Studies in gauge/string dualities

A Dissertation Presented

by

Diego Trancanelli

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Physics

Stony Brook University

May 2007

Stony Brook University

The Graduate School

Diego Trancanelli

We, the dissertation committee for the above candidate for the
Doctor of Philosophy degree,
hereby recommend acceptance of this dissertation.

Martin Roček
Professor, C. N. Yang Institute for Theoretical Physics, Stony Brook University
Dissertation Director

Peter van Nieuwenhuizen
Professor, C. N. Yang Institute for Theoretical Physics, Stony Brook University
Chairman of Dissertation

Warren Siegel
Professor, C. N. Yang Institute for Theoretical Physics, Stony Brook University

Axel Drees
Professor, Department of Physics, Stony Brook University

Michael Anderson
Professor, Department of Mathematics, Stony Brook University
Outside Member

This dissertation is accepted by the Graduate School.

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

Studies in gauge/string dualities

by

Diego Trancanelli

Doctor of Philosophy

in

Physics

Stony Brook University

2007

Dualities are among the most powerful tools in theoretical physics and, particularly, in string theory. The ones relating gauge field theories and theories of strings, the *gauge/string dualities*, are specially important and constitute the focus of this dissertation. We consider several topics in the context of two such dualities, the AdS/CFT correspondence and twistor string theory.

In the part about the AdS/CFT correspondence we start by studying the thermodynamics of type IIB superstrings on the maximally supersymmetric plane wave background, computing in particular the Hagedorn temperature of non-interacting strings and analyzing the limits of small and large background Ramond-Ramond flux.

We then consider a half-BPS supergravity solution whose moduli space can be mapped to the phase space of a gas of free fermions in a harmonic potential at finite temperature. We can match the ADM mass of the geometry with the thermal energy of the fermions and propose a way to also match the entropies in the two pictures.

The last chapter of this part is dedicated to the study of supersymmetric Wilson loops. We first introduce a large new family of loop operators preserving various amounts of supersymmetry, from two to sixteen supercharges. We then study a novel description of higher rank loops in terms of electrically charged D-branes. In particular, we compute correlation functions between such loops and chiral primary operators of $\mathcal{N} = 4$ super Yang-Mills theory and present the D-brane solutions corresponding to some examples of quarter-BPS loop operators.

In the part about twistor string theory we first extend the conjectured equivalence between perturbative $\mathcal{N} = 4$ super Yang-Mills and the topological B-model on $\mathbb{CP}^{3|4}$ to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal quiver gauge theories. This is achieved by orbifolding the fermionic directions of the supertwistor space. We also consider some explicit quivers and compute several scattering amplitudes.

Finally, we check the localization properties of some gravity amplitudes in twistor space and propose an extension of the twistor inspired MHV decomposition of Feynamn diagrams to the computation of tree level graviton scattering.

A Tatiana e Tomás

Contents

List of Figures **xiii**

Acknowledgements **xiv**

I Dualities between gauge and string theories 1

1 Overview **2**

2 The AdS/CFT correspondence **11**

2.1 Introduction 11

2.1.1 $\mathcal{N} = 4$ super Yang-Mills theory 14

2.1.2 Anti de Sitter space 15

2.1.3 The statement of the AdS/CFT correspondence 17

2.1.4 Matching the spectra 22

2.2 The plane wave limit of AdS/CFT 27

2.2.1 The Penrose limit 28

2.2.2 The plane wave σ-model 31

2.2.3 The gauge theory dual 33

2.3 Wilson loops in the AdS/CFT correspondence 38

2.3.1 Wilson loops in $\mathcal{N} = 4$ super Yang-Mills 39
2.3.2 Bulk description as minimal surfaces 45
2.3.3 Giant Wilson loops 51

3 Twistor string theory **59**
3.1 Introduction 59
3.2 Scattering amplitudes in twistor space 60
3.2.1 The spinor-helicity formalism 60
3.2.2 Tree level gluon amplitudes 64
3.2.3 Fourier transform to twistor space 66
3.2.4 Localization of amplitudes in twistor space 70
3.2.5 Supersymmetric extension 74
3.3 The B-model on $\mathbb{CP}^{3|4}$ and the twistor string 76
3.3.1 The D-instanton expansion 80
3.4 MHV decomposition of amplitudes 83

II Studies on the AdS/CFT correspondence **87**

4 String thermodynamics in a plane wave background **88**
4.1 Introduction 88
4.2 Hagedorn and AdS/CFT 89
4.2.1 Large f 93
4.2.2 Other issues 95
4.3 Superstring free energy 97
4.4 Modular properties of Z 102
4.5 The Hagedorn temperature 104

4.5.1 Expansion for small f 105
4.5.2 The large f limit 108
4.6 $AdS_3 \times S^3$ in NS-NS and R-R 3-form backgrounds 109

5 Half BPS geometries and free fermions 112

5.1 Introduction 112
5.2 Review of the LLM construction 115
5.3 1D fermions in the harmonic well 119
5.4 The hyperstar: low temperature regime 123
5.4.1 ADM form of the metric 125
5.4.2 Singularities of the metric 127
5.4.3 Fluxes and topology 130
5.4.4 ADM mass of the hyperstar 131
5.4.5 Angular momentum 144
5.4.6 Entropy 145
5.5 High temperature regime 147
5.5.1 Energy and angular momentum 154
5.6 Concluding remarks and outlook 156

6 Supersymmetric Wilson loops 158

6.1 A new class of supersymmetric Wilson loop 159
6.1.1 Definition and supersymmetry analysis 159
6.1.2 Topological twisting 164
6.1.3 Special loops 165
6.2 Correlators of giant Wilson loops and chiral primaries 178
6.2.1 Kaluza-Klein expansion 179

6.2.2 Operator product expansion of Wilson loops 182
6.2.3 Brane computation 186
6.2.4 The correlation functions from the normal matrix model . . . 192
6.2.5 Outlook 200
6.3 Quarter BPS Wilson loops 201
6.3.1 Wilson loop with insertions 202
6.3.2 Wilson loop wrapping a circle on S^5 213
6.3.3 Discussion 232

III Studies on twistor string theory 236

7 Fermionic orbifold of the twistor string 237

7.1 Introduction 237
7.2 Branes transverse to orbifold singularities 239
7.2.1 Open string sector 239
7.2.2 Closed string sector 241
7.3 The orbifold of the twistor string 243
7.3.1 The orbifold in the D5-brane sector 243
7.3.2 D1-branes 246
7.4 Explicit examples of orbifolds 249
7.4.1 An $\mathcal{N} = 1$ orbifold 249
7.4.2 An $\mathcal{N} = 2$ orbifold 256
7.5 Conclusion 259

8 A twistorial approach to gravity amplitudes 261

8.1 Introduction 261

8.2 A googly graviton amplitude 263

8.3 Disconnected MHV decomposition 269

8.3.1 The $\mathcal{M}^{(4)}(+,-,-,-)$ and $\mathcal{M}^{(5)}(+,-,-,-,-)$ amplitudes . . 269

8.3.2 The googly amplitude 273

8.4 Conclusion 275

A The string Hagedorn temperature **277**

B Alternative derivation of eq. (4.43) **280**

C Spherical harmonics and orthogonal polynomials **282**

References **286**

List of Figures

3.1 The two MHV graphs contributing to the vanishing four gluon amplitude $\mathcal{M}^{(4)}(1+, 2-, 3-, 4-)$. Notice that internal lines have opposite helicities at their two endpoints, and only MHV vertices are used to build the diagrams. 86

5.1 Droplet configuration in the low temperature limit. The greyscale ring around the Fermi level corresponds to a singular region of the spacetime. 124

5.2 Droplet configuration in the high temperature limit. 148

6.1 A depiction of string and D3-brane solutions. The solid line gives ρ as a function of θ for the string with boundary value of $\theta = \pi/3$. The D3-brane solution is represented by the dashed and dotted lines which are respectively ρ and u as functions of θ (for $\kappa = 1$). In both cases there are two solutions, a stable one with $0 \leq \theta \leq \theta_0$ (where ρ for the string and D3-brane are nearly indistinguishable) and an unstable one, with $\theta \leq \theta_0 \leq \pi$. The unstable D3-brane solution reaches the boundary of AdS not only at θ_0, but also at $\theta = \pi/2$, where u diverges, but then it turns back and closes smoothly on itself. 227

7.1 Regrouping of the D5-branes under the orbifold action: the kN branes are split into k stacks of N branes each. Note that the k stacks of branes are actually coincident at the point $\bar{\psi}_A = 0$ of the twistor space. 244

7.2 The stacks of D5 and D1-branes. D1-D5 strings are stretched between the two different kinds of branes. An interaction between the first and the second stack is also depicted. 247

7.3 The quiver for the D1-D5 brane system in the case $k = 3$. It has three nodes corresponding to the D5-branes indicated with N, and three nodes corresponding to the D1's. 248

7.4 The $\mathcal{N} = 1$ quiver for $k = 3$ and $a_A = (1, 1, 1, 0)$. 250

7.5 The double line notation: (a) scattering of four adjoint fields belonging to the same node; (b) scattering of two adjoint and two bi-fundamental fields; (c) scattering of four bi-fundamental fields with intermediate adjoint field; (d) same as in (c) but with intermediate bi-fundamental field. 252

7.6 The two Feynman diagrams that contribute to the tree level $(q\,q\,\tilde{q}\,\tilde{q})$ scattering. 254

7.7 The $\mathcal{N} = 2$ quiver for $k = 2$ and $a_A = (1, 1, 0, 0)$. 256

7.8 The two Feynman diagrams that contribute to the tree level $(\lambda\,q\,\tilde{q}\,\tilde{\lambda})$ scattering. 259

8.1 Two disconnected configurations contributing to $\mathcal{M}^{(n)}(1+, 2-, \ldots, n-)$. Each sphere represents a $\mathbb{CP}^1$, the sublocus of the twistor space where an MHV vertex is localized. The lines spanning between spheres are graviton propagators, with opposite helicities at the two endpoints. . 270

8.2 The $(+,-,-)$ graviton vertex. The external leg with momentum p is suitably extended off-shell and can be connected to a propagator. . . 270

8.3 The MHV diagrams contributing to the $\mathcal{M}^{(4)}(1+,2-,3-,4-)$ graviton amplitude. Note that, differently from the Yang-Mills case, here we do not require a fixed cyclic ordering of the external legs. 271

8.4 Two of the fifteen MHV diagrams contributing to the graviton amplitude $\mathcal{M}^{(5)}(1+,2-,3-,4-,5-)$. The remaining 12 graphs are obtained by appropriate relabeling of the external legs of these two graphs. . . 272

8.5 The $(+,+,-,-)$ graviton vertex. This vertex is non-holomorphic, for it depends not only on λ but also on $\tilde{\lambda}$. 273

8.6 Two of the nine MHV graphs in the $\mathcal{M}^{(5)}(1+,2+,3-,4-,5-)$ graviton amplitude. The other seven diagrams are obtained permuting the labels on the external legs. 275

Acknowledgements

I first of all wish to express my gratitude to my friends and collaborators Daniel, Riccardo, Simone, and also Costas, Manuela, and Nadav. Without them this work would have simply not been possible.

I am also happy to thank my advisor Martin Roček for his unwavering support and encouragement during the whole length of my studies at the YITP. I am grateful for the numerous and stimulating discussions we had and for sharing his deep and original insight on physics.

A special thank goes to Peter van Nieuwenhuizen, whom I really admire for his talent and dedication to science. He is also an extraordinary teacher and his courses on string theory and quantum field theory have taught me a great deal.

I have benefitted very much from interesting discussions with Leonardo Rastelli, Gordon Semenoff, and Warren Siegel.

I would also like to acknowledge Gianluca Grignani and Marta Orselli for their friendship, pleasant collaborations and conversations, and nice hospitality during several visits at the University of Perugia.

Infine il mio grazie più sentito e commosso a mia madre Lia, a Eny, Alessandra, Lina, Mirjiana, don Bruno e a tutti coloro che ci hanno accompagnato con la preghiera in questo lungo viaggio.

Part I

Dualities between gauge and string theories

Chapter 1

Overview

The notion of duality, *i.e.* the equivalence between two systems with different descriptions but same underlying physics, pervades our current understanding of string theory.

One of the first and probably most important discoveries of a duality in physics dates back to Dirac's observation that the Maxwell equations in the vacuum are invariant under the exchange of the electric and magnetic fields. To preserve this symmetry in the presence of sources, Dirac was led to define the concept of magnetic monopole. The magnetic charge m of the monopole, in order to produce a consistent theory at the quantum level, had to be quantized in units of the inverse of the electric charge e

$$m = \frac{2\pi k}{e}\,, \qquad k \in \mathbb{Z}\,.$$

Assuming that m and e are not both of order 1, exchanging electric and magnetic monopoles can then be regarded as the prototypical example of *weak/strong coupling duality*, or, in more modern language, of *S-duality*.

Under S-duality a theory with coupling constant g is mapped to a possibly different theory with coupling constant $1/g$. One might then be able to extract information about non-perturbative aspects of one theory by studying the weak coupling expansion of its S-dual.

Differently from S-duality, *T-duality* (and the closely related *mirror symmetry*) is perturbative in nature and acts on the geometric moduli of the background space, exchanging, for example, the compactification radius R of strings compactified on a circle with the inverse radius $1/R$.

As was realized during the so-called "second superstring revolution" in the mid nineties, the five superstring theories in ten dimensions are in fact connected by an intricate web of dualities and might just be different limits of a unique eleven-dimensional theory called *M-theory*. For example, type I superstrings and $SO(32)$ heterotic strings are a notable example of theories linked by S-duality. Two other theories, which will take the lion's share of this dissertation, namely type IIB superstrings and $\mathcal{N} = 4$ super Yang-Mills in four dimensions, exhibit selfduality properties under S-duality. On the other hand, T-duality relates the two type II theories, type IIA and type IIB, and the two heterotic theories, $SO(32)$ and $E_8 \times E_8$. Finally, type IIA and $E_8 \times E_8$ heterotic strings become eleven-dimensional at strong coupling, giving rise to M-theory.

A crucial ingredient in the development of this picture was the discovery of *D-branes*, topological defects which admit a dual interpretation as non-perturbative solitonic solutions of supergravity and as hypersurfaces in flat space where open fundamental strings can end.

The importance of D-branes in modern string theory is central and in great part motivated by the fact that they allow for the introduction of non-abelian gauge symmetry. In fact, Yang-Mills theories arise as the low energy limit of the dynamics of multiple coincident branes and quantum fields as the massless modes of the open strings living on the branes' world-volumes.

In virtue of the dual interpretation mentioned above, D-branes have also been essential in the construction of dualities between non-gravitational field theories and theories of strings, the *gauge/string dualities* which are the subject of this dissertation. Many of these dualities represent a powerful tool to explore regions of the moduli space of gauge theories which are not directly accessible by ordinary field theoretical techniques as, for example, the perturbative expansion in small parameters.

The most notable example in this class is given by the *Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence* discovered by J. Maldacena in 1997. This is one of the major breakthroughs in string theory in the last decade and the first explicit realization of the *holographic principle*: the idea that string theory, which is a theory of quantum gravity, has a dual description as a quantum field theory living on the boundary of the background space. According to the original and most studied form of this correspondence, maximally supersymmetric Yang-Mills theory in four dimensions, which is a conformal field theory, is conjectured to be the exact dual of type IIB superstrings living on the ten-dimensional space $AdS_5 \times S^5$.

What makes this duality so interesting is the fact that a non-trivial, strongly coupled limit of the gauge theory is mapped onto a solvable limit of the string theory. In fact, it turns out that, when one takes the limit of large number of colors N and large 't Hooft coupling $g_{YM}^2 N$ in the gauge theory, the corresponding string theory is well approximated by classical supergravity. This gives therefore a powerful tool to study strongly coupled systems where perturbative methods are not applicable. Of course $\mathcal{N} = 4$ super Yang-Mills is a highly supersymmetric and conformal theory which differs in many aspects from real world gauge theories such as QCD. The ultimate hope is however that the AdS/CFT correspondence may provide us with the tools and the insight necessary to understand strong interactions and find the string theory dual of QCD.

The fact that the AdS/CFT correspondence is a weak/strong coupling duality makes it interesting, but, at the same time, also difficult to prove (or disprove) because reliable computational techniques on the two sides of the duality do not have an overlapping domain of validity. It is therefore important to test this correspondence wherever possible and in this dissertation we discuss some of these tests.

The AdS/CFT correspondence has been extended to include theories will reduced supersymmetry, non-conformal theories, gauge theory duals of M-theory, and has also found some exciting phenomenological applications, as, for instance, the study of the strongly coupled plasma of quarks and gluons which is formed in heavy ion collisions in the RHIC experiment at BNL and will also be studied at CERN, once the LHC is turned on.

The second gauge/string duality studied in this dissertation is the conjectured equivalence, proposed by E. Witten in 2003, between *perturbative* $\mathcal{N} = 4$ super Yang-Mills and the D-instanton expansion of a particular version of topological string theory, the B-model on the supertwistor space $\mathbb{CP}^{3|4}$. This represents an interesting counterpart to the AdS/CFT correspondence, for it relates two *weakly coupled* theories. This *twistor string*, besides being a remarkable mathematical construction which is compelling in its own right, has also shed new light on previously known results in non-supersymmetric field theories and inspired very effective ways to reorganize the expansion in Feynman diagrams, allowing to compute numerous new scattering amplitudes, both at tree level and at higher orders, which are of crucial phenomenological importance, for example for the analysis of multi-jet production at LHC.

The AdS/CFT correspondence and twistor string theory are by no means the only interesting cases of fruitful interplay between gauge and string theory. There are many other important dualities relating gauge theories to string theories and M-theory which we do not discuss in this dissertation. For instance, the abundant observational evidence in cosmology that our universe has a small and positive cosmological constant naturally suggests to consider the analogue of the AdS/CFT correspondence for de Sitter spaces, the *dS/CFT correspondence*. Unfortunately this program, albeit its phenomenological relevance, is far less studied and understood than its AdS/CFT cousin, mainly because theories in de Sitter space cannot be supersymmetric and therefore are difficult to deal with.

A second example is the string inspired *Dijkgraaf-Vafa conjecture* that non perturbative quantities of a supersymmetric gauge theory, such as the glueball superpotential, may be computed from a simple matrix model with potential equal to the tree level superpotential of the gauge theory.

Finally, we mention the *Matrix theory* interpretation of M-theory in flat space, where, after compactifying on a torus some of the eleven flat directions, M-theory is described in terms of a quantum field theory.

In what follows we describe the content and organization of the present work.

Organization of the dissertation

This dissertation consists of three parts. In chapters 2 and 3 of this first part we present introductory accounts to the AdS/CFT correspondence and twistor string theory. The original material, published in [1]-[7], is contained in parts II and III.

The focus in part II is on the AdS/CFT correspondence. We start in chapter 4 by studying the thermodynamics of type IIB superstrings on the plane wave background, a maximally supersymmetric background obtained as a Penrose limit of $AdS_5 \times S^5$ [1]. In particular, we compute the partition function, whose modular properties are analyzed in detail, and the Hagedorn temperature for a gas of non-interacting strings. We show that the latter is a non-trivial function of the background Ramond-Ramond flux and of the string scale and we carefully investigate the limits of small and large flux. We also study the string thermodynamics in geometries that arise in D1-D5 systems such as $AdS_3 \times S^3 \times T^4$ backgrounds in the presence of NS-NS and R-R 3-forms.

Chapter 5 is based on [2]. There we consider solutions of type IIB supergravity which preserve half of the supersymmetries of the background. These geometries correspond to half-BPS chiral primary operators of $\mathcal{N} = 4$ super Yang-Mills and have a dual description in terms of non-interacting fermions in a harmonic potential. We investigate how turning on a temperature for these fermions affects the corresponding supergravity background and compare, in both limits of low and high temperature, the ADM mass of this solution with the thermal excitation energy of the dual fermions, finding agreement. We also argue how the solution develops a naked singularity, which should be resolved once α' string corrections are taken into account. These corrections are expected to give rise to a finite area horizon, allowing for a rigorous analysis of the entropy of the system and a comparison with the entropy of the dual fermions. In the limit of high temperature we find that the supergravity background resembles the metric of a dilute gas of D3-branes associated to the Coulomb branch of the conformal field theory.

The last chapter of part II, chapter 6, is dedicated to supersymmetric Wilson loops [3]-[5]. The study of these observables represents one of the main tests of the AdS/CFT correspondence, because for these operators one can resum the perturbative expansion of super Yang-Mills theory and extrapolate the results to strong coupling, thus providing a check of the corresponding string theory computation in the supergravity limit.

In section 6.1 we introduce a new large class of loop operators with non-trivial expectation value and preserving various amounts of supersymmetry, from two to sixteen supercharges. In this construction the scalar coupling is determined by the shape of the loop, which can be an arbitrary curve on a 3-sphere. Some previously known loops, notably the half-BPS circle, belong to this class, but we point out many more special cases not known before. We also remark that this class of loops might be related to a topologically twisted version of $\mathcal{N} = 4$ super Yang-Mills.

Sections 6.2 and 6.3 deal with the so-called "giant Wilson loops". According to the AdS/CFT dictionary, Wilson loop operators act as sources of fundamental strings extending in the bulk of $AdS_5 \times S^5$ and landing on the boundary along the loop. The expectation value of Wilson loops is then given by the minimal area swept by these strings. This description is appropriate for particles in the fundamental representation of the gauge group. For higher representations one needs to replace the strings with D3 and D5-branes for, respectively, the symmetric and antisymmetric representation. These branes pinch off at the boundary landing on the Wilson loop and carry an electric charge. The brane computation captures all the non-planar corrections to the string result at leading order in α'.

In section 6.2 we consider the expansion of a higher rank circular Wilson loop in a series of local operators and compute its correlation function with the chiral primaries of $\mathcal{N} = 4$ super Yang-Mills. We perform the computation both in the bulk picture, where the D-branes exchange light supergravity modes with the local operator on the boundary, and in the theory on the boundary. A peculiar feature of circular Wilson loops is the fact that the quantum field theory computation reduces to a matrix model computation. We find perfect agreement between the results coming from the D-branes and the matrix model.

In section 6.3 we study two examples of quarter-BPS loops where it is possible to find solutions for the D3-branes starting from first-order equations derived from the supersymmetry conditions. The first example is a straight half-BPS Wilson line with the insertion of two local half-BPS chiral primary operators. The second one is a circular loop which couples to three of the six scalars of the $\mathcal{N} = 4$ multiplet and whose expectation value is captured by a matrix model.

In part III we present two studies on twistor string theory. In chapter 7, inspired by the AdS/CFT correspondence, we extend the twistor construction to theories with reduced supersymmetry [6]. This is done by considering orbifolds of the $SU(4)$ R-symmetry group of $\mathcal{N} = 4$ super Yang-Mills, or, equivalently, of the fermionic directions of the supertwistor space $\mathbb{CP}^{3|4}$. The resulting gauge theories are $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal quiver theories. We test this construction by computing several scattering amplitudes for these theories and find agreement with the field theoretical expectations.

Yang-Mills scattering amplitudes in twistor space are localized on holomorphic curves, whose degree and genus is determined by the helicities of the scattering particles and by the number of loops. In chapter 8 we explore the possibility of extending the twistor construction to ordinary (non-conformal) gravity [7] and prove that localization properties also hold for the so-called "googly" graviton amplitudes, which are the simplest non-maximally helicity violating amplitudes. We also show, using the

KLT relations between closed and open string vertex operators, that for a particular subset of amplitudes one has factorization in terms of MHV vertices, but that novel ingredients are needed to reproduce generic amplitudes.

Note We have not included in this dissertation the work contained in [8], for it is not directly relevant to the topic of gauge/string dualities. There we found instantonic solutions to $\mathcal{N} = 1/2$ super Yang-Mills theory with matter fields. This theory arises when one deforms the usual $\mathcal{N} = 1$ superspace by considering a non-vanishing anticommutator for the fermionic coordinates, $\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}$. The motivation for introducing this deformation stems from string theory, in particular from superstrings propagating in a selfdual graviphoton background.

Chapter 2

The AdS/CFT correspondence

2.1 Introduction

It is an old and important idea in particle physics that gauge theories might have dual descriptions as string theories [9]. It was in fact observed by 't Hooft in 1974 [10] that one can reorganize the perturbative expansion of a $SU(N)$ gauge theory in a way that is very reminiscent of the string theory genus expansion, with the gauge theory Feynman diagrams seen as string world-sheets.

This is seen by using the *double line notation* where one associates oriented lines to color indices: fundamental and anti-fundamental indices are then represented by lines with opposite orientations, whereas adjoint indices are represented by a double line. Let's consider first a theory with only adjoint fields and with the coupling constant pulled in front of the action in an overall g^{-2} factor. A generic diagram will then scale as

$$(g^2)^{P-V} N^L \equiv \lambda^{P-V} N^{L-P+V} , \tag{2.1}$$

where P is the number of propagators, V the number of vertices, and L the number of closed loops. We have also introduced the *'t Hooft coupling*

$$\lambda \equiv g^2 N \, . \tag{2.2}$$

This diagram can be drawn on a simplicial Riemann surface of Euler number $\chi = V - P + L = 2 - 2h$, with h being the genus of the surface.[1] Eq. (2.1) can then be rewritten as $\lambda^{P-V} N^{2-2h}$ and the perturbative expansion of the partition function as a double sum

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} \sum_{n=0}^{\infty} c_n^{(h)} \lambda^n \, , \tag{2.3}$$

where $c_n^{(h)}$ depends on the Feynman diagrams at genus h. In the *planar* or *'t Hooft limit* of large N with fixed λ, the graphs that can be drawn an a sphere without crossing lines, the so-called *planar graphs*, clearly dominate. Adding fundamental matter means adding propagators with a single line and is therefore equivalent to introducing boundaries on the Riemann surface, so that $\chi = 2 - 2h - b$. In the planar limit these graphs are always suppressed with respect to the adjoint graphs of equal genus.

The first sum in the expansion (2.3) resembles the string genus expansion where loop diagrams are suppressed by a factor g_s^{2h-2} with respect to tree level diagrams, with g_s and h being now the string coupling and the genus of the world-sheet. The second sum on the other hand could be seen as the equivalent of an α' expansion. One is then led to regard the gauge theory graphs in the large N limit as defining the

[1] The surface associated to an $SU(N)$ gauge group is oriented. Other groups as $SO(N)$ and $Sp(N)$ would give rise to unoriented surfaces, for their adjoint representations are products of two fundamental ones, rather than a fundamental and an anti-fundamental.

world-sheet of some string with coupling

$$g_s = \frac{1}{N}\,, \tag{2.4}$$

so that this string is weakly coupled in the planar limit. Fundamental matter should correspond to an open string sector of the theory.

The ultimate application of this idea would be clearly to find the 't Hooft string dual to $SU(3)$ QCD. This has proved to be an extremely complicated problem which is still unresolved.

The first explicit realization of 't Hooft idea took place in 1997 when J. Maldacena [11] proposed that[2]

four-dimensional $SU(N)$ Yang-Mills theory with maximal $\mathcal{N}=4$ supersymmetry is exactly dual to type IIB superstring on the $AdS_5 \times S^5$ background with N units of Ramond-Ramond flux.

This represents also an example of *holographic duality*, where a theory of quantum gravity finds a dual description as a quantum field theory living on the boundary of the space.

Before motivating and discussing in more detail this conjecture, we present next the main ingredients of the AdS/CFT correspondence, namely $\mathcal{N}=4$ super Yang-Mills and some basic aspects of the AdS_5 space. A motivation based on matching global symmetries on the two sides of the duality will then become obvious. We shall then give a more direct argument based on the study of the near horizon geometry of D3-branes.

[2]This proposal has then been refined and clarified in [12]-[14]. By now there are numerous excellent reviews on this subject, see for example [14]-[18].

2.1.1 $\mathcal{N}=4$ super Yang-Mills theory

We start by reviewing some basic facts about $SU(N)$ $\mathcal{N}=4$ super Yang-Mills in four dimensions (see also, for example, [17]). This can be obtained by dimensionally reducing ten-dimensional $\mathcal{N}=1$ super Yang-Mills on a T^6 (thus preserving all the 16 supercharges of the original theory). The theory has only one multiplet, the gauge multiplet, composed by a gauge field A_μ (with $\mu = 0, \ldots, 3$), four Weyl fermions ψ_α^A (with $A = 1, \ldots, 4$ and $\alpha = 1, 2$), and six real scalars Φ^I. All these fields are adjoint under the gauge group $SU(N)$. There is also a global $SU(4) \simeq SO(6)$ R-symmetry under which the gauge field is a singlet, while the fermions and scalars transform respectively in the **4** and **6** representations.

The action (in Euclidean signature) reads

$$\begin{aligned} S \;=\; \frac{1}{g_{YM}^2} \int d^4x \,\mathrm{Tr} \Bigg(& \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{g_{YM}^2 \vartheta}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + D_\mu \Phi^I D^\mu \Phi^I + i \bar{\Psi} \Gamma^\mu D_\mu \Psi \\ & - \frac{1}{2} \left[\Phi^I, \Phi^J \right] \left[\Phi^I, \Phi^J \right] + i \bar{\Psi} \Gamma^I \left[\Phi^I, \Psi \right] \Bigg), \end{aligned} \tag{2.5}$$

where we have expressed the four Weyl fermions in terms of a single Majorana-Weyl spinor Ψ in ten dimensions, and Γ^μ and Γ^I are ten-dimensional 16×16 Dirac matrices. Here g_{YM} is the Yang-Mills coupling constant and we have allowed for a ϑ angle, which is relevant for non-trivial instantonic backgrounds.

This action is classically scale invariant since all terms in the Lagrangian have dimension 4. It is actually also conformal invariant, *i.e.* it is invariant under the $SO(4,2) \simeq SU(2,2)$ group formed by Poincaré, dilatations and special conformal transformations. This group combined with the 16 Poincaré supercharges inherited from ten-dimensional $\mathcal{N}=1$ SYM forms the larger (global and continuous) *super-*

conformal group $SU(2,2|4)$. This supergroup has, in addition to the 16 Poincaré supercharges Q^A_α and $\bar{Q}_{A\dot{\alpha}}$, also 16 *superconformal charges* S^A_α and $\bar{S}_{A\dot{\alpha}}$ stemming from the fact that the Poincaré supersymmetries and the special conformal transformations do not commute. The doubling of the number of supercharges is a typical feature of conformal theories.

The superconformal invariance persists also at the quantum level and the theory is UV finite (this does not prevent in any case wavefunction renormalization).[3] As a consequence the coupling constant g_{YM} is actually a non-running parameter which can be fixed to the desired value. Then $\mathcal{N}=4$ SYM is a unique theory defined only by the value of g_{YM} and the rank of the gauge group N.

2.1.2 Anti de Sitter space

The AdS_5 space is a 5-dimensional space with constant negative curvature which can be expressed in terms of 6 embedding coordinates X_i (with $i=-1,0,\ldots,4$) as a hyperboloid in $\mathbb{R}^{4,2}$

$$-X_{-1}^2 - X_0^2 + \sum_{k=1}^{4} X_k^2 = -R^2\,, \tag{2.6}$$

[3]The 1-loop β function turns out to be proportional to the factor [19]

$$\frac{11}{6}T(adj) - \frac{1}{3}\sum_A T(r_A) - \frac{1}{6}\sum_I T(r_I)\,,$$

where $T(r)$ is the Dynkin index of the representation r, and the indices A and I denote, respectively, Weyl fermions and complex scalars. Since all fields of the $\mathcal{N}=4$ gauge multiplet are in the adjoint one has

$$\beta \propto T(adj)\left(\frac{11}{6} - \frac{4}{3} - \frac{3}{6}\right) = 0\,.$$

The vanishing of the β function can be checked to hold also at two [20] and three loop level [21], and there are general arguments for all orders [22]-[24].

where R is the radius of the space. From this expression is clear that the isometry group of AdS_5 is $SO(4,2)$. Rewriting the embedding coordinates as

$$X_{-1} + X_4 = \frac{R}{z}\,, \qquad X_\mu = \frac{R}{z} x_\mu\,, \qquad \mu = 0, \ldots, 3\,, \tag{2.7}$$

the metric induced on the hypersurface (2.6) becomes

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + d\vec{x}^2\right)\,. \tag{2.8}$$

This is called *Poincaré patch* metric and $z \in [0, \infty)$ is called the *radial* coordinate of AdS_5. The boundary at spatial infinity is at $z = 0$ in these coordinates.

On the other hand, parameterizing the hypersurface as

$$X_{-1} = R\cosh\rho\cos t\,, \qquad X_0 = R\cosh\rho\sin t\,, \qquad X_k = R\sinh\rho\,\Omega_k\,, \tag{2.9}$$

with $\sum_{k=1}^{4} \Omega_k^2 = 1$, yields the *global coordinates* metric

$$ds^2 = R^2\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2\right)\,, \tag{2.10}$$

where $t \in (-\infty, \infty)$ is the *global time* of AdS_5.[4]

The Penrose diagram of AdS_5 is best understood from eq. (2.10) by taking out a factor of $\cosh^2\rho$ and defining $dx = d\rho/\cosh\rho$ [14]. One obtains a solid cylinder with boundary given by $S^3 \times \mathbb{R}$, where $\mathbb{R}$ is the time direction.[5] Light rays propagating

[4]Strictly speaking, defining t on an infinite line gives the universal covering of AdS_5. The hyperboloid is in fact already covered one time by taking $t \in [0, 2\pi)$, but in this way one would violate causality, having closed timelike curves for $\rho \to 0$.

[5]The $\mathbb{R}$ factor results from decompactifying the S^1 parameterized by t, as explained in the previous footnote.

in this cylinder can reach the boundary and bounce back in finite time, whereas massive particles moving along geodesics cannot. Further, the global metric (2.10) is defined over the whole space, the Poincaré metric (2.8) on the other hand covers only a "wedge" of the Penrose diagram contained between two horizons at $z = \infty$ and the boundary at $z = 0$.

2.1.3 The statement of the AdS/CFT correspondence

A first heuristic motivation for the AdS/CFT correspondence is based on the analysis of global symmetries. We have seen that $\mathcal{N} = 4$ SYM in the conformal phase has superconformal group $SU(2,2|4)$ with bosonic subgroup $SO(4,2) \times SO(6)$. This is precisely the isometry group of $AdS_5 \times S^5$. The $SO(4,2)$ isometry of AdS_5 acts in fact as the four-dimensional conformal group on the boundary, where $\mathcal{N} = 4$ SYM lives, and the $SO(6)$ isometry of the sphere can be identified with the R-symmetry group. Moreover, $AdS_5 \times S^5$ is a *maximally supersymmetric* background which realizes as symmetries of the vacuum all the 32 supersymmetries of type IIB strings. These can then be related to the 32 supercharges of the gauge theory. Finally, both $\mathcal{N} = 4$ SYM and type IIB strings exhibit a discrete Montonen-Olive $SL(2,\mathbb{Z})$ duality.[6]

A more direct motivation for the conjecture comes from studying systems of D-branes. These can be either regarded as hyperplanes where open strings end or, alternatively, as solitonic solutions of the supergravity equations of motion. The low energy limits in the two pictures should then produce two related theories.

Consider first type IIB strings in flat ten-dimensional Minkowski space and a

[6]For the gauge theory, this duality acts on the complex coupling constant $\tau \equiv \vartheta/2\pi + 4\pi i/g_{YM}^2$ as a modular transformation. In the string theory it acts on the axion-dilaton field $\tau \equiv C_0 + ie^{-\varphi}$ and is the remnant of the $SL(2,\mathbb{R})$ duality of type IIB supergravity, which is broken after taking into account stringy and quantum effects.

stack on N D3-branes. The branes act as boundary conditions for open strings, whose endpoints are confined to the branes' world-volumes, and as sources for closed strings, which can leave and cross the branes. Further, they carry N units of the self-dual 5-form charge and break half of the space-time supersymmetries. This description is valid when the effective loop expansion parameter g_sN is small.[7] For energies lower than the string scale $1/l_s$, one can integrate out massive states leaving a type IIB supergravity multiplet coming from the closed string sector and an $\mathcal{N} = 4$ gauge multiplet from the open strings [25]. The total low energy effective action for these fields is

$$S = S_{brane} + S_{bulk} + S_{int}\,, \tag{2.11}$$

where S_{brane} is four-dimensional $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$,[8] S_{bulk} is ten-dimensional type IIB supergravity and we have interaction terms between the two theories. Moreover, both S_{brane} and S_{bulk} contain higher derivative corrections. Taking now the $l_s = \sqrt{\alpha'} \to 0$ limit one can show that the interaction terms, being proportional to the coupling constant $\kappa \propto \alpha'^2 g_s$,[9] vanish, as well as the higher derivative terms. In this limit also the bulk action simplifies, becoming quadratic so that the closed strings in the bulk are free. This can be roughly seen by expanding the Hilbert-Einstein contribution in the bulk action in terms of the metric $g = \eta + \kappa h$ [14]

$$S_{bulk} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{g}\,\mathcal{R} + \ldots \sim \int d^{10}x\,\left[(\partial h)^2 + \kappa(\partial h)^2 h + \ldots\right] + \ldots\,. \tag{2.12}$$

[7] The string coupling g_s gets dressed by a factor N coming from the Chan-Paton factor of the string endpoint on the brane.

[8] The $U(1)$ factor can be shown to decouple so that one is eventually left with $SU(N)$.

[9] This relation is dictated by dimensional reasons, having $8\pi G_N = \kappa^2$ dimensions of $(mass)^{-8}$ and being $l_s = \sqrt{\alpha'}$ the only scale in the string theory.

One is thus left in the low energy limit with two decoupled theories, $\mathcal{N} = 4$ SYM on the branes and free gravity in the bulk. Notice that the open string degrees of freedom remain interacting when $\alpha' \to 0$ since the gauge theory coupling is $g_{YM}^2 = 4\pi g_s$.[10]

One can regard this system of branes in an alternative way, that, oppositely to the previous picture, is valid when $g_s N \gg 1$. A D3-brane can in fact be also seen as a *black 3-brane*, *i.e.* a solitonic solution of type IIB supergravity with mass and Ramond-Ramond charge related by a BPS condition. The solution for the metric in the string frame turns out to be

$$ds^2 = H^{-1/2} \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} \sum_{I=1}^{6} \left(dy^I\right)^2 , \tag{2.13}$$

where H is a harmonic function in the y^I coordinates. Assuming spherical symmetry one can write

$$H = 1 + \frac{R^4}{r^4} , \tag{2.14}$$

where r is the radial coordinate in the transverse directions $\sum_I \left(dy^I\right)^2 = dr^2 + r^2 d\Omega_5^2$, and

$$R^4 \equiv 4\pi g_s N \alpha'^2 \tag{2.15}$$

is the "charge" of the brane.[11] For $r \gg R$ we recover ten-dimensional Minkowski space, while the region $r < R$ is usually called *throat*. Moreover, the solution for the

[10] This follows from comparing the Yang-Mills action (2.5) with the D3-brane DBI action.

[11] It is easy to understand the result (2.15) from dimensional arguments. In fact R^4 is proportional to the Newton constant G_N, which scales as $g_s^2\, l_s^8$, and to $N\, T_{D3}$ which is the equivalent of mass in higher dimensions. The brane tension goes as $g_s^{-1}\, l_s^{-4}$ (and not as $g_s^{-2}\, l_s^{-4}$ because it is solitonic). Putting everything together one finds the behavior in eq. (2.15).

selfdual 5-form is

$$F_5 = (1+\star)\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dH^{-1}\,. \tag{2.16}$$

The dilaton is constant $e^{\varphi} = g_s$, so is the axion C_0. The NS-NS 2-form B_2 and the R-R 2-form C_2 are zero for a D3-brane.

Consider now the change of variable $z \equiv R^2/r$ and the near horizon limit $z \to \infty$. The metric (2.13) becomes

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu\right) + R^2 d\Omega_5^2\,, \tag{2.17}$$

which is the product geometry $AdS_5 \times S^5$. Notice that both AdS_5 and S^5 have the same radius, so that the total scalar curvature of the metric (2.17) vanishes. One very important feature of this background, as already mentioned at the beginning of this section, is that it is maximally supersymmetric: after the near horizon limit we restore the 16 supercharges broken by the brane.

If, besides taking the near horizon limit, we also take the low energy limit $\alpha' \to 0$, the string dynamics in the throat and the one in the asymptotic flat space get decoupled. In fact, closed strings in the flat space have in this limit very large wavelengths and do not see the throat, whereas strings in the throat do not have enough energy to climb out of it. Then one again ends up with two decoupled systems given by type IIB strings in $AdS_5 \times S^5$ and, again, free gravity in flat space.

Having obtained ten-dimensional free gravity both in the hyperplane picture and in the solitonic solution picture of the D-brane, we are led to identify the two other theories which resulted from the decoupling limit: four-dimensional $\mathcal{N} = 4$ SYM and type IIB strings on $AdS_5 \times S^5$. In particular, the relations between the parameters of the two theories are

$$g_{YM}^2 = 4\pi g_s\,, \qquad R = (4\pi g_s N)^{1/4}\,\sqrt{\alpha'} = \lambda^{1/4}\sqrt{\alpha'}\,, \tag{2.18}$$

and the rank of the gauge group N corresponds to the 5-form flux threading the S^5. With the identifications above the two theories are conjectured to be *exactly* equivalent at the full quantum level.

Special limits

The AdS/CFT duality gives useful information in limits where either the Yang-Mills or the string theory can be analyzed quantitatively. Because of difficulties in quantizing strings in background Ramond-Ramond fields, quantitative results for the string theory on $AdS_5 \times S^5$ are only known in some limits. The first one to be explored is the *supergravity limit* where IIB string theory coincides with classical type IIB supergravity. The next section 2.2 of this introduction and chapter 4 will be dedicated to a second limit, called the *plane wave limit*.

The supergravity limit is obtained by first taking the classical limit $g_s \to 0$, holding R constant. This projects onto tree level string theory. Then one takes the limit of large string tension, or large curvature. This is done by putting the effective string tension $R^2/\alpha' = \sqrt{4\pi g_s N} \to \infty$. This isolates the lowest energy modes of the string, which are the supergravity fields on the $AdS_5 \times S^5$ background.

On the Yang-Mills side, the first of these limits corresponds to taking $g_{YM}^2 \to 0$ and $N \to \infty$, while holding the 't Hooft coupling λ fixed. This is the 't Hooft large N (or planar) limit of the gauge theory discussed before. Then, the second limit,

$R^2/\alpha' \to \infty$, is equivalent to taking $\lambda \to \infty$. This gives the strongly coupled limit of the planar gauge theory.

The g_s loop expansion at the classical string level corresponds to N^{-k} corrections to planar results in the gauge theory, while the α' expansion at the classical supergravity level corresponds to $\lambda^{-k/2}$ corrections to the strong coupling results.

It is this fact, that a solvable limit of string theory is mapped onto a non-trivial limit of gauge theory which makes the AdS/CFT duality so interesting. At the same time, this has limited the checks of the conjecture to objects such as two and three point functions of chiral primary operators [26], which do not depend on the coupling constant and thus trivially extrapolate between weak and strong coupling, some anomalies [27][28], where dependence on the coupling constant is trivial, and also to the computations of expectation values of certain Wilson loops [29][30] and correlators of Wilson loops with chiral primary operators [31], which we will discuss in detail later.

2.1.4 Matching the spectra

We now illustrate the relation between the spectra of $\mathcal{N} = 4$ SYM and type IIB strings on $AdS_5 \times S^5$. We will see that every operator on the gauge theory side can be put in a one-to-one correspondence with a state in the bulk.

Gauge theory operators

Since $\mathcal{N} = 4$ SYM is a conformal theory where asymptotic states are not defined, with the term spectrum one indicates the collection of all local, gauge invariant operators

$\mathcal{O}(x)$ that are polynomial in the fields of the theory.[12] These operators are organized in infinite dimensional families which are the irreducible unitary representations of the $SU(2,2|4)$ superalgebra and are labeled by the quantum numbers of the bosonic subgroup $SO(3,1) \times SO(1,1) \times SU(4)_R$ of the superconformal group [32][33]. The labels for the Lorentz group are a pair of integer or half-integer numbers (s_+, s_-), the conformal dimension Δ is the label of $SO(1,1)$[13] and the Dynkin numbers $[r_1, r_2, r_3]$ the labels of the R-symmetry group (see also [17]).

The conformal dimension Δ is the eigenvalue of the dilatation operator and can be read off from two-point functions

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{x^{2\Delta}}\,. \tag{2.19}$$

In general Δ depends on the 't Hooft coupling, *i.e.* $\Delta = \Delta_0 + \gamma(\lambda)$, where Δ_0 is the classical (or engineering) dimension and γ is called the *anomalous dimension.*

We present now some of the commonly used nomenclature. *Conformal primary operators* are the operators annihilated by the generators K^μ of special conformal transformations and, similarly, *superconformal primary operators* are annihilated by the conformal supercharges S^A_α.[14] These are the operators with lowest dimension in a given representation and can be used to obtain *superconformal descendants* by acting on them with the supercharges Q^A_α. One can show that only symmetrized products of the scalar fields Φ^I can appear in superconformal primaries.[15] *Chiral primary*

[12] The gauge invariance condition is respected by taking traces over the gauge group and considering field strengths and covariant derivatives rather than the gauge fields. The dependence on the fields must be polynomial in order for the operators to have definite dimensions.

[13] Δ has to be non-negative in unitary representations.

[14] Every superconformal primary is also a conformal primary, but the converse is not true in general.

[15] This is easy to see by analyzing the transformation properties of the $\mathcal{N} = 4$ fields under Poincaré

operators (CPO) are the subset of superconformal primaries which are annihilated by some combination of Q's. These operators are in *shortened* or *BPS representations* and are of central importance because are protected from quantum corrections and do not renormalize. Then for chiral primaries $\Delta = \Delta_0$ at any order. A generic 1/2 BPS single-trace CPO with conformal dimension Δ preserves 8 Q's and 8 S's and can be expressed as

$$\mathcal{O}_\Delta(x) = C_{I_1 \cdots I_\Delta} \mathrm{Tr}(\Phi^{I_1} \ldots \Phi^{I_\Delta}) \,, \qquad (2.20)$$

where $C_{I_1 \cdots I_\Delta}$ is a $SO(6)$ symmetric traceless tensor. One can also consider multi-trace generalizations and operators preserving less supersymmetry, such as 1/4 and 1/8 BPS operators.[16] Using the complex basis $X \equiv \Phi^1 + i\Phi^2$, $Y = \Phi^3 + i\Phi^4$, and $Z = \Phi^5 + i\Phi^6$, eq. (2.20) becomes, for instance, $\mathcal{O}_\Delta = \mathcal{N}_\Delta \mathrm{Tr}\, Z^\Delta$, where $\mathcal{N}_\Delta$ is a normalization factor, and multi-trace operators are given by $\mathcal{O}_{\{\Delta_i, n_i\}} \sim \prod_i \left[\mathrm{Tr}\, Z^{\Delta_i}\right]^{n_i}$. The 1/2 BPS chiral primaries $\mathcal{O}_\Delta$ are in the $[0, \Delta, 0]$ (with $\Delta \geq 2$) representation of $SU(4)_R$.

Bulk modes

In the supergravity limit, it is thought that all operators in the Yang-Mills theory that are not protected by supersymmetry get infinitely large conformal dimensions and decouple from the spectrum. The protected operators are just those required to match the classical field degrees of freedom of IIB supergravity linearized about the

supercharges. Schematically

$$\{Q, \psi\} = F + [\Phi, \Phi] \,, \qquad \{Q, \bar{\psi}\} = D\Phi \,, \qquad [Q, \Phi] = \psi \,, \qquad [Q, F] = D\psi \,.$$

Then the only fields that are not Q-exact are the Φ^I. Moreover they have to enter in a symmetrized combination because the commutator between two Φ^I appears in the first transformation above.

[16] We remark though that 1/4 BPS operators have at least two traces and 1/8 BPS have three.

$AdS_5 \times S^5$ background [13].

This spectrum has been worked out in [34]. It is organized in multiplets of $SU(2,2|4)$ and it turns out that chiral primary operators correspond to the supergravity Kaluza-Klein modes after the reduction on the S^5

$$\varphi(x,y) = \sum_{\Delta=0}^{\infty} \varphi_\Delta(x)\, Y_\Delta(y)\,, \tag{2.21}$$

where φ is a generic supergravity field, x are the coordinates on AdS_5 and $Y_\Delta(y)$ are a basis of spherical harmonics on S^5. The supergravity fields receive mass contributions after the compactification on S^5, for example, the scalar modes have

$$m^2 = \Delta(\Delta - 4)\,, \qquad \Delta \geq 2\,. \tag{2.22}$$

Single-trace operators correspond to single-particle states in the bulk, whereas products of operators are either multi-particle states or bound states (if all operators are evaluated at the same point). For the complete dictionary between SYM descendants and supergravity fields see table 7 of [17]. We will come back to this point in much more detail in section 6.2.

Matrix model description of half-BPS operators

Here we briefly discuss how the dynamics of the half-BPS sector of $\mathcal{N} = 4$ SYM may be described in terms of a gauged matrix quantum mechanics model [35].[17] This description will be the focus of chapter 5.

[17]For earlier work see [36][37].

We start by considering the action for the complex scalar field Z on $S^3 \times \mathbb{R}$

$$S[Z(x)] = \frac{1}{2} \int d\Omega_3 \, dt \, \mathrm{Tr} \left(|D_\mu Z|^2 - |Z|^2 \right) \,, \tag{2.23}$$

where $D_\mu Z = \partial_\mu Z + [A_\mu, Z]$ and the mass term comes from the conformal coupling to the curvature of the S^3, which is taken to have unit radius. Since we are interested in reducing the action (2.23) to $\mathbb{R}$, we decompose Z in its KK modes on the sphere

$$Z(x) = \sum_{l=0}^{\infty} \sum_{m=1}^{(l+1)^2} Z^{l,m}(t) \, Y_{(0)}^{l,m}(\vec{x}) \,, \tag{2.24}$$

where $Y_{(0)}^{l,m}(\vec{x})$ are the scalar spherical harmonics on S^3. Their mass is given by $\Delta = l+1$ (with $l = 0, 1, \ldots$). The only mode respecting the BPS condition $\Delta = J = 1$ is then the s-wave of Z.

If one repeats this analysis also for the gauge field, one finds that only the s-wave of the temporal component A_t is contained in the BPS sector, whereas the spatial components have spherical harmonics with $\Delta = l + 2$ and are therefore expected to receive large anomalous dimensions and decouple. The field A_t is non-propagating and plays the role of a Lagrange multiplier enforcing Gauss law. After the reduction one has a matrix model in one dimension

$$S[Z(t)] = \frac{1}{2} \int dt \, \mathrm{Tr} \left(|D_t Z|^2 - |Z|^2 \right) \,. \tag{2.25}$$

Gauge fixing $A_t = 0$[18] and going to the eigenvalue basis $\{z_i\}$ (with $i = 1, \ldots, N$), the

[18]The effect of the gauge field does not disappear though, for one needs to require gauge invariance of the spectrum of states of the theory.

classical Hamiltonian of the system reads

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(|\dot{z}_i|^2 + |z_i|^2\right) . \tag{2.26}$$

Notice that this is a model of complex matrices, different from the usual Hermitian matrix quantum mechanics. It can be nonetheless argued that the creation operators for $Z^\dagger$ decouple in the half-BPS sector and one is only left with the creation operators for Z, effectively reducing the complex model to a Hermitian one [35]. Going to the eigenvalue basis introduces a Van der Monde determinant in the path integral and the eigenvalues behave as fermions in a harmonic potential [38]. For large N the eigenvalue distribution can be thought of as a distribution of incompressible "droplets" in the phase space of the fermions, with the ground state being a circle with radius of order N.

2.2 The plane wave limit of AdS/CFT

The AdS/CFT correspondence has been mostly studied in its "weakest" formulation of strong coupling, large curvature limit. The reason behind this is that to go beyond this limit one would need to solve the string theory side of the conjecture, which is difficult to deal with because of two main complications:

- the curved space $AdS_5 \times S^5$, where the strings live, gives rise to a world-sheet σ-model which is non-linear and therefore hard to integrate;
- the background R-R flux cannot be incorporated in the spinning string formalism, so that one has to use Green-Schwarz superstrings, which are notoriously complicated to quantize in a covariant way.

It is then interesting to look for special limits where one can handle these problems, and extract information beyond the supergravity approximation. One important example of such limits is a maximally supersymmetric background called the *plane wave background*.[19] It can be obtained as a *Penrose limit* of $AdS_5 \times S^5$ [39]-[43], and is such that the non-linear σ-model defined on it simplifies drastically and can actually be solved [44][45]. In [46] Berenstein, Maldacena, and Nastase (BMN) took the analogous limit in Yang-Mills theory, and proposed that a certain class of operators with large quantum numbers might be the gauge theory dual of the string spectrum on the plane wave. In this section we review this *plane wave/BMN duality*,[20] which will also be the central focus of chapter 4, where the string thermodynamics in this background will be studied.

2.2.1 The Penrose limit

The Penrose limit[21] is found by blowing up the neighborhood of a null geodesic of a given space-time. Every geometry admits such a limit, and, moreover, if this geometry is a solution of Einstein equations, then also the resulting plane wave is guaranteed to be a solution. One of the most important feature of the Penrose limit is that it never breaks any supersymmetry, so that the plane wave limit of a maximally supersymmetric background as $AdS_5 \times S^5$ is still maximally supersymmetric.[22]

The Penrose limit of $AdS_5 \times S^5$ can be obtained starting from the expression for

[19]This background admits 32 supercharges. In 10 dimensions the only maximally supersymmetric spaces are flat space, $AdS_5 \times S^5$, and the plane wave that we shall discuss in the following. Note that type IIA supergravity does not allow non-flat maximally supersymmetric solutions.

[20]A clear and detailed account of this topic can be found in [47].

[21]In supergravity contexts it is also sometimes called *Penrose-Gueven limit.*

[22]In some cases one might even enhance the amount of supersymmetry. For example, one of the possible Penrose limits of the orbifold $AdS_5 \times S^5/\mathbb{Z}_k$, the so called *discrete light-cone quantization (DLCQ)*, doubles the number of preserved supercharges from 16 to 32. In the case of the conifold $AdS_5 \times T^{1,1}$ the enhancement is from 8 to 32 supercharges.

the metric in global coordinates[23]

$$ds^2 = R^2 \left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2\right) , \quad (2.27)$$

and rescaling the coordinates as

$$r = R\sinh\rho\,, \qquad y = R\sin\theta\,, \qquad x^+ = \frac{t}{f}\,, \qquad x^- = fR^2(\phi - t)\,, \qquad (2.28)$$

where f is an arbitrary parameter introduced for dimensional reasons. Taking now the limit $R \to \infty$, while keeping r, y, and $x^\pm$ fixed, blows up a neighborhood of a null geodesic along the circumference parameterized by ϕ, with $\rho = \theta = 0$. This yields the plane-wave metric[24]

$$ds^2 = 2dx^+dx^- - f^2 \sum_{I=1}^{8} x_I^2\, dx^+dx^+ + \sum_{I=1}^{8} dx^I dx^I\,. \qquad (2.30)$$

Half of the x^I comes from the AdS_5 directions $r^2 = \sum_{I=1}^4 x_I^2$, and the remaining half from the S^5 directions $y^2 = \sum_{I=5}^8 x_I^2$, however after the Penrose limit they become indistinguishable. Note that the limit $f \to 0$ gives ten dimensional Minkowski space.

[23]For other space, such as $AdS_{4,7} \times S^{7,4}$, orbifolds, and $AdS_5 \times T^{1,1}$ see [47].

[24]This is a very special example of *pp-wave*. A generic pp-wave is given by

$$ds^2 = 2dx^+dx^- + F(x^+, x^I)dx^+dx^+ + \sum_{I=1}^{8} dx^I dx^I\,, \qquad (2.29)$$

and is called plane wave if $F(x^+, x^I) = \sum_{IJ} f_{IJ}(x^+)x^I x^J$. The metric (2.30) is sometimes called *homogeneous plane wave*, since it has $f_{IJ}(x^+)$ constant and proportional to δ_{IJ}. The pp-wave metric (2.29) admits a covariantly constant null Killing vector field and consequently does not receive α' corrections [48]. For the plane wave metric this vector field is also globally defined.

Under the Penrose limit the selfdual R-R 5-form reduces to the constant expression

$$F_{+1234} = F_{+5678} = 2f\,. \tag{2.31}$$

The dilaton $e^{\phi} = g_s$ is also constant.[25] Note that the metric (2.30) has an $SO(8)$ isometry which is broken down to $SO(4) \times SO(4)$ by the 5-form. There is an extra $\mathbb{Z}_2$ symmetry exchanging these two $SO(4)$ factors, and two non-compact $U(1)$ isometries corresponding to translational invariance along x^+ and x^-. Less manifest are 16 extra symmetries generated by Killing vectors obeying a pair of 4 dimensional Heisenberg algebras. These Killing vectors generate translations along the 8 transverse directions x^I accompanied by shifts in x^-, in such a way that the metric and 5-form remain invariant. Then the total bosonic symmetries are given by $[h(4) \times SO(4) \times U(1)]^2 \times \mathbb{Z}_2$ [47]. As already anticipated this background has also 32 Killing spinors. All these bosonic and fermionic generators form the supergroup $[PSU(2|2) \times U(1)]^2 \times \mathbb{Z}_2$, which is a Penrose contraction of the original superconformal group $PSU(2,2|4)$ of $AdS_5 \times S^5$.

Like the supergravity limit, eq. (2.30) is obtained from $AdS_5 \times S^5$ when the curvature is weak and the effective string tension is large, that is $R^2/\alpha' \to \infty$. However, this limit is taken asymmetrically, in a reference frame which has large angular momentum $J \sim R^2/\alpha'$ on S^5. In this way, the limit retains a particular subset of the the higher level string excitations. Those excitations are described by quantizing the string on the background (2.30) (2.31). This background has the advantage that non-interacting string theory can be quantized explicitly in the light-cone gauge and the energy spectrum can be obtained [44][45], as we discuss next.

[25] In the eqs. (2.30) and (2.31), throughout this review, and in chapter 4 we shall use the notation of [45]. In particular, $x^{\pm} = \frac{1}{\sqrt{2}}\left(x^9 \pm x^0\right)$.

2.2.2 The plane wave σ-model

The light-cone gauge fixing for the Green-Schwarz type IIB superstring consists in using the conformal symmetry to set $\sqrt{g}\, g_{ab} = \eta_{ab}$, and then in imposing

$$x^+ = p^+ \tau \,, \qquad \bar{\gamma}^+ \theta^{\mathcal{I}} = 0 \,, \tag{2.32}$$

to fix, respectively, the residual diffeomorphism invariance and the κ-symmetry. Here $\bar{\gamma}^+$ is a 16×16 Dirac matrix coming from the off-diagonal parts of the 32×32 matrix Γ^+ in ten dimensions, and $\theta^{\mathcal{I}}$ (with $\mathcal{I} = 1, 2$) represent two ten-dimensional Majorana-Weil spinors with the same chirality. After gauge fixing one is then left with 16 physical fermions describing on-shell space-time fermionic modes, and both θ^1 and θ^2 are in the same $SO(8)$ spinor representation, for example the $\mathbf{8_s}$. In the light-cone gauge the string σ-model on the plane wave (2.30) (2.31) becomes quadratic [44][45]

$$\mathcal{L} = \frac{1}{2} \left(\partial_+ x^I \partial_- x^I - m^2 x_I^2 \right) + i \left(\theta^1 \bar{\gamma}^- \partial_+ \theta^2 \bar{\gamma}^- \partial_- \theta^2 - 2m \theta^1 \bar{\gamma}^- \Pi \theta^2 \right) \,, \tag{2.33}$$

where $\partial_\pm = \partial_0 \pm \partial_1$, and Π is the product of four γ matrices.[26] The fields x^I and $\theta^{\mathcal{I}}$ are then free and massive with mass $m = f p^+$, the mass term for the fermions coming from the coupling to the R-R flux of the background (2.31). In the limit $m \to 0$ one recovers flat space. The Klein-Gordon equations of motion for x^I and $\theta^{\mathcal{I}}$ read

$$(\partial_+ \partial_- + m^2) x^I = 0 \,, \qquad \partial_+ \theta^1 - m \Pi \theta^2 = 0 \,, \qquad \partial_- \theta^2 - m \Pi \theta^1 = 0 \,, \tag{2.34}$$

[26] For more details about this notation see appendix A of [45].

and can be easily solved expanding in Fourier modes the fields and imposing appropriate boundary conditions along the world-sheet σ direction. The action (2.33) can then be canonically quantized in terms of the set of bosonic and fermionic creation and annihilation operators

$$\begin{aligned} [\bar{a}_0^I, a_0^J] &= \delta^{IJ}\,, \qquad [\bar{a}_m^{\mathcal{I}I}, a_n^{\mathcal{J}J}] = \delta_{mn}\delta^{\mathcal{I}\mathcal{J}}\delta^{IJ}\,, \\ \{\bar{\theta}_0^\alpha, \theta_0^\beta\} &= \tfrac{1}{4}(\gamma^+)^{\alpha\beta}\,, \qquad \{\bar{\eta}_m^{\mathcal{I}\alpha}, \eta_n^{\mathcal{J}\beta}\} = \tfrac{1}{2}(\gamma^+)^{\alpha\beta}\delta_{mn}\delta^{\mathcal{I}\mathcal{J}}\,, \end{aligned} \tag{2.35}$$

with $m, n \in \mathbb{Z}^+$, and $\alpha, \beta = 1, \ldots, 16$ are spinor $SO(8)$ indices. The light-cone Hamiltonian is given by the momentum conjugate to the light-cone time x^+ and reads

$$\begin{aligned} H &\equiv -p^- \\ &= f\left(a_0^I \bar{a}_0^I + 2\bar{\theta}_0\bar{\gamma}^-\Pi\theta_0 + 4\right) + \frac{1}{\alpha' p^+}\sum_{\mathcal{I}=1,2}\sum_{m=1}^{\infty}\omega_m\left(a_m^{\mathcal{I}I}\bar{a}_m^{\mathcal{I}I} + \eta_m^{\mathcal{I}}\bar{\gamma}^-\bar{\eta}_m^{\mathcal{I}}\right), \end{aligned} \tag{2.36}$$

where the frequencies are $\omega_m = \sqrt{m^2 + (\alpha' p^+ f)^2}$. The zero point energy exactly cancels out between the bosonic and the fermionic contributions, as expected from supersymmetry, and no regularization is required.

The vacuum $|0\rangle$ carries momentum p^+ and is defined as the state such that

$$\bar{a}_0^I|0\rangle = 0\,, \qquad \bar{a}_n^{\mathcal{I}I}|0\rangle = 0\,, \qquad \bar{\theta}_0^\alpha|0\rangle = 0\,, \qquad \bar{\eta}_n^{\mathcal{I}\alpha}|0\rangle = 0\,, \tag{2.37}$$

and a generic vector in the Fock space is obtained by acting on $|0\rangle$ with a_0^I, $a_n^{\mathcal{I}I}$, θ_0^α, and $\eta_n^{\mathcal{I}\alpha}$ (the indices $\mathcal{I} = 1, 2$ have the meaning of left and right sector). This can be restricted to the physical space by imposing on the states the level matching condition

$$N^1|\Psi_{phys}\rangle = N^2|\Psi_{phys}\rangle\,, \qquad N^{\mathcal{I}} = \sum_{n=1}^{\infty} n\left(a_n^{\mathcal{I}I}\bar{a}_n^{\mathcal{I}I} + \eta_n^{\mathcal{I}}\bar{\gamma}^-\bar{\eta}_n^{\mathcal{I}}\right)\,. \tag{2.38}$$

Consider first the non-zero modes. A generic single-string state is given by $a_n^{1I}a_n^{2J}|0\rangle$ and $\eta_n^{1\alpha}\eta_n^{2\beta}|0\rangle$ for bosonic modes, and by $a_n^{1I}\eta_n^{2\alpha}|0\rangle$ and $a_n^{2I}\eta_n^{1\alpha}|0\rangle$ for fermionic modes. Recalling the isometry group of the plane wave background and in order to make contact later with the corresponding gauge theory operators, it is useful to decompose the $SO(8)$ vector and spinor indices in $SO(4)\times SO(4)$ representations [45]. For example, if $I=(i,a)$ then $a_n^{1i}a_n^{2j}|0\rangle$ decomposes as $(\mathbf{4},\mathbf{1})\otimes(\mathbf{4},\mathbf{1}) = (\mathbf{1},\mathbf{1})\oplus(\mathbf{9},\mathbf{1})\oplus(\mathbf{6}^+,\mathbf{1})\oplus(\mathbf{6}^-,\mathbf{1})$. Similarly $a_n^{1a}a_n^{2b}|0\rangle$ decomposes as $(\mathbf{1},\mathbf{4})\otimes(\mathbf{1},\mathbf{4}) = (\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{9})\oplus(\mathbf{1},\mathbf{6}^+)\oplus(\mathbf{1},\mathbf{6}^-)$. Both $a_n^{1i}a_n^{2a}|0\rangle$ and $a_n^{1a}a_n^{2i}|0\rangle$ give $(\mathbf{4},\mathbf{1})\otimes(\mathbf{1},\mathbf{4}) = (\mathbf{4},\mathbf{4})$. The total number of states in this sector is then 64. This sector is not the same as the NS-NS sector of flat space since it contains combinations of the metric and the selfdual 5-form.

As for the zero-modes, notice that they are not constrained by the level matching condition and give rise in the low energy limit to a decoupled sector of the string spectrum which corresponds to supergravity modes on the plane wave background. For example, the modes coming from $a_0^Ia_0^J|0\rangle$ in the $SO(4)\times SO(4)$ decomposition mentioned above form $(\mathbf{1},\mathbf{1})\oplus(\mathbf{9},\mathbf{1})\oplus(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{9})\oplus(\mathbf{4},\mathbf{4})$, for a total of 36 states.

2.2.3 The gauge theory dual

The natural question to ask at this point is which is, if any, the gauge theory that is the holographic dual of the string living on the plane wave. Berenstein, Maldacena, and Nastase proposed in [46] a limit in $\mathcal{N}=4$ super Yang-Mills which is conjectured to the be analog of the Penrose limit.

To motivate the BMN proposal recall that the Penrose limit consists of blowing-up a geodesic along a combination of the azimuth ϕ of the S^5 and the global time t of AdS_5, see eq. (2.28). According to the AdS/CFT dictionary translations along t and rotations along ϕ correspond on the gauge theory side to the actions of the dilatation operator Δ and of a $U(1)$ generator in the $SO(6)$ R-symmetry group, which rotates two of the six scalars of the gauge multiplet and can be called J

$$i\frac{\partial}{\partial t} \leftrightarrow \Delta\,, \qquad i\frac{\partial}{\partial \phi} \leftrightarrow J\,. \tag{2.39}$$

One is then suggested to consider operators of $\mathcal{N}=4$ SYM with conformal dimension $\Delta \sim \sqrt{N}$ and R-charge $J \sim \sqrt{N}$, such that the momenta of the corresponding string state, identified by[27]

$$p^- \equiv \frac{f}{\sqrt{2}}(\Delta - J)\,, \qquad p^+ \equiv \frac{1}{\sqrt{2} f R^2}(\Delta + J)\,, \tag{2.40}$$

remain finite as $N \to \infty$ and $J \to \infty$ with the ratio J^2/N kept fixed in the limit. These operators with large R-charge J but finite anomalous dimension $\Delta - J$ are called *BMN operators*. They are in general non-BPS, but remain close to being BPS in the large N limit. For this reason they are sometimes said to be "near BPS".

[27]In string theory these are defined by $p^{\pm} = p_{\mp} = \frac{1}{2\pi\alpha'}\int d\sigma \partial_\tau x^{\pm}$ and have a simple form only in the light-cone gauge. In the gauge theory they are defined by the re-scaling of $p_{\pm} = \frac{1}{\sqrt{2}}(\Delta \mp J)$ needed to get the plane-wave limit.

Eigenstates of Δ and J and BMN operators

Recalling that p^- is the string Hamiltonian, the identification (2.40) implies then the equivalence between the string spectrum discussed in the previous section and the spectrum of the operator $\Delta - J$. This equivalence between the two spectra is best seen if one expresses the $\mathcal{N}=4$ multiplet in representations of the $[SO(4)\times U(1)]^2$ subgroup of $SO(4,2)\times SO(6)$. Recall that this is the isometry group of the plane wave (modulo the Heisenberg groups). On the gauge theory side it originates from the fact that eigenstates of Δ and J selects, respectively, a $U(1)_\Delta$ subgroup of $SO(4,2)$ and a $U(1)_J$ subgroup of $SO(6)_R$, breaking both $SO(4,2)$ and $SO(6)_R$ to $SO(4)\times U(1)$ factors.

If J is taken to be the generator of rotations in the Φ^5 and Φ^6 plane and $Z \equiv \Phi^5 + i\Phi^6$, then the $U(1)_J$ charge of Z is 1 and the charge of $Z^\dagger$ is -1. The remaining four scalars Φ^i have $J=0$, as the covariant derivative (gauge fields are trivial under R-symmetry). Half of the spinors have $J=1/2$ and the other half have $J=-1/2$.[28] On the other hand, the engineering dimension (namely the dimensions of the operators in the free theory) for all the scalars and the covariant derivative is $\Delta_0 = 1$, while for the fermions is $\Delta_0 = 3/2$. Considering for brevity only the bosonic fields one has then

$$[\Delta_0 - J](Z) = 0\,, \qquad [\Delta_0 - J](Z^\dagger) = 2\,, \qquad [\Delta_0 - J](\Phi^i) = 1\,, \qquad [\Delta_0 - J](D_\mu) = 1\,. \tag{2.41}$$

Both Z and $Z^\dagger$ are singlets of $SO(4)\times SO(4)$ (where the first group comes from $SO(4,2)$ and the second one comes from $SO(6)_R$), whereas Φ^i and D_μ are in the $(\mathbf{1},\mathbf{4})$ and $(\mathbf{4},\mathbf{1})$ representations, respectively.

[28] We call with the same symbol, J, both the operator and the eigenvalue, ditto for Δ.

BMN operators are completely specified in terms of their $\Delta_0 - J$ charge, their $SO(4) \times SO(4)$ representation, and the number of traces [47]. The simplest example of BMN operators is represented by single-trace chiral primaries $\mathcal{O}(x) = \mathcal{N}_J \mathrm{Tr}\, Z^J(x)$. These operators have $\Delta_0 = J$ and are clearly singlets of $SO(4) \times SO(4)$. They can be identified with the single-string vacuum $|0\rangle$ defined in eq. (2.37)

$$\mathcal{O}^J(0) \leftrightarrow |0\rangle \, . \tag{2.42}$$

The chiral-primary operators have protected conformal dimensions being 1/2 BPS. Then they are eigenstates of the full dilatation operator Δ and have $\Delta - J = 0$ exactly. The same is true for multi-trace operators, which, according to the BMN dictionary, should correspond to the multi-string vacuum.[29]

At the next level there are operators built out of Z's and one *impurity*, namely a field with $\Delta_0 - J = 1$. One example is $\mathcal{O}_i^J(x) = \mathcal{N}_{J,i} \mathrm{Tr}\, \Phi^i Z^J$. In total there are 8 bosonic (corresponding to the insertions of the 4 Φ^i and the 4 D_μ) and 8 fermionic single-trace operators at this level. These are all descendants of chiral primaries and are therefore also exact eigenstates of $\Delta - J = 1$. On the string theory side they correspond to $a_0^I|0\rangle$ and $\theta_0^\alpha|0\rangle$. One can similarly go on and consider operators with two or more impurities and match them to the appropriate string states. For more details see [47].

The plane-wave limit of the Yang-Mills theory has two parameters, g_{YM} and the ratio J^2/N which must be held fixed as $N \to \infty$. Combinations of them which appear

[29] More precisely, if, for example, $\mathcal{O}^r\mathcal{O}^s(x) = \mathcal{N}_{r,s} \mathrm{Tr}\, Z^r(x) \mathrm{Tr}\, Z^s(x)$ (with $r + s = J$), then the total momentum p^+ of the double-string vacuum is split between the two strings as rp^+ and sp^+.

naturally in the Yang-Mills perturbation theory are

$$\lambda' = \frac{g_{YM}^2 N}{J^2}\,, \tag{2.43}$$

and

$$g_2 = \frac{J^2}{N}\,. \tag{2.44}$$

It was shown in [49][50] that λ' governs the loop expansion in Yang-Mills theory and it also fixes the distance scale in string theory. Also, the constant g_2 is the effective string coupling in that it governs the loop expansion in string theory. It plays the same role in Yang-Mills theory where it weights Feynman graphs by the genus of the two dimensional surface on which they can be drawn without crossing lines. In light-cone quantization it is natural to consider states which have a fixed light-cone momentum p^+. In this case, we can easily see hat g_2 is related to string loops by using the second equation in (2.40) to trade the Yang-Mills parameters for the pair g_s and p^+, the light-cone momentum of the string,

$$\lambda' = \frac{2}{\left(f\alpha' p^+\right)^2}\,, \tag{2.45}$$

$$g_2 = g_{YM}^2 \frac{\left(f\alpha' p^+\right)^2}{2} = 2\pi g_s \left(f\alpha' p^+\right)^2\,. \tag{2.46}$$

The free string theory is obtained by putting $4\pi g_s = g_{YM}^2 \to 0$ in conjunction with the large N limit, with the combination $(f\alpha' p^+)$ non-zero and fixed. This is just the limit where λ' is held constant and g_2 is set to zero. In this limit, all quantities depend on the parameters g_{YM} and N only through the the combination $g_{YM}^2 N = \lambda$, the 't Hooft coupling. This means that free strings are described by the planar limit of Yang-Mills theory.

It has been checked explicitly that the spectrum of free strings is found in the conformal dimensions Δ of certain Yang-Mills operators computed from planar Feynman diagrams [46][51]-[53]. One-loop corrections to the string mass spectrum have similarly been related to computations on the gauge theory side [54]-[65].

2.3 Wilson loops in the AdS/CFT correspondence

A Wilson loop is a non-local gauge invariant operator whose expectation value gives the phase factor associated to a charged and heavy external particle moving along a contour $\mathcal{C}$

$$W(\mathcal{C}) = \mathrm{Tr}\, \mathcal{P} \exp\left(i \oint_{\mathcal{C}} A_\mu dx^\mu \right) , \tag{2.47}$$

where $\mathcal{P}$ means that the integral has to be path-ordered and the trace is over the representation, usually taken as the fundamental, of the external particle.[30] These operators are extremely interesting because play the role of order parameters and can therefore be used to study the confining/deconfining phases of a given theory.[31] In fact, it turns out that the potential $V(L)$ between two probe charges at a distance L is given by the expectation value of the rectangular Wilson loop defined by two parallel lines of length T and spatial separation L

$$\lim_{T\to\infty} \langle W \rangle = e^{-T\, V(L)} . \tag{2.48}$$

[30] In mathematical terms this is the trace of the holonomy of the gauge connection A after parallel transport along the curve $\mathcal{C}$. Notice also that we are probing an adjoint theory with fundamental objects, see Witten's lecture on Wilson lines in [66].

[31] For finite temperature the natural order parameter is a generalization of eq. (2.47), the *Polyakov loop*, where the contour is the compactified time direction in Euclidean signature S^1_β, and the period β is identified with the inverse of the temperature.

A confining phase exhibits a linear potential between the two charges, $V(L) \propto L$. The exponent in eq. (2.48) goes then like the area of the loop, a behavior referred to as *area law*. A non-confining phase on the other hand, like a Higgs phase where electric charges condense and $V(L)$ is constant, presents a *perimeter law* behavior. For a conformal theory the potential is Coulombic, as we shall see in the following.[32]

2.3.1 Wilson loops in $\mathcal{N} = 4$ super Yang-Mills

We extend now the gauge theory definition (2.47) to $\mathcal{N} = 4$ super Yang-Mills theory. As briefly mentioned above, the data which characterize a Wilson loop are the contour $\mathcal{C}$ along which one integrates the gauge connection and the representation $\mathcal{R}$ of the gauge group. In a supersymmetric context it is natural to define $\mathcal{C}$ in the superloop space parameterized by $\{x^\mu(\tau),\, y^I(\tau),\, \xi_A^\alpha(\tau)\}$, where x^μ and y^I are bosonic directions which couple to the gauge field A_μ and the six scalars Φ^I of the $\mathcal{N} = 4$ gauge multiplet, while ξ_A^α are fermionic directions associated to the spinors ψ_α^A. Gauge invariance requires the loop in the four x^μ directions to be closed, y^I and ξ_A^α on the other hand may be arbitrary. One can actually suppress the fermionic directions, since a loop operator containing the ψ_α^A is a supersymmetry descendant of a loop not containing them. Then a $\mathcal{N} = 4$ supersymmetric Wilson loop is naturally defined (in Euclidean signature)[33] as

$$W_{\mathcal{R}}(\mathcal{C}) = \frac{1}{\dim\mathcal{R}} \mathrm{Tr}_{\mathcal{R}}\, \mathcal{P} \exp \oint_{\mathcal{C}} d\tau \left[iA_\mu(\tau)\dot{x}^\mu + \Phi^I(\tau)\dot{y}^I \right] . \tag{2.49}$$

[32] Wilson loops have numerous other applications, also at the interface between physics and mathematics. For example, Wilson loops in Chern-Simons theory have been proven in [67] to compute knot invariants, such as the Jones polynomials.

[33] In Minkowski signature the i multiplying A_μ goes in front of the integral. Notice also that the exponent is not a pure phase.

The irreducible representations $\mathcal{R}$ of $SU(N)$ can be expressed in terms of Young diagrams. For the moment we take $\mathcal{R} = \Box$, the fundamental representation, but later on in section 2.3.3 we shall consider more general cases, which will be in fact one of the focuses of chapter 6.

Supersymmetry

Let us now study the invariance properties of the loop (2.49) under the Poincaré and conformal supersymmetry transformations of the gauge and scalar fields

$$\begin{aligned} \delta_Q\, A_\mu &= \bar{\Psi}\Gamma_\mu \epsilon_0\,, \qquad & \delta_Q\, \Phi^I &= \bar{\Psi}\Gamma^I \epsilon_0\,, \\ \delta_S\, A_\mu &= \bar{\Psi}\Gamma_\mu x^\nu \Gamma_\nu \epsilon_1\,, \qquad & \delta_S\, \Phi^I &= \bar{\Psi}\Gamma^I x^\nu \Gamma_\nu \epsilon_1\,. \end{aligned} \tag{2.50}$$

Here Ψ is the ten-dimensional spinor appearing in the action (2.5) and the transformation parameters $\epsilon_{0,1}$ are two ten-dimensional 16-components Majorana-Weyl fermions of opposite chirality. Focussing for the moment on the Poincaré supercharges one finds that $\delta_Q\, W(\mathcal{C}) = 0$ implies [68]

$$\left(i\Gamma^\mu \dot{x}_\mu + \Gamma^I \dot{y}^I\right)\epsilon_0 = 0\,. \tag{2.51}$$

For a fixed τ this equation has eight independent solutions if $\left(i\Gamma^\mu \dot{x}_\mu + \Gamma^I \dot{y}^I\right)$ squares to zero. This happens, recalling that the matrices Γ^μ and Γ^I anticommute, if

$$\dot{x}^2 - \dot{y}^2 = 0\,. \tag{2.52}$$

This is solved for $\dot{y}^I(\tau) = |\dot{x}|\Theta^I(\tau)$, where Θ^I is a unit vector on $\mathbb{R}^6$.[34] Then the Wilson loop (2.49) becomes (choosing, as already mentioned, $\mathcal{R} = \Box$)

$$W(\mathcal{C}) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\oint_{\mathcal{C}} d\tau\,\left[iA_\mu(\tau)\dot{x}^\mu + |\dot{x}|\Phi^I(\tau)\Theta^I(\tau)\right]\,. \tag{2.53}$$

Notice though that this solution, because of the dependence of Θ^I on τ, makes the Wilson loop (2.49) only *locally* supersymmetric. In order to have a globally supersymmetric loop one needs to require that the same Killing spinor be preserved at each point. Taking the unit vector Θ^I constant implies that the loop must be a straight line, $\ddot{x}^\mu = 0$, say along the time direction.[35] One can then always use the $SO(6)$ symmetry to rotate Θ^I in such a way that the loop will only couple to one scalar. Other ways of constructing non-trivial solutions to eq. (2.52) will the subject of section 6.1. There Θ^I will not be constant but will depend on the shape of loop in a non-trivial way, and the loop will in general couple to more than one scalar.

A similar computation can be carried over for the superconformal charges and one sees that the straight line Wilson loop preserves separately 8 Poincaré and 8 superconformal charges and is therefore 1/2 BPS.

The symmetry of the straight line loop is the $Osp(4^*|4)$ subgroup of $SU(2,2|4)$. Switching for a moment to Minkowski signature, it is easy to understand the origin of the bosonic subgroup. The conformal group $SO(4,2) \simeq SU(2,2)$ is broken down by the loop to $SO(1,2) \times SO(3) \simeq SU(1,1) \times SU(2) \simeq SO(4^*)$. The first factor consists of the three generators P, Δ, and K of the conformal group along the preferred

[34] In Minkowski signature eq. (2.52) reads $\dot{x}^2 + \dot{y}^2 = 0$, and one could also consider another non-trivial solution given by light-like loops $\dot{x}^2 = 0$ with $\dot{y}^I = 0$ for every I. These loops are effectively non-supersymmetric because they do not couple to the $\mathcal{N} = 4$ scalars. Light-like Wilson loops with cusps find vast applications in the computations of anomalous dimension of *twist operators* [69]-[72].

[35] The line has to be infinitely long for gauge invariance.

direction defined by the loop, while the second one is the group of rotations around the line. The R-symmetry group $SO(6) \simeq SU(4)$ is broken by a specific choice of Θ^I to $SO(5) \simeq Sp(4)$.

Next we briefly review the perturbative expansion of the loop (2.53) and introduce a second 1/2 BPS loop, the circular loop [29].

The circular loop in perturbation theory

Expanding the exponent in eq. (2.53) to second order and setting $\Theta^I = const.$, we find that the combined gauge and scalar propagators, defining $x_{(i)} \equiv x(\tau_i)$, read[36]

$$\left\langle \left(iA^a_\mu \dot{x}^\mu_{(1)} + |\dot{x}_{(1)}|\Theta^I \Phi^a_I \right) \left(iA^b_\rho \dot{x}^\rho_{(2)} + |\dot{x}_{(2)}|\Theta^J \Phi^b_J \right) \right\rangle = \frac{g^2 \delta^{ab}}{4\pi^2} \frac{|\dot{x}_{(1)}||\dot{x}_{(2)}| - \dot{x}_{(1)} \cdot \dot{x}_{(2)}}{|x_{(1)} - x_{(2)}|^2} \,. \tag{2.54}$$

For the infinite straight line it is obvious that this expression vanishes. Then one anticipates the expectation value of this loop to be 1, and we will in fact recover this result from the dual description in the bulk.

The straight line considered so far is not the only 1/2 BPS loop operator. A circular Wilson loop can be obtained from it by performing a conformal transformation, more precisely an inversion around the origin

$$x^\mu \to \frac{x^\mu}{x^2} \,. \tag{2.55}$$

[36] This expression is obtained in the Feynman gauge where, up to the index structure, the gauge and scalar propagator in configuration space are equal and given by $\frac{g^2}{4\pi^2}\frac{1}{x^2}$.
Inserting the propagator (2.54) in a smooth loop, one expects a priori a linear divergence when the two legs of the propagator approach each other. It was shown in [73] that, at any order in perturbation theory, this divergence in fact cancels between gauge and scalar fields provided that eq. (2.52) is satisfied. If the loop has cusps or self-intersections then an additional logarithmic divergence arises.

The resulting loop still preserves half of the supercharges, but a different combination than before, with Poincaré and superconformal charges mixed together [74]. The symmetry group is still $Osp(4|4^*)$, although realized in a less transparent way (see section 6.1).

One could think that the expectation value of the circular loop should also be trivial, but it turns out that this is not true [29], because of a conformal anomaly coming from mapping the point at infinity to the origin [30]. This can be understood considering that the transformation (2.55) is not a symmetry of $\mathbb{R}^4$, but only of S^4.

Taking the explicit parametrization for the circle $x^\mu = (\cos\tau, \sin\tau, 0, 0)$, one can readily see that the propagator (2.54) reduces to a constant

$$\left\langle \left(iA^a_\mu \dot{x}^\mu_{(1)} + |\dot{x}_{(1)}|\Theta^I \Phi^a_I\right) \left(iA^b_\rho \dot{x}^\rho_{(2)} + |\dot{x}_{(2)}|\Theta^J \Phi^b_J\right) \right\rangle = \frac{g^2 \delta^{ab}}{8\pi^2} \,. \tag{2.56}$$

It is then possible to sum the infinite class of Feynman diagrams without internal vertices, the so-called *ladder* or *rainbow diagrams*, and find a result valid for any value of the 't Hooft coupling λ. It is in fact believed that graphs with internal vertices cancel at any order in perturbation theory and one is then left only with free propagators [29]. This conjecture has been checked at order λ^2, where the graph with one internal 3-vertex cancels against the 1-loop self-energy correction of eq. (2.54). Explicit computations at higher orders have not yet been performed, but a formal, albeit incomplete, proof of the conjecture based on the conformal anomaly mentioned above has been presented in [30].[37]

[37] The argument goes roughly as follows. Starting from the straight line and performing an inversion the gauge propagator gets modified by a total derivative, which is equivalent to a gauge transformation. This transformation is singular at the origin, and computing the contribution from the singularity one can show that it is given by a matrix model. This is shown to hold at any order in perturbation theory.

The Gaussian matrix model

The fact that the propagator (2.54) loses the coordinate dependence for the circular loop suggests that it might be possible to map the problem of summing the ladder diagrams of the circle to a 0-dimensional matrix model [29]. Further, since interacting graphs are conjectured to cancel in the perturbative expansion we also expect this matrix model to be quadratic. This model is also taken to be Hermitian and is defined in terms of its partition function

$$Z = \int [dM] \exp\left(-\frac{2N}{\lambda}\mathrm{Tr}\, M^2\right) . \tag{2.57}$$

The expectation value of the circular Wilson loop is then given by [29][30]

$$\langle W \rangle = \left\langle \frac{1}{N}\mathrm{Tr}\, e^M \right\rangle = \frac{1}{Z}\int [dM] \exp\left(-\frac{2N}{\lambda}\mathrm{Tr}\, M^2\right)\frac{1}{N}\mathrm{Tr}\, e^M . \tag{2.58}$$

This expression is valid, up to instanton corrections [74], to all orders of $1/N$ and λ. Using classical methods in random matrix theory [38] one finds

$$\langle W \rangle = \frac{2}{\pi\lambda}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} dx\, \sqrt{\lambda - x^2}\, e^x = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) , \tag{2.59}$$

where the square root under the integral is the Wigner semi-circle distribution, and I_1 is a modified Bessel function.[38] This expression is valid for any value of λ. In particular we can compute the strong coupling limit of eq. (2.59), which is the

[38]This expression is obtained with a saddle point approximation in the planar limit of infinite N. It is also possible to solve the matrix model exactly, leaving N finite and thus including non-planar corrections, by using (Hermite) orthogonal polynomials [30]. The result turns out to be

$$\langle W \rangle = \frac{1}{N}\exp\left(\frac{\lambda}{8N}\right) L^1_{N-1}\left(-\frac{\lambda}{8N}\right) ,$$

interesting limit for a comparison with the string theory computation. For $\lambda \to \infty$ one has

$$\langle W \rangle \simeq \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}} \,. \tag{2.60}$$

This expression does not depend on the radius of the circle, as requested by conformal invariance, but only on the dimensionless parameter λ. In the asymptotic expansion of the Bessel function also appears another, subleading contribution which we did not include in the formula above and goes as $ie^{-\sqrt{\lambda}}$. This term is associated to instantonic correction and will be discussed in more detail in section 6.3.

2.3.2 Bulk description as minimal surfaces

Equipped with the strong coupling result (2.60) for the circular loop, we present now the dual, bulk description of Wilson loops as sources of fundamental strings in $AdS_5 \times S^5$.

It was proposed in [75][76] that a Wilson loop in the fundamental representation of the gauge group should be associated to a string extending on the bulk of $AdS_5 \times S^5$ and landing on the boundary along the contour of the loop. Then the expectation value of the loop operator is given by the (regularized) area of the minimal surface swept by the string. Here we motivate this proposal and discuss it in detail.[39]

As explained above, the Wilson loop gives the phase factor of an external probe in some representation of the gauge group that, so far, we have taken to be the

where L^1_{N-1} is a Laguerre polynomial. Expanding this expression for large N yields

$$\frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) + \frac{\lambda}{48N^2} I_2\left(\sqrt{\lambda}\right) + \frac{\lambda^2}{1280N^4} I_4\left(\sqrt{\lambda}\right) + \dots \,.$$

[39] A nice review can be found in [77].

fundamental one. In $\mathcal{N}=4$ super Yang-Mills there are no "quarks", but a fundamental particle can be mimicked in the following way. Consider a stack of $N+1$ D3-branes and then move one of them far away from the remaining N. There will be a very long string stretching in-between, which can be interpreted as a very heavy "W-boson". The endpoints of this string are the external non-dynamical sources that we call quarks. After taking the decoupling limit discussed in section 2.1.3, the string extends in the bulk of AdS_5 and lands on the loop on the boundary.

At the gauge theory level this consists in giving a large expectation value to one scalar and in breaking the gauge group $U(N+1) \to U(N) \times U(1)$. The resulting off-diagonal bosons are in the fundamental of $U(N)$. The amplitude for one of these W-bosons with mass m to go around a loop $\mathcal{C}$ of length $l(\mathcal{C})$ is given (in the limit of large m) by

$$\mathcal{A} \simeq e^{-m\, l(\mathcal{C})} \langle W(\mathcal{C}) \rangle \,. \tag{2.61}$$

According to the proposal in [75][76], this amplitude should be equal to the world-sheet area of the string associated to the contour $\mathcal{C}$

$$\mathcal{A} = \int [dX^\mu][dY^I][dh_{ab}] \exp\left(-\sqrt{\lambda}\, S[X,Y,h]\right) \,, \tag{2.62}$$

where one must impose the following boundary values on the fields: $X^\mu|_{\mathcal{C}} = x^\mu$, the coordinates parameterizing the loop, and $Y^I|_{\mathcal{C}} = y^I$, see eq. (2.49). Considering the limit of large λ the integral can be evaluated on the saddle point, where string fluctuations are suppressed and the action is the (minimal) area of the classical surface

$$\lim_{\lambda\to\infty} \mathcal{A} = e^{-\sqrt{\lambda}\, Area(\mathcal{C})} \,. \tag{2.63}$$

Then

$$\langle W(\mathcal{C})\rangle \simeq \exp\left[-\left(\sqrt{\lambda}\,Area(\mathcal{C}) - m\,l(\mathcal{C})\right)\right] . \tag{2.64}$$

The worldsheet area, being the string infinitely long, is formally infinite but is regularized subtracting the term $m\,l(\mathcal{C})$ A detailed discussion of this issue can be found in [73].

The classical surface has always the minimal area for a given set of boundary conditions and the counterterm $m\,l(\mathcal{C})$ is always larger than the minimal area. Then the renormalized area is always negative. As a consequence, the generic behavior for a Wilson loop at strong coupling is

$$\langle W(\mathcal{C})\rangle \simeq e^{\alpha\sqrt{\lambda}} , \tag{2.65}$$

where α is a non-negative constant.

We illustrate this in three important cases.

Infinite straight line

The simplest example where one can start testing this prescription is the infinite straight line, whose expectation value is expected to vanish.

To describe this system we use the Poincaré metric with cartesian coordinates on the boundary (the radius of AdS_5 is set equal to 1)

$$ds^2_{AdS} = \frac{1}{z^2}\left(dz^2 + \sum_{i=1}^{4} dx_i^2\right) . \tag{2.66}$$

Then we can take the loop along the x_1 direction, with $x_2 = x_3 = x_4 = 0$ and sitting at a fixed point Θ^I on the S^5. It is convenient to choose a static gauge where the world-sheet coordinates are taken to be $\{z, x_1\}$. The Nambu-Goto action for the fundamental string stretching in the interior of AdS_5 and landing along the loop is

$$\begin{aligned} S_{NG} &= \frac{1}{2\pi\alpha'}\int dz\, dx_1 \sqrt{\det\gamma} \\ &= \frac{1}{2\pi\alpha'}\int_{z_0}^{\infty} dz \int_0^{X_1} dx_1\, \frac{1}{z^2} = \frac{X_1\sqrt{\lambda}}{2\pi z_0}\,, \end{aligned} \tag{2.67}$$

where $\gamma_{ab} = G_{\mu\nu}\partial_a x^\mu \partial_b x^\nu$ is the induced world-sheet metric, X_1 is the length of the loop (which is eventually sent to infinity), and z_0 is a cut-off introduced to regularize the action.

As explained in [73], it is necessary to add an appropriate boundary term for the transverse scalar z. In fact it turns out that one needs to Legendre transform the action in such a way that it depends on p_z, the conjugate momentum of z, rather than on z itself

$$S_{bdy} = -\int_{\partial M} z\, p_z = -\int_{\partial M} z\, \frac{\partial \mathcal{L}_{NG}}{\delta z'}\,, \tag{2.68}$$

where the derivative of $\mathcal{L}_{NG}$ with respect to z' is computed considering a non-static gauge where the world-sheet coordinates are $\{\tilde{z}, x_1\}$, with $z = z(\tilde{z})$, and finally setting $z = \tilde{z}$. One has then

$$S_{bdy} = -\int_0^{X_1} dx_1\, \frac{\sqrt{\lambda}}{2\pi z}\Big|_{z=z_0} = -\frac{X_1\sqrt{\lambda}}{2\pi z_0}\,, \tag{2.69}$$

which exactly cancels eq. (2.67), giving $\langle W(line)\rangle = 0$.

Circular loop

The computation for the circle is very similar to the straight line case and we do not report it here. One finds again a divergent expression for the Nambu-Goto action

$$S_{NG} = \sqrt{\lambda}\left(\frac{1}{z_0} - 1\right) . \tag{2.70}$$

The first term is cancelled by the boundary condition for the scalar field and one is left with the second term, so that

$$\langle W(circle)\rangle = e^{\sqrt{\lambda}} . \tag{2.71}$$

This exactly reproduces the leading behavior of the strong coupling result (2.60) obtained from the matrix model on the boundary.

A more precise match with eq. (2.60) comes from considering the zero modes arising when gauge fixing the integral over the metrics. These give a factor of $\lambda^{1/4}$ for each zero mode integrated in the path integral. The number of zero-modes turns out to be three times the Euler character of the world-sheet, which has the topology of a disk and therefore $\chi = -1$ [78]. This account for the factor $\lambda^{-3/4}$ in eq. (2.60).

Other corrections come from considering string fluctuations around the classical surface [78]. They go in powers of $\lambda^{-1/2}$ and correspond to an α' expansion of the world-sheet σ-model.

Quark–antiquark potential

As last example we present the result for the potential between a static quark–antiquark pair [75][76].[40] This potential is given by the expectation value of two antiparallel straight lines along the Euclidean time, representing the positions on the boundary of the quark q and the antiquark $\bar{q}$. The length T of these lines is taken to be much larger than their spatial separation L. Both q and $\bar{q}$ sit at the *same* fixed point on the 5-sphere.[41]

One finds that, after subtracting the contribution coming from the infinite mass of the string connecting q and $\bar{q}$, the regularized world-sheet area yields a *non-confining* Coulomb potential [76]

$$S = -\frac{4\pi^2\sqrt{\lambda}}{\Gamma\left(1/4\right)^4 L}\,. \tag{2.72}$$

Notice that this potential scales as $\sqrt{\lambda}$, differently from the weak coupling result which scales as λ. The inversely proportional dependence on L is dictated by conformal invariance: L is in fact the only scale in the setup, and $1/L$ is the only way to achieve dimensions of energy. Any other dependence on the distance between q and $\bar{q}$ would require a second scale which cannot appear because of conformal invariance.

One way one could have anticipated this result is by thinking in terms of the warp factor of the AdS_5 metric $1/z^2$ [18]. Increasing the distance between q and $\bar{q}$ pushes the string into regions of greater z and therefore smaller $1/z^2$, decreasing the value of the proper renormalized area of the world-sheet.

[40] As already explained above, with the word "quark" one indicates in this context the endpoints of a long fundamental string connecting the stack on N D3-branes with the one D3-brane sitting far away from the others.

[41] This is a non-BPS configuration, where the Coulomb force between the two charges and the gradient of the transverse scalar field are both attractive and therefore generate a non-vanishing potential [75]. The configuration with q and $\bar{q}$ at antipodal point of S^5 is on the other hand BPS and does not give rise to a potential, because of a cancellation between two opposite forces.

2.3.3 Giant Wilson loops

So far we have always considered Wilson loops containing traces over the fundamental representation of the gauge group. We now move on to presenting the bulk description of *higher rank* loops, *i.e.* loops in representations other than the fundamental.[42] These loops will be the subject of sections 6.2 and 6.3.

The first guess for a bulk description of higher rank loops, coincident loops, or multiply wrapped loops would be to consider a set of coincident fundamental strings all landing along the loops on the boundary [79]. This approach presents serious technical difficulties arising from the fact that the string world-sheets develop conical singularities and branch cuts, whose locations need to be integrated over.

A more effective way to describe such loops has been first proposed in [80],[43] where it was suggested that a multiply wrapped loop might be associated to a D3-brane extending in the bulk and pinching off at the boundary landing on the loop. This proposal is based on merging two ideas: the first one is to generalize to the $AdS_5 \times S^5$ background the picture first put forward by Callan and Maldacena [83], that a fundamental string ending on a D3-brane in flat space can be described in terms of a curved D3-brane with a localized spike carrying a unit of electric flux, and the second one is the idea, known as *Emparan-Myers polarization effect* [84][85], that coincident strings can polarize into a single D-brane.

[42] We illustrate the explicit meaning of the symbol $\mathrm{Tr}_{\mathcal{R}}$ in the case of a Young diagram with two boxes. There are two such diagrams: the symmetric $\square\!\square$ and the antisymmetric $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$. The traces over these two representations are given by

$$\begin{aligned}
\mathrm{Tr}_{\square\!\square} M: &\qquad M^i_j M^j_i \to \frac{1}{2}\left(M^i_j M^j_i + M^j_j M^i_i\right) = \frac{1}{2}\left(\mathrm{Tr}\, M^2 + (\mathrm{Tr}\, M)^2\right) \,, \\
\mathrm{Tr}_{\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}} M: &\qquad M^i_j M^j_i \to \frac{1}{2}\left(M^i_j M^j_i - M^j_j M^i_i\right) = \frac{1}{2}\left(\mathrm{Tr}\, M^2 - (\mathrm{Tr}\, M)^2\right) \,.
\end{aligned}$$

[43] See also [81][82].

This brane picture has the advantage of automatically encoding the interactions between the coincident strings and yields all *non-planar contributions* to the expectation value of the higher rank Wilson loop [80].[44]

If we indicate with k the rank of the loop, the number of coincident loops or the number of windings, the D-brane must have k units of fundamental string charge dissolved in its world-volume. Further, it must preserve the supersymmetries and the $SO(1,2) \times SO(3) \times SO(5)$ isometry group of the gauge theory operator.

It turns out that there are two kinds of branes with these characteristics: an electrically charged D3-brane with $AdS_2 \times S^2$ world-volume and charge k, and a D5-brane with $AdS_2 \times S^4$ world-volume and, again, charge k. Both the D3 and D5-brane are 1/2 BPS and both have an AdS_2 factor which can be associated to the fundamental string world-sheet. The difference between them is that the D3 is completely contained in AdS_5, whereas the S^4 factor of the D5 is inside S^5.[45]

The holographic dictionary that has been established in [80][81][88] connects the D3-branes to Wilson loops in the rank k symmetric representation[46] and the D5-branes to the rank k antisymmetric one.[47]

This brane picture is very reminiscent of the *giant/dual giant graviton* picture for chiral primary operators [91]-[93]. A giant graviton is a D3-brane wrapping an $S^3 \subset S^5$, whereas a dual giant graviton wraps an $S^3 \subset AdS_5$. Both describe excitations

[44]For 't Hooft loops see [80] and for applications to finite temperature see [86].

[45]As a consequence, as long the D5-brane preserves an $SO(5)$ group, its volume is always proportional, independently on the shape of the loop on the boundary, to the area of the string world-sheet, the constant of proportionality being an universal factor [87].

[46]In the limit of $N \to \infty$ and $\lambda \to \infty$ it turns out that the symmetric Wilson loop coincides with the multiply wound one [89][90].

[47]By checking the brane results against matrix model computations, the connection between the D3-brane and the multiply wound loop and between the D5-brane and the antisymmetric loop was first seen in [80] and [81], respectively (see also [89][90]). A more formal proof of this dictionary was later given in [88], by using defect conformal field theory arguments.

with large angular momentum $J \sim N$. Inspired by this analogy, one can call the D3-brane *dual giant Wilson loop* and the D5 *giant Wilson loop*. The role of J is played in this context by the charge k.

The brane probe approximation discussed so far is valid when $k \sim N$ and breaks down if $k \gg N$. In that regime the brane backreacts on the geometry and the $AdS_5 \times S^5$ space is deformed into the supergravity solutions studied in [94][95].[48]

In section 6.2 we will make explicit use of the solutions for the D3 and D5-branes associated to higher rank loops. We therefore review them in detail in the following.

The D3-brane

We start with the D3-brane found in [80] and consider a circular Wilson loop of radius a placed on the boundary of AdS_5. The metric on AdS_5 can be written in polar coordinates as

$$ds^2_{AdS} = \frac{1}{z^2}\left(dz^2 + dr_1^2 + r_1^2 d\psi^2 + dr_2^2 + r_2^2 d\phi^2\right) . \tag{2.73}$$

The position of the loop is defined by $r_1 = a$ and $z = r_2 = 0$. As explained above, we look for a D3-brane which pinches off on this circle as $z \to 0$ and preserves a $SO(1,2) \times SO(3) \times SO(5)$ isometry.

The bulk action includes a DBI part and a Wess-Zumino term, which captures the coupling of the background Ramond-Ramond field to the brane

$$S_{D3} = T_{D3} \int \sqrt{\det(\gamma + 2\pi\alpha' F)} - T_{D3} \int P[C_{(4)}] \,, \tag{2.74}$$

[48] This is, again, very similar to what happens for giant gravitons when $J \sim N^2$. The resulting geometry in this case is the LLM "bubbling AdS" space which will be the focus of chapter 5.

where $T_{D3} = \frac{N}{2\pi^2}$ is the tension of the brane, γ is the induced metric, F the electromagnetic field strenght, and $P[C_{(4)}]$ is the pull-back of the 4-form

$$C_{(4)} = \frac{r_1 r_2}{z^4} dr_1 \wedge d\psi \wedge dr_2 \wedge d\phi \tag{2.75}$$

to the brane worldvolume. It turns out to be more convenient to use a new set of coordinates obtained by transforming $\{z,\, r_1,\, r_2\}$ into

$$z = \frac{a \sin\eta}{\cosh\rho - \sinh\rho\cos\theta}\,, \quad r_1 = \frac{a\cos\eta}{\cosh\rho - \sinh\rho\cos\theta}\,, \quad r_2 = \frac{a\sinh\rho\sin\theta}{\cosh\rho - \sinh\rho\cos\theta}\,. \tag{2.76}$$

In this coordinate system the metric on AdS_5 reads

$$ds^2_{AdS} = \frac{1}{\sin^2\eta}\left(d\eta^2 + \cos^2\eta\ d\psi^2 + d\rho^2 + \sinh^2\rho\ (d\theta^2 + \sin^2\theta\ d\phi^2)\right)\,, \tag{2.77}$$

where $\rho \in [0, \infty)$, $\theta \in [0, \pi]$, and $\eta \in [0, \pi/2]$. The Wilson loop is located at $\eta = \rho = 0$. One can pick a static gauge in which the worldvolume coordinates of the brane are identified with $\{\psi\,, \rho\,, \theta\,, \phi\}$ and the brane sits at a fixed point of the S^5 determined by the constant unit vector $\Theta^I \in \mathbb{R}^6$. The remaining coordinate can be seen as a scalar field, $\eta = \eta(\rho)$. Because of the symmetries of the problem the electromagnetic field has only one component, $F_{\psi\rho}(\rho)$. In this coordinates the DBI action in eq. (2.74) reads

$$S_{DBI} = 2N \int d\rho d\theta\, \frac{\sin\theta\sinh^2\rho}{\sin^4\eta}\sqrt{\cos^2\eta(1+\eta'^2) + (2\pi\alpha')^2\sin^4\eta F^2_{\psi\rho}}\,, \tag{2.78}$$

while the Wess-Zumino term is

$$S_{WZ} = -2N \int d\rho d\theta \, \frac{\cos\eta \sin\theta \sinh^2\rho}{\sin^4\eta} \left(\cos\eta + \eta' \sin\eta \, \frac{\sinh\rho - \cosh\rho\cos\theta}{\cosh\rho - \sinh\rho\cos\theta} \right). \quad (2.79)$$

The solution to the equations of motion reads [80]

$$\sin\eta = \frac{1}{\kappa}\sinh\rho\,, \qquad F_{\psi\rho} = \frac{ik\lambda}{8\pi N \sinh^2\rho}\,, \qquad \kappa = \frac{k\sqrt{\lambda}}{4N}. \quad (2.80)$$

From this solution one can see that k is not constrained for the D3-brane. In fact k determines the position of an $AdS_2 \times S^2$ foliation of AdS_5 which is a non-compact space. This has to be contrasted with the D5-brane case where we shall find a bound on k.

The bulk action has to be complemented with boundary terms for the worldvolume scalar η and for the electric field $F_{\psi\rho}$. These terms do not change the solution but alter the final value of the on-shell action which reads

$$S_{D3} = S_{DBI} + S_{WZ} + S_{\text{bdy}} = -2N(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa). \quad (2.81)$$

The expectation value of a Wilson loop in the rank k symmetric representation is then

$$\langle W_{S_k} \rangle = \exp\left[2N\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right)\right]. \quad (2.82)$$

For small κ this expression reproduces the result of k fundamental strings

$$\langle W_{S_k} \rangle \simeq e^{k\sqrt{\lambda}}. \quad (2.83)$$

The D5-brane

We now move on to reviewing the D5 solution found in [81], albeit in a different system of coordinates. We find convenient to take the $AdS_5 \times S^5$ metric as

$$\begin{aligned} ds^2 &= \cosh^2 u(d\zeta^2 + \sinh^2\zeta d\psi^2) + du^2 + \sinh^2 u(d\vartheta^2 + \sin^2\vartheta d\phi^2) + \\ &+ d\theta^2 + \sin^2\theta d\Omega_4^2 \,, \end{aligned} \tag{2.84}$$

where we have written the AdS_5 factor as an $AdS_2 \times S^2$ fibration. These coordinates are related to the usual Poincare patch by

$$\begin{aligned} r_1 &= \frac{a\cosh u \sinh\zeta}{\cosh u\cosh\zeta - \cos\vartheta\sinh u}\,, \qquad r_2 = \frac{a\sinh u\sin\vartheta}{\cosh u\cosh\zeta - \cos\vartheta\sinh u} \\ z &= \frac{a}{\cosh u\cosh\zeta - \cos\vartheta\sinh u}\,, \end{aligned} \tag{2.85}$$

where, as before, a denotes the radius of the Wilson loop. In these coordinates the Wilson loop is at $\zeta \to \infty$, $u = 0$ and it is parametrized by ψ. The selfdual 4-form potential can be taken to be

$$C_{(4)} = 4\left(\frac{u}{8} - \frac{1}{32}\sinh 4u\right) dH_2 \wedge d\Omega_2 - \left(\frac{3}{2}\,\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right) d\Omega_4\,, \tag{2.86}$$

where dH_2 denotes the volume element of the AdS_2 part of the metric.

Since we want to construct a D5-brane with $AdS_2 \times S^4$ worldvolume, it is natural to take a static gauge in which ζ, ψ and the coordinates of the $S^4 \subset S^5$ are the worldvolume coordinates. Furthermore we can take the following ansatz which preserves the $SO(1,2) \times SO(3) \times SO(5)$ symmetry of the Wilson loop

$$u = 0\,, \qquad \theta = const.\,, \tag{2.87}$$

and only the $F_{\psi\zeta}$ component of the worldvolume gauge field is turned on. With this ansatz the DBI and Wess-Zumino parts of the D5 action reduce to

$$\begin{aligned} S_{DBI} &= T_{D5} \int d^6\sigma \sqrt{\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \\ &= \frac{2N}{3\pi}\sqrt{\lambda} \int d\zeta \sinh\zeta \sin^4\theta \sqrt{1 + \frac{4\pi^2}{\lambda}\frac{F_{\psi\zeta}^2}{\sinh^2\zeta}}\,, \end{aligned} \tag{2.88}$$

and

$$\begin{aligned} S_{WZ} &= -2\pi\alpha' i\, T_{D5} \int F \wedge P[C_{(4)}] \\ &= \frac{4iN}{3} \int d\zeta\, F_{\psi\zeta} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta \right)\,, \end{aligned} \tag{2.89}$$

where we have used $T_{D5} = N\sqrt{\lambda}/8\pi^4$ and $\mathrm{vol}(\Omega_4) = 8\pi^2/3$. The equation of motion for the electric field states that the conjugate momentum is a constant equal to the number of fundamental string charge k dissolved in the D5-brane

$$\Pi \equiv \frac{-i}{2\pi}\frac{\delta\mathcal{L}}{\delta F_{\psi\zeta}} = \frac{2N}{3\pi}\frac{E\sin^4\theta}{\sqrt{1-E^2}} + \frac{2N}{3\pi}\left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta \right) = \ k\,, \tag{2.90}$$

where for convenience we have defined $E = \frac{-2\pi i}{\sqrt{\lambda}}\frac{F_{\psi\zeta}}{\sinh\zeta}$. This equation allows to determine the angle θ at which the D5 sits as a function of k

$$\theta_k - \sin\theta_k\, \cos\theta_k = \ \pi\,\frac{k}{N}\,, \tag{2.91}$$

while the electric field is given by $E = \cos\theta_k$. One can check that with this ansatz the equation of motion for u is also satisfied.

Notice that eq. (2.91) puts a bound on k, which has to be less than N. k in this case determines the location of the brane on S^5, which is a compact space. Another way to understand this bound is by recalling that a Young diagram of $SU(N)$ can have at most N boxes in every columns, otherwise gives a trivial representation.

All this is again very similar to the bound $J \leq N$ found for the momentum of a giant graviton.[49]

Adding the appropriate boundary terms for the electric field and the worldvolume scalars (see [81][87] for details) the on-shell action for the D5-brane becomes

$$S_{D5} = S_{DBI} + S_{WZ} + S_{\text{bdy}} = -\frac{2N}{3\pi}\sqrt{\lambda}\sin^3\theta_k\,, \tag{2.92}$$

so the expectation value of the Wilson loop in the rank k antisymmetric representation is given by

$$\langle W_{A_k}\rangle = \exp\left(\frac{2N}{3\pi}\sqrt{\lambda}\sin^3\theta_k\right)\,. \tag{2.93}$$

As previously noted in the literature, this result is consistent with the duality between the rank k and rank $N-k$ antisymmetric representations: indeed, it can be seen from eq. (2.91) that under $k \to N-k$ the angle θ_k goes into $\pi - \theta_k$. It can also be checked that in the limit $k/N \to 0$, in which the S^4 factor shrink to zero size, eq. (2.93) coincides with the action of k fundamental strings, as for small k/N eq. (2.91) gives $\theta_k^3 \sim 3\pi k/2N$, so that $\langle W_{A_k}\rangle \simeq \exp k\sqrt{\lambda}$.

[49]This phenomenon is known as *stringy exclusion principle* [91].

Chapter 3

Twistor string theory

3.1 Introduction

In 2003 Witten [96] proposed an interesting counterpart to the string description of *strongly coupled* $\mathcal{N} = 4$ super Yang-Mills discussed in the previous chapter. He discovered in fact a remarkable connection between *weakly coupled* $\mathcal{N} = 4$ super Yang-Mills theory and a particular version of topological string theory, the B-model on the twistor space $\mathbb{CP}^{3|4}$. More precisely, he pointed out that perturbative amplitudes of the gauge theory can be interpreted as a D-instanton expansion in the topological theory.

The interest in this twistor construction was mainly motivated by the possibility of developing very efficient computational techniques which allow to reorganize in a clever way the perturbative expansion of scattering amplitudes. In fact, after stripping out the color information, tree level Yang-Mills theory is effectively supersymmetric and therefore this proposal provides a new, suggestive approach to the study of Yang-Mills amplitudes. In particular some seemingly accidental properties of scattering amplitudes, like the holomorphicity of the renowned MHV Parke-Taylor formula for the scattering of gluons, receive a new elegant interpretation in terms of localization over certain subloci of the string target space $\mathbb{CP}^{3|4}$.

In this chapter we describe the basics of twistor string theory. We start with a discussion of localization properties of scattering amplitudes in twistor space and then present the corresponding string theory interpretation. Finally, we conclude with illustrating the twistor inspired *CSW formalism* for computing scattering amplitudes.[1]

3.2 Scattering amplitudes in twistor space

3.2.1 The spinor-helicity formalism

The complexified Lorentz group in four dimension, with signature $(+ - --)$, can be locally decomposed as $SO(3,1)_{\mathbb{C}} \simeq SL(2)_{\mathbb{C}} \times SL(2)_{\mathbb{C}}$, and its finite-dimensional representations are labeled by a pair of integer or half-integer numbers, (p, q). We indicate with λ_a (with $a = 1, 2$) a negative helicity spinor transforming as $\left(\frac{1}{2}, 0\right)$, and with $\tilde{\lambda}_{\dot{a}}$ (with $\dot{a} = 1, 2$) a positive helicity spinor transforming as $\left(0, \frac{1}{2}\right)$. Both λ and $\tilde{\lambda}$ are defined to be Grassmann-even objects, and because of this "twisted statistics" are called *twistors*. The undotted spinor indices are lowered and raised with the antisymmetric tensor ϵ_{ab} and its inverse ϵ^{ab} (obeying $\epsilon^{ab}\epsilon_{bc} = \delta^a_c$ and with $\epsilon_{12} = +1$)

$$\lambda^a = \epsilon^{ab}\lambda_b\,, \qquad \lambda_a = \epsilon_{ab}\lambda^b\,, \tag{3.1}$$

[1]Nice reviews of these topics, from which we heavily draw, can be found in [97][98]. See also Witten's lectures at PiTP 2004, which are available online at `http://www.admin.ias.edu/pitp/2004/schedule.html`.

and, similarly, the dotted indices are lowered and raised with $\epsilon_{\dot{a}\dot{b}}$ and $\epsilon^{\dot{a}\dot{b}}$. Given two spinors of the same chirality, λ and λ' or $\tilde{\lambda}$ and $\tilde{\lambda}'$, we can define antisymmetric inner products

$$\langle \lambda, \lambda' \rangle \equiv \epsilon_{ab} \lambda^a \lambda'^b \,, \qquad [\tilde{\lambda}, \tilde{\lambda}'] \equiv \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{\dot{a}} \tilde{\lambda}'^{\dot{b}} \,, \tag{3.2}$$

such that $\langle \lambda, \lambda' \rangle = -\langle \lambda', \lambda \rangle$ and $\langle \lambda, \lambda \rangle = 0$. Likewise for the opposite chirality spinors $[\tilde{\lambda}, \tilde{\lambda}'] = -[\tilde{\lambda}', \tilde{\lambda}]$ and $[\tilde{\lambda}, \tilde{\lambda}] = 0$.

The vector representation of $SO(3,1)_{\mathbb{C}}$ is the $\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right)$ and one can thus represent a momentum vector p_μ (with $\mu = 0, \ldots, 3$) in terms of a bi-spinor $p_{a\dot{a}}$

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}}\, p_\mu = p_0 + \vec{\sigma} \cdot \vec{p}, \tag{3.3}$$

with $\vec{\sigma}$ being the 2×2 Pauli matrices. From this identification follows that

$$p_\mu\, p^\mu = \det(p_{a\dot{a}}) = p_0^2 - \vec{p}^{\,2} \,, \tag{3.4}$$

and therefore p_μ is light-like if and only if $p_{a\dot{a}}$ has rank strictly less than 2. A generic rank 2 bi-spinor can be written as $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \mu_a \tilde{\mu}_{\dot{a}}$, whereas rank 1 bi-spinors are those that can be written in terms of a single pair of spinors λ_a and $\tilde{\lambda}_{\dot{a}}$ [2]

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \,. \tag{3.5}$$

[2] In Lorentzian signature $(+ - - -)$ momentum is real, so that $p_{a\dot{a}}$ is hermitian. This implies that $\tilde{\lambda} = \pm\bar{\lambda}$, where the $\pm$ signs are for future pointing and past pointing null vectors respectively, and therefore λ and $\tilde{\lambda}$ are not independent. In $(+ + - -)$ signature the Lorentz group, without any complexification, decomposes as $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, whose spinor representations are real, so that λ and $\tilde{\lambda}$ are now real and independent. In Euclidean signature $(+ + + +)$ one has $SO(4) \simeq SU(2) \times SU(2)$ with pseudoreal spinors. Hence a light-like vector cannot be real in Euclidean signature, but must be complex with λ and $\tilde{\lambda}$ complex and independent.

The spinors λ and $\tilde{\lambda}$ appearing in this formula have a simple physical meaning, being the wavefunctions of massless particles of helicity $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively. To see this, one can just write the Dirac equation for a particle of, say, helicity $-\frac{1}{2}$

$$0 = \sigma^{\mu}_{a\dot{a}} \partial_{\mu} \psi^{a}(x) = \frac{\partial}{\partial x^{a\dot{a}}} \psi^{a}(x) \,. \tag{3.6}$$

If $\psi^a(x) = w^a e^{ix_{a\dot{a}}p^{a\dot{a}}}$, one gets $ip_{a\dot{a}}w^a = 0$ and, using (3.5), $\lambda_a w^a = 0$, which implies $w^a = const.\, \lambda^a$.

It is useful to note that if p and p' are two light-like vectors given by $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ and $p'_{a\dot{a}} = \lambda'_a \tilde{\lambda}'_{\dot{a}}$, their scalar product can be expressed as

$$2\, p \cdot p' = \langle \lambda, \lambda' \rangle [\tilde{\lambda}, \tilde{\lambda}'] \,. \tag{3.7}$$

The relation (3.5) between a rank 1 bi-spinor and a pair of twistors is clearly not one-to-one, since given a light-like $p_{a\dot{a}}$ the spinors λ_a and $\tilde{\lambda}_{\dot{a}}$ are uniquely determined only modulo the scaling

$$\{\lambda, \tilde{\lambda}\} \rightarrow \{u\lambda, u^{-1}\tilde{\lambda}\} \,, \qquad u \in \mathbb{C}^* \,. \tag{3.8}$$

This means that there is no natural way to determine λ as a function of p.[3]

The wavefunction of massless particles of helicity $h = \pm 1$ can be also expressed in terms of twistors. Such particles are usually described by a momentum p_μ and a polarization vector ε_μ obeying the transversality constraint $\varepsilon_\mu \, p^\mu = 0$ and subject to the gauge transformation $\varepsilon^\mu \rightarrow \varepsilon^\mu + t\, p^\mu$, with t an arbitrary constant. Given a

[3] The precise statement is actually even stronger, not being possible to even pick a *continuous* way to determine λ as a function of p. This is due to a topological obstruction, namely the non-triviality of the Hopf line bundle formed by the possible λ's over the S^2 of directions of the light-like p.

momentum $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, it is possible to define such polarization vectors as

$$\varepsilon^{+}_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}\,, \qquad \varepsilon^{-}_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]}\,, \tag{3.9}$$

where the labels + and − refer clearly to the helicity of the vectors, and $\mu, \tilde{\mu}$ are arbitrary twistors which are only requested to not be proportional to λ and $\tilde{\lambda}$ respectively. It is clear that these definitions satisfy $p_{a\dot{a}}\,\varepsilon^{\pm,a\dot{a}} = 0$ and it is also possible to show that $\varepsilon^{\pm}$ are independent of the choice of μ and $\tilde{\mu}$ up to a gauge transformation.[4] To rigorously show that (3.9) describe particles with helicities ± 1 one must prove that the corresponding field strengths $F_{a\dot{a}b\dot{b}} = \epsilon_{ab}\tilde{f}_{\dot{a}\dot{b}} + \epsilon_{\dot{a}\dot{b}} f_{ab}$ are anti-selfdual (for $h = 1$) or selfdual (for $h = -1$). Substituting, for example, $A_{a\dot{a}} = \varepsilon^{+}_{a\dot{a}}\, e^{i x_{c\dot{c}} p^{c\dot{c}}}$, one finds $F_{a\dot{a}b\dot{b}} = \epsilon_{ab}\tilde{\lambda}_{\dot{a}}\tilde{\lambda}_{\dot{b}}\, e^{i x_{c\dot{c}} p^{c\dot{c}}}$ which is indeed an anti-selfdual form. A similar computation can be carried over for ε^{-}.

This construction suggests that it should be possible to express the scattering amplitudes of n massless particles in four dimensions in terms of the twistors and helicities associated to each particle

$$A^{(n)} = A^{(n)}(\lambda_1, \tilde{\lambda}_1, h_1; \ldots; \lambda_n, \tilde{\lambda}_n, h_n)\,, \tag{3.11}$$

rather than in terms of their momenta and wavefunctions, as the usual textbook prescription instructs us to do.[5] This formalism is called *spinor-helicity formalism*,

[4]Considering for example the case $h = +1$, one observes that μ lives on a two dimensional space and therefore its generic variation must be of the form $\delta\mu = \alpha\mu + \beta\lambda$, for some α and β. Under this variation one has

$$\varepsilon^{+}_{a\dot{a}} \to \varepsilon^{+}_{a\dot{a}} + \frac{\beta}{(1+\alpha)\langle \lambda, \mu \rangle} p_{a\dot{a}}\,, \tag{3.10}$$

which is exactly a gauge transformation of $\varepsilon^{+}_{a\dot{a}}$.

[5]It is possible to show that taking the modulus square of $A^{(n)}$ (which is the physical quantity necessary to compute cross-sections), one obtains expressions that only depend on the momenta and

and has the advantages of unifying the description of particles of different spin and of simplifying considerably the expressions for the amplitudes, as will be discussed presently [99][100].[6] In labeling the helicities all particles are taken to be outgoing. An amplitude with incoming as well as outgoing particles is obtained via crossing symmetry, that relates an incoming particle of given helicity to an outgoing particle of opposite helicity.

A scattering amplitude formulated in terms of twistors and helicities satisfy a homogeneity equation for each external particle ($i = 1, \ldots, n$)

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \tilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}} \right) A^{(n)} = -2h_i A^{(n)} \,, \tag{3.12}$$

which follows from the scaling properties of the wavefunctions under (3.8). Considering, for example, an amplitude with external gluons one can easily see from the definitions of the polarization vectors in eq. (3.9) that ε^+ scales as λ^{-2} and ε^- scales as λ^2, that is they scale with a power of $-2h$. This can be proved to hold also for particle of different helicity.

3.2.2 Tree level gluon amplitudes

We discuss now a particularly interesting class of amplitudes which will be the main focus of the rest of this introduction: tree level scatterings of gluons in Yang-Mills theory.[7] These amplitudes are of phenomenological relevance, since multijet production

no longer on the twistors, as required by Lorentz invariance. For example, a structure like $\langle \lambda_1, \lambda_2 \rangle$ gets squared to $|\langle \lambda_1, \lambda_2 \rangle|^2 = \langle \lambda_1, \lambda_2 \rangle [\tilde{\lambda}_1, \tilde{\lambda}_2] = 2p_1 \cdot p_2$.

[6] For a review see also [101].

[7] Notice that, if gluons are the only external particles in the process, these amplitudes are effectively identical to tree level gluon amplitudes in $\mathcal{N} = 4$ super Yang-Mills. This observation will be crucial in the following.

at LHC will be dominated by them.

Consider pure Yang-Mills theory with gauge group $U(N)$ and recall that tree level diagrams are planar and generate only single-trace interactions [10]. The gauge bosons are attached to the index loop in a definite cyclic order, say $1, 2, \ldots, n$, and the corresponding amplitude contains a color trace factor equal to $G = \mathrm{T}r\, T^1 T^2 \ldots T^n$. It is sufficient to compute one amplitude with a given cyclic order, and then sum over all possible permutations of external gluons to achieve Bose symmetry

$$A^{(n)} = ig^{n-2}(2\pi)^4\delta^4\left(\sum_{i=1}^{n}\lambda_a^i\tilde{\lambda}_{\dot{a}}^i\right) G\,\mathcal{M}^{(n)}(\lambda_1, \tilde{\lambda}_1, h_1; \ldots ; \lambda_n, \tilde{\lambda}_n, h_n) + permutations\,, \tag{3.13}$$

where $\mathcal{M}^{(n)}$ indicates a color stripped and cyclic ordered amplitude, g is the coupling constant of the theory, and we have included a delta function to enforce conservation of momentum.

Amplitudes with all and all but one gluons with the same helicity vanish identically [99]. To see this recall that a gluon 3-vertex carries a power of momentum, whereas a gluon 4-vertex does not. Then, schematically, the structure of an amplitude with n external gluons, m 4-vertices (with $m \leq n/2 - 1$), and $(n - m)$ 3-vertices is

$$\frac{(\varepsilon\cdot\varepsilon)^{m+1}(k\cdot\varepsilon)^{n-2(m+1)}}{(k\cdot k)^{n-m-3}}\,, \tag{3.14}$$

where the numerator contains the momenta and polarization vectors contracted in the vertices, and the denominator comes from the propagators. If $h_i = +1$ for all $i = 1, \ldots, n$, then the numerator always contains contractions of combinations of $\varepsilon_{a\dot{a}}^{+,\,i} = \frac{\mu_a\tilde{\lambda}_{\dot{a}}^i}{\langle\mu,\lambda^i\rangle}$ which vanish because $\langle\mu,\mu\rangle = 0$, and similarly if all helicities are negative. If all but one of the helicities are equal, say $h_1 = -1$ and $h_i = +1$ for $i = 2, \ldots, n$, one can always pick a gauge such that $\mu_a = \lambda_a^1$, then all contractions are again zero.

The first non-vanishing amplitudes are thus the ones with $n-2$ gluons of one helicity and 2 gluons with opposite helicity. They are called *maximally helicity violating* (MHV) amplitudes. The explicit expression for MHV amplitudes in the spinor-helicity formalism is remarkably simple [102][103].[8] If one takes all gluons to have positive helicity except the i-th and j-th ones, the color stripped amplitude reads

$$\mathcal{M}^{(n)}(1+,\ldots,i-,\ldots,j-,\ldots,n+) = \frac{\langle\lambda_i,\lambda_j\rangle^4}{\prod_{k=1}^{n}\langle\lambda_k,\lambda_{k+1}\rangle}\,. \tag{3.15}$$

This amplitude is said to be holomorphic, because it only depends on λ and not on $\tilde{\lambda}$ (see footnote 2). It has the correct homogeneity properties in each variable: it is homogeneous of degree -2 in λ_k for positive helicity gluons, and of degree $+2$ in λ_i and λ_j, as required by the condition (3.12). The anti-holomorphic amplitude with all but two gluons of negative helicity is obtained from (3.15) replacing $\langle\,,\,\rangle \to [\,,\,]$ and untilded twistors with tilded ones.

There is a profound reason behind the holomorphicity of eq. (3.15), and this will become manifest after transforming the amplitude to *twistor space*, as we discuss next.

3.2.3 Fourier transform to twistor space

Tree level amplitudes of external gluons as eq. (3.15) are invariant under conformal transformations. It is therefore interesting to write down the $SO(4,2)$ generators in terms of the twistors λ and $\tilde{\lambda}$. One finds that the conformal generators in this

[8]The same amplitudes written in the standard way in terms of momenta and polarization vectors are extremely complicated, even for $n=5$.

language have a rather exotic representation, and is suggested to rewrite the helicity amplitudes in a new set of variables, the *twistor space coordinates*, where the action of the conformal group is linearly realized.

The Lorentz generators in terms of λ and $\tilde{\lambda}$ are given by

$$J_{ab} = \frac{i}{2}\left(\lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda^a}\right) , \qquad \tilde{J}_{\dot{a}\dot{b}} = \frac{i}{2}\left(\tilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{b}}} + \tilde{\lambda}_{\dot{b}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}}\right) , \tag{3.16}$$

whereas translations are generated by

$$P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} , \tag{3.17}$$

which is a multiplicative operator. Also unusual is the second order form of the generator of special conformal transformations[9]

$$K_{a\dot{a}} = \frac{\partial^2}{\partial \lambda^a \partial \tilde{\lambda}^{\dot{a}}} , \tag{3.18}$$

and the inhomogeneous form of the dilatation operator[10]

$$D = \frac{i}{2}\left(\lambda^a \frac{\partial}{\partial \lambda^a} + \tilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} + 2\right) . \tag{3.19}$$

It is possible to show that these generators indeed annihilate the MHV amplitude (3.15) multiplied by the delta function of momentum conservation.[11]

[9]This is determined by the commutation relations $[D, P] = iP$ and $D, K = -iK$, so P has dimension $+1$ and K scales opposite to P. From eq. (3.17) it is natural to take λ and $\tilde{\lambda}$ to both have dimension $\frac{1}{2}$, and therefore K is guessed be a second order derivative in λ and $\tilde{\lambda}$.

[10]The value of the constant in the dilatation operator is found by requiring closure of the algebra $[K_{a\dot{a}}, P^{b\dot{b}}] = -i\left(\delta^b_a \tilde{J}^{\dot{b}}_{\dot{a}} + \delta^{\dot{b}}_{\dot{a}} J^b_a + \delta^{\dot{b}}_{\dot{a}}\delta^b_a D\right)$.

[11]Let's consider for example the dilatation operator. The numerator of the amplitude has the delta function with dimension $D = -4$ and the factor $\langle\lambda_i, \lambda_j\rangle^4$ with $D = +4$, so that D commutes

It is possible to recast the expressions (3.16)-(3.19) in a more standard form by transforming $(\lambda_a, \tilde{\lambda}_{\dot{a}})$ to the twistor space coordinates $(\lambda_a, \mu_{\dot{a}})$ with

$$\tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial\mu^{\dot{a}}}\,, \qquad \frac{\partial}{\partial\tilde{\lambda}^{\dot{a}}} \to i\mu_{\dot{a}}\,. \tag{3.20}$$

This choice of transforming $\tilde{\lambda}$ rather than λ clearly breaks the symmetry between positive and negative helicities. This means that amplitudes with m positive and $(n-m)$ negative helicities have different descriptions in twistor space from amplitudes with m negative and $(n-m)$ positive helicities, whereas in the spinor-helicity formalism they would simply be related by complex conjugation. In some instances such amplitudes are also called with different names: for example, if $\mathcal{M}^{(n)}(-,-,+,\dots,+)$ is an MHV amplitude, one calls *googly* the "dual" amplitude $\mathcal{M}^{(n)}(+,+,-,\dots,-)$.

In terms of the twistor space coordinates, the conformal generators become all homogeneous and first order

$$\begin{aligned} J_{ab} &= \tfrac{i}{2}\left(\lambda_a\tfrac{\partial}{\partial\lambda^b} + \lambda_b\tfrac{\partial}{\partial\lambda^a}\right)\,, \qquad \tilde{J}_{\dot{a}\dot{b}} = \tfrac{i}{2}\left(\mu_{\dot{a}}\tfrac{\partial}{\partial\mu^{\dot{b}}} + \mu_{\dot{b}}\tfrac{\partial}{\partial\mu^{\dot{a}}}\right)\,, \\ P_{a\dot{a}} &= i\lambda_a\tfrac{\partial}{\partial\mu^{\dot{a}}}\,, \qquad K_{a\dot{a}} = i\mu_{\dot{a}}\tfrac{\partial}{\partial\tilde{\lambda}^a}\,, \qquad D = \tfrac{i}{2}\left(\lambda^a\tfrac{\partial}{\partial\lambda^a} - \mu^{\dot{a}}\tfrac{\partial}{\partial\mu^{\dot{a}}} + 2\right)\,. \end{aligned} \tag{3.21}$$

This is a linear realization of the conformal group in Minkowski space $SO(4,2) \simeq SU(2,2)$. The group $SU(2,2)$, or its complexification $SL(4,\mathbb{C})$, has an obvious 4 dimensional representation, which is generated by 15 traceless 4×4 matrices corresponding to the 15 operators in the formula above, and which acts on the coordinates $Z^I = (\lambda^a, \mu^{\dot{a}})$ (with $I = 1,\dots,4$) spanning the *twistor space* $\mathbb{T} = \mathbb{C}^4$ [104].[12]

---

with the numerator. The denominator has, for each particle, a $D_k = -2$ coming from the second power of λ_k which is cancelled by the constant in the definition (3.19).

[12] In $(++--)$ signature the conformal group is $SO(3,3) \simeq SL(4,\mathbb{R})$, the coordinates Z^I are real and $\mathbb{T} = \mathbb{R}^4$. In Euclidean signature the conformal group is $SO(5,1) \simeq SU^*(4)$, the non-compact

The homogeneity condition (3.12) reads in these coordinates

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} + \mu_i^{\dot{a}} \frac{\partial}{\partial \mu_i^{\dot{a}}}\right) \tilde{A}^{(n)} = -(2h_i + 2)\tilde{A}^{(n)}\,, \tag{3.22}$$

where $\tilde{A}^{(n)}$ is the amplitude expressed in terms of Z^I. This conditions implies that scattering amplitudes in twistor space are homogeneous functions of the coordinates Z_i^I of degree $-2h_i - 2$. Notice that λ and μ transform in the same way under (3.8), so that we are allowed to identify two sets of Z^I that differ by a rescaling $Z^I \to uZ^I$, with $u \in \mathbb{C}^*$, and throw away the point $Z^I = 0$. It is then natural to consider λ and μ as living in the projectivization of the twistor space $\mathbb{PT}$, namely $\mathbb{CP}^3$ in Lorentzian signature or $\mathbb{RP}^3$ in $(++--)$ signature.[13] The Z^I are called *homogeneous* coordinates. On $\mathbb{PT}$ there is a natural volume form $\Omega = \epsilon_{IJKL} Z^I dZ^J dZ^K dZ^L$ of degree four. The projectivization of the space $\mathbb{T} \to \mathbb{PT}$ is the analog of quotienting by the gauge group the original theory expressed in terms of polarization vectors.

The concrete way to enforce the transformation (3.20) at the level of the scattering amplitudes is to perform a Fourier transform. This is best understood if one Wick rotates first from the $(+---)$ to the $(++--)$ signature, where $\tilde{\lambda}$ is real (see the discussion in the footnote 2).[14] Then one has for a generic function $f(\tilde{\lambda})$

$$\tilde{f}(\mu) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2}\, e^{i[\mu,\tilde{\lambda}]} f(\tilde{\lambda})\,, \tag{3.23}$$

version of $SU(4)$, and the twistor space is again a copy of $\mathbb{C}^4$.

[13]The space $\mathbb{CP}^N$ is the N-dimensional complex space spanned by $N+1$ complex coordinates Z^I, not all zero, subject to the identification $\{Z^I\} \simeq \{uZ^I\}$, with $u \in \mathbb{C}^*$. For $N = 1$ it is just the ordinary 2-sphere. The real projective space $\mathbb{RP}^3$ is a 3-sphere with antipodal points identified, $S^3/\mathbb{Z}_2$.

[14]This is allowed when one considers only tree level amplitudes.

and for a scattering amplitude

$$\tilde{A}^{(n)}(\lambda_i, \mu_i, h_i) = \int \prod_{k=1}^{n} \frac{d^2\tilde{\lambda}_k}{(2\pi)^2}\, e^{i[\mu_k, \tilde{\lambda}_k]} A^{(n)}(\lambda_i, \tilde{\lambda}_i, h_i)\,. \tag{3.24}$$

In Lorentzian signature, where $\tilde{\lambda}$ is complex, the integrals in eqs. (3.23) and (3.24) can be sometimes interpreted as contour integrals, if a suitable contour exists. The rigorous approach to defining the twistor transform in this case relies on the more formal methods of sheaf cohomology [105].

More details on the geometry of twistor space and on the twistor transform can be found in [106]-[108]. For a more introductory account of these topics see chapter 33 of [109].

3.2.4 Localization of amplitudes in twistor space

After performing the Fourier transform (3.24), the n external gluons of a scattering amplitude, defined by λ_i and $\tilde{\lambda}_i$, are mapped to n points P_i in the projective twistor space. The homogeneous coordinates of these points are $Z_i^I = (\lambda_a^i, \tilde{\lambda}_{\dot{a}}^i)$. From an empirical study of some simple cases, it turns out that these are very special sets of points. The claim is that if the P_i are randomly distributed, then the corresponding amplitude will vanish. In order to have a non-zero amplitude the P_i must be localized on specific subloci of the space. These subloci are algebraic curves in $\mathbb{PT}$.[15] The conjecture put forward in [96] is that the degree d and genus g of these curves are

[15] We work mostly in the $(++--)$ signature, where the transform (3.24) has a more natural interpretation and the scattering amplitudes are ordinary functions. Then an algebraic curve Σ on $\mathbb{RP}^3$ is defined as the zero set of a collection of polynomial equations with real coefficients in the real homogeneous coordinates Z^I. The simplest type of algebraic curve are *complete intersections*, defined as the zero set of two homogeneous polynomials $F(Z^I) = 0$ and $G(Z^I) = 0$. If the degrees of F and G are d_1 and d_2, then the degree of the curve is $d = d_1\, d_2$. More general cases, as for example the *twisted cubic*, are defined in terms of more than two polynomials [110].

given by

$$d = q - 1 + l\,, \qquad g \leq l\,, \tag{3.25}$$

where q is the number of external gluons with negative helicity, and l is the number of loops in the amplitude.

The easiest example to study are the MHV amplitudes (3.15) (considering only one cyclic ordering and neglecting the color factor G)

$$A^{(n)}(i^-, j^-) = ig^{n-2}(2\pi)^4\delta^4\left(\sum_{k=1}^{n} \lambda_a^k \tilde{\lambda}_{\dot{a}}^k\right) \frac{\langle \lambda_i, \lambda_j \rangle^4}{\prod_{k=1}^{n} \langle \lambda_k, \lambda_{k+1} \rangle}\,. \tag{3.26}$$

Rewriting the delta function in integral form

$$(2\pi)^4\delta^4\left(\sum_{k=1}^{n} \lambda_a^k \tilde{\lambda}_{\dot{a}}^k\right) = \int d^4x\, e^{ix_{a\dot{a}}\sum_{k=1}^{n} \lambda_k^a \tilde{\lambda}_k^{\dot{a}}}\,, \tag{3.27}$$

it is easy to perform the Fourier transform

$$\int \prod_{k=1}^{n} \frac{d^2\tilde{\lambda}_k}{(2\pi)^2}\, e^{i[\mu_k, \tilde{\lambda}_k]} \int d^4x\, e^{ix_{a\dot{a}}\sum_{k=1}^{n} \lambda_k^a \tilde{\lambda}_k^{\dot{a}}} \frac{\langle \lambda_i, \lambda_j \rangle^4}{\prod_{k=1}^{n} \langle \lambda_k, \lambda_{k+1} \rangle}\,, \tag{3.28}$$

and obtain [111]

$$\tilde{A}^{(n)}(i^-, j^-) = ig^{n-2} \int d^4x \prod_{k=1}^{n} \delta^2\left(\mu_{k,\dot{a}} + x_{a\dot{a}}\lambda_k^a\right) \frac{\langle \lambda_i, \lambda_j \rangle^4}{\prod_{k=1}^{n} \langle \lambda_k, \lambda_{k+1} \rangle}\,. \tag{3.29}$$

This result depends crucially on (3.15) being holomorphic, so that the integrals on the $\tilde{\lambda}_k$ only act on the exponential factors giving a product of delta functions.

The last formula is interpreted by saying that, for a given *modulus* $x_{a\dot{a}}$, the MHV amplitude (3.15) localizes in twistor space on the locus given by the zero set of the argument of the delta function

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0\,, \qquad \dot{a} = 1, 2\,. \tag{3.30}$$

These equations represent a complete intersection of two linear polynomials, the total degree of the curve being $d = 1$ and the genus $g = \frac{(d-1)(d-2)}{2}$ being zero. This curve represents a straight line in $\mathbb{RP}^3$ along which the n points P_i are localized, and the integral in x is an integral over the moduli space of all such lines. This is consistent with the conjecture (3.25), since for a tree level MHV amplitude one has $q = 2$ and $l = 0$. In the complex case the conditions (3.30) would just give $\mathbb{CP}^1$ (take, for example, $x_{a\dot{a}} = 0$, then one has a curve spanned by (λ^1, λ^2) which are just the homogeneous coordinates on $\mathbb{CP}^1$). This is an example of *holomorphic curve* in $\mathbb{CP}^3$, whose area, computed with the volume form Ω introduce above, is equal to $2\pi d$. In this case $d = 1$ and therefore this is a curve with minimal area among all non-trivial holomorphic curves, and it is associated to the minimal non-vanishing Yang-Mills amplitudes, the MHV amplitudes.

Assuming that eq. (3.25) is true, one can also understand from a different perspective why amplitudes with all and all but one gluons of the same helicity vanish: if $q = 0$ and $l = 0$ one would obtain a curve of degree $d = -1$, which does not exist, whereas for $q = 1$ and $l = 0$ one would obtain a point. The gluons would then be all coincident in twistor space, meaning $\lambda_i = \lambda_j$ and $p_i \cdot p_j = 0$ for all i and j. For $n \geq 4$ this would not allow the definition of non-trivial kinematic invariants, implying a vanishing amplitude.

Googly amplitudes A tree level googly amplitude (with, for example, $n = 5$ and $q = 3$) would give a degree 2 curve in twistor space. In this case, being the amplitude anti-holomorphic rather than holomorphic as in eq. (3.15), it is difficult to perform the Fourier transform (3.28).[16] It is anyway possible to bypass doing the Fourier transform, observing that the expected configuration of 5 points in $\mathbb{RP}^3$ should be given by the complete intersection

$$\sum_{I=1}^{4} a_I Z^I = 0\,, \qquad \sum_{I,J=1}^{4} b_{IJ} Z^I Z^J = 0\,, \qquad a_I, b_{IJ} \in \mathbb{R}\,. \tag{3.31}$$

This defines the most generic curve with $d = d_1 d_2 = 2$ and $g = \frac{(d-1)(d-2)}{2} = 0$. Assuming, without loss of generality, $a_4 \neq 0$ this reduces to the equation for a conic section in $\mathbb{RP}^2$ [17]

$$\sum_{I,J=1}^{3} c_{IJ} Z^I Z^J = 0\,, \qquad c_{IJ} \in \mathbb{R}\,. \tag{3.32}$$

To verify the conjecture in this case is then sufficient to prove that the 5 points are contained *i)* in a common $\mathbb{RP}^2$ and *ii)* in a common conic section of this space. The condition *ii)* is trivial since it is always possible to choose the 6 coefficients $c_{IJ} = c_{JI}$ such that the 5 equations $\sum_{I,J=1}^{3} c_{IJ} Z_i^I Z_i^J = 0$ (with $i = 1, \ldots, 5$) are satisfied. To verify *i)* one observes that, given 4 points $Q_\sigma \in \mathbb{RP}^3$ with coordinates Z_σ^I ($\sigma = 1, \ldots, 4$) they are contained in a common $\mathbb{RP}^2$ if and only if the Z_σ^I are linearly dependent, that is if and only if $K \equiv \det Z_\sigma^I = \epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$. On has then to prove that

$$K_{ijkl}(\lambda, \mu)\, \tilde{A}^{(5)} = 0\,, \tag{3.33}$$

[16] This computation would of corse be exactly as in (3.28) in the "dual" twistor space obtained by transforming $(\lambda, \tilde{\lambda}) \to (\tilde{\mu}, \tilde{\lambda})$.

[17] That this represents a conic section can be easily seen in the affine coordinates $x = Z^2/Z^1$ and $y = Z^3/Z^1$ defined in the patch where $Z^1 \neq 0$. In this coordinates eq. (3.32) becomes a quadratic equation in $\mathbb{R}^2$.

where $K_{ijkl}(\lambda,\mu)$ indicates the operator K specialized to any 4 of the 5 points P_i associated to the external gluons. Formally transforming back to momentum space $Z^I = (\lambda_a, -i\partial/\partial\tilde{\lambda}^{\dot{a}})$ this condition becomes a differential equation

$$K_{ijkl}\left(\lambda, -i\frac{\partial}{\partial\tilde{\lambda}}\right) A^{(5)} = \frac{1}{4}\sum_{\sigma_1,\ldots,\sigma_4} \epsilon_{\sigma_1\cdots\sigma_4}\langle\lambda_{\sigma_1},\lambda_{\sigma_2}\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial^2}{\partial\tilde{\lambda}^{\dot{a}}_{\sigma_3}\partial\tilde{\lambda}^{\dot{b}}_{\sigma_4}}A^{(5)} = 0\,. \tag{3.34}$$

This can be explicitly verified to hold (for details see [96]). We will later apply this same trick in proving localization for a googly gravity amplitude with 5 external gravitons.

3.2.5 Supersymmetric extension

So far supersymmetry has not played any role, since we have only considered tree level amplitudes with external gluons. Before introducing the string interpretation of the localization of amplitudes in twistor space, it is however necessary to extend the previous discussion to particles carrying supersymmetric quantum numbers. In particular we will be interested in $\mathcal{N}=4$ super Yang-Mills theory.

In addition to the commuting spinors λ_a and $\tilde{\lambda}_{\dot{a}}$, one describes each external particle by a spinless Grassmann-odd variable η_A, with $A = 1,\ldots,4$, of dimension zero and transforming in the **4** of the R-symmetry group $SU(4)_R$. The helicity operator is then

$$h = 1 - \frac{1}{2}\sum_{A=1}^{4}\eta_A\frac{\partial}{\partial\eta_A}\,. \tag{3.35}$$

This operator counts how many η are present in an amplitude, and a term of k-th order in η_i for some i describes a process in which the i-th particle has helicity $1-k/2$.

Amplitudes in the supersymmetric case depend also on the η variable of each particle

$$A^{(n)} = A^{(n)}(\lambda_1, \tilde{\lambda}_1, \eta_1, h_1; \ldots; \lambda_n, \tilde{\lambda}_n, \eta_n, h_n)\,, \tag{3.36}$$

and obey the homogeneity condition

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \tilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}} - \eta_i^A \frac{\partial}{\partial \eta_i^A} + 2\right) A^{(n)} = 0\,. \tag{3.37}$$

The (15) R-symmetry and the (32) fermionic generators of the superconformal group $PSU(2,2|4)$ of $\mathcal{N}=4$ super Yang-Mills read in terms of λ, $\tilde{\lambda}$, and η

$$\begin{gathered} R^A_B = \eta_A \tfrac{\partial}{\partial \eta_B} - \tfrac{1}{4}\delta^A_B \eta_C \tfrac{\partial}{\partial \eta_C}\,, \\ \tilde{\lambda}^{\dot{a}} \tfrac{\partial}{\partial \eta_A}\,, \qquad \eta_A \tfrac{\partial}{\partial \tilde{\lambda}^{\dot{a}}}\,, \qquad \lambda^a \eta_A\,, \qquad \tfrac{\partial^2}{\partial \lambda^a \partial \eta_A}\,, \end{gathered} \tag{3.38}$$

and are linearized by the transformation (3.20) along with

$$\eta_A \to i\frac{\partial}{\partial \psi^A}\,, \qquad \frac{\partial}{\partial \eta_A} \to i\psi^A\,. \tag{3.39}$$

The twistor space is now a supermanifold spanned by the four bosonic coordinates Z^I and their four fermionic counterparts ψ^A. This space is therefore $\mathbb{C}^{4|4}$ or, depending on the signature, $\mathbb{R}^{4|4}$.

The homogeneity condition for the amplitudes becomes

$$\left(Z_i^I \frac{\partial}{\partial Z_i^I} + \psi_i^A \frac{\partial}{\partial \psi_i^A}\right) \tilde{A}^{(n)}(Z_i; \psi_i) = 0\,, \tag{3.40}$$

suggesting a natural projectivization of the space to $\mathbb{CP}^{3|4}$ or $\mathbb{RP}^{3|4}$. This means that Z^I and ψ^A are subject to the identification $\{Z^I, \psi^A\} \simeq \{uZ^I, u\psi^A\}$, with $u \in \mathbb{C}^*$, and provide a linear realization of $PSU(4|4)$ on $\mathbb{CP}^{3|4}$.[18] The volume form is now $\Omega_0 = dZ^1 dZ^2 dZ^3 dZ^4 d\psi^1 d\psi^2 d\psi^3 d\psi^4$.

The super MHV amplitudes in the spinor-helicity formalism are given by (suppressing again the color factor G and considering only one ordering of the external legs)

$$A^{(n)} = ig^{n-2}(2\pi)^4 \delta^4 \left(\sum_{i=1}^{n} \lambda_a^i \tilde{\lambda}_{\dot{a}}^i \right) \delta^8 \left(\sum_{j=1}^{n} \lambda_b^j \eta_A^j \right) \prod_{k=1}^{n} \frac{1}{\langle \lambda_k, \lambda_{k+1} \rangle}\,, \tag{3.41}$$

and, after the super Fourier transform to twistor space, become

$$\tilde{A}^{(n)} = ig^{n-2} \int d^4x\, d^8\theta \prod_{i=1}^{n} \delta^2 \left(\mu_{i,\dot{a}} + x_{a\dot{a}} \lambda_i^a \right) \delta^4 \left(\psi_i^A + \theta_a^A \lambda_i^a \right) \prod_{k=1}^{n} \frac{1}{\langle \lambda_k, \lambda_{k+1} \rangle}\,, \tag{3.42}$$

where θ_a^A are fermionic moduli associated to η_A. The interpretation of this result is very similar to the bosonic case. Now the amplitudes are localized on the sublocus of the supertwistor space defined by the equations

$$\mu_{\dot{a}} + x_{a\dot{a}} \lambda^a = 0\,, \qquad \psi^A + \theta_a^A \lambda^a = 0\,. \tag{3.43}$$

3.3 The B-model on $\mathbb{CP}^{3|4}$ and the twistor string

We now move on to illustrating the string theory interpretation of these result, as proposed in [96].[19]

[18] Recall that this is the $\mathcal{N} = 4$ superconformal group. In chapter 10 we will use this observation to orbifold the twistor string, and reduce the amount of supersymmetry of the twistor construction.

[19] Interesting alternatives to Witten's construction have been put forward in [112] starting from conventional open strings propagating in $\mathbb{CP}^{3|4}$, in [113], where a mirror symmetric A-model version is considered, and in [114], where ADHM twistors are introduced. Conformal supergravity has also been studied in this approach in [115][116].

We start by motivating the appearance of one of the main ingredients of the proposal, namely the twistor space $\mathbb{CP}^{3|4}$. We have seen above that this supermanifold arises naturally as the space parameterized by the coordinates Z^I and ψ^A, with $I, A = 1, \ldots, 4$. There are other features of this space that are crucial in the following construction and that in large part depend on the fact that the number of fermionic dimension is exactly 4. Most importantly, a $\mathbb{CP}^{3|M}$ is Calabi-Yau if and only if $M = 4$. This can be seen[20] by considering $\mathbb{CP}^{3|M}$ as the sublocus of $\mathbb{C}^{4|M}$ defined by

$$\left(\sum_{I=1}^{4} \left| Z^I \right|^2 + \sum_{A=1}^{M} \left| \psi^A \right|^2 = r \right) \Big/ U(1) \tag{3.44}$$

where r is a positive constant playing the role of a Kähler class and $U(1)$ is a phase transformation acting as $\{Z^I, \psi^A\} \to e^{i\alpha}\{Z^I, \psi^A\}$. The holomorphic measure on $\mathbb{C}^{4|M}$, $\Omega_0 = dZ^1 \cdots dZ^4 d\psi^1 \cdots d\psi^M$, is invariant under this $U(1)$ only if it contains the same number of bosonic and fermionic coordinates,[21] so that $M = 4$. The measure Ω_0 descends to a globally defined holomorphic $(3, 0|4)$-form on $\mathbb{CP}^{3|4}$

$$\Omega = \frac{1}{(4!)^2} \epsilon_{IJKL} \epsilon_{ABCD} Z^I dZ^J dZ^K dZ^L d\psi^A d\psi^B d\psi^C d\psi^D \,, \tag{3.45}$$

which is also $SU(2, 2|4)$ invariant, hinting at a possible relation with $\mathcal{N} = 4$ SYM.[22] This ensures that the space is indeed Calabi-Yau and allows for the definition of a topological B-twist of an $\mathcal{N} = (2, 2)$ world-sheet σ-model in the fields Z^I and ψ^A with $\mathbb{CP}^{3|4}$ as target [117].

[20] An alternative way to prove this is of course by checking the Ricci flatness of the super Fubini-Study metric.

[21] Recall that if a Grassmann-odd field ψ transforms as $\psi \to e^{1\alpha}\psi$, then $d\psi \to e^{-i\alpha} d\psi$. This can be understood, for example, from the definition of Grassmann-odd integration, $\int d\psi \, \psi = 1$.

[22] This can be seen as another reason why $\mathcal{N} = 4$ SYM is special: only for $\mathcal{N} = 4$, *i.e.* for $M = 4$ in the notation above, there exists a topological B-model with a twistor target space.

Aiming at a connection with a Yang-Mills theory, we are interested in the open string sector of this B-model [118]. This sector is defined by a BRST invariant boundary condition which is given by a stack of N D5-branes. These are *almost space-filling* branes placed at $\bar{\psi}_{\bar{A}} = 0$.[23] We will see that the selfdual part of $\mathcal{N} = 4$ SYM with gauge group $U(N)$ will be reproduced by the world-volume action of these branes.

The world-volume action is a holomorphic super Chern-Simons theory [118]

$$S = \int_{D5} \Omega \wedge \mathrm{Tr}\left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) , \tag{3.46}$$

where $\mathcal{A} = \mathcal{A}_{\bar{I}} d\bar{Z}^{\bar{I}}$ is an antiholomorphic $(0,1)$-form[24] with values in the adjoint representation of $U(N)$ and Ω is the volume form (3.45), so that the integrand is a $(3,3)$-form which we can integrate over the target space. The superfield expansion of this holomorphic connection $\mathcal{A}$ reads

$$\begin{aligned} \mathcal{A}(z, \bar{z}, \psi) &= A(z, \bar{z}) + \psi^A \chi_A(z, \bar{z}) + \frac{1}{2!} \psi^A \psi^B \phi_{AB}(z, \bar{z}) \\ &\quad + \frac{1}{3!} \epsilon_{ABCD} \psi^A \psi^B \psi^C \tilde{\chi}^D(z, \bar{z}) + \frac{1}{4!} \epsilon_{ABCD} \psi^A \psi^B \psi^C \psi^D G(z, \bar{z}) \, . \end{aligned} \tag{3.47}$$

It turns out that the components of $\mathcal{A}$ are charged under the symmetry

$$\Sigma : \qquad Z^I \rightarrow Z^I \, , \qquad \psi^A \rightarrow e^{i\beta} \psi^A \tag{3.48}$$

as $\Sigma(A, \chi, \phi, \tilde{\chi}, G) = (0, -1, -2, -3, -4)$. This is an anomalous symmetry since it

[23]This almost space-filling requirement is dictated by the connection with the $\mathcal{N} = 4$ gauge multiplet, as will become clear presently. Moreover, on a more formal level, locating the brane at $\bar{\psi}_{\bar{A}} = 0$ also bypasses some problems related to the definition of differential forms on supermanifolds.

[24]In general the expansion of $\mathcal{A}$ could contain a piece $\mathcal{A}_{\bar{A}} d\bar{\psi}^{\bar{A}}$, but this drops out here because of the almost space-filling condition.

does not leave Ω invariant. It originates from the fact that the full R-symmetry group in twistor space is not $SU(4)_R$ but $U(4)_R$. This discrepancy between the R-symmetry groups of $\mathbb{CP}^{3|4}$ and $\mathcal{N}=4$ SYM causes the holomorphic Chern-Simons action (3.46) to only reproduce the selfdual part of the Yang-Mills action. To see this one can integrate out the fermionic coordinates to get

$$\begin{aligned} S &= \int_{\mathbb{CP}^3} \Omega' \wedge \mathrm{Tr}\Big[G\wedge(\bar{\partial}A + A\wedge A) + \tilde{\chi}^A \wedge \bar{D}\chi_A \\ &\qquad + \frac{1}{4}\epsilon^{ABCD}\phi_{AB}\wedge\bar{D}\phi_{CD} + \frac{1}{2}\epsilon^{ABCD}\chi_A\wedge\chi_B\wedge\phi_{CD}\Big], \end{aligned} \tag{3.49}$$

where Ω' is the bosonic reduction of Ω. After performing a Penrose transform [104], the twistor space fields with charge Σ are mapped into space-time fields of helicity $h = 1-\Sigma/2$, and eq. (3.47) almost[25] yields the field content of the $\mathcal{N}=4$ vector multiplet. In particular, A corresponds to an anti-selfdual gauge field of helicity $+1$, χ_A to a positive chirality spinor, ψ_{AB} to six real scalars, $\tilde{\chi}^A$ to a negative chirality spinor, and G to a selfdual 2-form of helicity -1. Then the action (3.49) reproduces the selfdual truncation of the $\mathcal{N}=4$ SYM action [119]

$$\begin{aligned} S &= \int d^4x\, \mathrm{Tr}\Big[\frac{1}{2}G^{ab}F_{ab} + \tilde{\chi}^{Aa}D_{a\dot{a}}\chi^{\dot{a}}_A \\ &\qquad + \frac{1}{8}\epsilon^{ABCD}\phi_{AB}D_{a\dot{a}}D^{a\dot{a}}\phi_{CD} + \frac{1}{4}\epsilon^{ABCD}\psi_{AB}\chi^{\dot{a}}_A\chi_{D\dot{a}}\Big]. \end{aligned} \tag{3.50}$$

This action has charge $\Sigma=-4$. The full $\mathcal{N}=4$ SYM action has also a term with $\Sigma=-8$, which can be introduced in this construction by considering non-perturbative instanton corrections.

[25]The differences are that A is anti-selfdual and G is a 2-form.

3.3.1 The D-instanton expansion

The instanton corrections needed to reproduce the anti-seldual completion of the action (3.49) come from Euclidean D1-branes wrapping holomorphic curves C_0 in $\mathbb{CP}^{3|4}$. The moduli characterizing these instantons are a $U(1)$ gauge field defined on the D1-brane world-volume and the position of C_0 inside $\mathbb{CP}^{3|4}$. To reproduce tree level amplitudes one just needs D1-branes wrapping genus zero curves, as seen from eq. (3.25). For these curves the $U(1)$ field does not play any relevant role, and one just needs to integrate the effective action of the instantons over the position of C_0, namely over the moduli space of holomorphic curves in twistor space.

Here we only consider the simplest case of genus $g = 0$ and degree $d = 1$.[26] We recall that the explicit map is

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0\,, \qquad \psi^A + \theta^A_a \lambda^a = 0\,. \tag{3.51}$$

Here the coordinates of $\mathbb{CP}^3$ are decomposed as $Z^I = (\lambda^a, \mu_{\dot{a}})$ and $x_{a\dot{a}}$, θ^A_a are the moduli of the embedding. In [96] the D1-instanton is initially placed at $\psi^A = 0$ and the dependence on the fermionic coordinates is then restored through an integration over the moduli space.

Tree level gauge theory scattering amplitudes are computed by considering the effective action for the *D1-D5 strings*

$$I_{D1-D5} = \int_{D1} dz\, \beta \bar{D} \alpha\,, \tag{3.52}$$

where α and β are fermions which carry respectively fundamental and anti-fundamental

[26] This is, as already mentioned, the case of tree level scattering amplitudes. Amplitudes with l loops receive contributions also from curves of genus $g \leq l$ according to the conjecture (3.25).

gauge group indices. They correspond to strings stretching from the D1 to the D5-brane and viceversa. The covariant derivative is $\bar{D} = \bar{\partial} + \mathcal{A}$. The action (3.52) contains the interaction term

$$\Delta I_{D1-D5} = \int_{D1} \mathrm{Tr}\, J\mathcal{A} = \int_{D1} J^a{}_b \mathcal{A}^b{}_a \,, \tag{3.53}$$

where $J^a{}_b = \alpha^a \beta_b$. Scattering amplitudes are obtained by taking correlation functions of the currents J's in the background of the superfield $\mathcal{A}$ and integrating them over the moduli space of a D1-instanton of appropriate degree. This degree is determined by the sum over the Σ symmetry (3.48) charges of the n external states

$$d = -\frac{1}{4}\sum_{i=1}^{n} \Sigma_i - 1\,. \tag{3.54}$$

In the particular case of external gluons, this corresponds to $d = q-1$, where q is the number of negative helicity gluons. Explicitly, when $d = 1$ one has for the n-point scattering amplitude

$$A^{(n)} = \int d^8\theta\, w_1 \ldots w_n \langle J^{a_1}_{\;1} \ldots J^{a_n}_{\;n} \rangle\,, \tag{3.55}$$

where the w_i are the wave functions of the external states and are given essentially by the coefficient of that state in the superfield expansion. Explicit amplitudes computations making use of this prescription can be found in [96][120].

The coupling constant

We now discuss the issue of how the coupling constant may arise in the theory. From a four-dimensional field theoretical point of view, the completion of the selfdual Yang-

Mills action $I_{SD} = \int d^4x \,\mathrm{Tr}\,(GF)$ is given by

$$I_{YM} = \int d^4x \,\mathrm{Tr}\left(GF - \frac{g_{YM}^2}{2}G^2\right). \tag{3.56}$$

Therefore, in the topological B-model one expects the coupling constant to originate from the D1-instanton expansion. This is rather surprising since now the YM perturbative coupling seems to come from non-perturbative sectors of the theory. One natural way to introduce a free parameter in the amplitude (3.55) is to weigh it by a factor $(e^{-I_{D1}})^d$, where I_{D1} is the action for a D1-instanton of degree $d = 1$. This has been already remarked in [96]. Another way to achieve this is to consider the coupling of the D1 to the closed sector of the B-model. This was first realized in [115] where a new field b, a (1,1) form in twistor space, was introduced. It has a minimal coupling to the D1 world-volume

$$I_b = \int_{D1} b_{I\bar{J}}\, dZ^I \wedge d\bar{Z}^{\bar{J}}. \tag{3.57}$$

This field is not present in the perturbative analysis of the B-model. The necessity of it was also recently rediscussed in [121] as a non-perturbative correction to Kodaira-Spencer theory [122]. For a D1 sitting at (x, θ) in the moduli space the coupling (3.57) directly defines the conformal supergravity superfield $\mathcal{W}(x,\theta) = \int_{D1_{(x,\theta)}} b$. The lowest component can be interpreted as a dilaton φ. As a consequence of the coupling (3.57), in a vacuum with expectation value $\langle\varphi\rangle = c$, an amplitude will be weighted by a factor $(e^{-c})^d$. This is reminiscent of ordinary string theory where the coupling constant comes from the dilaton expectation value.

In summary, we assume the contribution of the D1-instanton to an amplitude to be equal to $(g^2)^d$, where g^2 might come from $e^{-I_{D1}}$ or e^{-c}.

To get the standard normalization of the scattering amplitudes, one also needs to rescale each component of the superfield $\mathcal{A}$ by a factor $g^{1+\frac{1}{2}\Sigma}$, where Σ is the charge under the symmetry (3.48). For instance, A goes to gA, χ to $\sqrt{g}\chi$, and so on. In the end, the overall coupling constant in a tree-level n-point amplitude is

$$\left(\prod_{i=1}^{n} g^{1+\frac{1}{2}\Sigma_i}\right)(g^2)^{-\frac{1}{4}\sum_i \Sigma_i - 1} = g^{n-2}\,. \tag{3.58}$$

3.4 MHV decomposition of amplitudes

A priori one would expect a tree level Yang-Mills amplitude with q negative helicity gluons to receive contributions not only from $d = q-1$ genus zero curves but also from all possible decompositions in disconnected curves C_i of degree d_i such that $\sum_i d_i = q-1$.

An explicit calculation of the connected contribution to all googly amplitudes $\mathcal{M}^{(n)}(+,+,-,\ldots,-)$ was performed in [120] by integrating over the moduli space of connected curves with genus zero and degree 2. Surprisingly the result correctly reproduces the previously known amplitudes *without* the need to include any disconnected configuration. On the other hand, Cachazo, Svrček, and Witten considered in [123] the limit of totally disconnected configurations, that is $q-1$ curves of degree 1, and, quite amazingly, this is also enough to reproduce all the googly amplitudes [124] and likely all the tree level Yang-Mills amplitudes.

It was later on proved in [125] that in fact the connected and disconnected prescriptions are equivalent, and one can in principle use either one of them to compute scattering amplitudes. It is nonetheless drastically more convenient to use the disconnected (or CSW) prescription.[27] This formalism has been successfully extended in recent years to a variety of amplitude computations, both at tree level[28] and loops.[29] One attempt to extend it to graviton amplitudes will be the focus of chapter 8 of this dissertation [7].[30]

The basic idea of the CSW formalism is that an interaction point in Minkowski space is mapped in twistor space to a line [104], *i.e.* a linearly embedded copy of $\mathbb{CP}^1$, and so is an MHV amplitude, as is explained above. Then one is suggested to identify local vertices in Feynman diagrams with MHV amplitudes. In order to do so one needs to suitably extend off-shell some of the external legs of the amplitude, which can be interpreted as propagators. Arbitrary graph can thus be decomposed in MHV vertices connected, according to the CSW prescription, by scalar propagators $1/k^2$.

The off-shell continuation to momenta $p^2 \neq 0$ is defined in the following way. Recalling that MHV amplitudes in Yang-Mills theory are holomorphic, only an "off-shell" λ_a needs to be defined. We can extract λ_a from an on-shell momentum $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ by picking an arbitrary anti-holomorphic spinor $\eta^{\dot{a}}$ and contracting it with $p_{a\dot{a}}$. This gives λ_a up to a normalization factor which scales out in amplitudes which are

[27] An explicit example of the power of this method has been given in [123], where a considerably simpler form of $\mathcal{M}^{(n)}(-,-,-,+,\ldots,+)$, previously computed in [126], was obtained.

[28] The CSW method has been further refined and developed in [127]-[135], and has been applied to tree level amplitudes containing scalars and fundamental fermions [98][136]-[138], Higgs bosons [139], electro-weak currents [140], and massive particles [141].

[29] The literature in this case is very vast. Some of the early papers dealing with the MHV decomposition of loop amplitudes are [142]-[147].

[30] Further work in this direction can be found in [148]-[151].

homogeneous in λ_a[31]

$$\lambda_a^p = \frac{p_{a\dot{a}}\eta^{\dot{a}}}{[\tilde{\lambda}^p, \eta]} \,, \tag{3.59}$$

where the label p indicates that this is an off-shell spinor associated to a momentum $p_{a\dot{a}}$.

The concrete algorithm for decomposing a tree level amplitude with v vertices is to draw all possible tree graphs of v vertices and $v-1$ propagators, assign opposite helicities to the two ends of all internal lines, and distribute the external gluons among the vertices while preserving cyclic ordering. One is then instructed to keep only the MHV graphs, which are the ones whose vertices have precisely two negative helicity gluons emanating from them.

Example The simplest example to consider is the vanishing four gluon amplitude $\mathcal{M}^{(4)}(+,-,-,-)$ [123]. This receives contributions from the two diagrams in fig. 3.1. The first graph gives (recall the expression (3.15) for MHV amplitudes)

$$\frac{\langle\lambda^2,\lambda^p\rangle^4}{\langle\lambda^1,\lambda^2\rangle\langle\lambda^2,\lambda^p\rangle\langle\lambda^p,\lambda^1\rangle}\frac{1}{p^2}\frac{\langle\lambda^3,\lambda^4\rangle^4}{\langle\lambda^3,\lambda^4\rangle\langle\lambda^4,\lambda^p\rangle\langle\lambda^p,\lambda^3\rangle} \,, \tag{3.60}$$

where $\lambda_a^p = p_{a\dot{a}}\eta^{\dot{a}}$ is the off-shell spinor associated to the momentum of the internal line $p = p_1 + p_2 = -p_3 - p_4$. The contribution of the second graph is obtained from the equation above exchanging particles 2 and 4

$$\frac{\langle\lambda^q,\lambda^4\rangle^4}{\langle\lambda^1,\lambda^q\rangle\langle\lambda^q,\lambda^4\rangle\langle\lambda^4,\lambda^1\rangle}\frac{1}{q^2}\frac{\langle\lambda^2,\lambda^3\rangle^4}{\langle\lambda^2,\lambda^3\rangle\langle\lambda^3,\lambda^q\rangle\langle\lambda^q,\lambda^2\rangle} \,. \tag{3.61}$$

[31] This is a fortunate fact since the anti-holomorphic spinor $\tilde{\lambda}_{\dot{a}}$ in the denominator is undefined. We will come back to this point in chapter 9, while discussing a possible MHV decomposition of gravity amplitudes, which, differently from Yang-Mills theory, are generically not holomorphic.

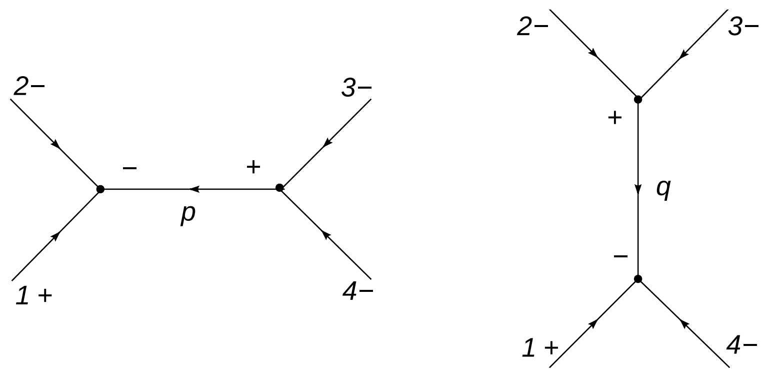


Figure 3.1: The two MHV graphs contributing to the vanishing four gluon amplitude $\mathcal{M}^{(4)}(1+, 2-, 3-, 4-)$. Notice that internal lines have opposite helicities at their two endpoints, and only MHV vertices are used to build the diagrams.

Defining $\phi_i \equiv \lambda_i^{\dot{a}} \eta_{\dot{a}}$ (the normalization factor of the definition (3.59) scales out and we neglect it), the sum of the two graphs is

$$-\frac{\phi_1^4}{\phi_1\phi_2\phi_3\phi_4}\left(\frac{\langle\lambda^3, \lambda^4\rangle}{[\tilde{\lambda}^2, \tilde{\lambda}^1]} - \frac{\langle\lambda^3, \lambda^2\rangle}{[\tilde{\lambda}^4, \tilde{\lambda}^1]}\right) , \tag{3.62}$$

which is zero because of conservation of momentum.

Part II

Studies on the AdS/CFT correspondence

Chapter 4

String thermodynamics in a plane wave background

4.1 Introduction

Understanding the finite temperature states of string theory is essential for many of its potential applications, particularly the study of black holes and early universe cosmology. One of the fascinating features exhibited by string theories is their *Hagedorn behavior*, that is the exponential growth of their densities of states with energy [152]. For their thermodynamics this leads to either a limiting, *Hagedorn temperature* beyond which an ensemble of strings cannot be heated or perhaps a phase transition to a state which is better described by degrees of freedom other than strings [153].

The existence of a Hagedorn temperature is well-established for all consistent non-interacting string theories on Minkowski space.[1] Recently, it has been noted

[1] The computation of this temperature for both the bosonic string and the superstring in flat space is reviewed in appendix A.

that the non-interacting type IIB superstring can be solved explicitly on a maximally supersymmetric plane-wave background [44][45]. This gives a background other than flat space where some of the ideas of string theory can be tested.

In this chapter we shall examine the thermodynamic states of string theory in the plane-wave background and give a derivation of the corresponding Hagedorn temperature.[2] In particular, we show that the Hagedorn temperature is a monotonically increasing function of the parameter $|f|\sqrt{\alpha'}$ where f is the Ramond-Ramond flux. In the following, we also clarify some of the issues related to the Hagedorn temperature in the limits of small and large f, and provide some comments on the interpretation of the Hagedorn behavior in the limit of Yang-Mills theory which is dual to the string theory.

4.2 Hagedorn and AdS/CFT

The string partition function in the canonical ensemble is the trace of the Boltzmann distribution

$$Z(\beta, f) \equiv e^{-\beta F(\beta,f)} = \mathrm{Tr}\left(e^{-\beta p^0}\right) , \tag{4.1}$$

where $F(\beta, f)$ is the Helmholtz free energy. Here the trace is over all physical multi-string states. The rest frame energy is given by $p^0 = \frac{1}{\sqrt{2}}(p^+ - p^-)$.

Note that, we could, as was done in [155], introduce a separate parameter for p^+ and p^- and study the theory with two parameters

$$\tilde{Z}(a, b, f) = \mathrm{Tr}\left(e^{-ap^+ + bp^-}\right) . \tag{4.2}$$

[2]For related work see also [154]-[158].

However, there is a symmetry of the theory which rescales $p^+ \to p^+/\Lambda$, $p^- \to p^-\Lambda$, and $f \to f\Lambda$, implying that $\tilde{Z}(a,b,f)$ is equal to $\tilde{Z}(\sqrt{ab}, \sqrt{ab}, f\sqrt{a/b})$. Thus, by computing $Z(\beta, f) = \tilde{Z}(\beta/\sqrt{2}, \beta/\sqrt{2}, f)$ we can deduce $\tilde{Z}(a,b,f)$ by identifying $\beta = \sqrt{2ab}$ and replacing $f \to f\sqrt{a/b}$.

For the free string theory, we can compute the Helmholtz free energy in eq. (4.1) exactly. This should then coincide with the free energy of Yang-Mills theory obtained from eq. (4.1) by taking a trace over Yang-Mills states with the momenta identified in eq. (2.40) and where the 't Hooft large N limit is taken.

In perturbation theory, the free energy would therefore be found as the sum of all orders in planar connected vacuum Feynman diagrams. However, at each order, these diagrams are proportional to N^2 and therefore diverge in the large N limit. On the other hand, the string theory free energy which we compute is not of order N^2, instead it is of order one. The reason for this discrepancy is that perturbation theory describes the deconfined phase of the gauge theory where the number of physical degrees of freedom is indeed of order N^2, and is only valid if the temperature is greater than the deconfinement transition temperature. That is not the regime described by free strings which rather exist only in the confined phase, found at temperatures below the deconfinement transition and where the number of degrees of freedom is not of order N^2 at large N, but is of order the number of color singlet operators which, at a given energy, is roughly constant with N. In fact, it is reasonable to identify the Hagedorn temperature, at which a description of the theory by free strings ceases to be meaningful, as the deconfinement transition temperature [159].

At this point, as clarification, we should note that this conformally invariant Yang-Mills theory when it is quantized on $\mathbb{R}^3 \times \mathbb{R}^1$ does not have a confining phase. It is always in a conformally invariant deconfined phase with a Coulomb-like force law for gauge theory interactions. However, the correct dual of the superstring is Yang-Mills theory with radial quantization, that is, it should be quantized on the space $S^3 \times \mathbb{R}^1$ which can be obtianed from $\mathbb{R}^3 \times \mathbb{R}^1$ by a conformal transformation. It is the energy on the space $S^3 \times \mathbb{R}^1$ which is dual to the string energy and is in fact given by the conformal dimension Δ of operators of Yang-Mills theory on the original space $\mathbb{R}^3 \times \mathbb{R}^1$. From this point of view, the discreteness of the spectrum of Δ comes from the fact that S^3 has finite volume. Further, when Δ is used as the Hamiltonian, the finite temperature Yang-Mills theory lives on the space $S^3 \times S^1$ where the time direction is Euclidean and has been periodically identified, $X^0 \sim X^0 + \beta$, with the appropriate antiperiodic boundary condition for fermions.

Even on this space, since the volume is finite, one does not expect a confinement-deconfinement phase transition when N is finite. This transition could only occur at infinite N. However, it is just the infinite N limit that must be taken to obtain strings on the plane-wave background. In this limit, the Yang-Mills theory could have a phase transition corresponding to confinement-deconfinement as the temperature is varied. An order parameter for such a phase transition is the Polyakov loop [160][161]

$$\left\langle \mathrm{Tr}\mathcal{P}e^{i\oint_{S^1} A} \right\rangle ,$$

which, in this adjoint gauge theory, transforms under a certain discrete large gauge symmetry related to confinement. There are many examples of gauge theories where this order parameter can be explicitly seen to characterize confinement [162]-[165]. For example, the one-dimensional non-Abelian coulomb gas studied in [166][167] has a deconfiment transition only at infinite N, corresponding to a re-arrangement of the distribution of eigenvalues of the unitary matrix $\mathcal{P}e^{i\oint_{S^1} A}$, analogous to that which is well known to occur at large N in unitary matrix models [168]. If, as is suggested in [159], the deconfinement and Hagedorn behaviors can be identified, the existence of a Hagedorn temperature in the string theory dual is a confirmation of the existence of a deconfinement transition in the Yang-Mills theory, at least in the planar limit which is dual to free strings.

We shall indeed find that, in the limit where the string coupling g_s is put to zero, there is a Hagedorn temperature for all finite values of the parameter f of the background, implying that the planar Yang-Mills theory indeed has a confinement-deconfinement phase transition. The string theory analysis gives the value of the transition temperature for the gauge theory.

It is interesting to contrast the situation of the plane-wave background to that in AdS/CFT before the plane wave limit is taken. In the latter case, the Hagedorn spectrum for the operator Δ appears in Yang-Mills theory as the exponentially increasing multiplicity of an infinite tower of operators which are gauge invariant traces of local products of the fields. When N is infinite, products of all sizes are independent operators. When the 't Hooft coupling λ is small and Δ of these operators deviates little from the tree level values, the number of operators with a given value of Δ can be counted [169], and it indeed grows exponentially with increasing Δ, producing a Hagedorn spectrum for Yang-Mills theory quantized on $S^3 \times \mathbb{R}^1$. Thus, we would expect large N Yang-Mills theory to have a Hagedorn temperature if λ is small enough. When λ gets large, the anomalous dimensions of operators get large and they begin to decouple from the low-lying spectrum.

At very large λ the dynamics is that of classical supergravity, perhaps with stringy corrections which are suppressed by factors of $1/\sqrt{\lambda}$. It is known that supergravity with an asymptotically AdS geometry has a Hawking-Page phase transition [170] between an AdS black hole state, which can be interpreted as the deconfined phase, and one which is AdS space with periodic euclidean time, which can be interpreted as the confined phase. Indeed, the fact that the free energy of the black hole is of order N^2, whereas in the periodic AdS space it is of order one is in line with this interpretation [13][171]. One could then speculate that the Hagedorn behavior which is seen in weakly coupled planar Yang-Mills theory evolves to the Hawking-Page transition of supergravity with periodic Euclidean time as λ goes from zero to infinity, and further that this corresponds to the deconfinement phase transition. It is also clear that the temperature where the Hawking-Page transition occurs is proportional to the radius of curvature of the AdS space, $T_H \sim R/\alpha' \sim \lambda^{1/4}$ and it actually becomes large as $\lambda \to \infty$, as we expect.

In contrast, the partition function of the limit of Yang-Mills theory which corresponds to the plane wave background would be the trace over states of the exponential of the operator

$$Z = \mathrm{Tr}\exp\left(-\frac{\beta^2}{\alpha'}\frac{(\Delta+J)}{2\beta f\sqrt{\lambda}} - \beta f\frac{\Delta-J}{2}\right) . \tag{4.3}$$

We see that the parameters indeed appear naturally in the combinations β^2/α' and βf.

4.2.1 Large f

In the limit where f is large for fixed β and λ, the states which dominate the partition sum are those with $\Delta = J$. They are just the single and multi-trace chiral primary operators, $\mathrm{Tr}\,(Z_1^J)\mathrm{Tr}\,(Z_2^J)...\mathrm{Tr}\,(Z_k^J)$, whose conformal dimensions $\Delta = \sum J_i = J$ are protected by supersymmetry. They correspond to single and multi-string states where the string is in its lowest state, the string state which is described by the supersymmetric vacuum of the worldsheet sigma model.

To find the partition functions, we can think of the number of times J_1 appears in the product of traces as the occupation number n_{J_1}, the number of strings which are in the state with quantum number J_1. J_1 can have both integers and half integers values. The contribution of this state to total J is $J_1 n_{J_1}$. We enforce Bose statistics by summing over all occupation numbers of all states, to get the partition function and the free energy

$$\begin{aligned}
Z &= e^{-\beta F} = \prod_{J=1/2,1,3/2\ldots}^{\infty} \sum_{n_J=0}^{\infty} \exp\left(-\frac{\beta n_J J}{\alpha' f\sqrt{\lambda}}\right) = \prod_{J=1/2,1,3/2\ldots}^{\infty} \frac{1}{1-e^{-\frac{\beta J}{\alpha' f\sqrt{\lambda}}}}\,, \\
F &= \frac{1}{\beta} \sum_{J=1/2,1,3/2\ldots}^{\infty} \ln\left(1-e^{-\frac{\beta J}{\alpha' f\sqrt{\lambda}}}\right) = -\frac{1}{\beta}\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{n\beta p}{2\alpha' f\sqrt{\lambda}}} \\
&= -\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{e^{\frac{n\beta}{2\alpha' f\sqrt{\lambda}}}-1} = -\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{2\alpha' f\sqrt{\lambda}}{\beta n^2} = -\frac{\pi^2\alpha' f\sqrt{\lambda}}{3\beta^2}\,, \qquad (4.4)
\end{aligned}$$

where $p = 2J$, and in the last step we have taken the large f limit. We will see that this coincides with the large f limit of the string partition function which we will find in the following section. To do this, we need to identify the infinite length of the X^- direction. We can do this by examining the quantization of P^+. J and Δ can be integers and, for fermions, half-integers, but in all cases the sum $\Delta + J$ are integers. Consequently, P^+ should be of the form $\sqrt{2}\pi \cdot \text{integer}/L$. From this we identify the infinite length in the X^9 direction as $L = 2\pi\alpha' f\sqrt{\lambda}$. Then the large f limit in eq. (4.4) is

$$F \to -\frac{\pi L}{6\beta^2}\,.$$

String interactions

One could speculate about what happens when string interactions are switched on. In the asymptotically AdS space, which is dual to Yang-Mills theory with finite J, string interactions are restored by relaxing the large N limit. This produce a cutoff of order N^2 on the number of independent traces of local operators, and therefore it should cut off the Hagedorn behavior – at least the counting of independent operators in weakly coupled Yang-Mills theory no longer produces a Hagedorn spectrum. Commensurate with this, we do not expect a deconfinement phase transition in Yang-Mills theory in the finite volume of $S^3 \times S^1$ if N is finite. This makes the prediction that interacting strings do not have a finite temperature phase transition on an asymptotically AdS space.

On the other hand, to obtain the plane-wave background, we should always take the limit $N \to \infty$. This would suggest that we always have a Hagedorn spectrum of traces of local operators, the main question being whether their quantum number $\Delta - J$ remains finite when both coupling constants, λ' and g_2, are non-zero. It is known that when $\Delta - J$ depends on g_2, it is shifted by a small amount when g_2 is small [55][57]. Thus, we can speculate that, as long as g_2 is small enough, the Hagedorn behavior indeed persists when string interactions are present.

4.2.2 Other issues

In the Yang-Mills partition function which is eq. (4.1) with the momenta (2.40), we should take $R/\sqrt{\alpha'} \to \infty$ while holding the temperature $\beta/\sqrt{\alpha'}$ fixed. The states which contribute in the trace are those which have finite $\Delta - J$. On the other hand, $\Delta + J$, and therefore both Δ and J, get arbitrarily large as $R/\sqrt{\alpha'} \to \infty$. One might question whether this limit is sensible. In the usual limit, p^+ and p^- are held constant when N is taken to infinity. Here, instead, the temperature is held constant and it is not a priori clear that holding the temperature constant and finite actually samples the states of the Yang-Mills theory which coincide with the string states. It would be interesting to find a way to check this directly. Unfortunately, the standard perturbative computation using path integrals is only valid in the deconfined phase which occurs at high temperatures, where the confined states that we find in string theory would be difficult to detect.

An important issue is the possible existence of zero modes of p^+. Any protected operator for which $J^2/N \to 0$ as $N \to \infty$ are zero modes of p^+. Some of these are just at the $p^+ = 0$ edge of the continuum spectrum and are included in our analysis. These are the operators $\mathrm{Tr}\,(Z^J)$ where J is not taken to infinity fast enough as N is taken to infinity. There are also other operators, such as the protected operators in the dilaton supermultiplet which have finite non-zero $\Delta - J$, and for which $\Delta + J$ are finite in the limit as $N \to \infty$, so that $p^+ = 0$. These could be considered as discrete zero modes of p^+, which seem to have no analog in the light-cone string spectrum. This would seem to be a mismatch between the string and Yang-Mills spectra.

There has recently been some discussion on the Hagedorn behavior of pp-wave strings [154][155], and also in discrete light cone quantization [156]. It is well known that when string theories are placed in a background electric NS B field or in a metric, the Hagedorn temperature depends on the parameters of the background [172][173][174]. Here we shall find that also the R-R flux (2.31) felt by a string in the pp-wave metric modifies the Hagedorn temperature. We shall also clarify some of the issues related to the small and large f limit of the Hagedorn temperature, providing results that, even if in qualitative agreement with those of [154][155], differ quantitatively. We shall then study the thermodynamic behavior of strings in geometries that arise in D1-D5 systems as $AdS_3 \times S^3 \times T^4$ with NS-NS and R-R 3-form backgrounds [175][46][156]. It would be intersting to rederive our results by means of a path integral procedure and then generalize them to higher genera [176].

4.3 Superstring free energy

The free energy of a gas of non-interacting superstrings is given by summing the free energies of free particles over all of the particle species in the string spectrum.[3] Each boson in the spectrum contributes

$$F_b = \frac{1}{\beta}\mathrm{Tr}\,\ln\left(1 - e^{-\beta p^0}\right) = -\sum_{n=1}^{\infty}\frac{1}{n\beta}\mathrm{Tr}\;e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)}\,, \tag{4.5}$$

where p^0 and $p^{\pm}$ are the energy and light-cone momenta of the particle. Similarly, each fermion contributes

$$F_f = -\frac{1}{\beta}\mathrm{Tr}\,\ln\left(1 + e^{-\beta p^0}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n\beta}\mathrm{Tr}\,e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)}\,. \tag{4.6}$$

We emphasize that the trace in each case is over the spectrum of single particle states, rather than multi-particle states. The total free energy is given by summing (4.5) and (4.6) over the particles which appear in the string spectrum. Because of supersymmetry, most of the string spectrum has paired fermionic and bosonic states,

[3]For a derivation of this formula see [177].

so that we can take the average of the two expressions

$$F_{susy} = -\sum_{n=1}^{\infty} \frac{1-(-1)^n}{2n\beta} \mathrm{Tr}\ e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)} . \tag{4.7}$$

However, there is one set of states in the spectrum which will turn out not to have a superpartner, they are the lowest energy excitations which are bosons and have vanishing light-cone Hamiltonian p^- and arbitrary p^+. To take these bosons into account, eq. (4.7) must be amended to read

$$F = -\sum_{n=1,\mathrm{odd}}^{\infty} \frac{1}{n\beta} \mathrm{Tr}_{\ (p^-<0)}\ e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)} - \sum_{n=1}^{\infty} \frac{1}{n\beta} \mathrm{Tr}_{\ (p^-=0)}\ e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)} . \tag{4.8}$$

Summing these operators over the spectrum of the operators p^- and p^+, which are found in light-cone quantization of the string, should yield the free energy. The last term is easily evaluated by noting that the measure for the trace over p^+ is $\frac{L}{\sqrt{2}\pi} \int_0^\infty dp^+$, where L is the (infinite) length of the 9-th dimension. We combine the odd integer sum in the last term with the first term. This removes the constraint on the spectrum in that term. Then,

$$\begin{aligned} F &= -\sum_{n=1,\mathrm{odd}}^{\infty} \frac{1}{n\beta} \mathrm{Tr}\ e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)} - \frac{L}{\pi\beta^2} \sum_{n=2,\mathrm{even}}^{\infty} \frac{1}{n^2} \\ &= -\sum_{n=1,\mathrm{odd}}^{\infty} \frac{1}{n\beta} \mathrm{Tr}\ e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)} - \frac{L\pi}{24\beta^2} . \end{aligned} \tag{4.9}$$

To proceed, we must examine the string spectrum.

The Green-Schwarz type IIB superstring can be quantized in the light-cone gauge, as reviewed in chapter 2. We recall here the explicit form of the light-cone Hamiltonian

(2.36), which is

$$
\begin{aligned}
H &\equiv -p^- \\
&= f\left(a_0^I \bar{a}_0^I + 2\bar{\theta}_0 \bar{\gamma}^- \Pi \theta_0 + 4\right) + \frac{1}{\alpha' p^+} \sum_{\mathcal{I}=1,2} \sum_{m=1}^{\infty} \sqrt{m^2 + (\alpha' p^+ f)^2} \left(a_m^{\mathcal{I}I} \bar{a}_m^{\mathcal{I}I} + \eta_m^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_m^{\mathcal{I}}\right) \\
&= f\left(N_0^B + N_0^F + 4\right) + \frac{1}{\alpha' p^+} \sum_{\mathcal{I}=1}^{2} \sum_{m=1}^{\infty} \sqrt{m^2 + (\alpha' p^+ f)^2} \left(N_{\mathcal{I}m}^B + N_{\mathcal{I}m}^F\right) . \qquad (4.10)
\end{aligned}
$$

The level matching condition $N_1 = N_2$ also has to be enforced by introducing an integration over the Lagrange multiplier τ_1. Explicitly eq. (4.9) reads

$$
\begin{aligned}
F &= -\sum_{n=1,\mathrm{odd}}^{\infty} \frac{L}{4\pi^2\alpha'} \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \prod_{I=1}^{8} \sum_{N_0^{B,I}=0}^{\infty} \sum_{n_R,n_L=0}^{4} \sum_{N_{1,2m}^{B,I}=0}^{\infty} \sum_{N_{1,2m}^{F}=0}^{8} \times \\
&\times \left[e^{2\pi i \tau_1 \sum_{m=1}^{\infty} m\left(N_{1m}^{B,I} + N_{1m}^F - N_{2m}^{B,I} - N_{2m}^F\right)} e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}} e^{-\frac{n\beta f N_0^{B,I}}{\sqrt{2}}} \binom{4}{n_R} \binom{4}{n_L} \right. \\
&\qquad \left. e^{-\frac{n\beta f}{\sqrt{2}}(-n_R + n_L + 4)} \binom{8}{N_{1m}^F} \binom{8}{N_{2m}^F} e^{-\sum_{m=1}^{\infty} R_m \left(N_{1m}^{B,I} + N_{1m}^F + N_{2m}^{B,I} + N_{2m}^F\right)} \right] - \frac{L\pi}{24\beta^2} ,
\end{aligned}
\qquad (4.11)
$$

where L is the length of the longitudinal direction, $N_0^F = -n_R + n_L$, and

$$
R_m = 2\pi\tau_2 \sqrt{m^2 + \mu^2} , \qquad \tau_2 = \frac{n\beta}{2\sqrt{2}\pi\alpha' p^+} , \qquad \mu = \alpha' p^+ f = \frac{n\beta f}{2\sqrt{2}\pi\tau_2} . \qquad (4.12)
$$

Due to the anticommutation relations of the creation/annihilation fermion operators, the degeneracy of a state with $n_{R,\,L}$ fermions is given by the binomial coefficient $\binom{4}{n_{R,\,L}}$. Analogously the occupation number $N_{\mathcal{I}m}^F$ for the fermion non-zero modes, which have eight independent components, runs from 0 to 8 and the degeneracy is given by the binomial coefficient $\left(\begin{array}{c} 8 \\ N^F_{\mathcal{I}m} \end{array}\right)$. Summing over the zero-modes, the free energy can be written as

$$
\begin{aligned}
F \;&=\; -\sum_{n=1,\mathrm{odd}}^{\infty} \frac{L}{4\pi^2\alpha'} \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \Bigg[e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}} \left(1 - e^{-\frac{n\beta f}{\sqrt{2}}}\right)^{-8} \times \\
&\times\; e^{-\frac{4n\beta f}{\sqrt{2}}} \left(1 + e^{-\frac{n\beta f}{\sqrt{2}}}\right)^4 \left(1 + e^{\frac{n\beta f}{\sqrt{2}}}\right)^4 \left| G\left(\tau_1, \tau_2, \frac{n\beta f}{\sqrt{2}2\pi\tau_2}\right) \right|^2 \Bigg] - \frac{L\pi}{24\beta^2}\,, \qquad (4.13)
\end{aligned}
$$

where G is given by

$$
G(\tau_1,\tau_2,\mu) = \prod_{I=1}^{8} \sum_{N^{B,I}_{1m}=0}^{\infty} \sum_{N^F_{1m}=0}^{8} \left(\begin{array}{c} 8 \\ N^F_{1m} \end{array}\right) e^{2\pi i\tau_1 \sum_{m=1}^{\infty} m\left(N^{B,I}_{1m}+N^F_{1m}\right)} e^{-\sum_{m=1}^{\infty} R_m\left(N^{B,I}_{1m}+N^F_{1m}\right)}\,. \tag{4.14}
$$

Performing the sums over the occupation numbers the generating function becomes

$$
G(\tau_1,\tau_2,\mu) = \prod_{m=1}^{\infty} \left(\frac{1 + e^{-2\pi\tau_2\sqrt{m^2+\mu^2} + 2\pi i\tau_1 m}}{1 - e^{-2\pi\tau_2\sqrt{m^2+\mu^2} + 2\pi i\tau_1 m}} \right)^8 , \tag{4.15}
$$

so that the free energy reads

$$
F = -\sum_{n=1,\mathrm{odd}}^{\infty} \frac{L}{4\pi^2\alpha'} \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}} \prod_{m=-\infty}^{\infty} \left(\frac{1 + e^{-2\pi\tau_2\sqrt{m^2+\mu^2} + 2\pi i\tau_1 m}}{1 - e^{-2\pi\tau_2\sqrt{m^2+\mu^2} + 2\pi i\tau_1 m}} \right)^8 - \frac{L\pi}{24\beta^2}\,. \tag{4.16}
$$

This equation is **not** in agreement with eq. (3.3) of [154], because it differs by the contribution of the zero light-cone energy mode. The limit $f \to \infty$ of eq. (4.16) can be easily computed since $G(\tau_1,\tau_2,\mu) \to 1$ in this limit so that F becomes

$$
F = -\frac{\pi L}{6\beta^2}\,, \tag{4.17}
$$

which coincides with the free energy of the dual gauge theory computed above in this limit, see eq. (4.4). Note that this is the free energy density of a gas of massless particles in two dimensions. Indeed, the lowest energy states of the string are massless chiral bosons which propagate down the two spacetime dimensional axis of the pp-wave space made of the X^+ and X^- directions. Furthermore they are chiral, in that their spectrum is composed entirely of left-moving particles. The spectrum of these particles is protected by supersymmetry, so we expect that this limit of the partition function is not corrected by string interactions.

We shall now extract information directly from eq. (4.16) instead of turning to the path integral approach as in [154]. To compute the Hagedorn temperature we need to estimate the asymptotic behavior of the product in eq. (4.16). This will be crudely estimated in this section. A more precise estimate will be obtained in the next section by using its modular transformations properties [178]. Consider the function defined by

$$Z(\tau_1, \tau_2, \mu) \equiv \prod_{m=-\infty}^{\infty} \left(\frac{1 + e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}}{1 - e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}} \right) . \tag{4.18}$$

This diverges only when τ_1, τ_2, and βf vanish, let us then consider these limits by taking first $\tau_1 = 0$, and then $\tau_2 \to 0$. For $\tau_1 = 0$ it reads

$$\begin{aligned} Z(0, \tau_2, \mu) &= \exp\left\{ \sum_{m=-\infty}^{\infty} \ln\left(\frac{1 + e^{-2\pi\tau_2\sqrt{m^2+\mu^2}}}{1 - e^{-2\pi\tau_2\sqrt{m^2+\mu^2}}} \right) \right\} \\ &= \exp\left\{ - \sum_{m=-\infty}^{\infty} \sum_{p=1}^{\infty} \left[\frac{(-1)^p}{p} - \frac{1}{p} \right] e^{-2\pi\tau_2 p\sqrt{m^2+\mu^2}} \right\} . \end{aligned} \tag{4.19}$$

Using the integral identity [179]

$$e^{-2\sqrt{ab}} \frac{1}{2}\sqrt{\frac{\pi}{a}} = \int_0^\infty e^{-at^2 - \frac{b}{t^2}} dt \,, \qquad (a, b > 0) \tag{4.20}$$

one can write

$$Z = \exp\left\{+2\sum_{m=-\infty}^{\infty}\sum_{p_{odd}=1}^{\infty}\frac{1}{p}\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2-\frac{\pi^2\tau_2^2p^2m^2}{t^2}-\frac{n^2\beta^2f^2p^2}{8t^2}}dt\right\}. \tag{4.21}$$

In the limit of interest $\tau_2 \to 0$, the sum over m may be approximated by an integral $\sum_m \simeq \int_{-\infty}^{\infty} dm$. The integration over m is Gaussian and can be readily performed. The leading behavior in the $\tau_2 \to 0$ limit then is

$$Z \simeq \exp\left\{\frac{n\beta f}{\sqrt{2}\pi\tau_2}\sum_{p=1}^{\infty}\frac{[1-(-1)^p]}{p}K_1\left(\frac{n\beta f p}{\sqrt{2}}\right)\right\}, \tag{4.22}$$

where K_1 is the modified Bessel function. Using the series expansion of the Bessel function it is easy to see that the leading term of eq. (4.22) in the limit $\beta f \to 0$ reproduces the expected flat space behavior. A more precise derivation of this result will be obtained in the next section.

4.4 Modular properties of Z

Consider the function defined by

$$Z_{a,b}(\tau_1,\tau_2,x) \equiv \prod_{m=-\infty}^{\infty}\left(1-e^{-2\pi\tau_2\sqrt{x^2+(m+b)^2}+2\pi i\tau_1(m+b)+2\pi i a}\right). \tag{4.23}$$

The partition function (4.18) is given by the ratio

$$Z\left(\tau_1,\tau_2,\frac{n\beta f}{2\pi\sqrt{2}\tau_2}\right) = \frac{Z_{\frac{1}{2},0}\left(\tau_1,\tau_2,\frac{n\beta f}{2\pi\sqrt{2}\tau_2}\right)}{Z_{0,0}\left(\tau_1,\tau_2,\frac{n\beta f}{2\pi\sqrt{2}\tau_2}\right)}. \tag{4.24}$$

It will turn out to be useful to define

$$\Delta_b(x) \equiv -\frac{1}{2\pi^2}\sum_{p=1}^{\infty}\cos(2\pi bp)\int_0^{\infty} ds\ e^{-p^2 s-\frac{\pi^2 x^2}{s}} = -\frac{x}{\pi}\sum_{p=1}^{\infty}\frac{\cos(2\pi bp)}{p}K_1\left(2\pi xp\right)\ . \tag{4.25}$$

The quantity $\Delta_b(x)$ corresponds to the zero-energy (Casimir energy) of a 2D complex scalar boson ϕ of mass m with the twisted boundary condition $\phi(\tau,\sigma+\pi) = e^{2\pi ib}\phi(\tau,\sigma)$. In the massless limit this zero energy correctly reproduces the familiar value

$$\lim_{x\to 0}\Delta_b(x) = \frac{1}{24}-\frac{1}{8}(2b-1)^2. \tag{4.26}$$

Following the appendix A of [178] it is not difficult to derive the modular property of eq. (4.23)

$$\ln Z_{a,b}(\tau_1,\tau_2,x) = \ln Z_{-b,a}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{x}{|\tau|}\right) - 2\pi\tau_2\Delta_b(x) + 2\pi\frac{\tau_2}{|\tau|^2}\Delta_a\left(\frac{x}{|\tau|}\right)\ . \tag{4.27}$$

As a consequence, the transformation properties of eq. (4.18) are

$$\begin{aligned}\ln Z\left(\tau_1,\tau_2,\frac{n\beta f}{\sqrt{2}2\pi\tau_2}\right) &= \ln Z_{0,\frac{1}{2}}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right) - \\ -\ln Z_{0,0}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right) &+ 2\pi\frac{\tau_2}{|\tau|^2}\left[\Delta_{\frac{1}{2}}\left(\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right) - \Delta_0\left(\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right)\right]\ .\end{aligned} \tag{4.28}$$

From the definition of the Casimir energy (4.25) the last two terms in the equation above read

$$\frac{n\beta f}{\sqrt{2}\pi|\tau|}\sum_{p=1}^{\infty}\frac{[1-(-1)^p]}{p}K_1\left(\frac{n\beta f p|\tau|}{\sqrt{2}\tau_2}\right)\ . \tag{4.29}$$

In the limit $\tau_1 \to 0$ and $\tau_2 \to 0$ the first two terms in eq. (4.28) behave smoothly whereas the second two give precisely the behavior found in eq. (4.22).

4.5 The Hagedorn temperature

The asymptotic value of the free energy (4.13) then is

$$F \sim \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{8\pi^2\alpha'} L \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}} \exp\left\{ \frac{8n\beta f}{\sqrt{2}\pi\tau_2} \sum_{p=1}^{\infty} \frac{[1-(-1)^p]}{p} K_1\left(\frac{n\beta f p}{\sqrt{2}}\right) \right\} . \tag{4.30}$$

The biggest value of β for which this expression diverges in the $\tau_2 \to 0$ limit is obtained by taking the $n = 1$ mode. When the exponent in the integrand of eq. (4.30) vanishes, F starts to diverge so that the Hagedorn temperature is defined by the equation

$$\frac{\beta_H^2}{4\pi\alpha'} = \frac{8\beta_H f}{\sqrt{2}\pi} \sum_{p=1}^{\infty} \frac{[1-(-1)^p]}{p} K_1\left(\frac{\beta_H f p}{\sqrt{2}}\right) . \tag{4.31}$$

Taking the derivative with respect to f one gets

$$\frac{\partial\beta_H}{\partial f} = -\frac{8\alpha'|f|\beta_H \sum_{p=1}^{\infty} [1-(-1)^p]\, K_0\left(\frac{\beta_H f p}{\sqrt{2}}\right)}{1 + 8\alpha' f^2 \sum_{p=1}^{\infty} [1-(-1)^p]\, K_0\left(\frac{\beta_H f p}{\sqrt{2}}\right)} . \tag{4.32}$$

The r.h.s. of this equation is always negative thus β_H is a decreasing function of $|f|\sqrt{\alpha'}$ and consequently T_H is an increasing function of $|f|\sqrt{\alpha'}$.

We shall now study the behavior of eq. (4.31) in the small and large f limit. For small f it is necessary to rewrite it as a power series in βf and then solve for β. This will be rigorously done in the next section and it will allow us to derive the correct result for the Hagedorn temperature at small f. For large f the behavior of eq. (4.31) is much easier to extract, and it should reproduce the dual gauge theory behavior.

4.5.1 Expansion for small f

To rewrite eq. (4.31) as a series expansion in f, we shall use the Mellin transform procedure. The series

$$S_b(x) = \sum_{p=1}^{\infty} \frac{1}{p} K_1(xp) \tag{4.33}$$

can in fact be rewritten as a power series in x by means of a Mellin transformation. The Mellin transform of $S_b(x)$ reads

$$M(s) = \int_0^{\infty} dx\, x^{s-1} S_b(x) = \sum_{p=1}^{\infty} \int_0^{\infty} dx \int_0^{\infty} \frac{dt}{4t^2} x^s\, e^{-t-\frac{x^2p^2}{4t}} . \tag{4.34}$$

Changing the integration variable x to $y = x^2p^2/(4t)$, $M(s)$ becomes

$$M(s) = \sum_{p=1}^{\infty} \int_0^{\infty} \frac{dy}{8} \left(\frac{2}{p}\right)^{s+1} y^{(s-1)/2} e^{-y} \int_0^{\infty} \frac{dt}{t^2} t^{(s+1)/2} e^{-t} . \tag{4.35}$$

The Mellin transform $M(s)$ exists provided the integrals over y and t are bounded for some $s > k$ with $k > 0$. In our case the integrals can be done for $s > 1$ and $M(s)$ is

$$M(s) = 2^{s-2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \zeta(s+1) . \tag{4.36}$$

The inversion of the Mellin transform gives back the function $S_b(x)$ and is accomplished by means of the inversion integral

$$S_b(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} ds\, M(s) x^{-s} , \tag{4.37}$$

where $C > k = 1$. The integral is well defined and to compute it we must close the contour and use the residue theorem. For this purpose it is convenient to change the argument of $\zeta(s+1)$ in the integrand as [179]

$$\zeta(s+1) = \pi^{s+1/2}\frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\zeta(-s)\,. \tag{4.38}$$

Therefore

$$S_b(x) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} ds \left(\frac{2\pi}{x}\right)^s \frac{\sqrt{\pi}}{4}\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{s-1}{2}\right)\zeta(-s)\,. \tag{4.39}$$

The contour can now be closed on the left so that the poles are at $s = 1, 0, -1, 1-2k, \ldots$ for $k = 2, 3, \ldots$ The residues can be easily computed and the result is

$$\begin{aligned} S_b(x) &= \frac{\pi^2}{6x} - \frac{\pi}{2} + \frac{x}{8}\left(1 - 2\gamma + 2\ln\frac{4\pi}{x}\right) \\ &+ \sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\left(\frac{x}{2\pi}\right)^{2k-1}\frac{\sqrt{\pi}}{2}\Gamma\left(k-\frac{1}{2}\right)\zeta(2k-1)\,, \end{aligned} \tag{4.40}$$

where γ is the Euler constant. Analogously one can rewrite the series

$$S_f(x) = \sum_{p=1}^{\infty}\frac{(-1)^p}{p}K_1(xp) \tag{4.41}$$

as

$$\begin{aligned} S_f(x) &= -\frac{\pi^2}{12x} + \frac{x}{8}\left(1 - 2\gamma + 2\ln\frac{\pi}{x}\right) \\ &+ \sum_{k=2}^{\infty}\frac{(-1)^k}{k!}(2^{2k-1}-1)\left(\frac{x}{2\pi}\right)^{2k-1}\frac{\sqrt{\pi}}{2}\Gamma\left(k-\frac{1}{2}\right)\zeta(2k-1)\,. \end{aligned} \tag{4.42}$$

The series appearing in the formula for the Hagedorn temperature (4.31) can then be rewritten as

$$
\begin{aligned}
S_b(x) - S_f(x) &= \sum_{p=1}^{\infty} \frac{1-(-1)^p}{p} K_1(xp) \\
&= \frac{\pi^2}{4x} - \frac{\pi}{2} + \frac{x}{2}\ln 2 - \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}(2^{2k-1}-2)\left(\frac{x}{2\pi}\right)^{2k-1} \frac{\sqrt{\pi}}{2}\Gamma\left(k-\frac{1}{2}\right)\zeta(2k-1)\,.
\end{aligned}
\tag{4.43}
$$

In appendix B we present an alternative derivation of this result.

Using these results for the series difference in eq. (4.31) one can derive the following formula for the Hagedorn temperature in the limit of small f

$$
\frac{\beta_H^2}{4\pi\alpha'} = 2\pi - \frac{4\beta_H f}{\sqrt{2}} + \frac{2\beta_H^2 f^2 \ln 2}{\pi} - \sum_{k=2}^{\infty} \frac{(-1)^k(2^{2k}-4)4\sqrt{\pi}}{k!}\left(\frac{\beta_H f}{2\pi\sqrt{2}}\right)^{2k} \Gamma(k-\frac{1}{2})\zeta(2k-1)\,.
\tag{4.44}
$$

Keeping only the two leading terms in the expansion of eq. (4.44) we get

$$
\beta_H^2(1-8\alpha' f^2 \ln 2) = 8\pi^2\alpha' - \frac{16\pi\alpha'\beta_H f}{\sqrt{2}}
\tag{4.45}
$$

The Hagedorn temperature then is

$$
T_H = \frac{1}{2\pi\sqrt{2\alpha'}}\left(1 + 2\sqrt{\alpha'}f + 2(1-2\ln 2)\alpha' f^2\right)\,.
\tag{4.46}
$$

As in [154] the Hagedorn temperature increases for small values of $f^2\alpha'$ but the second term differs from the one derived in [154] by a factor of $4\pi\sqrt{2}$.

In the flat space limit $f \to 0$ we recover the well known superstring Hagedorn temperature

$$T_H = \frac{1}{\beta_H} = \frac{1}{2\pi\sqrt{2\alpha'}} \,. \tag{4.47}$$

4.5.2 The large f limit

Let us now consider the large f behavior of eq. (4.31). It is particularly interesting to examine this limit because it is in this limit pp-wave that type IIB string theory is supposed to be dual to a subsector of a particular Yang-Mills theory [46]. For large value of f, the most relevant contribution to the series of the modified Bessel function K_1 is given by taking $p = 1$ in eq. (4.31). For large value of its argument the Bessel function in fact can be approximated by

$$K_1\left(\frac{\beta_H f p}{\sqrt{2}}\right) \sim \sqrt{\frac{\pi}{\sqrt{2}\beta_H f p}} \exp\left(-\frac{\beta_H f p}{\sqrt{2}}\right) , \tag{4.48}$$

so that terms with higher values of p are exponentially suppressed. The eq. (4.31) for the Hagedorn temperature then becomes

$$\frac{\beta_H^2}{4\pi\alpha'} = 8\sqrt{\frac{\beta_H f\sqrt{2}}{\pi}} \exp\left(-\frac{\beta_H f}{\sqrt{2}}\right) \to 0 \,, \qquad f \to \infty \,. \tag{4.49}$$

The rapid vanishing of the Bessel function in the large f limit implies that the Hagedorn temperature increases with f and for very large f is pushed toward infinity. This means that in this regime there is no Hagedorn transition at any finite temperature but instead the Hagedorn temperature is a limiting temperature. This is expected since the large f limit should indeed reproduce the gauge theory behavior.

4.6 $AdS_3 \times S^3$ in NS-NS and R-R 3-form backgrounds

The limit that gives the metric (2.30) in the $AdS_5 \times S^5$ geometry can be taken also in other geometries. As a particular case one can consider the $AdS_3 \times S^3$ geometry [180][175][46]. In this case the radii of AdS_3 and S^3 are the same and the computation is identical to the one we did above for $AdS_5 \times S^5$. It is interesting to consider a situation with a mixture of NS-NS and R-R 3-form field strengths. The six dimensional plane-wave metric is

$$ds^2 = 2dx^+dx^- - f^2\vec{y}^2dx^+dx^+ + d\vec{y}^2 \tag{4.50}$$

and the 3-form is given by

$$F^{NS}_{+12} = F^{NS}_{+34} = C_1 f \cos\alpha \,, \qquad F^{R}_{+12} = F^{R}_{+34} = C_2 f \sin\alpha \,, \tag{4.51}$$

where $\vec{y}$ parametrizes a point on T^4, and α is a fixed parameter which allows us to interpolate between the purely NS-NS background $\alpha = 0$ and the purely R-R background $\alpha = \pi/2$. C_1 and C_2 are constants depending on the string coupling and the normalization of the NS-NS and R-R field strenghts. In addition to the six coordinates in eq. (4.50) we have four additional directions which we can take to be T^4.

The light-cone Hamiltonian is

$$H = \sum_{n=-\infty}^{\infty} N_n \sqrt{f^2 \sin\alpha^2 + \left(f\cos\alpha + \frac{n}{\alpha' p^+}\right)^2} + 2\frac{L_0^{T^4} + \bar{L}_0^{T^4}}{\alpha' p^+} \,, \tag{4.52}$$

where the first term takes into account the massive bosons and fermions, and the second term takes into account the massless ones.

The computation of the free energy is similar to the one we performed in the previous section for the $AdS_5 \times S^5$ geometry. It reads

$$
\begin{aligned}
F &= -2^5 \sum_{n_{odd}}^{\infty} \frac{L}{8\pi^2\alpha'} \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^2 e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}} \prod_{m=1}^{\infty} \left|\frac{1+e^{2\pi i\tau m}}{1-e^{2\pi i\tau m}}\right|^8 \times \\
&\times \prod_{m=-\infty}^{\infty} \left\{ \frac{1+\exp\left[-2\pi\tau_2\sqrt{\left(\frac{nf\beta\sin\alpha}{2\sqrt{2}\pi\tau_2}\right)^2 + \left(m+\frac{n\beta f\cos\alpha}{2\sqrt{2}\pi\tau_2}\right)^2} + 2\pi i\tau_1 m\right]}{1-\exp\left[-2\pi\tau_2\sqrt{\left(\frac{nf\beta\sin\alpha}{2\sqrt{2}\pi\tau_2}\right)^2 + \left(m+\frac{n\beta f\cos\alpha}{2\sqrt{2}\pi\tau_2}\right)^2} + 2\pi i\tau_1 m\right]} \right\}^4 .
\end{aligned}
\tag{4.53}
$$

Here, because of the fermion zero modes, the ground state is degenerate and the free energy can be computed using F_{susy} defined in eq. (4.7).

The modular properties of the partition function in eq. (4.53) can be derived as done above. Consider

$$
Z(\tau_1,\tau_2,x) = \prod_{m=-\infty}^{\infty} \left(\frac{1+e^{-2\pi\tau_2\sqrt{x^2+(m+b)^2}+2\pi i\tau_1(m+b)+2\pi i a}}{1-e^{-2\pi\tau_2\sqrt{x^2+(m+b)^2}+2\pi i\tau_1(m+b)+2\pi i a}}\right) |\theta_4(0,2\tau)|^{-2} . \tag{4.54}
$$

In our case $a=0$, $b=\frac{n\beta f\cos\alpha}{2\sqrt{2}\pi\tau_2}$, and $x=\frac{n\beta f\sin\alpha}{2\sqrt{2}\pi\tau_2}$. Eq. (4.54) can be rewritten in terms of the definition (4.23) as

$$
Z(\tau_1,\tau_2,x) = \frac{Z_{\frac{1}{2},b}(\tau_1,\tau_2,x)}{Z_{0,b}(\tau_1,\tau_2,x)} |\theta_4(0,2\tau)|^{-2} . \tag{4.55}
$$

From the modular property of $Z_{a,b}(\tau_1,\tau_2,x)$, eq. (4.27), it follows that

$$
\ln Z\left(\tau_1,\tau_2,\frac{n\beta f\sin\alpha}{\sqrt{2}2\pi\tau_2}\right) = \ln Z_{-b,\frac{1}{2}}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta f\sin\alpha|\tau|}{2\pi\sqrt{2}\tau_2}\right) - \ln Z_{-b,0}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta\sin\alpha f|\tau|}{2\pi\sqrt{2}\tau_2}\right)
$$

$$+2\pi\tfrac{\tau_2}{|\tau|^2}\left[\Delta_{\frac{1}{2}}\left(\tfrac{n\beta f\sin\alpha|\tau|}{2\pi\sqrt{2}\tau_2}\right)-\Delta_0\left(\tfrac{n\beta f\sin\alpha|\tau|}{2\pi\sqrt{2}\tau_2}\right)\right]-2\ln\left|\theta_2(0,-\tfrac{1}{2\tau})\right|+\ln 2|\tau|\,. \quad (4.56)$$

The first two terms in eq. (4.56) behave smoothly in the $\tau_1 \to 0$, $\tau_2 \to 0$ limit. Moreover

$$\left|\theta_2(0,-\frac{1}{2\tau})\right| \to \exp\left(-\frac{\pi\tau_2}{4|\tau|^2}\right)\,.$$

Consequently, taking into account the definition of the Casimir energies, for the Hagedorn temperature we get

$$\frac{\beta_H^2}{4\pi\alpha'} = \frac{4\beta_H f\sin\alpha}{\sqrt{2}\pi}\sum_{p=1}^{\infty}\frac{[1-(-1)^p]}{p}K_1\left(\frac{p\beta_H f\sin\alpha}{\sqrt{2}}\right)+\pi\,. \quad (4.57)$$

It is interesting to note that this equation depends on the angle α only through $f\sin\alpha$, the R-R field strength.

Keeping only the two leading terms in the expansion for small f of eq. (4.57), we get the Hagedorn temperature

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}}\left(1+\sqrt{\alpha'}f\sin\alpha+(1-2\ln 2)\alpha' f^2\sin^2\alpha\right)\,. \quad (4.58)$$

In the case of purely NS-NS background, corresponding to $\alpha = 0$, we recover the well known superstring Hagedorn temperature for the flat background.

Chapter 5

Half BPS geometries and free fermions

5.1 Introduction

According to the AdS/CFT correspondence, deformations of AdS geometries should map to states in the dual CFT living at the boundary of AdS. Recently a concrete realization of this map has been found for the important sector of 1/2 BPS operators of $\mathcal{N} = 4$ super Yang-Mills. These operators, as reviewed in chapter 2, have conformal dimension Δ equal to the $U(1)_R$ charge and form a decoupled sector of $\mathcal{N} = 4$ super Yang-Mills which can be efficiently described by a gauged quantum mechanics matrix model with harmonic oscillator potential. The matrix model is well known to be completely integrable. The main reason behind integrability is that, in the eigenvalue basis, the eigenvalues behave as fermions in a harmonic potential. In the semiclassical limit the 1/2 BPS states can be depicted as droplets of fermions in a two-dimensional phase space. One expects then the following AdS/CFT dictionary. Small ripples above the Fermi sea correspond to graviton excitations of $AdS_5 \times S^5$. Small holes below the Fermi energy correspond to giant gravitons, while small droplets of fermions outside the Fermi sea map to dual giant gravitons.[1]

Remarkably this whole picture has found an impressive confirmation through the explicit construction of the full moduli space of 1/2 BPS IIB supergravity solutions discovered by Lin, Lunin and Maldacena (LLM) [184].[2] The phase space distribution of the matrix model eigenvalues is in one-to-one correspondence with IIB supergravity backgrounds which preserve half of the supersymmetry. Moreover the two-dimensional phase space of the fermions has an interesting physical embedding in the space-time geometry. At the quantum level the incompressibility of the droplets in phase space (due to Fermi-Dirac statistics) corresponds in the dual supergravity side to the requirement that the Ramond-Ramond five-form flux is quantized. The whole family of half-BPS geometries can be constructed in terms of an auxiliary function z which also determines the fermion distribution. The regularity of the supergravity background amounts to requiring a suitable boundary condition on the auxiliary function. The AdS "bubbling geometries" are therefore in general smooth supergravity backgrounds.

The fermions discussed so far are characterized by having a step-function distribution in the two-dimensional phase space. They can be seen therefore as fermions at zero "temperature". It is then natural to investigate how turning on the temperature affects the supergravity solution. The fermion at non-zero temperature are described by a Fermi-Dirac distribution. The corresponding AdS "bubbling" solu-

[1]All this is very reminiscent of both old and recent works on $c = 1$ string theory and its matrix model reinterpretation [181][182] (for a recent review see [183]).

[2]For related work see [185]-[195]. An attempt to achieve a similar description for the 1/4 BPS case can be found in [196].

tion has been first obtained in [197] and further studied in [198] where it was given the name of *hyperstar*. This supergravity background can be thought of as resulting from a coarse graining process of smooth 1/2 BPS geometries. The fermion distribution of the hyperstar fails to satisfy the boundary conditions necessary to obtain a smooth gravity solution. Quite generally when the smoothness condition is not satisfied naked singularities occur [199][200]. One expects that α' string corrections will modify the geometry in proximity of the singularity and that a horizon will be generated [201][202]. This class of singular supergravity solution can therefore be regarded as incipient black holes.[3]

The duality between fermion distributions and supergravity solutions at zero temperature suggests that the thermodynamic properties of the fermion gas at finite temperature should agree with the corresponding quantities in the supergravity side. In particular, one expects agreement between the thermal excitation energy of the fermions and the ADM mass of the supergravity solution. We will check that this is indeed the case in the two opposite regimes of low and high temperature.

As we have already remarked, the hyperstar geometry is singular. The singularity is resolved quantum mechanically through the appearance of a finite area horizon. One can then use the Bekenstein-Hawking formula to compute the associated entropy. By placing a stretched horizon in the hyperstar geometry we propose a way to match the supergravity entropy with the thermal entropy of the fermions in the low temperature regime, up to a numerical factor.

Similarly we investigate the opposite regime of high temperature. In this limit the Fermi-Dirac distribution reduces to the classical Boltzmann distribution. We find

[3] An example of singular LLM solution is the *superstar* [203], which has been investigated from this point of view in [204][205].

that in this regime the metric is reminiscent of the so called dilute gas limit of LLM configurations associated to the Coulomb branch of super Yang-Mills.

5.2 Review of the LLM construction

In this section we briefly review the LLM construction [184] of 1/2 BPS IIB supergravity backgrounds. These solutions correspond, in the dual gauge theory, to states satisfying the BPS condition $\Delta = J$, where Δ is the corresponding conformal dimension and J is a particular $U(1)$ charge of the $SO(6)$ R-symmetry group of $\mathcal{N} = 4$ super Yang-Mills. By selecting one generator of this $SO(6)$ we obtain a theory with $SO(4) \times SO(4) \times \mathbb{R}$ bosonic symmetry. In the dual supergravity description we look therefore for solutions with this isometry group. Assuming that the axion and dilaton are constant and that only the selfdual five-form field strength is turned on, the Ansatz for the background is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu + e^{H+G} d\Omega_3^2 + e^{H-G} d\tilde{\Omega}_3^2 \,, \\ F_{(5)} &= F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3 \,, \end{aligned} \tag{5.1}$$

where the Greek indices μ, ν run over $0, \ldots, 3$. The two three-spheres S^3 and $\tilde{S}^3$ in the metric make the $SO(4) \times SO(4)$ isometries manifest. The additional $\mathbb{R}$ isometry corresponds to the Hamiltonian $\Delta - J$.

For a background to be 1/2 BPS there should exist a solution to the Killing spinor equation. Analyzing this equation, LLM were able to prove that the generic 1/2 BPS

IIB supergravity background takes the form

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \,, \quad (5.2)$$

$$h^{-2} = 2y \cosh G \,, \qquad z = \frac{1}{2} \tanh G \,, \quad (5.3)$$

$$y\partial_y V_i = \epsilon_{ij}\partial_j z \,, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z \quad (5.4)$$

$$F = dB_t \wedge (dt + V) + B_t dV + d\hat{B} \,,$$

$$\tilde{F} = d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}} \,, \quad (5.5)$$

$$B_t = -\frac{1}{4} y^2 e^{2G} \,, \qquad \tilde{B}_t = -\frac{1}{4} y^2 e^{-2G} \,, \quad (5.6)$$

$$d\hat{B} = -\frac{1}{4} y^3 *_3 d\left(\frac{z + \frac{1}{2}}{y^2}\right) \,, \qquad d\hat{\tilde{B}} = -\frac{1}{4} y^3 *_3 d\left(\frac{z - \frac{1}{2}}{y^2}\right) \,, \quad (5.7)$$

where $i = 1, 2$, and $\star_3$ is the Hodge dual operator for the flat three-dimensional space parameterized by x_1, x_2, and y. Remarkably, the solution is completely specified in terms of a single auxiliary function $z(x_1, x_2, y)$ which satisfies the linear differential equation

$$\partial_i\partial_i z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0 \,. \quad (5.8)$$

It is important to note that at $y = 0$ the product of the radii of the two privileged three-spheres is zero. Therefore, to avoid singular geometries, the auxiliary function z must satisfy a suitable boundary condition. This smoothness condition turns out to be $z = \pm 1/2$ on the boundary plane $y = 0$. In the limit $z \to 1/2$ the $\tilde{S}^3$ sphere shrinks to zero while the other three-sphere remains finite. The reverse statement applies when $z \to -1/2$. It is conventional to assign black and white colors respectively to the $z = -1/2$ and $z = 1/2$ points in the (x_1, x_2) plane. If $\mathcal{D}$ denotes a black region in this plane, the energy of the associated supergravity solution has the simple expression

$$\Delta = J = \int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar}\, \frac{1}{2} \frac{(x_1^2 + x_2^2)}{\hbar} - \frac{1}{2} \left(\int_{\mathcal{D}} \frac{d^2x}{2\pi h} \right)^2 . \tag{5.9}$$

The $\mathbb{R}^2$ plane has then a natural interpretation as the phase space of one-dimensional fermions in a harmonic potential. This nicely matches the matrix model description in the dual CFT side [35]. It emerges a beautiful picture of the moduli space of 1/2 BPS geometries of IIB supergravity in terms of configurations of droplets of fermions on the (x_1, x_2) plane. Note that the fundamental equation (5.8) has the symmetry $z \to -z$ which simply exchanges the S^3 and $\tilde{S}^3$ in the solution. In a field theory description of the fermions, this symmetry amounts to a particle-hole duality.

The quantization condition on the total area $\mathcal{A}$ of the droplets is related to the five-form flux N as follows

$$\frac{\mathcal{A}}{2\pi\hbar} = N \,, \tag{5.10}$$

with $\hbar = 2\pi l_p^4$. The flux N coincides with the number of fermions. The simplest configuration in phase space is a black circular droplet of radius $R_0 = \sqrt{2\hbar N}$ and the associated geometry is $AdS_5 \times S^5$ with N units of the five-form flux. This background has $\Delta = J = 0$ and corresponds to the fermion ground state. The boundary of the droplet can be thought of as the Fermi level of the fermions. The S^5 of the background is obtained by fibering the $\tilde{S}^3$ sphere on a two-dimensional surface Σ_2 in the (x_1, x_2, y) space which encircles the droplet. One can easily obtain configurations with an arbitrary number of S^5's by adding other droplets. If we deform the circular droplet to configurations with different shapes but same area, we obtain backgrounds with $AdS_5 \times S^5$ asymptotics.

The fundamental equation (5.8) can be rewritten as a Laplace equation for the quantity $\Phi = z/y^2$ in a six-dimensional space with spherical symmetry in four of the coordinates. The coordinate y corresponds to the radial direction in the four-dimensional subspace. This observation reduces the task of finding the full solution $z(x_1, x_2, y)$ of eq. (5.8) to a well known initial-value problem. Once the boundary condition $z(x_1, x_2, 0)$ on the $y = 0$ plane is specified, the solution is

$$z(x_1, x_2, y) = \frac{y^2}{\pi} \int_{\mathbb{R}^2} \frac{z(x_1', x_2', 0) dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2} \,. \tag{5.11}$$

We can similarly get

$$V_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{\pi} \int_{\mathbb{R}^2} \frac{z(x_1', x_2', 0)(x_j - x_j') dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2} \,. \tag{5.12}$$

Since we are going to consider only droplet configurations with radial symmetry, it will be convenient to rewrite the above formulas in polar coordinates $(x_1, x_2) \to (R, \phi)$. It is easy to see that in this case $V_R = V_1 \cos\phi + V_2 \sin\phi = 0$. Defining $V \equiv V_\phi = R(-V_1 \sin\phi + V_2 \cos\phi)$ the differential equations relating z and V (5.4) read

$$y\, \partial_y V \; = \; -R\, \partial_R\, z \,, \qquad \frac{1}{R}\, \partial_R\, V \; = \; \frac{1}{y}\, \partial_y\, z \,. \tag{5.13}$$

Rewriting eq. (5.11) and eq. (5.12) in polar coordinates yields

$$z(R, y) = -\int z(R', 0) \frac{\partial}{\partial R'} z_0(R, y; R') dR' \,, \tag{5.14}$$

$$V(R, y) = \int z(R', 0) g_V(R, y; R') dR' \,, \tag{5.15}$$

where

$$z_0(R,y;R') = \frac{R^2 - R'^2 + y^2}{2\left[(R^2+R'^2+y^2)^2 - 4R^2R'^2\right]^{1/2}}\,, \tag{5.16}$$

$$g_V(R,y;R') = \frac{-2R^2R'(R^2 - R'^2 + y^2)}{\left[(R^2+R'^2+y^2)^2 - 4R^2R'^2\right]^{3/2}}\,. \tag{5.17}$$

We remark that z_0 is the LLM function corresponding to a circular droplet. Indeed in this case $z(R',0) = 1/2\,\mathrm{sign}(R'-R_0)$ and using eq. (5.14) one obtains $z(R,y) = z_0(R,y;R_0)$. As previously anticipated such a configuration gives rise to the $AdS_5 \times S^5$ solution. In fact performing the following change of coordinates [184]

$$y = R_0 \sinh\rho \sin\theta\,, \qquad R = R_0 \cosh\rho\cos\theta\,, \qquad \phi = \tilde{\phi} + t\,, \tag{5.18}$$

one recovers the $AdS_5 \times S^5$ metric in standard global form

$$ds^2 = R_0\big(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\tilde{\Omega}_3^2\big)\,. \tag{5.19}$$

A variation of the method described so far can be similarly applied to obtain 1/2 BPS M-theory backgrounds with $AdS_{4,7} \times S^{7,4}$ asymptotics [184]. In this case the geometry is in one-to-one correspondence with solutions of a three-dimensional Toda equation, which plays the same role as eq. (5.8).

5.3 1D fermions in the harmonic well

In this section we review the basics of the thermodynamics of one-dimensional fermions in a harmonic potential. In what follows, we will consistently adopt units in which

$\hbar = k_B = 1$. We consider a gas of N non-interacting fermions with hamiltonian

$$H(p,q) = \frac{1}{2}(p^2 + q^2) \tag{5.20}$$

in thermodynamic equilibrium at a given temperature T. For large N, we adopt the semi-classical approximation in which the energy is taken to be a continuous variable. The probability distribution as a function of the energy $H(p,q) = \epsilon$ is given by the Fermi-Dirac distribution:

$$n_{FD}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T} + 1}\,, \tag{5.21}$$

where μ is the Fermi energy. This is determined by the normalization condition

$$\int_0^\infty \frac{d\epsilon}{e^{(\epsilon-\mu)/T} + 1} = N\,, \tag{5.22}$$

which gives

$$\mu = T\ln(e^{N/T} - 1)\,. \tag{5.23}$$

We will first consider the limit of very small temperature T, or more precisely $N/T \gg 1$. In this limit, the Fermi level becomes

$$\mu = N + \mathcal{O}(Te^{-N/T})\,, \tag{5.24}$$

and the total energy of the Fermi gas is given by

$$E = \int_0^\infty \frac{\epsilon \, d\epsilon}{e^{(\epsilon-\mu)/T}+1}\,, \tag{5.25}$$

which for small T can be evaluated by means of the Sommerfeld expansion [206]

$$I = \int_0^\infty \frac{f(\epsilon)d\epsilon}{e^{(\epsilon-\mu)/T}+1} = \int_0^\mu f(\epsilon)d\epsilon + \frac{\pi^2}{6}T^2 f'(\mu) + \frac{7\pi^4}{360}T^4 f'''(\mu) + \mathcal{O}(T^6)\,. \tag{5.26}$$

This gives

$$E \simeq \frac{N^2}{2} + \frac{\pi^2}{6}T^2\,. \tag{5.27}$$

The first term is clearly the ground state energy of the N fermions, so we expect the dual gravity solution to have a mass (and angular momentum) difference of $\Delta = \frac{\pi^2}{6}T^2$ with respect to the $AdS_5 \times S^5$ background. It is worth noting that in eq. (5.27) we only neglect exponentially suppressed terms. In fact, since $f(\epsilon) = \epsilon$, there are no power series corrections to the energy beyond T^2. This is a specific feature of the 1D harmonic oscillator, and we will recover it in the energy[4] and angular momentum of the hyperstar in the low T limit.

To evaluate the entropy of the fermion gas, it is convenient to first obtain the free energy F of the system. This is computed from the partition function Z, which in the continuous limit we are considering reads

$$Z = \exp\left[-\frac{N\mu}{T} + \int_0^\infty d\epsilon \, \ln(1+e^{-(\epsilon-\mu)/T})\right]. \tag{5.28}$$

[4]Modulo a subtlety involving T^4 terms to be discussed later.

One can verify that this expression for the partition function is correct by checking that the relation $E = -\frac{\partial}{\partial\beta}\ln Z$ (where $\beta = 1/T$) is satisfied. Using the definition $F = -T\ln Z$ one obtains the free energy

$$F = N\mu - T\int_0^\infty d\epsilon \ln(1+e^{-(\epsilon-\mu)/T}) = N\mu - \int_0^\infty d\epsilon\frac{\epsilon}{e^{(\epsilon-\mu)/T}+1} = N\mu - E. \quad (5.29)$$

The entropy is then given by the relation $F = E - TS$ from which we get

$$S = \frac{2E - N\mu}{T}. \quad (5.30)$$

For small T, using eq. (5.24) and eq. (5.27) one gets

$$S \simeq \frac{\pi^2}{3}T\,, \quad (5.31)$$

where again only exponetially small terms are neglected.

We now consider the opposite limit of very high temperature $N/T \ll 1$. In this limit, the Fermi distribution clearly reduces to the Boltzmann density

$$n_{FD}(\epsilon) \to n_B(\epsilon) = N\beta\, e^{-\beta\epsilon} \quad (5.32)$$

where $\beta = 1/T$. The total energy in this approximation is $E = NT$. The entropy can be obtained from eq. (5.30) (which is valid for any temperature) using the large T approximation $\mu \simeq T\ln N/T$, and reads

$$S \simeq N\ln T + 2N - N\ln N. \quad (5.33)$$

5.4 The hyperstar: low temperature regime

We now introduce the 1/2 BPS geometry dual to the Fermi-Dirac gas described in the previous section [197]. This solution was named *hyperstar* in [198].

A given $z(R,0)$ corresponds to a fermion density $n(R)$ in the phase space via the relation

$$z(R,0) = \frac{1}{2} - n(R)\,. \tag{5.34}$$

For example, the $AdS_5 \times S^5$ solution is associated to the step function density $n_0 = \vartheta(R - R_0)$, which can be viewed as the zero temperature limit of the Fermi-Dirac distribution (5.21). One can turn on the temperature on the fermion side by replacing n_0 with $n_{FD}(R)$ and construct the corresponding supergravity background by using eqs. (5.14), (5.15) and (5.34). It is important to remark that the temperature we are turning on is the temperature in the "auxiliary" description of the free fermion gas. It is not a temperature of the supergravity solution or of the dual gauge theory. Indeed, we remain in the supersymmetric 1/2 BPS sector. It would be interesting to understand better what corresponds to this temperature on the gravity and gauge theory side. For the time being, we regard T just as a deformation parameter of the $AdS_5 \times S^5$ background.

For low temperatures, the solution is a small perturbation of the circular droplet. In fact in this limit the fermion configuration in the (x_1, x_2) looks like a black disk with the boundary slightly "blurred", as shown in fig. 5.1. In the low temperature

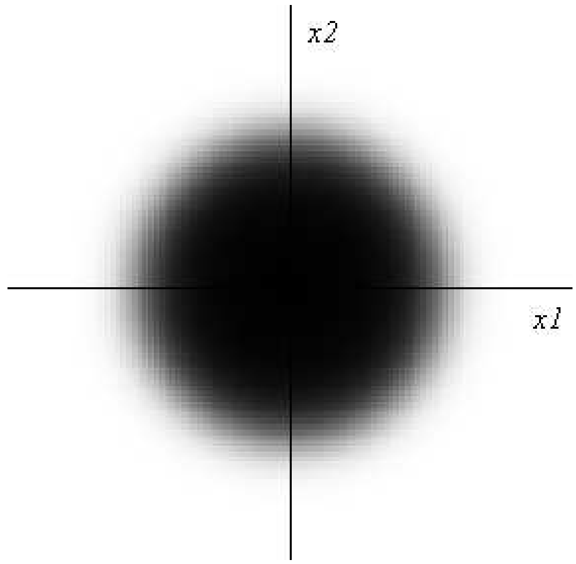


Figure 5.1: Droplet configuration in the low temperature limit. The greyscale ring around the Fermi level corresponds to a singular region of the spacetime.

limit the expressions (5.14) and (5.15) can be obtained analytically as follows [197]

$$\begin{aligned} z^T_{FD}(R,y) &= \frac{1}{2} + \int_0^\infty n_{FD}(R')\frac{\partial}{\partial R'}z_0(R,y;R')dR' \\ &= z_0(R,y;R_0) + \frac{\pi^2}{6}T^2\left[\frac{\partial^2}{\partial\epsilon^2}z_0(R,y;R_0=\sqrt{2\epsilon})\right]_{\epsilon=\frac{R_0^2}{2}} + \mathcal{O}(T^4)\,, \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} V^T_{FD}(R,y) &= -\int_0^\infty n_{FD}(R')g_V(R,y;R')dR' \\ &= V_0(R,y;R_0) - \frac{\pi^2}{6}T^2\left[\frac{\partial}{\partial\epsilon}\left(\frac{g_V(R,y;R_0=\sqrt{2\epsilon})}{\sqrt{2\epsilon}}\right)\right]_{\epsilon=\frac{R_0^2}{2}} + \mathcal{O}(T^4)\,, \end{aligned} \tag{5.36}$$

where we have used the Sommerfeld expansion (5.26) and where

$$V_0(R, y; R_0) = \frac{1}{2}\left(\frac{R^2 + R_0^2 + y^2}{[(R^2 + R_0^2 + y^2)^2 - 4R^2R_0^2]^{1/2}} - 1\right) \tag{5.37}$$

corresponds to the $AdS_5 \times S^5$ background. In these expressions $R_0 = \sqrt{2N}$ is the radius of the droplet in the phase space at $T = 0$, and N is the number of fermions. It is easy to check that eqs. (5.35) and (5.36) satisfy the differential equations (5.13).

5.4.1 ADM form of the metric

In order to compute the mass and the angular momentum associated to the hyperstar, it is convenient to perform the following change of coordinates

$$R = L^2\left(1 + \frac{r^2}{L^2}\right)^{1/2}\cos\theta\,, \qquad y = L\,r\,\sin\theta\,, \qquad \phi \;=\; \tilde{\phi} + \frac{t}{L}\,, \tag{5.38}$$

and we also rescale $t \to L\,t$ to have conventional units. Here $(t\,,\, r\,,\, \Omega_3)$ parameterize the asymptotic AdS_5 (in global coordinates), whereas $(\theta\,,\, \tilde{\phi}\,,\, \tilde{\Omega}_3)$ span the asymptotic S^5. Of course, L is the radius of both the AdS_5 and the S^5. It is related to the radius R_0 used by [184] via $R_0 = L^2$. In this system of coordinates the metric can be rewritten in ADM form as

$$ds^2 \;=\; -\mathcal{N}^2dt^2 + g_{\theta\theta}\left(\frac{dr^2}{r^2 + L^2} + d\theta^2\right) + g_{\tilde{\phi}\tilde{\phi}}\left(d\tilde{\phi} + \mathcal{N}^{\tilde{\phi}}dt\right)^2 + g_{\Omega_3\Omega_3}\,d\Omega_3^2 + g_{\tilde{\Omega}_3\tilde{\Omega}_3}\,d\tilde{\Omega}_3^2\,, \tag{5.39}$$

where $\mathcal{N}$ is the *lapse function* and $\mathcal{N}^{\tilde{\phi}}$ is the *shift vector*.

Introducing the expansion parameter

$$\gamma \equiv \frac{2\pi^2 T^2}{3L^8} = \frac{\pi^2 T^2}{6N^2}\,, \tag{5.40}$$

and using the explicit expressions for z^T_{FD} and V^T_{FD}

$$\begin{aligned}
z^T_{FD}(R,y) &= \frac{R^2 - R_0^2 + y^2}{2\left[(R^2+R_0^2+y^2)^2 - 4R^2R_0^2\right]^{1/2}} + \\
&\quad + \frac{2R_0^4 y^2\big((R_0^2+y^2)^2 + R^2(R_0^2-y^2) - 2R^4\big)}{\left[(R^2+R_0^2+y^2)^2 - 4R^2R_0^2\right]^{5/2}}\gamma + \mathcal{O}(\gamma^2)\,, \\
V^T_{FD}(R,y) &= \frac{1}{2}\left(\frac{R^2+R_0^2+y^2}{\left[(R^2+R_0^2+y^2)^2 - 4R^2R_0^2\right]^{1/2}} - 1\right) + \\
&\quad + \frac{2R_0^4 R^2\big((R^2-R_0^2)^2 - y^2(R^2+R_0^2+2y^2)\big)}{\left[(R^2+R_0^2+y^2)^2 - 4R^2R_0^2\right]^{5/2}}\gamma + \mathcal{O}(\gamma^2)\,,
\end{aligned} \tag{5.41}$$

one obtains, upon implementing eq. (5.38), the components of the metric, which we present here up to $\mathcal{O}(\gamma^2)$ terms[5]

$$\begin{aligned}
\mathcal{N}^2 &= \left(1 + \frac{r^2}{L^2}\right)\left[1 - \gamma L^2 F_1(r,\theta)\right], \\
\mathcal{N}^{\tilde{\phi}} &= \gamma \frac{2L(r^2+L^2)(r^2+L^2\cos^2\theta)}{(r^2+L^2\sin^2\theta)^3}\,, \\
g_{\tilde{\phi}\tilde{\phi}} &= L^2\cos^2\theta\left[1 + \gamma L^2 F_1(r,\theta)\right], \\
g_{\theta\theta} &= L^2\left[1 + \gamma L^2 \frac{r^2 - L^2\sin^2\theta}{r^2+L^2\sin^2\theta}F_2(r,\theta)\right],
\end{aligned}$$

[5]We notice that our expression for $g_{\tilde{\phi}\tilde{\phi}}$ differs from the one reported in [197].

$$g_{\Omega_3\Omega_3} = r^2\Big[1-\gamma L^2 F_2(r,\theta)\Big]\,,$$

$$g_{\tilde{\Omega}_3\tilde{\Omega}_3} = L^2\sin^2\theta\Big[1+\gamma L^2 F_2(r,\theta)\Big]\,, \tag{5.42}$$

where

$$F_1(r,\theta) = \frac{(3\cos^2\theta-1)r^4+3L^2\cos^4\theta\, r^2+L^4(2\cos^4\theta+\sin^2\theta)}{(r^2+L^2\sin^2\theta)^3}\,,$$

$$F_2(r,\theta) = F_1(r,\theta)-2\frac{\sin^2\theta\, r^2+L^4}{(r^2+L^2\sin^2\theta)^3}\,. \tag{5.43}$$

A general property of LLM distributions with compact support is that the corresponding geometries are asymptotically $AdS_5\times S^5$. One can check that this remains true for the metric (5.42). This is consistent with the fact that the droplet of fig. 5.1 is effectively confined in a finite region of the phase space. From the expressions in eq. (5.43), we also notice that the Sommerfeld expansion seems no longer reliable in a region around the point $r=\theta=0$. The appearance of eventual singularities will be discussed in the following section.

5.4.2 Singularities of the metric

The study of singularities for LLM geometries was undertaken in [198]-[200]. There it was shown that all singularities appearing in the LLM supergravity solutions are naked and fall into two classes, namely timelike and null. While the former are considered highly pathological due to the presence of closed timelike curves, the latter are not. In fact, for 1/2 BPS geometries with null singularity, the underlying fermion density function $n(R)$ always takes values in the region $n(R)\in[0,1]$. This is the case both for the hyperstar and the superstar solutions.

To verify the presence of a singularity in the geometry, one should find a curvature invariant which diverges. The first non-trivial invariant to consider is $R^2_{MN} = R_{MN}R^{MN}$, $(M, N = 0, \ldots, 9)$, since the Ricci scalar R vanishes. Indeed, as a result of the Weyl invariance of the classical theory, the trace of the matter stress-tensor is identically zero. In order to check the consistency of the metric (5.42) we explicitly verified that this is the case.

There are two ways to perform the computation of R^2_{MN}. The direct approach involves the explicit form of the metric, while the indirect one makes use of the field equations of type IIB supergravity. In this case, the knowledge of the five form field strength $F_{(5)}$ will suffice

$$R_{MN} = \frac{1}{5!}\left(\frac{5}{2}F_M{}^{P_1P_2P_3P_4}F_{NP_1P_2P_3P_4} - \frac{1}{4}g_{MN}F^2\right) . \tag{5.44}$$

Suppose now we use the approximate solution for the metric, whose explicit form was given in the previous section, eq. (5.42). A lengthy calculation gives

$$R^2_{MN} = \frac{160}{L^4} + \gamma\,\frac{P_6(r, \cos\theta)}{L^2(r^2 + L^2\sin^2\theta)^4} + \mathcal{O}(\gamma^2) . \tag{5.45}$$

In this γ expansion, the first term corresponds to $AdS_5 \times S^5$ and is, of course, finite. The linear term in γ is, however, potentially divergent. Here $P_6(r, \cos\theta)$ is a sixth order polynomial in both r and $\cos\theta$ and goes to zero as $(r^2 + L^2\sin^2\theta)$ when $r \to 0$ and $\theta \to 0$. The square of the Ricci tensor is therefore divergent when $r = 0$ and $\theta = 0$. It is interesting to note here that this is exactly the singular behavior one sees in Kerr black holes. Due to their angular momentum, the collapsing region is not a point but a zero-thickness ring. The Kretschmann invariant $K \sim R_{MNRS}R^{MNRS}$, for instance, for a Kerr black hole with mass M and angular momentum $J = Ma$, is

$$K = M^2 \frac{Q_6(r, \cos\theta)}{(r^2 + a^2 \cos^2\theta)^6}\,, \tag{5.46}$$

where $Q_6(r, \cos\theta)$ also indicates a sixth order polynomial having the same behavior as $P_6(r, \cos\theta)$ in the vicinity of $r = 0$ and $\theta = 0$. This is suggestive of the existence of an event horizon in the hyperstar geometry which may manifest itself through α'-corrections to the supergravity solution.

This is not however the result one would have anticipated. From the form of the metric in LLM coordinates, it is quite natural to expect a singularity at $y = 0$. Using eq. (5.38), we can see that this corresponds to $r = 0$ or $\theta = 0$ in asymptotic $AdS_5 \times S^5$ coordinates. On the other hand, the singular region appearing in eq. (5.45) is mapped to $(R = L, y = 0)$, which is just the Fermi surface of the fermions. We expect the singularity to be at least smeared over an extended region around the Fermi energy, since there the fermion density is less than one, see fig. 5.1.

What is therefore the true singular region of the hyperstar? We can try to address this question in a quite general fashion valid for all LLM geometries. We simply need to know the behavior of the functions $z(R, y)$ and $V(R, y)$ in proximity of $y = 0$. We can distinguish two different cases depending on whether $z_0(R, 0) = \lim_{y\to 0} z(R, y)$ is independent of the radial coordinate R or not. In what follows we will focus on the latter, since this is the case of the hyperstar.

We would like to find $z(R, y)$ and $V(R, y)$ in terms of an expansion in y or functions of y, such that the differential equations (5.13) will be order by order satisfied. It turns out that the appropriate Ansatz is the following

$$\begin{aligned} z(R,y) &= z_0(R,0) + f_1(R)\, y^2 \ln y + \dots\,, \\ V(R,y) &= V_0(R,0) + g_1(R)\, \ln y + \dots\,. \end{aligned} \tag{5.47}$$

The functions $f_i(R)$ and $g_i(R)$ are determined to each order from the same differential equation (5.13). For the case that concerns us here we have $f_1(R) = -\frac{1}{2R}\partial_R(R\partial_R z_0(R,0))$, $V_0(R,0) = -\frac{1}{2}R\partial_R z_0$, and $g_1(R) = -R\partial_R z_0$. It is now easy to find the complete solution for the metric and the five-form field strength in this region and subsequently calculate R^2_{MN}, using either of the methods indicated above. We find

$$R^2_{MN} = \frac{h_1(z_0(R,0))}{y^2} + h_2(z_0(R,0), f_1(R))\, \ln y + \dots\,, \tag{5.48}$$

where $h_1(z_0(R,0))$ and $h_2(z_0(R,0), f_1(R))$ are non-zero functions of the variables indicated. Indeed we see that the leading term is divergent at $y = 0$, as expected. We must therefore conclude that this is the singular region of the hyperstar, and that we cannot rely on the Sommerfeld expansion (5.26) for calculations in the small y region. This will be important later for computing the entropy through the Bekenstein-Hawking formula.

5.4.3 Fluxes and topology

To check the consistency of the hyperstar solution, we can verify that the flux of the five form $F_{(5)}$ remains equal to N, independently from the temperature T of the fermion gas. From general considerations this has to be expected, since the temperature can be viewed as a tunable continuous parameter and as such it cannot modify the flux which is a topological constraint. At zero temperature, *i.e.* for the $AdS_5 \times S^5$ solution, using the explicit expressions for the field strength in the LLM solution and the change of coordinates eq. (5.38) one obtains

$$F^{(0)}_{(5)} = \frac{r^3}{L}\, dt \wedge dr \wedge d\Omega_3 + 2N \sin^3\theta \cos\theta \, d\theta \wedge d\tilde{\phi} \wedge d\tilde{\Omega}_3 \,. \tag{5.49}$$

The flux is computed by integrating $F^{(0)}_{(5)\theta\tilde{\phi}\tilde{\Omega}_3}$ over the S^5, and including the appropriate normalization is equal to N. To check that temperature perturbations do not alter the flux, one has to verify that corrections to $F^{(0)}_{(5)\theta\tilde{\phi}\tilde{\Omega}_3}$ vanish when integrated over the five sphere. Up to second order in the temperature, these take the form

$$\begin{aligned} F^{(1)}_{(5)\theta\tilde{\phi}\tilde{\Omega}_3} &= \gamma \, \sin^3\theta \cos\theta \, \frac{L^6 \, p_6(r, \cos\theta)}{(r^2 + L^2 \sin^2\theta)^4} \,, \\ F^{(2)}_{(5)\theta\tilde{\phi}\tilde{\Omega}_3} &= \gamma^2 \, \sin^3\theta \cos\theta \, \frac{L^{10} \, p_{10}(r, \cos\theta)}{(r^2 + L^2 \sin^2\theta)^8} \,, \end{aligned} \tag{5.50}$$

where p_6 and p_{10} are polynomials of degree 6 and 10 respectively. Although the explicit formulas look rather involved, the integration over θ can be carried out exactly at arbitrary r and indeed yields

$$\int_0^{\pi/2} d\theta \, F^{(1)}_{(5)\theta\tilde{\phi}\tilde{\Omega}_3} = \int_0^{\pi/2} d\theta \, F^{(2)}_{(5)\theta\tilde{\phi}\tilde{\Omega}_3} = 0 \,. \tag{5.51}$$

5.4.4 ADM mass of the hyperstar

In this section we present a systematic derivation of the ADM mass of the hyperstar solution. The natural expectation is that this mass should coincide with the thermal energy of the auxiliary fermion gas system.

The Einstein-Hilbert action in a d-dimensional spacetime is

$$S_{grav} = \frac{1}{16\pi G_d} \int_{\mathcal{M}} d^d x \sqrt{-g}\,(R - 2\Lambda) - \frac{1}{8\pi G_d} \oint_{\partial\mathcal{M}} d^{d-1} x \sqrt{-\gamma}\,\Theta\,, \tag{5.52}$$

where we included the Gibbons-Hawking boundary term and $\gamma_{\mu\nu}$ is the metric on the $(d-1)$-dimensional timelike boundary. Following [207], the quasi-local stress-tensor can be computed by the variation of the gravitational action with respect to the boundary metric

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{grav}}{\delta \gamma_{\mu\nu}}. \tag{5.53}$$

Using eq. (5.52) this is

$$T^{\mu\nu} = \frac{1}{8\pi G_d} \left(\Theta^{\mu\nu} - \Theta \gamma^{\mu\nu}\right)\,. \tag{5.54}$$

In the previous expression we have introduced the extrinsic curvature of the $(d-1)$-dimensional timelike boundary embedded in $\mathcal{M}$

$$\Theta^{\mu\nu} = -\frac{1}{2} \nabla^{(\mu}_{(g)} \hat{n}^{\nu)}\,, \tag{5.55}$$

and we denoted the corresponding trace by Θ. The covariant derivative is taken with respect to the metric $g_{\mu\nu}$ of the full spacetime and $\hat{n}^\nu$ is the unit normal to the boundary. The stress-tensor (5.53) generically diverges as we approach the boundary $\partial\mathcal{M}$ when the spacetime is asymptotically AdS. In the context of the AdS/CFT correspondence we can view the gravitational quasi-local stress-tensor as the expectation value of the stress-tensor in the associated conformal field theory. The divergences get then a natural interpretation as standard ultraviolet divergences in quantum field theory [208]. We can regularize the theory by adding suitable counterterms to the original stress-tensor

$$T^{\mu\nu} = \frac{1}{8\pi G_d}\left(\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\right). \tag{5.56}$$

The counterterms are consistently constructed using only the boundary metric $\gamma_{\mu\nu}$ and its covariant derivatives and are (almost) uniquely determined by requiring a cancellation of the divergences and general covariance (for a review see [209]). The boundary metric $\gamma_{\mu\nu}$ can be written in the ADM form

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -\mathcal{N}_\Sigma^2 dt^2 + \sigma_{ab}(dx^a + \mathcal{N}_\Sigma^a dt)(dx^b + \mathcal{N}_\Sigma^b dt)\,, \tag{5.57}$$

where Σ is a surface of constant t inside $\partial\mathcal{M}$. Conserved charges are obtained by integrating $T^{\mu\nu}$ over a spacelike hypersurface at infinity. A finite expression for the mass is obtained substituting the regularized stress-energy tensor in the following formula

$$M = \int_\Sigma d^{d-2}x\,\sqrt{\sigma}\mathcal{N}_\Sigma u^\mu T_{\mu\nu}u^\nu, \tag{5.58}$$

where u^μ is the timelike unit normal to Σ. For instance, the application of this method to the five-dimensional AdS-Schwarzschild black hole

$$ds^2 = -\left[\frac{r^2}{L^2} + 1 - \left(\frac{r_0}{r}\right)^2\right]dt^2 + \frac{dr^2}{\left[\frac{r^2}{L^2} + 1 - \left(\frac{r_0}{r}\right)^2\right]} + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)\,, \tag{5.59}$$

yields [208]

$$M = \frac{3\pi l^2}{32G_5} + \frac{3\pi r_0^2}{8G_5}\,. \tag{5.60}$$

The first term, which is present also when the black hole disappears, corresponds to the Casimir energy of the vacuum in the dual CFT.

It would be nice to have a similar counterterm method directly in a ten-dimensional setting. Unfortunately, extending the program of holographic renormalization to ten-dimensional metrics with $AdS_5 \times S^5$ asymptotics seems problematic [210]. We are therefore forced to use alternative approaches. In the first one, we will determine the relevant components of the stress-tensor relative to some reference geometry following [211]. The second approach is the so called background subtraction method [212]. In both cases one has to carefully match the asymptotic geometry of the supergravity solution with that of a reference background. Neither of the methods can reproduce the Casimir energy of the associated CFT. However this will not be a problem in our case since we are interested in computing the energy difference between the 1/2 BPS supergravity solution and the $AdS_5 \times S^5$ ground state.

We now proceed to compute the mass of the hyperstar (5.42) as a series expansion in the small parameter $\gamma \equiv \frac{\pi^2 T^2}{6N^2}$. This mass should agree with the energy of the free fermion gas, eq. (5.27). We will first consider the leading order in γ and comment on γ^2 orders in a later section.

First approach

Following [211], we obtain the stress-tensor associated with the metric (5.42) relative to the $AdS_5 \times S^5$ background metric $g^0_{\mu\nu}$. We need to require that the difference between the two metrics falls off suitably fast for large radius. Explicitly we want that

$$g_{rr} - g^0_{rr} = o(1/r^6)\,, \qquad g_{ra} - g^0_{ra} = o(1/r^5)\,, \tag{5.61}$$

where $o(1/r^n)$ means that these differences go to zero more rapidly than $1/r^n$ and the index a runs over all the coordinates except r. To satisfy such requirement we implement an appropriate change of coordinates $(r, \theta) \to (\tilde{r}, \tilde{\theta})$, which we presently discuss. The effect of using these new coordinates is to make the leading asymptotic perturbations of the metric all in components parallel to the boundary directions. Then the line element becomes

$$ds^2 = g^0_{\mu\nu} dx^\mu dx^\nu + \frac{\hat{T}_{ab}}{\tilde{r}^2} dx^a dx^b + \dots \,, \tag{5.62}$$

from which one can read off the stress-tensor up to a multiplicative constant depending only on the space-time dimensions.

The first step is therefore to find a coordinate system such that the metric satisfies eq. (5.61). We consider the Ansatz

$$r = \tilde{r} + \frac{f_1}{\tilde{r}} + \frac{f_2}{\tilde{r}^3} \,, \qquad \cos\theta = \cos\tilde{\theta} + \frac{f_3}{\tilde{r}^2} + \frac{f_4}{\tilde{r}^4} \,, \tag{5.63}$$

where the $f_i = f_i(\tilde{\theta})$ $(i = 1, \dots, 4)$ are functions to be determined in order to adjust the asymptotics of the metric.

In terms of the new variables $\tilde{r}$ and $\tilde{\theta}$, the $g_{\tilde{r}\tilde{r}}$ component of the metric has an expansion for large $\tilde{r}$ which differs from the background reference metric $g^0_{\tilde{r}\tilde{r}} = (1 + \tilde{r}/L^2)^{-1}$ by terms containing the f_i. The $1/\tilde{r}^4$ term can be eliminated by tuning $f_1 = \frac{1}{4}(3\cos\tilde{\theta}^2 - 1)L^2\gamma$, and similarly the $1/\tilde{r}^6$ with an appropriate choice of f_2. The first constraint in eq. (5.61) is then satisfied. Analogously, f_3 and f_4 are fixed by requiring the vanishing of the $1/\tilde{r}^3$ and $1/\tilde{r}^5$ terms in $g_{\tilde{r}\tilde{\theta}}$, which appears after changing variables according to eq. (5.63). Once the f_i are fixed, one can verify that the other components of the metric coincide with $g^0_{\mu\nu}$ up to orders $\mathcal{O}\left(\tilde{r}^{-2}\right)$. The only disagreement is found in g_{tt} and $g_{\Omega_3\Omega_3}$, which contain a term at order $\mathcal{O}\left(\tilde{r}^{0}\right)$

$$\gamma(3\cos^2\tilde{\theta}-1)\,, \tag{5.64}$$

which, nonetheless, vanishes upon integration over the S^5 (including the appropriate measure). We notice that the same factor already appeared at leading order in the asymptotic expansion of the metric perturbation, see eq. (5.43).

From eq. (5.62) and the explicit expression for

$$g_{tt} = -\mathcal{N}^2 + g_{\tilde{\phi}\tilde{\phi}}\left(\mathcal{N}^{\tilde{\phi}}\right)^2 \tag{5.65}$$

one can read off the time-time component of the stress-tensor

$$\begin{aligned}\hat{T}_{tt} &= \left(g_{tt}(\tilde{r},\tilde{\theta})+1+\frac{\tilde{r}^2}{L^2}\right)\tilde{r}^2\\ &= \frac{\gamma}{8}\left(4(3\cos^2\tilde{\theta}-1)\tilde{r}^2+L^2\big(11-39\cos^2\tilde{\theta}+60\cos^4\tilde{\theta}\big)\right)+\mathcal{O}\left(\frac{1}{\tilde{r}^2}\right).\end{aligned} \tag{5.66}$$

This expression has to be integrated at the spacelike boundary in order to give the mass

$$\begin{aligned}M &= \frac{4}{16\pi G_{10}}\int\hat{\mu}\,\hat{T}_{tt}\\ &= \frac{4}{16\pi G_{10}}L^5(2\pi)(2\pi^2)^2\int_0^{\pi/2}d\theta\cos\theta\sin^3\theta\,\hat{T}_{tt}\,,\end{aligned} \tag{5.67}$$

where $G_{10}=\frac{\pi^4L^8}{2N^2}$ and $\hat{\mu}=\tilde{r}^{-3}\sqrt{g_{\tilde{\phi}\tilde{\phi}}\,g_{\theta\theta}\,g^3_{\Omega\Omega}\,g^3_{\tilde{\Omega}\tilde{\Omega}}}$ is the integration measure. The final result for the mass is

$$M = \frac{L^7}{4}\gamma = \frac{\pi^2}{6L}T^2\,, \tag{5.68}$$

which agrees with the thermal excitation energy of the N fermions above the ground state, eq. (5.27). The extra L in the denominator comes from the rescaling of the time variable already discussed. It is important to remark that in obtaining these expressions we have consistently worked at order γ. We will comment on the significance of higher order terms in a later section.

The superstar

As a further check of the validity of the procedure just discussed, we also apply it to the so-called *superstar*, a family of asymptotically $AdS_5 \times S^5$ backgrounds discovered in [203] and further studied from the LLM perspective in [204][199][205] [198]. The extremal 1/2 BPS superstar metric is governed by two parameters, the flux N of the 5-form through the S^5 and one of the three angular momenta on the S^5, J_3, which coincides with the energy Δ because of the BPS condition. Explicitly the metric can be written as [199]

$$\begin{aligned} ds^2 \;\; = \;\; & -\frac{1}{G}\left(\cos^2\theta + \frac{r^2}{L^2}G^2\right)dt^2 + \frac{L^2 H}{G}\sin^2\theta d\phi^2 + 2\frac{L}{G}\sin^2\theta dt d\phi + \\ & + G\left(\frac{dr^2}{f} + r^2 d\Omega_3^2\right) + L^2 G d\theta^2 + \frac{L^2}{G}\cos^2\theta d\tilde{\Omega}_3^2\,, \end{aligned} \tag{5.69}$$

with

$$f = 1 + H\frac{r^2}{L^2}\,, \qquad G = \sqrt{\sin^2\theta + H\cos^2\theta}\,, \qquad H = 1 + \frac{2L^2\Delta}{N^2r^2} \equiv 1 + \frac{Q}{r^2}\,. \tag{5.70}$$

Also in this example we want to satisfy the fall off conditions (5.61). By choosing an appropriate coordinate system as in eq. (5.63) it is easy to see that

$$\begin{aligned}\hat{T}_{tt} &= \frac{Q}{4L^2}\big(2 - 3\cos^2\tilde{\theta}\big)^2\tilde{r}^2 + \\ &+ \frac{Q}{64L^2}\big((6 - 36\cos^2\tilde{\theta})L^2 - (4 + 15\cos^2\tilde{\theta} - 24\cos^4\tilde{\theta})Q\big) + \mathcal{O}\left(\frac{1}{\tilde{r}}\right).\end{aligned} \tag{5.71}$$

The expression for the mass is then

$$M = \frac{4}{16\pi G_{10}}\int \hat{\mu}\,\hat{T}_{tt} = \frac{4}{16\pi G_{10}}L^5\pi^5 Q = \frac{\Delta}{L}\,, \tag{5.72}$$

which, up to the L coming from the rescaling of the time, is exactly the energy of the geometry. Note that we have again neglected contributions quadratic in Q in the stress-tensor (5.71).

Second approach: Background subtraction

We now discuss the second approach [212] for computing the mass of the hyperstar. In the background subtraction prescription the ADM mass is obtained by integrating the quasi-local energy $\mathcal{N}(K - K_0)$ over the $(d-2)$-dimensional spacelike hypersurface

Σ at radial infinity

$$M = \frac{1}{8\pi G_{10}} \int_{\Sigma} \mu \mathcal{N} (K - K_0). \tag{5.73}$$

To obtain M one needs $K^{\mu\nu}$, the extrinsic curvature of Σ embedded in a constant time hypersurface

$$K^{\mu\nu} = -\frac{1}{2} \nabla_{(h)}^{(\mu} \hat{r}^{\nu)} \, . \tag{5.74}$$

Now the covariant derivative is calculated with respect to the metric $h_{\mu\nu}$ of the constant time hypersurface, and $\hat{r}^\nu = g_{rr}^{-1/2} \delta_r^\nu$. In eq. (5.73) K and K_0 are the traces of the extrinsic curvature of the spacetime and of the reference background respectively, and μ is the measure on Σ.

In this case we also need to carefully tune the components of the boundary metric with those of the $AdS_5 \times S^5$ background by performing an asymptotic coordinate transformation as in the Ansatz (5.63). Let us first consider the extremal superstar solution in its five-dimensional reduction to understand which fall-off requirements we need to impose. The mass of this solution was first obtained in [213]. The line element reads

$$ds^2 = -H^{2/3} f dt^2 + H^{1/3} \left(f^{-1} dr^2 + r^2 d\Omega_3^2 \right) \, , \tag{5.75}$$

where H and f are defined as in eq. (5.70). The parameter Q appearing in eq. (5.70) is the five-dimensional electric charge and corresponds to the angular momentum J in the ten-dimensional uplifting of the superstar solution (5.69), see also [214]. We perform the following change of variable on the solution

$$\tilde{r}^2 = r^2 H^{1/3}\,, \tag{5.76}$$

which asymptotically amounts to

$$r = \tilde{r} - \frac{Q}{6\tilde{r}}\,. \tag{5.77}$$

A posteriori one can verify that additional higher order terms in eq. (5.77) do not modify the final answer for the mass. After this transformation, the difference between the components of the boundary metric $g_{\Omega_3\Omega_3}$ and the global AdS_5 background becomes of order $\mathcal{O}(\tilde{r}^{-4})$. An explicit calculation of the extrinsic curvature yields

$$K = -\frac{3}{L} - \frac{3L}{2\tilde{r}^2} + \left(\frac{3L^3}{8} - \frac{Q^2}{3L} + LQ\right)\frac{1}{\tilde{r}^4} + \mathcal{O}\left(\frac{1}{\tilde{r}^6}\right)\,. \tag{5.78}$$

To obtain a finite mass, we need to subtract the extrinsic curvature of AdS_5

$$K_0 = -\frac{3}{L}\left(1 + \frac{L^2}{\tilde{r}^2}\right)^{1/2}\,. \tag{5.79}$$

Using

$$M = \frac{1}{8\pi G_5}\int \mu\,\mathcal{N}\,(K - K_0) \tag{5.80}$$

with $G_5 = \pi/4$,[6] we obtain the well known result

$$M = Q\,. \tag{5.81}$$

[6]In this example we use the units of [213].

One can easily verify that if we had not implemented the transformation (5.77) we would have gotten

$$K = -\frac{3}{L} - \frac{3L}{2\tilde{r}^2} + \left(\frac{3L^3}{8} - \frac{Q^2}{3L} + \frac{3LQ}{2}\right)\frac{1}{\tilde{r}^4} + \mathcal{O}\left(\frac{1}{\tilde{r}^6}\right), \tag{5.82}$$

and correspondingly the incorrect result $M = \frac{3}{2}Q$. As in the ten-dimensional example (5.69), we have again neglected a term proportional to Q^2.

We now proceed similarly with the hyperstar solution using f_1, f_2 to fix the asymptotic behavior of $g_{\Omega_3\Omega_3}$ and analogously f_3, f_4 to fix $g_{\tilde{\Omega}_3\tilde{\Omega}_3}$.[7] Having four parameters at our disposal we require that $\delta g_{\Omega_3\Omega_3}$ and $\delta g_{\tilde{\Omega}_3\tilde{\Omega}_3}$ are of order $\mathcal{O}(\tilde{r}^{-4})$. With this choice we obtain $\delta g_{\theta\theta} = \mathcal{O}(\tilde{r}^{-2})$, while for the other component of the boundary metric $g_{\tilde{\phi}\tilde{\phi}}$ we have $\delta g_{\tilde{\phi}\tilde{\phi}} = \gamma L^4(-1 + 3\cos(\tilde{\theta}))$ which integrates to zero on the S^5. After having implemented this coordinate transformation, we can compute the extrinsic curvature to linear order in γ obtaining

$$\begin{aligned} K \;&=\; -\frac{3}{L} - \frac{L}{2\tilde{r}^2}\Big(3 - 7\big(3\cos^2\tilde{\theta} - 1\big)\gamma\Big) + \\ &\quad + \frac{L^3}{8\tilde{r}^4}\Big(3 + 4\big(28 - 159\cos^2\tilde{\theta} + 174\cos^4\tilde{\theta}\big)\gamma\Big) + \mathcal{O}\left(\frac{1}{\tilde{r}^5}\right). \end{aligned} \tag{5.83}$$

Subtracting the extrinsic curvature contribution of the background

$$K_0 = -\frac{3}{L}\left(1 + \frac{L^2}{\tilde{r}^2}\right)^{1/2} = -\frac{3}{L} - \frac{3L}{2\tilde{r}^2} + \frac{3L^3}{8\tilde{r}^4} + \mathcal{O}\left(\frac{1}{\tilde{r}^5}\right), \tag{5.84}$$

[7] We could have also chosen to use the parameters f_i to fix the other components $g_{\phi\phi}$,$g_{\theta\theta}$ of the boundary metric. This ambiguity alters only the quadratic contribution to the mass which, as will be discussed, is not physical.

and using the ADM mass formula, eq. (5.73), we obtain

$$M = \frac{L^7}{4}\gamma = \frac{\pi^2}{6L}T^2 \,, \tag{5.85}$$

which is again the expected result.

Contributions to the mass of order γ^2

It remains to discuss the relevance of the quadratic terms in γ that we have so far consistently neglected. According to the discussion following eq. (5.27), we would not expect contributions to the mass at orders higher than $\gamma \sim T^2$. We now check whether this is the case. Using the expressions at order $\gamma^2 \sim T^4$ for z^T_{FD} and V^T_{FD}

$$\begin{aligned}
z_{FD}^{T\,(2)}(R,y) &= \gamma^2 \frac{84 R_0^8\, y^2}{5\left[(R^2+R_0^2+y^2)^2 - 4R^2R_0^2\right]^{9/2}} \cdot \\
&\quad \cdot \Big((R_0^2+y^2)^4 + R^2(R_0^2-11y^2)(R_0^2+y^2)^2 + \\
&\qquad + 3R^4(3R_0^4+3R_0^2y^2-2y^4) + R^6(11R_0^2+14y^2) - 4R^8 \Big)\,, \\
V_{FD}^{T\,(2)}(R,y) &= \gamma^2 \frac{84 R_0^8\, R^2}{5\left[(R^2+R_0^2+y^2)^2 - 4R^2R_0^2\right]^{9/2}} \cdot \\
&\quad \cdot \Big((R_0^2-4y^2)(R_0^2+y^2)^3 - R^2(4R_0^6+9R_0^4y^2-9R_0^2y^4-14y^6) + \\
&\qquad + R^4(6R_0^2+21R_0^2y^2+6y^4) - R^6(4R_0^2+11y^2) + R^8 \Big)\,,
\end{aligned} \tag{5.86}$$

it is straightforward to write down the corresponding asymptotic expression for large r of the hyperstar metric, which is not particularly illuminating and, therefore, we do not present it.

It is not difficult to see that, differently from what expected, there seems to be a non-vanishing contribution to the mass proportional to γ^2.[8] The exact coefficient of this term depends on the procedure used to compute it. In the first approach discussed above there is a quadratic contribution to the stress-tensor

$$\hat{T}^{(2)}_{tt} = -\frac{\gamma^2}{16}\left(19 - 159\cos^2\tilde{\theta} + 216\cos^4\tilde{\theta}\right), \tag{5.87}$$

and to the mass

$$M^{(2)} = -\frac{L^7}{32}\gamma^2 = -\frac{\pi^4}{72L^9}T^4\,. \tag{5.88}$$

The method of background subtraction gives

$$K^{(2)} = -\gamma^2\frac{L^3}{8\tilde{r}^4}\left(69 - 540\cos^2\tilde{\theta} + 747\cos^4\tilde{\theta}\right), \tag{5.89}$$

and

$$M^{(2)} = -\frac{125L^7}{128}\gamma^2 = -\frac{125\pi^4}{288L^9}T^4\,. \tag{5.90}$$

The presence of this term and its scheme dependence are, however, not completely surprising and have already been discussed in the literature. In computing the super-star mass, both in five and ten dimensions, we already encountered a similar issue, see eqs. (5.71) and (5.78). Indeed, retaining the Q^2 terms in the computation of the mass, one would obtain a non-linear BPS condition $M \simeq Q - \frac{Q^2}{3L^2}$ [215]. This

[8]On the other hand, the angular momentum, which can be obtained from $\mathcal{N}^{\tilde{\phi}} \sim \frac{\gamma}{\tilde{r}^2} + \frac{\gamma^2}{\tilde{r}^4} + \ldots$, does not receive corrections beyond $\mathcal{O}(\gamma)$.

relation clearly conflicts with the expectation $M \geq |Q|$. One can nevertheless recover the usual linear BPS condition by including appropriate finite counterterms related to scalar fields [216]. This discussion can be generalized to the three-charged AdS_5 black hole. It has been observed in [217] that terms quadratic in the charges are related to a trace anomaly of the stress-tensor. This anomaly stems from a renormalization scheme which violates the asymptotic isometry group of AdS_5 and can be removed by adding to the action the finite counterterm proposed in [216].

In the light of these examples, we therefore consider eqs. (5.88) and (5.90) as spurious. They should be eliminated by a convenient choice of counterterms, although we do not know how to carry out this procedure directly in ten dimensions.

Orders beyond γ^2 do not contribute to the mass of the solution, because they fall off too fast at radial infinity.

5.4.5 Angular momentum

As a check of the BPS condition for the hyperstar solution, we now calculate the associated angular momentum. This computation is most easily done in a five-dimensional setting. The ten-dimensional angular momentum J coincides with the electric charge Q of the $U(1)$ gauge field A coming from dimensional reduction on the S^5. The gauge field can be read off from the term $g_{\tilde{\phi}\tilde{\phi}}(d\tilde{\phi}+\mathcal{N}^{\tilde{\phi}}dt)^2$ in the ADM metric and therefore coincides with the shift vector

$$A = \mathcal{N}^{\tilde{\phi}}\, dt = \frac{2L}{r^2}\gamma\, dt + \mathcal{O}\left(\frac{1}{r^4}\right) . \tag{5.91}$$

The associated charge (angular momentum) is then[9]

$$J = \frac{L^2}{16\pi G_5} \int_{S^3_\infty} \star_5 \, dA = \gamma \frac{L^8}{4} = \frac{\pi^2}{6} T^2 \,, \tag{5.92}$$

where $\star_5$ is the five-dimensional Hodge star operator. In our normalization the five-dimensional Newton constant is $G_5 = G_{10}/Vol(S^5) = 2\pi/L^5$. The L^2 factor in eq. (5.92) is necessary for obtaining conventional units. Comparing J with the mass formula eq. (5.68), we obtain the BPS relation $M = J/L$.

5.4.6 Entropy

In the previous sections we have found agreement between the ADM mass and the thermal energy of the fermions. Since the Fermi gas has non-vanishing entropy at non-zero temperature, we expect the same to occur for the supergravity solution. We would like to understand how this entropy arises geometrically in the case of the hyperstar. Although the solution we are considering seems to have a naked singularity, it is expected that α' corrections to the equations of motion might generate a finite-area *stretched horizon.* With these corrections we can think of the hyperstar as a legitimate black hole.

In the presence of an event horizon, the entropy of a gravitational solution in d dimensions is given by the celebrated Bekenstein-Hawking formula

$$S = \frac{\mathcal{A}}{4G_d} \tag{5.93}$$

where $\mathcal{A}$ is the area of the horizon. In our case the entropy is still given by eq. (5.93)

[9]Looking at the five-dimensional gauged supergravity action one would expect a contribution to the charge of the type $\int_{S^3_\infty} A \wedge F$. This term is nonetheless subleading and vanishes at radial infinity.

but now $\mathcal{A} = \mathcal{A}_{sh}$ is the area of the stretched horizon.

Since we do not know the explicit form of the α' corrections, the location of the stretched horizon is inherently ambiguous. Therefore we expect to reproduce the fermion entropy up to a numerical coefficient.

As we already discussed, the $y = 0$ plane is a null singular region. It is reasonable to assume that the α' corrections will generate a horizon at $y_{sh} \simeq 0 + \mathcal{O}(\alpha')$. We therefore need to compute the area of the $y = y_{sh}$ plane, with $y_{sh} \simeq \alpha' = g_s^{-1/2}\, l_p^2 \sim g_s^{-1/2}$ in units where $\hbar = 1$. This area turns out to be finite. The metric in LLM coordinates for fixed t and y reads

$$ds^2|_{t\,,\,y=fixed} = -h^{-2}V^2 d\phi^2 + h^2(dR^2 + R^2 d\phi^2) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2\,, \qquad (5.94)$$

so that the integration measure is

$$\mu = \sqrt{h^4 R^2 - V^2}\, y^3 \simeq h^2\, R\, y^3 \simeq \left(\frac{1}{4} - z_0^2\right)^{\frac{1}{2}} R\, y^2\,, \qquad (5.95)$$

where we have assumed the expansion (5.47), so that the term $V^2 \sim \ln^2 y$ can be neglected for small y against $h^4 \sim y^{-2}$ and $z \simeq z_0$. By restricting the measure (5.95) to $y = y_{sh}$, the Bekenstein-Hawking formula yields

$$\begin{aligned} S &= \frac{\mathcal{A}_{sh}(y = y_{sh})}{4G_{10}} = \frac{2\pi(2\pi^2)^2}{4 \cdot 2\pi^4} y_{sh}^2 \int_0^\infty RdR \left(\frac{1}{4} - z_0^2\right)^{1/2} \\ &\simeq c \int_0^\infty RdR\sqrt{n(1-n)}\,, \end{aligned} \qquad (5.96)$$

where c is a numerical constant. For the hyperstar $n = n_{FD}$ so that

$$\sqrt{n_{FD}(1 - n_{FD})} = \frac{e^{\frac{\beta}{2}(\epsilon-\mu)}}{1 + e^{\beta(\epsilon-\mu)}}\,, \tag{5.97}$$

with $\epsilon = R^2/2$ and $\mu = T\left(e^{N/T} - 1\right)$. Using eq. (5.96) we obtain

$$\begin{aligned} S &\simeq c\int_0^\infty d\epsilon \frac{e^{\frac{\beta}{2}(\epsilon-\mu)}}{1 + e^{\beta(\epsilon-\mu)}} \\ &= 2\,c\,T\left(\frac{\pi}{2} - \arctan e^{-\frac{\beta\mu}{2}}\right) = 2\,c\,T\left(\frac{\pi}{2} - \arctan \frac{1}{\sqrt{e^{N/T} - 1}}\right)\,. \end{aligned} \tag{5.98}$$

In the low temperature approximation this yield

$$S \propto T\left(1 + \mathcal{O}(e^{-N/T})\right)\,. \tag{5.99}$$

Therefore the entropy is proportional to T, as expected from eq. (5.31), up to corrections which are exponentially suppressed for $N \gg T$.

In the high temperature limit, however, eq. (5.98) does not seem to reproduce the logarithmic behavior of the Boltzmann entropy. In this limit, which will be studied in the next section, the assumptions and the approximations which led to eq. (5.96) might not be valid since T is not a small parameter.

5.5 High temperature regime

We now move to consider the high temperature regime. In this limit the Fermi-Dirac distribution reduces to the classical Boltzmann distribution. Correspondingly, the droplet spreads over a larger part of the $y = 0$ plane and the singular greyscale region is not confined inside a thin ring anymore, as shown in fig. 5.2 .

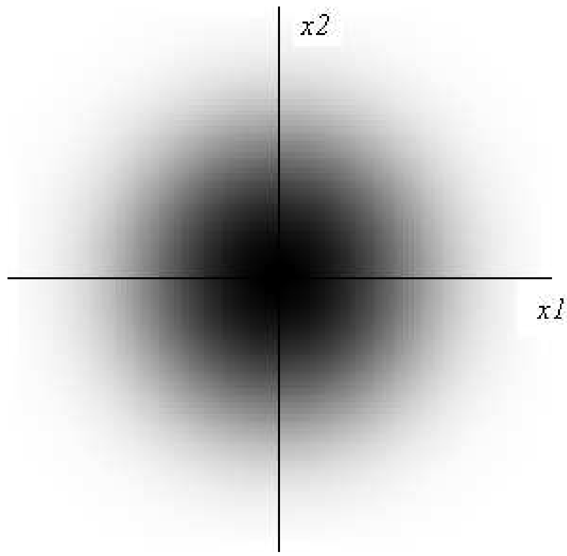


Figure 5.2: Droplet configuration in the high temperature limit.

The auxiliary function z can be computed in this regime as

$$z_B^T(R,y) \quad = \quad \frac{1}{2} + \int_0^\infty n_B(R')\frac{\partial}{\partial R'} z_0(R,y;R')dR'\,, \tag{5.100}$$

where

$$n_B(R) = N\beta e^{-\beta \frac{R^2}{2}}\,, \tag{5.101}$$

and $\beta = 1/T$. Making the change of variable $R'^2/2 = \epsilon$ and using the explicit expression for $z_0(R,y;R')$ we can write the integral as

$$z_B^T(R,y) = \frac{1}{2} - 2y^2 N\beta\, I_z(\beta,R,y)\,, \tag{5.102}$$

where we have defined

$$I_z(\beta, R, y) \equiv \int_0^\infty d\epsilon \; e^{-\beta\epsilon} \frac{(2\epsilon + y^2 + R^2)}{[(2\epsilon + y^2 + R^2)^2 - 8R^2\epsilon]^{3/2}}. \tag{5.103}$$

The high temperature limit corresponds to the small β region. Therefore we want to find an approximate expression for $I_z(\beta, R, y)$ near $\beta = 0$. It is easy to verify that $I_z(0, R, y) = \frac{1}{2y^2}$, and also that

$$\frac{\partial I_z}{\partial \beta} = -\int_0^\infty d\epsilon \; \frac{\epsilon \; e^{-\beta\epsilon}(2\epsilon + R^2 + y^2)}{[(2\epsilon + y^2 + R^2)^2 - 8R^2\epsilon]^{3/2}} \tag{5.104}$$

diverges as $\ln\beta$ in proximity of $\beta = 0$, because the integrand goes like $e^{-\beta\epsilon}/\epsilon$ for large ϵ. This suggests a low β expansion of the form

$$\begin{aligned} I_z(\beta, R, y) &= I_z(0, R, y) + A\beta\ln\beta + ... \\ z_B^T(R, y) &= \frac{1}{2} - 2y^2 N\beta\left(\frac{1}{2y^2} + A\beta\ln\beta + ...\right). \end{aligned} \tag{5.105}$$

Since it is not possible to compute explicitly $\frac{\partial I_z}{\partial \beta}$ for $\beta \to 0$ because of the divergence, to find its small β behavior we find it useful to first regulate the integral by considering the quantity

$$\frac{\partial I_z}{\partial \beta} + \frac{1}{4}\int_0^\infty d\epsilon \; \frac{\epsilon \, e^{-\beta\epsilon}}{\epsilon^2 + (R^2 + y^2)^2}. \tag{5.106}$$

The new piece

$$I_0 \equiv -\frac{1}{4}\int_0^\infty d\epsilon \; \frac{\epsilon \, e^{-\beta\epsilon}}{\epsilon^2 + (R^2 + y^2)^2} \tag{5.107}$$

has the same divergence structure of $\frac{\partial I_z}{\partial \beta}$ and its value is known for finite β in terms of the Sine and Cosine Integral functions $Si(x)$, $Ci(x)$. The corresponding small β expansion can be given explicitely as

$$I_0 = \frac{\gamma}{4} + \frac{1}{4}\ln[\beta(R^2 + y^2)] - \frac{\pi}{8}\beta(R^2 + y^2) + \mathcal{O}(\beta^2 \ln \beta)\,, \tag{5.108}$$

where γ is the Euler-Mascheroni constant. The combination (5.106) is by construction convergent for any β, and has a well defined $\beta \to 0$ limit which can be easily computed analytically

$$\left(\frac{\partial I_z}{\partial \beta} - I_0\right)\Big|_{\beta=0} = \frac{1}{4}\left(1 - \frac{R^2}{y^2}\right) - \frac{\ln 2}{4} + \frac{1}{4}\ln y^2 - \frac{1}{4}\ln(R^2 + y^2)\,. \tag{5.109}$$

Using eq. (5.108) we obtain the high temperature expansion of $\frac{\partial I_z}{\partial \beta}$

$$\frac{\partial I_z}{\partial \beta} = \frac{1}{4}\ln\beta + \frac{1}{4}\left(\gamma - \ln 2 + 1 - \frac{R^2}{y^2} + \ln y^2\right) + \mathcal{O}(\beta \ln \beta)\,, \tag{5.110}$$

and integrating in β we can finally get

$$I_z(\beta, R, y) = \frac{1}{2y^2} + \frac{1}{4}\beta\ln\beta + \frac{\beta}{4}\left(\gamma - \ln 2 - \frac{R^2}{y^2} + \ln y^2\right) + \mathcal{O}(\beta^2 \ln \beta). \tag{5.111}$$

The corresponding high temperature limit of z_B^T, keeping only the first two orders, reads then as follows

$$z_B^T(R, y) = \frac{1}{2} - N\beta - \frac{Ny^2}{2}\beta^2 \ln\beta + \mathcal{O}(\beta^2). \tag{5.112}$$

To obtain the metric we need to find also the function $V^T(R, y)$. Starting from eq. (5.15) and inserting the Boltzmann distribution we arrive at

$$V_B^T(R,y) = \int_0^\infty d\epsilon\, N\beta e^{-\beta\epsilon} \frac{2R^2(2\epsilon + y^2 + R^2)}{[(2\epsilon + y^2 + R^2)^2 - 8R^2\epsilon]^{3/2}} \equiv 2N\beta R^2 I_V(\beta, R, y)\,. \tag{5.113}$$

One can verify that $I_V(0,R,y)$ vanishes and that $\frac{\partial I_V}{\partial \beta}$ diverges logarithmically in the $\beta \to 0$ limit. We can proceed similarly as before by regulating $\frac{\partial I_V}{\partial \beta}$ with an appropriate "reference" integral, to finally obtain

$$I_V(\beta, R, y) = -\frac{1}{4}\beta \ln\beta - \frac{\beta}{4}\left(\gamma - \ln 2 + 1 + \ln y^2\right) + \mathcal{O}(\beta^2 \ln \beta). \tag{5.114}$$

In proximity of $\beta = 0$ the leading contribution to V_B^T is therefore

$$V_B^T(R,y) = -\frac{1}{2} N R^2 \beta^2 \ln\beta + \mathcal{O}(\beta^2). \tag{5.115}$$

The expressions for z_B^T and V_B^T consistently satisfy eq. (5.13). Note that z_B^T does not depend on R and that similarly V_B^T does not depend on y. This fact is nevertheless an artefact of the approximation we made. To study its limits of validity, we can look at eqs. (5.112) and (5.115) and require that the corrections are small. From this we can infer the conditions

$$y^2 \ll \frac{1}{\beta \ln\beta}\,, \qquad R^2 \ll \frac{1}{N\beta^2 \ln\beta}. \tag{5.116}$$

We can now find the metric at first order in the low β expansion, *i.e.* $z_B^T = 1/2 - N\beta$ and $V_B^T = 0$. The metric in the LLM coordinates is quickly computed and reads

$$ds^2 = \frac{y}{\sqrt{N\beta}}\left(-dt^2 + d\Omega_3^2\right) + \frac{\sqrt{N\beta}}{y}\left(dy^2 + y^2 d\tilde{\Omega}_3^2 + dR^2 + R^2 d\phi^2\right). \qquad (5.117)$$

Rescaling the coordinates as

$$\tilde{t} = (N\beta)^{-1/4} t\,, \qquad \tilde{y} = (N\beta)^{1/4} y\,, \qquad \tilde{R} = (N\beta)^{1/4} R\,, \qquad (5.118)$$

brings the metric into the form

$$ds^2 = (N\beta)^{1/4}\tilde{y}\left(-d\tilde{t}^2 + \frac{1}{\sqrt{N\beta}} d\Omega_3^2\right) + \frac{1}{(N\beta)^{1/4}\tilde{y}}\left(d\tilde{y}^2 + \tilde{y}^2 d\tilde{\Omega}_3^2 + d\tilde{R}^2 + \tilde{R}^2 d\phi^2\right). \qquad (5.119)$$

This form of the metric closely resembles the dilute gas approximation limit studied in [184]. There, one considers a configuration of droplets with area A_i in the (x_1, x_2) plane, and send the distance between the droplets to infinity by the rescaling

$$x \to \lambda\tilde{x}\,, \qquad x' \to \lambda\tilde{x}\,, \qquad y \to \lambda\tilde{y}\,, \qquad \lambda \to \infty\,, \qquad (5.120)$$

while keeping the droplets areas A_i fixed. The corresponding metric reads

$$ds^2 = H^{-1/2}\left[-d\tilde{t}^2 + \lambda^2 d\Omega_3^2\right] + H^{1/2}\left[d\tilde{y}^2 + \tilde{y}^2 d\tilde{\Omega}_3^2 + dx^i dx^i\right], \qquad (5.121)$$

where the harmonic function H is

$$H = \frac{1}{\pi}\sum_i \frac{A_i}{[(\tilde{x} - \tilde{x}_i')^2 + \tilde{y}^2]^2}\,. \qquad (5.122)$$

Thus the metric (5.121) can be viewed as a multi-center solution for a stack of separated D3-branes, and corresponds to the $SO(4)$ invariant sector of the Coulomb branch of the gauge theory.

Upon the identification $\lambda = (N\beta)^{-1/4}$, one can see that the dilute gas limit $\lambda \to \infty$ is similar to the high temperature regime $\beta \to 0$ of the thermal solution eq. (5.119). This is perhaps not surprising since in the high temperature limit the fermion density goes to zero. We also notice that a continuum version of eq. (5.122) with $A_i \equiv d^2\tilde{x}'/\sqrt{N\beta}$ gives

$$H = \frac{1}{\pi} \int \frac{d^2\tilde{x}'}{\sqrt{N\beta}} \frac{1}{[(\tilde{x}-\tilde{x}')^2 + \tilde{y}^2]^2} = \frac{1}{\sqrt{N\beta}\,\tilde{y}^2}\,, \tag{5.123}$$

which is what we would expect in order to match eq. (5.119) with eq. (5.121).

Taking into account the next to leading order corrections for z_B^T and V_B^T in eq. (5.112) and (5.115), we obtain the metric

$$\begin{aligned} ds^2 &= H^{-1/2}\left[-dt^2 + (1-N\beta)d\tilde{\Omega}_3^2\right] + H^{1/2}\left[dy^2 + (1+N\beta)y^2 d\Omega_3^2 + dR^2 + R^2 d\phi^2\right] \\ &\quad + \sqrt{N}R^2 y \beta^{3/2} \ln\beta dt d\phi \end{aligned} \tag{5.124}$$

where

$$H = \frac{N\beta}{y^2} - \frac{N^2\beta^2}{y^2} + \frac{1}{2}\beta^2 N \ln\beta. \tag{5.125}$$

At this order we have a non-vanishing V_B^T and this determines the presence of the mixed term $g_{\phi t}$ in the metric.

5.5.1 Energy and angular momentum

We remark that the region of validity of the approximations made so far does not allow us to use the metrics (5.117) and (5.124) in the asymptotic region $R^2+y^2 \gg 1$, because of the conditions (5.116). Therefore, to compute the energy of the hyperstar in the high temperature regime, we need to find the form of the metric in the complementary region of validity. The new metric will be trustable in the asymptotic region and will allow a calculation of the energy with the methods already discussed. To this end, it is convenient to first introduce polar coordinates in the (x_1, x_2, y) space

$$R = u \cos\vartheta\,, \qquad y = u \sin\vartheta\,. \tag{5.126}$$

Then one can evaluate eqs. (5.102) and (5.113) in an expansion for $u \gg 1$ while keeping T fixed but large (such that we are in the Boltzmann regime). The integrals involved in the expansion can be readily computed analytically and one ends up with the result

$$\begin{aligned} z_B^T(u,\vartheta) &= \frac{1}{2} - 2N\sin^2\vartheta\frac{1}{u^2} - 8\,NT\sin^2\vartheta(3\cos^2\vartheta-1)\frac{1}{u^4} \\ &\quad -48\,NT^2\sin^2\vartheta(10\cos^4\vartheta - 8\cos^2\vartheta+1)\frac{1}{u^6} + \mathcal{O}\left(\frac{1}{u^8}\right)\,, \qquad (5.127) \\ V_B^T(u,\vartheta) &= 2N\cos^2\vartheta\frac{1}{u^2} + 8\,NT\cos^2\vartheta(3\cos^2\vartheta-2)\frac{1}{u^4} \\ &\quad +48\,NT^2\cos^2\vartheta(10\cos^4\vartheta-12\cos^2\vartheta+3)\frac{1}{u^6} + \mathcal{O}\left(\frac{1}{u^8}\right)\,, \qquad (5.128) \end{aligned}$$

where in the expansion we have kept only terms which contribute to the mass and angular momentum. One can now go to the $AdS_5 \times S^5$ coordinates via the change of variables given in eq. (5.38) and use z_B^T and V_B^T to obtain the asymptotic form of the metric. The explicit expressions are somewhat lengthy and we will not report them here in detail. The computation of M and J follows the same lines of the one given in detail for the low temperature regime. Particularly straightforward is the evaluation of the angular momentum, which can be read off from the shift vector $\mathcal{N}^{\tilde{\phi}}$. The explicit calculation gives

$$\mathcal{N}^{\tilde{\phi}} = \left(\frac{4T}{L^3} - L\right)\frac{1}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) = \frac{2L}{N^2 r^2}\left(NT - \frac{N^2}{2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (5.129)$$

where we have used the relation $N = L^4/2$ which holds in our units. Viewing $\mathcal{N}^{\tilde{\phi}} dt \equiv A$ as a gauge field in five dimensions, the angular momentum is equal to the corresponding electric charge, as explained in the previous section. The result is then

$$J = NT - \frac{N^2}{2}. \quad (5.130)$$

This is indeed what we would have expected, since NT is the energy for a gas of N particles with Boltzmann density and $N^2/2$ is the ground state energy of the N fermions.

To compute the mass, we used both methods described in the previous section. Once again, quadratic terms in the charge $(NT - N^2/2)$ appear in the calculation, with different coefficients in the two methods. The linear term is however scheme independent and gives the correct result

$$M = \left(NT - \frac{N^2}{2}\right)/L\,. \quad (5.131)$$

5.6 Concluding remarks and outlook

In this chapter of the dissertation we explored the thermodynamic properties of a 1/2 BPS IIB supergravity solution called hyperstar. This background was first obtained in [197] by thermal coarse-graining of the bubbling AdS geometry found in [184]. The hyperstar is in correspondence with a distribution of free fermions in thermodynamic equilibrium at temperature T, living on a two-dimensional phase space contained in the ten-dimensional geometry.

We studied both limits of low and high temperature. In the former case, the fermions obey the Fermi-Dirac distribution and the supergravity background is obtained from the LLM Ansatz by means of a Sommerfeld expansion. We found agreement between the energy of the fermions and the ADM mass of the supergravity, modulo a subtlety involving T^4 terms which we discussed in the main text. We also proposed a way to match the entropy of the fermions with the entropy of the hyperstar in the low temperature limit. String α' corrections are expected to generate a finite area stretched horizon, lifting the naked singularity of the hyperstar to a true black hole singularity.

In the classical limit of high temperature, we found the explicit form of the metric of the supergravity background and we observed how this metric resembles the metric of a dilute gas of D3-branes, which corresponds to the $SO(4)$ invariant sector of the Coulomb branch of the CFT. We also computed the associated mass and angular momentum.

It would be interesting to push this study further. An important point, as already remarked, would be to understand better the meaning of the temperature for the supergravity solution. On a more fundamental level, it is worthwile to understand the exact relation between a thermalized solution like the hyperstar and the matrix model description of the 1/2 BPS sector of the dual CFT, extending considerations already made in [198].

Another issue is whether the appearance of the naked singularity in the hyperstar can be understood in terms of a distribution of giant gravitons, as is the case for the superstar [203].

The LLM geometries, upon dimensional reduction to five dimension, can be seen as interesting generalizations of AdS 1/2 BPS extremal black holes [218]. It would then be interesting to obtain the explicit dimensional reduction to five dimension of the hyperstar. In this setting one could use the powerful methods of holographic renormalization to carry out the computation of the ADM mass. Then one could prove in a rigorous way that the quadratic contributions to the mass are effectively spurious and can be eliminated within an appropriate renormalization scheme.

Finally, we would like to mention that the LLM construction has been extended to other BPS sectors of type IIB supergravity, see, for instance, [219] for the 1/4 BPS sector. In this case, one modifies the LLM Ansatz in order to accomodate an axion-dilaton field which breaks the supersymmetry by half. The effect of this field is to introduce a deficit angle in the "phase space". One could try to understand whether this phase space can be useful to study the mass and entropy of the corresponding supergravity geometry.

Chapter 6

Supersymmetric Wilson loops

This chapter contains three different studies on Wilson loop operators in the AdS/CFT correspondence. We start in the next section by introducing a large new class of supersymmetric loop operators. Given an arbitrary curve on a three-dimensional sphere we define a certain scalar coupling so that the loop preserves at least two supercharges. We present many explicit examples of loops not known before, providing a wide arena for possible calculations, both on the gauge theory side and in string theory, which may lead to further tests of the AdS/CFT correspondence.

In section 6.2 we study correlation functions of circular Wilson loops in higher dimensional representations with chiral primary operators of $\mathcal{N} = 4$ super Yang-Mills theory. This is done using the relation discussed in chapter 2 between higher rank Wilson loops in gauge theory and D-branes with electric fluxes in supergravity. We verify our results with a matrix model computation, finding perfect agreement in both the symmetric and the antisymmetric case.

Finally, in section 6.3 we present D3-brane solutions describing some 1/4 BPS loops. In one case, where the loop is conjectured to be given by a Gaussian matrix model, the action of the brane correctly reproduces the expectation value of the Wilson loop including all $1/N$ corrections at large λ. As in the corresponding string solution, here too we find two classical solutions, one stable and one not. The unstable one contributes exponentially small corrections that agree with the matrix model calculation.

6.1 A new class of supersymmetric Wilson loop

6.1.1 Definition and supersymmetry analysis

As explained in detail in chapter 2, a necessary requirement for a Wilson loop to be supersymmetric is that the norm of the vector Θ^I be one. But that alone leads only to local supersymmetry. If one considers the supersymmetry variation of the loop, then at every point along the loop one finds another condition for preserved supersymmetry. Only if all those conditions commute, will the loop be globally supersymmetric.

One simple way to satisfy this is if at every point one finds the same equation. This happens in the case of the straight line, where $\dot{x}^\mu$ is a constant vector and one takes also Θ^I to be a constant. This idea was generalized in a very ingenious way by Zarembo [68], who assigned for every tangent vector in $\mathbb{R}^4$ a unit vector in $\mathbb{R}^6$ (*i.e.* a 6×4 matrix $M^I{}_\mu$) and took $|\dot{x}|\Theta^I = M^I{}_\mu \dot{x}^\mu$. That construction guarantees that if a curve is contained within a one-dimensional linear subspace of $\mathbb{R}^4$ it is 1/2 BPS. Inside a 2-plane it will be 1/4 BPS, inside $\mathbb{R}^3$ it's 1/8 BPS and a generic curve is 1/16 BPS.

An amazing fact about those loops is that their expectation value seems to be trivial (the degree of rigor of this statement depends on the amount of supersymmetry)

[220]-[222]. But this is also a crucial shortcoming of this construction. As seen in chapter 2, one of the most interesting Wilson loop observables is the circle with a coupling to a single scalar, whose expectation value is a non-trivial function of both the rank of the gauge group N and of the 't Hooft coupling λ. This Wilson loop preserves 1/2 of the supersymmetries, but is not given by the above construction. Recently some 1/4 BPS loops were described that also do not have trivial expectation values, rather the values of all those loops seem to be described by a 0-dimensional matrix model [223].

Here we present a generalization of Zarembo's construction. This generalization will allow for loops which do not have a trivial expectation value and in a certain limit reproduce many of Zarembo's loops.

Consider an arbitrary curve on S^3, which we usually take as the unit sphere in flat $\mathbb{R}^4$ in the Euclidean gauge theory. This can also be the spatial slice for the Lorentzian theory on $S^3\times\mathbb{R}$. The basic ingredient in our proposal for the supersymmetric Wilson loop are the invariant one-forms on the group manifold $SU(2) = S^3$ (here we follow the conventions of [224]). In the flat coordinates x^μ satisfying $x^2 = 1$ they read

$$\begin{aligned}
\sigma_1^{R,L} &= 2\left[\pm(x^2dx^3 - x^3dx^2) + (x^4dx^1 - x^1dx^4)\right] \,, \\
\sigma_2^{R,L} &= 2\left[\pm(x^3dx^1 - x^1dx^3) + (x^4dx^2 - x^2dx^4)\right] \,, \\
\sigma_3^{R,L} &= 2\left[\pm(x^1dx^2 - x^2dx^1) + (x^4dx^3 - x^3dx^4)\right] \,,
\end{aligned} \tag{6.1}$$

where σ_i^R are the right (or left-invariant) one-forms and σ_i^L are the left (or right-invariant) one-forms. These are respectively dual to left (right) invariant vector fields ξ_i^R (ξ_i^L) generating right (left) group actions. We can now use either σ_i^R or σ_i^L to define a natural coupling to three of the scalars, say Φ^1, Φ^2, Φ^3. We choose to use the right one-forms. Our Ansatz for the supersymmetric Wilson loop on S^3 is then

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\oint\left(iA + \frac{1}{2}\sigma_i^R M^i{}_I\Phi^I\right), \tag{6.2}$$

where for convenience we write the integral in form notation. The 3×6 matrix $M^i{}_I$ specifies which three scalars the loop will couple to and satisfies that $MM^\top$ is the 3×3 unit matrix. When we need an explicit choice of M we take $M^1{}_1 = M^2{}_2 = M^3{}_3 = 1$ and all other entries zero. As a consistency check, we can see that the coupling to the scalars satisfies $\Theta^I\Theta^I = 1$, which follows from the condition on M and the property

$$\sigma_i^R\sigma_i^R = 4dx^\mu dx_\mu\,. \tag{6.3}$$

For the analysis of the supersymmetries preserved by these loops it is convenient to switch from forms to tangent vectors and write the pull-backs of the σ's in terms of $\mathbf{x} = ix^1\tau_1 + ix^2\tau_2 + ix^3\tau_3 + x^4\mathbb{I}$, the $SU(2)$ matrix representing x^μ, where τ_i are the Pauli matrices and $\mathbb{I}$ is the 2×2 identity matrix, as

$$\hat\sigma_i^R = -i\,\mathrm{tr}\,(\tau_i\,\mathbf{x}^\dagger\,\dot{\mathbf{x}})\,,\qquad \hat\sigma_i^L = -i\,\mathrm{tr}\,(\tau_i\,\dot{\mathbf{x}}\,\mathbf{x}^\dagger)\,. \tag{6.4}$$

The trace here is over this $SU(2)$.

We can now show that our Ansatz (6.2) leads to a supersymmetric Wilson loop. The supersymmetry variation of the Wilson loop is proportional to

$$\delta W \simeq \left(i\dot x^\mu\gamma_\mu + \frac{1}{2}\hat\sigma_i^R M^i{}_I\rho^I\gamma^5\right)\epsilon(x)\,. \tag{6.5}$$

Here we use a four-dimensional notation rather than the ten-dimensional one of chapter 2, then γ_μ and ρ^I are respectively the gamma matrices of $SO(4)$ and $SO(6)$, the Poincaré and R-symmetry groups and they commute with each-other. $\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$ is the four dimensional chirality matrix and $\epsilon(x)$ is a conformal Killing spinor given by two arbitrary constant spinors (which are also spinors of the R-symmetry group)

$$\epsilon = \epsilon_0 + x^\mu\gamma_\mu\epsilon_1\,. \tag{6.6}$$

We rearrange the variation of the loop as

$$\delta W \simeq i\dot{x}^\mu x^\nu\gamma_{\mu\nu}\epsilon_1 + \frac{1}{2}\hat{\sigma}_i^R M^i{}_I\rho^I\gamma^5\epsilon_0 - x^\eta\gamma_\eta\left(i\dot{x}^\mu x^\nu\gamma_{\mu\nu}\epsilon_0 + \frac{1}{2}\hat{\sigma}_i^R M^i{}_I\rho^I\gamma^5\epsilon_1\right). \tag{6.7}$$

Note now that the action of the γ's on a chiral spinor $\epsilon^\pm = \frac{1}{2}(1\pm\gamma^5)\epsilon$ can be expressed in terms of the Pauli matrices, allowing one to write

$$i\dot{x}^\mu x^\nu\gamma_{\mu\nu}\epsilon^\mp = \pm\frac{1}{2}\tau^i\hat{\sigma}_i^{R,L}\epsilon^\mp\,. \tag{6.8}$$

The first two terms in eq. (6.7) can then be decomposed by their chirality as

$$\delta W \simeq \frac{1}{2}\left(\hat{\sigma}_i^R(\tau^i\epsilon_1^- - M^i{}_I\rho^I\epsilon_0^-) - (\hat{\sigma}_i^L\tau^i\epsilon_1^+ - \hat{\sigma}_i^R M^i{}_I\rho^I\epsilon_0^+)\right), \tag{6.9}$$

and a similar expression holds for the last two terms.

Thus for a generic curve on S^3, when there are no linear relation between the six $\hat{\sigma}_i^{R,L}$ and $x^\eta\gamma_\eta$ is non-trivial, the only solution to the supersymmetry equation is for

$$\tau^i\epsilon_1^- = M^i{}_I\rho^I\epsilon_0^-\,,\qquad \epsilon_1^+ = \epsilon_0^+ = 0\,. \tag{6.10}$$

For special curves, when the pull-backs of the forms are not independent, there will be more solutions and the Wilson loops will preserve more supersymmetry. We will demonstrate this in some special cases below.

To solve this set of equations let us choose the matrix M that identifies i with I. Then we can eliminate for example ϵ_0^- from eq. (6.10) to get

$$i\tau_1\epsilon_1^- = -\rho_{23}\epsilon_1^- \,, \qquad i\tau_2\epsilon_1^- = -\rho_{31}\epsilon_1^- \,, \qquad i\tau_3\epsilon_1^- = -\rho_{12}\epsilon_1^- \,. \tag{6.11}$$

This is a set of constraints that are consistent with each other. However it is easy to see that only two of them are independent since the commutator of any two gives the remaining equation. With two independent projectors we are thus left with two independent components of ϵ_1^-, while ϵ_0^- depends on ϵ_1^-. So we conclude that for a generic curve on S^3 the Wilson loop preserves 1/16 of the original supersymmetries.

We would now like to explicitly find the two combinations of $\bar{Q}$ and $\bar{S}$ which leave the Wilson loop invariant. Notice that in singling out three of the scalars we are breaking the R-symmetry group $SU(4)$ down to $SU(2)_A \times SU(2)_B$, where $SU(2)_A$ corresponds to rotations of Φ^1, Φ^2, Φ^3 while $SU(2)_B$ rotates Φ^4, Φ^5, Φ^6. Then we recognize that the operators appearing in eq. (6.11) are just the generators of $SU(2)_R$ and $SU(2)_A$, and the above equations simply state that ϵ_1^- is a singlet of the diagonal sum of $SU(2)_R$ and $SU(2)_A$, while it's a doublet of $SU(2)_B$. More explicitly, we can always choose a basis in which ρ^i act as Pauli matrices on the $SU(2)_A$ indices, such that the equations above become

$$(\tau_k^R + \tau_k^A)\epsilon_1^- = 0, \qquad k = 1, 2, 3. \tag{6.12}$$

If we split the $SU(4)$ index in ϵ_1^- as

$$\epsilon^A_{1,\dot\alpha} = \epsilon^a_{1,\dot\alpha\dot a}, \tag{6.13}$$

where $\dot a$ and a are respectively $SU(2)_A$ and $SU(2)_B$ indices, then the solution to eq. (6.12) can be written as

$$\epsilon_1^a = \varepsilon^{\dot\alpha\dot a}\epsilon^a_{1,\dot\alpha\dot a}. \tag{6.14}$$

Using any of the equations in (6.10) we can determine ϵ_0

$$\epsilon_0^- = \tau_3^R\rho^3\epsilon_1^- = \tau_3^R\tau_3^A\epsilon_1^- = -\epsilon_1^-, \tag{6.15}$$

where in the last equality we used eq. (6.12). Our conclusion is then that the Wilson loops we introduced preserve the two supercharges

$$\bar{\mathcal{Q}}^a = \varepsilon^{\dot\alpha\dot a}\left(\bar Q^a_{\dot\alpha\dot a} - \bar S^a_{\dot\alpha\dot a}\right). \tag{6.16}$$

6.1.2 Topological twisting

This construction hints at a possible connection with a topologically twisted version of $\mathcal{N}=4$ SYM. The twisting consists in replacing $SU(2)_R$ with the diagonal sum of $SU(2)_R$ and $SU(2)_A$, which we denote as $SU(2)_{R'}$, so that the twisted Lorentz group is $SU(2)_L \times SU(2)_{R'}$. This twisting[1] was first considered in [226] and further studied in [225]. After the twisting the supercharges decompose under $SU(2)_L \times SU(2)_{R'} \times$

[1]It is case ii) in [225]. A different twisting, namely case i), was considered in [222] in the context of Zarembo's supersymmetric Wilson loops.

$SU(2)_B$ as

$$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})\rightarrow(\mathbf{2},\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3},\mathbf{2})+(\mathbf{1},\mathbf{1},\mathbf{2}). \tag{6.17}$$

From the above it is clear that the two supercharges $\mathcal{Q}^a$ are in the $(\mathbf{1},\mathbf{1},\mathbf{2})$, and therefore they become scalar after the twisting. As usual, one would then like to regard them as BRST charges, and the Wilson loops as observables in their cohomology. To this purpose, the first thing we should check is if those charges square to zero. For this computation we need to use the anticommutator

$$\left\{\bar{Q}_{\dot{\alpha}A},\bar{S}^B_{\dot{\beta}}\right\}=\delta^B_A\,L_{\dot{\alpha}\dot{\beta}}+\varepsilon_{\dot{\alpha}\dot{\beta}}\left(T^B{}_A-\frac{1}{2}\delta^B_A D\right)\,, \tag{6.18}$$

where $L_{\dot{\alpha}\dot{\beta}}$, $T^B{}_A$ and D are respectively the $SU(2)_R$, $SU(4)$ and dilatation generators. Splitting the indices as above $\bar{Q}_{\dot{\alpha}A}=\bar{Q}^a_{\dot{\alpha}i}$ and breaking the R-symmetry generators as

$$T^B{}_A=\varepsilon^{ab}T_{ij}+\varepsilon_{ij}T^{ab}, \tag{6.19}$$

we find that

$$\left\{\bar{\mathcal{Q}}^a,\bar{\mathcal{Q}}^b\right\}=-4T^{ab}. \tag{6.20}$$

So the scalar supercharges do not square to zero but rather the square is proportional to the $SU(2)_B$ generators.

6.1.3 Special loops

There are some special loops that preserve more supersymmetries.

1) Large S^2 If we restrict the loop to lie on an S^2 defined by, say, $x^4 = 0$, then the left and right forms are no longer independent, rather

$$\sigma_i^L = -\sigma_i^R = -2\epsilon_{ijk}x^j\,dx^k\,. \tag{6.21}$$

Then eq. (6.7) has more solutions. In additional to the previous ones, it is also solved by

$$\tau^i\epsilon_1^+ = -M^i{}_I\rho^I\epsilon_0^+\,. \tag{6.22}$$

Combining the two chiralities, this can be written as

$$i\gamma_{jk}\epsilon_1 = \epsilon_{ijk}M^i{}_I\rho^I\gamma^5\epsilon_0\,. \tag{6.23}$$

Contrary to the general S^3 case in eq. (6.10), we see that now the constraints are not chiral. One can solve them in the same way as described above, but we now get two copies of the solution, one for each chirality. The generic Wilson loop on S^2 will therefore preserve 1/8 of the supersymmetries. The four supercharges can be written explicitly as

$$\mathcal{Q}^a = \varepsilon^{\alpha i}\left(Q^a_{\alpha i} + S^a_{\alpha i}\right), \qquad \bar{\mathcal{Q}}^a = \varepsilon^{\dot\alpha i}\left(\bar{Q}^a_{\dot\alpha i} - \bar{S}^a_{\dot\alpha i}\right). \tag{6.24}$$

We can also determine what is the bosonic subgroup of $SO(5,1)\times SO(6)_R$ which leave these loops invariant. As for the general S^3 case, we have the symmetry under the group $SU(2)_B \subset SO(6)_R$ which rotates Φ^4, Φ^5, Φ^6. Moreover, there is an extra $U(1)$ symmetry generated by

$$\frac{1}{2}\left(P_4 - K_4\right), \tag{6.25}$$

where $P_\mu = -i\partial_\mu$ and $K_\mu = -i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu)$. Invariance under this generator follows from the fact that these loops satisfy $x^4 = 0$.

There is an interesting property of the loops on S^2 involving the replacement of the gauge and scalar couplings. Consider an arbitrary smooth curve on S^2 which is nowhere a geodesic and parameterized by $\vec{x}(s)$, and let us take $|\dot{x}| = 1$. The scalar couplings is given by the standard cross product in three dimensions as $\vec{\Theta}(s) = \dot{\vec{x}} \times \vec{x}$. Those are also unit vectors in $\mathbb{R}^3$, so we can consider also a loop whose shape is given by $\vec{\Theta}$. A simple calculation shows that the scalar couplings for the new loop is proportional to $\vec{x}$.

This suggests the existence of a duality between the scalar and vector couplings and it is tempting to speculate that it will extend to a duality between the embedding of the string in the dual description into the AdS_5 and S^5 parts of the geometry.

2) Large circle By this construction a maximal circle couples only to a single scalar. For example, a circle in the $(1,2)$ plane couples only to Φ^3. Studying the supersymmetry variation leads to the single constraint

$$\rho^3 \gamma^5 \epsilon_0 = i\gamma_{12}\epsilon_1, \tag{6.26}$$

so the loop preserves 16 (8 chiral and 8 anti-chiral) combinations of Q and S. This is the most studied 1/2 BPS circular Wilson loop. Using eq. (6.26) we may write down the sixteen supercharges as

$$\mathcal{Q}_A = i\gamma_{12} Q_A + \left(\rho^3 S\right)_A , \qquad \bar{\mathcal{Q}}^A = i\gamma_{12} \bar{Q}^A - \left(\rho^3 \bar{S}\right)^A , \tag{6.27}$$

where $A = 1, \ldots, 4$ and for simplicity we have omitted Lorentz indices. Furthermore, it is not difficult to show that the 1/2 BPS circle also preserves the bosonic group

$SO(1,2)\times SU(2)\times SO(5)$. Here, the $SO(5)\subset SO(6)_R$ simply follows from the fact that the loop couples to a single scalar. The $SU(2)$ symmetry is generated by

$$\frac{1}{2}\left(P_3-K_3\right), \qquad \frac{1}{2}\left(P_4-K_4\right), \qquad J_{34}, \tag{6.28}$$

where $J_{\mu\nu}=i(x_\mu\partial_\nu-x_\nu\partial_\mu)$ are the Lorentz generators. Finally, the $SO(1,2)$ symmetry is the Möbius group in the $(1,2)$ plane generated by

$$\frac{1}{2}\left(P_1+K_1\right), \qquad \frac{1}{2}\left(P_2+K_2\right), \qquad J_{12}. \tag{6.29}$$

All these bosonic symmetries, together with the above supercharges, form the supergroup $Osp(4^\star|4)$.

3) Latitude Consider a non-maximal circle on S^2 (a latitude) parameterized by

$$x^\mu=(\sin\theta_0\cos t\,,\sin\theta_0\sin t\,,\cos\theta_0\,,0)\,. \tag{6.30}$$

This is essentially the same Wilson loop operator considered in [223], except that by a conformal transformation we moved the circle from the equator to a parallel.[2] Such a loop couples to three scalars, but it can be shown that it gives only two independent constraints. Indeed, asking that the supersymmetry variation vanishes at every point only requires the two equations

$$\begin{aligned}\cos\theta_0\big(\gamma_{12}+\rho_{12}\big)\epsilon_1&=0\,,\\ \rho^3\gamma^5\epsilon_0&=\big[i\gamma_{12}+\gamma_3\rho^2\gamma^5\cos\theta_0(\gamma_{23}+\rho^{23})\big]\epsilon_1\,.\end{aligned} \tag{6.31}$$

[2]Also compared to [223], θ_0 is replaced here by $\pi/2-\theta_0$.

If $\cos\theta_0 \neq 0$, one has two independent constraints and the loop preserves 1/4 of the supersymmetries. In the special case $\cos\theta_0 = 0$, as expected from eq. (6.30), one recovers the 1/2 BPS maximal circle. One may solve the constraints (6.31) as described in the previous section by viewing γ_i and ρ_i as Pauli matrices acting on Lorentz and $SU(2)_A$ indices respectively. Then one can write the solutions to the first line of eq. (6.31) as

$$\begin{aligned} \epsilon^a_{1,(1)} &= \epsilon^a_{01} - \epsilon^a_{10} = \varepsilon^{\alpha\dot{a}}\epsilon^a_{1,\alpha\dot{a}}\,, \\ \epsilon^a_{1,(2)} &= \epsilon^a_{01} + \epsilon^a_{10} = \tau_1^{\alpha\dot{a}}\epsilon^a_{1,\alpha\dot{a}}\,, \end{aligned} \tag{6.32}$$

and similarly for the other chirality. The ϵ_0 spinors can be obtained from the second line of the contraints, for example for the positive chirality they read

$$\begin{aligned} \epsilon^a_{0,(1)} &= -\epsilon^a_{1,(1)} \\ \epsilon^a_{0,(2)} &= -\epsilon^a_{1,(2)} - 2\cos\theta_0\,\epsilon^a_{1,(1)}\,. \end{aligned} \tag{6.33}$$

Therefore the eight supercharges preserved by the loop may be written as

$$\begin{aligned} \mathcal{Q}^a_{(1)} &= \varepsilon^{\alpha\dot{a}}\left(Q^a_{\alpha\dot{a}} - S^a_{\alpha\dot{a}}\right), \qquad & \mathcal{Q}^a_{(2)} &= \tau_1^{\alpha\dot{a}}\left(Q^a_{\alpha\dot{a}} - S^a_{\alpha\dot{a}}\right) + 2\cos\theta_0\,\varepsilon^{\alpha\dot{a}}Q^a_{\alpha\dot{a}}\,, \\ \bar{\mathcal{Q}}^a_{(1)} &= \varepsilon^{\dot{\alpha}\dot{a}}\left(\bar{Q}^a_{\dot{\alpha}\dot{a}} - \bar{S}^a_{\dot{\alpha}\dot{a}}\right), \qquad & \mathcal{Q}^a_{(2)} &= \tau_1^{\dot{\alpha}\dot{a}}\left(\bar{Q}^a_{\dot{\alpha}\dot{a}} - \bar{S}^a_{\dot{\alpha}\dot{a}}\right) + 2\cos\theta_0\,\varepsilon^{\dot{\alpha}\dot{a}}\bar{Q}^a_{\dot{\alpha}\dot{a}}\,. \end{aligned} \tag{6.34}$$

One can also determine the bosonic group preserved by this non-maximal circle, which turns out to be $SU(2) \times SU(2)_B \times U(1)$. Besides the obvious $SU(2)_B$ symmetry, the other $SU(2)$ is generated by[3]

$$\frac{1}{2}\left(P_3 - K_3 - 2\cos\theta_0\, D\right), \qquad \frac{1}{2}\left(P_4 - K_4\right), \qquad J_{34} + \frac{1}{2}\cos\theta_0\left(P_4 + K_4\right), \tag{6.35}$$

where $D = -ix^\mu\partial_\mu$ is the dilatation generator. Finally, the $U(1)$ symmetry mixes Lorentz and R-symmetry generators, and is given by

$$J_{12} + J_{12}^A\,, \tag{6.36}$$

where J_{12}^A is a generator of $SU(2)_A$. This follows from the fact that the loop coordinates x^μ and the scalar couplings Θ^I satisfy the equation $x^2\Theta^1 - x^1\Theta^2 = 0$.

This loop also seems to be given by a Gaussian matrix model with the only modification that the coupling g^2 is replaced by $g^2\sin^2\theta_0$.

4) Two longitudes Inside a large S^2 consider a loop made of two arcs of length π connected at an arbitrary angle δ, *i.e.* two longitudes on the sphere. We can parametrize the loop in the following way

$$\begin{aligned} &x^\mu = (\sin t,\, 0,\, \cos t,\, 0)\,, && 0 \le t \le \pi\,, \\ &x^\mu = (-\cos\delta\sin t,\, -\sin\delta\sin t,\, \cos t,\, 0)\,, && \pi \le t \le 2\pi\,. \end{aligned} \tag{6.37}$$

The corresponding Wilson loop operator will couple to Φ^2 along the first arc and to $-\Phi^2\cos\delta + \Phi^1\sin\delta$ along the second one. It is straightforward to study the supersymmetry variation of this operator. Each arc, being (half) a maximal circle, is 1/2

[3]The 1/4-BPS circle of [223] preserves, as the maximal circle, the $SU(2)$ given by eq. (6.28). Our loop is obtained from that in [223] by a dilatation and a translation along x^3, after which the generators become the ones in eq. (6.35).

BPS and will produce a single constraint

$$\begin{aligned}
&\text{First arc:} && \rho^2\gamma^5\epsilon_0 = i\gamma_{31}\epsilon_1\,, \\
&\text{Second arc:} && (\rho^2\gamma^5\cos\delta - \rho^1\gamma^5\sin\delta)\epsilon_0 = i(\gamma_{31}\cos\delta - \gamma_{23}\sin\delta)\epsilon_1\,.
\end{aligned} \tag{6.38}$$

Combining the two equations, we see that the system has to satisy, as long as $\sin\delta \neq 0$,

$$\rho^2\gamma^5\epsilon_0 = i\gamma_{31}\epsilon_1\,, \qquad \rho^1\gamma^5\epsilon_0 = i\gamma_{23}\epsilon_1\,. \tag{6.39}$$

These constraints are of course consistent and therefore the loop will preserve 1/4 of the supersymmetries. When $\sin\delta = 0$, the second equation in (6.39) disappears and the loop becomes 1/2 BPS (in the case $\delta = \pi$, it is just the maximal circle discussed above, while in the case $\delta = 0$ the loop is made of two coincident half circles with opposite orientations). The same conditions apply also when one adds more circles or half-circles that all intersect at the north and south poles.

To solve the above constraints, we can proceed as usual by first eliminating ϵ_0. This gives the equation

$$(-i\gamma_{12} + \tau_3^A)\epsilon_1 = 0. \tag{6.40}$$

For a generic loop we had three such equations (for the anti-chiral spinor), which meant that the only solution had to be a singlet of the diagonal $SU(2)_R + SU(2)_A$ group. Here we find only one such equation for each of the chiralities, such that a $U(1)$ charge (τ_3^{total}) has to vanish. So in addition to the singlet, this constraint allows one of the states of the triplet.

Explicitly, we can write the two solutions with positive chirality as

$$\begin{aligned}
\epsilon^a_{1,(1)} &= \epsilon^a_{1,01} - \epsilon^a_{1,10} = \varepsilon^{\alpha\dot\alpha}\epsilon^a_{1,\alpha\dot\alpha} \\
\epsilon^a_{1,(2)} &= \epsilon^a_{1,01} + \epsilon^a_{1,10} = \tau_1^{\alpha\dot\alpha}\epsilon^a_{1,\alpha\dot\alpha}\,,
\end{aligned} \tag{6.41}$$

and similarly for the negative chirality. From the equation $\rho^3\gamma^5\epsilon_0 = i\gamma_{12}\epsilon_1$ we can then get

$$\epsilon_{0,(1)} = \gamma^5\epsilon_{1,(1)}\,, \qquad \epsilon_{0,(2)} = -\gamma^5\epsilon_{1,(2)}\,. \tag{6.42}$$

Thus the eight supercharges which annihilate the Wilson loop made of two longitudes are

$$\begin{aligned}
\mathcal{Q}^a_{(1)} &= \varepsilon^{\alpha\dot\alpha}\left(Q^a_{\alpha\dot\alpha} + S^a_{\alpha\dot\alpha}\right), &\qquad \mathcal{Q}^a_{(2)} &= \tau_1^{\alpha\dot\alpha}\left(Q^a_{\alpha\dot\alpha} - S^a_{\alpha\dot\alpha}\right), \\
\bar{\mathcal{Q}}^a_{(2)} &= \varepsilon^{\dot\alpha\dot\alpha}\left(\bar Q^a_{\dot\alpha\dot\alpha} - \bar S^a_{\dot\alpha\dot\alpha}\right), &\qquad \mathcal{Q}^a_{(2)} &= \tau_1^{\dot\alpha\dot\alpha}\left(\bar Q^a_{\dot\alpha\dot\alpha} + \bar S^a_{\dot\alpha\dot\alpha}\right).
\end{aligned} \tag{6.43}$$

The loop also preserves the bosonic symmetry group $U(1)\times SO(4)$. To see this, notice that each longitude is half a maximal circle and therefore it preserves an $SU(2)\times SO(5)$ symmetry (the group $SO(1,2)$ is broken by the fact that they are half circles). For the first arc one has exactly the same generators described above for the maximal circle in the $(1,2)$ plane, while for the second arc the generators is a δ-dependent rotation of those. One can then see that the only symmetries preserved simultaneously by both arcs are the $U(1)$ generated by $\frac{1}{2}(P_4 - K_4)$ and the $SO(4)\subset SO(6)_R$ rotating $\Phi^3, \Phi^4, \Phi^5, \Phi^6$.

This example has many interesting features. By a stereographic projection it is mapped to a cusp in the plane, where along each of the rays the scalar coupling is constant. This is an operator of the class constructed in [68] and has trivial expectation value. The observable on S^2 is not trivial, rather

$$W \simeq \begin{cases} 1 + \frac{g^2 N}{8\pi^2}\, \delta(2\pi - \delta)\,, & g^2 N \ll 1\,, \\ \exp \sqrt{\frac{g^2 N}{\pi^2}\, \delta(2\pi - \delta)}\,, & g^2 N \gg 1\,, \end{cases} \tag{6.44}$$

Note that, as in the latitude case, the only modification from the circle at $\delta = \pi$ is the rescaling of the coupling both at weak and at strong coupling by the same factor. But here the perturbative calculation does not seem as simple as before.

5) Hopf fibers Consider the following parametrization of the three sphere

$$\begin{aligned} x^1 &= -\sin\frac{\theta}{2}\sin\frac{\psi - \phi}{2}\,, \qquad & x^2 &= \sin\frac{\theta}{2}\cos\frac{\psi - \phi}{2}\,, \\ x^3 &= \cos\frac{\theta}{2}\sin\frac{\psi + \phi}{2}\,, \qquad & x^4 &= \cos\frac{\theta}{2}\cos\frac{\psi + \phi}{2}\,, \end{aligned} \tag{6.45}$$

where the range of the Euler angles is $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ and $0 \le \psi \le 4\pi$. In these coordinates the metric of the S^3 reads

$$ds^2 = \frac{1}{4}\left(d\theta^2 + \sin\theta^2 d\phi^2 + (d\psi + \cos\theta\, d\phi)^2\right). \tag{6.46}$$

This is the Hopf fibration of the three sphere, namely the S^3 is written as an S^1 fibration over S^2. The fiber is parameterized by ψ, while the base S^2 by (θ, ϕ). Consider now a Wilson loop along a generic fiber. This loop sits at constant $(\theta, \phi) = (\theta_0, \phi_0)$, while ψ varies along the curve. Such operator couples to a single scalar, namely Φ^3. An easy way to see this is to write the left invariant one-forms (6.1) in terms of the Euler angles

$$\begin{aligned}
\sigma_1^R &= -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi\,, \\
\sigma_2^R &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi\,, \\
\sigma_3^R &= d\psi + \cos\theta\, d\phi\,.
\end{aligned} \tag{6.47}$$

If θ and ϕ are constant, σ_1^R and σ_2^R will clearly not enter in the Wilson loop (6.2). An equivalent way to say this is that this curve only follows the vector field $\xi_3^R = \partial_\psi$ dual to σ_3^R. A single loop like this is 1/2 BPS, it is a maximal circle. In fact, studying the supersymmetry variation, one finds the following constraints on the two chiralities

$$\rho^3 \epsilon_0^- = \tau_3^R \epsilon_1^-\,, \qquad \rho^3 \epsilon_0^+ = \hat{\sigma}_i^L \tau^i \epsilon_1^+\,. \tag{6.48}$$

where $\hat{\sigma}_i^L$ is the pullback of the left-forms along the curve. Using the explicit expression for them in terms of the Euler angles

$$\begin{aligned}
\sigma_1^L &= \sin\phi\, d\theta - \cos\phi \sin\theta\, d\psi\,, \\
\sigma_2^L &= \cos\phi\, d\theta + \sin\phi \sin\theta\, d\psi\,, \\
\sigma_3^L &= d\phi + \cos\theta\, d\psi\,,
\end{aligned} \tag{6.49}$$

we get $\hat{\sigma}_i^L(\theta_0, \phi_0)\tau^i = \cos\theta_0 \tau^3 - \sin\theta_0(\cos\phi_0 \tau^1 - \sin\phi_0 \tau^2)$.

Take now a system made of several of these fibers. Notice that the anti-chiral part of the constraint (6.48) is independent of (θ_0, ϕ_0), therefore the system made of any number of fibers preserves the same 8 anti-chiral combination of $\bar{Q}$ and $\bar{S}$ as the single fiber. However, the 8 chiral supersymmetries are broken when two or more fibers are put together. Indeed, if we add a second fiber at (θ_1, ϕ_1), then we have two separate chiral constraints

$$\rho^3 \epsilon_0^+ = \hat{\sigma}_i^L(\theta_0, \phi_0)\sigma^i \epsilon_1^+ \,, \qquad \rho^3 \epsilon_0^+ = \hat{\sigma}_i^L(\theta_1, \phi_1)\tau^i \epsilon_1^+ \,. \tag{6.50}$$

To check when those two constraints are compatible, we subtract the two equations and check if the resulting matrix has vanishing eigenvalues

$$\det\left(\hat{\sigma}_i^L(\theta_0, \phi_0)\tau^i - \hat{\sigma}_i^L(\theta_1, \phi_1)\tau^i\right) = -2\left(1 - \cos\theta_0\cos\theta_1 - \sin\theta_0\sin\theta_1\cos(\phi_0 - \phi_1)\right). \tag{6.51}$$

This is zero only when $\theta_0 = \theta_1$ and $\phi_0 = \phi_1$. Therefore the combined system of two or more Hopf fibers does not preserve any of the chiral supercharges, but it does preserve the 8 anti-chiral supersymmetries defined by

$$\rho^3 \epsilon_0^- = -i\gamma_{12}\epsilon_1^- \,. \tag{6.52}$$

The corresponding supercharges preserved by the system will be essentially the same as the ones preserved by the 1/2 BPS maximal circle, except that we only select one chirality

$$\bar{\mathcal{Q}}^A = i\gamma_{12}\bar{Q}^A - \left(\rho^3\bar{S}\right)^A \,. \tag{6.53}$$

As for the bosonic symmetries, notice that a single fiber, being a maximal circle, preserves the group $SO(2,1) \times SU(2) \times SO(5)$. Keeping the coordinate system fixed, however, the explicit generators of this group will depend in general on (θ_0, ϕ_0). One can construct however one symmetry generator which is independent of the position of the fiber, which is simply

$$J_3^R = \frac{1}{2}\left(J_{12} - J_{34}\right). \tag{6.54}$$

The corresponding $U(1)$ will therefore be a symmetry for a system made of any number of fibers. Besides this, we have of course the $SO(5)$ symmetry following from the fact that these fibers only couple to one scalar. So we conclude that the bosonic group preserved by the Hopf fibers is $U(1) \times SO(5)$.

6) Flat connection over Hopf-base Consider a curve parameterized by the Euler angles θ and ϕ, which form the base of the Hopf fibration. Along the fibers we choose

$$\psi(s) = -\int_0^s ds'\,\dot{\phi}(s')\cos\theta(s')\,, \tag{6.55}$$

which guarantees that $\hat{\sigma}_3^R$ vanishes. A generic curve of this form will break all the chiral supersymmetries, and for the anti-chiral ones will introduce the constraints

$$\rho^2\epsilon_0^+ = \tau_2^R\epsilon_1^+\,, \qquad \rho^1\epsilon_0^+ = \tau_1^R\epsilon_1^+\,. \tag{6.56}$$

This is the anti-chiral part of eq. (6.39), and consequently the loop will preserve the anti-chiral supersymmetries in eq. (6.43)

$$\bar{\mathcal{Q}}^a_{(2)} = \varepsilon^{\dot\alpha i}\left(\bar{Q}^a_{\dot\alpha i} - \bar{S}^a_{\dot\alpha i}\right), \qquad \mathcal{Q}^a_{(2)} = \tau_1^{\dot\alpha i}\left(\bar{Q}^a_{\dot\alpha i} + \bar{S}^a_{\dot\alpha i}\right). \tag{6.57}$$

The example of the longitudes is a special case of those loops where the entire loop is contained within an S^2, so in addition to those four anti-chiral supercharges, it preserves also four chiral supercharges. To relate them explicitly, of the Euler angles, only θ varies along the two arcs of eq. (6.37) while ϕ and ψ are kept fixed with $\psi+\phi=\pi$, $\psi+\phi=3\pi$ or $\psi+\phi=5\pi$.

Note that in this construction there is an integral condition, that the values of ψ at the beginning and end of the curve are equal, so it closes.

Such a loop preserves an $SO(4)$ symmetry rotating Φ^3, Φ^4, Φ^5 and Φ^6.

7) Infinitesimal loops If a loop is concentrated entirely near one point, say $x^4=1$, one does not see the curvature of the sphere anymore. The left and right forms then become exact differentials

$$\sigma_i^{R,L} \sim 2dx_i\,, \qquad i=1,2,3\,. \tag{6.58}$$

So the Wilson loops (6.2) reduce to the ones constructed by Zarembo in [68]. The supersymmetry variation of such a loop is proportional to

$$\delta W \simeq \dot{x}^i\left(i\gamma_i + M^i{}_I\rho^I\gamma^5\right)\left(\epsilon_0+\gamma^4\epsilon_1\right). \tag{6.59}$$

The variations that will annihilate these loops will satisfy three constraints, which for our usual choice of $M^i{}_I$ are

$$\left(\gamma^i+\rho^i\gamma^5\right)\epsilon_0 = 0\,, \tag{6.60}$$

and the same for ϵ_1. Those loops generically preserve two combinations of Q and $\bar{Q}$ and two combinations of S and $\bar{S}$. If the curve is restricted further to lay only in a 2-plane or a line near $x^4=1$, the supersymmetry will be further enhanced.

This should explain why in this case the expectation value of the loops vanishes.

The planar loops come from infinitesimal ones on S^3, so it is quite natural that their VEV's vanish. This also explains why the construction of the D3-brane solution dual to the Wilson loop in this limit is singular (see section 6.3).

6.2 Correlators of giant Wilson loops and chiral primaries

A small circular Wilson loop, when probed from a distance much larger than its characteristic size, can be expanded in a series of local operators of different conformal dimension [227]. The operators which are allowed to appear in the expansions must preserve the same symmetries of eq. (2.53) and therefore must be bosonic, gauge invariant and $SO(5)$ invariant. The conformal dimension of some of these operators is not protected by the superconformal algebra and therefore they receive large anomalous dimensions and decouple in the strong coupling regime. An important class of operators which have protected dimensions and appear in the operator product expansion are the chiral primary operators introduced in chapter 2. The correlator with a local operator can then be read off from the expansion of the Wilson loop [227].[4]

In this section of the dissertation, we use the D3 and D5-branes described in section 2.3.3 to compute the correlation function between a circular Wilson loop in a higher representation and a chiral primary operator in the fundamental representation. We do this by studying the coupling to the brane worldvolume of the supergravity modes dual to the chiral primaries. These modes propagate from the insertion of the local operator on the boundary to the brane worldvolume in the bulk.

We first review the operator product expansion of the Wilson loop and how to

[4] For a nice review see [77].

compute correlation functions when the Wilson loop is described in terms of a fundamental string worldsheet. To evaluate this one needs to study the harmonic expansion on S^5 of the bulk fields which couple to the worldsheet. We then replace the fundamental string with the D3 and D5-branes. We start by investigating the symmetric case first and we then move on to the analysis of the antisymmetric one. Besides checking that our results reproduce for small k the well-known string limit, we also compare them against the expressions coming from the normal matrix model introduced in this context in [89], and find perfect agreement both in the symmetric and antisymmetric case.

6.2.1 Kaluza-Klein expansion

In this section we review the expansion in spherical harmonics for type IIB supergravity on $AdS_5 \times S^5$ [34], and identify the bulk excitations associated to turning on a chiral primary operator in the dual $\mathcal{N} = 4$ gauge theory [26]. These will be later used to construct the coupling of the various supergravity modes to the D3 and D5-branes.

The Einstein equations of type IIB supergravity read[5]

$$R_{mn} = \frac{1}{96} F_{mijkl} F_n{}^{ijkl} , \tag{6.61}$$

where the 5-form field strength $F_{(5)}$ is self-dual. In the Poincaré patch, the $AdS_5 \times S^5$

[5] In our conventions Latin indices run over the whole ten-dimensional manifold, while Greek indices $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ run over AdS_5 and S^5 respectively. We also choose units in which $R_{AdS_5} = R_{S^5} = 1$.

solution reads

$$ds^2 = \tfrac{1}{z^2}\left(dz^2 + d\vec{x}^{\,2}\right) + d\Omega_5^2\,, \tag{6.62}$$

$$\bar{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = -4\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\,, \qquad \bar{F}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = -4\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\,. \tag{6.63}$$

The fluctuations around the background geometry can be parametrized as follows

$$G_{mn} = g_{mn} + h_{mn}\,, \tag{6.64}$$

$$h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{h_2}{5} g_{\alpha\beta}\,, \qquad g^{\alpha\beta} h_{(\alpha\beta)} = 0\,, \tag{6.65}$$

$$h_{\mu\nu} = h'_{\mu\nu} - \frac{h_2}{3} g_{\mu\nu}\,, \qquad g^{\mu\nu} h'_{(\mu\nu)} = 0\,, \tag{6.66}$$

$$F = \bar{F} + \delta F\,, \qquad \delta F_{ijklm} = 5\nabla_{[i} a_{jklm]}\,, \tag{6.67}$$

where h_2 is the trace of the metric on the 5-sphere, $h_2 \equiv h_{\alpha\beta} g^{\alpha\beta}$. Note that the fields $h_{\mu\nu}$ and $h'_{\mu\nu}$ are related by a $d = 5$ Weyl shift. To identify the bulk excitation in AdS_5 we expand the fluctuations as follows[6]

$$h'_{\mu\nu} = \sum h'^{I}_{\mu\nu}(x) Y^I(y)\,, \tag{6.68}$$

$$h_2 = \sum h_2^I(x) Y^I(y)\,, \tag{6.69}$$

$$a_{\mu_1\mu_2\mu_3\mu_4} = \sum a^I_{\mu_1\mu_2\mu_3\mu_4}(x) Y^I(y)\,, \tag{6.70}$$

$$a_{\alpha_1\alpha_2\alpha_3\alpha_4} = -4 \sum \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4} b^I(x) \nabla^\alpha Y^I(y)\,, \tag{6.71}$$

where x and y refer to the AdS_5 and S^5 coordinates respectively, and Y^I are scalar

[6] We do not consider the harmonic expansion of $h_{(\alpha\beta)}$ as this fluctuation is related to $Q^2\bar{Q}^2$ descendants of chiral primaries in the dual super Yang-Mills theory.

spherical harmonics on the 5-sphere which satisfy[7]

$$\nabla^\alpha \nabla_\alpha Y^I = -\Delta(\Delta+4) Y^I \,. \tag{6.72}$$

Spherical harmonics on S^5 can be classified in terms of the $SO(6) \simeq SU(4)$ R-symmetry group. In particular scalar harmonics belong to the $[0, \Delta, 0]$ representation. The fields h_2 and b appear coupled in the linearized equation of motions. Their equations can be diagonalized introducing the linear combinations [26]

$$s^I = \frac{1}{20(\Delta+2)}[h_2^I - 10(\Delta+4) b^I] \,, \tag{6.73}$$

$$t^I = \frac{1}{20(\Delta+2)}[h_2^I + 10\Delta b^I] \,, \tag{6.74}$$

which obey the equations of motion

$$\nabla_\mu \nabla^\mu \, s^I = \Delta(\Delta-4) \, s^I \,, \tag{6.75}$$

$$\nabla_\mu \nabla^\mu \, t^I = (\Delta+4)(\Delta+8) \, t^I \,. \tag{6.76}$$

A scalar field in AdS with $m^2 = \Delta(\Delta-4)$ (with $\Delta \geq 2$) transforming in the $[0, \Delta, 0]$ representation corresponds to a chiral primary operator $\mathcal{O}_\Delta$ of conformal dimension Δ. Therefore, to linear order, the scalar field s^I corresponds to chiral primaries in the dual gauge theory. On the other hand, the scalars t^I are associated to their descendants, which we do not consider here.

[7]We include a brief review of spherical harmonics in the appendix C.

The linear solutions to the equations of motion turn out to be [26]

$$h_{\mu\nu} = -\frac{6}{5}\Delta\, s\, g_{\mu\nu} + \frac{4}{\Delta+1}\nabla_{(\mu}\nabla_{\nu)}s\,, \tag{6.77}$$

$$h_{\alpha\beta} = 2\Delta\, s\, g_{\alpha\beta}\,, \tag{6.78}$$

$$a_{\mu_1\mu_2\mu_3\mu_4} = 4\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5}b\,, \tag{6.79}$$

$$a_{\alpha_1\alpha_2\alpha_3\alpha_4} = -4\sum_I \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4}b^I(x)\nabla^\alpha Y^I(y)\,, \tag{6.80}$$

where $s = \sum s^I Y^I$ and $b = \sum b^I Y^I$. Using eq. (6.65) and the solution (6.78) one can identify $h_2 = 10\,\Delta\, s$. Setting $t^I = 0$ in eq. (6.74), one can then deduce $s = -b$.

6.2.2 Operator product expansion of Wilson loops

The Wilson loop operator can be expanded in terms of local operators when probed from distances much larger than its characteristic size a. For the circular Wilson loop with radius a we can write [227]

$$W(\mathcal{C}) = \langle W(\mathcal{C})\rangle \left(1 + \sum_n c_{(n)}a^{\Delta_{(n)}}\mathcal{O}_{(n)}\right). \tag{6.81}$$

In this expression $\mathcal{O}_{(n)}$ is a local gauge invariant operator with conformal dimension $\Delta_{(n)}$, and the sum over n runs over both the primary operators and their descendants. This operator product expansion must be invariant under the symmetries preserved by the Wilson loop. The 1/2 BPS circular loop has $\Theta^I(\tau) = \Theta^I = const.$, and therefore preserves a $SO(5)$ subgroup of the original $SO(6)$ R-symmetry group. The operators appearing in the OPE expansion must therefore contain $SO(5)$ singlets in the $SO(6) \to SO(5)$ decomposition. For example, at level $\Delta = 2$ we can consider the chiral primary operator $\mathcal{O}_2^A = C_{IJ}^A \mathrm{Tr}\Phi^I\Phi^J$, where C_{IJ}^A is a $SO(6)$ symmetric traceless tensor. Under $SO(6) \to SO(5)$ it decomposes as $\mathbf{20} \to \mathbf{1} + \mathbf{5} + \mathbf{14}$ and therefore, containing a singlet, it will appear in the OPE of the Wilson operator. A similar analysis can be performed for higher dimension operators, which in general will contain covariant derivatives, gauge field-strenghts and the fermions of the $\mathcal{N} = 4$ multiplet. Some of them will get large anomalous dimension in the strong coupling limit and therefore will decouple. The generic expansion looks as follows

$$\begin{aligned}\frac{W(\mathcal{C})}{\langle W(\mathcal{C})\rangle} &= 1 + c_{(2)}\, a^2\, Y_A^{(2)}(\Theta)\mathcal{N}_2 C^A_{IJ}\mathrm{Tr}\left(\Phi^I\Phi^J\right) \\ &\quad + c_{(3)}\, a^3\, Y_A^{(3)}(\Theta)\mathcal{N}_3 C^A_{IJK}\mathrm{Tr}\left(\Phi^I\Phi^J\Phi^K\right) + c_{(4)}\, a^3\mathrm{Tr}\left(\Theta^I X^I F_+\right) + \dots\,,\end{aligned} \tag{6.82}$$

where $Y_A^{(n)}(\theta)$ are spherical harmonics and $\mathcal{N}_n$ are normalization constants.

The coefficients appearing in the OPE expansion can be read off from the large distance behavior of the two point correlator of the Wilson loop and the local operators

$$\frac{\langle W(\mathcal{C})\mathcal{O}^{(n)}(x)\rangle}{\langle W(\mathcal{C})\rangle} = c_{(n)}\frac{a^{\Delta_{(n)}}}{L^{2\Delta_{(n)}}} + \dots\,, \tag{6.83}$$

where it is assumed that the loop radius a is much smaller than the distance L from the point of insertion of the local operator. Here we shall focus only on chiral primary operators $\mathcal{O}^A_\Delta = C^A_{I_1\cdots I_\Delta}\mathrm{Tr}(\Phi^{I_1}\dots\Phi^{I_\Delta})$.[8] These belong to short representations of the superconformal algebra, have protected conformal dimensions, and appear at all orders in the expansion (6.82).

In the AdS/CFT correspondence the chiral primary operators are dual to supergravity modes: $\mathcal{O}_\Delta$ corresponds to a scalar of mass $m^2 = \Delta(\Delta - 4)$, which is a

[8] We take the traces of the chiral primaries in the fundamental representation.

combination of the trace of the metric and the R-R 4-form over S^5, as we reviewed above.

We now briefly discuss the procedure for computing the correlation function of these operators with a Wilson loop in the strong coupling regime. The coupling to the string worldsheet of the supergravity mode dual to $\mathcal{O}_\Delta$ is given by a vertex operator V_Δ, which can be determined by expanding the string action to linear order in the fluctuation $h_{\mu\nu}$

$$\begin{aligned} S &= \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{\det\left(G_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu\right)} \\ &= \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{\det\left(g_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu\right)}\left(1+\frac{1}{2}(g_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu)^{-1}h_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu+\dots\right). \end{aligned} \tag{6.84}$$

The fluctuation of the metric $h_{\mu\nu}$ on AdS_5 is given in eq. (6.77). We write the scalar s^I in terms of a source s_0^I located at the boundary

$$s^I(\vec{x},z)=\int d^4\vec{x}\,'G_\Delta(\vec{x}\,';\vec{x},z)s_0^I(\vec{x}\,'), \tag{6.85}$$

where $G_\Delta(\vec{x}\,';\vec{x},z)$ is the bulk-to-boundary propagator which describes the propagation of the supergravity mode from the insertion point $\vec{x}\,'$ of the chiral primary operator to the point $(\vec{x},z)$ on the string worldsheet

$$G_\Delta(\vec{x}\,';\vec{x},z)=c\left(\frac{z}{z^2+|\vec{x}-\vec{x}\,'|^2}\right)^\Delta. \tag{6.86}$$

The constant $c=\frac{\Delta+1}{2^{2-\Delta/2}N\sqrt{\Delta}}$ is fixed by requiring the unit normalization of the 2-point function [227]. Since we are probing the Wilson loop from a distance L much larger than its radius a we can approximate

$$G_\Delta(\vec{x}\,';\vec{x},z) \simeq c\,\frac{z^\Delta}{L^{2\Delta}}\,, \qquad \partial_z s^I \simeq \frac{\Delta}{z} s^I\,, \qquad \partial_z^2 s^I \simeq \frac{\Delta(\Delta-1)}{z^2} s^I. \tag{6.87}$$

The relevant Christoffel symbols are readily computed to be

$$\Gamma^z_{\mu\nu} = z g_{\mu\nu} - \frac{2}{z}\delta^z_\mu \delta^z_\nu\,, \tag{6.88}$$

so that one finally has

$$h^I_{\mu\nu} \simeq -2\Delta\ g_{\mu\nu} s^I + \frac{4\Delta}{z^2}\delta^z_\mu\delta^z_\nu s^I\,. \tag{6.89}$$

Inserting this result into eq. (6.84), the coupling to the worldsheet is found to be [227]

$$\frac{1}{2\pi\alpha'}\int d\mathcal{A}\,(-2\Delta\, s)\frac{z^2}{a^2} \equiv \frac{1}{2\pi\alpha'}\int d\mathcal{A}\, V_\Delta\, s\,. \tag{6.90}$$

In this expression $d\mathcal{A}$ is the area element of the classical string. The correlation function is obtained from functionally differentiating the previous formula with respect to the source s_0

$$\begin{aligned}\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(\vec{x}_0)\rangle}{\langle W(\mathcal{C})\rangle} &= -Y^I(\theta)\frac{\delta}{\delta s_0(\vec{x}_0)}\frac{1}{2\pi\alpha'}\int d\mathcal{A}\, d^4x'\, V_\Delta\, G_\Delta(\vec{x}\,';\vec{x},z) s^I_0(\vec{x}\,') \\ &= -Y^I(\theta)\frac{1}{2\pi\alpha'}\int d\mathcal{A}\ V^\Delta G_\Delta(\vec{x}_0;\vec{x},z)\,. \end{aligned} \tag{6.91}$$

One obtains in the approximations of eq. (6.87)

$$\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(\vec{x}_0)\rangle}{\langle W(\mathcal{C})\rangle} = 2^{\Delta/2-1}\frac{\sqrt{\Delta\lambda}}{N}\frac{a^\Delta}{L^{2\Delta}}\,. \tag{6.92}$$

6.2.3 Brane computation

We now move on to studying the operator product expansion of Wilson loops in higher dimensional representations. We analyze the rank k symmetric representation first. In the bulk this is described by the D3-brane discussed in section 2.3.3. The first step in the computation consists in finding the vertex operator to be integrated over the world-volume of the brane.

Coupling of the D3-brane to the chiral primaries

The linearized coupling of the scalar s^I to the brane can be found by expanding the induced metric on the brane around the $AdS_5 \times S^5$ background g_{mn} and keeping the first order term in the fluctuation h_{mn}. Since the brane lies completely in AdS_5 we can write

$$\begin{aligned} S_{DBI} &= T_{D3} \int d^4\sigma \sqrt{\det\left(G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + 2\pi\alpha' F_{ab}\right)} \\ &= T_{D3} \int d^4\sigma \sqrt{\det\left(g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + 2\pi\alpha' F_{ab}\right)} \cdot \\ &\quad \cdot \left(1 + \frac{1}{2}\left(g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + 2\pi\alpha' F_{ab}\right)^{-1} h_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma + \dots\right). \end{aligned} \tag{6.93}$$

Here a, b are the brane world-volume indices.

The coupling of s^I to the 4-form in the Wess-Zumino term is obtained by replacing $C_{(4)} \to C_{(4)} + a_{(4)}$, where, using eq. (6.79) and the approximation (6.87), the fluctuation $a_{(4)}$ is

$$a^I_{\mu_1\dots\mu_4} \simeq -4\epsilon_{\mu_1\dots\mu_4 z}\partial^z s^I \simeq -4\Delta\, z\, \epsilon_{\mu_1\dots\mu_4 z} s^I\,, \tag{6.94}$$

so that

$$S^{(1)}_{WZ} = -T_{D3}\int P[a_{(4)}] = 4\,T_{D3}\,\Delta\int P[C_{(4)}]s\,, \tag{6.95}$$

where $s = \sum s^I Y^I$.

We use now the explicit solution to the equations of motion (2.80) to evaluate the on-shell value of the fluctuations (6.93) and (6.95). The first order in the fluctuation in eq. (6.93) turns out to be

$$S^{(1)}_{DBI} = 4N\Delta\kappa^2 \int d\rho d\theta\,\frac{\sin\theta}{\sinh^2\rho}\left(-1-2\kappa^2+\frac{1-\sinh^2\rho(\kappa^{-2}-\sin^2\theta)}{(\cosh\rho-\sinh\rho\cos\theta)^2}\right)s. \tag{6.96}$$

Similarly, the Wess-Zumino term reads

$$S^{(1)}_{WZ} = 8N\Delta\kappa^4 \int d\rho d\theta\,\frac{\sin\theta}{\sinh^2\rho}\left(1+\frac{1}{\kappa^2}\frac{\sinh^3\rho-\sinh\rho\cosh^2\rho}{\cosh\rho-\sinh\rho\cos\theta}\cos\theta\right)s. \tag{6.97}$$

The final result for the action is then

$$S^{(1)}_{D3} = S^{(1)}_{DBI}+S^{(1)}_{WZ} = -4N\Delta\int_0^{\sinh^{-1}\kappa} d\rho\int_0^{\pi} d\theta\,\frac{\sin\theta}{(\cosh\rho-\sinh\rho\cos\theta)^2}s. \tag{6.98}$$

The correlation function

The prescription for computing the correlation function between the Wilson loop and the chiral primary operator is to functionally differentiate the action (6.98) with respect to the source s^I_0 (see eq. (6.85))

$$\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(L)\rangle}{\langle W(\mathcal{C})\rangle} = -\frac{\delta S^{(1)}_{D3}}{\delta s_0}\Big|_{s_0=0}. \tag{6.99}$$

We approximate the bulk-to-boundary propagator with $c\frac{z^\Delta}{L^{2\Delta}}$ and use for z the expression (2.76). This yields

$$\frac{\langle W(\mathcal{C})\mathcal{O}_\Delta(L)\rangle}{\langle W(\mathcal{C})\rangle} \simeq \frac{a^\Delta}{L^{2\Delta}}\frac{4N\Delta}{\kappa^\Delta}\,c\int_0^{\sinh^{-1}\kappa} d\rho\,\sinh^\Delta\rho\int_0^\pi d\theta\frac{\sin\theta}{(\cosh\rho-\sinh\rho\cos\theta)^{2+\Delta}}. \tag{6.100}$$

We are neglecting terms of higher order in $\frac{a}{L^2}$.

After performing the two integrals, the final result for the coefficients of the operator product expansion turns out to be remarkably simple

$$c_{S_k,\Delta}=\frac{2^{\Delta/2+1}}{\sqrt{\Delta}}\sinh(\Delta\sinh^{-1}\kappa). \tag{6.101}$$

Interestingly enough, this can be expressed in terms of Chebyshev polynomials with imaginary argument

$$c_{S_k,\Delta}=\frac{(-1)^{\Delta/2}2^{\Delta/2+1}}{\sqrt{\Delta}}\cdot\begin{cases}-iV_\Delta(i\kappa) & \text{for } \Delta \text{ even}\\ T_\Delta(i\kappa) & \text{for } \Delta \text{ odd}\,,\end{cases} \tag{6.102}$$

where we have used the identities $T_n(x)=\cos(n\cos^{-1}x)$ and $V_n(x)=\sin(n\cos^{-1}x)$.

The string limit is recovered when $\kappa\to 0$. In this regime the S^2 in the brane world-volume shrinks to zero size and the D3 reduces effectively to a fundamental string with AdS_2 worldsheet. The coefficients (6.101) become

$$c_{S_k,\Delta}\simeq 2^{\Delta/2+1}\sqrt{\Delta}\,\kappa=2^{\Delta/2-1}\,\frac{\sqrt{\Delta\lambda}}{N}\,k\,, \tag{6.103}$$

in perfect agreement with the result (6.92) found originally in [227].

The D5-brane

We consider now the rank k antisymmetric case, corresponding in the bulk to the D5-brane presented in section 2.3.3.

The coupling of the KK scalars s^I to the D5 world-volume can be obtained along the same lines of the D3 calculation above. However, besides the fluctuation of the AdS_5 part of the metric $h_{\mu\nu}$, we also need the fluctuation of the metric in the S^5 direction $h_{\alpha\beta}$, as well as the fluctuation of the 4-form $a_{(4)}$ along the S^4. The explicit expressions can be found in section 6.2.1. In particular, in this coordinates the 4-form over the S^5 is

$$a_{\sigma_1\sigma_2\sigma_3\sigma_4} = 4\sin^4\theta\,\mu(\Omega_4)\sum s^I\,\partial_\theta Y^I\,, \tag{6.104}$$

where $\sigma_1,\ldots,\sigma_4$ are the coordinates on the S^4 and $\mu(\Omega_4) = \sin^3\sigma_1\sin^2\sigma_2\sin\sigma_3$ is the corresponding measure. Differently from the D3 case, now the S^5 spherical harmonics Y^I play an active role in the computation since the D5-brane extends into the 5-sphere. The explicit form of the harmonics is given in appendix C.

The variation of the DBI part of the action to first order in the fluctuations $h_{\mu\nu}$ and $h_{\alpha\beta}$ reads

$$S^{(1)}_{DBI} = \frac{T_{D5}}{2}\int\sqrt{\det(\gamma_{ab}+2\pi\alpha' F_{ab})}\,\big(\gamma_{ab}+2\pi\alpha' F_{ab}\big)^{-1}\big(h_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + h_{\alpha\beta}\partial_a X^\alpha\partial_b X^\beta\big). \tag{6.105}$$

Using the explicit solution reviewed in section 2.3.3, it is easy to compute the matrix $\gamma_{ab}+2\pi\alpha' F_{ab}$. Plugging in the explicit expressions for the fluctuations and using the fact that on the D5 solution we have $z = a/\cosh\zeta$ (this follows from the change of coordinate (2.85) after setting $u = 0$), we get after some computations

$$S^{(1)}_{DBI} = \pi T_{D5} \int d\zeta d\sigma_1 \dots d\sigma_4 \mu(\Omega_4) \sinh\zeta \sin^5\theta_k \left(-\frac{4\Delta}{\cosh^2\zeta \sin^2\theta_k} + 8\Delta \right) s^I Y^I \,. \tag{6.106}$$

Performing the integration over the S^4, only the $SO(5)$ invariant spherical harmonics are selected, namely the harmonics which depends on θ_k only, and we get

$$S^{(1)}_{DBI} = \frac{N\sqrt{\lambda}}{3\pi} \int d\zeta \sinh\zeta \sin^5\theta_k \left(-\frac{4\Delta}{\cosh^2\zeta \sin^2\theta_k} + 8\Delta \right) s^\Delta Y^{\Delta,0}(\theta_k) \,, \tag{6.107}$$

where the suffix on the harmonic indicates that all the quantum numbers except one were set to zero by the integration over the 4-sphere. As reviewed in appendix C, these S^4 invariant harmonics can be explicitely written as

$$Y^{\Delta,0}(\theta_k) = \mathcal{N}_\Delta \, C^{(2)}_\Delta(\cos\theta_k) \,, \tag{6.108}$$

where $C^{(2)}_\Delta(\cos\theta_k)$ are Gegenbauer polynomials, and $\mathcal{N}_\Delta$ is a normalization constant necessary to have orthonormality.

The linear coupling coming from the Wess-Zumino part of the action (2.89) can be obtained using the expression for the 4-form fluctuation in eq. (6.104), and after integrating over the S^4 as above, we get

$$S^{(1)}_{WZ} = \frac{8N\sqrt{\lambda}}{3\pi} \int d\zeta \sinh\zeta \sin^4\theta_k \cos\theta_k s^\Delta \, \partial_{\theta_k} Y^{\Delta,0}(\theta_k). \tag{6.109}$$

The correlation function

The correlator between the rank k antisymmetric Wilson loop and chiral primary operator $\mathcal{O}_\Delta(L)$ can now be computed plugging eq. (6.85) into eqs. (6.106) and (6.109), and differentiating with respect to the source s_0^Δ. As before, the bulk-to-boundary propagator can be approximated by $c\,z^\Delta/L^{2\Delta}$. Recalling that on the D5 solution $z = a/\cosh\zeta$, the ζ-integrals can be readily computed and we get

$$\begin{aligned}\frac{\langle W_{A_k}\,\mathcal{O}_\Delta(L)\rangle}{\langle W_{A_k}\rangle} \quad = \quad & \frac{a^\Delta}{L^{2\Delta}}\Bigg[\frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k Y^{\Delta,0}(\theta_k) - \\ & -\frac{2^{\Delta/2+1}\sqrt{\lambda}(\Delta+1)}{3\pi\sqrt{\Delta}(\Delta-1)}\sin^5\theta_k\left(\Delta Y^{\Delta,0}(\theta_k)+\frac{\cos\theta_k}{\sin\theta_k}\partial_{\theta_k}Y^{\Delta,0}(\theta_k)\right)\Bigg].\end{aligned} \tag{6.110}$$

Using the formula (C.16) for the derivatives of Gegenbauer polynomials, we obtain

$$\Delta Y^{\Delta,0}(\theta_k)+\frac{\cos\theta_k}{\sin\theta_k}\partial_{\theta_k}Y^{\Delta,0}(\theta_k) = \frac{\mathcal{N}_\Delta}{\sin^2\theta_k}\big(\Delta C_\Delta^{(2)}(\cos\theta_k) - (\Delta+3)\cos\theta_k\, C_{\Delta-1}^{(2)}(\cos\theta_k)\big). \tag{6.111}$$

The correlation function (6.110) can then be written as

$$\begin{aligned}\frac{\langle W_{A_k}\,\mathcal{O}_\Delta(L)\rangle}{\langle W_{A_k}\rangle} \quad = \quad & \frac{a^\Delta}{L^{2\Delta}}Y^{\Delta,0}(0)\Bigg[\frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k\cdot \\ & \cdot\frac{6(\Delta-2)!}{(\Delta+2)!}\left(2(\Delta+1)\cos\theta_k\, C_{\Delta-1}^{(2)}(\cos\theta_k) - \Delta C_\Delta^{(2)}(\cos\theta_k)\right)\Bigg],\end{aligned} \tag{6.112}$$

where we have factorized out the spherical harmonic evaluated at $\theta = 0$, $Y^{\Delta,0}(0) = \mathcal{N}_\Delta \frac{(\Delta+3)!}{6\Delta!}$.[9] The OPE coefficient $c_{A_k,\Delta}$ we aim to compute is the expression in square brackets. Using the recurrence relation eq. (C.17), we find that this expression can be written in the compact form

$$c_{A_k,\Delta} = \frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k \,\frac{6(\Delta-2)!}{(\Delta+1)!}\,C^{(2)}_{\Delta-2}(\cos\theta_k). \tag{6.113}$$

This is our final result for the correlation function of rank k antisymmetric Wilson loops and chiral primaries. In the next section, we will see that this result exactly matches the one obtained from the normal matrix model. As a check, one can verify that this expression reduces to the string result of [227] in the limit $k/N \to 0$, by using $\theta_k^3 \sim 3\pi k/2N$ and eq. (C.18) from the appendix.

6.2.4 The correlation functions from the normal matrix model

The conjecture that the expectation value of a circular loop is captured by the Hermitian matrix model (2.58) extends to higher rank Wilson loops as well. The result for the multiply wound Wilson loop has been obtained in [80], whereas [90] and [81][90] contain the computations for, respectively, the symmetric and antisymmetric representations.

When computing the correlation function between a Wilson loop and a chiral primary operator one can substitute the Hermitian model (2.58) with a complex one by introducing a second matrix M_{Im} and defining $z = M + iM_{\text{Im}}$. In [89] it was shown that, for certain representations of the Wilson loop (the multi-winding

[9] The OPE coefficient does not include a factor coming from the spherical harmonic evaluated at the unit 6-vector Θ^I appearing in eq. (2.53). After a rotation, this vector can always be set to $\Theta^I = (1, 0, \ldots, 0)$ which corresponds to the north pole of S^5, *i.e.* $\theta = 0$.

and the antisymmetric), the complex matrix model is equivalent to a normal matrix model, which is a complex model where the matrix is constrained to commute with its conjugate.[10] In the normal matrix model the expression for the Wilson loop reads

$$\langle W_{\mathcal{R}} \rangle = \frac{1}{\mathcal{Z}_N} \int_{[z,\bar{z}]=0} [d^2 z] \exp\left(-\frac{2N}{\lambda} \mathrm{Tr} z\bar{z} \right) \frac{1}{\dim \mathcal{R}} \mathrm{Tr}_{\mathcal{R}} e^{\frac{1}{\sqrt{2}}(z+\bar{z}) - \frac{\lambda}{8N}}. \tag{6.114}$$

For large N, the eigenvalues of this model are distributed in incompressible droplets in the complex plane. This leads to interpreting the complex plane as the phase space of free fermions, in analogy with the matrix quantum mechanics describing chiral primary operators [35][184]. For example, the Wilson loop in the fundamental representation has an eigenvalue distribution given by a circular droplet with constant density[11]

$$\rho(z) = \begin{cases} \frac{2}{\pi\lambda} & |z| < \sqrt{\frac{\lambda}{2}} \\ 0 & |z| > \sqrt{\frac{\lambda}{2}}\,. \end{cases} \tag{6.115}$$

In [89] it was also shown that the correlation function between a Wilson loop in the fundamental representation and a chiral primary operator is given by[12]

$$\langle W_{\Box} \mathcal{O}_{\Delta} \rangle = \frac{2^{\Delta/2}}{\mathcal{Z}_N} \int_{[z,\bar{z}]=0} [d^2 z] \exp\left(-\mathrm{Tr}\, z\bar{z}\right) \frac{1}{N} \mathrm{Tr}_{\Box}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z}) - \frac{\lambda}{8N}} \frac{1}{\sqrt{\Delta N^{\Delta}}} \mathrm{Tr}\, z^{\Delta} \tag{6.116}$$

We now use the normal matrix model to check our results for the coefficients of the operator product expansion of higher rank Wilson loops.

---

[10] This is required in order to be able to diagonalize simultaneously the two matrices M and M_{Im}. This allows to use the eigenvalue basis for solving the model.

[11] Projecting eq. (6.115) into the real axis one recovers the Wigner semi-circle distribution.

[12] The factor $2^{\Delta/2}$ instead of the $2^{-\Delta/2}$ of [89] is set in order to have normalizations consistent with [227].

The symmetric case

We start by reproducing the result (6.101) for $c_{S_k,\Delta}$ using the normal matrix model. According to the holographic dictionary put forward in [88], we are interested in the correlator between a Wilson loop in the rank k symmetric representation and the chiral primary operator $\mathcal{O}_\Delta = \frac{1}{\sqrt{\Delta N^\Delta}}\mathrm{Tr}Z^\Delta$. In the limit of large N and large λ the symmetric representation Wilson loop W_{S_k} effectively coincides with the multiply wound fundamental loop $W_\square^{(k)}$, as was shown in [89][90]. Therefore we limit ourselves to the simpler case of computing $\langle W_\square^{(k)}\mathcal{O}_\Delta\rangle$, where k is the winding number and corresponds in the brane probe picture to the number of fundamental strings dissolved in the brane.

We start from eq. (4.7) of [89], where we replace everywhere $\lambda \to k^2\lambda$

$$\langle W_\square^{(k)}\mathcal{O}_\Delta\rangle = \frac{2^{\Delta/2+1}e^{k^2\lambda/8N}}{k\sqrt{\Delta\lambda}}\oint \frac{dw}{2\pi i}w^\Delta e^{k\sqrt{\lambda}w/2}\left(1+\frac{k\sqrt{\lambda}}{2Nw}\right)^N\left[\left(1+\frac{k\sqrt{\lambda}}{2Nw}\right)^\Delta - 1\right]. \tag{6.117}$$

The large winding limit consists in taking $N \to \infty$ while keeping $\kappa \equiv \frac{k\sqrt{\lambda}}{4N}$ fixed. In this limit the integral can be evaluated around the saddle point of the terms proportional to N and k

$$\partial_w\left(\frac{k\sqrt{\lambda}}{2}w + N\log\left(1+\frac{k\sqrt{\lambda}}{2Nw}\right)\right) = 0\,, \tag{6.118}$$

which yields

$$w_\star = \sqrt{1+\kappa^2} - \kappa. \tag{6.119}$$

Inserting w_* in eq. (6.117) and using

$$\sqrt{1+\kappa^2}+\kappa = \exp(\sinh^{-1}\kappa)\,, \qquad \sqrt{1+\kappa^2}-\kappa = \exp(-\sinh^{-1}\kappa)\,, \tag{6.120}$$

it is easy to see that

$$\langle W_\Box^{(k)} \mathcal{O}_\Delta \rangle = \frac{2^{\Delta/2}}{2N\kappa\sqrt{\Delta}}\, 2\sinh(\Delta \sinh^{-1}\kappa) e^{2N(\kappa\sqrt{1+\kappa^2}+\sinh^{-1}\kappa)}. \tag{6.121}$$

To get a properly normalized expression one still needs to divide eq. (6.121) by

$$\langle W_\Box^{(k)} \rangle = \frac{1}{2N\kappa} e^{2N(\kappa\sqrt{1+\kappa^2}+\sinh^{-1}\kappa)}. \tag{6.122}$$

The final result coincides with eq. (6.101), which we obtained from the brane picture.

The antisymmetric case

To compute the OPE coefficients of Wilson loops in the rank k antisymmetric representation, we have to evaluate the following correlator in the normal matrix model [89]

$$\langle W_{A_k}\, \mathcal{O}_\Delta \rangle = \frac{2^{\Delta/2}\, e^{k\lambda/8N}}{\mathcal{Z}_N N^{\Delta/2}\sqrt{\Delta}} \int_{[z,\bar{z}]=0} [d^2 z] e^{-\mathrm{Tr}(z\bar{z})} \frac{1}{\mathrm{dim} A_k} \mathrm{Tr}_{A_k} e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\, \mathrm{Tr} z^\Delta. \tag{6.123}$$

This matrix integral can actually be solved exactly, as was shown in [89], and similarly to the case of the fundamental representation, it can be written as a k-dimensional contour integral. However, it does not seem to be easy to take the large N and large k limit with k/N fixed from such an expression. Here we follow a different approach to get the above correlator in this limit. First, as in [90], we find it convenient to rewrite the trace in the antisymmetric representation using the corresponding generating function

$$\mathrm{Tr}_{A_k} e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})} = \oint \frac{dt}{2\pi i} t^{k-1} \exp\left[\mathrm{Tr}\log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]. \tag{6.124}$$

Since we expect the correlator to be real, it is also convenient to replace $\mathrm{Tr}z^\Delta \to \frac{1}{2}(\mathrm{Tr}z^\Delta + \mathrm{Tr}\bar{z}^\Delta)$. The idea is then to view the insertion of the chiral primary $\mathrm{Tr}z^\Delta$ in eq. (6.123) as a small perturbation of the gaussian potential, by writing

$$\begin{aligned}
&\int_{[z,\bar{z}]=0} [d^2z] e^{-\mathrm{Tr}(z\bar{z})} \mathrm{Tr}z^\Delta \, e^{\mathrm{Tr}\log(1+\frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})})} = \\
&\quad = \mathcal{Z}_N \frac{\partial}{\partial\alpha}\left(\frac{1}{\mathcal{Z}_N(\alpha)} \int_{[z,\bar{z}]=0} [d^2z] e^{-\mathrm{Tr}(z\bar{z}) + \frac{\alpha}{2}(\mathrm{Tr}z^\Delta + \mathrm{Tr}\bar{z}^\Delta)} e^{\mathrm{Tr}\log(1+\frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})})}\right)\Bigg|_{\alpha=0} \\
&\quad \equiv \mathcal{Z}_N \frac{\partial}{\partial\alpha}\left\langle \exp\left[\mathrm{Tr}\log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]\right\rangle_\alpha \Bigg|_{\alpha=0},
\end{aligned} \tag{6.125}$$

where we have introduced an α-dependent partition function

$$\mathcal{Z}_N(\alpha) = \int_{[z,\bar{z}]=0} [d^2z] e^{-\mathrm{Tr}(z\bar{z}) + \frac{\alpha}{2}(\mathrm{Tr}z^\Delta + \mathrm{Tr}\bar{z}^\Delta)}\,, \tag{6.126}$$

and we have used that $\mathcal{Z}_N(\alpha) = \mathcal{Z}_N + \mathcal{O}(\alpha^2)$.[13] The problem is now to evaluate the correlation function (6.125) in the normal matrix model with the deformed potential

$$V(z,\bar{z}) = -\mathrm{Tr}z\bar{z} + \frac{\alpha}{2}\mathrm{Tr}(z^\Delta + \bar{z}^\Delta)\,. \tag{6.127}$$

Normal models with potentials of this kind were previously studied in the literature, for a recent account see for example [228][229]. To solve the model at large N, one

[13] This follows from the fact that in the matrix model with gaussian potential $\langle \mathrm{Tr}z^\Delta \rangle = \langle \mathrm{Tr}\bar{z}^\Delta \rangle = 0$.

can as usual go to the eigenvalue basis at the expenses of introducing a Vandermonde factor, and determine the eigenvalue density $\rho_\alpha(z,\bar{z})$ in the continuum limit. The density is found by solving the saddle point equation[14]

$$z - \frac{\Delta\alpha}{2}\bar{z}^{\Delta-1} = N\int d^2z'\,\frac{\rho_\alpha(z',\bar{z}')}{\bar{z}-\bar{z}'}\,, \tag{6.128}$$

where the term in the right hand side comes from the Vandermonde factor. Once the density is known, the correlation function in eq. (6.125) becomes

$$\left\langle \exp\left[\mathrm{Tr}\log\left(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]\right\rangle_\alpha \to \exp\left[N\int d^2z\,\rho_\alpha(z,\bar{z})\log\left(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]. \tag{6.129}$$

It is known that for potentials of the kind $V(z,\bar{z}) = -z\bar{z} + f(z) + \bar{f}(\bar{z})$, the density is a constant (equal to $\frac{1}{N\pi}$ in the normalizations we are using here) inside a certain droplet in the complex plane and zero outside. For the gaussian potential, as reviewed previously, the droplet is just a circle of radius $\sqrt{N}$ (to compare with eq. (6.115), one has to rescale $z \to \sqrt{\frac{\lambda}{2N}}z$), while the term proportional to α induces a deformation of the circle which preserves its total area (since we do not change the number of eigenvalues). It is not difficult to find the shape of the droplet which solves the saddle point equation (6.128) at leading order in α. It is convenient to work in polar coordinates $z = re^{i\phi}$. The curve which bounds the droplet can then be written at first order as

$$r(\phi) = \sqrt{N}\big(1 + \alpha\, f(\phi)\big). \tag{6.130}$$

[14] As in [90], the term $\exp\left[\mathrm{Tr}\log\left(1+\frac{1}{t}\,e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]$ does not modify the saddle point equation at leading order at large N.

Clearly $f(\phi)$ has to be periodic, and may be written as

$$f(\phi) = \sum_{n=1}^{\infty} a_n \cos n\phi\,, \tag{6.131}$$

where only cosines appear because of the symmetry of the potential (6.127) under $z \leftrightarrow \bar{z}$, and the mode with $n = 0$ is excluded by requiring the area to be preserved. The saddle point equation now reads

$$r\, e^{i\phi} - \frac{\Delta\alpha}{2} r^{\Delta-1}\, e^{-i(\Delta-1)\phi} = \frac{1}{\pi}\int_0^{2\pi} d\phi' \int_0^{\sqrt{N}(1+\alpha\, f(\phi'))} dr'\, \frac{r'}{r\, e^{-i\phi} - r'\, e^{-i\phi'}}. \tag{6.132}$$

Expanding the integral at first order in α and plugging in the Fourier expansion (6.131), we see that this equation is satisfied if $a_\Delta = N^{\Delta/2-1}\,\frac{\Delta}{2}$ and all other a_n vanish, so we find that the shape of the deformed droplet is given by the curve

$$r(\phi) = \sqrt{N}\left(1 + \frac{\alpha}{2}\,\Delta\, N^{\Delta/2-1} \cos \Delta\phi\right). \tag{6.133}$$

Before moving on to compute eq. (6.125), we can check the validity of the method by applying it to the computation of the correlator (6.116) when the Wilson loop is in the fundamental representation. In this case, following the same steps as above, in the large N limit we arrive at

$$\begin{aligned}
\langle W_\square \mathcal{O}_\Delta \rangle &= \frac{2^{\Delta/2}}{\sqrt{\Delta} N^{\Delta/2}} \frac{\partial}{\partial\alpha} \int d^2 z\, \rho_\alpha(z,\bar{z})\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\Big|_{\alpha=0} \\
&= \frac{2^{\Delta/2}}{\sqrt{\Delta} N^{\Delta/2}} \frac{\partial}{\partial\alpha} \frac{1}{N\pi} \int_0^{2\pi} d\phi \int_0^{\sqrt{N}(1+\frac{\alpha}{2}\,\Delta\, N^{\Delta/2-1}\cos\Delta\phi)} dr\, r\, e^{\sqrt{\frac{\lambda}{N}} r\cos\phi}\Big|_{\alpha=0} \\
&= \frac{2^{\Delta/2}\sqrt{\Delta}}{N} \frac{1}{2\pi} \int_0^{2\pi} d\phi\, e^{\sqrt{\lambda}\cos\phi} \cos\Delta\phi = \frac{2^{\Delta/2}\sqrt{\Delta}}{N} I_\Delta(\sqrt{\lambda})\,,
\end{aligned} \tag{6.134}$$

which is the result first found in [31] and the correct large N limit of the exact formula (6.117) (with $k = 1$).

Going back to the antisymmetric representation, we have to evaluate

$$\begin{aligned}
&\oint \frac{dt}{2\pi i} t^{k-1} \frac{\partial}{\partial\alpha} \exp\left[N \int d^2z\, \rho_\alpha(z,\bar{z}) \log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]\Bigg|_{\alpha=0} \\
&= \oint \frac{dt}{2\pi i} t^{k-1} \frac{\partial}{\partial\alpha} \left[N \int d^2z\, \rho_\alpha(z,\bar{z}) \log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right]\Bigg|_{\alpha=0} \times \\
&\qquad \times \exp\left[N \int d^2z\, \rho_0(z,\bar{z}) \log\left(1 + \frac{1}{t}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\bar{z})}\right)\right], \qquad (6.135)
\end{aligned}$$

where in the last line ρ_0 is just the circular droplet density. Since the exponent is independent of α, the t integral can be evaluated in the supergravity limit of large λ exactly as in [90]. We first make a change of variables $t = e^{\sqrt{\lambda}w}$, then the saddle point of the exponent is found to be

$$w_\star = \cos\theta_k\,, \qquad (6.136)$$

where θ_k is defined as in eq. (2.91). The exponent in eq. (6.135) gives a term proportional to the expectation value of the Wilson loop, while the prefactor is evaluated at the saddle point. After dividing by $\langle W_{A_k}\rangle$, the OPE coefficient can then be obtained as

$$\begin{aligned}
\frac{\langle W_{A_k}\, \mathcal{O}_\Delta\rangle}{\langle W_{A_k}\rangle} &= \frac{2^{\Delta/2}}{\sqrt{\Delta} N^{\Delta/2}} \frac{\partial}{\partial\alpha} \left[N \int d^2z\, \rho_\alpha(z,\bar{z}) \log\left(1 +\, e^{\frac{1}{2}\sqrt{\lambda}\left(\frac{z+\bar{z}}{\sqrt{N}} - 2\cos\theta_k\right)}\right)\right]\Bigg|_{\alpha=0} \\
&\simeq \frac{2^{\Delta/2} N\sqrt{\lambda}}{\sqrt{\Delta} N^{\Delta/2}} \frac{\partial}{\partial\alpha} \frac{2}{\pi} \int_0^{\theta_k} d\phi \int_{\frac{\cos\theta_k}{\cos\phi}}^{1+\frac{\alpha}{2}\Delta N^{\Delta/2-1}\cos\Delta\phi} dr\, r(r\cos\phi - \cos\theta_k)\Big|_{\alpha=0}, \qquad (6.137)
\end{aligned}$$

where the lower limit in the r integral comes from the fact that in the large λ limit the integral has support only in the region $r\cos\phi \geq \cos\theta_k$.[15] After doing the derivative, eq. (6.137) gives the final result

$$\frac{\langle W_{A_k}\,\mathcal{O}_\Delta\rangle}{\langle W_{A_k}\rangle} = \frac{2^{\Delta/2}\sqrt{\Delta\lambda}}{\pi}\int_0^{\theta_k} d\phi\,\cos\Delta\phi\,(\cos\phi-\cos\theta_k)\,. \tag{6.138}$$

Remarkably, the integral in eq. (6.138) precisely reproduces the Gegenbauer polynomials arising in the bulk computation, and the final result is

$$\frac{\langle W_{A_k}\,\mathcal{O}_\Delta\rangle}{\langle W_{A_k}\rangle} = \frac{2^{\Delta/2}}{3\pi}\sqrt{\Delta\lambda}\sin^3\theta_k\,\frac{6(\Delta-2)!}{(\Delta+1)!}\,C^{(2)}_{\Delta-2}(\cos\theta_k)\,, \tag{6.139}$$

which exactly matches the D5 computation of the OPE coefficient.

6.2.5 Outlook

In this section we computed the correlation function between a higher rank Wilson loop and a chiral primary operator in the fundamental representation using branes with electric fluxes. Following the proposal of [80][81][88], we considered a D3-brane for the rank k symmetric case and a D5-brane for the antisymmetric one. We then checked our results with the normal matrix model discussed in [89], finding perfect agreement in both cases.

We focussed on chiral primary operators but it should not be difficult to extend our computation to operators corresponding to other supergravity modes. For example, the KK modes of the dilaton are necessary to compute correlation functions of Wilson loops and $\mathrm{Tr}\,\Phi^\Delta F_+^2$.

It would be worthwhile to study more general representations of both the Wilson

[15] The upper limit in the ϕ integral is rigorously $\theta_k + \mathcal{O}(\alpha)$, but it is easy to see that the correction does not contribute at first order in α.

loop and the chiral primary operator. A particularly interesting issue to address would be understanding from our brane picture the selection rule found in [89]: for Wilson loops in the rank k antisymmetric representation the only non-vanishing correlators involve chiral primaries with traces over Young diagrams with at most k hooks. Another direction to pursue may be considering the correlation function between higher dimensional Wilson loop and a chiral primary operator with $\Delta \sim N$. In the bulk this would require to study the bulk-to-bulk exchange of supergravity degrees of freedom between the electric branes describing the Wilson loop and the (dual) giant gravitons associated with the chiral primary.

6.3 Quarter BPS Wilson loops

In this section we continue the investigation of the D-brane description of higher rank Wilson loops. In particular we study some systems where it is possible to find solutions for the D3-branes starting from first-order equations derived from the supersymmetry conditions. All the examples we present are Wilson loop operators which preserve 1/4 of the supersymmetry generators.

First we consider the system of a straight 1/2 BPS Wilson loop with the insertion of two 1/2 BPS local operators such that the combined system preserves 1/4 of the supercharges. If there was only the Wilson loop, the D3-brane would have been the one of [80], while if only the local operators were present, that would have involved the original giant gravitons [91]-[93]. This combined system of a Wilson loop and a local operator was presented in [230], where it was shown to be supersymmetric and the relevant string solution was found. In section 6.3.1 we present the D3-brane solution preserving the same supersymmetries, and interpolating between the giant

Wilson loop of [80] near the boundary and a giant graviton in the center of AdS_5.

The second system, which will be described in section 6.3.2, involves circular Wilson loops which couple to three of the $\mathcal{N} = 4$ scalars. This system, first presented in [231] (as a generalization of an example in [68]) and studied further in [223], also preserves 1/4 of the supersymmetries, but with a different combination of generators than in the previous example. At two loop order in the perturbative expansion the interacting graphs (in the Feynman gauge) cancel, which led to the conjecture that only ladder/rainbow diagrams contribute to these operators. All those diagrams combine nicely into a matrix model which was then compared with the string calculation in $AdS_5 \times S^5$. As we shall review in section 6.3.2 below, the results agreed including a subleading term, a world-sheet instanton, which matched a correction to the asymptotic expansion of the matrix model at strong coupling.

The calculation using D3-branes is applicable for a Wilson loop in a symmetric representation whose rank k is of order N. At large λ the analog observable in the matrix model agrees with the single-trace multiply wrapped loop which is given by a function of the ratio k/N and thus the D3-brane calculation captures non-planar corrections to the usual string calculation. We are able to compare the matrix model and the D-brane calculation for arbitrary k/N and find an agreement and a check of the aforementioned conjecture to all orders in $1/N$.

6.3.1 Wilson loop with insertions

Setup

We consider here a Wilson loop operator in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $S^3 \times \mathbb{R}$, where the line is the time direction with Lorentzian signature. The loop will be comprised of one line in the time direction along a point on S^3, and another line going in the opposite direction at the antipodal point. In addition we will include the insertion of local operators at the infinite past and infinite future.

Under the exponential map (after Wick-rotation), the space is mapped to flat $\mathbb{R}^4$ and the two lines to a single line through the origin with the local insertions at the origin and at infinity. Without the insertions this Wilson loop preserves half the supersymmetries of the vacuum, and we will consider the local insertions to also be 1/2 BPS, Z^J, where $Z = \Phi_1 + i\Phi_2$ is a complex scalar field (if Z^J is at the origin the charge has to be absorbed by $\bar{Z}^J$ at infinity). Note that the insertions are not gauge invariant, since they are not traced over, and transform in the adjoint representation of the gauge group. The entire configuration is nonetheless gauge invariant, because of the presence of the Wilson loop. This guarantees that in the string picture the charge generated by the local operator is carried by the open string (or D-brane) representing the Wilson loop, and not by another supergravity field.

Recall that the 1/2 BPS Wilson loop contains also a coupling to one of the scalars. For the combined system to be 1/4 BPS, this scalar has to be orthogonal to Z and $\bar{Z}$, so we take it to be Φ_3. Formally we can write the Wilson loop as

$$W_{Z^J} = \mathrm{Tr}\,\mathcal{P}\left[Z^J(-\infty)e^{i\int_{-\infty}^{\infty}(A_t(t,0)+\Phi_3(t,0))dt}\bar{Z}^J(\infty)e^{i\int_{\infty}^{-\infty}(A_t(t,\pi)+\Phi_3(t,\pi))dt}\right]. \qquad (6.140)$$

The arguments of A_t and Φ_3 are the time and the two points on S^3 given by an angle at 0 and π.

String solution

This Wilson loop was studied in [230] as were some non-supersymmetric generalizations of it and they were related to a certain spin-chain system. There it was proven that eq. (6.140), which was the pseudo-vacuum of the spin-chain system, is supersymmetric. Also the string solution describing it at large J and large 't Hooft coupling λ was given. We review it here.

Take the following metric for $AdS_5 \times S^2$ (the other directions on S^5 do not play any role and we do not write them explicitly)

$$\frac{ds^2}{L^2} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho \left(d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right) + d\theta^2 + \sin^2\theta d\phi^2\,. \tag{6.141}$$

L is the radius of curvature related to the 't Hooft coupling and the string tension by $L^4 = \lambda\,\alpha'^2$. The Wilson loop should reach the boundary at $\chi = 0$ and $\chi = \pi$. At those points it should approach $\theta = 0$ on the S^2, which is the direction corresponding to Φ_3. In the bulk the string should rotate around this sphere carrying the angular momentum related to Z^J.

The solution to the string equations of motion which satisfies these conditions is

$$\phi = t\,, \qquad \sin\theta = \frac{1}{\cosh\rho}\,. \tag{6.142}$$

There are two parts to the string: at $\chi = 0$ and at $\chi = \pi$. They are continuously connected to each other beyond $\rho = 0$. For a full derivation of the solution see [230].

Some interesting issues arise when studying the analog system in Euclidean signature. Those were discussed in [232].

Supersymmetry analysis

A precise counting of the supersymmetries preserved by the string solution (6.142) was performed in [230]. Here we briefly review that computation.

The number of supersymmetries preserved by the string is equal to the number of independent solutions to the equation $\Gamma\epsilon = \epsilon$. The κ-symmetry projector Γ is given by

$$\Gamma = \frac{1}{\sqrt{-\det g}}\,\partial_t x^\mu \partial_\rho x^\nu \gamma_\mu \gamma_\nu K\,, \tag{6.143}$$

where g is the induced metric on the world-sheet parameterized by t and ρ, K acts by complex conjugation, and $\gamma_\mu = e^a_\mu \Gamma_a$ with Γ_a constant tangent space gamma-matrices. The dependence of the Killing spinors ϵ on the relevant coordinates of the metric (6.141) is

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1} e^{-\frac{i}{2}t\,\Gamma_\star\Gamma_0} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5} e^{\frac{1}{2}\phi\,\Gamma_{56}} \epsilon_0\,, \tag{6.144}$$

where $\Gamma_\star = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4$ is the product of all gamma-matrices in the AdS_5 directions and ϵ_0 is any constant chiral complex 16-component spinor. The spinors ϵ solve the Killing equation

$$\left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab} + \frac{i}{2L}\Gamma_\star\gamma_\mu\right)\epsilon = 0\,. \tag{6.145}$$

Inserting the solution (6.142) into the expression (6.143) it is easy to see that Γ does not depend on t. The only place where t appears is in the exponent of the Killing spinors. Since the projection equation has to hold for all t and ρ we eliminate this dependence by imposing the condition

$$\Gamma_\star\Gamma_{056}\epsilon_0 = i\epsilon_0\,, \tag{6.146}$$

so that the Killing spinors become

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\epsilon_0\,. \tag{6.147}$$

After some manipulation the action of the projector can be written as

$$\Gamma\epsilon = -e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\,\Gamma_{01}K\epsilon_0\,, \tag{6.148}$$

so the projector equation is solved by all constant spinors satisfying

$$\Gamma_{01}K\epsilon_0 = -\epsilon_0\,. \tag{6.149}$$

It is easy to verify that the two conditions (6.146) and (6.149) are consistent with each-other, so there are eight linearly independent real solutions to this equation. Thus the string solution preserves 1/4 of the supersymmetries.

D3-brane solution

We look now for the D3-brane solution associated to this Wilson loops with insertions. The loop is in the time direction, as reviewed above, and preserves an $SO(3)\times SO(3)$ symmetry, the first being part of the AdS_5 isometry and the other coming from the S^5. It is convenient to use the metric (6.141) and fix a static gauge where t, ρ, ϑ and φ are the world-volume coordinates on the D3-brane. The Ansatz is then

$$\chi = \chi(\rho)\,, \qquad \theta = \theta(\rho)\,, \qquad \phi = t\,. \tag{6.150}$$

The brane action consists of a Dirac-Born-Infeld (DBI) part and of a Wess-Zumino

(WZ) term, which captures the coupling to the background R-R form

$$S = T_{D3}\int e^{-\Phi}\sqrt{-\det(g+2\pi\alpha' F)} - T_{D3}\int P[C_4]\,, \tag{6.151}$$

where $T_{D3} = \frac{N}{2\pi^2 L^4}$ is the brane tension and $P[C_4]$ denotes the pullback of the 4-form to the brane world-volume. The solution should include a non-zero electric field, carrying k units of flux associated to the Wilson loop. Because of the symmetry of the system, it will be in the direction $F_{t\rho}(\rho)$.

With the ansatz above the DBI action reads (in the following we absorb a factor of $2\pi\alpha'/L^2$ in the definition of $F_{t\rho}$)

$$S_{DBI} = \frac{2N}{\pi}\int dt\, d\rho\, \sinh^2\rho \sin^2\chi \sqrt{(\cosh^2\rho - \sin^2\theta)(1+\sinh^2\rho\,\chi'^2+\theta'^2) - F_{t\rho}^2}\,, \tag{6.152}$$

whereas the WZ term is given by

$$S_{WZ} = \frac{2N}{\pi}\int dt\, d\rho\, \sinh^4\rho \sin^2\chi\,\chi'\,, \tag{6.153}$$

and the relative sign between these two terms in the action is positive. In these formulas the $'$ denotes a derivative with respect to ρ.

It is rather complicated to solve the equations of motion coming from this action. Instead of trying to do this, we write down the supersymmetry equations derived from requiring κ-symmetry. These are first-order rather than second-order and can be integrated easily.

The κ-symmetry projector associated with the D3-brane embedding is (see for example [233])

$$\Gamma = \mathcal{L}_{DBI}^{-1} \left(\Gamma_{(4)} + L^2 F_{t\rho} \Gamma_{(2)} K \right) I \,, \tag{6.154}$$

where K acts by complex conjugation, I by multiplication by $-i$, and

$$\begin{aligned} \Gamma_{(4)} &= \partial_t x^\mu \partial_\rho x^\nu \partial_\vartheta x^\xi \partial_\varphi x^\zeta \gamma_\mu \gamma_\nu \gamma_\xi \gamma_\zeta = (\gamma_t + \gamma_\phi)(\gamma_\rho + \chi' \gamma_\chi + \theta' \gamma_\theta) \gamma_\vartheta \gamma_\varphi \,, \\ \Gamma_{(2)} &= \partial_\vartheta x^\mu \partial_\varphi x^\nu \gamma_\mu \gamma_\nu = \gamma_\vartheta \gamma_\varphi \,, \end{aligned} \tag{6.155}$$

with $\gamma_\mu = e_\mu^a \Gamma_a$. Using the vielbeins

$$\begin{gathered} e^0 = L \cosh\rho \, dt \,, \qquad e^1 = L \, d\rho \,, \qquad e^2 = L \sinh\rho \, d\chi \,, \\ e^3 = L \sinh\rho \sin\chi \, d\vartheta \,, \qquad e^4 = L \sinh\rho \sin\chi \sin\vartheta \, d\varphi \,, \\ e^5 = L \, d\theta \,, \qquad e^6 = L \sin\theta \, d\phi \,, \end{gathered} \tag{6.156}$$

and the Ansatz (6.150), the projectors $\Gamma_{(4)}$ and $\Gamma_{(2)}$ can be explicitly written as

$$\begin{aligned} \Gamma_{(4)} &= L^2 (\cosh\rho \Gamma_0 + \sin\theta \Gamma_6)(\Gamma_1 + \sinh\rho \, \chi' \, \Gamma_2 + \theta' \, \Gamma_5) \, \Gamma_{(2)} \,, \\ \Gamma_{(2)} &= L^2 \sinh^2\rho \sin^2\chi \sin\vartheta \, \Gamma_{34} \,. \end{aligned} \tag{6.157}$$

Adding the dependence on the other coordinates into eq. (6.144), the Killing spinors for the metric (6.141) are

$$\epsilon = e^{-\frac{i}{2}\rho \, \Gamma_\star \Gamma_1} e^{-\frac{i}{2} t \, \Gamma_\star \Gamma_0} e^{\frac{1}{2}\chi \, \Gamma_{12}} e^{\frac{1}{2}\vartheta \, \Gamma_{23}} e^{\frac{1}{2}\varphi \, \Gamma_{34}} e^{-\frac{i}{2}\theta \, \Gamma_\star \Gamma_5} e^{\frac{1}{2}\phi \, \Gamma_{56}} \epsilon_0 \,. \tag{6.158}$$

From the supersymmetry analysis in the string case, we know that the constant spinors ϵ_0 satisfy the conditions (6.146) and (6.149)

$$K\epsilon_0 = -\Gamma_{01}\epsilon_0\,, \qquad \Gamma_6\epsilon_0 = -i\Gamma_{12345}\epsilon_0\,. \tag{6.159}$$

Plugging $\phi = t$ in the expression (6.158) and using the second constraint in the equation above, the Killing spinors may be rewritten as

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{\frac{1}{2}\chi\,\Gamma_{12}}M\epsilon_0\,, \tag{6.160}$$

where

$$M = e^{\frac{1}{2}\vartheta\,\Gamma_{23}}e^{\frac{1}{2}\varphi\,\Gamma_{34}}. \tag{6.161}$$

The differential equations we are looking for will come from considering the projector equation

$$\Gamma\epsilon = \epsilon\,. \tag{6.162}$$

To simplify it, we move the matrix $e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{\frac{1}{2}\chi\,\Gamma_{12}}$ to the left of the projector Γ, using some gamma-matrix algebra and applying the constraints (6.159) (note that ϵ_0 and $M\epsilon_0$ satisfy the same constraints). In this way we get a set of 8 differential equations in θ, χ and $F_{t\rho}$ (on the left we indicate the gamma-matrix structure the equations come from)

$$
\begin{aligned}
&\Gamma_{0345}: && 0 = F_{t\rho}\sinh\rho\cos\chi\sin\theta - \theta'(\cosh^2\rho - \sin^2\theta)\\
&\Gamma_{\star}\Gamma_5: && 0 = F_{t\rho}\sinh\rho\sin\chi\sin\theta - \chi'\sinh^2\rho\sin\theta\cos\theta\\
&\Gamma_{0234}: && 0 = (\cosh^2\rho - \sin^2\theta)\sin\chi + \chi'\cosh\rho\sinh\rho\cos\chi\cos^2\theta\\
&\Gamma_{12}: && 0 = F_{t\rho}\sinh\rho\cos\chi\cos\theta + \theta'\sin\theta\cos\theta + \cosh\rho\sinh\rho\\
&\Gamma_{15}: && 0 = \chi'\cosh\rho\sinh\rho\cos\chi\sin\theta\cos\theta - \theta'\cosh\rho\sinh\rho\sin\chi + \sin\chi\sin\theta\cos\theta\\
&\Gamma_{25}: && 0 = F_{t\rho}\cosh\rho\sin\theta + \cos\chi\sin\theta\cos\theta - \chi'\cosh\rho\sinh\rho\sin\chi\sin\theta\cos\theta\\
& && \quad -\theta'\cosh\rho\sinh\rho\cos\chi\\
&\Gamma_{0134}: && 0 = F_{t\rho}\cosh\rho\cos\theta + (\cosh^2\rho - \sin^2\theta)\cos\chi - \chi'\cosh\rho\sinh\rho\sin\chi\cos^2\theta\\
&1: && 1 = -L^4\mathcal{L}_{DBI}^{-1}\sinh^2\rho\sin^2\chi\sin\vartheta\left(F_{t\rho}\sinh\rho\sin\chi\cos\theta + \chi'\sinh^2\rho\sin^2\theta\right).
\end{aligned}
\tag{6.163}
$$

One can solve for θ', χ' and $F_{t\rho}$ using for instance the first three equations. Once these are solved, the remaining five are automatically satisfied. The first three equations give

$$
\begin{aligned}
\theta' &= -\tan\theta\tanh\rho\,,\\
\chi'\,\cot\chi &= -\frac{\cosh^2\rho - \sin^2\theta}{\cosh\rho\sinh\rho\cos^2\theta}\,,\\
F_{t\rho} &= -\frac{\cosh^2\rho - \sin^2\theta}{\cosh\rho\cos\theta\cos\chi}\,.
\end{aligned}
\tag{6.164}
$$

The solution to the first equation is

$$
\sin\theta = \frac{C_1}{\cosh\rho}. \tag{6.165}
$$

The integration constant C_1 is related (in a complicated way) to the amount of angular momentum carried by the brane. After plugging the solution for θ into the expressions for χ' and $F_{t\rho}$, we find

$$\chi' \cot\chi = -\frac{\cosh^4\rho - C_1^2}{\sinh\rho\cosh\rho(\cosh^2\rho - C_1^2)}\,, \tag{6.166}$$

which is solved by

$$\sin\chi = C_2\,\frac{\coth\rho}{\sqrt{\cosh^2\rho - C_1^2}}\,, \tag{6.167}$$

with C_2 a second integration constant. Finally the electric field is

$$F_{t\rho} = -\frac{\cosh^4\rho - C_1^2}{\cosh^2\rho\sqrt{\cosh^2\rho - C_1^2 - C_2^2\coth^2\rho}}\,. \tag{6.168}$$

Plugging the BPS equations (6.164) into the DBI action (6.152), the square root simplifies to

$$\sqrt{(\cosh^2\rho - \sin^2\theta)(1 + \sinh^2\rho\,\chi'^2 + \theta'^2) - F_{t\rho}^2} = (\cosh^2\rho - \sin^2\theta)\frac{\tanh\rho\tan\chi}{\cos^2\theta}\,. \tag{6.169}$$

It is then straightforward to check that the solutions (6.165), (6.167) and (6.168) satisfy the brane equations of motion stemming from eqs. (6.152) and (6.153).

Conserved charges

The solution has two integration constants C_1 and C_2 which are related to the two conserved charges carried by the brane: the rank of the symmetric representation (or the number of windings of the Wilson loop) k, and the angular momentum J around the S^2 in the S^5.

The first charge k is the conjugate momentum to the gauge field after integrating over ϑ and φ

$$k = \Pi = \frac{2\pi\alpha'}{L^2}\, T_{D3} \int d\vartheta\, d\varphi\, \frac{\delta\mathcal{L}}{\delta F_{t\rho}} = \frac{4N}{\sqrt{\lambda}} C_2 \,. \tag{6.170}$$

Then

$$C_2 = \frac{k\sqrt{\lambda}}{4N} \equiv \kappa \,. \tag{6.171}$$

If we take $\kappa \to 0$ we recover the string solution (6.142). Notice that the electric field $F_{t\rho}$ does not vanish in this limit.

The other conserved charge carried by the brane is the angular momentum J

$$\begin{aligned} J &= 2\, T_{D3} \int d\vartheta\, d\varphi\, d\rho\, \frac{\delta\mathcal{L}}{\delta\dot{\phi}} \\ &= -\frac{4N}{\pi} \int d\rho\, \frac{\sinh^2\rho \sin^2\chi \sin^2\theta\, (1 + \sinh^2\rho\, \chi'^2 + \theta'^2)}{\sqrt{(\cosh^2\rho - \sin^2\theta)(1 + \sinh^2\rho\, \chi'^2 + \theta'^2) - F_{t\rho}^2}} \,. \end{aligned} \tag{6.172}$$

Here the range of the ρ integral is $[\operatorname{arccosh} C_1, \infty)$, which like in the case of the string, covers only half the world-volume, with $\chi < \pi/2$. A multiplicative factor of 2 was included to account for the other branch with $\chi > \pi/2$.

Plugging the explicit solutions in this expression it is easy to see that $J \to 0$ when $C_1 \to 0$. In this limit the brane does not rotate along the S^2 and the solution reduces to

$$\begin{aligned} \sin\theta &= 0 \,, \\ \sin\chi\, \sinh\rho &= \kappa \,, \\ F_{t\rho} &= -\frac{\cosh\rho}{\cos\chi} \,, \end{aligned} \tag{6.173}$$

which is, after a conformal transformation, the same as the 1/2 BPS brane of [80].

The energy gets contributions from the DBI action and the Wess-Zumino term

$$\begin{aligned}
E_{DBI} &= 2\,T_{D3}\int d\vartheta\, d\varphi\, d\rho\, \frac{\delta\mathcal{L}_{DBI}}{\delta\dot{t}} \\
&= \frac{4N}{\pi}\int d\rho\, \frac{\sinh^2\rho\sin^2\chi\cosh^2\rho\,(1+\sinh^2\rho\,\chi'^2+\theta'^2)}{\sqrt{(\cosh^2\rho-\sin^2\theta)(1+\sinh^2\rho\,\chi'^2+\theta'^2)-F_{t\rho}^2}}\,, \\
E_{WZ} &= 2\,T_{D3}\int d\vartheta\, d\varphi\, d\rho\, \frac{\delta\mathcal{L}_{WZ}}{\delta\dot{t}} = \frac{4N}{\pi}\int d\rho\, \sinh^4\rho\sin^2\chi\,\chi'\,.
\end{aligned} \tag{6.174}$$

In addition one has to add a total derivative term, which serves as a Legendre transform from the gauge field coordinate to the conjugate momentum Π, which is the correct canonical variable in this problem [80]. This is

$$E_{L.T.} = \frac{2L^2}{2\pi\alpha'}\int d\rho\,\Pi\,F_{t\rho} = -\frac{4N}{\pi}\kappa\int d\rho\,\frac{\cosh^2\rho-\sin^2\theta}{\cosh\rho\cos\theta\cos\chi}\,. \tag{6.175}$$

Plugging the BPS equations in the formulas (6.172), (6.174) and (6.175) above, one can see that

$$E_{DBI}+E_{WZ}+E_{L.T.}+J = \frac{4N}{\pi}\int d\rho\,\frac{\cosh^2\rho-\sin^2\theta}{\cosh\rho\cos\chi}\left[\sinh\rho\sin\chi-\frac{\kappa}{\cos\theta}\right]. \tag{6.176}$$

Using the explicit solution it is easy to check that the term in square brackets vanishes, so we get $E=-J=|J|$.

6.3.2 Wilson loop wrapping a circle on S^5

Setup

In this section we shall look at a family of circular Wilson loops that couple to three of the six scalars of the $\mathcal{N}=4$ multiplet. These operators were presented in [231] and studied in detail in [223]. As for the loop in the previous section, the S^5 part of the gravity calculation will reduce to an S^2 subspace. The difference will be that here the couplings to the scalars are smeared around the loop and not localized at two points.

While the Wilson loop follows a curve on the boundary of AdS_5 parameterized by

$$x^1 = R\cos s\,, \qquad x^2 = R\sin s\,, \tag{6.177}$$

the scalar to which it couples is given by the linear combination

$$\Phi(s) = \Phi_3\cos\theta_0 + \sin\theta_0(\Phi_1\cos s + \Phi_2\sin s)\,, \tag{6.178}$$

with an arbitrary fixed parameter θ_0. The loop may be written (in Euclidean signature) as

$$W_{\theta_0} = \mathrm{Tr}\,\mathcal{P}\exp\left[\oint (iA_\mu(s)\dot{x}^\mu + |\dot{x}|\Phi(s))\,ds\right]. \tag{6.179}$$

In the special case of $\theta_0 = 0$ this is the usual 1/2 BPS circle, while for $\theta_0 = \pi/2$ this is a special case of the supersymmetric Wilson loops constructed by Zarembo [68].

It was shown in [223] that up to order $(g^2_{YM}N)^2$ all interacting graphs in the Feynman gauge cancel and the only contribution comes from ladder diagrams where the propagator is a constant proportional to $\cos^2\theta_0$. This naturally led to the conjecture that the expectation value of this Wilson loop is given by the same matrix model as the 1/2 BPS one [29][30] with the replacement of the coupling λ by $\lambda' = \lambda\cos^2\theta_0$. This gives the prediction

$$\langle W_{\theta_0}\rangle = \frac{1}{N}L^1_{N-1}\left(-\frac{\lambda'}{4N}\right)\exp\left[\frac{\lambda'}{8N}\right], \tag{6.180}$$

where L^1_{N-1} is a Laguerre polynomial. In [223] only the planar limit of this expression

$$\langle W_{\theta_0,\,\mathrm{planar}} \rangle = \frac{2}{\sqrt{\lambda'}} I_1\left(\sqrt{\lambda'}\right) , \tag{6.181}$$

was considered (I_1 is a modified Bessel function). String theory provided exact agreement with the strong coupling expansion of this expression, as we shall review shortly. Furthermore, the same rescaling was observed in the computation of correlation functions between this 1/4 BPS loop and chiral primary operators [234].

In the present calculation we want to capture a different limit, beyond the planar one. We consider a multiply wrapped Wilson loop, or a loop in the k-th symmetric representation,[16] keeping the quantity $\kappa' \equiv k\sqrt{\lambda'}/4N$ fixed while taking both N and λ to infinity. This is the limit that was discussed in [80][81][89][90] for the 1/2 BPS loop, and in this limit the matrix model reduces to

$$\langle W_{\kappa'} \rangle = \exp\left[2N\left(\kappa'\sqrt{1+\kappa'^2} + \operatorname{arcsinh}\kappa'\right)\right] . \tag{6.182}$$

There is also a subleading contribution, that we did not include in the formula above, obtained by replacing $\kappa' \to -\kappa'$. The appearance of this term can be explained by the fact that perturbation theory should be invariant under $\lambda' \to e^{2i\pi}\lambda'$. At strong coupling the expectation value of the Wilson loop depends on $\sqrt{\lambda'}$, so that an extra term with $\kappa' \to -\kappa'$ is needed. In the planar approximation this subleading term reduces to $e^{-\sqrt{\lambda'}}$, which appears in the large λ' expansion of the Bessel function in eq. (6.181).

Later in this section we will be able to construct a D3-brane which is dual to the

[16] It has been proven in [89][90] that in the matrix model the multiply wound loop $W^{(k)}$ and the totally symmetric operator W_{S_k} coincide in the strong coupling regime.

multiply wrapped 1/4 BPS Wilson loop and we will recover eq. (6.182) from supergravity. This computation will also produce the subleading contribution discussed above, which will correspond to an unstable D3-brane solution.

String solution

To write the relevant string solutions in the dual supergravity picture we use the following metric on $AdS_5 \times S^2$ (as in the previous example we drop the directions on S^5 which do not play a role here)

$$\frac{ds^2}{L^2} = -d\chi^2 + \cos^2\chi(d\rho^2 + \sinh^2\rho\, d\psi^2) + \sin^2\chi(d\sigma^2 + \sinh^2\sigma\, d\varphi^2) + d\theta^2 + \sin^2\theta d\phi^2\,. \tag{6.183}$$

This metric has Lorentzian signature, which is somewhat more natural for the supersymmetry analysis, but later we will also use the Euclidean version obtained by Wick rotating $\chi \to iu$ and $\sigma \to i\vartheta$

$$\frac{ds^2}{L^2} = du^2 + \cosh^2 u(d\rho^2 + \sinh^2\rho\, d\psi^2) + \sinh^2 u(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + d\theta^2 + \sin^2\theta d\phi^2\,. \tag{6.184}$$

Note that in the Lorentzian case the χ coordinate foliates AdS_5 by $\mathbb{H}_2 \times \mathbb{H}_2$ surfaces ($\mathbb{H}_2$ is the two-dimensional hyperbolic space, or Euclidean AdS_2), while in the Euclidean case u foliates it into $\mathbb{H}_2 \times S^2$ surfaces.

The string describing the Wilson loop (6.179) will be at $\chi = 0$ (or $u = 0$) and should end at $\rho \to \infty$ along a circle parameterized by ψ. As we go along this circle we should also move along a circle on S^2, the parallel at angle θ_0 spanned by the angle ϕ. We take the ansatz where along the entire world-sheet we equate ψ and ϕ. As mentioned, the asymptotic value of θ should be θ_0. In [223] two solutions with these boundary conditions were found

$$\phi = \psi \,, \qquad \sinh\rho(\sigma) = \frac{1}{\sinh\sigma}\,, \qquad \sin\theta = \frac{1}{\cosh(\sigma_0 \pm \sigma)}\,. \tag{6.185}$$

Here σ is a world-sheet coordinate and σ_0 is related to the boundary value of θ by

$$\sin\theta_0 = \frac{1}{\cosh\sigma_0}\,. \tag{6.186}$$

One can eliminate σ from the previous equations to find the relation

$$\cosh\rho\cos\theta\sin\theta_0 - \sinh\rho\sin\theta\cos\theta_0 = \pm\sin\theta_0. \tag{6.187}$$

The two sign choices correspond to surfaces extending over the north and south pole of S^2 respectively. The classical action for the two cases is equal to

$$S = \mp\cos\theta_0\sqrt{\lambda} = \mp\sqrt{\lambda'}\,. \tag{6.188}$$

The dominant contribution has negative action and corresponds to the surface extended over less than half a sphere. That solution is stable, while the one extending over the other pole has positive action and three unstable modes.

These two solutions were interpreted in [223] as corresponding to the two saddle points in the asymptotic expansion of the Bessel function (6.181)

$$\langle W_{\theta_0,\,\mathrm{planar}}\rangle \underset{\lambda'\to\infty}{\longrightarrow} \frac{\sqrt{2}}{\sqrt{\pi}\lambda'^{3/4}}\left[e^{\sqrt{\lambda'}}\left(1+\mathcal{O}(1/\sqrt{\lambda'})\right) - ie^{-\sqrt{\lambda'}}\left(1+\mathcal{O}(1/\sqrt{\lambda'})\right)\right]\,. \tag{6.189}$$

Furthermore, it was shown there that considering the limit of large λ, while keeping small λ' and integrating over the three modes that are massless for $\lambda' = 0$, yields an identical result to the full planar expression from the matrix model (6.181), including all α' corrections.

The counting of the supersymmetries for the solutions (6.185) goes very similarly to the counting presented in the previous section for the loop with insertions. The dependence of the Killing spinors on the relevant components of the metric (6.183) is[17]

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1} e^{\frac{1}{2}\psi\,\Gamma_{12}} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5} e^{\frac{1}{2}\phi\,\Gamma_{56}} \epsilon_0\,, \tag{6.190}$$

while the constraints analogous to eqs. (6.146) and (6.149) are now

$$(\Gamma_{12} + \Gamma_{56})\epsilon_0 = 0\,, \tag{6.191}$$

and

$$K\epsilon_0 = -\Big(\cos\theta_0\,\Gamma_{13} + \sin\theta_0\Gamma_{16}\Big)\epsilon_0\,. \tag{6.192}$$

The two conditions (6.191) and (6.192) are compatible and therefore the two string solutions (6.185) preserve one quarter of the supersymmetries, as does the operator W_{θ_0} in the dual gauge theory.

D3-brane solution

We now move on to the construction of the 1/4 BPS D3-brane which describes the circular Wilson loop in the k-th symmetric representation $W_{\kappa'}$. The supersymmetry analysis will be presented in Lorentzian signature (6.183) to avoid defining the Killing spinors in Euclidean space. The resulting brane has extra factors of i in the projector

[17] This is presented in greater detail in the following, see eq. (6.194), together with the vielbeins (6.193).

equations and an over-critical electric field. Moreover it does not seem to correspond to a Wilson loop operator in the gauge theory, but to a higher-dimensional observable. Still we find this way of performing the calculation useful. After presenting the solution we will switch to Euclidean signature (6.184), where the solution will not suffer from those problems and will be perfectly well defined.

We parameterize the brane world-volume by $\{\rho,\, \psi,\, \sigma,\, \varphi\}$. The 1/2 BPS brane has constant $\chi = \arcsin\kappa$ and $\theta = 0$. A natural ansatz for the 1/4 BPS brane is then to take $\chi = \chi(\rho)$, $\theta = \theta(\rho)$ and identify ψ with ϕ. This is consistent with the symmetries of the loop. The asymptotic value of θ at $\rho = \infty$ should be θ_0. To carry the k units of flux represented by the Wilson loop operator, we switch on an electric field $F_{\rho\psi}(\rho)$.

Note that the dependence of θ and χ on ρ explicitly breaks the AdS_2 isometry. This fact makes it difficult to guess a simple ansatz for the solution to the equations of motion of the brane. As in the previous example we will then proceed by looking at the first-order supersymmetry equations which follow from requiring κ-symmetry on the brane world-volume.

We begin by constructing the Killing spinors associated to the $AdS_5 \times S^2$ metric (6.183) with Lorentzian signature. The vielbeins relevant for the D3-brane solution are

$$\begin{gathered} e^0 = L\, d\chi\,, \qquad e^1 = L\cos\chi\, d\rho\,, \qquad e^2 = L\cos\chi\sinh\rho\, d\psi\,, \\ e^3 = L\sin\chi\, d\sigma\,, \qquad e^4 = L\sin\chi\sinh\sigma\, d\varphi\,, \\ e^5 = L\, d\theta\,, \qquad e^6 = L\sin\theta\, d\phi\,. \end{gathered} \tag{6.193}$$

Using the same notation of section 6.3.1, the Killing spinors may then be written as (adding the dependence on ϑ and φ to eq. (6.190))

$$\epsilon = e^{-\frac{i}{2}\chi\Gamma_*\Gamma_0} e^{-\frac{i}{2}\rho\Gamma_*\Gamma_1} e^{\frac{1}{2}\psi\Gamma_{12}} e^{-\frac{1}{2}\sigma\Gamma_{03}} e^{\frac{1}{2}\varphi\Gamma_{34}} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5} e^{\frac{1}{2}\phi\,\Gamma_{56}} \epsilon_0 \,. \tag{6.194}$$

The DBI Lagrangian reads (with the sign in the square root appropriate for a brane with Euclidean world-volume and with $F_{\rho\psi}$ containing a factor of $2\pi\alpha'/L^2$)

$$\mathcal{L}_{DBI} = L^4 \sin^2\chi \sinh\sigma \sqrt{(-\chi'^2 + \theta'^2 + \cos^2\chi)(\cos^2\chi \sinh^2\rho + \sin^2\theta) + F_{\rho\psi}^2}\,, \tag{6.195}$$

and the projector associated with the D3-brane is

$$\Gamma = \mathcal{L}_{DBI}^{-1} \left(i\Gamma_{(4)} - L^2 F_{\rho\psi} \Gamma_{(2)} K \right) I\,, \tag{6.196}$$

where, again, K acts by complex conjugation, I by multiplication by $-i$ and

$$\begin{aligned}
\Gamma_{(4)} &= (\gamma_\rho + \chi'\,\gamma_\chi + \theta'\gamma_\theta)(\gamma_\psi + \gamma_\phi)\,\Gamma_{(2)} \\
&= L^2(\cos\chi\Gamma_1 + \chi'\Gamma_0 + \theta'\Gamma_5)(\cos\chi\sinh\rho\Gamma_2 + \sin\theta\Gamma_6)\,\Gamma_{(2)} \\
\Gamma_{(2)} &= \gamma_\sigma\gamma_\varphi = L^2 \sin^2\chi \sinh\sigma\,\Gamma_{34}\,.
\end{aligned} \tag{6.197}$$

Note that the projector Γ does not depend on ψ. As for the string case we can eliminate the dependence on ψ in the projection equation by imposing

$$(\Gamma_{12} + \Gamma_{56})\epsilon_0 = 0\,, \tag{6.198}$$

and then we impose also the condition

$$K\epsilon_0 = -(\cos\theta_0\Gamma_{12} + \sin\theta_0\Gamma_{16})\epsilon_0\,, \tag{6.199}$$

which both follow from the analysis of the supersymmetries of the string (6.192). The brane solution will then preserve the same quarter of supersymmetries as the string and as the gauge theory observable.

Because of the isometry of the system the factor of

$$M \equiv e^{-\frac{1}{2}\sigma\Gamma_{03}} e^{\frac{1}{2}\varphi\Gamma_{34}} \tag{6.200}$$

commutes with those two constraints, so ϵ_0 and $M\epsilon_0$ satisfy the same conditions.

Using some gamma-matrix algebra and applying the constraints above, we move the matrix $e^{-\frac{i}{2}\chi\Gamma_\star\Gamma_0} e^{-\frac{i}{2}\rho\Gamma_\star\Gamma_1} e^{-\frac{i}{2}\theta\Gamma_\star\Gamma_5}$ to the left of Γ in the projection equation. In this way one gets a set of 8 first order differential equations for θ, χ and $F_{\rho\psi}$ (indicating which gamma-matrix combination leads to them)

$$
\begin{aligned}
\Gamma_{0234}: &\qquad 0 = iF_{\rho\psi}\sin\chi\sin\theta\sin\theta_0 + \chi'\sinh\rho(\cos^2\chi - \sin^2\theta) + \cosh\rho\sin\chi\cos\chi\sin^2\theta \\
\Gamma_\star\Gamma_5: &\qquad 0 = iF_{\rho\psi}\sin\chi\sin\theta\cos\theta_0 - \chi'\cosh\rho\sin\theta\cos\theta + \sinh\rho\sin\chi\cos\chi\sin\theta\cos\theta \\
\Gamma_{1234}: &\qquad 0 = iF_{\rho\psi}\sinh\rho\cos\chi\sin\theta\sin\theta_0 + iF_{\rho\psi}\cosh\rho\cos\chi\cos\theta\cos\theta_0 - \\
&\qquad\quad -\chi'\sinh^2\rho\sin\chi\cos\chi - \theta'\sin\theta\cos\theta + \sinh\rho\cosh\rho\cos^2\chi \\
\Gamma_{2345}: &\qquad 0 = iF_{\rho\psi}\sinh\rho\cos\chi\sin\theta\cos\theta_0 - iF_{\rho\psi}\cosh\rho\cos\chi\cos\theta\sin\theta_0 - \\
&\qquad\quad -\theta'\sinh\rho\cosh\rho\cos^2\chi - \cos^2\chi\sin\theta\cos\theta \\
\Gamma_{01}: &\qquad 0 = iF_{\rho\psi}\sinh\rho\cos\chi\cos\theta\cos\theta_0 + iF_{\rho\psi}\cosh\rho\cos\chi\sin\theta\sin\theta_0 - \\
&\qquad\quad -\chi'\sinh\rho\cosh\rho\sin\chi\cos\chi + \cos^2\chi(\sinh^2\rho + \sin^2\theta) \\
\Gamma_{05}: &\qquad 0 = iF_{\rho\psi}\sinh\rho\cos\chi\cos\theta\sin\theta_0 - iF_{\rho\psi}\cosh\rho\cos\chi\sin\theta\cos\theta_0 \\
&\qquad\quad +\theta'(\sinh^2\rho\cos^2\chi + \sin^2\theta) \\
\Gamma_{15}: &\qquad 0 = iF_{\rho\psi}\sin\chi\cos\theta\sin\theta_0 - \chi'\sinh\rho\sin\theta\cos\theta + \\
&\qquad\quad +\theta'\sinh\rho\sin\chi\cos\chi + \cosh\rho\sin\chi\cos\chi\sin\theta\cos\theta \\
1: &\qquad 1 = -iL^4\mathcal{L}_{DBI}^{-1}\sin^2\chi\sinh\sigma\left(iF_{\rho\psi}\sin\chi\cos\theta\cos\theta_0 + \chi'\cosh\rho\sin^2\theta\right. \\
&\qquad\quad \left.-\sinh\rho\sin\chi\cos\chi\sin^2\theta\right)\,.
\end{aligned}
\tag{6.201}
$$

The 1/2 BPS solution

$$
\chi' = 0\,, \qquad F_{\rho\psi} = i\cos\chi\sinh\rho \tag{6.202}
$$

can be recovered by setting $\theta = \theta_0 = 0$ in eq. (6.201).

The equations (6.201) are all consistent with each other. One can solve any three of them, the remaining ones being automatically satisfied. The first three, for example, lead to the equations

$$\theta' = A\,\cos^2\chi\cos\theta\,, \qquad \chi' = A\,\sin\chi\cos\chi\sin\theta\,, \tag{6.203}$$

$$F_{\rho\psi} = -i\frac{\cos\chi\cos\theta}{\cos\theta_0}(A\,\cosh\rho\sin\theta - \sinh\rho)\,, \tag{6.204}$$

where

$$A = \frac{\sinh\rho\cos\theta\sin\theta_0 - \cosh\rho\sin\theta\cos\theta_0}{(\cos^2\chi - \sin^2\theta)\sinh\rho\cos\theta_0 + \cosh\rho\sin\theta\cos\theta\sin\theta_0}\,. \tag{6.205}$$

Taking the ratio of θ' and χ' yields

$$\sin\chi\cos\theta = C\,, \tag{6.206}$$

where C is an integration constant. Inserting this solution into the expression for θ' and solving the resulting differential equation gives

$$\cos\chi\,(\cosh\rho\cos\theta\sin\theta_0 - \sinh\rho\sin\theta\cos\theta_0) = D\,. \tag{6.207}$$

On the AdS_5 side of the ansatz at $\rho = 0$ the circle parameterized by ψ shrinks to a point. For the solution not to be singular at that point, the same has to happen also on the S^5 side, since $\phi = \psi$. The solution will be regular at $\rho = 0$ only if at that point $\sin\theta = 0$, which then gives D in terms of θ_0 and C as

$$D = \pm\sin\theta_0\sqrt{1 - C^2}\,, \tag{6.208}$$

where the $+$, $-$ signs correspond respectively to taking either $\theta = 0$ or $\theta = \pi$ at

$\rho = 0$, or, in other words, to wrapping the brane around the northern or the southern hemisphere of S^2. Notice that in the string limit ($\chi \to 0$, or $C \to 0$) the expression (6.207) reduces to the string solution (6.187).

These solutions in Lorentzian space are unphysical. The world-volume of the brane is Euclidean, but the electric field is over-critical, leading to an imaginary action. Furthermore, the branes do not end along curves on the boundary, but along higher-dimensional surfaces, and do not provide a holographic description of Wilson loops.

Therefore we analytically continue those solutions to Euclidean signature, where the resulting branes will provide a good holographic dual of Wilson loop operators. We take the Wick rotation

$$\chi = i\,u\,, \qquad \sigma = i\,\vartheta\,. \tag{6.209}$$

In these coordinates, the Euclidean AdS_5 is written as an $\mathbb{H}_2 \times S^2$ fibration as in eq. (6.184). The solution (6.206) and (6.207) in Lorentzian signature now becomes

$$\sinh u \cos\theta = c\,, \tag{6.210}$$

and

$$\cosh u\,(\cosh\rho\cos\theta\sin\theta_0 - \sinh\rho\sin\theta\cos\theta_0) = d\,. \tag{6.211}$$

Similarly to the Lorentzian case the solution is smooth at $\rho = 0$ only for

$$d = \pm\sin\theta_0\sqrt{1+c^2}\,. \tag{6.212}$$

The implicit equation (6.211) is solved for ρ as a function θ by

$$\begin{aligned}
\sinh\rho &= \operatorname{sign}(\theta_0-\theta)\frac{\sin\theta\sin\theta_0\left(\sqrt{1+c^2}\cos\theta_0+\cos\theta\sqrt{1+\frac{c^2\cos^2\theta_0}{\cos^2\theta}}\right)}{\cosh u(\cos^2\theta-\cos^2\theta_0)}\\
\cosh\rho &= \operatorname{sign}(\theta_0-\theta)\frac{\sqrt{1+c^2}\cos\theta\sin^2\theta_0+\sin^2\theta\cos\theta_0\sqrt{1+\frac{c^2\cos^2\theta_0}{\cos^2\theta}}}{\cosh u(\cos^2\theta-\cos^2\theta_0)}\,.
\end{aligned} \tag{6.213}$$

The sign function allows us to write in a single expression the two solutions corresponding to a brane wrapping over the north or south poles of the S^2. We will assume, without loss of generality, that $\theta_0 \leq \pi/2$.

Given that the solution may be written explicitly as a function of θ, it makes sense to use it, instead of ρ, as one of the world-volume coordinate. Thus the world-volume is parameterized by $\{\theta,\,\psi,\,\vartheta,\,\varphi\}$ and $\rho=\rho(\theta)$ and $u=u(\theta)$ are given by the solutions above. This parametrization will be singular in the 1/2 BPS limit, where $\theta=0$, but that solution is very simple, with arbitrary ρ and constant $u=\operatorname{arcsinh} c$.

The DBI action in this signature reads

$$S_{DBI}=4N\int d\theta\,\sinh^2 u\,\sqrt{(\cosh^2 u\,\rho'^2+u'^2+1)(\cosh^2 u\sinh^2\rho+\sin^2\theta)+F_{\theta\psi}^2}\,, \tag{6.214}$$

while the Wess-Zumino term can be written as

$$S_{WZ}=4N\int d\theta\,\rho'\sinh\rho\left(\frac{u}{2}-\frac{1}{2}\sinh u\cosh u-\sinh^3 u\cosh u\right)\,. \tag{6.215}$$

To obtain these expressions we have integrated over ψ and S^2. Now the $'$ stands for the derivative with respect to θ. We have checked that the solutions found above satisfy the equations of motion coming from eqs. (6.214) and (6.215).

In figure 6.1 we have plotted ρ and u as functions of θ for a D3-brane solution and for comparison also ρ for the analog string solution (in which case $u = 0$). There are two solution, both reaching infinite ρ at θ_0 (for the example pictured we took the values $\theta_0 = \pi/3$ and $\kappa = 1$). The stable solution then goes to $\rho = 0$ at $\theta = 0$, while the unstable solution goes to $\rho = 0$ at $\theta = \pi$.

Note that in the case of the unstable D3-brane solution the coordinate u diverges at the equator $\theta = \pi/2$, as can also be seen from eq. (6.210). This means that the D3-brane reaches the boundary of AdS at that point, and gets reflected back into the interior (after changing the sign of u). One could choose to truncate the surface there and consider either half of the solution. But in the dual gauge theory that will not correspond to a Wilson loop vacuum expectation value. Rather, the D3-brane extending from θ_0 to $\theta = \pi/2$ will be the correlator between the Wilson loop and a two-dimensional surface operator located where the brane reaches the boundary (the surface spanned by $\{\vartheta, \varphi\}$, the radius of the ψ circle shrinks to a point there). The other part of the solution, from $\theta = \pi/2$ to $\theta = \pi$ is the vacuum expectation value of the surface operator itself, with no Wilson loop insertion.

As usual the BPS equations simplify the square root in the DBI action, which in this case reduces to

$$S_{DBI} = 4N \int d\theta \left| F_{\theta\psi} \frac{\cos\theta_0}{\cos\theta} \right| \sinh^3 u \,. \tag{6.216}$$

This fact can be used to check the conservation of Π, the momentum conjugate to the gauge field A_ψ

$$\Pi = -i\,\frac{2\pi\alpha'}{L^2}\,T_{D3} \int d\vartheta\, d\varphi \frac{\delta\mathcal{L}_{DBI}}{\delta F_{\theta\psi}} = \pm\frac{4N}{\sqrt{\lambda}} \left| \frac{c}{\cos\theta_0} \right| \equiv \pm k, \tag{6.217}$$

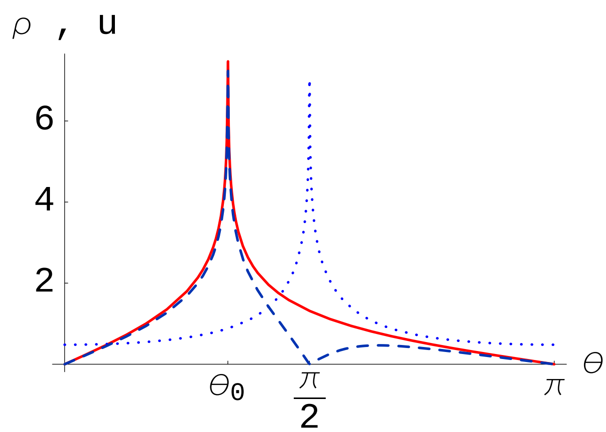


Figure 6.1: A depiction of string and D3-brane solutions. The solid line gives ρ as a function of θ for the string with boundary value of $\theta = \pi/3$. The D3-brane solution is represented by the dashed and dotted lines which are respectively ρ and u as functions of θ (for $\kappa = 1$). In both cases there are two solutions, a stable one with $0 \leq \theta \leq \theta_0$ (where ρ for the string and D3-brane are nearly indistinguishable) and an unstable one, with $\theta \leq \theta_0 \leq \pi$. The unstable D3-brane solution reaches the boundary of AdS not only at θ_0, but also at $\theta = \pi/2$, where u diverges, but then it turns back and closes smoothly on itself.

where the two signs correspond to the two solutions. This implies that

$$|c| = \kappa |\cos\theta_0| . \tag{6.218}$$

To compute the full on-shell action we have to supplement the DBI and WZ bulk contributions with total derivatives and boundary terms associated to the electric field $F_{\theta\psi}$ and the scalar $\rho(\theta)$. These are given respectively by[18]

$$S_{L.T.} = -i\,\frac{L^2}{2\pi\alpha'}\int d\theta\, d\psi\, \Pi\, F_{\theta\psi}\,, \tag{6.219}$$

and

$$-P_\rho\Big|_{\theta\to\theta_0} = -T_{D3}\int d\psi\, d\vartheta\, d\varphi\,\frac{\delta(\mathcal{L}_{DBI}+\mathcal{L}_{WZ})}{\delta\rho'}\Big|_{\theta\to\theta_0} . \tag{6.220}$$

where P_ρ is the momentum conjugate to ρ.

The boundary term for ρ can be motivated as follows. Let us consider the AdS_5 metric in the Poincaré patch

$$ds^2 = \frac{1}{z^2}(dz^2 + dr_1^2 + r_1^2 d\varphi_1^2 + dr_2^2 + r_2^2 d\varphi_2^2)\,. \tag{6.221}$$

The transformation relating z to our coordinates is

$$z = \frac{1}{\cosh u \cosh\rho - \cos\vartheta \sinh u}\,, \tag{6.222}$$

so that the $\rho \to \infty$ region corresponds to $z = 0$. In the Poincaré patch the boundary term associated to z has the form of a Legendre transform evaluated at the boundary

[18]For a general discussion on the role of boundary terms see [73][80].

of AdS_5 [73][80]

$$-\int z\, p_z\Big|_{z\sim 0}\,. \tag{6.223}$$

Using eq. (6.222) it is immediate to verify that in proximity of the boundary $z\, p_z \sim p_\rho$. This justifies the form of the boundary term for ρ.

Now we can evaluate the on-shell action. The bulk and boundary contributions diverge as we approach the boundary of AdS_5, *i.e.* in the limit $\theta \to \theta_0$. We can regularize these divergences by introducing a cut-off at $\theta_0 - \epsilon$. This leads to the following expression for the regularized DBI action

$$\begin{aligned}
S_{DBI} \;&=\; T_{D3}\int_0^{\theta_0-\epsilon} d\theta \int d\psi\, d\vartheta\, d\varphi\, \mathcal{L}_{DBI} = 4N\kappa^3 \sin\theta_0 \frac{\sqrt{1+c^2}}{\epsilon} \\
&+2N\kappa^3\,\frac{\sin\theta_0\tan\theta_0}{\sqrt{1+c^2}} - N\kappa \sec^2\theta_0(2+8c^2-4c^2\cos^2\theta_0)\sqrt{1+c^2\cos^2\theta_0} \\
&+2N\kappa\sec^3\theta_0\sqrt{1+c^2}\;\left(1+3c^2-6c^2\cos^2\theta_0+2c^2\cos^4\theta_0\right) \\
&-2N\sec^4\theta_0\left(1-4c^2\sin^2\theta_0-8c^4\sin^2\theta_0\right)\log\left(\frac{c+\sqrt{1+c^2}}{c\,\cos\theta_0+\sqrt{1+c^2\cos^2\theta_0}}\right) \\
&+8N\kappa^3\sin\theta_0\tan\theta_0\sqrt{1+c^2}\Big(2\log\epsilon - 2\log(\cos\theta_0\sin\theta_0) - \log\left(1+c^2\right) \\
&+\log\Big(\cos^2\theta_0\left(1+2c^2\right)+2\cos\theta_0\sqrt{1+c^2}\sqrt{1+c^2\cos^2\theta_0}+1\Big)\Big)\,.
\end{aligned} \tag{6.224}$$

The Legendre transform of the gauge field is written as the integral over the total derivative

$$\begin{aligned}
S_{L.T.} \;&=\; -i\,\frac{L^2}{2\pi\alpha'}\int_0^{\theta_0-\epsilon} d\theta\int d\psi\,\Pi\, F_{\theta\psi} \\
&=\; 4N\kappa\sin\theta_0\frac{\sqrt{1+c^2}}{\epsilon} - 2Nc\frac{3+\kappa^2+2c^2}{\sqrt{1+c^2}}\,,
\end{aligned} \tag{6.225}$$

and the boundary term for ρ is

$$-P_\rho\Big|_{\theta_0-\epsilon} = -2N\sin\theta_0(\kappa\sqrt{1+\kappa^2}+\operatorname{arcsinh}\kappa)\,\frac{\sqrt{1+c^2}}{\epsilon\,\sqrt{1+\kappa^2}} \\ -4N\sec\theta_0(\kappa\sqrt{1+\kappa^2}+\operatorname{arcsinh}\kappa)\,\frac{2\kappa^2-(1+4\kappa^2-\kappa^4)\cos^2\theta_0-2c^4}{4(1+\kappa^2)^{3/2}\sqrt{1+c^2}}\,. \tag{6.226}$$

Finally the regularized Wess-Zumino term turns out to be

$$S_{WZ} = -2N\left(c\sqrt{1+c^2}+\operatorname{arcsinh}c\right) - S_{DBI} - S_{L.T.} + P_\rho\,. \tag{6.227}$$

Those expressions are much more complicated than the 1/2 BPS case, where those three terms are

$$\begin{aligned} S_{DBI} &= 4N\int d\rho\sinh\rho\cosh u\sinh^3 u\,, \\ S_{WZ} &= 4N\int d\rho\sinh\rho\left(\frac{u}{2}-\frac{1}{2}\sinh u\cosh u-\sinh^3 u\cosh u\right)\,, \\ S_{L.T.} &= 4N\int d\rho\sinh\rho\cosh u\,. \end{aligned} \tag{6.228}$$

The boundary term for ρ just removes the divergence from the upper limit of ρ integration, giving -1 from the lower limit. Summing up all the contributions and using $\sinh u = \kappa$ gives the full on-shell action [80]

$$S = -2N\left(\kappa\sqrt{1+\kappa^2}+\operatorname{arcsinh}\kappa\right)\,. \tag{6.229}$$

While the regularized expressions for the 1/4 BPS loop are much more complicated, the sum of the bulk and boundary terms is exactly the same with the replacement of κ by c

$$S_{\text{total}} = -2N\left(c\sqrt{1+c^2} + \operatorname{arcsinh} c\right). \tag{6.230}$$

Recall that c is related to the number of units of flux carried by the brane, or the dimension of the representation of the Wilson loop by eq. (6.218)

$$c = \kappa' = \frac{k\cos\theta_0\sqrt{\lambda}}{4N}. \tag{6.231}$$

Therefore this stable solution will contribute to the expectation value of the 1/4 BPS Wilson loop at strong coupling

$$\langle W_{\kappa'}\rangle = \exp\left[2N\left(\kappa'\sqrt{1+\kappa'^2} + \operatorname{arcsinh}\kappa'\right)\right]. \tag{6.232}$$

This is the same result as can be derived from the matrix model observable (either the multiply wrapped loop [80] or the symmetric one [89][90]) in this limit. This serves as a confirmation that the matrix model correctly captures the 1/4 BPS loop including all $1/N$ corrections at large N and large λ.

An analogous computation can be done for the unstable branch, where the range of integration for the coordinate θ is $[\theta_0+\epsilon, \pi]$ and P_ρ is evaluated at $\theta_0+\epsilon$. The final result is exactly as above, except for the overall sign

$$S^{(\text{unstable})}_{\text{total}} = 2N\left(\kappa'\sqrt{1+\kappa'^2} + \operatorname{arcsinh}\kappa'\right). \tag{6.233}$$

As in the case of the string solution reviewed before, this should correspond to an exponentially small correction to the expectation value of the Wilson loop when doing the asymptotic expansion at large λ.

In addition to the 1/2 BPS limit, with $\theta_0 = 0$, there is another interesting limiting case, of $\theta_0 = \pi/2$ studied by Zarembo [68]. In that case the two D3-brane solutions, whose actions always have the opposite signs, are degenerate. Both have vanishing action, and in fact there are more than two solutions, rather a whole family parameterized by an S^3. But, unfortunately, looking at the solutions at this limit we find that they do not provide a good description for the Wilson loop. If we consider finite κ, then from eq. (6.218), the constant c vanishes and by eq. (6.210), also $u = 0$. Therefore the D3-brane shrinks to a two-dimensional surface and therefore the higher-derivative corrections to the DBI action cannot be ignored.

If instead we keep c finite in that limit, then κ will diverge, leading to a smooth D3-brane solution. But now as θ goes to θ_0 both ρ and u diverge, meaning that the brane ends along a 3-dimensional surface on the boundary, rather than the Wilson loop.

6.3.3 Discussion

We have presented some solutions for D3-branes in $AdS_5 \times S^5$, which are dual to certain 1/4 BPS Wilson loop operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The first example was a combined system of a loop with two local insertions made from complex scalar fields. Without the insertions the loop itself would have been 1/2 BPS and the trace of the local insertions is also 1/2 BPS, while the combined system preserves 1/4 of the supersymmetries. The second system was a family of Wilson loops with couplings to three of the scalars in a way that also preserves eight supercharges.

It is by now a standard feature of the AdS/CFT correspondence that very long operators in the gauge theory map to "giant" D-brane objects rather than to fundamental strings or supergravity modes. In our case the D3-branes should describe the Wilson loops in a high-dimensional symmetric representation, where the rank of the representation k is of order N. In the example of the loop with insertions we were able to calculate the energy and angular momentum and they agreed with each other, as would be expected, but there was no special feature arising from the fact that the loop is in a certain representation.

In the second example we were able to compare the result of the AdS calculation to a matrix model conjectured to describe those 1/4 BPS loops [29][30][223]. The value of those loops at large N and large λ in a symmetric representation is known (and coincides with the single-trace multiply-wrapped loop). We found that the classical action for the D3-brane correctly reproduces the expected result, which includes an infinite series of $1/N$ corrections to the planar string expression. Furthermore, we have found two solutions with the same boundary conditions, in exact analogy with the strings describing the loop in the fundamental representation. The second solution, which contributes an exponentially small correction to the Wilson loop in the supergravity limit is the brane analog of a world-sheet instanton. Such contributions are expected, since the string expansion is asymptotic in $1/\sqrt{\lambda}$.

The geometry of this second D3-brane solution is very interesting. Starting from the boundary of AdS, where it originates along the Wilson loop, it moves into the bulk, turns back, goes again to the boundary, gets reflected back into the interior, and closes off smoothly on itself. If we chose not to continue the solution, it would end on a two-dimensional surface on the boundary. So this part of the solution would describe the correlator of a Wilson loop and a surface operator which are non-trivially linked. It would be very interesting to understand further the nature of this surface operator. The connection between Wilson loops and surface operators may not be so surprising given that they may both be described by branes in the bulk (see *e.g.* [235][88][236]).

As discussed at the end of the last section, one would like also to consider a special limit of these loops, when $\theta_0 = \pi/2$. This limit is particularly interesting because of a comment made at the end of [223], where it was noticed that one may take λ large while keeping $\lambda' = \lambda \cos\theta_0$ small, in a way similar to the BMN limit [46]. When considering the string solution in that limit, the mass of the string modes becomes much larger than the mass of the three broken zero modes (those parameterizing the S^3 mentioned above). Ignoring all the stringy modes and integrating only over those three leads to the full result of the planar matrix model, including all α' (or $1/\sqrt{\lambda'}$) corrections. It would be extremely interesting if we were able to repeat the calculation here and find the exact expression including all $1/N$ and $1/\sqrt{\lambda'}$ corrections. Recall that those corrections would not be the same for the loop in the symmetric representation and for the multiply-wound loop. So this calculation would be a very good check of the recent identification of the D3-brane with the loop in the symmetric representation [88]. Unfortunately, as explained before, in this limit the D3-brane degenerates and does not provide a good description of those Wilson loops.

The loops studied in this section are not the most general 1/4 BPS Wilson loops, all our examples had a circular geometry, which is not required. Many other loops were described in [68], and there are probably even more. The string solutions describing those loops were studied by Dymarsky et al. [222], and perhaps there is a general classification of the relevant branes along the lines of [237].

After studying the probe brane in the $AdS_5 \times S^5$ background it is natural to consider the back-reaction of the brane on the geometry. This was pursued in the 1/2

BPS case in [94] and [95], where all the relevant metrics could be related to Young-tableaux, thus giving a correspondence between the representations of the Wilson loop and the associated metrics. It would be interesting to try to find the metrics in this case too, though this system has far less symmetry making it a much harder problem.

Finally, one can go further to a system which is only 1/8 BPS, by looking at the correlators of those Wilson loops with chiral primary local operators. Amazingly, in the gauge theory those also seem to be captured fully by ladder diagrams and may be reduced to some matrix model [234]. In the case of the 1/2 BPS loop this was checked in AdS using a string [227][31] and D-branes [4], as reported is section 6.2. For the 1/4 BPS loop this was done with a string in [234] and would be interesting to repeat this calculation with D3-branes.

Part III

Studies on twistor string theory

Chapter 7

Fermionic orbifold of the twistor string

7.1 Introduction

In this chapter of the dissertation we address the problem of constructing twistor theories with reduced supersymmetry. In fact, surprising and elegant as the duality presented in chapter 3 undoubtedly is, it has two obvious shortcomings. The first one is that in its original formulation it applies only to maximally supersymmetric gauge theories.[1] The second one is that superconformal invariance is automatically built-in by virtue of the twistor formalism. These features seems to make the original construction unfit for describing more realistic gauge theories.

Here we consider an extension of Witten's correspondence to a class of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetric gauge theories. As we do not know how to relax the

[1] Examples of twistor-inspired computations of amplitudes in theories with less supersymmetry have nonetheless appeared in the literature. See for example [238].

requirement of superconformal invariance, natural candidates are the superconformal quiver theories analyzed in [239][240], which we will recover from the twistor string.[2]

The procedure followed in [239] and [240] was to start with a parent $\mathcal{N} = 4$ super Yang-Mills theory and retrieve the superconformal daughter theories with reduced supersymmetry by orbifolding the $SU(4)_R$ symmetry rotating the supercharges.[3] These are quiver theories with bifundamental matter fields. In the present case of twistor strings, the $SU(4)_R$ symmetry is part of the isometry group of $\mathbb{CP}^{3|4}$. Thus, before Penrose transforming, this operation has a natural interpretation as a fermionic orbifold of the twistor string's target space.

Although it is not clear a priori what the meaning of a fermionic orbifold is, one is immediately tempted to establish a connection with the standard lore about D-branes tranverse to bosonic orbifold singularities [242], and their realization via geometric engineering [243]. In the case of $\mathcal{N} = 2$ superconformal theories engineered from type IIB superstrings, the moduli space of superconformal couplings is known to admit a duality group whose action is inherited from the S-duality of IIB superstrings.[4] It would be interesting to identify these moduli spaces in the twistor string theory.

We start by briefly recalling some basic facts about D-branes on (bosonic) orbifolds, and then we present our proposal for an orbifold of the fermionic directions of $\mathbb{CP}^{3|4}$. We discuss the role played by both D5 and D1-branes, and present, in the two final sections of this chapter, two classes of quiver theories obtained from this procedure.

[2]A related work pursuing this same direction can be found in [241].

[3]Strictly speaking, this is not really an orbifold in the conventional sense of the word, as one is not gauging the discrete symmetry in spacetime.

[4]Alternatively, it can be seen as the affine Weyl group acting on the primitive roots of affine ADE groups [244][240].

7.2 Branes transverse to orbifold singularities

Placing a set of D-branes transverse to a $\mathbb{C}^n/\Gamma$ orbifold (where $n = 2, 3$, and Γ is a discrete subgroup of $SU(n)$) gives rise to four dimensional gauge theories with $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry living in the brane worldvolume. To obtain their massless spectrum one has to consider the appropriate orbifold action on fields in both the open and closed string sectors. The open string sector contributes the field content of the gauge theories, which can be encoded in quiver diagrams. The closed string sector in turn contributes the necessary moduli which deform the transverse orbifold singularity to a smooth space. We briefly review this construction below, focusing in the case of abelian Γ for simplicity.

7.2.1 Open string sector

In the open string sector, Γ acts on the orbifolded transverse coordinates and on the Chan-Paton factors of open strings in the worldvolume directions. To get conformal theories one needs Γ to act on the Chan-Paton factors in (arbitrary number of copies of) the regular representation $\mathcal{R}$ of Γ. Choosing N copies of $\mathcal{R}$ amounts to considering $N|\Gamma|$ D3-branes before the projection, where $|\Gamma|$ is the order of Γ. At this stage one then has $U(N|\Gamma|)$ gauge symmetry in the worldvolume.

In the worldvolume directions Γ acts on the open string Chan-Paton factors matrices λ only. This action is specified by a matrix $\gamma_\Gamma \in \mathcal{R}$, and the invariant states satisfy

$$\gamma_\Gamma \lambda \gamma_\Gamma^{-1} = \lambda \,. \tag{7.1}$$

Now using elementary group theory the regular representation can be decomposed in irreducible representations as

$$\mathcal{R} = \oplus_{\mathbf{i}} n_{\mathbf{i}} \mathcal{R}_{\mathbf{i}} \,, \tag{7.2}$$

where $n_{\mathbf{i}} = \dim(\mathcal{R}_{\mathbf{i}}) = 1$ for abelian Γ. Acting on λ as in eq. (7.1) this decomposition projects out the fields whose Chan-Paton indices are not connected by one of the irreducible $\mathcal{R}_{\mathbf{i}}$'s. Taking N copies of $\mathcal{R}$ the gauge group will therefore be broken[5] to $F = \prod_{\mathbf{i} \in irreps} U(N)$. Each of these unitary groups with its gauge multiplet has an associated node in the quiver diagram.

In the orbifolded directions Γ acts on the Chan-Paton matrices through an element of the regular representation γ_Γ, and on space indices through the defining n dimensional representation $G_\Gamma^{n\times n}$, in such a way that

$$G_\Gamma^{n\times n}(\Psi(\mathbf{i})) = \Psi(\gamma_\Gamma(\mathbf{i})) \,, \tag{7.3}$$

$\Psi(\mathbf{i})$ being a label for the $\mathbf{i}$-th D3-brane in the orbifolded transverse space. This means that the action of the space group is correlated with the action on the Chan-Paton factors. The fields surviving the projection (7.3) can be obtained from the decomposition

$$\begin{aligned} \mathrm{Hom}(\mathcal{R}, G_\Gamma^{n\times n} \otimes \mathcal{R}) &= \bigoplus_{\mathbf{i},\mathbf{j}} \left[\mathrm{Hom}(\mathcal{R}_{\mathbf{i}}, G_\Gamma^{n\times n} \otimes \mathcal{R}_{\mathbf{j}}) \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)\right] \\ &= \bigoplus_{\mathbf{i},\mathbf{j}} a_{\mathbf{ij}} \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N) \,, \end{aligned} \tag{7.4}$$

where again $\mathbf{i}$ runs over irreducible representations, and $a_{\mathbf{ij}}$ are the Clebsch-Gordan coefficients in the decomposition of the tensor product. Physically these are $a_{\mathbf{ij}}$ chiral

[5]Actually there is a $U(1)$ subgroup acting trivially on the fields. It can be seen as the motion of the center of mass coordinate of the D-branes. Therefore the effective gauge group is $G = F/U(1)$.

multiplets transforming in bifundamental representations as

$$\oplus a_{\mathbf{ij}}(\mathbf{N}, \bar{\mathbf{N}})\,. \tag{7.5}$$

The quiver diagram has $a_{\mathbf{ij}}$ oriented links between nodes $\mathbf{i}$ and $\mathbf{j}$. For $\mathcal{N} = 2$ quivers $a_{\mathbf{ij}} = a_{\mathbf{ji}}$, which makes the links non-oriented. Each of them represents an $\mathcal{N} = 2$ hypermultiplet.

Finally, for $\mathbb{C}^2/\Gamma$ orbifolds one has two non-orbifolded transverse directions. Γ acts on these fields as in eq. (7.1). They provide the adjoint chiral superfields which together with the gauge multiplets complete the $\mathcal{N} = 2$ vector multiplet.

7.2.2 Closed string sector

In the closed string sector there are no Chan-Paton factors, and one can follow the ordinary orbifold techniques to find the spectrum. There are, in addition to the usual untwisted sector, $|\Gamma| - 1$ twisted sectors which play a crucial role in resolving the singularity. The untwisted sector is just the Kaluza-Klein reduction of the ten dimensional supergravity multiplet on $\mathbb{C}^2/\Gamma$ (for $\mathcal{N} = 2$) or on $\mathbb{C}^3/\Gamma$ (for $\mathcal{N} = 1$), together with the usual matter multiplets. In the large volume limit the moduli fields from the $|\Gamma| - 1$ twisted sectors can be seen to arise by wrapping the various form fields on the exceptional cycles of the blown-up singularity. For $\mathcal{N} = 2$ the blow-up is hyper-Kähler, whereas for $\mathcal{N} = 1$ it is only Kähler. In the first case there are moduli $b_{\mathbf{i}} = \int_{S^2_{\mathbf{i}}} B$ and $\vec{\zeta}_{\mathbf{i}} = \int_{S^2_{\mathbf{i}}} \vec{\omega}$, where for IIA B is the NS-NS B-field,[6] and $\vec{\omega}$ is the triplet of Kähler forms on the blow up. In the Kähler case $b_{\mathbf{i}} = \int_{S^2_{\mathbf{i}}} B$ and $\zeta_{\mathbf{i}} = \int_{S^2_{\mathbf{i}}} \omega$, where ω is the Kähler form. In the first case the combination $b_{\mathbf{i}} + i\vec{\zeta}_{\mathbf{i}}$ encodes (in the large

[6] For IIB one can in addition have the two-form from the R-R sector.

volume limit) the deformations of Kähler and complex structure of the resolution. Because of $\mathcal{N}=2$ supersymmetry the resolution is hyper-Kähler, and these two are related by the $SO(3)$ symmetry that rotates $\vec{\omega}$. In the second case $b_{\mathbf{i}}+i\zeta_{\mathbf{i}}$ parameterize the deformations of the complexified Kähler structure.[7]

Of course one has in addition to these moduli scalars various p-form fields from the twisted sectors. The twisted fields couple to open fields in the brane low effective action via Chern-Simons couplings. In the presence of the orbifold, closed fields from the k-th twisted sector C_k couple naturally to the $U(1)$ part of the field strength of the D-brane whose Chan-Paton factor is twisted by γ_Γ. Their supersymmetric completion involves terms which couple as Fayet-Iliopoulos parameters in the effective gauge theory on the brane world-volume. When these are non-zero the gauge symmetry is completely broken, and the Higgs branch of the world-volume theory is the resolved transverse space.

Taking into account all the twisted moduli one can write the full stringy quantum volume of the exceptional cycles of the geometry as $V_{\mathbf{i}}=\left(b_{\mathbf{i}}^2+|\vec{\zeta}_{\mathbf{i}}|^2\right)^{1/2}$ in the $\mathcal{N}=2$ case and $V_{\mathbf{i}}=(b_{\mathbf{i}}^2+\zeta_{\mathbf{i}}^2)^{1/2}$ in the $\mathcal{N}=1$ case. At the orbifold point $\zeta_{\mathbf{i}}=0$, and one can write the coupling constant of the $\mathbf{i}$-th gauge group as $1/(g_{YM\,\mathbf{i}})^2=V_{\mathbf{i}}/g_s$, where g_s is the string coupling constant. In type IIB one has also $c_{\mathbf{i}}=\int_{S^2_{\mathbf{i}}}B_R$, which plays the role of a theta angle in the gauge theory. One can then write the complexified couplings as $\tau_{\mathbf{i}}=\theta_{\mathbf{i}}+i/(g_{YM\,\mathbf{i}})^2=c_{\mathbf{i}}+b_{\mathbf{i}}\tau$, with $\tau=g_s^{-1}$. The S-duality of type IIB superstrings, which acts on B_{NS} and B_R manifests itself as a duality in the moduli space of couplings.

[7]In this case, in addition to B there are further moduli from the R-R sector: a hypermultiplet for type IIB or a vector multiplet for type IIA.

7.3 The orbifold of the twistor string

7.3.1 The orbifold in the D5-brane sector

The procedure reviewed in the previous section can be applied to the twistor string of [96] in order to reduce the $\mathcal{N} = 4$ supersymmetry. The homogeneous coordinates of $\mathbb{CP}^{3|4}$ provide a linear realization of $PSU(4|4)$, which is the $\mathcal{N} = 4$ superconformal group. It is therefore natural to use twistors to study conformal theories. To reduce supersymmetry, we can orbifold the fermionic directions of the super-twistor space. Physically, this amounts to orbifolding the $SU(4)_R$ R-symmetry, which is the Fermi-Fermi subgroup of $PSU(4|4)$. As reviewed in chapter 3, in the twistor theory of [96] a set of D5-branes is placed at $\bar{\psi}_A = 0$. In analogy with the conventional case, one possible interpretation is to view the orbifold as acting in the $\bar{\psi}_A$ directions, which are transverse to the D5-branes. This induces an action on the ψ^A which will be the one considered in the following.

Explicitly, we choose an action of the orbifold $\Gamma \in \mathbb{Z}_k$ under which the fermionic coordinates transform as

$$\psi^A \to e^{2\pi i a_A/k} \psi^A \,, \tag{7.6}$$

with the condition on the charges $\sum_A a_A = 0 \pmod{k}$, so that $\Gamma \in SU(4)_R$. The holomorphic volume form Ω on $\mathbb{CP}^{3|4}$ (see eq. (3.45)) is invariant under (7.6). This implies that the super-orbifold is still Calabi-Yau. This is crucial for the consistency of the B-model [117].

We consider a stack of kN D5-branes in the covering space. The orbifold action regroups the branes in k stacks of N branes each, as shown in figure 7.1. It is thus

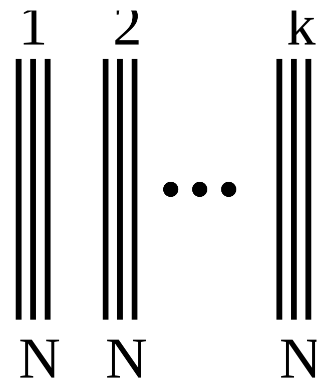


Figure 7.1: Regrouping of the D5-branes under the orbifold action: the kN branes are split into k stacks of N branes each. Note that the k stacks of branes are actually coincident at the point $\bar{\psi}_A = 0$ of the twistor space.

convenient to decompose the $U(kN)$ adjoint index into $A^a{}_b = A^{I\mathbf{i}}{}_{J\mathbf{j}}$, where $I, J = 1, \ldots, N$ label the brane within a stack and $\mathbf{i}, \mathbf{j} = 1, \ldots, k$ label the stacks.

An explicit representation of Γ is

$$\mathcal{R} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}_{kN \times kN} , \tag{7.7}$$

where $r_{\mathbf{i}} = e^{2\pi i \mathbf{i}/k}$ is a $N \times N$ matrix acting on the $\mathbf{i}$-th node of the associated quiver. The orbifold projection is enforced by requiring invariance of the components of the superfield $\mathcal{A}$ under the action of Γ. R-symmetry invariance of the superfield implies that (7.6) induces a conjugate transformation on the fermionic indices of the components. Combined with the action on the Chan-Paton factors given by (7.7), this gives the following orbifold action on a generic component (with $n = 0, \ldots, 4$

fermionic indices)[8]

$$(\Phi_{A_1,\ldots,A_n})^{I\mathbf{i}}{}_{J\mathbf{j}} \to e^{2\pi i(\mathbf{i}-\mathbf{j}-a_{A_1}-\ldots-a_{A_n})/k}(\Phi_{A_1,\ldots,A_n})^{I\mathbf{i}}{}_{J\mathbf{j}}\,. \tag{7.8}$$

For example, the lowest component of the superfield (3.47), which physically represents the positive helicity gluon, transforms as

$$A^{I\mathbf{i}}{}_{J\mathbf{j}} \to e^{2\pi i(\mathbf{i}-\mathbf{j})/k}A^{I\mathbf{i}}{}_{J\mathbf{j}}\,. \tag{7.9}$$

Invariance requires $\mathbf{i}=\mathbf{j}$, so that the gauge group is broken to $U(kN) \to [U(N)]^k$. Similarly, the positive helicity gluino χ_A transforms as

$$(\chi_A)^{I\mathbf{i}}{}_{J\mathbf{j}} \to e^{2\pi i(\mathbf{i}-\mathbf{j}-a_A)/k}(\chi_A)^{I\mathbf{i}}{}_{J\mathbf{j}}\,, \tag{7.10}$$

so that in this case one needs to enforce $\mathbf{i}=\mathbf{j}+a_A$. Depending on the value of the charge a_A, the field χ_A becomes either a gaugino or a bi-fundamental quark. One proceeds analogously with the other components of $\mathcal{A}$. One can picture the field content in a quiver diagram with k nodes corresponding to the k gauge groups and bi-fundamental matter as lines connecting pairs of nodes.

The choice of the discrete group one quotients by determines the amount of supersymmetry preserved by the orbifold. For generic Γ the supersymmetry is completely broken, while for $\Gamma \in \mathbb{Z}_k \subset SU(2)_R$ and $\Gamma \in \mathbb{Z}_k \subset SU(3)_R$ one has respectively $\mathcal{N}=2$ and $\mathcal{N}=1$ [239].

So far we have only focused on the D5-brane sector. In the following section we

[8]In this notation, for instance, $\Phi_{A_1A_2A_3A_4} = \frac{1}{4!}\epsilon_{A_1A_2A_3A_4}G$, where G is the highest component of $\mathcal{A}$.

tackle the problem of orbifolding the D1-instantons.

7.3.2 D1-branes

As already explained in chapter 3, the holomorphic Chern-Simons action on $\mathbb{CP}^{3|4}$ only reproduces the selfdual truncation of $\mathcal{N}=4$ super Yang-Mills. A non-perturbative correction to the B-model is needed in order to recover the non-selfdual part of the gauge theory. These new non-perturbative degrees of freedom are D1-branes wrapped on holomorphic cycles inside the supermanifold. These branes are D-instantons whose instanton number is given by the degree d of the map.

We proceed now to the analysis of the orbifold action on the D1-instanton sector. For this action to be faithful on the Chan-Paton factors of the D1's, we need to start with k D1-branes. To begin with, we locate them at $\psi^A=0$. As in [96], the fermionic dependence will be restored in the end through integration over the moduli space. Considering k D1-branes, the effective action (3.52) gets changed into

$$I_{D1-D5}=\int_{D1} dz\,(\beta^r_{I\mathbf{i}}\bar{\partial}\alpha^{I\mathbf{i}}_r+\beta^r_{I\mathbf{i}}\mathcal{B}_r{}^s\alpha^{I\mathbf{i}}_s+\beta^r_{I\mathbf{i}}\mathcal{A}^{I\mathbf{i}}{}_{J\mathbf{j}}\alpha^{J\mathbf{j}}_r)\,, \tag{7.11}$$

where $r=1,\ldots,k$ is a $U(k)$ index which labels the D1's. For instance, $\alpha^{I\mathbf{i}}_r$ is a string stretching from the r-th D1-brane to the I-th D5-brane inside the $\mathbf{i}$-th stack. In eq. (7.11) $\mathcal{B}_r{}^s$ is the $U(k)$ gauge field on the world-volume of the D1-branes. The action of the orbifold breaks $U(k)\to[U(1)]^k$. The D1-D5 strings α and β transform as

$$\begin{aligned}\alpha^{I\mathbf{i}}_r&\to e^{2\pi i(\mathbf{i}-r)/k}\alpha^{I\mathbf{i}}_r\,,\\ \beta^r_{I\mathbf{i}}&\to e^{2\pi i(r-\mathbf{i})/k}\beta^r_{I\mathbf{i}}\,.\end{aligned} \tag{7.12}$$

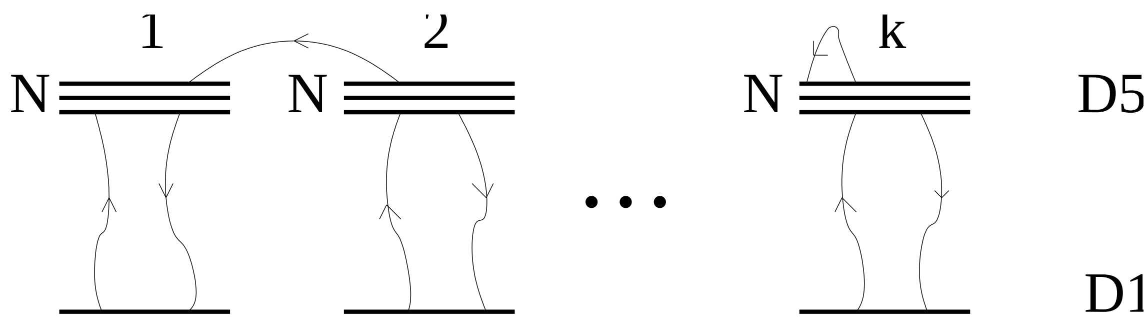


Figure 7.2: The stacks of D5 and D1-branes. D1-D5 strings are stretched between the two different kinds of branes. An interaction between the first and the second stack is also depicted.

Invariance under the orbifold action requires $\mathbf{i} = r$. This implies that the D1-D5 strings only stretch between the $\mathbf{i}$-th D1-brane and the D5-branes in the $\mathbf{i}$-th stack. This is shown in figure 7.2 and, in quiver language, for the specific example of $k = 3$, in figure 7.3.

The $U(1)$ fields living on the D1-branes and the bi-fundamental matter connecting them will not be considered in the following, although they are depicted in figure 7.3. The $U(1)$ bundles over curves of genus $g < 2$ do not have moduli as remarked in [96] and do not play a role in the computation of amplitudes. Further, it seems natural to neglect the bi-fundamental fields since in general, when the branes move away from $\psi^A = 0$, they should correspond to massive states.

The stack of k D1-branes can move away from the orbifold fixed point as one full regular brane. In the covering space, the k D1-branes are located in points related by the Γ action in the orbifold directions, whereas they coincide in the others. In particular, they coincide along the bosonic subspace and therefore the bosonic world-volume is the same for all of them. Since the branes cannot move independently we have only one set of moduli (x, θ) for the whole system. However this is not the

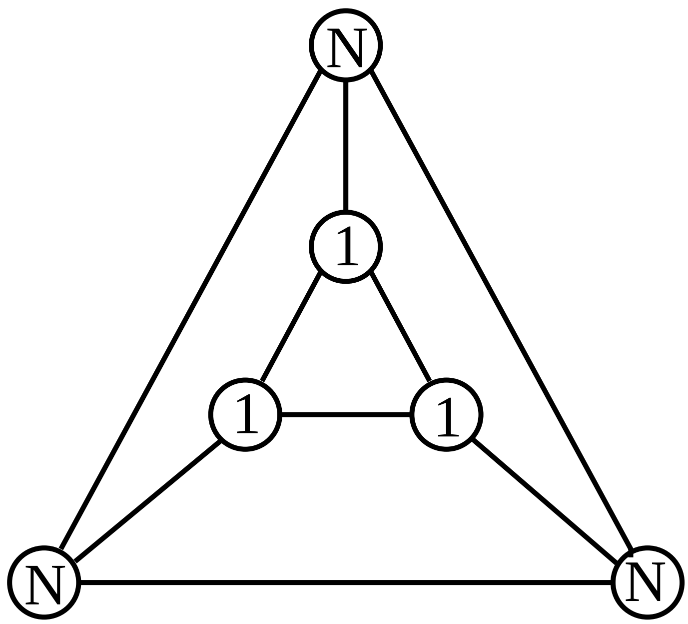


Figure 7.3: The quiver for the D1-D5 brane system in the case $k = 3$. It has three nodes corresponding to the D5-branes indicated with N, and three nodes corresponding to the D1's.

complete story. We do not fully explore the richness of the orbifold construction. If the branes were coincident at the fixed point of the fermionic coordinates, then there would be no constraint on their motion along the remaining directions and one would have an extended moduli space $\{(x_r, \theta_r)\}$ with k sets of parameters. We will not study this explicitly but we will limit ourselves to some comments. Having k independent D1-branes allows one to consider k independent minimal couplings (3.57) to the b field. This might be worth studying because it could provide a mechanism to generate k independent coupling constants, one at each node. Since in the usual case the moduli space of couplings is related to twisted sectors, it will be valuable to clarify this issue further by studying the closed sector of the B-model and check whether it contains twisted states.

We now illustrate what discussed so far in two explicit examples, and compute some amplitudes in these orbifold theories.

7.4 Explicit examples of orbifolds

7.4.1 An $\mathcal{N} = 1$ orbifold

We start by considering the case $\Gamma \in \mathbb{Z}_k \subset SU(3)_R$. Then $SU(4)_R$ is broken into $U(1)_R$, yielding $\mathcal{N} = 1$. We consider a particularly simple example, in which $k = 3$ and $a_A = (1, 1, 1, 0)$, see eq. (7.6). The gauge group is decomposed into $[U(N)]^3$ and the corresponding quiver diagram is depicted in figure 7.4.

The gauge sector contains three $\mathcal{N} = 1$ vector multiplets

$$A_{\mathbf{i}}\,, \qquad G_{\mathbf{i}}\,, \qquad \lambda_{\mathbf{i}}\,, \qquad \tilde{\lambda}_{\mathbf{i}}\,, \tag{7.13}$$

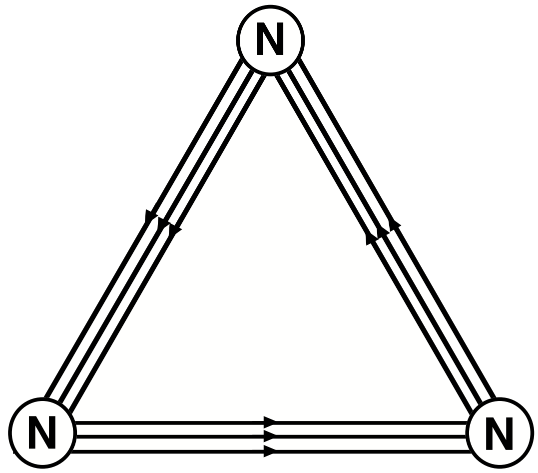


Figure 7.4: The $\mathcal{N}=1$ quiver for $k=3$ and $a_A=(1,1,1,0)$.

where $A_{\mathbf{i}} \equiv A^{I\mathbf{i}}{}_{J\mathbf{i}}$, $G_{\mathbf{i}} \equiv G^{I\mathbf{i}}{}_{J\mathbf{i}}$, $\lambda^a_{\mathbf{i}} \equiv (\chi_4)^{I\mathbf{i}}{}_{J\mathbf{i}}$, $\tilde{\lambda}^a_{\mathbf{i}} \equiv (\tilde{\chi}_4)^{I\mathbf{i}}{}_{J\mathbf{i}}$. The index $\mathbf{i}=1,2,3$ labels the nodes of the quiver.

On the other hand, the matter sector consists of three $\mathcal{N}=1$ chiral multiplets for each pair of nodes

$$q^{\mu}_{\mathbf{i},\mathbf{i+1}}\,, \qquad \tilde{q}^{\mu}_{\mathbf{i+1},\mathbf{i}}\,, \qquad \phi^{\mu}_{\mathbf{i},\mathbf{i+1}}\,, \qquad \tilde{\phi}^{\mu}_{\mathbf{i+1},\mathbf{i}}\,, \tag{7.14}$$

where now the index μ runs from 1 to 3. Here a subscript $\mathbf{i},\mathbf{j}$ indicates that the field has fundamental index in the $\mathbf{i}$-th node and anti-fundamental in the $\mathbf{j}$-th node. The quarks $q^{\mu}_{\mathbf{i},\mathbf{i+1}}$ and the anti-quarks $\tilde{q}^{\mu}_{\mathbf{i+1},\mathbf{i}}$ come from $(\chi^{\mu})^{I\mathbf{i}}{}_{J,\mathbf{i+1}}$ and $(\tilde{\chi}^{\mu})^{I,\mathbf{i+1}}_{\;\;J\mathbf{i}}$. The scalars $\phi^{\mu}_{\mathbf{i},\mathbf{i+1}}$ and $\tilde{\phi}^{\mu}_{\mathbf{i+1},\mathbf{i}}$ come from $(\phi_{\mu 4})^{I\mathbf{i}}{}_{J,\mathbf{i+1}}$ and $\epsilon^{\mu\nu\rho}(\phi_{\nu\rho})^{I,\mathbf{i+1}}_{\;\;J\mathbf{i}}$. The gauge theory with this field content is superconformal [239][240].

In terms of these fields the action (3.49) becomes

$$\mathcal{S} \;=\; \sum_{\mathbf{i}=1}^{3} \int_{\mathbb{CP}^3} \Omega' \wedge \mathrm{Tr}\,\Big[G_{\mathbf{i}} \wedge (\bar{\partial} A_{\mathbf{i}} + A_{\mathbf{i}} \wedge A_{\mathbf{i}}) + \tilde{\lambda}_{\mathbf{i}} \wedge \bar{D}_{\mathbf{i}} \lambda_{\mathbf{i}} + \tilde{q}_{\mu\,\mathbf{i+1},\mathbf{i}} \wedge \bar{D}_{\mathbf{i}}\, q^{\mu}_{\mathbf{i},\mathbf{i+1}}$$

$$+ \quad \tilde{\phi}_{\mu\,\mathbf{i+1,i}} \wedge \bar{D}_{\mathbf{i}}\, \phi^{\mu}_{\mathbf{i,i+1}} + \epsilon_{\mu\nu\rho}\, q^{\mu}_{\mathbf{i,i+1}} \wedge q^{\nu}_{\mathbf{i+1,i+2}} \wedge \phi^{\rho}_{\mathbf{i+2,i}} + \lambda_{\mathbf{i}} \wedge q^{\mu}_{\mathbf{i,i+1}} \wedge \tilde{\phi}_{\mu\,\mathbf{i+1,i}} \big] \,. \tag{7.15}$$

The interaction term (3.53) reads after the orbifold projection

$$\begin{aligned} \Delta I_{D1-D5} = \int_{D1} \mathrm{Tr}\, J\mathcal{A} = \int_{D1} J^{I\mathbf{i}}{}_{J\mathbf{j}} \mathcal{A}^{J\mathbf{j}}{}_{I\mathbf{i}} \to \\ \to \sum_{\mathbf{i}=1}^{3} \int_{D1} \mathrm{Tr}\, \big[J_{\mathbf{i}} A_{\mathbf{i}} + \psi^4 J_{\mathbf{i}} \lambda_{\mathbf{i}} + \psi^{\mu} J_{\mathbf{i+1,i}} q_{\mu\,\mathbf{i,i+1}} + \frac{1}{2} \epsilon_{\mu\nu\rho} \psi^{\mu} \psi^{\nu} J_{\mathbf{i,i+1}} \tilde{\phi}^{\rho}_{\mathbf{i+1,i}} \\ + \psi^{\mu} \psi^4 J_{\mathbf{i+1,i}} \phi_{\mu\,\mathbf{i,i+1}} + \frac{1}{3!} \epsilon_{\mu\nu\rho} \psi^{\mu} \psi^{\nu} \psi^{\rho} J_{\mathbf{i}} \tilde{\lambda}_{\mathbf{i}} \\ + \frac{1}{2} \epsilon_{\mu\nu\rho} \psi^{\mu} \psi^{\nu} \psi^4 J_{\mathbf{i,i+1}} \tilde{q}^{\rho}_{\mathbf{i+1,i}} + \psi^1 \psi^2 \psi^3 \psi^4 J_{\mathbf{i}} G_{\mathbf{i}} \big] \,, \end{aligned} \tag{7.16}$$

with $J_{\mathbf{i}} \equiv J^{J\mathbf{i}}_{I\mathbf{i}} = \alpha^{J\mathbf{i}} \beta_{I\mathbf{i}}$ and $J_{\mathbf{i,i+1}} \equiv J^{J,\mathbf{i+1}}_{I\mathbf{i}} = \alpha^{J,\mathbf{i+1}} \beta_{I\mathbf{i}}$. A convenient way to keep track of the group theory factors is to use a double line notation, where one assigns a different type of oriented line to the fundamental index of each node. For instance, an adjoint field is represented by two lines of the same type and opposite orientation, while a bi-fundamental field has two lines of different type and opposite orienation. For example, in the $\mathcal{N} = 1$ case discussed here there are three types of lines corresponding to the three nodes of the quiver in figure 7.4. Some examples are shown in figure 7.5.

MHV Amplitudes for the $\mathcal{N} = 1$ orbifold

In order to calculate MHV amplitudes in these theories we need to follow the general prescription given in [96] and [111]. This prescription is applicable also in this case since we do not allow the k D1-branes to move independently, as already discussed. Saturation of the fermionic degrees of freedom requires eight θ's. We consequently

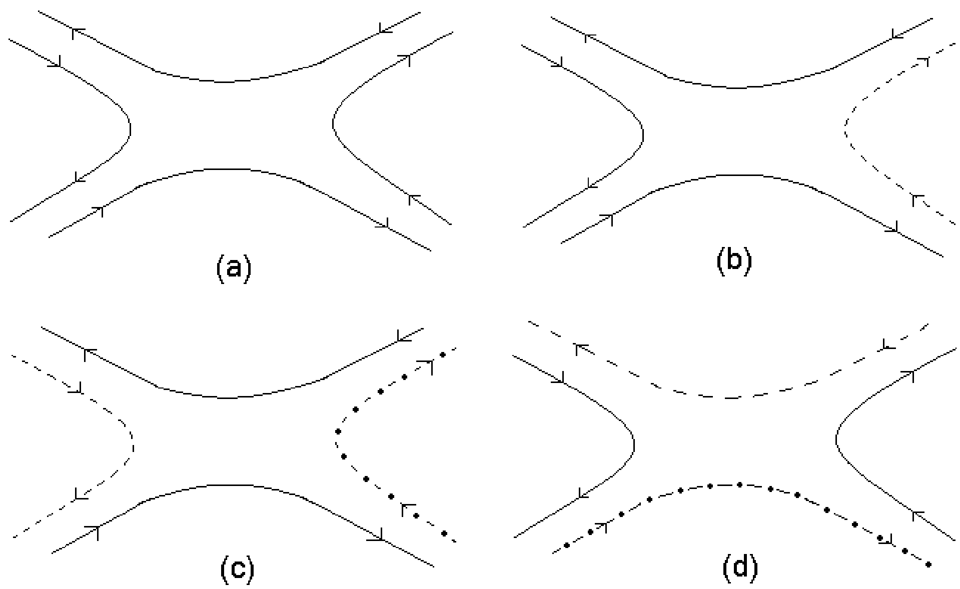


Figure 7.5: The double line notation: (a) scattering of four adjoint fields belonging to the same node; (b) scattering of two adjoint and two bi-fundamental fields; (c) scattering of four bi-fundamental fields with intermediate adjoint field; (d) same as in (c) but with intermediate bi-fundamental field.

have as many different MHV amplitudes, as there are possible products of terms from the superfield expansion giving eight θ's. The denominator of these analytic amplitudes is provided by the current correlation functions. The latters also provide the appropriate group structure. Note that here we should be a little bit more careful than usual, since part of the gauge theory trace is implicit in the summation of the indices $\mathbf{i}$ which belong to the fundamental representation. We should therefore make sure that we consider only meaningful products of currents, that correspond to single trace terms for each MHV analytic amplitude, for instance products like $J_{\mathbf{i}} J_{\mathbf{i},\mathbf{i+1}} J_{\mathbf{i+1},\mathbf{i}} J_{\mathbf{i}}$ for a four point amplitude of the form $(\lambda\, \tilde{q} q\, \tilde{\lambda})$. The possible group theory contractions are easily estabilished by drawing the diagrams in double-line notation.

It is now straightforward to proceed to the computation of specific amplitudes of interest. One could rewrite the $\mathcal{N} = 4$ superfield expansion eq. (3.47) in $SU(3) \times U(1)_R$ notation which is manifestly $\mathcal{N} = 1$ invariant. Recalling that the momentum structure of the amplitudes is solely determined by the form of this expansion, we deduce that analytic amplitudes in the $\mathcal{N} = 1$ orbifold theory bear an identical spinor product structure to the ones of $\mathcal{N} = 4$ SYM in $SU(3) \times U(1)_R$ notation. This is in complete accordance with field theoretical considerations, since the Lagrangian description of these theories is identical apart from their group structure. We will make this point clearer with several examples. Note, however, that in what follows we will omit group indices[9] and coupling constants, since we stay at the point in the moduli space where all the gauge couplings are equal.

[9]As usual we strip out the gauge group theory factor.

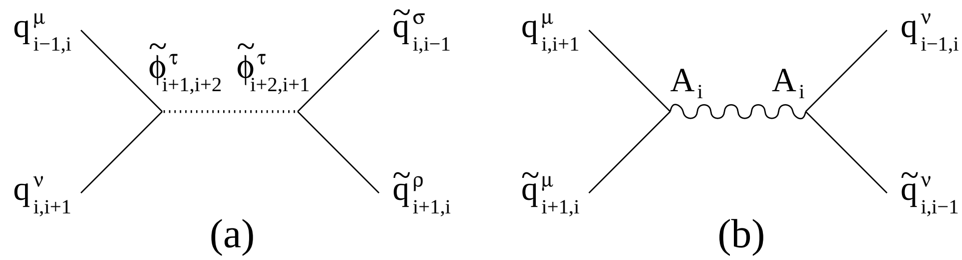


Figure 7.6: The two Feynman diagrams that contribute to the tree level $(q\,q\,\tilde{q}\,\tilde{q})$ scattering.

Example 1: Amplitudes $(A\ldots A\,G\,G)$**,** $(A\ldots A\,G\,\lambda\,\tilde{\lambda})$**, and** $(A\ldots A\,G\,q\,\tilde{q})$ These amplitudes have been extensively considered in the literature [102][103]. As a first trivial check of the above, we compute the standard four gluon amplitude $(A_{\mathbf{i}}A_{\mathbf{i}}G_{\mathbf{i}}G_{\mathbf{i}})$, with $\mathbf{i}=1,2,3$. Following the usual prescription, and using eq. (3.47), we have

$$\begin{aligned}\mathcal{M}^{(4)}(AAGG) &= \int d^8\theta(\psi_3^1\psi_3^2\psi_3^3\psi_3^4)(\psi_4^1\psi_4^2\psi_4^3\psi_4^4)\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\\ &= \frac{\langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \end{aligned}\tag{7.17}$$

This is the familiar formula for MHV scattering in $\mathcal{N}=4$ SYM. In the same way, one can compute amplitudes of the type $(A\ldots A\,G\,G)$, $(A\ldots A\,G\,\lambda\,\tilde{\lambda})$, $(A\ldots A\,G\,q\,\tilde{q})$, and $(A\ldots A\,\lambda\,\tilde{\lambda}\,q\,\tilde{q})$.

Example 2: Amplitudes $(q\,q\,\tilde{q}\,\tilde{q})$ **and** $(\lambda\,q\,\tilde{q}\,\tilde{\lambda})$ These amplitudes have, as previously mentioned, the same spinor product structure $\mathcal{N}=4$ SYM has. Yet, they are far more interesting cases to study. The reason is that they consist of two subamplitudes, shown in the case $(q\,q\,\tilde{q}\,\tilde{q})$ in figure 7.6. They depend on both gluon and scalar particle exchange. Let us now concentrate on $(q^{\rho}_{\mathbf{i-1,i}}\,q^{\sigma}_{\mathbf{i,i+1}}\,\tilde{q}^{\kappa}_{\mathbf{i+1,i}}\,\tilde{q}^{\lambda}_{\mathbf{i,i-1}})$. The other case $(\lambda\,q\,\tilde{q}\,\tilde{\lambda})$ can be computed in a similar manner. The two subamplitudes in figure 7.6

correspond in double line notation to diagrams (d) and (c) in figure 7.5

$$\mathcal{M}^{(4)}(qq\tilde{q}\tilde{q}) = \int d^8\theta \frac{1}{4} \psi_1^\rho \psi^\sigma{}_2 (\psi_3^4 \psi_3^\mu \psi_3^\nu)(\psi_4^4 \psi_4^\pi \psi_4^\tau) \epsilon_{\mu\nu\kappa}\epsilon_{\pi\tau\lambda} \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} . \quad (7.18)$$

Integration over θ^4 is straightforward and yields $\langle 34\rangle$. Then, we must sum over all possible contractions of momenta upon integration over the fermionic part of the space. There are three distinct contractions

$$(a) \int d^6\,\theta \psi_1^\rho \psi_2^\sigma \psi_3^\mu \psi_3^\nu \psi_4^\pi \psi_4^\tau + \left\{{}^{\mu\leftrightarrow\nu}_{\pi\leftrightarrow\tau}\right\} = \delta^{\rho\sigma}(\delta^{\mu\tau}\delta^{\nu\pi} - \delta^{\mu\pi}\delta^{\nu\tau})\langle 12\rangle\langle 34\rangle^2 ,$$

$$(b) \int d^6\,\theta \psi_1^\rho \psi_2^\sigma \psi_3^\mu \psi_3^\nu \psi_4^\pi \psi_4^\tau + \left\{{}^{\mu\leftrightarrow\nu}_{\pi\leftrightarrow\tau}\right\} = \big(\delta^{\rho\sigma}(\delta^{\mu\pi}\delta^{\nu\tau} - \delta^{\mu\tau}\delta^{\nu\pi}) - \epsilon^{\rho\mu\nu}\epsilon^{\sigma\pi\tau}\big)\langle 23\rangle\langle 34\rangle\langle 41\rangle ,$$

$$(c) \int d^6\,\theta \psi_1^\rho \psi_2^\sigma \psi_3^\mu \psi_3^\nu \psi_4^\pi \psi_4^\tau + \left\{{}^{\mu\leftrightarrow\nu}_{\pi\leftrightarrow\tau}\right\} = \big(\delta^{\rho\sigma}(\delta^{\mu\pi}\delta^{\nu\tau} - \delta^{\mu\tau}\delta^{\nu\pi}) - \epsilon^{\rho\pi\tau}\epsilon^{\sigma\mu\nu}\big)\langle 13\rangle\langle 24\rangle\langle 34\rangle . \quad (7.19)$$

The three spinor product structures in (7.19), are related through the Schouten identity $\langle pq\rangle\langle rs\rangle + \langle qr\rangle\langle ps\rangle + \langle rp\rangle\langle qs\rangle = 0$. Use of this identity reveals the two independent structures that we were expecting. Explicitly, we have

$$\int d^6\,\theta \psi_1^\rho \psi_2^\sigma \psi_3^\mu \psi_3^\nu \psi_4^\pi \psi_4^\tau = -\epsilon^{\rho\pi\tau}\epsilon^{\sigma\mu\nu}\langle 12\rangle\langle 34\rangle^2 + (\epsilon^{\rho\pi\tau}\epsilon^{\sigma\mu\nu} - \epsilon^{\rho\mu\nu}\epsilon^{\sigma\pi\tau})\langle 23\rangle\langle 34\rangle\langle 41\rangle . \quad (7.20)$$

Inserting eq. (7.20) into eq. (7.18), we obtain

$$\mathcal{M}^{(4)}(qq\tilde{q}\tilde{q}) = -\delta^\rho{}_\lambda \delta^\sigma{}_\kappa \frac{\langle 34\rangle^2}{\langle 23\rangle\langle 41\rangle} - \epsilon^{\rho\sigma\tau}\epsilon_{\tau\kappa\lambda}\frac{\langle 34\rangle}{\langle 12\rangle} . \quad (7.21)$$

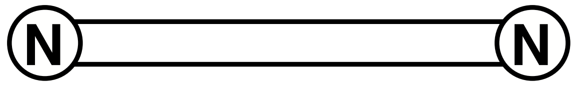


Figure 7.7: The $\mathcal{N}=2$ quiver for $k=2$ and $a_A=(1,1,0,0)$.

It is easy to see that this result is in agreement with the field theory predictions. An important remark is now in order. As we can also see in figure 7.6, there are two types of contributions to this scattering process. One of them comes from a Yukawa type interaction term while the other comes from the usual matter-gluon interaction. In general these two interaction terms would be weighted with the appropriate independent coupling constant. It would be interesting to check if the consistency of the twistor method constraints the couplings to be in the conformal region of the moduli space. Amplitudes like the one considered in this example might provide some insight on how to move away from the point in the moduli space where all the couplings are equal.

7.4.2 An $\mathcal{N}=2$ orbifold

We now move on to the case in which $\Gamma \in \mathbb{Z}_k \subset SU(2)_R$. This breaks $SU(4)_R \to SU(2)_R$, thus giving $\mathcal{N}=2$. For simplicity, we investigate the particular choice of $k=2$ and $a_A=(1,1,0,0)$, see eq. (7.6). The fields surviving the orbifold projection are organized into two $\mathcal{N}=2$ vector multiplets and two hypermultiplets. The resulting gauge group is $U(N)\times U(N)$. The associated quiver diagram has two nodes and two links and is given in figure 7.7. As in the previous $\mathcal{N}=1$ case, this theory is superconformal. The field content of the gauge sector is

$$A_{\mathbf{i}}\,, \qquad G_{\mathbf{i}}\,, \qquad \lambda^a_{\mathbf{i}}\,, \qquad \tilde{\lambda}^a_{\mathbf{i}}\,, \qquad \phi^m_{\mathbf{i}}\,, \tag{7.22}$$

where $A_{\mathbf{i}} \equiv A^{I\mathbf{i}}{}_{J\mathbf{i}}$, $G_{\mathbf{i}} \equiv G^{I\mathbf{i}}{}_{J\mathbf{i}}$, $\lambda^a_{\mathbf{i}} \equiv (\chi_{3,4})^{I\mathbf{i}}{}_{J\mathbf{i}}$, $\tilde{\lambda}^a_{\mathbf{i}} \equiv (\tilde{\chi}_{3,4})^{I\mathbf{i}}{}_{J\mathbf{i}}$, and $\phi^m_{\mathbf{i}} \equiv (\phi_{12,34})^{I\mathbf{i}}{}_{J\mathbf{i}}$. The nodes of the quiver are labelled by $\mathbf{i} = 1, 2$. The index a is the $SU(2)_R$ index, whereas m labels the two real components of the complex scalar field. The matter sector has two bifundamental hypermultiplets

$$q^{\mu}_{\mathbf{i},\mathbf{i+1}}\,, \qquad \tilde{q}^{\mu}_{\mathbf{i+1},\mathbf{i}}\,, \qquad H^{\mu}_{\mathbf{i},\mathbf{i+1}}\,, \qquad \tilde{H}^{\mu}_{\mathbf{i+1},\mathbf{i}}\,, \tag{7.23}$$

with the index $\mu = 1, 2$ labeling the hypermultiplets. The quarks $q^{\mu}_{\mathbf{i},\mathbf{i+1}}$ are given by $(\chi^{1,2})^{I\mathbf{i}}{}_{J,\mathbf{i+1}}$ and the anti-quarks $\tilde{q}^{\mu}_{\mathbf{i+1},\mathbf{i}}$ by $(\tilde{\chi}^{1,2})^{I,\mathbf{i+1}}{}_{J\mathbf{i}}$. The four scalars H^{μ} and $\tilde{H}^{\mu}$ come from $\phi_{13}, \phi_{14}, \phi_{23}$, and ϕ_{24}.

The projected action and the D1-D5 interaction term can be obtained in similarly to the previous $\mathcal{N} = 1$ case.

MHV Amplitudes for the $\mathcal{N} = 2$ orbifold

Example 1: Amplitudes like $(A \ldots A\, G\, G)$, $(A \ldots A\, G, \lambda\, \tilde{\lambda})$, and $(A \ldots A\, G\, q\, \tilde{q})$
We consider the scattering process between the following particles $(A\, A\, G\, \lambda^a\, \tilde{\lambda}^c)$. According to the twistor string prescription, we should compute

$$\mathcal{M}^{(5)}(AAG\lambda\tilde{\lambda}) = \int d^8\theta\, \psi^1_3\psi^2_3\psi^3_3\psi^4_3\psi^a_4\psi^1_5\psi^2_5\psi^b_5\epsilon_{bc}\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{7.24}$$

There are only two possible contractions between the different ψ's, which yield

$$\begin{aligned}\mathcal{M}^{(5)}(AAG\lambda\tilde{\lambda}) &= (\delta^{3b}\delta^{4a} - \delta^{3a}\delta^{4b})\epsilon_{bc}\frac{\langle 35\rangle^3\langle 34\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \\ &= \delta^{ac}(\delta^{4c} - \delta^{3c})\frac{\langle 35\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 45\rangle\langle 51\rangle}\,. \end{aligned}\tag{7.25}$$

As we see, we recovered the familiar result.

Example 2: Amplitude $(\lambda\, q\, \tilde{q}\, \tilde{\lambda})$ We will now apply the same method in order to compute the scattering amplitude $(\lambda^a\, q^\mu\, \tilde{q}^\rho\, \tilde{\lambda}^c)$ between a gluino-antigluino pair and a quark-antiquark one. To this end, we need to calculate the following integral

$$\mathcal{M}^{(4)}(\lambda q \tilde{q} \tilde{\lambda}) = \int d^8\theta\, \psi_1^a \psi^{\mu}{}_2 (\psi_3^3 \psi_3^4 \psi_3^\nu)(\psi_4^1 \psi_4^2 \psi_4^b)\epsilon_{\nu\rho}\epsilon_{bc} \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{7.26}$$

where $a, b, c = 3, 4$ and $\mu, \nu, \rho = 1, 2$. To perform the integration we need to sum over all the possible contractions between the fermionic coordinates of supertwistor space. In this example we can split the fermions into two groups, with no contractions among fermions belonging to different groups. In each group, fermions can be contracted in two different ways

$$\begin{aligned} (a) \int d^4\,\theta \psi_1^a \psi_3^3 \psi_3^4 \psi_4^b &= \delta^{a3}\delta^{4b}\langle 13\rangle\langle 34\rangle\,, \\ (b) \int d^4\,\theta \psi_1^a \psi_3^3 \psi_3^4 \psi_4^b &= -\delta^{a4}\delta^{b3}\langle 13\rangle\langle 34\rangle\,, \end{aligned} \tag{7.27}$$

and

$$\begin{aligned} (a) \int d^4\,\theta \psi_2^\mu \psi_3^\nu \psi_4^1 \psi_4^2 &= \delta^{\mu 2}\delta^{\nu 1}\langle 24\rangle\langle 34\rangle\,, \\ (b) \int d^4\,\theta \psi_2^\mu \psi_3^\nu \psi_4^1 \psi_4^2 &= -\delta^{\mu 1}\delta^{\nu 2}\langle 24\rangle\langle 34\rangle\,. \end{aligned} \tag{7.28}$$

We then substitute (7.27) and (7.28) into (7.26) and use the Schouten identity to obtain

$$\mathcal{M}^{(4)}(\lambda q \tilde{q} \tilde{\lambda}) = \delta^{ac}\delta^{\mu\rho}(\delta^{\mu 2} - \delta^{\mu 1})(\delta^{a3} - \delta^{a4}) \left(\frac{\langle 34\rangle^2}{\langle 23\rangle\langle 41\rangle} - \frac{\langle 34\rangle}{\langle 12\rangle} \right)\,. \tag{7.29}$$

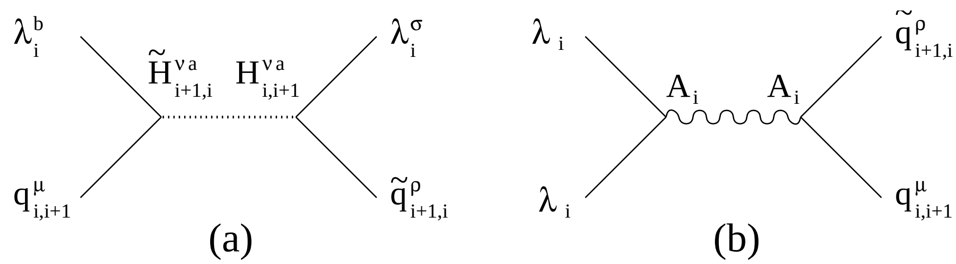


Figure 7.8: The two Feynman diagrams that contribute to the tree level $(\lambda\, q\, \tilde{q}\, \tilde{\lambda})$ scattering.

We see from the form of the result that there are exactly two distinct spinor product structures. They reflect contributions to the scattering process from two different types of interactions: the former is the standard quark-gluon interaction and the latter is of Yukawa type. Figure 7.8 shows the corresponding Feynman diagrams. This is in accordance with the usual field theory calculations.

Other amplitudes, with quarks or scalars as external particles, can be computed in a similar fashion. They usually retain the feature of receiving contributions from multiple interaction processes/vertices.

7.5 Conclusion

In this chapter of the dissertation we have investigated $\mathbb{Z}_k$ fermionic orbifolds of the topological B-model on $\mathbb{CP}^{3|4}$ to reduce the amount of supersymmetry of the dual $\mathcal{N} = 4$ super Yang-Mills theory. This was allowed by the fact that the Fermi-Fermi part of the $PSU(4|4)$ isometry group of $\mathbb{CP}^{3|4}$ is precisely the $SU(4)_R$ R-symmetry. We have discussed how the projection acts on both the D5 and the D1-branes. As examples we have considered $\mathcal{N} = 1$ and $\mathcal{N} = 2$ orbifolds and obtained the corresponding quiver theories. Several amplitudes have been computed and shown to agree with the field theory results.

We have worked only at the point in the moduli space where all gauge coupling constants are equal. It would be interesting to study the full moduli space of superconformal couplings and understand its interpretation in twistor string theory. Some indications on the origin of these moduli have been given in discussing the action of the orbifold on the D1-brane sector. Since these moduli are usually interpreted as coming from twisted fields it would be worth studying the closed string sector and identify the twisted states. The fermionic orbifold does not have an obvious geometrical meaning. The study of the twisted sector may be useful to shed some light on the geometrical interpretation of the orbifold.

Chapter 8

A twistorial approach to gravity amplitudes

8.1 Introduction

In this final chapter we discuss a twistorial approach to the computation of graviton amplitudes in ordinary (non-superconformal gravity). The closed string sector of the B-model on $\mathbb{CP}^{3|4}$ should presumably describe $\mathcal{N}=4$ conformal supergravity, which at tree level reduces to conformal gravity. Ordinary gravity amplitudes would be related not to the closed sector of the B-model on $\mathbb{CP}^{3|4}$ but to that of a yet unknown topological twistor string theory which probably describes $\mathcal{N}=8$ supergravity. Even though the correct framework for studying gravity has not been completely established, some preliminary indications on localization of tree level gravity amplitudes can be given. Some initial analysis of the MHV case was already anticipated in [96]. The crucial difference with respect to YM is that the n graviton MHV amplitude is *not holomorphic* in the spinor helicity variables in Minkowski space. This non-holomorphic dependence is nonetheless very simple, namely polynomial. The polynomial dependence implies that MHV gravity amplitudes are supported again on $d = 1$ curves, but now with a multiple derivative of a delta-function, as we discuss in the following.

It is natural to investigate if this behavior persists for non-MHV cases. In the next section we check the simplest non-trivial case, namely the googly amplitude $\mathcal{M}^{(5)}(+,+,-,-,-)$. Constructing a suitable differential operator which annihilates the amplitude, we verify that this is supported on a connected degree 2 curve of genus zero. This is similar to what happens for the corresponding googly YM amplitude, with the difference that we now have a derivative of a delta-function support.

This does not exclude *a priori* the presence of disconnected contributions and we comment on the possibility of a MHV decomposition of gravity amplitudes. Note that even without knowing the underlying string theory, having a MHV-like diagrammatic expansion would dramatically simplify the calculation of gravity amplitudes, which are notoriously complicated and in many cases not known in closed form.

The vertices are built using the MHV prescription for YM and the KLT relations, which in general express closed string amplitudes as a sum of products of open string amplitudes, in the field theory limit [245]. Differently from the gauge theory case it is not possible to construct MHV gravity diagrams using only holomorphic vertices. The only diagrams which can be built using holomorphic vertices correspond to amplitudes of the form $\mathcal{M}^{(n)}(+,-,\ldots,-)$. As in YM these are known to vanish. Using the completely disconnected prescription we verify that the MHV diagrams for $\mathcal{M}^{(4)}(+,-,-,-)$ and $\mathcal{M}^{(5)}(+,-,-,-,-)$ sum to zero. More problematic is an MHV construction for the other gravity amplitudes. Already the first not vanishing googly amplitude $\mathcal{M}^{(5)}(+,+,-,-,-)$ involves a *non* holomorphic 4 vertex. The naïve application of the MHV prescription of [123] to this amplitude seems to fail. In particular the result is not covariant. It is not quite clear whether this failure is due to the special features of gravity (*e.g.*, lack of conformal invariance) which may lead to the inequivalence of connected and disconnected prescriptions. If this were the case one should sum over both connected and disconnected configurations in the corresponding string theory. Another possibility would be that our off-shell extension needs to be modified.

8.2 A googly graviton amplitude

Starting from the observation that a closed string vertex operator factorizes into the product of two open string vertices, Kawai, Lewellen, and Tye [245] were able to derive a set of formulas relating closed string amplitudes to open string ones. In the low-energy limit these formulas imply a similar factorization of gravity amplitudes as products of two gauge theory amplitudes.

By direct use of the KLT relations it has therefore been possible [246] to obtain compact expressions for several tree-level gravity amplitudes, which would have been much more difficult to compute diagrammatically, considering the complexity of perturbative gravity.[1] A nice review of this topic is given in [247].

Following [246] we denote the amplitude for n external gravitons with momenta $p_1, \ldots, p_n$ and helicities $h_1, \ldots, h_n$ by $\mathcal{M}^{(n)}(1h_1, \ldots, nh_n)$. Similarly to the gluon case, the amplitude vanishes if more than $n-2$ gravitons have the same helicity. The first non-trivial amplitude describes the scattering of 2 gravitons with one helicity and

[1] The Hilbert-Einstein Lagrangian generates an infinite number of interaction vertices, which have complicated expressions containing hundreds of terms. The graviton propagator is also very complicated, compared to gauge theory. As a striking example of this complexity, consider that a gravity five loop diagram produces 10^{30} terms, even before performing any integration.

$n-2$ gravitons with the opposite one. Using the same terminology as in chapter 3, the amplitude with $q=2$ negative helicity gravitons is called maximally helicity violating (MHV), whereas the amplitude with $q=n-2$ negative helicity gravitons is called googly. Following custom we will use the abbreviated notation for the contraction of two spinors $\langle ij\rangle \equiv \epsilon_{ab}\lambda_i^a\lambda_j^b$ and $[ij] \equiv \epsilon_{\dot a\dot b}\tilde\lambda_i^{\dot a}\tilde\lambda_j^{\dot b}$.

The explicit expression in the MHV case of $n=5$, $q=2$ gravitons is [246]

$$\mathcal{M}^{(5)}(1-,2-,3+,4+,5+) = -4i\,(8\pi G_N)^{\frac{3}{2}}\,\frac{\langle 12\rangle^8}{\prod_{i=1}^4\prod_{j=i+1}^5\langle ij\rangle}\mathcal{E}(1,2,3,4)\,, \qquad (8.1)$$

where $\mathcal{E}(1,2,3,4) = \frac{1}{4i}\left([12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]\right)$. This amplitude is of the form

$$\mathcal{M}^{(5)}(1-,2-,3+,4+,5+) = \sum_{\alpha=1,2} R_\alpha(\lambda_i)P_\alpha(\tilde\lambda_i)\,, \qquad (8.2)$$

where the R's are rational functions and the P's are polynomials. Even though (8.2) is not holomorphic in λ as the Parke-Taylor formula (3.15), it splits in two parts, each of them displaying a simple polynomial dependence on $\tilde\lambda$. This generalizes to all MHV gravity amplitudes. As already shown in [96], the twistor transform of

$$A^{(5)}(\lambda_i,\tilde\lambda_i) = i(2\pi)^4\delta^4\left(\sum_i \lambda_i^a\tilde\lambda_i^{\dot a}\right)\mathcal{M}^{(5)}(1-,2-,3+,4+,5+) \qquad (8.3)$$

yields

$$\begin{aligned}\tilde A^{(5)}(\lambda_i,\mu_i) &= i\int d^4x\int\frac{d^2\tilde\lambda_1}{(2\pi)^2}\cdots\frac{d^2\tilde\lambda_5}{(2\pi)^2}\,e^{i\sum_{i=1}^5\tilde\lambda_i^{\dot a}(\mu_{i\dot a}+x_{a\dot a}\lambda_i^a)}\mathcal{M}^{(5)}(\lambda_i,\tilde\lambda_i)\\ &= i\sum_{\alpha=1,2}R_\alpha(\lambda_i)P_\alpha\left(i\frac{\partial}{\partial\mu_{i\dot a}}\right)\int d^4x\prod_{i=1}^5\delta^2(\mu_{i\dot a}+x_{a\dot a}\lambda_i^a)\,. \end{aligned} \qquad (8.4)$$

The twistor transformed amplitude is thus supported on a curve of degree $d = 1$ and genus $g = 0$, via a polynomial in derivatives of the delta function.

Now we move on to the googly amplitude, which is obtained by switching the λ's and the $\tilde{\lambda}$'s in (8.1)[2]

$$\begin{aligned}\mathcal{M}^{(5)}(1+,2+,3-,4-,5-) &= \left[\mathcal{M}^{(5)}(1-,2-,3+,4+,5+)\right]^* = \sum_{\alpha=1,2} P^*_\alpha(\lambda_i) R^*_\alpha(\tilde{\lambda}_i) \\ &= (8\pi G_N)^{\frac{3}{2}} \left(\frac{\langle 12\rangle\langle 34\rangle[12]^8}{[12][13][15][24][25][34][35][45]} + \frac{\langle 23\rangle\langle 41\rangle[12]^8}{[13][14][15][23][24][25][35][45]} \right). \end{aligned} \tag{8.5}$$

This amplitude obeys for each $i = 1, \ldots, 5$ a homogeneity condition as in eq. (3.12)

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \tilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}}\right) \mathcal{M}^{(5)} = -2h_i \mathcal{M}^{(5)} \tag{8.6}$$

where $h_i = \pm 2$ is the helicity of the i-th graviton.

The transform to twistor space of

$$A^{(5)}(\lambda_i, \tilde{\lambda}_i) = i(2\pi)^4 \delta^4\left(\sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}}\right) \mathcal{M}^{(5)}(1+,2+,3-,4-,5-) \tag{8.7}$$

would be

$$\tilde{A}^{(5)}(\lambda_i, \mu_i) = i \sum_{\alpha=1,2} P^*_\alpha(\lambda_i) \int d^4x \int \frac{d^2\tilde{\lambda}_1}{(2\pi)^2} \cdots \frac{d^2\tilde{\lambda}_5}{(2\pi)^2}\, e^{i\sum_{i=1}^5 \tilde{\lambda}_i^{\dot{a}}(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a)} R^*_\alpha(\tilde{\lambda}_i)\,. \tag{8.8}$$

The homogeneity condition in twistor space, which can be obtained from (8.6) by

[2] In Lorentz signature this amounts to a parity transformation since $\tilde{\lambda} = \pm\bar{\lambda}$, as explained in the footnote 2 of chapter 3.

performing the transformation (3.20), reads

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} + \mu_{i\dot{a}} \frac{\partial}{\partial \mu_{i\dot{a}}}\right) \tilde{A}^{(5)} = (-2h_i - 2)\tilde{A}^{(5)} \,. \tag{8.9}$$

According to (3.25), we expect $\tilde{A}^{(5)}$ to be supported on a $d = 2$, $g = 0$ curve in twistor space. Since the $\tilde{\lambda}$ dependence of (8.5) is through rational functions, it is not easy to perform explicitly the twistor transform and check this conjecture. Witten proposed an alternative way to avoid this cumbersome computation [96]. This method was described in chapter 3, and is based on the introduction of operators which control if a set of given points lies on a common curve embedded in twistor space. These operators are algebraic in the (λ, μ) space, and become differential once transformed back to the $(\lambda, \tilde{\lambda})$ space.

The relevant operator for the $n = 5$, $q = 3$ case is

$$K_{ijkl} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L \,, \tag{8.10}$$

where Z_i^I are homogeneous coordinates in $\mathbb{CP}^3$, namely $Z_i^I = (\lambda_i^1, \lambda_i^2, \mu_{i1}, \mu_{i2})$, for the i-th graviton ($i = 1, \ldots, 5$). To go to the $(\lambda, \tilde{\lambda})$ space, one simply replaces $\mu_{i\dot{a}}$ with $-i\frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}}$. We introduce the notation

$$\{ij\} = \epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \tilde{\lambda}_i^{\dot{a}} \partial \tilde{\lambda}_j^{\dot{b}}} \,. \tag{8.11}$$

The differential operator in $(\lambda, \tilde{\lambda})$ space is thus expressed as

$$K_{ijkl} = \langle ij \rangle \{kl\} - \langle ik \rangle \{jl\} - \langle jl \rangle \{ik\} + \langle il \rangle \{jk\} + \langle kl \rangle \{ij\} - \langle jk \rangle \{li\} \,. \tag{8.12}$$

If the amplitude is supported on a $d = 2$, $g = 0$ curve through a delta function, then one expects that $K_{ijkl} A^{(5)}(\lambda, \tilde{\lambda}) = 0$. This is indeed what happens for the $n = 5$, $q = 3$ tree-level gluon amplitude, as verified in [96]. What we are actually going to prove for the graviton amplitude is that

$$K_{ijkl} K_{i'j'k'l'} A^{(5)} = 0 \,. \tag{8.13}$$

This means that we still have a localization on a $d = 2$, $g = 0$ curve but now via a derivative of the delta function. This is somewhat similar to what happens in the 1-loop gluon amplitude analyzed in [96].

A useful simplification in checking (8.13) is achieved by using the manifest Poincaré invariance of both K and $A^{(5)}(\lambda, \tilde{\lambda})$. The Lorentz transformations are given by $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, with the first $SL(2, \mathbb{R})$ acting on the λ's and the second one on the $\tilde{\lambda}$'s. Translations act on the μ's as $\mu_{i\dot{a}} \to \mu_{i\dot{a}} + x_{a\dot{a}} \lambda_i^a$. It is thus possible to fix two points in twistor space Z_i, Z_j to convenient values: λ_i and λ_j can be fixed by use of $SL(2, \mathbb{R})$ plus a scaling allowed by (8.9), whereas $\mu_{i\dot{a}}$ and $\mu_{j\dot{a}}$ are fixed by the translations. We can choose for example to fix $Z_3 = (1, 0, 0, 0)$ and $Z_4 = (0, 1, 0, 0)$. This means $\lambda_3 = (1, 0)$, $\lambda_4 = (0, 1)$ and $\mu_3 = \mu_4 = (0, 0)$. The delta function of momentum conservation enforces

$$\tilde{\lambda}_3^{\dot{a}} = - \sum_{j=1,2,5} \lambda_j^1 \tilde{\lambda}_j^{\dot{a}} \,, \qquad \tilde{\lambda}_4^{\dot{a}} = - \sum_{j=1,2,5} \lambda_j^2 \tilde{\lambda}_j^{\dot{a}} \,. \tag{8.14}$$

By substituting (8.14) in (8.5) we obtain a "fixed" amplitude $A^{(5)}_{fix}$, which is function only of $\lambda_i, \tilde{\lambda}_i$ with $i = 1, 2, 5$. We find that the dependence of $A^{(5)}_{fix}$ on the $\tilde{\lambda}$'s is only through the bilinears $a \equiv [12]$, $b \equiv [15]$, and $c \equiv [25]$. The crucial property of $A^{(5)}_{fix}$ is that

$$\left(a\frac{\partial}{\partial a}+b\frac{\partial}{\partial b}+c\frac{\partial}{\partial c}\right)A^{(5)}_{fix}=0\,. \tag{8.15}$$

This follows directly from the observation that the original amplitude (8.5) is homogeneous of degree 0 in the antiholomorphic bilinears. Since eq. (8.14) is linearly homogeneous in the $\tilde{\lambda}$'s, the fixed amplitude is still homogeneous of degree 0 in a, b, c.

After fixing Z_3 and Z_4, eq. (8.12) can also be expressed in terms of a, b and c. Defining an operator $\hat{O}\equiv\left(a\frac{\partial}{\partial a}+b\frac{\partial}{\partial b}+c\frac{\partial}{\partial c}+1\right)$ we find

$$K_{1234}=-\tfrac{\partial}{\partial a}\hat{O}\,,\qquad K_{1345}=-\tfrac{\partial}{\partial b}\hat{O}\,,\qquad K_{2345}=-\tfrac{\partial}{\partial c}\hat{O}\,,$$

$$K_{1235}=-\left(\lambda_5^2\tfrac{\partial}{\partial a}-\lambda_2^2\tfrac{\partial}{\partial b}+\lambda_1^2\tfrac{\partial}{\partial c}\right)\hat{O}\,,\qquad K_{1245}=-\left(-\lambda_5^1\tfrac{\partial}{\partial a}+\lambda_2^1\tfrac{\partial}{\partial b}-\lambda_1^1\tfrac{\partial}{\partial c}\right)\hat{O}\,. \tag{8.16}$$

These are the only independent operators up to permutations. Since $A^{(5)}_{fix}$ is homogeneous of degree zero, $\hat{O}A^{(5)}_{fix}=A^{(5)}_{fix}$, and it follows that no component of K annihilates the amplitude. However from eq. (8.16) it can be seen that $K_{ijkl}A^{(5)}_{fix}$ is homogeneous of degree -1 in a, b, and c for every i, j, k, l, and thus it will be annihilated by the operator $\hat{O}$. From this observation we conclude

$$K_{ijkl}K_{i'j'k'l'}A^{(5)}_{fix}=0 \tag{8.17}$$

for any choice of i, j, k, l and i', j', k', l'.

8.3 Disconnected MHV decomposition

So far we have investigated the possibility for a twistor transformed gravity amplitude to be localized on connected curves whose degree and genus are given by (3.25). In the gauge theory context of [96], a certain string interpretation suggests that also disconnected curves may play a role in the computation of amplitudes, and that a connected contribution might be decomposed into disconnected pieces. An amplitude supported on a degree 2 curve can, for example, receive contributions from configurations with two disconnected degree 1 curves. Although one expects a contribution from all the possible decompositions, in [123] it was shown, as already explained in chapter 3, that tree level gauge theory amplitudes can be obtained by taking the completely disconnected configuration only. Inspired by what happens in the gauge theory, we try to check if a similar decomposition holds for gravity as well.

In the following we present the 3 and 4 graviton vertices given by the $\mathcal{M}^{(3)}(+,-,-)$ and $\mathcal{M}^{(4)}(+,+,-,-)$ MHV amplitudes and we try to apply this procedure to some simple gravity amplitudes, including the $n=5$ googly one studied in the previous section.

8.3.1 The $\mathcal{M}^{(4)}(+,-,-,-)$ and $\mathcal{M}^{(5)}(+,-,-,-,-)$ amplitudes

Amplitudes of the type $\mathcal{M}^{(n)}(1+,2-,\ldots,n-)$ should correspond to the twistor space diagrams in fig. 8.1. As already stated, these are known to vanish. Each $\mathbb{CP}^1$ represents a $(+,-,-)$ vertex, also depicted in fig. 8.2. This vertex is obtained by suitably extending the vanishing $\mathcal{M}^{(3)}(+,-,-)$ graviton amplitude off-shell, and is

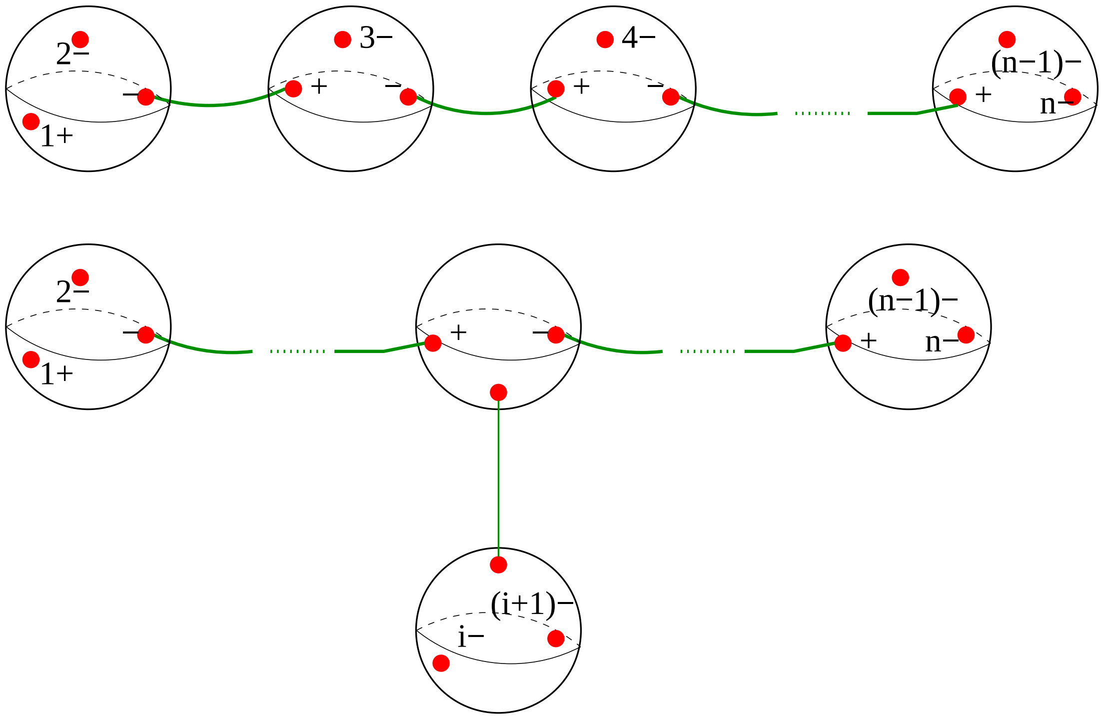


Figure 8.1: Two disconnected configurations contributing to $\mathcal{M}^{(n)}(1+, 2-, \ldots, n-)$. Each sphere represents a $\mathbb{CP}^1$, the sublocus of the twistor space where an MHV vertex is localized. The lines spanning between spheres are graviton propagators, with opposite helicities at the two endpoints.

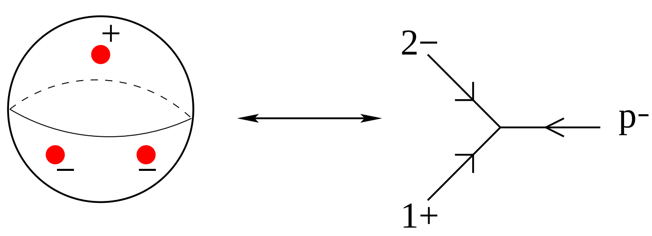


Figure 8.2: The $(+,-,-)$ graviton vertex. The external leg with momentum p is suitably extended off-shell and can be connected to a propagator.

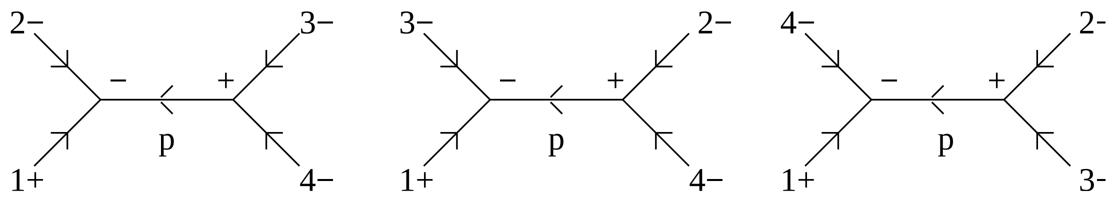


Figure 8.3: The MHV diagrams contributing to the $\mathcal{M}^{(4)}(1+, 2-, 3-, 4-)$ graviton amplitude. Note that, differently from the Yang-Mills case, here we do not require a fixed cyclic ordering of the external legs.

formally given by the square of the corresponding gluon amplitude [245].[3] The off-shell extension of the twistor λ_p corresponding to an off-shell momentum p has been given in [123] and described in chapter 3. It amounts to defining

$$\lambda_{pa} = \frac{p_{a\dot{a}}\eta^{\dot{a}}}{[\tilde{\lambda}_p, \eta]}\,, \tag{8.18}$$

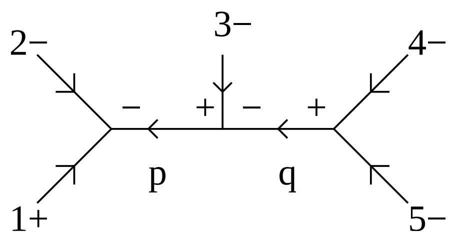
where $\eta^{\dot{a}}$ is an arbitrary spinor. The normalization factor is needed in order to have a consistent on-shell limit, and it can be dropped if the amplitude is homogeneous in the λ_p. The off-shell extension of the 3 graviton amplitude is therefore

$$\mathcal{M}^{(3)} = \left(\frac{\langle 2, p\rangle^4}{\langle 1, 2\rangle\langle 2, p\rangle\langle p, 1\rangle}\right)^2 . \tag{8.19}$$

Here we start focusing on $\mathcal{M}^{(4)}(1+, 2-, 3-, 4-)$. This is computed using the MHV diagrams shown in fig. 8.3. The contribution of the first graph in fig. 8.3 is given by

$$\frac{\langle 2p\rangle^8}{(\langle 12\rangle\langle 2p\rangle\langle p1\rangle)^2}\frac{1}{p^2}\frac{\langle 34\rangle^8}{(\langle p3\rangle\langle 34\rangle\langle 4p\rangle)^2} = \frac{\phi_1^6}{\phi_2^2\phi_3^2\phi_4^2}\frac{\langle 12\rangle\langle 34\rangle^2}{[12]}\,, \tag{8.20}$$

[3]The general KLT factorization formula relating closed and open string amplitudes reads $\mathcal{M}^{(n)}_{closed} \sim \sum_{p,p'} \mathcal{M}^{(n)}_{open}(p)\tilde{\mathcal{M}}^{(n)}_{open}(p')e^{i\pi F(p,p')}$ where p and p' are different orderings of the n external legs. In the $n = 3$ case the phase factor $e^{i\pi F(p,p')}$ drops out yielding $\mathcal{M}^{(3)}_{closed} \sim \mathcal{M}^{(3)}_{open}\tilde{\mathcal{M}}^{(3)}_{open}$. In the $\alpha' \to 0$ limit this translates to a similar relation between gravity and gauge theory amplitudes.



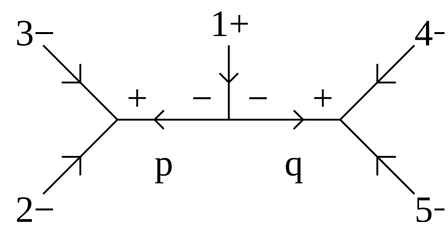


Figure 8.4: Two of the fifteen MHV diagrams contributing to the graviton amplitude $\mathcal{M}^{(5)}(1+, 2-, 3-, 4-, 5-)$. The remaining 12 graphs are obtained by appropriate relabeling of the external legs of these two graphs.

where we have used $\lambda_{pa} = -\lambda_{1a}\phi_1 - \lambda_{2a}\phi_2 = \lambda_{3a}\phi_3 + \lambda_{4a}\phi_4$, with $\phi_i = \tilde{\lambda}_{i\dot{a}}\eta^{\dot{a}}$. The remaining two diagrams are obtained by appropriately permuting the external labels. Using momentum conservation in the form of $\sum_{i=1}^{4}\langle yi\rangle[iz] = 0$ (where λ_y and $\tilde{\lambda}_z$ are arbitrary spinors), the final result can be arranged as

$$\mathcal{M}(1+, 2-, 3-, 4-) = \frac{\phi_1^6}{\phi_2^2\phi_3^2\phi_4^2}\Big(\langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle\Big)\frac{\langle 42\rangle}{[13]}. \tag{8.21}$$

This vanishes by virtue of the Schouten identity $\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0$ which is valid for any four spinors.

Moving now to $\mathcal{M}^{(5)}(1+, 2-, 3-, 4-, 5-)$ we need to consider graphs of the type given in fig. 8.4. The first diagram gives

$$\frac{\phi_1^6}{\phi_2^2\phi_3^2\phi_4^2\phi_5^2}\frac{\langle 12\rangle\langle 45\rangle\left(\langle 34\rangle\phi_4 + \langle 35\rangle\phi_5\right)^6}{[12][45]\left(\langle 13\rangle\phi_1 + \langle 23\rangle\phi_2\right)^4}, \tag{8.22}$$

where we have extended both λ_{pa} and λ_{qa} off-shell using the same spinor $\eta^{\dot{a}}$. This diagram yields 12 contributions once one takes into account all inequivalent exchanges

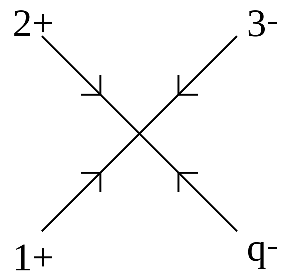


Figure 8.5: The $(+,+,-,-)$ graviton vertex. This vertex is non-holomorphic, for it depends not only on λ but also on $\tilde{\lambda}$.

of the negative helicity external gravitons. The second graph gives

$$\frac{\phi_1^6}{\phi_2^2\phi_3^2\phi_4^2\phi_5^2}\frac{\langle 23\rangle\langle 45\rangle(\langle 12\rangle\phi_2+\langle 13\rangle\phi_3)^4}{[23][45](\langle 14\rangle\phi_4+\langle 15\rangle\phi_5)^2}\,, \tag{8.23}$$

and two other terms obtained by permutations. Imposing momentum conservation, with some computer assistance one can verify that the sum of the 12 contributions coming from (8.22) and the 3 contributions coming from (8.23) vanishes as expected.

We stress here the holomorphicity of (8.19), which is the only vertex appearing in this kind of graphs. In this regard these computations do not differ from the gluon case, but in general non-vanishing graviton amplitudes are non-holomorphic, and it is therefore interesting to ask whether the MHV decomposition also holds in those cases.

8.3.2 The googly amplitude

We now come to the investigation of disconnected contribution to the amplitude $\mathcal{M}^{(5)}(1+,2+,3-,4-,5-)$. In the construction of the MHV graphs one also needs here the 4 graviton vertex depicted in fig. 8.5. The expression for the 4 graviton amplitude was first obtained in [246] and is given by

$$\mathcal{M}(1+,2+,3-,q-) = \frac{\langle 3q\rangle^8}{\langle 12\rangle\langle 13\rangle\langle 1q\rangle\langle 23\rangle\langle 2q\rangle\langle 3q\rangle}\frac{[3q]}{\langle 12\rangle}\,. \tag{8.24}$$

One immediately notices that this expression is not holomorphic and this is in strong contrast with the 3 graviton vertex (8.19) and all the gluon MHV vertices. Naively, an off-shell extension of (8.24) would require a redefinition of $\tilde{\lambda}^{\dot{a}}$ whenever it appears in an internal line. Hermiticity suggests to take the complex conjugate of (8.18) so to have

$$\tilde{\lambda}_{p\dot{a}} = \frac{p_{a\dot{a}}\xi^a}{\langle \lambda_p, \xi\rangle} \tag{8.25}$$

where $\xi = \eta^*$. Using this prescription one gets for the first graph in fig. 8.6

$$\frac{\phi_1^6}{\phi_3^2}\frac{\langle 13\rangle\langle 45\rangle^7[45]}{\langle 25\rangle\langle 24\rangle[13](\langle 25\rangle\phi_2 - \langle 54\rangle\phi_4)(\langle 24\rangle\phi_2 + \langle 54\rangle\phi_4)(\langle 25\rangle\phi_5 + \langle 24\rangle\phi_4)^2}\,, \tag{8.26}$$

and for the second graph

$$\frac{1}{\phi_3^2\phi_4^2(\phi_3\tilde{\phi}_3 + \phi_4\tilde{\phi}_4)}\frac{\langle 34\rangle(\langle 15\rangle\phi_1 + \langle 25\rangle\phi_2)^7([15]\tilde{\phi}_1 + [25]\tilde{\phi}_2)}{\langle 15\rangle\langle 25\rangle\langle 12\rangle^2[34](\langle 12\rangle\phi_2 + \langle 15\rangle\phi_5)(\langle 25\rangle\phi_5 - \langle 12\rangle\phi_1)}\,, \tag{8.27}$$

where $\tilde{\phi}_i = \lambda_{ia}\xi^a$. The factor $\phi_3\tilde{\phi}_3 + \phi_4\tilde{\phi}_4 = [\tilde{\lambda}_p, \eta]\langle\lambda_p, \xi\rangle$ comes from the normalization of (8.18) and (8.25) which does not cancel in this case. One can get all the other seven graphs by permutation of the external labels as usual. The expected result for this amplitude is given in (8.5), which some computer algebra showed not to match with the one following from (8.26) and (8.27). Moreover, the result depends on η. Therefore the prescription seems to fail in this case. One possible reason for this result might

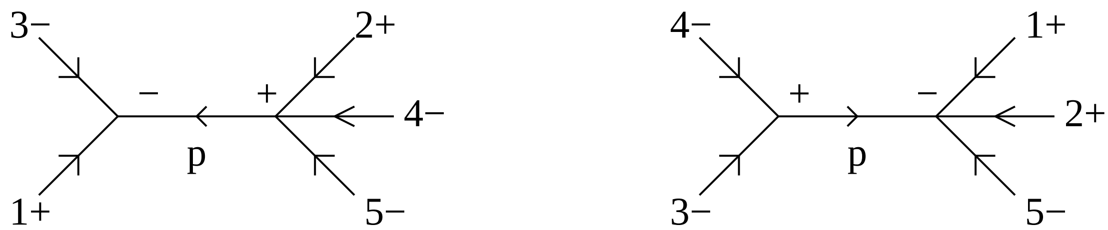


Figure 8.6: Two of the nine MHV graphs in the $\mathcal{M}^{(5)}(1+, 2+, 3-, 4-, 5-)$ graviton amplitude. The other seven diagrams are obtained permuting the labels on the external legs.

be that the heuristic proof of covariance given in [123] might not be generalizable in the presence of non-holomorphic vertices.

8.4 Conclusion

In this last chapter of the dissertation we have presented an attempt to extrapolate the twistor construction of [96] to ordinary gravity. We have checked that the simplest non-trivial gravity quantity, namely the five graviton googly amplitude, confirms the expectations of [96], and is indeed supported on a connected degree 2 curve in twistor space, just as the corresponding amplitude in the gauge theory.[4] There are however important differences between the two. In the simplest, MHV case, these stem from the fact that gravity amplitudes contain extra delta-function derivatives in twistor space variables, or equivalently they are not holomorphic in Minkowski space variables. It is clearly desirable to confirm that such behavior persists for further, non-MHV graviton amplitudes.

In a complementary approach to the computation presented in section 8.2, we

[4]The computation does not exclude additional contributions coming from disconnected, lower degree curves.

have further tried to calculate tree level graviton amplitudes by using MHV subamplitudes as vertices (computed from the gauge theory quantities via KLT relations, and suitably continuing them off-shell), in the spirit of the prescription given in [123] for gauge theories. Although it is possible that such a generalization might be feasible in principle, it is clear from our results that novel ingredients are necessary to correctly reproduce non-trivial gravity amplitudes.

We nevertheless find encouraging that two graviton amplitudes $\mathcal{M}^{(4)}(+,-,-,-)$ and $\mathcal{M}^{(5)}(+,-,-,-,-)$ vanish when computed from MHV vertices. We are aware that these are very special cases. Indeed, $\mathcal{M}^{(n)}(+,-,\dots,-)$ amplitudes involve only trivalent MHV vertices, which are holomorphic even in the graviton case. Unfortunately, the four-valent graviton MHV vertex is not holomorphic. We believe that this non-holomorphicity is an important reason for the failure of the MHV prescription to correctly reproduce the 5 graviton googly amplitude discussed in this note.

We must emphasize that the twistor string theory underlying an eventually successful version of such a construction might have nothing to do with the one of [96], or even there might be no such theory at all. Indeed, the closed string sector of the model of [96] is expected to be a kind of instanton expansion around $\mathcal{N}=4$ self-dual superconformal gravity. General Relativity is most definitely not conformally invariant, and therefore it should be related to a different model. The first computation presented in this chapter seems to suggest that there could be some localization in twistor space, and the disconnected prescription could provide an explicit and computable "instanton" expansion around some "self-dual" theory. In this respect, we think that the non-holomorphicity of higher MHV vertices could provide a hint about which could be the right theory to expand around.

Appendix A

The string Hagedorn temperature

In this appendix we recall the computation of the Hagedorn temperature in flat space both for the bosonic string and the superstring on the light-cone.[1]

One can compute the Hagedorn temperature from the entropy in the microcanonical ensemble

$$S(E) = \ln \Omega(E)\,, \tag{A.1}$$

where the Boltzmann constant k_B is set to 1, and $\Omega(E)$ is the number of microstates accessible to the system at energy E. Viewing the gas of strings as a collection of harmonic oscillators, the energy E is related to the number operator N of the oscillators, and $\Omega(E)$ is given by the partition of N, that is by the set of positive numbers that add up to N.

We start with the bosonic open string in the light-cone gauge. We then need to compute $p_{24}(N)$, the partition of N for oscillators that can vibrate in the 24 transverse light-cone directions. This is given, for large N, or, equivalently, for high energy E,

[1] The bosonic case is presented, for example, in chapter 16 of [248]. For a general review of string thermodynamics see [249][250].

by the approximate expression [248]

$$p_{24}(N) \simeq e^{4\pi\sqrt{N}}\,. \tag{A.2}$$

The bosonic open string spectrum of strings with no spatial momentum is given by the mass formula $\alpha'\, M^2 = N - 1 \simeq N$, from which follows $E \simeq \sqrt{N/\alpha'}$. Then one obtains from eq. (A.1)

$$S(E) = 4\pi\sqrt{\alpha'}E\,, \tag{A.3}$$

and the (inverse of the) Hagedorn temperature

$$\frac{1}{T_H} = \frac{\partial S}{\partial E} = 4\pi\sqrt{\alpha'}\,. \tag{A.4}$$

This result means that, in this high energy approximation, one can arbitrarily increase the energy of the gas of strings, while maintaining their temperature fixed and equal to the limiting temperature T_H.[2] For closed bosonic strings with no spatial momentum, upon enforcing the level matching condition $N = \tilde{N}$, one has $\alpha'\, M^2 = 4(N-1) \simeq 4N$, and $E \simeq 2\sqrt{N/\alpha'}$. Now the number of states is $\Omega(E) = p_{24}(N)\, p_{24}(\tilde{N})$, and thus the entropy and Hagedorn temperature are exactly as in eqs. (A.3) and (A.4).

For open superstrings in the light-cone, one needs the partition of N for oscillators with 8 bosonic and 8 fermionic modes

$$p_{8|8}(N) \simeq e^{2\pi\sqrt{2N}}\,. \tag{A.5}$$

[2]Sometimes the Hagedorn temperature can alternatively be interpreted as the critical temperature of a phase transition.

For large N, the energy is $E \simeq \sqrt{N/\alpha'}$ in both the NS and the R sector and one has

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}}\,, \tag{A.6}$$

so that the Hagedorn temperature for the superstring is a factor of $\sqrt{2}$ larger than in the bosonic case. The same is true for the closed superstring.

Appendix B

Alternative derivation of eq. (4.43)

The expression (4.43) for the difference $S_b(x) - S_f(x)$ can be obtained from a completely different procedure than the one in chapter 4, using the formula

$$S_b(x) - S_f(x) = -\frac{d}{dx}\left(x^2 \int_0^{\pi/x} dt' \int_0^{t'} dt \sum_{p=1}^{\infty} K_0\left(xp\right)\cos pxt\right) . \tag{B.1}$$

In virtue of the fact that [179]

$$\begin{aligned}\sum_{p=1}^{\infty} K_0\left(xp\right)\cos pxt = \frac{1}{2}\left(\gamma + \ln\frac{x}{4\pi}\right) + \frac{\pi}{2x\sqrt{1+t^2}} + \\ + \frac{\pi}{2}\sum_{l=1}^{\infty}\left(\frac{1}{\sqrt{x^2+(2l\pi - tx)^2}} - \frac{1}{2l\pi}\right) + \frac{\pi}{2}\sum_{l=1}^{\infty}\left(\frac{1}{\sqrt{x^2+(2l\pi + tx)^2}} - \frac{1}{2l\pi}\right) ,\end{aligned}$$

eq. (B.1) becomes

$$\begin{aligned}S_b(x) - S_f(x) = -\frac{\pi^2}{4x} - \frac{\pi}{2} + \frac{\pi}{2}\left(\frac{\sqrt{x^2+\pi^2}}{x} + \frac{x}{\pi + \sqrt{x^2+\pi^2}}\right) - \\ -\pi x \sum_{l=1}^{\infty}\left(\frac{1}{2l\pi + \sqrt{x^2+(2l\pi)^2}} - \frac{1}{(2l+1)\pi + \sqrt{x^2+(2l+1)^2\pi^2}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi^2}{4x} + \frac{\pi}{2} - \pi x \sum_{l=0}^{\infty} \left(\frac{1}{2l\pi + \sqrt{x^2 + (2l\pi)^2}} - \frac{1}{(2l+1)\pi + \sqrt{x^2 + (2l+1)^2\pi^2}} \right) \\
&= \frac{\pi^2}{4x} - \frac{\pi}{2} - \pi x \sum_{k=1}^{\infty} \frac{(-1)^k}{\pi k + \sqrt{x^2 + \pi^2 k^2}} \,. \qquad \text{(B.2)}
\end{aligned}
$$

Expanding for small values of x, it is easy to prove that eq. (B.2) becomes precisely eq. (4.43)

$$
S_b(x) - S_f(x) = \frac{\pi^2}{4x} - \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\sqrt{\pi}} \Gamma\left(k - \frac{1}{2}\right) \left(\frac{x}{\pi}\right)^{2k-1} T_{2k-1} \,, \qquad \text{(B.3)}
$$

where $T_s = (2^{1-s} - 1)\,\zeta(s)$ and $T_1 = -\ln 2$ [251].

Appendix C

Spherical harmonics and orthogonal polynomials

In this appendix we collect some facts about spherical harmonics and orthogonal polynomials that we have used in the computation of the correlators between giant Wilson loops and chiral primaries in section 6.2. We follow the treatments of [252][253].

Spherical harmonics in d dimensions are eigenfunctions of the Laplacian on the unit d-sphere

$$\nabla^2_{(d)} Y^I(\Omega) = \lambda Y^I(\Omega)\,, \tag{C.1}$$

where the Laplacian is

$$\nabla^2_{(d)} = \frac{1}{\sqrt{\det g}}\,\partial_i\,\sqrt{\det g}\,g^{ij}\,\partial_j\,, \tag{C.2}$$

with the metric given by $g_{ij} = \mathrm{diag}(1, \sin^2\theta_d(1, \sin^2\theta_{d-1}(\ldots)))$. The integer multi-index $I = (l_d, \dots, l_1)$ satisfies

$$l_d \geq l_{d-1} \geq \cdots \geq l_2 \geq |l_1| \,. \tag{C.3}$$

The general solution to eq. (C.1) is

$$Y^{l_d,\dots,l_1}(\theta_d, \dots, \theta_1) = \frac{e^{i\,l_1\theta_1}}{\sqrt{2\pi}} \prod_{n=2}^{d} {}_n\bar{P}_{l_n}^{\,l_{n-1}}(\theta_n) \,, \tag{C.4}$$

where we defined

$${}_n\bar{P}_L^{\,l}(\theta) = {}_nc_L^{\,l} (\sin\theta)^{-(n-2)/2} P_{L+(n-2)/2}^{-(l+(n-2)/2)}(\cos\theta) \,. \tag{C.5}$$

In this expression $P_n^{-m}(x)$ is the Legendre function of the first kind and the constant

$${}_nc_L^{\,l} = \left[\frac{(2L+n-1)(L+l+n-2)!}{2(L-l)!}\right]^{1/2} \tag{C.6}$$

is chosen to ensure the orthonormalization condition

$$\int \mu(\Omega_d)\; Y^I Y^{I'} = \delta^{II'} \,, \tag{C.7}$$

where $\mu(\Omega_d)$ is the measure over S^d. The integration over S^{d-1} selects only $SO(d)$ invariant harmonics

$$\int \mu(\Omega_{d-1}) \sum_I Y^I = \sum_{l_d} Y^{l_d,\,0,\dots,0} \,. \tag{C.8}$$

The eigenvalue λ depends only on $l_d \equiv \Delta$ because of the $O(d+1)$ symmetry of the problem and it can be found by studying the action of the Laplacian on $SO(d-1)$

invariant spherical harmonics

$$\left(\frac{1}{\sin^{d-1}\theta_d}\frac{\partial}{\partial\theta_d}\,\sin^{d-1}\theta_d\,\frac{\partial}{\partial\theta_d}\right)Y^{\Delta,0}(\Omega)=\lambda_\Delta Y^{\Delta,0}(\Omega)\,. \tag{C.9}$$

After the change of variable $x = \cos\theta_d$, this is recognized to be the Gegenbauer equation

$$\left((1-x^2)\frac{\partial^2}{\partial x^2}-d\,x\frac{\partial}{\partial x}\right)Y^{\Delta,0}(x)=\lambda_\Delta Y^{\Delta,0}(x)\,. \tag{C.10}$$

The solution to this equation is

$$\lambda_\Delta=-\Delta(\Delta+d-1)\,,\qquad Y^{\Delta,0}(x)=\mathcal{N}_\Delta\,C_\Delta^{\left(\frac{d-1}{2}\right)}(x)\,, \tag{C.11}$$

where $C_\Delta^{\left(\frac{d-1}{2}\right)}$ are Gegenbauer polynomials and the constant $\mathcal{N}_\Delta$ can be obtained from the orthonormality of the $Y^{\Delta,0}$'s

$$\mathcal{N}_\Delta=\left[\frac{\Delta!(2\Delta+d-1)\left[\Gamma\left(\frac{d-1}{2}\right)\right]^2\Gamma\left(\frac{d}{2}\right)}{2^{4-d}\pi^{\frac{d+2}{2}}\Gamma(\Delta+d-1)}\right]^{1/2}\,. \tag{C.12}$$

The Gegenbauer polynomials $C_\Delta^{(\lambda)}(x)$ are a generalization of the Legendre polynomials and can be obtained from the following generating function

$$\frac{1}{(1-2xt+t^2)^\lambda}=\sum_{\Delta=0}^{\infty}C_\Delta^{(\lambda)}(x)\,t^\Delta. \tag{C.13}$$

We list the first few of them

$$C_0^{(\lambda)}(x)=1\,,$$

$$
\begin{aligned}
C_1^{(\lambda)}(x) &= 2\lambda x\,, \\
C_2^{(\lambda)}(x) &= -\lambda + 2\lambda(1+\lambda)x^2\,, \\
C_3^{(\lambda)}(x) &= -2\lambda(1+\lambda)x + \frac{4}{3}\lambda(1+\lambda)(2+\lambda)x^3\,.
\end{aligned}
\tag{C.14}
$$

They satisfy the normalization condition

$$
\int_{-1}^{1} dx(1-x^2)^{\lambda-1/2}\left[C_\Delta^{(\lambda)}\right]^2 = 2^{1-2\lambda}\pi\frac{\Gamma(\Delta+2\lambda)}{(\Delta+\lambda)\Gamma^2(\lambda)\Gamma(\Delta+1)}
\tag{C.15}
$$

for $\lambda > -1/2$.

We have used the following formula for the derivative of a Gegenbauer polynomial

$$
(1-x^2)\partial_x C_\Delta^{(\lambda)}(x) = -\Delta\, x\, C_\Delta^{(\lambda)}(x) + (\Delta+2\lambda-1)C_{\Delta-1}^{(\lambda)}(x)\,,
\tag{C.16}
$$

and the following recurrence relation

$$
\Delta\, C_\Delta^{(\lambda)}(x) = 2(\Delta+\lambda-1)x\, C_{\Delta-1}^{(\lambda)}(x) - (\Delta+2\lambda-2)C_{\Delta-2}^{(\lambda)}(x)\,.
\tag{C.17}
$$

We have also used that

$$
C_\Delta^{\left(\frac{d-1}{2}\right)}(1) = \frac{(\Delta+d-2)!}{\Delta!(d-2)!}\,.
\tag{C.18}
$$

Bibliography

[1] G. Grignani, M. Orselli, G. W. Semenoff and D. Trancanelli, "The superstring Hagedorn temperature in a pp-wave background," JHEP **0306**, 006 (2003) [arXiv:hep-th/0301186].

[2] S. Giombi, M. Kulaxizi, R. Ricci and D. Trancanelli, "Half-BPS geometries and thermodynamics of free fermions," JHEP **0701**, 067 (2007) [arXiv:hep-th/0512101].

[3] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, "More supersymmetric Wilson loops," arXiv:0704.2237 [hep-th].

[4] S. Giombi, R. Ricci and D. Trancanelli, "Operator product expansion of higher rank Wilson loops from D-branes and matrix models," JHEP **0610**, 045 (2006) [arXiv:hep-th/0608077].

[5] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, "On the D3-brane description of some 1/4 BPS Wilson loops," JHEP **0704**, 008 (2007) [arXiv:hep-th/0612168].

[6] S. Giombi, M. Kulaxizi, R. Ricci, D. Robles-Llana, D. Trancanelli and K. Zoubos, "Orbifolding the twistor string," Nucl. Phys. B **719**, 234 (2005) [arXiv:hep-th/0411171].

[7] S. Giombi, R. Ricci, D. Robles-Llana and D. Trancanelli, "A note on twistor gravity amplitudes," JHEP **0407**, 059 (2004) [arXiv:hep-th/0405086].

[8] S. Giombi, R. Ricci, D. Robles-Llana and D. Trancanelli, "Instantons and matter in N = 1/2 supersymmetric gauge theory," JHEP **0510**, 021 (2005) [arXiv:hep-th/0505077].

[9] A. M. Polyakov, "Gauge Fields And Strings," Harwood, 1990.

[10] G. 't Hooft, "A planar diagram theory for strong interactions," Nucl. Phys. B **72**, 461 (1974).

[11] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[13] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[14] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[15] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS/CFT," Int. J. Mod. Phys. A **14**, 3597 (1999) [arXiv:hep-th/9902131].

[16] I. R. Klebanov, "TASI lectures: Introduction to the AdS/CFT correspondence," arXiv:hep-th/0009139.

[17] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," arXiv:hep-th/0201253.

[18] J. M. Maldacena, "Lectures on AdS/CFT," arXiv:hep-th/0309246.

[19] S. Ferrara and B. Zumino, "Supergauge Invariant Yang-Mills Theories," Nucl. Phys. B **79**, 413 (1974).

[20] D. R. T. Jones, "Charge Renormalization In A Supersymmetric Yang-Mills Theory," Phys. Lett. B **72**, 199 (1977).

[21] M. T. Grisaru, M. Rocek and W. Siegel, "Zero Three Loop Beta Function In N=4 Superyang-Mills Theory," Phys. Rev. Lett. **45**, 1063 (1980).

[22] M. F. Sohnius and P. C. West, "Conformal Invariance In N=4 Supersymmetric Yang-Mills Theory," Phys. Lett. B **100**, 245 (1981).

[23] M. T. Grisaru and W. Siegel, "Supergraphity. 2. Manifestly Covariant Rules And Higher Loop Finiteness," Nucl. Phys. B **201**, 292 (1982) [Erratum-ibid. B **206**, 496 (1982)].

[24] L. Brink, O. Lindgren and B. E. W. Nilsson, "N=4 Yang-Mills Theory On The Light Cone," Nucl. Phys. B **212**, 401 (1983); "The Ultraviolet Finiteness Of The N=4 Yang-Mills Theory," Phys. Lett. B **123**, 323 (1983).

[25] E. Witten, "Bound states of strings and p-branes," Nucl. Phys. B **460**, 335 (1996) [arXiv:hep-th/9510135].

[26] S. M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, "Three-point functions of chiral operators in D = 4, N = 4 SYM at large N," Adv. Theor. Math. Phys. **2**, 697 (1998) [arXiv:hep-th/9806074].

[27] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, "Correlation functions in the CFT(d)/AdS($d+1$) correspondence," Nucl. Phys. B **546**, 96 (1999) [arXiv:hep-th/9804058].

[28] G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, "R-current correlators in N = 4 super Yang-Mills theory from anti-de Sitter supergravity," Nucl. Phys. B **540**, 247 (1999) [arXiv:hep-th/9805105].

[29] J. K. Erickson, G. W. Semenoff and K. Zarembo, "Wilson loops in N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **582**, 155 (2000) [arXiv:hep-th/0003055].

[30] N. Drukker and D. J. Gross, "An Exact Prediction Of N = 4 Susym Theory For String Theory," J. Math. Phys. **42**, 2896 (2001) [arXiv:hep-th/0010274].

[31] G. W. Semenoff and K. Zarembo, "More exact predictions of SUSYM for string theory," Nucl. Phys. B **616**, 34 (2001) [arXiv:hep-th/0106015].

[32] V. K. Dobrev and V. B. Petkova, "All Positive Energy Unitary Irreducible Representations Of Extended Conformal Supersymmetry," Phys. Lett. B **162**, 127 (1985).

[33] S. Ferrara, "Superspace representations of SU(2,2/N) superalgebras and multiplet shortening," arXiv:hep-th/0002141.

[34] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, "The Mass Spectrum Of Chiral N=2 D = 10 Supergravity On S**5," Phys. Rev. D **32**, 389 (1985).

[35] D. Berenstein, "A toy model for the AdS/CFT correspondence," JHEP **0407**, 018 (2004) [arXiv:hep-th/0403110].

[36] V. Balasubramanian, M. Berkooz, A. Naqvi and M. J. Strassler, "Giant gravitons in conformal field theory," JHEP **0204**, 034 (2002) [arXiv:hep-th/0107119].

[37] S. Corley, A. Jevicki and S. Ramgoolam, "Exact correlators of giant gravitons from dual N = 4 SYM theory," Adv. Theor. Math. Phys. **5**, 809 (2002) [arXiv:hep-th/0111222].

[38] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[39] R. Gueven, "Plane wave limits and T-duality," Phys. Lett. B **482**, 255 (2000) [arXiv:hep-th/0005061].

[40] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "Penrose limits and maximal supersymmetry," Class. Quant. Grav. **19**, L87 (2002) [arXiv:hep-th/0201081].

[41] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, "Penrose limits, supergravity and brane dynamics," Class. Quant. Grav. **19**, 4753 (2002) [arXiv:hep-th/0202111].

[42] M. Blau and M. O'Loughlin, "Homogeneous plane waves," Nucl. Phys. B **654**, 135 (2003) [arXiv:hep-th/0212135].

[43] M. Blau, M. O'Loughlin, G. Papadopoulos and A. A. Tseytlin, "Solvable models of strings in homogeneous plane wave backgrounds," Nucl. Phys. B **673**, 57 (2003) [arXiv:hep-th/0304198].

[44] R. R. Metsaev, "Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background," Nucl. Phys. B **625**, 70 (2002) [arXiv:hep-th/0112044].

[45] R. R. Metsaev and A. A. Tseytlin, "Exactly solvable model of superstring in plane wave Ramond-Ramond background," Phys. Rev. D **65**, 126004 (2002) [arXiv:hep-th/0202109].

[46] D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].

[47] D. Sadri and M. M. Sheikh-Jabbari, "The plane-wave / super Yang-Mills duality," Rev. Mod. Phys. **76**, 853 (2004) [arXiv:hep-th/0310119].

[48] G. T. Horowitz and A. R. Steif, "Space-Time Singularities in String Theory," Phys. Rev. Lett. **64**, 260 (1990).

[49] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, "A new double-scaling limit of N = 4 super Yang-Mills theory and PP-wave strings," Nucl. Phys. B **643**, 3 (2002) [arXiv:hep-th/0205033].

[50] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, "PP-wave string interactions from perturbative Yang-Mills theory," JHEP **0207**, 017 (2002) [arXiv:hep-th/0205089].

[51] D. J. Gross, A. Mikhailov and R. Roiban, "Operators with large R charge in N = 4 Yang-Mills theory," Annals Phys. **301**, 31 (2002) [arXiv:hep-th/0205066].

[52] A. Santambrogio and D. Zanon, "Exact anomalous dimensions of N = 4 Yang-Mills operators with large R charge," Phys. Lett. B **545**, 425 (2002) [arXiv:hep-th/0206079].

[53] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," JHEP **0303**, 013 (2003) [arXiv:hep-th/0212208].

[54] M. Spradlin and A. Volovich, "Superstring interactions in a pp-wave background," Phys. Rev. D **66**, 086004 (2002) [arXiv:hep-th/0204146]; "Superstring interactions in a pp-wave background. II," JHEP **0301**, 036 (2003) [arXiv:hep-th/0206073].

[55] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, "BMN correlators and operator mixing in N = 4 super Yang-Mills theory," Nucl. Phys. B **650**, 125 (2003) [arXiv:hep-th/0208178].

[56] D. J. Gross, A. Mikhailov and R. Roiban, "A calculation of the plane wave string Hamiltonian from N = 4 super-Yang-Mills theory," JHEP **0305**, 025 (2003) [arXiv:hep-th/0208231].

[57] N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, "Operator mixing and the BMN correspondence," JHEP **0210**, 068 (2002) [arXiv:hep-th/0209002].

[58] N. Beisert, "BMN operators and superconformal symmetry," Nucl. Phys. B **659**, 79 (2003) [arXiv:hep-th/0211032].

[59] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, "BMN gauge theory as a quantum mechanical system," Phys. Lett. B **558**, 229 (2003) [arXiv:hep-th/0212269].

[60] J. Pearson, M. Spradlin, D. Vaman, H. L. Verlinde and A. Volovich, "Tracing the string: BMN correspondence at finite J**2/N," JHEP **0305**, 022 (2003) [arXiv:hep-th/0210102].

[61] J. Gomis, S. Moriyama and J. w. Park, "SYM description of SFT Hamiltonian in a pp-wave background," Nucl. Phys. B **659**, 179 (2003) [arXiv:hep-th/0210153].

[62] A. Pankiewicz and B. J. Stefanski, "pp-wave light-cone superstring field theory," Nucl. Phys. B **657**, 79 (2003) [arXiv:hep-th/0210246].

[63] Y. H. He, J. H. Schwarz, M. Spradlin and A. Volovich, "Explicit formulas for Neumann coefficients in the plane-wave geometry," Phys. Rev. D **67**, 086005 (2003) [arXiv:hep-th/0211198].

[64] C. S. Chu and V. V. Khoze, "Correspondence between the 3-point BMN correlators and the 3-string vertex on the pp-wave," JHEP **0304**, 014 (2003) [arXiv:hep-th/0301036].

[65] R. Roiban, M. Spradlin and A. Volovich, "On light-cone SFT contact terms in a plane wave," JHEP **0310**, 055 (2003) [arXiv:hep-th/0211220].

[66] P. Deligne *et al.*, "Quantum fields and strings: A course for mathematicians. Vol. 1, 2," *Providence, USA: AMS (1999) 1-1501*

[67] E. Witten, "Quantum field theory and the Jones polynomial," Commun. Math. Phys. **121**, 351 (1989).

[68] K. Zarembo, "Supersymmetric Wilson loops," Nucl. Phys. B **643**, 157 (2002) [arXiv:hep-th/0205160].

[69] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "A semi-classical limit of the gauge/string correspondence," Nucl. Phys. B **636**, 99 (2002) [arXiv:hep-th/0204051].

[70] M. Kruczenski, "A note on twist two operators in N = 4 SYM and Wilson loops in Minkowski signature," JHEP **0212**, 024 (2002) [arXiv:hep-th/0210115]; "Wilson loops and anomalous dimensions in cascading theories," Phys. Rev. D **69**, 106002 (2004) [arXiv:hep-th/0310030].

[71] Y. Makeenko, "Light-cone Wilson loops and the string / gauge correspondence," JHEP **0301**, 007 (2003) [arXiv:hep-th/0210256]. Y. Makeenko, P. Olesen and G. W. Semenoff, "Cusped SYM Wilson loop at two loops and beyond," Nucl. Phys. B **748**, 170 (2006) [arXiv:hep-th/0602100].

[72] A. V. Belitsky, A. S. Gorsky and G. P. Korchemsky, "Gauge / string duality for QCD conformal operators," Nucl. Phys. B **667**, 3 (2003) [arXiv:hep-th/0304028].

[73] N. Drukker, D. J. Gross and H. Ooguri, "Wilson loops and minimal surfaces," Phys. Rev. D **60**, 125006 (1999) [arXiv:hep-th/9904191].

[74] M. Bianchi, M. B. Green and S. Kovacs, "Instantons and BPS Wilson loops," arXiv:hep-th/0107028; "Instanton corrections to circular Wilson loops in N = 4 supersymmetric Yang-Mills," JHEP **0204**, 040 (2002) [arXiv:hep-th/0202003].

[75] S. J. Rey and J. T. Yee, "Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity," Eur. Phys. J. C **22**, 379 (2001) [arXiv:hep-th/9803001].

[76] J. M. Maldacena, "Wilson loops in large N field theories," Phys. Rev. Lett. **80**, 4859 (1998) [arXiv:hep-th/9803002].

[77] G. W. Semenoff and K. Zarembo, "Wilson loops in SYM theory: From weak to strong coupling," Nucl. Phys. Proc. Suppl. **108**, 106 (2002) [arXiv:hep-th/0202156].

[78] N. Drukker, D. J. Gross and A. A. Tseytlin, "Green-Schwarz string in AdS(5) x S(5): Semiclassical partition function," JHEP **0004**, 021 (2000) [arXiv:hep-th/0001204].

[79] D. J. Gross and H. Ooguri, "Aspects of large N gauge theory dynamics as seen by string theory," Phys. Rev. D **58**, 106002 (1998) [arXiv:hep-th/9805129].

[80] N. Drukker and B. Fiol, "All-genus calculation of Wilson loops using D-branes," JHEP **0502**, 010 (2005) [arXiv:hep-th/0501109].

[81] S. Yamaguchi, "Wilson loops of anti-symmetric representation and D5-branes," JHEP **0605**, 037 (2006) [arXiv:hep-th/0603208].

[82] D. Rodriguez-Gomez, "Computing Wilson lines with dielectric branes," Nucl. Phys. B **752**, 316 (2006) [arXiv:hep-th/0604031].

[83] C. G. . Callan and J. M. Maldacena, "Brane dynamics from the Born-Infeld action," Nucl. Phys. B **513**, 198 (1998) [arXiv:hep-th/9708147].

[84] R. Emparan, "Born-Infeld strings tunneling to D-branes," Phys. Lett. B **423**, 71 (1998) [arXiv:hep-th/9711106].

[85] R. C. Myers, "Dielectric-branes," JHEP **9912**, 022 (1999) [arXiv:hep-th/9910053].

[86] S. A. Hartnoll and S. Prem Kumar, "Multiply wound Polyakov loops at strong coupling," Phys. Rev. D **74**, 026001 (2006) [arXiv:hep-th/0603190].

[87] S. A. Hartnoll, "Two universal results for Wilson loops at strong coupling," Phys. Rev. D **74**, 066006 (2006) [arXiv:hep-th/0606178].

[88] J. Gomis and F. Passerini, "Holographic Wilson loops," JHEP **0608**, 074 (2006) [arXiv:hep-th/0604007]; "Wilson loops as D3-branes," JHEP **0701**, 097 (2007) [arXiv:hep-th/0612022].

[89] K. Okuyama and G. W. Semenoff, "Wilson Loops In N = 4 Sym And Fermion Droplets," JHEP **0606**, 057 (2006) [arXiv:hep-th/0604209].

[90] S. A. Hartnoll and S. P. Kumar, "Higher rank Wilson loops from a matrix model," JHEP **0608**, 026 (2006) [arXiv:hep-th/0605027].

[91] J. McGreevy, L. Susskind and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," JHEP **0006**, 008 (2000) [hep-th/0003075].

[92] M. T. Grisaru, R. C. Myers and O. Tafjord, "SUSY and Goliath," JHEP **0008**, 040 (2000) [hep-th/0008015].

[93] A. Hashimoto, S. Hirano and N. Itzhaki, "Large branes in AdS and their field theory dual," JHEP **0008**, 051 (2000) [hep-th/0008016].

[94] S. Yamaguchi, "Bubbling geometries for half BPS Wilson lines," Int. J. Mod. Phys. A **22**, 1353 (2007) [arXiv:hep-th/0601089].

[95] O. Lunin, "On gravitational description of Wilson lines," JHEP **0606**, 026 (2006) [arXiv:hep-th/0604133].

[96] E. Witten, "Perturbative gauge theory as a string theory in twistor space," Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171].

[97] F. Cachazo and P. Svrcek, "Lectures on twistor strings and perturbative Yang-Mills theory," PoS **RTN2005**, 004 (2005) [arXiv:hep-th/0504194].

[98] V. V. Khoze, "Gauge theory amplitudes, scalar graphs and twistor space," arXiv:hep-th/0408233.

[99] F. A. Berends and W. Giele, "The Six Gluon Process As An Example Of Weyl-Van Der Waerden Spinor Calculus," Nucl. Phys. B **294**, 700 (1987).

[100] G. Chalmers and W. Siegel, "Simplifying algebra in Feynman graphs. I: Spinors," Phys. Rev. D **59**, 045012 (1999) [arXiv:hep-ph/9708251]; "Simplifying algebra in Feynman graphs. II: Spinor helicity from the spacecone," Phys. Rev. D **59**, 045013 (1999) [arXiv:hep-ph/9801220]; "Simplifying algebra in Feynman graphs. III: Massive vectors," Phys. Rev. D **63**, 125027 (2001) [arXiv:hep-th/0101025].

[101] L. J. Dixon, "Calculating scattering amplitudes efficiently," TASI Lectures 1995, arXiv:hep-ph/9601359.

[102] S. J. Parke and T. R. Taylor, "An Amplitude For N Gluon Scattering," Phys. Rev. Lett. **56**, 2459 (1986).

[103] F. A. Berends and W. T. Giele, "Recursive Calculations for Processes with n Gluons," Nucl. Phys. B **306**, 759 (1988).

[104] R. Penrose, "Twistor algebra," J. Math. Phys. **8**, 345 (1967).

[105] R. Penrose, "Twistor Quantization And Curved Space-Time," Int. J. Theor. Phys. **1**, 61 (1968).

[106] R. Penrose, "The Nonlinear Graviton," Gen. Rel. Grav. **7**, 171 (1976).

[107] R. Penrose, "The Central programme of twistor theory," Chaos Solitons Fractals **10**, 581 (1999).

[108] S. A. Huggett and K. P. Tod, "An Introduction To Twistor Theory," *Cambridge, Uk: Univ. Pr. (1985) 145 P. (London Mathematical Society Student Texts, 4)*

[109] R. Penrose, "The Road To Reality," *New York: Alfred A. Knopf (2005) 1099 P.*

[110] J. Harris, "Algebraic Geometry," *New York: Springer-Verlag New York, Inc. (1992) 328 P. (Graduate Texts in Mathematics)*

[111] V. P. Nair, "A current algebra for some gauge theory amplitudes," Phys. Lett. B **214**, 215 (1988).

[112] N. Berkovits, “An alternative string theory in twistor space for N = 4 super-Yang-Mills,” Phys. Rev. Lett. **93**, 011601 (2004) [arXiv:hep-th/0402045]; N. Berkovits and L. Motl, “Cubic twistorial string field theory,” JHEP **0404**, 056 (2004) [arXiv:hep-th/0403187].

[113] A. Neitzke and C. Vafa, “N = 2 strings and the twistorial Calabi-Yau,” arXiv:hep-th/0402128; N. Nekrasov, H. Ooguri and C. Vafa, “S-duality and topological strings,” JHEP **0410**, 009 (2004) [arXiv:hep-th/0403167]; M. Aganagic and C. Vafa, “Mirror symmetry and supermanifolds,” arXiv:hep-th/0403192.

[114] W. Siegel, “Untwisting the twistor superstring,” arXiv:hep-th/0404255.

[115] N. Berkovits and E. Witten, “Conformal supergravity in twistor-string theory,” JHEP **0408**, 009 (2004) [arXiv:hep-th/0406051].

[116] C. h. Ahn, “N = 1 conformal supergravity and twistor-string theory,” JHEP **0410**, 064 (2004) [arXiv:hep-th/0409195].

[117] E. Witten, “Mirror manifolds and topological field theory,” arXiv:hep-th/9112056.

[118] E. Witten, “Chern-Simons Gauge Theory As A String Theory,” Prog. Math. **133**, 637 (1995) [arXiv:hep-th/9207094].

[119] W. Siegel, “N=2, N=4 string theory is selfdual N=4 Yang-Mills theory,” Phys. Rev. D **46**, 3235 (1992); “Selfdual N=8 supergravity as closed N=2 (N=4) strings,” Phys. Rev. D **47**, 2504 (1993) [arXiv:hep-th/9207043].

[120] R. Roiban, M. Spradlin and A. Volovich, “A googly amplitude from the B-model in twistor space,” JHEP **0404**, 012 (2004) [arXiv:hep-th/0402016];

R. Roiban and A. Volovich, "All googly amplitudes from the B-model in twistor space," Phys. Rev. Lett. **93**, 131602 (2004) [arXiv:hep-th/0402121]; R. Roiban, M. Spradlin and A. Volovich, "On the tree-level S-matrix of Yang-Mills theory," Phys. Rev. D **70**, 026009 (2004) [arXiv:hep-th/0403190].

[121] R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, "Topological M-theory as unification of form theories of gravity," Adv. Theor. Math. Phys. **9**, 603 (2005) [arXiv:hep-th/0411073].

[122] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[123] F. Cachazo, P. Svrcek and E. Witten, "MHV vertices and tree amplitudes in gauge theory," JHEP **0409**, 006 (2004) [arXiv:hep-th/0403047].

[124] C. J. Zhu, "The googly amplitudes in gauge theory," JHEP **0404**, 032 (2004) [arXiv:hep-th/0403115].

[125] S. Gukov, L. Motl and A. Neitzke, "Equivalence of twistor prescriptions for super Yang-Mills," arXiv:hep-th/0404085.

[126] D. A. Kosower, "Light Cone Recurrence Relations for QCD Amplitudes," Nucl. Phys. B **335**, 23 (1990).

[127] E. Witten, "Parity invariance for strings in twistor space," Adv. Theor. Math. Phys. **8**, 779 (2004) [arXiv:hep-th/0403199].

[128] J. B. Wu and C. J. Zhu, "MHV vertices and scattering amplitudes in gauge theory," JHEP **0407**, 032 (2004) [arXiv:hep-th/0406085].

[129] I. Bena, Z. Bern and D. A. Kosower, "Twistor-space recursive formulation of gauge theory amplitudes," Phys. Rev. D **71**, 045008 (2005) [arXiv:hep-th/0406133].

[130] D. A. Kosower, "Next-to-maximal helicity violating amplitudes in gauge theory," Phys. Rev. D **71**, 045007 (2005) [arXiv:hep-th/0406175].

[131] Y. Abe, V. P. Nair and M. I. Park, "Multigluon amplitudes, N = 4 constraints and the WZW model," Phys. Rev. D **71**, 025002 (2005) [arXiv:hep-th/0408191].

[132] F. Cachazo, P. Svrcek and E. Witten, "Gauge theory amplitudes in twistor space and holomorphic anomaly," JHEP **0410**, 077 (2004) [arXiv:hep-th/0409245].

[133] R. Britto, F. Cachazo and B. Feng, "New recursion relations for tree amplitudes of gluons," Nucl. Phys. B **715**, 499 (2005) [arXiv:hep-th/0412308]; R. Britto, F. Cachazo, B. Feng and E. Witten, "Direct proof of tree-level recursion relation in Yang-Mills theory," Phys. Rev. Lett. **94**, 181602 (2005) [arXiv:hep-th/0501052].

[134] T. G. Birthwright, E. W. N. Glover, V. V. Khoze and P. Marquard, "Multi-gluon collinear limits from MHV diagrams," JHEP **0505**, 013 (2005) [arXiv:hep-ph/0503063]; "Collinear limits in QCD from MHV rules," JHEP **0507**, 068 (2005) [arXiv:hep-ph/0505219].

[135] M. x. Luo and C. k. Wen, "Recursion relations for tree amplitudes in super gauge theories," JHEP **0503**, 004 (2005) [arXiv:hep-th/0501121].

[136] G. Georgiou and V. V. Khoze, "Tree amplitudes in gauge theory as scalar MHV diagrams," JHEP **0405**, 070 (2004) [arXiv:hep-th/0404072].

[137] G. Georgiou, E. W. N. Glover and V. V. Khoze, "Non-MHV tree amplitudes in gauge theory," JHEP **0407**, 048 (2004) [arXiv:hep-th/0407027].

[138] J. B. Wu and C. J. Zhu, "MHV vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos," JHEP **0409**, 063 (2004) [arXiv:hep-th/0406146]; X. Su and J. B. Wu, "Six-quark amplitudes from fermionic MHV vertices," Mod. Phys. Lett. A **20**, 1065 (2005) [arXiv:hep-th/0409228].

[139] L. J. Dixon, E. W. N. Glover and V. V. Khoze, "MHV rules for Higgs plus multi-gluon amplitudes," JHEP **0412**, 015 (2004) [arXiv:hep-th/0411092]; S. D. Badger, E. W. N. Glover and V. V. Khoze, "MHV rules for Higgs plus multi-parton amplitudes," JHEP **0503**, 023 (2005) [arXiv:hep-th/0412275].

[140] Z. Bern, D. Forde, D. A. Kosower and P. Mastrolia, "Twistor-inspired construction of electroweak vector boson currents," Phys. Rev. D **72**, 025006 (2005) [arXiv:hep-ph/0412167].

[141] S. D. Badger, E. W. N. Glover, V. V. Khoze and P. Svrcek, "Recursion relations for gauge theory amplitudes with massive particles," JHEP **0507**, 025 (2005) [arXiv:hep-th/0504159]; S. D. Badger, E. W. N. Glover and V. V. Khoze, "Recursion relations for gauge theory amplitudes with massive vector bosons and fermions," JHEP **0601**, 066 (2006) [arXiv:hep-th/0507161].

[142] F. Cachazo, P. Svrcek and E. Witten, "Twistor space structure of one-loop amplitudes in gauge theory," JHEP **0410**, 074 (2004) [arXiv:hep-th/0406177].

[143] A. Brandhuber, B. J. Spence and G. Travaglini, "One-loop gauge theory amplitudes in N = 4 super Yang-Mills from MHV vertices," Nucl. Phys. B **706**, 150 (2005) [arXiv:hep-th/0407214].

[144] M. x. Luo and C. k. Wen, "One-loop maximal helicity violating amplitudes in N = 4 super Yang-Mills theories," JHEP **0411**, 004 (2004) [arXiv:hep-th/0410045]; "Systematics of one-loop scattering amplitudes in N = 4 super Yang-Mills theories," Phys. Lett. B **609**, 86 (2005) [arXiv:hep-th/0410118].

[145] I. Bena, Z. Bern, D. A. Kosower and R. Roiban, "Loops in twistor space," Phys. Rev. D **71**, 106010 (2005) [arXiv:hep-th/0410054].

[146] F. Cachazo, "Holomorphic anomaly of unitarity cuts and one-loop gauge theory amplitudes," arXiv:hep-th/0410077; R. Britto, F. Cachazo and B. Feng, "Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts," Phys. Rev. D **71**, 025012 (2005) [arXiv:hep-th/0410179]; "Coplanarity in twistor space of N = 4 next-to-MHV one-loop amplitude coefficients," Phys. Lett. B **611**, 167 (2005) [arXiv:hep-th/0411107]; "Generalized unitarity and one-loop amplitudes in N = 4 super-Yang-Mills," Nucl. Phys. B **725**, 275 (2005) [arXiv:hep-th/0412103].

[147] Z. Bern, V. Del Duca, L. J. Dixon and D. A. Kosower, "All non-maximally-helicity-violating one-loop seven-gluon amplitudes in N = 4 super-Yang-Mills theory," Phys. Rev. D **71**, 045006 (2005) [arXiv:hep-th/0410224]; Z. Bern, L. J. Dixon and D. A. Kosower, "All next-to-maximally helicity-violating one-loop gluon amplitudes in N = 4 super-Yang-Mills theory," Phys. Rev. D **72**, 045014 (2005) [arXiv:hep-th/0412210].

[148] V. P. Nair, "A note on MHV amplitudes for gravitons," Phys. Rev. D **71**, 121701 (2005) [arXiv:hep-th/0501143].

[149] J. Bedford, A. Brandhuber, B. J. Spence and G. Travaglini, “A recursion relation for gravity amplitudes,” Nucl. Phys. B **721**, 98 (2005) [arXiv:hep-th/0502146].

[150] F. Cachazo and P. Svrcek, “Tree level recursion relations in general relativity,” arXiv:hep-th/0502160.

[151] Y. Abe, “An interpretation of multigraviton amplitudes,” Phys. Lett. B **623**, 126 (2005) [arXiv:hep-th/0504174].

[152] R. Hagedorn, “Statistical Thermodynamics Of Strong Interactions At High-Energies,” Nuovo Cim. Suppl. **3**, 147 (1965).

[153] K. Huang and S. Weinberg, “Ultimate Temperature And The Early Universe,” Phys. Rev. Lett. **25**, 895 (1970).

[154] L. A. Pando Zayas and D. Vaman, “Strings in RR plane wave background at finite temperature,” Phys. Rev. D **67**, 106006 (2003) [arXiv:hep-th/0208066].

[155] B. R. Greene, K. Schalm and G. Shiu, “On the Hagedorn behaviour of pp-wave strings and N = 4 SYM theory at finite R-charge density,” Nucl. Phys. B **652**, 105 (2003) [arXiv:hep-th/0208163].

[156] Y. Sugawara, “Thermal amplitudes in DLCQ superstrings on pp-waves,” Nucl. Phys. B **650**, 75 (2003) [arXiv:hep-th/0209145].

[157] R. C. Brower, D. A. Lowe and C. I. Tan, “Hagedorn transition for strings on pp-waves and tori with chemical potentials,” Nucl. Phys. B **652**, 127 (2003) [arXiv:hep-th/0211201].

[158] Y. Sugawara, "Thermal partition function of superstring on compactified pp-wave," Nucl. Phys. B **661**, 191 (2003) [arXiv:hep-th/0301035].

[159] J. J. Atick and E. Witten, "The Hagedorn Transition And The Number Of Degrees Of Freedom Of String Theory," Nucl. Phys. B **310**, 291 (1988).

[160] A. M. Polyakov, "Thermal Properties Of Gauge Fields And Quark Liberation," Phys. Lett. B **72**, 477 (1978).

[161] L. Susskind, "Lattice Models Of Quark Confinement At High Temperature," Phys. Rev. D **20**, 2610 (1979).

[162] B. Svetitsky and L. G. Yaffe, "Critical Behavior At Finite Temperature Confinement Transitions," Nucl. Phys. B **210**, 423 (1982).

[163] G. Grignani, G. W. Semenoff and P. Sodano, "Confinement - deconfinement transition in three-dimensional QED," Phys. Rev. D **53**, 7157 (1996) [arXiv:hep-th/9504105].

[164] G. Grignani, G. W. Semenoff, P. Sodano and O. Tirkkonen, "Charge Screening in the Finite Temperature Schwinger Model," Int. J. Mod. Phys. A **11**, 4103 (1996) [arXiv:hep-th/9511110].

[165] G. Grignani, G. W. Semenoff, P. Sodano and O. Tirkkonen, "Charge Screening and Confinement in the Hot 3D-QED," Nucl. Phys. B **473**, 143 (1996) [arXiv:hep-th/9512048].

[166] G. W. Semenoff, O. Tirkkonen and K. Zarembo, "Exact solution of the one-dimensional non-Abelian Coulomb gas at large N," Phys. Rev. Lett. **77**, 2174 (1996) [arXiv:hep-th/9605172].

[167] G. W. Semenoff and K. Zarembo, "Adjoint non-Abelian Coulomb gas at large N," Nucl. Phys. B **480**, 317 (1996) [arXiv:hep-th/9606117].

[168] D. J. Gross and E. Witten, "Possible Third Order Phase Transition In The Large N Lattice Gauge Theory," Phys. Rev. D **21**, 446 (1980).

[169] A. M. Polyakov, "Gauge fields and space-time," Int. J. Mod. Phys. A **17S1**, 119 (2002) [arXiv:hep-th/0110196].

[170] S. W. Hawking and D. N. Page, "Thermodynamics Of Black Holes In Anti-De Sitter Space," Commun. Math. Phys. **87**, 577 (1983).

[171] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].

[172] G. Grignani, M. Orselli and G. W. Semenoff, "Matrix strings in a B-field," JHEP **0107**, 004 (2001) [arXiv:hep-th/0104112].

[173] G. Grignani, M. Orselli and G. W. Semenoff, "The target space dependence of the Hagedorn temperature," JHEP **0111**, 058 (2001) [arXiv:hep-th/0110152].

[174] G. De Risi, G. Grignani and M. Orselli, "Space / time noncommutativity in string theories without background electric field," JHEP **0212**, 031 (2002) [arXiv:hep-th/0211056].

[175] J. G. Russo and A. A. Tseytlin, "On solvable models of type IIB superstring in NS-NS and R-R plane wave backgrounds," JHEP **0204**, 021 (2002) [arXiv:hep-th/0202179].

[176] G. Grignani, P. Orland, L. D. Paniak and G. W. Semenoff, "Matrix theory interpretation of DLCQ string worldsheets," Phys. Rev. Lett. **85**, 3343 (2000) [arXiv:hep-th/0004194].

[177] G. Grignani and G. W. Semenoff, "Thermodynamic partition function of matrix superstrings," Nucl. Phys. B **561**, 243 (1999) [arXiv:hep-th/9903246].

[178] T. Takayanagi, "Modular invariance of strings on pp-waves with RR-flux," JHEP **0212**, 022 (2002) [arXiv:hep-th/0206010].

[179] I. S. Gradstheyn and I. M. Ryzhik, "Table of Integrals, Series and Products," *New York: Academic Press (2000)*

[180] C. R. Nappi and E. Witten, "A WZW model based on a nonsemisimple group," Phys. Rev. Lett. **71**, 3751 (1993) [arXiv:hep-th/9310112].

[181] I. R. Klebanov, "String theory in two-dimensions," arXiv:hep-th/9108019.

[182] J. McGreevy and H. L. Verlinde, "Strings from tachyons: The c = 1 matrix reloaded," JHEP **0312**, 054 (2003) [arXiv:hep-th/0304224].

[183] Y. Nakayama, "Liouville field theory: A decade after the revolution," Int. J. Mod. Phys. A **19**, 2771 (2004) [arXiv:hep-th/0402009].

[184] H. Lin, O. Lunin and J. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," JHEP **0410**, 025 (2004) [arXiv:hep-th/0409174].

[185] D. Martelli and J. F. Morales, "Bubbling AdS(3)," JHEP **0502**, 048 (2005) [arXiv:hep-th/0412136].

[186] G. Mandal, "Fermions from half-BPS supergravity," JHEP **0508**, 052 (2005) [arXiv:hep-th/0502104].

[187] P. Horava and P. G. Shepard, "Topology changing transitions in bubbling geometries," JHEP **0502**, 063 (2005) [arXiv:hep-th/0502127].

[188] Y. Takayama and K. Yoshida, "Bubbling 1/2 BPS geometries and Penrose limits," Phys. Rev. D **72**, 066014 (2005) [arXiv:hep-th/0503057].

[189] L. Grant, L. Maoz, J. Marsano, K. Papadodimas and V. S. Rychkov, "Minisuperspace quantization of 'bubbling AdS' and free fermion droplets," JHEP **0508**, 025 (2005) [arXiv:hep-th/0505079].

[190] A. Ghodsi, A. E. Mosaffa, O. Saremi and M. M. Sheikh-Jabbari, "LLL vs. LLM: Half BPS sector of N = 4 SYM equals to quantum Hall system," Nucl. Phys. B **729**, 467 (2005) [arXiv:hep-th/0505129].

[191] S. Mukhi and M. Smedback, "Bubbling orientifolds," JHEP **0508**, 005 (2005) [arXiv:hep-th/0506059].

[192] Y. Takayama and A. Tsuchiya, "Complex matrix model and fermion phase space for bubbling AdS geometries," JHEP **0510**, 004 (2005) [arXiv:hep-th/0507070].

[193] L. Maoz and V. S. Rychkov, "Geometry quantization from supergravity: The case of 'bubbling AdS'," JHEP **0508**, 096 (2005) [arXiv:hep-th/0508059].

[194] H. Lin and J. M. Maldacena, "Fivebranes from gauge theory," Phys. Rev. D **74**, 084014 (2006) [arXiv:hep-th/0509235].

[195] A. E. Mosaffa and M. M. Sheikh-Jabbari, “On classification of the bubbling geometries,” JHEP **0604**, 045 (2006) [arXiv:hep-th/0602270].

[196] A. Donos, “A description of 1/4 BPS configurations in minimal type IIB SUGRA,” Phys. Rev. D **75**, 025010 (2007) [arXiv:hep-th/0606199].

[197] A. Buchel, “Coarse-graining 1/2 BPS geometries of type IIB supergravity,” Int. J. Mod. Phys. A **21**, 3495 (2006) [arXiv:hep-th/0409271].

[198] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, “The library of Babel: On the origin of gravitational thermodynamics,” JHEP **0512**, 006 (2005) [arXiv:hep-th/0508023].

[199] M. M. Caldarelli, D. Klemm and P. J. Silva, “Chronology protection in anti-de Sitter,” Class. Quant. Grav. **22**, 3461 (2005) [arXiv:hep-th/0411203].

[200] G. Milanesi and M. O Loughlin “Singularities and closed time-Like curves in Type IIB 1/2 BPS Geometries” JHEP **0509**, 008 (2005) [arXiv:hep-th/0507056].

[201] A. Dabholkar, “Exact counting of black hole microstates,” Phys. Rev. Lett. **94**, 241301 (2005) [arXiv:hep-th/0409148]; A. Dabholkar, R. Kallosh and A. Maloney, “A stringy cloak for a classical singularity,” JHEP **0412**, 059 (2004) [arXiv:hep-th/0410076].

[202] O. Lunin and S. D. Mathur, “Statistical interpretation of Bekenstein entropy for systems with a stretched horizon,” Phys. Rev. Lett. **88**, 211303 (2002) [arXiv:hep-th/0202072].

[203] R. C. Myers and O. Tafjord, “Superstars and giant gravitons,” JHEP **0111**, 009 (2001) [arXiv:hep-th/0109127].

[204] N. V. Suryanarayana, “Half-BPS giants, free fermions and microstates of superstars,” JHEP **0601**, 082 (2006) [arXiv:hep-th/0411145].

[205] P. G. Shepard, “Black hole statistics from holography,” JHEP **0510**, 072 (2005) [arXiv:hep-th/0507260].

[206] L. D. Landau and E. M. Lifshitz, “Statistical Physics: Part 1,” *Pergamon Press, (1969)*

[207] J. D. Brown and J. W. York, “Quasilocal energy and conserved charges derived from the gravitational action,” Phys. Rev. D **47**, 1407 (1993).

[208] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].

[209] K. Skenderis, “Lecture notes on holographic renormalization,” Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].

[210] M. Taylor-Robinson, “Higher-dimensional formulation of counterterms,” arXiv:hep-th/0110142.

[211] R. C. Myers, “Stress tensors and Casimir energies in the AdS/CFT correspondence,” Phys. Rev. D **60**, 046002 (1999) [arXiv:hep-th/9903203].

[212] S. W. Hawking and G. T. Horowitz, “The Gravitational Hamiltonian, action, entropy and surface terms,” Class. Quant. Grav. **13**, 1487 (1996) [arXiv:gr-qc/9501014].

[213] K. Behrndt, M. Cvetic and W. A. Sabra, “Non-extreme black holes of five dimensional N = 2 AdS supergravity,” Nucl. Phys. B **553**, 317 (1999) [arXiv:hep-th/9810227].

[214] M. Cvetic *et al.*, "Embedding AdS black holes in ten and eleven dimensions," Nucl. Phys. B **558**, 96 (1999) [arXiv:hep-th/9903214].

[215] A. Buchel and L. A. Pando Zayas, "Hagedorn vs. Hawking-Page transition in string theory," Phys. Rev. D **68**, 066012 (2003) [arXiv:hep-th/0305179].

[216] J. T. Liu and W. A. Sabra, "Mass in anti-de Sitter spaces," Phys. Rev. D **72**, 064021 (2005) [arXiv:hep-th/0405171].

[217] M. M. Caldarelli and P. J. Silva, "Giant gravitons in AdS/CFT. I: Matrix model and back reaction," JHEP **0408**, 029 (2004) [arXiv:hep-th/0406096].

[218] Z. W. Chong, H. Lu and C. N. Pope, "BPS geometries and AdS bubbles," Phys. Lett. B **614**, 96 (2005) [arXiv:hep-th/0412221].

[219] J. T. Liu, D. Vaman and W. Y. Wen, "Bubbling 1/4 BPS solutions in type IIB and supergravity reductions on S**n x S**n," Nucl. Phys. B **739**, 285 (2006) [arXiv:hep-th/0412043].

[220] Z. Guralnik and B. Kulik, "Properties of chiral Wilson loops," JHEP **0401**, 065 (2004) [hep-th/0309118].

[221] Z. Guralnik, S. Kovacs and B. Kulik, "Less is more: Non-renormalization theorems from lower dimensional superspace," Int. J. Mod. Phys. A **20**, 4546 (2005) [hep-th/0409091].

[222] A. Dymarsky, S. Gubser, Z. Guralnik and J. M. Maldacena, "Calibrated surfaces and supersymmetric Wilson loops," JHEP **0609**, 057 (2006) [hep-th/0604058].

[223] N. Drukker, "1/4 BPS circular loops, unstable world-sheet instantons and the matrix model," JHEP **0609**, 004 (2006) [arXiv:hep-th/0605151].

[224] J. P. Gauntlett and J. A. Harvey, "S duality and the dyon spectrum in $N = 2$ superYang-Mills theory," Nucl. Phys. B **463**, 287 (1996) [hep-th/9508156].

[225] C. Vafa and E. Witten, "A Strong coupling test of S duality," Nucl. Phys. B **431**, 3 (1994) [hep-th/9408074].

[226] J. P. Yamron, "Topological actions from twisted supersymmetric theories," Phys. Lett. B **213**, 325 (1988).

[227] D. Berenstein, R. Corrado, W. Fischler and J. M. Maldacena, "The operator product expansion for Wilson loops and surfaces in the large N limit," Phys. Rev. D **59**, 105023 (1999) [arXiv:hep-th/9809188].

[228] P. Wiegmann and A. Zabrodin, "Large N expansion for normal and complex matrix ensembles," arXiv:hep-th/0309253.

[229] A. Zabrodin, "New applications of non-Hermitian random matrices," Annales Henri Poincare **4**, S851 (2003) [arXiv:cond-mat/0210331].

[230] N. Drukker and S. Kawamoto, "Small deformations of supersymmetric Wilson loops and open spin-chains," JHEP **0607**, 024 (2006) [hep-th/0604124].

[231] N. Drukker and B. Fiol, "On the integrability of Wilson loops in AdS(5) x S**5: Some periodic ansatze," JHEP **0601**, 056 (2006) [arXiv:hep-th/0506058].

[232] A. Miwa and T. Yoneya, "Holography of Wilson-loop expectation values with local operator insertions," JHEP **0612**, 060 (2006) [arXiv:hep-th/0609007].

[233] K. Skenderis and M. Taylor, "Branes in AdS and pp-wave spacetimes," JHEP **0206**, 025 (2002) [hep-th/0204054].

[234] G. W. Semenoff and D. Young, "Exact 1/4 BPS loop - Chiral primary correlator," Phys. Lett. B **643**, 195 (2006) [hep-th/0609158].

[235] N. R. Constable, J. Erdmenger, Z. Guralnik and I. Kirsch, "Intersecting D3 branes and holography," Phys. Rev. D **68**, 106007 (2003) [hep-th/0211222].

[236] S. Gukov and E. Witten, "Gauge Theory, Ramification, And The Geometric Langlands Program," hep-th/0612073.

[237] A. Mikhailov, "Giant gravitons from holomorphic surfaces," JHEP **0011**, 027 (2000) [hep-th/0010206].

[238] J. Bedford, A. Brandhuber, B. J. Spence and G. Travaglini, "A twistor approach to one-loop amplitudes in N = 1 supersymmetric Yang-Mills theory," Nucl. Phys. B **706**, 100 (2005) [arXiv:hep-th/0410280]; "Non-supersymmetric loop amplitudes and MHV vertices," Nucl. Phys. B **712**, 59 (2005) [arXiv:hep-th/0412108].

[239] S. Kachru and E. Silverstein, "4d conformal theories and strings on orbifolds," Phys. Rev. Lett. **80**, 4855 (1998) [arXiv:hep-th/9802183].

[240] A. E. Lawrence, N. Nekrasov and C. Vafa, "On conformal field theories in four dimensions," Nucl. Phys. B **533**, 199 (1998) [arXiv:hep-th/9803015].

[241] J. Park and S. J. Rey, "Supertwistor orbifolds: Gauge theory amplitudes and topological strings," JHEP **0412**, 017 (2004) [arXiv:hep-th/0411123].

[242] M. R. Douglas and G. W. Moore, "D-branes, Quivers, and ALE Instantons," arXiv:hep-th/9603167; M. R. Douglas, B. R. Greene and D. R. Morrison,

"Orbifold resolution by D-branes," Nucl. Phys. B **506**, 84 (1997) [arXiv:hep-th/9704151].

[243] F. Cachazo, S. Katz and C. Vafa, "Geometric transitions and N = 1 quiver theories," arXiv:hep-th/0108120; F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, "A geometric unification of dualities," Nucl. Phys. B **628**, 3 (2002) [arXiv:hep-th/0110028].

[244] E. Witten, "Solutions of four-dimensional field theories via M-theory," Nucl. Phys. B **500**, 3 (1997) [arXiv:hep-th/9703166]; S. Katz, P. Mayr and C. Vafa, "Mirror symmetry and exact solution of 4D N = 2 gauge theories. I," Adv. Theor. Math. Phys. **1**, 53 (1998) [arXiv:hep-th/9706110].

[245] H. Kawai, D. C. Lewellen and S. H. H. Tye, "A Relation Between Tree Amplitudes Of Closed And Open Strings," Nucl. Phys. B **269**, 1 (1986).

[246] F. A. Berends, W. T. Giele and H. Kuijf, "On relations between multi - gluon and multigraviton scattering," Phys. Lett. B **211**, 91 (1988).

[247] Z. Bern, "Perturbative quantum gravity and its relation to gauge theory," Living Rev. Rel. **5**, 5 (2002) [arXiv:gr-qc/0206071].

[248] B. Zwiebach, "A first course in string theory," *Cambridge, UK: Univ. Pr. (2004) 558 P.*

[249] N. Deo, S. Jain and C. I. Tan, "The ideal gas of strings," SPIRES entry.

[250] S. A. Abel, J. L. F. Barbon, I. I. Kogan and E. Rabinovici, "String thermodynamics in D-brane backgrounds," JHEP **9904**, 015 (1999) [arXiv:hep-th/9902058].

[251] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, "Integrals And Series," *New York: Gordon and Breach Science Publishers (1986)*

[252] A. Higuchi, "Symmetric tensor spherical harmonics on the N sphere and their application to the de Sitter group SO(N,1)," J. Math. Phys. **28**, 1553 (1987) [Erratum-ibid. **43**, 6385 (2002)].

[253] G. Szego, "Orthogonal Polynomials," *4th ed. Providence, RI: Amer. Math. Soc. (1975)*