $\widetilde{\rho} = 0$. In section 3, we employ the method proposed in [17] in order to determine the relevant noncommutativity matrix for the ρ–deformation. Having acquired the open string data, we move on to map them to the closed string fields (g, B). This procedure is illustrated in section 4 where we derive the ρ–deformed flat space geometry up to third order in the deformation parameter. In section 5 we proceed with considerations on the DBI action which provide us with the gravity dual of the ρ–deformed theory to the same order. Finally, section 6, contains an extensive discussion on the ideas set forth in this note.

3.2 The Leigh Strassler Deformation

Not long after it was realized that $\mathcal{N} = 4$ SU(N) Super Yang Mills theory is finite (see e.g. [82] for an account), it became clear that it might not be the only four dimensional theory with that property [83, 84, 85, 86, 87]. It was however almost ten years later, when Leigh and Strassler undertook a systematic study of marginal deformations of $\mathcal{N} = 4$ and indeed showed that there exists a whole class of $\mathcal{N} = 1$ supersymmetric gauge theories satisfying both the requirements for conformal invariance and finiteness [19]. More precisely, they showed that the $\mathcal{N} = 4$ theory admits a three–complex–parameter family of marginal deformations which preserve $\mathcal{N} = 1$ supersymmetry and are described by the following superpotential:

$$\mathcal{W} = ih\mathrm{Tr}\left[\left(e^{i\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\beta}\Phi_1\Phi_3\Phi_2\right) + \rho\left(\Phi_1^3 + \Phi_2^3 + \Phi_3^3\right)\right] \tag{3.1}$$

where Φ^I with $I = 1, 2, 3$ are the three chiral superfields of the theory. Together with the gauge coupling g_{YM}, the complex parameters (h, β, ρ) that appear in the superpotential constitute the four couplings of the theory.

While it is clear at the classical level that these deformations are marginal — since all operators of the component Lagrangian have classical mass dimension equal to four — this is not necessarily true quantum mechanically. Leigh and Strassler realized that by using the constraints of $\mathcal{N} = 1$ supersymmetry and the exact NSVZ beta–functions [88, 89, 90] written in terms of the various anomalous dimensions of the theory, it was possible to express the conditions for conformal invariance of the quantum theory, through linearly dependent equations which were therefore likely to have nontrivial solutions. In this way, they were able to demonstrate that the deformation of (3.1) is truly marginal at the quantum level, so long as the four couplings of the theory satisfy a single complex constraint $\gamma(g_{YM}, \kappa, \beta, \rho) = 0$. In other words, there exists a three–complex–dimensional surface $\gamma(g_{YM}, \kappa, \beta, \rho) = 0$ in the space of couplings, where both beta functions and anomalous dimensions vanish and thus the $\mathcal{N} = 1$ gauge theories mentioned above are indeed conformally invariant. In general, the function γ is not known beyond two–loops [91, 92, 93, 30, 94] in perturbation theory, where it reads:

$$|h|^2 \left[\frac{1}{2} \left(|q|^2 + \frac{1}{|q|^2} \right) - \frac{1}{N^2} \left| q - \frac{1}{q} \right|^2 + |\rho|^2 \left(\frac{N^2 - 4}{2N^2} \right) \right] = g_{YM}^2 \tag{3.2}$$

with q defined as $q = \mathrm{e}^{i\beta}$ and N the number of colours of the gauge theory.

For the β–deformed gauge theory, *i.e.*, obtained by setting $\rho = 0$ in the superpotential of equation (3.1), the Leigh–Strassler constraint at two loops can be written as:

$$|h|^2\left[\frac{1}{2}\left(|q|^2+\frac{1}{|q|^2}\right)-\frac{1}{N^2}\left|q-\frac{1}{q}\right|^2\right]=g_{YM}^2 \tag{3.3}$$

In this case, one immediately notices that when $\beta=\beta_{\mathbb{R}}\in\mathbb{R}$ therefore $|q|=1$, (3.3) reduces to:

$$|h|^2\left[1-\frac{1}{N^2}\left|q-\frac{1}{q}\right|^2\right]=|h|^2\left(1-\frac{4}{N^2}\sin^2\beta_{\mathbb{R}}\right)=g_{YM}^2 \tag{3.4}$$

which in the large N limit yields: $|h|^2=g_{YM}^2$. Despite the fact that this result was obtained from the two–loop expression of the conformal invariance condition, it has been shown to be true to all orders in perturbation theory in the planar limit [29] (see also [31, 94]). Actually the author of [29] went even further and showed that all planar amplitudes in the $\beta=\beta_{\mathbb{R}}\in\mathbb{R}$ theory are proportional to their $\mathcal{N}=4$ counterparts, thus explicitly proving finiteness and conformal invariance. The proof made use of an existing proposal [14] for an equivalent "noncommutative" realization of the theory. For the more general case of complex $\beta=\beta_{\mathbb{R}}+i\beta_{\mathbb{I}}$, equation (3.3) in the planar limit reads:

$$\frac{1}{2}|h|^2\left(|q|^2+\frac{1}{|q|^2}\right)=|h|^2\cosh\left(2\beta_{\mathbb{I}}\right)=g_{YM}^2 \tag{3.5}$$

It is then evident that the coupling constant h receives corrections with respect to its $\mathcal{N}=4$ SYM value. Nevertheless, diagrammatic analysis [29] showed that all planar amplitudes with external gluons are equal to those of the $\mathcal{N}=4$ theory up to a five–loop level. To this order and beyond, it is most likely that the planar equivalence between the parent theory and its deformation will break down. For (more) recent investigation on β–deformations from the gauge theory point of view see [30, 94, 32, 34, 95, 96, 33].

Special points along the deformation occur when β is a root of unity. These points have been studied early on [97, 61] with a dual interpretation as orbifolds with discrete torsion. The marginally deformed theories have been further explored in [98, 99, 100], and several remarkable properties have been demonstrated. In particular it was shown that as expected, the S–duality of $\mathcal{N} = 4$ extends to their space of vacua, and that, again for special values of β, there are also new Higgs branches on moduli space. These are mapped by S–duality to completely new, confining branches which appear only at the quantum level. Furthermore, at large N the Higgs and confining branches can be argued to be described by Little String Theory [100]. Finally, the possibility of an underlying integrable structure for the deformed theories in analogy with $\mathcal{N} = 4$ SYM, was investigated at special values of the deformation parameter in [26, 101] and for generic β in [27, 28, 102, 103].

3.2.1 Marginal deformations and gauge/gravity duality

A natural place to explore theories that arise as marginal deformations of $\mathcal{N} = 4$ SU(N) SYM is the AdS/CFT correspondence where the strong coupling regime of the undeformed theory is realized as weakly coupled supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Due to superconformal symmetry, the dual gravitational description of these theories is expected to be of the form: $\mathrm{AdS}_5 \times \tilde{\mathrm{S}}^5$ with $\tilde{\mathrm{S}}^5$ a sphere deformed by the presence of additional NS–NS and RR fluxes. Indeed in [104], where the dual background was constructed to second order in the deformation parameters, it was shown that apart from the already present five–form flux one should also turn on (complexified) three–form flux $G_{(3)}$ along the S^5.

Essential progress however in this direction was only recently achieved through the work of Lunin and Maldacena [14]. The authors of [14] succeeded in finding the *exact* gravity dual of the β–deformed gauge theory.

In this case, apart from the $U(1)_R$ R–symmetry the theory preserves two global $U(1)$s, which act on the superfields in the following way:

$$\begin{aligned} U(1)_1: &\quad (\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, \mathrm{e}^{i\alpha_1}\Phi_2, \mathrm{e}^{-i\alpha_1}\Phi_3) \\ U(1)_2: &\quad (\Phi_1, \Phi_2, \Phi_3) \to (\mathrm{e}^{-i\alpha_2}\Phi_1, \mathrm{e}^{i\alpha_2}\Phi_2, \Phi_3) \end{aligned} \tag{3.6}$$

The main idea underlying the solution generating technique proposed in [14], was the natural expectation that the two U(1) symmetries preserved by the deformation would be realized geometrically in the dual gravity solution. For $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ their prescription amounts to performing an $SL(2,\mathbb{R})$ transformation on the complexified Kähler modulus τ of the two torus associated with the U(1) symmetries in question. The specific element of $SL(2,\mathbb{R})$ under consideration is: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$. It is chosen so as to ensure that the new solution will present no singularities as long as the original one is non–singular and its sole free parameter c is naturally identified with the real deformation parameter $\beta_{\mathbb{R}}$ of the gauge theory.

Later on, the method of Lunin and Maldacena was reformulated [40] in terms of the action of a T–duality group element on the background matrix $E = g + B$ providing a significantly easier way of obtaining the new solutions. In particular, it was shown [40] that one can embed the $SL(2,\mathbb{R})$ that acts on the Kähler modulus into the T–duality group $O(3,3,\mathbb{R})$ in the following way:

$$\mathcal{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{\Gamma} & \mathbf{1} \end{pmatrix} \quad \text{where now} \quad \mathbf{\Gamma} \equiv \begin{pmatrix} 0 & -\beta_{\mathbb{R}} & \beta_{\mathbb{R}} \\ \beta_{\mathbb{R}} & 0 & -\beta_{\mathbb{R}} \\ -\beta_{\mathbb{R}} & \beta_{\mathbb{R}} & 0 \end{pmatrix} \tag{3.7}$$

where $\mathbf{1}$ and $\mathbf{0}$ represent the 3×3 identity and zero matrices respectively. Suppose then that $E_0 = g_0 + B_0$ denotes the part of the original supergravity background along the U(1) isometry directions which are to be deformed. Acting on E_0 with the T–duality group element $\mathcal{T}$ of (3.7) one obtains the NS–NS fields of the deformed solution in terms of E_0 and Γ according to:

$$\begin{aligned} E &= \frac{1}{E_0^{-1} + \mathbf{\Gamma}} \\ e^{2\Phi} &= e^{2\Phi_0} \det(1 + E_0 \mathbf{\Gamma}) \equiv e^{2\Phi_0} G \end{aligned} \tag{3.8}$$

The RR-fields of the background can be computed using the T–duality transformation rules of [42, 43, 44, 45, 46], however the details of this transformation need not concern us here. As an example, let us consider ten–dimensional flat space parametrized as:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{\mu=1}^{3} \mathrm{d}x^\mu \mathrm{d}x_\mu + \sum_{i=1}^{3} (\mathrm{d}r_i^2 + r_i^2 \mathrm{d}\varphi_i^2) \tag{3.9}$$

In this case E_0 will contain the components of the flat metric along the polar angles

φ_i. Applying equations (3.8) yields:

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \sum_{\mu=1}^{3} \mathrm{d}x^\mu \mathrm{d}x_\mu + \sum_{i=1}^{3}(\mathrm{d}r_i^2 + G r_i^2 \mathrm{d}\varphi_i^2) + \beta_{\mathbb{R}} G r_1^2 r_2^2 r_3^2 \left(\sum_{i=1}^{3} \mathrm{d}\varphi_i\right)^2 \\
e^{2\Phi} &= G, \quad G^{-1} = 1 + \beta_{\mathbb{R}}^2 \left(\sum_{i\neq j} r_i^2 r_j^2\right), \quad B = \beta_{\mathbb{R}} G \left(\sum_{i\neq j} r_i^2 r_j^2 \mathrm{d}\varphi_i \mathrm{d}\varphi_j\right)
\end{aligned} \tag{3.10}$$

This is the deformed flat space geometry where by placing D3–branes at the origin and taking the near horizon limit, one obtains the gravity dual to the β–deformed gauge theory. Alternatively, the latter background can be constructed by applying (3.8) on $\mathrm{AdS}_5 \times \mathrm{S}^5$representing the dual gravitational description of the undeformed parent $\mathcal{N} = 4$ theory:

$$\begin{aligned}
\mathrm{d}s^2 =& \mathrm{R}^2(\mathrm{d}s^2_{\mathrm{AdS}_5} + \mathrm{d}s^2_5), \quad \text{where :} \quad \mathrm{d}s^2_5 = \sum_i (\mathrm{d}\mu_i^2 + G\mu_i^2 \mathrm{d}\varphi_i^2) + \hat{\beta} G \mu_1^2\mu_2^2\mu_3^2 (\sum_i \mathrm{d}\varphi_i)^2 \\
& e^{2\Phi} = e^{2\Phi_0} G, \quad G^{-1} = 1 + \hat{\beta}^2 (\sum_{i\neq j} \mu_i^2 \mu_j^2), \qquad \hat{\beta} = \mathrm{R}^2 \beta_{\mathbb{R}}, \qquad \mathrm{R}^4 = 4\pi e^{\Phi_0}\mathrm{N} \\
& B = \hat{\beta} R^2 G \left(\sum_{i\neq j} \mu_i^2 \mu_j^2 \mathrm{d}\varphi_i \mathrm{d}\varphi_j\right) \quad C_2 = -\beta_{\mathbb{R}}(16\pi N)\omega_1 (\sum_i \mathrm{d}\varphi_i) \\
& F_5 = (16\pi N)(\omega_{\mathrm{AdS}_5} + G\omega_{\mathrm{S}^5}), \quad \omega_{\mathrm{S}^5} = \mathrm{d}\omega_1 \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \mathrm{d}\varphi_3, \quad \omega_{\mathrm{AdS}_5} = \mathrm{d}\omega_4
\end{aligned} \tag{3.11}$$

Reformulating the Lunin–Maldacena generating solution technique in terms of the T–dualty group action, made especially transparent its relation to similar methods employed in the context of noncommutative gauge theories [2]. In the previous chapter, we identified the prescriptions used in these cases and particularly showed that matrix Γ of (3.7) is precisely the noncommutativity matrix Θ associated with the de-

[2] In fact, evidence relating marginal deformations and noncommutativity was given earlier both at strong [61] and weak [105, 106] coupling.

formation of the transverse space. Moreover, we proposed a method for determining Θ based solely on information from the gauge theory Lagrangian and basic notions of AdS/CFT. The main idea was to think of the matter fields as coordinates $(z^I, \bar{z}^{\bar{I}})$ parametrizing the space transverse to the D3–brane where the gauge theory lives. Then, reality properties, global symmetries and marginality constrained the form of Θ to be:

$$\Theta_\beta = a \begin{pmatrix} 0 & z_1 z_2 & -z_1 z_3 & 0 & -z_1 \bar{z}_2 & z_1 \bar{z}_3 \\ -z_1 z_2 & 0 & z_2 z_3 & \bar{z}_1 z_2 & 0 & -z_2 \bar{z}_3 \\ z_3 z_1 & -z_2 z_3 & 0 & -\bar{z}_1 z_3 & \bar{z}_2 z_3 & 0 \\ 0 & -\bar{z}_1 z_2 & \bar{z}_1 z_3 & 0 & \bar{z}_1 \bar{z}_2 & -\bar{z}_1 \bar{z}_3 \\ z_1 \bar{z}_2 & 0 & -\bar{z}_2 z_3 & -\bar{z}_1 \bar{z}_2 & 0 & \bar{z}_2 \bar{z}_3 \\ -\bar{z}_3 z_1 & \bar{z}_3 z_2 & 0 & \bar{z}_1 \bar{z}_3 & -\bar{z}_2 \bar{z}_3 & 0 \end{pmatrix} \tag{3.12}$$

with $a = 2\sin\beta_{\mathbb{R}}$. While it may seem that Θ_β is dissimilar from matrix Γ of (3.7), a coordinate transformation from $(z^I, \bar{z}^{\bar{I}})$ to polar coordinates (r_i, φ_i) on $\mathbb{R}^6$ reveals that:

$$\Theta_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & a \\ 0 & 0 & 0 & a & 0 & -a \\ 0 & 0 & 0 & -a & a & 0 \end{pmatrix} \tag{3.13}$$

thereby proving their identification [3].

For the general case of complex β, to obtain the dual background, one needs to perform an additional $SL(2,\mathbb{R})_s$ transformation on the solution corresponding to $\beta_{\mathbb{R}}$.

[3]Note that the Lunin–Maldacena technique is valid for small $\beta_{\mathbb{R}}$ in which case $a = 2\beta_{\mathbb{R}}$

By $SL(2,\mathbb{R})_s$ we denote here the $SL(2,\mathbb{R})$ symmetry of ten dimensional type IIB supergravity which acts nontrivially on the compexified scalar and two–form fields of the theory. Being a symmetry of the equations of motion it can be used to generate distinct solutions. Subsequent work on the subject of the β–deformed gauge theories has provided further checks of the AdS/CFT correspondence [20, 21, 22, 23, 24, 25] whereas generalizations as well as applications of the solution generating technique introduced in [14] were considered in [36, 28, 25, 37, 38, 39].

3.2.2 Special points along the general Leigh–Strassler deformation

In this article we will be mainly interested in the ρ–deformed gauge theories. In this case — when $\rho \neq 0$ — the theory does not preserve additional U(1) symmetries, it is however invariant under a global discrete symmetry $\mathbb{Z}_3 \times \mathbb{Z}_3$ acting on the superfields as:

$$\begin{aligned} \mathbb{Z}_{3(1)}: &\quad (\Phi_1, \Phi_2, \Phi_3) \to (\Phi_3, \Phi_1, \Phi_2) \\ \mathbb{Z}_{3(2)}: &\quad (\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, \mathrm{e}^{\frac{i2\pi}{3}}\Phi_2, \mathrm{e}^{\frac{-i2\pi}{3}}\Phi_3) \end{aligned} \tag{3.14}$$

As previously mentioned, the presence of global U(1)s is crucial in the solution generating technique of Lunin and Maldacena which is therefore not applicable here. In fact, the exact gravity dual for this case is still unknown. Despite however that the absence of extra continuous symmetries makes the cases of $\rho = 0$ and $\rho \neq 0$ radically different, there exist special points along the space of couplings where the two theories are not only similar but actually equivalent.

As first pointed out in [61] — see also [101, 107]— it is possible to start with either the set $(\beta, \rho) = (\beta, 0)$ or $(\beta, \rho) = (0, \rho)$, and via a field redefinition reach a point in the deformation space with $(\widetilde{\beta} \neq 0, \widetilde{\rho} \neq 0)$. The final point will obviously not represent the most general deformation, since the new couplings $\widetilde{\beta}$ and $\widetilde{\rho}$ will be given in terms of the original parameter. In other words, there will exist a function $f(\widetilde{\beta}, \widetilde{\rho}) = 0$ relating the two. Furthermore, requiring that the field redefinition be the result of a unitary transformation imposes a restriction on the original value of the coupling; be it β or ρ. In particular, suppose that we consider the marginally deformed theory at the point $(\beta, \rho = 0)$ and then take:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \to \begin{pmatrix} A & A & A \\ B & \omega B & \omega^2 B \\ C & \omega^2 C & \omega C \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \tag{3.15}$$

with Φ_I the three chiral superfields and $\omega = \mathrm{e}^{i2\pi/3}$ the third root of unity. Note here that since the deformation enters only in the superpotential, it suffices to consider transformations that affect the chiral fields independently from the antichiral ones. In other words, we do not expect mixing between holomorphic and antiholomorphic pieces. If we furthermore impose the following conditions on the free parameters A,B,C: $|A| = |B| = |C| = \frac{1}{\sqrt{3}}$ and $ABC = \pm \frac{i\lambda}{3\sqrt{1+2\cos 2\beta}}$ with $\lambda \in \mathbb{C}$, we find that the original β–deformed gauge theory is equivalent to the marginally deformed $\mathcal{N} = 4$ SYM theory with coupling constants:

$$\widetilde{\rho} = \pm \frac{2\sin\beta}{3\sqrt{1+2\cos 2\beta}} \quad \text{and} \quad \mathrm{e}^{i\widetilde{\beta}} = \pm \frac{2\cos\left(\beta - \frac{\pi}{6}\right)}{\sqrt{1+2\cos 2\beta}} \tag{3.16}$$

provided that $\beta = \beta_{\mathbb{R}} + i\beta_{\mathbb{I}}$ satisfies the following equation:

$$4\cos 2\beta_{\mathbb{R}} \cos 2\beta_{\mathbb{I}} + 4\cos^2 2\beta_{\mathbb{R}} + 4\cos^2 2\beta_{\mathbb{I}} - 3\,(1+3\lambda) = 0 \tag{3.17}$$

Solutions to (3.17) define special regions in the coupling constant space where the Leigh–Strassler theory with generic β and $\rho = 0$ is equivalent to a theory with both $\widetilde{\beta}$ and $\widetilde{\rho}$ nonvanishing but constrained to satisfy a specific relation dictated from (3.16). It is worth remarking here that there is no solution of (3.16) and (3.17) for which *both* β and $\widetilde{\beta}$ are real. This is particularly interesting, because it is only for the β–deformed gauge theory with $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ that a precise connection with noncommutativity is possible. It is natural to wonder whether distinct unitary field redefinitions of a type similar to (3.15) could take us from different β's to different $\widetilde{\beta}$ and $\widetilde{\rho}$. It is however not hard to deduce that up to a phase in $\widetilde{\rho}$ — which can be reabsorbed in the definition of the coupling constant h — and a sign in $\widetilde{\beta}$, all such unitary transformations share the same starting point (3.17) and lead to the same theory (3.16).

In an analogous manner one can find specific values of ρ for which the theory with $\beta = 0$ is equivalent to another one with both couplings $\widetilde{\beta}$ and $\widetilde{\rho}$ turned on. Detailed analysis in this case shows in fact that such a mapping is possible for *any* original value of ρ with parameters $\widetilde{\rho}$ and $\widetilde{\beta}$ given by:

$$\widetilde{\rho}^2 = -\frac{\rho^2}{\rho^2+3}, \quad \text{and} \quad \sin^2\widetilde{\beta} = -\widetilde{\rho}^2 = \frac{\rho^2}{\rho^2+3} \tag{3.18}$$

The precise field redefinition through which this is achieved, is of the form of (3.15):

$$\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix} \rightarrow \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix} \tag{3.19}$$

Note here again that $\widetilde{\beta} = \widetilde{\beta}_{\mathbb{R}} \in \mathbb{R}$ if and only if $\rho \in \mathbb{R}$ which implies that $\widetilde{\rho} \in \mathbb{I}$. If one additionaly assumes that $\widetilde{\beta}_{\mathbb{R}} \in \mathbb{R} \ll 1$ then the deformed theory with $\beta = 0$ and $\rho = q_1 \in \mathbb{R}$ is equivalent to a theory with $2\sin\widetilde{\beta} = \pm 2\frac{q_1}{\sqrt{3}}$ and $\widetilde{\rho} = \pm i\frac{q_1}{\sqrt{3}} \in \mathbb{I}$. In section 3, we will see that this particular point in the deformation space naturally shows up in the noncommutative description of the moduli space. This will provide us with a non–trivial check on the consistency of the noncommutative interpretation.

So far we have looked at special points in the space of couplings which can be studied at the level of the gauge theory lagrangian. There are however a couple of interesting observations one can additionaly make on the basis of the Leigh–Strassler constraint as this is given in equation (3.2). Notice first that (3.2) reduces in the planar limit to:

$$|h|^2\left[\frac{1}{2}\left(|q|^2 + \frac{1}{|q|^2}\right) + \frac{1}{2}|\rho|^2\right] = |h|^2\left[\cosh\left(2\beta_{\mathbb{I}}\right) + \frac{1}{2}|\rho|^2\right] = g_{YM}^2 \tag{3.20}$$

This implies that when $\rho \neq 0$ the coupling constant h at the conformal fixed point will be different from g_{YM}, in contrast to what happens for $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$. In this sense, turning on ρ is similar to turning on the imaginary part of $\beta = \beta_{\mathbb{I}}$. Yet, there seems to exist a particular point in the deformation space for which $h = g_{YM}$ continues to hold in the large N limit. This occurs when:

$$\cosh{(2\beta_{\mathbb{I}})} + \frac{1}{2}|\rho|^2 = 1 \Rightarrow \beta_{\mathbb{I}} = \frac{1}{2}\arg\cosh{(1-\frac{|\rho|^2}{2})} \tag{3.21}$$

Closer inspection however of (3.21) reveals that it has no possible solutions, assuming $\beta_{\mathbb{I}} \in \mathbb{R}$ and $|\rho| > 0$. This implies that despite appearances, there is no special point for which $h = g_{YM}$ at two loops in the planar limit. Naturally, one expects that an analogous equation relating the two couplings, for which $h = g_{YM}$ at large N, may arise at any order in perturbation theory. What is not clear of course, is whether it will generically have any solutions or not.

3.3 Marginal deformations and Noncommutativity

In chapter 2 we showed that for the β–deformed gauge theory it is possible to construct a noncommutativity matrix Θ encoding in a precise manner information on the moduli space of the theory. This construction is very simple and is based on fundamental properties of the gauge theory and AdS/CFT. In what follows we will adopt the reasoning outlined in the previous chapter, in order to determine a noncommutativity matrix for the ρ–deformation. We set $\beta = 0$ for the time being and later on discuss how to incorporate $\beta \neq 0$.

Our starting point is the F–term constraints:

$$\begin{aligned} &\Phi_1\Phi_2 = \Phi_2\Phi_1 + \rho\Phi_3^2, \quad \Phi_2\Phi_3 = \Phi_3\Phi_2 + \rho\Phi_1^2, \quad \Phi_3\Phi_1 = \Phi_1\Phi_3 + \rho\Phi_2^2 \\ &\overline{\Phi}_1\overline{\Phi}_2 = \overline{\Phi}_2\overline{\Phi}_1 - \overline{\rho}\overline{\Phi}_3^2, \quad \overline{\Phi}_2\overline{\Phi}_3 = \overline{\Phi}_3\overline{\Phi}_2 - \overline{\rho}\overline{\Phi}_1^2, \quad \overline{\Phi}_3\overline{\Phi}_1 = \overline{\Phi}_1\overline{\Phi}_3 - \overline{\rho}\overline{\Phi}_2^2 \end{aligned} \tag{3.22}$$

from which we read the holomorphic and antiholomorphic parts of Θ interpreting the eigenvalues of these matrices in the large N limit, as noncommuting coordinates parametrizing the space transverse to the worldvolume of the D3–brane. More precisely we have:

$$\begin{aligned} &[z_1, z_2] = \rho z_3^2, \quad [z_2, z_3] = \rho z_1^2, \quad [z_3, z_1] = \rho z_2^2 \\ &[\overline{z}_1, \overline{z}_2] = -\overline{\rho z}_3^2, \quad [\overline{z}_2, \overline{z}_3] = -\overline{\rho z}_1^2, \quad [\overline{z}_3, \overline{z}_1] = -\overline{\rho z}_2^2 \end{aligned} \tag{3.23}$$

Following again the steps of chapter 2, we would like to assume that there exists a star product between some *commuting* variables $z^I, \overline{z}^{\bar{I}}$ which leads to commutation relations analogous to (3.23), so that we can write for instance: $i\Theta^{12} = [z_1, z_2]_* = z_1 * z_2 - z_2 * z_1 = \rho z_3^2$. This enables us to define a noncommutativity matrix which although position dependent, its entries are ordinary commuting objects. Then, under a change of coordinates Θ will transform as a contravariant antisymmetric tensor field. We therefore write Θ in matrix form as:

$$\Theta = \begin{pmatrix} 0 & i\rho z_3^2 & -i\rho z_2^2 & & & \\ -i\rho z_3^2 & 0 & i\rho z_1^2 & & ? & \\ i\rho z_2^2 & -i\rho z_1^2 & 0 & & & \\ & & & 0 & -i\overline{\rho}\overline{z}_3^2 & i\overline{\rho}\overline{z}_2^2 \\ & ? & & i\overline{\rho}\overline{z}_3^2 & 0 & -i\overline{\rho}\overline{z}_1^2 \\ & & & -i\overline{\rho}\overline{z}_2^2 & i\overline{\rho}\overline{z}_1^2 & 0 \end{pmatrix} \tag{3.24}$$

It is clear from (3.24) that the F–term constraints determine the (2,0) and (0,2) parts of Θ. D–terms will in principle specify the (1,1) piecies of the noncommutativity matrix. However, as demonstrated in section 2.4, there is an alternative indirect way of acquiring the information pertaining to D–terms. Recall that for the β–deformed gauge theory it was possible to fully determine Θ by imposing certain simple conditions on its form — namely definite reality properties, symmetries and marginality. If and only if, there exists a choice for the $\Theta^{I\bar{I}}$ components of the noncommutativity matrix and the parameter ρ which respects these requirements, can we hope to describe the deformation in noncommutative terms [4]. We will see in the following that this is indeed the case here.

Let us first find out what are the possible (1,1) pieces of Θ which respect the symmetries of the theory. Consider for instance the commutator $[z^1, \overline{z}^{\bar{2}}] = i\Theta^{1\bar{2}}(z, \overline{z})$. We easily see that: $[z^1, \overline{z}^{\bar{2}}] \xrightarrow{\mathbb{Z}_{3(2)}} \mathrm{e}^{-\frac{i2\pi}{3}}[z^1, \overline{z}^{\bar{2}}]$. This constrains $\Theta^{1\bar{2}}$ to either vanish or be a combination of any of the following: $\overline{z}^{\bar{1}}z^3, \overline{z}^{\bar{3}}z^2, z^1\overline{z}^{\bar{2}}$. Note that all of the choices displayed are also invariant under the other discrete symmetry of the theory $\mathbb{Z}_{3(1)}$ as they should. In a similar fashion, one can determine all the other possible components of $\Theta^{I\bar{J}}$. Overall, this yields a plethora of potential noncommutativity parameters. Transforming however Θ to spherical coordinates [5] and requiring that it be real, transverse to and independent of the radial direction r, uniquely determines Θ. To be more precise, there remain *two* different possibilities for Θ^{IJ}. One of them is valid for $\rho \equiv -q_1 \in \mathbb{R}$:

$$\Theta_1 = iq_1 \begin{pmatrix} 0 & z_3^2 & -z_2^2 & 0 & -z_3\bar{z}_1+z_2\bar{z}_3 & z_2\bar{z}_1-z_3\bar{z}_2 \\ -z_3^2 & 0 & z_1^2 & \bar{z}_2 z_3-z_1\bar{z}_3 & 0 & -z_1\bar{z}_2+z_3\bar{z}_1 \\ z_2^2 & -z_1^2 & 0 & -z_2\bar{z}_3+z_1\bar{z}_2 & z_1\bar{z}_3-z_2\bar{z}_1 & 0 \\ 0 & -z_3\bar{z}_2+z_1\bar{z}_3 & z_2\bar{z}_3-z_1\bar{z}_2 & 0 & -\bar{z}_3^2 & \bar{z}_2^2 \\ z_3\bar{z}_1-z_2\bar{z}_3 & 0 & -z_1\bar{z}_3+z_2\bar{z}_1 & \bar{z}_3^2 & 0 & -\bar{z}_1^2 \\ -z_2\bar{z}_1+z_3\bar{z}_2 & z_1\bar{z}_2-z_3\bar{z}_1 & 0 & -\bar{z}_2^2 & \bar{z}_1^2 & 0 \end{pmatrix} \tag{3.25}$$

[4]Note for instance, that this description was not valid for the β–deformed theory when $\beta \in \mathbb{I}$.
[5]Refer to appendix B for the noncommutativity matrix in different coordinate systems.

and the other one, for $\rho \equiv iq_2$ with $q_2 \in \mathbb{R}$:

$$\Theta_2 = q_2 \begin{pmatrix} 0 & z_3^2 & -z_2^2 & 0 & z_3\overline{z}_1+z_2\overline{z}_3 & -z_2\overline{z}_1-z_3\overline{z}_2 \\ -z_3^2 & 0 & z_1^2 & -\overline{z}_2 z_3-z_1\overline{z}_3 & 0 & z_1\overline{z}_2+z_3\overline{z}_1 \\ z_2^2 & -z_1^2 & 0 & z_2\overline{z}_3+z_1\overline{z}_2 & -z_1\overline{z}_3-z_2\overline{z}_1 & 0 \\ 0 & z_3\overline{z}_2+z_1\overline{z}_3 & -z_2\overline{z}_3-z_1\overline{z}_2 & 0 & \overline{z}_3^2 & -\overline{z}_2^2 \\ -z_3\overline{z}_1-z_2\overline{z}_3 & 0 & z_1\overline{z}_3+z_2\overline{z}_1 & -\overline{z}_3^2 & 0 & \overline{z}_1^2 \\ z_2\overline{z}_1+z_3\overline{z}_2 & -z_1\overline{z}_2-z_3\overline{z}_1 & 0 & \overline{z}_2^2 & -\overline{z}_1^2 & 0 \end{pmatrix} \tag{3.26}$$

Combining the two into $\Theta_\rho = \Theta_1 + \Theta_2$ we define a unique noncommutativity matrix Θ describing the ρ–deformation for general complex $\rho = (-q_1 + iq_2) \in \mathbb{C}$. This presumably indicates that a noncommutative description of the transverse space is valid thoughout the whole of the ρ parameter space, contrary to what happens for the β–deformed gauge theory.

Let us now examine Θ in order to determine its properties. Recall that the noncommutativity parameter for the β–deformed theory, turned out to be position independent along isometry directions of the metric. This was crucial for employing the Lunin–Maldacena generating technique. In this case, we obviously do not expect Θ to be constant along isometry directions since we know that the ρ–deformed theory does not respect any other global U(1) symmetries except for the R–symmetry. Indeed, Θ is of a highly nontrivial form even when written in spherical coordinates (See appendix B). We may however hope to find a coordinate system for which Θ is position independent, even if not along isometry directions [6]. Recall that we already presented what we believe are the two necessary and sufficient conditions for this to occur:

$$\left.\begin{aligned} \partial_i\Theta^{ij} = 0 \\ T^{[ijk]} \equiv \Theta^{il}\partial_l\Theta^{jk} + \Theta^{kl}\partial_l\Theta^{ij} + \Theta^{jl}\partial_l\Theta^{ki} = 0 \end{aligned}\right\} \Rightarrow \quad T^{[ijk]} = \partial_l(\Theta^{l[i}\Theta^{jk]}) = 0 \tag{3.27}$$

[6]This is the case for the nongeometric Q–space [58, 59, 60], for instance.

It is actually easy to see that although the first condition is satisfied by the noncommutativity matrix thus determined, the second one is not. This implies that contrary to the β–deformation, there doesn't exist a coordinate system in which Θ can be put in constant form [7]. More importantly however, nonassociativity makes the task of explicitly constructing an appropriate star product for the scalar fields a rather non–trivial one. In fact, we have not so far been able to find a star product so as to rewrite the Lagrangian of the ρ–deformed gauge theory as that of the $\mathcal{N} = 4$ Lagrangian with the usual product between the matter content of the theory replaced by the star product. We will further address this issue in relation to the applicability of the method used in this article in section 7.

Finally, note that failure of associativity stems particularly from the (1,1) parts of the noncommutativity matrix thus naturally challenging our method for determining them. There exists however what we believe to be a highly non–trivial check that we have constructed the correct Θ describing the deformation. We saw in the previous section, that for some special points in the space of couplings of the marginally deformed theory, one can move from a theory where either β or ρ (but not both) is turned on, to a theory where both couplings are nonvanishing. The whole analysis as well as the appropriate field redefinitions which took us from one point to the other in the deformation space, relied on the holomorphicity of the superpotential. It would thus appear quite improbable that we would be able to see it happening in this context. In principle however, one would expect that if the deformation is indeed described from an open string theory perspective as a noncommutative deformation of the transverse space, then at these special points Θ should transform

[7] Note that this does not of course exclude the possibility of finding a reference frame for which this is true. Integrability however will be lost.

under a change of coordinates from Θ_β or Θ_ρ to $\Theta = \Theta_{\widetilde{\beta}} + \Theta_{\widetilde{\rho}}$. Moreover, one might hope that the coordinate transformation which would make this possible would be the precise analog of the field redefinition applied to the gauge theory. Note however that in the case of the β–deformation, it is only for $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ that a noncommutative description — with parameter $a = 2\sin\beta_{\mathbb{R}}$ — is valid. This implies that we can apply the above consistency check if and only if both the original and final points in the coupling constant space involve a real parameter $\beta_{\mathbb{R}}$. A glance at the previous section will convince us that this indeed occurs: starting with $\rho = q_1 \in \mathbb{R}$ and $\beta = 0$ one can reach a point with $\widetilde{\rho} = \frac{iq_1}{\sqrt{3}} \in \mathbb{I}$ and $\widetilde{a} = 2\sin\widetilde{\beta} = \frac{2q_1}{\sqrt{3}} \in \mathbb{R}$. In fact it is quite straightforward to check that a coordinate transformation according to (3.19) leads us from $\Theta_\rho = -\Theta_1$ to $\Theta = \Theta_{\widetilde{a}=\frac{2q_1}{\sqrt{3}}} + \Theta_{\widetilde{\rho}=\frac{iq_1}{\sqrt{3}}}$. Furthermore, it appears that this case exhausts all possible coordinate changes that relate noncommutativity matrices corresponding to different parameters of the Leigh–Strassler deformation. We take this result as evidence that both our prescription for determining the (1,1) parts of Θ as well as the very interpretation of the deformation in noncommutative terms are indeed justified.

3.4 The Seiberg–Witten equations and the deformed flat space solution.

In the previous section, we saw how the deformation of the superpotential affects the moduli space of the gauge theory at large N. In particular, the six dimensional flat space with metric $\mathcal{G}_{IJ}$ of the $\mathcal{N} = 4$ theory is promoted to a noncommutative space characterized now by the set $\mathcal{G}_{IJ}$ and Θ^{IJ}. Both metric and noncommutativity parameter are mainly determined from the Lagrangian of the theory; the former is read off from the kinetic term of the scalars while the latter from their potential.

Since an SU(N) gauge theory can be realized as the low energy limit of open strings attached on a stack of D3–branes, the set $(\mathcal{G}_{flat}, \Theta)$ describes the geometry of the transverse space as seen by the open strings in the limit of large N and $\alpha' \to 0$. We will thus refer to $(\mathcal{G}_{flat}, \Theta)$ as the *open* string parameters .

On the other hand, any theory of open strings necessarily contains closed strings. Closed strings however perceive the geometry quite differently from open strings. In fact, it was shown in [15, 16] that target space noncommutativity from the point of view of open strings corresponds to turning on a B–field from the viewpoint of closed strings. The set (g, B), with g the closed string metric, are the *closed* string parameters that describe the same geometry. In this context, (g, B) represent the deformed flat space solution into which D3–branes are immersed [8]. Suppose now that we are given a set of equations relating the two groups of data. Then — provided that the open string parameters determined in the previous section exactly and fully describe the deformation — we could specify the closed string fields (g, B) of the deformed flat space geometry for free, i.e. without having to solve the type IIB differential equations of motion [108].

Equations relating open and closed string parameters indeed exist in the literature [109, 110, 16, 15]:

$$
\begin{aligned}
g + B &= \frac{1}{\mathcal{G}^{-1} + \Theta} \\
g_s &= G_s \sqrt{\frac{\det \mathcal{G}^{-1}}{\det (\mathcal{G}^{-1} + \Theta)}} = G_s \sqrt{\frac{1}{\det (1 + \Theta \mathcal{G})}}
\end{aligned}
\tag{3.28}
$$

[8] We are obviously interested here in the limit where open and closed strings are decoupled from each other.

where G_s, g_s denote the corresponding open/closed string couplings [9]. They were however considered in a situation somewhat different from the one discussed in this article, namely for a *flat* D–brane embedded in *flat* background space with a *constant* B–field turned on *along* its worldvolume [16, 15]. It was under these circumstances that, the presence of the background B–field was shown to deform the algebra of functions on the worldvolume of the brane into that of a noncommutative Moyal type of algebra, where Θ is a c–number. While it is natural to ask what happens in situations where the B–field is not constant, technical difficulties have hindered progress in this direction. In the order of increasing complexity, two cases can be considered: the case of a closed $\mathrm{d}B = 0$ though not necessarily constant two–form field B and the case of nonvanishing NS–NS three form flux $H = \mathrm{d}B$ in a curved background. In [111] the former case was explored and the Moyal deformation of the algebra of functions on the brane worldvolume, was shown to naturally extend to the Kontsevich star product deformation [112]. The authors of [113] — see also [114, 115, 116] — undertook the study of the most general case where $H = \mathrm{d}B \neq 0$. They considered a special class of closed string backgrounds, called parallelizable, and expanded the background fields in Taylor series. It was then possible to perturbatively analyze n–point string amplitudes on the disk and obtain — in a first order expansion — the appropriate generalization of (3.28). In fact, it turned out that equation (3.28) is still valid for a weakly varying nonclosed B–field even though the corresponding algebra of functions is now both noncommutative and nonassociative.

In this letter, we want to apply the above formulas in a situation where the B–field lies in the *transverse* space to the D3–brane. Despite the fact that this case has not

[9] Note that $G_s = 1$ for the ρ–deformation.

been explicitly studied in the literature [10] one expects by T–duality that equations (3.28) should continue to hold. If our reasoning thus far is correct and (3.28) indeed provide the relation between open and closed string parameters in this setup, the resulting closed string fields (g_s, g, B) will constitute a new supergravity solution, i.e. the deformed flat space solution where D3–branes embedded.

There exists a natural place where we can test these ideas prior to checking whether the type IIB field equations for the set (g_s, g, B) are satisfied. Recall that both the gravity dual and the corresponding deformed flat space background are known for the β–deformed gauge theory [14]. The open string data $(\mathcal{G}_{flat}, \Theta_{\beta_{\mathbb{R}}})$ describing the β–deformation were determined in the previous chapter, where the noncommutativity parameter was found to be constant despite the fact that the associated NS–NS three form flux was non zero. It is easy to show that applying (3.28) to the open string parameters $(\mathcal{G}_{flat}, \Theta_{\beta_{\mathbb{R}}})$ one recovers the deformed flat space geometry found by Lunin and Maldacena in [14]. This may appear as a surprise unless one observes the extraordinary similarity between (3.28) and the T–duality transformation rules of (3.8). In fact, these equations are identical in this case although the interpretation of the variables involved is essentially different. We will return to this point again in the following section.

Having tested our ideas in the context of the β–deformation, we proceed to check whether one can specify the appropriate background for the ρ–deformation as well. It actually turns out that the closed string data (g_s, g, B) determined in the fashion described above, satisfy the supergravity equations of motion only up to *third* order in the deformation parameter ρ. The result is at least perplexing — whereas it doesn't

[10] Mainly because a constant B–field in the transverse space can be gauged away leaving no trace on the geometry.

completely invalidate our considerations it indicates the presence of some kind of flow in them. In fact, the discrepancy at higher orders can be traced in a variety of reasons. Note however that the breakdown of (3.28) in the case of a nonassociative deformation seems to be the most plausible one, since nonassociativity manifests itself at second order in the deformation parameter. We postpone further discussion on this issue until section 6 and close this section by presenting the detailed — though not particularly illuminating — form of the solution to this order [11].

[11] See appendix B for the precise definition of the variables x, x_i, y, y_i which appear here.

Metric g:

$$g_{r_1r_1} = 1 - \left(r_2^2\left[(q_1x_3 - q_2y_1)^2 + (q_1y - q_2x_2)^2\right] + r_3^2\left[(q_1y - q_2x_3)^2 + (q_1x_2 - q_2y_1)^2\right]\right)$$

$$g_{r_2r_2} = 1 - \left(r_1^2\left[(q_1x_3 - q_2y_2)^2 + (q_1y - q_2x_1)^2\right] + r_3^2\left[(q_1y - q_2x_3)^2 + (q_1x_1 - q_2y_2)^2\right]\right)$$

$$g_{r_3r_3} = 1 - \left(r_1^2\left[(q_1x_2 - q_2y_3)^2 + (q_1y - q_2x_1)^2\right] + r_2^2\left[(q_1y - q_2x_2)^2 + (q_1x_1 - q_2y_3)^2\right]\right)$$

$$g_{\varphi_1\varphi_1} = r_1^2\left[1 - \left(r_2^2\left[(q_1x_1 - q_2y_3)^2 + (q_1y_2 - q_2x)^2\right] + r_3^2\left[(q_1y_3 - q_2x)^2 + (q_1x_1 - q_2y_2)^2\right]\right)\right]$$

$$g_{\varphi_2\varphi_2} = r_2^2\left[1 - \left(r_1^2\left[(q_1x_2 - q_2y_3)^2 + (q_1y_1 - q_2x)^2\right] + r_3^2\left[(q_1y_3 - q_2x)^2 + (q_1x_2 - q_2y_1)^2\right]\right)\right]$$

$$g_{\varphi_3\varphi_3} = r_3^2\left[1 - \left(r_1^2\left[(q_1x_3 - q_2y_2)^2 + (q_1y_1 - q_2x)^2\right] + r_2^2\left[(q_1y_2 - q_2x)^2 + (q_1x_3 - q_2y_1)^2\right]\right)\right]$$

$$g_{r_1r_2} = r_1r_2\left[(q_1x_3 - q_2y_1)(q_1x_3 - q_2y_2) + (q_1y - q_2x_1)(q_1y - q_2x_2)\right]$$

$$g_{r_1r_3} = r_1r_3\left[(q_1x_2 - q_2y_1)(q_1x_2 - q_2y_3) + (q_1y - q_2x_3)(q_1y - q_2x_1)\right]$$

$$\begin{aligned} g_{r_1\varphi_1} = r_1 \big(& r_2^2\left[(q_1x_3 - q_2y_1)(q_2x - q_1y_2) + (q_2x_2 - q_1y)(q_1x_1 - q_2y_3)\right] + \\ + \ & r_3^2\left[(q_2x_3 - q_1y)(q_1x_1 - q_2y_2) + (q_1x_2 - q_2y_1)(q_2x - q_1y_3)\right]\big) \end{aligned}$$

$$g_{r_1\varphi_2} = r_1r_2^2\left[(q_2x - q_1y_1)(-q_1x_3 + q_2y_1) + (q_2x_2 - q_1y)(-q_1x_2 + q_2y_3)\right]$$

$$g_{r_1\varphi_3} = r_1r_3^2\left[(q_2x - q_1y_1)(-q_1x_2 + q_2y_1) + (q_2x_3 - q_1y)(-q_1x_3 + q_2y_2)\right]$$

$$g_{r_2r_3} = r_2r_3\left[(q_1x_1 - q_2y_3)(q_1x_1 - q_2y_2) + (q_1y - q_2x_3)(q_1y - q_2x_2)\right]$$

$$g_{r_2\varphi_1} = r_2r_1^2\left[(q_2x - q_1y_2)(-q_1x_3 + q_2y_2) + (q_2x_1 - q_1y)(-q_1x_1 + q_2y_3)\right]$$

$$\begin{aligned} g_{r_2\varphi_2} = r_2 \big(& r_1^2\left[(q_1x_3 - q_2y_2)(q_2x - q_1y_1) + (q_2x_1 - q_1y)(q_1x_2 - q_2y_3)\right] + \\ + \ & r_3^2\left[(q_2x_3 - q_1y)(q_1x_2 - q_2y_1) + (q_1x_1 - q_2y_2)(q_2x - q_1y_3)\right]\big) \end{aligned}$$

$$g_{r_2\varphi_3} = r_2r_3^2\left[(q_2x_3 - q_1y)(-q_1x_3 + q_2y_1) + (q_2x - q_1y_2)(-q_1x_1 + q_2y_2)\right]$$

$$g_{r_3\varphi_1} = r_3r_1^2\left[(q_2x_1 - q_1y)(-q_1x_1 + q_2y_2) + (q_2x - q_1y_3)(-q_1x_2 + q_2y_3)\right]$$

$$g_{r_3\varphi_2} = r_3r_2^2\left[(q_2x_2 - q_1y)(-q_1x_2 + q_2y_1) + (q_2x - q_1y_3)(-q_1x_1 + q_2y_3)\right]$$

$$\begin{aligned} g_{r_3\varphi_3} = r_3 \big(& r_2^2\left[(q_1x_3 - q_2y_1)(q_2x_2 - q_1y) + (q_2x - q_1y_2)(q_1x_1 - q_2y_3)\right] + \\ + \ & r_1^2\left[(q_2x_1 - q_1y)(q_1x_3 - q_2y_2) + (q_1x_2 - q_2y_3)(q_2x - q_1y_1)\right]\big) \end{aligned}$$

$$g_{\varphi_1\varphi_2} = r_1^2r_2^2\left[(q_2x - q_1y_1)(-q_1x + q_2x) + (q_2y_3 - q_1x_2)(-q_1x_1 + q_2y_3)\right]$$

$$g_{\varphi_1\varphi_3} = r_1^2r_3^2\left[(q_2x - q_1y_1)(-q_1y_3 + q_2x) + (q_2y_2 - q_1x_3)(-q_1x_1 + q_2y_2)\right]$$

Dilaton:

$$
\begin{aligned}
e^{2\Phi} &= G \\
G &= 1 + r_1^2 \left[(q_1 y - q_2 x_1)^2 + (q_1 y_1 - q_2 x)^2 + (q_1 x_3 - q_2 y_2)^2 + (q_1 x_2 - q_2 y_3)^2\right] + \\
&\quad + r_2^2 \left[(q_1 y - q_2 x_2)^2 + (q_1 y_2 - q_2 x)^2 + (q_1 x_1 - q_2 y_3)^2 + (q_1 x_3 - q_2 y_1)^2\right] + \\
&\quad + r_3^2 \left[(q_1 y - q_2 x_3)^2 + (q_1 x_2 - q_2 y_1)^2 + (q_1 x_1 - q_2 y_2)^2 + (q_1 y_3 - q_2 x)^2\right]
\end{aligned}
\tag{3.30}
$$

B–field:

$$
\begin{aligned}
&B_{r_1 r_2} = r_3(q_2 x_3 - q_1 y) && B_{r_2 r_3} = r_1(q_2 x_1 - q_1 y) && B_{r_3 r_1} = r_2(q_2 x_2 - q_1 y) \\
&B_{r_1 \varphi_2} = -r_2 r_3(q_1 x_2 - q_2 y_1) && B_{r_1 \varphi_3} = r_2 r_3(q_1 x_3 - q_2 y_1) && B_{r_2 \varphi_1} = r_1 r_3(q_1 x_1 - q_2 y_2) \\
&B_{r_2 \varphi_3} = -r_1 r_3(q_1 x_3 - q_2 y_2) && B_{r_3 \varphi_1} = -r_1 r_2(q_1 x_1 - q_2 y_3) && B_{r_3 \varphi_2} = r_1 r_2(q_1 x_2 - q_2 y_3) \\
&B_{\varphi_1 \varphi_2} = r_1 r_2 r_3(q_2 x - q_1 y_3) && B_{\varphi_2 \varphi_3} = r_1 r_2 r_3(q_2 x - q_1 y_1) && B_{\varphi_3 \varphi_1} = r_1 r_2 r_3(q_2 x - q_1 y_2)
\end{aligned}
\tag{3.31}
$$

3.5 D–branes in deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the near horizon geometry.

In this section we will address the issue of finding the gravity dual of the ρ–deformed gauge theory. To this end, it is helpful to first consider the β–deformation. As mentioned in the previous section, the T–duality transformation rules (3.8) with which the Lunin–Maldacena solution was constructed are identical in form to (3.28). Recall from section 2, that in order to obtain the dual background in this case one must use:

$$E_0 = g_{\mathrm{AdS}_5\times\mathrm{S}^5} \quad \text{and} \quad \Gamma = \Theta_{\beta_{\mathbb{R}}} \tag{3.32}$$

Suppose now that we want to interpret these variables according to (3.28). We would obviously have to think of $g_{\mathrm{AdS}_5\times\mathrm{S}^5}$ as the *open* string metric $\mathcal{G}_{\mathrm{AdS}_5\times\mathrm{S}^5}$ whereas of Γ as $\Theta_{\beta_{\mathbb{R}}}$. In this sense, $(G_s = g^2_{YM}, \mathcal{G}_{\mathrm{AdS}_5\times\mathrm{S}^5}, \Theta_{\beta_{\mathbb{R}}})$ would encode the geometry as seen at large N by the open strings attached on a D3–brane embedded in the Lunin–Maldacena (3.11) background.

In other words, consider a stack of N D3–branes in the deformed flat space geometry of (3.10). The near horizon limit of this configuration is the gravity dual of the Leigh–Strassler marginal deformation with $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ and $\rho = 0$. A probe D3–brane propagating near the stack will then be described by the DBI action written either in terms of the closed $(\widetilde{g}_s, \widetilde{g}, \widetilde{B})$ or of the open $(G_s, \mathcal{G}_{\mathrm{AdS}_5\times\mathrm{S}^5}, \Theta_\beta)$ string fields. However, the action of a single D3–brane seperated from a collection of (N-1) other branes can also be obtained by integrating out the massive open strings stretched between the probe and the source. Indeed, as expected according to [76, 77, 78, 79, 80, 81], the DBI action describing the motion of a D3–brane in this background should in the large N limit coincide with the leading IR part of the quantum effective action of the β–deformed theory obtained by keeping the U(1) external fields and integrating over the massive ones.

In this spirit, it does not seem surprising that the appropriate open string data in this case, are the metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$and the noncommutativity parameter $\Theta_{\beta_{\mathbb{R}}}$. In fact, the action of the β–deformed gauge theory can be written as that of the parent $\mathcal{N} = 4$ theory with the product of the matter fields replaced by a star product associated to $\Theta_{\beta_{\mathbb{R}}}$. Moreover, as conjectured in [14, 94, 31] and later proven in [29], all planar amplitudes are equal to their $\mathcal{N}=4$ counterparts up to an overall phase factor. This suggests that the iterative structure of the large N β–deformed gauge theory amplitudes, when $\beta=\beta_{\mathbb{R}}\in\mathbb{R}$, is identical to that of the $\mathcal{N}=4$ SYM theory. It is then not hard to imagine that the quantum effective action mentioned above will be analogous to that of the undeformed theory with the only difference being some phase factors coming from the noncommutative deformation of the product. Subsequently, the open string fields appearing in the DBI form of the effective action of the $\mathcal{N}=4$ theory $(G_s,\mathcal{G},\Theta=0)$ will be promoted to $(G_s,\mathcal{G},\Theta_{\beta_{\mathbb{R}}})$.

It is natural to wonder whether a similar situation could apply to the ρ–deformation as well. The analysis of section 3 may lead us to think that this is most likely *not* the case. Even if we succeeded in writing the action of the theory in question, as the $\mathcal{N}=4$ action with a star product between the matter fields, it is difficult to understand how planar equivalence between the two would be achieved with the deformation being both noncommutative and nonassociative. In fact, the proof given in [29] specifically relied on the associativity of the star product for the β–deformation. Nevertheless, nonassociativity is a second order effect in ρ and in view of the results of the previous section, one might hope that a solution to this order could be obtained in this case, too.

To explicitly check this we can directly use the second order expansion of (3.28):

$$\begin{aligned} g &= \mathcal{G}+\mathcal{G}\Theta\mathcal{G}\Theta\mathcal{G}+\mathcal{O}(\rho^4) \\ B &= -\mathcal{G}\Theta\mathcal{G}+\mathcal{O}(\rho^3) \\ G^{-1} &= 1+\mathrm{Tr}\left[\mathcal{G}\Theta-\frac{1}{2}\mathcal{G}\Theta\mathcal{G}\Theta\right]+\mathcal{O}(\rho^4) \end{aligned} \tag{3.33}$$

The equations above, provide a relation between the open string parameters of the deformed theory and the NS–NS string fields of the dual geometry. Since there is no obvious way to extract information on the associated RR–fluxes, we resort to the type IIB equations of motion in order to specify them. Fortunately, assuming no warp factor and making the usual ansatz for the five form field strength [12]:

$$\begin{aligned} \mathrm{d}s^2_{10} &= \mathrm{d}s^2_{AdS_5} + \mathrm{d}s^2_{\mathrm{S}^5} \\ F_5 &= f(\omega_{AdS_5} + \omega_{\widetilde{S}^5}) \end{aligned} \tag{3.34}$$

allows us to directly solve for the RR three form flux F_3:

$$\begin{aligned} F_3 &= -f^{-1}\mathrm{d}\star_5 e^{-2\Phi} H_3 \\ H_3 &= f^{-1}\mathrm{d}\star_5 F_3 \Rightarrow \mathrm{d}\left[B - f^{-1}\star_5 F_3\right] = 0 \end{aligned} \tag{3.35}$$

Note that to this order $F_3 = f \star_5 B$ which greatly simplifies calculations [13]. One can then indeed show that the type IIB equations are simultaneously satisfied up to third order in the deformation parameter, for the following set of fields [14]:

[12] We refer the reader to the appendix for the necessary definitions of the parameters involved as well as the type IIB field equations [108] in five dimensions.

[13] This is intimately connected to the fact that $\delta_{\mathrm{S}^5}\Omega_\rho = 0$ where $\Omega_\rho = \mathcal{G}\Theta\mathcal{G}$ denotes the form on S^5 associated to the bivector Θ_ρ. It is clear from (3.33) that $B = -\Omega$ to this order. It is worth remarking that $F_3 = f \star B$ is exact for the β–deformed theory where both $d\Omega_\beta = 0$ and $\delta_{\mathrm{S}^5}\Omega_\beta = 0$ hold.

[14] We set $R = 1$ where R the radius of AdS_5.

Dilaton:

$$
\begin{aligned}
e^{2\Phi} &= e^{2\Phi_0} G \\
G^{-1} &= 1 + q_1^2 \left(y^2 + s_\alpha^2 x_1^2 + (c_\alpha^2 + s_\alpha^2 c_\theta^2) x_2^2 + (c_\alpha^2 + s_\alpha^2 s_\theta^2) x_3^2 + c_\alpha^2 y_1^2 + s_\alpha^2 s_\theta^2 y_2^2 + s_\alpha^2 c_\theta^2 y_3^2 \right) + \\
&\quad + 2 q_1 q_2 \left(xy + (c_\alpha^2 - s_\alpha^2) x_1 y_1 - \left(c_\theta^2 + s_\theta^2 (c_\alpha^2 - s_\alpha^2) \right) x_2 y_2 - \left(s_\theta^2 + c_\theta^2 (c_\alpha^2 - s_\alpha^2) \right) \right) \\
&\quad + q_1^2 \left(x^2 + c_\alpha^2 x_1^2 + s_\alpha^2 s_\theta^2 x_2^2 + s_\alpha^2 c_\theta^2 x_3^2 + s_\alpha^2 y_1^2 + \left(c_\alpha^2 + s_\alpha^2 c_\theta^2 \right) y_2^2 + \left(c_\alpha^2 + s_\alpha^2 s_\theta^2 \right) y_3^2 \right)
\end{aligned}
\tag{3.36}
$$

B–field:

$$
\begin{aligned}
& B_{\alpha\theta} = s_\alpha \left(q_1 y + q_2 x \right) \quad B_{\alpha\varphi_1} = 0 \quad B_{\alpha\varphi_2} = s_\alpha s_\theta c_\theta \left(q_1 x_2 - q_2 y_2 \right) \\
& B_{\alpha\varphi_3} = s_\alpha s_\theta c_\theta \left(-q_1 x_3 + q_2 y_3 \right) \quad B_{\theta\varphi_1} = c_\alpha s_\alpha^2 \left(q_1 x_1 - q_2 y_1 \right) \\
& B_{\theta\varphi_2} = -c_\alpha s_\alpha^2 s_\theta^2 \left(q_1 x_2 - q_2 y_2 \right) \quad B_{\theta\varphi_3} = -c_\alpha s_\alpha^2 c_\theta^2 \left(q_1 x_3 - q_2 y_3 \right) \\
& B_{\varphi_1\varphi_2} = -c_\alpha s_\alpha^2 s_\theta c_\theta \left(q_1 y_3 + q_2 x_3 \right) \quad B_{\varphi_2\varphi_3} = -c_\alpha s_\alpha^2 s_\theta c_\theta \left(q_1 y_1 + q_2 x_1 \right) \\
& \qquad B_{\varphi_3\varphi_1} = -c_\alpha s_\alpha^2 s_\theta c_\theta \left(q_1 y_2 + q_2 x_2 \right)
\end{aligned}
\tag{3.37}
$$

F_3 and F_5–form flux:

$$
\begin{aligned}
F_3 &= \star_{S^5} \Omega \quad \text{with} \quad \Omega \equiv \mathcal{G}_{ik} \mathcal{G}_{jl} \Theta^{kl} dx^i \wedge dx^j \\
F_5 &= f(\omega_{AdS_5} + G \omega_{\mathrm{S}^5})
\end{aligned}
\tag{3.38}
$$

Metric g:

$$
\begin{aligned}
g_{\alpha\alpha} &= 1 - q_1^2\left(c_\theta^2 x_2^2 + s_\theta^2 x_3^2 + y^2\right) + 2q_1q_2\left(-xy + c_\theta^2 x_2y_2 + s_\theta^2 x_3y_3\right) - q_2^2\left(x^2 + c_\theta^2 y_2^2 + s_\theta^2 y_3^2\right)\\
g_{\theta\theta} &= s_\alpha^2\left[1 - q_1^2\left(y^2 + s_\alpha^2 x_1^2 + c_\alpha^2\left(s_\theta^2 x_2^2 + c_\theta^2 x_3^2\right)\right) - 2q_1q_2\left(xy - s_\alpha^2 x_1y_1 - c_\alpha^2\left(s_\theta^2 x_2y_2 + c_\theta^2 x_3y_3\right)\right) -\right.\\
&\quad \left. -q_2^2\left(x^2 + s_\alpha^2 y_1^2 + c_\alpha^2\left(s_\theta^2 y_2^2 + c_\theta^2 y_3^2\right)\right)\right]\\
g_{\varphi_1\varphi_1} &= c_\alpha^2\left[1 - s_\alpha^2\left(q_1^2\left(x_1^2 + s_\theta^2 y_2^2 + c_\theta^2 y_3^2\right) + 2q_1q_2\left(-x_1y_1 + s_\theta^2 x_2y_2 + c_\theta^2 x_3y_3\right) +\right.\right.\\
&\qquad\qquad \left.\left. +q_2^2\left(y_1^2 + s_\theta^2 x_2^2 + c_\theta^2 x_3^2\right)\right)\right]\\
g_{\varphi_2\varphi_2} &= s_\alpha^2 s_\theta^2\left[1 - q_1^2\left(x_2^2(c_\alpha^2 + s_\alpha^2 c_\theta^2) + c_\alpha^2 y_1^2 + s_\alpha^2 c_\theta^2 y_3\right) +\right.\\
&\quad \left. +2q_1q_2\left(-c_\alpha^2 x_1y_1 + (c_\alpha^2 + s_\alpha^2 c_\theta^2)x_2y_2 - s_\alpha^2 c_\theta^2 x_3^2 y_3^2\right) - q_2^2\left(c_\alpha^2 x_1^2 + s_\alpha^2 c_\theta^2 x_3^2 + (c_\alpha^2 + s_\alpha^2 c_\theta^2)y_2^2\right)\right]\\
g_{\varphi_3\varphi_3} &= s_\alpha^2 c_\theta^2\left[1 - q_1^2\left(c_\alpha^2 y_1^2 + s_\alpha^2 s_\theta^2 y_2^2 + \left(c_\alpha^2 + s_\alpha^2 s_\theta^2\right)x_2^2\right) +\right.\\
&\quad \left. +2q_1q_2\left(-c_\alpha^2 x_1y_1 - s_\alpha^2 s_\theta^2 x_2y_2 + \left(c_\alpha^2 + s_\alpha^2 s_\theta^2\right)x_3y_3\right) - q_2^2\left(c_\alpha^2 x_1^2 + s_\alpha^2 s_\theta^2 x_2^2 + \left(c_\alpha^2 + s_\alpha^2 s_\theta^2\right)y_3^2\right)\right]\\
g_{\alpha\theta} &= c_\alpha s_\alpha c_\theta s_\theta\left[q_1^2\left(x_2^2 - x_3^2\right) + 2q_1q_2\left(-x_2y_2 + x_3y_3\right) + q_2^2\left(y_2^2 - y_3^2\right)\right]\\
g_{\alpha\varphi_1} &= c_\alpha s_\alpha\left[q_1^2\left(x_1y + s_\theta^2 x_3y_2 c_\theta^2 x_2y_3\right) + q_1q_2\left(xx_1 - yy_1 + x_2x_3 - y_2y_3\right)\right) -\\
&\quad \left. -q_2^2\left(xy_1 + c_\theta^2 x_3y_2 + s_\theta^2 x_2y_3\right)\right]\\
g_{\alpha\varphi_2} &= c_\alpha s_\alpha s_\theta^2\left[-q_1^2(x_2y + x_3y_1) + q_1q_2(-xx_2 - x_1x_3 + yy_2 + y_1y_3) + q_2^2(xy_2 + x_1y_3)\right]\\
g_{\alpha\varphi_3} &= c_\alpha s_\alpha s_\theta^2\left[-q_1^2(x_2y + x_3y_1) + q_1q_2(-xx_2 - x_1x_3 + yy_2 + y_1y_3) + q_2^2(xy_2 + x_1y_3)\right]\\
g_{\theta\varphi_1} &= c_\alpha^2 s_\alpha^2 c_\theta s_\theta(q_1^2 + q_2^2)(x_3y_2 - x_2y_3)\\
g_{\theta\varphi_2} &= s_\alpha^2 s_\theta c_\theta\left[q_2^2\left(xy_2 + s_\alpha^2 x_3y_1 + c_\alpha^2 x_1y_3\right) - q_1q_2\left(xx_2 - yy_2 + x_1x_3 - y_1y_3\right) -\right.\\
&\quad \left. -q_1^2\left(x_1y + c_\alpha^2 x_3y_1 + s_\alpha^2 x_1y_3\right)\right]\\
g_{\theta\varphi_3} &= s_\alpha^2 s_\theta c_\theta\left[q_1^2\left(x_3y + c_\alpha^2 x_2y_1 + s_\alpha^2 x_1y_2\right) + q_1q_2\left(x_1x_2 - y_1y_2 + xx_3 - yy_3\right) -\right.\\
&\quad \left. -q_2^2\left(xy_3 + s_\alpha^2 x_2y_1 + c_\alpha^2 x_1y_2\right)\right]\\
g_{\varphi_1\varphi_2} &= c_\alpha^2 s_\alpha^2 s_\theta^2(q_1^2 + q_2^2)(x_1x_2 + y_1y_2)\\
g_{\varphi_2\varphi_3} &= c_\alpha^2 s_\alpha^2 c_\theta^2(q_1^2 + q_2^2)(x_1x_3 + y_1y_3)\\
g_{\varphi_3\varphi_1} &= s_\alpha^4 c_\theta^2 s_\theta^2(q_1^2 + q_2^2)(x_2x_3 + y_2y_3)
\end{aligned}
\tag{3.39}
$$

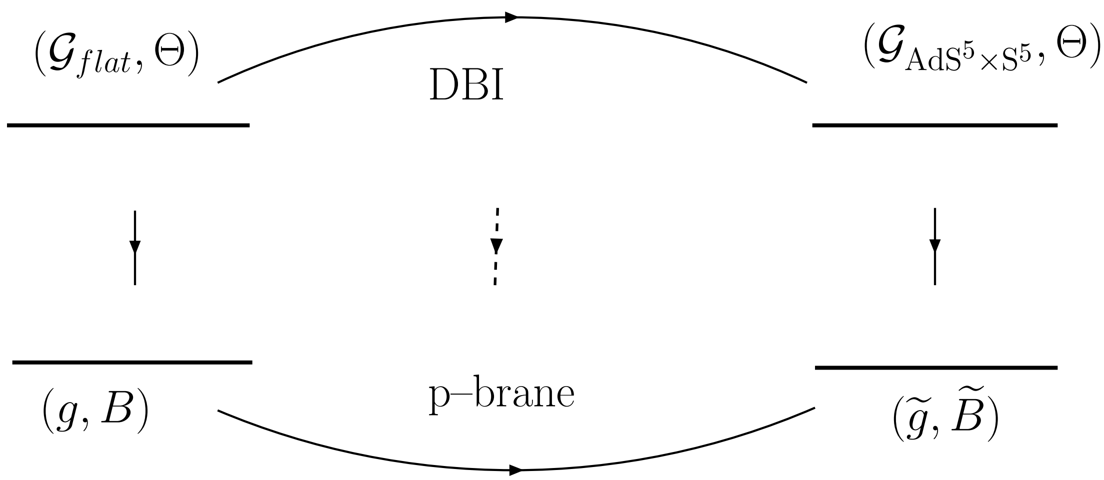

Figure 3.1: Open vs Closed string fields: a way to understand deformations of the original AdS/CFT proposal.

3.6 Discussion

In the previous sections we set forth some new ideas that helped us obtain new gravity solutions up to third order in the deformation paremeter ρ. Moreover, we observed that the same method provided the exact supergravity backgrounds related to the β–deformed gauge theory. We view these results as evidence that supports the basic ideas of our proposal which can be nicely summarized in Fig. 3.6. In this section we would like to discuss its obscure points; the ones that possibly underlie its failure to provide the solution to all orders in the deformation parameter.

The method proposed in this note can be divided into three steps. Let us seperately consider the issues that arise in each one. The starting point consists of determining the appropriate open string data. In this, we mainly use information from the gauge theory Lagrangian. Obviously, the fact that the noncommutativity matrix specified in this case, fails to preserve the property of associativity is quite displeasing. In particular, it is not at all clear how to define an appropriate star product. Subsequently, there is no obvious way in which one can rewrite the Lagrangian of the ρ–deformed theory in terms of the $\mathcal{N} = 4$ SYM Lagrangian with a modified product between the matter fields. In fact, most of the star products that we attempted to define produced extra terms in the action, to second and third order in the deformation parameter ρ. Interestingly enough though, all the additional terms were essentially of the same form as the ones coming from the β and ρ deformation themselves. We hope to explore this point further in the future.

A related issue is that of the D–terms. We showed in section 2.4, that we can rewrite these terms in the $\mathcal{N} = 4$ theory as a sum of the F–terms with a potential term involving the commutator between holomorphic and antiholomorphic matter fields:

$$\mathrm{Tr}[\Phi_I, \tilde{\Phi}^I][\Phi_J, \tilde{\Phi}^J] = \mathrm{Tr}[\Phi_I, \Phi_J][\tilde{\Phi}^I, \tilde{\Phi}^J] + \mathrm{Tr}[\Phi_I, \tilde{\Phi}^J][\Phi_J, \tilde{\Phi}^I] \tag{3.40}$$

It was then clear that should we wish to only deform the F–terms of the potential, we must appropriately alter the commutator: $[\Phi_I, \tilde{\Phi}^J]$. For the β–deformed gauge theory, the $(1,1)$ pieces of the noncommutativity matrix precisely ensured that the D–terms remained unaffected by the deformation according to (3.40). The lack of a star product in the case of the ρ–deformation however, makes it impossible to perform this consistency check.

The next step of the method proposed herein, consisted in mapping the open string parameters to the closed string ones. The precise mapping was formulated through the equations of (3.28) which as discussed in section 5, were derived under particularly different conditions than the ones considered in this paper. Their validity in this case is therefore naturally disputable, even more so, in view of the nonassociativity of Θ.

We believe that this actually constitutes the most plausible reason for the failure of our proposal to produce the exact supergravity background whereas at the same time explains why the method works to this order where nonassociativity precisely comes into play. In fact, it seems that when T^{ijk} of (3.27) is nonvanishing, both Θ and $T = \Theta\partial\Theta$ are necessary for defining the deformation. A natural generalization of (3.28) would then relate $(\mathcal{G}, \Theta, T = \Theta\partial\Theta)$ to $(g, B, H = dB)$ and presumably provide the deformed flat space solution to all orders in the deformation.

Note that we do not necessarily maintain that it would directly solve the problem of finding the dual gravity background as well. It may very well be that nonassociativity spoils the planar equivalence between the $\mathcal{N} = 4$ theory and its deformation. This would obviously be reflected on the form of the quantum effective action and therefore of the DBI, making it difficult to determine the relevant open string data [15].

It is also worth remarking that (3.28) provides a relation only between the NS–NS fields of the open and closed string backgrounds. Information pertaining to the RR–fields is however essential, especially for determining the dual gravitational solution. In fact, in section 6 we had to rely on a particular ansatz for the metric and the five form flux in order to fully specify the background. In this light, it may seem plausible that a different ansatz — a warp factor in front of the AdS part of the metric, in particular — could grant us the solution to all orders in the deformation parameter. The presence of a non–trivial warp factor may actually be related to the deviation of the coupling constant h in (3.1) from its original g_{YM} value. Indeed, in the case of the β–deformation, a warp factor is absent from the solution when $\beta \in \mathbb{R}$ and

[15] Actually it is possible to imagine that this can occur independently from the nonassociativity of the product. Yet it would make it hard to understand why the procedure employed in this article would give a true solution to any order in the deformation.

$h = g_{YM}$, while it is not when $\beta_{\mathbb{I}} \neq 0$ and the Leigh–Strassler constraint indicates that $h \neq g_{YM}$. This fact therefore represents another possible explanation as to why our method fails to give an exact solution [16].

Finally, we would like to note that throughout this article we considered the Leigh–Strassler marginal deformation at the point $\beta = 0$. It is however natural to think that quantum corrections will generically generate a β–like term since no obvious symmetry argument could prohibit it. In this sense it may seem rather significant to incorporate a nonvanishing β in our discussion. This is actually not difficult to do, provided that $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$. In this case, we can define $\Theta = \Theta_{\beta_{\mathbb{R}}} + \Theta_{\rho}$ and follow the method outlined in this note. The result is straightforward but unfortunately does not give any further insight into the higher order corrections of the background. The case of generic $\beta \in \mathbb{C}$ is more interesting but also more difficult to study. A noncommutative description of the deformation is not valid in this case and one relies on the $SL(2,\mathbb{R})_s$ symmetry of the supergravity equations of motion in order to construct the dual solution [14]. Consequently, there is no obvious way to incorporate a complex β in our method.

The reason that makes the case of complex β worthwhile to explore further, is that according to the analysis of section 2, there exist some special points in the deformation space which can take us from a theory of generic β and $\rho = 0$, to a marginal deformation where both $\widetilde{\rho}$ and $\widetilde{\beta}$ with $\widetilde{\beta} \in \mathbb{C}$ are non vanishing. Since the gravity dual in the former case is known, investigating the solution at these points may provide useful information on how to extend our results to all orders in the deformation parameters.

[16] Note that we examined the case of a warp factor of the form G^n with $n \in \mathbb{R}$ and G of (3.36) but found no value of n for which the supergravity equations were satisfied to third order in the deformation parameter.

Chapter 4

Conclusions and Open Problems

In this work, we studied the Leigh–Strassler marginal deformations of $\mathcal{N} = 4$ SYM. At first, we focused on the much better understood case of the β–deformed gauge theory. We established a connection with noncommutativity and explored its role within the solution generating technique of Lunin and Maldacena. Next, we turned to the ρ–deformed theory and found gravity solutions corresponding to the associated flat space deformation and the AdS/CFT dual of the gauge theory, up to third order in the deformation parameter. We achieved this by using the techniques developed for the β–deformed theory so as to relate the deformation to a noncommutative deformation of the transverse space. Having obtained the open string parameters $(\mathcal{G}, \Theta)$ encoding the geometry of the moduli space, we determined the corresponding closed string fields (g, B) with the use of a well–known mapping between the two. The most remarkable feature of our computation, is its almost purely algebraic nature.

There are various possibilities for future work in the context of exactly marginal deformations of $\mathcal{N} = 4$. They range from addressing the questions raised in the previous chapter, to establishing a precise connection with generalized complex geometry

[117, 118], investigating the role of the $\mathrm{SL}(2,\mathbb{R})$ symmetry and relating our results to previous work on the same subject [104].

Several more issues should however be addressed, if this work is to provide a general framework for discussing the gravity duals of gauge theories.

A curious feature of our approach for instance, is that supersymmetry does not play any central role in it. Indeed the whole discussion so far has solely relied on the commutation relations between the scalar fields of the theory. When however supersymmetry is preserved, scalars are accompanied by their fermionic superpartners and it is obvious that similar (anti)commutation relations will be obeyed by the fermions alone as well as between the scalars and the fermions of the theory. It seems plausible to us that information pertaining to these (anti)commutation relations is hidden in the RR sector of the theory [119, 120, 121]. It would therefore be of great importance to study it in a similar fashion.

A related question, that seems not to have been investigated in the literature, is the presence of constant RR–flux in $\mathrm{AdS}_5 \times \mathrm{S}^5$space and its possible relation to nonanticommutativity — from the point of view of the gauge theory living on a D3–brane embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Such a relation could possibly uncover the hidden structure of the IR effective $\mathcal{N} = 4$ SYM Lagrangian and its connection to the DBI.

Another interesting direction for study is the $\mathcal{N} = 1^*$ gauge theory, obtained by adding mass terms to the $\mathcal{N} = 4$ SYM superpotential. This theory has been investigated in a number of works [122, 123, 75] where the notion of a noncommutative transverse space is touched upon yet has not been made precise. A curious question, is whether the dual description of these theories can be reached by placing D3–branes in some deformed geometry and taking the near horizon limit. If so, various aspects of our work could be relevant in this study. Similar considerations could then be applicable to orbifold deformations of $\mathcal{N} = 4$ SYM as well as to noncommutative gauge theories.

In summary, the ideas set forth in this note represent alternative means into investigating deformations of the AdS/CFT correspondence. Obviously, a number of issues should be resolved before they can provide a concrete proposal for constructing new supergravity backgrounds. We do however believe that they open up a path that leads to a better understanding of gauge/gravity duality, which we hope to further explore in the future.

Appendix A

Noncommutative gauge theories

The most direct application of the ideas discussed in chapter 2 is to consider the Lunin–Maldacena prescription in order to obtain the gravity duals of noncommutative gauge theories with β–type noncommutativity [1]. This simply means that we wish to think of Θ^{ij} or rather $\mathbf{\Gamma}$ of (2.9) as a noncommutativity matrix along the worldvolume of the D3–brane [2]. Provided a decoupling limit exists [3], we can use the solution generating technique reviewed in section 2.2, to either deform the p–brane solution itself, or the near horizon geometry directly. For reasons of uniformity, we decided to adhere to the latter prescription in what follows. In four dimensional Euclidean space, Θ^{ij} can be written in complex coordinates as:

$$\begin{aligned} [z_i, z_j] =& ibz_iz_j, \quad [\overline{z}_i, \overline{z}_j] = ib\overline{z}_i\overline{z}_j, \quad [z_i, \overline{z}_j] = -ibz_i\overline{z}_j \\ &\text{for} \quad i < j \quad \text{and} \quad \text{i,j=1,2} \end{aligned} \tag{A.1}$$

[1]Similar considerations in the context of the Maldacena–Nunez background appeared in [36, 124].
[2]Obviously the same procedure can be applied to all branes in a fashion similar to [52, 53, 54].
[3]One can actually check this by either calculating the graviton absorption cross–section or the potential that gravitons feel due to the presence of the D–brane [125].

As we already saw in the previous section transforming to polar coordinates yields a constant noncommutativity parameter along the two–torus:

$$[\phi_1, \phi_2] = ib, \quad [\rho_1, \rho_2] = [\rho_i, \phi_j] = 0 \quad i, j = 1, 2 \tag{A.2}$$

Constructing a matrix out of these relations is a fairly obvious step which leads us to matrix $\mathbf{\Gamma}$ appearing in (2.6). We can therefore directly apply the associated T–duality transform (2.6) on the $\mathrm{AdS}_5 \times \mathrm{S}^5$geometry. The relevant part of the background matrix is:

$$E = u^2 R^2 \begin{pmatrix} \rho_1^2 & 0 \\ 0 & \rho_2^2 \end{pmatrix} \tag{A.3}$$

and substituting into (2.7) we find:

$$\begin{aligned} ds^2_{str} &= ds^2_{\widetilde{AdS}} + ds^2_{S^5}, \quad \text{where} \quad ds^2_{\widetilde{AdS}} = u^2 R^2 (d\rho_1^2 + d\rho_2^2 + G(\rho_1^2 d\phi_1^2 + \rho_2^2 d\phi_2^2)) \\ B &= \hat{b} R^2 G \rho_1^2 \rho_2^2 u^4 d\phi_1 \wedge d\phi_2, \quad e^{2\Phi} = G e^{2\Phi_0} \\ G &= \frac{1}{1 + \hat{b}^2 \rho_1^2 \rho_2^2 u^4}, \quad \hat{b} = R^2 b \\ F_3 &= -3(4\pi \mathrm{N}) b u^3 \rho_1 \rho_2 d\rho_1 \wedge d\rho_2 \wedge du, \quad F_5 = 4\pi \mathrm{N} (\omega_{\widetilde{AdS}} + \omega_{S^5}) \end{aligned} \tag{A.4}$$

with the RR–fields computed using the T–duality rules of [42, 43, 44, 45, 46]. Note here that the effect of noncommutativity is important for large radial directions but negligible for small ones. The same behaviour has been observed in the case of the Melvin Universe [57, 54]. It seems natural therefore to expect that manifestations of this spatial nonhomogeneity will be similar to those described in [57]. It would be interesting for this purpose to explore the instanton, monopole and vortex solutions of the theory. In the Melvin–twist gauge theory the corresponding analysis showed

[57] that although the length of the magnetic monopole is position dependent, its mass agrees with the ordinary SYM monopole solution. It is plausible that study of the β–type noncommutative gauge theory along these lines will lead to analogous results. In addition, it is important to investigate the stability properties of the above solution, since the background in question may generically break supersymmetry (see e.g. [126, 127] for a discussion on this point). We would like now to proceed and consider the same type of deformation in Lorentzian signature but before doing so, let us make a few remarks regarding the action of the gauge theory dual to (A.4).

Clearly, knowledge of an appropriate star product is more often than not necessary in order to specify the action that describes a noncommutative gauge theory. In the case illustrated above, Θ^{ij} is position dependent and it is then known that a suitable product is the one defined by Kontsevich in [128]. Naively one would then think that the action of the gauge theory is obtained by simply replacing the ordinary product of functions with the star product. The latter product is however not compatible with the Leibnitz rule so that one should actually employ what is referred to as the "frame formalism" introduced in [129]. Alternatively, one can take advantage of the fact that Θ^{ij} is constant in polar coordinates and specify a Moyal–like product of functions. The precise mapping between this product and the one defined by Kontsevich should then be found, which would however *not* be the result of a simple change of coordinates. This procedure has been carried out explicitly in a number of cases [130, 57, 54] and we refer the reader to these papers for details.

Let us now move on to consider the β–type deformation on a four–dimensional spacetime with Lorentzian signature. Performing a wick rotation according to $z \to ix^+, \overline{z} \to ix^-$ along with $b \to ib$ we can write the commutation relations of equation

(A.1) as:

$$[x^+, z] = ibx^+z, \quad [x^-, \overline{z}] = ibx^-\overline{z}, \quad [x^+, \overline{z}] = ibx^+\overline{z}, \quad [x^-, z] = ibx^-z$$
$$\text{with} \quad [z, \overline{z}] = [x^+, x^-] = 0 \tag{A.5}$$

We therefore see that in this case we have to deal with a temporal noncommutativity parameter. In general, field theories on spaces with time–like noncommutativity $\Theta^{0i} \neq 0$ are acausal [131, 132] whereas their quantum counterparts are not unitary. A decoupled *field theory* limit for D–branes in this case does not exist. It was however found in [131, 132] that a scaling limit where the closed string sector can be separated from the open string one is indeed possible. Massive open strings do not decouple in this limit which thus defines a noncommutative open string theory (NCOS) rather than a field theory. Several aspects of these NCOS theories are explored in [133, 134, 135, 136, 137, 138].

The precise analysis of which types of noncommutativity lead to unitary theories and which not, was carried out in [139] along the lines of [140]. There it was shown that a necessary condition for unitarity is that the following inner product between external momenta is positive definite:

$$p \diamond p \equiv -p_\mu \Theta^{\mu\sigma} \mathcal{G}_{\sigma\tau} \Theta^{\tau\nu} p_\nu > 0 \tag{A.6}$$

where $\mathcal{G}$ is the background metric for the open strings and the corresponding field theory. Let us therefore evaluate this quantity for the β–like noncommutativity under consideration here. It is easier if we first perform a coordinate transformation to go from coordinates (t, x_1, x_2, x_3) to (τ, θ, r, ϕ) defined as: $t = \tau \cosh\theta$, $x_1 = \tau \sinh\theta$, $x_2 = r\cos\phi$ and $x_3 = r\sin\phi$. Here $\tau\epsilon(-\infty, \infty)$,$r\epsilon[0, \infty)$ whereas θ can be chosen compact or non compact. This transformation will bring the commutation relations to the form [4]:

$$[\theta,\phi] = ib \quad \text{and} \quad [\tau,r] = [r,\phi] = [\tau,\phi] = [\tau,\theta] = [r,\theta] = 0 \tag{A.7}$$

and substituting into (A.6) we obtain: $p \diamond p = b^2(p_\theta^2 r^2 + p_\phi^2 \tau^2)$ which is clearly positive definite. Can we therefore deduce that the β–type noncommutative deformation describes a unitary field theory? To be precise, the unitarity requirement of (A.6) is proven for a position independent noncommutativity parameter turned on in flat space. In our case, as soon as we go to a reference frame where Θ is constant, the corresponding spacetime exhibits a time–dependent behaviour. It is therefore ambiguous what the meaning of unitarity is in this context.

It may be interesting however to address these issues through the dual gravity description of this theory. Let us therefore apply the T–duality transformation rules in order to construct this background. Alternatively, we can wick rotate the Euclidean solution of equation (A.4) according to $\rho_1 \to i\tau, \phi_1 \to i\theta, b \to ib$. Either way we obtain

[4]These coordinates cover half of $\mathbb{R}^{1,3}$[127].

[5]:

$$
\begin{aligned}
ds^2_{str} &= ds^2_{\widetilde{AdS}} + ds^2_{S^5}, \quad \text{where} \quad ds^2_{\widetilde{AdS}} = u^2R^2(-d\tau^2 + dr^2 + G(\tau^2 d\theta^2 + r^2 d\phi^2)) \\
B &= \hat{b}R^2G\tau^2r^2u^4 d\theta \wedge d\phi, \quad e^{2\Phi} = Ge^{2\Phi_0} \\
G &= \frac{1}{1+\hat{b}^2\tau^2r^2u^4}, \quad \hat{b} = R^2 b \\
F_3 &= -3(4\pi \mathrm{N})bu^3\tau r d\tau \wedge dr \wedge du, \quad F_5 = 4\pi\mathrm{N}(\omega_{\widetilde{AdS}} + \omega_{S^5})
\end{aligned}
\tag{A.8}
$$

Note again that equation (A.8) defines a time dependent background dual to a noncommutative theory which can be thought of as living either in flat space with temporal time–dependent noncommutativity parameter or in a time–dependent background which is noncommutative only along some of the spatial directions. Similar time–dependent configurations were explored in [141, 142, 143, 144]. For the case of compact θ with $\theta \sim \theta + 2\pi$ and rational parameter β, the gravity solution (A.8) corresponds to the near horizon geometry of a D3–brane immersed in a time–dependent background that admits an orbifold description [127, 145, 146]. The latter deformation of flat space can be recovered from flat space with the same technique [127]:

$$
\begin{aligned}
ds^2 &= -d\tau^2 + dr^2 + \frac{\tau^2}{1+b^2\tau^2r^2}d\theta^2 + \frac{r^2}{1+b^2\tau^2r^2}d\phi^2 \\
e^{2\Phi} &= \frac{1}{1+b^2\tau^2r^2} \\
B &= -\frac{b\tau^2r^2}{1+b^2\tau^2r^2}d\theta\wedge d\phi
\end{aligned}
\tag{A.9}
$$

The background indicated above presents an interesting time evolution noted in [127].

[5]Note that wick rotation in these coordinates is different from the usual case. As a result, the background is well defined and does not seem to indicate the need for another kind of scaling limit. We therefore expect that indeed this supergravity solution is dual to a field theory.

In particular, it appears to be periodically changing for the designated choices of θ and β. At $\tau = -\infty$ it is described via the orbifold $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta=2\pi} \times [\mathbb{C}/\mathbb{Z}_N]$ (i.e. orbifold by the boost $\Delta = 2\pi$) which gradually evolves to $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta=2\pi} \times \mathbb{C}$ at time $\tau = 0$. Then the reverse process begins until it reaches the original orbifold description at $\tau = \infty$. In complete analogy, the spacetime of equation (A.8) shows a periodic evolution with the effects of noncommutativity becoming most important at $\tau = \pm\infty$ but negligible at $\tau = 0$ where the geometry tends to $\mathrm{AdS}_5 \times \mathrm{S}^5$.

This completes our discussion of noncommutative gauge theories. We have clearly here only alluded to a number of issues regarding these theories and noncommutative spacetimes in general. It would certainly be of interest to explore these issues further in the future.

Appendix B

The noncommutativity matrix

Here we present the noncommutativity matrix in polar coordinates (r_i, φ_i) with $i = 1, 2, 3$ on $\mathbb{R}^6$. We assume that Θ_ρ is given in terms of commuting variables $(z, \overline{z})$ and that we can follow the transformation rules of contravariant tensors when changing coordinate systems, namely:

$$\Theta^{i'j'} = \frac{\partial x'^{i'}}{\partial x^i} \frac{\partial x'^{j'}}{\partial x^j} \Theta^{ij} \tag{B.1}$$

Rescalling q_i of (3.25) and (3.26) as $q_i \to 2q_i$ then yields:

$$\Theta_\rho = \begin{pmatrix} 0 & -(q_2x_3-q_1y)r_3 & (q_2x_2-q_1y)r_2 & 0 & \frac{(q_1x_2-q_2y_1)r_3}{r_2} & \frac{(q_1x_3-q_2y_1)r_2}{r_3} \\ (q_2x_3-q_1y)r_3 & 0 & -(q_2x_1-q_1y)r_1 & -\frac{(q_1x_1-q_2y_2)r_3}{r_1} & 0 & \frac{(q_1x_3-q_2y_2)r_1}{r_3} \\ -(q_2x_2-q_1y)r_2 & (q_2x_1-q_1y)r_2 & 0 & \frac{(q_1x_1-q_2y_3)r_2}{r_1} & -\frac{(q_1x_2-q_2y_3)r_1}{r_2} & 0 \\ 0 & \frac{(q_1x_1-q_2y_2)r_3}{r_1} & -\frac{(q_1x_1-q_2y_3)r_2}{r_1} & 0 & -\frac{(q2x-q1y3)r_3}{r_1r_2} & \frac{(q_2x-q_1y_2)r_2}{r_1r_3} \\ -\frac{(q_1x_2-q_2y_1)r_3}{r_2} & 0 & \frac{(q_1x_2-q_2y_3)r_1}{r_2} & \frac{(q_2x-q_1y_3)r_3}{r_1r_2} & 0 & -\frac{(q_2x-q_1y_1)r_1}{r_2r_3} \\ \frac{(q_1x_3-q_2y_1)r_2}{r_3} & -\frac{(q_1x_3-q_2y_2)r_1}{r_3} & 0 & -\frac{(q_2x-q_1y_2)r_2}{r_1r_3} & \frac{(q_2x-q_1y_1)r_1}{r_2r_3} & 0 \end{pmatrix} \tag{B.2}$$

where to keep the expressions compact, we defined variables x, x_i and y, y_i according to:

$$x_1 = -C_1 r_1 + C_2 r_2 + C_3 r_3 \quad x_2 = C_1 r_1 - C_2 r_2 + C_3 r_3 \quad x_3 = C_1 r_1 + C_2 r_2 - C_3 r_3$$
$$y_1 = -S_1 r_1 + S_2 r_2 + S_3 r_3 \quad y_2 = S_1 r_1 - S_2 r_2 + S_3 r_3 \quad y_3 = S_1 r_1 + S_2 r_2 - S_3 r_3$$
$$x = C_1 r_1 + C_2 r_2 + C_3 r_3 \qquad y = S_1 r_1 + S_2 r_2 + S_3 r_3 \tag{B.3}$$

whereas S_i, C_i represent the following triginometric functions:

$$S_1 = \sin(\varphi_2 + \varphi_3 - 2\varphi_1), \quad S_2 = \sin(\varphi_3 + \varphi_1 - 2\varphi_2), \quad S_3 = \sin(\varphi_1 + \varphi_2 - 2\varphi_3)$$
$$C_1 = \cos(\varphi_2 + \varphi_3 - 2\varphi_1), \quad C_2 = \cos(\varphi_3 + \varphi_1 - 2\varphi_2), \quad C_3 = \cos(\varphi_1 + \varphi_2 - 2\varphi_3) \tag{B.4}$$

The discrete symmetry $\mathbb{Z}_{3(1)} \times \mathbb{Z}_{3(2)}$ along with the $U(1)_R$ are particularly transparent in this form. Observe first that under $\mathbb{Z}_{3(1)}$:

$$\begin{gathered} \mathbb{Z}_{3(1)}: \quad (x_1, x_2, x_3, y_1, y_2, y_3) \to (x_3, x_1, x_2, y_3, y_1, y_2) \\ \text{while} \quad (x, y) \to (x, y) \end{gathered} \tag{B.5}$$

Then it is easy to see for example, that $\Theta_\rho^{r_1\varphi_2} = \frac{(q_1 x_2 - q_2 y_1) r_3}{r_2} \to \Theta_\rho^{r_3\varphi_1} = \frac{(q_1 x_1 - q_2 y_3) r_2}{r_1}$. The action of $\mathbb{Z}_{3(2)}$ is equally simple transforming the polar angles φ_i as:

$$\mathbb{Z}_{3(2)}: \quad (\varphi_1, \varphi_2, \varphi_3) \to (\varphi_1, \varphi_2 + \frac{2\pi}{3}, \varphi_3 - \frac{2\pi}{3}) \tag{B.6}$$

thus leaving invariant the trigonometric functions S_i, C_i which depend on the following combinations: $\sigma_i \equiv \frac{1}{3}(\varphi_{i+1} + \varphi_{i+2} - 2\varphi_1)$. Morever, note that Θ_ρ is independent of

$\psi = \frac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3)$ therefore respects the $U(1)_R$ R–symmetry of the theory.

In a similar manner, one obtains the noncommutativity matrix Θ_ρ in spherical coordinates denoted as $(r, \alpha, \theta, \varphi_1, \varphi_2, \varphi_3)$.

$$\begin{aligned} z_1 &= r\cos\alpha \mathrm{e}^{i\phi_1}, \quad z_2 = r\sin\alpha\sin\theta \mathrm{e}^{i\phi_2}, \quad z_3 = r\sin\alpha\cos\theta \mathrm{e}^{i\phi_3} \\ \overline{z}_1 &= r\cos\alpha \mathrm{e}^{-i\phi_1}, \quad \overline{z}_2 = r\sin\alpha\sin\theta \mathrm{e}^{-i\phi_2}, \quad \overline{z}_3 = r\sin\alpha\cos\theta \mathrm{e}^{-i\phi_3} \end{aligned} \tag{B.7}$$

where it reads[1]:

$$\Theta_\rho = \begin{pmatrix} 0 & -\frac{q_2 x + q_1 y}{s_\alpha} & 0 & \frac{c_\theta(-q_1 x_2 + q_2 y_2)}{s_\alpha s_\theta} & \frac{s_\theta(q_1 x_3 - q_2 y_3)}{s_\alpha c_\theta} \\ \frac{q_2 x + q_1 y}{s_\alpha} & 0 & \frac{-q_1 x_1 + q_2 y_1}{c_\alpha} & \frac{c_\alpha(q_1 x_2 - q_2 y_2)}{s_\alpha^2} & \frac{c_\alpha(q_1 x_3 - q_2 y_3)}{s_\alpha^2} \\ 0 & \frac{q_1 x_1 - q_2 y_1}{c_\alpha} & 0 & \frac{c_\theta(q_2 x_3 + q_1 y_3)}{c_\alpha c_\theta} & -\frac{s_\theta(q_2 x_2 + q_1 y_2)}{c_\alpha c_\theta} \\ \frac{c_\theta(q_1 x_2 - q_2 y_2)}{s_\alpha s_\theta} & \frac{c_\alpha(-q_1 x_2 + q_2 y_2)}{s_\alpha^2} & -\frac{c_\theta(q_2 x_3 + q_1 y_3)}{c_\alpha c_\theta} & 0 & \frac{c_\alpha(q_2 x_1 + q_1 y_1)}{s_\alpha^2 s_\theta c_\theta} \\ -\frac{s_\theta(q_1 x_3 - q_2 y_3)}{s_\alpha c_\theta} & \frac{c_\alpha(-q_1 x_3 + q_2 y_3)}{s_\alpha^2} & \frac{s_\theta(q_2 x_2 + q_1 y_2)}{c_\alpha c_\theta} & -\frac{c_\alpha(q_2 x_1 + q_1 y_1)}{s_\alpha^2 s_\theta c_\theta} & 0 \end{pmatrix} \tag{B.8}$$

Note that Θ_ρ is now a five–dimensional matrix along the S^5 and that variables x, x_i, y, y_i appearing in (B.8) are rescaled by 1/r. In other words we have here defined:

$$\begin{aligned} x_1 &= (-c_\alpha C_1 + s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3), \quad x_2 = (c_\alpha C_1 - s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3) \\ x_3 &= (c_\alpha C_1 + s_\alpha s_\theta C_2 - s_\alpha c_\theta C_3), \quad x = (c_\alpha C_1 + s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3) \\ y_1 &= (-c_\alpha S_1 + s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3), \quad y_2 = (c_\alpha S_1 - s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3) \\ y_3 &= (c_\alpha S_1 + s_\alpha s_\theta S_2 - s_\alpha c_\theta S_3), \quad y = (c_\alpha S_1 + s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3) \end{aligned} \tag{B.9}$$

It is then clear that Θ_ρ is independent of the radial direction r.

[1]We use here the following abbreviations: $s_\alpha = \sin\alpha, c_\alpha = \cos\alpha, s_\theta = \sin\theta, c_\theta = \cos\theta$.

Appendix C

RR–fields and supergravity equations of motion

As mentioned previously, although the procedure proposed in this article gives us the solution for the NS–NS fields of the geometry for free, it does not produce any information on the RR–ones. We thus have to compute them using the supergravity equations of motions [108]. We employ the following ansatz [1]:

$$\begin{aligned} \mathrm{d}s^2_{10} &= \mathrm{d}s^2_{AdS_5} + \mathrm{d}s^2_5 \\ C = 0 \quad F_5 &= f(\omega_{AdS_5} + \omega_{\widetilde{\mathrm{S}}^5}) \end{aligned} \tag{C.1}$$

where f is the appropriate normalization coefficient for the flux which in this case reduces to $f = 16\pi\mathrm{N}$ and $\omega_{AdS_5}, \omega_{\widetilde{\mathrm{S}}^5}$ are the volume elements of the corresponding

[1]Note that the vanishing axion condition can be deduced from the other two in equation (C.1).

parts of the $\mathrm{AdS}_5 \times \widetilde{\mathrm{S}}^5$ geometry. Then the supergravity field equations reduce to:

$$
\begin{aligned}
D^2 e^{-2\Phi} &= -\frac{1}{6}\left(F_3^2 - e^{-2\Phi} H_3^2\right) \\
F_3 &= -f^{-1} \mathrm{d} \star_5 e^{-2\Phi} H_3 \\
H_3 &= f^{-1} \mathrm{d} \star_5 F_3 \\
R_{MN} &= -2 D_M D_N \Phi - \frac{1}{4} g_{MN} D^2 \Phi + \frac{1}{2} g_{MN} \partial_R \Phi \partial^R \Phi + \\
&\quad + \frac{1}{96} e^{2\Phi} F_{MPQR} F_N^{PQR} + \frac{1}{4}(H_{MPQ} H_N^{PQ} + e^{2\Phi} F_{MPQ} F_N^{PQ}) - \frac{1}{48} g_{MN}(H_3^2 + e^{2\Phi} F_3^2)
\end{aligned}
\tag{C.2}
$$

where M, N represent five dimensional indices on the compact piece of the geometry whereas $\star_5$ denotes the Hodge star on the same manifold.

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