Bundles of Irreducible Clifford Modules and the Existence of Spin Structures

A Dissertation presented

by

Xuan Chen

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2017

Stony Brook University

The Graduate School

Xuan Chen

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

H. Blaine Lawson - Dissertation Advisor
Professor, Department of Mathematics

Claude LeBrun - Chairperson of Defense
Professor, Department of Mathematics

Marie-Louise Michelsohn
Professor, Department of Mathematics

Martin Rocek
Professor, C.N. Yang Institute for Theoretical Physics

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

Bundles of Irreducible Clifford Modules and the Existence of Spin Structures

by

Xuan Chen

Doctor of Philosophy

in

Mathematics

Stony Brook University

2017

It is known that if an oriented riemannian manifold is spin, then there exists a bundle of real Clifford modules which are pointwise irreducible. Similarly, if the manifold is spin^c, then there exists a bundle of complex pointwise irreducible Clifford modules. However, the converse of these statements is only partly true, and this thesis explores it in various cases.

Contents

Acknowledgements

I would like to thank my advisor, Blaine Lawson, for his help and support over the years, for patiently listening to my naive thoughts and enlightening me, and for encouraging me when I encountered frustrations.

I want also to thank Claude LeBrun, Marie-Louise Michelsohn, and Martin Rocek, for kindly being a part of my defense committee. I also benefited from many excellent courses and talks over the past years, and I want to express my gratefulness.

I am indebted to my parents for their love and support, and for not being at their side.

1 Introduction

In differential geometry and topology, there is a notion of spin manifold. We denote by Spin_n the connected double covering group of SO_n (and therefore, the universal covering group in case $n \geq 3$). An oriented riemannian n-manifold X is called spin if the principal bundle of its oriented orthonormal frames $P_{\mathrm{SO}}(X)$ has a fibre-wise non-trivial double covering. The double covering bundle can be given the structure of a principal Spin_n-bundle, and is denoted $P_{\mathrm{Spin}}(X)$; $P_{\mathrm{Spin}}(X)$, together with the covering map, is called a spin structure.

On the other hand, every oriented riemannian n-manifold (X, g) has a real Clifford bundle, denoted $\mathrm{Cl}(X)$. Each fibre $\mathrm{Cl}_x(X)$ of $\mathrm{Cl}(X)$ is formed by "products" of tangent vectors at x, subject to the relation $\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} = -2g(\mathbf{u}, \mathbf{v}) \cdot \mathrm{Id}$, $\forall \mathbf{u}, \mathbf{v} \in T_x(X)$ where the "$\cdot$" is the product operation called Clifford multiplication. This is a bundle of (associative) algebras.

If X is spin, then there are vector bundle(s) over X, called the irreducible spinor bundles, that are fibre-wise irreducible $\mathrm{Cl}(X)$ representations, which do not come from SO representations. There is also a complex analogue of this story: if we replace Spin_n by $\mathrm{Spin}_n^c := \mathrm{Spin}_n \times_{\mathbb{Z}_2} \mathrm{U}_1$ in a certain manner (see the next section), then we can define a Spin^c-manifold, and if X is a Spin^c-manifold, then there are complex vector bundle(s) over X, that are fibre-wise irreducible representations of $\mathbb{C}\mathrm{l}(X)$, the complexification of $\mathrm{Cl}(X)$.

The purpose of this article is to look into a "folk theorem", which says that the converse of the above is also true. More specifically, if there is a vector bundle (or complex vector bundle in the complex case) over X that is a fibre-wise irreducible representation of $\mathrm{Cl}(X)$ (or $\mathbb{C}\mathrm{l}(X)$, respectively), does that guarantee that X is spin (or Spin^c, respectively)? As we will see, in the complex case (section 3), the answer is yes, but in the real case, it depends on the dimension of X, due to the structure of the Clifford algebras. If the dimension $n = \dim X$ is congruent to $0, 6$ or 7 mod 8 (section 4), then the answer is yes. Otherwise (section 5), the answer is no and we will show that a Spin^c-structure guarantees (but is not necessary for) the existence of a bundle of real irreducible Clifford modules. Moreover, in case n is congruent to $2, 3$ or 4 mod 8 (section 6), an even weaker structure which we call Spin^h-structure will guarantee the existence of a bundle of real irreducible Clifford modules.

2 Conventions and background

For a reference, see [**1**, Sections I.1, I.4, I.5, II.2, II.3 and Appendix D].

Throughout this article, all tensor products are over $\mathbb{R}$ unless otherwise stated.

Definition 1 *Let $k = \mathbb{R}$ or $\mathbb{C}$. A k-algebra $\mathbb{A}$ with unit is a k-vector space with an associative and distributive multiplication, and the multiplicative identity is denoted by* 1.

Let S be a k-vector space and $\mathbb{A}$ be a k-algebra with unit 1. S is called a (left) $\mathbb{A}$-module if there is a k-bilinear map

$$\rho : \mathbb{A} \times S \to S$$

satisfying $\rho(a, \rho(b, s)) = \rho(ab, s)$ and $\rho(1, s) = s$ for any $a, b \in \mathbb{A}$ and $s \in S$. If S and V are $\mathbb{A}$-modules and V is a k-vector subspace of S, then V is called a ($\mathbb{A}$-)submodule of S. If S is an $\mathbb{A}$-module and all submodules of S are either the zero k-vector space or S itself, then S is called an irreducible $\mathbb{A}$-module.

In this article, we will usually take $k = \mathbb{R}$ and $\mathbb{A} = \mathrm{Cl}_n$ (see below), or

$k = \mathbb{C}$ and $\mathbb{A} = \mathbb{C}\mathrm{l}_n$ for some n. In case $k = \mathbb{R}$, S will sometimes be a complex vector space taken with its underlying real structure.

Denote by $k(N)$ the algebra of N by N matrices with entries in k.

Let $\mathbb{R}^n$ be equipped with a fixed inner product and an orientation, and Cl_n be the real Clifford algebra on $\mathbb{R}^n$, formed by "products" of elements in $\mathbb{R}^n$ subject to the relation $u \cdot v + v \cdot u = -2g(u, v) \cdot \mathrm{Id}$ for any $u, v \in \mathbb{R}^n$, where g is the inner product on $\mathbb{R}^n$, and the "$\cdot$" is called Clifford multiplication. Formally, $\mathrm{Cl}_n \equiv (\Sigma_{r=0}^{\infty} \otimes^r \mathbb{R}^n)/I$, where I is the ideal in $\Sigma_{r=0}^{\infty} \otimes^r \mathbb{R}^n$ generated by all elements of the form $v \otimes v + g(v, v)\mathrm{Id}$ for $v \in \mathbb{R}^n$. Let $\mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n \otimes \mathbb{C}$ be the complex Clifford algebra.

By a real (irreducible) Clifford module, we mean an (irreducible) Cl_n-module for some n; by a complex (irreducible) Clifford module, we mean an (irreducible) $\mathbb{C}\mathrm{l}_n$-module. In particular, we say a $\mathbb{C}\mathrm{l}_n$-module is real irreducible if it is irreducible as a Cl_n-module.

To understand the real or complex irreducible Clifford modules, it's essential to know the structure of the Clifford algebras Cl_n and $\mathbb{C}\mathrm{l}_n$ for every n. It turns out that they are all of the form $k(N)$ or of the form $k(N) \oplus k(N)$ for some number N and some field k among $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, and they follow certain "periodicity". A detailed discussion is given in [**1**, I.4], from which we will keep handy the periodicity formulae $\mathrm{Cl}_{n+8} \cong \mathrm{Cl}_n \otimes \mathbb{R}(16)$, $\mathbb{C}\mathrm{l}_{n+8} \cong \mathbb{C}\mathrm{l}_n \otimes \mathbb{R}(16)$ $(\forall n \geq 0)$ and the structure of following Clifford algebras.

n	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathbb{C}\mathrm{l}_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

It's worth mentioning that the field being $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ plays a key role in the discussions of sections 4, 5, and 6 in this article.

From the structure of $\mathbb{C}\mathrm{l}_n$, we can see that the irreducible $\mathbb{C}\mathrm{l}_n$-modules all have complex dimension $N = 2^{[\frac{n}{2}]}$, and up to isomorphism, there are two of them when n is odd, and a unique one when n is even. Let $\omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]} e_1 e_2 \dots e_n \in \mathbb{C}\mathrm{l}_n$ denote the complex volume element, where $(e_1, e_2, \dots, e_n)$ is an oriented orthonormal basis of $\mathbb{R}^n$. The complex volume element is well defined, i.e., independent of the choice of the oriented orthonormal basis. In case n is odd, there are two inequivalent irreducible $\mathbb{C}\mathrm{l}_n$-modules, and $\omega_{\mathbb{C}}$ acts on one of them as 1, and on the other as -1, thus distinguishing the two inequivalent complex irreducible Clifford modules. Once we pin down a Clifford module M, for any φ in the Clifford algebra and $m \in M$, we interchangeably use "$\varphi \cdot m$", "$\mu(\varphi, m)$", or simply "φm" to denote the Clifford action.

The above are pointwise. Now we move on to a manifold. Unless otherwise stated, let X be a compact, connected and oriented riemannian n-manifold. Let $\mathrm{Cl}(X)$ be the Clifford bundle of its tangent bundle, that is, each fibre $\mathrm{Cl}_x(X)$ of $\mathrm{Cl}(X)$ is formed by "products" of tangent vectors at x, subject to the relation $\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} = -2g(\mathbf{u}, \mathbf{v}) \cdot \mathrm{Id}$, $\forall \mathbf{u}, \mathbf{v} \in T_x(X)$ where the "$\cdot$" is the product operation called Clifford multiplication and g is the riemannian metric on TX. Formally, $\mathrm{Cl}(X) \equiv (\Sigma_{r=0}^{\infty} \otimes^r TX)/I$, where I is the bundle of ideals in $\Sigma_{r=0}^{\infty} \otimes^r TX$ generated by all elements of the form $\mathbf{v} \otimes \mathbf{v} + g(\mathbf{v}, \mathbf{v})1$ for $\mathbf{v} \in TX$. Let $\mathbb{C}\mathrm{l}(X) = \mathrm{Cl}(X) \otimes \mathbb{C}$ be the complex Clifford bundle. There is a (unique) global section called the complex volume element and denoted $\omega_{\mathbb{C}}(X) \in \mathbb{C}\mathrm{l}(X)$. At each point $x \in X$, it is defined as $\omega_{\mathbb{C}}|_x(X) = i^{[\frac{n+1}{2}]} e_1 e_2 \dots e_n$ for any oriented orthonormal frame $(e_1, e_2, \dots, e_n)$ of $T_x X$, which is independent of the choice of the oriented orthonormal frame, and so $\omega_{\mathbb{C}}$ is a well-defined a global section of $\mathbb{C}\mathrm{l}(X)$.

Definition 2 *Let $k = \mathbb{R}$ or $\mathbb{C}$, and $\mathbb{A}$ be a k-vector bundle over X. Then $\mathbb{A}$ is called a* <u>bundle of k-algebras</u> *, if $\mathbb{A}_x$ is a k-algebra for any $x \in X$, and if the algebra operations are continuous on X.*

Suppose $\mathbb{A}$ is a bundle of k-algebras over X. A k-vector bundle $\$$ over X is called a <u>bundle of $\mathbb{A}$-modules</u> *over X if there is a continuous bundle map*

$$\rho : \mathbb{A} \otimes_k \$ \to \$$$

(where $\otimes_k$ means tensor product as k-vector bundles over X) that makes $\$_x$ an $\mathbb{A}_x$-module for each $x \in X$, and in this case, $\$$ is called <u>irreducible</u> *if it is* fibre-wise *irreducible, namely if $\$_x$ is an irreducible $\mathbb{A}_x$-module for any $x \in X$.*

The bundles $\mathrm{Cl}(X)$ and $\mathbb{C}\mathrm{l}(X)$ are bundles of algebras over $\mathbb{R}$ and $\mathbb{C}$ respectively. A bundle of $\mathrm{Cl}(X)$-modules is also called a bundle of (real) Clifford modules, and a bundle of $\mathbb{C}\mathrm{l}(X)$-modules is also called a bundle of complex Clifford modules. The notion of bundles of Clifford modules look very similar to that of spinor bundles.

Note that the notion of irreducibility here is fibre-wise, which is stronger than being "globally irreducible". And it's possible that for some bundle of algebras $\mathbb{A}$, a bundle of irreducible $\mathbb{A}$-modules may not exist. For example, when $\mathbb{A} = \mathbb{C}\mathrm{l}(X)$, in the next section, we show that a bundle of irreducible $\mathbb{C}\mathrm{l}(X)$-modules exists if and only if X is Spin^c. Here we recall the notion of a Spin^c-manifold.

Definition 3 *Denote* $\mathrm{Spin}_n^c = \mathrm{Spin}_n \times_{\mathbb{Z}_2} \mathrm{U}_1$, *where* U_1 *is the first unitary group. It is a multiplicative subgroup of $\mathbb{C}\mathrm{l}_n$. There are well defined maps*

$$\mathrm{Spin}_n^c \to \mathrm{Spin}_n/\mathbb{Z}_2 \equiv \mathrm{SO}_n$$

and

$$\mathrm{Spin}_n^c \to \mathrm{U}_1/\mathbb{Z}_2 \equiv \mathrm{U}_1$$

.

A $\underline{\mathrm{Spin}^c\text{-}structure}$ *on an oriented riemannian n-mainfold X is a principal* Spin_n^c*-bundle* $P_{\mathrm{Spin}^c}(X)$ *and a principal* U_1*-bundle* $P_{\mathrm{U}_1}(X)$ *over X, together with a* Spin_n^c*-equivariant bundle map* $P_{\mathrm{Spin}^c}(X) \to P_{\mathrm{SO}}(X) \times P_{\mathrm{U}_1}(X)$ *where* $P_{\mathrm{SO}}(X)$ *is the orthonormal frame bundle on X.*

Note also that definition 2 does not say a rank N bundle of $\mathrm{Cl}(X)$-modules could be trivialized locally, say on some small open set $U \subseteq X$, by some *continuous* sections $s_1, \ldots, s_N$ so that at each point $x \in U$, the action ρ is a prescribed Clifford action μ under some basis of $\mathrm{Cl}_x(X)$ and the basis $(s_1, \ldots, s_N)$ of $\not{S}_x$ (namely, can we "smash" U to the single point x so that the "smashing" commutes with Clifford multiplications). However, this is true. We first recall the formal terms of trivialization.

Let P_{SO} denote the oriented orthonormal frame bundle of X. Associated to any $e = (e_1, e_2, \ldots, e_n) \in P_{\mathrm{SO}}\big|_x$ at some point $x \in X$, there is a linear coordinate map

$$\begin{aligned} L^e : T_x X &\to \mathbb{R}^n \\ v &\mapsto \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \end{aligned}$$

where $v = \sum_i e_i a_i$. This extends uniquely to an algebra morphism $\mathrm{Cl}_x(X) \to \mathrm{Cl}_n$ which we also denote as L^e. Similarly, let N be the k-rank ($k = \mathbb{R}$ or $\mathbb{C}$) of some k-vector bundle $\not{S}$, then associated to any tuple $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$

k-linearly independent in $\not{S}_x$, there is a linear coordinate map

$$\begin{aligned} L^{\varepsilon} : \not{S}_x &\to k^N \\ s &\mapsto \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} \end{aligned}$$

where $s = \sum_i \varepsilon_i w_i$.

Theorem 1 *If $\not{S}$ is a bundle of $\mathrm{Cl}(X)$-modules, then there exists a Cl_n-module M (together with $\mu : \mathrm{Cl}_n \otimes M \to M$), and for any $x_0 \in X$, there is a neighbourhood U of x_0, a continuous section e of $P_{\mathrm{SO}}|_U$, and a continuous section ε of pointwise bases of $\not{S}$ over U, so that the diagram*

$$\begin{tikzcd} \mathrm{Cl}_x(X) \otimes \not{S}_x \arrow[r, "\rho_x"] \arrow[d, "L^{e(x)} \otimes L^{\varepsilon(x)}"'] & \not{S}_x \arrow[d, "L^{\varepsilon(x)}"'] \\ \mathrm{Cl}_n \otimes M \arrow[r, "\mu"'] & M \end{tikzcd}$$

commutes for any $x \in U$.

The proof will be put off.

3 The complex case

If the manifold is Spin^c, then there exists an irreducible complex spinor bundle, which is proved to be([**1**, Proposition II.3.8] for the real version) a bundle of complex irreducible Clifford modules. The following main result (Theorem 2) asserts that the converse is also true, namely, if there exists a bundle of complex irreducible Clifford modules $\not{S}$ on X, then X has a Spin^c-structure canonically associated to $\not{S}$. The principal Spin^c-bundle of this Spin^c-structure is constructed explicitly as a subbundle of the frame bundle of $\not{S}$. Moreover (in Theorem 3), $\not{S}$ is the "unique" spinor bundle associated to this Spin^c-structure, in the following sense. When $n \equiv \dim X$ is even, denote by M the unique (up to equivalence) complex irreducible Cl_n-module; when n is odd, the volume element $\omega_{\mathbb{C}}(X)$ in $\mathbb{C}\mathrm{l}(X)$ acts on $\not{S}$ by either 1 or -1, and we denote by M the complex irreducible Cl_n-module on which $\omega_{\mathbb{C}}$ acts by 1 or -1 accordingly. Then up to equivalence $\not{S}$ determines the Spin^c structure as well as the complex irreducible Cl_n-module M, and we will prove that $\not{S}$ is the spinor bundle constructed from the Spin^c structure and M.

Theorem 2 *Suppose there exists a bundle of complex irreducible* $\mathrm{Cl}(X)$-*modules, denoted as* $\not{S}$. *Then* X *admits a* Spin^c-*structure canonically associated to* $\not{S}$.

Lemma 1 *For any real volume element* $vol \in \Lambda^{2N}_{\mathbb{R}}\mathbb{C}^N$ *and any element* $g \in \mathrm{Spin}_n$, *the action of* g *always preserves* vol.

Proof: There exists a hermitian metric $(\cdot,\cdot)$ on $\mathbb{C}^N$, so that all unit vectors in $\mathbb{R}^n \subseteq \mathrm{Cl}_n$ act on $\mathbb{C}^N$ by isometries ([**1**, Prop. I.5.16]). Fixing a unitary basis under this hermitian metric, then g acts on $\mathbb{C}^N$ as a $N \times N$ complex matrix $\mathbf{g}$. Now we have $\mathbf{g} \in U_N$, since g, as a product of unit vectors in $\mathbb{R}^n$, acts on $\mathbb{C}^N$ by a unitary map. Thus $\det_{\mathbb{R}}(\mathbf{g}) = \det_{\mathbb{C}}(\mathbf{g})\det_{\mathbb{C}}(\overline{\mathbf{g}}) = 1$. So $g \in \mathrm{SL}(\mathbb{R}^{2N})$. The proof is complete. $\square$

Proof of Theorem 2:

Since $\not{S}$ is a representation of $\mathrm{Cl}(X)$, for any $e \in P_{\mathrm{SO}}\big|_x$, there must be a complex basis ε of $\$_x$ such that, if we identify $\mathrm{Cl}_x(X)$ with Cl_n through L^e and identify $\$_x$ with $\mathbb{C}^N$ through L^ε, then we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}_x(X) \otimes \$_x & \xrightarrow{\rho_x} & \$_x \\ {\scriptstyle L^e \otimes L^\varepsilon} \downarrow & & \downarrow {\scriptstyle L^\varepsilon} \\ \mathrm{Cl}_n \otimes \mathbb{C}^N & \xrightarrow[\mu]{} & \mathbb{C}^N \end{array}$$

Here $\mathbb{C}^N$, together with the action μ, is considered a fixed irreducible complex Cl_n-module (with the standard complex structure), obtained as follows. In case $n \equiv \dim X$ is even, all complex irreducible Cl_n-modules are isomorphic, and then fix $(\mathbb{C}^N, \mu)$ to be any one of them. In case n is odd, the global (continuous) section $\omega_{\mathbb{C}}(X)$ of $\mathbb{C}\mathrm{l}(X)$ either acts on $\$$ as 1 or as -1, since $\$$ is irreducible and X is connected. Fix $(\mathbb{C}^N, \mu)$ to be any complex irreducible Cl_n-module where $\omega_{\mathbb{C}}$ acts as 1 (or -1) if $\omega_{\mathbb{C}}(X)$ acts on $\$$ as 1 (resp. as -1).

Notational Convention Once we fix μ as above, any $\varphi \in \mathrm{Cl}_n$ then acts on $\mathbb{C}^N$ as an $N \times N$ complex matrix, denoted by $\boldsymbol{\varphi}$. For $g \in \mathrm{Spin}_n \subseteq \mathrm{Cl}_n$, similarly, its representation matrix is denoted by $\mathbf{g}$.

Definition 4 *The ε and e in the above commutative diagram are said to be* associated *to each other.*

Since $\$$ is a complex vector bundle of rank N, we can choose a real volume form $vol \in \Lambda^{2N}_{\mathbb{R}}(\$)$ that agrees with the canonical orientation given by the complex structure J of $\$$. If a basis ε satisfies $\varepsilon_1 \wedge J\varepsilon_1 \wedge \varepsilon_2 \wedge J\varepsilon_2 \wedge \ldots \varepsilon_N \wedge J\varepsilon_N = vol$, then we call ε a *unit volume frame/basis.*

Now we define

$$P_{\mathrm{Spin}^c} \equiv \bigcup_{x \in X} \left\{ \begin{array}{c} \text{all unit volume bases } \varepsilon \text{ of } \$_x \\ \text{so that there exists } e \in P_{\mathrm{SO}}\big|_x \text{ associated to } \varepsilon \end{array} \right\}$$

In Proposition 2, we will see this definition is essentially independent of the choice of vol and the particular μ in the isomorphism class of Clifford modules.

We claim this is the principal Spin^c-bundle we are looking for, by the following steps. Let the map

$$\pi_0 : \mathrm{Spin}_n \to \mathrm{SO}(n)$$

denote the double cover map, and

$$\pi : P_{\mathrm{SO}} \to X$$

denote the oriented orthogonal frame bundle. Also recall the notational convention before the statement of Definition 4: once we fix the complex irreducible Clifford module $(\mathbb{C}^N, \mu)$, we denote the matrix of the action of a Clifford (or spin, in particular) element φ on $(\mathbb{C}^N, \mu)$ by the corresponding bold-faced letter $\boldsymbol{\varphi}$.

Proposition 1 *Let e, $e' \in P_{\mathrm{SO}}|_x$, so $e = e'h$, for a unique $h \in \mathrm{SO}(n)$. This means, writing $h = (h_{ij})$, then $e_j = \sum_i e'_i h_{ij}$. Choose $g \in \mathrm{Spin}_n$ such that $\pi_0(g) = h$. Now suppose ε and ε' are unit volume frames of $\slashed{S}_x$, and that ε is associated to e, then ε' is associated to e' if and only if $\varepsilon = \lambda\varepsilon'\mathbf{g}$ for some $\lambda \in S^1 \subseteq \mathbb{C}$, which means $\varepsilon_j = \lambda \sum_i \varepsilon'_i g_{ij}$ if we write $\mathbf{g} = (g_{ij})$ (i.e., $\varepsilon'\mathbf{g}$ means treating $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_N)$ as a row vector and doing matrix multiplication).*

Proof:

$$\begin{tikzcd}
& \mathrm{Cl}_x(X) \otimes \slashed{S}_x \arrow[dl, "L^e \otimes L^\varepsilon"'] \arrow[dr, "L^{e'} \otimes L^{\varepsilon'}"] & \\
\mathrm{Cl}_n \otimes \mathbb{C}^N \arrow[rr] \arrow[d, "\mu"'] & & \mathrm{Cl}_n \otimes \mathbb{C}^N \arrow[d, "?"] \\
\mathbb{C}^N \arrow[rr] & & \mathbb{C}^N \\
& \slashed{S}_x \arrow[ul, "L^\varepsilon"] \arrow[ur, "L^{\varepsilon'}"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
& \phi \otimes s \arrow[dl, "L^e \otimes L^\varepsilon"'] \arrow[dr, "L^{e'} \otimes L^{\varepsilon'}"] & \\
\varphi \otimes \sigma \arrow[rr] \arrow[d, "\mu"'] & & g\varphi g^{-1} \otimes \alpha\sigma \arrow[d, "?"] \\
\mu(\varphi \otimes \sigma) \arrow[rr] & & \alpha\mu(\varphi \otimes \sigma) \\
& \rho(\phi \otimes s) \arrow[ul, "L^\varepsilon"] \arrow[ur, "L^{\varepsilon'}"'] &
\end{tikzcd}
$$

Let $\varepsilon = \varepsilon'\alpha$ for some $\alpha \in \mathbb{C}(N)$. Since both ε' and ε are unit volume, we see $\alpha \in \mathrm{SL}(\mathbb{R}^{2N})$ as a real matrix.

If a vector $v \in T_xX$ is expressed as $v = \sum_i a_i e_i = \sum_i a'_i e'_i$, then the column vectors $a = (a_1, a_2, \ldots, a_n)^{tr}$ and $a' = (a'_1, a'_2, \ldots, a'_n)^{tr}$ satisfy $a' = ha$. Treating $a, a' \in \mathbb{R}^n \subseteq \mathrm{Cl}_n$, then $a' = ha = \mathrm{Ad}_g(a)$ since $\pi_0(g) = h$, namely $L^{e'}(v) = \mathrm{Ad}_g(L^e(v))$, for any $v \in T_xX$. Extending this to $\mathrm{Cl}_x(X)$, we see $L^{e'}(\phi) = \mathrm{Ad}_g(L^e(\phi)) = gL^e(\phi)g^{-1}$ for any $\phi \in \mathrm{Cl}_x(X)$ (c.f. [**1**, Section II.3]).

Suppose an element $\phi \otimes s$ in $\mathrm{Cl}_x(X) \otimes \$_x$ has coordinate $\varphi \otimes \sigma$ under the trivialization $L^e \otimes L^\varepsilon$, then the image $\rho(\phi \otimes s)$ should have coordinate $\mu(\varphi \otimes \sigma)$ under the trivialization L^ε. Under the new basis (e', ε'), $\phi \otimes s$ should have coordinate $(g\varphi g^{-1}) \otimes (\alpha\sigma)$, and under ε', $\rho(\phi \otimes s)$ has coordinate $\alpha\mu(\varphi \otimes \sigma)$.

Now, ε' is associated to e' if and only if $g\varphi g^{-1}\alpha\sigma = \alpha\varphi\sigma$ (i.e., if and only if the question-marked map in the diagram is the Clifford module action) for any $\varphi \in \mathrm{Cl}(n)$ and $\sigma \in \mathbb{C}^N$. Using matrix notations, this is written $\mathbf{g}\boldsymbol{\varphi}\mathbf{g}^{-1}\alpha = \alpha\boldsymbol{\varphi}$, or $\mathbf{g}^{-1}\alpha\boldsymbol{\varphi} = \boldsymbol{\varphi}\mathbf{g}^{-1}\alpha$, for any $\varphi \in \mathrm{Cl}_n$, namely $\mathbf{g}^{-1}\alpha$ commutes with $\boldsymbol{\varphi}$ for any $\varphi \in \mathrm{Cl}_n$. Thus for any eigenvalue λ of $\mathbf{g}^{-1}\alpha$, $\ker(\mathbf{g}^{-1}\alpha - \lambda)$ is invariant under Cl_n. Since the Clifford representation is irreducible, $\mathbf{g}^{-1}\alpha = \lambda \cdot Id$. Since both $\mathbf{g}$ and α are in $\mathrm{SL}(\mathbb{R}^{2N})$, we see $\lambda \cdot Id \in \mathrm{SL}(\mathbb{R}^{2N})$ as a real matrix. Now

$$1 = \det_{\mathbb{R}}(\lambda \cdot Id) = \det_{\mathbb{C}}(\lambda \cdot Id)\det_{\mathbb{C}}(\overline{\lambda \cdot Id}) = \lambda^N \overline{\lambda}^N = |\lambda|^{2N}, \text{ so } |\lambda| = 1 \qquad \square$$

Proposition 2 *Spin^c acts transitively and freely on the fibres of the bundle $P_{\mathrm{Spin}^c} \to X$, i.e, P_{Spin^c} is a principal Spin^c-bundle on X. Up to principal bundle isomorphism, this bundle is independent of the choice of the volume form chosen and the choice of the particular Clifford module action μ within an equivalence class. Moreover, if $\not{S}_1$ and $\not{S}_2$ are isomorphic as bundles of complex irreducible Clifford modules, namely there is a complex vector bundle isomorphism $f : \not{S}_1 \to \not{S}_2$ so that the diagram*

$$\begin{tikzcd} \mathrm{Cl}(X) \otimes \not{S}_1 \arrow[r, "\rho_1"] \arrow[d, "\mathrm{Id}\otimes f"'] & \not{S}_1 \arrow[d, "f"'] \\ \mathrm{Cl}(X) \otimes \not{S}_2 \arrow[r, "\rho_2"] & \not{S}_2 \end{tikzcd}$$

commutes, then $\not{S}_1$ and $\not{S}_2$ define the same principal Spin^c-bundle up to equivalence.

Proof: Let the action be

$$\mathrm{Spin}^c \times P_{\mathrm{Spin}^c} \to P_{\mathrm{Spin}^c}$$

$$\big([g, \lambda], \varepsilon\big) \mapsto \lambda\varepsilon\mathbf{g}$$

Free : If $\lambda\varepsilon\mathbf{g} = \varepsilon$ for some ε, then $\lambda\mathbf{g} = 1$. We have to prove $[g, \lambda] = 1$. For simplicity, let's identify $g \in \mathrm{Spin}_n$ with $[g, 1] \in \mathrm{Spin}^c$, and identify $\lambda \in \mathbb{C}$ with $1 \otimes \lambda \in \mathrm{Cl}_n \otimes \mathbb{C} = \mathbb{C}\mathrm{l}_n$, so we say $g \in \mathrm{Spin}_n \subseteq \mathrm{Spin}^c \subseteq \mathbb{C}\mathrm{l}_n$ and $\lambda \in \mathbb{C} \subseteq \mathbb{C}\mathrm{l}_n$. Now we know $\lambda\mathbf{g} = 1 \in \mathbb{C}(N)$. If we can prove $\lambda g = 1 \in \mathbb{C}\mathrm{l}_n$, then we have $g = \dfrac{1}{\lambda} \in \mathrm{Spin}_n \cap S^1 = \{\pm 1\}$ ([**2**, Claim 2.2.4]), and therefore $[g, \lambda] = 1$. Now we only need to prove $\lambda g = 1 \in \mathbb{C}\mathrm{l}_n$, given that $\lambda\mathbf{g} = 1 \in \mathbb{C}(N)$.

If n is even, then $\mathbb{C}\mathrm{l}_n \cong \mathbb{C}(N)$, and the complex irreducible representation $(\mathbb{C}^N, \mu)$ of $\mathbb{C}\mathrm{l}_n$ is faithful. Recall that $\mathbf{g}$ is the matrix of g under this representation, so $\lambda\mathbf{g} = 1$ is the same as $\lambda g = 1$.

Now we look at the case when n is odd. Since $\lambda\mathbf{g} = 1$, we see g acts on the complex irreducible Clifford module $(\mathbb{C}^N, \mu)$ by $\dfrac{1}{\lambda}$. To prove $g = \dfrac{1}{\lambda} \in \mathbb{C}\mathrm{l}_n$, we need to prove that g acts on any complex irreducible Clifford module $(\mathbb{C}^N, \hat{\mu})$ by $\dfrac{1}{\lambda}$. This is true because ([**1**, Section I.5]) the even part $\mathbb{C}\mathrm{l}_n^0 \cong \mathbb{C}\mathrm{l}_{n-1} \cong \mathbb{C}(N)$ and therefore the restriction of the actions μ and $\hat{\mu}$ on $\mathrm{Spin}_n \subseteq \mathbb{C}\mathrm{l}_n^0$ have to be equivalent.

Transitive : For any ε and ε' in the same fibre, let ε and ε' be associated to e and e' respectively, then there is an $h \in \mathrm{SO}(n)$, so that $e' = eh$. Pick $g \in \mathrm{Spin}_n$ so that $\pi_0(g) = h$, then there exists a $\lambda \in S^1$ so that $\varepsilon' = \lambda\varepsilon g$ by Proposition 1.

Independent of the volume :

Suppose we have two different volume forms vol_1 and vol_2, both in the correct orientation, then there is a positive function f on X so that $vol_2 = f^{2N} vol_1$. Denote the thus-defined Spin^c-bundles P_1 and P_2, then both P_1 and P_2 are principal Spin^c-bundles. For any $\varepsilon \in P_1$ associated to some tangent bundle frame e and unit volume under vol_1, $f\varepsilon$ (pointwise multiplication) is also associated to e, and is unit volume under vol_2, so $f\varepsilon \in P_2$. Conversely, for any $\varepsilon \in P_2$, we have $\dfrac{1}{f}\varepsilon \in P_1$. Note that multiplying by f is a scalar multiplication fibrewise, which is central and in particular, commuting with Spin^c, so this gives a Spin^c-equivariant bundle isomorphism $P_1 \cong P_2$.

Independent of the particular μ in an equivalence class :

Suppose $(\mathbb{C}^N, \mu)$ and $(\mathbb{C}^N, \hat{\mu})$ are equivalent as complex irreducible Cl_n-

modules, namely, there is an $A \in \mathrm{GL}_N(\mathbb{C})$ so that the diagram

$$\begin{tikzcd} \mathrm{Cl}_n \otimes \mathbb{C}^N \arrow[r, "\mu"] \arrow[d, "\mathrm{Id}\otimes A"] & \mathbb{C}^N \arrow[d, "A"'] \\ \mathrm{Cl}_n \otimes \mathbb{C}^N \arrow[r, "\hat{\mu}"] & \mathbb{C}^N \end{tikzcd}$$

commutes. Denote by P the principal Spin^c-bundles defined using μ and a volume form vol. Denote by $\hat{P}$ the principal Spin^c-bundles defined using $\hat{\mu}$ and the volume form $vol/\det_{\mathbb{R}}(A)$. We have proved the choice of the volume form in the correct orientation does not change the principal Spin^c-bundle up to equivalence, so we only need to show $P \cong \hat{P}$ as principal Spin^c-bundles.

Now take any $x \in X$. In the diagram

$$\begin{tikzcd} \mathrm{Cl}_n \otimes \mathbb{C}^N \arrow[r, "\mu"'] \arrow[dd, bend right=60, "\mathrm{Id}\otimes A"'] & \mathbb{C}^N \arrow[dd, bend left=60, "A"] \\ \mathrm{Cl}_x(X) \otimes \not{\$}_x \arrow[r, "\rho_x"] \arrow[u, "L^e\otimes L^\varepsilon"'] \arrow[d, "L^{\hat{e}}\otimes L^{\hat{\varepsilon}}"'] & \not{\$}_x \arrow[u, "L^\varepsilon"] \arrow[d, "L^{\hat{\varepsilon}}"] \\ \mathrm{Cl}_n \otimes \mathbb{C}^N \arrow[r, "\hat{\mu}"] & \mathbb{C}^N \end{tikzcd}$$

let $\hat{\varepsilon} = \varepsilon A^{-1}$, so the triangles on the left and on the right each commutes. The equivalence of μ and $\hat{\mu}$ says the rectangle involving $\mathrm{Id} \otimes A$, μ, $\hat{\mu}$ and A commutes. So the upper rectangle of the diagram commutes if and only if the lower rectangle commutes. Together with the choice of the volume forms above, this implies $\varepsilon \in P$ if and only if $\hat{\varepsilon} = \varepsilon A^{-1} \in \hat{P}$. So we get an abstract bundle isomorphism $P \to \hat{P}$, $\varepsilon \mapsto \hat{\varepsilon} = \varepsilon A^{-1}$. For $[g, \lambda] \in \mathrm{Spin}^c$, denote the matrix of g using μ by $\mathbf{g}$, and the matrix using $\hat{\mu}$ by $\hat{\mathbf{g}}$. Then $\hat{\mathbf{g}} = A\mathbf{g}A^{-1}$. In P, $[g, \lambda]$ acts on ε and gives $\lambda\varepsilon\mathbf{g}$. In $\hat{P}$, $[g, \lambda]$ acts on $\hat{\varepsilon}$ and gives $\lambda\hat{\varepsilon}\hat{\mathbf{g}}$. If $\hat{\varepsilon} = \varepsilon A^{-1}$, then $\lambda\hat{\varepsilon}\hat{\mathbf{g}} = \lambda\varepsilon A^{-1}A\mathbf{g}A^{-1} = \lambda\varepsilon\mathbf{g}A^{-1}$. So the actions of Spin^c on P and $\hat{P}$ commute with the above isomorphism $P \cong \hat{P}$, namely, P and $\hat{P}$ are isomorphic as principal Spin^c-bundles.

Independent of the choice among equivalent bundles of Clifford modules :

Let $f : \not{S}_1 \to \not{S}_2$ be an isomorphism of bundles of Clifford modules, as in the statement of the proposition. Choose any volume form vol_2 of $\not{S}_2$ in the correct orientation. Since f is an isomorphism of complex vector bundles, $vol_2 \equiv f^*(vol_1)$ is a volume form of $\not{S}_1$ in the correct orientation. Let P_1 be the principal Spin^c-bundle obtained from $\not{S}_1$ using vol_1 and an appropriate μ, P_2 from $\not{S}_2$ using vol_2 and the same μ. For any frame $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ of $\not{S}_1$ at any point in X, define $f_*(\varepsilon) = (f(\varepsilon_1), f(\varepsilon_2), \ldots, f(\varepsilon_N))$. Since f is $\mathrm{Cl}(X)$-equivariant, we see $\varepsilon \in P_1$ if and only if $f_*(\varepsilon) \in P_2$. Since f is complex linear, $f_*(\varepsilon\mathbf{g}) = f_*(\varepsilon)\mathbf{g}$ for any $g \in \mathrm{Spin}^c$. Therefore $f_* : P_1 \to P_2$ is a Spin^c-equivariant isomorphism. $\square$

We have now obtained the principal Spin^c-bundle P_{Spin^c}. In order to get the principal Spin^c-structure on X, we will relate P_{Spin^c} to P_{SO} in Proposition 3, and to a principal S^1-bundle P_{U_1} (the canonical U_1-bundle) in Proposition 4.

Lemma 2 *Let G be a Lie group, A be a closed normal Lie subgroup of G, and $H = G/A$. Let P_G be a principal G-bundle on X. Then $P_H \equiv P_G/A$ is a principal H-bundle on X, and P_G is a principal A-bundle on P_H. Moreover, the quotient map $q : P_G \to P_H$ is G-equivariant.*

Proof: For any A-orbit of P_G, called pA for some $p \in P_G$, and any $g \in G$, we have $Ag = gA$ because A is normal in G, so $(pA)g = pgA$ is again an A-orbit. Therefore G acts on P_H (which respect the fibres of $P_H \to X$), and $A \subseteq G$ acts on P_H by identity, so H acts on the fibres of P_H. Around each point in X, there is a small neighbourhood U, that $P_G|_U \cong U \times G$ as G-bundles. Passing to the quotient, we see $P_H|_U \cong U \times H$ as H-bundles. So H acts on the fibres of P_H freely and transitively, i.e., P_H is a principal H-bundle on X. Since A acts on P_G freely, P_G is a principal A-bundle on P_H. The quotient map q is G-equivariant because the diagram

$$\begin{tikzcd} P_G|_U \arrow[r, "\cong"] \arrow[d, "q"] & U \times G \arrow[d] \\ P_H|_U \arrow[r, "\cong"] & U \times H \end{tikzcd}$$

commutes. □

Proposition 3 *The space P_{Spin^c} is a principal S^1-bundle over P_{SO}, and the projection map $P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}}$ is Spin^c-equivariant, namely, $[g,z] \in \mathrm{Spin}^c$ acting on P_{Spin^c} commutes with $\pi_0(g)$ acting on P_{SO}.*

Proof: Define $q : P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}}$ by $q(\varepsilon) = e$ for any ε associated to e. According to Proposition 1, Spin^c acts on P_{Spin^c} and the diagram

$$\begin{tikzcd} P_{\mathrm{Spin}^c} \arrow[r, "{[g,\lambda]}"] \arrow[d, "q"] & P_{\mathrm{Spin}^c} \arrow[d, "q"] \\ P_{\mathrm{SO}} \arrow[r, "\pi_0(g)"] & P_{\mathrm{SO}} \end{tikzcd}$$

commutes for any $[g,\lambda] \in \mathrm{Spin}^c$. If ε and ε' lie in the same fibre $q^{-1}(e)$, then $\varepsilon' = [g,\lambda]\varepsilon$ for some $[g,\lambda] \in \mathrm{Spin}^c$. Plugging these into the above diagram and we get

$$\begin{tikzcd} \varepsilon \arrow[r, mapsto, "{[g,\lambda]}"] \arrow[d, mapsto, "q"] & \varepsilon' \arrow[d, mapsto, "q"] \\ e \arrow[r, mapsto, "\pi_0(g)"] & e \end{tikzcd}$$

and therefore $\pi_0(g) = 1 \in \mathrm{SO}_n$, so $[g,\lambda] \in S^1 \subseteq \mathrm{Spin}^c$. Now apply Lemma 2 with $G = \mathrm{Spin}^c$ and $A = S^1$, we see P_{Spin^c} is a principal S^1-bundle on P_{SO} and the quotient map q is Spin^c-equivariant. □

Next we want to define the canonical U_1-bundle for the Spin^c-structure.

We see P_{Spin^c} is a principal S^1-bundle over P_{SO}; let $\hat{E}$ be the associated complex line bundle over P_{SO}. If we do get a Spin^c-structure, $\hat{E} \otimes_{\mathbb{C}} \hat{E}$ should be obtained by pulling back the complex line bundle associate to the canonical U_1-bundle through $\pi : P_{\mathrm{SO}} \to X$. That indicates we could try to prove first that, $\hat{E} \otimes_{\mathbb{C}} \hat{E}$ can be trivialized over $\pi^{-1}(x), \forall x$, then prove $\hat{E} \otimes_{\mathbb{C}} \hat{E}$ is pulled back from a bundle over X. We will almost do that, but to circumvent the subtleties, we use the following lemma.

Lemma 3 *Suppose $\pi : P_G \to X$ is a principal G-bundle. Let $\hat{F} \to P_G$ be a complex vector bundle, and suppose that G acts on it G-equivariantly by vector-bundle isomorphisms. Then $\hat{F} = \pi^* F$ for some bundle $F \to X$.*

Proof: Let $F = \hat{F}/G$. We first prove that F is a vector bundle on X. For $v \in \hat{F}$, denote $[v] = \{gv : g \in G\}$, i.e., the G-orbit of v, then $[v] \in F$. If $[v_1] = [v_2]$, then $v_1, v_2 \in \hat{F}$ lie in the same G-orbit, and thus the same fibre of $\hat{F} \to X$, say at $x \in X$. Sending $[v_1]$ to x, we get a map $F \to X$. If $[v_1], [v_2] \in F$ lie above the same point x in X, since G acts transitively on the fibres of P_G, for any $e \in \pi^{-1}(x)$ we can choose representatives $v_1, v_2 \in \hat{F}$ that lie above e, and this choice is unique because G acts freely on P_G and by bundle isomorphisms on $\hat{F}$. Then define $[v_1] + [v_2] = [v_1 + v_2]$ and $c[v_1] = [cv_1]$ for any scalar c. This is independent of the choice of $e \in \pi^{-1}(x)$, because G acts on $\hat{F}$ be vector-bundle isomorphisms. Therefore F is a vector bundle on X.

Now $\pi^* F$ consists of elements $([v], e)$ where $[v] \in F$ and $e \in P_G$ lie above the same point in X. As above, there is a unique the representative v of $[v]$ that lies in the fibre $\hat{F}_e$. Mapping $([v], e)$ to v, we get a bundle isomorphism $\pi^* F \cong \hat{F}$. $\square$

Proposition 4 *There is a principal* U_1*-bundle, denoted* P_{U_1}*, on* X*, together with a* Spin^c*-equivariant bundle map* $P_{\mathrm{Spin}^c} \to P_{U_1}$.

Here "Spin^c-equivariant" means $[g, z] \in \mathrm{Spin}^c$ acting on P_{Spin^c} should commute with z^2 acting on P_{U_1} ([**1**, Appendix D]).

Proof: We first claim that SO_n acts on $\hat{E} \otimes_{\mathbb{C}} \hat{E} \to P_{\mathrm{SO}}$ by line-bundle isomorphism, in an SO_n-equivariant manner. For any $h \in \mathrm{SO}_n$, there exist $g' = -g'' \in \mathrm{Spin}_n$ that $\pi_0(g') = \pi_0(g'') = h$. Recall that Spin^c acts on P_{Spin^c}, and thus on $\hat{E}$. For $\varepsilon \otimes \varepsilon' \in \hat{E} \otimes_{\mathbb{C}} \hat{E}$, we see $([g'', 1]\varepsilon) \otimes ([g'', 1]\varepsilon')$ $= ([-g', 1]\varepsilon) \otimes ([-g', 1]\varepsilon') = ([g', 1]\varepsilon) \otimes ([g', 1]\varepsilon')$, so the action of $h \in SO_n$ on $\hat{E} \otimes_{\mathbb{C}} \hat{E}$ given by $\varepsilon \otimes \varepsilon' \mapsto ([g', 1]\varepsilon) \otimes ([g', 1]\varepsilon')$ is well defined.

This action is SO_n-equivariant: as in Proposition 1, h carries e to $e' = eh^{-1}$, so if $\varepsilon \otimes \varepsilon'$ lies above e, then $([g', 1]\varepsilon) \otimes ([g', 1]\varepsilon')$ lies above e'. Every SO_n acts on $\hat{E} \otimes_{\mathbb{C}} \hat{E}$ line-bundle isomorphism, so the by lemma, we get a complex line bundle $E \to X$ so that $\hat{E} \otimes_{\mathbb{C}} \hat{E} = \pi^* E$. Take P_{U_1} to be the unitary frame of E.

Let's look at the composed mapping $\hat{E} \to \hat{E} \otimes_{\mathbb{C}} \hat{E} \to E$, where the first mapping is the diagonal map. This map, restricted to the unitary frames, gives the Spin^c-equivariant map $P_{\mathrm{Spin}^c} \to P_{U_1}$. □

Summarizing the above propositions, the bundle $P_{\mathrm{Spin}^c} \to X$, together with the Spin^c-equivariant bundle map $P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}} \times P_{U_1}$, gives us a Spin^c-structure on X. This finishes the proof of Theorem 2. □

In Theorem 2, we saw that any equivalence class $[\not{S}]$ of bundles of complex irreducible Clifford modules, where $[\cdot]$ means the equivalence class, determines an equivalence class of complex irreducible Cl_n-modules (since X is connected an $\omega_{\mathbb{C}}(X)$ is a section of $\mathbb{C}\mathrm{l}(X)$) and an equivalence class of principal Spin^c-bundles (Proposition 1.2). Denote the pair by $\alpha([\not{S}]) \equiv ([P_{\mathrm{Spin}^c}], [\mu])$, where $(\mathbb{C}^N, \mu)$ is any complex irreducible Cl_n-module in the appropriate equivalence class. Conversely, given a principal Spin^c-bundle P_{Spin^c} and a complex irreducible Cl_n-module $(\mathbb{C}^N, \mu)$, we can construct the complex irreducible spinor bundle $\not{S} \equiv P_{\mathrm{Spin}^c} \times_{\mathrm{Spin}^c} \mathbb{C}^N$, where $\mathrm{Spin}^c \subseteq \mathbb{C}\mathrm{l}_n$ acts on $\mathbb{C}^N$ by $\mu \otimes \mathbb{C}$. This descends to equivalence classes, i.e., we define $\beta([P_{\mathrm{Spin}^c}], [\mu]) \equiv [\not{S}]$.

Theorem 3 *The above maps α and β are inverse of each other.*

Proof: To show $\alpha \circ \beta = \mathrm{Id}$, we start with particular representatives P'_{Spin^c} and μ', and let $\not{S}$ be the associated spinor bundle, so $[\not{S}] = \beta([P'_{\mathrm{Spin}^c}], [\mu'])$. To find $\alpha[\not{S}]$, we need to construct the principal Spin^c-bundle P_{Spin^c}. If $\omega_{\mathbb{C}}$ acts on $(\mathbb{C}^N, \mu')$ by 1 (or -1, respectively), then $\omega_{\mathbb{C}}(X)$ acts on $\not{S}$ by 1 (resp. -1) ([**1**, Proposition II.3.8]). As in Theorem 2, we can choose any μ in the same class as μ' is, so we can choose $\mu = \mu'$ in particular. Then $P_{\mathrm{Spin}^c} \subseteq P'_{\mathrm{Spin}^c}$, so they are equal since they are both principal Spin^c-bundles. Therefore $\alpha \circ \beta = \mathrm{Id}$.

Next we show $\beta \circ \alpha = \mathrm{Id}$. For any bundle $\not{S}$ of complex irreducible Clifford modules, let $\alpha([\not{S}]) = ([P_{\mathrm{Spin}^c}], [\mu])$. Fix representatives P_{Spin^c} and μ, and denote by $\not{S}'$ the spinor bundle associated to P_{Spin^c} and $(\mathbb{C}^N, \mu)$. We want to show $\not{S}' \cong \not{S}$ as bundles of complex Clifford modules. For any $\varepsilon \in P_{\mathrm{Spin}^c}$ and $s \in \not{S}$ lying above the same point in X, the coordinate of s under ε is $\sigma \equiv L^{\varepsilon}(s) \in \mathbb{C}^N$. Now let $f(s) = [\varepsilon, \sigma] \in P_{\mathrm{Spin}^c} \times_{\mathrm{Spin}^c} \mathbb{C}^N = \not{S}'$. We want to say $f : \not{S} \to \not{S}'$ is a $\mathrm{Cl}(X)$-equivariant isomorphism. First, f is well-defined: if we choose $\varepsilon' = \lambda\varepsilon\mathbf{g}$, then the new coordinate of s is $\sigma' \equiv L^{\varepsilon'}(s) = \dfrac{1}{\lambda}\mathbf{g}^{-1}\sigma$, so $[\varepsilon', \sigma']$ $= [\lambda\varepsilon\mathbf{g}, \dfrac{1}{\lambda}\mathbf{g}^{-1}\sigma] = [\varepsilon, \sigma]$. Second, f is injective: if $f(s) = f(s') = [\varepsilon, \sigma]$, then under the frame ε, both s and s' have the same coordinate $L^\varepsilon(s) = L^\varepsilon(s') = \sigma$, so $s = s'$. The fact that f is $\mathrm{Cl}(X)$-equivariant follows from the $\mathrm{Cl}(X)$ action of $\not{S}$ given in [**1**, Proposition II.3.8]. Since both $\not{S}$ and $\not{S}'$ are irreducible, f must be an isomorphism. $\square$

4 The real case in dimension 0, 6, 7 mod 8

Theorem 2 primarily says that, the existence of a complex bundle of irreducible $\mathrm{Cl}(X)$-modules is equivalent to the existence of a Spin^c-structure, and such bundles are spinor bundles. However, the real analogue is only true when n is congruent to $0, 6$ or 7 mod 8, where the proof is completely analogous to that of Proposition 1. As before, X is a compact oriented Riemannian manifold of dimension n.

Theorem 4 *Assume n is congruent to $0, 6$ or 7 mod 8. Suppose there exists a bundle $\not{S}$ on X of real irreducible Clifford modules, then X is spin.*

Proof: Let N denote the real rank of $\not{S}$. Fix a principal S^0-bundle associated to the line bundle $\Lambda^{top}\not{S}$, denoted by S (which plays the role of the volume form, whether $\not{S}$ is orientable or not). For a frame $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ of $\not{S}_x$, ε is called *unit volume* if $\varepsilon_1 \wedge \varepsilon_2 \wedge \ldots \wedge \varepsilon_N \in S$.

As before, define

$$P_{\mathrm{Spin}} \equiv \bigcup_{x \in X} \left\{ \begin{array}{c} \text{all unit volume bases } \varepsilon \text{ of } \not{S}_x \\ \text{so that there exists } e \in P_{\mathrm{SO}}\big|_x \text{ associated to } \varepsilon \end{array} \right\}$$

And we can make the parallel argument as follows.

Proposition 5 *Let e, $e' \in P_{\mathrm{SO}}|_x$, so $e = e'h$, for a unique $h \in \mathrm{SO}(n)$. Choose*

$g \in \mathrm{Spin}_n$ *such that* $\pi_0(g) = h$. *Now suppose* ε *and* ε' *are unit volume frames of* $\$_x$, *and that* ε *is associated to* e, *then* ε' *is associated to* e' *if and only if* $\varepsilon = \pm\varepsilon'\mathbf{g}$, *where* $\mathbf{g}$ *is now the real* $N \times N$ *matrix that represents* g.

Proof:

$$\begin{tikzcd}
& \mathrm{Cl}_x(X) \otimes \$_x \arrow[dl, "L^e \otimes L^\varepsilon"'] \arrow[dr, "L^{e'} \otimes L^{\varepsilon'}"] & \\
\mathrm{Cl}_n \otimes \mathbb{R}^N \arrow[rr] \arrow[d, "\mu"'] & & \mathrm{Cl}_n \otimes \mathbb{R}^N \arrow[d, "?"] \\
\mathbb{R}^N \arrow[rr] & & \mathbb{R}^N \\
& \$_x \arrow[ul, "L^\varepsilon"] \arrow[ur, "L^{\varepsilon'}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \phi \otimes s \arrow[dl, "L^e \otimes L^\varepsilon"'] \arrow[dr, "L^{e'} \otimes L^{\varepsilon'}"] & \\
\varphi \otimes \sigma \arrow[rr] \arrow[d, "\mu"'] & & g\varphi g^{-1} \otimes \alpha\sigma \arrow[d, "?"] \\
\mu(\varphi \otimes \sigma) \arrow[rr] & & \alpha\mu(\varphi \otimes \sigma) \\
& \rho(\phi \otimes s) \arrow[ul, "L^\varepsilon"] \arrow[ur, "L^{\varepsilon'}"'] &
\end{tikzcd}$$

Let $\varepsilon = \varepsilon'\alpha$ for some $\alpha \in \mathbb{R}(N)$. Just as in the proof of Proposition 1, we have the diagram above, and ε' is associated to e' if and only if $g\varphi g^{-1}\alpha\sigma = \alpha\varphi\sigma$ for any $\varphi \in \mathrm{Cl}(n)$ and $\sigma \in \mathbb{R}^N$. Using matrix notations, this is written $\mathbf{g}\boldsymbol{\varphi}\mathbf{g}^{-1}\alpha = \alpha\boldsymbol{\varphi}$, or $\mathbf{g}^{-1}\alpha\boldsymbol{\varphi} = \boldsymbol{\varphi}\mathbf{g}^{-1}\alpha$, for any $\varphi \in \mathrm{Cl}_n$, namely $\mathbf{g}^{-1}\alpha$ commutes with $\boldsymbol{\varphi}$ for any $\varphi \in \mathrm{Cl}_n$. In case n is congruent to $0, 6$ or 7 mod 8, Cl_n is isomorphic to (not in correspondence) $\mathbb{R}(N)$ or $\mathbb{R}(N) \oplus \mathbb{R}(N)$, and $\{\boldsymbol{\varphi} : \varphi \in \mathrm{Cl}_n\}$ is $\mathbb{R}(N)$. If $\mathbf{g}^{-1}\alpha$ commutes with either $\mathbb{R}(N)$, then it has to be a real scalar, call it λ. Now both ε and ε' are both unit volume, so $\lambda = \pm 1$. $\square$

The proof that the projection $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$ is Spin-equivariant is a duplication of the complex case. So we proved Theorem 4. $\square$

5 The real case: other dimensions

In the other cases the statement is not true, namely, a real bundle of irreducible $\mathrm{Cl}(X)$-modules does not guarantee the existence of a Spin structure. Note that every spinor bundle is of course a bundle of Clifford modules. In fact, if n is congruent to $1, 2, 3, 4$ or 5 mod 8, then X admits a real bundle of irreducible Clifford modules if and only if X is Spin^c, whereas X may not be Spin.

Proposition 6 *Suppose $n = \dim X$ is congruent to $1, 2, 3, 4$ or 5 mod 8. If there exists a bundle $\not{S}$ of complex irreducible Clifford modules over X, then $\not{S}$ is a bundle of real irreducible Clifford modules over X by dropping the complex structure.*

Proof: $\not{S}$ is obviously a bundle of real Clifford modules. The only thing to prove is that $\not{S}$ is real irreducible. We prove this by looking at its rank. Since Cl_n is an algebra with identity, it suffices to show that real irreducible Cl_n-modules and have the same dimension as complex irreducible ones have when $n \equiv 1, 2, 3, 4$ or 5 mod 8.

If $n \equiv 1$ or 5 mod 8, $\mathrm{Cl}_n \cong \mathbb{C}(N)$ and $\mathbb{C}\mathrm{l}_n \cong \mathbb{C}(N) \oplus \mathbb{C}(N)$, so both real and complex irreducible Cl_n-modules have real dimension $2N$.

If $n \equiv 2$ or 4 mod 8, $\mathrm{Cl}_n \cong \mathbb{H}(N)$ and $\mathbb{C}\mathrm{l}_n \cong \mathbb{C}(2N)$, so both real and complex irreducible Cl_n-modules have real dimension $4N$.

If $n \equiv 3$ mod 8, $\mathrm{Cl}_n \cong \mathbb{H}(N) \oplus \mathbb{H}(N)$ and $\mathbb{C}\mathrm{l}_n \cong \mathbb{C}(2N) \oplus \mathbb{C}(2N)$, so both real and complex irreducible Cl_n-modules have real dimension $4N$. $\square$

Since every Spin^c manifold admits a bundle of complex irreducible Clifford modules, but may or may not be Spin, the above proposition has the following immediate consequence.

Corollary *Let X be a compact* Spin^c *manifolds of real dimension n congruent to $1, 2, 3, 4$ or 5 mod 8, with $w_2(X) \neq 0$. Then X has a bundle of real irreducible Clifford modules, but X is not* Spin.

Example The manifolds $\mathbb{CP}^2$ and $S^1 \times \mathbb{CP}^2$ are Spin^c but not Spin. If X is a compact Spin^c manifold but not Spin, and if Y is a compact Spin manifold, then $X \times Y$ is compact and Spin^c but not Spin. On each of the above manifolds, as long as the dimension of the manifold is $1, 2, 3, 4$ or 5 mod 8, there exists a bundle of real irreducible Clifford modules but the manifold is not Spin.

Proof:

We know the second Stiefel-Whitney class $w_2(\mathbb{CP}^2) \neq 0$ and $w_2(\mathbb{CP}^2)$ is the mod 2 reduction of $c_1(\mathbb{CP}^2)$, so $\mathbb{CP}^2$ is Spin^c but not Spin.

If $X = S^1 \times \mathbb{CP}^2$, since the full Stiefel-Whitney class $w(S^1) = 1$, $w_2(X) \neq 0$, and X is not Spin. Since $w_2(\mathbb{CP}^2)$ is $c_1(\mathbb{CP}^2)$ mod 2, pulling this back to X we see that $w_2(X)$ comes from the integral class, namely the pull back of $c_1(\mathbb{CP}^2)$ to X, by the mod 2 reduction. Therefore X is Spin^c.

If X is Spin^c but not Spin, and Y is Spin, then X embeds in $X \times Y$ and so $X \times Y$ is not Spin. Since both X and Y are Spin^c, so is $X \times Y$.

The rest follows from the corollary. $\square$

6 The real case and Spin^h-manifolds

In Proposition 6 of the previous section the key things we are using are that

• $\mathbb{C} \otimes \mathbb{C} \equiv \mathbb{C} \oplus \mathbb{C}$, so a real irreducible $\mathbb{C}$-module and a complex irreducible $\mathbb{C}$-module have the same real dimension ($= 2$).

• $\mathbb{H} \otimes \mathbb{C} \equiv \mathbb{C}(2)$, so a real irreducible $\mathbb{H}$-module and a complex irreducible $\mathbb{H}$-module have the same real dimension ($= 4$).

These guarantee that a bundle complex irreducible Clifford modules is also real irreducible in case the Clifford algebra is complex or quaternionic.

Note we also know

• $\mathbb{H} \otimes \mathbb{H} \equiv \mathbb{R}(4)$, so a real irreducible $\mathbb{H}$-module and a quaternionic irreducible $\mathbb{H}$-module (where the quaternionic scalar multiplication and the $\mathbb{H}$-module action act on different sides of the module) have the same real dimension ($= 4$). This can be more easily observed by noting that $\mathbb{H}$ can act on $\mathbb{H}$ from the left, or from the right by inverse, and these two actions commute.

This indicates that, in case the Clifford algebra is quaternionic, we may only need something weaker than a Spin^c-structure to guarantee the existence of a bundle of real irreducible Clifford modules. Let's introduce the following definiton analogous to the definition of the Spin^c-structure.

Definition 5 *Denote* $\mathrm{Spin}_n^h = \mathrm{Spin}_n \times_{\mathbb{Z}_2} \mathrm{Sp}_1$*, where* Sp_1 *is the group of unit quaternions. Then* Spin_n^h *is a group and lies in the algebra* $\mathrm{Cl}_n \otimes \mathbb{H}$*. There are well defined maps*

$$\mathrm{Spin}_n^h \to \mathrm{Spin}_n/\mathbb{Z}_2 \equiv \mathrm{SO}_n$$

and

$$\mathrm{Spin}_n^h \to \mathrm{Sp}_1/\mathbb{Z}_2 \equiv \mathrm{SO}_3$$

.

A <u>Spin^h*-structure*</u> *on an oriented riemannian* n*-mainfold* X *is a principal* Spin_n^h*-bundle* $P_{\mathrm{Spin}^h}(X)$ *and a principal* SO_3*-bundle* $P_{\mathrm{SO}_3}(X)$ *over* X*, together* *with a* Spin^h_n*-equivariant bundle map* $P_{\mathrm{Spin}^h}(X) \to P_{\mathrm{SO}}(X) \times P_{\mathrm{SO}_3}(X)$ *where* $P_{\mathrm{SO}}(X)$ *is the orthonormal frame bundle on* X*.*

Proposition 7 *Suppose* $n = \dim X$ *is congruent to* $2, 3$ *or* 4 *mod* 8*. If* X *admits a* Spin^h*-structure, then there is a bundle of real irreducible Clifford modules over* X*.*

Proof: Recall that if a rank n vector bundle E is spin, then $\mathrm{Cl}(E) = P_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}} \mathrm{Cl}_n$, where Ad is the adjoint (or conjugate) action of the Spin_n over Cl_n, namely $\mathrm{Ad}_g(\varphi) = g\varphi g^{-1}$ for $g \in \mathrm{Spin}_n$, $\varphi \in \mathrm{Cl}_n$. Note that since Ad factors through SO_n, we can define a similar action of Spin^h_n on Cl_n by $\mathrm{Ad}^h_{[g,q]} = g\varphi g^{-1}$ for $[g, q] \in \mathrm{Spin}^h_n$, where $g \in \mathrm{Spin}_n$ and $q \in \mathrm{Sp}_1$. Now if a rank n vector bundle E is Spin^h, then $\mathrm{Cl}(E) = P_{\mathrm{Spin}^h}(E) \times_{\mathrm{Ad}^h} \mathrm{Cl}_n$.

Suppose an n-manifold X is Spin^h, and M is an irreducible ($\mathrm{Cl}_n \otimes \mathbb{H}$)-module. Let $E = P_{\mathrm{Spin}^h}(X) \times_{\mathrm{Spin}^h_n} M$. We then proceed as in [**1**, Proposition II.3.8]. The diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}^h}(X) \times Cl_n \times M & \xrightarrow{\mu} & P_{\mathrm{Spin}^h}(X) \times M \\ \downarrow{\scriptstyle \rho_{[g,q]}} & & \downarrow{\scriptstyle \rho_{[g,q]}} \\ P_{\mathrm{Spin}^h}(X) \times Cl_n \times M & \xrightarrow[\mu]{} & P_{\mathrm{Spin}^h}(X) \times M \end{array}$$

given by

$$\begin{array}{ccc} (p, \varphi, m) & \longrightarrow & (p, \varphi m) \\ \downarrow & & \downarrow \\ (p[g,q]^{-1}, g\varphi g^{-1}, [g,q]m) & \longrightarrow & (p[g,q]^{-1}, [g,q]\varphi m) \end{array}$$

commutes (to justify the last line of the second diagram, note that the $\mathbb{H}$ component and the Cl_n component in $\mathrm{Cl}_n \otimes \mathbb{H}$ commute with each other, namely $[g, 1][1, q] = [g, q] = [1, q][g, 1]$). Therefore μ descends to a mapping

$$\mathrm{Cl}(X) \oplus E \to E$$

which makes E a $\mathrm{Cl}(X)$-module.

In case $n = \dim X$ is $2, 3$ or 4 mod 8, note that Cl_n is either $\mathbb{H}(N)$ or $\mathbb{H}(N) \oplus \mathbb{H}(N)$ for some number N, and $\mathrm{Cl}_n \otimes \mathbb{H}$ is either $\mathbb{R}(4N)$ or $\mathbb{R}(4N) \oplus \mathbb{R}(4N)$ respectively. Therefore, M as an irreducible $(\mathrm{Cl}_n \otimes \mathbb{H})$-module must be isomorphic to $\mathbb{R}^{4N}$, and have real dimension $4N$. So E is rank $4N$ as a real vector bundle. But we proved E is a bundle of $\mathrm{Cl}(X)$-modules, and sin note that the (real) dimension of M is just the dimension of an irreducible (real) Cl_n-module. So E is a bundle of $\mathrm{Cl}(X)$-modules, and an irreducible one should have rank $4N$, so E must be irreducible. $\square$

Now we want to find some examples of Spin^h-mainfold that are not Spin^c. We need the following topological characterization first.

Proposition 8 *An oriented connected (riemannian) manifold X is Spin^h if and only if there exists a rank 3 real vector bundle E over X, so that $\mathbf{w}_2(TX) = \mathbf{w}_2(E)$.*

Proof: Given the exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}^h_n \to \mathrm{SO}_n \times \mathrm{SO}_3 \to 1$$

of coefficient groups, we have the exact sequence

$$H^1(X; \mathrm{Spin}^h_n) \xrightarrow{\xi} H^1(X; \mathrm{SO}_n) \oplus H^1(X; \mathrm{SO}_3) \xrightarrow{\mathbf{w}_2 + \mathbf{w}_2} H^2(X; \mathbb{Z}_2)\ .$$

Now X admits a Spin^h-structure if and only if there is a principal Spin^h-bundle $P_{\mathrm{Spin}^h}(X)$ and a principal SO_3-bundle E so that $\xi(P_{\mathrm{Spin}^h}(X)) = P_{\mathrm{SO}}(X) \oplus E$, and by the exact sequence above, the existence of $P_{\mathrm{Spin}^h}(X)$ is equivalent to $\mathbf{w}_2(P_{\mathrm{SO}}(X)) + \mathbf{w}_2(E) = 0$, namely $\mathbf{w}_2(X) = \mathbf{w}_2(E)$. $\square$

Example Let $X = \mathrm{SU}_3/\mathrm{SO}_3$ be the oriented manifold whose only non-zero mod 2 cohomology classes are $1, \mathbf{w}_2, \mathbf{w}_3$ and $\mathbf{w}_2 \cdot \mathbf{w}_3$ [**1**, Appendix D], and according to Landweber and Stong, is not Spin^c. We claim X is Spin^h.

Proof: Let E be the rank 3 vector bundle associated to the SO_3-bundle defined by the natural projection $\mathrm{SU}_3 \to X$. We claim $\mathbf{w}_2(E) = \mathbf{w}_2(X)$. Since the only non-zero element in $H^2(X;\mathbb{Z}_2)$ is $\mathbf{w}_2(X)$, we only need to show $\mathbf{w}_2(E) \neq 0$. Note $\mathbf{w}_1(E) = 0$ since $H^1(X;\mathbb{Z}_2) = 0$. If $\mathbf{w}_2(E) = 0$, then E is Spin, namely, there exists a double cover $P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E) = \mathrm{SU}_3$. This contradicts the fact that $\pi_1(\mathrm{SU}_3) = 0$. $\square$

7 Proof of Theorem 1

Proof of Theorem 1: Notice that $\mathrm{Cl}_n = I_1 \oplus I_2 \oplus \ldots \oplus I_M$ for some non-zero "minimal" left ideals $I_1, I_2, \ldots, I_M$, where "minimal" means the left ideal I_i is irreducible considered as a Cl_n-module. In case $\mathrm{Cl}_n \cong k(N)$, we have $\mathrm{Cl}_n = I_1 \oplus I_2 \oplus \cdots \oplus I_N$ where $I_1 \cong I_2 \cong \cdots \cong I_N \cong k^N$ is the unique irreducible Cl_n-module up to equivalence. In case $\mathrm{Cl}_n \cong k(N) \oplus k(N)$, we have $\mathrm{Cl}_n = I_1 \oplus I_2 \oplus \cdots \oplus I_{2N}$ where $I_1 \cong I_2 \cong \cdots \cong I_N \cong k^N$ and $I_{N+1} \cong I_{N+2} \cong \cdots \cong I_{2N} \cong k^N$ are the two inequivalent Cl_n-modules. If we have a local trivialization $\mathrm{Cl}(X)|_U \cong \mathrm{Cl}_n \times U$, then also use $I_1, I_2, \ldots, I_M$ to denote the subbundles.

We first prove that, in case $\not{S}$ is reducible, locally there is a (continuous) decomposition $\not{S} = \not{S}_1 \oplus \not{S}_2 \oplus \ldots \oplus \not{S}_l$ into irreducible Clifford bundles (may not be possible globally). Then we prove the theorem in case $\not{S}$ is irreducible. Everything done below is local.

First we decompose $\not{S}$ locally. Denote the rank of $\not{S}$ by r. At each point $x_0 \in X$, there are linearly independent vectors $s_1(x_0), s_2(x_0), \ldots, s_r(x_0) \in \not{S}_{x_0}$. Trivialize $\mathrm{Cl}(X)$ on a neighbourhood U of x_0. Extend $s_1, s_2, \ldots, s_r$ continuously to U, shrinking U if necessary, so that they are pointwise linearly independent on U. For each $s_j(x_0)$, since $1 \in \mathrm{Cl}_x(X)$ acts on it as identity, the subspace $\mathrm{Cl}_{x_0}(X)s_j(x_0) = \{\phi s_j(x_0) : \phi \in \mathrm{Cl}_{x_0}(X)\}$ is non-zero. Since $\mathrm{Cl}_{x_0}(X) = (I_1)_{x_0} \oplus (I_2)_{x_0} \oplus \ldots \oplus (I_M)_{x_0}$, there exists an i_j so that $(I_{i_j})_{x_0} s_j(x_0)$ is not the zero space. Shrinking U if necessary, we see $I_{i_j} s_j \subseteq \not{S}$ is a non-zero subbundle of $\not{S}$ on U, and an irreducible bundle of Clifford modules. Denote this bundle on U by S_j. Then $\not{S} = \sum_{1 \le j \le r} S_j$. Since each S_j is irreducible, we can choose a sub-collection $\not{S}_i$ of $\{S_j\}_{1 \le j \le r}$ so that $\sum \not{S}_{i x_0}$ is a direct sum and is $\$_{x_0}$. Since the rank of $\sum \$_i \subseteq \$$ is locally minimal, $\$$ is the direct sum of the $\$_i$'s in a neighbourhood of x_0.

Now we can suppose $\$$ is irreducible. As before, there is a left ideal I of $\mathrm{Cl}(X)|_U$, so that $I\$ = \$$ and I itself is a bundle irreducible Clifford modules over U. Now choose any continuous sections s of $\$$ on U, so that $s(x_0) \neq 0$. Then $I_{x_0} \to \$_{x_0}$, $A \mapsto As(x_0)$ is a vector-space isomorphism. Shrinking U, we get a vector-bundle isomorphism $I \xrightarrow{\cdot s} \$$. Choose a constant section ϵ of basis of I so that the diagram

$$\begin{tikzcd}
\mathrm{Cl}_x(X) \otimes I_x \arrow[r, "\rho_x"] \arrow[d, "L^{\epsilon(x)} \otimes L^{\epsilon(x)}"'] & I_x \arrow[d, "L^{\epsilon(x)}"'] \\
\mathrm{Cl}_n \otimes M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

commutes. Denote the image of ϵ under $I \xrightarrow{\cdot s} \$$ by ε, then the diagram in the theorem commutes. $\square$

Reference

[1] H. Blaine Lawson and Marie-Louise Michelsohn, *Spin Geometry*, Princeton University Press, 1990

[2] John W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Princeton University Press, 1996