Monetary Risk Measure with Their Applications

In Portfolio Management

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Quantifying the risk of the uncertainty in the future value of a portfolio is a key task in risk management. For decades many researchers have been trying to formalize sophisticated risk measures and apply them in the real financial world. In view of that this dissertation is dedicated to investigate the development of risk measures along with the applications in risk management, particularly in partial hedging under stochastic interest rate.

In this dissertation we assess the partial hedging problems by formulating hedging strategies that minimize conditional value-at-risk (CVaR) of the portfolio loss under stochastic interest rate environment. The combination of stochastic interest and CVaR hedging method makes the valuing approach more complex than the existing model with constant interest rate. We take up two issues in searching the optimal CVaR hedging strategy: given the initial capital constraint we minimize the CVaR of the portfolio loss; by prescribing a bound on the risk, we

also minimize the hedging cost. As an illustration of this hedging technique we derive hedging strategies for a European call option with the Black Scholes setting under HJM framework; explicit formulas are presented. We also investigate CVaR hedging problems by using the real financial data.

The last chapter in this dissertation investigate nominal and robust portfolio optimization by employing difference version of CVaR as the risk measure. We assume that the return only known to follow a distribution set. High frequency data is used to test the performance of CVaR optimization and Worst CVaR optimization with contrast to a equally weighted protfolio.

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List of Abbreviations

Var	Value-at-Risk
CVaR	Conditional Value-at-Risk
HJM	Heath-Jarrow-Morton

1. Introduction

Quantifying the risk of the uncertainty in the future value of a portfolio is a key task in risk management. For decades many researchers have been trying to formalize sophisticated risk measures and apply them in the real financial world. In view of that this dissertation is dedicated to investigate the development of risk measures along with the applications in risk management, particularly in partial hedging under stochastic interest rate.

The problem of pricing and hedging of a contingent claim is well understood in the context of a complete market. Given a sufficient allocation of initial wealth, every contingent claim can be replicated by a self-financing trading strategy, thereby being hedged perfectly. The cost of replication defines the price of the claim, and can be computed as the expectation of the claim under a unique equivalent martingale measure. The main characteristic of partial hedging is that it allows investors to allocate a smaller amount of initial capital than in the case of perfect hedging, while still managing the risk in a systematic way.

Different partial hedging approaches have been proposed and examined in the literature. The well-known examples are quadratic hedging and quantile hedging using Value-at-Risk. The performance of partial hedging mostly relies on the selection of the underlying risk measure. In this dissertation we introduce a partial hedging strategy by using Conditional value-at-risk (CVaR). We formulate hedging strategies that minimize CVaR of the portfolio loss under stochastic interest rate environment. The combination of stochastic interest and CVaR hedging method makes the valuing approach more complex than the existing model with constant interest rate. We take up two issues in this work: given the initial capital constraint we minimize the CVaR of the loss L; by prescribing a bound on the risk, we minimize the cost of the target portfolio. As an illustration of this hedging technique we derive hedging strategies for a European call option with the Black Scholes setting under HJM framework; explicit formulas are derived. We also investigate CVaR hedging problems by using the real financial data.

2. Monetary Measure of Risk

Risk measures are quantitative tools developed to determine the minimum capital reserves that need to be maintained by financial institutions so as to ensure their financial stability. In 1952 Markowitz first adopted variance as the measure of risk. Value-at-Risk (VaR) is a popular measure of downside risk in financial risk management since the middle of 1990s, and defined as the maximum amount of loss that can be observed for a given confidence level in the determined time interval. For a given probability distribution, VaR is basically a quantile estimation written as

$$VaR_{\alpha}(X) = F^{-1}(\alpha)$$

VaR is an important risk measure and required by the Basel Accord. However, it has been criticized in recent years in terms of three aspects: first, VaR is not sub-additive in the general distribution cases; also, VaR may exhibit multiple local extreme values for discrete distributions and hard to be optimized; moreover, VaR only represents a percentile of a probability distribution, and does not fully grasp the information of the uncertainty beyond.

The criticisms of VaR motivated researchers to formalize better measures of risk. ADEH took the first line of research by introducing the ideal of “Thinking Coherently” in 1997. In 1999 ADEH introduced “Coherent Risk Measures” where the consistency conditions were illustrated. This is considered a major contribution in risk measurement research. New risk measures that satisfy these consistency conditions and as easy to compute as VaR are constructed, for example, Conditional Value at Risk (CVaR) by Uryasev and Rockafeller in 1999 and Expected Shortfall (ES) by Acerbi et. Al. in 2000. Another significant contribution in this area was done by Follmer and Schied in 2002 with the introduction of “Convex Risk Measures”. Convex risk measure forms a theoretical framework going one step further in terms of reflecting real market conditions. This group of risk measures drop the positive homogeneity axiom of coherent risk measures in order to reflect the liquidy risk in financial markets. Later in 2004 Bion-Nadal defined conditional convex risk measures, which allows random variable to be values of risk measure, and evolves the convex risk measures by integrating the asymmetric information theory in risk measurement.

Dynamic risk measure is a sequence of conditional risk measures adapted to the underlying filtration. A crucial question in the dynamical framework is how risk evaluations at different time periods are interrelated. One of todays most used notion of time consistency is strong time consistency, which can be characterized by additivity of the acceptance sets and penalty funtions, and also by a supermartingale property of the risk process and the penalty function process, see [11,13].

An outline of this section is as follows: section 2 review the notion of a measure of risk, its proper axioms, and some representation results concerning them; section 3 we extend the measures to a conditional framework. Dynamic risk measures in both continuous time and discrete time settings are investigated; time consistency as an important property for dynamic risk measures is also presented. As an illustration of all theories discussed in this section, we introduce the dynamic entropic risk measure and its application with a pension fund setting.

2.1 Axioms and Acceptance Set

The uncertainty in the future value of a portfolio is usually described by a function $X:\Omega \to \mathbb{R}$, where Ω represents a fixed set of scenarios, and X can be interpreted as the discounted value of a portfolio which is the risk. We aim to determine $\rho(X)$, which quantifies the risk and can serve as the minimal amount of capital that makes the position acceptable. Let χ denotes a linear space of functions X, and $\chi \subset L^0(\Omega, F, P)$. It will be seen late that we often restrict ourselves to $L^p(\Omega, F, P)$, $p \in [1, +\infty]$.

Definition 2.1 A mapping ρ: $X \to \mathbb{R}$ is called a monetary risk measure if it satisfies the following conditions for all $X, Y \in \chi$.

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Cash invariance: If $m \in R$, then $\rho(X + m) = \rho(X) - m$

Monotonicity implies that if a portfolio has higher returns for all possible states of nature relative to another portfolio, risk associated to this portfolio is naturally lower. Translation invariance assures that a risk measure is expressed in currency terms by saying that when an amount of cash is added to the portfolio as a risk free investment, the risk of the portfolio decreases by the same amount.

Definition 2.2 A monetary measure of risk ρ: $X \to \mathbb{R}$ is called a coherent measure of risk if it satisfies the condition of

- Positive Homogeneity: If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$,
- Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

Positive homogeneity implies that there is a linear relation between the position size and the associated risk of the portfolio. Subadditivity states that a merger does not create extra risk. Convex risk measure below drops the positive homogeneity axiom in order to reflect the liquidity risk in financial market.

Definition 2.3 A monetary risk measure ρ: X $\rightarrow \mathbb{R}$ is called a convex measure of risk if it satisfies the property of

- Convexity: ρ (λ X+ (1−λ)Y) ≤ λ ρ(X) + (1−λ)ρ(Y), for 0 ≤ λ≤ 1.

It is immediately clear that coherent risk measure is indeed a special case of convex risk measure. That is, when the liquidity risk is taken as 0, every coherent risk measure is convex.

Proposition 2.4. For a risk measure $\rho : X \rightarrow \mathbb{R}$ we have

$$\rho(X + \rho(X)) = 0 \text{ for } X \in \chi, \text{ where } \left|\rho(X)\right| < +\infty. \quad (1)$$

If additionally positive homogeneous, then it holds that

(1) $\rho(0) = 0$, i.e., ρ is normalized,

(2) $\rho(c) = -c$ for all $c \in °$.

<u>*Proof*</u>. Appendix 1

We formally introduce the concept of acceptance with respect to its risk measure ρ

$$A_\rho : \ = \{\mathrm{X} \in \chi \mid \rho(\mathrm{X}) \leq 0\} \quad (2)$$

The set A_ρ is called the acceptance set of acceptable future net value, where ρ(X) is considered the smallest amount of money that would have to be added to X to make it acceptable. The following propositions summarize the relations between monetary measures of risk and their acceptance sets.

Proposition 2.5. Suppose that ρ is a monetary measure of risk with its acceptance set $\mathrm{A} := \mathrm{A}_\rho$.

(a) A is non-empty, closed with respect to the supremum norm $\|\cdot\|$, and satisfies

the following two conditions:

$$\inf\{m \in \mathbb{R} \mid m \in \mathrm{A}\} > -\infty.$$

$$\mathrm{X} \in \mathrm{A}, Y \in \chi, Y \geq \mathrm{X} \Rightarrow Y \in \mathrm{A}.$$

(b) ρ can be recovered from A:

$$\rho(X) = \inf\{m \in \mathbb{R} \mid m + \mathrm{X} \in \mathrm{A}\}, \tag{3}$$

(c) ρ is a convex risk measure if and only if A is convex,

(d) ρ is positively homogeneous if and only if A is a cone. In particular, ρ is coherent if and only if A is a convex cone.

Proof. Appendix 2

(3) takes a given set $\mathrm{A} \subset \chi$ of acceptable positions as the initial object, i.e., for $\mathrm{X} \in \chi$, we define the capital requirement as the minimal amount m for which $m + \mathrm{X}$ becomes acceptable.

2.2 Structure Theorems and Robust Representation

To find out whether a risk measure is coherent or not, we need to characterize the class of coherent risk measures. Here we investigate how a coherent risk measure can be constructed. The following theorem is usually a useful tool.

Theorem 2.6 Let p and q be such that $p \in [1,+\infty)$ and $1/p+1/q=1$. A mapping $\rho : L^p(\Omega, F, P) \to \mathbb{R}$ is a coherent risk measure satisfying the $L^p - Fatou$ property if and only if there exists a convex $L^q(P) - closed$ and $L^q - bounded$ set ς of probability measures that are absolutely continuous with respect to P such that

$$\rho(X) = \sup_{\S \in \varsigma} E_{\S}(-X), X \in L^p(\Omega, F, P). \tag{4}$$

<u>*Proof*:</u> (Inoue, 2003)

<u>Recall:</u> A risk measure $\rho : L^p(\Omega, F, P) \to °$, $p \in [1,+\infty)$, is said to satisfy the $L^p - Fatou$ property if for each bounded sequence $(X_n)_{n \in •} \subset L^p(\Omega, F, P)$ and $X \in L^p(\Omega, F, P)$ such that $X_n \xrightarrow[n\to\infty]{L^p} X$ **,** we have $\rho(X) \le \liminf_{n\to\infty} \rho(X_n)$. The remaining of this section will introduce a dual representation.

The structure theory was extended to the convex case by Foller and Schied and some others. Readers are referred to [33,34,35,37]

Let $\mathrm{M}_1(P) = \mathrm{M}_1(\Omega, F)$ denote the set of all probability measures Q on (Ω, F) which are absolutely continuous with respect to P. Moreover, and let $\mathrm{M}_{1,f}(P) =: \mathrm{M}_{1,f}(\Omega, F)$ represent the set of all finitely additive set functions Q (Q: $F \to [0,1]$) that are normalized to Q[Ω]=1, and continuous with respect to P so that Q[Z]=0 when P[Z]=0.

Let $\alpha: M_{1,f}(P) \to \mathbb{R} \cup \{+\infty\}$ be any functional such that

$$\min_{Q \in M_{1,f}(P)} \alpha(Q) \in \mathbb{R} \tag{5}$$

For each $Q \in M_{1,f}(P)$ with $\alpha(Q) < \infty$, the functional $X \mapsto E_Q[-X] - \alpha(Q)$ is convex, monotone, and cash invariant on L^{∞}, and these three properties are preserved when taking the supreme over $Q \in M_{1,f}(P)$. Thus,

$$p(X) \coloneqq \sup_{Q \in M_{1,f}(P)} \left(E_Q[-X] - \alpha(Q)\right) \tag{6}$$

defines a convex measure of risk on L^{∞} such that

$$p(0) = -\min_{Q \in M_{1,f}(P)} \alpha(Q) \tag{7}$$

The functional α is called a penalty function for risk measure ρ on $M_{1,f}$(P).

Every convex measure of risk on L^{∞} is of form (6). (8) gives a variational formula for the minimal penalty function α_{min}.

$$\rho\,(X) = \sup_{Q \in M_{1,f}(P)} (E_Q[-X] - \alpha_{min}(Q)) \tag{8}$$

The minimal penalty function $\alpha_{\min}(Q)$ is given by

$$\alpha_{min}(Q) = \sup_{X \in \chi}\left(E_Q[-X] - \rho(X)\right) \tag{9}$$

Theorem 2.7 Suppose $\rho : L^{\infty} \to \mathbb{R}$ is a convex measure of risk. Then the following conditions are equivalent.

a) ρ can be represented by some penalty function on $M_1(P)$,

b) ρ can be represented by the restriction of the minimal penalty function α_{min} to $M_1(P)$:

$$p(X) = \sup_{Q \in M_1(P)} (E_Q[-X] - \alpha_{min}(Q)), \in L^{\infty}\ , \tag{10}$$

c) ρ is continuous from above: If X_n] X P - a.s. then $\rho\left(X_n\right)$Z $\rho(X)$,

d) ρ has the "Fatou property": For any bounded sequence (X_n) which converges P - a.s. to some X, $\rho(X) \le \liminf_{n \uparrow \infty} \rho(X_n)$.

The robust representation form for coherent risk measures is a special case of the robust representation form for convex risk measures.

Corollary 2.8 The minimal penalty function α_{min} of a coherent measure of risk ρ takes only the values 0 and +∞. In particular,

$$\rho(X) = \max_{Q \in \varsigma_{\max}} E_Q[-X] \tag{11}$$

Where $\varsigma_{\max} \coloneqq \{Q \in M_{1,f}(P) | \alpha_{min}(Q) = 0\}$ is the largest set ς for which robust representation of (11) holds.

Proof.

Due to the positive homogeneity of ρ, its minimal penalty function satisfies

$$\alpha_{min}(Q) = \sup_{X \in L^\infty} \left(E_Q[-X] - \rho(X) \right) = \sup_{X \in L^\infty} \left(E_Q[-\lambda X] - \rho(\lambda X) \right) = \lambda \alpha_{min}(Q)$$

for all $Q \in M_{1,f}(P)$ and $\lambda > 0$. Hence, α_{min} can only take the values 0 and +∞.

2.3 Dynamic Risk Measure

Let $T \in \mathbb{N} \bigcup \{\infty\}$ be the time horizon; $T: =\{0,...,T\}$ for $T < \infty$, and �$: = \mathbb{N}_0$ for $T = \infty$. Denote t as a fixed stopping time given information available at time t, $t \in T$. Introduce a sub-σ-algebra F_t in addition to the probability space (Ω, F, P); F_t can be interpreted as knowledge about the underlying risk at time t such that $F_0 = \{\phi, \Omega\}$, $F = F_T$ for $T<\infty$, and $F = \sigma(\mathrm{U}_{t \geq 0} F_t)$ for $T=\infty$.

Define the spaces $A := \left\{ k \in L^0 \mid \|k\|_A < \infty \right\}$ and $B := \left\{ k \in A \mid \Delta k_t \geq 0 \forall t \in \mathbb{N} \right\}$, where $\Delta k_t := k_t - k_{t-1}$, $k_{-1} = 0$, and $\|k\|_{A^1} := E\left[\sum_{t \in \bullet} |\Delta k_t| \right]$. Define also the bilinear form on $L^\infty \times A$, i.e., $\langle X, k \rangle := E\left[\sum_{t \in \bullet} X_t \Delta k_t \right]$. By introducing a projection $\pi_t(X)_\tau := 1_{\{t \leq \tau\}} X_{t \wedge T}$, $t \leq \tau < T$, we can obtain the above in a conditional setting. In Particular, $L_t^\infty := \pi_t L^\infty$, $A_t := \pi_t A$, $B_t := \pi_t B$, and $\langle X, k \rangle_t := E\left[\sum_{t \in [t,T]} X_t \Delta k_t \mid F_t \right]$. Finally, we define the set of density processes as $¢_t := \left\{ k \in B_t \mid \langle 1, k \rangle_t = 1 \right\}$.

2.3.1 Risk Measures in Conditional Framework

A natural extension of a static risk measure is given by a conditional risk measure, which integrates the information available at the time of risk assessment. Intuitively, a conditional measure of risk should be a mapping from χ to the set of F_t-measurable random variables. For $t \in T, L_t^\infty := L^\infty(\Omega, F_t, P)$ is the space of all essentially bounded F_t measurable random variables. All equalities and inequalities between random variables and between sets are understood to hold P almost surely.

Consider risk measures defined on the set L^∞. In the dynamical setting, a conditional risk measure ρ_t assigns to each terminal payoff X an F_t- measurable random variable $\rho(X|F_t)$. Rigorous definitions of conditional risk measure are given below.

Definition 2.1 A mapping $\rho(\cdot|F_t): L^\infty(\Omega, F, P) \to L_t^\infty$ is a conditional risk measure if it satisfies the following three axioms.

- Monotonicity: if $\rho(X|F_t) \geq \rho(Y|F_t)$ for all X, Y$\in L^\infty(\Omega, F, P)$ with X ≤ Y,
- Conditional Translation Invariance: $\rho(X + H|F_t) \geq \rho(X|F_t) - H$ for X$\in L^\infty(\Omega, F, P)$ and H$\in L^\infty(\Omega, F_t, P)$.

Definition 2.2 A conditional risk measure $\rho(\cdot|F_t): L^\infty(\Omega, F, P) \to L_t^\infty$ is coherent if it satisfies the following properties:

- Subadditivity: $\rho(X + Y|F_t) \leq \rho(X|F_t) + \rho(Y|F_t)$ for X,Y$\in L^\infty(\Omega, F, P)$,
- Conditional positive homogeneity: $\rho(\Lambda X|F_t) = \Lambda\rho(X|F_t)$ if $\Lambda \in L^\infty(\Omega, F, P)$ and X $\in L^\infty(\Omega, F, P)$.

Definition 2.3 A conditional risk measure $\rho(\cdot|F_t): L^\infty(\Omega, F, P) \to L_t^\infty$ is convex if for all X, Y$\in L^\infty(\Omega, F, P)$ such that $0 \le \Lambda \le 1$ it holds that

- Convexity: $\rho(\Lambda X + (1-\Lambda)Y|F_t) \le \Lambda\rho(X|F_t) - (1-\Lambda)\rho(Y|F_t)$.

The static risk measure we investigated in the previous section is actually a trivial case of the conditional one. Assume that there is no additional information about risk available at the time of risk assessment at time t, that is, $F_t\{\varnothing, \Omega\}$, then for every X$\in\chi$, $\rho(X)$ is F_t -measurable as a constant, and thus $\rho = \rho(\cdot|F_t)$.

In a conditional setting, we can also define an acceptance set A of all bounded adapted stochastic processes that are \`acceptable'. In order to be \`acceptable', X should feature a non-positive conditional risk measure. Formally,

$$A_t = \{X \in L^\infty | \rho(X|F_t) \le 0\} \tag{12}$$

By construction, ρ_t can be uniquely determined as

$$\rho_t^A(X) = \inf\{Y \in L_t^\infty | X + Y \in A_t\} \tag{13}$$

The minimal penalty function $\alpha_{min}(Q)$ for each Q$\in$M$_1(P)$ can be written as,

$$\alpha_t^{min}(Q) = \text{Q-ess}\sup_{X\in A_t} E[-X|F_t]$$

(14)

A conditional convex risk measure ρ_t takes the robust representation in the form of

$$\rho_t(X) = ess\sup_{Q\in\varsigma}(E_Q[-X|F_t] - \alpha_t^{min}(Q)) \tag{15}$$

For simplicity, we will denote $\rho(X|F_t)$ as $\rho_t(X)$ in the following dynamical work.

2.3.2 Continuous Time Dynamic Risk Measure for Final Payments

Dynamic risk measure is a sequence of conditional risk measures adapted to the underlying filtration. Therefore, the positions we are going to rate now are stochastic processes $(X_t)_{t\in T}$ rather than random variables. We first restrict ourselves to ones with final payments only, that is, $X_t = 0$ for $t \neq T$. Under this circumstances $(X_t)_{t\in T}$ can actually be identified with the random variable X_T, such that nothing changed from a practical point of view and the set of positons χ is a subset of $L^0(\Omega, F, P)$. Define $T_- = [0, T-1)$.

Definition 2.4 A mapping $\rho : \Omega \times T_- \times \chi \ni (\omega, t, \mathrm{X}) \mapsto \rho(\omega, t, \mathrm{X}) = \rho_t(\mathrm{X})(\omega) \in \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) is called a dynamic risk measure for final payments if the following conditions are met:

- the process $(\rho_t(\mathrm{X}))_{t\in T_-}$ is $(F_t)_{t\in T_-}$-adapted,
- ρ is monotone: $\rho_t(X) < \rho_t(Y), t\in T_-$, if $X, Y \in \chi$ and $X > Y$,
- ρ is dynamic translation invariant: it holds that $\rho_t(X+Y) = \rho_t(X) - Y$ for $t \in T_-, \mathrm{X}, Y \in \chi$ such that Y is F_t-measurable.

Definition 2.5 A dynamic risk measure for final payments $\rho : \Omega \times T_- \times \chi \to \overline{\mathbb{R}}$ is coherent if it satisfies the following properties:

- Subadditivity: $\rho_t(X+Y) \leq \rho_t(X) + \rho_t(Y)$ if $t \in T_-$, and $\mathrm{X}, Y \in \chi$,
- Dynamic positive homogeneity: $\rho_t(\Lambda \mathrm{X}) = \Lambda \rho_t(\mathrm{X})$ for $t \in T_-, \mathrm{X} \in \chi$ and F_t-measurable $\Lambda \in \chi$ with $\Lambda \geq 0$.

Definition 2.6 A dynamic risk measure for final payments $\rho : \Omega \times T_- \times \chi \to \overline{\mathbb{R}}$ is called convex if for $t \in T_-$ and F_t-measurable random variable $\Lambda \in \chi$ such that $0 \leq \Lambda \leq 1$ we have

$$\rho_t(\Lambda X+(1-\Lambda)Y)\le \Lambda\rho_t(X)+(1-\Lambda)\rho_t(Y),\ X,Y\in\chi. \tag{16}$$

Suppose that $\rho:\Omega\times T_-\times\chi\to\bar{\circ}$ is a dynamic risk measure. Fix $t\in T_-$, we can then write the acceptance set as

$$\mathrm{A}_t^\rho=\{\mathrm{X}\in\chi\mid\rho_t(\mathrm{X})\le 0\} \tag{17}$$

The following two theorems define the penalty functions for dynamic convex and coherent risk measures.

Theorem 2.7 Let p and q be such that $p\in[1,+\infty)$ and $1/p+1/q=1$. If $\rho:\Omega\times T_-\times L^p(\Omega,F,\circ)\to\bar{\circ}$ is a convex dynamic risk measure for final payments, then there exists a mapping $\alpha:\Omega\times T_-\times\varsigma_q\to(-\infty,+\infty]$ such that

$$\rho_t(X)=\operatorname*{ess\,sup}_{\S\in\varsigma_q}(E_Q(-\mathrm{X}\mid F_t)-\alpha_t(Q)),\mathrm{X}\in L^p(\Omega,F,P). \tag{18}$$

where $\varsigma_q=\{Q= \ P\mid dQ/dP\in L^p(\Omega,F,P)\}$. More precisely, α is given by

$$\alpha_t(Q)=\operatorname*{ess\,sup}_{\mathrm{X}\in A_t^p}E_Q(-X\mid F_t). \tag{19}$$

Theorem 2.8 Let p and q be such that $p\in[1,+\infty)$ and $1/p+1/q=1$. If $\rho:\Omega\times T_-\times L^p(\Omega,F,\circ)\to\bar{\circ}$ is a coherent dynamic risk measure for final payments, then for every $t\in T_-$, there exists a convex $L^q(P)$-closed set ς_q^t such that

$$\rho_t(X)=\operatorname*{ess\,sup}_{Q\in\varsigma_q^t}E_Q(-\mathrm{X}\mid F_t),\mathrm{X}\in L^p(\Omega,F,P). \tag{20}$$

More precisely, the set ς_q^t is the Legendre-Fenchel transform of ρ_t given by

$$\alpha_t(Q)=\operatorname*{ess\,sup}_{\mathrm{X}\in A_t^p}E_Q(-X\mid F_t). \tag{21}$$

2.3.3 Discrete Time Dynamic Risk Measure for processes

Have knowledge about the dynamic risk measure for final payments, we are are ready to consider that for general random cash balance processes. We mainly focus on a dynamic and consistent approach of risk measurement for random processes. The definitions are actually a generalization of that in subsection 3.2. Here we set $T_{-} = \{0,1,...,T-1\}$.

Definition 2.9 A mapping $\rho:\Omega\times T_{-}\times\chi\ni(\omega,t,\mathrm{X})$ a $\rho(\omega,t,\mathrm{X}) = \rho_t(\mathrm{X})(\omega)\in\overline{\circ}$ is called a dynamic risk measure if the following conditions are met:

- the process $(\rho_t(\mathrm{X}))_{t\in T_{-}}$ is $(F_t)_{t\in T_{-}}$-adapted,
- ρ is independent of the past: for $t\in T_{-}$ and $\mathrm{X}\in\chi, \rho_t(\mathrm{X})$ does not depend on $X_0, X_1, ..., X_{t-1}$,
- ρ is monotone: $\rho_t(X)\le\rho_t(Y), t\in T_{-}$, if $X, Y\in\chi$ and $X\ge Y$,
- ρ is dynamic translation invariant: it holds that

$$\rho_t(X+Y) = \rho_t(X) - \sum_{n=t}^{T}\frac{Y_n}{(1+r)^{n-t}} \tag{22}$$

for $t\in T_{-}, \mathrm{X}, Y\in\chi$ such that $Y=(0,...,0,Y_t,...,Y_T)$ and $\sum_{n=t}^{T} Y_n/(1+r)^{n-t}$ is F_t-measurable.

Definition 2.10 A dynamic risk measure $\rho:\Omega\times T_{-}\times\chi\to\overline{\circ}$ is called coherent if it satisfies the following properties:

- Subadditivity: $\rho_t(X+Y)\le\rho_t(X)+\rho_t(Y), X, Y\in\chi, t\in T_{-}$,
- Dynamic positive homogeneity: $\rho_t(\Lambda\mathrm{X}) = \Lambda\rho_t(\mathrm{X})$ for $\mathrm{X}\in\chi$ and F_t-measurable $\Lambda\in\chi$ with $\Lambda\ge 0$.

Definition 2.11 A dynamic risk measure $\rho : \Omega \times T_{-} \times \chi \to \overline{\mathbb{R}}$ is convex if

$$\rho_t(\Lambda X + (1-\Lambda)Y) \le \Lambda \rho_t(X) + (1-\Lambda)\rho_t(Y), \tag{23}$$

for $X, Y \in \chi, t \in T_{-}$ and $\Lambda \in L^{\infty}(\Omega, F, P)$ with $0 \le \Lambda \le 1$.

The only property that needs a comment is the independence of the past. For any fixed $t \in T_{-}$ and a position X, $\rho_t(X)$ is evaluated at time t, so all payments $X_0, X_1, ..., X_{t-1}$, which have already passed, cannot influence the value of $\rho_t(X)$.

2.3.4 Time Consistent Property

A crucial question in the dynamical framework is how risk evaluations at different times are interrelated. Proposition 3.12 summarize the conditional a dynamic convext risk measure should meet.

Proposition 2.12 A dynamic convex risk measure $(\rho_t)_{t\in T}$ is time consistent if and only if any of the following conditions holds:

1) For all $t\in T$ such that $t < T$, and for all $X, Y \in L^\infty$:

$$\rho_{t+1}(X) \le \rho_{t+1}(Y) \text{ P - a.s.} \Rightarrow \rho_t(X) \le \rho_t(Y) \text{ P - a.s.} \tag{24}$$

2) For all $t\in T$ such that $t < T$ and for all $X, Y \in L^\infty$:

$$\rho_{t+1}(X) = \rho_{t+1}(Y) \text{P - a.s.} \Rightarrow \rho_t(X) = \rho_t(Y) \text{ P - a.s.} \tag{25}$$

3) $(\rho_t)_{t\in T}$ is recursive, i.e., for all $t, s \ge 0$ such that $t,\ t+s\in T$,

$$\rho_t = \rho_t(-\rho_{t+s}) \text{ P - a.s.} \tag{26}$$

1) explains that if a position at some future time (ex. t+1) is preferable to the other one, it is also preferable at time t. This directly implies 2). By translation invariance we can verify the one step excursiveness, and the reclusiveness property follows the induction hypothesis on s. There are many ways to characterize time consistency. In this work we follow strong time consistency characterized by the additivity of the acceptance sets and penalty functions.

If we restrict a conditional convex risk measure p_t to the space L^∞_{t+s} for some s ≥0, the corresponding acceptance set is given by

$$A_{t,t+s} := \{X \in L^\infty_{t+s} \,|\, \rho_t(X) \le 0 \text{ P - a.s. }\} \tag{27}$$

and the minimal penalty function by

$$\alpha^{min}_{t,t+s}(Q) := Q\text{-} \operatorname*{ess\,sup}_{X\in A_{t,t+s}} E_Q[-X \mid F_t], \quad Q\in M_1(P) \tag{28}$$

Theorem 2.13 Let p and q be such that $p \in [1,+\infty)$ and $1/p+1/q=1$. Then $\rho : \Omega \times T_{-} \times \chi^{p} \to \overline{\circ}$ is a time-consistent convex dynamic risk measure that has the L^{p}-Fatou property if and only if it is of the form

$$\rho_t(X) = \underset{Q \in \varsigma_q}{ess\sup}(E_Q(-\sum_{n=t}^{T} \frac{X_n}{(1+r)^{n-t}} \mid F_t) - \alpha_t(Q)), \tag{29}$$

where

$$\alpha_t(Q) = \underset{X \in \chi^p}{ess\sup} E_Q(-\sum_{n=t}^{T} \frac{X_n}{(1+r)^{n-t}} - \rho_t(X) \mid F_t), \tag{30}$$

$$\varsigma_q = \{Q = \ P \mid \frac{dQ}{dP} \in L^p(\Omega, F, P)\} \tag{31}$$

2.4 Illustration: Entropic Risk Measures

This section is designed to illustrate the theories presented in Section 3. The pension fund exercise demonstrates the performance of dynamic risk measure in continuous time setting for final payments and in discrete time for random processes.

2.4.1 Dynamic Entropic Risk Measures

The entropic risk measure is one of the most appealing convex risk measures due to its explicit characterization. It is a convex risk measure related to the exponential utility function $u(x) = 1 - e^{-\gamma x}$ and defined on L^{∞}. Such a measure admits a representation under P. That is, given two probability measures Q and P such that Q<<P, The relative entropy (or Kullback-Leibler divergence) of a probability measure Q with respect to a reference probability measure P is given by:

$$H(Q|P) = E_Q\left[\log\frac{dQ}{dP}\right] = E\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right], \tag{32}$$

where $\frac{dQ}{dP}$ is a Radon-Nikodym derivative. If the absolute continuity condition is not satisfied, $H(Q|P)$ is then defined to be ∞.

The relative entropy demonstrates the distance from a probability measure Q to a reference probability measure P. The non-negative function $H(Q|P)$ could be a well-suited candidate for a penalty function as it is minimal for the measure that is closest to P. We take $\frac{1}{\gamma}H(Q|P) = \alpha(Q)$. When risk preferences are characterized by an exponential utility function, γ is a coefficient to account for risk aversion; otherwise, γ measures the degree of distrust the agent puts in P.

The static entropic risk measure is written as:

$$\rho^{e}(x) = \sup_{Q\in M}\left\{E_Q[-X] - \frac{1}{\gamma}H[Q|P]\right\} \tag{33}$$

In this setting, ρ^e is a collection of mappings from $L^{\infty}(F_T) \to \mathbb{R}$. For any probability measure Q,

$$\alpha(Q) = \frac{1}{\gamma}H(Q|P) = \sup_{x\in L^{\infty}}\left(E_Q[-X] - \frac{1}{\gamma}\log E\left[e^{-\gamma X}\right]\right) \tag{34}$$

The supermum is attained by letting $X = -\frac{1}{\gamma} log \frac{dQ}{dP}$ if $Q \ll P$, and the corresponding risk measure thus takes the form of:

$$\rho^{e}(X) = \frac{1}{\gamma} \log E_Q\left[e^{-\gamma X}\right] \tag{35}$$

Given time horizon $T = \{0,1,\ldots,T\}$, the conditional entropic risk measure is a family of mappings from $L^\infty(F_T) \to L^\infty(F_t)$. Naturally, we define

$$\rho_t^e(X) = \frac{1}{\gamma} \log E[\mathrm{e}^{-\gamma^X} \mid F_t] \tag{36}$$

The dynamic entropic risk measure is a family of successive conditional entropic risk measures, which is strongly time-consistent.

<u>*Proof.*</u>

$$\rho_t^e\left(-\rho_{t+1}^e(X)\right) = \frac{1}{\gamma} \log E\left[e^{\gamma \frac{1}{\gamma} log E\left[e^{-\gamma^X} \middle| F_{t+1}\right]} \middle| F_t\right]$$

$$= \frac{1}{\gamma} \log E\left[E[e^{-\gamma^X} \middle| F_{t+1}\right] | F_t]$$

By the law of iterated expectations, this reduces to

$$\frac{1}{\gamma} \log E\left[E[e^{-\gamma^X} \middle| F_{t+1}\right] | F_t] = \frac{1}{\gamma} \log E\left[e^{-\gamma^X} \middle| F_t\right] = \rho_t^e(X)$$

2.4.2 Dynamic Entropic Risk Measure for Random Processes

Define the entropy of a density process $k \in ¢_{t,\theta}$. We aim to investigate the conditional entropic risk measure for bounded discrete-time random processes, i.e. $p_t^e : L^\infty \to L^\infty(F_t)$

Definition 2.1 The entropy $H_t(a) of\ a\ \in Q_t\ is\ given\ by$

$$H_t(a) = E[\sum_{s=t}^{T} \Delta a_s \log((T+1-t)\Delta a_s)] \quad , \tag{37}$$

for all t=S,…,T.

For any density process $a \in Q_t$,

$$\frac{1}{\gamma} H_t(a) = \sup_{X \in L_t^\infty} \{\langle -X, a\rangle - \frac{1}{\gamma}\log(\frac{1}{T+1-t}\sum_{s=t}^{T} E[e^{-\gamma^X}|F_t])\} \tag{38}$$

The supremum is attained by letting $X_s = -\frac{1}{\gamma}\log((T+1-t)\Delta a_s)$. The corresponding risk measure takes thus the form of:

$$\rho_t^e(X) = \frac{1}{\gamma}\log(\frac{1}{T+1-t}\sum_{s=t}^{T} E[e^{-\gamma^X}|F_t]) \tag{39}$$

The dynamic entropy risk measure for random processes is also strongly time-consistent.

<u>*Proof.*</u> $X - p_{t+1}^e(X)$

$$\rho_t^e(X - \rho_{t+1}^e(X)) = \frac{1}{\gamma}\log\left(\frac{1}{T+1-t}E\left[e^{-\gamma^X}|F_t\right] + \frac{T-t}{T+1-t}E\left[e^{\gamma^{\rho_{t+1,}^e(X)}}|F_t\right]\right)$$

Substitution yields

$$\frac{1}{\gamma}\log\left(\frac{1}{T+1-t}E\left[e^{-\gamma X_t}|F_t\right] + \frac{1}{T+1-t}E\left[e^{\log(\frac{1}{T-t}\sum_{s=t+1}^{T}E[e^{-\gamma X_s}|F_{t+1}])}|F_t\right]\right)$$

This finally reduces to

$$\frac{1}{\gamma}\log(\frac{1}{T+1-t}E[e^{-\gamma X_t}|F_t] + \frac{1}{T+1-t}\sum_{s=t+1}^{T} E[e^{-\gamma X_s}|F_t]) =$$

$$\frac{1}{\gamma}\log\left(\frac{1}{T+1-t}\sum_{s=t}^{T} E[e^{-\gamma X_s}|F_t]\right) = \rho_t^e(X)$$

2.4.3 Entropic Risk Measure with Brownian Motion

In this work we consider the discrete time dynamic entropic risk measure adapted to the filtration $(F_t)_{t=s}^{t=T}$ that is generated by a random walk process; and the continuous-time entropic risk measure adapted to the augmented continuous-time filtration $(F_t)_{t\in[s,T]}$ that is generated by Brownian motion W($\sigma(W_s, s \le t)$ and the P-null sets of F). It is well-known that under the assumption of Brownian filtration, the dynamic entropic risk measure arises as a solution to backward stochastic differential equations (BSDE). Recall the definition of a BSDE

Definitions 2.2 Let $W = (W_t)_{t\in[S,T]}$ be a d-dimensional Brownian motion defined on $(\Omega, F, (F_t)_{t\in[S,T]}, P)$. Further, let $X \in L^2(F_T)$ be a terminal condition and g be a $P \otimes B(\mathbb{R}) \otimes B(\mathbb{R}^d)$-measurable coefficient, where P denotes the predictable σ-algebra and $B(\mathbb{R})$ and $B(\mathbb{R}^d)$ are the Borel σ-algebras on $\mathbb{R}$ and $\mathbb{R}^d$, respectively. An adapted solution for the BSDE associated with (g, X) is a pair of progressively measurable processes $(Y_t, Z_t)_{t\le T}$ with values in $\mathbb{R}\times\mathbb{R}^d$ such that $E[\sup_{t\le T}\left|Y_t^2\right|] < \infty$, and $E[\int_s^T \left|Z_s\right|^2 ds] < \infty$, $s \le t \le T$,

$$Y_t = X + \int_t^T g\left(s, Y_s, Z_s\right) ds - \int_\tau^T Z_s dW_s \tag{40}$$

The uniqueness and existence of a solution is formulated in terms of conditions on the coefficient or driver g. The dynamic entropic risk measure under Brownian filtration assumption arises as a solution to BSDE is summarized below.

Theorem 2.3 The dynamic entropic risk measure $(p_t^e(X))_{t\in[S,T]}$ is a solution of the following BSDE with quadratic coefficient $g(t, Z_t) = \frac{\gamma}{2}\|Z_t\|^2$ and terminal bounded condition $X\epsilon L^\infty(F_T)$. With $\gamma > 0$ and $(W_t)_{t\in[S,T]}$ a d-dimensional Brownian motion,

$$-d\rho_t^e(X) = \frac{\gamma}{2} Z_t^2 dt - Z_t dW_t, \quad \rho_T^e(X) = -X \tag{41}$$

The g-coefficient associated with the dynamic entropic risk measure is of the quadratic growth type, i.e.:$g(t, Z_t) = \frac{\gamma}{2} \|Z_t\|^2 \leq \widehat{K}(1 + \|Z_t\|^2) d\mathrm{P}\times dt\ a.s.$,for some constant K. Under the additional condition $\left|\frac{\partial g(t,Z_t)}{\partial Z_t}\right| \leq \widehat{K}(1 + \|Z_t\|) d\mathrm{P}\times dt\ a.s.$ for some constant $\widehat{K}$, Kobylanski (2000) proved that there exists a unique solution of (41) such that Y is bounded. Table 1 summarize the numerical scheme to solve the case d=1.

Table 1 Numerical scheme for the case d=1 (Peng, 2009)

Suppose that a discretization $(X_1^n, \dots, X_n^n)$ of X_t is given.

1. Let $n \in \mathbb{N}$. Simulate a sequence $\{\in_i^n\}_{i=1,\dots,n}$ of Bernoulli variables such that

$$\mathbb{P}[\in_i^n = 1] = \mathbb{P}[\in_i^n = -1] = \frac{1}{2}$$

2. Compute $W_k^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \epsilon_i^n$ and $\Delta W_k^n = W_{k+1}^n - W_k^n$.

3. ξ^n is F_n^n -measurable and there exists some function ϕ^n such that $\xi^n = \phi^n(\in_1^n, \in_2^n, \dots, \in_n^n)$.

Put $Y_n^n = \xi^n$ and $Z_n^n = 0$.

4. Consider now the following equation:

$$Y_{n-1}^n = Y_n^n + g_{n-1}^n(n-1, X_{n-1}^n, Y_{n-1}^n, Z_{n-1}^n)\frac{1}{n} - Z_{n-1}^n \Delta W_n^n.$$

a) We describe a procedure to solve for Z_{n-1}^n. First, define Y_n^+ and Y_n^- as follows:

$$Y_n^+ = \phi^n(\in_1^n, \dots, \in_{n-1}^n, 1) \text{ and } Y_n^- = \phi^n(\in_1^n, \dots, \in_{n-1}^n, -1).$$

b) Z_{n-1}^n is now the unique solution of the following set of equations:

$$Y_{n-1}^n = Y_n^+ + g_{n-1}^n(n-1, X_{n-1}^n, Y_{n-1}^n, Z_{n-1}^n)\frac{1}{n} - Z_{n-1}^n \frac{1}{\sqrt{n}}$$

$$Y_{n-1}^{n} = Y_n^- + g_{n-1}^{n}(n-1, X_{n-1}^{n}, Y_{n-1}^{n}, Z_{n-1}^{n})\frac{1}{n} + Z_{n-1}^{n}\frac{1}{\sqrt{n}}$$

c) This yields $Z_{n-1}^{n} = \sqrt{n}\ \frac{Y_n^+ - Y_n^-}{2}$. (I don't know how to make that middle dot)

5. Solve for Y_{n-1}^{n}

6. Consider now the following equation for $k = n-2, \dots, 1$

$$Y_k^n = Y_{k+1}^n + g_k^n(k, X_k^n, Y_k^n, Z_k^n)\frac{1}{n} - Z_k^n \Delta W_k^n$$

a) Solve for Z_k^n by defining (same procedure as under 4):

$Y_{k+1}^+ = \phi^n(\in_1^n, \dots, \in_k^n, 1)$ and $Y_{k+1}^- = \phi^n(\in_1^n, \dots, \in_k^n, -1)$

b) Solve for Y_k^n

Remark 7.1

1. $W^n \to W$ as $n \to \infty$

2. $g^n(k, X^n, Y^n, Z^n) \to g(t, X, Y, Z)$ as $n \to \infty$

3. Convergence result (provided that Y is bounded):

$(\mathrm{Y}^n, \int_t^T Z_s^n dW_s^n) \to (Y, \int_t^T Z_s dW_s)$ as $n \to \infty$

In what follows, we setup a numerical example to illustrate the ideas of the entropic risk measure we just discussed.

2.5 Numerical Example: A Pension Fund

Consider a pension fund setting. Let F_{t+1}^N denote the nominal funding level, which reflects the relative value of the pension fund assets and nominal liabilities in one year; and S_{t+1}^N be the pension fund surplus ratio, $S_{t+1}^N = F_{t+1}^N - 1$. We have the following restriction,

$$p(F_{t+1}^N < 1) \leq 2.5\% \Rightarrow P(S_{t+1}^N < 0) \leq 2.5\%$$

which states that the probability of underfunding in one year from now is less than 2.5%. While the nominal funding level is commonly employed as a supervisory monitoring ratio, we formulate our supervision rule in terms of the surplus ratio S_{t+1}^N. The reason is that in such a way we do not need to worry about the impact of future indexation on the risk assessment, because the liabilities are modelled as if they are fully indexed to price inflation. The indexation buffer is the difference between the nominal funding level and the real funding level. It seems reasonable to define the surplus ration in terms of real liabilities since the participants' main concern is the purchasing power of the pension plan. The risk measure is then the smallest amount of money that need to be put aside to make next period's surplus ration acceptable.

For simplicity, we assume that the model of the surplus ration only has one source of time-independent uncertain, for instance, stock market movements; any other sources of uncertainty are fully hedged, absent or reinsure. Suppose that the uncertainty is fully driven by a one-dimensional Brownian motion. That is,

$$dF_t = \sigma F_t dW_t, \quad F_s = h$$

Where

F_t : real funding level at time t

σ : standard deviation of the real funding level

W_t : one-dimensional standard Brownian motion at time t.

The constant σ can be considered as an abstract number that captures uncertainty in a very simple way, which summarizes the pension fund exposure to the Brownian motion. Its value very much depends on the pension fund characteristics. We somehow set σ=10%.

According to Ito's Lemma, the dynamics of dS_t are given by dF_t, so the surplus ration process $(S_t)_{t\in[S,T]}$ we are interested in is given by $(F_t)_{t\in[S,T]} - 1_{[S,T]}$.

Given the model of the surplus ratio, we try to measure its risk dynamically. The risk is considered as a valuation of the final surplus ratio and the surplus ratio process. We consider a simulation run length of one year and 5000 trajectories of Brownian motion.

2.5.1 Risk measurement for the Final Surplus Ratio

The final ratio S_T is described by a simple diffusion process. The dynamic entropic risk measure is given by the solution of a one-dimensional BSDE. In order to employ (41) we need the dynamics of

$$de^{-\gamma S_t} = -\gamma e^{-\gamma S_t} dS_t + \frac{1}{2}\gamma^2 e^{-\gamma S_t} d[S_t, S_t]$$

The entropic risk measure for the final payment occurred at time T is given by $\frac{1}{\gamma}\log E[e^{-\gamma ST}|F_T] = -S_T$, and the remaining measures are obtained by means of a BSDE with $g(t, Z_t) = \frac{\gamma}{2}\|Z_t\|^2$. This BSDE is Markovian and its g-coefficient grows less than quadratically in Z_t. We implement this one-dimensional BSED by following Table 1. By taking risk aversion parameter γ=8, we obtain the following figure 1-3.

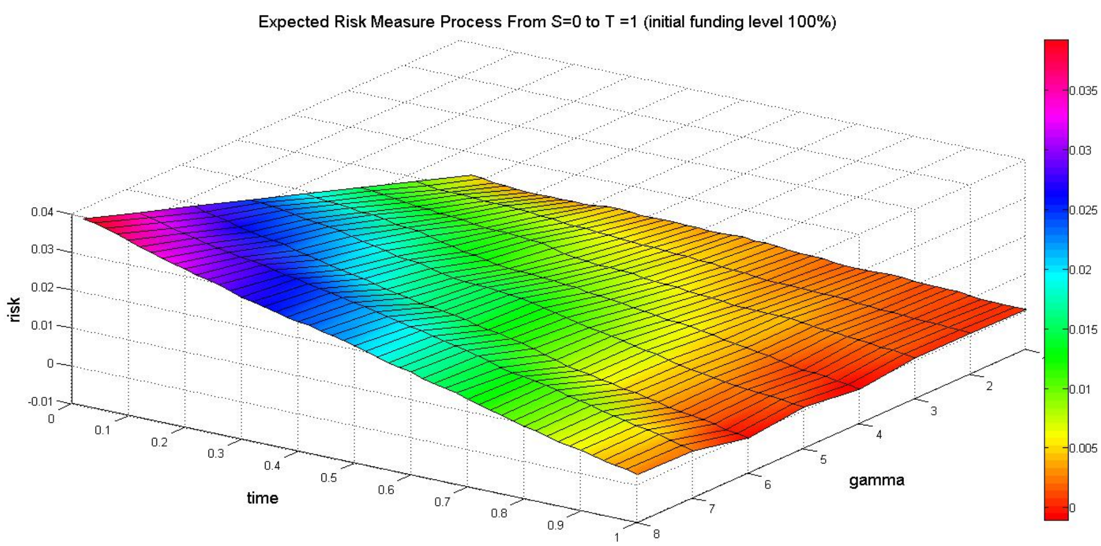

Figure 1

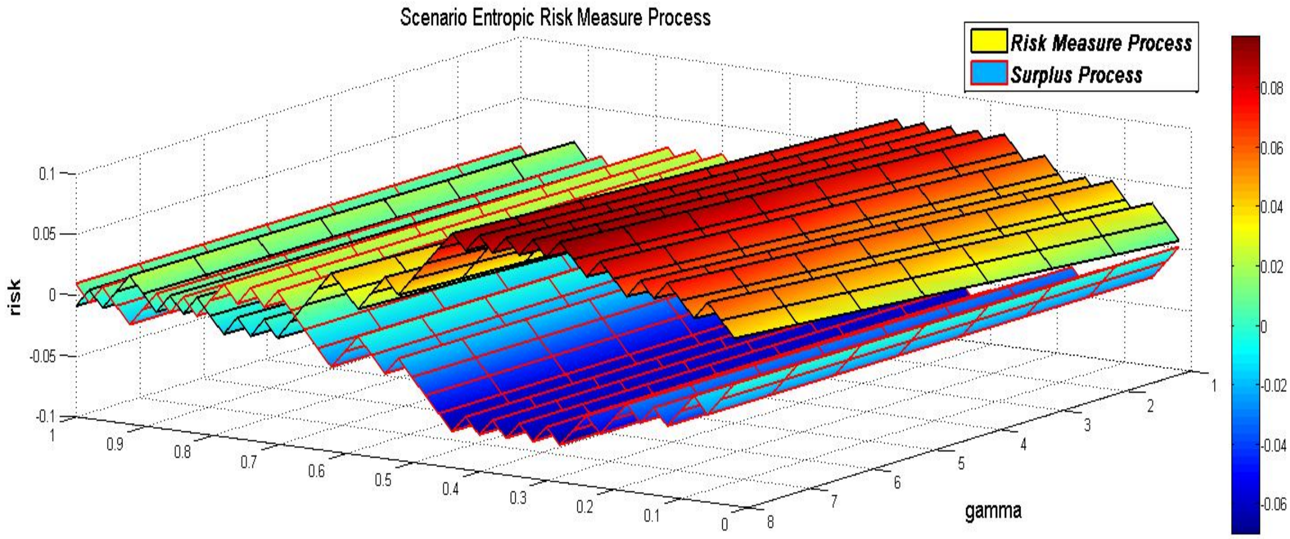

Figure 2

From Figure 1 we observe that the expected risk measure process is strongly time consistent, and satisfies a supermartingale property; that is, for every t, we have $E[\rho_{t+1}(S_T)] \leq \rho_t(S_T)$. Figure 2 plot a particular trajectory of the surplus ratio process and its associated risk measure process. It can be observed that they follow the same pattern but with different directions. The risk assessment goes downwards when the incoming information is positive.

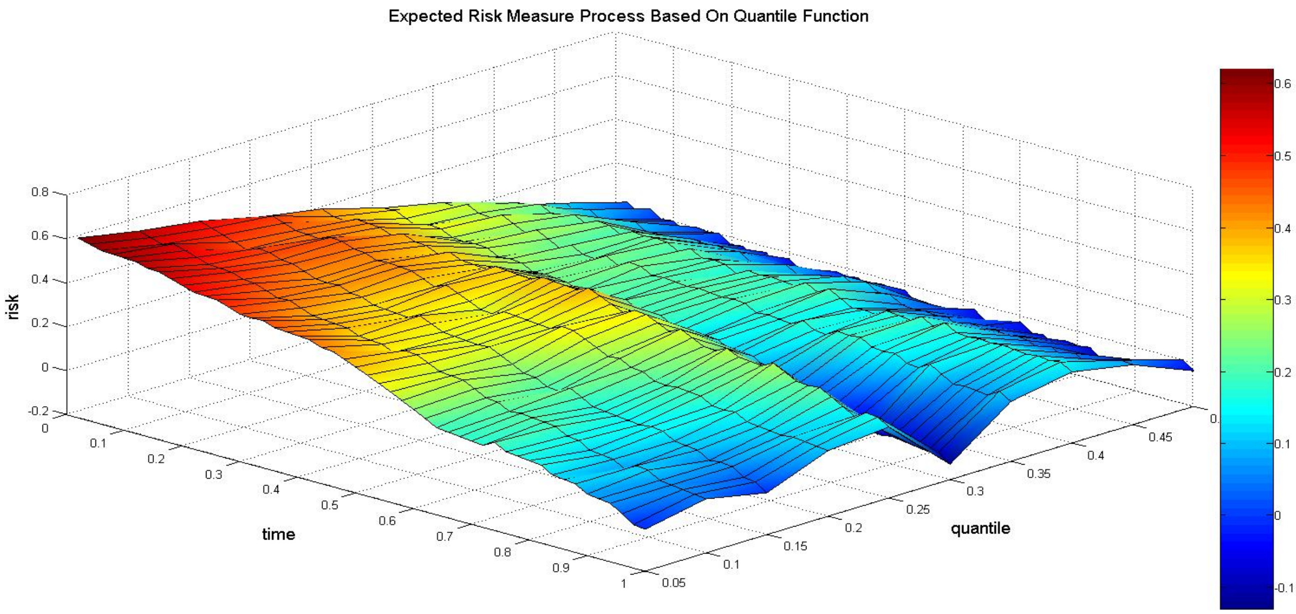

Figure 3

Figure 3 plots the expected risk measure based on the negative of the quantile function of S_1, i.e. $F^{-1}(q) = \inf\{S_1 \in L^2 \,|\, q \le F(S_1)\}$, where F is the cumulative distribution function of S_1. We can see that the VaR is not as robust as the entropic risk measure.

2.5.2 Risk Measurement for the Surplus Ratio Process

In the case of dynamic risk measurement for the surplus ratio process, we use the dynamic risk measure (39) in section 4.2, which values the surplus ratio at each time when the new information arrives. For simplicity, let us assume that the dynamic entropic risk measure assigns a uniform weight to each $E[e^{-\gamma X_s}|F_t]$; one can alternatively vary the weights for different time instances by considering that some time periods are more important than the others. The value of the dynamic risk measure for random processes is then obtained by taking the natural logarithm (scaled by $\frac{1}{\gamma}$) of the average value of $e^{-\gamma Y_t^s}$, where $Y_t^s = \frac{1}{\gamma}\log E\left[e^{-\gamma S_s}|F_t\right]$ can be computed by means of a suitable BSDE. By taking σ=10% and γ= 8, we generate Figure 4-5.

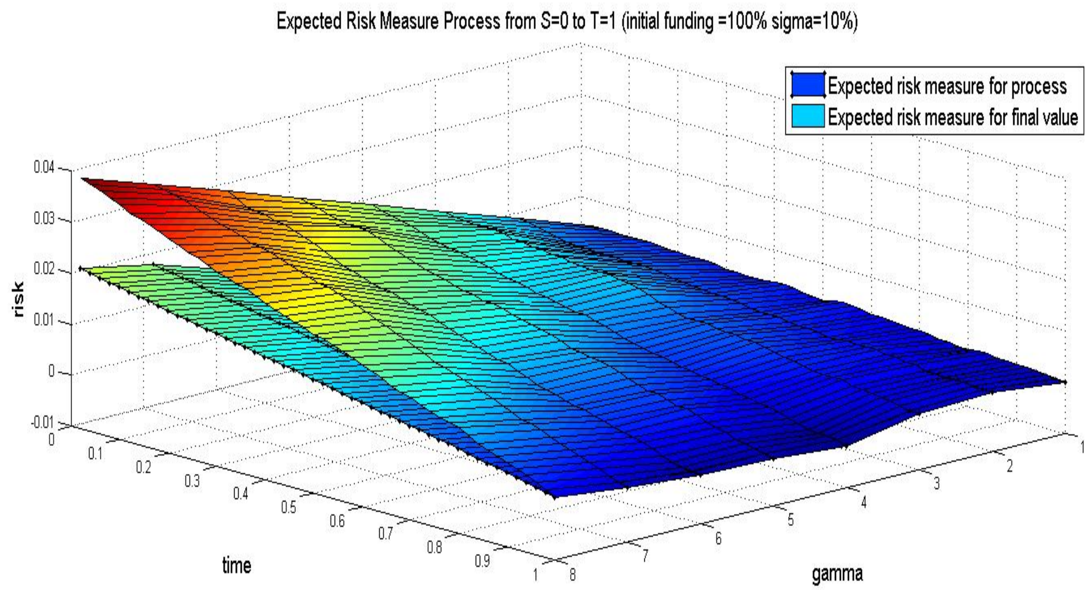

Figure 4

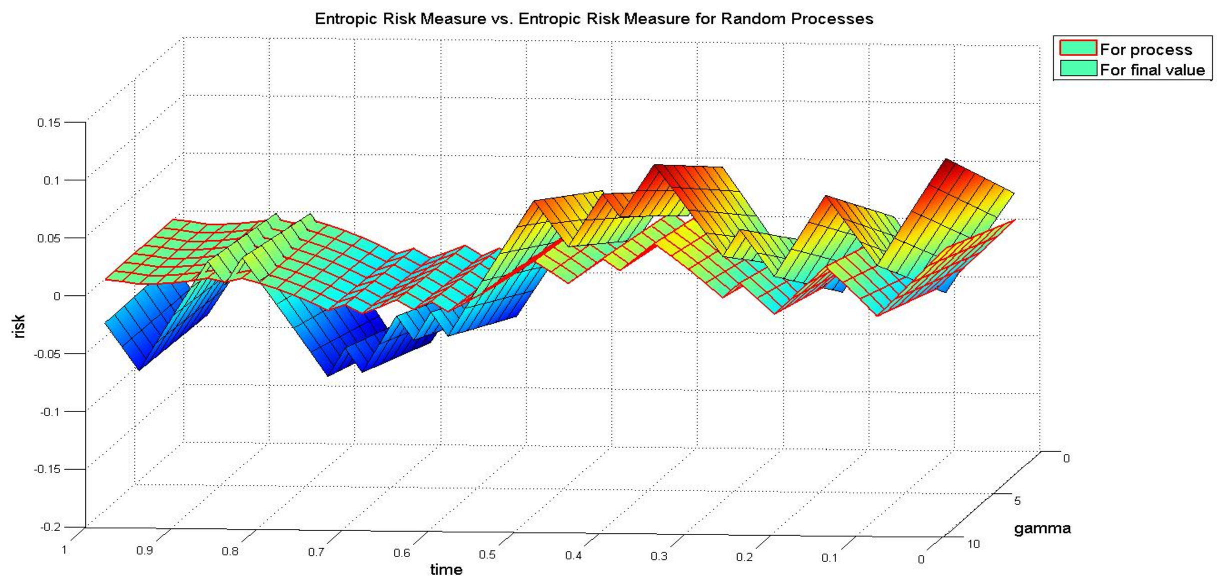

Figure 5

Figure 4 and Figure 5 plot dynamic risk measurement of the surplus ratio process, where the surplus ratio is evaluated at each point t in time when new information arrives. As it can be observed, the expected risk measure for the whole process is smaller than the expected risk measure of the final surplus ratio only. The reason is that $\frac{1}{T+1-t}\sum_{s=t}^{T} E[e^{-\gamma S_s} | F_t] \le E[e^{-\gamma S_s} | F_t]$. In addition, the measurement of the process is more stable than only measuring the final surplus ratio when new information is gathered; this can be evident from the standard deviation of the risk measurement. For the risk measure process of a particular trajectory of the surplus ratio in Figure 5, we see that due to the less volatility of the measurement for the surplus ratio process, the risk measure of the final surplus ratio is almost shifted upwards.

3. Partial Hedging

Typically a perfect hedge will not be admitted with a contingent claim in an incomplete financial market model. In order to stay on the safe side, we can use superhedging. However, both from a practical and from a theoretical point of view, the required cost is usually too high. Thus, relaxing the requirements would be natural

To stay on the safe side with high probability, let us consider the strategies of partial hedging. In other words, the purpose is to find the optimal hedging strategies, while subject to a given cost constraint. For a given claim H, we are going to reduce the construction of the strategies to a optimization problem of superhedging for another modified claim $\widetilde{\mathrm{H}}$. It would be the solution to a static optimization problem, which is Neyman-Pearson type. $\widetilde{\mathrm{H}}$ will typically have the form $\widetilde{\mathrm{H}} = \mathrm{H} \cdot \mathrm{I}_{\mathrm{A}}$, which is the form of a knock-out option. At current stage, we only take into account the probability that a shortfall occurs. The size of the shortfall if it does occur is not considered here.

Different partial hedging approaches have been proposed and examined in the literature. For example, quadratic hedging, which minimizes the expectation of a quadratic error (L^2). Such a measure has been criticized for the consequence of penalizing both loss and profit equally, also for not being able to capture the heavy-tailness phenomena in the financial market. Quantile hedging based on a dynamic version of VaR is probably the most-known approach. Recently Melnikov proposed CVaR partial hedging. In this section, we will discuss partial hedging more comprehensively and quantify the downside risk with respect to an acceptance set for positions that are suitably hedged. If the acceptability above is defined by shortfall risk, we need to solve the problem of constructing efficient strategies. That is, to minimize the risk subject to a given cost constraint.

3.1 Problem Setup

Consider a portfolio consists of a risky asset S and a default-free bond R. Let $\{S(t)\}_{t\in[0,T]}$ represents the price of the asset S at time index t, and $\{R(t)\}_{t\in[0,T]}$ represents the price of the bond R at time t. We assume that both $S(t)$ and $R(t)$ are F_t measurable. Define $\xi_S(t)$ and $\xi_R(t)$ as the number of shares held on S and R at time $t\in[0,T]$. The value of the portfolio is given by:

$$V(t)=\xi_S(t)S(t)+\xi_R(t)R(t), t\in[0,T].$$

Let $\xi(t):=(\xi_S(t),\xi_B(t))$ be a self-financing strategy, then $V(t)$ satisfies the following stochastic differential equation:

$$V(t)=V(0)+\int_0^t \xi_S(s)dS(s)+\int_0^t \xi_R(s)dR(s), t\in[0,T].$$

A strategy $\{\xi_t\}_{t\in[0,T]}$ is admissible if

$$V(t)\geq 0, \text{ for all } t\in[0,T], \mathrm{P}-a.s.$$

In the context of a complete market, every contingent claim can be replicated by a self-financing strategy $\{\xi^*(t)\}_{t\in[0,T]}$. Let $H(T)$, $(H(T)\geq 0)$, be the payoff of a contingent claim at time T; $H(T)$ is a F_T measurable random variable. The cost of the replication defines the price of the contingent claim: let § be the unique equivalent martingale measure, the value of the claim can be replicated by $\{\xi^*(t)\}_{t\in[0,T]}$ which requires an initial amount of

$$V^*(0)=\xi_S^*(0)S(0)+\xi_B^*(0)B(0)=E^{\S}[R(T)^{-1}H(T)].$$

Such a strategy is served as a perfect hedge for the claim $H(T)$. In the case of partial hedging it only requires a smaller initial amount $V(0)$ of no larger than ν, that is,

$$V(0)\leq \nu < E^{\S}[R(T)^{-1}H(T)].$$

What would be the optimal partial hedge that can be achieved?

An investor who shorts the contingent claim $H(T)$ wants to construct a portfolio $\{V(t)\}_{t\in[0,T]}$ with a purpose of hedging the potential loss $L = H(T) - V(T)$ at maturity time T. Note that in a complete market if the initial value $V(0)$ is smaller than the price of the claim, then $L \neq 0$. Our goal is to find the most efficient strategy such that the risk is controlled.

3.2 Quantile Hedging

Quantile hedging is probably the most studied approach. The first papers appearing in this research area were Follmer (1999,2000) and Krutchenko (2001). Later Melnikov (2004,2005,2008) applied the quantile hedging technique on pricing equity-linked life insurance products.

Quantile hedging maximizes the probability that a hedge is successful by applying a dynamic version of the static Varlue-at-Risk (VaR). VaR has been adopted as a standard risk measure in the financial industry. It has a number of deficiencies recognized by financial professionals. More recently,

Efficient Hedging

Suppose in an arbitrage-free market model, and denote a discounted European claim as H such that

$$\pi_{\mathrm{sup}}(\mathrm{H}) = sup_{P^* \in \mathcal{P}} E^*[H] < \infty$$

there exists a self-financing trading strategy. The value process $\mathrm{V}^{\uparrow}$ of the strategy satisfies

$$\mathrm{V}_{\mathrm{T}}^{\uparrow} \geq \mathrm{H} \quad \mathrm{P} - \mathrm{a.s}$$

There exists obligation from sale of H. However, the seller of H can cover almost any possible obligation by using the superhedging strategy above. Thus, the seller can eliminate the corresponding risk completely. $\pi_{\mathrm{sup}}(\mathrm{H})$ is the smallest amount for such a superhedging strategy to be available. From a practical point of view, this cost is often too high. Moreover, from a theoretical point of view, if H is not attainable then $\pi_{\mathrm{sup}}(\mathrm{H})$, which is the price for H, is too high because it would permit arbitrage. Even if H is attainable, by using a replicating strategy for H, a complete elimination of risk will consume the whole proceeds from the sale of H, and we will lost any opportunity of making a profit along with the elimination of risk.

Therefore, let us consider the case that the seller is unwilling to put up the initial capital which is required by a superhedge. In this case, the seller is ready to accept some risk. Then the question is: what is the optimal partial hedge strategy that can be used under a given smaller amount of capital constraint? In order to make this more precisely, the seller's attitude towards risk should be expressed by some criterions. In the following sections, we will study several of such criteria.

In this section, our goal is to construct a optimal strategy which maximizes the probability of a successful hedge, while statisfies a given constraint on the initial capital.

Let us fix an initial amount of capital

$$\upsilon < \pi_{\sup}(H)$$

We need to construct a self-financing trading strategy, and the value process would maximize the probability

$$P[V_T \geq H]$$

among all those strategies which have initial investment V_0 bounded by υ and subject to the bounds $V_t \geq 0$ for $t = 0, \ldots, T$. The second constraint amounts to admissibility as follows[?]:

Definition 3.1. A self-financing trading strategy is called an admissible strategy if its value process satisfies $V_T \geq 0$.

Let V^* be the value process of an admissible strategy $\bar{\xi}^*$ and satisfies

$$P[V_T^* \geq H] = \max P[V_T \geq H]$$

the maximum here is taken over all value processes V of all possible admissible strategies under the constraint

$$V_0 \leq \upsilon$$

Note that without the constraint of admissibility the problem above is well posed

Now let us take a look at the idea of quantile hedging with respect to a Value at Risk criterion: we only take into account the probability of a shortfall, not the size of the loss when a shortfall actually occurs. The shortfall probability is mainly focused here and may be reasonable in cases where we are trying to avoid a loss by any means. In the next section, we consider other optimality criteria which, from an economic point of view, are usually more appropriate. However, some central ideas are expressed quite clearly in the present context in term s of the mathematical techniques.

First, before passing to the general incomplete case, let us consider the particularly transparent case for a complete market model. The set

$$\{V_T \geq H\}$$

is called the success set corresponding to the value process V with respect to an admissible strategy. As a first preliminary step, we reduce the problem to another problem which constructs a success set of maximal probability.

Proposition 2. [] Let P^* denote the unique equivalent martingale measure in a complete market model, and assume that $A^* \in \mathcal{F}_T$ maximizes the probability $P[A]$ among all sets $A \in \mathcal{F}_T$ satisfying the constraint

$$E^*[H \cdot I_A] \leq \upsilon$$

Then the replicating strategy $\bar{\xi}^*$ of the knock-out option

$$H^* \coloneqq H \cdot I_{A^*}$$

sovles the optimization problem defined by

$$P[V_T^* \geq H] = \max P[V_T \geq H]$$

and

$$V_0 \leq \upsilon$$

and A^* coincides up to P-null sets with the success set of $\bar{\xi}^*$.

Proof. As a first step, let V be the value process of any admissible strategy such that $V_0 \leq \upsilon$. We denote by $A \coloneqq \{V_T \geq H\}$ the corresponding success set. Admissibility yields that $V_T \geq H \cdot I_A$. Moreover, the results of Section 5.3 imply that V is a P^*-martingale. Hence, we obtain that

$$E^*[H \cdot I_A] \leq E^*[V_T] = V_0 \leq \upsilon$$

Therefore, A fulfills the constraints $E^*[H \cdot I_A] \leq \upsilon$ and it follows that

$$P[A] \leq P[A^*]$$

As a second step, we consider the trading strategy $\bar{\xi}^*$ and its value process V^*. Clearly, $\bar{\xi}^*$ is admissible, and its success set satisfies

$$\{V_T^* \geq H\} = \{H \cdot I_{A^*} \geq H\} \supseteq A^*$$

On the other hand, the first part of the proof yields that

$$P[V_T^* \geq H] \leq P[A^*]$$

It follows that the two sets A^* and $\{V_T^* \geq H\}$ coincide up to P-null stes. In particular, $\bar{\xi}^*$ is an optimal strategy.

Next, we will construct the optimal success set A^*. The existence of A^* was assumed in the proposition above. To solve this problem we use the Neyman-Pearson lemma. To this end, let us introduce the measure Q^* that is given by

$$\frac{dQ^*}{dP^*} \coloneqq \frac{H}{E^*[H]}$$

and write the constraint $E^*[H \cdot I_A] \leq \upsilon$ as

$$Q^*[A] \leq \alpha \coloneqq \frac{\upsilon}{E^*[H]}$$

under the constraint $Q^*[A] \leq \alpha$, an optimal success set would maximize the probability $P[A]$. Let us denote the generalized density of P with respect to Q^* as dP/dQ^*, in terms of the Lebesgue decomposition as constructed above. Hence, we can defined the level as follows:

$$c^* := \inf\{c \geq 0 \mid Q^*\left[\frac{dP}{dQ^*} > c \cdot E^*[H]\right] \leq \alpha\}$$

and the set

$$A^* := \left\{\frac{dP}{dQ^*} > c^* \cdot E^*[H]\right\} = \{\frac{dP}{dP^*} > c^* \cdot H\}$$

Proposition 3. (Follmer,2000) If the set A^* in

$$A^* := \left\{\frac{dP}{dQ^*} > c^* \cdot E^*[H]\right\} = \{\frac{dP}{dP^*} > c^* \cdot H\}$$

Satisfies

$$Q^*[A^*] = \alpha$$

then A^* maximizes the probability $P[A]$ over all $A \in \mathcal{F}_T$ satisfying the constraint

$$E^*[H \cdot I_A] \leq \upsilon$$

Proof. The condition $E^*[H \cdot I_A] \leq \upsilon$ is equivalent to $Q^*[A] \leq \alpha = Q^*[A^*]$. Thus, the particular form of the set A^* in $A^* := \left\{\frac{dP}{dQ^*} > c^* \cdot E^*[H]\right\} = \{\frac{dP}{dP^*} > c^* \cdot H\}$ and the Neyman-Pearson lemma in the form of Proposition A.28 imply that $P[A] \leq P[A^*]$.

Therefore, we have the following result (Follmer, 2000):

Corollary 3.4. Denote by P^* the unique equivalent martingale measure in a complete market model, and assume that the set A^* of $A^* := \left\{\frac{dP}{dQ^*} > c^* \cdot E^*[H]\right\} = \{\frac{dP}{dP^*} > c^* \cdot H\}$

satisfies

$$Q^*[A^*] = \alpha$$

Then the optimal strategy solving

$$P[V_T^* \geq H] = \max P[V_T \geq H]$$

and

$$V_0 \leq \upsilon$$

is given by the replicating strategy of the knock-out option $H^* = H \cdot I_{A^*}$.

The solution to the optimization problem

$$P[V_T^* \geq H] = \max P[V_T \geq H]$$

and

$$V_0 \leq \upsilon$$

still have the assumption that the set A^* satisfies $Q^*[A^*] = \alpha$. Clearly, if

$$P\left[\frac{dP}{dP^*} = c^* \cdot H\right] = 0$$

then this condition is satisfied. However, in general it would not be possible to construct any set A whose Q^*-probability is α exactly. In such a case, we can replace the indicator function I_{A^*} of the critical region A^* by a randomized test according to the Neyman-Pearson theorem. That is, by an $\mathcal{F}_T$-measurable $[0,1]$-valued function ψ. Let us denote $\mathcal{R}$ the class of all randomized tests. Consider the following optimization problem:

$$E[\psi^*] = \max \{E[\psi] | \psi \in \mathcal{R} \text{ and } E_{Q^*}[\psi] \leq \alpha\}$$

where Q^* is the measure given by $\frac{dQ^*}{dP^*} \coloneqq \frac{H}{E^*[H]}$, $\alpha = \upsilon / E^*[H]$ as in $Q^*[A] \leq \alpha \coloneqq \frac{\upsilon}{E^*[H]}$.

According to the generalized Neyman-Pearson lemma, the solution is given by

$$\psi^* = I_{\{\frac{dP}{dP^*} > c^* \cdot H\}} + \gamma \cdot I_{\{\frac{dP}{dP^*} = c^* \cdot H\}}$$

where c^* is defined by $c^* \coloneqq \inf\{c \geq 0 \mid Q^*\left[\frac{dP}{dQ^*} > c \cdot E^*[H]\right] \leq \alpha\}$ and γ is chosen in a way such that $E_{Q^*}[\psi^*] = \alpha$ satiesfies, i.e.,

$$\gamma = \frac{\alpha - Q^*[\frac{dP}{dP^*} > c^* \cdot H]}{Q^*[\frac{dP}{dP^*} = c^* \cdot H]} \text{ in case } P[\frac{dP}{dP^*} = c^* \cdot H] \neq 0$$

Definition 3.5. (Follmer, 2000) Let V be the value process of an admissible strategy $\bar{\xi}$. The success ration of ξ is defined as the randomized test

$$\psi_V = I_{\{V_T \geq H\}} + \frac{V_T}{H} \cdot I_{\{V_T < H\}}$$

The success set $\{V_T \geq H\}$ of V coincides with the set $\{\psi_V = 1\}$. In the original problem's extended version , now we are constructing a strategy which will maximize the expected success ration $E[\psi_V]$ subject to the cost constraint $V_0 \leq \upsilon$ under the measure P:

Theorem 6 (Follmer, 2000) Suppose that P^* is the unique equivalent martingale measure in a complete market model. Let ψ^* be given by $\psi^* = I_{\left\{\frac{dP}{dP^*} > c^* \cdot H\right\}} + \gamma \cdot I_{\left\{\frac{dP}{dP^*} = c^* \cdot H\right\}}$, and denote by $\bar{\xi}^*$ a replicating strategy for the discounted claim $H^* = H \cdot \psi^*$. Then the success ration ψ_{V^*} of $\bar{\xi}^*$ maximizes the expected success ratio $E[\psi_V]$ among all admissible strategies with initial investment $V_0 \leq \upsilon$. Moreover, the optimal success ration ψ_{V^*} is P-a.s. equal to ψ^*.

Here we do not prove this theorem, since it is a special case of the theorem introduced below. Once the optimal randomized test ψ^* is determined by using the generalized Neyman-Pearson lemma, the proof would be similar to one of Corollary 4. The condition

$$P\left[\frac{dP}{dP^*} = c^* \cdot H\right] = 0$$

means that $\psi^* = I_{A^*}$ with A^* as in $A^* := \left\{\frac{dP}{dQ^*} > c^* \cdot E^*[H]\right\} = \{\frac{dP}{dP^*} > c^* \cdot H\}$. In this case, the strategy $\bar{\xi}^*$ reduces to the strategy described in Corollary 3.4.

Let us now consider the general case of an arbitrage-free model which is possibly in incomplete market, that is, the set $\mathcal{P}$ of equivalent martingale measures is no longer assumed to consist of a single element. It is assumed only that

$$\mathcal{P} \neq \emptyset$$

In this setting, our goal is to look for an admissible strategy and the success ration ψ_{V^*} satisfies the following condition

$$E[\psi_{V^*}] = \max E[\psi_V]$$

where on the right-hand side we maximize over all admissible strategies under an initial investment constraint

$$V_0 \leq \upsilon$$

Theorem 7. (Follmer, 2000) There exists a randomized test ψ^* such that

$$\sup_{P^* \in \mathcal{P}} E^*[H \cdot \psi^*] = \upsilon$$

and which maximizes $E[\psi]$ among all $\psi \in \mathcal{R}$ subject to the constraints

$$E^*[H \cdot \psi] \leq \upsilon \text{ for all } P^* \in \mathcal{P}$$

Moreover, the superhedging strategy for the modified claim

$$H^* = H \cdot \psi^*$$

with initial investment $\pi_{\sup}(H^*)$ solves the problem

$$E[\psi_{V^*}] = \max E[\psi_V]$$

and

$$V_0 \leq \upsilon$$

Proof. Denote by $\mathcal{R}_0$ the set of all $\psi \in \mathcal{R}$ which satisfy the constraints $E^*[H \cdot \psi] \leq \upsilon$ for all $P^* \in \mathcal{P}$, and take a sequence $\psi_n \in \mathcal{R}_0$ such that

$$E[\psi_n] \longrightarrow \sup_{\psi \in \mathcal{R}_0} E[\psi] \text{ as } n \uparrow \infty$$

we have a sequence of convex combinations $\widetilde{\psi_n} \in \text{conv}\{\psi_n, \psi_{n+1}, \dots\}$ converging P-a.s. to a function $\widetilde{\psi} \in \mathcal{R}$. Clearly, $\widetilde{\psi_n} \in \mathcal{R}_0$ for each n. Hence, Fatou's lemma yields that

$$E^*\left[H\widetilde{\psi}\right] \leq \lim_{n\uparrow\infty} \inf E^*\left[H\widetilde{\psi_n}\right] \leq \upsilon \text{ for all } P^* \in \mathcal{P}$$

and it follows that $\widetilde{\psi} \in \mathcal{R}_0$. Moreover,

$$E\left[\widetilde{\psi}\right] = \lim_{n\uparrow\infty} E\left[\widetilde{\psi_n}\right] = \lim_{n\uparrow\infty} E[\psi_n] = \sup_{\psi \in \mathcal{R}_0} E[\psi]$$

So $\psi^* := \widetilde{\psi}$ is the desired maximize.

We must also show that $\sup_{P^* \in \mathcal{P}} E^*[H \cdot \psi^*] = \upsilon$ holds. To this end, note first that $P[\psi^* = 1] = 1$ is impossible due to our assumption $\upsilon < \pi_{\sup}(H)$. Hence, if $\sup_{P^* \in \mathcal{P}} E^*[H \cdot \psi^*] < \upsilon$, then we can find some $\varepsilon > 0$ such that $\psi_\varepsilon \coloneqq \varepsilon + (1 - \varepsilon)\psi^* \in \mathcal{R}_0$, and the expectation $E[\psi_\varepsilon]$ must be strictly larger than $E[\psi^*]$. This, however, contradicts the maximality of $E[\psi^*]$.

Now let $\bar{\xi}$ be any admissible strategy whose value process V satisfies $V_0 \leq \upsilon$. If ψ_V denotes the corresponding success ratio, then

$$H \cdot \psi_V = H \wedge V_T \leq V_T$$

The $\mathcal{P}$-martingale property of V yields that for all $P^* \in \mathcal{P}$,

$$E^*[H \cdot \psi_V] \leq E^*[V_T] = V_0 \leq \upsilon$$

Therefore, ψ_V is contained in $\mathcal{R}_0$ and it follows that

$$E[\psi_V] \leq E[\psi^*]$$

Consider the superhedging strategy $\overline{\xi^*}$ of $H^* = H \cdot \psi^*$ and denote by V^* its value process. Clearly, $\overline{\xi^*}$ is an admissible strategy. Moreover,

$$V_0^* = \pi_{\sup}(H^*) = \sup_{P^* \in \mathcal{P}} E^*[H \cdot \psi^*] = \upsilon$$

Thus, $E[\psi_V] \leq E[\psi^*]$ yields that ψ_{V^*} satisfies

$$E[\psi_{V^*}] \leq E[\psi^*]$$

On the other hand, V_T^* dominates H^*, so

$$H \cdot \psi_{V^*} = H \wedge V_T^* \geq H \wedge H^* = H \cdot \psi^*$$

Therefore, ψ_{V^*} dominates ψ^* on the set $\{H > 0\}$. Moreover, any success ratio is equal to one on $\{H = 0\}$, and we obtain that $\psi_{V^*} \geq \psi^*$ P-almost surely. According to $E[\psi_{V^*}] \leq E[\psi^*]$, this can only happen if the two randomized tests ψ_{V^*} and ψ^* coincide P-almost everywhere. This proves that $\overline{\xi^*}$ solves the hedging problem

$$E[\psi_{V^*}] = \max E[\psi_V]$$

and

$$V_0 \leq \upsilon.$$

3.3 CVaR Hedging

Conditional Value-at-Risk (CVaR) has attracted much attention in recent years. It is is a risk measure that is a superior alternative to VaR in that it conveys information about the average loss exceeds the VaR level and also satisfies the sub-additive property. Melnikov (2012) suggested addressing partial hedging problem by employing conditional Varlue-at-Risk (CVaR).

Let $(\Omega, \{\mathrm{F}_t\}_{t\in[0,T]}, \mathrm{F}, \mathrm{P})$ be a standard probability space and L be a F -measurable random variable characterizing the loss. Assume that $E^{\mathrm{P}}[L] < \infty$.

Recall that the VaR of the loss L at a confidence level $\varepsilon \in (0,1)$ is defined as:

$$VaR_\varepsilon(L) = \inf\{x : \varepsilon(L > x) \le 1-\varepsilon\},$$

and CVaR at a confidence level $\varepsilon \in (0,1)$ is defined as:

$$CVaR_\varepsilon(L) = \frac{1}{1-\varepsilon}\left[E^{\mathrm{P}}[L1_{\{L\ge VaR_\varepsilon(L)\}}] + VaR_\varepsilon(L)(1-\varepsilon-\mathrm{P}(L \ge VaR_\varepsilon(L)))\right].$$

If the cumulative distribution function $F_L(l) := \mathrm{P}(L \le l)$ is continuous, VaR is simply the inverse function of F_L, and CVaR can be rewritten as:

$$CVaR_\varepsilon(L) = E[L \mid L \ge VaR_\varepsilon(L)],$$

which equals to the expected shortfall (ES), and

$$CVaR_\varepsilon(L) = \frac{1}{1-\varepsilon}\int_\varepsilon^1 VaR_\varepsilon(L)du,$$

which equals to the average value-at-risk (AVaR).

There are convenient methods of computing and estimating CVaR. Rockafellar (2000) showed the possibility of computing both VaR and CVaR simultaneously by introducing an auxiliary function:

$$F_{\varepsilon}(z) = z + \frac{1}{1-\varepsilon} E^{\varepsilon}[(L-z)^{+}].$$

CVaR of the loss L is the solution of the minimization problem:

$$CVaR_{\varepsilon}(L) = \min_{z\in\mathbb{R}} F_{\varepsilon}(x,z)$$

Consider a portfolio consists of a risky asset S and a default-free bond R. Let $\{S(t)\}_{t\in[0,T]}$ represents the price of the asset S at time index t, and $\{R(t)\}_{t\in[0,T]}$ represents the price of the bond R at time t. We assume that both $S(t)$ and $R(t)$ are $\mathcal{F}_t$ measurable. Define $\xi_S(t)$ and $\xi_R(t)$ as the number of shares held on S and R at time $t\in[0,T]$. The value of the portfolio is given by:

$$V(t) = \xi_S(t)S(t) + \xi_R(t)R(t), t\in[0,T].$$

Let $\xi(t) := (\xi_S(t), \xi_B(t))$ be a self-financing strategy, then $V(t)$ satisfies the following stochastic differential equation:

$$V(t) = V(0) + \int_0^t \xi_S(s)dS(s) + \int_0^t \xi_R(s)dR(s), t\in[0,T].$$

A strategy $\{\xi_t\}_{t\in[0,T]}$ is admissible if

$$V(t) \geq 0, \text{ for all } t\in[0,T], \text{ P}-a.s.$$

In the context of a complete market, every contingent claim can be replicated by a self-financing strategy $\{\xi^*(t)\}_{t\in[0,T]}$. Let $H(T)$, $(H(T)\geq 0)$, be the payoff of a contingent claim at time T; $H(T)$ is a $\mathcal{F}_T$ measurable random variable. The cost of the replication defines the price of the contingent claim: let § be the unique equivalent martingale measure, the value of the claim can be replicated by $\{\xi^*(t)\}_{t\in[0,T]}$ which requires an initial amount of

$$V^*(0) = \xi_S^*(0)S(0) + \xi_B^*(0)B(0) = E^{\S}[R(T)^{-1}H(T)].$$

Such a strategy is served as a perfect hedge for the claim $H(T)$. In the case of partial hedging it only requires a smaller initial amount $V(0)$ of no larger than ν, that is,

$$V(0) \le \nu < E^{\S}[R(T)^{-1}H(T)].$$

What would be the optimal partial hedge that can be achieved?

An investor who shorts the contingent claim $H(T)$ wants to construct a portfolio $\{V(t)\}_{t\in[0,T]}$ with a purpose of hedging the potential loss $L = H(T) - V(T)$ at maturity time T. Note that in a complete market if the initial value $V(0)$ is smaller than the price of the claim, then $L \ne 0$. Our goal is to find the most efficient strategy such that the risk is controlled. This paper takes two issues in locating optimal strategy: one is to minimize the CVaR of the loss L, with respect to the constraint that the initial cost $V(0)$ is smaller than some number ν:

$$\begin{aligned} &\min_{V(0),\xi} CVaR_{\varepsilon}(H(T) - V(T)) \\ &\text{s.t. } V(0) \le \nu \end{aligned}$$

where $\nu < E^{\mathbb{Q}}[R(T)^{-1}H(T)]$; the other one is to minimize the hedging cost $V(0)$ so that the CVaR is less than or equal to a number c:

$$\begin{aligned} &\min_{V(0),\xi} V(0) = E^{\mathbb{Q}}[R(T)^{-1}V(T)] \\ &\text{s.t. } CVaR_{\varepsilon}(H(T) - V(T)) \le c \end{aligned}$$

These two problems are discussed in the following sections.

3.3.1 Minimizing CVaR

We first consider the CVaR minimizing problem. According to the CVaR representation we have

$$\begin{aligned}&\min_{V(0),\xi}\min_{z\in\mathbb{R}} z+\frac{1}{1-\varepsilon}E^{P}[(H(T)-V(T)-z)^{+}]\\&\text{s.t. } V(0)\le v\end{aligned}$$

Melnikov (2012) shows that we can interchange the order of two minimization problems:

$$\begin{aligned}&\min_{z\in\mathbb{R}} z+\frac{1}{1-\varepsilon}\min_{V(0),\xi} E^{P}[(H(T)-V(T)-z)^{+}]\\&\text{s.t. } V(0)\le v\end{aligned}$$

For $z\ge 0$, we have $(H(T)-V(T)-z)^{+}=((H(T)-z)^{+}-V(T))^{+}$; then we can write (5) to be

$$\begin{aligned}&\min_{V(0),\xi} E^{\mathrm{P}}[((H(T)-z)^{+}-V(T))^{+}]\\&\text{s.t. } V(0)\le v\end{aligned}$$

In this work we quantify the loss of an undiscounted portfolio. This is different from Follmer (1999), Gao (2011) and Melnikov (2012). Despite the fact that we want to control the risk of a hedging portfolio at maturity T rather than its current value, dealing with CVaR of a discounted portfolio would result in computational inefficiency. To see this, take a call option $H(T)=(S(T)-K)^{+}$ as an example, which we will see later: by taking the discount factor into account, $(H(T)-z)^{+}$ is replaced by

$$(R(T)^{-1}H(T)-z)^{+}=R(T)^{-1}(H(T)-zR(T))^{+}=R(T)^{-1}(S(T)-K-zR(T))^{+},$$

where $S(T)-zR(T)$ is a linear combination of two log normal random variables, and difficult to generate a closed solution.

To solve the optimization problem, we suggest applying the measure transformation method to the discounted problem before addressing the undiscounted one. The discounted problem was well-studied by Follmer and Leukert (2000):

Theorem 3.3.1. Let $H(T)$ be the payoff of a contingent claim, then the optimal hedging strategy $(V^*(0), \xi^*)$ of the shortfall minimization problem:

$$\begin{aligned} &\min_{V(0),\xi} E^{p}[R(T)^{-1}(H(T)-V(T))^{+}] \\ &\text{s.t. } V(0) \le \nu < E^{\mathbb{Q}}[R(T)^{-1}H(T)] \end{aligned}$$

is the perfect hedge for the claim $H^*(T) = H(T)\psi^*$, where

$$\begin{aligned} \psi^* &= 1_{\{\frac{dP}{d\mathbb{Q}} > a^*\}} + \gamma 1_{\{\frac{dP}{d\mathbb{Q}} = a^*\}} \\ a^* &= \inf\{a \ge 0 : E^{\mathbb{Q}}[R(T)^{-1}H(T)1_{\{\frac{dP}{d\mathbb{Q}} > a\}}] \le \nu\} \\ \gamma &= \frac{\nu - E^{\mathbb{Q}}[R(T)^{-1}H(T)1_{\{\frac{dP}{d\mathbb{Q}} > a^*\}}]}{E^{\mathbb{Q}}[R(T)^{-1}H(T)1_{\{\frac{dP}{d\mathbb{Q}} = a^*\}}]} \end{aligned}$$

By Theorem 1, we can generate the following theorem for the undiscounted process:

Theorem 3.3.2. Let $H(T)$ be the payoff of a contingent claim, then the optimal hedging strategy $(V^*(0), \xi^*)$ of the shortfall minimization problem:

$$\begin{aligned} &\min_{V(0),\xi} E^{P}[(H(T)-V(T))^{+}] \\ &\text{s.t. } V(0) \le \nu < E^{\S}[R(T)^{-1}H(T)] \end{aligned}$$

is the perfect hedge for the claim $H^*(T) = H(T)\psi^*$, where

$$\begin{aligned} \psi^* &= 1_{\{R(T)\frac{dP}{d\S} > a^*\}} + \gamma 1_{\{R(T)\frac{dP}{d\S} = a^*\}} \\ a^* &= \inf\{a \ge 0 : E^{\S}[R(T)^{-1}H(T)1_{\{R(T)\frac{dP}{d\S} > a\}}] \le \nu\} \text{Proof} \\ \gamma &= \frac{\nu - E^{\S}[R(T)^{-1}H(T)1_{\{R(T)\frac{dP}{d\S} > a^*\}}]}{E^{\S}[R(T)^{-1}H(T)1_{\{R(T)\frac{dP}{d\S} = a^*\}}]} \end{aligned}$$

Define

$$\frac{dP^*}{dP} = \frac{R(T)}{E^{P}[R(T)]},$$

then

$$E^{\mathrm{P}}[(H(T)-V(T))^+] = E[R(T)]E^{\mathrm{P}^*}[R(T)^{-1}(H(T)-V(T))^+].$$

To minimize (\ref{eq: shortfall p*}) we apply Theorem 1, then

$$\frac{d\mathrm{P}^*}{d\mathbb{Q}} = \frac{R(T)}{E^{\mathrm{P}}[R(T)]}\frac{d\mathrm{P}}{d\mathbb{Q}},$$

finishes the proof.

We can apply Theorem 2 to solve problem (5) by letting $\tilde{H}(T) := (H(T)-z)^+$; and that gives the solution to (2). Based on the result in Melnikov (2012), we conclude that:

Theorem 3.3.3. The optimal strategy $(V(0)^*, \xi^*)$ for the CVaR minimization problem is a perfect hedge for the contingent claim $(H-z^*)^+\psi^*(z^*)$, where $\psi^*(z)$ is given by

$$\begin{aligned}
\psi^*(z) &= 1_{\{R(T)\frac{d\mathrm{P}}{d\mathbb{Q}} > a^*(z)\}} + \gamma(z)1_{\{R(T)\frac{d\mathrm{P}}{d\mathbb{Q}} = a^*(z)\}} \\
a^*(z) &= \inf\{a \ge 0 : E^{\mathbb{Q}}[R(T)^{-1}(H(T)-z)^+ 1_{\{R(T)\frac{d\mathrm{P}}{d\mathbb{Q}} > a\}}] \le \nu\} \\
\gamma(z) &= \frac{\nu - E^{\mathbb{Q}}[R(T)^{-1}(H(T)-z)^+ 1_{\{R(T)\frac{d\mathrm{P}}{d\mathbb{Q}} > a^*(z)\}}]}{E^{\mathbb{Q}}[R(T)^{-1}(H(T)-z)^+ 1_{\{R(T)\frac{d\mathrm{P}}{d\mathbb{Q}} = a^*(z)\}}]}
\end{aligned}$$

and z^* is the solution of minimization problem:

$$\min_{z>0} c(z) = \begin{cases} z + \dfrac{1}{1-\varepsilon}E^{\mathrm{P}}[(H(T)-z)^+(1-\psi^*(z))] & \text{if } 0 \le z < \hat{z} \\ z & \text{if } z \ge \hat{z} \end{cases}$$

$\hat{z}$ is the solution of

$$E^{\mathbb{Q}}[R(T)^{-1}(H(T)-z)^+] = \nu$$

Note that only when $z \ge 0$ can we use the result of (5) to solve (2). Nonethless, we do not need to worry about the case when $z < 0$ because we assume that for any U that is close enough to 1, the optimal $\hat{z}$ that equals to the VaR_ε of the portfolio is always nonnegative.

3.3.2 Minimizing hedging costs

In this subsection we address the hedging costs minimization problem. Let us take the shortfall optimization problem as the point of departure.

Theorem 3.3.4. Let $H(T)$ be the payoff of a contingent claim, then the optimal hedging strategy $(V^*(0),\xi^*)$ of the shortfall minimization problem:

$$\begin{aligned}&\min_{V(0),\xi} V(0) = E^{\S}[R(T)^{-1}V(T)]\\ &\text{s.t. } E^{\mathrm{P}}[(H(T)-V(T))^+] \le c < E^{\mathrm{P}}[H(T)]\end{aligned}$$

is the perfect hedge for the claim $H^*(T) = H(T)(1-\psi^*)$, where

$$\begin{aligned}\psi^* &= 1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP}>a^*\}} + \gamma 1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP}=a^*\}}\\ a^* &= \inf\{a \ge 0 : E^P[H(T)1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP}>a\}}] \le c\}\\ \gamma &= \frac{c - E^P[H(T)1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP}>a^*\}}]}{E^P[H(T)1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP}=a^*\}}]}\end{aligned}$$

Proof

Follow the proof of Theorem 3.3.3, we reformulate the problem to be:

$$\begin{aligned}&\min_{V(0),\xi} V(0) = E^{\mathbb{Q}}[R(T)^{-1}V(T)]\\ &\text{s.t. } E^{p^*}[R(T)^{-1}(H(T)-V(T))^+] \le \frac{c}{E^p[R(T)]}\end{aligned}$$

where

$$\frac{d\mathrm{P}^*}{d\mathrm{P}} = \frac{R(T)}{E^{\mathrm{P}}[R(T)]}$$

As in (Follmer, 1999} and (Follmer, 2004), we rewrite the problem by employing a random test ψ :

$$\min_{\psi\in f[0,1]} E^{\mathbb{Q}}[R(T)^{-1}H(T)(1-\psi)]$$
$$\text{s.t. } E^{P^*}[R(T)^{-1}H(T)\psi] \le \frac{c}{E^P[R(T)]}$$

To see this, first we assume that $\hat{V}(T)$ is the solution of problem(\ref{eq:minHC 3}) and $\tilde{\psi}$ is the solution of problem (\ref{eq:minHC 4}), then $\tilde{V}(T) := H(T)(1-\tilde{\psi})$ satisfies the constraint. Thus, we have

$$E^{\mathbb{Q}}[R(T)^{-1}H(T)(1-\tilde{\psi})] = E^{\mathbb{Q}}[R(T)^{-1}\tilde{V}(T)] \ge E^{\mathbb{Q}}[R(T)^{-1}\hat{V}(T)].$$

On the other side $\hat{\psi} := (1-\hat{V}(T)/H(T))1_{\{\hat{V}(T)\le H(T)\}} \in [0,1]$ satisfies the original constraint and

$$E^{\mathbb{Q}}[R(T)^{-1}H(T)(1-\tilde{\psi})] \le E^{\mathbb{Q}}[R(T)^{-1}H(T)(1-\hat{\psi})] \le E^{\mathbb{Q}}[R(T)^{-1}\hat{V}(T)].$$

Apply Neyman-Pearson's lemma we can solve the shall fall minimization problem. See (follmer, 2004) for details. The solution is given by ψ^*.

By following lemma 2.5 and lemma 2.6 in \cite{melnikov2012dynamic} we have

Theorem 5. If $E^P[H] > c(1-\varepsilon)$ and $E^{\mathrm{P}}[(H-c)^+] > 0$,

then the optimal strategy $(V(0)^*, \xi^*)$ for the hedging costs minimization problem (\ref{eq:minHC 1}) is a perfect hedge for the contingent claim $(H-z^*)^+(1-\psi^*(z^*))$, where $\psi^*(z)$ is given by

$$\psi^*(z) = 1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP} > a^*(z)\}} + \gamma(z)1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP} = a^*(z)\}}$$
$$a^*(z) = \inf\{a \ge 0 : E^P[(H(T)-z)^+ 1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP} > a\}}] \le (c-z)(1-\varepsilon)\}$$
$$\gamma(z) = \frac{(c-z)(1-\varepsilon) - E^P[(H(T)-z)^+ 1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{d�} > a^*(z)\}}]}{E^P[(H(T)-z)^+ 1_{\{R(T)^{-1}\frac{d\mathbb{Q}}{dP} = a^*(z)\}}]}$$

and z^* is the solution of

$$\min_{z\in[0,c]} E^{\mathbb{Q}}[R(T)^{-1}(H-z)^+(1-\psi^*(z))].$$

3.4 Option Pricing under HJM

HJM is widely accepted as the most general framework derived by directly modeling the dynamics of instantaneous forward-rates. In this subsection we tackle the option pricing problem under HJM framework. Take a European call option as an example, the objective function for pricing the option is formulated as:

$$V(0) = E^{\mathbb{Q}}[R(T)^{-1}(S(T)-K)^{+}],$$

where T is the maturity and K is the strike price of the option. We start by introducing the basic assumptions concerning the financial setup.

The setup is similar to (amin,1992) and (gao2011). Fix a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is the real-world probability measure. Let $\{B_1^{\mathbb{P}}(t)\}_{t\in[0,T]}$ be a standard Brownian motion defined on $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathcal{F}, \mathbb{P})$. For a given continuous initial forward rate curve $\{f(0,t)\}_{t\in[0,T]}$, we assume that the forward rate process follows Itô's formula dynamics

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dB_1^{\mathbb{P}}(u),$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are drift and volatility processes, respectively.

The spot interest rate at time t, $\{r(t)\}_{t\in[0,T]}$ is given by the instantaneous forward rate of a forward contract, i.e.,

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(u,t)du + \int_0^t \sigma(u,t)dB_1^{\mathbb{P}}(u).$$

For every maturity time T, the forward price $\{P(t,T)\}_{t\in[0,T]}$ can be written as

$$\begin{aligned}
&P(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \\
&= P(0,T) + \int_0^t P(u,T)\left(r(u) - \alpha^*(u,T) + \frac{1}{2}\sigma^*(u,T)^2\right)du - \int_0^t P(u,T)\sigma^*(u,T)dB_1^{\mathbb{P}}(u),
\end{aligned}$$

where,

$$\sigma^*(t,T) = \int_t^T \sigma(t,u)du,$$
$$\alpha^*(t,T) = \int_t^T \alpha(t,u)du.$$

The dynamic of the price process $\{R(t)\}_{t\in[0,T]}$ for a zero-coupon bond is described by

$$\begin{aligned} R(t) &= \exp\Big(\int_0^t r(u)du\Big) \\ &= \frac{1}{P(0,t)}\exp\Big(\int_0^t \alpha^*(s,t)ds + \int_0^t \sigma^*(s,t)dB_1^{\mathrm{P}}(s)\Big). \end{aligned}$$

Consider the return process of the risky-asset S. Given the probability space $(\Omega, \{\mathrm{F}_t\}_{t\in[0,T]}, \mathrm{F}, \mathrm{P})$, where $\{\mathrm{F}_t\}_{t\in[0,T]}$ is the augmented filtration driven by two independent Brownian motions $\{(B_1^{\mathrm{P}}(t), B_2^{\mathrm{P}}(t))\}_{t\in[0,T]}$ initialized at zero. The dynamic of the asset price is governed by the stochastic differential equation:

$$S(t) = S(0) + \int_0^t (\mu(u)+r(u))S(u)du + \int_0^t \sigma_1(u)S(u)dB_1^{\mathrm{P}}(u) + \int_0^t \sigma_2(u)S(u)dB_2^{\mathrm{P}}(u)$$

where $\{\mu(t)\}_{t\in[0,T]}$ denotes the excess return process without randomness. The summation of μ and r represents the expected growth rate.

Let $\mathbb{Q}$ be a probability measure equivalent to P. The Radon-Nikodym derivative $\frac{d\mathbb{Q}}{dP}$ transforms the real-world measure P into the risk-neutral measure $\mathbb{Q}$ with the assumption of no-arbitrage, i.e.,

$$\frac{d\mathbb{Q}}{dP} = \exp\Big(-\frac{1}{2}\int_0^T \theta_1(u)^2 du - \frac{1}{2}\int_0^T \theta_2(u)^2 du + \int_0^T \theta_1(u)dB_1^P(u) + \int_0^T \theta_2(u)dB_2^P(u)\Big).$$

By Girsanov's theorem, the processes:

$$B_1^{\S}(t) = B_1^{\mathrm{P}}(t) - \int_0^t \theta_1(u)du,$$
$$B_2^{\S}(t) = B_2^{\mathrm{P}}(t) - \int_0^t \theta_2(u)du,$$

are two independent $\mathbb{Q}$-Brownian motions.

As both $\{B(t)^{-1}S(t)\}_{t\in[0,T]}$ and $\{B(t)^{-1}P(t,T)\}_{t\in[0,T]}$ are martingales under $\mathbb{Q}$, we have

$$-\alpha^*(t,T)+\frac{1}{2}\sigma^*(t,T)^2-\sigma^*(t,T)\theta_1(t)=0,$$
$$\mu(t)+\sigma_1(t)\theta_1(t)+\sigma_2(t)\theta_2(t)=0.$$

Thus,

$$\theta_1(t)=\frac{-\alpha^*(t,T)+\frac{1}{2}\sigma^*(t,T)^2}{\sigma^*(t,T)},$$
$$\theta_2(t)=\frac{-\mu(t)\sigma^*(t,T)+\sigma_1(t)\alpha^*(t,T)-\frac{1}{2}\sigma_1(t)\sigma^*(t,T)^2}{\sigma_2(t)\sigma^*(t,T)}.$$

From the preceding we obtain the explicit representations of $S(t)$ and $R(t)$ under $\mathbb{Q}$:

$$S(t)=S(0)R(t)\exp\Big(-\frac{1}{2}\int_0^t(\sigma_1^2(u)+\sigma_2^2(u))du+\int_0^t\sigma_1(u)B_1^{\S}(u)+\int_0^t\sigma_2(u)B_2^{\S}(u)\Big),$$
$$R(T)=\frac{1}{P(0,T)}\exp\Big(\frac{1}{2}\int_0^T\sigma^*(u,T)^2du+\int_0^T\sigma^*(u,T)dB_1^{\S}(u)\Big).$$

Recall the option pricing problem:

$$V_0=E^{\S}[R(T)^{-1}(S(T)-K)^+],$$

Based on (\ref{eq:QS}) and (\ref{eq:QR}), we derive the following solution:

$$E^{\S}[R(T)^{-1}(S(T)-K)^+]=S(0)N(d_1)-KP(0,T)N(d_2),$$

where

$$d_1=\frac{\log\Big(\frac{S(0)}{P(0,T)K}\Big)+\frac{1}{2}\int_0^T\Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2+\sigma_2(u)^2\Big)du}{\sqrt{\int_0^T\Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2+\sigma_2(u)^2\Big)du}},$$
$$d_2=\frac{\log\Big(\frac{S(0)}{P(0,T)K}\Big)-\frac{1}{2}\int_0^T\Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2+\sigma_2(u)^2\Big)du}{\sqrt{\int_0^T\Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2+\sigma_2(u)^2\Big)du}}.$$

3.5 CVaR Hedging Under HJM Framework

In this subsection we integrate HJM methodology and CVaR hedging results from Chapter 2. Again, take the European call option as an example, whose payoff is given by $H(T) = (S(T) - K)^{+}$, where K is the strike price. The problem formulation is as discussed in the section 3.4.

We assume that the probability measures are atomless; this implies that the component $1_{\{R(T)\frac{dP}{d\S} = a^{*}(z)\}}$ in Theorem 3 and Theorem 5 can be ignored. To apply Theorem 3 and Theorem 5 we need to compute the following three functions:

$$
\begin{aligned}
f_1(z,a) &= E^{\mathbb{Q}}[R(T)^{-1}(H(T) - z)^{+}1_{\{R(T)\frac{dP}{d\mathbb{Q}} > a\}}],\\
f_2(z,a) &= E^{p}\Big[(H(T) - z)^{+}1_{\{R(T)\frac{dP}{d\mathbb{Q}} \le a\}}\Big],\\
f_3(z) &= E^{\mathbb{Q}}[R(T)^{-1}(H(T) - z)^{+}].
\end{aligned}
$$

The optimization problem in theorem 3 can be rewritten as:

$$
\min_{z>0} c(z) = \begin{cases} z + \dfrac{1}{1-\varepsilon} f_2(z, a^{*}(z)) & \text{if } 0 \le z < \hat{z} \\ z & \text{if } z \ge \hat{z} \end{cases}
$$

where

$$
a^{*}(z) = \inf\{a \ge 0 : f_1(z,a) \le \nu\},
$$

and

$\hat{z}$ is the solution of $f_3(z) = \nu$. Theorem 5 can be reformulated as:

$$
\min_{z \in [0,c]} f_1(z, a^{*}(z)^{-1}),
$$

where

$$
a^{*}(z) = \inf\{a \ge 0 : f_2(z, a^{-1}) \le (c - z)(1 - \varepsilon)\}.
$$

We assume that $z > 0$, therefore $((S(T)-K)^+ - z)^+ = (S(T)-K-z)^+$. Then for a call option we have

$$f_1^{call}(z,a) = E^{\mathbb{Q}}[R(T)^{-1}(S(T)-K-z)^+ 1_{\{R(T)\frac{dP}{d\mathbb{Q}}>a\}}],$$
$$f_2^{call}(z,a) = E^{P}\left[(S(T)-K-z)^+ 1_{\{R(T)\frac{dP}{d\mathbb{Q}}\le a\}}\right],$$
$$f_3^{call}(z) = E^{\mathbb{Q}}[R(T)^{-1}(S(T)-K-z)^+].$$

Recall that $f_3^{call}(z)$ is the option pricing formula which has already been studied in section 3.4:

$$f_3^{call}(z) = S(0)N(d_1(z)) - (K+z)P(0,T)N(d_2(z))$$

where

$$d_1(z) = \frac{\log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) + \frac{1}{2}\int_0^T \Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2 + \sigma_2(u)^2\Big)du}{\sqrt{\int_0^T \Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2 + \sigma_2(u)^2\Big)du}}$$
$$d_2(z) = \frac{\log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) - \frac{1}{2}\int_0^T \Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2 + \sigma_2(u)^2\Big)du}{\sqrt{\int_0^T \Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2 + \sigma_2(u)^2\Big)du}}$$

Similarly, we can obtain the closed form solution for f_1 and f_2. Let $N_2(x,y) := N(x)N(y)$ be the distribution function of two independent standard normal random variables, we have

$$f_1^{call}(z,a) = S(0)N_2(\Sigma^{-1/2}\eta_1(z,a)) - (K+z)P(0,T)N_2(\Sigma^{-1/2}\eta_2(z,a)),$$

where

$$\eta_1(z,a) = \begin{pmatrix} -\log(aP(0,T)) + \Sigma_{12} - \frac{1}{2}\Sigma_{11} \\ \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) + \frac{1}{2}\Sigma_{22} \end{pmatrix},$$
$$\eta_2(z,a) = \begin{pmatrix} -\log(aP(0,T)) - \frac{1}{2}\Sigma_{11} \\ \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) - \frac{1}{2}\Sigma_{22} \end{pmatrix},$$

Σ is a 2×2 positive definite matrix given by

$$\Sigma_{11} = \int_0^T ((\theta_1(u) - \sigma^*(u,T))^2 + \theta_2(u)^2)du$$
$$\Sigma_{22} = \int_0^T (\sigma^*(u,T) + \sigma_1(u))^2 + \sigma_2(u)^2 dun$$
$$\Sigma_{12} = \Sigma_{21} = -\int_0^T (\theta_1(u) - \sigma^*(u,T))(\sigma^*(u,T) + \sigma_1(u))du - \int_0^T \theta_2(u)\sigma_2(u)du.$$

And

$$f_2^{call}(z,a) = \frac{S(0)}{P(0,T)} e^A N_2(\tilde{\Sigma}^{-1/2}\zeta_1(z,a)) - (K+z)N_2(\tilde{\Sigma}^{-1/2}\zeta_2(z,a)),$$

where

$$\zeta_1(z,a) = \begin{pmatrix} \log(aP(0,T)) - A - \frac{1}{2}\Sigma_{11} \\ \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) + A + \frac{1}{2}\Sigma_{22} \end{pmatrix},$$

$$\zeta_2(z,a) = \begin{pmatrix} \log(aP(0,T)) - \int_0^T \Big(\alpha^*(u,T) + \frac{1}{2}(\theta_1(u)^2 + \theta_2(u)^2)\Big)du \\ \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) + \int_0^T \Big(\mu(u) + \alpha^*(u,T) - \frac{1}{2}(\sigma_1(u)^2 + \sigma_2(u)^2)\Big)du \end{pmatrix},$$

$$A = \int_0^T \Big(\mu(u) + \alpha^*(u,T) + \frac{1}{2}\sigma^*(u,T)^2 + \sigma^*(u,T)\sigma_1(u)\Big)du$$

and

$$\tilde{\Sigma} := \begin{pmatrix} \Sigma_{11} & -\Sigma_{12} \\ -\Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

3.6 Numerical Example

In this section we apply our results from previous sections to the one factor Hull-White model in which the short rate dynamic is modeled by

$$dr(t) = (b(t) - ar(t))dt + \sigma_f dB_1^{\mathrm{P}}(t),$$

where a and σ_f are the parameters regulating mean reverting and volatility respectively, and $b(t)$ is determined by:

$$b(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma_f^2}{2a}(1 - e^{-2at}).$$

By solving the differential equation we obtain

$$r(t) = f(0,t) + \frac{\sigma^2}{2a}(1 - e^{-at})^2 + \sigma_f \int_0^t e^{a(u-t)} dB_1^{\mathrm{P}}(u).$$

Note that $f(0,t)$ is generally fitted by an initial yield curve. In this example since we consider 3-month short term option, $f(0,t)$ can be simplify assumed to be a constant. For the spot rate data we download overnight ICE LIBOR rate for US dollar from Bloomberg, ranging from 3/11/2012 to 3/10/2013. The generalized method of moments (GMM) is adopted to fit a and σ_f, see (park, 2004) for detail. For the stock price, we download the daily returns of S\&P 500 index from 'Yahoo! finance', ranging from 3/11/2012 to 3/10/2013. Maximum likelihood method is employed to estimate both μ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. The correlation between the two Brownian motions is assumed to be 0.5.

For the call option of S\&P 500 index of three different maturities: $T = 30, 60, 90$, given the strike price $K = S_0$, we tackle the CVaR minimization problem and the hedging costs minimization problem based on the results in section \ref{sec: 3.1} and \ref{sec: 3.2}. The results are shown in figure \ref{fig:1} and figure \ref{fig:2}.

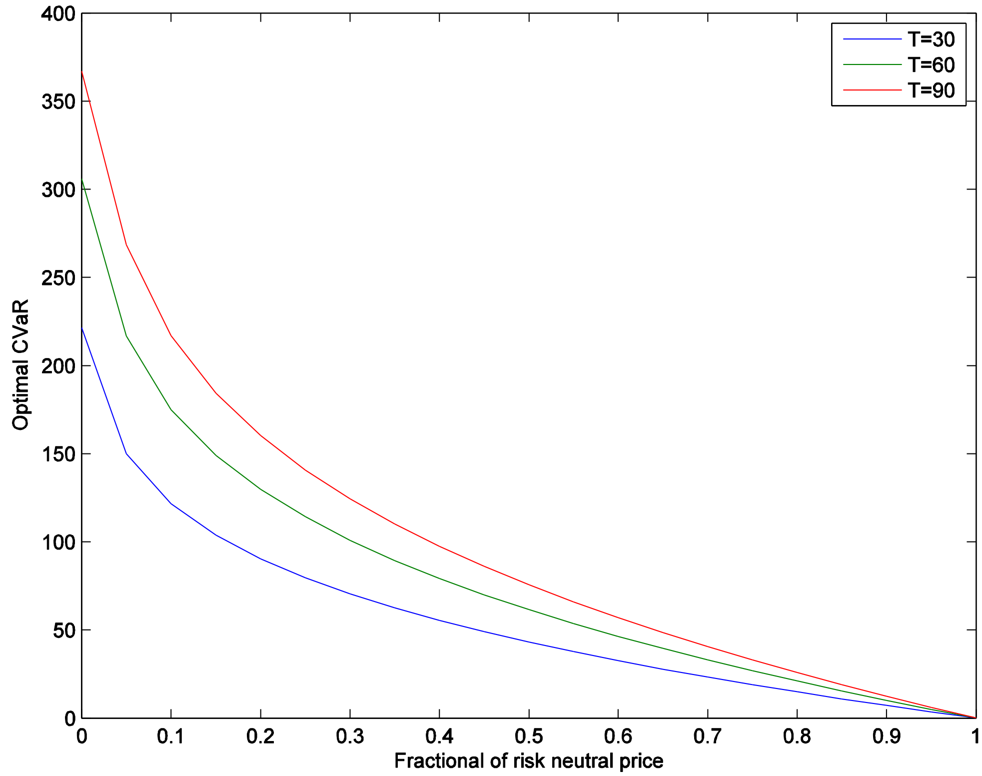
400
350
300
250
200
150
100
50
0
Optimal CVaR
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Fractional of risk neutral price
T=30
T=60
T=90

Figure 6

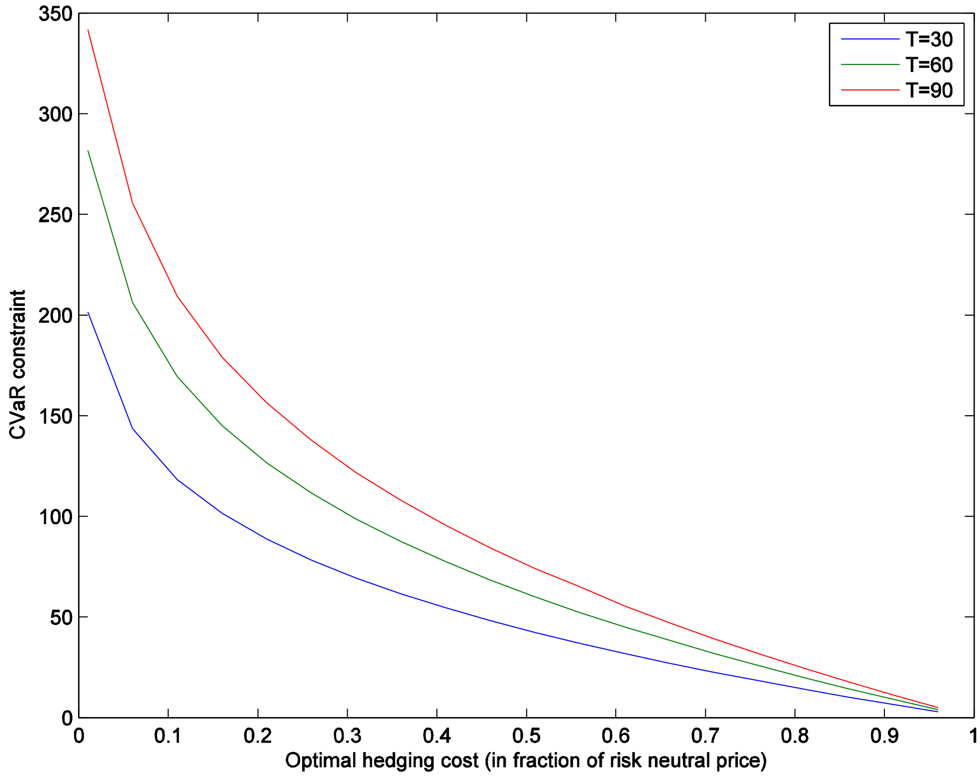

Figure 7

Figure 6 plots the optimal CVaRs with respect to the constraint on the initial costs ν, which is given by the fraction of the risk neutral price. One can observe that the optimal CVaRs decrease to zero when ν equals to the risk neutral price, meaning that the call option is hedged perfectly. CVaRs reach to the maximum values when the fraction equals to zero, implying that the investor is exposed to the full risk. The optimal CVaR comes near zero when the allocation of initial wealth approaches to the risk neutral price in the case of perfect hedging. The optimal CVaRs also increases as the the maturity T increases, showing that the risk is larger for portfolios held longer.

Figure 7 plots the optimal hedging costs according to the constraint c on CVaR. We can see that figure \ref{fig:2} demonstrates similar trends as in figure \ref{fig:1}. When c reaches to 350, the resulting risk is close the the maximum CVaR, and its corresponding optimal hedging cost equals to zero. When the CVaR value is fixed at zero, we obtain the perfect hedging. The minimal optimal hedge equals to the risk neutral price.

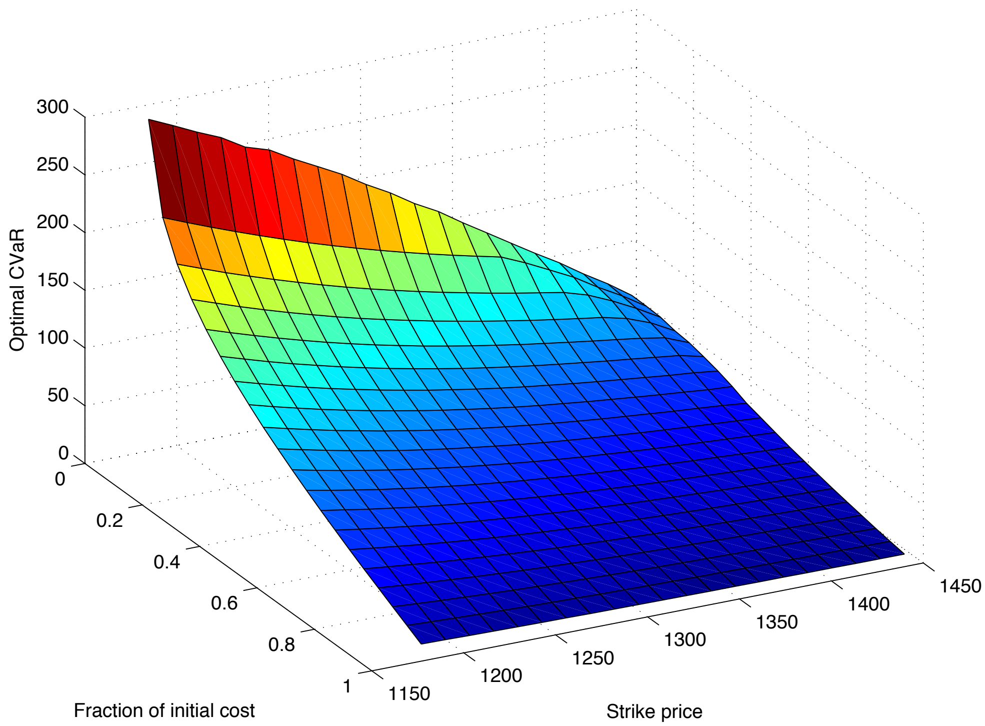

Figure 8

Figure 8 plots the optimal CVaRs with respect to the constraint on the initial costs ν for options with various price. We can observe that the optimal CVaRs decrease to zero when ν equals to the risk neutral price. CVaRs reach to the maximum values when the fraction equals to zero. When the strike price is extreme high, the optimal CVaR comes near zero as such an call option is very unlikely to be exercised. Also, options with higher price become less sensitive to the amount of initial capital being available.

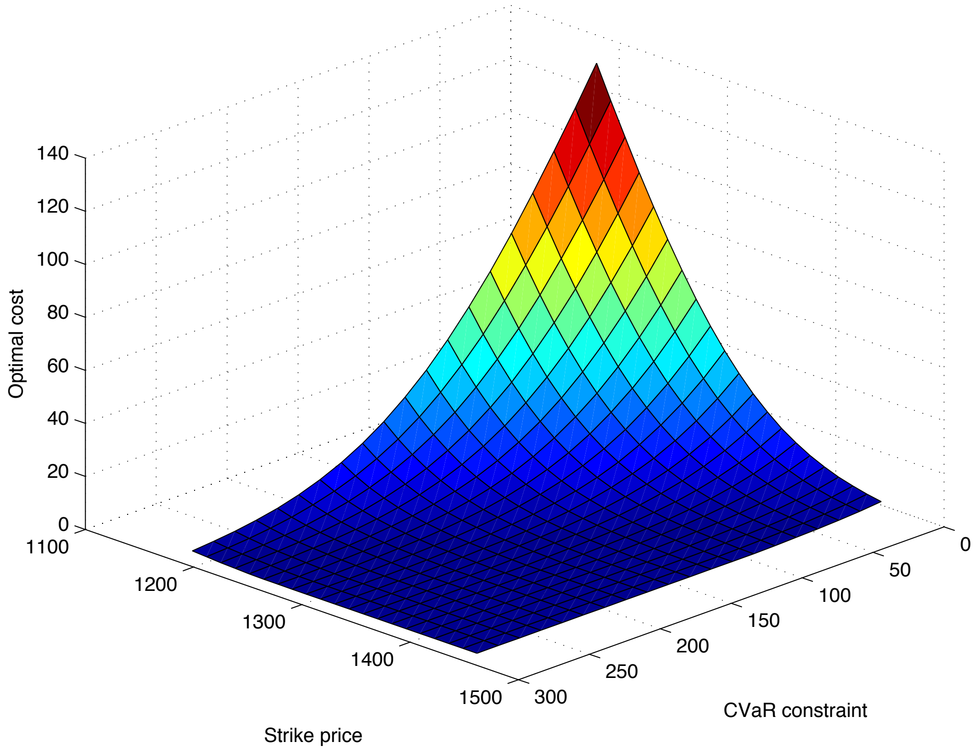

Figure 9

Figure 9 plots the optimal hedging costs of optional with difference strike price, according to the constraint *c* on CVaR. We can see that When *c* is approaching to 0, the resulting hedging cost get close to the maximum CVaR. When the CVaR value is fixed at zero, we obtain the perfect hedging. The minimal optimal hedge equals to the risk neutral price.

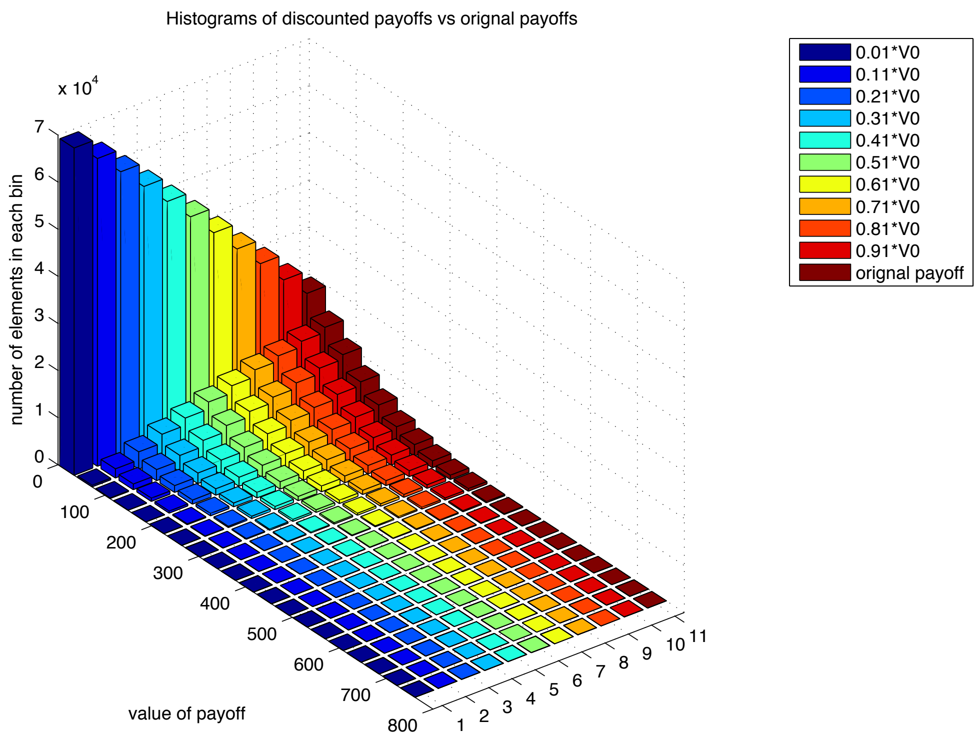

Figure 10

Figure 10 shows the relationship between the original contingent claim and the fractional claim we perform perfect hedge on. Each colour corresponds to a particular risk budget. We see that when the risk budge is close to the amount required by perfect hedge, the distribution of the fractional claim concentrates to that of H(T).

3.7 The Neyman – Pearson Lemma

Neyman-Pearson Lemma plays an important rule in deriving theorems in Chater 3. Let P and Q be two probability measures on $(\Omega, \mathcal{F})$ such that

$$\mathrm{P[A]} = \mathrm{P[A} \cap \mathrm{N]} + \int_{\mathrm{A}} \frac{\mathrm{dP}}{\mathrm{dQ}} \mathrm{dQ}, \quad \mathrm{A} \in \mathcal{F}$$

The following theorem (Follmer, 2000) shows the Lebesgue decomposition of P with respect to Q.

Theorem 1. For any two probability measures Q and P on $(\Omega, \mathcal{F})$, there exists a set $\mathrm{N} \in \mathrm{F}$ with $\mathrm{Q}\,[\mathrm{N}] = 0$ and a $\mathcal{F}$– measurable function $\varphi \geq 0$ such that

$$\mathrm{P[A]} = \mathrm{P[A} \cap \mathrm{N]} + \int_{\mathrm{A}} \varphi \mathrm{dQ} \quad \text{for all } \mathrm{A} \in \mathcal{F}$$

One writes

$$\frac{\mathrm{dP}}{\mathrm{dQ}} := \begin{cases} \varphi & \text{on } \mathrm{N}^{c} \\ +\infty & \text{on } \mathrm{N} \end{cases}$$

For fixed $\mathrm{c} \geq 0$, we have

$$\mathrm{A}^{0} := \left\{ \frac{\mathrm{dP}}{\mathrm{dQ}} > c \right\}$$

here we use $\frac{\mathrm{dP}}{\mathrm{dQ}} = \infty$ on N.

The Neyman-Pearson lemma is stated as follows(Follmer, 2000):

Proposition 1 (Neyman – Pearson lemma). If $\mathrm{A} \in \mathcal{F}$ is such that $\mathrm{Q[A]} \leq \mathrm{Q[A^0]}$, then $\mathrm{P[A]} \leq \mathrm{P[A^0]}$.

Proof. Let $\mathrm{F} := \mathrm{I}_{\mathrm{A}^0} - \mathrm{I}_{\mathrm{A}}$. Then $\mathrm{F} \geq 0$ on N, and $\mathrm{F} \cdot \left(\frac{\mathrm{dP}}{\mathrm{dQ}} - \mathrm{c} \right) \geq 0$. Hence

$$P[A^0] - P[A] = \int F dP = \int_N F\, dP + \int F \frac{dP}{dQ} dQ \geq c \int F dQ = c(Q[A^0] - Q[A])$$

the proposition is proved.

Remark 1. From statistical test theory, A^0 can be viewed as the likelihood quotient test of Q (the null hypothesis) against P (the alternative hypothesis). That is, the null hypothesis Q is rejected if the outcome ω of an test is in A^0. In such a statistical test, there are two kinds of error that can occur. An error is called type 1 error if the null hypotheses is rejected, although the true probability is Q. Similarly, an error is called type 2 error if the null hypothesis is not rejected, despite that the true probability is P. $Q[A^0]$ is the probability of a type 1 error and is usually called the significance level or the size of the statistical test A^0. Then $P[(A^0)^c]$ is the probability of a type 2 error. The power of the test A^0 is given by the complementary probability $P[A^0] = 1 - P[(A^0)^c]$. Therefore, we can compare the likelihood quotient test to the set A in the above proposition, which can be viewed as another statistical test. Hence, we can restate the proposition as follows: the maximal power of a likelihood quotient test can be obtained on its significance level.

Indicator functions of sets have only two values: 0 and 1. The proposition stated above can be generalized by considering $\mathcal{F}$-measurable function $\psi: \Omega \to [0,1]$ and Let $\mathcal{R}$ be the set which contains all such functions.

Theorem 2. (Follmer, 2000) Let $\Pi := \frac{1}{2}(P + Q)$, and define the density $\varphi := dP/dQ$ as above.

(a) Take $c \geq 0$, and suppose that $\psi^0 \in \mathcal{R}$ satisfies Π- a.s.

$$\psi^0 = \begin{cases} 1 & \text{on } \{\varphi > c\} \\ 0 & \text{on } \{\varphi < c\} \end{cases}$$

Then for any $\psi \in \mathcal{R}$,

$$\int \psi dQ \leq \int \psi^0 dQ \Rightarrow \int \psi dP \leq \int \psi^0\, dP$$

(b) For any $\alpha_0 \in (0,1)$ there is some $\psi^0 \in \mathcal{R}$ of the form

$$\psi^0 = \begin{cases} 1 & \text{on } \{\varphi > c\} \\ 0 & \text{on } \{\varphi < c\} \end{cases}$$

such that $\int \psi^0\, dQ = \alpha_0$. More precisely, if c is an $(1-\alpha_0)$-quantile of φ under Q, we can define ψ^0 by

$$\psi^0 = I_{\{\varphi > c\}} + \kappa I_{\{\varphi = c\}}$$

Where κ is defined as

$$\kappa := \begin{cases} 0 & \text{if } Q[\varphi = c] = 0 \\ \dfrac{\alpha_0 - Q[\varphi > c]}{Q[\varphi = c]} & \textit{otherwise} \end{cases}$$

(c) Any $\psi^0 \in \mathcal{R}$ satisfying

$$\int \psi dQ \leq \int \psi^0 dQ \Rightarrow \int \psi dP \leq \int \psi^0\, dP$$

is of the form

$$\psi^0 = \begin{cases} 1 & \text{on } \{\varphi > c\} \\ 0 & \text{on } \{\varphi < c\} \end{cases}$$

For some $c \geq 0$.

Proof. (a): Take $F := \psi^0 - \psi$ and repeat the proof of Proposition A. 28.

(b): Let F denote the distribution function of φ under Q. Then $Q[\varphi > c] = 1 - F(c) \leq \alpha_0$ and

$$Q[\varphi = c] = F(c) - F(c-) \geq F(c) - 1 + \alpha_0 = \alpha_0 - Q[\varphi > c]$$

Hence $0 \leq \kappa \leq 1$ and ψ^0 belongs to $\mathcal{R}$. The fact that $\int \psi^0 dQ = \alpha_0$ is obvious.

(c): Suppose that ψ^* satisfies

$$\int \psi\, dQ \leq \int \psi^*\, dQ \Rightarrow \int \psi\, dP \leq \int \psi^*\, dP$$

The case in which $\alpha_0 \coloneqq \int \psi^* dQ$ equals 0 or 1 are trivial. For $0 < \alpha_0 < 1$, we can take ψ^0 as in part (b). Then $\alpha_0 \coloneqq \int \psi^* dQ = \int \psi^0 dQ$. One also has that $\int \psi^* \, dP = \int \psi^0 \, dP$, as can be seen by applying

$$\int \psi dQ \leq \int \psi^0 dQ \implies \int \psi dP \leq \int \psi^0 \, dP$$

to both ψ^* and ψ^0 with reversed roles. Hence, for $f \coloneqq \psi^0 - \psi^*$ and $N = \{\varphi = \infty\}$,

$$0 = \int f \, dP - c \int f dQ = \int_N f dP + \int \ f \cdot (\varphi - c) dQ$$

But

$$\psi^0 = \begin{cases} 1 & \text{on } \{\varphi > c\} \\ 0 & \text{on } \{\varphi < c\} \end{cases}$$

implies that both $f \geq 0$ P-a.s. on N, and $f \cdot (\varphi - c) \geq 0$ Q-a.s. Hence f vanishes Π-a.s. on $\{\varphi \neq c\}$.

Remark 2. In Remark 1, a randomized statistical test is given by an element ψ of $\mathcal{R}$. Let ω be the outcome of a statistical test, and denote the probability $p \coloneqq \psi(\omega)$, which is the probability of rejection of the null hypothesis. Power and significant level of a randomized test are given. We call a test of the following form

$$\psi^0 = \begin{cases} 1 & \text{on } \{\varphi > c\} \\ 0 & \text{on } \{\varphi < c\} \end{cases}$$

a generalized likelihood quotient test. Hence, we can state the general Neyman-Pearson lemma in Theorem 2 as follows: On the significance level a randomized test will have its maximal power, and it is satisfied if and only if the randomized test is a generalized likelihood quotient test.

3.8 Concluding Remarks

The purpose of this section is to construct CVaR hedging strategies under stochastic interest rate environment. We first generalize the results presented by (melnikov, 2012) and then add the stochastic interest rate component. Modeling the stochastic movements of interest rates is essential in pricing and hedging interest-rate-sensitive price; the result of this paper can be applied on pricing a variety of equity-linked life insurance products, for example. The Neyman Pearson lemma shows that the optimal strategy of CVaR hedging is a perfect hedge for an adjusted claim.

To illustrate this hedging technique we consider a European call option, explicit formulas for solving CVaR minimization problems under HJM framework are derived. Note that the geometric Brownian motion is employed in monitoring the asset dynamics. As the risk of Brownian motion is completely determined by its variance, the advantage of CVaR is not shown in this thin-tailed model. For future work we consider improving CVaR hedging by adopting heavy-tailed models and adding jump components.

4. Nominal and Robust Portfolio Optimization

Conditional Value-at-Risk (CVaR) has attracted much attention in recent years. Rockafellar and Uryasev (2000, 2002) formulated CVaR minimization problem, which can be solved by minimizing a more tractable auxiliary function without predetermining the corresponding VaR, and VaR can be calculated as a by-product at the same time. Such a formulation usually results in convex programs, and even linear programs; since then CVaR is becoming more and more popular in financial management. Konno, Waki and Yuuki (2002) explained the significance of using CVaR in reducing downside risk in portfolio optimization.

Recently, many researchers have paid more attention to the issue of lack of robustness of CVaR optimization. It is well-known that the portfolio decision is very sensitive to the mean and the covariance matrix, thus one cannot neglect the modelling risk arises due to the uncertainty of the underlying probability distribution. The distribution uncertainty occurs in many situations, for example, when there are limited data samples possessed or the data samples are not stable. Therefore, instead of the precise information on the mean and the covariance matrix of asset returns, some types of uncertainties are introduced, such as polytopic uncertainty, box uncertainty and ellipsoidal uncertainty, in the parameters involved in the mean and the covariance matrix, and then translated the problem into semidefinite programs that can be solved by interior-point algorithms developed in recent years. El Ghaoui (2003) studied the robust portfolio optimization using worst-case VaR.

This section introduces the concept of worst-case CVaR and followed by investigating the minimization of worst-case CVaR associated with mixture distribution uncertainty. At the end of this section we present the application of worst-case CVaR to robust portfolio optimization together with an illustrative numerical example.

4.1 Worst-Case Conditional Value-at-Risk

The Average Value at Risk (AVaR), also called conditional value-at-risk or expected shortfall, has become very popular and displaced the traditional VaR due to its coherence. AVaR represents the information about the magnitude of the losses larger than the VaR level. The AVaR at tail probability ε is defined as the average of the VaRs which are larger than the VaR at tail probability α. Refer to Chapter Three for formal definition.

Let x denotes a decision vector ($x \in \mathrm{X}$), and $l(x, y)$ represents the loss with random vector y. Assume that y follows a continuous distribution with its density function denoted as $f(\cdot)$. We also assume $E(|l(x, y)| < \infty$ for each $x \in \mathrm{X}$, so that CVaR and worst-case CVaR are probably defined. The probability of $l(x, y)$, for any given x, not exceeding a threshold α is given as

$$WCVaR_{\beta}(x) \triangleq \sup_{p(\cdot)\in P} CVaR_{\beta}(x) = \min_{\alpha\in R} \max_{\lambda\in\Lambda} \sum_{i=1}^{3} \lambda_i F_{\beta}^{i}(x,\alpha)$$

F_{β}^{i} is the axillary function in calculating CVaR, i denotes the ith likelihood function with weight λ_i.

As we pointed out that the coherent risk measure CVaR is a coherent risk measure, so does worst-case CVaR.

4.2 Numerical Example

In this subsection, we consider a numerical example to illustrate the robust portfolio optimization problems. We select fifty tickers from S&P 500, and sample the data for the frequency of 5 minute.

Plot a sample return data as in figure 11, we roughly observed that the behavior of returns is not consistent among different time periods. The return behaviours of other tickers are very similar. If we divide our observations into three sub-intervals, the expected mean and variance of returns of one period are very different from the others. Therefore, we assume that uncertain vector y is sampled by three corresponding likelihood function, i.e., rather than using a nominal distribution, we use a mixture distribution (f1, f2, f3, each with corresponding normalized weights (using sample days of sub-interval)).

Consequently, we might consider CVaR not reliable since the underlying assumption that the probability distribution is precisely known to be a nominal one is violated. We therefore perform a worst-case CVaR minimization, which will lead to robust portfolio optimization. Let μ is the minimum expected portfolio return, y^i denotes the expected value of y with respect to its interval-likelihood distribution; f(x,y) denote the loss associated with decision vector x and random vector y.

Let θ be the solution of formula (1.1), our task is:

$$\min_{(\alpha,\theta)\in R\times R}\{\theta : \sum_{i=1}^{3}\lambda_i F_\beta^i(x,\alpha) \le \theta, \forall\lambda\in\Lambda\} \tag{1.1}$$

s.t.

e^T* r= Wo;

$$\underline{X} = (-1,\dots,-1) \le X \le \overline{X}(1,\dots,1);$$

$$X^T y^i \ge \mu,$$

$$\alpha + \frac{1}{1-\beta}(\Pi^i)^T u^i \le \theta$$

where $\Pi^i = (\Pi_1^i, \ldots, \Pi_s^i)$, Π_k^i is the probability according to the k---th sample with respect to the i-th likelihood distribution.; we introduce u^i as the auxiliary vector; $u_k^i \ge f(x, y_k^i) - \alpha$; 6) $u_k^i \ge 0$, k=1,2,3. For minimum expected return μ=0. 0055, and CI level (β)=95%, we obtain the following equity curves.

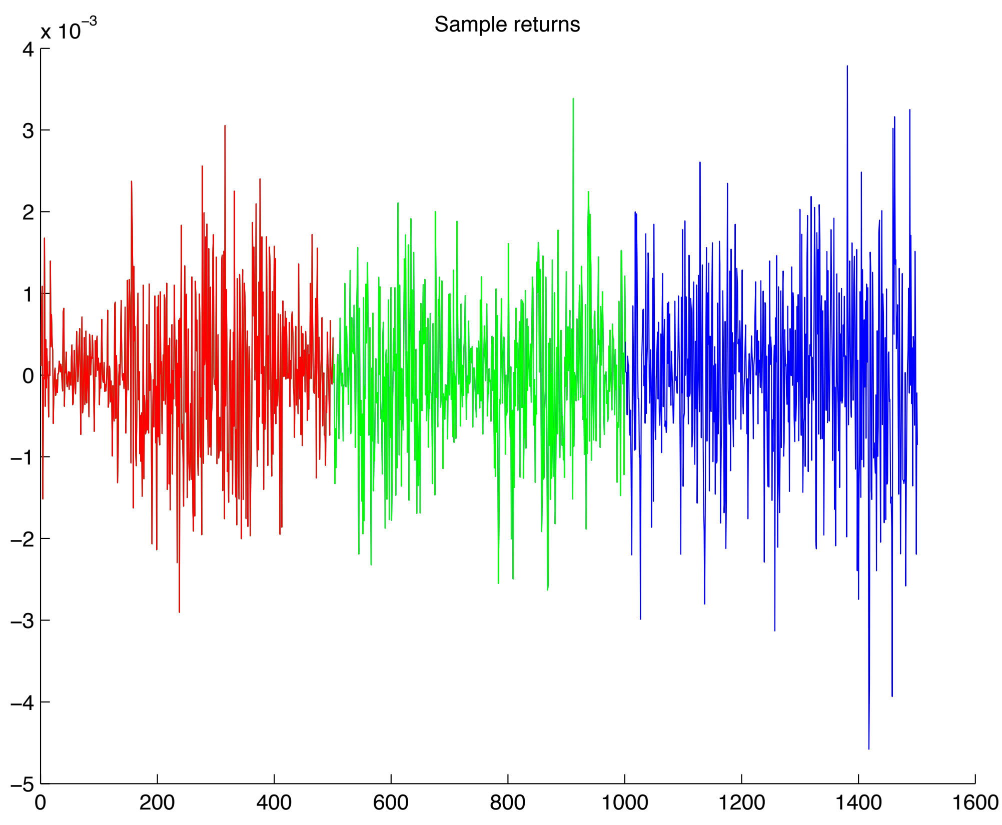

Figure 11

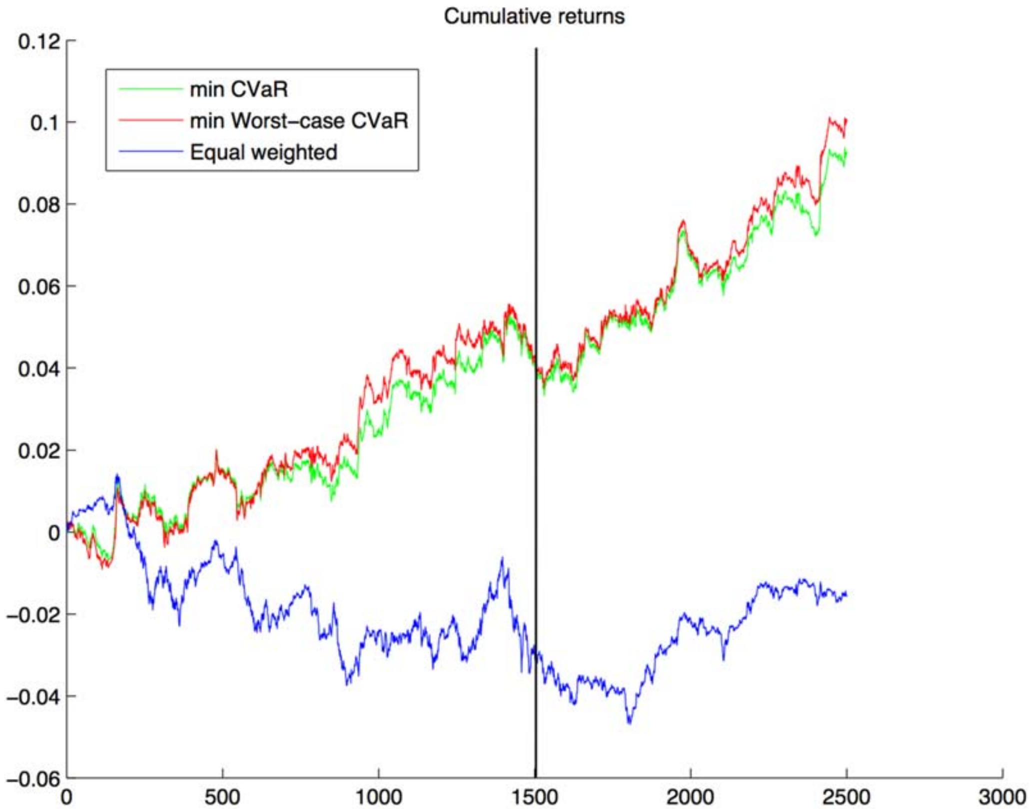

Figure 12

The above figure illustrates the evolution of the values of the robust optimal portfolio (via WCVaR) and the nominal optimal portfolio (via CVaR). It shows that the robust optimal portfolio almost always outperforms the nominal optimal portfolio.

Appendix

Appendix 1

Fix $X \in \chi$ such that $|\rho(X)| < +\infty$. By translation invariance one has

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.$$

If ρ is positive homogeneous, then $\rho(0) = \rho(n \cdot 0) = n\rho(0)$, so $\rho(0) = 0$,

Therefore, $\rho(c) = \rho(0) - c = -c, c \in {}^\circ$ since $\rho(X + \rho(X)) = 0$.

Appendix 2

(a) a) is straightforward.

(b) cash invariance implies that for $\mathrm{X} \in \chi$,

$$
\begin{aligned}
\inf\{m \in {}^{\circ} \mid m + \mathrm{X} \in \mathrm{A}_{p}\} &= \inf\{m \in {}^{\circ} \mid \rho(m + X) \leq 0\} \\
&= \inf\{m \in {}^{\circ} \mid \rho(X) \leq m\} \\
&= \rho(X)
\end{aligned}
$$

(c) A is clearly convex if ρ is a convex measure of risk. The converse will follow from Proposition 2.6 together with (0.6).

(d) Clearly, positive homogeneity of ρ implies that A is a cone. The converse follows as in (c).

Appendix 3

(a) It is straightforward to verify that ρ_{A} satisfies cash invariance and monotonicity. We show next that ρ_{A} takes only finite values. To this end, fix some Y in the non-empty set A. For $\mathrm{X} \in \chi$ given, there exists a finite number m with $m + \mathrm{X} > Y$, for X and Y are both bounded. Then

$$\rho_{\mathrm{A}}(X) - m = \rho_{\mathrm{A}}(m + X) \leq \rho_{\mathrm{A}}(Y) \leq 0,$$

and hence $\rho_{\mathrm{A}}(X) \leq m < \infty$. Note that (2) is equivalent to $\rho_{\mathrm{A}}(0) > -\infty$.

To show that $\rho_{\mathrm{A}}(\mathrm{X}) > -\infty$ for arbitrary $\mathrm{X} \in \chi$, we take m' such that $\mathrm{X} + m' \leq 0$ and conclude by monotonicity and cash invariance that $\rho_{\mathrm{A}}(\mathrm{X}) \geq \rho_{\mathrm{A}}(0) + m' > -\infty$.

(b) Suppose that $\mathrm{X}_1, \mathrm{X}_2 \in \chi$ and that $m_1, m_2 \in °$ are such that $m_i + \mathrm{X}_i \in \mathrm{A}$. If $\lambda \in [0,1]$, then the convexity of A implies that $\lambda(m_1 + \mathrm{X}_1) + (1-\lambda)(m_2 + \mathrm{X}_2) \in \mathrm{A}$. Thus, by the cash invariance of ρ_{A},

$$\begin{aligned} 0 &\geq \rho_{\mathrm{A}}(\lambda(m_1 + \mathrm{X}_1) + (1-\lambda)(m_2 + \mathrm{X}_2)) \\ &= \rho_{\mathrm{A}}(\lambda \mathrm{X}_1 + (1-\lambda)\mathrm{X}_2) - (\lambda m_1 + (1-\lambda)m_2), \end{aligned}$$

and the convexity of ρ_{A} follows.

(c) As in the proof of convexity, we obtain that $\rho_{\mathrm{A}}(\lambda \mathrm{X}) \leq \lambda \rho_{\mathrm{A}}(\mathrm{X})$ for $\lambda \geq 0$ if A is a cone. To prove the converse inequality, let $m < \rho_{\mathrm{A}}(\mathrm{X})$. Then $m + \mathrm{X} \notin \mathrm{A}$. Thus, $\lambda m < \rho_{\mathrm{A}}(\lambda \mathrm{X})$, and (c) follows.

(d) The inclusion $\mathrm{A} \in \mathrm{A}_{\rho\mathrm{A}}$ is obvious, and Proposition 1.5 implies that A is $\|\cdot\| - closed$ as soon as $\mathrm{A} = \mathrm{A}_{\rho\mathrm{A}}$. Conversely, assume that A is $\|\cdot\| - closed$. We have to show that $X \notin A$, there is some $\lambda \in (0,1)$ such that $\lambda m + (1-\lambda)X \notin A$. Thus, $0 \le \rho_A(\lambda m + (1-\lambda)X) = \rho_A((1-\lambda)X) - \lambda m$. Since ρ_A is a monetary measure of risk, we have

$$|\rho_A((1-\lambda)X) - \rho_A(X)| \le \lambda \|X\|.$$

Hence,

$$\rho_A(X) \ge \rho_A((1-\lambda)X) - \lambda \|X\| \ge \lambda(m - \|X\|) > 0$$

Appendix 4

Here we review the valuation of European call option under HJM model. Recall that the price of the option is given by:

$$\begin{aligned} V_0 &= E^{\S}[R(T)^{-1}(S(T)-K)^{+}] \\ &= E^{\S}[R(T)^{-1}S(T)1_{\{S(T)>K\}}] - E^{\S}[R(T)^{-1}K1_{\{S(T)>K\}}]. \end{aligned}$$

First define

$$\frac{d\S_1}{d\S} := \exp\Big(-\frac{1}{2}\int_0^T (\sigma_1(u)^2+\sigma_2(u)^2)du + \int_0^T \sigma_1(u)dB_1^{\S}(u) + \int_0^T \sigma_2(u)dB_2^{\S}(u)\Big),$$

then under we have

$$\begin{aligned} B_1^{\S}(t) &= B_1^{\S_1}(t) + \int_0^t \sigma_1(u)du, \\ B_2^{\S}(t) &= B_2^{\S_1}(t) + \int_0^t \sigma_2(u)du, \end{aligned}$$

where $B_1^{\S_1}$ and $B_2^{\S_1}$ are two independent Brownian motions under $\S_1$.

Thus

$$\begin{aligned} S(T) &= S(0)R(T)\exp\Big(-\frac{1}{2}\int_0^T (\sigma_1(u)^2+\sigma_2(u)^2)du + \int_0^T \sigma_1(u)dB_1^{\S}(u) + \int_0^T \sigma_2(u)dB_2^{\S}(u)\Big) \\ &= \frac{S(0)}{P(0,T)}\exp\Big(\frac{1}{2}\int_0^T \Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2 + \sigma_2(u)^2\Big)du + \int_0^T (\sigma^*(u,T)+\sigma_1(u))dB_1^{\S_1}(u) + \int_0^T \sigma_2(u)dB_2^{\S_1}(u)\Big). \end{aligned}$$

The event $S(T)>K$ implies that

$$\begin{aligned} &\int_0^T (\sigma^*(u,T)+\sigma_1(u))dB_1^{\S_1}(u) + \int_0^T \sigma_2(u)dB_2^{\S_1}(u) \\ &> \log\Big(\frac{P(0,T)K}{S(0)}\Big) - \frac{1}{2}\int_0^T \Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2 + \sigma_2(u)^2\Big)du. \end{aligned}$$

Thus we have

$$E^{\S}[R(T)^{-1}S(T)1_{\{S(T)>K\}}] = S(0)E^{\S_1}[1_{\{S(T)>K\}}] = S(0)N(d_1),$$

On the other side, define

$$\frac{d\S_2}{d\S} := \exp\left(-\int_0^T \sigma^*(u,T)dB_1^{\S}(u) - \frac{1}{2}\int_0^T \sigma^*(u,T)^2 du\right),$$

Then we have

$$B_1^{\S}(t) = B_1^{\S_2}(t) - \int_0^t \sigma^*(u,T)du$$
$$B_2^{\S}(t) = B_2^{\S_2}(t)$$

where $B_1^{\S_2}$ and $B_2^{\S_2}$ are two independent Brownian motions under $\S_2$. Thus

$$S(T) = \frac{S(0)}{P(0,T)}\exp\left(\frac{1}{2}\int_0^T (\sigma^*(u,T)^2 - \sigma_1(u)^2 - \sigma_2(u)^2)du + \int_0^t (\sigma^*(u,T) + \sigma_1(u))dB_1^{\S}(u) + \int_0^t \sigma_2(u)dB_2^{\S}(u)\right)$$
$$= \frac{S(0)}{P(0,T)}\exp\left(-\frac{1}{2}\int_0^T \left(\left(\sigma^*(u,T) + \sigma_1(u)\right)^2 + \sigma_2(u)^2\right)du + \int_0^t (\sigma^*(u,T) + \sigma_1(u))dB_1^{\S_2}(u) + \int_0^t \sigma_2(u)dB_2^{\S_2}(u)\right).$$

The event $S(T) > K$ implies that

$$\int_0^T (\sigma^*(u,T) + \sigma_1(u))dB_1^{\S_2}(u) + \int_0^T \sigma_2(u)dB_2^{\S_2}(u)$$
$$> \log\left(\frac{P(0,T)K}{S(0)}\right) + \frac{1}{2}\int_0^T \left(\left(\sigma^*(u,T) + \sigma_1(u)\right)^2 + \sigma_2(u)^2\right)du.$$

Then

$$E^{\S}[R(T)^{-1}K1_{\{S(T)>K\}}] = KP(0,T)N(d_2)$$

Appendix 5

We can decompose f_1^{call} to be:

$$f_1^{call}(z,a) = E^{\S}[R(T)^{-1}(S(T)-K-z)^{+}1_{\{R(T)\frac{d\mathrm{P}}{d\S}>a\}}]$$
$$= S(0)E^{\S_1}[1_{\{S(T)>K+z\}}1_{\{R(T)\frac{d\mathrm{P}}{d\S}>a\}}] - (K+z)P(0,T)E^{\S_2}\left[1_{S(T)>K+z}1_{\{R(T)\frac{d\mathrm{P}}{d\S}>a\}}\right],$$

where $\S_1$ and $\S_2$ are defined in the previous section. Then apply (\ref{eq:Q1B1}) and (\ref{eq:Q1B2}) we obtain the dynamic of $R(T)d\mathrm{P}/d\S$ under $\S_1$:

$$R(T)\frac{d\mathrm{P}}{d\S} = \frac{1}{P(0,T)}\exp\Big(\frac{1}{2}\int_0^T(\sigma^*(u,T)^2-\theta_1(u)^2-\theta_2(u)^2)du$$
$$+\int_0^T(\sigma^*(u,T)-\theta_1(u))dB_1^{\S}(u)-\int_0^T\theta_2(u)dB_2^{\S}(u)\Big)$$
$$= \frac{1}{P(0,T)}\exp\Big(\Sigma_{12}-\frac{1}{2}\Sigma_{11}-W_1\Big),$$

where Σ_{11}, Σ_{12} are given by (\ref{eq:Sigma11}), (\ref{eq:Sigma12}), and

$$W_1 :\overset{(d)}{=} \int_0^T(\theta_1(u)-\sigma^*(u,T))dB_1^{\S_1}(u)+\int_0^T\theta_2(u)dB_2^{\S_1}(u)$$
$$< -\log(aP(0,T))+\frac{1}{2}\int_0^T(\sigma^*(u,T)^2-\theta_1(u)^2-\theta_2(u)^2+2(\sigma^*(u,T)-\theta_1(u))\sigma_1(u)-2\theta_2(u)\sigma_2(u))du.$$

Therefore $R(T)d\mathrm{P}/d\S > a$ implies

$$W_1 < -\log(aP(0,T))+\Sigma_{12}-\frac{1}{2}\Sigma_{11}.$$

On the other side, recall $S(T)>K+z$ implies

$$W_2 :\overset{(d)}{=} -\int_0^T(\sigma^*(u,T)+\sigma_1(u))dB_1^{\S_1}(u)-\int_0^T\sigma_2(u)dB_2^{\S_1}(u)$$
$$< \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big)+\frac{1}{2}\int_0^T\Big(\big(\sigma^*(u,T)+\sigma_1(u)\big)^2+\sigma_2(u)^2\Big)du = \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big)+\frac{1}{2}\Sigma_{22},$$

Note that (W_1, W_2) are binormal distributed with zero mean and covariance matrix Σ. It is easy to compute that the entries of Σ

$$
\begin{aligned}
\Sigma_{11} &= E[W_1^2] = \int_0^T ((\theta_1(u) - \sigma^*(u,T))^2 + \theta_2(u)^2) du, \\
\Sigma_{22} &= E[W_2^2] = \int_0^T (\sigma^*(u,T) + \sigma_1(u))^2 + \sigma_2(u)^2 du, \\
\Sigma_{12} &= \Sigma_{21} = E[W_1 W_2] = -\int_0^T (\theta_1(u) - \sigma^*(u,T))(\sigma^*(u,T) + \sigma_1(u)) du - \int_0^T \theta_2(u)\sigma_2(u) du.
\end{aligned}
$$

Thus we have:

$$S(0)E^{\S_1}[1_{S(T)>K+z} 1_{\{R(T)\frac{dP}{d\S}>c\}}] = S(0)N_2(\Sigma^{-1/2}\eta_1(z,a)),$$

where $\eta_1(z,a)$ is given by (\ref{eq:eta1}). Similarly under $\S_2$ we have

$$
\begin{aligned}
R(T)\frac{dP}{d\S} &= \frac{1}{P(0,T)} \exp\Big(-\frac{1}{2}\int_0^T ((\theta_1(u) - \sigma^*(u,T))^2 + \theta_2(u)^2) du + \\
&\int_0^T (\sigma^*(u,T) - \theta_1(u)) dB_1^{\S_2}(u) - \int_0^T \theta_2(u) dB_2^{\S_2}(u)\Big) \\
&= \frac{1}{P(0,T)} \exp\Big(-\frac{1}{2}\Sigma_{11} - W_1\Big),
\end{aligned}
$$

and that

$R(T)dP/d\S > a$ implies

$W_1 < -\log(aP(0,T)) - \frac{1}{2}\Sigma_{11}$.And recall that $S(T) > K + z$ implies

$$
\begin{aligned}
W_2 &< \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) - \frac{1}{2}\int_0^T \Big(\big(\sigma^*(u,T) + \sigma_1(u)\big)^2 + \sigma_2(u)^2\Big) du \\
&= \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) - \frac{1}{2}\Sigma_{22}.
\end{aligned}
$$

Note that here we keep use (W_1, W_2) to denote a bivariate normal random vector under $\S_2$ instead of $\S_1$ for simplicity. Thus we have

$$(K+z)P(0,T)E^{\S_2}\big[1_{S(T)>K+z} 1_{\{R(T)\frac{dP}{d\S}>c\}}] = (K+z)P(0,T)N_2(\Sigma^{-1/2}\eta_2(z,a)),$$

Appendix 6

Recall that the dynamic of stock price process under market measure P is given by

$$
\begin{aligned}
S(t) &= S(0)R(t)\exp\Big(\int_0^t \Big(\mu(u) - \frac{1}{2}(\sigma_1(u)^2 + \sigma_2(u)^2)\Big)du + \int_0^t \sigma_1(u)dB_1^{\mathrm{P}}(u) + \int_0^t \sigma_2(u)B_2^{\mathrm{P}}(u)\Big) \\
&= \frac{S(0)}{P(0,t)}\exp\Big(\int_0^t \Big(\mu(u) + \alpha^*(u,t) - \frac{1}{2}(\sigma_1(u)^2 + \sigma_2(u)^2)\Big)du + \int_0^t (\sigma^*(u,t) + \sigma_1(u))dB_1^{\mathrm{P}}(u) + \int_0^t \sigma_2(u)B_2^{\mathrm{P}}(u)\Big).
\end{aligned}
$$

Consider the decomposition of f_2^{call}:

$$
f_2^{call}(z,a) = E^{\mathrm{P}}\Big[S(T)1_{\{S(T)>K+z\}}1_{\{R(T)\frac{d\mathrm{P}}{d\S}\le a\}}\Big] - (K+z)E^{\mathrm{P}}\Big[1_{\{S(T)>K+z\}}1_{\{R(T)\frac{d\mathrm{P}}{d\S}\le a\}}\Big].
$$

Here to define a new measure $\tilde{\mathrm{P}}$:

$$
\frac{d\tilde{\mathrm{P}}}{d\mathrm{P}} := \exp\Big(\int_0^T (\sigma^*(u,T) + \sigma_1(u))dB_1^{\mathrm{P}}(u) + \int_0^T \sigma_2(u)B_2^{\mathrm{P}}(u) - \frac{1}{2}\int_0^T (\sigma^*(u,T) + \sigma_1(u))^2 + \sigma_2(u)^2)du\Big).
$$

Then we have

$$
\begin{aligned}
E^{\mathrm{P}}\Big[S(T)1_{\{S(T)>K+z\}}1_{\{R(T)\frac{d\mathrm{P}}{d\S}\le a\}}\Big] &= \frac{S(0)}{P(0,T)}\exp\Big(\int_0^T \Big(\mu(u) + \alpha^*(u,T) + \frac{1}{2}\sigma^*(u,T)^2 + \sigma^*(u,T)\sigma_1(u)\Big)du\Big) \\
E^{\tilde{\mathrm{P}}}\Big[1_{S(T)>K+z}1_{\{R(T)\frac{d\mathrm{P}}{d\S}\le a\}}\Big] &= \frac{S(0)}{P(0,T)}e^A E^{\tilde{\mathrm{P}}}\Big[1_{\{S(T)>K+z\}}1_{\{R(T)\frac{d\mathrm{P}}{d\S}\le a\}}\Big],
\end{aligned}
$$

where A is given by (\ref{eq:constA}).

By Girsanov theorem we have

$$
\begin{aligned}
B_1^{\mathrm{P}}(t) &= B_1^{\tilde{\mathrm{P}}}(t) + \int_0^t (\sigma^*(u,T) + \sigma_1(u))du, \\
B_2^{\mathrm{P}}(t) &= B_2^{\tilde{\mathrm{P}}}(t) + \int_0^t \sigma_2(u)du.
\end{aligned}
$$

where $B_1^{\tilde{\mathrm{P}}}$ and $B_2^{\tilde{\mathrm{P}}}$ are two independent Brownian motions under $\tilde{\mathrm{P}}$. Then $S(T)$ and $R(T)d\mathrm{P}/\S$ becomes:

$$
\begin{aligned}
S(T) &= \frac{S(0)}{P(0,T)} \exp\Big(\int_0^T \Big(\mu(u) + \alpha^*(u,T) - \frac{1}{2}(\sigma_1(u)^2 + \sigma_2(u)^2)\Big)du \\
&+ \int_0^T (\sigma^*(u,T) + \sigma_1(u))dB_1^{\mathrm{P}}(u) + \int_0^T \sigma_2(u) B_2^{\mathrm{P}}(u)\Big) \\
&= \frac{S(0)}{P(0,T)} \exp\Big(A + \frac{1}{2}\Sigma_{22} - W_2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
R(T)\frac{d\mathrm{P}}{d\S} &= \frac{1}{P(0,T)} \exp\Big(\int_0^T \Big(\alpha^*(u,T) + \frac{1}{2}(\theta_1(u)^2 + \theta_2(u)^2)\Big)du \\
&- \int_0^T (\theta_1(u) - \sigma^*(u,T))dB_1^{\mathrm{P}}(u) - \int_0^T \theta_2(u) dB_2^{\mathrm{P}}(u)\Big) \\
&= \frac{1}{P(0,T)} \exp\Big(A + \frac{1}{2}\Sigma_{11} - W_1\Big)
\end{aligned}
$$

Thus $R(T)d\mathrm{P}/d\S \le a$ implies

$$
-W_1 \le \log(aP(0,T)) - A - \frac{1}{2}\Sigma_{11},
$$

and $S(T) > K + z$ implies

$$
W_2 < \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) + A + \frac{1}{2}\Sigma_{22}.
$$

It is easy to see that $(W_1, W_2) \sim N(0, \tilde{\Sigma})$.

Then we have

$$
E^{\mathrm{P}}\Big[S(T)1_{S(T)>K+z}1_{\{R(T)\frac{d\mathrm{P}}{d\S}\le a\}}\Big] = \frac{S(0)}{P(0,T)} e^A N_2(\tilde{\Sigma}^{1/2}\zeta_1(z,a)),
$$

where $\zeta_1(z,a)$ is given by (\ref{eq:zeta1}). On the other side we do not need to change the measure. For $\{R(T)d\mathrm{P}/d\S \le a\}$ we have

$$
-W_1 \le \log(aP(0,T)) - \int_0^T \Big(\alpha^*(u,T) + \frac{1}{2}(\theta_1(u)^2 + \theta_2(u)^2)\Big)du,
$$

and for $S(T) > K + z$ we have

$$
W_2 < \log\Big(\frac{S(0)}{P(0,T)(K+z)}\Big) + \int_0^T \Big(\mu(u) + \alpha^*(u,T) - \frac{1}{2}(\sigma_1(u)^2 + \sigma_2(u)^2)\Big)du.
$$

Finally we will get

$$(K+z)E^{p}\left[1_{S(T)>K+z}1_{\{R(T)\frac{dp}{d\mathbb{Q}}\leq a\}}\right]=(K+z)N_{2}(\tilde{\Sigma}^{-1/2}\zeta_{2}(z,a)),$$

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