Covariant quantization of the superstring

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Abstract of the Dissertation

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Quantization of the manifestly space-time supersymmetric string theory has been possible only in the light-cone gauge. Covariant quantization is expected to be a stronger calculational tool to investigate various aspect of superstring theory. But covariant quantization of the superstring has been unsolved for over 20 years mainly because of the presence of infinite tower of ghosts. We give here new BRST operator of the superstring in which we successfully treat all the needed infinite tower of ghosts and using it we show how to calculate some lower points amplitudes in a manifestly covariant manner.

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Chapter 1

1st quantized BRST

1.1 Prologue

The advantages of supersymmetry are somewhat obscured in the Ramond-Neveu-Schwarz formalism, as is the case for supersymmetric particle theories when not formulated in superspace. For example, cancellations of divergences are not obvious, and amplitudes with many fermions are difficult to calculate.

Some of these problems were resolved with the Green-Schwarz formalism, but it proved difficult to quantize except in the lightcone gauge, where some manifest supersymmetry is retained in trade for the loss of some manifest Lorentz invariance. (Similar remarks apply to the Casalbuoni-Brink-Schwarz superparticle.) For example, higher-point diagrams of any type are difficult to calculate because longitudinal polarizations and momenta introduce nonlinearities, and in particular cancellation of anomalies (or any ϵ-tensor contribution) is difficult to check.

Covariant quantization of the Green-Schwarz action was attempted [1]. A class of derivative gauges was introduced that led to a pyramid of ghosts. Counting arguments showed that the conformal anomaly canceled, and summation of ghost determinants agreed with the lightcone result due to the "identity" $1-2+3-... = 1/4$. Unfortunately, due to a noninvertible transformation the gauge-fixed action found by this method proved not to be invariant under the Becchi-Rouet-Stora-Tyutin transformations derived by the same method [2]. This problem already appeared for the Casalbuoni-Brink-Schwarz superparticle.

In the meantime, an alternative approach to the quantum superparticle was developed [3], based on adding extra dimensions to the lightcone, a method that had successfully given free gauge-invariant actions for arbitrary representations of the Poincaré group in arbitrary dimensions [4]. This approach directly gave a BRST operator with the right cohomology. Using the relation between this BRST operator and Zinn-Justin-Batalin-Vilkovisky first-quantization [5], a manifestly supersymmetric classical mechanics action for this superparticle followed, including a BRST-invariant gauge-fixed action [6]. A crucial difference from the previous method was that "nonminimal" fields were required: There was necessarily a "pyramid" of ghosts, not just a linear tower. However, because of a required Fierz identity, the method of adding extra dimensions could not be directly applied to the lightcone Green-Schwarz superstring.

Various alternatives for a manifestly supersymmetric superstring have since been tried; the most successful is the pure spinor formalism [7]. It has proven somewhat more useful than RNS or lightcone GS approaches in calculating tree amplitudes [8]; its application to loop amplitudes is in progress [9]. If the formalism for all loops is developed, it should provide a simpler proof of finiteness, which previously required a combination of RNS and lightcone GS results (and equivalence of the two approaches). The pure spinor approach has two main shortcomings:

The first problem is the lack of a manifestly supersymmetric (and Lorentz covariant) path-integration measure. This is a problem in all known superspace approaches to first-quantizing superparticles and superstrings. One consequence is that Green functions (or the effective action in the superparticle case) are not manifestly supersymmetric off shell. Another is that gauge fixing the string field theory (with ghost fields) is not simple. We will not address this problem here.

The other problem is that the pure-spinor BRST operator lacks the c and b ghosts associated with the usual 2D coordinate invariances (and their associated Virasoro constraints). This is directly related to the lack of a corresponding action with worldsheet metric; the action is known only in the conformal gauge. Furthermore, the moduli that are the remnants of the metric in the conformal gauge must be inserted by hand. Another consequence of the lack of these ghosts as fundamental variables is that they must be reconstructed as complicated composite operators for use as insertions in loop diagrams. The (gauge-fixed) action, BRST operator, moduli, and operator insertions are thus separate postulates of the formalism, rather than all following from a gauge-invariant action as in other formalisms.

In this paper we will formulate the superstring with the ghost structure indicated by the original attempt of [1] and the successful treatment of the superparticle in [3]: the usual c and b ghosts, and a pyramid of spinors labeled by ghost number and generation. The main result is the BRST operator (from which the gauge-invariant action follows), which takes the form

$$Q_{sstring} = U\left(\int c\, T \;+\; \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi|_{>} \right) U^{-1} \tag{1.1}$$

with

$$U = e^{\int \tilde{\theta} D}\; e^{i\int R^a|_{>} P_a^{(\pm)}}\; e^{\int (R^{\oplus}+\theta\tilde{\gamma}^{\oplus}\theta/2)|_{>} b} \tag{1.2}$$

where T is essentially the energy-momentum tensor, D and P are the usual "covariant derivatives" in the affine Lie algebra of the classical superstring, $\tilde{\theta}$ is a certain linear combination of ghost θ's, R^i are certain expressions quadratic in θ's, π is conjugate to θ, $\tilde{\gamma}^{\oplus}$ is a ghost partner to the gamma matrices γ^a (which act only on θ and π), and $|_{>}$ picks out the ghost contributions. The gauge-fixed Hamiltonian is just $\{Q, \int b\} = \int T$. The unitary transformations are necessary because they change the Hilbert space, and so cannot be dropped: A simple analog is the BRST operator for the spinning (Dirac) particle in an external gauge field:

$$Q_{Dirac} = e^{c\gamma^a\nabla_a/\gamma^{\oplus}}(\gamma^{\oplus 2}b)e^{-c\gamma^a\nabla_a/\gamma^{\oplus}} = \gamma^{\oplus 2}b + \gamma^{\oplus}\gamma^a\nabla_a - \tfrac{1}{2}c(\gamma^a\nabla_a)^2$$

where $1/\gamma^{\oplus}$ doesn't exist on the correct Hilbert space, but cancels when the "unitary" transformation is evaluated.

We begin in chapter 2 by reviewing the free superparticle, which has already been quantized (and its BRST cohomology checked) in this approach. Because of the similarity of the algebra of super Yang-Mills [10] to that of the superstring [11], in chapter 3 we couple this superparticle to external super

Yang-Mills superfields. We use an almost identical method to derive the BRST operator for the superstring in section 4. We finish with our conclusions in chapter 5. (Mathematical details are relegated to the Appendix A.)

1.2 Review of free superparticle

We will start from the free super BRST operator derived in [3]. The generic BRST operator for arbitrary fields (massless, or massive by dimensional reduction) is constructed by starting with a representation of the lightcone SO(D−2) (which defines the theory) and adding 4 bosonic and 4 fermionic dimensions to obtain a covariant representation, including all auxiliary fields and ghosts. (This is somewhat redundant for bosons, but necessary for fermions.) The resulting generators S^{AB} of OSp(D,2|4) spin carry vector indices A, B that are separated into the usual SO(D−1,1) indices a, b and the rest as

$$A \ = \ (+, -, a; \mu, \tilde{\mu}) = (+, -, i), \qquad \mu \ = \ (\oplus, \ominus) \tag{1.3}$$

where $+, -$ belong to an SO(1,1) subgroup and $\mu, \tilde{\mu}$ to two Sp(2)'s, of which only the diagonal subgroup will be useful. The BRST operator then takes the generic form

$$Q'_{free} \ = \ \tfrac{1}{2} c \, \Box \ + \ S^{\oplus a} \partial_a \ + \ S^{\oplus\oplus} b \ + \ S^{\tilde{\oplus}-} \qquad (\, \Box = \partial^a \partial_a \,) \tag{1.4}$$

In the case of the superparticle, the spin operators are

$$S^{AB} \ = \ - \tfrac{1}{4} \, \bar{\eta} \Gamma^{[A} \Gamma^{B\}} \eta \tag{1.5}$$

in terms of self-conjugate variables η, which arose from the usual self-conjugate SO(D−2) fermionic spinor of lightcone superspace. We decompose the OSp(D,2|4) gamma-matrices Γ^A in terms of those of the subgroup SO(1,1) and those (γ)

of the subgroup OSp(D−1,1|4) as

$$\Gamma^i \;=\; \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix}, \quad \Gamma^+ \;=\; \begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix}, \quad \Gamma^- \;=\; \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix} \tag{1.6}$$

with (anti)commutation relations

$$\{\gamma^a, \gamma^b\} \;=\; -\,2\eta^{ab}, \qquad \eta^{ab} = (-+++\cdots)$$

$$\{\gamma^a, \gamma^\mu\} \;=\; \{\gamma^a, \tilde{\gamma}^\mu\} \;=\; 0 \tag{1.7}$$

$$[\gamma^\mu, \gamma^\nu] \;=\; [\tilde{\gamma}^\mu, \tilde{\gamma}^\nu] \;=\; 2C^{\mu\nu}, \quad [\gamma^\mu, \tilde{\gamma}^\nu] \;=\; 0 \tag{1.8}$$

where $C^{\mu\nu}$ is the Sp(2) metric with convention

$$C^{\oplus\ominus} \;=\; C_{\ominus\oplus} \;=\; i \;=\; -\,C^{\ominus\oplus} \;=\; -\,C_{\oplus\ominus} \tag{1.9}$$

and we have denoted $\gamma^{\tilde{\mu}} \equiv \tilde{\gamma}^\mu$ for legibility. The generalization of the fermionic superspace coordinate θ and its conjugate momentum appear through the analogous decomposition

$$\eta \;=\; \begin{pmatrix} \pi \\ \theta \end{pmatrix}, \qquad \pi \;=\; \frac{\partial}{\partial\theta} \tag{1.10}$$

We begin with a chiral (Weyl) spinor η, and multiplication by any Γ changes the chirality: not just Γ^a (γ^a) as usual, but also $\Gamma^\pm$, which shows that π and θ have opposite chirality (as expected, since they are conjugate), and Γ^μ (γ^μ).

This BRST operator is supersymmetric and also has an infinite pyramid

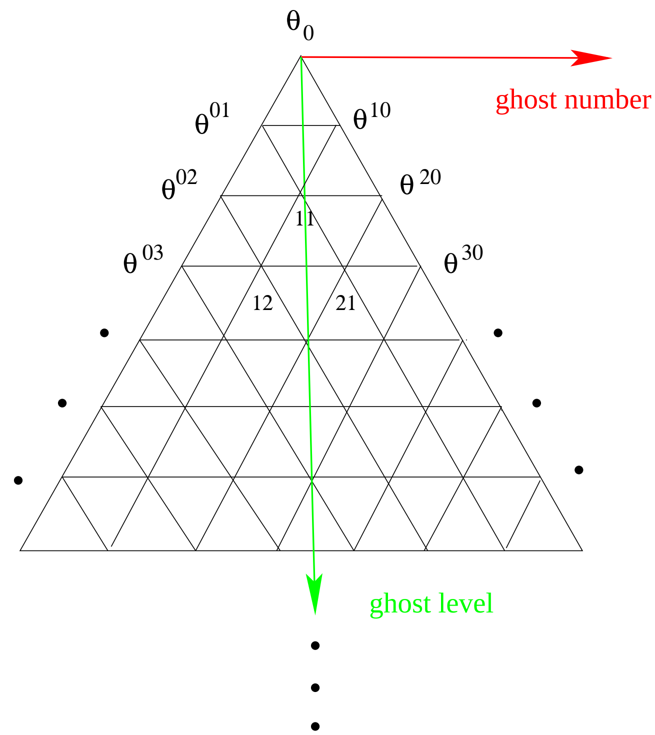

Figure 1.1: infinite pyramid of ghosts

of ghosts. To see these ghosts we need to define creation and annihilation operators from γ^μ and $\tilde{\gamma}^\mu$ as follows:

$$\gamma^\mu \;=\; a^\mu \;+\; a^{\dagger\mu} \quad , \quad \tilde{\gamma}^\mu \;=\; i(a^\mu \;-\; a^{\dagger\mu}) \tag{1.11}$$

$$[a^\mu, a^{\dagger\nu}] \;=\; C^{\mu\nu} \tag{1.12}$$

Then θ can be expanded giving the usual physical supersymmetry fermionic coordinate θ_0 at the top of the infinite pyramid of ghosts:

$$
\begin{aligned}
|^{p,q}\rangle &\equiv i^{\frac{(p+q)(p+q+1)}{2}} \frac{1}{\sqrt{p!}\sqrt{q!}} (a^{\dagger\oplus})^p (a^{\dagger\ominus})^q \ |0\rangle \\
\langle_{p,q}| &\equiv (-i)^{\frac{(p+q)(p+q+1)}{2}} \frac{1}{\sqrt{p!}\sqrt{q!}} \langle 0| \ (a_\oplus)^p (a_\ominus)^q \\
\langle_{p,q}|^{r,s}\rangle &= \delta^r_p \delta^s_q \\
\theta^{p,q} &\equiv \langle\theta|^{p,q}\rangle \ = \ \theta^{p,q\dagger} \\
\pi_{p,q} &\equiv \langle_{p,q}|\pi\rangle
\end{aligned}
\qquad (1.13)
$$

where

$$
\theta_0 \ \equiv \ \theta^{0,0}. \qquad (1.14)
$$

A power of i has been inserted to make $\theta^{p,q}$ real: The product of n real fermions gets a sign $(-1)^{n(n-1)/2}$ under Hermitian conjugation, because of the reverse ordering. The ghost a's and $a^\dagger$'s are fermions, because they take fermions to bosons, and vice versa (in contrast to ordinary γ matrices, which take fermions to fermions). Thus $\theta^{p,q}$ is the product of $p+q+1$ fermions, including $\langle\theta|$ itself. Then $\pi_{p,q}$ is not necessarily Hermitian, but has been defined to give 0 or 1 in graded commutators. (But $\pi^{p,q}$ is always Hermitian, like $|\pi\rangle$ and π_0.) We will sometimes also use a notation where $\theta^{p,q}$ carries instead p $\oplus$'s and q $\ominus$'s: For example, $\theta^{1,0} \equiv \theta^\oplus$. Note that the ghosts alternate in both statistics and chirality with each ghost level.

So in superspace notation the free super BRST operator is

$$
Q'_{free} \ = \ \tfrac{1}{2} c \Box \ - \tfrac{1}{2}\bar{\pi}\gamma^{\oplus 2}\theta b \ + \ \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus \pi \ - \tfrac{i}{2}\bar{\pi}\gamma^\oplus \not{p}\theta, \qquad \not{p} \ \equiv \ -i\partial_a\gamma^a \qquad (1.15)
$$

We can make a unitary transformation on Q'_{free} to give a convenient form with which to work. Specifically, the unitary transformation

$$Q_{free} \ = \ U_0 Q'_{free} U_0^\dagger \tag{1.16}$$

with

$$U_0 \ = \ e^{\bar{\theta}\tilde{\gamma}^{\oplus}\theta b/2} \tag{1.17}$$

gives Q_{free} in terms of the supersymmetry generator q_0, spinor covariant derivative d_0 and all their nonminimal versions:

$$Q_{free} \ = \ \tfrac{1}{2}c\,\Box - 2\bar{\pi}a^{\dagger\oplus}a^{\oplus}\theta b - \tfrac{i}{2}\bar{q}a^{\dagger\oplus}d \tag{1.18}$$

$$q \ = \ \pi - \not{p}\theta, \quad d \ = \ \pi + \not{p}\theta \tag{1.19}$$

Actually, q_0 is the only part of q that does not appear in this form of the BRST operator: Because of the creation and annihilation operators, θ_0 and π_0 appear only as their supersymmetry invariant combination d_0. Thus the supersymmetry generator that anticommutes with this form of Q is just the usual one q_0. (This can also be derived in a straightforward way by starting with the lightcone q.) Then the supersymmetry generator for Q'_{free} can be obtained by inverting the unitary transformation on q_0:

$$\begin{aligned} q'_0 \ &= \ U_0^\dagger q_0 U_0 \\ &= \ \pi_0 - \not{p}\theta_0 - \theta^{\oplus}b \end{aligned} \tag{1.20}$$

1.3 Interacting superparticle

The (D=3,4,6,10) superparticle BRST operator in a super Yang-Mills background (with constant superfield strength) is closely related to the superstring BRST operator. The introduction of the SYM background can be established by gauge covariantizing the super covariant derivatives p_a and $d_{0\alpha}$:

$$p_a \longrightarrow \nabla_a \tag{1.21}$$

$$d_0 \longrightarrow \nabla_0 \tag{1.22}$$

Then the graded algebra among the covariant derivatives is [10]

$$[\nabla_a, \nabla_b] = F_{ab} \tag{1.23}$$

$$\{\nabla_{0\alpha}, \nabla_{0\beta}\} = 2\gamma_{a\alpha\beta}\nabla^a \tag{1.24}$$

$$[\nabla_{0\alpha}, \nabla_a] = \gamma_{a\alpha\beta}W^\beta \tag{1.25}$$

The Bianchi identity from the above algebra gives

$$\gamma_{a\alpha\beta}[\nabla^a, W^\beta] = 0 \tag{1.26}$$

and the D=3,4,6,10 dimensional gamma matrix (which is symmetric in those cases) identity

$$\gamma_{a(\alpha\beta}{\gamma^a}_{\gamma)}{}^{\delta} = 0. \tag{1.27}$$

We begin at linear order in the fields, where the background satisfies the equations of motion

$$\{\nabla_\alpha, W^\alpha\} \;=\; 0 \tag{1.28}$$

$$[\nabla^a, F_{ab}] \;=\; 0 \tag{1.29}$$

1.3.1 Constant YM background

One way to build this interacting super BRST operator is by considering an ordinary constant YM background first, and next supersymmetrizing it by including a constant fermionic field strength (not yet superfield) $\mathring{w}^\alpha$. Then we extend the result to a nonconstant SYM background in the next subsection.

Making the gauge choice

$$\mathring{A}_a = \tfrac{i}{2} x^b \mathring{F}_{ba}$$

for constant field strength, the super BRST operator can be written in the form

$$\mathring{Q}'_{YMB} \;=\; Q'_{free} \;+\; \tfrac{1}{2}\mathring{F}_{ab}V^{ab} \tag{1.30}$$

We then find

$$V^{ab} \;=\; \tfrac{i}{2}cx^{[a}p^{b]} \;+\; \tfrac{i}{2}R^{\oplus}R^{[a}p^{b]} -\; (c+R^{\oplus})\;\bar{\pi}\gamma^{ab}\theta \;+\; \tfrac{1}{4}(x+R)^{[a}\bar{\pi}\gamma^{\oplus}\gamma^{b]}\theta \tag{1.31}$$

in terms of an expression R^i defined below, where we use the notation

$$C^{[a}D^{b]} \;\;\equiv\;\; C^aD^b - C^bD^a \tag{1.32}$$

$$\gamma^{ab} \;\;\equiv\;\; -\tfrac{1}{4}\gamma^{[a}\gamma^{b]} \tag{1.33}$$

We can also write

$$
\begin{gathered}
V^{ab} = V^{\oplus ab} \\
V^{ijk} = i(x^i x^j + R^i R^j)p^k + \tfrac{1}{2}(x+R)^{(i}\bar{\pi}\gamma^{j]}\gamma^k\theta \\
x^{\oplus} = c, \;\; p^{\oplus} = 0
\end{gathered}
\tag{1.34}
$$

(There is further antisymmetrization in the last two indices upon contraction with F, following the graded symmetrization in the first two indices shown above: The tensor V^{ijk} has mixed symmetry.)

The expression R^i is given by

$$
R^i(\theta) \equiv \tfrac{1}{2}\bar{\theta}\mathcal{O}\gamma^i\theta \tag{1.35}
$$

where the operator $\mathcal{O}$ is defined to satisfy

$$
\begin{aligned}
[\gamma^{\oplus}, \mathcal{O}] &= 0 \\
\{\tilde{\gamma}^{\oplus}, \mathcal{O}\} &= 2\gamma^{\oplus} \\
[a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus}, \mathcal{O}] &= 0 \\
\langle 0|\mathcal{O}\tilde{\gamma}^{\oplus} &= -i\langle 0|\tilde{\gamma}^{\oplus} = \langle 0|\gamma^{\oplus}
\end{aligned}
\tag{1.36}
$$

As an explicit form of $\mathcal{O}$ we find

$$
\mathcal{O} = \frac{1}{2}\left\{\frac{1}{\tilde{\gamma}^{\oplus}}, \gamma^{\oplus}\right\} \tag{1.37}
$$

where

$$\frac{1}{\tilde{\gamma}^{\oplus}} = \sum_{p=0}^{\infty}\Bigg[\Theta_{N_\oplus - N_\ominus}\frac{N_\oplus !}{(N_\oplus + p + 1)!}\; i a_\oplus (i a_\oplus a_\ominus)^p - a^{\dagger\ominus}(-i a^{\dagger\oplus} a^{\dagger\ominus})^p \Theta_{N_\ominus - N_\oplus}\frac{N_\ominus !}{(N_\ominus + p + 1)!}\Bigg] \tag{1.38}$$

with

$$\Theta_x = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad , \qquad N_\mu = a^{\dagger\mu} a_\mu$$

(not summed over μ). This representation satisfies (1.36) if we regularize indefinite norm states. (See the Appendices for details.)

1.3.2 Constant SYM background

From this $\mathring{Q}'_{YMB}$ we can construct a BRST operator for a supersymmetric constant SYM background $\mathring{Q}'_{SYMB}$ in the form

$$\mathring{Q}'_{SYMB} = Q'_{free} + \tfrac{1}{2}\mathring{F}_{ab} V^{ab} + \mathring{w}^\alpha V_\alpha \tag{1.39}$$

In addition to first-quantized transformations we take q'_0 (1.20) to also generate the second-quantized transformations of $\mathring{w}^\alpha$ and $\mathring{F}_{ab}$

$$\{q'_{0\beta}, \mathring{w}^\alpha\} = \gamma^{ab}{}_\beta{}^\alpha \mathring{F}_{ab} \tag{1.40}$$

$$\{q'_0, \mathring{F}_{ab}\} = 0 \tag{1.41}$$

so that they cancel up to a gauge transformation (generated by Q'_{free}):

$$\{q'_0,\ \overset{\circ}{Q}'_{SYMB}\} \quad = \quad \{Q'_{free}, \Psi\} \tag{1.42}$$

We then have

$$\tfrac{1}{2}\overset{\circ}{F}_{ab}\ \{q'_{0\beta}, V^{ab}\}\ -\ \overset{\circ}{w}{}^{\alpha}[q'_{0\beta}, V_\alpha]\ =\ -\overset{\circ}{F}_{ab}\ \gamma^{ab}{}_{\beta}{}^{\alpha}\ V_\alpha\ +\ \{Q'_{free}, \Psi_\beta\} \tag{1.43}$$

This is true if we define V_α by

$$\{q'_{0\beta}, V^{ab}\}|_{\gamma^{ab}}\ \equiv\ -2\gamma^{ab}{}_{\beta}{}^{\alpha}\ V_\alpha \tag{1.44}$$

which means we define V_α from the left-hand side by selecting only terms with an explicit γ^{ab}.

With this definition we find V_α and Ψ_α

$$V_\alpha \quad \equiv \quad -(c + R^{\oplus})\ q'_{0\alpha} \tag{1.45}$$

$$\Psi_\alpha \quad \equiv \quad -\tfrac{i}{2}(x^b +\ R^b)(\gamma^a_{\alpha\beta}\theta^\beta_0\overset{\circ}{F}_{ab} - 2\gamma_{b\alpha\beta}\overset{\circ}{w}{}^{\beta}) +\ \tfrac{1}{2}(\gamma^b_{\alpha\beta}\theta^\beta_0 + \gamma^b_{\alpha\beta}\tilde{\theta}^\beta_0)iR^a\overset{\circ}{F}_{ab} \tag{1.46}$$

where

$$\tilde{\theta}\ =\ -i\langle 0|\mathcal{O}|\theta\rangle\ =\ \langle 0|e^{ia_\oplus a_\ominus} - 1|\theta\rangle \tag{1.47}$$

which contains all nonminimal ghost-number-zero ghosts.

To obtain a gauge independent and explicitly supersymmetric expression we perform a unitary transformation

$$U_1\ =\ e^{\Lambda}$$

where

$$\begin{aligned}
\Lambda \ \ = \ \ & iR^b \ (\ \theta_0\gamma_b\overset{\circ}{w} + \ \tilde{\theta}\gamma_b\overset{\circ}{w} + \ \tfrac{1}{2}\theta_0\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac} + \ \tfrac{1}{2}\tilde{\theta}\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac} \\
& \qquad + \ \tfrac{1}{2}\tilde{\theta}\gamma_b\gamma^{ac}\tilde{\theta}\overset{\circ}{F}_{ac} \ - \ \tfrac{1}{2}\theta_0\gamma^c\tilde{\theta}\overset{\circ}{F}_{bc}) \\
& - \theta_0\gamma^b\tilde{\theta} \ (\ \tfrac{1}{3}\theta_0\gamma_b\overset{\circ}{w} + \ \tfrac{2}{3} \ \tilde{\theta}\gamma_b\overset{\circ}{w} + \ \tfrac{1}{4}\theta_0\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac} \\
& \qquad + \ \tfrac{3}{4} \ \tilde{\theta}\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac} + \ \tfrac{5}{12} \ \tilde{\theta}\gamma_b\gamma^{ac}\tilde{\theta}\overset{\circ}{F}_{ac})
\end{aligned} \tag{1.48}$$

After another unitary transformation U_0 (1.17), $\overset{\circ}{Q}{}'_{SYMB}$ becomes (at the linearized level)

$$\begin{aligned}
\overset{\circ}{Q}_{SYMB} \ \ = \ \ & \tfrac{1}{2} \ (c + R^{\oplus} + \tfrac{1}{2}\bar{\theta}\tilde{\gamma}^{\oplus}\theta \ )|_{>}(\ \Box \ - \ W\nabla_0 \ - \ \bar{\pi}\gamma^{ab}\theta|_{>} \ F_{ab} \ ) \\
- \ \ & 2 \ \bar{\pi}a^{\dagger\oplus}a^{\oplus}\theta|_{>}b \ + \ \tfrac{1}{4} \ \bar{\pi}\tilde{\gamma}^{\oplus}\pi|_{>} \\
- \ \ & \tfrac{i}{2} \ (\nabla_a \ + \tilde{\theta}\gamma_a W \ + \tfrac{1}{2} \ \tilde{\theta}\gamma_a\{W,\nabla_0\}\tilde{\theta} \ ) \ \bar{\pi}\gamma^{\oplus}\gamma^a\theta|_{>} \\
- \ \ & \tfrac{1}{2} \ \bar{\pi}^{\oplus}\nabla_0 \ - \ \tfrac{1}{2} \ (\nabla_a \ + \tilde{\theta}\gamma_a W \ + \tfrac{1}{2} \ \tilde{\theta}\gamma_a\{W,\nabla_0\}\tilde{\theta} \ - \ \tfrac{i}{2}R^b|_{>}F_{ab}) \ \bar{\theta}^{\oplus}\gamma^a\nabla_0 \\
- \ \ & \tfrac{1}{3} \ \bar{q}^{\oplus}\gamma^a\tilde{\theta} \ \tilde{\theta}\gamma_a W \ + \ \tfrac{5}{24} \ \bar{q}^{\oplus}\gamma^a\tilde{\theta} \ \tilde{\theta}\gamma_a\{W,\nabla_0\}\tilde{\theta} \\
+ \ \ & \tfrac{i}{4} \ \bar{q}^{\oplus}\gamma^b\tilde{\theta} \ R^a|_{>} \ F_{ab} \\
+ \ \ & \tfrac{1}{4} \ [\ i\nabla_b R^b|_{>}, -i\nabla_a \ \bar{\pi}\gamma^{\oplus}\gamma^a\theta|_{>}] \\
+ \ \ & R^{\oplus}|_{>}P_a(\tilde{\theta}\gamma^a W \ + \ \tfrac{1}{2}\tilde{\theta}\gamma^a\{W,\nabla_0\}\tilde{\theta}) \\
+ \ \ & \tfrac{1}{2\cdot 3!} \ \big[\ i\nabla_c R^c|_{>}, [\ i\nabla_b R^b|_{>}, -i\nabla_a \ \bar{\pi}\gamma^{\oplus}\gamma^a\theta|_{>}]\big]
\end{aligned} \tag{1.49}$$

where $|_{>}$ means that we drop θ_0 contributions, and

$$\square = -\nabla^a \nabla_a \tag{1.50}$$

$$\nabla_a = p_a + A_a \tag{1.51}$$

$$\nabla_{0\alpha} = \pi_{0\alpha} + (\not{p}\theta_0)_\alpha + A_\alpha \tag{1.52}$$

$$q^{\oplus\alpha} = \pi^{\oplus\alpha} + (\not{p}\theta^\oplus)^\alpha \tag{1.53}$$

The superfields have the θ_0 expansions

$$F_{ab} = \mathring{F}_{ab} \tag{1.54}$$

$$W^\alpha = \mathring{w}^\alpha + (\gamma^{ab}\theta_0)^\alpha \mathring{F}_{ab} \tag{1.55}$$

$$A_a = \mathring{A}_a + \bar{\theta}_0\gamma_a\mathring{w} + \tfrac{1}{2}\bar{\theta}_0\gamma_a\gamma^{bc}\theta_0 \mathring{F}_{bc} \tag{1.56}$$

$$A_\alpha = (\gamma^a\theta_0)_\alpha\mathring{A}_a + \tfrac{2}{3}(\gamma^a\theta_0)_\alpha \bar{\theta}_0\gamma_a\mathring{w} + \tfrac{1}{4}(\gamma^a\theta_0)_\alpha \bar{\theta}_0\gamma_a\gamma^{bc}\theta_0 \mathring{F}_{bc} \tag{1.57}$$

in the gauge

$$\mathring{A}_a = \tfrac{i}{2}x^b \mathring{F}_{ba} \tag{1.58}$$

$$\mathring{A}_\alpha = 0 \tag{1.59}$$

used above, but (1.49) is manifestly gauge independent and supersymmetric.

1.3.3 Arbitrary SYM background

After making a final unitary transformation

$$U_3 = e^{(R^\oplus + \theta\tilde{\gamma}^\oplus\theta/2)|_{>b}}$$

the above BRST operator can be written in the simple form

$$Q''_{SYMB} = \; U \left[\; \tfrac{1}{2} c \left(\; \Box - W\nabla_0 - \bar{\pi}\gamma^{ab}\theta|_> \; F_{ab} \; \right) + \tfrac{1}{4} \; \bar{\pi}\tilde{\gamma}^{\oplus}\pi|_> \; \right] U^{-1}$$

$$U = e^{\tilde{\theta}\nabla_0} \; e^{iR^a|_>\nabla_a} \; e^{(R^{\oplus}+\theta\tilde{\gamma}^{\oplus}\theta/2)|_> b} \qquad (1.60)$$

which can be applied directly to the case of an arbitrary, nonlinear SYM background.

In fact, the nilpotence of this BRST operator does not seem to require that the background be on shell. This is contradictory to the usual result that any description of linearized "quantum" Yang-Mills in a Yang-Mills background must have the background on shell, since nonabelian gauge invariance relates kinetic and interaction terms [12]. (Similar remarks apply to any nonabelian gauge theory, such as gravity or strings.) This paradox is probably due to the fact that we have not required an "integrability" condition on the background: For a generic self-interacting field theory, an action (or ZJBV action) of the form

$$S = \tfrac{1}{2}\phi^j\phi^i K_{ij} + \tfrac{1}{6}\phi^k\phi^j\phi^i V_{ijk} + ... \qquad (1.61)$$

results in the kinetic operator (or BRST operator) in a background

$$Q_{ij} = K_{ij} + \phi^k V_{kij} + ... \qquad (1.62)$$

From S we can see that $K, V, ...$ must be totally (graded) symmetric. In Q, this condition on K is seen to follow simply from hermiticity, but the condition on V is not so obvious. Since we are ultimately concerned with the BRST operator for the superstring without background, and are using the SYM case in a background only as an analogy, we will not consider this obscurity further here.

As explained in the Introduction, in the above expression for the BRST operator (1.60) we are not allowed to remove the exponential factors, since that would lead to a trivial result. This fact can be understood already in the free case: The BRST operator that would result from dropping the background and exponentials has the wrong cohomology, since the remaining two terms have no dependence on θ_0, so one would obtain an ordinary superfield satisfying only the Klein-Gordon equation. In this case the exponentials are required for Q to be regularizable: Certain poorly defined quantities cancel upon their expansion. (See Appendices A.2-A.3.)

1.4 Superstring

The superstring is described by a 2D field theory whose algebra of covariant derivatives (currents) resembles that of interacting particle covariant derivatives for a constant SYM background:

$$\begin{aligned}
\{D_\alpha^{(\pm)}(1), D_\beta^{(\pm)}(2)\} &= 2\delta(2-1)\gamma^a_{\alpha\beta}P_a^{(\pm)}(1) \\
[D_\alpha^{(\pm)}(1), P_a^{(\pm)}(2)] &= 2\delta(2-1)\gamma_{a\alpha\beta}\Omega^{(\pm)\beta}(1) \\
\{D_\alpha^{(\pm)}(1), \Omega^{(\pm)\beta}(2)\} &= \pm i\delta'(2-1)\delta_\alpha^\beta \\
[P_a^{(\pm)}(1), P_b^{(\pm)}(2)] &= \pm i\delta'(2-1)\eta_{ab} \\
[P^{(\pm)}, \Omega^{(\pm)}] &= \{\Omega^{(\pm)}, \Omega^{(\pm)}\} \;=\; 0
\end{aligned} \tag{1.63}$$

where

$$\begin{aligned}
D_\alpha^{(\pm)} &= \pi_{0\alpha} + (\gamma^a\theta_0)_\alpha \hat{P}_a^{(\pm)} \pm i\tfrac{1}{2}(\gamma^a\theta_0)_\alpha\theta_0\gamma_a\theta_0' \\
P_a^{(\pm)} &= \hat{P}_a^{(\pm)} \pm i\theta_0\gamma_a\theta_0' \\
\Omega^{(\pm)\alpha} &= \pm i\theta_0'
\end{aligned} \tag{1.64}$$

and

$$\hat{P}^{(\pm)} = \tfrac{1}{\sqrt{2}}\left(i\frac{\delta}{\delta X} \pm X'\right) \tag{1.65}$$

in the Hamltonian formalism correspond to the left(right)-moving combinations of P_0 and P_1 of the first-order formalism after using the equation of motion for P_1 (see below). (In the definitions above, $(\pm)$'s on π and θ are understood.) Also, $'$ means a σ derivative as usual. D, P, Ω (anti)commute with the supersymmetry generator

$$q_\alpha \;=\; q_\alpha^{(+)} + q_\alpha^{(-)}, \quad q_\alpha^{(\pm)} \;=\; \int \pi_{0\alpha} - (\gamma^a\theta_0)_\alpha \hat{P}_a^{(\pm)} \mp i\tfrac{1}{6}(\gamma^a\theta_0)_\alpha\theta_0\gamma_a\theta_0' \quad (1.66)$$

So we can see the analogy between the covariant derivatives of the free superstring and the superparticle with SYM background.

$$(D_\alpha, P_a, \Omega^\alpha) \leftrightarrow (\nabla_\alpha, \nabla_a, W^\alpha) \quad (1.67)$$

as well as the less precise analogy

$$' \leftrightarrow F_{ab} \quad (1.68)$$

1.4.1 BRST

Now we can guess the result for the superstring BRST operator from the result of the superparticle in a constant SYM background:

$$Q_{sstring} = U \left(\int c\; T \;+\; \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi|_{>} \right) U^{-1} \quad (1.69)$$

where

$$U = e^{\int \tilde{\theta} D}\; e^{\int iR^a|_{>}P_a}\; e^{\int (R^{\oplus}+\theta\tilde{\gamma}^{\oplus}\theta/2)|_{>}b} \quad (1.70)$$

and $Q_{sstring} = Q^{(+)}_{sstring} + Q^{(-)}_{sstring}$. From now on we will suppress the σ-integral symbol for convenience. Since θ_0 and π_0 appear only in D, P, and T, this Q is automatically supersymmetric under the above supersymmetry generator.

There are two major differences in T as compared to the superparticle:

Firstly the string has a $c'b$ ghost contribution. Secondly the superstring has ΩD as an analog of $W\nabla_0 - \bar{\pi}\gamma^{ab}\theta F_{ab}$, from the correspondence above. So our trial form of T is

$$\begin{aligned} T^{(\pm)} &= \tfrac{1}{2}\,\Box_\pm \mp i\left\{c'b + \bar{\theta}'\pi + w^\pm(\bar{\theta}\pi)'\right. \\ &\quad \left. + A_1^\pm\left[\bar{\theta}(a^{\dagger\oplus}a_\oplus - a^{\dagger\ominus}a_\ominus)\pi\right]' + A_2^\pm\left[\bar{\theta}(a^{\dagger\oplus}a_\oplus + a^{\dagger\ominus}a_\ominus)\pi\right]'\right\} \end{aligned} \tag{1.71}$$

where $\Box_\pm = -\hat{P}^{(\pm)a}\hat{P}_a^{(\pm)}$. (The true energy-momentum tensor is actually $T \mp i(cb)'$.) The constants $w^\pm$, $A_1^\pm$ and $A_2^\pm$ will be determined by 3 conditions: (1) The conformal weight of $\bar{\pi}\tilde{\gamma}^\oplus\pi$ should be 1. (2) The conformal anomaly should cancel in $D = 10$. (3) θ_0 should have conformal weight 0 due to supersymmetry. (The A_1 term is ghost number, while the A_2 term is ghost level.)

Satisfying these constraints we find

$$\begin{aligned} A_1^\pm &= 1, \quad A_2^\pm = w^\pm = 0 \\ \Rightarrow \quad T^{(\pm)} &= \tfrac{1}{2}\,\Box_\pm \mp i\left\{ c'b + \bar{\theta}'\pi + \left[\bar{\theta}(a^{\dagger\oplus}a_\oplus - a^{\dagger\ominus}a_\ominus)\pi\right]' \right\} \end{aligned} \tag{1.72}$$

and the gauge-fixed Hamiltonian is $\{Q, \int b^{(+)} + b^{(-)}\} = \int T^{(+)} + T^{(-)}$.

This Q has four interesting quantum numbers:(1) ghost number; (2) conformal weight, which is "momentum number" (1 for $P, b, \pi, '$) minus ghost number; (3) (10D) engineering dimension (-1 for x, c, $-\frac{1}{2}$ for θ, 2 for $'$); and (4) a mysterious "field weight", which is 1 for all fields, but for which we attribute a 1 for $\tilde{\gamma}^\oplus$. (Thus, Q is quadratic in momenta and primes, and cubic in fields and $\tilde{\gamma}^\oplus$'s.)

From now on let's concentrate on one chirality. After expanding the exponential factor and regularizing (as explained in Appendix A.3) we find

$$\begin{aligned}
Q^{(+)}_{sstring} \;=\;& \left(c + R^{\oplus} + \tfrac{1}{2}\bar\theta\tilde\gamma^{\oplus}\theta\;\right)|_{>} \\
&\qquad\quad \times \left(\;\tfrac{1}{2}\Box_{+}\; - ic'b \;-\; i\bar\theta'\pi \;-\; i[\bar\theta(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]'\right) \\
&- \;2\;\bar\pi a^{\dagger\oplus}a^{\oplus}\theta|_{>}b \;+\; \tfrac{1}{4}\;\bar\pi\tilde\gamma^{\oplus}\pi|_{>} \\
&- \;\tfrac{i}{2}\;\left(\hat P_a \;+ i\theta_0\gamma_a\theta'_0 \;+ 2i\tilde\theta\gamma_a\theta'_0 \;+\; i\tilde\theta\gamma_a\tilde\theta' \;-\tfrac{1}{2}\;R'_a|_{>}\right)\;\bar\pi\gamma^{\oplus}\gamma^{a}\theta|_{>} \\
&- \;\tfrac{1}{2}\;\bar\pi^{\oplus}\left(\pi_0 + (\gamma^a\theta_0)\hat P_a + i\tfrac{1}{2}(\gamma^a\theta_0)\theta_0\gamma_a\theta'_0\;\right) \\
&- \;\tfrac{1}{2}\;\left(\hat P_a \;+ i\theta_0\gamma_a\theta'_0 \;+ 2i\tilde\theta\gamma_a\theta'_0 \;+\; i\tilde\theta\gamma_a\tilde\theta' \;-\tfrac{1}{2}\;R'_a|_{>}\right) \\
&\qquad \times\bar\theta^{\oplus}\gamma^{a}\left(\pi_0 + (\gamma^b\theta_0)\hat P_b + i\tfrac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta'_0\;\right) \\
&- \;\tfrac{1}{3}\;\bar q^{\oplus}\gamma^{a}\tilde\theta\left(2i\tilde\theta\gamma_a\theta'_0 \;+\; i\tfrac{5}{4}\;\tilde\theta\gamma_a\tilde\theta' - \tfrac{3}{4}\;R'_a|_{>}\right) \\
&+ \;\tfrac{1}{2}\;\left(\hat P_a \;+ i\theta_0\gamma_a\theta'_0 \;+ 2i\tilde\theta\gamma_a\theta'_0 \;+\; i\tilde\theta\gamma_a\tilde\theta' \;-\; \tfrac{1}{2}R'_a|_{>}\right)^2 R^{\oplus}|_{>} \\
&+ \;ic'b\;\left(R^{\oplus} + \tfrac{1}{2}\bar\theta\tilde\gamma^{\oplus}\theta\right)|_{>}
\end{aligned} \tag{1.73}$$

This Q satisfies $Q^2 = 0$, as can be checked directly.

1.4.2 Constraints

The constraints of the gauge-invariant action (see following subsection) can be obtained directly from the BRST operator by taking its (graded) commutator and keeping just ghost-number-zero terms: The Virasoro constraints $\mathcal{A}$ follow as usual from b (with the gauge-fixed action from $\int b$), while generalizations $\mathcal{B}$ of the $\gamma{\cdot}pd$ constraint (κ symmetry generator) follow from $\theta^{p,p+1}$, and first-class generalizations $\mathcal{E}$ of the second-class constraint d follow from $\pi^{p,p+1}$ [6]:

$$\mathcal{A} = \tfrac{1}{2}\Box_{+} - i\sum_{q=0}^{\infty}\bar{\theta}'^{q,q}\pi_{q,q} \tag{1.74}$$

$$\mathcal{B}_0 = \gamma^a\Pi\mathcal{P}_a(\boldsymbol{d}_0+\pi^{1,1}) - 2\theta^{1,1}\mathcal{A} + 2\vartheta^0(\tfrac{1}{2}\mathcal{P}^2+\mathcal{A}) \tag{1.75}$$

$$\mathcal{B}_p = \gamma^a\Pi\mathcal{P}_a(\pi^{p,p}+\pi^{p+1,p+1}) + 2(\theta^{p,p}-\theta^{p+1,p+1})\mathcal{A} + 2\vartheta^p(\tfrac{1}{2}\mathcal{P}^2+\mathcal{A}) \tag{1.76}$$

$$\mathcal{E}_0 = \boldsymbol{d}_0 - \pi^{1,1} + \gamma^a\Pi\mathcal{P}_a\theta^{1,1} \tag{1.77}$$

$$\mathcal{E}_p = \Pi(\pi^{p,p}-\pi^{p+1,p+1}) + \gamma^a\Pi\mathcal{P}_a(\theta^{p,p}+\theta^{p+1,p+1}) \tag{1.78}$$

where

$$\begin{aligned}
\mathcal{P}_a &\equiv \hat{P}_a + i\theta_0\gamma_a\theta_0' + 2i\tilde{\theta}\gamma_a\theta_0' + i\tilde{\theta}\gamma_a\tilde{\theta}' - \tfrac{1}{2}\tilde{R}_a'|_{>} \\
\boldsymbol{d}_0 &\equiv \pi_0 + \gamma^a\hat{P}_a\theta_0 + i\tfrac{1}{2}(\gamma^a\theta_0)\theta_0\gamma_a\theta_0' + \tfrac{2}{3}\gamma^a\tilde{\theta}\left(2i\tilde{\theta}\gamma_a\theta_0' + i\tfrac{5}{4}\tilde{\theta}\gamma_a\tilde{\theta}' - \tfrac{3}{4}\tilde{R}_a'|_{>}\right) \\
\vartheta^p &\equiv 2\theta^{1,1} + 2\theta^{2,2} + \cdots + 2\theta^{p-1,p-1} + \theta^{p,p} - \theta^{p+1,p+1} - 2\theta^{p+2,p+2} - 2\theta^{p+3,p+3} - \cdots
\end{aligned} \tag{1.79}$$

and $\tilde{R}^a$ indicates that only ghost-number-zero θ's are selected.

Since this procedure requires the component expression, we explicitly use the projection operator

$$\Pi = \frac{1}{\tilde{\gamma}^{\oplus}}(\tilde{\gamma}^{\oplus})_{reg} \tag{1.80}$$

as, e.g., $\Pi|\theta\rangle$, in some terms, as explained in Appendix A.3. Also, because $R^{\oplus}$ only interacts with $\pi\Pi$ we express $[\pi^{p,p+1}, R^{\oplus}]$ as a projected expression ϑ^p. (For the full expression, see Appendix A.4.)

These constraints are closed classically (after regularization: see Appendix

A.4)

$$
\begin{aligned}
[\mathcal{A}(1), \mathcal{A}(2)] &= -\delta'(2-1)[\mathcal{A}(1)+\mathcal{A}(2)] \\
[\mathcal{A}(1), \mathcal{E}_p(2)] &= -\delta'(2-1)\mathcal{E}_p(1) \\
[\mathcal{A}(1), \mathcal{B}_p(2)] &= -\delta'(2-1)[\mathcal{B}_p(1)+\mathcal{B}_p(2)] \\
\{\mathcal{E}_p(1), \mathcal{E}_q(2)\} &= 0 \\
\{\mathcal{B}_p^{\alpha}(1), \mathcal{B}_q^{\beta}(2)\} &= 8\delta'(2-1)\vartheta^{p\alpha}\mathcal{P}_a(1)\gamma^{a\beta\delta}E_{q\delta}(2) \\
&\quad +4\delta'(2-1)\gamma_a^{\alpha\lambda}(\pi^{p,p}+\pi^{p+1,p+1})_{\lambda}(1)\gamma^{a\beta\delta}E_{q\delta}(2) \\
&\quad -2\delta'(2-1)(\theta^{p,p}-\theta^{p+1,p+1}+\vartheta^{p})^{\alpha}\mathcal{B}_q^{\beta}[(1)+(2)] \\
&\quad +((p,\alpha,1)\leftrightarrow(q,\beta,2)) \\
\{\mathcal{E}_{p\alpha}(1), \mathcal{B}_q^{\delta}(2)\} &= -4\delta_{\alpha}^{\beta}\delta(2-1)\mathcal{A}(1) \\
&\quad -2\delta'(2-1)(\theta^{q,q}-\theta^{q+1,q+1}+\vartheta^{p})^{\beta}\mathcal{E}_{p\alpha}(2) \qquad (1.81)
\end{aligned}
$$

where

$$
E_q \equiv \sum_r i^{r-q}\frac{1}{\sqrt{r}}(\Theta_{r-q-1}-\Theta_{r-q-2})\mathcal{E}_{r-1}
$$

1.4.3 Action

From this Hamiltonian form of the BRST operator for the superstring we can find the ZJBV form [5]:

$$
Q^{ZJBV} = \tfrac{i}{2}\dot{\phi}^A\phi_A - H - \check{\phi}_A\{\phi^A, Q^H] \qquad (1.82)
$$

where

$$
\begin{aligned}
[\phi^A, \phi^B\} &= \Omega^{AB} \\
\phi^A = \Omega^{AB}\phi_B, \qquad & \phi_B = \phi^A \Omega_{AB}
\end{aligned}
\tag{1.83}
$$

and $\check{\phi}_A$ is the antifield which is canonically conjugate to the field ϕ^A by the antibracket. ZJBV is useful for Lagrangian quantization, but since Q is sufficient for Hamiltonian quantization, we leave the details for Appendix A.5.

Constraints appearing in the gauge-invariant Hamiltonian have ghost number 0; their ghosts have ghost number 1; their antighosts have ghost number -1; the antifields of their antighosts have ghost number 0, and we can identify them in the ZJBV BRST operator with the Lagrange multipliers of the gauge-invariant Hamiltonian. (More generally, we interpret all the negative-ghost-number fields as antifields.) Let

$$
\begin{aligned}
\Phi_{p,p+1} &\equiv (-1)^{p+1} i \tfrac{1}{2}\sqrt{p+1}\,\check{\theta}_{p,p+1} \\
\Psi_{p,p+1} &\equiv (-1)^{p+1} i \tfrac{1}{2}\sqrt{p+1}\,\check{\pi}_{p,p+1} \\
\check{b} &\equiv g
\end{aligned}
\tag{1.84}
$$

and similarly for their antifields. Then we find the gauge invariant action in Hamiltonian form S_H either from the usual Hamiltonian procedure (using the constraints of the previous subsection), or as the antifield-free part of $Q^{ZJBV}_{sstring}$:

$$
S_H = -\dot{X}\hat{P}^0 + i\sum_{\pm,p}\dot{\theta}^{p,p}\pi_{p,p} - \sum_{\pm} g_\pm \mathcal{A}_\pm + \sum_{\pm,p\geq 0}\left(\tilde{\Phi}_{\pm,p,p+1}\mathcal{E}_p + \tilde{\Psi}_{\pm,p,p+1}\mathcal{B}_p\right)
\tag{1.85}
$$

(Again, for each sign $\pm$ we use fermions of the corresponding chirality.)

If we consider only the quadratic terms and the $\mathcal{A}$ term, keeping only the physical fields, and introducing $\hat{P}^1$ as an independent variable, we find the first-order, $2D$ world-sheet covariant form [11] (with world-sheet metric $\eta_{mn} = (-+)$)

$$S_0^{phys} \;=\; \hat{P}^m \partial_m X - \tfrac{1}{2} g_{mn} \hat{P}^m \hat{P}^n + i\sqrt{2} \sum_{\pm} \partial_\pm \theta_0^\pm \pi_0^\pm \tag{1.86}$$

where $\theta_0^\pm \equiv \theta_{0L,R}$, $\pi_0^\pm \equiv \pi_{0L,R}$, and $\partial_\pm \equiv \frac{1}{\sqrt{2}}(e_0{}^m \pm e_1{}^m)\partial_m$. By introducing supersymmetric variables

$$\begin{aligned}
P^m &= \hat{P}^m + \epsilon^{mn}(\eta^+_{(0)n} - \eta^-_{(0)n}) \\
D_0^\pm &= \pi_0^\pm + [\hat{P}^\pm \pm \tfrac{1}{2}(\eta^+_{(0)\mp} - \eta^-_{(0)\mp})] \cdot \gamma \theta_0^\pm \\
\eta^\pm_{(0)m} &\equiv \tfrac{i}{\sqrt{2}}(\partial_m \theta_0^\pm) \gamma \theta_0^\pm
\end{aligned} \tag{1.87}$$

(where we suppress spacetime indices for simplicity) and plugging this into (1.86) we find

$$\begin{aligned}
S_0^{phys} \;=\; & -\tfrac{1}{2} g_{mn} P^m P^n + P^m [\partial_m X - (\eta^+_{(0)m} + \eta^-_{(0)m})] \\
& -\epsilon^{mn}[(\partial_m X)\cdot(\eta^+_{(0)n} + \eta^-_{(0)n}) - \eta^+_{(0)m}\eta^-_{(0)n}] + i\sqrt{2}\sum_{\pm} \partial_\pm \theta_0^\pm D_0^\pm
\end{aligned} \tag{1.88}$$

Except for the last term this is the Green-Schwarz supersting action. To extend this redefinition to the whole action one can further define (we use p, q for ghost level and $(p), (q)$ when there is confusion with world sheet indices

$l, m, n)$

$$
\begin{aligned}
\mathcal{P}^m &= \hat{P}^m + \epsilon^{mn}(\eta^+_{(0)n} - \eta^-_{(0)n}) + \epsilon^{mn}(\chi^+_n - \chi^-_n) \\
\mathcal{D}^\pm_0 &= \pi^\pm_0 + \left\{[\hat{P}^\pm \pm \tfrac{1}{2}(\eta^+_{(0)\mp} - \eta^-_{(0)\mp})] \cdot \gamma\theta^\pm_0 \mp \tfrac{2}{3}(\xi^+_\mp - \xi^-_\mp) \cdot \gamma\tilde{\theta}^\pm\right\} \\
\mathcal{D}^\pm_p &= \pi^\pm_p + \mathcal{P}^\pm \cdot \gamma\theta^\pm_p \qquad\qquad (p \geq 1) \\
\pi^\pm_p &\equiv \pi^{p,p}_{L,R} \\
\theta^\pm_p &\equiv \theta^{p,p}_{L,R} \\
\vartheta^\pm_p &\equiv \vartheta^p_{L,R} \\
\chi^\pm_m &\equiv \tfrac{1}{\sqrt{2}}\left(2i(\partial_m\theta^\pm_0)\gamma\tilde{\theta}^\pm - i(\partial_m\tilde{\theta}^\pm)\gamma\tilde{\theta}^\pm + \tfrac{1}{2}\partial_m\tilde{R}^\pm\right) \\
\xi^\pm_m &\equiv \tfrac{1}{\sqrt{2}}\left(2i(\partial_m\theta^\pm_0)\gamma\tilde{\theta}^\pm - i\tfrac{5}{4}(\partial_m\tilde{\theta}^\pm)\gamma\tilde{\theta}^\pm + \tfrac{3}{4}\partial_m\tilde{R}^\pm\right) \\
\eta^\pm_{(p)m} &\equiv \tfrac{i}{\sqrt{2}}(\partial_m\theta^\pm_p)\gamma\theta^\pm_p \\
\Phi_{p\pm} &\equiv \Phi^{L,R}_{p,p+1} \\
\Psi_{p\pm} &\equiv \Psi^{L,R}_{p,p+1}
\end{aligned}
\qquad (1.89)
$$

Then our manifestly worldsheet-covariant action reads

$$
S_0 = \tilde{S}_{GS} + i\sqrt{2}\sum_{\pm,p\geq 0}\partial_\pm\theta^\pm_p\mathcal{D}^\pm_p + S_A \qquad (1.90)
$$

where S_A consists of Lagrange multipliers times all the (first-class) constraints other than Virasoro

$$
S_A = \sum_{\pm,p\geq 0}\left(\Psi_{p\pm}\mathcal{B}^\pm_p + \Phi_{p\pm}\mathcal{E}^\pm_p\right) \qquad (1.91)
$$

and $\tilde{S}_{GS}$ is an extension of the usual GS action to the fields $\theta^\pm_p$ at nonzero ghost levels

$$
\begin{aligned}
\tilde{S}_{GS} &= -\tfrac{1}{2} g_{mn} \mathcal{P}^m \mathcal{P}^n \\
&\quad + \sum_{\pm} \mathcal{P}^{\pm} \left[\partial_{\pm} X - (\eta^{+}_{(0)\pm} + \eta^{-}_{(0)\pm}) + \sum_{p\geq 1} \eta^{\pm}_{(p)\pm} \pm (\chi^{+}_{\pm} - \chi^{-}_{\pm}) \right] \\
&\quad - \epsilon^{mn} \left[(\partial_m X) \cdot \left\{ (\eta^{+}_{(0)n} - \eta^{-}_{(0)n}) + (\chi^{+}_{n} - \chi^{+}_{n}) \right\} + \eta^{+}_{(0)m} \eta^{-}_{(0)n} \right] \\
&\quad + \tfrac{2}{3} \sum_{\pm} \pm \tfrac{i}{\sqrt{2}} (\partial_{\pm} \theta^{\pm}_{0} \gamma \tilde{\theta}^{\pm})(\xi^{+}_{\mp} - \xi^{-}_{\mp})
\end{aligned}
\tag{1.92}
$$

This is a first-order action in terms of the coordinates $X, \theta^{\pm}_{p}$, momenta $\mathcal{P}^m, \mathcal{D}^{\pm}_{p}$, worldsheet metric g_{mn}, and Lagrange multipliers $\Phi_{p\pm}, \Psi_{p,\pm}$. Now $\mathcal{E}_p$ and $\mathcal{B}_p$ are expressed in terms of these new variables as

$$
\begin{aligned}
\mathcal{E}^{\pm}_{p} &= \mathcal{D}^{\pm}_{p} - \mathcal{D}^{\pm}_{p+1} + 2\mathcal{P}^{\pm} \cdot \gamma \theta^{\pm}_{p+1} \\
\mathcal{B}^{\pm}_{p} &= \mathcal{P}^{\pm} \cdot \gamma (\mathcal{D}^{\pm}_{p} + \mathcal{D}^{\pm}_{p+1} - 2\mathcal{P}^{\pm} \cdot \gamma \theta^{\pm}_{p+1}) \\
&+ (\Theta_{p-1} \theta^{\pm}_{p} - \theta^{\pm}_{p+1} + \vartheta^{\pm}_{p}) \times \Big\{ \mathcal{P}^{\pm 2} - [\mathcal{P}^{\pm} \mp (\eta^{+}_{(0)\mp} - \eta^{-}_{(0)\mp} + \chi^{+}_{\mp} - \chi^{-}_{\mp})]^2 \\
&\quad \pm \tfrac{1}{\sqrt{2}} \Big[\sum_{q\geq 1} i \partial_{\pm} \theta^{\pm}_{q} \left(\mathcal{D}^{\pm}_{q} - \mathcal{P}^{\pm} \cdot \gamma \theta^{\pm}_{q} \right) + i \partial_{\pm} \theta^{\pm}_{0} \\
&\quad \times \Big(\mathcal{D}^{\pm}_{0} - \Big\{ [\hat{P}^{\pm} \pm \tfrac{1}{2} (\eta^{+}_{(0)\mp} - \eta^{-}_{(0)\mp})] \cdot \gamma \theta^{\pm}_{0} \mp \tfrac{2}{3} (\xi^{+}_{\mp} - \xi^{-}_{\mp}) \cdot \gamma \tilde{\theta}^{\pm} \Big\} \Big) \ \Big] \quad \Big\}
\end{aligned}
\tag{1.93}
$$

Elimination of $\mathcal{P}^1$ by its equation of motion reproduces the previous Hamiltonian form of the action except for terms quadratic in $\mathcal{E}$, which can be eliminated by a redefinition of Φ. The gauge-fixed action with ghosts is most easily obtained from the Hamiltonian formalism as $H = \{Q, \int b\} = \int T$: Then (with the full $\theta^{p,q}$)

$$S_{GF} \;=\; \hat{P}^m \partial_m X - \tfrac{1}{2}\eta_{mn}\hat{P}^m\hat{P}^n + i\sqrt{2}\sum_{\pm}\partial_{\pm}c^{\pm}b_{\pm\pm} + i\sqrt{2}\sum_{\pm}\partial_{\pm}\theta^{\pm}\pi^{\pm} \quad (1.94)$$

1.5 Summary of 1st quantized BRST

We have given a gauge-invariant action for the superstring and its corresponding BRST operator. The BRST-invariant gauge-fixed action is the obvious quadratic expression following from $\{Q, \int b\}$ (and is thus BRST invariant since $Q^2 = 0$). This is sufficient to perform S-matrix calculations (with vertex operators of the type given for the superparticle above), but a naive application would require a measure that breaks manifest supersymmetry. (For example, solving for the cohomology of the superparticle with this BRST operator in [3] required using the equivalent of the lightcone gauge.) In principle, a covariant measure that avoids picture changing altogether (in particular, for the bosonic ghosts) can be found by methods similar to those used in [4]; we hope to return to this problem. The cohomology of this BRST operator should also be checked: The massless level follows from the previous analysis for the superparticle; the massive levels should follow from a similar lightcone analysis.

Chapter 2

Scattering amplitudes

2.1 Prologue

Many formalisms have been introduced for calculating scattering amplitudes for superstrings. The most practical of these have been (covariant) Ramond-Neveu-Schwarz (RNS) [13], (lightcone) Green-Schwarz (GS) [14], hybrid RNS-GS (H) [15], and pure spinor (PS) [7][16]. All of these have (at least) two important defects:

(1) Some kind of insertion is required. It may be separate from the vertices, or may be combined with some vertices to put them into different "pictures". The result is to complicate the calculations or destroy manifest symmetry. (The only exception is tree graphs with external bosons only, where such methods make cyclic symmetry more obscure but avoid producing extra terms that cancel.)

(2) Supersymmetry is not completely manifest. The most serious case is RNS, where fermion vertices are much more complicated than boson (because the

spinors are not free fields, so in practice noncovariant exponentials of bosons must be used), and sums over spin structures (periodic/antiperiodic boundary conditions) must be performed in loops. In the GS and H cases there is partial supersymmetry (and partial 10D Lorentz invariance), which complicates vertices for the "longitudinal" directions, which are required for general higher-point calculations; for this reason we will not consider GS and H in detail. The most symmetric is PS, which has only an integration measure that is explicitly dependent on the spinor coordinates.

In a previous paper [17] we introduced a new formalism for the superstring (based on a similar one for the superparticle [3]) using an infinite pyramid of ghosts for the spinor coordinate (GP) [1]. A derivation was also given from a covariant action. (The RNS action is not spacetime-supersymmetry covariant. The GS action [18] has defied covariant quantization [2]. The H and PS formalisms do not follow from the quantization of an action with general worldsheet metric.) The Becchi-Rouet-Stora-Tyutin operator found there was rather complicated, but fortunately none of the results of our previous paper will be needed explicitly here for calculation, but only for justification of the validity of our approach. In fact, the gauge-fixed action and massless vertex operators were guessed much earlier [11]. (An early attempt to apply them to amplitude calculations failed because spinor ghosts were not included [19].) The fact that these simple rules can be applied so naively hints that perhaps an even simpler formalism exists that implies the same rules.

There are (at least) two new conceptual results in this paper (in addition to the explicit calculations), both of which involve the treatment of zero-modes. These allow us to evaluate trees and loops without evaluating explicit integrals or (super)traces over these zero-modes, thereby solving the above two problems:

(1) In loop calculations we infrared regularize the worldsheet propagators. In principle one should do this anyway, since IR divergences are notorious in two dimensions, especially for 2D conformal field theories, but usually such problems are avoided by examining only IR-safe quantities. In our case such a regularization allows a simple counting of the infinite number of zero-modes arising from the ghost pyramid (including those from the physical spinor), with the only result being the introduction of factors of 1/4 due to the usual summation $1 - 2 + 3 - ... = 1/4$. (Regularization of x zero-modes is unnecessary; it only replaces the momentum-conservation δ-function with a sharp Gaussian.)

(2) In tree graphs these zero-modes do not appear separately, having been absorbed into the definition of the (first-quantized) vacuum. Specifically, since we do not perform explicit integration over spinor zero-modes, we also do not need to define measure factors for such integrations, make insertions of operators (essentially Dirac δ-functions in those modes) to kill those modes, nor use operators of different pictures to hide such insertions. We do not make special manipulations to deal with such modes; care of them is taken automatically by naively ignoring them. Although we do not analyze this vacuum (or other) state in detail here (we effectively work with the old Heisbenberg matrix mechanics, ignoring Schrödinger wave functions), we explain why such behavior is implied by the standard N=1 superspace formulation of the vector multiplet.

The net result of these ideas is that the calculational rules are the most naive generalization of the rules of the bosonic string: (1) The b and c ghosts appear in the same way, affecting only the measure. (2) The spinor ghosts serve only to ensure correct counting of zero-modes, and give an extra factor of $1/4$ to any trace of γ-matrices. (3) IR regularization takes care of all (physical and ghost) spinor zero-modes. (4) The vertex operator for the massless states generalizes the bosonic-string one just by adding the same spin terms as in ordinary field theory or supergraphs (to include the spinor vertices), taking into account the stringy generalization of the algebra of covariant derivatives [11].

Consequently, for the case of tree graphs with external vectors only, our rules are almost identical to (R)NS calculations in the $\mathcal{F}_1$ picture. We explain the advantage of this picture and why it is more relevant to the superstring.

As an interesting side result, we show how the $\partial\theta$ terms in the $DP\Omega$ current algebra arise already in the superparticle.

2.2 Rules

2.2.1 Vertex operators

We now present the main result of this paper, the rules themselves, with examples later. (Derivations are given in the Appendices.) Here we will calculate amplitudes with only massless external states. (We also concentrate on open strings, but the results generalize in the usual way to closed.)

To a limited extent first-quantization can be applied to particles as well as to strings: It gives only one-particle irreducible graphs (vertices at the tree level), whereas for the string it gives complete S-matrix amplitudes by duality (for given loop level and external states). However, the methods are almost identical, particularly since the superparticle is the zero-modes of the superstring.

The vertex operators follow from the results of our previous paper [17] but are basically those of [11] with a small modification from ghosts (as expected from the integrated vertex operators of PS [7][16]):

$$V = A^A(x, \theta) J_A$$

where A^A are superfields and J_A are 2D currents:

$$A^A = (A_\alpha, A^a, W^\alpha, F^{ab})$$

$$J_A = (\Omega^\alpha, P_a, D_\alpha, \hat{S}_{ab})$$

where J_A have zero-modes j_A, of which only p_a and d_α act nontrivially on A^A.

D, P, Ω are the currents of [11], while $\hat{S}$ is the Lorentz current of the θ ghosts ("superspin"). (Appendix B.1 gives the relation of vertex operators between Lagrangian and Hamiltonian formalisms.)

As for the bosonic string, the integrated vertex operator is $\int V$ and the unintegrated one is cV; the b and c ghosts work in exactly the same way, to keep the measure conformal. (We could also add a term $\alpha'(\partial_a A^a)\partial c$ to the unintegrated vertex operator to avoid having to apply $\partial_a A^a = 0$ [20].)

The external-state superfields and the currents can be expanded in θ for evaluation in terms of 2D Green functions of the fundamental variables: For example, the vertex for just the vector is then

$$V_B = A_a(x)\partial x^a + \tfrac{1}{2}F^{ab}(x)S_{ba}$$

where S is the Lorentz current of all θ's, physical and ghost. There are also terms higher-order in θ, but in the absence of external fermions there are no π's to cancel the extra θ's, so such terms won't contribute. Because of its universality, this form is useful for comparison to other formalisms.

2.2.2 Current algebra

However, when calculating general amplitudes (including fermions), it is more convenient to expand neither the currents nor superfields (thus manifesting supersymmetry). This requires rules for evaluating products of arbitrary numbers of currents. Although this problem is generally intractable for arbitrary representations of arbitrary current algebras, in our case it is relatively simple: (1) $\hat{S}$ doesn't act on the superfields. It is quadratic in free fields, so the matrix element of any product of such currents is simply the sum of products of loops of them (in 2D perturbation theory), from contracting the (ghost) θ of one with the π of the next. Each such loop contributes the trace of the product of the γ matrices that appear sandwiched between θ and π in $\hat{S}_{ab} = \theta\gamma_{ab}\pi|_>$ (where "$|_>$" means to restrict to ghosts).

(2) The remaining currents D_α, P_a, and Ω^α form a separate algebra. Although their "loops" are more complicated (since D is cubic in free fields), the structure constants are so simple that no loop contains more than 4 currents: only the combinations PP, $D\Omega$, DDP, or $DDDD$. Since D and P (but not Ω) can also act on superfields, the matrix element of such currents and superfields reduces to the sum of products of these 4 types with strings of D and P acting on superfields.

The loops are:

$$\begin{aligned}
\langle P_a(1)P_b(2)\rangle &= -\eta_{ab}G''_x(1,2)\\
\langle D_\alpha(1)\Omega^\beta(2)\rangle &= -i\delta_\alpha^\beta G'_\theta(1,2)\\
\langle D_\alpha(1)D_\beta(2)P_a(3)\rangle &= -i\gamma_{a\alpha\beta}[2G_\theta(2,3)G'_\theta(1,3) - 2G_\theta(1,3)G'_\theta(3,2)\\
&\quad +G_\theta(1,2)(G''_x(1,3)+G''_x(2,3))]\\
\langle D_\alpha(1)D_\beta(2)D_\gamma(3)D_\delta(4)\rangle &= 2iG'_\theta(1,2)G_\theta(1,3)G_\theta(1,4)(\gamma^a_{\alpha\gamma}\gamma_{a\delta\beta} - \gamma^a_{\alpha\delta}\gamma_{\gamma\beta})\\
&\quad +2iG'_\theta(1,2)G_\theta(2,3)G_\theta(2,4)(\gamma^a_{\beta\delta}\gamma_{\gamma\alpha} - \gamma^a_{\beta\gamma}\gamma_{a\delta\alpha})\\
&\quad +iG''_x(1,2)\left[G_\theta(1,3)G_\theta(2,4)\gamma^a_{\alpha\gamma}\gamma_{a\beta\delta}\right.\\
&\quad\left. - G_\theta(2,3)G_\theta(1,4)\gamma^a_{\beta\gamma}\gamma_{a\alpha\delta}\right] + \text{perm.}
\end{aligned} \tag{2.1}$$

where $\langle\quad\rangle$ refers to fully contracted operator products, and "1" means "z_1", etc. We have distinguished the x and $\theta\pi$ Green functions (G_x and G_θ) because only G_θ gives zero-mode corrections, which is explained in detail in Appendix B.5. For N string loops, G is a genus-N Green function: for trees, $G'_x(z_1-z_2) = -iG_\theta(z_1-z_2) = -\frac{1}{z_1-z_2}$ and $G'''_x(z_1-z_2) = -iG'_\theta(z_1-z_2) = \frac{1}{(z_1-z_2)^2}$; at 1 string loop they are Jacobi theta functions and their derivatives; etc.

The action of the currents on the fields is given by considering all possible symmetrizations of the D's. Any symmetrization of 2 D's (acting on a field) gives

$$D_{(\alpha}(1)D_{\beta)}(2) \quad \to \quad G_\theta(1,2)\gamma^a_{\alpha\beta}[P_a(1)+P_a(2)] \tag{2.2}$$

This reduces any string of currents to sums of strings of P's times antisymmetrized strings of D's, which are evaluated as

$$D_{[\alpha}(1)\cdots D_{\beta]}(2)P_a(3)\cdots P_b(4)A(5) = G_\theta(1,5)\cdots G'_x(4,5)(d_{[\alpha}\cdots d_{\beta]}p_a\cdots p_b A)(5) \tag{2.3}$$

where $p_a = -i\partial_a$, and we can replace $\pi_\alpha = \partial/\partial\theta^\alpha$ with the usual supersymmetry covariant derivative d_α in such antisymmetrizations since final results can always be evaluated at $\theta = 0$ by supersymmetry.

By 10D dimensional analysis, any $\hat{S}$ loop is dimensionless, while any $DP\Omega$ loop has dimension 2. This implies (contrary to expectations, but well known from the bosonic case) that each $DP\Omega$ loop carries an extra factor of *the inverse of* α'. (In the particle case, there is instead an inverse of z.) Thus, the maximum number of $DP\Omega$ loops gives the lowest power in momenta, and each loop less gives two more powers of momenta. (One way to see the dimensional analysis is to note that each current acting on a superfield gives a

G'_x or G_θ. The same is true in a $DP\Omega$ loop, except that 2 currents "close" the loop to give a G''_x or G'_θ. Thus, each $DP\Omega$ loop introduces an extra factor of G''_x(or G'_θ)$/(G_\theta)^2$. On the other hand, closing an $\hat{S}$ loop gives a $(G_\theta)^2$ instead of G'_θ, so such loops give no extra factor.)

Finally, there is the usual momentum dependence coming from Green functions connecting the superfields to each other, from their x dependence only: For the usual plane waves,

$$\langle A(1) \cdots A(N) \rangle = A \cdots A \; e^{-\sum_{i<j} k_i \cdot k_j G_x(i,j)} \tag{2.4}$$

with units $\alpha' = 1/2$ for the string.

2.2.3 Component expansion

The final result for an amplitude is given as a "kinematic factor" times a scalar function of momentum invariants, expressed as an integral over the worldsheet positions of the vertices. The kinematic factor is expressed, by the above procedure, as a sum of products of superfields, representing external state wave functions. The string rules have effectively already performed covariant θ integration, so these superfields may be evaluated at $\theta = 0$. (As in the usual superspace methods, where θ expansion and integration is replaced by the action on the "Lagrangian" of the product of all supersymmetry covariant derivatives d_α, supersymmetry guarantees that all θ dependence cancels, up to total x derivatives.)

The evaluation of spinor derivatives follows from the (linearized) constraints on the gauge covariant superspace derivatives, and their Bianchi identities [10]. The result is

$$
\begin{aligned}
d_{(\alpha}A_{\beta)} &= 2\gamma^a_{\alpha\beta}A_a \\
d_\alpha A_a - \partial_a A_\alpha &= 2\gamma_{a\alpha\beta}W^\beta \\
d_\alpha W^\beta &= \tfrac{1}{2}\gamma^{ab}{}_\alpha{}^\beta F_{ba} \\
d_\alpha F^{ab} &= 2i\gamma^{[a}_{\alpha\beta}\partial^{b]}W^\beta
\end{aligned} \tag{2.5}
$$

The result is also (linearized) gauge invariant (except for $\partial_a A^a = 0$, as explained above), so one may use a Wess-Zumino gauge where $A_\alpha = 0$ at $\theta = 0$. (A review of gauge covariant derivatives appears in Appendix B.3.)

2.2.4 IR regularization

In evaluation of tree graphs there is the usual δ-function for conservation of total momentum from the zero-modes of x, but θ effectively has no zero-modes: The effect of the θ ghosts is to mimic GS where, unlike momentum, the 8 surviving fermionic variables of the lightcone are self-conjugate, and thus have no vanishing eigenvalues. Thus there is no residual integration over θ zero-modes (unlike PS).

In loops there is the usual summation over θ zero-modes in the sum over all states, but the ghosts again mimic GS by effectively reducing the number to 8 from the physical 32 (θ and conjugate π), using the sum

$$1 - 2 + 3 - 4 + ... = 1/4$$

when counting the number of θ's at successive ghost levels (alternating in statistics). Application of this rule requires infrared regularization of the 2D Green functions to "remove" the zero-modes: The factor in the partition function from these zero-modes is the IR regulator ϵ to the power $16 \times 1/4 = 4$ (from the 16-valued spinor index on the θ's). Since the $\theta\pi$ Green function goes as $1/\epsilon$ (+ the usual finite expression + $\mathcal{O}(\epsilon)$), the amplitude vanishes until 4-point. Thus the power of the regulator counts zero-modes.

The $1/4$ rule also applies in γ-matrix algebra. Amplitudes involve traces of products of γ-matrices. These matrices are the same at each ghost level (except that chirality, as well as statistics, alternates with ghost level), so the net effect of the ghosts appears only when taking a trace: Applying the usual γ-matrix identities, the trace is reduced to $str(I) = 16 \times 1/4 = 4$, again reproducing GS. The difference from GS is that the γ-matrices are for 10 dimensions, so the result is Lorentz covariant, and the usual 10D Levi-Civita tensor is produced (where appropriate) instead of spurious 8D ϵ-tensors. For example, anomalies can be found from 6-point graphs. (Details of the regularized Green functions are given in Appendix B.5.)

2.3 Trees

2.3.1 RNS pictures

We begin by proving that the trees with external bosons are identical to those obtained from (the NS sector of) RNS. This is most obvious in the $\mathcal{F}_1$ picture. Although this picture was the original one to be used in (R)NS amplitude calculations, it was immediately replaced with the $\mathcal{F}_2$ picture [21]. We refer here to the picture for the physical coordinates (x, ψ), and not just the ghosts: For example, vector vertices have always been $\partial x + ...$ except for two ψ vertices, while in the $\mathcal{F}_1$ picture all vertices are $\partial x + ...$. In the proof of equivalence [21], starting from the $\mathcal{F}_1$ picture, one pulls factors of (the $\pm 1/2$ modes of) $G = \psi \cdot \partial x$ (worldsheet supersymmetry generator) off of two unintegrated $\partial x + ...$ vertices to turn them into ψ vertices, then collides the G's to produce (the 0 mode of) a worldsheet energy-momentum tensor T, which gives a constant acting on a physical state. (With ghosts the approach is similar, with G replaced with the picture-changing operator, which is simply the operator product of the gauge-fixed G with e^{ϕ} in terms of the bosonized ghost ϕ.) The resulting rules are then the same as the rules for the bosonic string, including the factors of c for the three unintegrated vertices, except that the ∂x vertex has the extra spin term. The β and γ ghosts are completely ignored; the vacuum used is in what is usually called the "-1 picture", so the zero-modes of γ (or ϕ) are already eliminated. (What is usually called "picture changing" in the modern covariant formalism would start with the $\mathcal{F}_2$ picture, introduce two factors of picture changing times inverse picture changing, use the picture changing to change the two ψ vertices, and use the inverses to change the initial and final vacuua. Unfortunately, the inverse has an overall factor of c, so in the new vacuum $\langle \gamma\gamma c \rangle \sim 1$ [22], and the γ's pick out the ψ terms again in two of the unintegrated vertices $\gamma\psi + c(\partial x + ...)$. Thus such transformations preserve the $\mathcal{F}_2$ picture as far as the physical sector is concerned.)

Historically, the $\mathcal{F}_1$ picture was introduced first because: (1) It is more similar to the bosonic string, and (2) cyclic symmetry is manifest (no need to bother with picture changing). The $\mathcal{F}_2$ picture was then chosen because the physical-state conditions were more obvious. Although in modern language the BRST conditions are clear in either picture, it's interesting to examine the differences in the pictures if the ghosts are ignored, since the ghosts differ in different formulations of the superstring, but all formulations have similar integrated vertices. Then the ground state of the $\mathcal{F}_2$ picture is the "physical" tachyon, at $m^2 = -1/2$, while in the $\mathcal{F}_1$ picture it's an "unphysical" tachyon at $m^2 = -1$. Furthermore, the $\mathcal{F}_1$ picture has an additional "ancestor" trajectory 1/2 unit higher than the leading physical trajectory. These "disadvantages" were noticed in the days before Gliozzi-Scherk-Olive projection. On the other hand, for the superstring this projection eliminates the "physical" tachyon as well as the ancestor trajectory. So the only remaining additional unphysical state of the $\mathcal{F}_1$ picture is its vacuum, while GSO projection has eliminated the vacuum of the $\mathcal{F}_2$ picture altogether! This suggests that any comparison of the RNS formulation to others would be easier in the $\mathcal{F}_1$ picture.

2.3.2 For bosons only

The proof of equivalence of the $\mathcal{F}_1$ vector trees to the vector trees of our formalism is then simple: One only has to note that the operator algebra of the vertices is identical. But the vertices are identical in form; only the explicit representation of the spin current is different. So one only has to check the equivalence of the two current algebras. Since they are both (10D) Lorentz currents, quadratic in free fields, this means just checking that the central charge is the same. (The same method has been used for comparing PS to the $\mathcal{F}_2$ picture [8].) The reason the result for the central charge is the same is that the GP result is the same as the GS result: The γ-matrix algebra is the same except for a trace, which is 1/4 as big in the lightcone as for a covariant spinor, but GP again gets a factor of 1/4 from summing over ghosts. (As we'll see below, similar arguments apply in loops, unless one gathers enough spin currents to produce a Levi-Civita tensor.)

The calculations in the $\mathcal{F}_1$ picture (and GP) are somewhat harder than the $\mathcal{F}_2$ picture because two vertices have been replaced with ones that generate more terms, which cancel. Also, RNS bosonic trees are simpler than PS or GP because integration over the vector fermion ψ^a effectively does all γ-matrix algebra. However, tree amplitudes with fermions are much harder in RNS than PS or GP (and increase in difficulty as the number of fermions increases). PS is still simpler than GP, because θ integration takes the place of the change in the two vertices, and so also avoids generating extra terms. So, for trees RNS is the easiest for pure bosons, PS is easiest with fermions, and GP is a bit harder than both. However, GP requires fewer rules, since all vertices are the same, so it produces more terms at an intermediate stage but is easier to "program". This feature is a peculiarity of tree graphs: At the loop level we'll see that GP maintains the simplest rules, while RNS produces extra terms that cancel (because supersymmetry is not manifest).

At first sight these rules for GP might seem peculiar because there is no explicit integral over spinor zero-modes, as expected in known superspace approaches. The answer can be seen from examining the simpler (and better understood) case of 4D N=1 super Yang-Mills. Since the vacuum of the open bosonic string can be identified with a constant Yang-Mills ghost (or gauge parameter), we examine the ghost superfield ϕ, and look at $\phi = 1$, a supersymmetric condition. Since this superfield is chiral, and supergraphs prefer unconstrained superfields, we write $\phi = \bar{d}^2\chi$ in terms of a general complex superfield χ. Then clearly $\chi = \bar{\theta}^2$ in our case. This is still supersymmetric because of the gauge invariance $\delta\chi = \bar{d}_{\dot{\alpha}}\lambda^{\dot{\alpha}}$. Furthermore, this χ has a nice norm, $\int d^4\theta\ |\chi|^2 = 1$. In Hilbert-space notation we thus write the norm and supersymmetry as

$$\langle 0|0\rangle = 1, \qquad q_\alpha|0\rangle = Q|\lambda\rangle_\alpha$$

so the vacuum is supersymmetry invariant up to a BRST triviality, and the norm includes zero-mode integration, but the extra zero-modes are absorbed by the vacuua, and no insertions are required. (We could also use $|\lambda\rangle_\alpha = \Lambda_\alpha|0\rangle$ to define $\hat{q}_\alpha = q_\alpha - [Q, \Lambda_\alpha]$, $\hat{q}_\alpha|0\rangle = 0$.) This supersymmetry of the vacuum is enough to ensure the amplitudes transform correctly, since the vertex operators are superfields times supersymmetry invariant currents, and the vacuum and vertex operators (integrated and unintegrated) are BRST invariant. (The unintegrated vertex operators we have used are BRST invariant only after including terms higher-order in ghost θ's, which don't contribute to amplitudes for massless external states, and probably not for massive ones either, because of the absence of ghost π's to cancel them.) The fact that the vacuum is "half-way" up in the θ expansion was also found for the expansion in the spinor ghost coordinates in a lightcone analysis of the BRST cohomology for the GP superparticle [3]. Note that this choice of vacuum is relevant only for trees; at 1 loop one effectively does a (super)trace over all states rather than a vacuum expectation value, so the vacuum is irrelevant. (We assume a similar situation will occur at higher loops, but we have not checked yet.) As we will see below, one important affect on this vacuum choice for trees, which does not affect loops, is that:

For tree graphs only, background fields are always evaluated in the Wess-Zumino gauge.

If this vacuum structure can be better understood, it might be possible to find an analog of the $\mathcal{F}_2$ picture for GP, avoiding the production of extra canceling terms, making it the simplest formalism even for trees. As an attempt at formulating such a picture, one can consider this picture for RNS: For pure bosons, two vertices must be in the "-1 picture", so it is convenient to consider those two as the initial and final states, using the vertex operators on the initial and final vacuua. Generalizing only those 2 states to include fermions, we can write their vertex operators as in GP, but now identifying the currents with

$$D_\alpha = e^{-\phi/2} S_\alpha, \quad P_a = e^{-\phi}\psi_a, \quad \Omega^\alpha = e^{-3\phi/2} S^\alpha$$

The first two are the usual for the spinor (in the $-1/2$ picture) and vector (in the -1 picture), while the last can be identified as that for the spinor in the $-3/2$ picture (also with conformal weight 1) if we use the "supersymmetric gauge"

$$W^{\alpha} \sim \gamma^{a\alpha\beta} \partial_{a} A_{\beta}$$

instead of the WZ gauge. (Hitting $cA_{\alpha}\Omega^{\alpha}$ with picture changing produces $cW^{\alpha}D_{\alpha}$.) These currents satisfy almost the same algebra as the usual ones (including $DP \sim \Omega$, to leading order); the only exceptions are ΩP and $\Omega\Omega$. As a guess for the GP analog, we can then try to construct a new $DP\Omega$ for this picture that depends only on the ghosts. Unfortunately (the simplest guess for) this construction seems not to work, apparently because the dependence on the WZ gauge hasn't been eliminated, and is incompatible with the supersymmetric gauge.

2.3.3 General 3-point

As explained in Section 2, we prefer the superfield formalism for the calculation of amplitudes with fermions. This includes the all-vector amplitude in the same calculation. The only nonvanishing operator products for the 3-point tree, after applying the Landau gauge condition ($\partial \cdot A = 0$), are:

$$
\begin{aligned}
&A. \quad \langle P_a(1)P_b(2)\rangle \times P_c(3)A^a(1)A^b(2)A^c(3) + \text{permutations} \\
&B. \quad (P_c(3)A^a(1))(P_a(1)A^b(2))(P_b(2)A^c(3)) + \text{perm.} \\
&C. \quad \langle P_a(1)P_b(2)\rangle \times D_\alpha(3)A^a(1)A^b(2)W^\alpha(3) + \text{perm.} \\
&D. \quad \langle D_\alpha(1)D_\beta(2)P_a(3)\rangle \times W^\alpha(1)W^\beta(2)A^a(3) + \text{perm.} \\
&E. \quad \langle D_\beta(2)\Omega^\gamma(3)\rangle \times D_\alpha(1)W^\alpha(1)W^\beta(2)A_\gamma(3) + \text{perm.} \\
&F. \quad (D_\gamma(3)W^\alpha(1))(D_\alpha(1)W^\beta(2))(D_\beta(2)W^\gamma(3)) + \text{perm.} \\
&G. \quad \langle \hat{S}_{ab}(1)\hat{S}_{cd}(2)\rangle \times P_e(3)F^{ab}(1)F^{cd}(2)A^e(3) + \text{perm.} \\
&H. \quad \langle \hat{S}_{ab}(1)\hat{S}_{cd}(2)\hat{S}_{ef}(3)\rangle \times F^{ab}(1)F^{cd}(2)F^{ef}(3) \qquad (2.6)
\end{aligned}
$$

where the $\hat{S}$ contraction is the usual γ trace.

The other contributions, like $(\langle D_\alpha\Omega^\beta\rangle P)\cdot AW^\alpha A_\beta$, $(\langle SS\rangle D_\alpha)\cdot FFW^\alpha$, $(PPD_\alpha)\cdot AAW^\alpha$ and $(PD_\alpha D_\beta)AWW$, all vanish using $k_i \cdot k_j = 0, \not{k}W = 0$ in the Wess-Zumino gauge. We give some details of the calculation in Appendix B.6.

Notice that F and H combine to give the GP sum $1 - 2 + 3 - 4 \cdots = 1/4$. From these combinations we find the manifestly supersymmetric 3-point tree amplitudes for vectors and spinors

$$
\begin{aligned}
A_3^{tree} \quad = \quad & k_1 \cdot A(3)A(1) \cdot A(2) + k_3 \cdot A(2)A(1) \cdot A(3) + k_2 \cdot A(1)A(2) \cdot A(3) \\
& + iA(1) \cdot W(2)\gamma W(3) + iA(2) \cdot W(3)\gamma W(1) + iA(3) \cdot W(1)\gamma W(2)
\end{aligned} \tag{2.7}
$$

where $A(i)$ are the vectors and $W(i)$ the spinors. (Note that we use the usual anticommuting fields for the spinors; numerical evaluation involves fermionic functional differentiation, replacing these fields with the usual commuting wave functions, and may introduce signs if not all terms have the same ordering.)

This result applies to both the superparticle and superstring. In the string case there is also a factor of $1/(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)$ from the Green functions, but this is canceled as usual with the inverse factor from the conformal measure obtained from $\langle c(1)c(2)c(3)\rangle$.

2.4 IR regularization

2.4.1 Zero modes

The kinematic factor in supersymmetric amplitudes is closely related to the spinor zero-mode problem, which is the most important problem in the Lorentz covariant superparticle and superstring. If we naively integrate over zero-modes of the infinite pyramid of spinors with no vertex attached, we find $0 \cdot \infty^2 \cdot 0^3 \cdot \infty^4 \cdot 0^5 \cdot \infty^6 \cdots$. So we need to regularize the zero-mode integration. In Appendix B.5 we derive the 2D Green function with a 2D regularization mass, but it turns out that the zero-mode behavior of the 2D Green function is exactly that of the 1D one. So we will concentrate on the 1D case here. To do this IR regularization we introduce small mass terms in the superparticle free action (for 1D "proper time" coordinate z)

$$X^a(-\partial_z^2 + \xi^2)X_a, \qquad -i\pi(\partial_z + \epsilon)\theta \tag{2.8}$$

Now we can fix the measure of zero-modes for X and θ without ambiguity. For X, neglecting the Laplacian term, which vanishes for zero-modes,

$$\begin{aligned}
\lim_{\xi\to 0} \int d^D X_0 \; e^{-T\xi^2 X_0^2/2 - i(\sum k)\cdot X_0} &= \lim_{\xi\to 0} \left(\frac{2\pi}{T\xi^2}\right)^{D/2} e^{-(\sum k)^2/2T\xi^2} \\
&= (2\pi)^D \delta^D\left(\sum k\right) \\
&= \int d^D X_0 \; e^{-i(\sum k)\cdot X_0}
\end{aligned}$$

where T is the range of z (at 1 loop, the period). Here we used $\lim_{\xi\to 0} e^{-x^2/2\xi^2}/\sqrt{\xi} =$

$\sqrt{2\pi}\delta(x)$. Therefore our zero-mode measure for X is

$$\int dX_0 = \lim_{\xi\to 0}\left(\frac{2\pi}{T\xi^2}\right)^{D/2} \tag{2.9}$$

However, this bosonic zero-mode does not appear explicitly, since this always gives momentum conservation thanks to the vertex operators.

Similarly for θ we see

$$\int d\theta\ d\pi\ e^{iT\epsilon\pi\theta} = (i\epsilon T)^{\pm 2^{(D-2)/2}} \tag{2.10}$$

where "$\pm$" stands for fermionic and bosonic spinor respectively. Then our zero-mode measure for a spinor is

$$\int d\theta\ d\pi = \lim_{\epsilon\to 0}(i\epsilon T)^{\pm 2^{(D-2)/2}} \tag{2.11}$$

In our case we have an infinite pyramid of spinors and hence we get

$$\begin{aligned}\int d\theta\ d\pi &= \lim_{\epsilon\to 0}(i\epsilon T)^{(2^{(D-2)/2})(1-2+3-4+\cdots)} \\ &= \lim_{\epsilon\to 0}(i\epsilon T)^{2^{(D-6)/2}}\end{aligned} \tag{2.12}$$

where we used coherent-state regularization for the ambiguous sum $1-2+\cdots=$

1/4:

$$\begin{aligned}
tr[(N+1)(-1)^N] &= \int \frac{d^2z}{\pi} e^{-z^*z} \langle z|(a^\dagger a + 1)(-1)^{a^\dagger a}|z\rangle \\
&= \int \frac{d^2z}{\pi} e^{-|z|^2} (\langle z|a^\dagger a| - z\rangle + \langle z| - z\rangle) \\
&= \int \frac{d^2z}{\pi} (-|z|^2 + 1) e^{-2|z|^2} \\
&= -\tfrac{1}{4} + \tfrac{1}{2} = \tfrac{1}{4}
\end{aligned} \tag{2.13}$$

More intuitively

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 \cdots \\
\frac{1}{(1+x)^2} &= 1 - 2x + 3x^3 - 4x^4 + \cdots
\end{aligned}$$

so at $x = 1$ we get 1/2 and 1/4 respectively.

Therefore in $D = 10$ we get effectively ϵ^4 for zero-modes. So our complete spinor measure with non-zero modes is

$$\mathcal{D}\theta\ \mathcal{D}\pi\ (iT\epsilon)^4 \tag{2.14}$$

The significant role of this effective power will be clear after we discuss the Green function.

The regularization $1 - 2 + 3 - 4 + \cdots = 1/4$ explains how we can get a physical $SO(8)$ spinor contribution out of 2 covariant 16-component spinors π and θ. Because we cannot project covariant spinors into physical spinors in a covariant way, we need to add infinitely many ghosts to achieve this 1/4 reduction in amplitudes.

2.4.2 Regularized Green functions

We summarize the results of Appendix B.5 here. We find the regularized 1D Green functions for x and θ

$$
\begin{aligned}
G^x(z) &= \frac{1}{2\xi}\frac{\cosh[\xi(|z|-T/2)]}{\sinh(\xi T/2)} \\
G^\theta(z) &= i(-\partial_z+\epsilon)\left[\frac{1}{2\epsilon}\frac{\cosh[\epsilon(|z|-T/2)]}{\sinh(\epsilon T/2)}\right]
\end{aligned}
\tag{2.15}
$$

The ϵ correction to ∂_z in G^θ is nontrivial because it multiplies a Green function with a $1/\epsilon$ term.

It is convenient to expand the Green functions in ϵ when we calculate scattering amplitudes:

$$
\begin{aligned}
G^x &= \frac{1}{\xi^2 T}+\sum_{n=0}^{\infty} G^x_n \xi^n \\
G^\theta &= \frac{i}{\epsilon T}+\sum_{n=0}^{\infty} G^\theta_n \epsilon^n
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
G^x_0 &= \frac{T}{12}+\frac{|z|(|z|-T)}{2T}=\frac{T}{12}+G^x_{un} \\
G^\theta_0 &= \frac{i}{2}Sign(z)-i\frac{z}{T}=G^\theta_{un}
\end{aligned}
\tag{2.17}
$$

and G^x_{un} and G^θ_{un} are the usual $1D$ Green functions with periodic boundary conditions, normalized to $G_{un}(0)=0$. The extra constant will not contribute to massless amplitudes because of derivatives and $k^2=0$.

Because the mass (re)moves zero-modes, the usual fudges of the massless

Green functions are eliminated: There is no freedom to add constants (dependent on T, but not z) to G, and the δ function in its equation of motion is not modified to $\delta(z) - 1/T$ to preserve "charge conservation". But the latter property is restored upon expansion in the regulator:

$$\begin{aligned} G^x &= \frac{1}{\xi^2 T} + \Delta G^x, \quad (-\partial^2 + \xi^2)G^x = \delta \quad \Rightarrow \quad (-\partial^2 + \xi^2)\Delta G^x = \delta - \frac{1}{T} \\ G^\theta &= \frac{i}{\epsilon T} + \Delta G^\theta, \quad -i(\partial + \epsilon)G^\theta = \delta \quad \Rightarrow \quad -i(\partial + \epsilon)\Delta G^\theta = \delta - \frac{1}{T} \end{aligned} \tag{2.18}$$

Similarly we will do this expansion for superstring Green functions. The details are given in Appendix B.5. However, expansion of G^x is unnecessary, because in vertex operators X appears only as $\dot{X}$ (and as an argument of the superfields), and any contraction involving this vertex operator is always finite. (The derivative kills the potentially divergent $1/\xi^2$ term.) For this reason X regularization gives only energy-momentum conservation and is irrelevant to amplitude corrections. But ϵ expansion of G^θ is crucial, as we will see in the next section.

2.5 Loops

2.5.1 N<4 super

Here we give simple examples. The only differences from standard first-quantization of a loop of a scalar particle or bosonic string can be associated with "kinematic factors" that may also depend on the positions of the vertices on the worldline/sheet. (For a summary of the standard analysis of the other factors, see Appendix B.4.)

Collecting the results of the zero-mode measure and the Green function zero-mode behavior, *the amplitude is zeroth order in ϵ.*

Since there is an ϵ^4 in the measure, we should pick up an ϵ^{-4} in the integrand of the path integral. For example, one sub-diagram of the N-point 1-loop amplitude is proportional to

$$\oint d\epsilon \; \epsilon^3 \; G^\theta(1,2) G^\theta(2,3) \cdots G^\theta(N,1) \tag{2.19}$$

Then to evaluate this amplitude we should expand each G^θ and collect terms with ϵ^{-4}.

We now notice that every G^θ gives $i/\epsilon T$. For $N < 4$ there are not enough powers of ϵ^{-1} and so their amplitudes just vanish.

There is no zero-mode behavior for any contraction involving ∂X because of the derivative. Therefore P contractions start to contribute only at $N = 5$ (a black dot in Fig. 2.1).

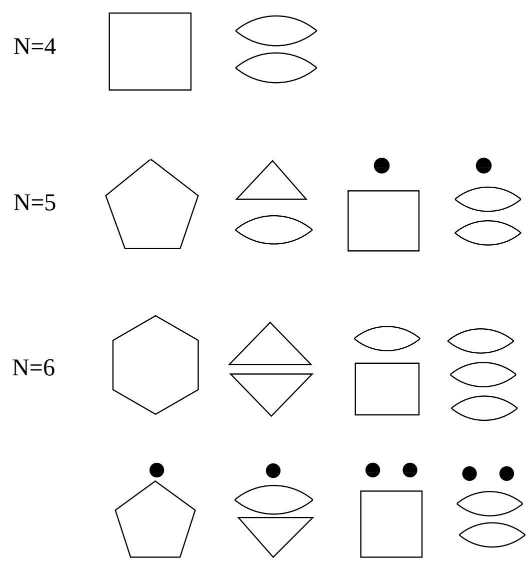

Figure 2.1: Schematic diagrams for various contractions

2.5.2 N=4 vector only

The first nonvanishing amplitude is at $N = 4$. However, this is just the case where every G^θ from the S_{ab}'s contributes $i/\epsilon T$. So the integration is trivially done for K_4 and only its spin algebra matters. There are two kinds of diagrams: the case where all 4 points are connected, and the case where each pair of points is connected separately (Fig. 2.1). These two diagrams have opposite sign. Each closed contraction should be traced over all ghost pyramid spinors to give $1 - 2 + 3 - 4 + \cdots = 1/4$. Therefore we get for K_4, omitting external field factors,

$$\begin{aligned} K_4 &= -\frac{1}{4}\left[tr(\gamma^{ab}\gamma^{cd}\gamma^{ef}\gamma^{gh}) + 5 \text{ permutations }\right] \\ &\quad +\frac{1}{16}\left[tr(\gamma^{ab}\gamma^{cd})tr(\gamma^{ef}\gamma^{gh}) + 2 \text{ permutations }\right] \end{aligned} \tag{2.20}$$

Using the Mathematica code Tracer.m we evaluate this gamma-matrix trace to find

$$
\begin{aligned}
K_4 \quad = \quad & \tfrac{1}{2}(\delta^{bc}\delta^{de}\delta^{fg}\delta^{ha} + \delta^{be}\delta^{cf}\delta^{dg}\delta^{ha} + \delta^{ae}\delta^{fg}\delta^{ch}\delta^{bd} \ + 45 \text{ terms} \\
& \quad \text{from antisymmetrizing each pair of indices } \ [ab][cd][ef][gh]) \\
& -\tfrac{1}{2}\left[(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})(\delta^{eg}\delta^{fh} - \delta^{eh}\delta^{fg})\right. \\
& \quad +(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be})(\delta^{cg}\delta^{dh} - \delta^{ch}\delta^{dg}) \\
& \quad \left. + \ (\delta^{ag}\delta^{bh} - \delta^{ah}\delta^{bg})(\delta^{ce}\delta^{df} - \delta^{cf}\delta^{de})\right] \qquad (2.21)
\end{aligned}
$$

This is the well-known kinematic factor for both tree and 1-loop. We can also express this results in terms of F as [23]

$$
\begin{aligned}
& F^{ac}(1)F^{b}{}_{c}(2)F_{a}{}^{d}(3)F_{bd}(4) - \frac{1}{8}F^{ab}(1)F_{ab}(2)F^{cd}(3)F_{cd}(4) \\
& -\frac{1}{4}F^{ab}(1)F^{cd}(2)[F_{ab}(3)F_{cd}(4) - 2F_{ac}(3)F_{bd}(4)] \qquad (2.22)
\end{aligned}
$$

which can be interpreted as “graviton”, “dilaton”, and “axion” as far as Lorentz (and not gauge) structure is concerned. (In the nonplanar case, it actually corresponds to those poles for color singlets in the 1+2=3+4 channel.)

2.5.3 N=4 super

Here we again prefer the superfield formalism as explained in Section 2. However, the 4-point one-loop case is dramatically simplified due to the IR regularization. Consider the 4 types of fully contracted operators again and then notice that they can have only limited $1/\epsilon$ factors, since each G^θ gives such a factor while $G^{\theta\prime}$ doesn't:

$$\begin{aligned}
P_a(z_1)P_b(z_2) &: \mathcal{O}(\epsilon^0) \\
D_\alpha(z_1)\Omega^\beta(z_2) &: \mathcal{O}(\epsilon^0) \\
D_\alpha(z_1)D_\beta(z_2)P_a(z_3) &: \mathcal{O}(\epsilon^{-1}) \\
D_\alpha(z_1)D_\beta(z_2)D_\gamma(z_3)D_\delta(z_4) &: \mathcal{O}(\epsilon^{-2}) \\
(\hat{S})^n &: \mathcal{O}(\epsilon^{-n})
\end{aligned} \tag{2.23}$$

This means that except for $\hat{S}$ they appear at best from the 6-point at 1 loop. So the only contractions for this amplitude are from $\hat{S}^4$ and $\hat{S}^2 d^2$. We also need to consider the case where 4 d's act on the superfields. Then we can directly write down the kinematic factor for the manifestly supersymmetric, 4-point, 1-loop amplitude

$$\begin{aligned}
&\tfrac{1}{4!} d_{[\alpha} d_\beta d_\gamma d_{\delta]} W^\alpha(1) W^\beta(2) W^\gamma(3) W^\delta(4) \\
&+\tfrac{3}{32} tr(\gamma^{ab}\gamma^{cd}) d_{[\alpha} d_{\beta]} F_{ab}(1) F_{cd}(2) W^\alpha(3) W^\beta(4) + \text{perm.} \\
&+\hat{K}_4(F^4)
\end{aligned} \tag{2.24}$$

$\hat{K}_4$ is the same as K_4 above except that the (super)traces don't include the physical π, θ. Of course, this missing contribution comes from $(d_\beta W^\alpha)$ $(d_\alpha W^\beta)$ $(d_\delta W^\gamma)$ $(d_\gamma W^\delta)$ plus different permutations of the d's. Also, the missing contribution for the $tr(\gamma^{ab}\gamma^{cd})$ terms comes from $-\frac{1}{4} W^\alpha (d_{[\alpha} d_{\delta]} W^\beta) W^\gamma (d_{[\gamma} d_{\beta]}) W^\delta$ plus different permutations. Note that this result already has the same form as the 4D N=1 supergraph calculation for N=4 super Yang-Mills [24] (if we rewrite it in Majorana notation for comparison), where there $tr(I) = 4$ already, so $\hat{S}$ terms are unnecessary to produce $str(I) = 16 \times 1/4 = 4$. (There the d^4 comes from overall θ integration, the d's of the W's being killed by loop-θ integration.)

We give here the fermion part of the result of (2.24) and leave details to Appendix B.6.

$$
\begin{aligned}
K_4^{FFBB} &= -\tfrac{i}{2} W(1)\gamma_{ab}\gamma_c\partial_d W(2) F^{cd}(3)F^{ab}(4) + 3 \leftrightarrow 4 \\
&= \tfrac{i}{2} W(1)\gamma_{abc}\partial_d W(2) F^{ab}(3)F^{cd}(4) + iW(1)\gamma_a\partial_b W(2)F^{ac}(3)F_c{}^b(4) + 3 \leftrightarrow 4 \\
K_4^{FFFF} &= -4k_1 \cdot k_4 \; W(1)\gamma W(2) \cdot W(3)\gamma W(4) + 2 \leftrightarrow 4
\end{aligned}
\tag{2.25}
$$

where $\gamma_{abc} = \frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}$. (The $[abc]$ means to sum over permutations with signs to antisymmetrize.) The second form of the FFBB amplitude can be interpreted as "axion" and "traceless graviton" terms. (Using the fermion field equation and symmetry, the former term is totally antisymmetric in $abcd$ and a total curl on the fermions, as the FF factor is then for the bosons, while the latter term is symmetric and traceless in ab.) We have written these amplitudes in manifestly gauge invariant form. Note that the complete 4-point amplitude is totally symmetric in all 4 external lines. (This was clear from the original form (2.24).) This means that not only are the specific cases listed above separately symmetric between boson lines and between fermion lines (if we had used wave functions instead of fermionic fields then they would be antisymmetric), but the amplitudes for other arrangements of fermions and bosons are obtained simply by permutation. The usual representations are given in Appendix B.6.

2.5.4 N>4 vector only

In principle there is no difficulty to evaluate higher-point diagrams. Some new terms occur compared to the $N = 4$ case. First of all, ∂X can contribute from one vertex, acting on a field, which is indicated by a black dot in Fig. 2.1. (All the other vertices contribute contractions between $\theta(z_i)$ and $\pi(z_j)$ from S.) Terms of the Green function higher-order in the ϵ expansion start to appear and thus K_N has z_i dependence. We give a schematic diagram for various types of contractions in Fig. 2.1. Notice that our diagram exactly coincides with earlier covariant RNS results [25]. There can also be corrections from the fermion partition function because of regularization. For example, this correction in the 6-point amplitude is proportional to $\theta_1'''(0|i\tau)/\theta_1'(0|i\tau)$ (see Appendix B.4.2).

2.5.5 N=5 vector only

First we will consider the part of the amplitude that doesn't have a black dot in Fig. 2.1. Let's call the graphs without and with a black dot K_N^a and K_N^b respectively. Since the 5-point amplitude has 5 sides we should choose G_0^θ from exactly one side. This is true for both the pentagon and triangle + ellipse graphs. The difference between them is the gamma matrix trace factor. So we can write down the part of K_5^a for a given group-index ordering (the

kth vector has $\theta\gamma^{a_k b_k}\pi$) as:

$$K_5^a \;\; = \;\; -\frac{1}{4}[\; G_0^\theta(z_2 - z_1)tr(\gamma^{a_1 b_1}\gamma^{a_2 b_2}\gamma^{a_3 b_3}\gamma^{a_4 b_4}\gamma^{a_5 b_5}) + 23 \text{ permutations }]$$
$$+\frac{1}{16}[\; G_0^\theta(z_2 - z_1)tr(\gamma^{a_1 b_1}\gamma^{a_2 b_2}\gamma^{a_3 b_3})tr(\gamma^{a_4 b_4}\gamma^{a_5 b_5}) + 11 \text{ permutations }] \tag{2.26}$$

Then we can write

$$K_5^b \;\; = \;\; \sum_{j=2}^{5} k_j^{a_1} G^{x\prime}(z_1 - z_j) K_4(2,3,4,5) + 4 \text{ permutations} \tag{2.27}$$

where K_4 was given in subsection 5.2. K_5^a and K_5^b complete the 5-point planar amplitude. Totally antisymmetric ϵ-tensor terms vanish because the 5 external momenta are not independent. Notice that the light-cone GS calculation reduces to our results after heavy algebra [26], and RNS needs a spin-structure sum to produce this result [25].

We postpone the $N \geq 6$-point amplitudes to another paper, which will be interesting because of the anomaly cancellation issue. One good thing in our covariant formalism is that we have a totally antisymmetric ϵ-tensor naturally in the hexagon amplitude, where we have enough momenta to have a nonvanishing result, contrary to the 5-point case.

2.6 Open problems

There are many avenues of further study, in particular:

(1) Many types of diagrams can be calculated. At the tree level, diagrams with many fermions have not yet been explicitly evaluated in any formalism. New algebraic methods for the current algebra might be useful. At the 1-loop level, little has been done with fermions or higher-point functions. Alternative IR regularization schemes could be considered. The 2-loop 4-vector calculation would be a good test, and nothing more than that has been done at 2 loops, and nothing at all at higher loops.

(2) The Hilbert space needs to be studied covariantly, especially the vacuum, to completely justify the naive manipulations we have made for tree graphs. It would be useful to find the relation of these methods to supergraphs, where explicit zero-mode integrations appear (both in loops, corresponding to π zero-modes, and an overall integral for θ zero-modes.) Massive vertex operators for physical states are expected to also be relatively simple, as the spinor ghosts should appear again in a minimal way (as opposed to the more complicated structure of the BRST operator). The analogy to second-quantized ghost pyramids (e.g., for higher-rank forms) might be useful: There ghosts beyond the first generation (i.e., the usual Faddeev-Popov ghosts) appear only at 1 loop, to define the measure.

(3) Closer relations to other formulations might exist. An analog to the $\mathcal{F}_2$ picture of RNS might further simplify tree calculations. The many similarities with PS suggests it might be a particular gauge choice of GP that truncates the ghost spectrum.

Appendix A

A.1 Sp(2) components

The matrix elements of the Sp(2) operators are

$$\langle p,q|\gamma^{\oplus}|r,s\rangle \;=\; C^{r,s}_{p,q}(\sqrt{p}\;\delta_{p,r+1}\delta_{q,s}\;+\;i\sqrt{q+1}\;\delta_{p,r}\delta_{q+1,s}) \tag{A.1}$$

$$\langle p,q|\tilde{\gamma}^{\oplus}|r,s\rangle \;=\; C^{r,s}_{p,q}(-i\sqrt{p}\;\delta_{p,r+1}\delta_{q,s}\;-\;\sqrt{q+1}\;\delta_{p,r}\delta_{q+1,s}) \tag{A.2}$$

where $C^{r,s}_{p,q} = i^{\frac{(r+s)(r+s+1)-(p+q)(p+q+1)}{2}}$. From these we can find "inverse" operators, especially

$$\langle p,q|\frac{1}{\tilde{\gamma}^{\oplus}}|r,s\rangle = \; i^{r-p}C^{r,s}_{p,q}\sqrt{\frac{p!s!}{q!r!}}\;\delta_{q-p+r-s,1}\;[\Theta_{p-q}\Theta_{s-q}\;-\;\Theta_{s-r}\Theta_{p-r}] \tag{A.3}$$

It satisfies

$$\tilde{\gamma}^{\oplus}\frac{1}{\tilde{\gamma}^{\oplus}}\tilde{\gamma}^{\oplus} = \tilde{\gamma}^{\oplus}, \qquad \tilde{\gamma}^{\oplus}\frac{1}{\tilde{\gamma}^{\oplus}} \to I, \qquad \frac{1}{\tilde{\gamma}^{\oplus}}\tilde{\gamma}^{\oplus} \to I \tag{A.4}$$

The arrows means there is cancellation among the multiplied matrix elements to the infinite ghost level. The subtle point of this cancellation will be studied in Appendix A.2.

Then we find that $\gamma^{\oplus}$ and $\frac{1}{\tilde{\gamma}^{\oplus}}$ don't (anti)commute but give

$$\begin{aligned}\langle p,q|\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\}|r,s\rangle &= i^{r-p+1}C^{r,s}_{p,q}\sqrt{\frac{p!s!}{q!r!}}\;\delta_{q-p+r-s,0}[(\Theta_{p-q}+\Theta_{p-q-1})(\Theta_{s-q}+\Theta_{s-q-1}) \\ &\quad -(\Theta_{s-r-1}+\Theta_{s-r})(\Theta_{p-r-1}+\Theta_{p-r})]\end{aligned} \tag{A.5}$$

$$\begin{aligned}\langle p,q|[\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}]|r,s\rangle &= i^{r-p+1}C^{r,s}_{p,q}\sqrt{\frac{p!s!}{q!r!}}\;\delta_{q-p+r-s,0}[(\Theta_{p-q}-\Theta_{p-q-1})(\Theta_{s-q}+\Theta_{s-q-1}) \\ &\quad -(\Theta_{s-r-1}-\Theta_{s-r})(\Theta_{p-r-1}+\Theta_{p-r})] \\ &= 2i\delta_{p,q}\delta_{r,s}\end{aligned} \tag{A.6}$$

Some interesting and useful commutators are

$$\langle p,q|[\gamma^{\oplus},\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\}]|r,s\rangle = -4i^{r-p}C^{r,s}_{p,q}(\sqrt{r+1}\delta_{p,q}\delta_{r+1,s}+\sqrt{p}\delta_{p,q+1}\delta_{r,s}) \tag{A.7}$$

$$\begin{aligned}\langle p,q|[\gamma^{\oplus}\gamma^{\oplus},\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\}]|r,s\rangle &= -8i^{r-p+1}C^{r,s}_{p,q}(\sqrt{(r+2)(r+1)}\delta_{p,q}\delta_{r+2,s} \\ &+ \sqrt{p(p-1)}\delta_{p,q+2}\delta_{r,s}-2\sqrt{p(r+1)}\delta_{p,q+1}\delta_{r+1,s})\end{aligned} \tag{A.8}$$

$$\begin{aligned}\langle p,q|[\gamma^{\oplus}\gamma^{\oplus},\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\}]\gamma^{\oplus}|r,s\rangle &= 16i^{r-p}C^{r,s}_{p,q}(\sqrt{(r+3)(r+2)(r+1)}\delta_{p,q}\delta_{r+3,s} \\ &+ \sqrt{p(p-1)(r+1)}\delta_{p,q+2}\delta_{r+1,s} \\ &- 2\sqrt{p(r+2)(r+1)}\delta_{p,q+1}\delta_{r+1,s})\end{aligned} \tag{A.9}$$

Using (A.1),(A.2) and (A.5) we find

$$\begin{aligned}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta &= \sum_{pq}[\sqrt{p(p-1)}\ (-1)^{p+1} i^{p+q+1}\ \bar{\pi}^{p,q}\ \theta^{q,p-2} \\ &\quad + 2\sqrt{p(q+1)}\ (-1)^{q+1} i^{p+q+1}\ \bar{\pi}^{p,q}\ \theta^{q+1,p-1} \\ &\quad + \sqrt{(q+1)(q+2)}\ (-1)^{p+1} i^{p+q+1}\ \bar{\pi}^{p,q}\ \theta^{q+2,p}]\end{aligned} \quad (A.10)$$

$$\tfrac{1}{2}\bar{\pi}\tilde{\gamma}^{\oplus}\pi = \sum_{pq}\sqrt{p}\ (-1)^{q+1}\ \bar{\pi}^{p,q}\ \pi^{q,p-1} \quad (A.11)$$

$$\begin{aligned}\bar{\pi}\gamma^{\oplus}\gamma^{a}\theta &= i\sum_{pq}[\sqrt{p}\ (-1)^{p}\ \bar{\pi}^{p,q}\ \gamma^{a}\ \theta^{q,p-1} \\ &\quad + \sqrt{q+1}\ (-1)^{q+1}\ \bar{\pi}^{p,q}\ \gamma^{a}\ \theta^{q+1,p}]\end{aligned} \quad (A.12)$$

$$\begin{aligned}R^{a} &= \tfrac{1}{4}\ \bar{\theta}\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\}\gamma^{a}\theta \\ &= \tfrac{1}{4}\sum_{pqr}(-1)^{(q-p+r+1)r-qp-p} i^{q-p+1}\sqrt{\frac{p!(q-p+r)!}{q!r!}}\bar{\theta}^{p,q}\ \gamma^{a}\ \theta^{q-p+r,r} \\ &\quad [(\Theta_{p-q}+\Theta_{p-q-1})(\Theta_{r-p}+\Theta_{r-p-1}) \\ &\quad - (\Theta_{q-p-1}+\Theta_{q-p})(\Theta_{p-r-1}+\Theta_{p-r})]\end{aligned} \quad (A.13)$$

$$\begin{aligned}R^{\oplus} &= \tfrac{1}{4}\ \bar{\theta}\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\}\gamma^{\oplus}\theta \\ &= \tfrac{1}{4}\sum_{pqr}(-1)^{(q-p+r+2)r-qp-p}\sqrt{\frac{p!(q-p+r+1)!}{q!r!}}\bar{\theta}^{p,q}\ \theta^{q-p+r+1,r} \\ &\quad [(\Theta_{p-q}+\Theta_{p-q-1})(\Theta_{r-p+1}+2\Theta_{r-p}+\Theta_{r-p-1}) \\ &\quad - (\Theta_{q-p-1}+\Theta_{q-p})(\Theta_{p-r-2}+2\Theta_{p-r-1}+\Theta_{p-r})]\end{aligned} \quad (A.14)$$

The component fields defined above satisfy

$$\{\pi_{p,q}, \theta^{r,s}] \;=\; \delta_p^r \delta_q^s \tag{A.15}$$

A.2 Subtle points in Sp(2) operators

In this appendix we explain some subtle points about $\frac{1}{\tilde{\gamma}^{\oplus}}$, due to the infinite dimensional structure of Sp(2) operators.

Consider the commutator

$$\{\ \bar{\pi}\tilde{\gamma^{\oplus}}\theta\ ,\ [\ \overset{(1)}{\theta}\overset{(0)}{\theta}\ ,\ R^{a}\ ]\ \} \tag{A.16}$$

where

$$\begin{aligned}
\overset{(0)}{\theta} &= \langle 0|e^{ia_{\oplus}a_{\ominus}}|\theta\rangle \\
&= \theta_0\ +\ \tilde{\theta} \\
&= \theta_0\ +\ \theta^{\oplus\ominus}\ +\ \theta^{\oplus\oplus\ominus\ominus}\ +\ \theta^{\oplus\oplus\oplus\ominus\ominus\ominus}\ +\ \cdots \\
\overset{(1)}{\theta} &= \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})|\theta\rangle \\
&= 2i(\theta^{\oplus}\ -\ \sqrt{2}\theta^{\oplus\oplus\ominus}\ +\sqrt{3}\theta^{\oplus\oplus\oplus\ominus\ominus}\ -\ \sqrt{4}\theta^{\oplus\oplus\oplus\oplus\ominus\ominus\ominus}\ +\ \cdots)
\end{aligned} \tag{A.17}$$

We will also need:

for $n \geq 0$

$$\begin{aligned}
\overset{(n)}{\theta} &= \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})^{n}|\theta\rangle \\
&= \sum_{k=0}^{\infty}\ (-1)^{k(n+k+1)} i^{\frac{n(n+1)}{2}}(1+\Theta_{n-\frac{1}{2}})\sqrt{\frac{(n+k)!}{k!}}\theta^{n+k,k}
\end{aligned} \tag{A.18}$$

for $n < 0$

$$\begin{aligned}
\overset{(n)}{\theta} &= \langle 0|e^{ia_\oplus a_\ominus}(ia^{\dagger\oplus})^{|n|}|\theta\rangle \\
&= \sum_{k=0}^{\infty} (-1)^{k(|n|+k+1)} i^{\frac{|n|(|n|+1)}{2}} (1+\Theta_{|n|-\frac{1}{2}}) \sqrt{\frac{k!}{(|n|+k)!}}\, \theta^{k,|n|+k}
\end{aligned} \tag{A.19}$$

The above double commutator should vanish because the inner one involves only θ. However, if we apply the Jacobi identity we see

$$\begin{aligned}
\{\bar{\pi}\tilde{\gamma^\oplus}\pi, [\overset{(1)}{\theta}\overset{(0)}{\theta}, R^a]\} &= 0 = [\{\bar{\pi}\tilde{\gamma^\oplus}\pi, \overset{(1)}{\theta}\overset{(0)}{\theta}\}, R^a] - \{\overset{(1)}{\theta}\overset{(0)}{\theta}, [\bar{\pi}\tilde{\gamma^\oplus}\pi, R^a]\} \\
&= [\tilde{0}, R^a] - 2\,\{\overset{(1)}{\theta}\overset{(0)}{\theta}, \bar{\pi}\gamma^\oplus\gamma^a\theta\} \\
&= [\tilde{0}, R^a] + 2\ \overset{(1)}{\theta}\gamma^a\overset{(1)}{\theta} + 4\ \overset{(0)}{\theta}\gamma^a\overset{(2)}{\theta}
\end{aligned} \tag{A.20}$$

where we have introduced the scalar $\tilde{0}$ defined by

$$\begin{aligned}
[\tfrac{1}{2}\bar{\pi}\gamma^\oplus\pi, \overset{(n)}{\theta}]\ \ (n \geq -1)\ &\equiv\ -i^{\frac{n(n+1)}{2}+n}(1+\Theta_{|n|-\frac{1}{2}})\sqrt{\frac{(n+1)!}{0!}}(\bar{\pi}^{n+1,0} - \bar{\pi}^{n+1,0}) \\
&+(-1)^n i^{\frac{n(n+1)}{2}+n}(1+\Theta_{|n|-\frac{1}{2}})\sqrt{\frac{(n+2)!}{1!}}(\bar{\pi}^{n+2,1} - \bar{\pi}^{n+2,1}) \\
&+\ \cdots \\
&+(-1)^{(k+2)(n+k+2)-1} i^{\frac{n(n+1)}{2}+n}(1+\Theta_{|n|-\frac{1}{2}})\sqrt{\frac{(n+k+1)!}{k!}}(\bar{\pi}^{n+k+1,k} - \bar{\pi}^{n+k+1,k}) \\
&+\ \cdots \\
&\equiv \tilde{0}\ \to\ 0
\end{aligned} \tag{A.21}$$

and

$$[\overset{(n)}{\theta}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta] \;=\; -\, i^{n}(1+\Theta_{n-\frac{1}{2}})\gamma^{a}\overset{(n+1)}{\theta} \tag{A.22}$$

Unfortunately, $\tilde{0}$ is not zero in the presence of R^a. Let's assume the collective $\overset{(n)}{\theta}_k$ has $k+1$ terms instead of an infinite number of terms. Then $[\frac{1}{2}\bar{\pi}\gamma^{\oplus}\pi,\ \overset{(n)}{\theta}_k]$ gives

$$(-1)^{(k+2)(n+k+2)-1} i^{\frac{n(n+1)}{2}+n}(1+\Theta_{|n|-\frac{1}{2}})\sqrt{\frac{(n+k+1)!}{k!}}\,\bar{\pi}^{n+k+1,k}$$

Finally, one can see that

$$(-1)^{(k+2)(n+k+2)-1} i^{\frac{n(n+1)}{2}+n}(1+\Theta_{|n|-\frac{1}{2}})\sqrt{\frac{(n+k+1)!}{k!}}[\bar{\pi}^{n+k+1,k}, R^a]$$

gives $-\, i^{n}(1+\Theta_{|n|-\frac{1}{2}})\overset{(n+1)}{\theta}_k$. So if we make the collective $\overset{(n)}{\theta}_k$ have an infinite number of terms by $k\to\infty$ then $\tilde{0}$ produces a nonzero result in the commutator with R^a. This exactly cancels the remaining terms in (A.20) to make the double commutator consistent.

Keeping this subtle point in mind let's consider the following transformation

$$\begin{aligned}
Q'_{free} \;&=\; e^{iR^a p_a}\tilde{Q}_{free}e^{-iR^a p_a} \\
&=\; e^{iR^a p_a}(\tfrac{1}{2}c\,\Box \;-\; \tfrac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta b \;+\; \tilde{0}b \;+[-iR^a p_a, -\tfrac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta \;+\; \tilde{0}]b \\
&\quad +\; \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi \;+\; [-iR^a p_a, \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi] \;+\; \tfrac{1}{2}[-iR^a p_a, [-iR^a p_a, \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi]] \\
&\quad -\; \tfrac{i}{2}\bar{\pi}\gamma^{\oplus}\not{p}\theta \;+\; [-iR^a p_a, -\tfrac{i}{2}\bar{\pi}\gamma^{\oplus}\not{p}\theta])e^{-iR^a p_a}
\end{aligned} \tag{A.23}$$

where $\tilde{0}b$ vanishes in Q'_{free} but will give a nontrivial contribution in the presence of R^a. We will determine this term from nilpotency of Q'_{free}.

In terms of $R^\oplus$ and $\overset{(n)}{\theta}$, $\tilde{Q}_{free}$ is

$$\begin{aligned}\tilde{Q}_{free} &= \tfrac{1}{2}c\,\Box \ - \tfrac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta b - \overset{(2)}{\theta}\ \gamma^a\ \overset{(0)}{\theta}\ p_a b \ + \ \tfrac{1}{2}\ \overset{(1)}{\theta}\ \gamma^a\ \overset{(1)}{\theta}\ p_a b \\ & \quad + \ \tilde{0}b \ + \ [-iR^a, \tilde{0}]p_a b \ + \ \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\pi \ - \ \tilde{0}^a p_a \\ & \quad + \ \tfrac{1}{2}[-iR^a p_a, -\tilde{0}^b p_b] \ + \ \tfrac{1}{2}R^\oplus\Box \end{aligned} \tag{A.24}$$

where

$$[-iR^a p_a, -\tfrac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta]b \ = \ - \ \overset{(2)}{\theta}\ \gamma^a\ \overset{(0)}{\theta}\ p_a b \ + \ \tfrac{1}{2}\ \overset{(1)}{\theta}\ \gamma^a\ \overset{(1)}{\theta}\ p_a b \tag{A.25}$$

and we have defined the vector $\tilde{0}^a$

$$\tilde{0}^a \equiv -[-iR^a, \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\pi] + \tfrac{i}{2}\bar{\pi}\gamma^\oplus\gamma^a\theta \tag{A.26}$$

The nilpotency of Q'_{free} implies

$$-\{-\tilde{0}^a, \tfrac{1}{2}R^\oplus\}p_a\Box \ = \ \tfrac{1}{2}(-\ \overset{(2)}{\theta}\ \gamma^a\ \overset{(0)}{\theta}\ + \tfrac{1}{2}\ \overset{(1)}{\theta}\ \gamma^a\ \overset{(1)}{\theta}\ + [-iR^a, \tilde{0}])p_a\Box \tag{A.27}$$

$$\begin{aligned}[\tfrac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta, \tfrac{1}{2}[-iR^a p_a, -\tilde{0}^b p_b]]\ b &= -[-\tfrac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta \ + \ \tilde{0}, \tfrac{1}{2}R^\oplus]\Box\ b \\ &= \tfrac{1}{4}(3i\ \overset{(1)}{\theta}\overset{(2)}{\theta}\ - 2i\ \overset{(0)}{\theta}\overset{(3)}{\theta}\ - 2[\tilde{0}, R^\oplus])\Box\ b \end{aligned} \tag{A.28}$$

From (A.26) we can find $\tilde{0}^a$ as

$$\tilde{0}^a = \frac{i}{4} \overset{(1)}{\pi}_\infty \gamma^a \overset{(0)}{\theta} - \frac{i}{4} \overset{(0)}{\pi}_\infty \gamma^a \overset{(1)}{\theta} \tag{A.29}$$

where

$$\overset{(n)}{\chi}_\infty \equiv \lim_{k\to\infty} (-1)^{k(n+k+1)} (-i)^{\frac{n(n+1)}{2}} \sqrt{\frac{(n+k+1)!}{k!}} \chi^{n+k,k} \tag{A.30}$$

for $\chi = \theta, \pi$. Inserting this into (A.27) we find

$$[iR^a, \tilde{0}] = \overset{(1)}{\theta} \gamma^a \overset{(1)}{\theta} \tag{A.31}$$

$$[\tilde{0}, R^\oplus] = 2i \overset{(1)}{\theta} \overset{(2)}{\theta} \tag{A.32}$$

One solution for $\tilde{0}$ is

$$\tilde{0} = \overset{(1)}{\pi}_\infty \overset{(1)}{\theta} \tag{A.33}$$

Now $\tilde{Q}_{free}$ is nilpotent, as it should be. However, the origin of collective nonminimal fields is that R^a produces $\bar{\pi}\gamma^\oplus\gamma^a\theta$ using $[\bar{\pi}\tilde{\gamma}^\oplus\pi, \overset{(n)}{\theta}] \to 0$. (The key feature of R^a is $\overset{(n)}{\theta}$, see (A.13).) And this implies that R^a and $R^\oplus$ commute with $\bar{\pi}\gamma^\oplus\gamma^\oplus\theta$ only up to these collective nonminimal fields (see (A.7)). So these collective fields are just mathematical objects to compensate terms like $[R^i, \tilde{0}]$. Therefore, a physical (but not mathematical) equivalent is to drop $\tilde{0}^{(a)}$ and $\overset{(n)}{\theta}$ and regard R^i as terms commuting with $\bar{\pi}\gamma^\oplus\gamma^\oplus\theta$ in (A.24). The resulting $\tilde{Q}_{free}$ is simply

$$\tilde{Q}_{free} = \frac{1}{2} c \Box - \frac{1}{2} \bar{\pi}\gamma^\oplus\gamma^\oplus\theta b + \frac{1}{4} \bar{\pi}\tilde{\gamma}^\oplus\pi + \frac{1}{2} R^\oplus \Box \tag{A.34}$$

with $[\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta, R^{\oplus}] \sim 0$.

Our results for $\mathring{Q}'_{YMB}$(1.30), $\mathring{Q}_{SYMB}$(1.49) and Q''_{SYMB}(1.60) all reflect this prescription. $\tilde{\theta}$ in these BRST operators will produce only $\pi^{\oplus}$, $\theta^{\oplus}$ and $\theta^{\oplus\oplus}b$ ($\theta^{\oplus\oplus}b$ will only appear in (1.30)) dropping $\overset{(1)}{\theta}$ and $\overset{(2)}{\theta}\ b$. If we want to be rigorous $\overset{(n)}{\theta}$ and $\tilde{0}^{(a)}$ should be kept and the commutator $[R^i, \bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta]$ should be calculated for both to compensate terms from $\tilde{0}^{(a)}$.

If we consider this mathematical rigor for Q'_{YMB}(1.30) we find

$$
\begin{aligned}
Q_{YMB}^{collective} \;=\;& e^{iR^a\nabla_a}\big[\ \tfrac{1}{2}c(\Box - \bar{\pi}\gamma^{ab}\theta F_{ab}) \;+\; \tfrac{1}{2}R^{\oplus}(\Box - \bar{\pi}\gamma^{ab}\theta F_{ab}) \\
&+\; \tfrac{1}{2D}[iR^a, \tilde{0}_a](\Box - \bar{\pi}\gamma^{ab}\theta F_{ab}) \;-\; \tfrac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta b \;+\; \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi \\
&+\; \tilde{0}b \;-\; \overset{(2)}{\theta}\ \gamma^a\ \overset{(0)}{\theta}\ \nabla_a b \;+\; \tfrac{1}{2}\ \overset{(1)}{\theta}\ \gamma^a\ \overset{(1)}{\theta}\ \nabla_a b \;-\; [iR^a, \tilde{0}]\nabla_a b \\
&-\; \tilde{0}^a\nabla_a \\
&-\; \tfrac{1}{2}\{(cb - \tfrac{1}{2}) + R^{\oplus}b + \tfrac{1}{2D}[iR^a, \tilde{0}_a]b\} \\
&\quad \times\ \bar{\theta}\frac{1}{\tilde{\gamma}^{\oplus}}\gamma^{ab}\theta[F_{ab}, \nabla_c](-\ \overset{(2)}{\theta}\ \gamma^c\ \overset{(0)}{\theta} \;+\; \tfrac{1}{2}\ \overset{(1)}{\theta}\ \gamma^c\ \overset{(1)}{\theta} \;-\; [iR^c, \tilde{0}]) \\
&+\; \tfrac{1}{2}\{c + R^{\oplus} + \tfrac{1}{2D}[iR^a, \tilde{0}_a]\} \\
&\quad \times\ \bar{\theta}\frac{1}{\tilde{\gamma}^{\oplus}}\gamma^{ab}\theta[F_{ab}, \nabla_c]\tilde{0}^c\ \big]\ e^{-iR^a\nabla_a}\ \big|_{\text{linear in } F,\ [\nabla, F]=0}
\end{aligned}
\tag{A.35}
$$

where

$$
\begin{aligned}
D \;&=\; \text{dimension of space-time} \quad \text{(10 here)} \\
\text{and}\ \ \Box \;&=\; -\nabla^a\nabla_a
\end{aligned}
\tag{A.36}
$$

If we drop $\overset{(n)}{\theta}$ and $\tilde{0}^{(a)}$ and regard $[\gamma^{\oplus}, \frac{1}{\tilde{\gamma}^{\oplus}}] = 0$ (which is equivalent to $[R^i, \bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta]] = 0$) we come back to Q'_{YMB}(1.30). The last four lines do not contribute to Q'_{YMB}

but are there for nonconstant Yang-Mills background.

A.3 Regularization

In this appendix we will consider a regularization procedure which will give the prescription of the previous appendix. The motivation is the fact that $[\bar{\pi}\tilde{\gamma}^{\oplus}\pi, \overset{(n)}{\theta}] \to 0$. However, this does not exactly vanish, but leaves a piece of $\overset{(n+1)}{\pi}{}_{\infty}$. This remnant gives a nontrivial contribution in the presence of $\frac{1}{\tilde{\gamma}^{\oplus}}$, i.e.,

$$\begin{aligned} \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})^n\tilde{\gamma}^{\oplus}|\theta\rangle &\to 0 \\ \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})^n\tilde{\gamma}^{\oplus}\frac{1}{\tilde{\gamma}^{\oplus}}|\theta\rangle &\to \overset{(n)}{\theta} \end{aligned} \tag{A.37}$$

Now if we introduce some regularization parameter z as

$$\langle p,q|\tilde{\gamma}^{\oplus}|r,s\rangle \overset{regularized}{\longrightarrow} z^{p+r+s+r}\langle p,q|\tilde{\gamma}^{\oplus}|r,s\rangle, \quad z \to 1 \tag{A.38}$$

then (A.37) becomes

$$\begin{aligned} \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})^n\tilde{\gamma}^{\oplus}|\theta\rangle_k &\to z^{2n+2k+1}\ \theta^{n+k+1,k} \\ \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})^n\tilde{\gamma}^{\oplus}\frac{1}{\tilde{\gamma}^{\oplus}}|\theta\rangle_k &\to z^{2n+2k+1}\overset{(n)}{\theta}{}_k \end{aligned} \tag{A.39}$$

where k means the collective field has $k+1$ terms. The regularization is to send k to infinity with fixed z (<1). Then the operator $\tilde{\gamma}^{\oplus}\frac{1}{\tilde{\gamma}^{\oplus}}$ is a projection operator which will project out the collective fields. The free terms in any $Q_{backgroud}$ all have this projection operator which goes to the identity in the absence of $\frac{1}{\tilde{\gamma}^{\oplus}}$.

$$
\begin{aligned}
& -\tfrac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta b \;+\; \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi \quad - \quad i\tfrac{1}{2}\nabla_a\bar{\pi}\gamma^{\oplus}\gamma^{a}\theta \\
& \qquad\qquad \rightarrow \quad -\tfrac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\Pi\theta b \;+\; \tfrac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\Pi\pi \;-i\tfrac{1}{2}\nabla_a\bar{\pi}\gamma^{\oplus}\gamma^{a}\Pi\theta
\end{aligned}
\tag{A.40}
$$

The arrow means inserting the projection operator and dropping collective fields after expansion of exponential factors.

If a collective field is truncated, i.e., with incomplete beginning or ending components, then we cannot remove it by this regularization procedure, but we can still avoid its contribution. This situation occurs when we consider the commutator

$$
\{\eta\pi^{p,p+1},[\nabla_0\tilde{\theta},\bar{\pi}\gamma^{\oplus}\gamma^{a}\theta]\} = \{\eta\pi^{p,p+1},-i\nabla_0\gamma^{a}\theta^{\oplus}\} = 0 \quad (p>0) \tag{A.41}
$$

where η is a constant fermionic field. This implies

$$
[\nabla_0\tilde{\theta},(\pi^{p,p}+\pi^{p+1,p+1})\eta] = 0 \tag{A.42}
$$

The second argument in the commutator is an example of a truncated collective field. Actually, this commutator indeed vanishes if we consider $\overset{(1)}{\theta}$, which we drop in regularization. So for consistent regularization we take this as vanishing.

This fact is applied for closure of the algebra

$$
[\eta\theta^{q,q+1},\{\,[\eta'\pi^{p,p+1},Q_R],Q_R\}] \tag{A.43}
$$

where Q_R is any version of the BRST operator including R^i. After canceling ghost-number-nonzero components this commutator reduces to

$$[\eta' C_p^-, \eta C_q^+] \tag{A.44}$$

where $C_p^\pm$ will be superstring constraints if we use $Q_{sstring}$. But the original commutator is just

$$\tfrac{1}{2}[\eta\theta^{q,q+1},\ [\eta'\pi^{p,p+1}, \{Q_R, Q_R\}]] \tag{A.45}$$

and it is just zero due to nilpotency of Q_R. However, $\eta' C_p^-$ has a term like $\eta\gamma^a(\pi^{p,p} + \pi^{p+1,p+1})$, which comes from $[\eta'\pi^{p,p+1}, \bar{\pi}\gamma^\oplus\gamma^a\theta]$. But this combination of $\pi^{p,p}$ is just an example of truncated collective fields. This truncated collective field will interact with $\tilde{\theta}$ in ηC_q^+ giving a nonzero contribution in this apparently vanishing commutator. This should be canceled by a $[\eta'\pi^{p,p+1}, \overset{(1)}{\theta}]$ contribution, which we projected out by regularization. What this means is that for consistent regularization we should take the commutator with truncated collective fields, $\tilde{\theta}$, R^i as vanishing. For example, we should take $[(\pi^{p,p} + \pi^{p+1,p+1}), \tilde{\theta}]$ as zero but we should calculate $\{(\theta^{p,p} + \theta^{p+1,p+1}), (\pi^{p,p} + \pi^{p+1,p+1})\}$ in $[\eta' C_p^-, \eta C_p^+]$, both of which come from $\bar{\pi}\gamma^\oplus\gamma^a\theta$.

A.4 Closure of constraints

First of all, if one directly calculates $[\pi^{p,p+1}, R^{\oplus}]$ one gets

$$\begin{aligned} [\pi^{p,p+1}, R^{\oplus}] &= -\tfrac{3}{4}\sum_r (-1)^p\sqrt{p+1}\theta^{r,r}(\Theta_{r-p}+2\Theta_{r-p-1}+\Theta_{r-p-2}) \\ &\quad +\tfrac{1}{4}\sum_r (-1)^p\sqrt{p+1}\theta^{r,r}[(\Theta_{p-r+1}+2\Theta_{p-r}+\Theta_{p-r-1}) \\ &\equiv -\tfrac{3}{4}A+\tfrac{1}{4}B \\ &= -\tfrac{1}{2}(A-B)-\tfrac{1}{4}(A+B) \end{aligned} \tag{A.46}$$

But $A+B$ is just $(-1)^p\sqrt{p+1}\ \overset{(0)}{\theta}$, and it will vanish when it acts on the projection operator Π. Also,

$$A-B \;=\; -2(-1)^p\sqrt{p+1}\vartheta_p$$

in terms of ϑ (1.79). When we calculate closure of the constraints, there are two types of terms related to the above expression, i.e.,

$$\{[\eta\pi^{p,p+1}, R^{\oplus}], [\eta'\theta^{q,q+1}, \bar{\pi}\tilde{\gamma}^{\oplus}\pi]\}$$

and

$$\{[\eta\pi^{p,p+1}, R^{\oplus}], [\eta'\pi^{q,q+1}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta]\}$$
$$\{[\eta\pi^{p,p+1}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta], [\eta'\pi^{q,q+1}, R^{\oplus}]\}$$

where η and η' are constant spinors.

The first type is always zero due to the symmetry of $\gamma^{\oplus}$ and $\tilde{\gamma}^{\oplus}$. If $p \neq q$,

$p \neq q+1$ and $p+1 \neq q$ the second type cancels because of the $A-B$ sign difference. If $p=q$ we can express $[\eta\pi^{p,p+1}, R^{\oplus}]$ as just $-\sqrt{p+1}\eta(\theta^{p,p} - \theta^{p+1,p+1})$. If $p=q+1$ or $p+1=q$ we can use $\sqrt{p+1}\eta(\theta^{p,p} - \theta^{p+1,p+1})$. This becomes clearer in a simpler situation,

$$\{R^{\oplus}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta\} = 0$$

Then we have

$$\{\eta'\pi^{q,q+1}, [\eta\pi^{p,p+1}, \{R^{\oplus}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta\}]\} = 0$$

But this implies

$$[\{\eta\pi^{p,p+1}, R^{\oplus}\}, \{\eta'\pi^{q,q+1}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta\}] + [\{\eta'\pi^{q,q+1}, R^{\oplus}\}, \{\eta\pi^{p,p+1}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta\}] = 0$$

This is one part of the closure of constraints. A similar analysis shows

$$\begin{aligned} & [\ \eta\hat{P}_a\gamma^a(\theta^{p,p} + \theta^{p+1,p+1}),\ \eta'\hat{P}_a\gamma^a(\theta^{p,p} + \theta^{p+1,p+1})\ ] \\ \sim\ & [\ \eta(\pi^{p,p} - \pi^{p+1,p+1}),\ \tfrac{i}{4}\tilde{R}_a\ ]\ \eta'\gamma^a(\theta^{p,p} + \theta^{p+1,p+1}) \\ & +\eta\gamma^a(\theta^{p,p} + \theta^{p+1,p+1})\ [\ \tfrac{i}{4}\tilde{R}_a, \eta'(\pi^{p,p} - \pi^{p+1,p+1})\ ] \end{aligned} \tag{A.47}$$

where $\sim$ means we should drop truncated collective fields.

Secondly, from the fact that $[\tilde{\theta}, \bar{\pi}\gamma^{\oplus}\gamma^{a}\theta] = \overset{(1)}{\theta}\ -i\ \theta^{\oplus} \sim -i\ \theta^{\oplus}$ due to regularization when $\tilde{\theta}$ acts on terms from $\bar{\pi}\gamma^{\oplus}\gamma^{a}\theta$, we should change the sign of $\tilde{\theta}$. (More precisely, they are all zero except for $\pi^{1,1}$ from $\bar{\pi}\gamma^{\oplus}\gamma^{a}\theta$. For only this term we can see this sign-change effect as explained in the previous appendix.) This fact was implicitly expressed with the projection operator Π in the constraints.

Finally, one should be cautious about $\{\bar{\pi}\gamma^{\oplus}\gamma^{a}\theta\ ,\ \bar{\pi}\gamma^{\oplus}\gamma^{b}\theta\} = (-1)(-2)g^{ab}$ $\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta$. The "$-$" sign comes from the fact that OSp(2) gamma matrices (and therefore $a^{\dagger}$ and a) anticommute with ordinary gamma matrices. This gives an additional sign when one calculates terms like $[\eta\gamma^{a}\theta^{p,p}\ ,\ \eta'\gamma^{b}\pi^{p,p}]$.

A.5 ZJBV form of BRST

The ZJBV form of the BRST operator follows from the Hamiltonian form of the BRST operator

$$Q^{ZJBV} = \tfrac{i}{2}\dot{\phi}^A \phi_A - \check{\phi}_A\{\phi^A, Q^H] \tag{A.48}$$

Then in our case,

$$Q^{ZJBV}_{sstring} = -\dot{X}\cdot P^0 \; + \sum_{\pm} Q^{(\pm)ZJBV}_{sstring} \tag{A.49}$$

where, e.g., for the $(+)$ term (for $(-)$, just add a $-$ for each $'$)

$$\begin{aligned}
Q^{ZJBV}_{sstring} \;\; = \;\; & \dot{c}b \; + \; \dot{\theta}\pi \\
+ \;\; & (\check{X}_a - \check{P}'_a)\Big[\Big(c + \tfrac{1}{2}\bar{\theta}\tilde{\gamma}^{\oplus}\theta\Big)\,\hat{P}^a \;-\; \tfrac{i}{2}\bar{\pi}\gamma^{\oplus}\gamma^a\theta \\
& -\Big(i\theta_0\gamma_a\theta'_0 \;+2i\tilde{\theta}\gamma_a\theta'_0 \;+\; i\tilde{\theta}\gamma_a\tilde{\theta}' \;-\tfrac{1}{2}\;R'_a|_>\Big)\,R^{\oplus} + \; i\tfrac{1}{2}\bar{\theta}^{\oplus}\theta_0\;\theta_0\gamma^a\theta'_0 \\
& -\tfrac{1}{2}\bar{\theta}^{\oplus}\gamma^b\gamma^a\theta_0\,\Big(i\tfrac{1}{2}\theta_0\gamma_b\theta'_0 \;+\; 2i\tilde{\theta}\gamma_b\theta'_0 \;+\; i\tilde{\theta}\gamma_b\tilde{\theta}' - \tfrac{1}{2}\;R'_a|_>\Big) \\
& -\tfrac{1}{3}\bar{\theta}^{\oplus}\gamma^a\gamma^b\tilde{\theta}\,\Big(2i\tilde{\theta}\gamma_a\theta'_0 \;+\; i\tfrac{5}{4}\;\tilde{\theta}\gamma_a\tilde{\theta}' - \tfrac{3}{4}\;R'_a|_>\Big)\Big] \\
- \;\; & \check{c}\left[icc' + 2\;\bar{\pi}a^{\dagger\oplus}a^{\oplus}\theta|_>\right] \\
- \;\; & \check{b}\Big(\;-\tfrac{1}{2}\hat{P}^2 \;-2ic'b \;-icb' \;-\; i\bar{\theta}'\pi \;-\; i[\bar{\theta}(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]'\Big)
\end{aligned}$$

$$
\begin{aligned}
+ & \sum_{p>1} \check{\theta}_{p,p+1} \Big[\ \big\{ i \left(c + R^{\oplus} + \tfrac{1}{2}\bar{\theta}\tilde{\gamma}^{\oplus}\theta \ \right)|_{>} \theta'^{p,p+1} \big\} \\
& - 2i[(p+1) - \sqrt{p+1}\frac{\sqrt{p}+\sqrt{p+2}}{2}]\Pi\theta^{p+1,p} b \\
& +(-1)^{p+1} i \tfrac{1}{2}\sqrt{p+1}\Pi(\pi^{p,p} - \pi^{p+1,p+1}) \\
& +(-1)^{p+1} i \tfrac{1}{2}\sqrt{p+1}\gamma^a \Pi(\theta^{p,p} + \theta^{p+1,p+1})\mathcal{P}_a \Big] \\
- & \ \tfrac{i}{2} \ \check{\theta}_{0,1} \Big\{ \pi_0 + (\gamma^a\theta_0)\hat{P}_a + i\tfrac{1}{2}(\gamma^a\theta_0)\theta_0\gamma_a\theta'_0 \\
& + \tfrac{2}{3}\gamma^a\tilde{\theta}\left(2i\tilde{\theta}\gamma_a\theta'_0 \ + \ i\tfrac{5}{4} \ \tilde{\theta}\gamma_a\tilde{\theta}' - \tfrac{3}{4} \ R'_a|_{>} \right) - \pi^{1,1} + \gamma^a \Pi \theta^{1,1}\mathcal{P}_a \Big\} \\
- & \sum_{p>1} \check{\pi}_{p,p+1} \Big[\big\{ i \left(c + R^{\oplus} + \tfrac{1}{2}\bar{\theta}\tilde{\gamma}^{\oplus}\theta \ \right)|_{>} \pi^{p,p+1} \big\}' \\
& +(-1)^{p+1} i\sqrt{p+1}\vartheta^p \Big\{ -\tfrac{1}{2}\mathcal{P}^2 + \tfrac{1}{2}\hat{P}^2 + i\bar{\theta}'\pi + i[\bar{\theta}(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]' \Big\} \\
& + \tfrac{1}{2} R^{\oplus}|_{>} \sum_r (-1)^{r^2+p+1}\sqrt{\frac{p+1}{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-p-1} + \Theta_{r-p-2})\mathcal{P}_a \\
& - (-1)^{p+1} i \sqrt{p+1}(\theta^{p,p} - \theta^{p+1,p+1}) \\
& \quad \times \left(\ \tfrac{1}{2}\hat{P}^2 \ + \ i\bar{\theta}'\pi \ + \ i[\bar{\theta}(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]' \right) \\
& + 2i[(p+1) - \sqrt{p+1}\frac{\sqrt{p}+\sqrt{p+2}}{2}]\pi^{p+1,p} b \\
& -(-1)^{p+1} i \tfrac{1}{2}\sqrt{p+1}\gamma^a \Pi(\pi^{p,p} + \pi^{p+1,p+1})\mathcal{P}_a \\
& + \tfrac{1}{4} \sum_r (-1)^{r^2+p+1}\sqrt{\frac{p+1}{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-p-1} + \Theta_{r-p-2}) \\
& \quad \times \left(-\bar{\pi}\gamma^{\oplus}\gamma^a\theta|_{>} + i\bar{\theta}^{\oplus}\gamma^a \left(\pi_0 + (\gamma^b\theta_0)\hat{P}_b + i\tfrac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta'_0 \ \right) + i\bar{q}^{\oplus}\gamma_a\tilde{\theta} \right) \Big] \\
- & \ \tfrac{i}{2} \ \check{\pi}_{0,1} \Big\{ \gamma^a \left(\pi_0 + (\gamma^b\theta_0)\hat{P}_b + i\tfrac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta'_0 \ \right) \\
& \quad \times \left(\hat{P}_a \ + i\theta_0\gamma_a\theta'_0 \ + 2i\tilde{\theta}\gamma_a\theta'_0 \ + \ i\tilde{\theta}\gamma_a\tilde{\theta}' - \tfrac{1}{2} \ R'_a|_{>} \ \right) \\
& + \tfrac{2}{3}\gamma^b\gamma^a\tilde{\theta}\left(2i\tilde{\theta}\gamma_a\theta'_0 \ + \ i\tfrac{5}{4} \ \tilde{\theta}\gamma_a\tilde{\theta}' - \tfrac{3}{4} \ R'_a|_{>} \right) \\
& \quad \times \left(\hat{P}_b \ + i\theta_0\gamma_b\theta'_0 \ + 2i\tilde{\theta}\gamma_b\theta'_0 \ + \ i\tilde{\theta}\gamma_b\tilde{\theta}' - \tfrac{1}{2} \ R'_b|_{>} \ \right) \\
& -\gamma^a\Pi\pi^{1,1}\left(\hat{P}_a \ + i\theta_0\gamma_a\theta'_0 \ + 2i\tilde{\theta}\gamma_a\theta'_0 \ + \ i\tilde{\theta}\gamma_a\tilde{\theta}' - \tfrac{1}{2} \ R'_a|_{>} \ \right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\theta^{1,1}\left(\ -\tfrac{1}{2}\hat{P}^2 \ - \ i\bar{\theta}'\pi \ - \ i[\bar{\theta}(a^{\dagger\oplus}a_\oplus - a^{\dagger\ominus}a_\ominus)\pi]'\right) \\
&+2\vartheta^0 \times \Bigg\{-\tfrac{1}{2}\left(\hat{P}_a \ + i\theta_0\gamma_a\theta_0' \ + 2i\tilde{\theta}\gamma_a\theta_0' \ + \ i\tilde{\theta}\gamma_a\tilde{\theta}' \ - \tfrac{1}{2} \ R_a'|_>\right)^2 \\
&\quad +\tfrac{1}{2}\hat{P}^2 \ + \ i\bar{\theta}'\pi \ + \ i[\bar{\theta}(a^{\dagger\oplus}a_\oplus - a^{\dagger\ominus}a_\ominus)\pi]'\Bigg\} \\
&+\tfrac{1}{2}R^\oplus|_> \sum_r (-1)^{r^2+1}\frac{1}{\sqrt{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-1}+\Theta_{r-2})\mathcal{P}_a \\
&-\tfrac{i}{4}\sum_r (-1)^{r^2+1}\tfrac{1}{\sqrt{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-1}+\Theta_{r-2}) \\
&\quad \times\left(-\bar{\pi}\gamma^\oplus\gamma^a\theta|_> + i\bar{\theta}^\oplus\gamma^a\left(\pi_0 + (\gamma^b\theta_0)\hat{P}_b + i\tfrac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta_0' \ \right) + i\bar{q}^\oplus\gamma_a\tilde{\theta}\right)\Bigg\} \\
- \quad &\sum_{q\neq p+1}\check{\theta}_{p,q}[\theta^{p,q}, Q_{sstring}\} \\
- \quad &\sum_{q\neq p+1}\check{\pi}_{p,q}[\pi^{p,q}, Q_{sstring}\}
\end{aligned} \tag{A.50}
$$

where

$$
\Theta_x = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{A.51}
$$

Appendix B

B.1 Hamiltonian to Lagrangian

B.1.1 Superparticle

In our previous paper we constructed the BRST operator for the superparticle and superstring in a super Yang-Mills background [17]. From the BRST operator we can get the gauge fixed Hamiltonian:

$$\begin{aligned} H_{GF}^{particle} &= \{b, Q_\theta\} \\ &= -\tfrac{1}{2}\Box + W^\alpha \nabla_\alpha + \tfrac{1}{2} F^{ab}\theta\gamma_{ba}\pi|_> \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} \Box &= -(p_a + A_a)^2, \qquad \eta_{ab} = \delta_{ab} \\ \nabla_\alpha &= d_{0\alpha} + A_\alpha \\ d_{0\alpha} &= \pi_{0\alpha} + (\not{p}\theta_0)_\alpha, \quad \pi_{0\alpha} = \partial/\partial\theta_0^\alpha \\ \gamma^{ab} &= -\tfrac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a), \qquad \{\gamma^a, \gamma^b\} = 2\delta^{ab} \end{aligned}$$

and ∇_α, ∇_a are the graded covariant derivatives.

Notice that π and θ are shorthand notation for $\pi_{p,q}$ and $\theta^{p,q}$, where $p-q$ is the ghost number and $p+q$ is the ghost level. (Even level and odd level correspond to fermion and boson respectively.) The expression "$|_>$" means "ghosts only".

Now we go to the Lagrangian form of the action for x. To obtain complete results for the amplitude rules, we need to keep terms in the Hamiltonian quadratic in the background fields. This has two unusual consequences: In the Lagrangian, (1) all these terms will become linear (as familiar from the bosonic case), and (2) such terms new to the supersymmetric case will appear only with $\dot\theta$.

Neglecting $ib\dot c$ and $F \cdot \hat s$ we see

$$\begin{aligned} & -p \cdot \dot x + i\pi\dot\theta + \tfrac{1}{2}(p+A)^2 + W^\alpha[\pi_{0\alpha} + p^a(\gamma_a\theta_0)_\alpha + A_\alpha] \\ \Rightarrow \quad & -\tfrac{1}{2}\dot x^2 + i\pi\dot\theta + A\cdot\dot x - iA_\alpha\dot\theta^\alpha + W^\alpha[\pi_{0\alpha} + (\dot x - A)\cdot(\gamma\theta_0)_\alpha - \tfrac{1}{2}(\gamma\theta)_\alpha \cdot W\gamma\theta] \end{aligned}$$

By redefining $\pi_{0\alpha} \Rightarrow \pi_{0\alpha} + A\cdot(\gamma\theta_0)_\alpha + \frac{1}{2}(\gamma\theta_0)_\alpha \cdot W\gamma\theta_0$ (deformed only with gauge fields) we get

$$-\tfrac{1}{2}\dot x^2 + i\pi\dot\theta - A_\alpha i\dot\theta_0^\alpha + A\cdot(\dot x + i\theta_0\gamma\dot\theta_0) + W^\alpha[\pi_{0\alpha} + \dot x\cdot(\gamma\theta_0)_\alpha + \tfrac{i}{2}(\gamma\theta_0)_\alpha\cdot\theta_0\gamma\dot\theta_0]$$

The background terms then give the vertex operator

$$\begin{aligned} V &= A^A j_A \\ &= A_\alpha\omega^\alpha + A^a p_a + W^\alpha d_\alpha + \tfrac{1}{2}F^{ab}\hat s_{ba} \end{aligned} \qquad \text{(B.2)}$$

where

$$
\begin{aligned}
d_\alpha &= \pi_{0\alpha} + \dot{x}\cdot(\gamma\theta_0)_\alpha + \tfrac{i}{2}(\gamma\theta_0)_\alpha \cdot \theta_0\gamma\dot{\theta}_0 \\
p_a &= \dot{x} + i\theta_0\gamma\dot{\theta}_0 \\
\omega^\alpha &= -i\dot{\theta}_0^\alpha \\
\hat{s}_{ab} &= \theta\gamma_{ab}\pi|_>
\end{aligned}
\tag{B.3}
$$

The fact that $\dot{\theta}$ vanishes by its free field equations is related to the fact that its contraction with π gives a $\delta(z)$, canceling a (spacetime) propagator, and thus contracting two 3-point vertices into a 4-point vertex. Thus, they originate from terms in the Hamiltonian quadratic in background fields. The string vertex operator is the same, with the z derivative replaced with the left- or right-handed worldsheet derivative.

In our previous paper [17] the background coupling had additional terms involving the expression R^a, quadratic in ghost θ's. These terms never contribute to amplitudes because there are no ghost π's to cancel them. ($\hat{s}$ has a ghost π, but together with a ghost θ.) This is also true for the superstring.

B.1.2 Superstring

Like the case of the superparticle, the gauge fixed action for the superstring comes from $\{\int b, Q_{sstring}\}$, adding first-order terms: without background,

$$S_{GF} = \int d^2z\ \hat{P}^m \partial_m X - \tfrac{1}{2}\eta_{mn}\hat{P}^m\hat{P}^n + i\sqrt{2}\sum_{\pm}\partial_{\pm}c^{\pm}b_{\pm\pm} + i\sqrt{2}\sum_{\pm}\partial_{\pm}\theta^{\pm}\pi^{\pm}$$

The $\sqrt{2}$ comes from $\partial_{\pm} = (1/\sqrt{2})(\partial_0 \pm \partial_1)$.

We can introduce the background as for the particle case:

$$\begin{aligned} V &= A^A J_A \\ &= A_\alpha \Omega^\alpha + A^a P_a + W^\alpha D_\alpha + \tfrac{1}{2}F^{ab}\hat{S}_{ba} \end{aligned} \tag{B.4}$$

where

$$\begin{aligned} D_\alpha &= \pi_{0\alpha} + (\gamma^a\theta_0)_\alpha \partial X_a + i\tfrac{1}{2}(\gamma^a\theta_0)_\alpha \theta_0\gamma_a\partial\theta_0 \\ P_a &= \partial X_a + i\theta_0\gamma_a\partial\theta_0 \\ \Omega_\alpha &= -i\partial\theta_{0\alpha} \\ \hat{S}_{ab} &= \theta\gamma_{ab}\pi|_{>} \end{aligned} \tag{B.5}$$

B.2 Current algebra

The operator (affine Lie) algebra remains simple because the currents are no more than cubic in the fundamental variables:

$$J_A(z_1)J_B(z_2) = G'_{AB}(z_1-z_2)f_{AB}{}^C[z']J_C(z')+G''_{AB}(z_1-z_2)\boldsymbol{\eta}_{AB}+ :: J_A(z_1)J_B(z_2) :: \tag{B.6}$$

where J_A has zero-modes j_A, of which only p_a and d_α act nontrivially on A^A, and G_{AB} is the relevant Green function. For example, $G^{(n)}_{ab} = G^{(n)}_x$ and $G^{(n)}_{\alpha\beta} = G^{(n-1)}_\theta$. The various definitions are

$$\begin{aligned}
f_{\alpha\beta}{}^a[z]P_a(z) &\equiv \gamma^a_{\alpha\beta}[P_a(z_\alpha) + P_a(z_\beta)] \\
f_{a\alpha\beta}[z]\Omega^\beta(z) &\equiv 2\gamma_{a\alpha\beta}\Omega^\beta(z_a) \\
f_{\alpha a\beta}[z]\Omega^\beta(z) &\equiv -2\gamma_{a\alpha\beta}\Omega^\beta(z_a) \\
\boldsymbol{\eta}_\alpha{}^\beta &= -i\delta^\beta_\alpha \\
\boldsymbol{\eta}^\beta{}_\alpha &= i\delta^\beta_\alpha \\
\boldsymbol{\eta}_{ab} &= -\eta_{ab} \\
& \quad \text{otherwise vanish}
\end{aligned}$$

and

$$
\begin{aligned}
:: J_A(z_1) J_B(z_2) :: A^C &\equiv \; : J_A(z_1) J_B(z_2) : A^C \\
J_A(z_1) A^B(z_2) &= G'_{AB}(z_1 - z_2)(j_A A^B)(z_2) + :J_A(z_1) A^B(z_2): \\
A^A(z_1) A^B(z_2) &= e^{-k_1 \cdot k_2 G_x(z_1 - z_2)} :A^A(z_1) A^B(z_2): \\
:: J_A(z_1) J_B(z_2) :: J_C(z_3) &\equiv (-1)^{BC} G'_{AC}(z_1 - z_3) f_{AC}{}^D[z'] J_D(z') J_C(z_3) \\
&\quad + G'_{BC}(z_2 - z_3) f_{BC}{}^D[z'] J_A(z_1) J_D(z') \\
&\quad + (-1)^{BC} G'_{AC}(z_1 - z_3) f_{AC}{}^D[z'] :: J_D(z') J_C(z_3) :: \\
&\quad + G'_{BC}(z_2 - z_3) f_{BC}{}^D[z'] :: J_A(z_1) J_D(z') :: \\
&\quad + :: J_A(z_1) J_B(z_2) J_C(z_3) :: \qquad \text{(B.7)}
\end{aligned}
$$

It is then straightforward to get (2.1), which is all that is needed in amplitude calculations.

B.3 Component expansions

The θ expansion of the superfields follows directly from the constraints on the (super)field strengths

$$\begin{aligned}
[\nabla_a, \nabla_b] &= F_{ab} \\
\{\nabla_{0\alpha}, \nabla_{0\beta}\} &= 2\gamma_{a\alpha\beta}\nabla^a \\
[\nabla_{0\alpha}, \nabla_a] &= 2\gamma_{a\alpha\beta}W^\beta
\end{aligned}$$

and the Bianchi identities that follow from them.

Although in practice we perform component expansions by evaluating spinor derivatives at $\theta = 0$, we can also directly expand superfields in θ. In a Wess-Zumino gauge we have:

$$\begin{aligned}
F_{ab} &= \overset{\circ}{F}_{ab} \\
W^\alpha &= \overset{\circ}{W}{}^\alpha + \tfrac{1}{2}(\gamma^{ab}\theta_0)^\alpha \overset{\circ}{F}_{ab} \\
A_a &= \overset{\circ}{A}_a + 2\theta_0\gamma_a\overset{\circ}{W} + \tfrac{1}{2}\theta_0\gamma_a\gamma^{bc}\theta_0\,\overset{\circ}{F}_{bc} \\
A_\alpha &= (\gamma^a\theta_0)_\alpha\overset{\circ}{A}_a + \tfrac{4}{3}(\gamma^a\theta_0)_\alpha\theta_0\gamma_a\overset{\circ}{W} + \tfrac{1}{4}(\gamma^a\theta_0)_\alpha\theta_0\gamma_a\gamma^{bc}\theta_0\overset{\circ}{F}_{bc}
\end{aligned}$$

where $\circ$ indicates θ_0 independence, and we have expanded only to constant field strengths W and F, which is sufficient for lower-point diagrams because of the deficiency of π's. The vertex operator $V = V_B + V_F$ for the superparticle is then :

$$
\begin{aligned}
V_B &= i\mathring{A} \cdot \dot{x} + \tfrac{1}{2}\mathring{F}^{ab}\theta\gamma_{ba}\pi \\
V_F &= \mathring{W}^{\alpha}[\pi_{0\alpha} - i\dot{x} \cdot (\gamma\theta_0)_\alpha - \tfrac{i}{6}(\gamma_a\theta_0)_\alpha\theta_0\gamma^a\dot{\theta}_0]
\end{aligned}
\tag{B.8}
$$

Here $\theta\gamma_{ab}\pi$ includes the physical π_0, θ_0. (In terms of superfields and currents we hide this physical π_0, θ_0 in W^α and D_α. Then $\hat{S}$ has only ghost number non-zero π, θ.) Notice that the spinor vertex is the supersymmetry generator q_α, which will happen again in the superstring case. Inserting plane waves for the fields,

$$V_B = A_a(i\dot{x}^a + \theta\gamma^{ab}\pi k_b)e^{ik\cdot x} \tag{B.9}$$

$$V_F = w^\alpha[\pi_{0\alpha} - i\dot{x} \cdot (\gamma\theta_0)_\alpha - \tfrac{i}{6}(\gamma_a\theta_0)_\alpha\theta_0\gamma^a\dot{\theta}_0]e^{ik\cdot x} \tag{B.10}$$

The superstring vertices are essentially the same.

B.4 Loop review

B.4.1 Superparticle

Since our theory is 1st-quantized, we should calculate amplitudes in terms of worldline Green functions with periodic boundary conditions [27]. So our partition function with imaginary time is

$$\mathcal{N} \int_0^\infty \frac{dT}{T} \int \mathcal{D}X \; \mathcal{D}c \; \mathcal{D}b \; \mathcal{D}\theta \; \mathcal{D}\pi \; Tr \; e^{-\int_0^T dz \;\; L_\theta} \tag{B.11}$$

where

$$\mathcal{N} = \int \mathcal{D}P \; e^{-\int_0^T dz P^2/2} \tag{B.12}$$

and $\int dT/T$ comes from the Schwinger proper-time integral representation of the 1-loop vacuum energy $-Tr[\ln(-\Box)]$.

Since this is a 1-loop amplitude, we should impose periodic boundary conditions, on both X and θ (to preserve supersymmetry):

$$X(T) = X(0), \qquad \theta(T) = \theta(0) \tag{B.13}$$

This boundary condition also results in a supertrace naturally in the loop amplitude.

In this setting the color-ordered, N-point, 1-loop amplitude of the superparticle can be written as:

$$A_N = G_N \int_0^\infty dT \; (2\pi T)^{-D/2} \int_{0 \le z_r \le z_{r+1} \le z_N = T \le \infty} d^{N-1} z_i \; K_N \; e^{-\sum_{1 \le r < s \le N} k_r \cdot k_s G_{rs}} \tag{B.14}$$

The factor $(2\pi T)^{-D/2}$ comes from $\mathcal{N}\int\mathcal{D}Xe^{-\int\dot{X}^2/2}$. The z_i integration factors come from the Nth order expansion of the vertex operator. The worldline Green function $G(z_s - z_r)$ is given in (2.15). Examples of the kinematic factor K_N are given in section 5. The factor G_N is the trace of group generators in a given ordering.

B.4.2 Superstrings

The procedure is almost identical to the particle case. One difference is that our Green functions are now doubly periodic:

$$\begin{aligned} G^x(z) &= G^x(z+2\pi i) = G^x(z+T) \\ G^\theta(z) &= G^\theta(z+2\pi i) = G^\theta(z+T) \end{aligned} \tag{B.15}$$

Again this periodic boundary condition in both directions is required by supersymmetry. Also, there is a topological distinction among graphs, namely planar, nonplanar, and unorientable graphs. We will concentrate on the planar one here; the others follow from similar considerations.

We can write the color-ordered, N-point, 1-loop, superstring amplitude in a form identical to that of the particle case (B.14), but with the string Green function given in Appendix B.5. After the usual change of variables

$$\rho_i = e^{-z_i}, \qquad w = \rho_N = e^{-T} \tag{B.16}$$

we have

$$G_N \int_0^1 \frac{dw}{w} \left(\frac{-2\pi}{\ln w}\right)^{D/2} \int_{0\le w\le\rho_{r+1}\le\rho_r\le 1} \prod_{r=1}^{N-1} \frac{d\rho_r}{\rho_r} \; K_N \; e^{-\sum_{1\le r<s\le N} k_r\cdot k_s G_{rs}} \tag{B.17}$$

In the bosonic case there is a factor of $[f(w)]^{-D+2}$ coming from the partition function (and a w from the tachyon mass) for D X's and the 2 reparametrization ghosts b and c. In the supersymmetric case this is canceled (as in all superstring formulations) by an $[f(w)]^{(2^{D/2})(1-2+3-4+\cdots)} = [f(w)]^{2^{(D-4)/2}}$ which comes from the infinite pyramid of spinors, in $D = 10$. However, the regularization introduces corrections to the spinor partition function:

$$\prod_n (1-w^n)^4(1-w^n)^4 \Rightarrow \prod_n (1-w^{n+i\epsilon})^4(1-w^{n-i\epsilon})^4 \tag{B.18}$$

The ϵ expansion of this partition function gives corrections to amplitudes. For example, in the 6-point, 1-loop amplitude we expect a term $\sim (1/\epsilon)^6(\epsilon^2\theta_1'''/\theta_1')$, where the $(1/\epsilon)^6$ comes from 6 G^θ's and the $\epsilon^2\theta_1'''/\theta_1'$ comes from expansion of the spinor partition function.

B.5 Periodic Green functions

B.5.1 Second order

The general Fourier decomposition of a function in 2 dimensions with doubly periodic boundary conditions ($(x, y) \simeq (x + 2\pi, y) \simeq (x, y + 2\pi\tau)$) for real $\tau = T/2\pi$ is

$$G(x - x', y - y') = \sum_{n,m} G_{n,m}\ e^{in(x-x')+im(y-y')/\tau} \tag{B.19}$$

Then the $G_{n,m}$ for the Green function of the differential operator $-\partial_x^2 - \partial_y^2 + \epsilon^2$ is easily found to be

$$G_{n,m} = \frac{1}{2\pi\tau} \frac{1}{n^2 + \frac{m^2}{\tau^2} + \epsilon^2} \tag{B.20}$$

For simplicity we can set $x' = y' = 0$ by translational invariance. Using Schwinger proper-time parametrization we get

$$G(x, y) = \frac{1}{2\pi\tau} \sum_{n,m} \int_0^\infty ds\ e^{-s(n^2+m^2/\tau^2+\epsilon^2)+inx+imy/\tau} \tag{B.21}$$

Next using Jacobi's transform

$$\sum_m\ e^{-sm^2/\tau^2+imy/\tau} = \tau\sqrt{\frac{\pi}{s}} \sum_m\ e^{-(2\pi m-y/\tau)^2\tau^2/4s}$$

we get

$$G(x, y) = \frac{1}{2\pi} \sum_{n,m} \int_0^\infty ds\ \sqrt{\frac{\pi}{s}}\ e^{-(2\pi m-y/\tau)^2\tau^2/4s-s(n^2+\epsilon^2)+inx} \tag{B.22}$$

Then using

$$\int_0^\infty ds\ s^{\alpha-1}\ e^{-ps-q/s} = 2\left(\frac{q}{p}\right)^{\alpha/2} K_\alpha(2\sqrt{pq})$$

we get

$$G(x,y) = \frac{1}{\sqrt{2\pi}}\sum_{n,m}\left(\frac{(2\pi m\tau - y)^2}{n^2+\epsilon^2}\right)^{1/4} e^{inx} K_{1/2}\left(\sqrt{(2\pi m\tau - y)^2(n^2+\epsilon^2)}\right) \tag{B.23}$$

Also using $K_{1/2}(z) = \sqrt{\pi/2z}e^{-z}$ we get

$$\begin{aligned} G(x,y) &= \frac{1}{2}\sum_{n,m}\frac{1}{\sqrt{n^2+\epsilon^2}}e^{-|2\pi m\tau - y|\sqrt{n^2+\epsilon^2}+inx} \\ &= \frac{1}{2\epsilon}e^{-\epsilon|y|} + \frac{1}{2\epsilon}\sum_{m\neq 0} e^{-\epsilon|2\pi m\tau - y|} + \frac{1}{2}\sum_{n\neq 0}\frac{1}{\sqrt{n^2+\epsilon^2}}e^{-|y|\sqrt{n^2+\epsilon^2}+inx} \\ &\quad \frac{1}{2}\sum_{n,m\neq 0}\frac{1}{\sqrt{n^2+\epsilon^2}}e^{-|2\pi m\tau - y|\sqrt{n^2+\epsilon^2}+inx} \end{aligned} \tag{B.24}$$

Now let's transform each sum into a sum over positive integers only:

$$\begin{aligned} \frac{1}{2\epsilon}\sum_{m\neq 0} e^{-\epsilon|2\pi m\tau - y|} &= \frac{1}{\epsilon}\cosh(\epsilon y)\sum_{m=1}^\infty e^{-2\pi m\tau\epsilon} \\ &= \frac{1}{2\epsilon}e^{-\pi\tau\epsilon}\frac{\cosh(\epsilon y)}{\sinh(\pi\tau\epsilon)} \end{aligned} \tag{B.25}$$

$$\frac{1}{2}\sum_{n\neq 0}\frac{1}{\sqrt{n^2+\epsilon^2}}e^{-|y|\sqrt{n^2+\epsilon^2}\ +\ inx} = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{\sqrt{n^2+\epsilon^2}}(\lambda_n^n + c.c.) \tag{B.26}$$

$$\frac{1}{2}\sum_{n,m\neq 0}\frac{1}{\sqrt{n^2+\epsilon^2}}e^{-|2\pi m\tau - y|\sqrt{n^2+\epsilon^2}\ +\ inx} = \frac{1}{2}\sum_{n,m=1}^\infty \frac{1}{\sqrt{n^2+\epsilon^2}}w_n^{mn}(\rho_n^n + \rho_n^{-n} + c.c.) \tag{B.27}$$

where

$$\begin{aligned}
\ln \lambda_n &= i\left(x + i|y|\sqrt{1+\frac{\epsilon^2}{n^2}}\right) \\
\ln \rho_n &= i\left(x + iy\sqrt{1+\frac{\epsilon^2}{n^2}}\right) \\
w_n &= e^{-2\pi\tau\sqrt{1+\epsilon^2/n^2}}
\end{aligned} \tag{B.28}$$

We can subtract out G for the particle from $G(x,y)$, which includes the part divergent as $\epsilon \to 0$:

$$\frac{1}{2\epsilon}\frac{\cosh[\epsilon(|y|-\pi\tau)]}{\sinh(\pi\tau\epsilon)} \tag{B.29}$$

The remainder is

$$\frac{1}{2}\sum_{n=1}^{\infty}\tfrac{1}{\sqrt{n^2+\epsilon^2}}(\lambda_n^n + c.c.) + \frac{1}{2}\sum_{n,m=1}^{\infty}\tfrac{1}{\sqrt{n^2+\epsilon^2}}w_n^{mn}(\rho_n^n + \rho_n^{-n} + c.c) \tag{B.30}$$

In the $\epsilon \to 0$ limit ($w_n \to w = e^{-T}, \lambda_n \to \lambda, \rho_n \to \rho = e^{-z}, z = y - ix$), using

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = -\ln(1-x)$$

we get for the remainder

$$\begin{aligned}
&-\ln|1-\lambda| - \sum_{m=1}^{\infty}\ln|(1-w^m\rho)(1-w^m\rho^{-1})| \\
&= -\frac{[Re(z)]^2}{2T} - \frac{1}{2}\ln|\lambda| - \ln|f(w)^2| + G^x_{un}
\end{aligned}$$

where (assuming $|y| = y$ for simplicity)

$$
\begin{aligned}
G^x_{un}(z,T) &= -\ln\left|\frac{2\pi\theta_1(\frac{iz}{2\pi}|\frac{iT}{2\pi})}{\theta_1'(0|\frac{iT}{2\pi})}\right| + \frac{[Re(z)]^2}{2T} \\
\theta_1(\tfrac{iz}{2\pi}|\tfrac{iT}{2\pi}) &= -iw^{1/8}(\rho^{1/2}-\rho^{-1/2})\prod_{m=1}^{\infty}(1-w^m\rho)(1-w^m\rho^{-1})(1-w^m) \\
\theta_1'(0|\tfrac{iT}{2\pi}) &= 2\pi w^{1/8}f^3(w), \qquad f(w) = \prod_{m=1}^{\infty}(1-w^m)
\end{aligned}
$$

and G^x_{un} is the unregularized Green function with the usual T-dependent "constant" added to normalize its short distance behavior to be the same as that of the tree case (see, e.g., [28]).

Combining the two parts we get

$$
\begin{aligned}
G(\rho) &= \frac{1}{T\epsilon^2} + \frac{y^2}{2T} - \frac{y}{2} + \frac{T}{12} + \mathcal{O}(\epsilon) \\
&\quad + \frac{1}{2}Re\left(\frac{\ln^2\rho}{\ln w}\right) - \frac{1}{2}\ln|\rho| - \ln|[f(w)]^2| + G^x_{un} + \mathcal{O}(\epsilon) \\
&= \frac{1}{T\epsilon^2} - \frac{1}{12}\ln|w[f(w)]^{24}| + G^x_{un} + \mathcal{O}(\epsilon) \qquad \text{(B.31)}
\end{aligned}
$$

The first term is the zero-mode behavior, and the second term is a constant that won't contribute to massless amplitudes (because of derivatives and $k^2 = 0$; the non-f piece is the same as for the particle).

B.5.2 First order

The worldsheet Green function for θ can be obtained by differentiating that for X. However, to be careful about zero-modes some modification is needed.

For the 1st-order differential operator $-i(\partial_y - i\partial_x + \epsilon)$ we find the mode sum of the Green function

$$\begin{aligned} G^\theta &= \frac{i}{T}\sum_{m,n}\frac{-im/\tau + n + \epsilon}{m^2/\tau^2 + (n+\epsilon)^2}e^{inx+imy/\tau} \\ &= i(-\partial_y - i\partial_x + \epsilon)\frac{1}{2\pi\tau}\sum_{m,n}\frac{1}{m^2/\tau^2 + (n+\epsilon)^2}e^{inx+imy/\tau} \end{aligned}$$

where the ϵ in the numerator is nontrivial because the second-order Green function has a $1/\epsilon$ pole. The above sum is almost identical to the second-order case except for the change $n^2 + \epsilon^2 \Rightarrow (n+\epsilon)^2$. Therefore we can write the first-order Green function as

$$\begin{aligned} G^\theta &= i(-\partial_y - i\partial_x + \epsilon)\left[\frac{1}{2\epsilon}\frac{\cosh[\epsilon(|y| - \pi\tau)]}{\sinh(\pi\tau\epsilon)} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+\epsilon}(\lambda_n^n + c.c.)\right. \\ &\qquad \left. + \frac{1}{2}\sum_{n,m=1}^{\infty}\frac{1}{n+\epsilon}w_n^{mn}(\rho_n^n + \rho_n^{-n} + c.c.)\right] \end{aligned} \tag{B.32}$$

where

$$\begin{aligned} \ln\lambda_n &= i\left[x + i|y|\left(1 + \frac{\epsilon}{n}\right)\right] \\ \ln\rho_n &= i\left[x + iy\left(1 + \frac{\epsilon}{n}\right)\right] \\ w_n &= e^{-2\pi\tau(1+\epsilon/n)} \end{aligned}$$

Hence it has a less divergent leading term followed by the expected differentiated second-order Green function:

$$
\begin{aligned}
G^{\theta} &= \frac{i}{T\epsilon} + \sum_{n=0}^{\infty} G_n^{\theta} \epsilon^n \\
&= \frac{i}{T\epsilon} + i(\partial_y + i\partial_x) G_0^x + \mathcal{O}(\epsilon)
\end{aligned}
\tag{B.33}
$$

B.6 Super amplitudes

B.6.1 Super tree

In this section we give some details of the calculation of 3-point tree and 4-point 1-loop super amplitudes.

We will concentrate on terms which give fermion contributions. In (2.6) only C, D, and E give the AWW amplitude. Let's consider $A(1)W(2)W(3)$, for example. Using (2.1),(B.7) we get for the tree (no fermion zero-mode regularization)

$$\begin{aligned}
\langle P_a(1)P_b(2)\rangle &= -\eta_{ab}\tfrac{1}{(z_1-z_2)^2} \\
\langle D_\alpha(1)\Omega^\beta(2)\rangle &= \delta_\alpha^\beta\tfrac{1}{(z_1-z_2)^2} \\
\langle P_a(1)D_\alpha(2)D_\beta(3)\rangle &= -i\gamma_{a\alpha\beta}[\tfrac{2}{z_1-z_2}\tfrac{1}{(z_1-z_3)^2} - \tfrac{2}{z_1-z_3}\tfrac{1}{(z_2-z_1)^2} \\
&\quad -\tfrac{1}{z_2-z_3}(\tfrac{1}{(z_1-z_2)^2} + \tfrac{1}{(z_1-z_3)^2})]
\end{aligned} \tag{B.34}$$

Then we see in the Wess-Zumino gauge ($A_\alpha = \gamma^a_{\alpha\beta}\theta^\beta A^a + \mathcal{O}(\theta^2) + \cdots$, etc.)

$$
\begin{aligned}
C : (PPD) \quad &: \quad -\langle P_a(1)P_b(2)\rangle A^a(1)(D_\alpha(3)A^b(2))W^\alpha(3) \\
&= \quad -i\tfrac{1}{(z_1-z_2)^2}\tfrac{1}{z_3-z_2}A^a(1)(d_\alpha A_a(2))W^\alpha(3) \\
&= \quad -i\tfrac{1}{(z_1-z_2)^2}\tfrac{2}{z_3-z_2}A^a(1)W(2)\gamma_a W(3) \\
(PDP) \quad &: \quad -\langle P_a(1)P_b(3)\rangle A^a(1)(D_\alpha(2)A^b(3))W^\alpha(2) \\
&= \quad -i\tfrac{1}{(z_1-z_3)^2}\tfrac{2}{z_2-z_3}A^a(1)W(3)\gamma_a W(2) \\
D : (PDD) \quad &: \quad -\langle P_a(1)D_\alpha(2)D_\beta(3)\rangle A^a(1)W^\alpha(2)W^\beta(3) \\
&= \quad i[\tfrac{2}{z_1-z_2}\tfrac{1}{(z_1-z_3)^2} - \tfrac{2}{z_1-z_3}\tfrac{1}{(z_2-z_1)^2} \\
&\qquad -\tfrac{1}{z_2-z_3}(\tfrac{1}{(z_1-z_2)^2} + \tfrac{1}{(z_1-z_3)^2})]A(1)\cdot W(2)\gamma W(3) \\
E : (\Omega DD) \quad &: \quad \langle \Omega^\alpha(1)D_\beta(2)\rangle(D_\gamma(3)A_\alpha(1))W^\beta(2)W^\gamma(3) \\
&\qquad -\langle \Omega^\alpha(1)D_\gamma(3)\rangle(D_\beta(2)A_\alpha(1))W^\beta(2)W^\gamma(3) \\
&= \quad i\left[-\tfrac{1}{(z_1-z_2)^2}\tfrac{1}{z_3-z_1} + \tfrac{1}{(z_1-z_3)^2}\tfrac{1}{z_2-z_1}\right]A(1)\cdot W(2)\gamma W(3)
\end{aligned}
\tag{B.35}
$$

This reduces to (2.7).

B.6.2 1 loop : 2 fermions + 2 vectors

We will concentrate on the case where the fermions are at both ends. The other case can be easily obtained by permutation. There are two kinds of contributions: $W(dW)(dddW)W$ (with $WF(ddF)W$) and $W(dW)(dW)(ddW)$ (and corresponding $W(dW)F(dF)$). The W^2F^2 contribution gives a GP sum with the corresponding $W^2(dW)^2$ as usual. The explicit formula is

$$\begin{aligned}
A: &\quad - \quad W^\alpha(1)\left(d_\gamma W^\beta(2)\right)\left(\tfrac{1}{3!}d_{[\delta}d_\alpha d_{\beta]}W^\gamma(3)\right)W^\delta(4)\\
&\quad + \quad \tfrac{3}{16}tr(\gamma_{ab}\gamma_{cd})W^\alpha(1)F^{ab}(2)\left(\tfrac{1}{2!}d_{[\delta}d_{\alpha]}F^{cd}(3)\right)W^\delta(4)\\
B: &\quad - \quad W^\alpha(1)\left(\tfrac{1}{3!}d_{[\delta}d_\alpha d_{\gamma]}W^\beta(2)\right)\left(d_\beta W^\gamma(3)\right)W^\delta(4)\\
&\quad + \quad \tfrac{3}{16}tr(\gamma_{ab}\gamma_{cd})W^\alpha(1)\left(\tfrac{1}{2!}d_{[\delta}d_{\alpha]}F^{ab}(2)\right)F^{cd}(3)W^\delta(4)\\
C: &\quad - \quad W^\alpha(1)\left(d_\alpha W^\beta(2)\right)\left(d_\delta W^\gamma(3)\right)\left(\tfrac{1}{2!}d_{[\beta}d_{\gamma]}W^\delta(4)\right)\\
&\quad + \quad \tfrac{3}{16}tr(\gamma_{ab}\gamma_{cd})W^\alpha\left(d_\alpha W^\beta(2)\right)F^{ab}(3)\left(d_\beta F^{cd}(4)\right)\\
D: &\quad - \quad W^\alpha(1)\left(d_\delta W^\beta(2)\right)\left(d_\alpha W^\gamma(3)\right)\left(\tfrac{1}{2!}d_{[\gamma}d_{\beta]}W^\delta(4)\right)\\
&\quad + \quad \tfrac{3}{16}tr(\gamma_{ab}\gamma_{cd})W^\alpha(1)F^{ab}(2)\left(d_\alpha W^\gamma(3)\right)\left(d_\gamma F^{cd}(4)\right)\\
E: &\qquad\quad \left(d_\beta d_\gamma W^\alpha(1)\right)\left(d_\delta W^\beta(2)\right)\left(d_\alpha W^\gamma(3)\right)W^\delta(4)\\
&\quad - \quad \tfrac{3}{16}tr(\gamma_{ab}\gamma_{cd})\left(d_\beta F^{ab}(1)\right)\left(d_\delta W^\beta(2)\right)F^{cd}(3)W^\delta(4)\\
F: &\qquad\quad \left(d_\gamma d_\beta W^\alpha(1)\right)\left(d_\alpha W^\beta(2)\right)\left(d_\delta W^\gamma(3)\right)W^\delta(4)\\
&\quad - \quad \tfrac{3}{16}tr(\gamma_{ab}\gamma_{cd})\left(d_\gamma F^{ab}(1)\right)F^{cd}(2)\left(d_\delta W^\gamma(3)\right)W^\delta(4)
\end{aligned} \tag{B.36}$$

A and B vanish due to a GP sum. $C+D$ and $E+F$ give identical contributions, using integration by parts (momentum conservation) and the (free) W field equation $\partial\!\!\!/ W = 0$. The results are given in (2.25).

These results appear in the literature in forms where neither gauge invariance nor permutation symmetry (relating FFBB and FBFB) is manifest, which we now provide for comparison. When written in terms of each momentum and gauge field, the results are (before applying integration by parts)

$$\begin{aligned}
C: &\quad -\tfrac{1}{2}k_1\cdot k_2 W(1)\not{A}_2\not{k}_2\not{A}_3 W(4) + \tfrac{1}{2}A_3\cdot k_4 W(1)\not{A}_2\not{k}_2\not{k}_3 W(4) \\
D: &\quad -\tfrac{1}{2}k_1\cdot k_3 W(1)\not{A}_3\not{k}_3\not{A}_2 W(4) + \tfrac{1}{2}A_2\cdot k_4 W(1)\not{A}_3\not{k}_3\not{k}_2 W(4) \\
E: &\quad -\tfrac{1}{2}k_1\cdot k_3 W(4)\not{A}_2\not{k}_2\not{A}_3 W(1) + \tfrac{1}{2}A_3\cdot k_1 W(4)\not{A}_2\not{k}_2\not{k}_3 W(1) \\
F: &\quad -\tfrac{1}{2}k_1\cdot k_2 W(4)\not{A}_3\not{k}_3\not{A}_2 W(1) + \tfrac{1}{2}A_2\cdot k_1 W(4)\not{A}_3\not{k}_3\not{k}_2 W(1)
\end{aligned} \tag{B.37}$$

Each of $C+D$ and $E+F$ can then be re-expressed as

$$\begin{aligned}
&\tfrac{1}{2}k_1\cdot k_4\ W(1)\not{A}_3\not{k}_3\not{A}_2 W(4) \\
+&\ k_1\cdot k_4\ k_4\cdot A_2\ W(1)\not{A}_3 W(4) + k_1\cdot k_2\ A_2\cdot A_3\ W(1)\not{k}_2 W(4) \\
+&\ k_1\cdot A_2\ k_4\cdot A_3\ W(1)\not{k}_2 W(4) + k_1\cdot A_3\ k_4\cdot A_2\ W(1)\not{k}_3 W(4)
\end{aligned} \tag{B.38}$$

Another expression for each of C, D, E, F can be obtained by absorbing the second term into the first term, and the summed result is:

$$-\,k_1\cdot k_3\ W(1)\not{A}_3(\not{k}_3+\not{k}_4)\not{A}_2 W(4) - k_1\cdot k_2 W(1)\not{A}_2(\not{k}_2+\not{k}_4)\not{A}_3 W(4) \tag{B.39}$$

B.6.3 1 loop : 4 fermions

There are totally $\frac{1}{2} \cdot \binom{4}{2} \cdot 2 \cdot 2 = 3 \cdot 4 = 12$ terms (and also 12 corresponding $(dF)^2 W^2$ terms) contributing to the 4-fermion amplitude:

$$
\begin{aligned}
[\alpha\beta][\gamma\delta] \quad : \quad & \tfrac{1}{2!2!}(d_{[\delta}d_{\gamma]}W^\alpha)W^\beta(d_{[\beta}d_{\alpha]}W^\gamma)W^\delta \\
& -\tfrac{1}{2!2!}(d_{[\gamma}d_{\delta]}W^\alpha)W^\beta W^\gamma(d_{[\beta}d_{\alpha]}W^\delta) \\
& -\tfrac{1}{2!2!}W^\alpha(d_{[\delta}d_{\gamma]}W^\beta)(d_{[\alpha}d_{\beta]}W^\gamma)W^\delta \\
& +\tfrac{1}{2!2!}W^\alpha(d_{[\gamma}d_{\delta]}W^\beta)W^\gamma(d_{[\alpha}d_{\beta]}W^\delta) \\
[\alpha\gamma][\beta\delta] \quad : \quad & -\tfrac{1}{2!2!}(d_{[\delta}d_{\beta]}W^\alpha)(d_{[\gamma}d_{\alpha]}W^\beta)W^\gamma W^\delta \\
& +\tfrac{1}{2!2!}(d_{[\beta}d_{\delta]}W^\alpha)W^\beta W^\gamma(d_{[\gamma}d_{\alpha]}W^\delta) \\
& +\tfrac{1}{2!2!}W^\alpha(d_{[\alpha}d_{\gamma]}W^\beta)(d_{[\delta}d_{\beta]}W^\gamma)W^\delta \\
& -\tfrac{1}{2!2!}W^\alpha W^\beta(d_{[\beta}d_{\delta]}W^\gamma)(d_{[\alpha}d_{\gamma]}W^\delta) \\
[\alpha\delta][\beta\gamma] \quad : \quad & -\tfrac{1}{2!2!}W^\alpha(d_{[\alpha}d_{\delta]}W^\beta)W^\gamma(d_{[\gamma}d_{\beta]}W^\delta) \\
& +\tfrac{1}{2!2!}(d_{[\gamma}d_{\beta]}W^\alpha)(d_{[\delta}d_{\alpha]}W^\beta)W^\gamma W^\delta \\
& -\tfrac{1}{2!2!}(d_{[\beta}d_{\gamma]}W^\alpha)W^\beta(d_{[\delta}d_{\alpha]}W^\gamma)W^\delta \\
& +\tfrac{1}{2!2!}W^\alpha W^\beta(d_{[\alpha}d_{\delta]}W^\gamma)(d_{[\beta}d_{\gamma]}W^\delta)
\end{aligned}
\tag{B.40}
$$

For each term there are 4 terms, which come from $[d_\alpha d_\beta - \gamma^a_{\alpha\beta}(-i\partial_a)]^2$. Among them only the $\gamma\gamma$ term survives, and the others vanish due to a GP sum from corresponding $(dF)^2WW$ terms.

For each group two terms are equal to the other 2 terms, and the resultant

6 terms are

$$
\begin{aligned}
&2W(1)\gamma_a W(2)\ W(3)\gamma_b W(4)\ k_1^b k_3^a \\
&-2W(1)\gamma_a W(2)\ W(3)\gamma_b W(4)\ k_1^b k_4^a \\
&+2W(1)\gamma_a W(3)\ W(2)\gamma_b W(4)\ k_1^b k_2^a \\
&-2W(1)\gamma_a W(3)\ W(2)\gamma_b W(4)\ k_1^b k_4^a \\
&-2W(1)\gamma_a W(4)\ W(2)\gamma_b W(3)\ k_1^b k_3^a \\
&+2W(1)\gamma_a W(4)\ W(2)\gamma_b W(3)\ k_1^b k_2^a
\end{aligned}
\tag{B.41}
$$

Symmetry between any two fermion lines is somewhat obscure in this form. But there are Fierz identities which make it clear:

$$
\begin{aligned}
A: \quad & W(1)\gamma^a W(2)\ W(3)\gamma_a W(4) \\
= \quad & -W(1)\gamma^a W(3)\ W(4)\gamma_a W(2) - W(1)\gamma^a W(4)\ W(2)\gamma_a W(3)
\end{aligned}
$$

$$
\begin{aligned}
B: \quad & W(2)\gamma^c\gamma_d\gamma^a W(1)\ W(4)\gamma_a\gamma_b\gamma_c W(3) k_1^b k_3^d \\
= \quad & 4W(2)\gamma^a W(4)\ W(3)\gamma_a W(1) k_2\cdot k_3 - 4W(2)\gamma_a W(4)\ W(3)\gamma_b W(1) k_3^a k_2^b \\
& +12W(2)\gamma_a W(3)\ W(1)\gamma W(4) k_1^a k_3^b - 12W(2)\gamma^a W_3\ W(1)\gamma_a W(4) k_1\cdot k_3
\end{aligned}
$$

$$
\begin{aligned}
C: \quad & W(2)\gamma^c\gamma_d\gamma^a W(1)\ W(4)\gamma_a\gamma_b\gamma_c W(3)k_1^b k_3^d \\
= \quad & 8W(2)\gamma^a W(4)\ W(3)\gamma_a W(1)k_2\cdot k_3 - 8W(2)\gamma_a W(4)\ W(3)\gamma_b W(1)k_3^a k_2^b \\
& +16W(2)\gamma_a W(3)\ W(1)\gamma W(4)k_1^a k_3^b - 16W(2)\gamma^a W_3\ W(1)\gamma_a W(4)k_1\cdot k_3 \\
& -4W(2)\gamma_a W(1)\ W(4)\gamma_b W(3)k_3^a k_1^b
\end{aligned} \tag{B.42}
$$

Using the above identities we can rewrite (B.41) as

$$
4k_1\cdot k_2\ W(1)\gamma W(4)\cdot W(2)\gamma W(3) - 4k_1\cdot k_4\ W(1)\gamma W(2)\cdot W(3)\gamma W(4) \tag{B.43}
$$

Now symmetry in fermion lines can be checked using Fierz identity A.

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