Degenerate Maxima in Hamiltonian Systems

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Abstract of the Dissertation

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In this thesis we examine paths of Hamiltonians with fixed degenerate maxima. We show that a nonconstant loop of Hamiltonians with a fixed global maximum cannot be totally degenerate at the maximum. We use this to show that symplectic 4-manifolds admitting a nonconstant loop of Hamiltonians with a fixed global maximum must be uniruled. We also use this to find nontrivial elements of $\pi_1(Ham(M, \omega))$ with no Hofer length minimizing representatives.

Contents

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Chapter 1

Introduction

Symplectic manifolds have their origins as phase spaces in the Hamiltonian formulation of classical mechanics. One can realize a phase space as being a cotangent bundle T^*M, equipped with fibre coordinates p, and base coordinates q. Let $H_t : T^*M \to \mathbb{R}$ be a family of functions. Using the canonical 2-form $\omega = dp \wedge dq$, Hamilton's equations are equivalent to defining a vector field X_t^H by $dH_t(\cdot) = \omega(X_t^H, \cdot)$ whose flow is defined by $\frac{d}{dt}\phi_t^H = X_t^H \circ \phi_t^H$.

Symplectic geometry generalizes these structures on cotangent bundles to arbitrary manifolds equipped with a closed, nondegenerate, skew-symmetric 2-form. In this framework, symplectic structures have two aspects. One is dynamic and comes from the Hamiltonian flows they support, while the other is geometric and comes from the holomorphic curves they contain.

Throughout (M, ω) will be a closed, connected symplectic manifold. $H_t : M \to \mathbb{R}$ will be a smooth family of functions parameterized by $t \in [0, 1]$, and X_t^H along with ϕ_t^H will be defined as above. A symplectomorphism ψ is called

Hamiltonian if there is a path ϕ_t^H as defined above satisfying $\phi_1^H = \psi$. The collection of all such maps is a group denoted by $Ham(M, \omega)$.

In this thesis we examine Hamiltonian flows whose associated Hamiltonian generating functions have fixed maxima. By this we mean points $x \in M$ so that $H_t(x)$ is a maximum of H_t for each t. McDuff proves [6] several results for Hamiltonian loops for which the fixed global maxima are nondegenerate. The aim of this thesis is to extend these results to the degenerate case.

Theorem 1.0.1. *Let $\{\phi_t^H\}$, $t \in [0, 1]$, be a nonconstant loop in $Ham(M, \omega)$ based at Id, with $F_{max} \neq \emptyset$ its fixed global maximum set. For every $x_0 \in F_{max}$, we must have $D\phi_t^H(x_0) \neq Id$ for some value of t.*

Our proof requires that the maximum be global and uses methods of holomorphic curves. Of course a similar statement holds for global minima, by simply considering the function $-H_t$. This result allows us to then construct loops of Hamiltonians with fixed nondegenerate global maxima.

Definition 1.0.2. *A point $x_0 \in M$ is called a fixed local maximum on U of H_t if there exists a neighborhood $U \subset M$ of x_0 such that $H_t(x) \geq H_t(y)$ for all values of t and $\forall y \in U$. Similarly, $x_0 \in M$ is called a fixed global maximum if $H_t(x_0) \geq H_t(y)$ for all values of t and $\forall y \in M$.*

Definition 1.0.3. *A fixed maximum, $x_0 \in M$, is called nondegenerate at t_0 if $\frac{d}{dt}|_{t=t_0} D\phi_t^H(x_0)v \neq 0$ for all $0 \neq v \in T_{x_0}M$. It is called nondegenerate if it is nondegenerate for all time. It is called totally degenerate if $D\phi_t^H(x_0) = Id$ for all values of t.*

The degenerate maxima fall into three possible categories. A point could be nondegenerate for some value t_0, but degenerate at other times. In this case we give a method of homotoping the path to one which is nondegenerate for all time. A point could be degenerate for all time, but satisfy $D\phi_t^H(x_0) \neq Id$ for some value of t. In this case we construct a path homotopic to an iterate of ϕ_t^H which is nondegenerate at t_0. Finally, Theorem 1.0.1 shows that a global maximum cannot be totally degenerate for all time.

Combining these constructions with results of Slimowitz we obtain

Corollary 1.0.4. *Let M be a symplectic manifold with dim $M \leq 4$. If $\{\phi_t^H\}$ is any nonconstant Hamiltonian loop with $x_0 \in M$ a fixed global maximum, then there is a nonconstant loop ϕ_t^K with x_0 still a fixed global maximum, which is an effective S^1 action near x_0.*

If dim(M) =2, then we must have $M = S^2$, and the statement says very little as it is already known that such a ϕ_t^K exists. When dim$(M) = 4$, however, we offer a construction of the new loop, $\{\phi_t^K\}$. As mentioned, it will not necessarily be homotopic to the original loop $\{\phi_t^H\}$, but instead homotopic to an iterate of it. Whether there is a loop with a fixed nondegenerate global maximum homotopic to $\{\phi_t^H\}$ is an interesting question for future research. Nonetheless the result is still useful.

A symplectic manifold is called uniruled if some point class nonzero Gromov-Witten invariant does not vanish. More specifically this means there exist $a_2, \ldots, a_k \in H_*(M)$ so that

$$\langle pt, a_2, \ldots, a_k \rangle_{k,\beta}^M \neq 0 \text{ for some } 0 \neq \beta \in H_2^S(M).$$

Here pt is the point class in $H_0(M)$. McDuff uses the Seidel element and methods of relative Gromov-Witten invariants to show ([6], Theorem 1.1)

Theorem 1.0.5. *(McDuff) Suppose $Ham(M, \omega)$ contains a loop γ with a fixed global maximum near which γ is an effective circle action. Then (M, ω) is uniruled.*

McDuff relies heavily on the algebraic structure of the quantum homology of M as well as the invertibility of the Seidel element. The methods used in this thesis are largely inspired by the methods of [6], however we rely on the geometric structures of a certain Hamiltonian bundle over S^2, as opposed to the algebraic information it gives rise to. Combining Corollary 1 and Theorem 1.0.5, we have

Theorem 1.0.6. *If dim $M \leq 4$ and there exists a nonconstant loop of Hamiltonians $\{\phi_t^H\}$ with a fixed global maximum, then (M, ω) is uniruled.*

Given a path ϕ_t^H, $t \in [0,1]$, the Hofer length is defined as

$$\mathcal{L}(\phi_t^H) = \int_0^1 \Big(\max_x H_t(x) - \min_x H_t(x) \Big) dt.$$

One can use this to define a nondegenerate Finsler metric on the Hamiltonian group. While these appear to be very simple definitions, they are quite difficult to work with in practice. Many fundamental open questions remain regarding the geometry of Hamiltonian groups with this metric. Bialy and Polterovich [1] introduce the notion of Hofer geodesics and demonstrate various properties their associated Hamiltonian functions must satisfy, Lalonde and McDuff [2] investigate them further. In the following definition, a path is called regular if $\frac{d}{dt}\phi_t^H \neq 0$ for all t.

Definition 1.0.7. *Given an interval $I \subset \mathbb{R}$, a path $\{\phi_t\}_{t\in I}$ is called a* geodesic *if it is regular and if every $s \in I$ has a closed neighborhood $\mathcal{N}(s) = [a_s, b_s]$ in I such that the path $\{\phi_{\beta(t)}\}_{t\in\mathcal{N}(s)}$ is a local minimum of $\mathcal{L}$, where $\beta : \mathcal{N}(s) \to [0,1]$ is the linear reparameterization $\beta(t) = (t - a_s)/(b_s - a_s)$. Such a path is said to be locally length-minimizing at each moment.*

Theorem 1.0.8. *(Lalonde-McDuff) If a path ϕ_t, $t \in [0,1]$ is a Hofer length minimizing geodesic then its generating Hamiltonian has at least one fixed global minimum and one fixed global maximum.*

Many questions remain regarding when these geodesics exist and what other properties they must satisfy. Lalonde and McDuff ([2], Proposition 1.7) give an example of a Hamiltonian symplectomorphism $\phi : S^2 \to S^2$ which is not the endpoint of any Hofer length minimizing geodesic from the identity. In contrast, McDuff in [4] shows there is a neighborhood $U \subset Ham(M,\omega)$ of the identity such that every $\phi \in U$ can be joined to the identity by a Hofer length minimizing geodesic. We may combine Corollary with Theorem 1.0.6 and Theorem 1.0.8 to conclude

Theorem 1.0.9. *Let (M,ω) be a closed, connected symplectic 4-manifold, and suppose $\gamma \in \pi_1(Ham(M,\omega))$ is nontrivial. If there exists a representative $\{\phi_t^H\}$ of γ which is Hofer length minimizing, then (M,ω) is uniruled.*

Of course this says nothing if the Hamiltonian group is simply connected. McDuff ([5], Proposition 1.10) demonstrates that $\pi_1(Ham(M,\omega)) \neq 0$ if M

is a suitable two point blow up of any symplectic 4-manifold. Thus, if M is not uniruled (e.g. $\mathbb{T}^4$, a $K3$ surface, or a surface of general type), this two point blow up is a 4-manifold for which there are nontrivial elements of $\pi_1(Ham(M, \omega))$ having no Hofer length minimizing representatives.

Chapter 2 contains a discussion of positive and semipositive paths. It also contains the proofs needed for Corollary 1 assuming Theorem 1.0.1. Chapter 3 contains a discussion of the Hamiltonian fibrations used. As the machinery needed to prove Theorem 1.0.1 is discussed here, its proof is left to the end of this chapter.

Chapter 2

Positive and Semipositive Paths

2.1 Basic Notions

Let $\mathbb{R}^{2n}$ with basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ be equipped with the standard symplectic structure given by $\omega = \sum dx_i \wedge dy_i$ and the standard almost complex structure:

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The space of matrices which preserve ω are precisely those which satisfy $A^T JA = J$. This space is denoted $Sp(2n)$ and its Lie algebra $\mathbf{sp}(2n)$ consists of matrices which satisfy $JAJ = A^T$. Throughout, when in $\mathbb{R}^{2n}$ we will use these standard symplectic and almost complex structures and metric given by, $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.

A differentiable path in $Sp(2n)$ is called *positive* if it satisfies

$$\frac{d}{dt}A_t = JQ_tA_t$$

where Q_t is a positive definite symmetric matrix for each t. These paths are natural generalizations of circle actions near maxima of the corresponding time independent Hamiltonian. They correspond precisely to the flows of Hamiltonian functions of the form

$$H_t(x) = const - \frac{1}{2}\langle x, Q_t x\rangle.$$

The simplest example of such a path is the counter clockwise rotation $A_t = e^{2\pi kJt}$, with $k > 0$. Here $Q_t = 2\pi kI$. In the event that Q_t is symmetric, but only positive semidefinite (i.e., Q_t could have eigenvalues of zero for certain values of t), the path is called *semipositive.* The following is proved by Lalonde and McDuff in [3].

Lemma 2.1.1. *1. The set of positive paths is open in the C^1 topology.*

2. Any piecewise smooth positive path may be C^0 approximated by a positive path.

As stated in [3], the first result follows from the openness of the positive cone field, and the second from its convexity. The authors also give several characterizations of how positive paths may travel through $Sp(2n)$. Slimowitz furthers these results ([10], Theorems 3.3 & 4.1).

Theorem 2.1.2. *(Slimowitz) Let* $n = 1, 2$ *and let* $A_t \in Sp(2n)$ *be a positive loop. Then* A_t *can be homotoped through positive loops to a circle action.*

In principle this should be true in all dimensions, but the details have only been worked out for these cases. It is further shown ([10], Lemma 4.7) that

Lemma 2.1.3. *(Slimowitz) In* $Sp(4)$*, any two loops of matrices of the form*

$$\begin{pmatrix} e^{2\pi b_i J t} & 0 \\ 0 & e^{2\pi d_i J t} \end{pmatrix}$$

for $i = 1, 2$ *and* $t \in [0, 1]$ *are homotopic through positive loops provided* $b_i, d_i \geq 1$ *and* $b_1 + d_1 = b_2 + d_2$*.*

Recall that the action of a group G on a set X is called effective if for any g, there exists an x so that $g \cdot x \neq x$. This result shows that any positive loop in $Sp(4)$ may be homotoped through positive loops to an S^1 action which is also effective.

In both [3] and [10], the authors make use of the space $Conj$, defined as the space of conjugacy classes of elements in $Sp(2n)$ with the (non Hausdorff) quotient topology. This space is useful for the study of positive paths because the path $X^{-1} A_t X$ will remain positive for $X \in Sp(2n)$ if A_t began as one. Furthermore, it allows one to reduce the analysis of paths of matrices to paths of eigenvalues. The authors divide the space $Sp(2n)$ into generic regions consisting of matrices with only simple eigenvalues and other regions consisting of higher codimensional strata. A careful analysis is then made regarding ways generic paths move through the codimension 1 strata. The benefit of positivity over semipositivity here is that a generic positive path will always be transverse to this codimension 1 strata. Given the non Hausdorff nature of $Conj$ this fact becomes crucial to the arguments of Lalonde and McDuff as well as to those of Slimowitz.

Remark 2.1.4. *The following is an example of some of the difficulties that arise when the path is only semipositive. Because the path is symplectic, eigenvalues come in quadruplets of the form*

$$\lambda, \overline{\lambda}, 1/\lambda, 1/\overline{\lambda}$$

or pairs if either $\lambda \in \mathbb{R} - \{0\}$ *or* $|\lambda| = 1$ *([7], Lemma 2.20).*

Extending ω *and* J *to* $\mathbb{R}^{2n} \otimes \mathbb{C}$ *by complex linearity, a nondegenerate Hermitian symmetric form is defined by*

$$\beta(v, w) = -i\langle Jv, w\rangle = -i\omega(v, \overline{w}).$$

If v *is an eigenvector of the complexified matrix with eigenvector* $\lambda \in S^1$ *of multiplicity* 1*, then* $\beta(v, v) \in \mathbb{R} - \{0\}$*. For such vectors, assign the value* ± 1*, called a splitting number, corresponding to the sign of* $\beta(v, v)$*. Krein's Lemma is fundamental to many results obtained for positive paths.*

Lemma 2.1.5. *(Krein) Under a positive flow simple eigenvalues on* S^1 *labelled with* $+1$ *must move counter clockwise, and those labelled with* -1 *must move clockwise.*

The proof of Krein's Lemma can be slightly altered to show that if A_t *is a* *positive path with $A_{t_0} = Id$, then for some $\epsilon > 0$, the eigenvalues of A_t must be contained in $S^1 - \{1\}$ for $t_0 < t < t_0 + \epsilon$. This result very much relies on $Q_{t_0} > 0$. If Q_{t_0} is only positive semidefinite, then the path can travel within the fibre over 1 in $Conj$ through nilpotent matrices, making the analysis of the path very difficult, and different methods must be used in examining the behavior of such paths.*

2.2 Linearization Near Maxima

In our setting we wish to consider Hamiltonian functions on manifolds. Fixed maxima must be fixed points of the associated flow for all time (i.e., $\phi_t^H(x) = x, \forall t$). Choosing a Darboux chart around such a point x, H_t may be written as

$$H_t(x) = const - \frac{1}{2}\langle x, Q_t x\rangle + O(\|x\|^3) \tag{2.2.1}$$

and as before we will call the path (semi)positive if Q_t is positive (semi)definite. As the point x is only assumed to be a maximum, as opposed to a nondegenerate maximum, we may only assume $Q_t \geq 0$ and the flow of its linearization is a semipositive path. The first results deal with the case when $Q_{t_0} > 0$ for some t_0, and thus is a positive path for some ϵ time. We describe a method of "spreading out the positivity" to homotop our path to a new one which is positive on all of $[0, 1]$. We do so in such a way that, if x is a maximum of H_t on some set V, it will remain a maximum of the new Hamiltonian on V.

Lemma 2.2.1. *Let $\{\phi_t^H\} \subset Ham(M,\omega)$ for $t \in [0,1]$ be a path of Hamiltonians whose generating function, H_t, has a fixed local maximum at x_0. Let $-\frac{1}{2}\langle x, Q_t x\rangle$ be the quadratic part of H_t, and let $I^+ = \{t \in [0,1] | Q_t > 0\}$. If $\emptyset \neq I^+ \neq [0,1]$ choose $t_0 \in I^+$ and $t_1 \notin I^+$. Then the path may be homotoped through semipositive paths with fixed endpoints to a new one, $\{\phi_t^F\}$, whose quadratic part is positive in I^+ and in a $\delta > 0$ neighborhood of t_1. Furthermore, if x_0 was a maximum of H_t on a neighborhood V of x_0 for all t, it will remain a maximum of F_t on V for all t, and δ will depend only on the initial neighborhood such that $Q_t > 0$.*

Proof. For the purposes of this proof, consider t as a variable in $\mathbb{R}/\mathbb{Z}$. Let δ be such that $Q_t > 0$ for $|t - t_0| < \delta$. Note that we must have $|t_1 - t_0| \geq \delta$. Let a be smaller than any of the eigenvalues of Q_t for $|t - t_0| < \delta/2$ and let $b >> 1$. Define a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying: $\alpha'(r) \geq 0$, $\alpha(r) = br - a$ for $r < a/2b$, and $\alpha(r) = 0$ for $r > a/b$. Next consider the autonomous Hamiltonian function K defined on $\mathbb{R}^{2n}$ given by $K(x) = \alpha(\|x\|)\|x\|^2$. Also, let $\beta : [0,1] \to [0,1]$ be a smooth nonincreasing function which is 1, $\beta(t) > 0$ on $[0, \delta/3]$, and 0 and 0 near $\delta/2$.

For each value of $0 \leq s \leq 1$ define a function:

$$K_{s,t} = \begin{cases} 0 & \text{for } t < t_1 - \delta/2 \\ s\beta(|t - t_1|)K & \text{for } t_1 - \delta/2 \leq t \leq t_1 + \delta/2 \\ 0 & \text{for } t_1 + \delta/2 \leq t \leq t_0 - \delta/2 \\ -s\beta(|t - t_0|)K & \text{for } t_0 - \delta/2 \leq t \leq t_0 + \delta/2 \\ 0 & \text{for } t_0 + \delta/2 \leq t \end{cases}$$

For each value of s, this time-dependent function will generate a smooth path of Hamiltonians, $\{\psi^K_{s,t}\}$. Since any perturbation from the identity map is eventually undone, the path will satisfy $\psi^K_{s,0} = \psi^K_{s,1} = Id$, regardless of the values of t_0 and t_1, and thus will always be a loop.

We now consider the composition $\phi^F_{s,t} = \psi^K_{s,t} \circ \phi^H_t$, and note that the corresponding time dependent family of functions $F_{s,t}$ are given by the formula

$$F_{s,t} = K_{s,t} \# H_t = K_{s,t} + H_t \circ (\phi^K_{s,t})^{-1}. \tag{2.2.2}$$

We now claim that for a suitable choice of b, our path $\phi^F_{s,t}$ is positive for $t \in I^+$ and $|t - t_1| < \delta/3$. To show positivity, we need only show that Hamiltonian function has non-degenerate quadratic part at x_0. Working in local coordinates, we fix s and t with $|t - t_1| < \delta/3$, and choose $v \in \mathbb{R}^{2n}$, and take the limit

$$\begin{aligned} \lim_{r\to 0} \frac{\Big(K_{s,t}(rv) + H_t \circ (\psi^K_{s,t})^{-1}(rv)\Big)}{\|rv\|^2} &= -a\beta(|t-t_1|)s + \lim_{r\to 0} \frac{H_t \circ (\phi^K_{s,t})^{-1}(rv)}{\|rv\|^2} \\ &\leq -a\beta(|t-t_1|)s \\ &< 0 \end{aligned}$$

where the inequality and subsequent minus sign on the right are explained by our convention of the quadratic portion actually being negative semidefinite.

Calling Q'_t the quadratic portion of F_t, $Q'_t > 0$ for $|t - t_0| < \delta/2$, since a was chosen smaller than any of the eigenvalues of Q_t here. To see that $Q'_t > 0$ on the rest of I^+, we note that $\beta = 0$ in this region, and

$$\begin{aligned}\lim_{r\to 0}\frac{\Big(K_{s,t}(rv)+H_t\circ(\psi^K_{s,t})^{-1}(rv)\Big)}{\|rv\|^2} &= \lim_{r\to 0}\frac{H_t\circ(\phi^K_{s,t})^{-1}(rv)}{\|rv\|^2}\\ &< 0\end{aligned}$$

As s and t are fixed while taking this limit, the inequality holds by the definition of I^+.

Finally, our perturbed function will remain a maximum in V for $|t-t_1|<\delta$, and by choosing b large enough (the choice depends on the third order terms of the initial H_t), it will remain a maximum for $|t-t_0|<\delta$, as well. □

Proposition 2.2.2. *Let $\{\phi^H_t\}$ for $t\in[0,1]$ be a path of Hamiltonians based at Id with generating function H_t. Suppose x_0 is a maximum of H_t on some neighborhood V of x_0, for all t. Letting Q_t be as in Lemma 2.2.1, if $Q_{t_0}>0$ for some $0\le t_0\le 1$, then $\{\phi^H_t\}$ may be homotoped through semipositive paths with fixed endpoints to a new path $\{\phi^F_t\}$ whose associated quadratic portion is strictly positive for all $t\in[0,1]$. Furthermore x_0 will be a maximum of F_t on V for all t, as well.*

Proof. As the $\delta>0$ value from Lemma 2.2.1 depended only on the neighborhood of t_0 for which Q_t remained positive, we may carry out the process a finite number of times to homotop our path through semipositive paths with fixed endpoints to one which is positive for all t. Furthermore, by construction, x_0 remains a maximum on V throughout. □

We now consider a slightly more general circumstance. We consider the case when our path is degenerate for all time, but for some t_0, we have

$D\phi^H_{t_0}(x_0) \neq Id$.

Proposition 2.2.3. *Let $\{\phi^H_t\} \subset Ham(M,\omega)$ for $t \in [0,1]$ be a path of Hamiltonians with generating function H_t. Let x_0 be a maximum of H_t on some neighborhood V of x_0. If $D\phi^H_t \neq Id$ for some t_0, then there is a new path, $\{\phi^K_t\}$, whose associated Hamiltonian function, K_t, is nondegenerate at x_0 and for $t = t_0$. Furthermore, ϕ^K_t can be chosen to be homotopic to $\{(\phi^H_t)^m\}$ for some $m \leq 1 + dim(ker(D\phi^H_{t_0} - Id))$. Furthermore, x_0 will remain a maximum of K_t on V.*

Proof. Choose a Darboux chart within V, sending x_0 to $0 \in \mathbb{R}^{2n}$. Throughout, for convenience of notation, we explicitly work in $\mathbb{R}^{2n}$ and the linearization of ϕ^H_t. We refer to the linearization as the path $A_t \in Sp(2n,\mathbb{R})$, and note that it satisfies $A_0 = A_1 = Id$ and $\frac{d}{dt}A_t(x) = JQ_t(x)A_t(x)$ with $Q_t \geq 0$ and symmetric. Let t_0 be such that $Q_{t_0} \neq 0$.

Identify $\mathbb{R}^{2n} = E_0 \oplus E_1$ with $E_0 = ker(Q_{t_0})$ and E_1 the sum of eigenspaces of Q_{t_0} with nonzero eigenvalues. We first consider the case when $J(E_0) = E_0$. Choose $v \in E_1$ to be an eigenvector for Q_{t_0}, and let $0 \neq w \in E_0$. Split $\mathbb{R}^{2n} = \mathbb{R}^4 \oplus \mathbb{R}^{2n-4}$ with $\mathbb{R}^4$ spanned by $\{v, w, Jv, Jw\}$ and $\mathbb{R}^{2n-4} = (\mathbb{R}^4)^\omega$ its symplectic orthogonal. Define $B \in Sp(2n)$ by

$$BA^{-1}_{t_0}w = v, \qquad BA^{-1}_{t_0}Jw = Jv$$
$$BA^{-1}_{t_0}v = -w, \qquad BA^{-1}_{t_0}Jv = -Jw,$$
$$BA^{-1}_{t_0}|_{\mathbb{R}^{2n-4}} = Id.$$

Let $B_s \in Sp(\mathbb{R}^{2n})$ for $s \in [0,1]$ satisfy $B_0 = Id$ and $B_1 = B$, and let $f_s : \mathbb{R}^{2n} \to$

$\mathbb{R}^{2n}$ be a family of symplectomorphisms fixing the origin and supported in a small neighborhood of it satisfying $Df_s(0) = B_s$, where $Df_s(0) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the derivative at the origin. We may assume that x_0 is a maximum of H_t throughout the support of the family f_s. We now wish to consider the family of paths given by:

$$\phi^K_{s,t} = \phi^H_t(f_s^{-1}\phi^H_t f_s)$$

which will provide a homotopy from $\left(\phi^H_t\right)^2$ to $\phi^K_{1,t}$. For each s this will remain a Hamiltonian flow, and will be generated by:

$$K_{s,t} = H_t + H_t(f_s \circ (\phi^H_t)^{-1}).$$

By our choice of f, x_0 will remain a constant maximum of $\phi^K_{s,t}$ for all values of s. To determine the degeneracy of our maximum, we simply differentiate:

$$\begin{aligned}
\frac{d}{dt}D\phi^H_{s,t}(0) &= \frac{d}{dt}A_tB_s^{-1}A_tB_s \\
&= \dot{A}_tB_s^{-1}A_tB_s + A_tB_s^{-1}\dot{A}_tB_s \\
&= JQ_t(A_tB_s^{-1}A_tB_s) + A_tB_s^{-1}JQ_tA_tB_s \\
&= JQ_t(A_tB_s^{-1}A_tB_s) + A_tJB_s^TQ_tA_tB_s \\
&= JQ_t(A_tB_s^{-1}A_tB_s) + J(A_t^{-1})^TB_s^TQ_tA_tB_s \\
&= J\Big(Q_t + (B_sA_t^{-1})^TQ_t(B_sA_t^{-1})\Big)(A_tB_s^{-1}A_tB_s).
\end{aligned}$$

Since the matrix $Q_t + (B_sA_t^{-1})^TQ_t(B_sA_t^{-1})$ remains symmetric and non-negative for all values of s and t, we need check that it is nondegenerate in the v, w plane when $s = 1$ and $t = t_0$. For the moment rename $\Gamma = (B_1 A_{t_0}^{-1})^T Q_{t_0} (B_1 A_{t_0}^{-1})$. We compute:

$$\begin{aligned} \langle v + aw, (Q_{t_0} + \Gamma)(v + aw) \rangle &= \langle v, Q_{t_0} v \rangle + \langle v, \Gamma v \rangle + a \langle v, \Gamma w \rangle \quad (2.2.3) \\ &\quad + a \langle w, \Gamma v \rangle + a^2 \langle w, \Gamma w \rangle \\ &= (1 + a^2) \langle v, Q_{t_0} v \rangle \\ &> 0. \end{aligned}$$

To see that we created no new kernel, let $u \in \mathbb{R}^{2n}$. Then

$$\langle u, (Q_{t_0} + \Gamma) u \rangle = \langle u, Q_{t_0} u \rangle + \langle u, \Gamma u \rangle$$

with both matrices being nonnegative. Thus the sum can only be zero if $\langle u, Q_{t_0} u \rangle = 0$.

In the case when J does not preserve E_0, we may choose $w \in E_0$ so that $\langle Jw, Q_{t_0} Jw \rangle > 0$. In this case, setting $v = Jw$ we define

$$\begin{aligned} BA_{t_0}^{-1} w = v, \qquad BA_{t_0}^{-1} v = -w, \\ BA_{t_0}^{-1}|_{\mathbb{R}^{2n-2}} = Id. \end{aligned}$$

As equation 2.2.3 remains the same, the remainder of the proof remains identical to the previous case. □

Corollary 2.2.4. *Let $x_0 \in M$ be a fixed local maximum on U of a family of Hamiltonians H_t, such that $D\phi_t^H(x) \neq Id$ for some t_0. Then we may construct* *a new path ϕ_t^F which is positive for all $t \in [0,1]$ and such that if $H_t(x_0) \geq H_t(y)$ for some $y \in U$, then $F_t(x_0) \geq F_t(y)$. Furthermore, if ϕ_t^H was a loop, then so is ϕ_t^F.*

For the case $D\phi_t^H(x) \equiv Id$ we will need to restrict our paths to only those which are loops. The autonomous case is much more straightforward, even if not restricted to circle actions. The following result is Lemma 12.27 from [7] and the norm we use is the standard operator norm.

Proposition 2.2.5. *Let $x(t) = x(t+T)$ be a periodic solution of the differential equation $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. If*

$$T\Big(\sup_{t\in[0,T]} \|Df(x(t))\|\Big) < 1$$

then x is constant.

Recall that given H_t, a point x_0 is called *totally degenerate* if $D\phi_t^H(x_0) = Id$ for all time. Given an autonomous Hamiltonian, and referring to its set of totally degenerate points as D, this result shows that for every value of T, there is an entire neighborhood U_T of D satisfying $\phi_t^H(x) \neq x$ for any $x \in U_T - D$ and $0 < t \leq T$. Thus any loop of autonomous Hamiltonians with a fixed point $x \in M$ satisfying $D\phi_t^H(x) \equiv Id$, $\forall t$ must be the constant loop.

We now restrict ourselves to the case where H_t has a *fixed global maximum* as in Definition 1.0.2. We also restrict ourselves to the case when ϕ_t^H is a loop in $Ham(M, \omega)$. As stated in Theorem 1.0.1, a result analogous to Proposition 2.2.5 holds in the time-dependent case near a fixed global maximum of a Hamiltonian generating function. We postpone the proof of this theorem until

Chapter 3, but we have the following important corollary which we restate for the convenience of the reader.

Corollary 2.2.6. *Let M be a symplectic manifold with dim $M \leq 4$. If $\{\phi_t^H\}$ is any nonconstant Hamiltonian loop with $x_0 \in M$ a fixed global maximum, then there is a nonconstant loop ϕ_t^K with x_0 still a fixed global maximum, which is an effective S^1 action near x_0.*

Proof. In 2 dimensions, the very existence of the initial loop forces the manifold to be S^2. One may then consider a time-1 rotation coming from a standard height function with x_0 as the maximum is an S^1 action which will be effective near x_0. Thus we need only consider the 4 dimensional case.

By Theorem 1.0.1 we must have $D\phi_{t_0}^H(x_0) \neq Id$ for some value of t_0. By Proposition 2.2.3, we may construct a new loop $\{\phi_t^K\}$ so that x_0 remains a fixed global maximum of the associated Hamiltonian generating function K_t and such that the quadratic portion of K_t is positive definite at x_0 when $t = t_0$. By Proposition 2.2.2 we may homotop $\{\phi_t^K\}$ to a new loop $\{\phi_t^F\}$ with x_0 again remaining a fixed global maximum of F_t and so that F_t has quadratic portion which is positive definite at x_0 for all values of t. Thus the linearization is a positive loop and by Theorem 2.1.2 and Lemma 2.1.3, it is homotopic through positive loops to an effective circle action. Applying the following result of McDuff ([6], Lemma 2.22) finishes the proof.

Lemma 2.2.7. *(McDuff) Suppose the loop γ in $Ham(M, \omega)$ has a nondegenerate fixed maximum at x_0. Suppose also that the linearized flow at x_0 is homotopic through positive paths to a linear circle action. Then γ is homotopic* *through Hamiltonian loops with fixed maximum at* x_0 *to a loop* γ' *that is the given circle action near* x_0.

□

Chapter 3

Hamiltonian Fibrations

3.1 Hamiltonian Bundles Over S^2

Given (M, ω) and any loop of Hamiltonians, $\{\phi_t^H\}$, there is an associated Hamiltonian fibration $P \to S^2$ with fibre symplectomorphic to (M, ω). Throughout we use the standard almost complex structure on S^2, which we call j. Begin with two copies, $M \times D_\pm$, where $D_\pm$ denotes two different copies of the unit disk with opposite orientations. Define the equivalence relation:

$$P := M \times D_+ \cup M \times D_- / \sim, \;\; (\phi_t^H(x), e^{2\pi i t})_+ = (x, e^{2\pi i t})_-. \tag{3.1.1}$$

Since $\phi_0^H(x) = \phi_1^H(x) = x, \forall x \in M$, the two copies of D glue together along their boundaries to give a copy of S^2, and $P \to S^2$ will be a fibration with fibre diffeomorphic to M. Denote the projection map by $\pi : P \to S^2$. The vertical tangent bundle here is given by $T^{Vert}P = ker(D\pi) \subset TP$. Because the fibres are symplectic, they have Chern classes, and the bundle thus comes equipped with two canonical classes. c_1^{Vert} is the first Chern class of the vertical tangent bundle, and $[\tau] \in H_2(P)$ is the unique class satisfying

$$[\tau]|_{\pi^{-1}(z)} = [\omega] \text{ and } [\tau]^{n+1} = 0 \tag{3.1.2}$$

called the coupling class. P can be given a symplectic structure by defining the form

$$\Omega = \tau + k\pi^*\beta \tag{3.1.3}$$

for a symplectic form β on S^2 and k chosen sufficiently large. Here τ is the vertically closed connection 2-form determined by ω. Such bundles have proved incredibly fruitful in studying the relationship between a manifold's Hamiltonian dynamics and the algebraic structure of the manifold's quantum homology.

Ω restricted to the vertical tangent space to the fibres, $T^{Vert}P = kerD\pi$, is nondegenerate. Thus Ω gives a connection 2-form on P, and we have a well defined horizontal distribution, which we will denote by $T^{Hor}P$. To be more precise:

$$T_p^{Hor}P = \{v \in T_pP | \Omega(v,w) = 0, \forall w \in kerD\pi(p)\} \tag{3.1.4}$$

Definition 3.1.1. *([8] §8.2) An almost complex structure, $\widetilde{J} : TP \to TP$ will be called compatible with the fibration if the following conditions are met:*

1. *$\pi : P \to S^2$ is holomorphic*

2. $\widetilde{J}|_{T_z^{Ver}P}$ *is tamed by* $\omega, \forall z \in S^2$

3. $\widetilde{J}(T^{Hor}P) \subset T^{Hor}P$.

Such a bundle contains lots of sections. To see this, let $\mathcal{L}(M)$ denote the free loop space of M (i.e. the space of all unbased maps $S^1 \to M$), and $\mathcal{L}_0(M)$ the subset of contractible loops. Let ϕ_t^H be an arbitrary loop of Hamiltonians, and consider the map $M \to \mathcal{L}(M)$ given by $x \mapsto \{\phi_t^H(x)\}$ which sends each point to its orbit. This map is well defined and continuous. Thus if M is connected, it maps all of M into either $\mathcal{L}_0(M)$ or $\mathcal{L}(M) - \mathcal{L}_0(M)$. By the Arnold conjecture, there must exist at least one contractible orbit, hence they must all be contractible. Thus every closed orbit $\{\phi_t^H\}$ provides a section $S^2 \to P$. To see this explicitly, choose any $x \in M$. A contraction of the orbit $\{\phi_t^H(x)\}$ is a map from the unit disk $f : D \to M$ with $f(e^{2\pi i t}) = \phi_t^H(x)$. Thus an explicit formula for our section $s : S^2 \to P$ would be

$$s(r,t) = x \times (r,t), \text{ on } D_- \tag{3.1.5}$$

$$s(r,t) = f(r,t) \times (r,t), \text{ on } D_+.$$

In the event that x_0 is fixed by ϕ_t^H for all time, we may choose $f(r,t)$ to be constant. We will denote this constant section by s_0. In particular, if x_0 is a fixed global maximum of H_t, these sections have very nice properties: they are holomorphic with respect to suitable almost complex structures on P, and as will be shown in Lemma 3.2.1 below, they have minimal energy among all sections. These properties are crucial to our arguments.

3.2 Proof of Theorem 1.0.1

Let ϕ_t^H be a loop in $Ham(M,\omega)$ based at Id, with F_{max} the fixed global maxima set. We show that if there is an $x_0 \in F_{max}$ and $D\phi_t^H(x_0) \equiv 0$ for all time, then the loop must be constant.

Let $P \to S^2$ be the bundle constructed in equation 3.1.1. We define a symplectic form, Ω, on P by

$$\Omega_- := \omega + \delta d(r^2) \wedge dt, \text{ on } M \times D_- \tag{3.2.1}$$
$$\Omega_+ := \omega + \Big(\kappa(r^2,t)d(r^2) - d(\rho(r^2)H_t)\Big) \wedge dt \text{ on } M \times D_+$$

where we have used normalized polar coordinates (r,t) on D with $t := \theta/2\pi$. Here $\rho(r^2)$ is a nondecreasing function that equals 0 near 0 and 1 near 1, and $\delta > 0$ is some small constant. As long as $\kappa(r^2,t) = \delta$ near $r = 1$, these two forms will fit together to give a closed form on P. To be symplectic, Ω must be nondegenerate, but this can be seen to happen iff $\kappa(r^2,t) - \rho'(r^2)H_t(x) > 0, \forall (r,t) \in D_\pm$ and $x \in M$. Note here that varying κ in equation 3.2.1 does not affect the horizontal distribution defined in equation 3.1.4. Also note that a representative of the coupling class, τ from equation 3.1.2, can be explicitly seen as

$$\omega \text{ on } M \times D_-$$
$$\omega - d(\rho(r^2)H_t) \wedge dt \text{ on } M \times D_+ \quad .$$

By our choice of Ω, $T^{Hor}(M \times D_-)$ is spanned by ∂_r and ∂_t, and $T^{Hor}(M \times$

D_+) is spanned by the vectors ∂_r and $\partial_t - X_t^H$ at each point. Thus conditions (1) and (3) completely determine $\widetilde{J}$ on $T^{Hor}P$:

$$\widetilde{J}^{Hor}(\partial_r) = \partial_t,\ \widetilde{J}^{Hor}(\partial_t) = -\partial_r \text{ on } M \times D_-, \text{ and} \tag{3.2.2}$$
$$\widetilde{J}^{Hor}(\partial_r) = \partial_t - X_t^H,\ \widetilde{J}^{Hor}(\partial_t - X_t^H) = -\partial_r \text{ on } M \times D_+.$$

Because $\widetilde{J}$ is tamed by Ω, the bilinear form

$$g_{\widetilde{J}}(v,w) = \frac{1}{2}\big(\Omega(v, \widetilde{J}w) + \Omega(w, \widetilde{J}v)\big)$$

defines a Riemannian metric on P, with associated Levi-Civita connection, ∇. To obtain a connection which will preserve $\widetilde{J}$ we use ([8] § 3.1)

$$\widetilde{\nabla}_v X = \nabla_v X - \frac{1}{2}\widetilde{J}(\nabla_v \widetilde{J})X. \tag{3.2.3}$$

Note that if a point x is fixed by ϕ_t^H for all time, then x gives rise to a constant section $(r,t) \mapsto x \times (r,t)$ and $\partial_t - X_t^H = \partial_t$ along this section. The following result was also proved by McDuff ([6], Proposition 2.11) and a version was proved by McDuff and Tolman ([9], Lemma 3.1), as well. For completeness, we also include a proof here.

Lemma 3.2.1. *Suppose that $x_0 \in M$ is a fixed global maximum of the loop of Hamiltonians for all time, and consider the constant section $s_0 : (r,t) \mapsto x_0 \times (r,t)$. Then, given any $\widetilde{J}$ compatible with the fibration, the only holomorphic sections in class $[s_0]$ are constant ones, and are parameterized by elements of the fixed global maximum set, F_{max} for H_t.*

Proof. We use the symplectic form given by equation 3.2.1. At a point in the image of our section $u : S^2 \to P$, split $TP = T^{Vert}P \oplus T^{Hor}P$, and write elements of $T_{u(r,t)}P$ as $v + h$. We compute:

$$\begin{aligned}\Omega(v+h, \widetilde{J}(v+h)) &= \omega(v, \widetilde{J}v) + \Omega(h, \widetilde{J}h) \geq \Omega(h, \widetilde{J}h) \\ &\geq 2r\Big(\kappa(r^2,t) - \rho'(r^2)\max_{x\in M} K_t(x)\Big)dr \wedge dt(h, \widetilde{J}h).\end{aligned}$$

The first inequality is an equality only if the curve is horizontal, and the second is an equality only if the section is contained in $F_{max} \times S^2$. Since any other curve representing the same class as $[s_0]$ must have the same symplectic area, and we are finished. □

The compatibility conditions of Definition 3.1.1 do not determine $\widetilde{J}$ on $T^{Vert}P$, so we now construct one explicitly. In our case, we wish the almost complex structure we construct to be regular for a constant section through F_{max}.

First, choose a Darboux chart around $x_0 \in F_{max}$, call it U, and identify it with a neighborhood of $0 \in \mathbb{R}^{2n}$. Using the standard $\{x_i, y_i\}$ coordinates on $\mathbb{R}^{2n}$ and the standard J, choose an almost complex structure on M which is the pullback of J on U, and refer to it as J_0.

Take $\widetilde{J_0}^{Vert}$ on $T^{Vert}(M \times D_-)$ to be J_0. This forces $\widetilde{J}^{Vert} = (\phi_t^H)_* J_0$ on $M \times \partial D_+$, and we must extend this to the rest of $T^{Vert}(M \times D_+)$. In our coordinates on U, $(\phi_t^H)_* J_0$ will be given by conjugation by $D\phi_t^H(x)$, so that at a point x, we have

$$(\phi_t^H)_* J_0 = (D\phi_t^H(x))^{-1} \circ J_0 \circ D\phi_t^H(x)$$

where we have realized $D\phi_t^H(x)$ as a loop of maps $D\phi_t^H : U \to Sp(2n)$ based at the constant map $U \mapsto Id$. Since $D\phi_t^H(0) = Id$ for all time, we may also assume our initial neighborhood U is small enough that there is a loop of maps $Y_t : U \to \mathbf{sp}(2n)$ based at the constant map $U \mapsto 0$ satisfying

$$\exp(Y_t(x)) = D\phi_t^H(x)$$

where exp is the standard exponential map from $\mathbf{sp}(2n) \to Sp(2n)$. Letting $\beta : [0,1] \to [0,1]$ be a smooth, nondecreasing function which is 0 near 0 and 1 near 1, we may consider the family of maps $Y_{r,t} : U \to \mathbf{sp}(2n)$ given by $Y_{r,t}(x) = \beta(r) Y_t(x)$. Noting that by our choice of β, $Y_{r,t}(x) = 0$ for r close to 0, we may now consider this as a family of maps smoothly parameterized by D_+. We now extend our almost complex structure to all of $U \times D_+$ by the formula:

$$\widetilde{J}^{Vert}(x \times (r,t)) = (\exp(Y_{r,t}(x))^{-1} \circ J_0 \circ \exp(Y_{r,t}(x)), \qquad (3.2.4)$$

where $x \times (r,t) \in U \times D_+$.

Finally extend $\widetilde{J}^{Vert}$ to the rest of $T^{vert}(M \times D_+)$ in a way compatible with the fibration (see [8], §8.2), and take $\widetilde{J} = \widetilde{J}^{Vert} \oplus \widetilde{J}^{Hor}$. We now claim that the $\widetilde{J}$ just constructed is a regular almost complex structure for our constant maximum section.

Lemma 3.2.2. *Let* $\xi \in \Omega^0(S^2, s_0^*(TP))$ *be any vector field along* s_0*. Then* $\nabla_\xi \widetilde{J} = 0$.

Proof. Given a section ξ of TP defined in a neighborhood of $Im(s_0)$, we may write it as $v_\xi + h_\xi$ where v_ξ is a section of $T^{Vert}P$ and h_ξ a section of $T^{Hor}P$, both defined in a small neighborhood of $Im(s_0)$. We consider $\widetilde{J}^{Hor}$ and $\widetilde{J}^{Vert}$ separately.

A direct calculation shows that if h is tangent to $Im(s_0)$, then $\nabla_h \widetilde{J}^{Hor} = 0$. If $v \in T^{Vert}(M \times D_-)$, one clearly has $\nabla_v \widetilde{J}^{Hor} = 0$ along $x_0 \times D_-$. If $x_0 \times (r,t) \in x_0 \times D_+$, then because ∇ is Levi-Civita, we must have $\nabla_v(\partial_t - X) = a_{r,t}\partial_r$ and $\nabla_v \partial_r = -a_{r,t}(\partial_t - X)$ with $a_{r,t} \in \mathbb{R}$. But then using the identity

$$(\nabla_v \widetilde{J}^{Hor})(X) = \nabla_v(\widetilde{J}^{Hor}(X)) - \widetilde{J}^{Hor}(\nabla_v(X))$$

as well as the Leibniz rule, one can easily see that $\nabla_v \widetilde{J}^{Hor} = 0$ along $x_0 \times D_+$, as well. Thus we have $\nabla_\xi \widetilde{J}^{Hor} = 0$ for any $\xi \in \Omega^0(S^2, u^*(TP))$.

Similar rationale holds to show $\nabla_h \widetilde{J}^{Vert} = 0$ for h tangent to $Im(s_0)$, and $\nabla_v \widetilde{J}^{Vert} = 0$ for $v \in T^{Vert}P$ along $x_0 \times D_-$. Thus we need only concern ourselves with the value of $\nabla_v \widetilde{J}^{Vert}$ at points in $U \times D_+$ with U a neighborhood of x_0. Using Darboux coordinates, we may expand ϕ_t^H about x_0, and we have

$$\phi_t^H(x) = x + \sum_{i \leq j} A_{i,j}(t) x_i x_j + O(\|x\|^3) \tag{3.2.5}$$

with $A_{i,j}(t)$ a time-dependent loop of vectors in $\mathbb{R}^{2n}$ and the higher order terms also depending on time. Since x_0 is a totally degenerate maximum, we may write $|H_t(x) - H_t(0)| \leq C\|x\|^4$ in our neighborhood for some C, and thus $\|X_t^H(x)\| \leq C'\|x\|^3$ in our neighborhood for some C' since it is defined in terms of dH_t. We now use the fact that

$$\begin{aligned} X_t^H(\phi_{t_0}^H(x)) &= \frac{d}{dt}\phi_t^H(x)|_{t=t_0} \\ &= \sum_{i\leq j}\Big(\frac{d}{dt}A_{i,j}(t)|_{t=t_0}\Big)x_i x_j + O(\|x\|^3) \end{aligned}$$

for every $0 \leq t_0 \leq 1$. But in order for $\|\frac{d}{dt}\phi_t^H(x)\| = \|X_t^H(\phi_t^H(x))\| \leq C'\|x\|^3$, we must have each $A_{i,j}(t)$ a constant function of t. As $\phi_0^H = Id$, $A_{i,j}(t) = 0$ for all t. Thus,

$$\begin{aligned} \phi_t^H(x) &= x + O(\|x\|^3) \\ D\phi_t^H(x) &= Id + O(\|x\|^2). \end{aligned}$$

Since $D\phi_t^H(x) = \exp(Y_t(x))$ we have $\nabla_v \exp(Y_t(x)) = 0$, and it is easy to see that $\nabla_v \exp(Y_{r,t}(x)) = 0$ also. Finally, since $\exp(Y_{r,t}(x)) = Id$ along our section, we may say

$$\nabla_v\Big((\exp(Y_{r,t}(x))^{-1} \circ J_0 \circ \exp(Y_{r,t})\Big) = 0$$

along our section, as well. Thus $\nabla_\xi \widetilde{J}^{Vert} = 0$, for any $\xi \in \Omega^0(S^2, u^*(TP))$. □

Proposition 3.2.3. *Let $\phi_t^H, t \in [0,1]$ be a loop of Hamiltonians based at Id. Let F_{max} be the set of fixed global maxima of H_t, and suppose that $D\phi_t^H(x_0) \equiv$*

Id for some $x_0 \in F_{max}$ and for all values of t. Let s_0 denote the constant section through x_0 and let $\widetilde{J}$ be as constructed above. Then s_0 is a regular $\widetilde{J}$ holomorphic map.

Proof. For $\widetilde{J}$ to be regular for s_0, the differential,

$$D_{s_0} : \Omega^0(S^2, s_0^*(TP)) \to \Omega^{0,1}(S^2, s_0^*(TP))$$

which maps smooth sections of $s_0^*(TP)$ to $\widetilde{J}$ antiholomorphic $s_0^*(TP)$ valued 1-forms on S^2, must be surjective. An explicit formula for D_{s_0} evaluated at $\xi \in \Omega^0(S^2, s_0^*(TP))$ is given by:

$$D_{s_0}\xi = \frac{1}{2}\big(\widetilde{\nabla}\xi + \widetilde{J}(s_0)\widetilde{\nabla}\xi \circ j\big) + \frac{1}{4}N_{\widetilde{J}}(\xi, ds_0) \tag{3.2.6}$$

where $\widetilde{\nabla}$ is from equation 3.2.3 and $N_{\widetilde{J}}$ is the Nijenhuis tensor, see [8] Remark 3.1.2. Given $v_z \in T_zS^2$ this formula returns

$$(D_{s_0}\xi)v_z = \frac{1}{2}\big(\widetilde{\nabla}_{ds_0(z)(v_z)}\xi + \widetilde{J}(s_0)\widetilde{\nabla}_{ds_0(z)(jv_z)}\xi\big) + \frac{1}{4}N_{\widetilde{J}}(\xi, ds_0(z)(v_z)) \in u^*(TP).$$

As $\nabla_\xi \widetilde{J} = 0$ for all $\xi \in \Omega^0(S^2, s_0^*(TP))$, equation 3.2.3 becomes $\widetilde{\nabla} = \nabla$. A formula for $N_{\widetilde{J}}(X, Y)$ (which can be found in [8] Lemma C.7.1) is given by

$$N(X, Y) = (J\nabla_Y J - \nabla_{JY} J)X - (J\nabla_X J - \nabla_{JX} J)Y.$$

Thus $\nabla_\xi \widetilde{J} = 0$ also implies the Nijenhuis tensor vanishes, so that D_{s_0} reduces to

$$D_{s_0}\xi = \frac{1}{2}\big(\nabla\xi + \widetilde{J}(s_0)\nabla\xi \circ j\big). \tag{3.2.7}$$

The complex bundle $s_0^*(TP)$ splits as $TS^2 \oplus \nu_{s_0}$ with $\nu_{s_0} = s_0^*(T^{Vert}P)$. A trivialization for ν_{s_0} is given by the path $\{D\phi_t^H\}$, and we are assuming $D\phi_t^H(x_0) \equiv Id$. Furthermore, $\widetilde{J}$ along this section is the constant product $J_0 \times j$, and so the complex bundle $(\nu_{s_0}, \widetilde{J}^{Vert})$ is trivial, and the connection ∇ on this bundle is also trivial. Thus we may split $s_0^*(TP)$ as a sum of complex line bundles $\oplus_0^n = L_i$, with L_0 corresponding to TS^2 and we have $c_1(L_0) = 2$ and $c_1(L_i) = 0$ for $i \neq 0$.

Moreover by equation 3.2.7 D_{s_0} preserves this splitting. This shows that equation 3.2.7 gives the formula for the standard Cauchy-Riemann operator. The vertical portion of D_{s_0} acts on a trivial bundle, and we see that the vertical portion of D_{s_0} is surjective. □

Let $\mathcal{M}_1([s_0], \widetilde{J})$ be the space of equivalence classes $[u, z]$ of simple holomorphic sections in class $[s_0]$ with 1 marked point. Here two holomorphic section maps (u, z) and (u', z') are called equivalent if there is $f \in PSL(2, \mathbb{C})$ so that

$$u' = u \circ f \text{ and } f(z') = z.$$

Identify x_0 with its image over $0 \in D_+$. There is only one $\widetilde{J}$ holomorphic curve in class $[s_0]$ passing through x_0, and all other sections through x_0 have larger energy. Since all stable $\widetilde{J}$ holomorphic maps through x_0 must involve a section, there can be no bubbling.

We have the evaluation map

$$ev : \mathcal{M}_1([s_0], \widetilde{J}) \times S^2 \to P, \text{ by}$$

$$ev([u, z]) = u(z).$$

Given $\mathcal{M}_1(A, J)$, the moduli space of J holomorphic curves $u : S^2 \to M$ representing $A \in H_2(M)$ and a submanifold $X \subset M$, one may consider the space $ev^{-1}(X)$. This is referred to as the "cutdown" moduli space and consists of elements of $\mathcal{M}_1(A, J)$ which send the marked point to X. Referring to this space as $\mathcal{M}_1^{Cut}(A, J, X)$, in order to use such a cutdown moduli space, three conditions must be satisfied:

- $\mathcal{M}^{Cut}(A, J, X)$ must be compact
- Every curve in $\mathcal{M}^{Cut}(A, J, X)$ must be regular
- The differential of the evaluation map must be transverse to X.

We consider the cutdown moduli space given by $ev^{-1}((x_0, 0)) \subset \mathcal{M}_1([s_0], \widetilde{J})$ with $0 \in D_+$. Note that as $\mathcal{M}_1([s_0], \widetilde{J})$ had been quotiented out by $PSL(2, \mathbb{C})$, $\mathcal{M}_1^{Cut}([s_0], \widetilde{J}, (x_0, 0))$ consists of a single map.

The tangent space to $\mathcal{M}([s_0], \widetilde{J})$ can be identified with $kerD_u \subset \Omega^0(S^2, u^*(T^{Vert}P))$, and the differential of the evaluation map at the point (u, w) is given by

$$dev_{u,w}(\xi) = \xi(w).$$

This is surjective at $\mathcal{M}_1^{Cut}([s_0], \widetilde{J}, (x_0, 0))$ if, given any $v \in T_{s_0(w)}^{Vert}P$, there is $\xi \in \Omega^0(S^2, s_0^*(T^{Vert}P))$ satisfying

$$\xi(0) = v, \text{ and } D_{s_0}\xi = 0.$$

But as $s_0^*(T^{Vert}P)$ has been shown to be a trivial holomorphic bundle, we may choose ξ to be a constant section. One can see from equation 3.2.7 that $D_{s_0}(\xi) = 0$ if ξ is constant, and $dev_{s_0,0}$ must then be surjective.

As $dim(\mathcal{M}_1([s_0], \widetilde{J})) = 2n + 2c_1^{Vert}([s_0]) = 2n$, the fact that $\mathcal{M}_1([s_0], \widetilde{J})$ contains only constant sections through F_{max} shows that F_{max} must contain an entire neighborhood of x_0. This of course implies that given any x in the closure of this neighborhood, we must have $D\phi_t^H(x) \equiv Id$ for all time. Thus the set of points in F_{max} satisfying $D\phi_t^H(x) \equiv Id$ for all time is a $2n$ dimensional manifold, and thus must be equal to M. As ϕ_t^H was assumed to be nonconstant, this contradiction completes the proof of Theorem 1.0.1.

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