# Integrability of N=4 and N=2 Super Yang Mills

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by

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Abstract of the Thesis

# Integrability of N=4 and N=2 Super Yang Mills

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The present thesis is devoted to the study of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetric field theories. We construct the appropriate supesymmetric algebras and we examine the symmetries of the theories. We then derive the matrices of anomalous dimensions of local gauge invariant operators of the theories. It is also shown that these matrices can be identified with hamiltonians that describe spin chains. Finally, we also prove that the coupling of the gauge and hyper Lagrangians does not effect the anomalous dimensions of single trace composite operators which are built of scalar fields of $\mathcal{N} = 2$ gauge multiplet.

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# Chapter 1

# Introduction and Conclusions

Supersymmetric gauge theories are of great interest. Particularly, after the discovery of AdS/CFT correspondence which relates a $\mathcal{N} = 4$ Super Yang Mills theory to a Type IIB string theory, [1], there is growing interest in these theories. In this project, extended supersymmetric theories are examined and in particular the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ super Yang Mills (SYM) theories, and the $\mathcal{N} = 2$ hyper Lagrangian.

Firstly, we construct the super Poincaré algebra and we then proceed to the building of massless particle representations of the theory, which lead to the field content and eventually to the Lagrangians of the theories that will be examined, [2], [3]. The most remarkable characteristic of the $\mathcal{N} = 4$ SYM is its symmetries. The theory is conformally invariant even at the quantum level. So, its gauge coupling does not receive any quantum corrections. However, using the super-conformal symmetry of the theory we build local operators which are renormalized and have interesting properties. We also introduce the concepts of superconformal and chiral primary operators, see [3], [4] and [5].

The renormalization of local gauge invariant operators is then analysed. The main purpose of this project is the computation of the anomalous dimension of several scalar composite operators and the connection of the results to hamiltonians of integrable spin chains. This was first done for the $SO(6)$ sector of $\mathcal{N} = 4$ SYM by Minahan and Zarembo [6]. Here, we use the same techniques applied to the $SU(2)$ sector of the theory, [7], and conclude that the matrix of anomalous dimensions of opertors in $SU(2)$ sector is identified with the hamiltonian of the XXX Heisenberg spin chain. Then, Bethe equations can be used in order to diagonalize the hamiltonian and to find the basis where the renormalization of the considered operators is multiplicative. The eigenvalues correspond to the anomalous dimensions of the operators of this basis. The diagonalization of the Heisenberg hamiltonian can be performed for a small number of magnons[1] in the spin chain. But one can also diagonalize the hamiltonian in the thermodynamic limit, [7] and [5]. It is then observed that the eigenvalues correspond to the energy of a classical string of large angular momentum spinning in a subspace of $AdS_5 \times S^5$. Hence, we notice that there is a correspondence among string states and local operators of the super Yang Mills in the limit of large angular momentum.

We then follow a similar procedure in order to compute the anomalous dimension of local operators in $\mathcal{N} = 2$ SYM, [8]. The main difference from the previous analysis is that now the coupling constant will acquire quantum corrections that we must take into consideration when we renormalize the composite operators. In this case, the resulting matrix of the anomalous dimensions is identified with a hamiltonian of an XXZ spin chain. It is also found that when the $\mathcal{N} = 2$ SYM is coupled to the hyper Lagrangian the result remains the same. This happens because there are two contributions that cancel, one coming from the coupling constant and the other from the self energy renormalization.

[1] Magnon is an excitation of a spin state in the spin chain.

# Chapter 2

# Supersymmetric Field Theories

### 2.0.1 The supersymmetry algebra

The generators of Poincaré algebra, $SO(1,3)$ are $P_\mu$ and $M_{\mu\nu}$ which correspond to the translations and Lorentz transformations in Minkowski space-time with metric $\eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1)$, where $\mu,\nu = 0,1,2,3$. The generators satisfy

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}) \\ [M_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu) \\ [P_\mu, P_\nu] = 0.\end{aligned} \tag{2.1}$$

The field theories that we will study are also invariant under an internal symmetry group G, with generators that form a Lie algebra

$$[T_A, T_B] = f_{AB}^C T_C. \tag{2.2}$$

The Poincaré algebra can be extended by allowing anticommuting as well commuting generators, which are the $Q^i_\alpha$ and $\overline{Q}_{\dot\alpha j}$, where $\alpha, \dot\alpha = 1,2$ are spinor indices and $a, b = 1, ..., \mathcal{N}$ label the number of superymmetry charges. The Poincaré algebra is then supplemented by

$$
\begin{aligned}
&\{Q^a_\alpha, \overline{Q}^b_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^{ab} \\
&[P_\mu, Q^a_\alpha] = [P_\mu, \overline{Q}_{\dot\alpha b}] = 0 \\
&[Q^a_\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha{}^\beta Q^a_\beta \\
&[\overline{Q}_{\dot\alpha b}, M_{\mu\nu}] = (\overline{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \overline{Q}^{\dot\beta}_b \\
&\{Q^a_\alpha, Q^b_\beta\} = \varepsilon_{\alpha\beta} Z^{ab} \\
&\{\overline{Q}_{\dot\alpha a}, \overline{Q}_{\dot\beta b}\} = \varepsilon_{\dot\alpha\dot\beta} Z^\dagger_{ab} \\
&[Q^a_\alpha, T^A] = b^{Aa}{}_b Q^b_\alpha \\
&[T^A, \overline{Q}_{\dot\alpha a}] = b^{\dagger Aa}_b \overline{Q}_{\dot\alpha a} \\
&[Z^{ab}, X] = [Z^\dagger_{cd}, X] = 0\,,
\end{aligned}
\tag{2.3}
$$

where $b^{\dagger Ab}_a = b^{Ab}{}_a$. In Eq.(2.3), $X$ is any generator in the algebra so that $Z^{ab} = -Z^{ba}$ generate the center of the algebra.

The supersymmetry algebra (2.3) closes under the action of a group of automorphisms which, in our case, will be U($\mathcal{N}$) or a subgroup of it. These symmetries are called R-symmetries and act on the charges $Q^a_\alpha$ and $\overline{Q}_{\dot\alpha b}$ as

$$
Q^{a\prime}_\alpha = U^a{}_b Q^b_\alpha \qquad \overline{Q}'_{\dot\alpha b} = \overline{Q}_{\dot\alpha a} U^{\dagger a}_b\,.
$$

We can easily check that $Q^{a\prime}_\alpha$'s and $\overline{Q}'_{\dot\alpha b}$'s satisfy the supersymmetry algebra as well.

### 2.0.2 Massless irreducible representations

In order to find the massless representations of the supersymmetry algebra, we choose a Lorentz frame $q_\mu = (-k, 0, 0, k)$, with $q^2 = 0$. The anticommutation relation of $Q^{i\prime}_\alpha$'s and $\overline{Q}'_{\dot\alpha j}$'s then is

$$
\{Q^a_\alpha, \overline{Q}_{\dot\beta b}\} \;=\; 4k \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta^a_b
$$

When we choose $\alpha = \dot\beta = 2$ and $a = b$ and require that no zero norm states exist in the Hilbert space we conclude to $Q^{2i} = 0$. Then using the algebra we find that the central charges vanish for the massless representations

$$
\{Q^a_\alpha, Q^b_\beta\} \;=\; \{\overline{Q}_{\dot\alpha a}, \overline{Q}_{\dot\beta b}\} = 0\,.
$$

| Helicity $\leq 1$ | $\mathcal{N}$ =2 gauge | $\mathcal{N}$ =2 hyper | $\mathcal{N}$ =4 gauge |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 1/2 | 2 | 2 | 4 |
| 0 | 2 | 4 | 6 |
| −1/2 | 2 | 2 | 4 |
| −1 | 1 | 0 | 1 |

Table 2.1: The $\mathcal{N}$=2 and $\mathcal{N}$=4 multiplets.

We may define now fermionic creation and annihilation operators

$$\begin{aligned} \Gamma^{2a-1} &= \frac{1}{2\sqrt{k}}(Q_1^a + Q_{1,a}^*) \\ \Gamma_{2a}^\dagger &= (\Gamma^a)^\dagger = \frac{1}{2\sqrt{k}}(Q_1^a - Q_{1,a}^*)\,, \end{aligned} \tag{2.4}$$

which satisfy the $2\mathcal{N}$ dimensioal Clifford algebra

$$\begin{aligned} \{\Gamma^I, \Gamma_J^\dagger\} &= 2\delta_J^I \\ \{\Gamma^I, \Gamma^J\} &= \{\Gamma_I^\dagger, \Gamma_J^\dagger\} = 0, \end{aligned} \tag{2.5}$$

where $I, J = 1, ...2\mathcal{N}$. There is one irreducible representaion of the Clifford algebra which is represented by $2^{\mathcal{N}}$ states that are created by the action of $Q_{1,i}^*$ on the vacuum which is defined as

$$Q_1^a|\Omega_\lambda\rangle = 0.$$

The vacuum is the state of lowest helicity $\lambda$, $|\Omega_\lambda\rangle = |q_\mu, \lambda\rangle$. So, the states which constitute the multiplet are constructed as

$$Q_{a_1}^* \ldots Q_{a_n}^* |\Omega_\lambda\rangle\,. \tag{2.6}$$

This state has helicity $\lambda + \frac{n}{2}$ and the maximum helicity in the multiplet is $\lambda + \frac{\mathcal{N}}{2}$. The field theories that we will study are CPT invariant, meaning that the particle spectrum is symmetric in the change of sign of the helicity. CPT invariance therefore implies, in general, a doubling of the number of states. The multiplets, that we are interested in, i.e. $\mathcal{N} = 2$ and $\mathcal{N} = 4$ with $|\lambda| \leq 1$ appear in Table (1).

### 2.0.3 Field content and Lagrangians

In the previous analysis it was showed that the supersymmetry particle representations for $\mathcal{N}$ =2 and $\mathcal{N}$ =4, with maximum spin 1, consist of spin 1 gauge particles, spin 1/2 fermions and spin 0 scalars. As a result, the fields in supersymmetric theories with spin less or equal to 1 are spin 1 gauge fields, spin 1/2 Weyl fermion fields and spin 0 scalar fields, but these fields are restricted to enter in multiplets of the corresponding supersymmetry algebras.

For any $1 \leq \mathcal{N} \leq 4$, we have a gauge multiplet, which transforms under the adjoint representation of the gauge algebra, G. For $\mathcal{N}$ =4, this is a unique multiplet. For $\mathcal{N}$ =1 and $\mathcal{N}$ =2, we also have matter multiplets. For $\mathcal{N}$ =2, this is the chiral multiplet, and for $\mathcal{N}$ =2 we have the hypermultiplet, both of which transform under the fundamental representation $G$ of G. In our case, $\mathcal{N}$ =4 theory has $SU(N)$ gauge symmetry and the R symmetry is $SU(4)_R$. The $\mathcal{N}$ =2 gauge multiplet has $SU(N_c)$ gauge symmetry and the R symmetry is $SU(2)_R$. Finally, the $\mathcal{N}$ =2 hypermultiplet contains fields that transform under the fundamental of $SU(N_c)$.There is also another gauge group, which is the $SU(N_f)$, and the R symmetry is again $SU(2)_R$. The fields of the theories, that we will study, are the gauge field $A_\mu$, Weyl fermions $\psi$, $\Lambda^I$ $(I = 1, 2)$ and $\lambda^a$ $(a = 1, ..., 4)$ and scalar fields $\phi$, $Q^I$ $(I = 1, 2)$ and $X^i$ $(i = 1, ..., 6)$. The field content of the $\mathcal{N}$ =2 gauge and hyper multiplet and $\mathcal{N}$ =4 is:

- $\mathcal{N}$ =2 **Gauge Multiplet** $\{A_\mu,\ \Lambda^I,\ \phi\}$, where $A_\mu$ is the gauge field $\lambda^I$ are Weyl fermions, and $\phi$ is a complex scalar. The theory also possesses an $SU(2)_R$ symmetry, $A_\mu$ and $\phi$ are singlets, while $\Lambda^I$s, transform as a doublet ($I = 1, 2$ is an $SU(2)_R$ index).All the field transform under the adjoint of $SU(N_c)$

- $\mathcal{N}$ =2 **Hypermultiplet** $\{\psi,\ Q^I\}$, where $\psi$ are Weyl fermions and $Q^I$s are two complex scalars. All of them transform under the fundamental representation of $SU(N_c)$. Under $SU(2)_R$ symmetry, $\psi$ are singlets, while $Q^I$s transform as a doublet. The fields also transform under the fundamental represenation of $SU(N_f)$.

- $\mathcal{N}$ =4 **Gauge Multiplet** $\{A_\mu,\ \lambda^a\ X^i\}$, where $\lambda^a$, $a = 1, \cdots, 4$ are Weyl fermions and $X^i$, $i = 1, \cdots, 6$ are real scalars. Under $SU(4)_R$ symmetry, $A_\mu$ is a singlet, $\lambda^a$ is a **4** and the scalars $X^i$ are in the anti-symmetric representation, **6**. All the fields transform in the adjoint representation of $SU(N)$.

Supersymmetric Lagrangians comprise gauge, spin 1/2 fermion and scalar fields, (arranged in multiplets of the supersymmetry algebra). We will focus on local Lagrangians in which each term has a total of no more than two derivatives on all boson fields and no more than one derivative on all fermion fields, i.e. we will use the common Lagrangian for scalar fields, Dirac's Lagrangian for fermions and the non abelian generalization of Maxwell's Lagrangian for gauge fields. All renormalizable Lagrangians are of this form. The Lagrangians can be found either by checking their invariance under supersymmetric transformations or using superspace technique, which is a more powerful tool.

The action of the $\mathcal{N}$=2 super Yang Mills field theory is written in Euclidean space

$$S_{gauge,2} = \int d^4x \frac{1}{g^2} Tr\Big\{ \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\overline{\Lambda}_1 \overline{\sigma}^\mu D_\mu \Lambda^1 + i\Lambda^2 \sigma^\mu D_\mu \overline{\Lambda}_2 + (D_\mu \phi)^\dagger (D^\mu \phi) \qquad (2.7)$$
$$-i\sqrt{2}\epsilon_{IJ}\Lambda^I\Lambda^J\phi^\dagger - i\sqrt{2}\epsilon_{IJ}\overline{\Lambda}^I\overline{\Lambda}^J\phi - \frac{1}{2}[\phi, \phi^\dagger]^2 \Big\},$$

where, as it was noted previously, each field is in the adjoint representaion of $SU(N_c)$, thus $F = \sum_{A=1}^{N_c^2-1} F^A T^A$, where $F$ is an arbitrary field of the above action. The generators of the group are normalised such as

$$Tr(T^A T^B) = N\delta^{AB}. \qquad (2.8)$$

The covariant derivatives and the field strength are

$$D_\mu * = \partial_\mu * + ig[A_\mu, *] \qquad (2.9)$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu].$$

The $\epsilon_{IJ}$ are the Clebsch-Gordan coeffecients that relate the **2** of the $SU(2)_R$ and the $U(1)_R$, which the complex $\phi$ transforms under. The bare dimensions of the fields are

$$\Delta_0[A_\mu] = 1 \qquad (2.10)$$
$$\Delta_0[\phi_i] = 1$$
$$\Delta_0[\Lambda] = 3/2$$

The Lagrangian of the hypermultiplet is also written in the Euclidean space and reads

$$
\begin{aligned}
S_{hyper,2} = \frac{1}{g^2}\int d^4x \Big\{ & (D_\mu Q^I)^\dagger D^\mu Q_I + i\overline{\psi}\overline{\sigma}^\mu D_\mu \psi + i\tilde{\psi}\sigma^\mu D_\mu \overline{\tilde{\psi}} \\
& + i\sqrt{2}\tilde{\psi}\phi\psi + i\sqrt{2}\overline{\psi}\phi^\dagger\overline{\tilde{\psi}} - \frac{g^2}{2}(\overline{Q}_I Q^J)(\overline{Q}_J Q^I) \\
& - \frac{1}{2}\epsilon_{IJ}\epsilon^{KL}(\overline{Q}_L Q^J)(\overline{Q}_K Q^I) \\
& + i\sqrt{2}\overline{Q_I \Lambda}^I \overline{\tilde{\psi}} - ig\sqrt{2}\tilde{\psi}\Lambda_I Q^I \\
& + i\sqrt{2}e^{IJ}\overline{Q}_I \Lambda_J \psi - ig\sqrt{2}\epsilon_{IJ}\overline{\psi \Lambda}^I Q^J \\
& - 2\overline{Q}_I \phi^\dagger \phi Q^I \Big\}.
\end{aligned}
\tag{2.11}
$$

So, we have considered the coupling of the gauge and hypermultiplets. The fields $\psi$ and $Q^I$s tranform in the fundamental of $SU(N_c)$ and in the adjoint of the $SU(N_f)$. Finally, the Lagrangian for the $\mathcal{N}$ =4 super Yang Mills is

$$
\begin{aligned}
S_4 = \frac{1}{g^2}\int d^4x Tr \Big\{ & \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a + D_\mu X^i D^\mu X_i \\
& + C_i^{ab}\lambda_a [X^i, \lambda_b] + \bar{C}_{iab}\bar{\lambda}^a [X^i, \bar{\lambda}^b] - \frac{1}{2}\sum_{i,j}[X^i, X^j]^2 \Big\}
\end{aligned}
$$

The constants $C_i^{ab}$ and $C_{iab}$ are the Clebsch Gordan coeffecients connecting the **6** and the **4** (or $\mathbf{4}^*$) of $SU(4)_R$.

Besides the symmetries that are already noticed the action is also invariant under rescalings. Rescaling invariance and Poincaré symmetry combine into a larger symmetry, called conformal symmetry, and form the group $SO(2,4)$. However, the considered action is invariant under a more general symmetry, the super Poincaré symmetry. So, the combination of super Poincaré symmetry and conformal invariance leads to the superconformal symmetry, which will be described bellow. We have to point out that this symmetry is not broken at the quantum level, since the $\mathcal{N}$ =4 super Yang Mills theory has no ultraviolet divergences, hence the $\beta$ function of the theory is equal to zero.

### 2.0.4 Super-conformal symmetry

We start with the conformal symmetry, which forms the group $SO(2,4)$ with the follwing generators: the usual translations $P^\mu$ and the Lorentz transformations $M_{\mu\nu}$, but also the dilations $D$ and the special conformal transformations

$K^\mu$. The above generators satisfy the conformal algebra

$$\begin{aligned}
&[D, P_\mu] = -iP_\mu \\
&[D, K_\mu] = iK_\mu \\
&[D, M_{\mu\nu}] = 0 \\
&[P_\mu, K_\nu] = 2i(L_{\mu\nu} - \eta_{\mu\nu} D) \\
&[M_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\lambda} P_\nu - \eta_{\lambda\nu} P_\mu) \\
&[M_{\mu\nu}, K_\lambda] = -i(\eta_{\mu\lambda} K_\nu - \eta_{\lambda\nu} K_\mu) \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}).
\end{aligned} \tag{2.12}$$

So, despite the 4 $P_\mu$ and the 6 $M_{\mu\nu}$, we also have 4 $K_\mu$ and 1 $D$, the conformal group has therefore 15 generators. We now may redefine

$$\begin{aligned}
M_{\mu 5} &= \frac{1}{2}(P_\mu + K_\mu) \\
M_{4\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\
M_{54} &= D
\end{aligned} \tag{2.13}$$

with the metric $\eta_{IJ} = diag(-1, +1, +1, +1; +1, -1)$ $(I, J = 0, 1, ..., 5)$. Then, the conformal algebra (2.13) becomes

$$[M_{IJ}, M_{KL}] = i(\eta_{JK} M_{IL} - \eta_{JL} M_{IK} - \eta_{IK} M_{JL} + \eta_{IL} M_{JK}), \tag{2.14}$$

As a result, it is obvious now that the conformal generators generate the group $SO(2, 4)$.

The supercnformal algebra is constructed in a similar way that was used in order to enlarge the Poincaré algebra to the super-Poincaré. We first notice that the conformal group has two subgroups generated by $\{P_\mu, M_{\mu\nu}\}$ and $\{K_\mu, M_{\mu\nu}\}$. By extending these two subgroups we find the usual super-Poincaré algebra plus 16 more generators which will lead us to the superconformal algebra. The new fermionic generators $S_{\alpha a}$ are called conformal supercharges, and transform under the $\mathbf{4}^*$ of the $SU(4)_R$. While the $\bar{S}^{\dot{\alpha} a}$ transform under $\mathbf{4}$. Hence, the super-conformal algebra is

$$[K^\mu, Q^a_\alpha] = (\sigma^\mu)_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\bar S^a_{\dot\gamma}, \qquad [K^\mu, \bar Q_{\dot\alpha a}] = (\sigma^\mu)_{\beta\dot\alpha}\epsilon^{\beta\gamma}S_{\gamma a}, \tag{2.15}$$

$$[P^\mu, S_{\alpha a}] = (\sigma^\mu)_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\bar Q_{\dot\gamma a}, \qquad [P^\mu, \bar S^a_{\dot\alpha}] = (\sigma^\mu)_{\beta\dot\alpha}\epsilon^{\beta\gamma}Q^a_\gamma, \tag{2.16}$$

$$[M^{\mu\nu}, Q^a_\alpha] = -i(\sigma^{\mu\nu})_{\alpha\beta}\epsilon^{\beta\gamma}Q^a_\gamma, \qquad [M^{\mu\nu}, \bar Q_{\dot\alpha a}] = -i(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\bar Q_{\dot\gamma a}, \tag{2.17}$$

$$[M^{\mu\nu}, S_{\alpha a}] = -i(\sigma^{\mu\nu})_{\alpha\beta}\epsilon^{\beta\gamma}S_{\gamma a}, \qquad [M^{\mu\nu}, \bar S^a_{\dot\alpha}] = -i(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\bar S^a_{\dot\gamma}, \tag{2.18}$$

$$[D, Q^a_\alpha] = -\frac{i}{2}Q^a_\alpha, \qquad [D, \bar Q_{\dot\alpha a}] = -\frac{i}{2}\bar Q_{\dot\alpha a} \tag{2.19}$$

$$[D, S_{\alpha a}] = +\frac{i}{2}\,S_{\alpha a}, \qquad [D, \bar S^a_{\dot\alpha}] = +\frac{i}{2}\,\bar S^a_{\dot\alpha}, \tag{2.20}$$

$$\{Q^a_\alpha, \bar Q_{\dot\beta b}\} = (\sigma^\mu)_{\alpha\dot\beta}\delta^a{}_b P_\mu, \qquad \{S_{\alpha a}, \bar S^b_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta}\delta_a{}^b K_\mu. \tag{2.21}$$

The remaining commutation or anticommutation relations give zero, except for the anticommutation relations between $S_{\alpha a}$ and $Q^a_\alpha$ which

$$\{Q^a_\alpha, S_{\beta b}\} = (\sigma^{ij})^a{}_b\epsilon_{\alpha\beta}R_{ij} + i(\sigma^{\mu\nu})_{\alpha\beta}\delta^a{}_b M_{\mu\nu} - i\epsilon_{\alpha\beta}\delta^a{}_b D, \tag{2.22}$$

$$\{\bar Q_{\dot\alpha a}, \bar S^b_{\dot\beta}\} = -(\sigma^{ij})_a{}^b\epsilon_{\dot\alpha\dot\beta}R_{ij} + i(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\delta_a{}^b M_{\mu\nu} - i\epsilon_{\dot\alpha\dot\beta}\delta_a{}^b D\,, \tag{2.23}$$

where $R_{ij}$ $(i, j = 1, \ldots, 6)$ are the $SU(4)_R$ generators. All the supercharges transform as spinors under $SU(4)_R$, so

$$[R^a_b, Q^c_\alpha] = \delta^c_b Q^a_\alpha - \delta^a_b Q^c_\alpha, \qquad [R^a_b, \bar Q_{\dot\alpha c}] = -\delta^a_c \bar Q_{\dot\alpha b} + \delta^a_b \bar Q_{\dot\alpha c} \tag{2.24}$$

$$[R^a_b, S^\alpha_c] = -\delta^a_c S^\alpha_b + \delta^a_b S^\alpha_c, \qquad [R^a_b, \bar S^{\dot\alpha c}] = \delta^c_b \bar S^{\dot\alpha a} - \delta^a_b \bar S^{c\dot\alpha}, \tag{2.25}$$

where $(\sigma^{ij})^a{}_b R_{ij} = R^a_b$. We denote the three Cartan generators of the $SU(4)$ as $(R_{12}, R_{34}, R_{56})$, and the corresponding charges as $(J_1, J_2, J_3)$.

# Chapter 3

# The Anomalous Dimension of Composite Operators

## 3.1 Local operators

We will consider local, gauge invariant operators which are polynomial in the fields of the $\mathcal{N}$ =4 multiplet. Locality of the operators means that they are evaluated at some point of spacetime. In general, we will be interested in operators that are made by traces of the fields. So, we may have either single trace operators or products of traces. Both of them will be gauge invariant. In the present project we will examine operators that are made by scalars. In the $\mathcal{N}$ =4 case, we considerthe six real scalar fields $X^i$ which transform in the **6** of $SU(4)_R$. It is convenient to combine these fields into three complex scalar fields

$$\begin{aligned} Z &= X^1 + iX^2 \\ W &= X^3 + iX^4 \\ Y &= X^5 + iX^6, \end{aligned} \tag{3.1}$$

which transform under the R-symmetry as

$$\begin{aligned} [R_{12}, Z] &= Z \\ [R_{34}, W] &= W \\ [R_{56}, Y] &= Y \end{aligned} \tag{3.2}$$

The conjugate fields will have one more minus sign in the RHS, and all other commutators between $R_{ij}$ and $Z, W, Y$ are zero. The conserved charges of the generators are

$$(\Delta, m_1, m_2; J_1, J_2, J_3), \tag{3.3}$$

where $\Delta$ is the bare dimension of the operator, $m_1, m_2$ are the charges of the Lorentz group $SO(1,3) \simeq SU(2) \times SU(2)$ [1], and $J_1$ , $J_2$ and $J_3$ are the charges of the R symmetry [2].

Let us now consider a local operator in the field theory $\mathcal{O}(x)$. Under a scaling transformation $x \mapsto \Lambda x$, the local operator scales as $\mathcal{O}(x) \mapsto \Lambda^{-\Delta}\mathcal{O}(\Lambda x)$ where $\Delta$ is the bare dimension of $\mathcal{O}(x)$. The generator of these scalings is the dilatation generator $D$, which acts on the operator as

$$[D, \mathcal{O}(x)] = -i(\Delta - x\partial_x)\mathcal{O}(x). \tag{3.4}$$

When $x = 0$, it just counts the scaling dimension $[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0)$. Acting with $D$ on the commutator $[K_\mu, \mathcal{O}(0)]$ and using the Jacobi identity, we are led to

$$[D, [K_\mu, \mathcal{O}(0)]] = [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)]] = -i(\Delta - 1)[K_\mu, \mathcal{O}(0)], \tag{3.5}$$

the operator $[K_\mu, \mathcal{O}(0)]$ therefore has scaling dimension lower than that of $\mathcal{O}(0)$ by one. Hence, $K_\mu$ acts as a annihilation operator that lowers the dimension by one. This is also implied by Eqs.(2.13), where it is seen that $K_\mu$ has dimension -1. In similar way we observe that

$$[D, [P_\mu, \mathcal{O}(0)]] = [[D, P_\mu], \mathcal{O}(0)]+[P_\mu, [D, \mathcal{O}(0)]] = -i(\Delta+1)[K_\mu, \mathcal{O}(0)], \tag{3.6}$$

so the $P_\mu$ is a creation operator and $D$ plays the role of the Cartan generator of the conformal algebra. We can also check, in Eqs. (2.13), that $P_\mu$ has dimension $+1$. It is well known that operators with negative dimension do not exist in a unitary quantum field theory. Hence, if we keep acting with $K_\mu$ to any operator with definite dimension we will eventually get an operator of zero scaling dimension.

$$[K_\mu, \mathcal{O}(0)] = 0. \tag{3.7}$$

A non zero operator satisfying the above relation is defined as conformal pri-

[1] Each charge $m_1, m_2$ corresponds to each one of the two $SU(2)$s.

[2] $SU(4)_R$ has rank 3 (i.e. it has three Cartan generators), so it admits three conserved charges.

mary operator $\mathcal{O}$. Any descendant operator can be constructed by acting with $P_\mu$ on the conformal primary operator.

The action of conformal generators on a conformal primary operator $\mathcal{O}(0)$ at the point 0 is described by

$$\begin{aligned}
&[P_\mu, \mathcal{O}(0)] = -i\partial_\mu \mathcal{O}(0) \\
&[M_{\mu\nu}, \mathcal{O}(0)] = -i\Sigma^{(m_1,m_2)}_{\mu\nu}\mathcal{O}(0) \\
&[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0) \\
&[K_\mu, \mathcal{O}(0)] = 0
\end{aligned} \tag{3.8}$$

Here $\Sigma^{\mu\nu}$ is a Lorentz generator [3] acting on the indices of $\mathcal{O}$. Notice that the action of $\Sigma^{(m_1,m_2)}_{\mu\nu}$ yields that $\mathcal{O}$ is an $(m_1, m_2)$-tensor in the Lorentz indices.

Observing Eqs.(2.19) and (2.20), we see that the dimensions of $Q^a_\alpha$ and $S_{\beta b}$ are $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. So, succesive application of S on any operator will lower its dimension until it will reach zero. Thus, we can introduce the notion of superconformal primary operator $\mathcal{O}$, which is defined as

$$[S_{\beta b}, \mathcal{O}(0)] = [S^b_{\dot\beta}, \mathcal{O}(0)] = 0, \tag{3.9}$$

for all $\beta$, $\dot\beta$ and $b$. We may put a further restriction in order to construct a chiral primary operator

$$[Q^a_\alpha, \mathcal{O}(0)] = 0, \tag{3.10}$$

for some $\alpha$ and $a$. Descendant operators can be constructed by acting the rest of the $Q$ on the superconformal primary operator. This is the way that we can construct the superconformal multiplet. Each superconformal multiplet contains only one superconformal primary. We should notice that each superconformal primary is also a conformal primary [4], but the converse is not, always, true. In $\mathcal{N}$ =4 superconformal primary operators can be formed only by scalars. The simplest example are single trace operators of the form

$$Tr(X^{((i_1}...X^{i_n))}), \tag{3.11}$$

where the indices $i_1, ..., i_n = 1, ...6$ are symmetrized and traceless and the trace is taken over the gauge algebra.

[3] $\Sigma^{\mu\nu}$ is the spin part of the Lorentz generator. Since, we consider an operator at $x = 0$ the orbital part is zero.

[4] This is true since $K_\mu$ and $S_{\beta b}$ commute.

From the superconformal algebra, we have

$$[\{Q^a_\alpha, S_{\beta b}\}, \mathcal{O}(0)] = [(\sigma^{ij})^a{}_b \epsilon_{\alpha\beta} R_{ij} + i(\sigma^{\mu\nu})_{\alpha\beta} \delta^a{}_b M_{\mu\nu} - i\epsilon_{\alpha\beta}\delta^a{}_b D, \mathcal{O}(0)]. \quad (3.12)$$

Chiral primaries satisfying $[Q^a_\alpha, \mathcal{O}(0)] = 0$, make the LHS of (3.12) zero, because it can be rewritten as

$$\{Q^a_\alpha, [S_{\beta b}, \mathcal{O}(0)]\} + \{S_{\beta b}, [Q^a_\alpha, \mathcal{O}(0)]\} \quad (3.13)$$

using a graded Jacobi identity. Since, we consider that $\mathcal{O}(0)$ is a scalar, then $[M_{\mu\nu}, \mathcal{O}(0)]$ =0, so that the scaling dimension of $\mathcal{O}(0)$ $\Delta$ is related to its R-charge. Operators of the form $\mathcal{O}_1 = Tr(Z^{J_1})$, $\mathcal{O}_2 = Tr(W^{J_2})$, and $\mathcal{O}_3 = Tr(Y^{J_3})$ with R-charges $(J_1, 0, 0)$, $(0, J_2, 0)$, and $(0, 0, J_3)$ respectively, are chiral primary operators. Indeed, their scaling dimension is easily found by acting with $D$. So it is obvious that the first operator has bare dimension $J_1$ and the rest have $J_2$ and $J_3$ respectively. These three chiral primaries are annihilated by eight of the sixteen supercharges $Q^1_\alpha$, $Q^2_\alpha$, $Q_{\dot\alpha 3}$ and $Q_{\dot\alpha 4}$, so they are half-BPS operators. Since, the superconformal symmetry is not broken at the quantum level, their bare dimensions do not receive any quantum corrections but remain at their classical values at all orders in perturbation theory. Non-BPS operators, however, acquire non-trivial correction to their dimensions $\Delta$. In the following section, we will consider operators in the $SU(2)$ sector of $\mathcal{N}$ =4 theory. These operators consist only of $Z$ and $W$ scalar fields without their conjugate fields. This sector of operators is closed to all orders in perturbation theory under operator mixing. This is seen by the fact that operators of different bare dimension do not mix and by the conservation of Lorentz and R charges. Another common sector of composite operators that are built by the scalars of the $\mathcal{N} = 4$ super Yang Mills is the $SO(6)$ [5] sector. Operators in this sector may consist of all the real scalar fields of the $\mathcal{N} = 4$ theory. This sector is closed only to one loop order in perturbation theory.

### 3.1.1 Renormalization of composite operators

The renormalization of composite operators will be analysed in this section. We will study single trace operators which are built of scalar fields that belong either to $\mathcal{N} = 4$ or $\mathcal{N} = 2$ super Yang Mills gauge multiplets. As it was mentioned above, the gauge coupling constant $g$ of the $\mathcal{N} = 4$ theory is not renormalised, but non-BPS gauge invariant local composite operators are renormalised in general. So, the bare scaling dimension of a gauge invariant

[5] $SU(4)$ and $SO(6)$ are locally equal, and the **6** of $SU(4)$ is the fundamental representation of $SO(6)$.

operator acquires corrections at the quantum level, the so-called anomalous dimension.

In general, composite operators of the same bare dimension and the quantum numbers can be mixed by quantum corrections, if we work in a massless field theory. However in a massive field theory, it is also possible to have mixing of operators of different dimensions. Since, we have only considered massless theories we are not interested in this case. There is a basis of operators in which they are renormilized multiplicatively. We may then diagonalize the two point functions in this basis. The renormalized operator will be a rescaled version of the operator which consists of the bare fields

$$\mathcal{O}^A = Z^A{}_B \mathcal{O}^B_{\text{ren}}, \tag{3.14}$$

in a particular basis. The renormalization factor is found by studying the following correlator. The bare and the renormalized correlators are related by

$$\langle X^{i_L} \dots X^{i_{l+1}} X^{i_l} \dots X^{i_1} \mathcal{O}^A \rangle = Z_X^{L/2} Z_B^A \langle X_r^{i_L} \dots X_r^{i_{l+1}} X_r^{i_l} \dots \; X_r^{i_1} \mathcal{O}_r^B \rangle, \tag{3.15}$$

where $X^i = Z_X^{1/2} X_r^i$, such that the correlator $\langle X^i X^j \rangle$ to be finite. $Z$s depend on the ultraviolet cut off and the couling. Since, we may encounter operator mixing, the anomalous dimension of an operator is generalized to the matrix of anomalous dimension which is given by

$$\Gamma = \frac{dZ}{d \ln \Lambda} \cdot Z^{-1}. \tag{3.16}$$

By diagoanalizing $\Gamma$ we find operators which are multiplicatively renormalizable, with the anomalous dimension to be the corresponding eigenvalue. Then, the two point correlator reads

$$\langle \mathcal{O}'(x) \mathcal{O}'(y) \rangle = \frac{c}{|x-y|^{2(\Delta+\gamma)}}, \tag{3.17}$$

where $\mathcal{O}'$ is the operator that corresponds to an eigenvector of $\Gamma$ with eigenvalue $\gamma$, and c is a constant. The correlator depends only on $|x-y|$ because of translation and Lorentz invariance. The power is determined by the fact that it is a homogeneous function under rescaling. If the gauge coupling is small, then the quantum corrections are small enough (i.e. $\gamma \ll \Delta$) and the correlator can be expanded as

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{c}{|x-y|^{2\Delta}} (1 - \gamma ln(\Lambda^2 |x-y|^2) + ...), \tag{3.18}$$

where $\Lambda$ is the ultraviolet cutoff.

### 3.1.2 One-loop anomalous dimension in $\mathcal{N} = 4$ SYM

We initially consider operators in the $SO(6)$ sector of the general form

$$\mathcal{O}(x) = \frac{1}{N^{L/2}} Tr(X^{i_1}...X^{i_L}), \tag{3.19}$$

where $i_1, ..., i_L = 1, ...6$. We will now proceed to the calcuation of the matrix of anomalous dimension at one loop in large N limit. Note that only two of the $L$ legs are relevant in the current one-loop computation, and only the nearest-neighbour interactions are relevant since they give planar diagrams, while non-planar interactions are suppressed in the large-$N$ limit. At the one-loop, there are three diagrams that contribute to the two-point function: (a) self-energy, (b) gluon exchange, and (c) scalar four-vertex diagrams. However, here we only need to compute the four-scalar diagram. The vertex of this diagram is readily found by inspection of the last term of the $\mathcal{N} = 4$ super Yang Mills action. The vertex is $\frac{N^2}{2g^2}(2\delta_{i_l}^{j_{l+1}}\delta_{i_{l+1}}^{j_l} - \delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}} - \delta_{i_l i_{l+1}}\delta^{j_l j_{l+1}})$

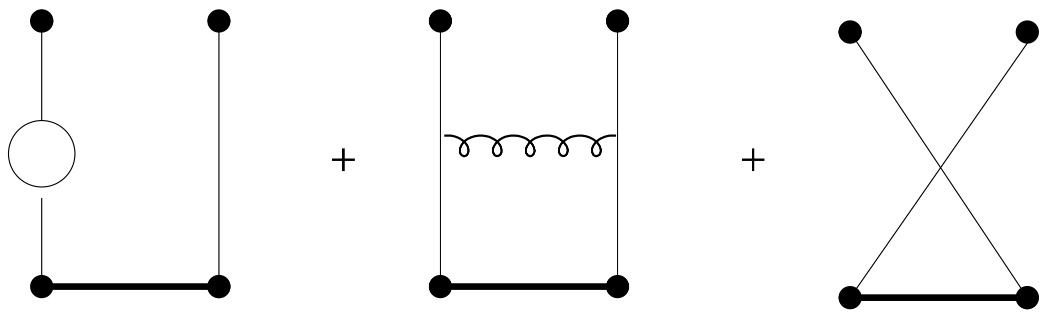

Figure 3.1: *One loop diagrams. The thick horizontal lines represent the composite operator*

It is convenient now to introduce the operators that act on the $SO(6)$ indices

$$\begin{aligned} I_{l,l+1}...\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}... &= ...\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}... \\ K_{l,l+1}...\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}... &= ...\delta_{i_l i_{l+1}}\delta^{j_l j_{l+1}}... \\ P_{l,l+1}...\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}... &= ...\delta_{i_l}^{j_{l+1}}\delta_{i_{l+1}}^{j_l}..... \end{aligned} \tag{3.20}$$

The four-scalar diagram yields a Z factor which is

$$Z^{\dots j_l j_{l+1} \dots}_{\dots i_l i_{l+1} \dots} = (I_{l,l+1} - \lambda(2P_{l,l+1} - K_{l,l+1} - I_{l,l+1}) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4}) \dots \delta^{j_l}_{i_l} \delta^{j_{l+1}}_{i_{l+1}} \dots \Rightarrow$$

$$Z^{\dots j_l j_{l+1} \dots}_{\dots i_l i_{l+1} \dots} = (I_{l,l+1} - \frac{\lambda}{16\pi^2} ln\Lambda(2P_{l,l+1} - K_{l,l+1} - I_{l,l+1})) \dots \delta^{j_l}_{i_l} \delta^{j_{l+1}}_{i_{l+1}} \dots, \quad (3.21)$$

where $\lambda \equiv Ng^2$ is 't Hooft parameter and the gauge indices are supressed. The other two diagrams are clearly diagonal in the $SO(6)$ indices. So, their contribution will be of the form

$$Z^{\dots j_l j_{l+1} \dots}_{\dots i_l i_{l+1} \dots} = (I_{l,l+1} + C \frac{\lambda}{16\pi^2} ln\Lambda I_{l,l+1}) \dots \delta^{j_l}_{i_l} \delta^{j_{l+1}}_{i_{l+1}} \dots, \quad (3.22)$$

where C is a constant to be determined. Thus, by adding the above results we find the matrix of anomalous dimensions for operators that are built from scalar fields of $\mathcal{N} = 4$ SYM is

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (C' I_{l,l+1} - 2P_{l,l+1} + K_{l,l+1}). \quad (3.23)$$

This result is valid in the more general sector, $SO(6)$. Any chiral primary operator is known to have $\Gamma = 0$. According to Eq.(3.11), chiral primaries are symmetric and traceless in the $SO(6)$ indices., so the action of $P_{l,l+1}$ on them will give one and the action of $K_{l,l+1}$ zero. Hence, $C' = 2$.

If we are now restricted to the $SU(2)$ sector, there is no contribution from the $K_{l,l+1}$ operator[6], so the matrix of the anomalous dimension reduces to

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} (I_{l,l+1} - P_{l,l+1}). \quad (3.24)$$

It is now noticed that this operator is identified with the Hamiltonian of the XXX Heisenberg spin chain. The operators in the $SU(2)$ sector are of the form $Tr(Z^{L-M} W^M)$. If we consider that $Z$ field corresponds to spin up state and $W$ field to spin down then the energies of various states correspond to the anomalous dimensions of the operators. Furthermore, the Hamilatonian can also be written as

$$H = \frac{\lambda}{8\pi^2} \sum_{l=1}^{n} (\frac{1}{2} - 2\boldsymbol{S}_l \cdot \boldsymbol{S}_{l+1}). \quad (3.25)$$

[6]In order for the $K_{l,l+1}$) operator not to give zero we should also include the complex scalar fields in the $SU(2)$ sector, but this is not the case.

The ground state has all spins up and corresponds to the chiral primary operator $Tr(Z^L)$. The excitations about the ground state correspond to operators which are not chiral primaries, so the receive quantum corrections to their bare dimension.

### 3.1.3 One-loop anomalous dimension in $\mathcal{N} = 2$ theories

We will firstly study the case of the gauge Lagrangian and then we will couple it to the hypermultiplet Lagrangian. The crucial difference from the $\mathcal{N} = 4$ super Yang Mills is that the coupling constant now receives quantum corrections. The relation of the bare and the remnormalized coupling constants is $g = Z_g g_r$. The $\beta$ function of the $\mathcal{N} = 2$ theories is

$$\beta(g) = -\frac{g^3}{16\pi^2} b_0, \tag{3.26}$$

where $b_0 = -N_f$ for the hyper-Lagrangian and $b_0 = 2N_c$ for the gauge Lagrangian. By the definition of $\beta$ function we have

$$\beta(g) \equiv \Lambda \frac{\partial}{\partial \Lambda} g_r = -g\Lambda \frac{\partial}{\partial \Lambda} \log Z_g = -\frac{g^3}{16\pi^2} b_0 \tag{3.27}$$

one can derive the expression for $Z_g$

$$Z_g = 1 - \frac{g^2 b_0}{16\pi^2} ln\Lambda. \tag{3.28}$$

We consider the operators of the form

$$\mathcal{O}(x) = \frac{1}{N^{L/2}} Tr(\phi^{i_1} ... \phi^{i_L}), \tag{3.29}$$

where $i_1, ..., i_L = 1, 2$. The diagrams that we now compute are the same as in Fig.(3.1). We follow the same procedure, so we focus only in the nearest-neighbors interaction in order to take only the planar diagrams which lead to the large $N_c$ limit. In the case of the gauge Lagrangian the self energy diagram vanishes, meaning that the fields are renormalized such that $Z_\phi^{1/2} = Z_g$. So, there are two remaining diagrams. The four-scalar vartex diagram is computed in exactly the same way as before and it yields

$$Z_{4\varphi, ... i_l i_{l+1} ...}^{... j_l j_{l+1} ...} = (I_{l,l+1} - \frac{\lambda}{16\pi^2} ln\Lambda(2P_{l,l+1} - K_{l,l+1} - I_{l,l+1})) ... \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} .... \tag{3.30}$$

For the diagram with gluon exchange we take

$$Z^{\dots j_l j_{l+1}\dots}_{gluon,\dots i_l i_{l+1}\dots} = (I_{l,l+1} - \frac{\lambda}{16\pi^2} ln\Lambda I_{l,l+1})\dots\delta^{j_l}_{i_l}\delta^{j_{l+1}}_{i_{l+1}}\dots. \tag{3.31}$$

Since, in our case the coupling constant runs, in order to find the one-loop correction to the correlator of the form

$$\langle\varphi^{i_L}\dots\varphi^{i_{l+1}}\varphi^{i_l}\dots\varphi^{i_1}\mathcal{O}\rangle = Z_\phi^{L/2} Z\langle\varphi_r^{i_L}\dots\varphi_r^{i_{l+1}}\varphi_r^{i_l}\dots\ \varphi_r^{i_1}\mathcal{O}_r\rangle, \tag{3.32}$$

we must multiply the RHS of Eq.(3.32) a factor of $Z_g$ for each field. In the case of the gauge Lagrangian, this will lead us to one more term that we should take into account, which is

$$Z^{\dots j_l j_{l+1}\dots}_{g,\dots i_l i_{l+1}\dots} = (I_{l,l+1} + \frac{\lambda}{8\pi^2} ln\Lambda I_{l,l+1})\dots\delta^{j_l}_{i_l}\delta^{j_{l+1}}_{i_{l+1}}\dots \tag{3.33}$$

By adding Eqs.(3.30), (3.31) and (3.33) we get

$$Z^{\dots j_l j_{l+1}\dots}_{\dots i_l i_{l+1}\dots} = (I_{l,l+1} + \frac{\lambda}{16\pi^2} ln\Lambda(K_{l,l+1} + 2I_{l,l+1} - 2P_{l,l+1}))\dots\delta^{j_l}_{i_l}\delta^{j_{l+1}}_{i_{l+1}}\dots. \tag{3.34}$$

So, the matrix of anomalous dimensions is the same as in the $\mathcal{N} = 4$ case (in the $SO(6)$ sector), but now, the indices of the scalar fields take only two values. Thus,

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}(2I_{l,l+1} - 2P_{l,l+1} + K_{l,l+1}). \tag{3.35}$$

It is noticed that operators which are built of products of the complex scalar field $\phi$ have zero anomalous dimensions, in a similar way with $SU(4)$ case. This suggest to take them as the ground state of the spin chain. We identify $\bar{\phi}$ with the spin up state and the $\phi$ with the spin down.

The matrix of anomalous dimensions can be written in the basis of sigma matrices $\sigma_\mu \equiv (\mathbf{1}, \sigma^x, \sigma^y, \sigma^z)$ In this basis the real scalar fields are represented by $\varphi_1 \to (1\ 0)$ and $\varphi_2 \to (0\ 1)$. The permutation operator is writen as

$$P^{j_l j_{l+1}}_{i_l i_{l+1}} = \frac{1}{2}\sum_{\mu=0}^{3}(\sigma^\mu)^{j_l}{}_{i_l}(\sigma^\mu)^{j_{l+1}}{}_{i_{l+1}}\ , \tag{3.36}$$

and the trace operator is

$$K_{i_l i_{l+1}}^{j_l j_{l+1}} = \frac{1}{2} \sum_{\mu=0}^{3} \alpha_\mu (\sigma^\mu)^{j_l}_{\ i_l} (\sigma^\mu)^{j_{l+1}}_{\ i_{l+1}} \tag{3.37}$$

where $\alpha_\mu = (1, 1, -1, 1)$. By changing basis, the matrix of anomalous dimensions is shown that is identical with the Hamiltonian of an XXZ spin chain:

$$\Gamma = -\frac{\lambda}{32\pi^2} \sum_{l=1}^{L} [(\sigma^x)_l (\sigma^x)_{l+1} + (\sigma^y)_l (\sigma^y)_{l+1} + 3\,((\sigma^z)_l (\sigma^z)_{l+1} - \mathbf{1}_l \mathbf{1}_{l+1})] \equiv H_{\text{XXZ}}. \tag{3.38}$$

We finally show that when we couple the $\mathcal{N} = 2$ gauge and hyper Lagrangian result remains the same. In this case, $\beta$ function reads

$$\beta(g) = -\frac{g^3}{16\pi^2}(2N_c - N_f), \tag{3.39}$$

This leads to a contribution to the renormalization factor that is associated with the renormalization of the coupling constant.

$$Z_{g,\ldots i_l i_{l+1}\ldots}^{\ldots j_l j_{l+1}\ldots} = (I_{l,l+1} + (\frac{g^2 N_c}{8\pi^2} - \frac{g^2 N_f}{16\pi^2}) ln\Lambda I_{l,l+1}) \ldots \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} \ldots \tag{3.40}$$

The term with $N_f$ will cancel with a term coming from the self energy diagram. In the hyper Lagrangian the scalar fields of the gauge multiplet couple to the fermions of the hypermultiplet through two Yukawa terms. These terms give a correction to the self energy of the scalar fields. So, the contribution of the diagram in Fig.(3.2) to the first diagram of Fig. (3.1) will give a $Z_{s.e.}$ factor

$$Z_{s.e.,\ldots i_l i_{l+1}\ldots}^{\ldots j_l j_{l+1}\ldots} = (I_{l,l+1} + \frac{g^2 N_f}{16\pi^2} ln\Lambda I_{l,l+1}) \ldots \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} \ldots \tag{3.41}$$

It is therefore clear that the two contributions that contain the $N_f$ cancel and the result remains uneffected.

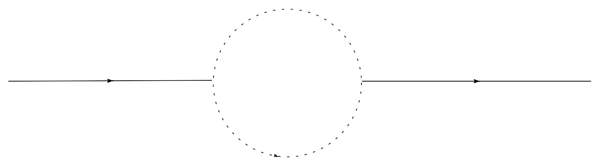

Figure 3.2: *The correction to the self energy of scalar fields due to the Yukawa interactions among the scalars of the gauge multiplet and the fermions of the hypermultiplet. This diagram contributes to the first diagram of Fig.(3.1)*

# Bibliography


[1] **J.M. Maldacena** “The large n limit of superconformal field theories and supergravity”, hep-th/9711200, 1997

[2] **P. van Nieuwenhuizen and P. West** “Principles of Supersymmetry and Supergravity”, Lecture Notes

[3] **E. D’Hoker and D.Z. Freedman** “Supersymmetric Gauge Theories and the AdS/CFT Corespondence”, hep-th/0201253, 2002

[4] **F.A. Dolan and H. Osborn**, “On Short and Semi-Short Representations for Four Dimensional Superconformal Symmetry”, hep-th/0209056, 2002

[5] **K. Okamura**, “Aspects of Integrability in AdS/CFT Duality”, hep-th/0803.3999, 2008

[6] **J.A. Minahan and K. Zarembo** “The Bethe Ansatz for N=4 Super Yang-Mills”, hep-th/0212208, 2003

[7] **J.A. Minahan**, “A brief introduction to the Bethe ansatz in N=4 super-Yang-Mills”, J. Phys. A: Math. Gen. **39** (2006)

[8] **P. Di Vecchia and A. Tanzini**, “N=2 Super Yang-Mills and the XXZ spin chain”, hep-th/0405262, 2004