Generalized isometries in superspace

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Abstract of the Dissertation

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$N = (2, 2)$ supersymmetric models are of interest for mathematicians and physicists and have been used extensively as a tool for the investigation of generalized Kähler geometry. In the sigma-model approach, it is convenient to formulate and manipulate sigma-models in superspace where essential geometric properties are captured by the generalized Kähler potential which gives rise to bihermitian geometry description. Recent developments in differential geometry show that one can also characterize these targets using structures that interpolate between complex and symplectic geometry and are defined on the sum $T \oplus T^*$.

The research work that will be presented here extends the set of known superspace tools for the manipulation of bihermitian / generalized Kähler geometries, namely, the gauging of isometries along directions that mix chiral and twisted chiral or semichiral multiplets.

Other results that will be presented relate to possible $N = (4, 4)$

supersymmetry in semichiral models and sigma models formulation on the sum $T \oplus T^*$.

To my parents

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Chapter 1

Introduction

1.1 Motivation: The string action

String theory [1–3] is a compelling framework for the quantization of gravity on the same footing as all other forces of nature, that is, as a microscopic theory of spin 2 particles. It overcomes the hurdles of nonrenormalizability in four dimensions by promoting point particles propagating along a timeline to strings which are maps from a two-dimensional *worldsheet*, Σ, to D-dimensional target space M^D

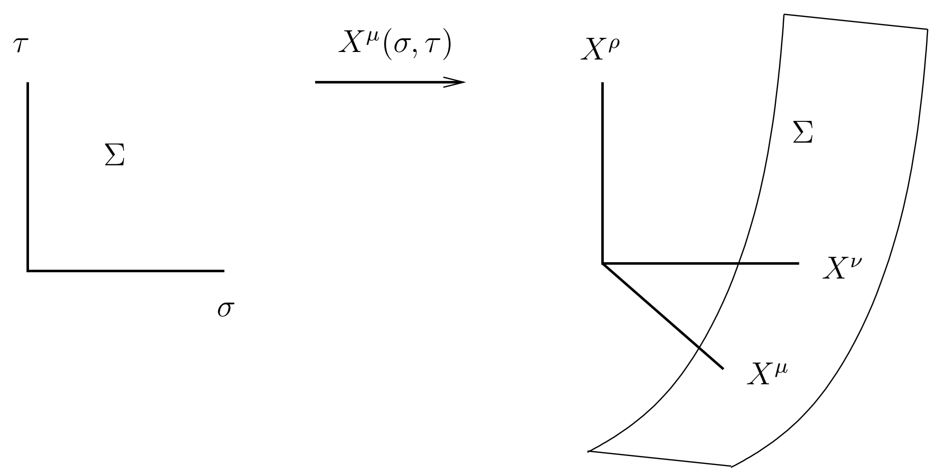

Figure 1.1: The worldsheet embedded in D dimensional target

The string action is a generalization of that of the particle, namely, the

Nambu-Goto action which minimizes this surface in M^D reads

$$S = -T \int_\Sigma d^2\sigma \, \sqrt{(X' \cdot \dot{X})^2 - \dot{X}^2 (X')^2} \, , \tag{1.1}$$

where we introduce the string tension $T = \frac{1}{2\pi\alpha'}$. This action is, classically, equivalent to the Polyakov action where a worldsheet metric $h_{\alpha\beta}$ is introduced

$$S = -\tfrac{T}{2} \int_\Sigma \sqrt{-h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \, , \tag{1.2}$$

.

In the conformal gauge $h = \mathrm{diag}(-1, 1)$ this gives a sigma-model

$$S = -\tfrac{T}{2} \int d^2\sigma \, \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu \, , \tag{1.3}$$

where $\partial = \partial_\sigma + \partial_\tau$.

The two dimensional string is therefore embedded in the D-dimensional target space through *nonlinear sigma-model*; which is a field theory whose fields are the coordinates of a Riemannian manifold.

Strings need not propagate on flat targets only. An interesting class of backgrounds that will be considered here are due to NS-NS sector and admit a metric $G_{\mu\nu}(X)$, an antisymmetric tensor which is a torsion potential $B_{\mu\nu}(X)$ and a dilaton coupling the Ricci scalar which gives an expansion parameter. These modes are matched with the massless spectrum of the closed string. A string propagating in such a background is therefore subject to the action

$$S = -\tfrac{T}{2} \int_\Sigma d^2\sigma \, \sqrt{-h} \big(h^{\alpha\beta} G_{\mu\nu}(X) + \epsilon^{\alpha\beta} B_{\mu\nu}(X) \big) \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' \Phi R^{(2)} \, . \tag{1.4}$$

Nonlinear sigma-models also appear in other branches of physics such as statistical physics where they appear as continuum limits with the for a spin system which preserves the target metric.

1.2 Supersymmetric sigma-models

To obtain a realistic spectrum which allows also fermionic modes one must also include worldsheet fermions ψ^μ [1]. Adding fermions to the Polyakov action we find

$$S = -\tfrac{T}{2}\int d^2\sigma\ \partial^\alpha X^\mu \partial_\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \ . \tag{1.5}$$

This model exhibits a symmetry with fermionic parameter mixing bosons and fermions, known as *supersymmetry* (*e. g.* [17]) that acts as follows

$$\delta X^\mu = \epsilon \psi^\mu \ , \ \delta \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \ , \tag{1.6}$$

and closes on-shell to give translations:

$$[\delta_{\epsilon^1}, \delta_{\epsilon^2}] = 2\bar{\epsilon}_1 \rho^\alpha \epsilon_2 \partial_\alpha \ . \tag{1.7}$$

Since there are two generators for this symmetry of opposite chiralities this symmetry is $N = (1,1)$ supersymmetry.

1.3 $N = (1,1)$ supersymmetry in superspace

A convenient way to write down manifestly $N = (1,1)$ supersymmetric actions is to endow space with anticommuting directions $\theta^\alpha, \ \alpha = \pm$ such that supersymmetry transformations are translations. In two dimensions, one can introduce real spinors

$$C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta} \ , \ C_{+-} = i \ , \ \theta_\alpha = \theta^\beta C_{\beta\alpha} \ , \ \theta^\alpha = C^{\alpha\beta}\theta_\beta \ . \tag{1.8}$$

The supersymmetry transformations $\delta_\epsilon = [-i\epsilon^\alpha Q_\alpha, \cdot]$ shifts the worldsheet coordinates

$$Q_\alpha = i\tfrac{\partial}{\partial\theta^\alpha} + \theta^\beta \partial_{\beta\alpha} \ , \ \delta_\epsilon \sigma^{\ddagger} = -i\epsilon^\pm\theta^\pm \ , \ \delta_\epsilon\theta^\pm = \epsilon^\pm \tag{1.9}$$

[1] This also mends the inconsistency due to tachyonic mode in the bosonic model

so that two consecutive transformations are a translation $Q_\pm^2 = i\partial_{\substack{+\\=}}$. Note that unlike $d = 4$ superspace there are no ∂_{+-} derivatives.

A superfield has a finite expansion in θ

$$\Phi^\mu = X^\mu + \theta^\alpha \psi^\mu_\alpha - \tfrac{i}{2}\theta^\alpha\theta_\alpha F^\mu \tag{1.10}$$

where we use $\theta^\alpha\theta_\alpha = 2i\theta^-\theta^+$. The highest term F^μ is an auxiliary as we shall soon see. Defining the supercovariant derivatives

$$D_\pm = -i(Q_\pm - 2\theta^\pm \partial_{\substack{+\\=}}) \ , \tag{1.11}$$

and $\cdot|$ as the truncation to $\theta = 0$ we write the expansion as

$$\Phi^\mu| = X^\mu \ , \ D_\pm \Phi^\mu| = \psi^\mu_\pm \ , \ D_+ D_- \Phi^\mu| = F^\mu \tag{1.12}$$

A superfield transforms under supersymmetry $\delta_\epsilon \Phi = -i\epsilon Q \Phi$ as

$$\begin{aligned} \delta_\epsilon X^\mu &= \epsilon\psi \\ \delta_\epsilon \psi^\mu_\alpha &= i(\epsilon^\beta \partial_{\beta\alpha} X^\mu - \epsilon_\alpha F^\mu) \\ \delta_\epsilon F^\mu &= \epsilon^\alpha \partial_{\alpha\beta} \psi^\beta \end{aligned} \tag{1.13}$$

which satisfy the supersymmetry algebra, *e. g.*

$$[\delta(\epsilon_1), \delta(\epsilon_2)] X^\mu = \delta(\epsilon_1)(\epsilon_2 \psi^\mu) - (1 \leftrightarrow 2) = -2i\epsilon_1^\alpha \epsilon_2^\beta \partial_{\alpha\beta} X^\mu \tag{1.14}$$

Using Berezin integration we write an action in superspace as

$$S = \int_\Sigma \int d^2\theta \mathcal{L} = \int_\Sigma D_+ D_- \mathcal{L}| \tag{1.15}$$

for example the action $\mathcal{L} = D_+ \Phi^\mu D_- \Phi_\mu$ gives

$$S = \int_\Sigma \partial_{+\!\!+} X^\mu \partial_= X_\mu + i\psi^{\mu\alpha} \partial_{\alpha\beta} \psi^\beta_\mu - F^\mu F_\mu \tag{1.16}$$

which, after elimination of the auxiliary fields F^μ is the string in flat background.

Superpotential terms, that contain no derivatives are also possible. These terms are reduced to components as follows

$$\begin{aligned} S &= \int_\Sigma \int d^2\theta \; D_+\Phi^\mu D_-\Phi_\mu + W(\Phi) \\ &= \int_\Sigma \partial_{\mp}X^\mu \partial_= X_\mu + i\psi^{\mu\alpha}\partial_{\alpha\beta}\psi^\beta_\mu - F^\mu F_\mu + W_\mu F^\mu + W_{\mu\nu}\psi^\mu_+\psi^\nu_- \end{aligned} \tag{1.17}$$

and, as their name suggest, contribute after elimination of the auxiliaries by their equation of motion a potential for the dynamic degrees of freedom

$$S = \int_\Sigma \partial_{\mp}X^\mu \partial_= X_\mu + i\psi^{\mu\alpha}\partial_{\alpha\beta}\psi^\beta_\mu + \tfrac{1}{4}W^\mu(\Phi)W_\mu(\Phi) + W_{\mu\nu}(\Phi)\psi^\mu_+\psi^\nu_- \tag{1.18}$$

Another important extension is the nonlinear sigma-model

$$S = \int_\Sigma \int d^2\theta \; D_+\Phi^\mu (g+b)_{\mu\nu} D_-\Phi^\nu \; . \tag{1.19}$$

The component reduction of this model interact, after elimination of the auxiliaries, with geometric features of the manifold with the metric g and the torsion potential b

$$S = \int_\Sigma \partial_{\mp}X^\mu \partial_= X^\nu (g+b)_{\mu\nu} + ig_{\mu\nu}\psi^\mu_+\nabla^{(+)}_=\psi^\nu_+ + ig_{\mu\nu}\psi^\mu_-\nabla^{(-)}_{\mp}\psi^\nu_- + \tfrac{1}{2}R^+_{\mu\nu\rho\sigma}\psi^\mu_+\psi^\nu_+\psi^\rho_-\psi^\sigma_- \tag{1.20}$$

where the covariant derivatives have torsional connections that are defined when we discuss $N=(2,2)$ supersymmetric backgrounds.

1.4 Complex Geometry

Enhancing nonlinear sigma-models with supersymmetry leads to a profound interplay between physics and geometry; in particular, targets with $N=(2,2)$ worldsheet supersymmetry are described by a pair of complex structures obeying torsionful flatness condition. In this section, we establish the mathematical preliminaries that will soon emerge out of physical constructions. A more rigorous introduction can be find in, *e. g.* [4].

A $2d$-dimensional manifold M covered by the atlas $\{\mathcal{U}_i\}$ is locally similar to $\mathbb{C}^d$ if transition functions on the overlap $\mathcal{U}_i \bigcap \mathcal{U}_j$ are analytic. To formulate these conditions in terms of structures, we define *almost complex structure* as an globally defined endomorphism on the tangent space TM satisfying

$$J^i_j:\ TM \to TM\ ,\ J^2 = -\mathbf{1}\ . \tag{1.21}$$

This is just a generalization of $i = \sqrt{-1}$ so that the projectors

$$P_\pm = \tfrac{1}{2}(\mathbf{1} \pm iJ)\ ,\ P_\pm^2 = P_\pm\ ,\ P_+P_- = P_-P_+ = \mathbf{0}\ . \tag{1.22}$$

project to (anti)holomorphic directions. The consistency of these structures (and their promotion to *complex structures*) is equivalent to the integrability of the almost complex structures

$$(1 \mp iJ)[(1 \pm iJ)X, (1 \pm iJ)Y]\ ,\ \forall X, Y \in TM\ . \tag{1.23}$$

The real part of this expression is the Nijenhuis tensor

$$\begin{aligned} \mathcal{N}(J)^\mu_{\lambda\nu} X^\nu Y^\lambda \partial_\mu &= [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \\ &= \left(J^\sigma{}_{[\lambda} J^\mu{}_{\nu],\sigma} + J^\mu{}_\sigma J^\sigma{}_{[\lambda,\nu]}\right) X^\nu Y^\lambda \partial_\mu\ . \end{aligned} \tag{1.24}$$

A manifold admitting a complex structure, that is, an integrable almost complex structure is called *complex manifold*. Given such a structure, it is possible to consistently split many key geometrical notions into (anti)holomorphic ones. In particular, coordinates are split into (anti)holomorphic directions $(\bar\mu), \mu$ and the differential operator is split $d = \partial + \bar\partial$, resulting in further classification of the de Rahm cohomology

$$H^{(r)} = \bigoplus_{p+q=r} H^{(p,q)}\ , \tag{1.25}$$

and the notion of Dolbeault cohomology which is with respect to the (anti)holomorphic differential.

Given an almost complex structure, one can write an hermitian metric g satisfying $gJ + J^T g = 0$ starting from any metric $\tilde g$

$$g = \tfrac{1}{2}(\tilde g + J^T \tilde g J) \tag{1.26}$$

This metric is a $(1,1)$-symmetric tensor, since for two (anti)holomorphic section the hermiticity condition gives, *e. g.*

$$g_{\mu\nu} = g\left(\tfrac{\partial}{\partial z^\mu}, \tfrac{\partial}{\partial z^\nu}\right) = g\left(J^\rho_\mu \tfrac{\partial}{\partial z^\rho}, J^\sigma_\nu \tfrac{\partial}{\partial z^\sigma}\right) = g\left(i\tfrac{\partial}{\partial z^\mu}, i\tfrac{\partial}{\partial z^\nu}\right) = -g_{\mu\nu} \tag{1.27}$$

using this metric we define the fundamental 2-form ω

$$\omega = g_{\mu\rho} J^\rho_\nu dx^\mu \wedge dx^\nu \in \Omega^{(1,1)} T^* M \ . \tag{1.28}$$

When this form is closed, $d\omega = 0$, the manifold is symplectic and is called a Kähler manifold. This statement is equivalent to covariant constancy of the complex structure due to the symmetry of the metric connection and the covariant constancy of the metric

$$\begin{aligned} d\omega &= \partial_\mu \omega_{\nu\rho}\ dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &= \nabla_\mu \omega_{\nu\rho}\ dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &= g_{\nu\lambda} \nabla_\mu J^\lambda_\rho\ dx^\mu \wedge dx^\nu \wedge dx^\rho \ . \end{aligned} \tag{1.29}$$

Writing the closure of the Kähler form in an explicit manner we have

$$\begin{aligned} -\,id\omega &= (\partial + \bar\partial) g_{\mu\bar\nu} dz^\mu \wedge d\bar z^{\bar\nu} \\ &= g_{\mu\bar\nu,\rho} dz^\rho \wedge dz^\mu \wedge d\bar z^{\bar\nu} + g_{\mu\bar\nu,\bar\rho} d\bar z^{\bar\rho} \wedge dz^\mu \wedge d\bar z^{\bar\nu} \\ &= \tfrac{1}{2}\left(g_{\mu\bar\nu,\rho} - g_{\rho\bar\nu,\mu}\right) dz^\rho \wedge dz^\mu \wedge d\bar z^{\bar\nu} + \tfrac{1}{2}\left(g_{\mu\bar\nu,\bar\rho} - g_{\mu\bar\rho,\bar\nu}\right) d\bar z^{\bar\rho} \wedge dz^\mu \wedge d\bar z^{\bar\nu} \\ &= 0 \end{aligned} \tag{1.30}$$

An immediate consequence of this condition is that locally, on the patch $\mathcal{U}_i$, the metric could be captured by a function $K_i(z, \bar z)$:

$$g_{\mu\bar\nu} = \partial_\mu \bar\partial_{\bar\nu} K_i \ , \tag{1.31}$$

which, in turn implies that the only connection pieces are

$$\Gamma^\mu_{\nu\rho} = g^{\mu\bar\lambda} g_{\nu\bar\lambda,\rho} \ , \ \Gamma^{\bar\mu}_{\bar\nu\bar\rho} = g^{\bar\mu\lambda} g_{\bar\nu\lambda,\bar\rho} \ , \tag{1.32}$$

and the manifold is thus torsion-free and has a $U(N)$ holonomy. A Ricci-flat Kähler manifold is a Calabi-Yau manifold and the holonomy group is further reduced to $SU(N)$.

For future purposes, we generalize this notion of integrability with respect to a single structure to mutual integrability of two structures. Future results will be expressed using the the Magri-Morosi concomitant [50–52] which is defined for two endomorphisms I and J of the tangent bundle TM of a manifold M as

$$\mathcal{M}(I,J)^i_{jk} := -\mathcal{M}(J,I)^i_{kj} = I^l{}_j J^i{}_{k,l} - J^l{}_k I^i{}_{j,l} - I^i{}_l J^l{}_{k,j} + J^i{}_l I^l{}_{j,k}\ . \tag{1.33}$$

This concomitant has previously been used when discussing supersymmetry algebra, *e. g.* in discussing $(1,0)$ and $(1,1)$ formulations of certain (p,q) sigma models in [52] and discussing generalized complex geometry for $(2,2)$ models in [42].

The Magri-Morosi concomitant relates to the simultaneous integrability of two structures and is a tensor only when $[I,J]=0$. More precisely, two commuting complex structures are simultaneously integrable if and only if their Magri-Morosi concomitant vanishes. The part antisymmetric in j,k is the Nijenhuis concomitant $\mathcal{N}(I,J)$; when $I=J$ this becomes the Nijenhuis tensor $\mathcal{N}(I)$. If $\mathcal{N}(I)=0$, then I is integrable.

Assuming that we have one I-connection $\nabla^{(I)}$ and one J-connection $\nabla^{(J)}$ differing only in the sign of the torsion $\Gamma^{(I/J)} = \Gamma^{(0)} \pm T$, we can rewrite $\mathcal{M}$ as

$$\begin{aligned}\mathcal{M}(I,J)^i_{jk} &= I^l{}_j \nabla^{(J)}_l J^i{}_k - J^l{}_k \nabla^{(I)}_l I^i{}_j - I^i{}_l \nabla^{(J)}_j J^l{}_k + J^i{}_l \nabla^{(I)}_k I^l{}_j - [I,J]^i_m \Gamma^{(J)\ m}_{jk} \\ &:= \widehat{\mathcal{M}}(I,J)^i_{jk} - [I,J]^i_m \Gamma^{(J)\ m}_{jk}\ .\end{aligned} \tag{1.34}$$

Finally, we note that in the special case when $I^i{}_j$ and $J^i{}_j$ are curl-free in the lower indices, the concomitant simplifies to

$$\mathcal{M}(I,J)^i_{jk} = (JI)^i_{j,k} - (IJ)^i_{k,j}\ . \tag{1.35}$$

1.5 $N=(2,2)$ supersymmetry

So far, we have written $N=(1,1)$ nonlinear sigma models whose target space was unrestricted and characterized by the metric $g_{\mu\nu}$ and the 2-form $b_{\mu\nu}$. In the following, we study the consequences of extending the supersymmetry to $N=(2,2)$.

Working in $N = (1,1)$ superspace, an Ansatz for the extra supersymmetry reads [20]

$$\delta'(\eta)\phi^\mu = \eta^+ J^\mu_{+\nu} D_+\phi^\nu + \eta^- J^\mu_{-\nu} D_-\phi^\nu \ , \tag{1.36}$$

(note that the $\pm$ indices on the transformation matrices $J^\mu_{\pm\nu}$ are not Lorentz indices.) To have this as an extra supersymmetry, we require both invariance of the model and the supersymmetry algebra to hold.

Requiring that these transformations are a symmetry of the action implies that $J_\pm$ are covariantly constant with the appropriate torsionfull connections and are hermitian with respect to the metric piece; *e. g.* for J_+

$$\begin{aligned}\delta'_+(D_+\Phi^\mu \mathcal{E}_{\mu\nu} D_-\Phi^\nu) &= -\eta^+ D_+\Phi^\mu D_+D_-\Phi^\nu (J^\rho_{+\mu} g_{\rho\nu} + g_{\mu\rho} J^\rho_{+\nu}) \\ &- \eta^+ D_+\Phi^\mu D_+\Phi^\nu D_-\Phi^\rho g_{\mu\sigma}(J^\sigma_{+\nu,\rho} + g^{\sigma\tau} J^\eta_{+\tau}\mathcal{E}_{\eta\rho,\nu} + g^{\sigma\tau}\mathcal{E}_{\tau\rho,\eta} J^\eta_{+\nu})\end{aligned} \tag{1.37}$$

Using the hermiticity condition

$$g_{\mu\rho} J^\rho_{+\nu} + J^\rho_{+\mu} g_{\rho\nu} = 0 \ , \tag{1.38}$$

which is required to eliminate the first we find the second line vanishes if

$$\nabla^{(\pm)}_\rho J^\mu_{\pm\nu} = J^\mu_{\pm\nu,\rho} + \Gamma^{\pm\mu}_{\rho\sigma} J^\sigma_{\pm\nu} - \Gamma^{\pm\sigma}_{\rho\nu} J^\mu_{\pm\sigma} = 0 \tag{1.39}$$

where we use the torsionful (Bismut) connections

$$\Gamma^{\pm\mu}_{\nu\rho} = \Gamma^\mu_{\nu\rho} \pm g^{\mu\sigma} H_{\sigma\nu\rho} \tag{1.40}$$

and the torsion 3-form is the field strength for the Kalb-Ramond 2-form

$$H = db \ , \ H_{\mu\nu\rho} = \tfrac{1}{2}(b_{\mu\nu,\rho} + b_{\nu\rho,\mu} + b_{\rho\mu,\nu}) \tag{1.41}$$

Assuming such invariance of the model, we require that these transformations are indeed supersymmetry transformations. For the same-index commutators we finds

$$[\delta'(\eta_1^\pm), \delta'(\eta_2^\pm)]\Phi^\mu = 2i\eta_1^\pm\eta_2^\pm J^\mu_{\pm\nu} J^\nu_{\pm\rho} \partial_{\substack{+\\=}}\Phi^\rho - \eta_1^\pm\eta_2^\pm \mathcal{N}(J_\pm)^\mu_{\nu\rho} D_\pm\phi^\nu D_\pm\Phi^\rho \tag{1.42}$$

implying that $J_\pm$ are complex structures (that is integrable structures satisfying $J_\pm^2 + 1 = 0$). The opposite sign commutator gives

$$[\delta'_+, \delta'_-]\Phi^\mu = \epsilon^+\epsilon^-[J_+, J_-]^\mu{}_\nu D_+D_-\Phi^\nu + \epsilon^+\epsilon_-\mathcal{M}(J_-, J_+)^\mu_{\rho\nu} D_+\Phi^\nu D_-\Phi^\rho \quad (1.43)$$

Using the covariant flatness condition (1.39) this expression could be simplified:

$$[\delta'_+, \delta'_-]\Phi^\mu = \epsilon^+\epsilon^-[J_+, J_-]^\mu{}_\nu \big(D_+D_-\Phi^\nu + \Gamma^{(-)\nu}_{\nu\rho} D_+\Phi^\nu D_-\Phi^\rho \big) \ , \quad (1.44)$$

which vanishes if the equation of motion is satisfied or when the complex structures commute.

1.5.1 $N = (2,2)$ supersymmetry in superspace

Models quadratic in derivatives can be written in $N = (2,2)$ superspace when the integrand is derivative-free. This makes the model manifestly $N = (2,2)$ supersymmetric (higher supersymmetry in superspace requires projective / harmonic superspace formulations (*e. g.* [60] and [61] respectively for review.) which are not as straightforward). We shall follow closely the notation of [18].

Extending $N = (1,1)$ superspace with two extra Grassmann directions $(\sigma^{\pm\!\!\pm}, \theta^\pm, \bar\theta^\pm)$ we define the supercovariant derivatives

$$\mathbb{D}_\pm = \partial_\pm + \tfrac{i}{2}\bar\theta^\pm \partial_{\pm\!\!\pm} \ , \ \bar{\mathbb{D}}_\pm = \bar\partial_\pm + \tfrac{i}{2}\theta^\pm \partial_{\pm\!\!\pm} \quad (1.45)$$

satisfying

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_{\pm\!\!\pm} \quad (1.46)$$

with all other commutators vanishing.

In a similar manner we define the supersymmetry generators

$$\mathbb{Q}_\pm = i\mathbb{D}_\pm + \theta^\pm \partial_{\pm\!\!\pm} \ , \ \bar{\mathbb{Q}}_\pm = i\bar{\mathbb{D}}_\pm + \bar\theta^\pm \partial_{\pm\!\!\pm} \quad (1.47)$$

such that all $\mathbb{Q}$ and $\mathbb{D}$ anticommutators vanish. Supersymmetry transformation on a superfield reads

$$\delta\Phi = i(\epsilon^\alpha \mathbb{Q}_\alpha + \bar\epsilon^\alpha \bar{\mathbb{Q}}_\alpha)\Phi \ . \quad (1.48)$$

Since the target geometry is obvious in $N = (1,1)$ superspace, we introduce a standard convention for this reduction, where

$$D_\alpha = \bar{\mathbb{D}}_\alpha + \mathbb{D}_\alpha \ , \ Q_\alpha = i(\mathbb{D}_\alpha - \bar{\mathbb{D}}_\alpha) \ . \tag{1.49}$$

A superspace measure using this convention reads

$$\int d^2\theta d^2\bar{\theta}\mathcal{L} = \mathbb{D}_+\mathbb{D}_-\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathcal{L}\big| = -\tfrac{1}{4}D_+D_-Q_+Q_-\mathcal{L}\big| \ . \tag{1.50}$$

Having described the basic features of $N = (2,2)$ superspace, we next turn to describe irreducible representation of $N = (2,2)$ supersymmetry and $N = (2,2)$ supersymmetric actions.

1.5.2 Constrained Superfields

Unlike $N = (1,1)$ superspace, in $N = (2,2)$ superspace one can find constraints consistent with the algebra (1.45). Historically [20], these constraints were obtained by analyzing properties of gauge multiplets, and we indeed follow this line of thought later on. We summarize these constraints in the following table which also defines conventions to be used throughout this manuscript

Table 1.1: Constrained $N = (2,2)$ superfields

Superfield	Constraint	Superfield	Constraint
Chiral	$\bar{\mathbb{D}}_\pm\phi = 0$	Left semichiral	$\bar{\mathbb{D}}_+\mathbb{X}_L = 0$
Antichiral	$\mathbb{D}_\pm\bar{\phi} = 0$	Anti-Left semichiral	$\mathbb{D}_+\bar{\mathbb{X}}_L = 0$
Twistedchiral	$\bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = 0$	Right semichiral	$\bar{\mathbb{D}}_-\mathbb{X}_R = 0$
Anti twistedchiral	$\mathbb{D}_+\bar{\chi} = \bar{\mathbb{D}}_-\bar{\chi} = 0$	Anti-Right semichiral	$\mathbb{D}_-\bar{\mathbb{X}}_R = 0$

Such superfields have simple expansion in superspace, *e. g.*

$$\phi| = X \ , \ \mathbb{D}_\pm\phi| = \psi_\pm \ , \ \mathbb{D}_+\mathbb{D}_-\phi| = F \ . \tag{1.51}$$

This defines two complex structures on the target through the reduction (1.49), namely,

$$Q_\alpha\Phi^\mu = J^\mu_{\alpha\nu}D_\alpha\Phi^\nu \ . \tag{1.52}$$

For (anti)(twisted)chiral superfields (satisfying two constraints) it is easy to see that the complex structures are diagonal

$$Q_{\pm}(\phi, \bar{\phi}, \chi, \bar{\chi}) = iD_{\pm}(\phi, -\bar{\phi}, \pm\chi, \pm\bar{\chi}) \tag{1.53}$$

and thus are an off-shell representation, (since $[J_+, J_-] = 0$ is equivalent, through (1.44), to $[\delta_+, \delta_-] = 0$). Decomposing

$$\mathrm{Ker}[J_+, J_-] = \mathrm{Ker}(J_+ + J_-) \oplus \mathrm{Ker}(J_+ - J_-) \tag{1.54}$$

we identify that these superfields parametrize the first two kernels, namely

$$(J_+ + J_-)\chi = (J_+ - J_-)\phi = 0 \ . \tag{1.55}$$

Semichiral superfields [19] satisfy less constraints:

$$Q_+\mathbb{X}_L = iD_+\mathbb{X}_L \ , \ Q_-\mathbb{X}_R = iD_-\mathbb{X}_R \ . \tag{1.56}$$

That is, left(right)-semichirals diagonalize $J_+(J_-)$. In the next section, we show that these superfields chart sectors where the complex structures do not commute $[J_+, J_-] \neq 0$. These superfields thus contain auxiliaries, *e. g.*

$$\begin{aligned} \mathbb{X}_L| &\equiv X_L \quad , \quad Q_-\mathbb{X}_L \equiv \Psi_{L-} \ , \\ \mathbb{X}_R| &\equiv X_R \quad , \quad Q_+\mathbb{X}_R \equiv \Psi_{R+} \ . \end{aligned} \tag{1.57}$$

The supersymmetry algebra implies that

$$\begin{aligned} Q_+\Psi_{L-} &= iD_+\Psi_{L-} \quad , \quad Q_-\Psi_{L-} = i\partial_{=}X_L \\ Q_-\Psi_{R+} &= iD_-\Psi_{R+} \quad , \quad Q_+\Psi_{R+} = i\partial_{+\!\!+}X_R \end{aligned} \tag{1.58}$$

Which allow the closure of the algebra off-shell. In [18] it was shown that these multiplets are sufficient to chart all bihermitian manifolds.

1.5.3 Matter Actions

Lagrange densities for nonlinear sigma-models in $N=(2,2)$ superspace are just real functions, the *generalized Kähler potential*:

$$S=\int_\Sigma\int d^4\theta\ K(\phi^\alpha,\bar\phi^{\bar\alpha},\chi^{\alpha'},\bar\chi^{\bar\alpha'},\mathbb{X}^a_L,\bar{\mathbb{X}}^{\bar a}_L,\mathbb{X}^{a'}_R,\bar{\mathbb{X}}^{\bar a'}_R)\ . \tag{1.59}$$

Superpotential terms are also possible and are consistent with chirality properties and reality; that is

$$S_{sp}=\int_\Sigma\int \mathbb{D}_+\mathbb{D}_-\ W(\phi)+\int_\Sigma\int \mathbb{D}_+\bar{\mathbb{D}}_-\ \tilde W(\chi)+c.c\ . \tag{1.60}$$

but are not relevant for our investigation of the sigma-model.

In the following, we describe the target space geometry encoded by the generalized Kähler potential. As mentioned, this is most conveniently done in $N=(1,1)$ superspace, where the metric and b-field are present explicitly.

Kähler Submanifolds

To explain the etymology, let us first look at a potential with chiral superfield:

$$K=K(\phi^a,\bar\phi^{\bar a})\ . \tag{1.61}$$

The reduction of such a $N=(2,2)$ model to $N=(1,1)$ superspace is achieved by pushing the Q-derivatives through. Defining a collective notation $\varphi^i=(\phi^a,\bar\phi^{\bar a})$ and noting that for chirals $J_+=J_-\equiv J$ we find

$$Q_+Q_-K=Q_+(K_iJ^i{}_jD_-\varphi^j)=K_{ij}(\delta^i{}_k\delta^j{}_l+J^i{}_kJ^j{}_l)D_+\varphi^kD_-\varphi^l\ . \tag{1.62}$$

That is, the Hermitian piece of the Hessian K_{ij} is the metric for the lowest bosonic components, and K is thus the Kähler potential for that manifold.

Similar consideration applies for a potential depending on twistedchiral superfields, where there is an additional overall sign since for such model $J_+=-J_-\equiv J$ so that in the collective notation $\tilde\varphi^i=(\chi^{a'},\bar\chi^{\bar a'})$ one finds

$$Q_+Q_-K=-Q_+(K_iJ^i{}_jD_-\tilde\varphi^j)=-K_{ij}(\delta^i{}_k\delta^j{}_l+J^i{}_jJ^j{}_l)D_+\tilde\varphi^kD_-\tilde\varphi^l\ . \tag{1.63}$$

1.5.4 Gates-Hull-Roček, Take One $[J_+, J_-] = 0$

A first example for non-Kähler geometry was worked out in a seminal paper by Gates, Hull and Roček [20]. After introducing twistedchiral superfields, one immediately finds that for a model containing both types one may find nonvanishing b-field. Working out explicitly the reduction to $N = (1,1)$ superspace with the collective notation $\varphi^i = (\phi^a, \bar\phi^{\bar a}, \chi^{a'}, \bar\chi^{\bar a'})$ we find

$$Q_+Q_-K(\phi^a, \bar\phi^{\bar a}, \chi^{a'}, \bar\chi^{\bar a'}) = K_{ij}(J^i_{+k}J^j_{-l} - \Pi^i{}_k\delta^j{}_l)D_+\varphi^k D_-\varphi^l \tag{1.64}$$

where $\Pi = J_+J_- = J_-J_+$ is the *local product*. This gives rise to b-field when twistedchirals are coupled to chirals, namely

$$K_{ij}(J^i_{+k}J^j_{-l} - \Pi^i{}_k\delta^j{}_l) = 2\begin{pmatrix} 0 & K_{a\bar a} & K_{aa'} & 0 \\ K_{a\bar a} & 0 & 0 & K_{\bar a\bar a'} \\ -K_{a\bar a} & 0 & 0 & -K_{a'\bar a'} \\ 0 & -K_{\bar a\bar a'} & -K_{a'\bar a'} & 0 \end{pmatrix} . \tag{1.65}$$

1.5.5 Gates-Hull-Roček, Take two: $[J_+, J_-] \neq 0$

To chart sectors where $[J_+, J_-] \neq 0$, it is required to use semichiral multiplets. In the reduction of generalized Kähler potential in semichirals to $N = (1,1)$ one finds auxiliary superfields, which are to be eliminated by their equation of motion.

Starting from the generalized Kähler potential

$$Q_+Q_-K(\mathbb{X}^A_L, \bar{\mathbb{X}}^{\bar A}_L, \mathbb{X}^{A'}_R, \bar{\mathbb{X}}^{\bar A'}_R) , \tag{1.66}$$

we use the indices $L = (A, \bar A)$, $R = (A', \bar A')$ and the canonical complex structure $J = \mathrm{diag}(i, -i)$ to find

$$\begin{aligned} Q_+Q_-K &= Q_+(D_-X^R JK_R + K_L\Psi^L_-) \\ &= -D_-\Psi^R_+ JK_R + \Psi^R_+ K_{RR} JD_-X^R + D_+X^L JK_{LR} JD_-X^R \\ &\quad +D_+X^L JK_{LL}\Psi^L_- + \Psi^R_+ K_{RL}\Psi^L_- + K_L JD_+\Psi^L_- \end{aligned} \tag{1.67}$$

Introducing the commutators $C_{LL} = [J, K_{LL}]$ and C_{RR} this expression simplifies, after integration by parts and extraction of the equations of motion for the auxiliaries to

$$
\begin{aligned}
= \; & \left(\Psi_+^R + (D_+X^L C_{LL} - D_+X^R K_{RL} J) K^{LR}\right) \\
& \quad K_{RL}\left(\Psi^L - K^{LR}(C_{RR} D_- X^R + J K_{RL} D_- X^L)\right) \\
& + D_+X^L (J K_{LR} J + C_{LL} K^{LR} C_{RR}) D_- X^R - D_+ X^R K_{RL} J K^{LR} J K_{RL} D_- X^L \\
& - D_+ X^R K_{RL} J K^{LR} C_{RR} D_- X^R + D_+ X^L C_{LL} K^{LR} J K_{RL} D_- X^L \qquad (1.68)
\end{aligned}
$$

where $K_{RL}K^{LR} = \delta^R{}_R$ and thus K_{LR} is assumed to be invertible. Complex structures are found by putting the auxiliaries on-shell, and are, indeed, non-commuting:

$$
J_+ = \begin{pmatrix} J & 0 \\ K^{RL} C_{LL} & K^{RL} J K_{LR} \end{pmatrix} \; , \; J_- = \begin{pmatrix} K^{LR} J K_{RL} & K^{LR} C_{RR} \\ 0 & J \end{pmatrix} . \qquad (1.69)
$$

Torsion arises quite naturally in these models; for example, starting from the quadratic generalized Kähler potential $\mathbb{X}_L \mathbb{X}_R + c.c$ we find the pure (though trivial) b-field

$$
D_+ \left(X_L, \bar{X}_L, X_R, \bar{X}_R\right) \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} D_- \begin{pmatrix} X_L \\ \bar{X}_L \\ X_R \\ \bar{X}_R \end{pmatrix} . \qquad (1.70)
$$

With all multiplets present, we add the (c)hiral and the (t)wistedchiral sectors to $g + b$ and find [18]

$$
\begin{aligned}
E_{LL} &= C_{LL} K^{LR} J_s K_{RL} \\
E_{LR} &= J_s K_{LR} J_s + C_{LL} K^{LR} C_{RR} \\
E_{Lc} &= K_{Lc} + J_s K_{Lc} J_c + C_{LL} K^{LR} C_{Rc} \\
E_{Lt} &= -K_{Lt} - J_s K_{Lt} J_t + C_{LL} K^{LR} A_{Rt} \\
E_{RL} &= -K_{RL} J_s K^{LR} J_s K_{RL} \\
E_{RR} &= -K_{RL} J_s K^{LR} C_{RR} \\
E_{Rc} &= K_{Rc} - K_{RL} J_s K^{LR} C_{Rc} \\
E_{Rt} &= -K_{Rt} - K_{RL} J_s K^{LR} A_{Rt} \\
E_{cL} &= C_{cL} K^{LR} J_s K_{RL} \\
E_{cR} &= J_c K_{cR} J_s + C_{cL} K^{LR} C_{RR}
\end{aligned}
$$

$$
\begin{aligned}
E_{cc} &= K_{cc} + J_c K_{cc} J_c + C_{cL} K^{LR} C_{Rc} \\
E_{ct} &= -K_{ct} - J_c K_{ct} J_t + C_{cL} K^{LR} A_{Rt} \\
E_{tL} &= C_{tL} K^{LR} J_s K_{RL} \\
E_{tR} &= J_t K_{tR} J_s + C_{tL} K^{LR} C_{RR} \\
E_{tc} &= K_{tc} + J_t K_{tc} J_c + C_{tL} K^{LR} C_{Rc} \\
E_{tt} &= -K_{tt} - J_t K_{tt} J_t + C_{tL} K^{LR} A_{Rt}
\end{aligned} \tag{1.71}
$$

where $A = \{K, J\}$ and the commutators C_{XY} generalize as expected. The complex structures for this model reads [18]

$$
J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ K^{RL} C_{LL} & K^{RL} J_s K_{LR} & K^{RL} C_{Lc} & K^{RL} C_{Lt} \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} \tag{1.72}
$$

and

$$
J_- = \begin{pmatrix} K^{LR} J_s K_{RL} & K^{LR} C_{RR} & K^{LR} C_{Rc} & K^{LR} A_{Rt} \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix} \tag{1.73}
$$

1.6 Gauging Isometries

1.6.1 Bosonic model

Isometries occur when one can identify a transformation leaving the geometric data invariant; that is, given a coordinate transformation

$$
\delta\phi^i = \lambda k^i \text{ such that } g_{ij}(\phi') = g_{ij}(\phi) \ , \tag{1.74}
$$

which implies

$$
k^k_{,i} g_{kj} + k^k_{,j} g_{ik} + k^k g_{ij,k} = \mathcal{L}_k g = \nabla_{(i} k_{j)} = 0 \ . \tag{1.75}
$$

In the presence of torsion, this requirement generalizes to an exact shift of the torsion potential

$$\mathcal{L}_k b = d\alpha \ , \ \mathcal{L}_k H = 0 \ . \tag{1.76}$$

The isometry could be gauged, that is, the model is invariant after promoting the rigid parameter $\lambda \to \lambda(\phi)$. This requires the covariantization of the derivatives

$$\partial_\alpha \phi^i \mathcal{E}_{ij} \partial^\alpha \phi^j \to (\partial_\alpha \phi^i + A_\alpha k^i) \mathcal{E}_{ij} (\partial^\alpha \phi^j + A^\alpha k^j) \tag{1.77}$$

with a connection transforming as $\delta A_\alpha = -\partial_\alpha \lambda$. This discussion is generalized to the nonabelian case when the parameter λ and the connection A are group-valued.

In the reminder of this section we shall first gauge isometries for Kähler sumbanifolds in $N = (2,2)$ superspace and introduce the appropriate gauge multiplets and their field strengths before going to $N = (1,1)$ and components.

1.6.2 Gauging in $N = (2,2)$ superspace

As in many cases, manipulations in $N = (2,2)$ superspace are simpler than those performed at the sigma model level. Derivative terms arise from the generalized Kähler potential and we therefore need to add degrees of freedom, the gauge multiplet, that mend the transformations of those terms. We now work out (non)abelian examples and discuss $N = (2,2)$ gauge multiplets [23] .

A Kähler submanifold parametrized, *e. g.* by the lowest components of (anti)chiral multiplets with indices separated in accordance (so that (anti) holomorphicity translates to (anti)chirality). Starting from a potential $K(\Phi^a, \bar{\Phi}^{\bar{a}})$, An isometry is manifest if there is a vector (with separated (anti)holomorphic components)

$$k^i \partial_i = k^a \partial_a + \bar{k}^{\bar{a}} \partial_{\bar{a}} \ , \tag{1.78}$$

which leaves the action is invariant, that is

$$\lambda \mathcal{L}_k K = \lambda (k^a \partial_a + \bar{k}^{\bar{a}} \partial_{\bar{a}}) K = f(\Phi) + \bar{f}(\bar{\Phi}) \ . \tag{1.79}$$

Gauging the isometry in a holomorphic manner means promoting λ to (anti) holomorphic components and, in the case $f(\Phi) = 0$ requiring

$$(\Lambda k^a \partial_a + \bar{\Lambda} \bar{k}^{\bar{a}} \partial_{\bar{a}}) K^{(g)} = 0 \ . \tag{1.80}$$

Using the rigid isometry we write

$$(\Lambda k^a \partial_a + \bar{\Lambda} \bar{k}^{\bar{a}} \partial_{\bar{a}}) K = \tfrac{i}{2}(\bar{\Lambda} - \Lambda) \mathcal{L}_{Jk} K \ , \ Jk = i(k^a \partial_a - \bar{k}^{\bar{a}} \partial_{\bar{a}}) \ . \tag{1.81}$$

Thus, adding a gauge field transforming as $\delta V = i(\bar{\Lambda} - \Lambda)$ the gauged potential

$$K^{(g)}(\Phi^a, \bar{\Phi}^{\bar{a}}, V) = \exp\big(- \tfrac{1}{2} V \mathcal{L}_{Jk} \big) K(\Phi^a, \bar{\Phi}^{\bar{a}}) \tag{1.82}$$

satisfies (1.80).

Abelian

A Kähler potential with an obvious abelian isometry is independent of one of the direction, *e. g.* for a Kähler potential in (anti)chirals we write

$$K(\phi, \bar{\phi}) = K(\phi + \bar{\phi}) \ , \ (\partial - \bar{\partial}) K = 0. \tag{1.83}$$

This potential is invariant under $\delta\phi = i\lambda$.

Gauging this symmetry means promoting λ to (anti)chiral superfields

$$\delta\phi = i\Lambda \ , \ \bar{\mathbb{D}}_{\pm}\Lambda = 0 \ ; \ \delta\bar{\phi} = -i\bar{\Lambda} \ , \ \mathbb{D}_{\pm}\bar{\Lambda} = 0 \ . \tag{1.84}$$

To ensure the invariance of the Kähler potential, we introduce a gauge multiplet such that $\delta V = i(\bar{\Lambda} - \Lambda)$ and therefore

$$\delta_{\Lambda, \bar{\Lambda}} K(\Phi + \bar{\Phi} + V) = 0 \ . \tag{1.85}$$

Writing gauge invariant field-strengths is immediate given the transformation properties of the gauge multiplet V. As no single derivative can knock out both Λ and $\bar{\Lambda}$, we write the Lorentz-neutral field-strengths

$$\tilde{F} = \bar{\mathbb{D}}_{+} \mathbb{D}_{-} V \ , \ \bar{\tilde{F}} = \mathbb{D}_{+} \bar{\mathbb{D}}_{-} V \ . \tag{1.86}$$

These field-strengths are thus (anti)twistedchiral; historically, this is how twistedchiral constraints were identified.

An identical discussion follows for isometries in a Kähler manifold parametrized by the lowest components of (anti)twistedchiral superfields.

$$K^{(g)} = K(\bar{\chi} + \chi + \tilde{V}) \ , \ \delta\chi = \tilde{\Lambda} \ , \ \delta\bar{\chi} = \bar{\tilde{\Lambda}} \ , \ \delta\tilde{V} = i(\bar{\tilde{\Lambda}} - \tilde{\Lambda}) \ . \tag{1.87}$$

The field-strengths for the semichiral gauge multiplet are thus (anti)chirals

$$F = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{V} \ , \ \bar{F} = \mathbb{D}_+\mathbb{D}_-\tilde{V} \ . \tag{1.88}$$

Nonabelian

Starting from a system with nonabelian gauge invariance; *e. g.* an $U(N)$ invariant model with fundamental matter $K(\bar{\Phi}^i\Phi_i)$, $i = 1, \cdots, N$ is invariant under rigid transformations

$$g(\Phi)_i = \exp(i\lambda_a T^a)_i{}^j\Phi_j \ , \ g(\bar{\Phi})^i = \bar{\Phi}^j\exp(-i\lambda_a T^a)_j{}^i \ . \tag{1.89}$$

with T^a are the $U(N)$ generators. Gauging is achieved using algebra valued gauge multiplet $K^{(g)} = K(\bar{\Phi}e^V\Phi)$ with the transformation properties (with all indices suppressed)

$$g(\Phi) = e^{i\Lambda}\Phi \ , \ g(\bar{\Phi}) = \bar{\Phi}^j e^{-i\bar{\Lambda}} \ , \ g(e^V) = e^{i\bar{\Lambda}}e^V e^{-i\Lambda} \ . \tag{1.90}$$

In a chiral representation the bared derivatives $\bar{\mathbb{D}}_\pm$ are covariant $\bar{\nabla}_\pm = \bar{\mathbb{D}}_\pm$ given the chirality of the parameter Λ

$$g(\bar{\nabla}_\pm\Phi) = g(\bar{\mathbb{D}}_\pm\Phi) = \bar{\mathbb{D}}_\pm e^{i\Lambda}\Phi = e^{i\Lambda}\bar{\mathbb{D}}_\pm\Phi = e^{i\Lambda}\bar{\nabla}\Phi \ , \tag{1.91}$$

whereas $\mathbb{D}_\pm$ are not. The latter are covariantized using the gauge multiplet $\nabla_\pm = e^{-V}\mathbb{D}_\pm e^V$:

$$g(\nabla_\pm\Phi) = g(e^{-V}\mathbb{D}_\pm e^V\Phi) = e^{i\Lambda}e^{-V}e^{-i\bar{\Lambda}}\mathbb{D}_\pm e^{i\bar{\Lambda}}e^V e^{-i\Lambda}e^{i\Lambda}\Phi = e^{i\Lambda}\nabla_\pm\Phi \ . \tag{1.92}$$

The field-strengths are (anti)twistedchiral and are obtained as anticommutators of covariant derivatives

$$\tilde{F} = i\{\bar{\nabla}_+, \nabla_-\} \ , \ \bar{\tilde{F}} = -i\{\bar{\nabla}_-, \nabla_+\} \ . \tag{1.93}$$

These field-strengths are covariant, *e. g.* $g(\tilde{F}) = e^{i\Lambda}\tilde{F}e^{-i\Lambda}$.

Likewise, the twistedchiral gauge multiplet enters as

$$K^{(g)} = K(\bar{\chi}e^{\tilde{V}}\chi) \ , \ g(e^{\tilde{V}}) = e^{i\bar{\tilde{\Lambda}}}e^{\tilde{V}}e^{-i\tilde{\Lambda}} \ . \tag{1.94}$$

In the twistedchiral representation, the covariantized derivatives read

$$\bar{\nabla}_+ = \bar{\mathbb{D}}_+ \ , \ \nabla_+ = e^{-\tilde{V}}\mathbb{D}_+e^{\tilde{V}} \ , \ \nabla_- = \mathbb{D}_- \ , \ \bar{\nabla}_- = e^{-\tilde{V}}\bar{\mathbb{D}}_-e^{\tilde{V}} \ , \tag{1.95}$$

and the (anti)chiral field-strengths are

$$F = i\{\bar{\nabla}_+, \bar{\nabla}_-\} \ , \ \bar{F} = -i\{\nabla_+, \nabla_-\} \ . \tag{1.96}$$

Actions in $N = (2, 2)$ superspace are extended to include the field strengths in the Kähler potential and (for the chiral gauge multiplet) twistedchiral superpotential terms (called D-terms for reasons that will be obvious as we go to components)

$$S = \int_\Sigma \left\{ \int d^4\theta K(\tilde{F}, \bar{\tilde{F}}, \phi, \bar{\phi}) + \int d^2\theta W(\phi) + \int d^2\tilde{\theta}\tilde{W}(\tilde{F}) + c.c \right\} \ . \tag{1.97}$$

1.6.3 $N = (1, 1)$ superspace and components

The gauge multiplet is unconstrained and is thus has a full 16 components expansion. Given its transformation properties it is convenient to gauge away many, removing a chiral and an antichiral multiplets from the expansion. To discuss gauge-invariant only degrees of freedom, it is convenient to go to $N = (1, 1)$ superspace in the Wess-Zumino gauge

$$V|_{\bar{\theta}} = 0 \ , \ Q_\pm V|_{\bar{\theta}} = 2A_\pm \ , \ Q_+Q_-V|_{\bar{\theta}} = 2d \ . \tag{1.98}$$

Using the definitions of field-strengths $\tilde{F}$ and $\bar{\tilde{F}}$ and identifying the field-strength for the $N = (1, 1)$ multiplet containing the connections $f = D_+A_- + D_-A_+$.

we find:

$$\tilde{F}| = \tfrac{1}{4}(D_+ + iQ_+)(D_- - iQ_-)V| = \tfrac{1}{2}(d - if)$$
$$\bar{\tilde{F}}| = \tfrac{1}{4}(D_+ - iQ_+)(D_- + iQ_-)V| = \tfrac{1}{2}(d + if) \ . \tag{1.99}$$

The reduction to components thus agrees with the number of degrees of freedom expected for V after gauge fixing where d is an $N = (1,1)$ scalar multiplet and f contains the gauge fields.

$$d| = \sigma \ , \ D_\pm d| = \psi_\pm \ , \ D_+ D_- d| = d$$
$$f| = \sigma' \ , \ D_\pm f| = \lambda_\pm \ , \ D_+ D_- f| = \partial_= A_{+\!\!+} - \partial_{+\!\!+} A_= = f \ . \tag{1.100}$$

Reducing the model with gauged abelian (anti)chiral isometry to $N = (1,1)$ superspace we find

$$Q_+ Q_- K(\bar{\phi}^i + \phi^i + V^i) = \mathcal{D}_+ \phi^i \mathcal{E}_{ij} \mathcal{D}_- \phi^j + 2d^i K_i \ , \tag{1.101}$$

where the $N = (1,1)$ covariantized superderivatives acts as

$$\mathcal{D}_\pm \phi^i = D_\pm \phi^i - iA^i_\pm \ , \ \mathcal{D}_\pm \bar{\phi}^i = D_\pm \bar{\phi}^i + iA^i_\pm \ . \tag{1.102}$$

and the last term is a moment map that are important when when gauged models are dualized and quotiented

In the nonabelian case, the covariantization of derivatives is achieved by covariantizing the $N = (1,1)$ derivatives only:

$$\bar{\nabla}_\pm = \tfrac{1}{2}(\mathcal{D}_\pm + iQ_\pm) \tag{1.103}$$

since the $N = (2,2)$ derivatives still anticommute $\{Q_+, Q_-\} = 0$, we decompose the field-strengths

$$\tilde{F} = \tfrac{1}{4}\left(\{\mathcal{D}_+, Q_-\} + \{\mathcal{D}_-, Q_+\}\right) + \tfrac{i}{4}\{\mathcal{D}_+, \mathcal{D}_-\}$$
$$\bar{\tilde{F}} = \tfrac{1}{4}\left(\{\mathcal{D}_+, Q_-\} + \{\mathcal{D}_-, Q_+\}\right) - \tfrac{i}{4}\{\mathcal{D}_+, \mathcal{D}_-\} \tag{1.104}$$

so that the field strength (the imaginary piece) is a commutator of $N = (1,1)$ covariant derivatives as expected.

When taken to components, gauge actions are consisted of a usual $N = (1, 1)$ scalar action and the gauge action

$$
\begin{aligned}
S &= -\int_\Sigma \int d^4\theta \; \tilde{F}\bar{\tilde{F}} \\
&= \int_\Sigma \left(\partial_{\#}\sigma\partial_{=}\sigma + i\partial_{=}\psi_+\psi_+ + i\partial_{\#}\psi_-\psi_- - D^2\right) \\
&\quad + \left(\partial_{\#}\sigma'\partial_{=}\sigma' + i\partial_{=}\lambda_+\lambda_+ + i\partial_{\#}\lambda_-\lambda_- - (\partial_{\#}A_{=} - \partial_{=}A_{\#})^2\right) \ .
\end{aligned}
\tag{1.105}
$$

Superpotential terms are also called D-terms since they contribute a potential for σ and $\psi_\pm$ through the auxiliary D:

$$
\int d^2\tilde{\theta} \; \tilde{W}(\tilde{F}) + c.c. = 2W'(\sigma)D + 2W''(\sigma)\psi_+\psi_- \ . \tag{1.106}
$$

1.6.4 T-duality and Buscher rules

Dualities are a key feature of string theory [1–3] and relate spectra and target geometries of seemingly different theories. One celebrated duality is T(oroidal)-duality [16, 24] relating closed strings propagating in a target with one direction compactified on a circle of radius R.

The compactified direction $X^c(\sigma, \tau)$ must satisfy

$$
X^c(\tau, \sigma + \pi) = X^c(\tau, \sigma) + 2\pi R\omega \tag{1.107}
$$

where ω is the winding number. Separating modes to Left (Right) movers we therefore find the contribution to the energy from momenta in the direction X^c in appropriate units

$$
M^2 = \left(\frac{n}{R}\right)^2 + (\omega R)^2 + \{\text{other terms}\} \tag{1.108}
$$

where n is the mode excited in the X^c direction excitation. It is therefore clear that this spectrum matches that of a string propagating in a background compactified on a circle with inverse radius $R \to 1/R$ where winding / momentum modes exchanged. Writing the momenta modes as

$$
\tfrac{1}{2}(p^c_L + p^c_R) = \tfrac{n}{R} \ , \ \tfrac{1}{2}(p^c_L - p^c_R) = \omega R \tag{1.109}
$$

we find that the duality takes $X_R \to -X_R$.

An alternative derivation of this this duality is achieved through a mother action [16, 24] that gives the two models when certain degrees of freedom are removed using their equation of motion. After gauging an isometry, the original model could be restored by constraining the field-strength to vanish. In $N=(1,1)$ superspace, we write a nonlinear sigma-model with a single gauged isometry along the 0-direction and such a constraint

$$\begin{aligned} S \;=\; \int_\Sigma \int d^2\theta \; & g_{00} A_+ A_- + A_+ \mathcal{E}_{0j} D_- \Phi^j + D_+ \Phi^i \mathcal{E}_{i0} A_- + D_+ \Phi^i \mathcal{E}_{ij} D_- \Phi^j \\ & + \Theta (D_+ A_- + D_- A_+) \;, \end{aligned} \tag{1.110}$$

where the indices were spit $\mu = (0, i)$. On a topologically trivial target, eliminating the multiplier Θ by its equation of motion restores the original model as it implies $A_\pm = D_\pm \Phi^0$.

This construction, however, allows another manipulation; namely, the elimination of the fields $A_\pm$ rather than Θ. Substituting the equations of motion

$$\begin{aligned} 0 &= A_+ + \tfrac{\mathcal{E}_{i0}}{g_{00}} D_+ \Phi^i - \tfrac{1}{g_{00}} D_+ \Theta \\ 0 &= A_- + \tfrac{\mathcal{E}_{0j}}{g_{00}} D_- \Phi^j + \tfrac{1}{g_{00}} D_- \Theta \;, \end{aligned} \tag{1.111}$$

we find a *dual* model in the coordinates $\tilde{\Phi} = (\Theta, \Phi^i)$

$$\tilde{S} = \int_\Sigma \int d^2\theta \; D_+ \tilde{\Phi}^\mu \tilde{\mathcal{E}}_{\mu\nu} D_- \tilde{\Phi}^\nu \tag{1.112}$$

with the dual geometry

$$\tilde{\mathcal{E}}_{00} = \frac{1}{g_{00}} \;,\; \tilde{\mathcal{E}}_{0j} = \frac{\mathcal{E}_{0j}}{g_{00}} \;,\; \tilde{\mathcal{E}}_{i0} = -\frac{\mathcal{E}_{i0}}{g_{00}} \;,\; \tilde{\mathcal{E}}_{ij} = \mathcal{E}_{ij} - \frac{\mathcal{E}_{i0}\mathcal{E}_{0j}}{g_{00}} \;, \tag{1.113}$$

or more explicitly,

$$\begin{aligned} \tilde{g}_{00} &= \frac{1}{g_{00}} \;,\; \tilde{g}_{0j} = \frac{b_{0j}}{g_{00}} \;,\; \tilde{b}_{0j} = \frac{g_{0j}}{g_{00}} \\ \tilde{g}_{ij} &= g_{ij} - \frac{g_{i0} g_{0j} + b_{i0} b_{0j}}{g_{00}} \;,\; \tilde{b}_{ij} = b_{ij} - \frac{g_{i0} b_{0j} + b_{i0} g_{0j}}{g_{00}} \;. \end{aligned} \tag{1.114}$$

The geometry of the dual model exchanges the roles of metric and the torsion potential mixing the direction of isometry with other directions and exchange the compactification radius for the isometry in the well known manner $R \to 1/R$. This is the celebrated duality exchanging momentum and winding modes for a closed string compactified on a circle.

In $N = (2,2)$ superspace, the constraints on the field-strengths for a chiral isometry could be integrated by parts to give (anti)twistedchiral constraints on the the Lagrange multipliers for the gauge multiplet V:

$$i(\lambda\tilde{F} - \bar{\lambda}\bar{\tilde{F}}) = i(\lambda\bar{\mathbb{D}}_+\mathbb{D}_-V - \bar{\lambda}\mathbb{D}_+\bar{\mathbb{D}}_-V) = (\chi + \bar{\chi})V \ . \tag{1.115}$$

The duality is therefore just a Legendre transformation in $N = (2,2)$ superspace; starting from the mother action for, *e. g.* the nonlinear sigma-model with an abelian isometry

$$S_M = \int_\Sigma \int d^4\theta \ K(\bar{\phi} + \phi + V, x) - (\chi + \bar{\chi})V \ , \tag{1.116}$$

and the dual action is obtained by substituting the equation of motion for V

$$\frac{\partial}{\partial V} K(\bar{\phi} + \phi + V, x) = \chi + \bar{\chi} \ . \tag{1.117}$$

Flat space: an example

The Kähler potential for $\mathbb{C}$ parametrized by chirals is $K = \bar{\phi}\phi$. Up to Kähler transformations we can write this potential as $K = \frac{1}{2}(\bar{\phi} + \phi)^2$ so that there is an obvious rigid isometry $\delta\phi = i\lambda$. The gauged models with a constrained field-strengths is therefore

$$S = \int_\Sigma \int d^4\theta \ \tfrac{1}{2}(\bar{\phi} + \phi + V)^2 - (\chi + \bar{\chi})V \ . \tag{1.118}$$

In $N = (2,2)$ superspace, the equation of motion for the gauge multiplet V reads

$$\chi + \bar{\chi} = \phi + \bar{\phi} + V \ , \tag{1.119}$$

and the dual action is therefore

$$\tilde{S} = -\int_\Sigma \int d^4\sigma \ \tfrac{1}{2}(\bar{\chi} + \chi)^2 \sim -\int_\Sigma \int d^4\sigma \ \bar{\chi}\chi \tag{1.120}$$

which is an alternative description of $\mathbb{C}$.

Likewise, since the constraints on the field-strengths for a gauge multiplet for twistedchiral isometry are (anti)chiral, the duality exchanges chirals and twistedchirals parameterizing the direction of isometry.

1.7 d-isometries and the $O(d, d, \mathbb{Z})$ group

When combined with transformations that preserve the target, the dualities due to gauged d commuting isometries give models related by elements of $O(d, d, \mathbb{Z})$. The element acts through an embedding in $O(D, D, \mathbb{Z})$ as described in [56].

Starting from the $O(D, D)$ metric $\hat{\eta}$

$$\hat{\eta} = \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix} , \tag{1.121}$$

where all blocks are $D \times D$, an element $\hat{Y} \in O(D, D)$ which preserves this metric satisfies

$$\hat{Y} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} , \ \hat{Y}\hat{\eta}\hat{Y}^T = \hat{\eta} \ , \ \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \mathbb{R}^{D \times D} \ .$$
$$\Rightarrow \hat{a}^T\hat{c} + \hat{c}^T\hat{a} = \hat{b}^T\hat{d} + \hat{d}^T\hat{b} = 1 \ , \ \hat{a}^T\hat{d} + \hat{c}^T\hat{b} = 0 \ . \tag{1.122}$$

The element $\hat{Y} \in O(D, D)$ is therefore generated by:

- b-shifts

$$\hat{B} = \begin{pmatrix} 1 & \hat{b} \\ 0 & 1 \end{pmatrix} , \ \hat{b} = -\hat{b}^T \ , \tag{1.123}$$

- $Gl(D)$ transformations

$$\hat{G} = \begin{pmatrix} \hat{a} & 0 \\ 0 & (\hat{a}^{-1})^T \end{pmatrix} , \ \hat{a}_i{}^j \in Gl(D) \ , \tag{1.124}$$

- Factorized dualities transformations which are written using a set of projectors P

$$\hat{T} = \begin{pmatrix} 1-P & P \\ P & 1-P \end{pmatrix} \ , \ P^2 = P \ . \tag{1.125}$$

In what follows, we shall construct general elements of $O(d,d,\mathbb{Z})$ using these operations; one convenient factorization for elements of $O(d,d,\mathbb{Z})$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^T & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^T & \alpha^T\beta \\ \gamma\alpha^T & \alpha^{-1}+\gamma\alpha^T\beta \end{pmatrix} \tag{1.126}$$

where $(\beta,\gamma) = -(\beta,\gamma)^T \in \mathbb{Z}^{d\times d}$ and $\alpha \in Gl(d,\mathbb{Z})$. One can, in fact, relax the condition $\alpha \in Gl(d,\mathbb{Z})$ to $\alpha \in Gl(d)$ as long as $\alpha^{-1} + \gamma\alpha^T\beta \in \mathbb{Z}^{d\times d}$.

A subgroup $O(d,d,\mathbb{Z}) \in O(D,D)$ acts on nonlinear sigma model in D dimensions with $d < D$ commuting isometries along a d-torus T^d in the following way:

1. Constant b-field shifts in T^d acting as

$$\mathcal{E}' = \mathcal{E} + b \ . \tag{1.127}$$

   These are allowed as the contribution to the torsion $db = 0$ is trivial. b is restricted to integer entries to respect the cycles of T^d.

2. $Gl(d,\mathbb{Z})$ transformations in T^d acting as

$$\mathcal{E}' = \hat{a}\mathcal{E}\hat{a}^T \ , \tag{1.128}$$

   which are harmless field-redefinitions, provided that they are inverted over the integers.

3. Dualities on T^d. These transformations actually restrict to the nontrivial subgroup to $O(d,d,\mathbb{Z}) \in O(D,D)$.

The action of this element on $\mathcal{E}$ could be conveniently expressed using an embedding of $O(d,d,\mathbb{Z})$ in $O(D,D,\mathbb{Z})$

$$\hat{a} = \mathrm{diag}(a,1) \ , \ \hat{b} = \mathrm{diag}(b,0) \ , \ \hat{c} = \mathrm{diag}(c,0) \ , \ \hat{d} = \mathrm{diag}(d,1) \ . \tag{1.129}$$

where the extra blocks are $(D-d)\times(D-d)$. We can write the action of an element of $O(d,d,\mathbb{Z})$ as a fractional linear transformation on $\mathcal{E}$

$$\mathcal{E}' = (\hat{a}\mathcal{E}+\hat{b})(\hat{c}\mathcal{E}+\hat{d})^{-1}\ . \tag{1.130}$$

1.8 Generalized Complex Geometry

An alternative discussion of the geometry of $N=(2,2)$ supersymmetric targets is carried out within the framework of generalized complex geometry [6, 40] which is an interpolation between symplectic and complex geometry best described by maps and operations on the direct sum $T\oplus T^*$. In what follows we introduce some key notions of this contemporary topic and present a manifestly $O(d,d,\mathbb{Z})$ covariant description of the nonlinear sigma-model.

1.8.1 Operations on $T\oplus T^*$

An element $A\in T\oplus T^*$ of a D-dimensional manifold M consists of a vector $a\in T$ and a 1-form $\alpha\in T^*$:

$$A = a\oplus\alpha \tag{1.131}$$

which we frequently write simply as

$$A = a+\alpha\ . \tag{1.132}$$

For two elements, $A,B\in T\oplus T^*$, there is a natural symmetric pairing

$$\langle A,B\rangle = \tfrac{1}{2}\left(i_a\beta + i_b\alpha\right) \tag{1.133}$$

which is preserved by the group $O(D,D)$, and a natural product (the Dorfman bracket / product):

$$A\circ B = [a,b]\oplus\left(\mathcal{L}_a\beta - i_b d\alpha\right)\ . \tag{1.134}$$

satisfying the Leibniz rule

$$A\circ(B\circ C) = (A\circ B)\circ C + B\circ(A\circ C) \tag{1.135}$$

which is easily verified using

$$\mathcal{L}_a = i_a d + d i_a \ , \ \mathcal{L}_{[a,b]} = [\mathcal{L}_a, \mathcal{L}_b] \leftrightarrow i_{[a,b]} = i_a d i_b - i_b d i_a + d i_a i_b + i_a i_b d \ . \quad (1.136)$$

The antisymmetrized brackets (Courant brackets) are defined using the Dorfman product

$$\begin{aligned}[A, B] &= \tfrac{1}{2}(A \circ B - B \circ A) \\ &= [a, b] + \mathcal{L}_a \beta - \mathcal{L}_b \alpha - \tfrac{1}{2} d(i_a \beta - i_b \alpha) \\ &= A \circ B - d \langle A, B \rangle \ . \end{aligned} \quad (1.137)$$

and satisfy the following identity

$$\mathrm{Jac}(A, B, C) = d\mathrm{Nij}(A, B, C) \quad (1.138)$$

where

$$\begin{aligned}\mathrm{Jac}(A, B, C) &= [[A, B], C] + \text{c.p.} \\ \mathrm{Nij}(A, B, C) &= \tfrac{1}{3} \langle [A, B], C \rangle + \text{c.p.} \ . \end{aligned} \quad (1.139)$$

These operations establish a *Courant Algebroid* which is a generalization of a tangent bundle. First, we define a *Lie Algebroid* as a vector bundle L on a manifold M with a Lie bracket $[,] : L \times L \to L$ and a map $a : L \to T$ satisfying

$$\begin{aligned} a([X, Y]) &= [a(X), a(Y)] \ , \ \forall X, Y \in C^\infty(L) \ , \ f \in C^\infty(M) \\ [X, fY] &= f[X, Y] + (a(X)f)Y \ . \end{aligned} \quad (1.140)$$

The last condition is a Leibniz condition. The tangent bundle T is thus a Lie algebroid.

A Courant algebroid is A bundle E equipped with the following operations

1. An anchor map $\pi : \ E \to T$.

2. A Lie bracket $[,] : \ E \times E \to E$.

3. A pairing (symmetric bilinear form) $\langle , \rangle : \ E \times E \to R$ inducing a differ-

ential map $\mathcal{D}$: $\langle \mathcal{D}f, A\rangle = \pi(A)f$, $\forall f \in C^\infty(M)$, $A \in C^\infty(E)$.

Satisfying the following conditions

1. $\pi[A, B] = [\pi A, \pi B]$.
2. $\mathrm{Jac}(A, B, C) = \mathcal{D}(\mathrm{Nij}(A, B, C))$.
3. $[A, fB] = f[A, B] + (\pi(A)f)B - \langle A, B\rangle \mathcal{D}f$.
4. $\pi \circ \mathcal{D} = 0$ that is $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$.
5. $\pi(A)\langle B, C\rangle = \langle [A, B] + \mathcal{D}\langle A, B\rangle, C\rangle + \langle B, [A, C] + \mathcal{D}\langle A, C\rangle\rangle$.

where $A, B, C \in C^\infty(E)$, $f, g \in C^\infty(M)$ and $\mathrm{Jac}(A, B, C), \mathrm{Nij}(A, B, C)$ are defined in (1.139).

As we have already mentioned, the natural product $\langle , \rangle$ is $O(D, D)$ invariant; it is therefore interesting to investigate how the brackets transform under (some) group elements. We follow [6] and address this using the *twisted* brackets.

Elements previously identified with b-shift elements act on sections of $T \oplus T^*$ by *twisting* with a 2-form B

$$e^B A = e^B(a + \alpha) = a + \alpha + i_a B \ . \tag{1.141}$$

Note that there is a sign difference between this convention and the one employed previously as $i_a B = a^\mu b_{\mu\nu} dx^\nu$.

The twisted Dorfman bracket is defined by

$$e^{\mathbf{b}} A \circ e^{\mathbf{b}} B = e^{\mathbf{b}}(A \circ_H B) \ \Rightarrow \ A \circ_H B = A \circ B + i_b i_a H \ , \tag{1.142}$$

where $H = d\,\mathbf{b}$ is a closed three form; the (twisted) Dorfman bracket may be antisymmetrized to give the (twisted) Courant bracket

$$[A, B]_H \equiv \tfrac{1}{2}(A \circ_H B - B \circ_H A) = [a, b] \oplus \big(\mathcal{L}_a \beta - \mathcal{L}_b \alpha - \tfrac{1}{2} d\,(i_a \beta - i_b \alpha) + i_b i_a H\,\big) \ . \tag{1.143}$$

We note

$$[A, B]_H = A \circ_H B - d\,\langle A, B\rangle \ . \tag{1.144}$$

We may also introduce a metric G on $T \oplus T^*$ that is positive definite and satisfies $G^2 = 1$; this metric can be expressed in terms of the ordinary metric g on T and the 2-form potential $\mathbf{b}$ by

$$\langle A, GB \rangle = \langle GA, B \rangle \equiv \left\langle g^{-1}(\alpha + i_a \mathbf{b}) \oplus ga, e^{\mathbf{b}} B \right\rangle \ . \tag{1.145}$$

Equivalently, if we write $A \in T \oplus T^*$ as a column vector, then

$$e^{\mathbf{b}} A = \begin{pmatrix} 1 & 0 \\ -\mathbf{b} & 1 \end{pmatrix} \begin{pmatrix} a \\ \alpha \end{pmatrix} , \tag{1.146}$$

and

$$G = \begin{pmatrix} -g^{-1}\mathbf{b} & g^{-1} \\ g - \mathbf{b} g^{-1} \mathbf{b} & \mathbf{b} g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathbf{b} & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathbf{b} & 1 \end{pmatrix} . \tag{1.147}$$

A metric reduces the structure group $O(D, D) \to O(D) \times O(D)$ which is its maximal compact subgroup.

Ordinary complex structures reduce the structure group of a Riemann manifold $O(D, \mathbb{R}) \to U(D/2, \mathbb{C})$. In a similar fashion to the treatment of complex structures in (1.4), we define a *Generalized almost complex structure* as an endomorphism on $\mathcal{J} : \ T \oplus T^* \to T \oplus T^*$ which is both complex and symplectic; that is:

$$\mathcal{J}^2 = -1 \ , \ \mathcal{J}^* = -\mathcal{J} \ . \tag{1.148}$$

This structure is promoted to *Generalized complex structure* if it is Courant integrable.

Complex and symplectic manifolds therefore admit generalized complex structures:

$$\mathcal{J}_c = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} , \ \mathcal{J}_s = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} . \tag{1.149}$$

For a Kähler manifold, both $\mathcal{J}_{c,s}$ are compatible choices as it is both complex and symplectic.

The last structure to be introduced within the context of generalized complex geometry is that of a *Generalized Kähler structure.* As we have already argued, the Courant algebroid supports the presence of a 2-form torsion potential just as the bihermitian structure (g, b, J_+, J_-) arising in $N = (2, 2)$ supersymmetric sigma-models. The analogue of this is the generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ which are two commuting generalized complex structures factorizing the generalized metric $\mathcal{J}_1\mathcal{J}_2 = -G$. Bihermitian and generalized Kähler structures are related in [6]. We introduce modified generalized Kähler structure which transforms covariantly under elements of $O(D,D)$ [2].

$$\mathcal{J}_{1,2} = \frac{1}{2}\begin{pmatrix} 1 & \\ \mathbf{b} & 1 \end{pmatrix}\begin{pmatrix} J_- \pm J_+ & (J_- \mp J_+)g^{-1} \\ g(J_- \mp J_+) & g(J_- \pm J_+)g^{-1} \end{pmatrix}\begin{pmatrix} 1 & \\ -\mathbf{b} & 1 \end{pmatrix} . \qquad (1.150)$$

Generalized Kähler geometry and the bihermitian geometry are therefore equivalent.

[2]Note that these structure are different than those introduced in [6].

Chapter 2

Some new results towards $O(d,d,\mathbb{Z})$ covariant formalism

The machinery of genarelized complex geometry suggests that sections of $T \oplus T^*$ are $O(d,d,\mathbb{Z})$ covariant. This is supported in the sigma-model approach in the doubled formalism [62–64] and by considering hamiltonian formalism [5, 42] where momenta are explicit. In the following, we describe unpublished work to formulate nonlinear sigma-models in a manifestly $O(d,d,\mathbb{Z})$ covariant formalism using the fundamental building blocks of generalized complex geometry.

The proposed formulation resembles known structures such as Chern Simons theory, String-field theory and WZW models, and is of potential interest as it may relate generalized complex geometry to representation theory [1]. Another advantage of this formalism is that the symplectic structure is not introduced by hand as in the Hamiltonian formalism but rather emerge naturally.

An $O(d,d,\mathbb{Z})$ covariant formalism may also simplify the worldsheet investigation of nongeometric backgrounds [35, 63, 65] and has also a potential to simplify ideas such as local $O(d,d,\mathbb{Z})$ transformations (*e. g.* [66, 67]).

[1] I thank J. Alm for clarifying this point and wish him well.

2.1 $O(d,d,\mathbb{Z})$ transformations of Complex Structures

The transformation properties of complex structures under T-duality have been worked out in [57]; in this section, we generalize these results to the full $O(d,d,\mathbb{Z})$ group and express these as a fractional-linear transformation. We first find a fractional-linear formulation for the transformation law of complex structures under factorized duality (section 2.1.1) which allows us to act with a general element of the group $O(d,d,\mathbb{Z})$ (section 2.1.2). Starting form a nonlinear σ-model in $N=(1,1)$ superspace with a D dimensional target space

$$S = \int_\Sigma D_+ D_- \; D_+\phi^i (g+\mathbf{b})_{ij} D_-\phi^j \;,\; i,j = 1,\cdots,D \;, \tag{2.1}$$

we separate d isometries and write the Lagrange-density as [56]

$$D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu + D_+\phi^a F^1_{a\nu} D_-\phi^\nu + D_+\phi^\mu F^2_{\mu b} D_-\phi^b + D_+\phi^a F_{ab} D_-\phi^b \tag{2.2}$$

where $\mu,\nu = 1,\cdots,d$ and $a,b = d+1,\cdots,D$.

Requiring that the model (2.2) admits an $N=(2,2)$ supersymmetry amounts to the existence of two integrable complex structures, $J_\pm$, on the target manifold $\mathcal{M}$ that are flat with respect to the torsionfull connections $\Gamma_\pm \pm \frac{1}{2} d\mathbf{b}$. The two extra supersymmetries act as:

$$\delta_\epsilon \phi^i = \epsilon J^i_{+j} D_+\phi^j \;,\; \delta_\eta \phi^i = \eta J^i_{-j} D_-\phi^j \;. \tag{2.3}$$

The $N=(2,2)$ superspace ingredients of such model could be easily read, given these complex structures [18–20]. Namely, we identify chiral components with the kernel $\ker(J_+ - J_-)$, twisted chiral components with $\ker(J_+ + J_-)$, and semichiral components from the *complement* to $\ker[J_+, J_-]$. These three types of multiplets are sufficient to describe any model with $N=(2,2)$ supersymmetry [18].

The extra supersymmetry transformations (2.3) dictates transformation under two of the generators of $O(d,d,\mathbb{Z})$, namely:

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \circ J_\pm = J_\pm \ , \ \begin{pmatrix} \alpha^T & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \circ J_\pm = \alpha^{-1} J_\pm \alpha \tag{2.4}$$

To find the action of an $O(d,d,\mathbb{Z})$ element on $J_\pm$ we, therefore, need to find only the action a factorized duality generator (1.125) and compose it to a general element of $O(d,d,\mathbb{Z})$.

2.1.1 The transformation a complex structure under factorized duality

We pick, without loss of generality, the factorized duality with $P = 1$, and generalize the results of [57] to dualization of tori rather than circles.

To dualize (2.2) we gauge (some of) its d isometries [16] $D_\pm\phi^i \to D_\pm\phi^i + V^i_\pm$ and choose the gauge $D_\pm\phi = 0$.

$$V^\mu_+ E_{\mu\nu} V^\nu_- + D_+\phi^a F^1_{a\nu} V^\nu_- + V^\mu_+ F^2_{\mu j} D_-\phi^j + D_+\phi^a F_{ab} D_-\phi^b - \tilde{\phi}_\mu (D_+ V^\mu_- + D_- V^\mu_+) \tag{2.5}$$

eliminating all the Lagrange multipliers $\tilde{\phi}_i$ by their equation of motion restores (2.2) while eliminating $V^i_\pm$ gives an action dualized on the corresponding circle. Dividing the complex structures $J_\pm$ into blocks as we do for $\mathcal{E}$ in (2.2)

$$J_\pm = \begin{pmatrix} \mathfrak{J}^\mu_{\pm\nu} & j^{2\mu}_{\ \pm b} \\ j^{1a}_{\ \pm\nu} & j^a_{\pm b} \end{pmatrix} \tag{2.6}$$

we write the transformation properties For the first order gauged action (2.5) under the symmetry δ_ϵ:

$$\delta_\epsilon \phi^a = \epsilon \left(j^{1a}_{\ +\nu} V^\nu_+ + j^a_{+b} D_+\phi^b \right) \tag{2.7}$$

$$\delta_\epsilon V^\mu_+ = D_+\epsilon \left(\mathfrak{J}^\mu_{+\nu} V^\nu_+ + j^{2\mu}_{\ +b} D_+\phi^b \right) \tag{2.8}$$

$$\delta_\epsilon \tilde{\phi}_\nu = -\epsilon \left[\left(\mathfrak{J}^\mu_{+\rho} V^\rho_+ + j^{2\mu}_{\ +b} D_+\phi^b \right) E_{\mu\nu} + \left(j^{1a}_{\ +\rho} V^\rho_+ + j^a_{+b} D_+\phi^b \right) F^1_{a\nu} \right] \ . \tag{2.9}$$

Substituting the equation of motion for V_-

$$V_+^\nu = -\left(D_+\phi^a F^1_{a\rho} + D_+\tilde{\phi}_\rho\right) E^{\rho\nu} \tag{2.10}$$

we write the transformation laws for $\phi, \tilde{\phi}$

$$\delta_\epsilon \phi \;=\; \epsilon\left(j_+ D_+\phi - j_+^1 (E^T)^{-1}\left((F^1)^T D_+\phi + D_+\tilde{\phi}\right)\right) \tag{2.11}$$

$$\delta_\epsilon \tilde{\phi} \;=\; -\epsilon\Big[E^T\left(-\mathfrak{J}_+ (E^T)^{-1}\left((F^1)^T D_+\phi + D_+\tilde{\phi}\right) + j_+^2 D_+\phi\right) \tag{2.12}$$

$$+\,(F^1)^T\left(-j_+^1 (E^T)^{-1}\left((F^1)^T D_+\phi + D_+\tilde{\phi}\right) + j_+ D_+\phi\right)\Big] \tag{2.13}$$

and obtain the dualized complex structure

$$\tilde{J}_+ = \begin{pmatrix} \tilde{\mathfrak{J}}_+ & \tilde{\mathfrak{J}}_+ (F^1)^T - (E^T j_+^2 + (F^1)^T j_+) \\ -j_+^1 (E^T)^{-1} & j_+ - j_+^1 (E^T)^{-1} (F^1)^T \end{pmatrix} \tag{2.14}$$

where $\tilde{\mathfrak{J}}_+ = (E^T \mathfrak{J}_+ + (F^1)^T j_+^1)(E^T)^{-1}$.

To relate the complex structure $\tilde{J}_+$ of (2.14) to the original complex structure J_+ through a fractional linear transformation we introduce the matrices

$$\hat{\Xi}^{D\times D} = \begin{pmatrix} 1^{d\times d} & 0 \\ 0 & 0 \end{pmatrix}\;,\;\hat{\Psi}^{D\times D} = \begin{pmatrix} 0^{d\times d} & 0 \\ 0 & 1 \end{pmatrix}\;, \tag{2.15}$$

and write the fractional linear transformation:

$$\tilde{J}_+ = (\hat{\Xi}\mathcal{E}^T - \hat{\Psi}) J_+ (\hat{\Xi}\mathcal{E}^T - \hat{\Psi})^{-1}\;. \tag{2.16}$$

A similar treatment of the complex structure J_- gives the transformation laws for the first-order action (2.5) under δ_η:

$$\delta_\eta \phi^a \;=\; \eta\left(j^{1a}_{\ -\nu} V^\nu_- + j^a_{-b} D_- \phi^b\right) \tag{2.17}$$

$$\delta_\eta V^\mu_- \;=\; D_-\eta\left(\mathfrak{J}^\mu_{-\nu} V^\nu_- + j^{2\mu}_{\ -b} D_- \phi^b\right) \tag{2.18}$$

$$\delta_\eta \tilde\phi_\mu \;=\; \eta\left[E_{\mu\nu}\left(\mathfrak{J}^\nu_{-\rho} V^\rho_- + j^{2\nu}_{\ -b} D_-\phi^b\right) + F^2_{\nu a}\left(j^{1a}_{\ -\rho} V^\rho_- + j^a_{-b} D_-\phi^b\right)\right] \;. \tag{2.19}$$

Using the equations of motion for V_- we find the transformed complex structure $\tilde J_-$:

$$\tilde J_- = \begin{pmatrix} \tilde{\mathfrak{J}}_- & (Ej^2_- + F^2 j_-) - \tilde{\mathfrak{J}}_- F^2 \\ j^1_- E^{-1} & j_- - j^1_- E^{-1} F^2 \end{pmatrix} , \tag{2.20}$$

where $\tilde{\mathfrak{J}}_- = (E\mathfrak{J}_- + F^2 j^1_-)E^{-1}$. As in (2.16) we can write

$$\tilde J_- = (\hat\Xi\mathcal{E} + \hat\Psi) J_- (\hat\Xi\mathcal{E} + \hat\Psi)^{-1} \tag{2.21}$$

2.1.2 The transformation of a complex structure under an element of $O(d,d,\mathbb{Z})$

Having described the action of a factorized duality on the complex structures, we follow that construction of an $O(d,d,\mathbb{Z})$ element (1.126) to identify the action of such an element on $J_\pm$.

Starting from $\mathcal{E}, J_+$, and J_-, we first shift with β using its embedding in $\mathbb{Z}^{D\times D}$:

$$\mathcal{E}^1 = \mathcal{E} + \hat\beta \ , \ J^1_\pm = J_\pm \ , \ \hat\beta = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} . \tag{2.22}$$

where

$$\{\mathcal{E}^1, J^1_\pm\} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \circ \{\mathcal{E}, J_\pm\} \tag{2.23}$$

etc next, we act with a $Gl(d)$ element

$$\mathcal{E}^2 = \hat\alpha^T(\mathcal{E}^0 + \hat\beta)\hat\alpha \ , \ J^2_\pm = \hat\alpha^{-1} J^0_\pm \hat\alpha \ , \ \hat\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.24}$$

The first duality gives

$$\mathcal{E}^3 \;=\; (\hat{\Psi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Xi})(\hat{\Xi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Psi})^{-1} \tag{2.25}$$

$$J_+^3 \;=\; \underbrace{(\hat{\Xi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})^T\hat{\alpha}-\hat{\Psi})\hat{\alpha}^{-1}}_{\lambda_+^3} J_+(\lambda_+^3)^{-1} \tag{2.26}$$

$$J_-^3 \;=\; \underbrace{(\hat{\Xi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\;\hat{\alpha}+\hat{\Psi})\hat{\alpha}^{-1}}_{\lambda_-^3} J_-(\lambda_-^3)^{-1} \tag{2.27}$$

acting with a another shift, where $\hat{\gamma}$ is embedded just as $\hat{\beta}$:

$$\mathcal{E}^4 = (\hat{\Psi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Xi})(\hat{\Xi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Psi})^{-1}+\hat{\gamma}\ ,\ J_\pm^4 = J_\pm^3\ . \tag{2.28}$$

Another T-duality gives the final expressions for $J_\pm$:

$$J_+^5 \;=\; \underbrace{\left[\hat{\Xi}\left((\hat{\Psi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Xi})(\hat{\Xi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Psi})^{-1}+\hat{\gamma}\right)^T-\hat{\Psi}\right]\lambda_+^3}_{\lambda_+^5} J_+(\lambda_+^5)^{-1}\ , \tag{2.29}$$

$$J_-^5 \;=\; \underbrace{\left[\hat{\Xi}\left((\hat{\Psi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Xi})(\hat{\Xi}\hat{\alpha}^T(\mathcal{E}+\hat{\beta})\hat{\alpha}+\hat{\Psi})^{-1}+\hat{\gamma}\right)\;+\hat{\Psi}\right]\lambda_-^3}_{\lambda_-^5} J_-(\lambda_-^5)^{-1} \tag{2.30}$$

Simplifying $\lambda_\pm^5$, we write the action of a general $O(d,d,\mathbb{Z})$ element on the complex structures as a fractional linear transformation

$$\tilde{J}_+ \;=\; (\hat{c}\mathcal{E}^T-\hat{d})J_+(\hat{c}\mathcal{E}^T-\hat{d})^{-1}\ , \tag{2.31}$$

$$\tilde{J}_- \;=\; (\hat{c}\mathcal{E}\;+\hat{d})J_-(\hat{c}\mathcal{E}\;+\hat{d})^{-1}\ . \tag{2.32}$$

This implies that $\mathcal{E}J_-$ $(\mathcal{E}^TJ_+)$ transforms as $\mathcal{E}$ $(\mathcal{E}^T)$, where the later follows from the constraints (1.122) on $\mathsf{g} \in O(d,d,\mathbb{Z})$:

$$\tilde{\mathcal{E}}^T = (\mathcal{E}^T\hat{c}^T + \hat{d}^T)^{-1}(\mathcal{E}^T\hat{a}^T + \hat{d}^T)(\hat{c}\mathcal{E}^T - \hat{d})(\hat{c}\mathcal{E}^T - \hat{d})^{-1} = -(\hat{a}\mathcal{E}^T - \hat{b})(\hat{c}\mathcal{E}^T - \hat{d})^{-1} \; . \tag{2.33}$$

2.2 Nonlinear sigma-models on $T \oplus T^*$

Consider a nonlinear sigma-model for maps $\phi(\sigma, \bar{\sigma})$ from a (compact) world-sheet Σ with complex coordinates σ to some target space M with a metric g and a local two-form potential $\mathbf{b}$:

$$S_\phi = \int_\Sigma \partial\phi^i \, \mathcal{E}_{ij} \, \bar{\partial}\phi^j = \int_\Sigma \partial\phi^i \, (g_{ij} + \mathbf{b}_{ij}) \, \bar{\partial}\phi^j \;, \; \mathcal{E} \equiv g + \mathbf{b} \;, \tag{2.34}$$

where $g, \mathbf{b}$ are pulled back from M to Σ with the map ϕ. As shown in [20] (up to trivial changes of notation), this can be written in a more global way in terms of a closed three-form $H = d\,\mathbf{b}$ (locally the exterior derivative of $\mathbf{b}$) by extending the map ϕ to a map $\hat{\phi} : \Sigma \times [0,1] \to M$ such that for $t = 0$, $\hat{\phi}$ maps to a single point on Σ, and for $t = 1$, $\hat{\phi}(\sigma, \bar{\sigma}, 1) = \phi(\sigma, \bar{\sigma})$. Then

$$S_\phi = \int_\Sigma \partial\phi^i \, g_{ij} \, \bar{\partial}\phi^j + \int_0^1 dt \int_\Sigma H_{ijk} \, \partial\hat{\phi}^i \bar{\partial}\hat{\phi}^j \dot{\hat{\phi}}^k \;, \tag{2.35}$$

where S depends in the usual way on the choice of the extension $\hat{\phi}$: the difference of two extensions $\hat{\phi}_{1,2}$ gives a compact three-fold, namely the suspension of Σ, and $S_1 - S_2$ is just the integral of the pullback of H to this 3-fold. We call the first term in (2.35) the kinetic term and the second term the WZ-term.

We can write this in terms of the natural objects of generalized geometry as follows: we introduce

$$\xi = \partial\phi^i \partial_i \oplus S_i d\phi^i \;, \; \bar{\xi} = \bar{\partial}\phi^i \partial_i \oplus \bar{S}_i d\phi^i \;, \; \xi_t = \dot{\hat{\phi}}^i \partial_i \oplus \hat{S}_{ti} d\phi^i \;, \tag{2.36}$$

where $S_i, \bar{S}_i$ are maps from Σ to T^* at the point $\phi(\sigma, \bar{\sigma})$.

2.2.1 The generalized kinetic term

Consider the natural object

$$S_G = \int_\Sigma \left\langle \xi, G\bar{\xi}\right\rangle \equiv \tfrac{1}{2}\int_\Sigma \partial\phi^i[(g-\mathbf{b}g^{-1}\mathbf{b})_{ij}\bar{\partial}\phi^j+(\mathbf{b}g^{-1})_i{}^j\bar{S}_j]+S_i[g^{ij}\bar{S}_j-(g^{-1}\mathbf{b})^i{}_j\bar{\partial}\phi^j]\ , \tag{2.37}$$

where as usual g^{ij} are the components of g^{-1}. This can be simplified:

$$S_G = \tfrac{1}{2}\int_\Sigma \partial\phi^i\, g_{ij}\, \bar{\partial}\phi^j + (S_i + \partial\phi^m \mathbf{b}_{mi})g^{ij}(\bar{S}_j - \mathbf{b}_{jk}\bar{\partial}\phi^k) \tag{2.38}$$

2.2.2 The generalized WZ-term

The tangent space component of the sections $\xi, \bar{\xi}$, and ξ_t are, by chain rule, pulled back derivatives; *e. g.*

$$\pi(\xi)A = \partial\phi^i\partial_i A = \partial A\ . \tag{2.39}$$

This allows us to calculate the Lie derivatives:

$$[\partial\phi^i\partial_i, \bar{\partial}\phi^j\partial_j] \ = \ [\partial,\bar{\partial}]\phi^i\partial_i = 0 \tag{2.40}$$

$$\mathcal{L}_{\partial\phi^i\partial_i}\bar{S}_j d\phi^j \ = \ \left(\partial\bar{S}_j + (\partial_j\partial\phi^i)\bar{S}_i\right) d\phi^j \tag{2.41}$$

$$\mathcal{L}_{\bar{\partial}\phi^i\partial_i}S_j d\phi^j \ = \ \left(\bar{\partial}S_j + (\partial_j\bar{\partial}\phi^i)S_i\right) d\phi^j \tag{2.42}$$

and the Courant bracket

$$[\xi,\bar{\xi}] = \left(\partial\bar{S}_j + (\partial_j\partial\phi^i)\bar{S}_i - \bar{\partial}S_j - (\partial_j\bar{\partial}\phi^i)S_i - \tfrac{1}{2}\partial_j(\partial\phi^i\bar{S}_i - \bar{\partial}\phi^i S_i)\right) d\phi^j \in T^*\ . \tag{2.43}$$

Pairing the bracket $[\xi,\bar{\xi}]$ with the section ξ_t [2] gives, using the pullback

[2] It is important to note that hitherto, the maps $\hat{S}_{ti}$ have played no role in the generalized WZ-term (2.45) since the Courant bracket (2.43) has no tangent space component.

$\dot{\hat{\phi}}^i\partial_i = \frac{d}{dt}$, total $\partial, \bar{\partial}$ and $\frac{d}{dt}$ derivatives:

$$S_{WZ} = -2\int_0^1 dt \int_\Sigma \langle \xi_t, [\xi, \bar{\xi}] \rangle \tag{2.44}$$

$$= -\int_0^1 dt \int_\Sigma \Big(\partial(\dot{\hat{\phi}}^i \bar{S}_i) - \bar{\partial}(\dot{\hat{\phi}}^i S_i) - \tfrac{1}{2}\tfrac{d}{dt}(\partial\hat{\phi}^i \bar{S}_i - \bar{\partial}\hat{\phi}^i S_i) \Big) \ . \tag{2.45}$$

Combining the kinetic and the WZ-terms we write the generalized nonlinear sigma-model

$$\begin{aligned} S_G + S_{WZ} &= \int_\Sigma \langle \xi, G\bar{\xi} \rangle - 2\int_0^1 dt\ \langle \xi_t, [\xi, \bar{\xi}] \rangle \\ &= \int_\Sigma \partial\phi^i \mathcal{E}_{ij} \bar{\partial}\phi^j + \tfrac{1}{2}(S_i + \partial\phi^k \mathcal{E}_{ki}) g^{ij} (\bar{S}_j - \mathcal{E}_{jl}\bar{\partial}\phi^l) \ . \end{aligned} \tag{2.46}$$

Integrating out the fields S and $\bar{S}$ we recover the original nonlinear sigma-model (2.34).

2.2.3 Field-redefinitions and frames for the action

In the WZW-like formulation (2.46) the WZ term introduces a Liouville-like piece (2.45) while all the geometric data is encoded in the kinetic term (2.37) through the generalized metric G.

It could be seen from (2.46) that any field-redefinition of the auxiliaries S_i or $\bar{S}_i$ would preserve the on-shell action (2.34). As we now show, some of these redefinitions modify the form of (2.46) so that it is still expressible in terms of familiar operations in $T \oplus T^*$ and reshuffle the geometric data of the target space (g, b) between the kinetic and the WZ-terms.

2.2.4 The 0-frame

The natural pairing on $T \oplus T^*$ is invariant under b-transformations generated by a 2-form $e^{\mathsf{b}}A = a + \alpha + i_a\mathsf{b}$

$$\langle e^{\mathsf{b}}A, e^{\mathsf{b}}C \rangle = \langle A, B \rangle + \tfrac{1}{2}(i_a i_c \mathsf{b} + i_c i_a \mathsf{b}) = \langle A, B \rangle \ , \ \forall \mathsf{b} \in \wedge^2 T^* \ . \tag{2.47}$$

Under this transformation the Courant bracket undergoes a twisting with the 3-form $d\mathsf{b}$:

$$[e^{\mathsf{b}}A, e^{\mathsf{b}}C] = e^{\mathsf{b}}[A, C]_{d\mathsf{b}} \ . \tag{2.48}$$

closed 2-forms therefore generate a symmetry of the off-shell action provided that the generalized metric transforms by conjugation $G \to e^{\mathsf{b}}Ge^{-\mathsf{b}}$. This is the obvious symmetry of the on shell action

$$\delta b = \lambda \ , \ \lambda \in \wedge^2 T^*, d\lambda = 0 \ . \tag{2.49}$$

Starting from the original WZW formulation (2.46), that will henceforth referred to as the *Courant frame*, we define the 0-*frame* for the sections

$$\xi^0 = e^{\mathbf{b}}\xi \ , \ \bar{\xi}^0 = e^{\mathbf{b}}\bar{\xi} \tag{2.50}$$

and the **b**-conjugated metric

$$G^0 = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} = e^{\mathbf{b}}Ge^{-\mathbf{b}} \ . \tag{2.51}$$

Writing the Lagrange density in terms of the sections ξ^0 we find

$$L = \left\langle \xi^0, G^0\bar{\xi}^0 \right\rangle - 2\int_0^1 dt \ \left\langle \xi_t, [\xi^0, \bar{\xi}^0]_{-db} \right\rangle \ , \tag{2.52}$$

where the torsion potential is due to the WZ-term. This field redefinition symmetrizes the equations of motion for S_i; whereas previously involved both $\mathcal{E}$ and $\mathcal{E}^T$,

$$S_i + \mathcal{E}^T_{ij}\partial\phi^j = \bar{S}_i - \mathcal{E}_{ij}\bar{\partial}\phi^j = 0 \tag{2.53}$$

now the equations of motion for S_i^0 and $\bar{S}_i^0$ depend only on the metric g:

$$S_i^0 + g_{ij}\partial\phi^j = \bar{S}_i^0 - g_{ij}\bar{\partial}\phi^j = 0 \ . \tag{2.54}$$

2.2.5 The g-frame

Another interesting symmetry of the natural pairing involves a **symmetric** map:

$$e^{\mathsf{g}}A = a + \alpha + \mathsf{g}(a) \to \left\langle e^{\mathsf{g}}A, e^{-\mathsf{g}}C \right\rangle = \langle A, C\rangle \ , \ \mathsf{g} : T \to T^* \ . \tag{2.55}$$

To find its effect on the WZ-term we pull $\partial, \bar\partial$ and $\frac{d}{dt}$ back to the worldsheet and calculate

$$2\left\langle \xi_t, [\mathsf{g}(\partial\phi), \bar\xi] \right\rangle \ = \ -\bar\partial(\mathsf{g}_{\mu\nu}\partial\phi^\mu\dot\phi^\nu) + \tfrac{1}{2}\tfrac{d}{dt}(\partial\phi^\mu \mathsf{g}_{\mu\nu}\bar\partial\phi^\nu) \tag{2.56}$$

$$-2\left\langle \xi_t, [\xi, \mathsf{g}(\bar\partial\phi)] \right\rangle \ = \ -\partial(\mathsf{g}_{\mu\nu}\bar\partial\phi^\mu\dot\phi^\nu) + \tfrac{1}{2}\tfrac{d}{dt}(\partial\phi^\mu \mathsf{g}_{\mu\nu}\bar\partial\phi^\nu) \ , \tag{2.57}$$

so up to total worldsheet derivatives

$$\left\langle \xi_t, [e^{\mathsf{g}}\xi, e^{-\mathsf{g}}\bar\xi] \right\rangle \ = \ \left\langle \xi_t, [\xi, \bar\xi] \right\rangle + \left\langle \xi_t, [\mathsf{g}(\partial\phi), \bar\xi] \right\rangle - \left\langle \xi_t, [\xi, \mathsf{g}(\bar\partial\phi)] \right\rangle \tag{2.58}$$

$$= \ \left\langle \xi_t, [\xi, \bar\xi] \right\rangle + \tfrac{1}{2}\tfrac{d}{dt}(\partial\phi^\mu \mathsf{g}_{\mu\nu}\bar\partial\phi^\nu) \ . \tag{2.59}$$

This symmetry does not result in a simple transformation of the Courant brackets as we find for the b-transform, but gives a tractable shift for the whole WZ term.

Introducing the *g-frame* sections

$$\xi^g = e^g\xi^0 \ , \ \bar\xi^g = e^{-g}\bar\xi^0 \tag{2.60}$$

and the conjugated generalized metric $G^g = e^{-g}G^0e^g$ we write

$$L = \left\langle \xi^g, G^g\bar\xi^g \right\rangle - 2\int_0^1 dt \ \left\langle \xi_t, [\xi^g, \bar\xi^g]_{-d\mathbf{b}} - d(\partial\phi g\bar\partial\phi) \right\rangle = \partial\phi\mathcal{E}\bar\partial\phi + \tfrac{1}{2}S^g g^{-1}\bar S^g \ . \tag{2.61}$$

Writing the kinetic term explicitly

$$\left\langle \xi^g, G^g\bar\xi^g \right\rangle = \tfrac{1}{2}\left(S^g g^{-1}\bar S^g - \partial\phi\bar S^g + \bar\partial\phi S^g\right) \ , \tag{2.62}$$

we find that now the Liouville term cancels between the the kinetic and the

WZ-terms and all the geometric data for the on-shell action lies in the WZ-term.

2.2.6 Σ-frames

So far, only symmetries of the natural pairing were considered as operations relating different frames. It is easy to see that under sign change for the cotangent piece

$$\xi \to \Sigma\xi \ , \ \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.63}$$

the basic operations transform as

$$\langle \Sigma A, \Sigma B \rangle = - \langle A, B \rangle \ , \ [\Sigma A, \Sigma B]_H = \Sigma [A, B]_{-H} \ . \tag{2.64}$$

We can therefore define the Courant$_\Sigma$, 0_Σ, and g_Σ-frames by acting with Σ on the corresponding sections and changing the overall sign of the action, the twisting and the shift.

We conclude with table (2.1) that summarizes all frames that were introduced:

2.2.7 $O(d, d, \mathbb{Z})$ transformations

In the previous section $b(g)$-transformations were reciprocated by twisting (shift) of the Courant brackets and conjugation of the generalized metric so that the transformed action in $T \oplus T^*$ differed by field-redefinitions. A key insight in writing these field-redefinitions in the WZW-like formulation is applying symmetries of the natural pairing to the sections $(\xi, \bar{\xi})$ and tracking down their action on the Courant brackets in the WZ-term.

In this section we study $O(D, D)$ transformations, which are the full structure group of $T \oplus T^*$. We find that, given an invariant d-torus $T^d \subseteq M$ in the target space, we may choose the auxiliaries $\hat{S}_t$ such that the WZ-term is invariant under the subgroup $O(d, d, \mathbb{Z})$. We then go to the generalized sigma-model where the action of these generators could be associated with standard operations in target space, namely, change of coordinates, constant shifts of the torsion potential and T-dualities along isometries.

Table 2.1: Frames for the off shell sigma-model on $T \oplus T^*$

Frame	Sections		Generalized Metric
Action			
Courant	ξ^C	$\bar{\xi}^C$	$G^C = e^{-\mathbf{b}} G^0 e^{\mathbf{b}}$
$L = \left\langle \xi^C, G^C \bar{\xi}^C \right\rangle - 2 \int_0^1 dt \left\langle \xi_t, [\xi^C, \bar{\xi}^C] \right\rangle$			
Courant$_\Sigma$	$\xi^{C_\Sigma} = \Sigma \xi^C$	$\bar{\xi}^{C_\Sigma} = \Sigma \bar{\xi}^C$	$G^C = \Sigma e^{-\mathbf{b}} G^0 e^{\mathbf{b}} \Sigma$
$L = -\left\langle \xi^{C_\Sigma}, G^{C_\Sigma} \bar{\xi}^{C_\Sigma} \right\rangle + 2 \int_0^1 dt \left\langle \xi_t, [\xi^{C_\Sigma}, \bar{\xi}^{C_\Sigma}] \right\rangle$			
0-frame	$\xi^0 = e^{\mathbf{b}} \xi^C$	$\bar{\xi}^0 = e^{\mathbf{b}} \bar{\xi}^C$	G^0
$L = \left\langle \xi^0, G^0 \bar{\xi}^0 \right\rangle - 2 \int_0^1 dt \left\langle \xi_t, [\xi^0, \bar{\xi}^0]_{-d\mathbf{b}} \right\rangle$			
0_Σ-frame	$\xi^{0_\Sigma} = \Sigma \xi^0 = e^{-\mathbf{b}} \xi^{C_\Sigma}$	$\bar{\xi}^{0_\Sigma} = \Sigma \bar{\xi}^0 = e^{-\mathbf{b}} \bar{\xi}^{C_\Sigma}$	$G^{0_\Sigma} = \Sigma G^0 \Sigma = -G^0$
$L = -\left\langle \xi^{0_\Sigma}, G^{0_\Sigma} \bar{\xi}^{0_\Sigma} \right\rangle + 2 \int_0^1 dt \left\langle \xi_t, [\xi^{0_\Sigma}, \bar{\xi}^{0_\Sigma}]_{d\mathbf{b}} \right\rangle$			
g-frame	$\xi^g = e^g \xi^0$	$\bar{\xi}^g = e^{-g} \bar{\xi}^0$	$G^g = e^{-g} G^0 e^g$
$L = \left\langle \xi^g, G^g \bar{\xi}^g \right\rangle - 2 \int_0^1 dt \left\langle \xi_t, [\xi^g, \bar{\xi}^g]_{-d\mathbf{b}} - d(\partial\phi g \bar{\partial}\phi) \right\rangle$			
g_Σ-frame	$\xi^{g_\Sigma} = \Sigma \xi^g = e^{-g} \xi^{0_\Sigma}$	$\bar{\xi}^{g_\Sigma} = \Sigma \bar{\xi}^g = e^g \bar{\xi}^{0_\Sigma}$	$G^{g_\Sigma} = \Sigma G^g \Sigma$
$L = -\left\langle \xi^{g_\Sigma}, G^{g_\Sigma} \bar{\xi}^{g_\Sigma} \right\rangle + 2 \int_0^1 dt \left\langle \xi_t, [\xi^{g_\Sigma}, \bar{\xi}^{g_\Sigma}]_{d\mathbf{b}} + d(\partial\phi g \bar{\partial}\phi) \right\rangle$			

Lastly, we match elements of $O(d,d,\mathbb{Z})$ which act by conjugation on the generalized metric with elements of $O(d,d,\mathbb{Z})$ that act on $\mathcal{E}$ with a fractional-linear transformation.

2.2.8 (Partial) Invariance of the WZ-term in the Courant frame

Both b-shifts and $Gl(D)$ transformations do not introduce contributions to tangent piece of the Courant bracket so invariance of the WZ-term with respect to these requires

$$[\hat{B}\xi,\hat{B}\bar{\xi}] = \hat{B}[\xi,\bar{\xi}]\ ,\ [\hat{G}\xi,\hat{G}\bar{\xi}] = \hat{G}[\xi,\bar{\xi}]\ , \tag{2.65}$$

which is compensated for by ξ_t transformation. This is shown in propositions 3.23 (b-shifts) and 3.24 ($Gl(D)$ transformations) in [6].

For the factorized duality, *e. g.* T_1, given an isometry along the ϕ^1 direction, $\partial_1\partial\phi^i = \partial_1 S_i = 0$, we calculate, using the pullback of ∂ to the worldsheet

$$\begin{aligned}[\hat{T}_1\xi,\hat{T}_1\bar{\xi}] &= [S_1\partial_1 + \partial\phi^a\partial_a + \partial\phi^1 d\phi^1 + S_a d\phi^a, \bar{S}_1\partial_1 + \bar{\partial}\phi^a\partial_a + \bar{\partial}\phi^1 d\phi^1 + \bar{S}_a d\phi^a] && (2.66)\\ &= (\partial\bar{S}_1 - \bar{\partial}S_1)\partial_1 - \tfrac{1}{2}\partial_i(\partial\phi^j\bar{S}_j - \bar{\partial}\phi^j S_j) && (2.67)\end{aligned}$$

which differs from the bracket $[\xi,\bar{\xi}]$ by a tangent piece. Setting $\hat{S}_{ti} = 0$ for all isometric directions cancels this contribution

$$\left\langle \xi_t, [\hat{T}_1\xi,\hat{T}_1\bar{\xi}]\right\rangle = \langle \xi_t, [\xi,\bar{\xi}]\rangle = \left\langle \hat{T}_1\xi_t, \hat{T}_1[\xi,\bar{\xi}]\right\rangle\ . \tag{2.68}$$

At first glance it appears as if the factorized duality transformations along an isometry are on different footing than B and G as they do not transform ξ_t, however, since $[\xi,\bar{\xi}]$ has no tangent space component B transformations on ξ_t are only formal.

The invariance with respect to d of the factorized duality generators and the periods of T^d restricts the nontrivial subgroup to $O(d,d,\mathbb{Z})$.

2.2.9 The transformed generalized sigma-model

The action of $O(d,d,\mathbb{Z})$ elements in $T \oplus T^*$ changes the geometry of the sigma-model obtained after integrating out the auxiliaries S_i and $\bar{S}_i$. We study the geometry of the new sigma-models, again, in the Courant frame where due to the $O(d,d,\mathbb{Z})$ invariance of the WZ-term all transformations are due to conjugation of the generalized metric G.

A related result was given in [58], where it was argued, though not within the context and framework of generalized complex geometry, that $G+1$ transform linearly under those $O(d,d,\mathbb{Z})$ transformations.

- b-transformations: Starting from the generalized metric (1.147) we conjugate G

$$\hat{B}G\hat{B}^{-1} = \begin{pmatrix} 1 & \\ \mathbf{b}+\hat{b} & 1 \end{pmatrix} \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} \begin{pmatrix} 1 & \\ -(\mathbf{b}+\hat{b}) & 1 \end{pmatrix} \tag{2.69}$$

  $\hat{B}$ therefore shifts the torsion potential by constant $\delta\mathbf{b} = \hat{b}$.

- $Gl(D)$ transformations: These transformations do not mix tangent and cotangent pieces and we can, therefore, study their action after integrating out the auxiliaries. The appropriate conjugation acts as

$$\mathcal{E} \to \hat{a}\mathcal{E}\hat{a}^T \tag{2.70}$$

  which rotates the fields ϕ^i.

- Factorized dualities: In the kinetic term, the factorized duality $\hat{T}_1$ swaps $\partial\phi^1 \leftrightarrow S_1$. We can therefore write, using the invariance of the WZ-term in the Courant frame and after integrating out the $2(D-1)$ of the auxiliaries S_a and $\bar{S}_a$

$$\left(\left\langle \xi, \hat{T}_1 G \hat{T}_1 \bar{\xi} \right\rangle - 2\int_0^1 dt \ \left\langle \xi_t, [\xi, \bar{\xi}]\right\rangle\right)\Big|_{S_a, \bar{S}_a} = \tag{2.71}$$

$$S_1 g_{11} \bar{S}_1 + S_1 \mathcal{E}_{1b} \bar{\partial}\phi^b + \partial\phi^a \mathcal{E}_{a1} \bar{S}_1 + \partial\phi^a \mathcal{E}_{ab} \bar{\partial}\phi^b \tag{2.72}$$

$$\tfrac{1}{2}(\partial\phi^1 + S_1 g_{11} + \partial\phi^a \mathcal{E}_{a1}) g^{11} (\bar{\partial}\phi^1 - g_{11}\bar{S}_1 - \mathcal{E}_{1b}\bar{\partial}\phi^b) \tag{2.73}$$

$$+\partial\phi^1 \bar{S}_1 - \bar{\partial}\phi^1 S_1 \ . \tag{2.74}$$

This expression differs from the ordinary mother action for gauged isometry along ϕ^1 [16, 19] by a term proportional to the equations of motion for S_1. Integrating the pair S_1 and $\bar{S}_1$ therefore gives a sigma-model whose target space geometry relates to the original geometry by T-duality along the ϕ^1 direction.

2.2.10 Linear and fractional-linear transformations

In [56] it was shown that elements of the subgroup $O(d, d, \mathbb{Z})$ act on the metric and the torsion potential through fractional linear transformations

$$\hat{Y} \circ \mathcal{E} = (\hat{a}\mathcal{E} + \hat{b})(\hat{c}\mathcal{E} + \hat{d})^{-1} = \tilde{\mathcal{E}} \ . \tag{2.75}$$

We now identify these transformations with similarity transformations of the generalized metric using the results of the previous section. An alternative derivation of these results is given in appendix A.

- b-shifts: An elements that shifts the torsion potential through fractional linear transformations is of the form

$$\begin{pmatrix} 1 & \hat{b} \\ 0 & 1 \end{pmatrix} \circ \mathcal{E} = \mathcal{E} + \hat{b} \ . \tag{2.76}$$

To act linearly on the generalized metric generating the same shift we identify this element with $\hat{\eta}\hat{B}\hat{\eta}$.

- $Gl(D)$ transformations: An element that rotates $\mathcal{E}$ through fractional linear transformations has the form

$$\begin{pmatrix} \hat{a} & 0 \\ 0 & (\hat{a}^{-1})^T \end{pmatrix} \circ \mathcal{E} = \hat{a}\mathcal{E}\hat{a}^T \ . \tag{2.77}$$

  To obtain this transformation though similarity of G we identify, just as for $\hat{B}$, this element with $\hat{\eta}\hat{G}\hat{\eta}$.

- Factorized duality: The element $\hat{T}_1$ gives a T-dual action when acting on $\mathcal{E}$ (as a fractional linear) or the generalized metric G (through similarity transformation).

We therefore identify the action of an $O(d,d,\mathbb{Z})$ element (2.75) on G

$$\hat{Y} \circ G = \hat{\eta}\hat{Y}\hat{\eta}G\hat{Y}^T \tag{2.78}$$

2.2.11 Lift to $N = (1,1)$ superspace

Insofar, our discussion was limited to Bosonic nonlinear sigma-models; it is, however, straightforward to endow these models with $N = (1,1)$ supersymmetry by formulating them in $N = (1,1)$ superspace. This requires the promotion of all fields to $N = (1,1)$ superfields, all derivatives to superderivatives $D_\pm$, and the grading of the algebraic operations on $T \oplus T^*$.

We formulate our model in $N = (1,1)$ superspace in terms of Fermionic sections $\xi_\pm = D_\pm\phi^i\partial_i \oplus S_{\pm i}d\phi^i \in T \oplus T^*$ and the Bosonic section ξ_t

$$S_{N=(1,1)} = \int_\Sigma \int D_+D_- \ \left(\langle\xi_+, G\xi_-\rangle - 2\int_0^1 dt \ \langle\xi_t, [\xi_+, \xi_-]\rangle \right) \ , \tag{2.79}$$

where the operations are graded

$$\langle A, B\rangle \ = \ \tfrac{1}{2}\left(i_a\beta + (-)^{F(A)F(B)}i_b\alpha\right) \tag{2.80}$$

$$[A, B] \ = \ [a, b\} + \mathcal{L}_a\beta - (-)^{F(A)F(B)}\mathcal{L}_b\alpha - \tfrac{1}{2}d\left(i_a\beta - (-)^{F(A)F(B)}i_b\alpha\right) \tag{2.81}$$

A lift to $N = (2,2)$ is a more involved and gives rise to a generalized Kähler structure.

Chapter 3

Gauging and dualities along generalized isometries

In a discussion limited to Kähler submanifolds, we encountered a consistent picture where isometries that mix (anti)chirals are gauged using gauge field with (anti)twistedchiral field-strengths and is therefore dualized to (anti) twistedchirals. Investigation of dualities on Kähler manifolds were carried out using these methods, *e. g.* to derive the physics proof for mirror symmetry [32, 36].

This picture is, however, incomplete as bihermitian targets are potentially torsional and should therefore allow also semichiral superfields and isometries mixing chirals and twistedchiral superfields. Analyzing the constraints on these superfields immediately leads to the postulate [14] that these are T-dual, as the constraint on, *e. g.* a left semichiral is the sum of a chiral and a twistedchiral

$$\bar{\mathbb{D}}_+\mathbb{X}_L = \bar{\mathbb{D}}_+(\phi + \chi) = 0\ . \tag{3.1}$$

In [26] and [27] we have demonstrated that this is indeed the case by constructing new gauge multiplets that are suited to gauging such isometries and carrying out the duality transformation. Similar work was introduced simultaneously in [28] where features of the multiplet gauging semichiral isometries were studied using a constraint-based approach.

The new multiplets are larger than those used to gauge (twisted)chiral isometries and contain two more $N = (1, 1)$ scalar multiplets; this is to provide enough (3) diffeomorphisms which are required as the smallest number of semichirals charting a patch is 4 [18].

The set of field-strengths for the new multiplets is therefore significantly larger than that previously encountered and consistes, in $N = (2,2)$ superspace, of an (anti)chiral and an (anti)twistedchiral field-strengths (for the multiplet guging semichiral isometries) and a set of *fermionic* semichirals for the multiplet gauging an isometry mixing (twisted)chirals. The latter is therefore dubbed the *Large Vector Multiplet (LVM)* as it contains auxiliary superfields.

After introducing the multiplets in $N = (2,2)$ superspace we study their reduction to $N = (1,1)$ superspace, matter couplings and the action for the semichiral multiplet.

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New $N = (2, 2)$ vector multiplets

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Abstract

We introduce two new $N = (2, 2)$ vector multiplets that couple naturally to generalized Kähler geometries. We describe their kinetic actions as well as their matter couplings both in $N = (2, 2)$ and $N = (1, 1)$ superspace.

1 Introduction

Generalized Kähler geometry has aroused considerable interest both among string theorists and mathematicians [1, 2, 55]. Recently, a number of groups have tried to construct quotients [3, 4, 5, 6]; however, it is unclear how general or useful the various proposals are. Experience has shown that supersymmetric σ-models are often a very helpful guide to finding the correct geometric concepts and framework for quotient constructions [7, 8]. In this paper, we take the first step in this direction; further results will be presented in [9].

The basic inspiration for our work is the interesting duality found in [10, 11]. As was shown in [8, 12], T-dualities arise when one gauges an isometry, and then constrains the field-strength of the corresponding gauge multiplet to vanish. Here we address the question: what are the gauge multiplets corresponding to the duality introduced in [10, 11]?

In section 2, we analyze the types of isometries that arise on generalized Kähler geometries which are suitable for gauging, and describe the corresponding multiplets in $N = (2, 2)$ superspace. In addition to the usual multiplets with chiral or twisted chiral gauge parameters, we find two new multiplets: one with semichiral gauge parameters, which we call the semichiral gauge multiplet, and one with a pair of gauge parameters, one chiral and one twisted chiral; the last has more gauge-invariant components than other multiplets, and hence we call it the large vector multiplet.

In section 3, we describe the $N = (1, 1)$ superspace content of these mulitplets; this exposes their physical content. We describe both multiplets and their couplings to matter, and discuss possible gauge actions for them. The component content of the various $N = (1, 1)$ multiplets that arise is well known and can be found in [13].

Throughout this paper we follow the conventions of [14].

2 Generalized Kähler geometry: $N = (2, 2)$ superspace

Generalized Kähler geometry (GKG) arises naturally as the target space of $N = (2, 2)$ supersymmetric σ-models. As shown in [14], such σ-models always admit a local description in $N = (2, 2)$ superspace in terms of complex chiral superfields ϕ, twisted chiral superfields χ and semichiral superfields $\mathbb{X}_L, \mathbb{X}_R$ [15]. These models have also been considered in $N=(1,1)$ superspace [16, 17].

These geometries may admit a variety of holomorphic isometries that can be gauged by different kinds of vector multiplets. We now itemize the basic types of isometries.

2.1 Isometries

The simplest isometries act on purely Kähler submanifolds of the generalized Kähler geometry, that is only on the chiral superfields ϕ or the twisted chiral superfields χ; for a single $U(1)$ isometry away from a fixed point, we may choose coordinates so that the Killing vectors take the form:

$$k_\phi = i(\partial_\phi - \partial_{\bar\phi}) \quad , \qquad k_\chi = i(\partial_\chi - \partial_{\bar\chi}) \quad . \tag{2.1}$$

In [10, 11], new isometries that mix chiral and twisted chiral superfields or act on semichiral superfields were discovered; we may take them to act as

$$\begin{aligned} k_{\phi\chi} &= i(\partial_\phi - \partial_{\bar\phi} - \partial_\chi + \partial_{\bar\chi}) \ , & (2.2)\\ k_{LR} &= i(\partial_L - \partial_{\bar L} - \partial_R + \partial_{\bar R}) \ , & (2.3)\end{aligned}$$

where $\partial_L = \frac{\partial}{\partial \mathbb{X}_L}$, *etc.* One might imagine more general isometries that act along an arbitrary vector field; however, compatibility with the constraints on the superfields (chiral and twisted chiral superfields are automatically semichiral but not vice-versa) allows us to restrict to the cases above; in particular, if the vector field has a component along k_ϕ, k_χ or $k_{\phi\chi}$, we can (locally) redefine $\mathbb{X}$ to eliminate any component along k_{LR}.

A general Lagrange density in $N=(2,2)$ superspace has the form:

$$K = K(\phi, \bar\phi, \chi, \bar\chi, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R) \tag{2.4}$$

For the four isometries listed above the corresponding invariant Lagrange densities are[1]:

$$k_\phi\, K(\phi+\bar\phi,\chi,\bar\chi,\mathbb{X}_L,\bar{\mathbb{X}}_L,\mathbb{X}_R,\bar{\mathbb{X}}_R)=0 \tag{2.5}$$

$$k_\chi\, K(\phi,\bar\phi,\chi+\bar\chi,\mathbb{X}_L,\bar{\mathbb{X}}_L,\mathbb{X}_R,\bar{\mathbb{X}}_R)=0 \tag{2.6}$$

$$k_{\phi\chi}\, K(\phi+\bar\phi,\chi+\bar\chi,i(\phi-\bar\phi+\chi-\bar\chi),\mathbb{X}_L,\bar{\mathbb{X}}_L,\mathbb{X}_R,\bar{\mathbb{X}}_R)=0 \tag{2.7}$$

$$k_{LR}\, K(\phi,\bar\phi,\chi,\bar\chi,\mathbb{X}_L+\bar{\mathbb{X}}_L,\mathbb{X}_R+\bar{\mathbb{X}}_R,i(\mathbb{X}_L-\bar{\mathbb{X}}_L+\mathbb{X}_R-\bar{\mathbb{X}}_R))=0 \tag{2.8}$$

In general, the isometries act on the coordinates with some constant parameter λ:

$$\delta z=[\lambda k,z]\quad , \tag{2.9}$$

where z is any of the coordinates $\phi,\chi,\mathbb{X}_L,\mathbb{X}_R$, etc.

2.2 Gauging and Vector Multiplets

We now promote the isometries to local gauge symmetries: the constant transformation parameter λ of (2.9) becomes a local parameter Λ that obeys the appropriate constraints.

$$\begin{aligned}
\delta_g\phi=i\Lambda &\quad\Rightarrow\quad \bar{\mathbb{D}}_\pm\Lambda=0\\
\delta_g\bar\phi=-i\bar\Lambda &\quad\Rightarrow\quad \mathbb{D}_\pm\bar\Lambda=0\\
\delta_g\chi=i\tilde\Lambda &\quad\Rightarrow\quad \bar{\mathbb{D}}_+\tilde\Lambda=\mathbb{D}_-\tilde\Lambda=0\\
\delta_g\bar\chi=-i\bar{\tilde\Lambda} &\quad\Rightarrow\quad \mathbb{D}_+\bar{\tilde\Lambda}=\bar{\mathbb{D}}_-\bar{\tilde\Lambda}=0\\
\delta_g\mathbb{X}_L=i\Lambda_L &\quad\Rightarrow\quad \bar{\mathbb{D}}_+\Lambda_L=0\\
\delta_g\mathbb{X}_R=i\Lambda_R &\quad\Rightarrow\quad \bar{\mathbb{D}}_-\Lambda_R=0\\
\delta_g\bar{\mathbb{X}}_L=-i\bar\Lambda_L &\quad\Rightarrow\quad \mathbb{D}_+\bar\Lambda_L=0\\
\delta_g\bar{\mathbb{X}}_R=-i\Lambda_R &\quad\Rightarrow\quad \mathbb{D}_-\bar\Lambda_R=0\ .
\end{aligned} \tag{2.10}$$

To ensure the invariance of the Lagrange densities (2.5-2.8) under the local transformations (2.10), we introduce the appropriate vector multiplets. For the isometries (2.5,2.6) these give the well known transformation properties

[1] Generally, isometries may leave the Lagrange density invariant only up to a (generalized) Kähler transformation [19, 14], but as our interest here is the structure of the vector multiplet, we are free to choose the simplest situation.

for the usual (un)twisted vector multiplets:

$$\begin{aligned}
\delta_g V^\phi = i(\bar\Lambda - \Lambda) \;&\Rightarrow\; \delta_g(\phi + \bar\phi + V^\phi) = 0 \\
\delta_g V^\chi = i(\bar{\tilde\Lambda} - \tilde\Lambda) \;&\Rightarrow\; \delta_g(\chi + \bar\chi + V^\chi) = 0 \ ,
\end{aligned} \tag{2.11}$$

whereas for generalized Kähler transformations we need to add *triplets* of vector multiplets.

For the the semichiral isometry k_{LR}, we introduce the vector multiplets:

$$\begin{aligned}
\delta_g \mathbb{V}^L = i(\bar\Lambda_L - \Lambda_L) \;&\Rightarrow\; \delta_g(\mathbb{X}_L + \bar{\mathbb{X}}_L + \mathbb{V}^L) = 0 \\
\delta_g \mathbb{V}^R = i(\bar\Lambda_R - \Lambda_R) \;&\Rightarrow\; \delta_g(\mathbb{X}_R + \bar{\mathbb{X}}_R + \mathbb{V}^R) = 0 \\
\delta_g \mathbb{V}' = \Lambda_L + \bar\Lambda_L + \Lambda_R + \bar\Lambda_R \;&\Rightarrow\; \delta_g(i(\mathbb{X}_L - \bar{\mathbb{X}}_L + \mathbb{X}_R - \bar{\mathbb{X}}_R) + \mathbb{V}') = 0
\end{aligned} \tag{2.12}$$

We refer to this multiplet as the semichiral vector multiplet.

For the $k_{\phi\chi}$ isometry we introduce the vector multiplets

$$\begin{aligned}
\delta_g V^\phi = i(\bar\Lambda - \Lambda) \;&\Rightarrow\; \delta_g(\phi + \bar\phi + V^\phi) = 0 \\
\delta_g V^\chi = i(\bar{\tilde\Lambda} - \tilde\Lambda) \;&\Rightarrow\; \delta_g(\chi + \bar\chi + V^\chi) = 0 \\
\delta_g V' = \Lambda + \bar\Lambda + \tilde\Lambda + \bar{\tilde\Lambda} \;&\Rightarrow\; \delta_g(i(\phi - \bar\phi + \chi - \bar\chi) + V') = 0 \ ,
\end{aligned} \tag{2.13}$$

and refer to this multiplet as the large vector multiplet due to the large number of gauge-invariant components that comprise it.

2.3 $N=(2,2)$ field-strengths

We now construct the $N=(2,2)$ gauge invariant field-strengths for the various multiplets introduced above.

2.3.1 The known field-strengths

The field-strengths for the usual vector multiplets are well known:

$$\begin{aligned}
\tilde W = i\,\mathbb{D}_-\bar{\mathbb{D}}_+ V^\phi \quad &, \quad \bar{\tilde W} = i\,\bar{\mathbb{D}}_-\mathbb{D}_+ V^\phi \ , \\
W = i\,\bar{\mathbb{D}}_-\bar{\mathbb{D}}_+ V^\chi \quad &, \quad \bar W = i\,\mathbb{D}_-\mathbb{D}_+ V^\chi \ .
\end{aligned} \tag{2.14}$$

Note that $\tilde W$, the field-strength for the chiral isometry is twisted chiral whereas W, the field-strength for the twisted chiral isometry, is chiral.

2.3.2 Semichiral field-strengths

To find the gauge-invariant field-strengths for the vector multiplet that gauges the semichiral isometry it is useful to introduce the complex combinations:

$$
\begin{aligned}
\mathbb{V} &= \frac{1}{2}(\mathbb{V}' + i(\mathbb{V}^L + \mathbb{V}^R)) \;\;\Rightarrow\;\; \delta_g \mathbb{V} = \Lambda_L + \Lambda_R \;, \\
\tilde{\mathbb{V}} &= \frac{1}{2}(\mathbb{V}' + i(\mathbb{V}^L - \mathbb{V}^R)) \;\;\Rightarrow\;\; \delta_g \tilde{\mathbb{V}} = \Lambda_L + \bar{\Lambda}_R \;.
\end{aligned}
\tag{2.15}
$$

Then the following complex field-strengths are gauge invariant:

$$
\begin{aligned}
\mathbb{F} &= \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathbb{V} \;\;, \quad \bar{\mathbb{F}} = -\mathbb{D}_+\mathbb{D}_-\bar{\mathbb{V}} \;, \\
\tilde{\mathbb{F}} &= \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{\mathbb{V}} \;\;, \quad \bar{\tilde{\mathbb{F}}} = -\mathbb{D}_+\bar{\mathbb{D}}_-\bar{\tilde{\mathbb{V}}} \;,
\end{aligned}
\tag{2.16}
$$

where $\mathbb{F}$ is chiral and $\tilde{\mathbb{F}}$ is twisted chiral.

2.3.3 Large Vector Multiplet field-strengths

As above it is useful to introduce the complex potentials:

$$
\begin{aligned}
V &= \frac{1}{2}[V' + i(V^\phi + V^\chi)] \;\;\Rightarrow\;\; \delta_g V = \Lambda + \tilde{\Lambda} \;, \\
\tilde{V} &= \frac{1}{2}[V' + i(V^\phi - V^\chi)] \;\;\Rightarrow\;\; \delta_g \tilde{V} = \Lambda + \bar{\tilde{\Lambda}} \;.
\end{aligned}
\tag{2.17}
$$

Because $(\tilde{\Lambda})\Lambda$ are (twisted)chiral respectively, the following complex spinor field-strengths are gauge invariant:

$$
\begin{aligned}
\mathbb{G}_+ &= \bar{\mathbb{D}}_+ V \;\;, \quad \bar{\mathbb{G}}_+ = \mathbb{D}_+ \bar{V} \;, \\
\mathbb{G}_- &= \bar{\mathbb{D}}_- \tilde{V} \;\;, \quad \bar{\mathbb{G}}_- = \mathbb{D}_- \bar{\tilde{V}} \;.
\end{aligned}
\tag{2.18}
$$

The higher dimension field-strengths can all be constructed from these spinor field-strengths:

$$
\begin{aligned}
W &= -i\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-V^\chi = \bar{\mathbb{D}}_+\mathbb{G}_- + \bar{\mathbb{D}}_-\mathbb{G}_+ \\
\bar{W} &= -i\mathbb{D}_+\mathbb{D}_-V^\chi = -(\mathbb{D}_+\bar{\mathbb{G}}_- + \mathbb{D}_-\bar{\mathbb{G}}_+) \\
\tilde{W} &= -i\mathbb{D}_+\bar{\mathbb{D}}_-V^\phi = \bar{\mathbb{D}}_+\bar{\mathbb{G}}_- + \mathbb{D}_-\mathbb{G}_+ \\
\bar{\tilde{W}} &= -i\bar{\mathbb{D}}_+\mathbb{D}_-V^\chi = -(\mathbb{D}_+\mathbb{G}_- + \bar{\mathbb{D}}_-\bar{\mathbb{G}}_+) \\
B &= -\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-(V' + iV^\phi) = \bar{\mathbb{D}}_-\mathbb{G}_+ - \bar{\mathbb{D}}_+\mathbb{G}_-
\end{aligned}
$$

$$
\begin{aligned}
\bar{B} &= \mathbb{D}_+\mathbb{D}_-(V' - iV^\phi) = -(\mathbb{D}_-\bar{\mathbb{G}}_+ - \mathbb{D}_+\bar{\mathbb{G}}_-) \\
\tilde{B} &= -\mathbb{D}_+\bar{\mathbb{D}}_-(V' - iV^\chi) = \mathbb{D}_-\mathbb{G}_+ - \bar{\mathbb{D}}_+\bar{\mathbb{G}}_- \\
\bar{\tilde{B}} &= \bar{\mathbb{D}}_+\mathbb{D}_-(V' + iV^\chi) = -(\bar{\mathbb{D}}_-\bar{\mathbb{G}}_+ - \mathbb{D}_+\mathbb{G}_-) \ ;
\end{aligned} \tag{2.19}
$$

the chirality properties of these field-strengths are summarized below:

Field-strength	Property
W, B	chiral
$\bar{W}, \bar{B}$	anti-chiral
$\tilde{W}, \tilde{B}$	twisted chiral
$\bar{\tilde{W}}, \bar{\tilde{B}}$	anti-twisted chiral

(2.20)

3 Gauge multiplets in $N=(1,1)$ superspace

To reveal the physical content of the gauge multiplets, we could go to components, but it is simpler and more informative to go to $N=(1,1)$ superspace. We expect to find spinor gauge connections and unconstrained superfields. As mentioned in the introduction, the component content of various $N=(1,1)$ multiplets can be found in [13].

The procedure for going to $N=(1,1)$ components is well-known; for a convenient review, see [14]. We write the $N=(2,2)$ derivatives $\mathbb{D}_\pm$ and their complex conjugates $\bar{\mathbb{D}}_\pm$ in terms of real $N=(1,1)$ derivatives $D_\pm$ and the generators $Q_\pm$ of the nonmanifest supersymmetries,

$$
\mathbb{D}_\pm = \frac{1}{2}(D_\pm - iQ_\pm) \quad , \qquad \bar{\mathbb{D}}_\pm = \frac{1}{2}(D_\pm + iQ_\pm) \ , \tag{3.1}
$$

and $N=(1,1)$ components of an unconstrained superfield Ψ as $\Psi| = \phi$, $Q_\pm\Psi| = \psi_\pm$, and $Q_+Q_-\Psi| = F$.

3.1 The semichiral vector multiplet

We first identify the $N=(1,1)$ components of the semichiral vector multiplet, and then describe various couplings to matter.

3.1.1 $N = (1,1)$ components of the gauge multiplet

We can find all the $N=(1,1)$ components of the semichiral gauge multiplet from the field strengths (2.16) except for the spinor connections $\Gamma_\pm$. The only linear combination of the gauge parameters Λ_R, Λ_L that does not enter algebraically in (2.12) is $(\Lambda_L + \bar{\Lambda}_L - \Lambda_R - \bar{\Lambda}_R)$, and hence the connections must transform as:

$$\delta_g \Gamma_\pm = \frac{1}{4} D_\pm (\Lambda_L + \bar{\Lambda}_L - \Lambda_R - \bar{\Lambda}_R)\Big| \ . \tag{3.2}$$

This allows us to determine the connections as:

$$\Gamma_+ = \left(\frac{1}{2} Q_+ \mathbb{V}^L - \frac{1}{4} D_+ \mathbb{V}'\right)\Big| \quad , \quad \Gamma_- = -\left(\frac{1}{2} Q_- \mathbb{V}^R - \frac{1}{4} D_- \mathbb{V}'\right)\Big| \ , \tag{3.3}$$

where the $D_\pm$ terms vanish in Wess-Zumino gauge. The gauge-invariant component fields are just the projections of the $N=(2,2)$ field-strengths (2.16) and the field-strength of the connection $\Gamma_\pm$:

$$f = i(D_+ \Gamma_- + D_- \Gamma_+) \ . \tag{3.4}$$

These are not all independent–they obey the Bianchi identity:

$$f = i\left(\mathbb{F} - \bar{\mathbb{F}} + \tilde{\mathbb{F}} - \bar{\tilde{\mathbb{F}}}\right)\Big| \ . \tag{3.5}$$

Thus this gauge multiplet is described by an $N=(1,1)$ gauge multiplet and three real unconstrained $N=(1,1)$ scalar superfields:

$$\hat{d}^1 = \left(\mathbb{F} + \bar{\mathbb{F}}\right)\big| \quad , \quad \hat{d}^2 = \left(\tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big| \quad , \quad \hat{d}^3 = i\left(\mathbb{F} - \bar{\mathbb{F}} - \tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big| \ . \tag{3.6}$$

Though not essential, the simplest way to find the $N=(1,1)$ reduction of various $N=(2,2)$ quantities is to go to a Wess-Zumino gauge, that is reducing the $N=(2,2)$ gauge parameters to a *single* $N=(1,1)$ gauge parameter by gauging away all $N=(1,1)$ components with algebraic gauge transformations. Here this means imposing

$$\begin{array}{lll} \mathbb{V}^L| = 0 \ , & (Q_+\mathbb{V}^L)| = 2\Gamma_+ \ , & (Q_-\mathbb{V}^L)| = 0 \ , \\ \mathbb{V}^R| = 0 \ , & (Q_+\mathbb{V}^R)| = 0 \ , & (Q_-\mathbb{V}^R)| = -2\Gamma_- \ , \\ \mathbb{V}'| = 0 \ , & (Q_+\mathbb{V}')| = 0 \ , & (Q_-\mathbb{V}')| = 0 \ , \end{array} \tag{3.7}$$

on the gauge multiplet and

$$\Lambda^L| = \bar{\Lambda}^L| = -\Lambda^R| = -\bar{\Lambda}^R| \ , \quad (Q_-\Lambda^L)| = (Q_-\bar{\Lambda}^L)| = (Q_+\Lambda^R)| = (Q_+\bar{\Lambda}^R)| = 0 \tag{3.8}$$

on the gauge parameters. This leads directly to:

$$(Q_+Q_-\mathbb{V}^L)| = 2i(\hat{d}^1 - \hat{d}^2) \ , \quad (Q_+Q_-\mathbb{V}^R)| = 2i(\hat{d}^1 + \hat{d}^2) \ , \quad (Q_+Q_-\mathbb{V}')| = 2i\hat{d}^3 \ . \tag{3.9}$$

3.1.2 Coupling to matter

We start from the gauged $N=(2,2)$ Lagrange density:

$$K_{\mathbb{X}} = K_{\mathbb{X}}\left(\mathbb{X}_L + \bar{\mathbb{X}}_L + \mathbb{V}^L, \mathbb{X}_R + \bar{\mathbb{X}}_R + \mathbb{V}^R, i(\mathbb{X}_L - \bar{\mathbb{X}}_L + \mathbb{X}_R - \bar{\mathbb{X}}_R) + \mathbb{V}'\right) \ . \tag{3.10}$$

In the Wess-Zumino gauge defined above, we have

$$X_{L(R)} = \mathbb{X}_{L(R)}| \ , \tag{3.11}$$

and $N=(1,1)$ spinor components:

$$\begin{aligned} (Q_+\mathbb{X}_L)| = iD_+X_L + \Gamma_+ \quad &, \quad (Q_-\mathbb{X}_L)| = \psi_- \ , \\ (Q_-\mathbb{X}_R)| = iD_-X_R - \Gamma_- \quad &, \quad (Q_+\mathbb{X}_R)| = \psi_+ \ . \end{aligned} \tag{3.12}$$

Then for the tuple X^i and the isometry vector k^i defined as

$$\begin{aligned} k^i \ &\equiv \ k_{\phi\chi} = k_{LR} = (i, -i, -i, i) \ , \\ X^i \ &= \ (X_L, \bar{X}_L, X_R, \bar{X}_R) \ , \end{aligned} \tag{3.13}$$

we write the gauge covariant derivative as it appears in [8]

$$\nabla_\pm X^i = D_\pm X^i - \Gamma_\pm k^i. \tag{3.14}$$

We can compute

$$\begin{aligned} (Q_+Q_-\mathbb{X}_L)| \ &= \ iD_+\psi_- + i(\hat{d}^1 - \hat{d}^2) + \hat{d}^3 \\ (Q_+Q_-\mathbb{X}_R)| \ &= \ -iD_+\psi_- + i(\hat{d}^1 + \hat{d}^2) + \hat{d}^3 \ . \end{aligned} \tag{3.15}$$

Using

$$\frac{\partial^2 K}{\partial X^i \partial X^j} k^i = 0 \quad \Rightarrow \quad \frac{\partial^2 K}{\partial X^i \partial X^j} D_\pm X^i = \frac{\partial^2 K}{\partial X^i \partial X^j} \nabla_\pm X^i \ , \tag{3.16}$$

we obtain the gauged $N\!=\!(1,1)$ Lagrange density

$$E_{ij} \nabla_+ X^i \nabla_- X^j + K_i L^i{}_\alpha \hat{d}^\alpha \ , \tag{3.17}$$

with:

$$L = \begin{pmatrix} i & -i & 1 \\ -i & i & 1 \\ i & i & 1 \\ -i & -i & 1 \end{pmatrix} \ . \tag{3.18}$$

Here $E = \frac{1}{2}(g + B)$ in the reduced Lagrange density is that same as for the ungauged σ-model [14, 18].

3.1.3 The vector multiplet action

Introducing the notation

$$\mathbb{F}^i \equiv (\mathbb{F}, \bar{\mathbb{F}}, \tilde{\mathbb{F}}, \bar{\tilde{\mathbb{F}}}) \quad , \qquad d^i \equiv (f, \hat{d}^1, \hat{d}^2, \hat{d}^3) \ , \tag{3.19}$$

and using the (twisted)chirality properties

$$\bar{\mathbb{D}}_\pm \mathbb{F} = \mathbb{D}_\pm \bar{\mathbb{F}} = \bar{\mathbb{D}}_+ \tilde{\mathbb{F}} = \mathbb{D}_- \tilde{\mathbb{F}} = \mathbb{D}_+ \bar{\tilde{\mathbb{F}}} = \bar{\mathbb{D}}_- \bar{\tilde{\mathbb{F}}} = 0 \ , \tag{3.20}$$

we find

$$(Q_\pm \mathbb{F}^i)| = J_\pm{}^i{}_j \, M^j{}_k (D_\pm \hat{d}^k) \ , \tag{3.21}$$

with

$$M = \frac{1}{4} \begin{pmatrix} -i & 2 & 0 & -i \\ i & 2 & 0 & i \\ -i & 0 & 2 & i \\ i & 0 & 2 & -i \end{pmatrix} \quad , \quad J_\pm \equiv \mathrm{diag}(i, -i, \pm i, \mp i) \ . \tag{3.22}$$

Starting from an $N{=}(2,2)$ action:

$$S_{\mathbb{X}} = \int d^2\xi\, D_+D_-Q_+Q_- \left(a\,\mathbb{F}\bar{\mathbb{F}} - b\,\tilde{\mathbb{F}}\bar{\tilde{\mathbb{F}}}\right) \tag{3.23}$$

we write the reduction to $N{=}(1,1)$ in terms of the gauge-invariant $N{=}(1,1)$ components $\hat{d}^i$:

$$S_{\mathbb{X}} = \frac{1}{2}\int d^2\xi\, D_+D_- \left(D_+\hat{d}^i\, D_-\hat{d}^j\, g_{ij}\right) \ , \tag{3.24}$$

where

$$g = \frac{1}{8}\begin{pmatrix} a+b & 0 & 0 & a-b \\ 0 & 4a & 0 & 0 \\ 0 & 0 & 4b & 0 \\ a-b & 0 & 0 & a+b \end{pmatrix} \ . \tag{3.25}$$

To obtain real and positive definite g we require $ab > 0$ which yields one $N = (1,1)$ gauge multiplet and three scalar multiplets. In particular, when $a = b$, we find the usual diagonal action.

Other gauge-invariant terms are possible; these are general superpotentials and have the form

$$S_P = \int i\mathbb{D}_+\mathbb{D}_-\, P_1(\mathbb{F}) + \int i\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\, \bar{P}_1(\bar{\mathbb{F}}) + \int i\mathbb{D}_+\bar{\mathbb{D}}_-\, P_2(\tilde{\mathbb{F}}) + \int i\bar{\mathbb{D}}_+\mathbb{D}_-\, \bar{P}_2(\bar{\tilde{\mathbb{F}}}) \ , \tag{3.26}$$

where P are holomorphic functions. These terms reduce trivially to give:

$$S_P = 2\int iD_+D_-\ \mathrm{Re}\Big(P_1(\tfrac{1}{2}\hat{d}^1 - \tfrac{i}{4}(f + \hat{d}^3)) + P_2(\tfrac{1}{2}\hat{d}^2 - \tfrac{i}{4}(f - \hat{d}^3))\Big) \ . \tag{3.27}$$

Particular examples of such superpotentials include mass and Fayet-Iliopoulos terms.

3.1.4 Linear terms

To perform T-duality transformations, one gauges an isometry, and then constrains the field-strength to vanish [8, 12]. We will discuss T-duality for generalized Kähler geometry in detail in [9]; it was introduced (without exploring the gauge aspects) in [10, 11]. Here we describe the $N = (2,2)$ superspace coupling and its reduction to $N{=}(1,1)$. We constrain the field-strengths to vanish using unconstrained complex Lagrange multiplier superfields $\Psi, \tilde{\Psi}$

$$\mathcal{L}_{linear} = \Psi\mathbb{F} + \bar{\Psi}\bar{\mathbb{F}} + \tilde{\Psi}\tilde{\mathbb{F}} + \bar{\tilde{\Psi}}\bar{\tilde{\mathbb{F}}} \ ; \tag{3.28}$$

integrating by parts, we can re-express this in terms of chiral and twisted chiral Lagrange multipliers $\phi = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\Psi$, $\chi = \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{\Psi}$ to obtain

$$\mathcal{L}_{linear} = \phi\mathbb{V} + \bar{\phi}\bar{\mathbb{V}} + \chi\tilde{\mathbb{V}} + \bar{\chi}\bar{\tilde{\mathbb{V}}} \ . \tag{3.29}$$

This reduces to an $N=(1,1)$ superspace Lagrange density (up to total derivative terms)

$$\begin{aligned} \mathcal{L}_{linear} \ =& \ \phi(i\hat{d}^3 - 2\hat{d}^1 + if) + \bar{\phi}(i\hat{d}^3 + 2\hat{d}^1 + if) \\ &+ \chi(i\hat{d}^3 + 2\hat{d}^2 - if) + \bar{\chi}(i\hat{d}^3 - 2\hat{d}^2 - if) \ , \end{aligned} \tag{3.30}$$

where $\phi, \bar{\phi}, \chi, \bar{\chi}$ are the obvious $N = (1,1)$ projections of the corresponding $N=(2,2)$ Lagrange multipliers. When we perform a T-duality transformation, we add this to the Lagrange density (3.17).

3.2 The Large Vector Multiplet

We now study the $N=(1,1)$ components of the large vector multiplet.

3.2.1 $N=(1,1)$ gauge invariants

Starting with the eight $N = (2,2)$ second-order gauge invariants (2.19), we descend to $N = (1,1)$ superspace and identify the $N = (1,1)$ gauge field-strength.

Imposing the condition that the $N=(1,1)$ gauge connection transforms as

$$\delta_g A_\pm = \frac{1}{4}D_\pm(\bar{\tilde{\Lambda}} + \tilde{\Lambda} - \bar{\Lambda} - \Lambda) \ , \tag{3.31}$$

we find the quantities

$$\begin{aligned} A_+ \ &= \ -\left(\frac{1}{4}Q_+(V^\phi - V^\chi)\right)\bigg| = \left(\frac{i}{4}Q_+(\tilde{V} - \bar{\tilde{V}})\right)\bigg| , \\ A_- \ &= \ -\left(\frac{1}{4}Q_-(V^\phi + V^\chi)\right)\bigg| = \left(\frac{i}{4}Q_-(V - \bar{V})\right)\bigg| ; \end{aligned} \tag{3.32}$$

of course, any gauge-invariant spinor may be added to $A_\pm$. It is useful to introduce the real and imaginary parts of $\mathbb{G}_\pm$:

$$\Xi^A_\pm = (\,\mathrm{Re}(\mathbb{G}_\pm)|\,,\mathrm{Im}(\mathbb{G}_\pm)|\,)\ . \tag{3.33}$$

These form a basis for the $N=(1,1)$ gauge-invariant spinors. The field-strength of the connection $A_\pm$

$$f = i(D_+A_- + D_-A_+) = i(Q_+\Xi^2_- + Q_-\Xi^2_+) \tag{3.34}$$

is manifestly gauge invariant. The remaining $N\!=\!(1,1)$ gauge-invariant scalars are:

$$\begin{aligned}\hat q^1 &= i(Q_-\Xi^1_+ - Q_+\Xi^1_-)\ ,\\ \hat q^2 &= i(Q_-\Xi^1_+ + Q_+\Xi^1_-)\ ,\\ \hat q^3 &= i(Q_-\Xi^2_+ - Q_+\Xi^2_-)\ .\end{aligned} \tag{3.35}$$

The decomposition of the $N\!=\!(2,2)$ invariants W, B is

$$F^i = \left.\begin{pmatrix} W\\ B\\ \bar W\\ \bar B\\ \tilde W\\ \tilde B\\ \bar{\tilde W}\\ \bar{\tilde B}\end{pmatrix}\right| = \frac{1}{2}\begin{pmatrix} -i & -i & 1 & 1 & 0 & 1 & 0 & i\\ i & -i & -1 & 1 & 1 & 0 & i & 0\\ i & i & 1 & 1 & 0 & 1 & 0 & -i\\ -i & i & -1 & 1 & 1 & 0 & -i & 0\\ -i & -i & -1 & 1 & -1 & 0 & 0 & -i\\ i & -i & 1 & 1 & 0 & -1 & -i & 0\\ i & i & -1 & 1 & -1 & 0 & 0 & i\\ -i & i & 1 & 1 & 0 & -1 & i & 0\end{pmatrix}\begin{pmatrix} iD_+\Xi^1_-\\ iD_-\Xi^1_+\\ iD_+\Xi^2_-\\ iD_-\Xi^2_+\\ \hat q^1\\ \hat q^2\\ \hat q^3\\ f\end{pmatrix}\ . \tag{3.36}$$

3.2.2 Matter couplings in $N\!=\!(1,1)$ superspace

We start from the gauged $N\!=\!(2,2)$ Lagrange density:

$$K_\phi\left(\phi+\bar\phi+V^\phi, \chi+\bar\chi+V^\chi, i(\phi-\bar\phi+\chi-\bar\chi)+V'\right)\ . \tag{3.37}$$

We reduce to $N\!=\!(1,1)$ superfields, which in the Wess-Zumino gauge

$$V^\phi| = 0\ ,\quad V^\chi| = 0\ ,\quad V'| = 0\ , \tag{3.38}$$

are simply

$$
\begin{aligned}
\phi| &= \phi \ , \\
\chi| &= \chi \ , \\
(Q_+\phi)| &= +iD_+\phi - (\Xi^1_+ + i\Xi^2_+) - A_+ \ , \\
(Q_+\chi)| &= +iD_+\chi - (\Xi^1_+ + i\Xi^2_+) + A_+ \ , \\
(Q_-\phi)| &= +iD_-\phi - (\Xi^1_- + i\Xi^2_-) - A_- \ , \\
(Q_-\chi)| &= -iD_-\chi + (\Xi^1_- - i\Xi^2_-) - A_- \ . \qquad (3.39)
\end{aligned}
$$

It is useful to introduce the notation

$$\varphi^i \ = \ (\phi, \bar{\phi}, \chi, \bar{\chi}) \qquad (3.40)$$

and the covariant derivatives

$$\nabla_\pm \varphi^i = D_\pm \varphi^i + A_\pm k^i \ . \qquad (3.41)$$

This gives

$$Q_\pm \varphi^i = J_\pm{}^i{}_j \nabla_\pm \varphi^j + \Xi^1_\pm J_\mp{}^i{}_j k^j + \Xi^2_\pm \Pi^i{}_j k^j \qquad (3.42)$$

and

$$
\begin{aligned}
2Q_+Q_-\varphi^i = \ & D_+(\Pi^i{}_j \nabla_- \varphi^j - \Xi^1_- k^i - 2\Xi^2_- J_-{}^i{}_j k^j) \\
& - D_-(\Pi^i{}_j \nabla_+ \varphi^j - \Xi^1_+ k^i - 2\Xi^2_+ J_+{}^i{}_j k^j) + 2\tilde{L}^i{}_\alpha \hat{q}^\alpha \qquad (3.43)
\end{aligned}
$$

where $\alpha = 1, 2, 3$ and

$$\tilde{L} = -\frac{i}{2} \begin{pmatrix} 2 & 0 & i \\ 2 & 0 & -i \\ 0 & 2 & i \\ 0 & 2 & -i \end{pmatrix} \qquad (3.44)$$

The $N\!=\!(1,1)$ superspace Lagrange density is (after integrating by parts and using the isometry)

$$
\begin{aligned}
\mathcal{L} = K_{ij} \Big[& -\tfrac{1}{2} \left(\nabla_+ \varphi^i \left(\Pi^j{}_l \nabla_- \varphi^l - 2\Xi^2_- J_-{}^j{}_l k^l \right) + \left(\Pi^i{}_k \nabla_+ \varphi^k - 2\Xi^2_+ J_+{}^i{}_k k^k \right) \nabla_- \varphi^j \right) \\
& + \left(J_+{}^i{}_k \nabla_+ \varphi^k + \Xi^1_+ J_-{}^i{}_k k^k + \Xi^2_+ \Pi^i{}_k k^k \right) \left(J_-{}^j{}_l \nabla_- \varphi^l + \Xi^1_- J_+{}^j{}_l k^l + \Xi^2_- \Pi^j{}_l k^l \right) \Big] \\
& + K_i \tilde{L}^i{}_\alpha \hat{q}^\alpha \ . \qquad (3.45)
\end{aligned}
$$

The large vector multiplet has the gauge-invariant spinors $\Xi^A_\pm$; it is useful to isolate their contribution to expose the underlying $N\!=\!(1,1)$ gauged nonlinear σ-model. We define the matrices:

$$E_{kl} = \tfrac{1}{2}K_{ij}\left(2J_+{}^i{}_kJ_-{}^j{}_l - \Pi^i{}_k\delta^j{}_l - \Pi^j{}_l\delta^i{}_k\right) \tag{3.46}$$

$$E_{Al} = \begin{pmatrix} K_{ij}J_-{}^i{}_kk^kJ_-{}^j{}_l \\ K_{ij}\left(J_+{}^i{}_kk^k\delta^j{}_l + \Pi^i{}_kk^kJ_-{}^j{}_l\right) \end{pmatrix} \tag{3.47}$$

$$E_{kA} = \left(K_{ij}J_+{}^i{}_kJ_+{}^j{}_lk^l\ ,\ K_{ij}\left(J_-{}^j{}_lk^l\delta^i{}_k + J_+{}^i{}_k\Pi^j{}_lk^l\right)\right) \tag{3.48}$$

$$E_{AB} = \begin{pmatrix} K_{ij}J_-{}^i{}_kk^kJ_+{}^j{}_lk^l & K_{ij}\Pi^i{}_kk^kJ_+{}^j{}_lk^l \\ K_{ij}J_-{}^i{}_kk^k\Pi^j{}_lk^l & K_{ij}\Pi^i{}_kk^k\Pi^j{}_lk^l \end{pmatrix} \tag{3.49}$$

We find

$$\begin{aligned}\mathcal{L} = &\left(\Xi^A_+ + \nabla_+\varphi^iE_{iC}E^{CA}\right)E_{AB}\left(\Xi^B_- + E^{BD}E_{Dj}\nabla_-\varphi^j\right) \\ &+\nabla_+\varphi^i\left(E_{ij} - E_{iA}E^{AB}E_{Bj}\right)\nabla_-\varphi^j + K_i\tilde{L}^i{}_\alpha\hat{q}^\alpha\end{aligned} \tag{3.50}$$

with E^{AB} the inverse of E_{AB}.

3.2.3 The vector multiplet action

A general $N\!=\!(2,2)$ action for the large multiplet can be written as

$$S_a = \int d^2\xi D_+D_-Q_+Q_-\left(F^iF^jg_{ij} + \mathbb{G}^A_+\mathbb{G}^B_-m_{AB}\right)\ , \tag{3.51}$$

where the ranges for indices are $i,j = 1,\cdots,8$; $AB = 1,2$, and the spinor invariants were arranged into tuples

$$\mathbb{G}^A_\pm = (\mathbb{G}_\pm, \bar{\mathbb{G}}_\pm)\ . \tag{3.52}$$

Other terms of the type $(\mathbb{D}_\pm,\bar{\mathbb{D}}_\pm)(\mathbb{G}_\pm,\bar{\mathbb{G}}_\pm)$ could be integrated by parts to give the W and B invariants. One could also add superpotential terms.

This action can be reduced to $N\!=\!(1,1)$ using the block-(twisted)chirality of F and the semichirality of $\mathbb{G}$. In general, one finds terms with higher derivatives; it does not seem possible to find a sensible kinetic action, but we leave a complete analysis for future work.

3.2.4 Linear terms

As discussed above for the semichiral vector multiplet, linear couplings of unconstrained Lagrange multiplier fields multiplying the field-strengths are needed to discuss T-duality. In $N=(2,2)$ superspace, we constrain the field-strengths $\mathbb{G}_\pm$ to vanish with unconstrained complex spinor Lagrange multiplier superfields $\Psi_\mp$:

$$\mathcal{L}_{linear} = i\left(\Psi_+\mathbb{G}_- + \Psi_-\mathbb{G}_+ + \bar{\Psi}_+\bar{\mathbb{G}}_- + \bar{\Psi}_-\bar{\mathbb{G}}_-\right) \ . \tag{3.53}$$

When we integrate by parts and define semichiral Lagrange multpliers $\mathbb{X}_{L,R} = -i\bar{\mathbb{D}}_\pm\Psi_\mp$, we find

$$\mathcal{L}_{linear} = \mathbb{X}_L V + \bar{\mathbb{X}}_L\bar{V} + \mathbb{X}_R\tilde{V} + \bar{\mathbb{X}}_R\bar{\tilde{V}} \ . \tag{3.54}$$

Reducing to $N=(1,1)$ supperspace, and defining $N=(1,1)$-components for the Lagrange multipliers as in (3.11,3.1.2) we find

$$\begin{aligned}\mathcal{L}_{linear} = \ & \psi_-\left(i\Xi^1_+ - \Xi^2_+\right) + \tfrac{1}{2}X_L\left((\hat{q}^2+\hat{q}^1) + i(f+\hat{q}^3)\right)\\ &+ \bar{\psi}_-\left(-i\Xi^1_+ - \Xi^2_+\right) + \tfrac{1}{2}\bar{X}_L\left(-(\hat{q}^2+\hat{q}^1) + i(f+\hat{q}^3)\right)\\ &+ \psi_+\left(-i\Xi^1_- + \Xi^2_-\right) + \tfrac{1}{2}X_R\left(-(\hat{q}^2-\hat{q}^1) - i(f-\hat{q}^1)\right)\\ &+ \bar{\psi}_+\left(i\Xi^1_+ - i\Xi^2_+\right) + \tfrac{1}{2}\bar{X}_R\left((\hat{q}^2-\hat{q}^1) - i(f-\hat{q}^1)\right) \ .\end{aligned} \tag{3.55}$$

We can easily integrate out $\psi_\pm$ and their complex conjugates; this $\Xi^A_\pm$ from the action. We are then left with the usual T-duality transformation as we shall discuss in [9].

Note:

As we were completing our work, we became aware of related work by S.J. Gates and W. Merrell; we thank them for agreeing to delay their work and post simultaneously.

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T-duality and Generalized Kähler Geometry

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Abstract

We use the new $N=(2,2)$ vector multiplets to clarify T-dualities for generalized Kähler geometries. Following the usual procedure, we gauge isometries of nonlinear σ-models and introduce Lagrange multipliers that constrain the field-strengths of the gauge fields to vanish. Integrating out the Lagrange multipliers leads to the original action, whereas integrating out the vector multiplets gives the dual action. The description is given both in $N=(2,2)$ and $N=(1,1)$ superspace.

1 Introduction

The basic inspiration for our work is the interesting duality found in [10, 11] for two dimensional nonlinear σ-models with $N = (2,2)$ supersymmetry and target space geometries that are not Kähler. As was shown in [8, 12], T-dualities arise when one gauges an isometry, and then constrains the field-strength of the corresponding gauge field to vanish. In this paper, we use the new vector multiplets introduced in [22, 24] to describe T-duality for generalized Kähler geometries (for a sampling of articles in the field, see [55]). We first work in $N = (2,2)$ superspace, and then reduce to $N = (1,1)$ superspace and find the usual T-duality of Buscher [20].

The plan of the paper is as follows: In the next section we briefly review T-duality in the pure Kähler case [20]. We then review the classes of isometries that generalized Kähler geometries admit. Next, we consider T-dualities along isometries in the kernel of the commutator of the left and right complex structures that mix chiral and twisted chiral multiplets [16]. Finally we describe T-dualities along isometries in the cokernel of the commutator, which act only on the semichiral multiplets[15].

We end with a brief conclusion.

2 Kähler geometry and T-duality

In this section, we briefly review isometries, gauging, and T-duality in $N = (2,2)$ and $N = (1,1)$ superspace for a system with chiral superfields ϕ^a and an $N = (2,2)$ superspace Lagrange density given by a Kähler potential $K(\phi^a, \bar{\phi}^a)$ [19, 12, 20]. For simplicity, we consider an isometry generated by a holomorphic Killing vector k that leaves the Kähler potential invariant[1]

$$k \equiv k^i \partial_i = k^a \partial_a + \bar{k}^a \bar{\partial}_a \quad , \quad \mathcal{L}_k K = 0 \ , \tag{2.1}$$

where $\varphi^i = \{\phi^a, \bar{\phi}^a\}$. The isometry is gauged using a multiplet V^ϕ to promote the constant (real) transformation parameter λ to a complex chiral superfields

[1]The general case when K is invariant only up to a Kähler transformation is discussed in detail in [19], and in the generalized Kähler case in [9].

Λ:

$$\lambda(k^a\partial_a+\bar{k}^a\bar{\partial}_a)K(\phi^a,\bar{\phi}^a)=0 \;\;\rightarrow\;\; \left(\Lambda k^a\partial_a+\bar{\Lambda}\bar{k}^a\bar{\partial}_a+\delta V^\phi\partial_{V^\phi}\right)K^{(g)}(\phi^a,\bar{\phi}^a,V^\phi)=0\,. \tag{2.2}$$

From (2.1), it follows that[2]

$$\left(\Lambda k^a\partial_a+\bar{\Lambda}\bar{k}^a\bar{\partial}_a\right)K(\phi^a,\bar{\phi}^a)=\tfrac{i}{2}(\bar{\Lambda}-\Lambda)\mathcal{L}_{Jk}K \quad , \quad Jk=i(k^a\partial_a-\bar{k}^a\bar{\partial}_a)\,. \tag{2.3}$$

Using the usual gauge transformation $\delta V^\phi = i(\bar{\Lambda}-\Lambda)$, we find the gauged action [19]:

$$K^{(g)}(\phi^a,\bar{\phi}^a,V^\phi)=\exp\left(-\tfrac{1}{2}V\mathcal{L}_{Jk}\right)K(\phi^a,\bar{\phi}^a)\,. \tag{2.4}$$

To find the T-dual model [12], we constrain the twisted chiral field-strength $\bar{\mathbb{D}}_+\mathbb{D}_-V^\phi$ to vanish. We impose this with a Legendre transformation of the density with a twisted chiral Lagrange multiplier χ:

$$K^{(g)}(\phi^a,\bar{\phi}^a,V^\phi)-(\chi+\bar{\chi})V^\phi\,. \tag{2.5}$$

In $N=(2,2)$ superspace, we find the T-dual Lagrange densities by integrating out either $\chi+\bar{\chi}$, which gives the original Kähler potential, or V^ϕ, which gives the T-dual potential $\tilde{K}(\chi+\bar{\chi},x^A)$ where x^A are "spectator" fields, *i.e.*, combinations of the φ^i that are inert under the action of the isometry (2.1). The geometric nature of the duality is made manifest when we descend to $N=(1,1)$ superspace. In Wess-Zumino gauge, the $N=(1,1)$ components of the multiplet V^ϕ and the covariant derivatives are

$$V^\phi|=0 \;\;,\quad Q_\pm V^\phi|=A_\pm \;\;,\quad i\,Q_+Q_-V^\phi|=d \;\;,\quad \nabla_\pm\varphi^i=D_\pm\varphi^i-A_\pm k^i \tag{2.6}$$

the constrained Lagrange density (2.5) becomes:

$$g_{ij}\nabla_+\varphi^i\nabla_-\varphi^j-i\,d(K_iJ^i_jk^j+(\chi+\bar{\chi}))+f(\chi-\bar{\chi}) \tag{2.7}$$

where $f=i(D_+A_-+D_-A_+)$ is the $N=(1,1)$ field-strength for the gauge fields, g_{ij} is the Kähler metric, and $K_iJ^i_jk^j\equiv\mathcal{L}_{Jk}K$ is proportional to the moment map when the Kähler potential is invariant (as discussed above, in general $\mathcal{L}_{Jk}K\rightarrow-\mu$). Integrating out the $N=(1,1)$ auxiliary superfield d sets

[2] When K is not invariant, $\mathcal{L}_{Jk}K$ becomes the moment map of the isometry: $\mathcal{L}_{Jk}K\rightarrow -\mu$.

$\chi+\bar\chi$ equal to the moment map. This can be solved either by expressing $\chi+\bar\chi$ as a function of φ^i, or by changing coordinates to $\chi+\bar\chi$ and a combination of φ^i algebraically independent of the moment map; the two procedures are related simply by a diffeomorphism. This gives the $N=(1,1)$ gauged Lagrange density with Lagrange multiplier $(\chi-\bar\chi)$ constraining the field-strength f:

$$\mathcal{L}_1 = g_{ij}\nabla_+\varphi^i\nabla_-\varphi^j + f(\chi-\bar\chi)\ . \tag{2.8}$$

Thus $N=(2,2)$ T-duality is the same as $N=(1,1)$ T-duality up to an accompanying diffeomorphism; this was originally proven by Buscher [20], but not explicitly spelled out.

3 T-duality for the generalized Kähler geometry

In a recent paper [22] we discussed gauge multiplets suitable for gauging isometries of generalized Kähler geometries. We found three distinct vector multiplets, corresponding to three distinct types of isometries: those along the kernel of either J_+-J_- or (equivalently) J_++J_-, those acting on both kernels, and those along the cokernel of the commutator $[J_+,J_-]$. The isometries can be expressed, following [10], in adapted coordinates:

$$k_\phi = i(\partial_\phi-\partial_{\bar\phi})\ , \tag{3.1}$$

$$k_{\phi\chi} = i(\partial_\phi-\partial_{\bar\phi}-\partial_\chi+\partial_{\bar\chi})\ , \tag{3.2}$$

$$k_{LR} = i(\partial_L-\partial_{\bar L}-\partial_R+\partial_{\bar R})\ . \tag{3.3}$$

If we assume that the generalized Kähler potential is invariant, the corresponding gauged actions are:

$$K_\phi = K_\phi\left(\phi+\bar\phi+V^\phi, x\right)\ , \tag{3.4}$$

$$K_{\phi\chi} = K_{\phi\chi}\left(\phi+\bar\phi+V^\phi, \chi+\bar\chi-V^\chi, i(\phi-\bar\phi+\chi-\bar\chi)+V', x\right)\ , \tag{3.5}$$

$$K_{\mathbb{X}} = K_{\mathbb{X}}\left(\mathbb{X}_L+\bar{\mathbb{X}}_L+\mathbb{V}^L, \mathbb{X}_R+\bar{\mathbb{X}}_R+\mathbb{V}^R, i(\mathbb{X}_L-\bar{\mathbb{X}}_L+\mathbb{X}_R-\bar{\mathbb{X}}_R)+\mathbb{V}', x\right)\ , \tag{3.6}$$

where x represents all possible spectator fields. The case (3.4) is essentially identical to the Kähler case above; aside from subtleties pertaining to the interpretation of the moment map, which will be discussed in [9], there are no new features. We now consider (3.5,3.6) in detail, and show that they again reduce to standard Buscher duality in $N=(1,1)$ superspace, along with some natural diffeomorphisms inherited from $N=(2,2)$ superspace. A more general discussion of isometries and moment maps will be given in [9].

3.1 T-duality for an isometry $k_{\phi\chi}$

For an invariant generalized Kähler potential K in adapted coordinates, the gauged action is (3.5). In the special circumstance when all the spectators are (twisted) chiral, we can give a nice geometric interpretation of the gauging analogous to the Kähler case above. In this case both complex structures are simultaneously diagonalizable; and the manifold has the **Bi**hermitian **L**ocal **P**roduct (BiLP) geometry defined in [18]. Using the invariance of K under $k_{\phi\chi}$, and using the complex structures $J_\pm$ and their product $\Pi = J_+J_-$

$$\begin{aligned}&[i(\Lambda\partial_\phi - \tilde{\Lambda}\partial_\chi) + c.c.]K \\ &\quad = \tfrac{i}{4}[(\bar{\Lambda}-\Lambda)\mathcal{L}_{(J_++J_-)k} + (\bar{\tilde{\Lambda}}-\tilde{\Lambda})\mathcal{L}_{(J_+-J_-)k} + i(\Lambda+\bar{\Lambda}-\tilde{\Lambda}-\bar{\tilde{\Lambda}})\mathcal{L}_{\Pi k}]K\end{aligned} \tag{3.7}$$

To gauge the isometry, we require

$$\begin{aligned}0 &= \delta V^\alpha \partial_{V^\alpha} K^{(g)} \\ &\quad + \tfrac{i}{4}[(\bar{\Lambda}-\Lambda)\mathcal{L}_{(J_++J_-)k} + (\bar{\tilde{\Lambda}}-\tilde{\Lambda})\mathcal{L}_{(J_+-J_-)k} + i(\Lambda+\bar{\Lambda}-\tilde{\Lambda}-\bar{\tilde{\Lambda}})\mathcal{L}_{\Pi k}]K^{(g)}\end{aligned} \tag{3.8}$$

The three superfields of the large vector multiplet [22] have the right gauge transformations to gauge this symmetry:[3]

$$\begin{aligned}&\delta V^\phi = i(\bar{\Lambda}-\Lambda) \ \ , \quad \delta V^\chi = i(\bar{\tilde{\Lambda}}-\tilde{\Lambda}) \ \ , \quad \delta V' = (-\Lambda-\bar{\Lambda}+\tilde{\Lambda}+\bar{\tilde{\Lambda}}) \\ &\quad \Rightarrow K^{(g)} = \exp\left(-\tfrac{1}{4}V^\phi\mathcal{L}_{(J_++J_-)k} - \tfrac{1}{4}V^\chi\mathcal{L}_{(J_+-J_-)k} - \tfrac{1}{4}V'\mathcal{L}_{\Pi k}\right) K \ .\end{aligned} \tag{3.9}$$

To find the T-dual, we introduce Lagrange multipliers that constrain the field strengths of the large vector multiplet to vanish. As discussed in [22], it

[3]Our conventions here, which are compatible with the inherent geometric objects $J_\pm, k$, are slightly different than those introduced in [22]; see Appendix B for the relation between the conventions.

is useful to introduce complex potentials for the field-strengths:

$$
\begin{aligned}
V_L &= \frac{1}{2}[-V' + i(V^\phi - V^\chi)] \quad \Rightarrow \quad \delta_g V_L = \Lambda - \tilde{\Lambda} \ , \\
V_R &= \frac{1}{2}[-V' + i(V^\phi + V^\chi)] \quad \Rightarrow \quad \delta_g V_R = \Lambda - \bar{\tilde{\Lambda}} \ .
\end{aligned} \tag{3.10}
$$

Since $(\tilde{\Lambda})\Lambda$ are respectively (twisted)chiral, these give the following gauge invariant complex spinor, semichiral, field-strengths:

$$
\begin{aligned}
\mathbb{G}_+ &= \bar{\mathbb{D}}_+ V_L \quad , \qquad \bar{\mathbb{G}}_+ = \mathbb{D}_+ \bar{V}_L \ , \\
\mathbb{G}_- &= \bar{\mathbb{D}}_- V_R \quad , \qquad \bar{\mathbb{G}}_- = \mathbb{D}_- \bar{V}_R \ .
\end{aligned} \tag{3.11}
$$

Using the chirality properties of the field-strengths we obtain the constrained $N=(2,2)$ generalized Kähler potential, as in (2.5), using semichiral Lagrange multipliers $\tilde{\mathbb{X}}$:

$$
K^{(g)} - \mathcal{L}_{const.} = K^{(g)} - \tfrac{1}{2}\tilde{\mathbb{X}}_L V_L - \tfrac{1}{2}\bar{\tilde{\mathbb{X}}}_L \bar{V}_L - \tfrac{1}{2}\tilde{\mathbb{X}}_R V_R - \tfrac{1}{2}\bar{\tilde{\mathbb{X}}}_R \bar{V}_R \ . \tag{3.12}
$$

This applies to the general case, not just BiLP geometries, though in general, we do not have a nice geometric form of $K^{(g)}$ (this will be discussed in [9]).

3.1.1 Reduction to $N=(1,1)$ superspace

Using the results of [22] (as summarized and clarified in Appendix A), we obtain the $N=(1,1)$ reduction of this action in the Wess-Zumino gauge; the part from $K^{(g)}$ is

$$
\begin{aligned}
\mathcal{L} \ = \ & \left(\Xi^A_+ + \nabla_+\varphi^i E_{iC}E^{CA}\right) E_{AB} \left(\Xi^B_- + E^{BD}E_{Dj}\nabla_-\varphi^j\right) \\
& + \nabla_+\varphi^i \left(E_{ij} - E_{iA}E^{AB}E_{Bj}\right) \nabla_-\varphi^j \\
& + iK_i k^j \left(\hat{q}^\phi (J^i_{+j} + J^i_{-j}) + \hat{q}^\chi (J^i_{+j} - J^i_{-j}) + \hat{q}'\Pi^i{}_j\right)
\end{aligned} \tag{3.13}
$$

where we introduce the matrices:

$$
E_{kl} \ = \ K_{ij}\left(J^i_{+k}J^j_{-l} - \tfrac{1}{2}\Pi^i{}_k\delta^j{}_l - \tfrac{1}{2}\Pi^j{}_l\delta^i{}_k\right) \tag{3.14}
$$

$$
E_{Al} \ = \ K_{ij}k^k \begin{pmatrix} J^i_{-k}J^j_{-l} \\ \Pi^i{}_k J^j_{-l} \end{pmatrix} \tag{3.15}
$$

$$E_{kA} = K_{ij}k^l\Big(J^i_{+k}J^j_{+l}\ ,\ J^i_{+k}\Pi^j{}_l\Big) \tag{3.16}$$

$$E_{AB} = K_{ij}k^k k^l \begin{pmatrix} J^i_{-k}J^j_{+l} & J^i_{-k}\Pi^j{}_l \\ \Pi^i{}_k J^j_{+l} & \Pi^i{}_k\Pi^j{}_l \end{pmatrix} \tag{3.17}$$

where the normalizations of the auxiliary fields $\Xi_\pm, \hat{q}$ as well as the field-strength f are given in Appendix A.

The constraint reduces to

$$\begin{aligned}
\mathcal{L}_{const.} &= \tilde{X}_L(i\hat{q}' - \tfrac{i}{2}f + \hat{q}^\phi - \hat{q}^\chi - iD_+\Xi^2_-) + \tilde{\psi}_-(+i\Xi^1_+ - \Xi^2_+) \\
&+ \bar{\tilde{X}}_L(i\hat{q}' - \tfrac{i}{2}f - \hat{q}^\phi + \hat{q}^\chi + iD_+\Xi^2_-) + \bar{\tilde{\psi}}_-(-i\Xi^1_+ - \Xi^2_+) \\
&+ \tilde{X}_R(i\hat{q}' + \tfrac{i}{2}f + \hat{q}^\phi + \hat{q}^\chi + iD_-\Xi^2_+) + \tilde{\psi}_+(-i\Xi^1_- + \Xi^2_-) \\
&+ \bar{\tilde{X}}_R(i\hat{q}' + \tfrac{i}{2}f - \hat{q}^\phi - \hat{q}^\chi - iD_-\Xi^2_+) + \bar{\tilde{\psi}}_+(+i\Xi^1_- + \Xi^2_-)\ ,
\end{aligned} \tag{3.18}$$

where $\tilde{X} = \tilde{\mathbb{X}}|$, $\tilde{\psi}_+ = Q_+\tilde{\mathbb{X}}_L|$ and $\tilde{\psi}_- = Q_-\tilde{\mathbb{X}}_R|$ are the $N\!=\!(1,1)$ components of the Lagrange multipliers $\tilde{\mathbb{X}}$.

3.1.2 T-duality for the large vector multiplet in $N = (1,1)$ superspace

Integrating out the auxiliaries $\tilde{\psi}_\pm$ simply constrains $\Xi^A_\pm$ to vanish, and we obtain the gauged Lagrange density:

$$\begin{aligned}
\mathcal{L} &= K_{ij}(J^i_{+k}J^j_{-l} - \tfrac{1}{2}\Pi^i{}_k\delta^j{}_l - \tfrac{1}{2}\delta^i{}_k\Pi_j{}^l)\nabla_+\varphi^k\nabla_-\varphi^l \\
&+ i\hat{q}^\phi(K_i(J^i_{+j} + J^i_{-j})k^j + i(\tilde{X}_L - \bar{\tilde{X}}_L + \tilde{X}_R - \bar{\tilde{X}}_R)) \\
&+ i\hat{q}^\chi(K_i(J^i_{+j} - J^i_{-j})k^j - i(\tilde{X}_L - \bar{\tilde{X}}_L - \tilde{X}_R + \bar{\tilde{X}}_R)) \\
&+ i\hat{q}'(K_i\Pi^i{}_j k^j - (\tilde{X}_L + \bar{\tilde{X}}_L + \tilde{X}_R + \bar{\tilde{X}}_R)) \\
&+ \tfrac{i}{2}f(\tilde{X}_L + \bar{\tilde{X}}_L - \tilde{X}_R - \bar{\tilde{X}}_R))
\end{aligned} \tag{3.19}$$

Imposing the equations of motion for $\hat{q}^\alpha$, which again just give diffeomorphisms, we obtain a gauged nonlinear σ-model with constrained field strength which proves that the dual geometries are indeed related by a Buscher duality.

3.2 T-duality along semichiral isometries k_{LR}

In the presence of semichiral superfields we can no longer decompose the action of the gauged isometry as in the BiLP case (3.7) and separate the rigid piece which acts on the Kähler potential with $\mathcal{L}_k$. An extensive treatment of non BiLP geometries is left for [9]. Making the notation of [22] compatible with the previous section we redefine the complex potentials[4] and reduce in the Wess-Zumino gauge:

$$\begin{array}{llll} \mathbb{V}^L| = 0 \ , & (Q_+\mathbb{V}^L)| = 2\Gamma_+ & (Q_-\mathbb{V}^L)| = 0 \ , & Q_+Q_-\mathbb{V}^L = -2i(\hat{d}^2 - \hat{d}^1) \\ \mathbb{V}^R| = 0 \ , & (Q_+\mathbb{V}^R)| = 0 & (Q_-\mathbb{V}^R)| = 2\Gamma_- \ , & Q_+Q_-\mathbb{V}^R = -2i(\hat{d}^2 + \hat{d}^1) \\ \mathbb{V}'\,| = 0 \ , & (Q_+\mathbb{V}'\,)| = 0 & (Q_-\mathbb{V}'\,)| = 0 \ , & Q_+Q_-\mathbb{V}' \ = -2i\hat{d}^3 \ . \end{array} \tag{3.20}$$

The $N=(1,1)$ gauge field-strength $f = i(D_+\Gamma_- + D_-\Gamma_+)$ obeys the Bianchi identity

$$i(\mathbb{F} - \bar{\mathbb{F}} + \tilde{\mathbb{F}} - \bar{\tilde{\mathbb{F}}})\Big| = f \tag{3.21}$$

(the $N=(2,2)$ field-strengths $\mathbb{F}, \tilde{\mathbb{F}}$ are given in Appendix B). Following [22] we write the constrained Lagrange density

$$K_{\mathbb{X}}\left(\mathbb{X}_L + \bar{\mathbb{X}}_L + \mathbb{V}^L, \mathbb{X}_R + \bar{\mathbb{X}}_R + \mathbb{V}^R, i(\mathbb{X}_L - \bar{\mathbb{X}}_L + \mathbb{X}_R - \bar{\mathbb{X}}_R) + \mathbb{V}'\right) - \tilde{\phi}\mathbb{V} - \bar{\tilde{\phi}}\bar{\mathbb{V}} - \tilde{\chi}\tilde{\mathbb{V}} - \bar{\tilde{\chi}}\bar{\tilde{\mathbb{V}}} \tag{3.22}$$

which reduces to $N=(1,1)$:

$$\begin{aligned} \mathcal{L} = E_{ij}\nabla_+\mathbb{X}^i\nabla_-\mathbb{X}^j &+ \hat{d}^1[(-i\partial_L - i\partial_{\bar{L}} + i\partial_R + i\partial_{\bar{R}})K + 2(\tilde{\phi} - \bar{\tilde{\phi}})] \\ &+ \hat{d}^2[(i\partial_L + i\partial_{\bar{L}} + i\partial_R + i\partial_{\bar{R}})K - 2(\tilde{\chi} - \bar{\tilde{\chi}})] \\ &+ \hat{d}^3[\tfrac{1}{2}(\partial_L - \partial_{\bar{L}} - \partial_R + \partial_{\bar{R}})K - i(\tilde{\phi} + \bar{\tilde{\phi}} + \tilde{\chi} + \bar{\tilde{\chi}})] \\ &+ f[-i\hat{\phi} - i\bar{\tilde{\phi}} + i\tilde{\chi} + i\bar{\tilde{\chi}}] \ , \end{aligned} \tag{3.23}$$

where $E_{ij} = (g_{ij} + B_{ij})$ is the metric and B-field of the generalized Kähler geometry as given in, *e.g.*, [14]. As in the previous section, we impose the equations of motion for $\hat{d}^\alpha$ to obtain the gauged nonlinear σ-model with the constraint on the field-strength f that we recognize as the hallmark of T-duality. Again, the $\hat{d}^\alpha$ equations of motion just give diffeomorphisms.

[4] See Appendix B for full details

4 Conclusions

We have used the gauge multiplets constructed in [22, 24] to investigate the duality between semichiral and (twisted) chiral superfields discovered in [10], and found that the dual geometries are related by Buscher duality. We demonstrated this in $N=(2,2)$ superspace where we gave the generalized Kähler potentials with gauged isometries. When we descended to $N=(1,1)$ superspace, the nature of the T-duality was clarified: we found a gauged nonlinear σ-model with a Lagrange multiplier constraining the field-strength of the gauge field as well as diffeomorphisms relating the generalized moment maps in the original geometry to natural coordinates in the dual geometry.

This work is part of an ongoing exploration of generalized complex geometry, using nonlinear σ models, and is therefore complimentary to the mathematical aspects of T-duality considered in [21]. The full construction of the moment maps and a geometric discussion of these results is left for future work [9].

Note:

After completing our work, we became aware of related results obtained by W. Merrell and D. Vaman.

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A Reduction to $N=(1,1)$ superspace for gauged BiLP geometries

In this appendix we review some of the results of [22] as they emerge from inherent geometric objects for BiLP geometries. The $N=(1,1)$ invariants system of [22] is slightly modified so that the reduction of the gauged Lagrange density (3.12) to $N=(1,1)$ is simpler in this context; namely, carrying out the reduction for the matter couplings piece will give convenient redefinitions for $N=(1,1)$ gauge invariants. Acting with $Q_\pm$ on the gauged action we can identify the connections $A_\pm$ that enter with $J_\pm k$ respectively:

$$\begin{aligned} Q_\pm K^{(g)} &= K^{(g)}_i \big(J^i_{\pm j} D_\pm \varphi^j - \tfrac{1}{4} Q_\pm (V^\phi + V^\chi) J^i_{+j} k^j \\ &\qquad -\tfrac{1}{4} Q_\pm (V^\phi - V^\chi) J^i_{-j} k^j - \tfrac{1}{4} Q_\pm V' \Pi^i{}_j k^j \big) \\ &= K^{(g)}_i \big(J^i_{\pm j} \nabla_\pm \varphi^j + \Xi^1_\pm J^i_{\mp j} k^j + \Xi^2_\pm \Pi^i{}_j k^j \big) \ . \end{aligned} \tag{A.1}$$

We find it useful to modify the $N=(1,1)$ notation of [22], introducing:

$$\hat{q}^\phi = -i\tfrac{1}{2}(Q_{[+}\Xi^1_{-]} - D_{[+}\Xi^2_{-]}) \ , \quad \hat{q}^\chi = -i\tfrac{1}{2}(Q_{(-}\Xi^1_{+)} + D_{(+}\Xi^2_{-)}) \ , \quad \hat{q}' = -i\tfrac{1}{2} Q_{[+}\Xi^2_{-]} \tag{A.2}$$

and the field-strength for the connections $A_\pm$

$$f = -iQ_{(+}\Xi^2_{-)} = i(D_+ A_- + D_- A_+) \tag{A.3}$$

which allows us to write the reduction for $Q_+ Q_- K^{(g)}$ in terms of the geometric objects:

$$\begin{aligned} &Q_+ Q_- K^{(g)} = \\ &\quad K^{(g)}_{ij} [(J^i_{+k} \nabla_+ \varphi^k + J^i_{-k} k^k \Xi^1_+ + \Pi^i{}_k k^k \Xi^2_+)(J^j_{-l} \nabla_- \varphi^k + J^j_{+l} k^l \Xi^1_- + \Pi^j{}_l k^l \Xi^2_-) \\ &\qquad - \tfrac{1}{2} (\delta^i{}_k \Pi^j{}_l + \Pi^i{}_k \delta^j{}_l) \nabla_+ \varphi^k \nabla_- \varphi^l] \\ &\quad + i K^{(g)}_i k^k (\hat{q}^\phi (J_+ + J_-)^i{}_k + \hat{q}^\chi (J_+ - J_-)^i{}_k + \hat{q}' \Pi^i{}_k) \end{aligned} \tag{A.4}$$

B Conventions and notation

The conversion between the notation of [22] and the current notation can be derived from changing some signs:

$$\left\{ \mathbb{V}', \mathbb{V}^R, V', V^R \right\} \to - \left\{ \mathbb{V}', \mathbb{V}^R, V', V^R \right\} \tag{B.1}$$

as well as

$$\{ \tilde{\Lambda}, \Lambda^R \} \to -\{ \tilde{\Lambda}, \Lambda^R \} \ . \tag{B.2}$$

These changes correct some unnatural conventions for the definitions of isometries.

We summarize the essential consequences here for both the large vector multiplet and the semichiral vector multiplet in the tables below.

<table>
<tr><th>Object</th><th>Old</th><th>New</th></tr>
<tr><td>δV^{ϕ}
δV^{χ}</td><td colspan="2">$i(\bar{\Lambda} - \Lambda)$
$i(\bar{\tilde{\Lambda}} - \tilde{\Lambda})$</td></tr>
<tr><td>$\delta V'$</td><td>$\Lambda + \bar{\Lambda} + \tilde{\Lambda} + \bar{\tilde{\Lambda}}$</td><td>$-\Lambda - \bar{\Lambda} + \tilde{\Lambda} + \bar{\tilde{\Lambda}}$</td></tr>
<tr><td>Complex potential
and variation (1)</td><td>$V = \frac{1}{2}(V' + i(V^{\phi} + V^{\chi}))$
$\delta V = \Lambda + \tilde{\Lambda}$</td><td>$V_L = \frac{1}{2}(-V' + i(V^{\phi} - V^{\chi}))$
$\delta V_L = \Lambda - \tilde{\Lambda}$</td></tr>
<tr><td>Complex potential
and variation (2)</td><td>$\tilde{V} = \frac{1}{2}(V' + i(V^{\phi} - V^{\chi}))$
$\delta V = \Lambda + \bar{\tilde{\Lambda}}$</td><td>$V_R = \frac{1}{2}(-V' + i(V^{\phi} + V^{\chi}))$
$\delta V_L = \Lambda - \bar{\tilde{\Lambda}}$</td></tr>
<tr><td>$N\!=\!(2,2)$
Gauge invariants</td><td>$\mathbb{G}_+ = \bar{\mathbb{D}}_+ V$
$\mathbb{G}_- = \bar{\mathbb{D}}_- \tilde{V}$
$\bar{\mathbb{G}}_+ = \mathbb{D}_+ \bar{V}$
$\bar{\mathbb{G}}_- = \mathbb{D}_- \bar{\tilde{V}}$</td><td>$\mathbb{G}_+ = \bar{\mathbb{D}}_+ V_L$
$\mathbb{G}_- = \bar{\mathbb{D}}_- V_R$
$\bar{\mathbb{G}}_+ = \mathbb{D}_+ \bar{V}_L$
$\bar{\mathbb{G}}_- = \mathbb{D}_- \bar{V}_R$</td></tr>
<tr><td>Decomposition
to $N\!=\!(1,1)$</td><td colspan="2">$\Xi^A_{\pm} = (\ \mathrm{Re}(\mathbb{G}_{\pm})|\ ,\ \mathrm{Im}(\mathbb{G}_{\pm})|\ )$
$D_{\pm}\Xi^A_{\mp}$</td></tr>
<tr><td>q-invariants:</td><td>$\hat{q}^1 = i(Q_-\Xi^1_+ - Q_+\Xi^1_-)$
$\hat{q}^2 = i(Q_-\Xi^1_+ + Q_+\Xi^1_-)$
$\hat{q}^3 = i(Q_-\Xi^2_+ - Q_+\Xi^2_-)$</td><td>$\hat{q}^{\phi} = -i\frac{1}{2}(Q_{[+}\Xi^1_{-]} - D_{[+}\Xi^2_{-]})$
$\hat{q}^{\chi} = -i\frac{1}{2}(Q_{(-}\Xi^1_{+)} + D_{(+}\Xi^2_{-)})$
$\hat{q}' = -i\frac{1}{2}(Q_+\Xi^2_- - Q_-\Xi^2_+)$</td></tr>
<tr><td>The field-strength f</td><td>$i(Q_+\Xi^2_- + Q_-\Xi^2_+)$</td><td>$-i(Q_+\Xi^2_- + Q_-\Xi^2_+)$</td></tr>
</table>

Table 1: Large vector multiplet conventions and definitions

Object	Old	New
$\delta\mathbb{V}^L$ $\delta\mathbb{V}^R$	$i(\bar{\Lambda}_L - \Lambda_L)$ $i(\bar{\Lambda}_R - \Lambda_R)$	
$\delta\mathbb{V}'$	$\Lambda_L + \bar{\Lambda}_L + \Lambda_R + \bar{\Lambda}_R$	$-\Lambda_L - \bar{\Lambda}_L + \Lambda_R + \bar{\Lambda}_R$
Complex potential and variation (1)	$\mathbb{V} = \frac{1}{2}(\mathbb{V}' + i(\mathbb{V}^L + \mathbb{V}^R))$ $\delta\mathbb{V} = \Lambda_L + \Lambda_R$	$\mathbb{V} = \frac{1}{2}(-\mathbb{V}' + i(\mathbb{V}^L - \mathbb{V}^R))$ $\delta\mathbb{V} = \Lambda_L - \Lambda_R$
Complex potential and variation (2)	$\tilde{\mathbb{V}} = \frac{1}{2}(\mathbb{V}' + i(\mathbb{V}^L - \mathbb{V}^R))$ $\delta\mathbb{V} = \Lambda_L + \bar{\Lambda}_R$	$\tilde{\mathbb{V}} = \frac{1}{2}(-\mathbb{V}' + i(\mathbb{V}^L + \mathbb{V}^R))$ $\delta\mathbb{V} = \Lambda_L - \bar{\Lambda}_R$
$N=(2,2)$ Gauge invariants	$\mathbb{F} = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathbb{V} \quad , \quad \bar{\mathbb{F}} = -\mathbb{D}_+\mathbb{D}_-\bar{\mathbb{V}}$ $\tilde{\mathbb{F}} = \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{\mathbb{V}} \quad , \quad \bar{\tilde{\mathbb{F}}} = -\mathbb{D}_+\bar{\mathbb{D}}_-\bar{\tilde{\mathbb{V}}}$	
$\hat{d}$-invariants	$\hat{d}^1 = \left(\mathbb{F} + \bar{\mathbb{F}}\right)\Big\vert \quad , \quad \hat{d}^2 = \Big(\tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\Big)\Big\vert \quad , \quad \hat{d}^3 = i\Big(\mathbb{F} - \bar{\mathbb{F}} - \tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\Big)\Big\vert$	
Gauge fields	$\Gamma_+ = \frac{1}{2}\left(Q_+\mathbb{V}^L - \frac{1}{2}D_+\mathbb{V}'\right)\big\vert$ $\Gamma_- = -\frac{1}{2}\left(Q_-\mathbb{V}^R - \frac{1}{2}D_-\mathbb{V}'\right)\big\vert$	$\Gamma_+ = \frac{1}{2}\left(Q_+\mathbb{V}^L + \frac{1}{2}D_+\mathbb{V}'\right)\big\vert$ $\Gamma_- = \frac{1}{2}\left(Q_-\mathbb{V}^R - \frac{1}{2}D_-\mathbb{V}'\right)\big\vert$
Bianchi identity	$i(\mathbb{F} - \bar{\mathbb{F}} + \tilde{\mathbb{F}} - \bar{\tilde{\mathbb{F}}})\Big\vert = f = i(D_+\Gamma_- + D_-\Gamma_+)$	

Table 2: Semichiral vector multiplet conventions and definitions

Chapter 4

Nonabelian multiplets

The new gauge multiplets can be used to covariantize $N = (2, 2)$ derivatives as in (1.5). This gives a nonabelian extension of previous results where field-strengths are proper anticommutators of covariant derivatives.

Inspecting the transformation properties of the new gauge multiplets, it is clear how their peculiar field-strengths arise. First, for the semichiral vector multiplet we find no pair of commuting covariant derivatives as in (1.5) and thus there are four field-strengths, an (anti)chiral and an (anti)twistedchiral while for the LVM we actually find two sets of covariant derivatives which differs by fermionic gauge covariant field-strengths.

Gauge actions are essential for the investigation of dualities. A far shot attempting to study mirrors for generalized Kähler manifold will require the consideration of quantum effects from such terms. We identify the $N = (1, 1)$ reduction of these field-strengths and overcome the risk of higher derivative on fermions in the LVM action by restricting to either of the sets.

While for the nonabelian case we have seen that consistent actions for the LVM must not mix the two sets of gauge invariants, this is not hold in the abelian case since we need not concern with anticommutators of the form $\{\mathbb{G}, \mathbb{G}\}$. we explore gauge actions that are not permitted for the large vector multiplet; namely, action where both types of field-strengths are present (that is, (un)hatted in the notation of [30]).

Other interesting terms are mass-like terms in $\mathbb{G}$'s which give superpotentials for the $N = (1, 1)$ scalar and field-strengths as well as possible kinetic terms for $\Xi^A_\pm$ and topologicl terms.

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Nonabelian Generalized Gauge Multiplets

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Abstract

We give the nonabelian extension of the newly discovered $N = (2, 2)$ two-dimensional vector multiplets. These can be used to gauge symmetries of sigma models on generalized Kähler geometries. Starting from the transformation rule for the nonabelian case we find covariant derivatives and gauge covariant field-strengths and write their actions in $N = (2, 2)$ and $N = (1, 1)$ superspace.

1 Introduction

Studying nonlinear sigma models whose target spaces are generalized Kähler geometries in $N=(2,2)$ superspace, we found new vector multiplets [22, 24, 23, 30] that can be used to gauge isometries which mix different types of superfields [10]. In this note, we extend these results to the nonabelian case; in particular we find the algebra of the gauge-covariant superspace derivatives.

The plan of the paper is as follows: In the next section, we review the abelian multiplets [22, 24]. In section 3, we discuss the nonabelian extensions of the large vector multiplet, which couples chiral and twisted chiral gauge symmetries [16]; we give the fundamental superfield gauge potentials, construct covariant derivatives as well as field strengths in $N=(2,2)$ superspace, reduce to $N=(1,1)$ superspace and discuss actions (*cf.* [30] for the abelian case). In section 4, we repeat this discussion for the semichiral vector multiplet [15]. We end with a few remarks. We follow the notation of [23].

2 Abelian vector multiplets

Until recently, two $N=(2,2)$ vector multiplets were known. Both are described by a single unconstrained scalar superfield V and differ by their gauge transformations: The chiral gauge multiplet transforms with a chiral gauge parameter ($\bar{\mathbb{D}}_\pm\Lambda=0$)

$$\delta V^\phi = i(\bar\Lambda-\Lambda)\ , \tag{2.1}$$

whereas the *twisted* chiral gauge multiplet transforms with a twisted chiral gauge parameter ($\bar{\mathbb{D}}_+\tilde\Lambda=\mathbb{D}_-\tilde\Lambda=0$)

$$\delta V^\chi = i(\bar{\tilde\Lambda}-\tilde\Lambda)\ . \tag{2.2}$$

These multiplets have gauge invariant twisted chiral and chiral field strengths, respectively:

$$\tilde W = \bar{\mathbb{D}}_+\mathbb{D}_-V^\phi\ ,\quad W=\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-V^\chi\ . \tag{2.3}$$

In [22], we introduced two new multiplets: the large vector multiplet (rewritten here in the conventions of [23])

$$\delta V^\phi = i(\bar\Lambda-\Lambda)\ \ ,\quad \delta V^\chi = i(\bar{\tilde\Lambda}-\tilde\Lambda)\ \ ,\quad \delta V' = (-\Lambda-\bar\Lambda+\tilde\Lambda+\bar{\tilde\Lambda})\ , \tag{2.4}$$

which transforms with chiral *and* twisted chiral parameters, and the semichiral vector multiplet (see also [24])

$$\delta\mathbb{V}^L = i(\bar\Lambda^L - \Lambda^L) \ , \quad \delta\mathbb{V}^R = i(\bar\Lambda^R - \Lambda^R) \ , \quad \delta\mathbb{V}' = (-\Lambda^L - \bar\Lambda^L + \Lambda^R + \bar\Lambda^R) \ , \tag{2.5}$$

whose gauge parameters are semichiral: $\bar{\mathbb{D}}_+\Lambda^L = \bar{\mathbb{D}}_-\Lambda^R = 0$. In both cases it is useful to introduce complex linear combinations with simple transformations.

For the large vector multiplet, one finds the combinations

$$\begin{aligned} V_L &= \tfrac{1}{2}(-V' + i(V^\phi - V^\chi)) \ \Rightarrow \ \delta V_L = \Lambda - \tilde\Lambda \ , \\ V_R &= \tfrac{1}{2}(-V' + i(V^\phi + V^\chi)) \ \Rightarrow \ \delta V_R = \Lambda - \bar{\tilde\Lambda} \ . \end{aligned} \tag{2.6}$$

Note that $V_{L,R}$ are constrained, as they have the same real part

$$V_L + \bar V_L = V_R + \bar V_R \ , \tag{2.7}$$

or equivalently,

$$V_L = V_R - iV^\chi \tag{2.8}$$

This constraint is preserved by the gauge transformations. The field-strengths of the large vector multiplet are semichiral spinors:

$$\mathbb{G}_+ = \bar{\mathbb{D}}_+ V_L \ , \quad \mathbb{G}_- = \bar{\mathbb{D}}_- V_R \ , \quad \bar{\mathbb{G}}_+ = \mathbb{D}_+ \bar V_L \ , \quad \bar{\mathbb{G}}_- = \mathbb{D}_- \bar V_R \ . \tag{2.9}$$

For the semichiral vector multiplet, we find similar combinations:

$$\mathbb{V} = \tfrac{1}{2}(-\mathbb{V}' + i(\mathbb{V}^L - \mathbb{V}^R)) \ \Rightarrow \ \delta\mathbb{V} = \Lambda_L - \Lambda_R \ , \tag{2.10}$$

$$\tilde{\mathbb{V}} = \tfrac{1}{2}(-\mathbb{V}' + i(\mathbb{V}^L + \mathbb{V}^R)) \ \Rightarrow \ \delta\tilde{\mathbb{V}} = \Lambda_L - \bar\Lambda_R \ . \tag{2.11}$$

As for the large vector multiplet, these combinations are constrained to have the same real part. The field-strengths of the semichiral vector multiplet are chiral and twisted chiral scalars:

$$\mathbb{F} = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathbb{V} \ , \quad \bar{\mathbb{F}} = -\mathbb{D}_+\mathbb{D}_-\bar{\mathbb{V}} \ , \quad \tilde{\mathbb{F}} = \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{\mathbb{V}} \ , \quad \bar{\tilde{\mathbb{F}}} = -\mathbb{D}_+\bar{\mathbb{D}}_-\bar{\tilde{\mathbb{V}}} \ . \tag{2.12}$$

3 Nonabelian Large Vector Multiplet

3.1 Covariant derivatives and field-strengths

The nonabelian generalizations of (2.1) and (2.2) are well known; for finite gauge transformations they are

$$g(\Lambda)e^{V^\phi} = e^{i\bar\Lambda}e^{V^\phi}e^{-i\Lambda} \ , \ \ g(\tilde\Lambda)e^{V^\chi} = e^{i\bar{\tilde\Lambda}}e^{V^\chi}e^{-i\tilde\Lambda} \ . \tag{3.1}$$

These clearly carry over for (2.4), *except* for the transformation of V', which cannot easily be generalized in a way compatible with the group property. Instead, we use the complex potential V_R, and postulate

$$g(\Lambda,\tilde\Lambda)e^{-iV_R} = e^{i\bar{\tilde\Lambda}}e^{-iV_R}e^{-i\Lambda} \ \Rightarrow \ g(\Lambda,\tilde\Lambda)e^{i\bar V_R} = e^{i\bar\Lambda}e^{i\bar V_R}e^{-i\tilde\Lambda} \ . \tag{3.2}$$

This choice is arbitrary, as we could have formulated (3.2) using the complex potential V_L; we define it by $e^{-iV_L} = e^{-V^\chi}e^{-iV_R}$ (*cf.* eq. (2.8)).

To avoid introducing extra degrees of freedom, we impose a gauge covariant reality condition:

$$e^{i\bar V_R} = e^{V^\phi}e^{iV_R}e^{V^\chi} \ , \tag{3.3}$$

which is compatible with (2.6), as it reduces to $i(\bar V_R - V_R) = V^\phi + V^\chi$ in the Abelian limit.

Covariant derivatives can be constructed in different representations appropriate to the matter fields they act on. Here we start with the chiral representation, which acts naturally on chiral superfields. In this representation, the covariant derivatives ∇ transform as

$$g(\Lambda)\nabla = e^{i\Lambda}\nabla e^{-i\Lambda} \ . \tag{3.4}$$

Other representations can be found by conjugating with the appropriate combinations of e^{V^ϕ}, e^{V^χ} and e^{iV_R}. Because the gauge parameter Λ is chiral, $\bar{\mathbb{D}}_\pm\Lambda = 0$, $\bar{\mathbb{D}}_\pm$ are already covariant

$$\bar\nabla_\pm = \bar{\mathbb{D}}_\pm \ , \tag{3.5}$$

that is, they transform as (3.4). Likewise, the usual expressions

$$\nabla_\pm = e^{-V^\phi}\mathbb{D}_\pm e^{V^\phi} \tag{3.6}$$

are covariant. However, because $\bar{\mathbb{D}}_+\tilde{\Lambda} = \mathbb{D}_-\tilde{\Lambda} = 0$, there are more covariantly transforming derivatives; a quick calculation shows that

$$\begin{aligned}
\hat{\nabla}_+ &= e^{iV_R}\mathbb{D}_+e^{-iV_R} \\
\hat{\bar{\nabla}}_- &= e^{iV_R}\bar{\mathbb{D}}_-e^{-iV_R} \\
\hat{\bar{\nabla}}_+ &= e^{iV_R}e^{V^\chi}\bar{\mathbb{D}}_+e^{-V^\chi}e^{-iV_R} \\
\hat{\nabla}_- &= e^{iV_R}e^{V^\chi}\mathbb{D}_-e^{-V^\chi}e^{-iV_R}
\end{aligned} \tag{3.7}$$

are also good covariant derivatives. The derivatives (3.5) and (3.6) are simple in chiral or antichiral representation. A twisted chiral representation is obtained after a similarity transformation with $e^{-iV_L} = e^{-V^\chi}e^{-iV_R}$; then the supercovariant derivatives $\hat{\nabla}$ become simple:

$$\begin{aligned}
e^{-iV_L}\hat{\nabla}_+e^{iV_L} &= e^{-V^\chi}\mathbb{D}_+e^{V^\chi} \\
e^{-iV_L}\hat{\bar{\nabla}}_-e^{iV_L} &= e^{-V^\chi}\bar{\mathbb{D}}_-e^{V^\chi} \\
e^{-iV_L}\hat{\bar{\nabla}}_+e^{iV_L} &= \bar{\mathbb{D}}_+ \\
e^{-iV_L}\hat{\nabla}_-e^{iV_L} &= \mathbb{D}_- \ .
\end{aligned} \tag{3.8}$$

In this representation, the derivatives ∇ are complicated.

The difference of two covariant derivatives is a covariant tensor, and thus is a field-strength. We write four spinor field-strengths that are the nonabelian generalizations of (2.9)[1]:

$$\begin{aligned}
\mathbb{G}_+ &= i(\hat{\bar{\nabla}}_+ - \bar{\nabla}_+) \ , \quad \mathbb{G}_- = i(\hat{\bar{\nabla}}_- - \bar{\nabla}_-) \ , \\
\bar{\mathbb{G}}_+ &= i(\hat{\nabla}_+ - \nabla_+) \ , \quad \bar{\mathbb{G}}_- = i(\hat{\nabla}_- - \nabla_-) \ .
\end{aligned} \tag{3.9}$$

We may shift the spinor covariant derivatives by these spinor field-strengths as we wish; indeed, such shifts play a crucial role in understanding the kinetic

[1]The signs in (3.9) may seem inconsistent with hermitian conjugation; however, hermitian conjugation also changes the representation these Lie algebra-valued quantities act on, whereas in (3.9) both $\mathbb{G}$ and $\bar{\mathbb{G}}$ act on the *same* representation. This results in an extra $(-)$ sign.

terms of the large vector multiplet.

These spinor field-strengths obey (nonlinear) semichiral constraints. Using the identity

$$\{\hat{\bar{\nabla}}_\pm + \bar{\nabla}_\pm \, , \hat{\bar{\nabla}}_\pm - \bar{\nabla}_\pm\} = \{\hat{\nabla}_\pm + \nabla_\pm \, , \hat{\nabla}_\pm - \nabla_\pm\} = 0 \ , \tag{3.10}$$

we find

$$\begin{aligned} (\hat{\bar{\nabla}}_\pm + \bar{\nabla}_\pm)\mathbb{G}_\pm = 0 \quad &\Leftrightarrow \quad \bar{\nabla}_\pm \mathbb{G}_\pm - \tfrac{i}{2}\{\mathbb{G}_\pm, \mathbb{G}_\pm\} = 0 \ , \\ (\hat{\nabla}_\pm + \nabla_\pm)\bar{\mathbb{G}}_\pm = 0 \quad &\Leftrightarrow \quad \nabla_\pm \bar{\mathbb{G}}_\pm - \tfrac{i}{2}\{\bar{\mathbb{G}}_\pm, \bar{\mathbb{G}}_\pm\} = 0 \ . \end{aligned} \tag{3.11}$$

Higher dimension field-strengths may be found by taking anticommutators of covariant derivatives (3.4). In general, each chirality choice ((twisted)(anti)chiral) has three possible field-strengths and one trivial anticommutator. For example

$$\{\nabla_+, \nabla_-\} = 0 \ \ , \tag{3.12}$$

whereas

$$\{\hat{\nabla}_+, \nabla_-\} \ \ , \quad \{\nabla_+, \hat{\nabla}_-\} \ \ , \ \ \{\hat{\nabla}_+, \hat{\nabla}_-\} \tag{3.13}$$

are nonvanishing. Using (3.9) and (3.12), we have:

$$\{\bar{\mathbb{G}}_+, \bar{\mathbb{G}}_-\} = \{\hat{\nabla}_+, \nabla_-\} + \{\nabla_+, \hat{\nabla}_-\} - \{\hat{\nabla}_+, \hat{\nabla}_-\} \ . \tag{3.14}$$

We thus find the independent field-strengths:

$$\begin{array}{lll} \text{chiral:} & \mathrm{F} = \{\hat{\bar{\nabla}}_+, \bar{\nabla}_-\} = -i\bar{\nabla}_-\mathbb{G}_+ \ , & \hat{\mathrm{F}} = \{\bar{\nabla}_+, \hat{\bar{\nabla}}_-\} = -i\bar{\nabla}_+\mathbb{G}_- \\ \text{antichiral:} & \bar{\mathrm{F}} = \{\hat{\nabla}_+, \nabla_-\} = -i\nabla_-\bar{\mathbb{G}}_+ \ , & \hat{\bar{\mathrm{F}}} = \{\nabla_+, \hat{\nabla}_-\} = -i\nabla_+\bar{\mathbb{G}}_- \\ \text{twisted chiral:} & \tilde{\mathrm{F}} = \{\hat{\bar{\nabla}}_+, \nabla_-\} = i\hat{\bar{\nabla}}_+\bar{\mathbb{G}}_- \ , & \hat{\tilde{\mathrm{F}}} = \{\bar{\nabla}_+, \hat{\nabla}_-\} = i\hat{\nabla}_-\mathbb{G}_+ \\ \text{twisted antichiral:} & \bar{\tilde{\mathrm{F}}} = \{\hat{\nabla}_+, \bar{\nabla}_-\} = i\hat{\nabla}_+\mathbb{G}_- \ , & \hat{\bar{\tilde{\mathrm{F}}}} = \{\nabla_+, \hat{\bar{\nabla}}_-\} = i\hat{\bar{\nabla}}_-\bar{\mathbb{G}}_+ \end{array} \tag{3.15}$$

The nonabelian field-strengths (3.15) match, in the abelian case [23], with combinations of the form

$$2\mathrm{F} = W + iB \ \ , \quad 2\hat{\mathrm{F}} = W - iB \ , \ etc. \tag{3.16}$$

Each field-strength has specific chirality properties that follow from its definition, *e.g.*,

$$\hat{\bar{\nabla}}_+ \mathrm{F} = \bar{\nabla}_- \mathrm{F} = 0 \quad , \quad \bar{\nabla}_+ \hat{\mathrm{F}} = \hat{\bar{\nabla}}_- \hat{\mathrm{F}} = 0 \ , \ etc. \tag{3.17}$$

3.2 Reduction to $N=(1,1)$ superspace

As in the abelian case, we decompose

$$\mathbb{D}_\pm = \tfrac{1}{2}(D_\pm - iQ_\pm) \ , \tag{3.18}$$

as well as

$$\mathrm{Re}\ \mathbb{G}_\pm|_{N=(1,1)} = \Xi^1_\pm \quad , \quad \mathrm{Im}\ \mathbb{G}_\pm|_{N=(1,1)} = \Xi^2_\pm \ . \tag{3.19}$$

The two sets of $N=(1,1)$ supercovariant derivatives decompose as[2]:

$$\begin{aligned} \nabla_\pm &= \tfrac{1}{2}(\mathcal{D}_\pm - iQ_\pm) \quad , \quad \hat{\nabla}_\pm = \tfrac{1}{2}(\hat{\mathcal{D}}_\pm - i\hat{Q}_\pm) \ , \\ \hat{\mathcal{D}}_\pm &= \mathcal{D}_\pm - 2i\Xi^1_\pm \quad , \quad \hat{Q}_\pm = Q_\pm - 2i\Xi^2_\pm \ . \end{aligned} \tag{3.20}$$

The hatted set differs from the unhatted set by covariant field redefinitions; note that the redefinition exchanging $\mathcal{D}$ and $\hat{\mathcal{D}}$ is a shift of the $N=(1,1)$ connections for $\mathcal{D}_\pm = D_\pm + iA_\pm$ by $2i\Xi^1_\pm$.

The field-strengths $F, \tilde{F}$ can be expressed $N=(1,1)$ superspace by acting with $\hat{\mathcal{D}}_+$, $\hat{Q}_+$, $\mathcal{D}_-$ and Q_- on $\Xi^{1,2}_\pm$ (*cf.* eq. (3.15)). We therefore define:

$$\check{\mathcal{D}}_+ = \hat{\mathcal{D}}_+ \quad , \quad \check{Q}_+ = \hat{Q}_+ \quad , \quad \check{\mathcal{D}}_- = \mathcal{D}_- \quad , \quad \check{Q}_- = Q_- \ . \tag{3.21}$$

From the spinor derivatives (3.21), we construct real $N=(1,1)$ scalars

$$\begin{aligned} q^\chi &= \{\check{\mathcal{D}}_+, \check{Q}_-\} = -i\check{Q}_{(+}\Xi^1_{-)} + i\check{\mathcal{D}}_{[+}\Xi^2_{-]} \\ q^\phi &= \{\check{\mathcal{D}}_-, \check{Q}_+\} = i\check{Q}_{[+}\Xi^1_{-]} - i\check{\mathcal{D}}_{(+}\Xi^2_{-)} \\ q' &= \{\check{Q}_+, \check{Q}_-\} = i\check{Q}_{[+}\Xi^2_{-]} + i\check{\mathcal{D}}_{(+}\Xi^1_{-)} \end{aligned}$$

as well as the field-strength

$$f \equiv \{\check{\mathcal{D}}_+, \check{\mathcal{D}}_-\} = i\check{Q}_{(+}\Xi^2_{-)} + i\check{\mathcal{D}}_{[+}\Xi^1_{-]} \tag{3.22}$$

[2]As is standard, in descending to $N=(1,1)$ superspace, we reduce the gauge parameter to a single $N=(1,1)$ superfield; this means we perform a partial (Wess-Zumino) gauge-fixing. As a result, $Q_\pm$ has no $N=(1,1)$ connection.

These conventions simplify the $N=(1,1)$ reduction of all unhatted field-strengths by eliminating both $i\mathcal{D}\Xi$ and $\{\Xi,\Xi\}$ terms:

$$
\begin{aligned}
4\mathrm{F}| \;&=\; f - q' + i(q^\chi + q^\phi) &(3.23)\\
4\bar{\mathrm{F}}| \;&=\; f - q' - i(q^\chi + q^\phi)\\
4\tilde{\mathrm{F}}| \;&=\; f + q' - i(q^\chi - q^\phi)\\
4\bar{\bar{\tilde{\mathrm{F}}}}| \;&=\; f + q' + i(q^\chi - q^\phi)\\
\hat{\mathrm{F}}| \;&=\; \bar{\bar{\tilde{\mathrm{F}}}} - i\check{\mathcal{D}}_+\Xi^1_- + \{\Xi^1_+,\Xi^1_-\} - \{\Xi^2_+,\Xi^2_-\} - i(i\check{\mathcal{D}}_+\Xi^2_- - \{\Xi^{(1}_+,\Xi^{2)}_-\})\\
\hat{\bar{\tilde{\mathrm{F}}}}| \;&=\; \tilde{\mathrm{F}} - i\check{\mathcal{D}}_+\Xi^1_- + \{\Xi^1_+,\Xi^1_-\} - \{\Xi^2_+,\Xi^2_-\} + i(i\check{\mathcal{D}}_+\Xi^2_- - \{\Xi^{(1}_+,\Xi^{2)}_-\})\\
\hat{\tilde{\tilde{\mathrm{F}}}}| \;&=\; \mathrm{F} + i\check{\mathcal{D}}_-\Xi^1_+ + \{\Xi^1_+,\Xi^1_-\} + \{\Xi^2_+,\Xi^2_-\} + i(i\check{\mathcal{D}}_-\Xi^2_+ - \{\Xi^{[1}_+,\Xi^{2]}_-\})\\
\hat{\bar{\bar{\tilde{\mathrm{F}}}}}| \;&=\; \bar{\mathrm{F}} + i\check{\mathcal{D}}_-\Xi^1_+ + \{\Xi^1_+,\Xi^1_-\} + \{\Xi^2_+,\Xi^2_-\} - i(i\check{\mathcal{D}}_-\Xi^2_+ - \{\Xi^{[1}_+,\Xi^{2]}_-\})
\end{aligned}
\tag{3.24}
$$

The $N=(1,1)$ fields (3.21) and (3.22) could be redefined [30] by real shifts of q^i and the connections $A_\pm$. The redefinitions

$$
\hat{q}^\chi = \{\mathcal{D}_+,\hat{Q}_-\} \quad,\quad \hat{q}^\phi = \{\hat{\mathcal{D}}_-,Q_+\} \quad,\quad \hat{q}' = \{Q_+,\hat{Q}_-\} \;\text{ and }\; \hat{f} = \{\mathcal{D}_+,\hat{\mathcal{D}}_-\} \tag{3.25}
$$

simplify the reduction to $N=(1,1)$ for the field-strengths $\hat{\mathrm{F}},\hat{\tilde{\tilde{\mathrm{F}}}}$ and introduces extra $i\mathcal{D}\Xi$ and $\{\Xi,\Xi\}$ terms into the reduction for the field-strengths $\mathrm{F},\tilde{\mathrm{F}}$.

An immediate consequence of the structure of these definitions is that we are able to remove both $i\mathcal{D}\Xi$ and $\{\Xi,\Xi\}$ terms from either the hatted set or the unhatted set of $N=(2,2)$ field-strengths, but not both simultaneously. This result greatly simplifies our discussion of actions for the large vector multiplet.

3.3 The action in $N=(1,1)$ superspace

3.3.1 Generalities

We descend to $N=(1,1)$ superspace by rewriting the measure in terms of $D_\pm, Q_\pm$ and explicitly evaluating the Q derivatives (see, *e.g.*, [23]). Starting with the $N=(2,2)$ superspace measure and an $N=(2,2)$ Lagrange density K, we write an action

$$
S = \int d^2\xi D_+D_-Q_+Q_- \; K \; . \tag{3.26}
$$

Since K is a gauge scalar, we are free to choose whether $Q(D)$-derivatives acts on K as $Q(\mathcal{D})$ or $\hat{Q}(\hat{\mathcal{D}})$.

This leads to a subtlety in descending to $N=(1,1)$ superspace. Unlike the abelian case, where one can simply exchange Q and D-derivatives using the complex structure

$$Q_\pm \varphi^i = J_\pm{}^i{}_j D_\pm \varphi^j \ , \tag{3.27}$$

the natural nonabelian field-strengths (3.15) have chirality properties with respect to different $N=(2,2)$ supercovariant derivatives. This results in a possible shift of the relation (3.27) with a spinor multiplet $\mathbb{G}$. For example, the action of $\check{Q}_\pm$ on the (anti)chiral field-strengths reads

$$\begin{array}{c||c|c}
 & \check{Q}_+ & \check{Q}_- \\
\hline\hline
\mathrm{F} & i\check{\mathcal{D}}_+\mathrm{F}| & i\check{\mathcal{D}}_-\mathrm{F}| \\
\bar{\mathrm{F}} & -i\check{\mathcal{D}}_+\bar{\mathrm{F}}| & -i\check{\mathcal{D}}_-\bar{\mathrm{F}}| \\
\hat{\mathrm{F}} & i\check{\mathcal{D}}_+\hat{\mathrm{F}}| - 2[\mathbb{G}_+,\hat{\mathrm{F}}]| & i\check{\mathcal{D}}_-\hat{\mathrm{F}}| + 2[\mathbb{G}_-,\hat{\mathrm{F}}]| \\
\hat{\bar{\mathrm{F}}} & -i\check{\mathcal{D}}_+\hat{\bar{\mathrm{F}}}| + 2[\bar{\mathbb{G}}_+,\hat{\bar{\mathrm{F}}}]| & -i\check{\mathcal{D}}_-\hat{\bar{\mathrm{F}}}| - 2[\bar{\mathbb{G}}_-,\hat{\bar{\mathrm{F}}}]|
\end{array} \tag{3.28}$$

For the action of $\check{Q}_+\check{Q}_-$ we find:

$$\begin{array}{c||c}
 & \check{Q}_+\check{Q}_- \\
\hline\hline
\mathrm{F} & \check{\mathcal{D}}_-\check{\mathcal{D}}_+\mathrm{F}| + i[q^\phi,\mathrm{F}]| \\
\bar{\mathrm{F}} & \check{\mathcal{D}}_-\check{\mathcal{D}}_+\bar{\mathrm{F}}| - i[q^\phi,\bar{\mathrm{F}}]| \\
\hat{\mathrm{F}} & -\{i\check{\mathcal{D}}_- + 2\mathbb{G}_-,[i\check{\mathcal{D}}_+ - 2\mathbb{G}_+,\hat{\mathrm{F}}]\} + [f+q'+iq^\chi - 2i\check{\mathcal{D}}_+\mathbb{G}_-,\hat{\mathrm{F}}]| \\
\hat{\bar{\mathrm{F}}} & -\{i\check{\mathcal{D}}_- + 2\bar{\mathbb{G}}_-,[i\check{\mathcal{D}}_+ - 2\bar{\mathbb{G}}_+,\hat{\bar{\mathrm{F}}}]\} - [f+q'-iq^\chi - 2i\check{\mathcal{D}}_+\bar{\mathbb{G}}_-,\hat{\bar{\mathrm{F}}}]|
\end{array} \tag{3.29}$$

where we used the anticommutators

$$\begin{aligned}
\{\check{Q}_+, i\check{\mathcal{D}}_- + 2\mathbb{G}_-\} &= f + q' + iq^\chi - 2i\check{\mathcal{D}}_+\mathbb{G}_- \\
-\{\check{Q}_+, i\check{\mathcal{D}}_- + 2\bar{\mathbb{G}}_-\} &= f + q' - iq^\chi - 2i\check{\mathcal{D}}_+\bar{\mathbb{G}}_-
\end{aligned} \tag{3.30}$$

3.3.2 Evaluating the actions

Physically sensible actions cannot have terms with higher derivatives on fermions. We now generalize the results presented in [30], where field redefinitions were found that allowed us to write down actions for (anti)chiral field-strengths and twisted (anti)chiral field-strengths.

Naïvely, quadratic terms in (anti)chiral field-strengths may appear in three flavors: $\mathrm{F}\bar{\mathrm{F}}$, $\hat{\mathrm{F}}\bar{\hat{\mathrm{F}}}$, and $\mathrm{F}\bar{\hat{\mathrm{F}}} + c.c.$ Using the results of the previous section, one can show that there exist field redefinitions that eliminate higher derivative terms for *either* of the first two but not the last $N=(2,2)$ action. Extending these results to the twisted (anti)chirals we find that sensible gauge actions in $N = (2,2)$ are either combinations of only hatted field-strengths or only unhatted ones.

From (3.15) and (3.17), we see that *e.g.*, the action[3] $K = \mathrm{Tr}(\mathrm{F}\bar{\mathrm{F}} - \tilde{\mathrm{F}}\bar{\tilde{\mathrm{F}}})$ is conveniently reduced to $N=(1,1)$ superspace by acting with Q-derivatives as Q_- and $\hat{Q}_+$:

$$\begin{aligned}
&\int D_+D_-Q_+Q_- \ \mathrm{Tr}(\mathrm{F}\bar{\mathrm{F}} - \tilde{\mathrm{F}}\bar{\tilde{\mathrm{F}}}) \\
&= \frac{1}{4}\int D_+D_- \ \mathrm{Tr}(\check{\mathcal{D}}_+(f-q')\check{\mathcal{D}}_-(f-q') + \check{\mathcal{D}}_+(q^\chi+q^\phi)\check{\mathcal{D}}_-(q^\chi+q^\phi) + \\
&\qquad \check{\mathcal{D}}_+(f+q')\check{\mathcal{D}}_-(f+q') + \check{\mathcal{D}}_+(q^\chi-q^\phi)\check{\mathcal{D}}_-(q^\chi-q^\phi) - 2[q',q^\chi]q^\phi) \\
&= \frac{1}{2}\int D_+D_- \ \mathrm{Tr}(\check{\mathcal{D}}_+f\check{\mathcal{D}}_-f + \check{\mathcal{D}}_+q^\chi\check{\mathcal{D}}_-q^\chi + \check{\mathcal{D}}_+q'\check{\mathcal{D}}_-q' + \check{\mathcal{D}}_+q^\phi\check{\mathcal{D}}_-q^\phi - [q',q^\chi]q^\phi) \ .
\end{aligned} \tag{3.31}$$

It is interesting to notice that the action (3.31), and in particular the scalar commutator term, is reminiscent of the action for $N = 2$ $d = 4$ super Yang-Mills theory. This suggests that the large vector multiplet action has $N = (4,4)$ supersymmetry. We leave this for future work.

Other possible contributions to the nonabelian large vector multiplet action originate from superpotentials which are encoded in four complex functions. Their reduction to $N=(1,1)$ superspace reads:

$$S_{sp} = 2\,\mathrm{Re}\int D_+D_- \left(P(\mathrm{F}) + \tilde{P}(\tilde{\mathrm{F}}) + \hat{P}(\hat{\mathrm{F}}) + \hat{\tilde{P}}(\hat{\tilde{\mathrm{F}}})\right)\Big| \ . \tag{3.32}$$

The criterion for consistent superpotential terms, which is the absence of terms of the form $\mathcal{D}_+\Xi_-\mathcal{D}_-\Xi_+$, is automatically met for *any* of the field redefinition required to make the kinetic terms consistent. In the abelian limit, terms involving $\mathrm{F}\hat{\mathrm{F}}$ are also chiral, but were excluded by a consistency condition found in [30]; here, these terms are not chiral and hence are automatically

[3]It is essential to have both $\mathrm{F}\bar{\mathrm{F}}$ and $\tilde{\mathrm{F}}\bar{\tilde{\mathrm{F}}}$ terms in the $N=(2,2)$ Lagrange-density to give dynamics all $N = (1,1)$ multiplets. In the abelian case, where terms of the form $\{\Xi,\Xi\}$ vanish, actions that contain only chiral or only twisted chiral multiplets are also possible. These actions are discussed in [30].

excluded.

The superpotential terms (3.32) give mass terms for the scalar multiplets as well actions for the spinor multiplets $\Xi_{\pm}^{1,2}$ (as in the abelian case [30]).

4 Nonabelian Semichiral Vector Multiplet

The strategy we follow for the semichiral vector multiplet is very similar to the one we use for the large vector multiplet. However, since the gauge parameters are semichiral, we find unique gauge covariant spinor derivatives and all the field-strengths arise in the usual way as (anti)commutators.

We take the nonabelian generalization of (2.5 , 2.10 , 2.11) to be:

$$\begin{aligned} g(\Lambda_L,\Lambda_R)e^{i\mathbb{V}} &= e^{i\Lambda_L}e^{i\mathbb{V}}e^{-i\Lambda_R} \\ g(\Lambda_L,\bar{\Lambda}_R)e^{i\tilde{\mathbb{V}}} &= e^{i\Lambda_L}e^{i\tilde{\mathbb{V}}}e^{-i\bar{\Lambda}_R} \\ g(\Lambda_L)e^{\mathbb{V}^L} &= e^{i\bar{\Lambda}_L}e^{\mathbb{V}^L}e^{-i\Lambda_L} \\ g(\Lambda_R)e^{\mathbb{V}^R} &= e^{i\bar{\Lambda}_R}e^{\mathbb{V}^R}e^{-i\Lambda_R} \ . \end{aligned} \tag{4.1}$$

As for the large vector multiplet, not all of these potentials are independent. We impose the gauge covariant constraint

$$e^{i\mathbb{V}} = e^{-\mathbb{V}^L}e^{i\bar{\tilde{\mathbb{V}}}} \tag{4.2}$$

as well as the gauge covariant reality constraint

$$e^{i\mathbb{V}} = e^{-\mathbb{V}^L}e^{i\bar{\mathbb{V}}}e^{\mathbb{V}^R} \ . \tag{4.3}$$

The covariant derivatives read (in the left semichiral representation):

$$\begin{aligned} &\bar{\nabla}_+ = \bar{\mathbb{D}}_+ \\ &\bar{\nabla}_- = e^{i\mathbb{V}}\bar{\mathbb{D}}_-e^{-i\mathbb{V}} \\ &\nabla_+ = e^{i\mathbb{V}}e^{i\bar{\tilde{\mathbb{V}}}}\mathbb{D}_+e^{-i\bar{\tilde{\mathbb{V}}}}e^{-i\mathbb{V}} = e^{i\tilde{\mathbb{V}}}e^{-i\bar{\mathbb{V}}}D_+e^{i\bar{\mathbb{V}}}e^{-i\tilde{\mathbb{V}}} = e^{-\mathbb{V}^L}D_+e^{\mathbb{V}^L} \\ &\nabla_- = e^{i\tilde{\mathbb{V}}}\mathbb{D}_-e^{-i\tilde{\mathbb{V}}} \end{aligned} \tag{4.4}$$

The nonabelian generalization of the field-strengths (2.12) is

$$\mathbb{F} = i\{\bar{\nabla}_+,\bar{\nabla}_-\} \qquad \bar{\mathbb{F}} = -i\{\nabla_+,\nabla_-\}$$

$$\tilde{\mathbb{F}} = i\{\bar{\nabla}_+, \nabla_-\} \qquad \bar{\tilde{\mathbb{F}}} = -i\{\nabla_+, \bar{\nabla}_-\} \ , \tag{4.5}$$

which are covariantly chiral and twisted chiral scalars.

4.1 Reduction to $N=(1,1)$ superspace

Having a single set of $N=(2,2)$ supercovariant derivatives, we introduce the $N=(1,1)$ derivatives

$$\nabla_\pm = \tfrac{1}{2}(\mathcal{D}_\pm - iQ_\pm) \ , \tag{4.6}$$

that give the $N=(1,1)$ field-strength for the connections

$$f = \{\mathcal{D}_+, \mathcal{D}_-\} = \{\nabla_+ + \bar{\nabla}_+, \nabla_- + \bar{\nabla}_-\} = -i\left(\mathbb{F} + \tilde{\mathbb{F}} - \bar{\mathbb{F}} - \bar{\tilde{\mathbb{F}}}\right)\Big| \ , \tag{4.7}$$

and three scalars that follow the notation of [22]

$$\hat{d}^1 = \left(\mathbb{F} + \bar{\mathbb{F}}\right)\big| \quad , \quad \hat{d}^2 = \left(\tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big| \quad , \quad \hat{d}^3 = i\left(\mathbb{F} - \bar{\mathbb{F}} - \tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big| \ . \tag{4.8}$$

In [22] we have obtained the D-term action, which is a simple sum of kinetic terms for the (twisted) chiral field-strengths. Using the chirality properties of the field-strengths

$$Q_\pm(\mathbb{F}, \bar{\mathbb{F}}, \tilde{\mathbb{F}}, \bar{\tilde{\mathbb{F}}}) = \mathcal{D}_\pm(i\mathbb{F}, -i\bar{\mathbb{F}}, \pm i\tilde{\mathbb{F}}, \mp i\bar{\tilde{\mathbb{F}}}) \ , \tag{4.9}$$

the nonabelian action

$$S_{\mathbb{X}} = \int d^2\xi\, D_+ D_- Q_+ Q_- \mathrm{Tr}\left(\mathbb{F}\bar{\mathbb{F}} - \tilde{\mathbb{F}}\bar{\tilde{\mathbb{F}}}\right) \tag{4.10}$$

reduces to the $N=(1,1)$ action

$$\begin{aligned} S_{\mathbb{X}} = \int d^2\xi D_+ D_- \ \mathrm{Tr}\ \Big(&\tfrac{1}{2}\mathcal{D}_+ f \mathcal{D}_- f + \tfrac{1}{2}\mathcal{D}_+\hat{d}^3\mathcal{D}_-\hat{d}^3 \\ &+\mathcal{D}_+\hat{d}^1\mathcal{D}_-\hat{d}^1 + \mathcal{D}_+\hat{d}^2\mathcal{D}_-\hat{d}^2 - [\hat{d}^1, \hat{d}^2]\hat{d}^3\Big) \ . \end{aligned} \tag{4.11}$$

In a similar fashion, we write the superpotential terms in $N=(1,1)$ superspace:

$$S_P = 2\ \mathrm{Re}\int D_+ D_- \left(P_1(\mathbb{F}) + P_2(\tilde{\mathbb{F}})\right)\Big| \ . \tag{4.12}$$

These terms include mass and Fayet-Illiopoulos terms.

5 Conclusion

In this paper we have extended the results of [22, 24, 30] to the nonabelian case. While the semichiral vector multiplet generalizes straightforwardly, the extension for the large vector multiplet gave rise to subtleties and ambiguities that were not present in the abelian case, namely, the different chirality conditions (3.15) that follow from the doubled set of supercovariant derivatives $\nabla, \hat{\nabla}$.

The nonabelian extension sheds light on the origins of some of the constraints on actions for the large vector multiplet [30]. In particular, in the nonabelian case, D-terms are further restricted to four possible kinetic terms and the restrictions on superpotential terms found in [30] are an immediate consequence of the incompatibility in chirality properties for $N=(2,2)$ field-strengths.

Our results should make it possible to give a complete description of the gauging of isometries of generalized Kähler geometries (*cf.*, [8]).

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Chapter 5

Abelian LVM action

While for the nonabelian case we have seen that consistent actions for the LVM must not mix the two sets of gauge invariants, this does not hold in the abelian case since we need not concern with anticommutators of the form $\{\mathbb{G}, \mathbb{G}\}$. we explore gauge actions that are not permitted for the abelian large vector multiplet; namely, action where both types of field-strengths are present (that is, (un)hatted in the notation of [30]).

Other interesting terms are mass-like terms in $\mathbb{G}$'s which give superpotentials for the $N = (1, 1)$ scalar and field-strengths as well as possible kinetic terms for $\Xi^A_\pm$ and topologicl terms.

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The Large Vector Multiplet Action

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Abstract

In this short note we discuss possible actions for the $d = 2$, $N = (2, 2)$ large vector multiplet of [22, 23]. We explore two scenarios that allow us to write kinetic and superpotential terms for the scalar field-strengths, and write kinetic terms for the spinor invariants that can introduce topological terms for the connections.

1 Introduction

Generalized Kähler manifolds, which are torsionful manifolds equipped with two complex structures [2], arise as target space for $d = 2$, $N=(2,2)$ σ-models with both twisted chiral and chiral superfields [16], or semichiral superfields [15]. These manifolds and their world-sheet origins are subject to growing interest for both mathematicians and string theorists [2, 55].

Recently, new $N=(2,2)$ multiplets were introduced [22, 24] to gauge isometries in generalized Kähler manifolds [6] and to show that the duality introduced in [10] is, in fact, T-duality [23, 25]; in particular it was shown that this T-duality relates (twisted) chiral multiplets $(\chi)\phi$ to semichiral multiplets $\mathbb{X}_{L,R}$,

$$K(\phi,\bar{\phi},\chi,\bar{\chi},x) \overset{\text{T-duality}}{\longleftrightarrow} \tilde{K}(\mathbb{X}_L,\mathbb{X}_R,\bar{\mathbb{X}}_L,\bar{\mathbb{X}}_R,x) \ , \tag{1.1}$$

where x are arbitrary spectator fields. In [22] it was shown that the action for the semichiral vector multiplet (gauging the RHS of eq. 1.1) corresponds to that of one ordinary vector multiplet field-strength and three scalar multiplets. However, the large vector multiplet (gauging the LHS of eq. 1.1) has four extra spinor multiplets that can complicate the construction of kinetic terms by introducing higher derivative actions.

In this note we address this problem. We find two possible types of actions, one consisting of chiral field-strengths and the other consisting of twisted chiral field-strengths where higher derivative terms are explicitly eliminated using particular field redefinitions. We then discuss possible superpotentials that can accompany those kinetic terms, as well as actions for the spinor invariants, which are found to be field theories with first derivatives for both bosons and fermions. Finally, we give the modified matter couplings due to the field redefinitions.

Throughout this paper we follow the notation of [23].

2 Review: The large vector multiplet in $N=(2,2)$ and $N=(1,1)$ superspace

We start our discussion with a Kähler potential for (twisted) chiral superfields with a gauged isometry

$$K(i(\phi-\bar{\phi})+V^{\phi}\ ,\ i(\chi-\bar{\chi})+V^{\chi}\ ,\ \phi+\bar{\phi}-\chi-\bar{\chi}+V')\ ; \tag{2.2}$$

we use the notation of [23] where the transformation properties for the three (real) superfields are

$$\delta V' = -\Lambda-\bar{\Lambda}+\tilde{\Lambda}+\bar{\tilde{\Lambda}} \quad , \quad \delta V^{\phi} = i(\bar{\Lambda}-\Lambda) \quad , \quad \delta V^{\chi} = i(\bar{\tilde{\Lambda}}-\tilde{\Lambda}) \ . \tag{2.3}$$

These combine to give the complex potentials:

$$\begin{aligned} V^L &= \tfrac{1}{2}(-V' + iV^\phi - iV^\chi) \quad \rightarrow \quad \delta V^L = \Lambda - \tilde{\Lambda} \ , \\ V^R &= \tfrac{1}{2}(-V' + iV^\phi + iV^\chi) \quad \rightarrow \quad \delta V^R = \Lambda - \bar{\tilde{\Lambda}} \ , \end{aligned} \qquad (2.4)$$

along with their complex conjugates, which are potentials for the semichiral gauge-invariant field-strengths

$$\mathbb{G}_+ = \bar{\mathbb{D}}_+ V^L \quad , \quad \mathbb{G}_- = \bar{\mathbb{D}}_- V^R \quad , \quad \bar{\mathbb{G}}_+ = \mathbb{D}_+ \bar{V}^L \quad , \quad \bar{\mathbb{G}}_- = \mathbb{D}_- \bar{V}^R \ , \qquad (2.5)$$

and eight chiral and twisted chiral field-strengths:

$$\begin{aligned} W &= i\bar{\mathbb{D}}_+\bar{\mathbb{D}}_- V^\chi \quad , \quad B = i\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-(-V' + iV^\phi) \ , \\ \bar{W} &= i\mathbb{D}_+\mathbb{D}_- V^\chi \quad , \quad \bar{B} = i\mathbb{D}_+\mathbb{D}_-(-V' - iV^\phi) \ , \\ \tilde{W} &= i\bar{\mathbb{D}}_+\mathbb{D}_- V^\phi \quad , \quad \tilde{B} = i\bar{\mathbb{D}}_+\mathbb{D}_-(-V' - iV^\chi) \ , \\ \bar{\tilde{W}} &= i\mathbb{D}_+\bar{\mathbb{D}}_- V^\phi \quad , \quad \bar{\tilde{B}} = i\mathbb{D}_+\bar{\mathbb{D}}_-(-V' + iV^\chi) \ . \end{aligned} \qquad (2.6)$$

These can be expressed as $N = (2,2)$ supercovariant derivatives on the invariant spinors:

$$\begin{aligned} W &= +(\bar{\mathbb{D}}_+\mathbb{G}_- + \bar{\mathbb{D}}_-\mathbb{G}_+) \quad , \quad B = i(\bar{\mathbb{D}}_+\mathbb{G}_- - \bar{\mathbb{D}}_-\mathbb{G}_+) \ , \\ \bar{W} &= -(\mathbb{D}_+\bar{\mathbb{G}}_- + \mathbb{D}_-\bar{\mathbb{G}}_+) \quad , \quad \bar{B} = i(\mathbb{D}_+\bar{\mathbb{G}}_- - \mathbb{D}_-\bar{\mathbb{G}}_+) \ , \\ \tilde{W} &= -(\bar{\mathbb{D}}_+\bar{\mathbb{G}}_- + \mathbb{D}_-\mathbb{G}_+) \quad , \quad \tilde{B} = i(\bar{\mathbb{D}}_+\bar{\mathbb{G}}_- - \mathbb{D}_-\mathbb{G}_+) \ , \\ \bar{\tilde{W}} &= +(\mathbb{D}_+\mathbb{G}_- + \bar{\mathbb{D}}_-\bar{\mathbb{G}}_+) \quad , \quad \bar{\tilde{B}} = i(\mathbb{D}_+\mathbb{G}_- - \bar{\mathbb{D}}_-\bar{\mathbb{G}}_+) \ . \end{aligned} \qquad (2.7)$$

The descent to $N = (1,1)$ uses the decompositions of the $N = (2,2)$ derivatives and fields:

$$\mathbb{D}_\pm = \tfrac{1}{2}(D_\pm - iQ_\pm) \quad \text{and} \quad \mathbb{G}_\pm| = \Xi^1_\pm + i\Xi^2_\pm \qquad (2.8)$$

where $|$ indicates projection to $N = (1,1)$. We use a real basis of $N = (1,1)$ gauge-invariant fields [23]:

$$\begin{aligned} \hat{q}^\phi &= -i\tfrac{1}{2}(Q_{[+}\Xi^1_{-]} - D_{[+}\Xi^2_{-]}) \quad , \quad iD_\pm\Xi^{1,2}_\mp \ , \\ \hat{q}^\chi &= -i\tfrac{1}{2}(Q_{(-}\Xi^1_{+)} + D_{(+}\Xi^2_{-)}) \quad , \quad \hat{q}' = -i\tfrac{1}{2}Q_{[+}\Xi^2_{-]} \ , \end{aligned} \qquad (2.9)$$

as well as the field-strength for the $N = (1,1)$ connections $A_\pm = \frac{1}{4}Q_\pm(V^\phi \pm V^\chi)$:

$$f = -i(Q_+\Xi^2_- + Q_-\Xi^2_+) = i(D_+A_- + D_-A_+) \ . \qquad (2.10)$$

The linear relations between the $N=(2,2)$ and $N=(1,1)$ invariants could be summarized in the matrix equation:

$$L = \left(\begin{array}{c} W \\ B \\ \bar{W} \\ \bar{B} \\ \tilde{W} \\ \tilde{B} \\ \bar{\tilde{W}} \\ \bar{\tilde{B}} \end{array}\right)\Bigg| = \frac{1}{2}\left(\begin{array}{cccccccc} -i & -i & 0 & 0 & 0 & -1 & 0 & -i \\ 1 & -1 & i & -i & -i & 0 & 1 & 0 \\ i & i & 0 & 0 & 0 & -1 & 0 & i \\ 1 & -1 & -i & i & i & 0 & 1 & 0 \\ i & i & 0 & 0 & 1 & 0 & 0 & -i \\ 1 & -1 & -i & -i & 0 & -i & -1 & 0 \\ -i & -i & 0 & 0 & 1 & 0 & 0 & i \\ 1 & -1 & i & i & 0 & i & -1 & 0 \end{array}\right)\left(\begin{array}{c} iD_+\Xi^1_- \\ iD_-\Xi^1_+ \\ 2iD_+\Xi^2_- \\ 2iD_-\Xi^2_+ \\ 2\hat{q}^\phi \\ 2\hat{q}^\chi \\ 2\hat{q}' \\ f \end{array}\right) = UL' \tag{2.11}$$

3 Possible candidates for large multiplet action

3.1 Naïve kinetic terms for $\hat{q}', \hat{q}^{\phi,\chi}$ and f

A generic, Lagrange density in the (twisted) chiral invariants has eight free parameters[1]. These correspond to two 2×2 Hermitian matrices $s_{c,t} = s^\dagger_{c,t}$:

$$\mathcal{L}_{kin} = (W,B)s_c\left(\begin{array}{c} \bar{W} \\ \bar{B} \end{array}\right) + (\tilde{W},\tilde{B})s_t\left(\begin{array}{c} \bar{\tilde{W}} \\ \bar{\tilde{B}} \end{array}\right) = L^i L^j S_{ij} \tag{3.12}$$

Reduction of such a density to $N=(1,1)$ is straightforward using the matrices S (eq. 3.12), U (eq. 2.11), and the complex structures $J_\pm = \mathrm{diag}(i,-i,\pm i,\mp i)$; after integration by parts we find:

$$\begin{aligned} \int D_+D_-Q_+Q_-\mathcal{L}_{kin} &= 2\int D_+D_-(U^T(J_+SJ_- - J_+J_-S)U)_{ij}D_+L'^i D_-L'^j \\ &= \int D_+D_-S'_{ij}D_+L'^i D_-L'^j \ . \end{aligned} \tag{3.13}$$

Terms of the form $D_{[\pm]}D_\pm\Xi^A_\mp$ lead to second derivatives on spinors and are, therefore, to be avoided. An Ansatz that leads to this desired result involves a particular choice of linear field redefinitions:

$$\begin{aligned} &L'' = U'L \quad , \quad U' = \left(\begin{array}{cc} 1 & 0 \\ \alpha & 1 \end{array}\right)U \quad , \quad L''_{1,2,3,4} = 0 \\ &A_\pm \to A_\pm + \alpha^{(\pm)}_B\Xi^B_\pm \quad , \quad \hat{q}^i \to \hat{q}^i + \beta^{i(\pm)}_B D_\pm\Xi^B_\mp \quad , \quad B = 1,2 \end{aligned} \tag{3.14}$$

[1]Terms mixing chiral and twisted chiral fields lead to total derivatives and could be interesting for global considerations.

and parameters $s_{c,t}$[2] Where all blocks of are 4×4.

The matrix U' is invertible and we therefore require, to propagate all field-strengths, that $J_+SJ_- - J_+J_-S$ has four vanishing eigenvalues. Diagonalizing S we find four pair of eigenvalues, two pairs due to s_c and two pairs due to s_t that brings us to the following classes of consistent kinetic terms:

- Chiral: $s_t = 0$, s_c is arbitrary hermitian.
- Twisted chiral: $s_c = 0$, s_t is arbitrary hermitian.
- Mixed: $\det s_c = \det s_t = 0$

One can verify by explicit substitution of the blocks of S'

$$S' = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{3.15}$$

that these choices indeed satisfy the condition for consistent field redefinition

$$\begin{pmatrix} 1 & \alpha^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \to A - BC^{-1}B^T = 0 \ . \tag{3.16}$$

Analysis of the nonabelian extension to the large vector multiplet introduces further restrictions on the large vector multiplet action [26], allowing only the mixed solution and restricting to (twisted) chiral combinations of the form $W \pm iB$ *etc*. We now present in detail the classes unique to the abelian case: the chiral and the twisted chiral solutions. The field redefinitions are listed in table 1.

An explicit form for these kinetic terms after field redefinition in $N=(1,1)$ superspace is obtained by reduction:

- Chiral:
  Writing the entries of s_c explicitly

  $$s_c = s = \begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix} \tag{3.17}$$

  we push $Q_\pm$ through and find:

  $$\int Q_+Q_- \left[(W,B) s_c \begin{pmatrix} \bar{W} \\ \bar{B} \end{pmatrix} \right] = \tag{3.18}$$

  $$\tfrac{1}{2} \int D_+(2\hat{q}_c^\phi, 2\hat{q}_c^\chi, 2\hat{q}_c', f_c) \begin{pmatrix} d & c & 0 & b \\ c & a & -b & 0 \\ 0 & -b & d & c \\ b & 0 & c & a \end{pmatrix} D_- \begin{pmatrix} 2\hat{q}_c^\phi \\ 2\hat{q}_c^\chi \\ 2\hat{q}_c' \\ f_c \end{pmatrix} \tag{3.19}$$

  where $\int$ is the $N=(1,1)$ measure $\int d^2zD_+D_-$.

[2] It is impossible to cancel all higher derivative terms solely by adjusting the parameters $s_{c,t}$.

Table 1: Field redefinitions for kinetic terms

	Chiral $s_t = 0$	Twisted chiral $s_c = 0$
$A_\pm \to$	$A^{(c)}_\pm = A_\pm + \Xi^1_\pm$	$A^{(t)}_\pm = A_\pm - \Xi^1_\pm$
$f \to$	$f_c = f + iD_{(+}\Xi^1_{-)}$	$f_t = f - iD_{(+}\Xi^1_{-)}$
$\hat{q}' \to$	$\hat{q}'_c = \hat{q}' + \frac{1}{2} iD_{[+}\Xi^1_{-]}$	$\hat{q}'_t = \hat{q}' - \frac{1}{2} iD_{[+}\Xi^1_{-]}$
$\hat{q}^\chi \to$	$\hat{q}^\chi_c = \hat{q}^\chi$	$\hat{q}^\chi_t = \hat{q}^\chi + iD_{(+}\Xi^2_{-)}$
$\hat{q}^\phi \to$	$\hat{q}^\phi_c = \hat{q}^\phi - iD_{[+}\Xi^2_{-]}$	$\hat{q}^\phi_t = \hat{q}^\phi$

- Twisted chiral:
  In a similar fashion, we set the entries $s_t = s$ and reduce the twisted chiral action to $N=(1,1)$:

$$\int Q_+ Q_- \left[(\tilde{W}, \tilde{B}) s_t \begin{pmatrix} \bar{\tilde{W}} \\ \bar{\tilde{B}} \end{pmatrix} \right] = \tag{3.20}$$
$$\frac{1}{2} \int D_+ (2\hat{q}^\phi_t, 2\hat{q}^\chi_t, 2\hat{q}'_t, f_t) \begin{pmatrix} -a & c & b & 0 \\ c & -d & 0 & -b \\ b & 0 & -d & c \\ 0 & -b & c & -a \end{pmatrix} D_- \begin{pmatrix} 2\hat{q}^\phi_t \\ 2\hat{q}^\chi_t \\ 2\hat{q}'_t \\ f_t \end{pmatrix} .$$

3.2 Mass-like terms

We now investigate terms of the form

$$\int i\bar{\mathbb{D}}_+\bar{\mathbb{D}}_- P_c(W,B) + \int i\bar{\mathbb{D}}_+\mathbb{D}_- P_t(\tilde{W},\tilde{B}) + \int i\mathbb{D}_+\mathbb{D}_- \bar{P}_c(\bar{W},\bar{B}) + \int i\mathbb{D}_+\bar{\mathbb{D}}_- \bar{P}_t(\bar{\tilde{W}},\bar{\tilde{B}}) \tag{3.21}$$

that arise naturally from a naïve mass term for the invariant semichiral spinors $\mathbb{G}_\pm$, *e.g.*,

$$\int \mathbb{D}_+\mathbb{D}_-\bar{\mathbb{D}}_+\bar{\mathbb{D}}_- \left(\mathbb{G}_+\mathbb{G}_-\right) = \int \mathbb{D}_+\mathbb{D}_- \left(\bar{\mathbb{D}}_-\mathbb{G}_+\bar{\mathbb{D}}_+\mathbb{G}_-\right) = \frac{1}{4}\int \mathbb{D}_+\mathbb{D}_- \left(W^2 + B^2\right) , \tag{3.22}$$

and reduce to $N=(1,1)$ superspace in a straightforward manner due to the (twisted) chirality of the field-strengths [22].

We construct sensible candidates by taking into account the field redefinitions of section 3.1, and requiring the absence of terms of the form

$$\int D_+D_-\left(D_+\Xi^A_- m_{AB} D_-\Xi^B_+\right) \ , \tag{3.23}$$

which give, when reduced to components, higher derivatives on spinors. As we now see, these terms can give:

- Superpotentials for $\hat{q}'_{c,t}, \hat{q}^{\phi,\chi}_{c,t}$ and $f_{c,t}$.
- Kinetic terms for $\Xi^A_\pm$ which are first order in derivatives.
- Topological terms.

3.2.1 Superpotentials for $\hat{q}', \hat{q}^{\phi,\chi}$ and f

After carrying out the field redefinitions of sec. 3.1 we have, in each scenario, four field-strengths that contain only the redefined $\hat{q}'_{c,t}, \hat{q}^{\phi,\chi}_{c,t}$ and $f_{c,t}$. We can, therefore, write in the chiral scenario any function for $P_c(W, B)$ (*c.f.*, eq. 3.21) which reduces to $N\!=\!(1,1)$ superspace as:

$$2\int iD_+D_-\mathrm{Re}\left(P_c(2\hat{q}^\chi_c + if_c, \hat{q}'_c - i\hat{q}^\phi_c)\right) \ . \tag{3.24}$$

In a similar fashion we write in the twisted chiral scenario a superpotential $P_t(\tilde{W}, \tilde{B})$ (*c.f.*, eq. 3.21) that reduces to $N\!=\!(1,1)$ superspace as:

$$2\int iD_+D_-\mathrm{Re}\left(P_t(2\hat{q}^\phi_t - if_t, \hat{q}'_t + i\hat{q}^\chi_t)\right) \ . \tag{3.25}$$

Particular examples of such superpotentials include mass and Fayet-Illiopoulos terms.

3.2.2 Kinetic and topological terms for $\Xi^A_\pm$

After chiral field redefinition, we can write combinations of the twisted chiral field strengths:

$$\begin{aligned}\tilde{W} + i\tilde{B} &= 2iD_+(i\Xi^1_- + \Xi^2_-) + \hat{q}^\phi_c + \hat{q}^\chi_c - i\hat{q}'_c - \tfrac{i}{2}f_c \ ,\\ \tilde{W} - i\tilde{B} &= 2iD_-(i\Xi^1_+ - \Xi^2_+) + \hat{q}^\phi_c - \hat{q}^\chi_c + i\hat{q}'_c - \tfrac{i}{2}f_c \ ,\end{aligned} \tag{3.26}$$

that separate left and right spinors, and introduce a generic, *"harmolomorphic"*[3], function for the twisted chirals:

$$P_t(\tilde{W}, \tilde{B}) = P_{t+}(\tilde{W} + i\tilde{B}) + P_{t-}(\tilde{W} - i\tilde{B}) \ , \tag{3.27}$$

[3]Since it obeys $\partial_{\bar{\tilde{W}}} P_t = \partial_{\bar{\tilde{B}}} P_t = \partial_{\tilde{W}+i\tilde{B}}\partial_{\tilde{W}-i\tilde{B}} P_t = 0$.

that contains no bad terms.

A quadratic function in $\tilde{W} \pm i\tilde{B}$ generates kinetic terms for the components of the spinors $\Xi^A_\pm$:

$$\Xi^A_\pm\big| = \Xi^A_\pm \quad , \quad D_\pm \Xi^A_\pm\big| = V^A_{\substack{+\\=}} \quad , \quad \begin{array}{c} D_- \Xi^A_+\big| = b^A \\ D_+ \Xi^A_-\big| = b'^A \end{array} \quad , \quad D_+ D_- \Xi^A_\pm\big| = \xi^A_\pm \ . \tag{3.28}$$

When reducing a term of the form, *e.g.*, $D_- \Xi^A_+ D_- \Xi^B_+$ appearing in $(\tilde{W} - i\tilde{B})^2$ we find[4]

$$D_+ D_- (D_- \Xi^A_+ D_- \Xi^B_+) = \partial_= V^{(A}_{+\!\!+} b^{B)} + \xi^{(A}_+ \partial_= \Xi^{B)}_+ \ . \tag{3.29}$$

Note that $(\tilde{W} \pm i\tilde{B})^2$ also introduce terms of the form $D_\pm \Xi^A_\mp f$, which, when reduced to components using

$$f| = f \quad , \quad D_\pm f| = \lambda_\pm \quad , \quad D_+ D_- f| = F = \partial_{+\!\!+} A_= - \partial_= A_{+\!\!+} \ , \tag{3.30}$$

gives terms such as bF which are topological.

In the twisted chiral scenario we write the combinations

$$\begin{aligned} W + iB &= 2iD_-(\Xi^2_+ - i\Xi^1_+) + \hat{q}^\phi_t - \hat{q}^\chi_t + i\hat{q}'_t - \tfrac{i}{2} f_t \ , \\ W - iB &= 2iD_+(\Xi^2_- - i\Xi^1_-) - \hat{q}^\phi_t - \hat{q}^\chi_t - i\hat{q}'_t - \tfrac{i}{2} f_t \ , \end{aligned} \tag{3.31}$$

for the chiral field-strengths which allows us to write kinetic terms for the spinor invariants such as

$$P_c(W, B) = P_{c+}(W + iB) + P_{c-}(W - iB) \ . \tag{3.32}$$

4 Matter couplings revised

In [22, 23] we discussed the reduction of the invariant Kähler potential (2.2) and wrote down the couplings of the large vector multiplet to (twisted) chiral matter. In particular, we wrote the gauge covariant derivative

$$\nabla_\pm \varphi^i = D_\pm \varphi^i - A_\pm k^i \quad , \quad \varphi^i = (\phi, \bar{\phi}, \chi, \bar{\chi}) \quad , \quad \mathcal{L}_k K = 0 \tag{4.33}$$

which couples φ^i minimally to the connections. When including the action terms, the field redefinitions of section 3.1 modify the matter couplings. In the chiral scenario, we change

$$\nabla_\pm \varphi^i - \Xi^1_\pm k^i = \nabla^c_\pm \varphi^i \tag{4.34}$$

[4] These terms can be interpreted as a twist of known ghost actions with $-3/2$ the ghost number [27]. I thank Martin Roček and Cumrum Vafa for pointing this out.

and keep the form of the reduced Lagrange density[5]

$$\begin{aligned}\mathcal{L}_{m(c)} \;=\; & \left(\Xi^A_+ + \nabla^c_+\varphi^i E_{iC}E^{CA}\right) E_{AB}\left(\Xi^B_- + E^{BD}E_{Dj}\nabla^c_-\varphi^j\right) \\ & +\nabla^c_+\varphi^i\left(E_{ij} - E_{iA}E^{AB}E_{Bj}\right)\nabla^c_-\varphi^j \\ & +iK_ik^k\left(\hat{q}^\phi_c(J^i_{+k}+J^i_{-k}) + \hat{q}^\chi_c(J^i_{+k}-J^i_{-k}) + \hat{q}'_c\Pi^i{}_k\right) \;, \end{aligned} \tag{4.35}$$

where we introduce the matrices

$$E_{kl} = K_{ij}\left(J^i_{+k}J^j_{-l} - \tfrac{1}{2}\Pi^i{}_k\delta^j{}_l - \tfrac{1}{2}\delta^i{}_k\Pi^j{}_l\right) \tag{4.36}$$

$$E_{Al} = K_{ij}\begin{pmatrix} (J^i_{+k}+J^i_{-k})J^j_{-l} - \Pi^i{}_k\delta^j{}_l \\ \Pi^i{}_kJ^j_{-l} + (J^i_{+k}+J^i_{-k})\delta^j{}_l \end{pmatrix} k^k \tag{4.37}$$

$$E_{kB} = K_{ij}\Big(J^i_{+k}(J^j_{+l}+J^j_{-l}) - \delta^i{}_k\Pi^j{}_l \;,\; J^i_{+k}\Pi^j{}_l + \delta^i{}_k(J^j_{+l}+J^j_{-l})\Big)k^l \tag{4.38}$$

$$E_{AB} = K_{ij}\begin{pmatrix} (J^i_{+k}+J^i_{-k})(J^j_{+l}+J^j_{-l}) & (J^i_{+k}+J^i_{-k})\Pi^j{}_l \\ \Pi^i{}_k(J^j_{+l}+J^j_{-l}) & \Pi^i{}_k\Pi^j{}_l \end{pmatrix} k^kk^l \tag{4.39}$$

and E^{AB} is the inverse of E_{AB}.

In the twisted chiral scenario we change $(A^{(c)}_\pm, \hat{q}'_c, \hat{q}^{\phi,\chi}_c) \to (A^{(t)}_\pm, \hat{q}'_t, \hat{q}^{\phi,\chi}_t)$ and modify the matrices

$$E_{Al} = K_{ij}\begin{pmatrix} -(J^i_{+k}-J^i_{-k})J^j_{-l} + \Pi^i{}_k\delta^j{}_l \\ \Pi^i{}_kJ^j_{-l} + (J^i_{+k}-J^i_{-k})\delta^j{}_l \end{pmatrix} k^k \tag{4.40}$$

$$E_{kA} = K_{ij}\Big(J^i_{+k}(J^j_{+l}-J^j_{-l}) + \delta^i{}_k\Pi^j{}_l \;,\; J^i_{+k}\Pi^j{}_l - \delta^i{}_k(J^j_{+l}-J^j_{-l})\Big)k^l \tag{4.41}$$

$$E_{AB} = K_{ij}\begin{pmatrix} -(J^i_{+k}-J^i_{-k})(J^j_{+l}-J^j_{-l}) & -(J^i_{+k}-J^i_{-k})\Pi^j{}_l \\ \Pi^i{}_k(J^j_{+l}-J^j_{-l}) & \Pi^i{}_k\Pi^j{}_l \end{pmatrix} k^kk^l \;. \tag{4.42}$$

It is useful to investigate the low-energy properties of such σ-models when the kinetic terms for $\hat{q}^{\phi,\chi}, \hat{q}'$ and f flow to zero (*e.g.*, [28]). We now show two simple cases where the Lagrange-densities due to the same Kähler potential (2.2) in the (twisted)chiral scenario are compatible:

- Quotient action: If there are no kinetic terms for $\Xi^{1,2}_\pm$ the two Lagrange-densities are related by field redefinitions:

  $$\hat{q}^\phi_c = \hat{q}^\phi_t - iD_{[+}\Xi^2_{-]} \;,\quad \hat{q}^\chi_c = \hat{q}^\chi_t - iD_{(+}\Xi^2_{-)} \;,\quad \hat{q}'_c = \hat{q}'_t + iD_{[+}\Xi^1_{-]} \;,\quad A^c_\pm = A^t_\pm + 2\Xi^1_\pm \;. \tag{4.43}$$

  When integrating both $\Xi^A_\pm$ and $A_\pm$ we therefore obtain the same quotients.

[5] If the density is invariant only up to generalized Kähler transformations we replace the Lie derivatives $(K_iJ^i_{\pm j}k^j \;,\; K_i\Pi^i{}_jk^j)$ with the moment maps $(-\mu_\pm \;,\; -\mu_\Pi)$ [9].

- T-duality: When adding a linear term that constrains the field-strengths to vanish

$$K \to K - \tfrac{1}{2}(\hat{\mathbb{X}}_L V_L + \bar{\hat{\mathbb{X}}}_L \bar{V}_L + \hat{\mathbb{X}}_R V_R + \bar{\hat{\mathbb{X}}}_R \bar{V}_R)\ . \tag{4.44}$$

we find [23] that $\Xi_{\pm}^{1,2}$ are also constrained to vanish, which, in both cases, gives the action:

$$\begin{aligned}
\mathcal{L} &= K_{ij}(J^i_{+k}J^j_{-l} - \tfrac{1}{2}\Pi^i{}_k\delta^j{}_l - \tfrac{1}{2}\delta^i{}_k\Pi_j{}^l)\nabla_+\varphi^k\nabla_-\varphi^l \\
&+ i\hat{q}^{\phi}(K_i(J^i_{+j} + J^i_{-j})k^j + i(\tilde{X}_L - \bar{\tilde{X}}_L + \tilde{X}_R - \bar{\tilde{X}}_R)) \\
&+ i\hat{q}^{\chi}(K_i(J^i_{+j} - J^i_{-j})k^j - i(\tilde{X}_L - \bar{\tilde{X}}_L - \tilde{X}_R + \bar{\tilde{X}}_R)) \\
&+ i\hat{q}'(K_i\Pi^i{}_jk^j - (\tilde{X}_L + \bar{\tilde{X}}_L + \tilde{X}_R + \bar{\tilde{X}}_R)) \\
&+ \tfrac{i}{2}f(\tilde{X}_L + \bar{\tilde{X}}_L - \tilde{X}_R - \bar{\tilde{X}}_R))
\end{aligned} \tag{4.45}$$

5 Conclusion

In this paper we presented two possible candidates, which are unique to the abelian case, for Kähler and superpotential terms for the large vector multiplet action where undesired higher derivative terms are removed by field redefinitions and choice of gauge invariants present in the Kähler potential. We then write possible kinetic terms for the spinor invariants which include twisted conformal field theories and topological terms. This work concludes our presentation of the $N = (2,2)$ multiplets [22, 23, 26] which is a step towards treating generalized Kähler geometry on similar footing as torsion-free complex manifolds that arise naturally as target spaces for supersymmetric σ-models. Future work along this avenue may include a full treatment of the moment maps and generalized Kähler quotients [9].

Another possible generalization resulting from these new gauge multiplets is the formulation of new gauge linear σ-models (GLSMs) with H-fluxes which can generalize results such as [29, 31] or mirror symmetry (*e.g.*, [28, 32]). It is important, however, to note that the IR flow for the spinor invariants action of sec. 3.1 must be studied first as it may lead to new physical degrees of freedom in the effective theory.

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Chapter 6

$N = (4,4)$ Supersymmetry for semichirals

$N = (4,4)$ supersymmetry requires further restrictions on the target geometry imposing quaternionic algebra on triplets of complex structures. As $N = (2,2)$ already exhausted all possibilities of derivatives appearing in the integrand, there is no straightforward superspace description giving $N = (4,4)$ supersymmetric models, though projective and harmonic superspace exist (*e. g.* [60] and [61] respectively for review.)

For Kähler submanifolds, $N = (4,4)$ supersymmetry restricts the structure to hyperkähler manifolds which is of great interest in both physics and mathematics [12]. In this paper, we studied the conditions for $N = (4,4)$ supersymmetry for a model with semichiral fields only. We write down an Ansatz that respects the semichirality condition and investigate the constraints on the structures due to extension to $N = (4,4)$ supersymmetry.

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Sigma models with off-shell $N = (4, 4)$ supersymmetry and noncommuting complex structures

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Abstract

We describe the conditions for extra supersymmetry in $N = (2, 2)$ supersymmetric nonlinear sigma models written in terms of semichiral superfields. We find that some of these models have additional off-shell supersymmetry. The $(4, 4)$ supersymmetry introduces geometrical structures on the target-space which are conveniently described in terms of Yano f-structures and Magri-Morosi concomitants. On-shell, we relate the new structures to the known bi-hypercomplex structures.

Contents

1 Introduction

The target-space geometry of two-dimensional supersymmetric nonlinear sigma models has been extensively discussed in the literature. In [16], and partly in [33], the general case including a B-field was described in $(1,1)$ superspace. For $(2,2)$ supersymmetry the target-space geometry was shown to be bihermitean, i.e., the metric is hermitean with respect to two complex structures $J_{(\pm)}$. Off-shell, a manifest $(2,2)$ formulation was only found when the complex structures commute[1]. Similar results hold for $(4,4)$ supersymmetry: A mani-

[1]Some other models with off-shell (2,2) supersymmetry were found in [15]–[11].

fest $(2,2)$ formulation was only found when (some of) the complex structures commute.

More recently the bihermitean geometry of [16] has been described as generalized Kähler geometry [2], a subclass of generalized complex geometry [36]. The intimate relation of this description to sigma models is elucidated in, e.g., [37]–[18]. In particular, as shown in [14], a complete $(2,2)$ superspace description of generalized Kähler geometry, including the case when the complex structures do not commute, requires semichiral fields [15] in addition to the chiral and twisted chiral fields; this had been conjectured but not proven by Sevrin and Troost [34]. The superspace lagrangian K is further shown to be a potential for the metric and B-field [40];

The bi-hypercomplex geometry of [16] has likewise been described as generalized hyperkähler geometry in [41] and [42].

In the present paper, we discuss models written in terms of semichiral fields only. We ask under which conditions such a model can carry $(4,4)$ supersymmetry. A limited class of such models was recently discussed in [43]. There the extra transformations were taken to be linear in the derivatives of the fields, and the target-space was restricted to be four-dimensional. It was found that no interesting solution for $N=(4,4)$ supersymmetry exists, but instead one can find an interesting solution for $N=(4,4)$ twisted supersymmetry. This implied that the target-space must have pseudo-hypercomplex geometry.

Some models including semichiral but no chiral or twisted chiral fields had been treated previously in [44]; they include additional auxiliary $(4,4)$ fields, and only become purely semichiral models on-shell.

Models with commuting complex structures, described by n chiral and m twisted chiral fields, have off-shell $(4,4)$ supersymmetry when $n=m$ and the Lagrangian K satisfies certain differential constraints [16]. Purely semichiral models have to have an equal number of left and right semichiral fields [15]. Here we find that for some such models whose Lagrangian again satisfies certain differential constraints, there is an off-shell algebra. This algebra has an interpretation in terms of an integrable Yano f-structure on $TM \oplus TM$, the sum of two copies of the tangent bundle of the target-space. We already know from [16] that a sigma model with $(4,4)$ supersymmetry has two quaternion-worth of complex structures, $J^A_{(\pm)}$, living on TM and we find that all of these structures fit together nicely. In particular we resolve the interplay between the various integrability conditions involving Nijenhuis tensors and Magri-Morosi concomitants.

The generalized Kähler potential for those semichiral models that are invariant under the off-shell algebra satisfy a constraint. This is analogous to the $(4,4)$ conditions in [16] which are realized for commuting complex structures by the $N=4$ twisted chiral multiplet. For a subclass of our models, we can give a geometric interpretation of the condition as a kind of hermiticity condition: a certain tensor is preserved by the f-structures.

We follow the method used in previous discussions of additional nonmanifest supersymmetries, e.g., in [16] and [19]. To study the additional symmetries, we make the most general ansatz compatible with the properties of the superfields, and then read off the constraints that follow from closure of the supersymmetry algebra and invariance of the action. The constraints from the algebra are discussed in section 3, the invariance of the action is presented in section 5. Often in these investigations field-equations arise and the algebra only closes on-shell. In section 4 we analyze off-shell closure while postponing the on-shell discussion to section 6.

2 Preliminaries

This section contains background material needed for the discussions in later sections.

The $(2,2)$ supersymmetry algebra for the covariant derivatives is given by

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_{\mathop{\pm}\limits^{\pm}}\ , \tag{2.1}$$

and the left and right semichiral fields $\mathbb{X}^{a,a'}$, and left and right anti-semichiral fields $\bar{\mathbb{X}}^{\bar{a},\bar{a}'}$ [15] satisfy

$$\bar{\mathbb{D}}_+\mathbb{X}^a = 0\ , \quad \bar{\mathbb{D}}_-\mathbb{X}^{a'} = 0, \quad \mathbb{D}_+\bar{\mathbb{X}}^{\bar{a}} = 0\ , \quad \mathbb{D}_-\bar{\mathbb{X}}^{\bar{a}'} = 0\ . \tag{2.2}$$

A useful collective notation, often used in previous papers, is $\mathbb{X}^L = (\mathbb{X}^a, \bar{\mathbb{X}}^{\bar{a}})$ and
$\mathbb{X}^R = (\mathbb{X}^{a'}, \bar{\mathbb{X}}^{\bar{a}'})$. When we need a notation for all of the fields we write $\mathbb{X}^i$ with $i = (L, R)$.

We shall consider the generalized Kähler potential K and the sigma model it defines through the action

$$S = \int d^2\xi \mathbb{D}^2\bar{\mathbb{D}}^2 \, K(\mathbb{X}^i) \; . \tag{2.3}$$

The target-space manifold $\mathcal{M}^{4d}$ coordinatized by the d left and d right semichiral fields (and their conjugates) carries bihermitean geometry. This means that there are two complex structures $J_{(\pm)}$, a metric g hermitean with respect to both of these and a closed three form H such that [16]

$$\begin{aligned}
&J^2_{(\pm)} = -\mathbb{1} \\
&\nabla^{(\pm)} J_{(\pm)} = 0, \quad \Gamma^{(\pm)} = \Gamma^0 \pm \frac{1}{2} g^{-1} H \\
&J^t_{(\pm)} g J_{(\pm)} = g \\
&d^c_+ \omega_+ + d^c_- \omega_- = 0, \quad H = d^c_+ \omega_+ = -d^c_- \omega_- \; ,
\end{aligned} \tag{2.4}$$

where Γ^0 is the Levi-Civita connection for the metric g and $d^c_{(\pm)} := J_{(\pm)}(d)$. The expression for d^c becomes most simple in complex coordinates: $d^c = i(\bar{\partial} - \partial)$.

In later sections we shall also need the explicit form of the complex structures: They are defined in terms of the matrices [14]

$$K_{LR} := \left(\begin{array}{cc} K_{aa'} & K_{a\bar{a}'} \\ K_{\bar{a}a'} & K_{\bar{a}\bar{a}'} \end{array} \right) \; , \tag{2.5}$$

and with $C := [j, K]$, they read

$$J_{(+)} = \left(\begin{array}{cc} j & 0 \\ K^{-1}_{RL} C_{LL} & K^{-1}_{RL} j K_{LR} \end{array} \right)$$

$$J_{(-)} = \left(\begin{array}{cc} K^{-1}_{LR} j K_{RL} & K^{-1}_{RL} C_{RR} \\ 0 & j \end{array} \right) \tag{2.6}$$

with j denoting a canonical $2d \times 2d$ complex structure

$$j := \left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) \; . \tag{2.7}$$

The description (2.4) applies to bihermitean geometry in general, which may be described using chiral, twisted chiral and semichiral fields [14]. A special feature of the case we are interested in here is that, although locally we may always write $H = dB$, for the model with only semichiral fields B is globally defined (away from type change loci [2]). For more aspects of the global structure of bihermitean geometry, see [45].

The data $(g, B, J_{(\pm)})$ in (2.4) may be packaged as structures on $TM \oplus T^*M$ in the form of generalized Kähler geometry [2].

3 Nonmanifest supersymmetries

3.1 Ansatz for non-manifest supersymmetries

Requiring that the derivatives are covariant with respect to the additional supersymmetries, e.g., $\bar{\mathbb{D}}_+(\delta\mathbb{X}^a) = \delta(\bar{\mathbb{D}}_+\mathbb{X}^a) = 0$, leads to the following general ansatz for $N = (4,4)$ supersymmetry:

$$\begin{aligned}
\delta\mathbb{X}^a &= \bar{\epsilon}^+\bar{\mathbb{D}}_+ f^a(\mathbb{X}^{L,R}, \bar{\mathbb{X}}^{L,R}) + g^a_b(\mathbb{X}^c)\bar{\epsilon}^-\bar{\mathbb{D}}_-\mathbb{X}^b + h^a_b(\mathbb{X}^c)\epsilon^-\mathbb{D}_-\mathbb{X}^b \ , \\
\delta\bar{\mathbb{X}}^{\bar{a}} &= \epsilon^+\mathbb{D}_+\bar{f}^{\bar{a}}(\mathbb{X}^{L,R}, \bar{\mathbb{X}}^{L,R}) + \bar{g}^{\bar{a}}_{\bar{b}}(\bar{\mathbb{X}}^{\bar{c}})\epsilon^-\mathbb{D}_-\bar{\mathbb{X}}^{\bar{b}} + \bar{h}^{\bar{a}}_{\bar{b}}(\bar{\mathbb{X}}^{\bar{c}})\bar{\epsilon}^-\bar{\mathbb{D}}_-\bar{\mathbb{X}}^{\bar{b}} \ , \\
\delta\mathbb{X}^{a'} &= \bar{\epsilon}^-\bar{\mathbb{D}}_-\tilde{f}^{a'}(\mathbb{X}^{L,R}, \bar{\mathbb{X}}^{L,R}) + \tilde{g}^{a'}_{b'}(\mathbb{X}^{c'})\bar{\epsilon}^+\bar{\mathbb{D}}_+\mathbb{X}^{b'} + \tilde{h}^{a'}_{b'}(\mathbb{X}^{c'})\epsilon^+\mathbb{D}_+\mathbb{X}^{b'} \ , \\
\delta\bar{\mathbb{X}}^{\bar{a}'} &= \epsilon^-\mathbb{D}_-\bar{\tilde{f}}^{\bar{a}'}(\mathbb{X}^{L,R}, \bar{\mathbb{X}}^{L,R}) + \bar{\tilde{g}}^{\bar{a}'}_{\bar{b}'}(\bar{\mathbb{X}}^{\bar{c}'})\epsilon^+\mathbb{D}_+\bar{\mathbb{X}}^{\bar{b}'} + \bar{\tilde{h}}^{\bar{a}'}_{\bar{b}'}(\bar{\mathbb{X}}^{\bar{c}'})\bar{\epsilon}^+\bar{\mathbb{D}}_+\bar{\mathbb{X}}^{\bar{b}'} \ ,
\end{aligned} \tag{3.1}$$

where $\epsilon^\pm$ are the transformation parameters. This ansatz is covariant under left and right holomorphic transformations, i.e., coordinate transformations of the form[2]

$$\begin{aligned}
&\mathbb{X}^a \to \mathbb{X}'^a(\mathbb{X}^b) \ , \quad \mathbb{X}^{\bar{a}} \to \mathbb{X}'^{\bar{a}}(\mathbb{X}^{\bar{b}}) \ , \\
&\mathbb{X}^{a'} \to \mathbb{X}'^{a'}(\mathbb{X}^{b'}) \ , \quad \mathbb{X}^{\bar{a}'} \to \mathbb{X}'^{\bar{a}'}(\mathbb{X}^{\bar{b}'}) \ .
\end{aligned} \tag{3.2}$$

[2]Strictly speaking, these are not the most general left and right holomorphic transformations, as they also preserve the choice of polarization, i.e., the separation into left and right coordinates.

A useful way of rewriting these nonmanifest transformations introduces the matrices $U^{(\pm)}$ and $V^{(\pm)}$ defined as

$$\bar{\delta}^{\pm}\mathbb{X} := \bar{\delta}^{\pm}\begin{pmatrix} \mathbb{X}^{a} \\ \bar{\mathbb{X}}^{\bar{a}} \\ \mathbb{X}^{a'} \\ \bar{\mathbb{X}}^{\bar{a}'} \end{pmatrix} = \bar{\delta}^{\pm}\begin{pmatrix} \mathbb{X}^{L} \\ \mathbb{X}^{R} \end{pmatrix} = U^{(\pm)}\bar{\epsilon}^{\pm}\bar{\mathbb{D}}_{\pm}\mathbb{X}\ , \qquad \delta^{\pm}\mathbb{X} = V^{(\pm)}\epsilon^{\pm}\mathbb{D}_{\pm}\mathbb{X} \tag{3.3}$$

where[3]

$$U^{(+)} = \begin{pmatrix} * & f^{a}_{\bar{b}} & f^{a}_{b'} & f^{a}_{\bar{b}'} \\ * & 0 & 0 & 0 \\ * & 0 & \tilde{g}^{a'}_{b'} & 0 \\ * & 0 & 0 & \bar{\tilde{h}}^{\bar{a}'}_{\bar{b}'} \end{pmatrix}, \quad U^{(-)} = \begin{pmatrix} g^{a}_{b} & 0 & * & 0 \\ 0 & \bar{h}^{\bar{a}}_{\bar{b}} & * & 0 \\ \tilde{f}^{a'}_{b} & \tilde{f}^{a'}_{\bar{b}} & * & \tilde{f}^{a'}_{\bar{b}'} \\ 0 & 0 & * & 0 \end{pmatrix} \tag{3.4}$$

and

$$V^{(\pm)} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \bar{U}^{(\pm)} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{3.5}$$

Here

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3.6}$$

Note that one column in each of the transformation matrices $U^{(\pm)}$ and $V^{(\pm)}$ is arbitrary. For the remainder of the paper, we set the arbitrary entries to zero. Doing so provides us with full integrability of the transformation matrices and an interpretation of the off-shell algebra in terms of Yano f-structures. The consequences of keeping the arbitrariness is discussed briefly in section 7.

[3] The fundamental tensorial objects are defined in (3.1). Additional covariant indices denote partial derivatives, e.g., $f^{a}_{i} := \partial_i f^{a}$, etc.

For later use, we introduce the projection operators $P^{\pm}, \hat{P}^{\pm}$:

$$P_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \hat{P}_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ ,$$

$$P_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \hat{P}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{3.7}$$

3.2 Magri-Morosi concomitant

To interpret the expressions we find below, we use the Magri-Morosi concomitant [46, 47] defined for two endomorphisms I and J of the tangent bundle TM of a manifold M as

$$\mathcal{M}(I,J)^i_{jk} := -\mathcal{M}(J,I)^i_{kj} = I^l{}_j J^i{}_{k,l} - J^l{}_k I^i{}_{j,l} - I^i{}_l J^l{}_{k,j} + J^i{}_l I^l{}_{j,k} \ . \tag{3.8}$$

This concomitant has previously been used when discussing supersymmetry algebra, e.g., in discussing $(1,0)$ and $(1,1)$ formulations of certain (p,q) sigma models in [48] and discussing generalized complex geometry for $(2,2)$ models in [38].

The Magri-Morosi concomitant relates to the simultaneous integrability of two structures and is a tensor only when $[I,J]=0$. More precisely, two commuting complex structures are simultaneously integrable if and only if their Magri-Morosi concomitant vanishes. The part antisymmetric in j,k is the Nijenhuis concomitant $\mathcal{N}(I,J)$; when $I=J$ this becomes the Nijenhuis tensor $\mathcal{N}(I)$. If $\mathcal{N}(I)=0$, then I is integrable.

Assuming that we have one I-connection $\nabla^{(I)}$ and one J-connection $\nabla^{(J)}$ differing only in the sign of the torsion $\Gamma^{(I/J)} = \Gamma^{(0)} \pm T$, we can rewrite $\mathcal{M}$ as

$$\begin{aligned} \mathcal{M}(I,J)^i_{jk} &= I^l{}_j \nabla^{(J)}_l J^i{}_k - J^l{}_k \nabla^{(I)}_l I^i{}_j - I^i{}_l \nabla^{(J)}_j J^l{}_k + J^i{}_l \nabla^{(I)}_k I^l{}_j - [I,J]^i_m \Gamma^{(J)\ m}_{jk} \\ &:= \widehat{\mathcal{M}}(I,J)^i_{jk} - [I,J]^i_m \Gamma^{(J)\ m}_{jk} \ . \end{aligned} \tag{3.9}$$

We shall need this version in section 6.4 below.

Finally, we note that in the special case when $I^i{}_j$ and $J^i{}_j$ are curl-free in the lower indices, the concomitant simplifies to

$$\mathcal{M}(I,J)^i_{jk} = (JI)^i_{j,k} - (IJ)^i_{k,j} \ . \tag{3.10}$$

3.3 Constraints from the supersymmetry algebra

Imposing the left-with-right commutator algebra for the ansatz (3.3) relates the Magri-Morosi concomitant of transformation matrices to the commutator of the same matrices as follows

$$\begin{aligned} [\bar\delta^\pm, \bar\delta^\mp]\mathbb{X}^i = 0 \quad &\Longleftrightarrow \quad \mathcal{M}(U^{(\pm)}, U^{(\mp)})^i_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\bar{\mathbb{D}}_\mp\mathbb{X}^k = [U^{(\pm)}, U^{(\mp)}]^i{}_m\bar{\mathbb{D}}_\pm\bar{\mathbb{D}}_\mp\mathbb{X}^m \ , \\ [\bar\delta^\pm, \delta^\mp]\mathbb{X}^i = 0 \quad &\Longleftrightarrow \quad \mathcal{M}(U^{(\pm)}, V^{(\mp)})^i_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\mathbb{D}_\mp\mathbb{X}^k = [U^{(\pm)}, V^{(\mp)}]^i{}_m\bar{\mathbb{D}}_\pm\mathbb{D}_\mp\mathbb{X}^m \ . \end{aligned} \tag{3.11}$$

These relations can be rewritten covariantly using $\widehat{\mathcal{M}}$ defined in (3.9) as

$$\begin{aligned} \widehat{\mathcal{M}}(U^{(\pm)}, U^{(\mp)})^i_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\bar{\mathbb{D}}_\mp\mathbb{X}^k &= [U^{(\pm)}, U^{(\mp)}]^i{}_m\left(\bar{\mathbb{D}}_\pm\bar{\mathbb{D}}_\mp\mathbb{X}^m + \Gamma^{(\mp)\ m}_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\bar{\mathbb{D}}_\mp\mathbb{X}^k\right) \\ &= [U^{(\pm)}, U^{(\mp)}]^i{}_m\bar{\nabla}^{(\mp)}_\pm\bar{\mathbb{D}}_\mp\mathbb{X}^m \ , \\ \widehat{\mathcal{M}}(U^{(\pm)}, V^{(\mp)})^i_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\mathbb{D}_\mp\mathbb{X}^k &= [U^{(\pm)}, V^{(\mp)}]^i{}_m\left(\bar{\mathbb{D}}_\pm\mathbb{D}_\mp\mathbb{X}^m + \Gamma^{(\mp)\ m}_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\mathbb{D}_\mp\mathbb{X}^k\right) \\ &= [U^{(\pm)}, V^{(\mp)}]^i{}_m\bar{\nabla}^{(\mp)}_\pm\mathbb{D}_\mp\mathbb{X}^m \ . \end{aligned} \tag{3.12}$$

In the last equalities we have identified the pullback of the covariant derivative, for use in the on-shell section. Note that constraints on the semichiral fields imply that some of the equations vanish trivially.

The constraints from the left-with-left and right-with-right part of the algebra involve the Nijenhuis tensor:

$$[\bar\delta^\pm, \bar\delta^\pm]\mathbb{X}^i = 0 \iff \mathcal{N}(U^{(\pm)})^i_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\bar{\mathbb{D}}_\pm\mathbb{X}^k = 0 \ . \tag{3.13}$$

Finally, using the algebra (2.1), the commutator $[\delta^\pm, \bar\delta^\pm]\mathbb{X}^i = i\bar\epsilon^\pm\epsilon^\pm\partial_{\substack{+\\=}}\mathbb{X}^i$ yields

$$\begin{aligned} \mathcal{M}(U^{(\pm)}, V^{(\pm)})^i_{jk}\bar{\mathbb{D}}_\pm\mathbb{X}^j\mathbb{D}_\pm\mathbb{X}^k \quad = \quad & \left[(UV)^{(\pm)i}_{\ j} + \delta^i_j\right]\bar{\mathbb{D}}_\pm\mathbb{D}_\pm\mathbb{X}^j \\ & + \left[(VU)^{(\pm)i}_{\ j} + \delta^i_j\right]\mathbb{D}_\pm\bar{\mathbb{D}}_\pm\mathbb{X}^j \ . \end{aligned} \tag{3.14}$$

4 Off-shell interpretation of the algebra constraints

In this section we analyze the constraints found in section 3.3, separating the conditions into algebraically independent parts.

4.1 The conditions for off-shell invariance

Off-shell, $\mathbb{DXDX}$ and $\mathbb{DDX}$ are independent structures and hence both sides in equation (3.11) and (3.14) must vanish independently. This gives the conditions

$$\begin{aligned}\mathcal{M}(U^{(+)},U^{(-)})^i_{jk} &= 0\ , \qquad j\neq a,\ k\neq a' \\ \mathcal{M}(U^{(+)},V^{(-)})^i_{jk} &= 0\ , \qquad j\neq a,\ k\neq \bar a' \\ \mathcal{M}(U^{(+)},V^{(+)})^i_{jk} &= 0\ , \qquad j\neq a,\ k\neq \bar a\ ,\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}[U^{(+)},U^{(-)}]^i{}_j &= 0\ , \qquad j\neq a,a' \\ [U^{(+)},V^{(-)}]^i{}_j &= 0\ , \qquad j\neq a,\bar a'\ ,\end{aligned} \tag{4.2}$$

and finally

$$(UV)^{(+)i}_{\ \ j} = -\delta^i_j\ , \quad j\neq \bar a\ , \quad (VU)^{(+)i}_{\ \ j} = -\delta^i_j\ , \quad j\neq a\ , \tag{4.3}$$

together with their complex conjugate equations. Setting the arbitrary entries in the transformation matrices to zero sets the undetermined columns in (4.3) to zero,

$$(UV)^{(+)i}_{\ \ \bar a} = (VU)^{(+)i}_{\ \ a} = (UV)^{(-)i}_{\ \ \bar a'} = (VU)^{(-)i}_{\ \ a'} = 0\ . \tag{4.4}$$

The constraint (3.13) implies that $U^{(\pm)}$ and $V^{(\pm)}$ are integrable on some subspace. When we impose (4.4), the integrability extends to the full space:

$$\mathcal{N}(U^{(\pm)})^i_{jk} = 0\ . \tag{4.5}$$

The conditions in (4.1) may be written as in (3.10) plus curl terms;

$$\mathcal{M}(U^{(\pm)},V^{(\pm)})^i_{kj} = (VU)^{(\pm)i}_{\ \ j,k} - (UV)^{(\pm)i}_{\ \ k,j} + U^{(\pm)l}_{\ \ j}V^{(\pm)i}_{\ \ [k,l]} - V^{(\pm)l}_{\ \ k}U^{(\pm)j}_{\ \ [j,l]} = 0\ . \tag{4.6}$$

The first two terms vanish due to (4.3). The form of the ansatz (3.4) reveals that most of the third and fourth terms also vanish identically. The remaining ones may be shown to be zero due to (4.3) and the integrability (4.5). As an example of the last statement consider

$$U^{(+)l}_{j} V^{(+)a'}_{[k,l]} - V^{(+)l}_{k} U^{(+)a'}_{[j,l]} \tag{4.7}$$

which is nonvanishing for $j, k = b', d'$ when it becomes

$$\tilde{h}^{a'}_{[b',c']} \tilde{g}^{c'}_{d'} - \tilde{g}^{a'}_{[d',c']} \tilde{h}^{c'}_{b'} \ . \tag{4.8}$$

A short calculation then shows that this combination is zero due to (4.3)

$$\tilde{h}^{a'}_{c'} \tilde{g}^{c'}_{d'} = \tilde{g}^{a'}_{c'} \tilde{h}^{c'}_{d'} = -\delta^{a'}_{d'} \ , \tag{4.9}$$

and (4.5)

$$\tilde{g}^{c'}_{[d'} \tilde{g}^{a'}_{b'],c'} - \tilde{g}^{a'}_{c'} \tilde{g}^{c'}_{[b',d']} = 0 \ . \tag{4.10}$$

In summary, off-shell we find the following algebraic constraints in all sectors not projected out by the semi-chiral constraints:

- The transformation matrices $U^{(\pm)}, V^{(\pm)}$ all commute.
- The products $U^{(\pm)} V^{(\pm)}$ and $V^{(\pm)} U^{(\pm)}$ equal minus one.
- The transformation matrices are all separately integrable.
- The Magri-Morosi concomitant vanishes for all two pairs of the transformation matrices. We showed that some of these, namely the last one in (4.1) relating $U^{(\pm)}$ with $V^{(\pm)}$, follow from the above three constraints.

The zeros in the arbitrary columns of the transformation matrices gives full integrability as in (4.5) and the relations (4.4). This makes the products $U^{(\pm)} V^{(\pm)}$ and $V^{(\pm)} U^{(\pm)}$ act as projection operators and we find a nice geometric interpretation in terms of f-structures.

4.2 A Yano f-structure

The fact that the matrices $U^{(\pm)}$ (and $V^{(\pm)}$) are degenerate and satisfy (4.3) and (4.4),

$$\begin{aligned} U^{(+)}V^{(+)} &= -\mathrm{diag}(1,0,1,1), \quad V^{(+)}U^{(+)} = -\mathrm{diag}(0,1,1,1), \\ U^{(-)}V^{(-)} &= -\mathrm{diag}(1,1,1,0), \quad V^{(-)}U^{(-)} = -\mathrm{diag}(1,1,0,1) \end{aligned} \tag{4.11}$$

prevents a direct interpretation in terms of complex structures on the tangent space TM. We are led to consider endomorphisms on $TM \oplus TM$ and the weaker f-structures instead. The following $8d \times 8d$ matrices are f-structures in the sense of Yano [49]:

$$\mathcal{F}_{(\pm)} := \begin{pmatrix} 0 & U^{(\pm)} \\ V^{(\pm)} & 0 \end{pmatrix} \quad \Longrightarrow \mathcal{F}^3_{(\pm)} + \mathcal{F}_{(\pm)} = 0 \ . \tag{4.12}$$

This follows directly from conditions in (4.3). Moreover, $-\mathcal{F}^2_{(\pm)}$ and $1+\mathcal{F}^2_{(\pm)}$ define integrable distributions, as can be shown using (4.3) and (4.5). More explicitly: Using the projectors (3.7), the conditions (4.11) may be written as

$$\hat{P}_\pm = 1 + V^{(\pm)}U^{(\pm)} \ , \quad P_\pm = 1 + U^{(\pm)}V^{(\pm)} \ . \tag{4.13}$$

Then we may define

$$m_{(\pm)} := 1 + \mathcal{F}^2_{(\pm)} = \begin{pmatrix} P_\pm & 0 \\ 0 & \hat{P}_\pm \end{pmatrix}, \quad l_{(\pm)} := -\mathcal{F}^2_{(\pm)} = \begin{pmatrix} 1-P_\pm & 0 \\ 0 & 1-\hat{P}_\pm \end{pmatrix}. \tag{4.14}$$

These fulfill

$$l_{(\pm)} + m_{(\pm)} = 1, \quad l^2_{(\pm)} = l_{(\pm)}, \quad m^2_{(\pm)} = m_{(\pm)} \ , \quad l_{(\pm)}m_{(\pm)} = 0 \tag{4.15}$$

and

$$\mathcal{F}_{(\pm)}l_{(\pm)} = l_{(\pm)}\mathcal{F}_{(\pm)} = \mathcal{F}_{(\pm)}, \quad m_{(\pm)}\mathcal{F}_{(\pm)} = \mathcal{F}_{(\pm)}m_{(\pm)} = 0. \tag{4.16}$$

The operators $l_{(\pm)}$ and $m_{(\pm)}$ applied to the tangent space at each point of the manifold are complementary projection operators and define complementary distributions in the sense of Yano: $\Lambda_\pm$, the first fundamental distribution, and $\Sigma_\pm$, the second fundametal distribution, corresponding to $l_\pm$ and $m_\pm$, of dimensions $6d$ and $2d$, respectively.

Let $\mathcal{N}_{\mathcal{F}_{(\pm)}}$ denote the Nijenhuis tensor for the f-structures $\mathcal{F}_{(\pm)}$. By a theorem of Ishihara and Yano [50] we have that

i. $\Lambda_\pm$ is integrable iff $m^i_{(\pm)l}\mathcal{N}^l_{\mathcal{F}_{(\pm)}jk} = 0$,

ii. $\Sigma_\pm$ is integrable iff $\mathcal{N}^i_{\mathcal{F}_{(\pm)}jk} m^j_{(\pm)l} m^k_{(\pm)m} = 0$.

From the definition of the f-structures in (4.12), one can derive that these two conditions are fulfilled. Hence, the distributions $\Lambda_\pm$ and $\Sigma_\pm$ are integrable.

4.3 Additional twisted supersymmetry

In a previous paper [43] we investigated the special case of four-dimensional target space and required the transformations (3.1) to be linear. There, it was found that no solution with interesting geometry exists which possesses additional supersymmetry. On the other hand, one could impose additional twisted linear supersymmetry $[\delta, \delta] = -\partial$ for a solution with interesting geometrical properties.

In the general case treated in this paper, we have found that additional supersymmetry can indeed be imposed. But also additional twisted supersymmetry could be considered. The difference would be that contraint (3.14) would receive a minus sign,

$$\begin{aligned}\mathcal{M}(U^{(\pm)}, V^{(\pm)})^i_{jk}\bar{\mathbb{D}}_\pm \mathbb{X}^j \mathbb{D}_\pm \mathbb{X}^k \quad = \quad & \left[(UV)^{(\pm)i}_{\ \ j} - \delta^i_j\right]\bar{\mathbb{D}}_\pm \mathbb{D}_\pm \mathbb{X}^j \\ & + \left[(VU)^{(\pm)i}_{\ \ j} - \delta^i_j\right]\mathbb{D}_\pm \bar{\mathbb{D}}_\pm \mathbb{X}^j \ . \end{aligned} \tag{4.17}$$

with the effect that the structure defined in (4.12) would be a f-structure of hyperbolic type,

$$\mathcal{F}_{(\pm)}(\mathcal{F}^2_{(\pm)} - 1) = 0, \tag{4.18}$$

that is, generalizations of product structures instead of complex structures.

5 Invariance of the action

The bihermitean geometry of [16] is derived from the (1,1) sigma model via two requirements: closure of the algebra and invariance of the action. More precisely, the supersymmetry algebra implies the existence of the complex structures, whereas invariance the action implies the bihermiticity of the metric and the covariant constancy of the complex structures. Similarly, for $(4,4)$ supersymmetry, the algebra implies that the transformations are given in terms of left and right hypercomplex structures whereas invariance of the action implies the metric is hermitean with respect to all of these structures and the left and right connections preserve the the left and right structures respectively. When, in later sections, we use the knowledge from [16] in understanding our algebra conditions on-shell we can thus use the existence of a hypercomplex structures freely, but only require them to be covariantly constant if we assume that the action is invariant.

At the manifest $(2,2)$ level the discussion of additional supersymmetries in the model with (anti)chiral fields (the hyperkähler case) follows similar lines [19]. Extra supersymmetries lead to new complex structures as part of the conditions for closure of the algebra and invariance of the $(2,2)$ action leads to to the requirement that they are covariantly constant and that the metric is bihermitean.

When the complex structures commute and the sigma model is describable in $(2,2)$ superspace using (an equal number of) chiral and twisted chiral superfields, $(4,4)$ supersymmetry comes at the price of extra conditions on the potential K [16]. This is also true for the linear-transformation model in [43]. We expect the same to be true here.

The action (2.3) is invariant under the supersymmetry transformations (3.3) provided that

$$\big(K_i U^{(+)i}{}_{[j}\big)_{\,k]} = 0, \quad j,k \neq a, \tag{5.1}$$

and analogously for $U^{(-)}$ and $V^{(\pm)}$. We can write this out as (3.1) a system of equations for K:

$$\big(K_a f^i_{[j} + K_{a'}\tilde{g}^{a'}_{[j} + K_{\bar{a}'}\bar{\tilde{h}}^{\bar{a}'}_{[j}\big)_{k]} = 0 \tag{5.2}$$

plus analogous relations from $U^{(-)}$ and $V^{(\pm)}$.

The conditions (5.2) (or (5.1)) have to be satisfied for the generalized Kähler potential K to allow $(4,4)$ supersymmetry in a model with noncommuting complex structures whose commutator has empty kernel. In this sense it plays a similar role to the Monge-Ampère equation for models with vanishing torsion.

In the four-dimensional case with linear twisted supersymmetry transformations, it turned out to be possible to solve (5.2), (see [43]) but this is much harder in general. However, when the curl of $\tilde{g}$ and $\tilde{h}$ vanish, the condition has an interpretation on $TM \oplus TM$ much like a hermiticity condition, which we now turn to.

We combine the Hessian K_{ij} of the Kähler potential into an antisymmetric tensor on $\mathfrak{B}$ on $TM \oplus TM$ as

$$\mathfrak{B} = \begin{pmatrix} 0 & K \\ -K^t & 0 \end{pmatrix}. \tag{5.3}$$

The relation (5.1) can be used to show that off-shell the f-structures (4.12) preserve $\mathfrak{B}$ on a subspace projected out by the second fundamental projection operators $l_{(\pm)}$ defined in (4.16),

$$l_{(\pm)}\, \mathcal{F}^t_{(\pm)} \mathfrak{B} \mathcal{F}_{(\pm)}\, l_{(\pm)} = l_{(\pm)}\, \mathfrak{B}\, l_{(\pm)}. \tag{5.4}$$

This may be easily verified using (4.3), which implies $V^t K^t U = -K$ (except for one column and one row).

6 On-shell interpretation of the algebra constraints

In this section we discuss two main issues: How the conditions derived in section 3.3 have a larger set of solutions on-shell, and the relation to the underlying (hermitean) bi-hypercomplex geometry derived in [16]. In spirit the treatment is similar to both the $(1,1)$ discussion in [16] of extended supersymmetry and to the hyperkähler derivation in [19]: In [16] it was found that the left and right complex structures had to commute to get off-shell closure since the algebra gives a term proportional to this commutator times the field-equations. In [19] it was found that field equations as well as conditions from the invariance of the action were needed for closure of the algebra of non-manifest additional supersymmetries.

Below we separate the conclusions we may draw from closure of the algebra only and those where in addition we need invariance of the action.

6.1 On-shell algebra

In this subsection we use a coordinate transformation to derive an explicit relation between the components of the transformation matrices and the underlying hypercomplex structure. The field equations that follow from the action (2.3) are

$$\bar{\mathbb{D}}_+ K_a = 0 \ , \quad \mathbb{D}_+ K_{\bar{a}} = 0 \ , \quad \bar{\mathbb{D}}_- K_{a'} = 0 \ , \quad \mathbb{D}_- K_{\bar{a}'} = 0 \ . \tag{6.1}$$

These imply that on-shell, K_a is a semichiral superfield on equal footing with $\mathbb{X}^a$; we may change coordinates to a left-holomorphic or right-holomorphic basis with coordinates $Z^A = \{\mathbb{X}^a, Y_a := K_a\}$ or $Z^{A'} = \{\mathbb{X}^{a'}, Y_{a'} := K_{a'}\}$, respectively [11]. In the left basis, the $\delta^+, \bar{\delta}^+$ transformations become very simple, whereas in the right basis, the $\delta^-, \bar{\delta}^-$ transformations become simple. Since $K(\mathbb{X}^a, \mathbb{X}^{a'})$ is the generating function for the transformation between the bases, on-shell it is sufficient to study the transformations that are simple in one particular basis.

The ansatz for the $\delta^+, \bar{\delta}^+$ transformations is simple in the left basis:

$$\delta^+ Z^A = 0 \ , \quad \delta^+ \bar{Z}^{\bar{A}} = \epsilon^+ \mathbb{D}_+ \bar{\mathbf{f}}^{\bar{A}} \ , \quad \bar{\delta}^+ Z^A = \bar{\epsilon}^+ \bar{\mathbb{D}}_+ \mathbf{f}^A \ , \quad \bar{\delta}^+ \bar{Z}^{\bar{A}} = 0 \ . \tag{6.2}$$

Closure of this part of the algebra is very simple; it implies

$$\mathbf{f}^A{}_{\bar{B}} \bar{\mathbf{f}}^{\bar{B}}{}_C = -\delta^A_C \ , \tag{6.3}$$

and

$$\mathbf{f}^A{}_{C[\bar{B}} \mathbf{f}^C{}_{\bar{D}]} = 0 \ , \tag{6.4}$$

where $\mathbf{f}^A{}_{\bar{B}}$ again denotes derivation with respect to $\bar{Z}^{\bar{B}}$. These are precisely the conditions found in section 10 of [19], and imply that

$$J^{(1)}_{(+)} = \begin{pmatrix} 0 & \mathbf{f}^A{}_{\bar{B}} \\ \bar{\mathbf{f}}^{\bar{A}}{}_B & 0 \end{pmatrix} \ , \quad J^{(2)}_{(+)} = \begin{pmatrix} 0 & i\mathbf{f}^A{}_{\bar{B}} \\ -i\bar{\mathbf{f}}^{\bar{A}}{}_B & 0 \end{pmatrix} \ , \quad J^{(3)}_{(+)} = \begin{pmatrix} i\mathbb{I} & 0 \\ 0 & -i\mathbb{I} \end{pmatrix} \tag{6.5}$$

generate an integrable hypercomplex structure. Similarly, in the right basis, the δ^- transformations generate a second integrable hypercomplex structure so that in total we get a bi-hypercomplex structure,

$$J^{(A)}_{(\pm)} J^{(B)}_{(\pm)} = -\delta^{AB} + \epsilon^{ABC} J^{(C)}_{(\pm)} \ . \tag{6.6}$$

We still need to impose the $[\delta^+, \delta^-]$ part of the algebra and want to compare to the off-shell transformations (3.1). For both of these tasks, we need to go back to the $\mathbb{X}^a, \mathbb{X}^{a'}$ coordinate basis. For illustrative purposes, we focus on $\bar\delta^+$. Comparing (3.1) and (6.2), we immediately find that on-shell

$$f^a(\mathbb{X}^i) = \mathbf{f}^a(\mathbb{X}^a, \bar{\mathbb{X}}^{\bar a}, K_a(\mathbb{X}^i), K_{\bar a}(\mathbb{X}^i)) \ . \tag{6.7}$$

Off-shell, f^a may differ from $\mathbf{f}^a$ by a factor Δf^a, which satisfies $\bar{\mathbb{D}}_+(\Delta f^a(\mathbb{X}^a, K_a(\mathbb{X}^i))) = 0$ on-shell. This gives an off-shell ambiguity in f^a. We also have (trivially) that $\bar\delta^+\bar{\mathbb{X}}^{\bar a} = 0$. Next we have

$$\bar\delta^+ \bar Y_{\bar a} = K_{\bar a b}\bar\delta^+\mathbb{X}^b + K_{\bar a R}\bar\delta^+\mathbb{X}^R = 0 \tag{6.8}$$

and

$$\bar\delta^+ Y_a := K_{ab}\bar\delta^+\mathbb{X}^b + K_{aR}\bar\delta^+\mathbb{X}^R = \bar\epsilon^+\bar{\mathbb{D}}_+ f_a = \bar\epsilon^+(f_{a\bar b}\bar{\mathbb{D}}_+\bar{\mathbb{X}}^{\bar b} + f_{aR}\bar{\mathbb{D}}_+\mathbb{X}^R) \ , \tag{6.9}$$

where $f_a(\mathbb{X}^i) := \mathbf{f}_a(\mathbb{X}^a, \bar{\mathbb{X}}^{\bar a}, K_a(\mathbb{X}^i), K_{\bar a}(\mathbb{X}^i))$. We can rewrite these equations as

$$K_{LR}\bar\delta^+\mathbb{X}^R = \bar\epsilon^+ \begin{pmatrix} f_{a\bar b}\bar{\mathbb{D}}_+\bar{\mathbb{X}}^{\bar b} + f_{aR}\bar{\mathbb{D}}_+\mathbb{X}^R - K_{ab}\bar{\mathbb{D}}_+ f^b \\ -K_{\bar a b}\bar{\mathbb{D}}_+ f^b \end{pmatrix} \ , \tag{6.10}$$

where the matrix K_{LR} is defined as in (2.5). Since K_{LR} is invertible, we can find the on-shell transformations $\bar\delta^+\mathbb{X}^R$. To find the corresponding functions $\tilde g^{a'}_{b'}$ and $\bar{\tilde h}^{\bar a'}_{\bar b'}$ in (3.1), since we are on-shell, we need to eliminate one type of term, e.g., $\bar{\mathbb{D}}_+\bar{\mathbb{X}}^{\bar b}$, using the field equations.[4] Then (6.10) becomes

$$K_{LR}\bar\delta^+\mathbb{X}^R = \bar\epsilon^+ \begin{pmatrix} -(f_{a\bar c} - K_{ab}f^b_{\bar c})(K^{-1})^{\bar c d}K_{dR} + f_{aR} - K_{ab}f^b_R \\ K_{\bar a b}f^b_{\bar c}(K^{-1})^{\bar c d}K_{dR} - K_{\bar a b}f^b_R \end{pmatrix} \bar{\mathbb{D}}_+\mathbb{X}^R \ . \tag{6.11}$$

[4]We assume that $K_{a\bar b}$ is invertible, otherwise, we would need to eliminate another type of term, but the net effect would be the same.

and we find

$$
\begin{aligned}
\tilde{g}^{e'}_{f'} &= (K^{-1})^{e'a}[f_{af'} - K_{ab}f^b_{f'} - (f_{a\bar{c}} - K_{ab}f^b_{\bar{c}})(K^{-1})^{\bar{c}d}K_{df'}] \\
&\quad -(K^{-1})^{e'\bar{a}}[K_{\bar{a}b}f^b_{f'} - K_{\bar{a}b}f^b_{\bar{c}}(K^{-1})^{\bar{c}d}K_{df'}] \ , \\
\bar{\tilde{h}}^{\bar{e}'}_{\bar{f}'} &= (K^{-1})^{\bar{e}'a}[f_{a\bar{f}'} - K_{ab}f^b_{\bar{f}'} - (f_{a\bar{c}} - K_{ab}f^b_{\bar{c}})(K^{-1})^{\bar{c}d}K_{d\bar{f}'}] \\
&\quad -(K^{-1})^{\bar{e}'\bar{a}}[K_{\bar{a}b}f^b_{\bar{f}'} - K_{\bar{a}b}f^b_{\bar{c}}(K^{-1})^{\bar{c}d}K_{d\bar{f}'}] \ ,
\end{aligned} \tag{6.12}
$$

as well as the constraints

$$
\begin{aligned}
0 &= (K^{-1})^{e'a}[f_{a\bar{f}'} - K_{ab}f^b_{\bar{f}'} - (f_{a\bar{c}} - K_{ab}f^b_{\bar{c}})(K^{-1})^{\bar{c}d}K_{d\bar{f}'}] \\
&\quad -(K^{-1})^{e'\bar{a}}[K_{\bar{a}b}f^b_{\bar{f}'} - K_{\bar{a}b}f^b_{\bar{c}}(K^{-1})^{\bar{c}d}K_{d\bar{f}'}] \ , \\
0 &= (K^{-1})^{\bar{e}'a}[f_{af'} - K_{ab}f^b_{f'} - (f_{a\bar{c}} - K_{ab}f^b_{\bar{c}})(K^{-1})^{\bar{c}d}K_{df'}] \\
&\quad -(K^{-1})^{\bar{e}'\bar{a}}[K_{\bar{a}b}f^b_{f'} - K_{\bar{a}b}f^b_{\bar{c}}(K^{-1})^{\bar{c}d}K_{df'}] \ .
\end{aligned} \tag{6.13}
$$

In a similar way, we can find g^a_b, h^a_b as well as their complex conjugates. The full set of relations will now be discussed in the original coordinates $\mathbb{X}^i$.

6.2 Closure modulo field-equations and relations from invariance of the action

Though conceptually simple, the final expressions that we found (6.12)–(6.13) are rather involved and complicate the discussion on the on-shell $[\delta^+, \delta^-]$ algebra. Here we present an alternative description that uses only $\mathbb{X}^i$ coordinates and relates directly to the bi-hypercomplex geometry of [16]. We start from the ansatz (3.1) and only use the field equations to show that the conditions from closure of the algebra have more solutions on-shell. Whereas in the previous subsection discussing the on-shell algebra, it was convenient to change coordinates, here it turns out to be convenient to change the basis for the covariant derivatives.

Recall the field equations (6.1)

$$
\begin{aligned}
&K_{ai}\bar{\mathbb{D}}_+\mathbb{X}^i = 0 \ , \qquad K_{a'i}\bar{\mathbb{D}}_-\mathbb{X}^{\bar{i}} = 0, \\
&K_{\bar{a}i}\mathbb{D}_+\mathbb{X}^i = 0 \ , \qquad K_{\bar{a}'i}\mathbb{D}_-\mathbb{X}^i = 0 \ .
\end{aligned} \tag{6.14}
$$

These equations are first order in spinorial derivatives. To be able to use them to understand the conditions (3.12) (3.14), which contain second order spinorial deivatives, we must differentiate (6.14). We are then faced with the task of relating the plus/minus connections to second and third derivatives of the generalized Kähler potential K. Since the metric is a nonlinear function of the Hessian of K, this is not easy. Instead we choose to express the on-shell condition in terms of the complex structures $J_{(\pm)}$ defined in (2.6) and use $\nabla^{(\pm)}J_{(\pm)} = 0$ to relate them to the connections (assuming invariance of the action).

We introduce a real basis for the spinor derivatives:

$$\mathbb{D}_\pm := \frac{1}{2}(D_\pm - iQ_\pm) \ , \tag{6.15}$$

then (6.14) becomes[5]

$$\begin{aligned} Q_+\mathbb{X}^R &= J^R_{(+)k}D_+\mathbb{X}^k \\ Q_-\mathbb{X}^L &= J^L_{(-)k}D_-\mathbb{X}^k \end{aligned} \tag{6.16}$$

where we have introduced (components of) the complex structures $J_{(\pm)}$ as defined in section 2.

The semichiral conditions rewritten in terms of the real operators (6.15) and (2.7) read

$$\begin{aligned} Q_+\mathbb{X}^L &= jD_+\mathbb{X}^L \\ Q_-\mathbb{X}^R &= jD_-\mathbb{X}^R \ . \end{aligned} \tag{6.17}$$

Combining this with (6.16) and (2.6) we find that on-shell

$$Q_\pm\mathbb{X} := Q_\pm \begin{pmatrix} \mathbb{X}^L \\ \mathbb{X}^R \end{pmatrix} = J_{(\pm)}D_\pm \begin{pmatrix} \mathbb{X}^L \\ \mathbb{X}^R \end{pmatrix} = J_{(\pm)}D_\pm\mathbb{X} \ , \tag{6.18}$$

which using (6.15) implies

$$\begin{aligned} \mathbb{D}_\pm\mathbb{X}^i &= \bar{\pi}^{(\pm)i}_k D_\pm\mathbb{X}^k \\ \bar{\mathbb{D}}_\pm\mathbb{X}^i &= \pi^{(\pm)i}_k D_\pm\mathbb{X}^k \end{aligned} \tag{6.19}$$

[5]Note however that we use full $(2,2)$ superfield expressions in, e.g., (6.16); we can reduce to $(1,1)$ superspace by restricting to superfields to depend only on half the spinor coordinates.

where we have introduced the projection operator

$$\pi := \frac{1}{2}\left(\mathbb{1} + iJ\right) \ , \tag{6.20}$$

and its complex conjugate.

6.3 Relations to bi-hypercomplex geometry

In subsection 6.1 we constructed the bi-hypercomplex structures directly in terms of the transformations of the left and right holomorphic coordinates, and related bi-hypercomplex structures to the f-structures implicitly by constructing the tensors in the ansatz (3.1) in terms of the same transformations. In this subsection we analyze the relation using the real basis; this makes some aspects clearer while complicating others.

From the $N = (1,1)$ analysis of [16] we know that when the model has $(4,4)$ supersymmetry there exists an $SU(2)$ worth of left and right complex structures $(J^{(1)}_{(\pm)}, J^{(2)}_{(\pm)}, J^{(3)}_{(\pm)})$ on the $4d$ dimensional space, satisfying the bi-hypercomplex algebra (6.6). We now relate the f-structures to $J^{(A)}_{(\pm)}$.

The complex structures $J_{(\pm)}$ are part of the $SU(2)$ worth of complex structures, and we set $J^{(3)}_{(\pm)} := J_{(\pm)}$. In the real basis (6.15), the additional supersymmetries take the form

$$\delta_s \mathbb{X} := \delta^{\pm}\mathbb{X} + \bar{\delta}^{\pm}\mathbb{X} = \frac{1}{2}\left[\left(J^{(1)}_{(\pm)} + iJ^{(2)}_{(\pm)}\right)\epsilon^{\pm} D_{\pm}\mathbb{X} + \left(J^{(1)}_{(\pm)} - iJ^{(2)}_{(\pm)}\right)\bar{\epsilon}^{\pm} D_{\pm}\mathbb{X}\right] , \tag{6.21}$$

Identifying (6.21) with (3.3) we deduce that

$$\begin{aligned} \frac{1}{2}\left(J^{(1)}_{(\pm)} - iJ^{(2)}_{(\pm)}\right) &= U^{(\pm)}\pi^{(\pm)} \ , \\ \frac{1}{2}\left(J^{(1)}_{(\pm)} + iJ^{(2)}_{(\pm)}\right) &= V^{(\pm)}\bar{\pi}^{(\pm)} \ . \end{aligned} \tag{6.22}$$

This relation implies

$$\begin{aligned} (UV)^{(\pm)}\bar{\pi}^{(\pm)} &= -\bar{\pi}^{(\pm)} \\ (VU)^{(\pm)}\pi^{(\pm)} &= -\pi^{(\pm)} \ . \end{aligned} \tag{6.23}$$

A further consequence of the algebra (6.6) is, e.g., that

$$U^{(\pm)}\pi^{(\pm)} = \bar{\pi}^{(\pm)}U^{(\pm)}\pi^{(\pm)} \;, \quad V^{(\pm)}\bar{\pi}^{(\pm)} = \pi^{(\pm)}V^{(\pm)}\bar{\pi}^{(\pm)} \;. \tag{6.24}$$

On $TM \oplus TM$ we have that

$$\frac{1}{2}\begin{pmatrix} 0 & J^{(1)}_{(\pm)} - iJ^{(2)}_{(\pm)} \\ J^{(1)}_{(\pm)} + iJ^{(2)}_{(\pm)} & 0 \end{pmatrix} = \mathcal{F}_{(\pm)}\begin{pmatrix} \bar{\pi}^{(\pm)} & 0 \\ 0 & \pi^{(\pm)} \end{pmatrix} =: \mathcal{F}_{(\pm)}\Pi_{(\pm)} \;, \tag{6.25}$$

and the relations (6.24) can be used to show that both sides square to -$\Pi_{(\pm)}$.

Finally, assuming that the action is invariant we have $\nabla^{(\pm)}J^{(A)}_{(\pm)} = 0$, (see (2.4)) which implies that

$$\begin{aligned} \nabla^{(\pm)}U^{(\pm)}\pi^{(\pm)} &= 0 \\ \nabla^{(\pm)}V^{(\pm)}\bar{\pi}^{(\pm)} &= 0 \;, \end{aligned} \tag{6.26}$$

The equations (6.22)–(6.25) expresses the relation between the bi-hypercomplex geometry and the extra supersymmetries (3.1). The relation does not seem to be one-to-one since only, e.g., $U^{(+)}\pi^{(+)}$ enters. However, the particular form (3.4) of $U^{(\pm)}$ may be used in combination with the explicit expressions (2.6) of $J_{(\pm)}$ to show that all of $U^{(\pm)}$ is in fact determined by $J^{(A)}_{(\pm)}$. This is evident from the explicit expressions for the components of $U^{(+)}$ in section 6.1.

6.4 On-shell interpretation of the constraints.

On-shell, there are more cases when the algebra of the extra supersymmetries close, in analogy to, e.g., models written in terms of (anti)chiral fields. To illustrate the line of argument we first discuss (3.14).

Modulo the curl-part,

$$U^{(\pm)l}_{\;j}V^{(\pm)i}_{[k,l]} - V^{(\pm)l}_{\;k}U^{(\pm)j}_{[j.l]} \tag{6.27}$$

we may use (4.6) to rewrite (3.14) as

$$- \quad \left[(VU)^{(\pm)i}_{\;\;j,k} - (UV)^{(\pm)i}_{\;\;k,j}\right]\bar{\mathbb{D}}_{\pm}\mathbb{X}^j\mathbb{D}_{\pm}\mathbb{X}^k \tag{6.28}$$

$$+ \quad \left[(UV)^{(\pm)i}_{\;\;j} + \delta^i_j\right]\bar{\mathbb{D}}_{+}\mathbb{D}_{\pm}\mathbb{X}^j + \left[(VU)^{(\pm)i}_{\;\;j} + \delta^i_j\right]\mathbb{D}_{\pm}\bar{\mathbb{D}}_{+}\mathbb{X}^j = 0. \tag{6.29}$$

Since the LHS is

$$\bar{\mathbb{D}}_\pm \left[(UV)^{(\pm)i}_{\ j} \mathbb{D}_\pm \mathbb{X}^j \right] + \mathbb{D}_\pm \left[(VU)^{(\pm)i}_{\ j} \bar{\mathbb{D}}_\pm \mathbb{X}^j \right] + \{\bar{\mathbb{D}}_\pm, \mathbb{D}_\pm\}\mathbb{X}^i \tag{6.30}$$

and we know from (6.19) and (6.23) that on-shell the square brackets become $-\mathbb{D}_\pm \mathbb{X}^i$ and $-\bar{\mathbb{D}}_\pm \mathbb{X}^i$ respectively, we see that the LHS vanishes on-shell. It remains to consider the terms in (6.27).

Writing the term out in full, including the derivatives, we have

$$\begin{aligned} & (U^{(\pm)l}_{\ j} V^{(\pm)i}_{\ [k,l]} - V^{(\pm)l}_{\ k} U^{(\pm)i}_{\ [j,l]}) \bar{\mathbb{D}}_\pm \mathbb{X}^j \mathbb{D}_\pm \mathbb{X}^k \\ = \;& (U^{(\pm)l}_{\ j} V^{(\pm)i}_{\ [k,l]} - V^{(\pm)l}_{\ k} U^{(\pm)i}_{\ [j,l]}) \pi^{(\pm)j}_{\ p} \bar{\pi}^{(\pm)k}_{\ q} D_\pm \mathbb{X}^p D_\pm \mathbb{X}^q \end{aligned} \tag{6.31}$$

Using the relations (6.22) and (6.24) it is possible to show that one can replace all the U's and V's by, e.g., combinations of $\pi^{(\pm)}$'s and $J^{(1)}_{(\pm)}$ yielding the following expression for the curl-terms:

$$\left(J^{(1)i}_{\ k} \mathcal{N}(\bar{\pi})^k_{rq} J^{(1)r}_{\ j} \pi^j_p - J^{(1)i}_{\ k} \mathcal{N}(\pi)^k_{rp} J^{(1)r}_{\ j} \bar{\pi}^j_q + \mathcal{N}(J^{(1)})^i_{jk} \pi^j_p \bar{\pi}^k_q \right) D\mathbb{X}^p D\mathbb{X}^q \ , \tag{6.32}$$

where the $(\pm)$-indices were omitted for clarity. The integrability of the $J^{(A)}_{(\pm)}$'s means that all the Nijenhuis-tensors and thus all of terms in (6.32) vanish. We thus see that on-shell (3.14) implies no new constraints.

Next we consider (3.12). Off-shell we had to set the terms with independent structures separately to zero (4.2). On-shell we find no conditions on the tensors if we also assume invariance of the action.

The RHS of (3.12) is

$$[U^{(+)}, U^{(-)}]^i_j \bar{\nabla}^{(-)}_+ \bar{\mathbb{D}}_- \mathbb{X}^j \ , \tag{6.33}$$

where $\nabla^{(-)}_\pm$ is the pull-back of the minus-covariant derivative $\nabla^{(-)}_i$ in the $\mathbb{D}_\pm$ basis. We want to avoid the off-shell conclusion that the commutator vanishes and observe that the commutator multiplies something that looks like a field equation. However, we have to use (6.18) to see if it actually vanishes on-shell.

In the remainder of this section, we use the conditions that follow from invariance of the action [16], which imply that the metric is hermitean with respect to all the complex structures and the connections $\Gamma^{(\pm)}$ preserve the hypercomplex structures[6] $J_{(\pm)}$: $\nabla^{(\pm)} J_{(\pm)} = 0$. A straightforward calculation shows that[7]

$$\bar{\nabla}_+^{(-)} \bar{\mathbb{D}}_- \mathbb{X}^i = -\frac{1}{2} \left\{ \pi^{(-)}, \pi^{(+)} \right\}^i_{\ k} \nabla_+^{(-)} D_- \mathbb{X}^k \ . \tag{6.34}$$

To lowest order, the RHS is proportional to the $(1,1)$ field equation. Since it is written in manifest $(2,2)$ form, one may expect that it also vanishes to all orders. In fact, the $(2,2)$ relation

$$\{Q_+, Q_-\} \mathbb{X}^i = 0 \ , \tag{6.35}$$

has the on-shell content

$$[J_{(-)}, J_{(+)}]^i_j \nabla_+^{(-)} D_- \mathbb{X}^j = 0 \ , \tag{6.36}$$

where again covariant constancy of the complex structures is used. Since the commutator is invertible in a model with only semichiral fields,

$$\nabla_+^{(-)} D_- \mathbb{X}^j = 0 \ , \tag{6.37}$$

and that the RHS of (6.34) vanishes.

Using the connections with skew torsion $T = \pm\frac{1}{2} dB$ we have from the definition (3.9) that the LHS of (3.12) is

$$\begin{aligned} &\widehat{\mathcal{M}}(U^{(+)}, U^{(-)})^i_{jk} \bar{\mathbb{D}}_+ \mathbb{X}^j \bar{\mathbb{D}}_- \mathbb{X}^k = \\ &\left(U_j^{(+)l} \nabla_l^{(-)} U_k^{(-)i} - U_k^{(-)l} \nabla_l^{(+)} U_j^{(+)i} - U_l^{(+)i} \nabla_j^{(-)} U_k^{(-)l} + U_l^{(-)i} \nabla_k^{(+)} U_j^{(+)l} \right) \bar{\mathbb{D}}_+ \mathbb{X}^j \bar{\mathbb{D}}_- \mathbb{X}^k \ . \end{aligned} \tag{6.38}$$

Given the results for the RHS, the appropriate projections of $\widehat{\mathcal{M}}(U^{(+)}, U^{(-)})$ thus have to vanish. However, we know from (6.19) that on-shell

$$\bar{\mathbb{D}}_+ \mathbb{X}^j \bar{\mathbb{D}}_- \mathbb{X}^k = \pi_l^{(+)j} D_+ \mathbb{X}^l \pi_s^{(-)k} D_- \mathbb{X}^s \ , \tag{6.39}$$

and invoking invariance of the action, we may use (6.26) to conclude that then indeed $\widehat{\mathcal{M}}(U^{(+)}, U^{(-)}) = 0$.

[6] This is equivalent to restricting the holonomy of the connections $\Gamma^{(\pm)}$ to a symplectic group.

[7] Here the operator $\nabla_\pm^{(-)}$ is the pullback in the $D_\pm$-basis.

In summary our result is very similar to the hyperkähler discussion in [19], we need to invoke invariance of the action to show that there are more solutions on-shell to the conditions from the algebra[8].

The only constraints we get on the transformation matrices on-shell for invariant actions are the integrability condition

$$\mathcal{N}(U^{(\pm)})^i_{jk}\pi_l^{(\pm)j}\pi_m^{(\pm)k}D_\pm\mathbb{X}^l D_\pm\mathbb{X}^m = 0 \ . \tag{6.40}$$

together with the identification (6.22).

7 Discussion

Throughout this paper, the arbitrary entries in the transformation matrices $U^{(\pm)}$ (and $V^{(\pm)}$) were set to zero. Off-shell, this has the advantages of yielding geometric structures on the full target-space. Keeping the arbitrariness would restrict the features (e.g., integrability) of these structures to certain subspaces.

We have identified new geometric structures on the target-space of sigma models written in terms of semichiral fields. These structures arise when we study additional off-shell supersymmetries. We have discussed the f-structures as living on the sum of two copies of the tangent bundle $TM \oplus TM$. Clearly one would like to identify the relation to generalized complex geometry on $TM \oplus T^*M$. Formally, this may be achieved using the existence of a metric [14]

$$g = \Omega[J_{(+)}, J_{(-)}] \ , \tag{7.1}$$

where

$$\Omega := \begin{pmatrix} 0 & 2iK_{LR} \\ -2iK_{RL} & 0 \end{pmatrix} \ . \tag{7.2}$$

We use g to relate TM and T^*M to write $\mathcal{F}$ as an f-structure on $TM \oplus T^*M$:

$$\tilde{\mathcal{F}} := \begin{pmatrix} 0 & Ug^{-1} \\ gV & 0 \end{pmatrix} \ . \tag{7.3}$$

We plan to return to the geometry of f-structures in the context of generalized complex geometry in a later publication.

[8] In hyperkähler case the the algebra *only* closes on-shell.

A related question concerns the condition for invariance of the action. As we have shown for a subclass of our transformations, this amounts to the conservation of an antisymmetric tensor $\mathfrak{B}$ on certain subspaces of $TM \oplus TM$ by the f-structures. Again, the corresponding object on $TM \oplus T^{*}M$ can be found using the metric g:

$$\tilde{\mathfrak{B}} = \begin{pmatrix} 0 & Kg^{-1} \\ -gK^{t} & 0 \end{pmatrix}. \tag{7.4}$$

It remains to clarify where this object fits into the generalized complex picture. This also ties in with the question of how the conditions for invariance that we have described relate to those found in [44], where $(4,4)$ models with auxiliary fields are discussed.

In the precursor to this article [43] where the nonmanifest transformations were linear and the target space was four-dimensional, there was no interesting solution with additional supersymmetry. Additional twisted supersymmetry could be imposed, however. The target-space was then seen to carry indefinite signature metric and vanishing three form H, the geometry being pseudo-hyperkähler. In the present paper, where the target space is $4d$-dimensional, the transformations close to an ordinary supersymmetry algebra if $d > 1$, i.e. the dimension of the target space is larger than four. This stems from the fact that a complex number a can never fulfill $a\bar{a} = -1$, whereas for a matrix A with complex conjugated components $\bar{A}$, this could indeed be fulfilled. We could also have considered a twisted supersymmetry in the general case. The result would have been hyperbolic f-structures, a generalization of the result in [43].

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