Algebraic Structures with Structure Constants, and Homotpical Algebra

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Abstract of the Dissertation

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We consider algebraic structures on vector spaces (or chain complexes) V with operations having any number, m, inputs, and any number, n, outputs, including m or n equal to 0. An operation with 0 inputs and n outputs means a choice of an element in the n-fold tensor product of V (for example, the unit of a commutative algebra), an operation with m inputs and 0 outputs means a linear map from the n-fold tensor product of V to the ground field (for example, a linear functional or a pairing), and an operation with 0 inputs and 0 outputs means an element of the ground field, ie a constant (for example, the volume of a manifold as part of an algebra structure its differential forms). The operations may involve a boundary map, so we call the homology classes of the constant operations "structure constants".

Such an algebraic structure is determined by a certain map. We study this map up to an algebraic version of homotopy, and show, for example, that if the maps defining two algebraic structures are homotopic, then they have equal structure constants.

We can also compare algebra structures expressed in different ways on different spaces, and transport (resolved) algebra structures on one space to algebra structures on another space, such that the structure constants only change by an overall scale factor. Given extra structure, we can give explicit formulas for the transported structures. Such extra structure always exists, which allows us to transport a structure on a chain complex to its homology by giving an explicit formula.

Table of Contents

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Chapter 1

Introduction

We consider algebraic structures on vector spaces (or chain complexes) V with operations having any number, m, inputs, and any number, n, outputs, including m or n equal to 0. An operation with 0 inputs and n outputs means a choice of an element in the n-fold tensor product of V (for example, the unit of a commutative algebra), an operation with m inputs and 0 outputs means a linear map from the n-fold tensor product of V to the ground field (for example, a linear functional or a pairing), and an operation with 0 inputs and 0 outputs means an element of the ground field, ie a constant (for example, the volume of a manifold as part of an algebra structure its differential forms). The operations may involve a boundary map, so we call the homology classes of the constant operations "structure constants".

One can sometimes do homotopical algebra á la [9] at the level of algebra structures on chain complexes using free resolutions. but with structures having both products and coproducts, this becomes impossible, because there are no free structures. However, as Dmitri Tamarkin discussed with Sullivan, operations themselves form a sort of algebras (with products given by, for example, composition). Also algebraic structures on chain complexes are determined by maps of algebras at the level of operations on the chain complex. Thus one can form resolutions at this level (this is the idea of an infinity version of an algebra). In [8], Dennis Sullivan describes how one can do homotopical algebra at this second level to give homotopy theoretic equivalences on algebra structures on chain complexes (though only for operations having positively many inputs and outputs). In this thesis, we add operations having 0 inputs or 0 outputs, along with constants, and we examine their role. We show (for admissible cases) that if the maps defining two algebraic structures are homotopic, then they have equal structure constants.

Following this scheme requires considering certain algebra structures on spaces of operations on chain complexes (which is required, for example, to write relations on operations). We show that if we consider only only certain types of operations (like compositions, tensor products, and so forth) then it becomes easy to transport (resolved) algebra structures on one chain complex to algebra structures on another chain complex, such that the structure constants are equal up to an overall scaling factor. We observe that equivalence of algebra structures on chain complexes via zig-zags implies homotopy equivalence of their structure maps.

Finally, given certain extra structure, we can give explicit formulas for the transported structures. Such extra structure always exists, which allows us to transport a structure on a chain complex to its homology by giving an explicit formula.

Chapter 2

Algebraic Structures, Homotopical Algebra, and Structure Constants

2.1 Spaces, Operations, and Combination Operations

We wish to consider algebraic structures on chain complexes up to certain homotopy theoretic equivalences. We think of an algebraic structures as the data of a chain complex, a space operations on the chain complex (which one may picture using planar trees with one internal vertex), and another space of operations–called combination operations–which give operations on the operations (eg composition of operations). One may imagine combination operations by how they act on operations (for example by composing outputs of a tree into inputs of another tree), or as trees themselves. Using this latter image, one sees combination operations can also be composed, for example, thus providing them with structure.

We think of a space of combinations operations as an algebra under composition and certain other operations. This algebra of combination operations may act on a space of operations, giving it algebraic structure, which in turn may act on a vector space, giving it algebraic structure. Thus just as we consider abstract groups instead of just groups of permutations, we may consider abstract algebras of combination operations (given algebraic structure by composition and other operations) and abstract algebras of operations (given algebraic structure by combination operations).

We allow, most generally, multilinear operations on chain complexes V with k inputs and l outputs, that is, linear maps of chain complexes

$$\underbrace{V\otimes\ldots\otimes V}_{k}\to\underbrace{V\otimes\ldots\otimes V}_{l}$$

where k and l are allowed to range over 0, 1, 2 and higher. A 0-to-k operation on V means a map from the zero-fold tensor product $V^{\otimes k}=\mathbb{K}$ (which is the chain complex with the ground field concentrated in degree zero and the zero differential), to $V^{\otimes k}$, or equivalently, a choice of an element in $V^{\otimes k}$. The l-to-0 operations are maps from $V^{\otimes l}$ to $\mathbb{K}$, that is, linear functionals on V, bilinear pairings on V and so forth. A 0-to-0 operation on V is a map from $\mathbb{K}$ to itself, or equivalently, a choice of an elements of $\mathbb{K}$, that is, a constants.

The k-fold tensor product of a chain complex V is the chain complex whose degree-n component consists of k-fold tensor products of elements of V whose degrees sum to n, with differential

$$d(v_1 \otimes \ldots \otimes v_k) := \sum_i \pm v_1 \otimes \ldots \otimes dv_i \otimes \ldots \otimes v_k$$

whose sign (as most other signs herein) is determined by the Koszul sign convention. The k-to-l operations on a fixed chain complex V also form a chain complex $Hom(V^{\otimes k}, V^{\otimes l})$, and it is graded by degree (which measures the difference between the sum of the degrees of the inputs and the sum of the degrees of the outputs), with differential given by the (graded) commutator with the differential on the input and output complexes, that is

$$\partial p := [d, p] := d \circ p \mp p \circ d$$

The space of all operations on a chain complex V is the set $End(V)$ of chain complexes $Hom(V^{\otimes k}, V^{\otimes l})$ operations, where k and l range over some family $\mathcal{O}$ of choices of pairs of k and l. We call $\mathcal{O}$ the set of operation types. Thus, in the presence of a set of operation types, $\mathcal{O}$, a space of operations is (by definition) a set of chain complexes $P = \{P(a) | a \in \mathcal{O}\}$ indexed by $\mathcal{O}$.

We consider only those combination operations, which input n operations and output 1 operation, where n is allowed to range over 0, 1, 2 and higher. An n-to-1 combination operation on a space of operations $P = \{P(a) | a \in \mathcal{O}\}$ consists of map of chain complexes

$$P(a_1) \otimes \ldots \otimes P(a_n) \to P(a_0)$$

for some fixed choices of inputs types $a_1, \ldots, a_n$ and output types a_0, chosen from $\mathcal{O}$. The tensor product $P(a_1) \otimes \ldots \otimes P(a_n)$ is a chain complex with a differential like before, denoted ∂,

$$\partial(p_1 \otimes \ldots \otimes p_n) := \sum_i \pm p_1 \otimes \ldots \otimes \partial p_n \otimes \ldots \otimes p_n$$

and the Hom space $Hom(P(a_1) \otimes \ldots \otimes P(a_n), P(a_0))$ is a chain complex with differential like before, denoted $\mathcal{D}$

$$\mathscr{D}(c) := [\partial, c] := \partial \circ c \mp c \circ \partial$$

The space of all combination operations on a space of operations P is the set $End(P)$ of chain complexes $Hom(P(a_1) \otimes \ldots \otimes P(a_n), P(a_0))$ where n ranges over $0, 1, 2, \ldots$, and $a_1, \ldots, a_n$ and a_0 range over $\mathcal{O}$. Thus the set $End(P)$ is indexed by the set $\mathcal{C}(\mathcal{O}) := \{\binom{a_1 \ldots a_n}{a_0} | a_1, \ldots, a_n, a_0 \in \mathcal{O}, n = 0, 1, 2, \ldots\}$ which we call the set of combination operation types, whose elements we denote with small capital letters, or by $\binom{a_1 \ldots a_n}{a_0}$ to display their input and output types. Thus, in the presence of a set of operations types $\mathcal{O}$, a space of combination operations C is (by definition) a set of chain complexes $C = \{C(\textsc{f}) | \textsc{f} \in \mathcal{C}(\mathcal{O})\}$.

We will soon observe that the space $End(P)$ of combination operations on a space of operations P has certain natural choices of operations on it, given by composition and certain other operations, and these satisfy certain relations. We use the spaces $End(P)$ as prototypes of a combination algebra. We say a space of combination operations C is a combination algebra, if it has operations analogous to those on the spaces $End(P)$, which satisfy the relations we know hold for all $End(P)$. These form a category with maps that commute with the operations.

We define an action of a combination algebra C on a space of operations to be a map of algebras $C \to End(P)$. Such a map gives P algebraic structure with operations furnished by C, which compose and interact in the ways operation on $End(P)$ do. If P has a distinguished action of C on it, then then P is an algebra of operations, and we call it a C algebra structure. These form a category with maps that commute with operations.

If V is a chain complex, then $End(V)$ is a space of operations on V, thus a combination algebra C may provide $End(V)$ with algebraic structure (by a map of combination algebras $C \to End(End(V))$). If $End(V)$ has distinguished C algebra structure, then an action of another C algebra P on V is a map of C algebras $P \to End(V)$. Such a map gives V algebraic structure with operations furnished by P, which interact in the manner operations in $End(V)$ and P interact as algebras with operations given by C. These, too, forma a category with maps that commute with operations. We call the C algebra map $P \to End(V)$ the algebra structure's structure map.

We can form free resolutions and do homological algebra at the level of algebras of operations; this allows us to define various homotopical equivalences between maps of algebras of operations. Since algebra structures are given by maps, we can study them up to the homotopical equivalences between their structure maps.

One notes that there aren't necessarily free resolutions at the level of algebras structures on a chain complex, since we aren't even assured of having free algebras when operation with multiple inputs and outputs are present. Thus we are forced, in a sense, to move up a level to algebras of operations.

We work over a fixed field $\mathbb{K}$ of characteristic 0. All tensor products are taken over $\mathbb{K}$.

2.2 Combination Algebras

We find it convenient to study combination algebras for an arbitrary set $\mathcal{O}$ of operation types.

Definition 2.1. *A set of* ***operation types****, denoted $\mathcal{O}$ is simply a set. In the presence of a set of operation types, we define*

$$\mathcal{C}(\mathcal{O}) := \{\binom{a_1 \dots a_n}{a_0} | a_1, \dots, a_n, a_0 \in \mathcal{O}, n = 0, 1, 2, \dots\}$$

to be the set of ***combination operation types*** *on operation types $\mathcal{O}$.*

Let $\mathrm{F} = \binom{a_1 \dots a_n}{a_0}$ be a type of combination operation on $\mathcal{O}$. We call $a_1, \dots, a_n$ the ***inputs*** *or* ***input types*** *of F, and a_0 the* ***outputs*** *or* ***output type*** *of F. We call the number of input types of F its* ***arity****. The arity zero types look like $\binom{}{a}$ for some a in $\mathcal{O}$.*

Consider the space of all combination operations $End(P)$ on a space of operations P (as in the previous section). Let f and g be two combination operations, of types $\mathrm{F} = \binom{a_1 \dots a_m}{a_0}$ and $\mathrm{G} = \binom{b_1 \dots b_n}{b_0}$, respectively. If the output b_0 of g and the i-th input a_i of f are equal, then we can compose g into the i-th slot of g to get a new operation

$$(f \;{}_i\circ\; g) := f \circ (1 \otimes \dots \otimes g \otimes \dots \otimes 1)$$

of type

$$F \;{}_i\circ\; G := \binom{a_1 \dots \otimes a_{i-1} b_1 \dots b_n a_{a+1} \dots a_m}{a_0}$$

We could also permute the m inputs of f with a permutation σ in the m-th permutation group S_m to get a new operation

$$f^\sigma$$

of type

$$F^\sigma := \binom{a_{\sigma^{-1}(1)} \cdots a_{\sigma^{-1}(m)}}{a_0}$$

For each operation type a, there is also an identity operation

$$1_a$$

of type

$$\binom{a}{a}$$

Proposition 2.1. *There is a certain space of relations that the operations of composition, permutation of inputs and units on a space of operations satisfy, universally.*

We call these relations the relations of a combination algebra, and we discuss them in Lemm 4.5 of the appendix. We define a combination algebra to be a set of chain complexes with analogous operations, which satisfy these relations, the relations of a combination algebra:

Definition 2.2. *In the presence of a set of operation types $\mathcal{O}$, a **combination algebra (on operation types $\mathcal{O}$)** is a set of chain complexes $C = \{C(\mathrm{F}) | \mathrm{F} \in \mathcal{C}(\mathcal{O})\}$ indexed by the various types of combination operations $\mathcal{C}(\mathcal{O})$, together with operations given by degree zero chain maps, as follows, which satisfy the relations in Proposition 4.5:*

1. *a **composition** for each $\mathrm{F}\ {}_i\circ\ \mathrm{G}$ that makes sense*

$$[\mathrm{F}\ {}_i\circ\ \mathrm{G}] : C(\mathrm{F}) \otimes C(\mathrm{G}) \to C(\mathrm{F}\ {}_i\circ\ \mathrm{G})$$

2. *a **permutation** for each F^σ that makes sense*

$$[\mathrm{F}]^\sigma : C(\mathrm{F}) \to C(\mathrm{F}^\sigma)$$

3. *and an operation called a **unit** for each operation type a*

$$\mathbb{K} \to C\binom{a}{a}$$

There is an analogous notion of a combination algebra without a differential. To emphasize that a combination algebra lacks a differential, we say graded combination algebra, and to emphasize that it does have a differential, we say differential graded (or dg) combination algebra.

We denote the differential on combination operations by $\mathcal{D}$.

We may consider the category of combination algebras on a fixed set $\mathcal{O}$ of operation types. A map of combination algebras $B \to C$ is a set of maps of chain complexes (or graded vector spaces if lacking a differential) $B(\mathrm{F}) \to C(\mathrm{F})$ which commute with all the operations

The most important facts to note at present is the following:

Proposition 2.2. *Combination algebras can be specified by generators and relations, and given the zero differential.*

One should also note that combination algebras have the usual notions of ideals, kernels, quotient algebras, subalgebras, isomorphisms, direct limits, and certain other. For full details see the appendix.

2.3 Algebras of Operations Over a Combination Algebra

Recall that a space of operations (in the presence of a set of operation types) is a set of chain complexes indexed by the set of operation types. We note the following tautology:

Proposition 2.3. *Let P be a space of operations. The space $End(P)$ of all combination operation on P has the structure of a combination algebra.*

Definition 2.3. *Given a space P of operations, we call the distinguished combination algebra structure on $End(P)$ given by composition, permutation of inputs and the units the **combination algebra of endomorphisms on** P. We only consider $End(P)$ with this structure.*

*An **action of a combination algebra** C **on a space of operations** P is a map of combination algebras $C \to End(P)$.*

*We say P is a **dg$C$ algebra** if P has a distinguished action of C on it. We may also say P is an algebra over C. Or if the action is clear from the context, we may simply say P is an algebra of operations.*

There are analogous definitions without differentials. We emphasize the difference by applying the graded or differential graded prefix, just as we dif for combination algebras.

Given a fixed combination algebra C we may form the category of all algebras over C, with or without differentials. These categories have the usual notions of ideals, differential ideals, kernels, quotient algebras, direct limits, and coproducts. Ignoring the differential, we can specify algebras over a combination algebra by generators and relations.

We find it convenient to work with algebras over combination algebras that have zero differential, due, in part, to the sequence of facts to follow, whose proofs are left to the appendix.

Proposition 2.4. *Let C be a combination algebra with zero differential, and let P be a dg$C$ algebra.*

The differential on P acts by derivations of the combination operations, and dg$C$ algbera structure on P induces a dg$C$ algbera structure on $H(P)$ (with zero differential), moreover a map $P \to Q$ of dg$C$ algebras induces a map of dg$C$ algebras $H(P) \to H(Q)$.

Proof. The failure of the differential ∂ on P to be a derivation of a combination operation c in C is measured by its boundary $\mathcal{D}c$. Since $\mathcal{D}$, and thus $\mathcal{D}c$,too, is zero, ∂ is a derivation.

The homology of P is an algebra over the homology of C, but $C = H(C)$, so $H(P)$ is again a dg$C$ algebra with zero differential.

The induced map is a map of dg$H(C)$ algebras, but $C = H(C)$, so it is a map of dg$C$ algebras. □

Definition 2.4. *Fix a set $\mathcal{O}$ of operation types.*

*A **free variable, free operations, free generators, etc.**, is an entity with an integer degree and operation type in $\mathcal{O}$.*

Let C be a combination algebra.

Let $\{x_\alpha\}$ be a set of free generators. A ***free graded C algebra on the set*** *$\{x_\alpha\}$ of free generators is a graded C algebra $\mathbb{K}[x_\alpha]$ (without a differential), together with a distinguished map $\{x_\alpha\} \to \mathbb{K}[x_\alpha]$ such that any map $\{x_\alpha\} \to Q$ to an algebra over C extends uniquely to a map of graded C algebras $\mathbb{K}[x_\alpha] \to Q$.*

Let P be an algebra over C and $\{x_\alpha\}$ a set of free generators. A ***free extension of P by the set*** *$\{x_\alpha\}$ is a graded C algebra $P[x_\alpha]$ (without a differential) together with a distinguished map of sets $\{x_\alpha\} \to P[x_\alpha]$ and a distinguished map of C algebras $P \to P[x_\alpha]$ such that any maps $\{x_\alpha\} \to Q$ and $P \to Q$ of sets and C algebras, respectively, extend uniquely over $P[x_\alpha]$ to a graded C algebra.*

Proposition 2.5. *Free algebras and free extensions exist, and are unique up to canonical isomorphism. We denote the free C algebra on the empty set by $\mathbb{K}$. The algebra $\mathbb{K}$ is initial in the category of graded C algebras, and free algebras are the same same as free extensions of $\mathbb{K}$.*

If C has zero differential, then the initial algebra $\mathbb{K}$ with the zero differential is a dgC algebra.

One may find it useful to note that $\mathbb{K}$ simply consists of the space of zero-to-one operations of C acted on by the higher operations in C (See Proposition 4.14).

The following is a key lemma that allows us to form resolutions of algebras of operations and do homotopical algebra:

Lemma 2.6. *Let C be a combination algebra with zero differential. Let P be a dgC algebra and $P[x_\alpha]$ a free extension of P to a graded C algebra.*

A differential on $P[x_a\alpha]$ extending the differential on P which makes $P[x_\alpha]$ into dgC algebra both determines and is determined by extensions of the differential over the free generators such that $\partial^2(x_\alpha) = 0$ for all generators x_α.

A dgC algebra map $f : P[x_\alpha] \to Q$ extending dgC algebra map $P \to Q$ determines and is determined by an extension of the algebra map f over the generators, such that $\partial f(x_\alpha) = \pm f\partial(x_\alpha)$ for all generators x_α.

Extensions of this flavor in other settings are Hirsch extensions.

2.4 Homotopical Algebra for Algebras Of Operations

Let C be a fixed combination algebra with zero differential.

We give two lemmas and a definition, which are analogous to the two lemmas and definition one gives to do a minimalist kind of homotopy theory in the context of free resolutions of modules over a ring, which in mock terms, are as follows:

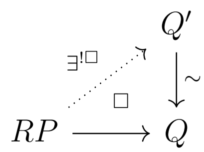
1. Mock Lemma 1: One can form free resolutions.

$$RP \xrightarrow{\sim} P$$

2. Mock Definition: A definition of homotopy for maps from a resolved space

$$RP \overset{\square}{\rightrightarrows} Q$$

3. Mock Lemma 2: One can lift a map from a free resolution over a quasi-isomorphism up to homotopy, and any two lifts are homotopic



It is a further, proposition that being homotopic gives an equivalence relation. This proposition is nontrivial in the context of algebras.

Using the above ingredients, one can define familiar notions of homotopy equivalence between objects and maps, and one can show familiar facts, like any two resolutions are homotopy equivalent, and maps between algebras lift to a unique homotopy class of maps between their resolutions.

Definition 2.5. *We define a map of dgC $P \to Q$ to be a* ***quasi-isomorphism*** *if it induces an isomorphism on homology. We decorate quasi-isomorphism with the symbol: $\sim$.*

We will see that a quasi-isomorphism between resolved algebras is a homotopy equivalence. This is akin to how a map of cell complexes, which induces isomorphisms on all homotopy groups, is a homotopy equivalence. Free resolutions are our version of cell complexes.

We now discuss these ingredients in the context of algebras of operation.

2.4.1 Free Triangular Extensions

We work over a fixed combination algebra C with zero differential.

We denote the simultaneous free extension by $\{x_\alpha\}$ and $\{x_\beta\}$ by $P[x_\alpha, x_\beta]$, which is canonically isomorphic to their sequential extensions $P[x_\alpha][x_\beta]$. We denote the simultaneous free extension by a sequence of free operations $\{x_{\alpha_1}\}, \{x_{\alpha_2}\}, \ldots$ by $P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$, which is canonically isomorphic to the direct limit of their sequential extensions $P \subset P[x_{\alpha_1}] \subset P[x_{\alpha_1}][x_{\alpha_2}] \subset \ldots$, thus we may benignly confuse the distinction. Let $\{x_\alpha\}$ be a set of free operations and $\{x_\alpha\} = \{x_{\alpha_1}\} \sqcup \{x_{\alpha_2}\} \sqcup \ldots$ a partition. We may write $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$ to connote this partition of its variables.

Definition 2.6. *Let P be a dgC algebra with differential d, and let $P[x_\alpha]$ be a free extension of P with a differential extending the differential on P. We call $P[x_\alpha]$ a* ***free triangular*** *extension of P if there is partition of its variables $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$ such that*

$$\begin{aligned}
&\{\partial x_{\alpha_1}\} \subset P \\
&\{\partial x_{\alpha_2}\} \subset P[x_{\alpha_1}] \\
&\{\partial x_{\alpha_3}\} \subset P[x_{\alpha_1}, x_{\alpha_2}] \\
&\vdots
\end{aligned}$$

We describe this situation by saying the differential takes generators into previous terms.

Since a free algebra is the same thing as a free extension of the initial algebra $\mathbb{K}$, it makes sense to say ***free triangular algebra*** *when $P = \mathbb{K}$.*

One may think of P as something like a topological space and $\{x_\alpha\}$ as cells whose boundaries are attached to $P[x_\alpha]$ by the boundary map ∂. One thinks of a free triangular extension $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$ of P as a space P with a first layer of cells $\{x_{\alpha_1}\}$ attached to P, a second layer of cells $\{x_{\alpha_2}\}$ attached to $P[x_\alpha]$, and so forth. One may think of the initial algebra $\mathbb{K}$ as the space of a point, and the unique map $\mathbb{K} \to P$ to every algebra P as a choice of base point. Thus every algebra has a base point, and in a sense, algebra maps preserve this base point. Free algebras and free extensions of $\mathbb{K}$ are the same thing, thus free triangular algebras are something like cell complexes built off a base point.

The following lemma may be thought of of as the algebraic analogue of replacing a map of topological spaces with a cofibration (ie a closed embedding), which for spaces one may do by repeatedly adding cells to the domain of a map and extending the map over these cells, until it becomes quasi-isomorphic in the sense of spaces. The cofibration is the inclusion of the original domain into the new domain (a closed embedding). One notes that the cylinder of a space is formed in this manner, by taking two copies of a space and the map that pancakes them together, then attaching cells between them the two copies.

An algebraic construction of this flavor dates back to Hirsch, and may be called the Hirsch extension theorem [3, 2]. This lemma replaces Mock Lemma 1, saying that every object has a free resolution:

Lemma 2.7 (Every map can be replaced by a cofibration). *Let $P \to Q$ be a map of dgC algebras. There is a free triangular extension $P[x_\alpha]$ of P and a dgC algebra map $P[x_\alpha] \to Q$ extending the original map to a quasi-isomorphism*

$$\begin{array}{ccc} P[x_\alpha] & \overset{\sim}{\dashrightarrow} & Q \\ \uparrow & \nearrow & \\ P & & \end{array}$$

Proof. Let $P \to Q$ be a dg$C$ algebra map.

One way to do this, which is not most efficient is as follows: first, we extend P the differential and the map so the map becomes surjective on homology. Then we extend them all again to kill the kernel on homology, creating a new kernel with new homology. We iterate this second step, and then take a direct limit. The differential on each extension maps generators into the previous algebra, so the limit is a map from a free triangular extension of P to Q, which we check induces an isomorphism on homology.

Consider the induced map of $P \to Q$ on homology. The cokernel of this induced map measures its failure to be surjective on homology. Lift the cokernel back to a subspace of the homology $H(Q)$ and pick a set of representatives $\{x'_{\alpha_1}\}$ of a linear generating set of the lifted space. Let $\{x_{\alpha_1}\}$ be an isomorphic copy of $\{x'_{\alpha_1}\}$. Form the free extension $P[x_{\alpha_1}]$ of P as graded C algebras. Extend the differential over $\{x_{\alpha_1}\}$ by zero. Since the differential squares to zero on $\{x_{\alpha_1}\}$, it extends uniquely to a differential on $P[x_{\alpha_1}]$ making $P[x_{\alpha_1}]$ a differential graded C algebra. We extend the map over $\{x_{\alpha_1}\}$ by sending it isomorphically to $\{x'_{\alpha_1}\}$. By Lemma 2.6 this induces a map of graded algebras on the extension. Since the differential commutes with the map on the free generators the map commutes with the differential by the same lemma. Thus $P[x_{\alpha_1}] \to Q$ is a dc$C$ algebra map, and its obviously surjective on homology.

Now consider the kernel of $P[x_{\alpha_1}] \to Q$ on homology. Take a representatives $\{x'_{\alpha_2}\}$ of a linear generating set of this kernel. The set $\{x'_{\alpha_2}\}$ consists of cycles, which map to cycles Y' in Q, each of which represents a zero homology class in $H(Q)$. Thus the elements of Y' are boundaries of some elements Y in Q. Let $\{x_{\alpha_2}\}$ be an isomorphic copy of $\{x'_{\alpha_2}\}$, shifted up or down one degree (opposite the direction of the differential). Extend the differential over x_{α_2} by shifting its elements back the other way to $\{x'_{\alpha_2}\}$. Since the differential squares to zero on $\{x_{\alpha_2}\}$, the differential determines differential on all of $P[\{x_{\alpha_1}\}][\{x_{\alpha_2}\}]$ by Lemma 2.6. Extend the map on $P[x_{\alpha_1}]$ over the set $\{x_{\alpha_2}\}$ by mapping elements of $\{x_{\alpha_2}\}$ to the elements of Y whose boundaries are the images of corresponding $\{x'_{\alpha_2}\}$. This determines a map of graded C algebras $P[x_{\alpha_1}][x_{\alpha_2}] \to Q$. Since this map extends the dg$C$ algebra map on the presvious extension, $P[x_{\alpha_1}] \to Q$ and commutes with the differential on the new generators $\{x_{\alpha_2}\}$, thus the map on $P[x_{\alpha_1}][x_{\alpha_2}]$ is a dg$C$ algebra map by the same lemma. We may regard this as a map $P[x_{\alpha_1}, x_{\alpha_2}] \to Q$. One checks that the kernel of the composite map $P[X_1] \to P[x_{\alpha_1}, x_{\alpha_2}] \to Q$ maps to zero in the homology of $P[x_{\alpha_1}, x_{\alpha_2}]$.

One iterates this second step to get an increasing sequence of dg$C$ algebras whose differentials send generators into previous terms, and with maps extending the previous maps to Q. The limiting algebra is a free triangular extension $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$ of P and with a map $P[x_\alpha] \to Q$. We check this map is a quasi-isomorphism.

The map $P[x_{\alpha_1}] \to P[x_\alpha] \to Q$ is surjective on homology, so $P[X] \to Q$ is, too. Suppose p represents a nonzero homology class in kernel of $P[x_\alpha] \to Q$ on homology. This p lives in some first finite extension $P[x_{\alpha_1}, \ldots, x_{\alpha_m}]$, so it must be in the kernel of the composite map from the next extension $P[x_{\alpha_1}, \ldots, x_{\alpha_m}, x_{\alpha_{m+1}}] \to P[x_\alpha] \to Q$ on homology (since we conveniently picked a linear generating set in the first step). This implies its homology class in $P[x_\alpha]$ is trivial, contradicting that it's non-trivial. So the kernel is zero. Thus $P[x_\alpha] \to Q$ is an isomorphism on homology, and the claim follows. □

One may think of the inclusion map $P \to P[x_\alpha]$ as the cofibration which replaces the original map $P \to Q$.

One may, of course, interpret the words "free resolution" of P by applying Lemma 2.7 to the map $\mathbb{K} \to P$ from the base point (initial algebra) to P, obtaining a free triangular algebra $\mathbb{K}[x_\alpha]$ and a quasi-isomorphism $\mathbb{K}[x_\alpha] \to P$. We can obtain a relative resolution by applying the lemma to the inclusion map of any subspace.

Definition 2.7. *A* ***free resolution*** *of a dgC algebra P is a free triangular algebra $\mathbb{K}[x_\alpha]$ with quasi-isomorphism $\mathbb{K}[x_\alpha] \xrightarrow{\sim} P$. A free resolution of a dgC algebra P* ***relative*** *to a dgC subalgebras $Q \subset P$ is a free triangular extension $Q[x_\alpha]$ and a quasi-isomorphism $Q[x_\alpha] \xrightarrow{\sim} P$.*

Corollary 2.8. *Free resolutions and free resolutions relative to subalgebras exist, by Lemma 2.7, and free resolutions are resolutions relative to the subalgebra determined by the initial algebra.*

2.4.2 Definition of Homotopy

The ability to turn maps into cofibrations (Lemma 2.7 enables us to define a notion of homotopy and relative homotopy. We will observe that homotopy and homotopy relative to the subalgebra determined by the initial algebra are really the same thing, which is something like saying homotopy is homotopy relative to the base point. Because of this, we mostly work rel a subalgebra.

Proposition 2.9. *Let $P[x_\alpha]$ be a free triangular extension of a dgC algebra P, and form the free graded C algebra $P[x_\alpha, y_\alpha] := P[x_\alpha, x_\alpha]$ where y_α denotes the second copy of x_α. We may regard $P[x_\alpha]$ as a subalgebra in two different ways, by $P[x_\alpha]$ and $P[y_\alpha]$*

Consider the map

$$m : P[x_\alpha, y_\alpha] \to P[x_\alpha]$$

determined by taking each copy of x_α to x_α.

There is a unique differential on $P[x_\alpha, y_\alpha]$ extending the differentials $P[x_\alpha]$ $P[y_\alpha]$ and it makes m a dgC algebra map.

Given two maps $f : P[x_\alpha] \to Q$ and $g : P[x_\alpha] \to Q$ that agree on P and regarding their domains as $P[x_\alpha]$ and $P[y_\alpha]$ respectively, the unique algebra map

$$f \wedge_P g : P[x_\alpha, y_\alpha] \to Q$$

extending both f and g is a dgC algebra map.

Proof. Since $P[x_\alpha]$ as above is a free triangular resolution, its variables have a partition $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$ with descending differential, and let $P[y_\alpha] = P[y_{\alpha_1}, y_{\alpha_2}, \ldots]$ be a copy.

We may form $P[x_{\alpha_1}, y_{\alpha_1}]$ as a graded C algebra. Since the original differentials map $\{x_{\alpha_1}\}$ and $\{y_{\alpha_1}\}$ to P and square to zero, they determine a differential on $P[x_{\alpha_1}, y_{\alpha_1}]$ by Lemma 2.6. The map $P[x_{\alpha_1}, y_{\alpha_1}] \to P[x_{\alpha_1}]$ induced by sending $\{x_{\alpha_1}\}$ and $\{y_{\alpha_1}\}$ identically to $\{x_{\alpha_1}\}$ commutes with the differential on $\{x_{\alpha_1}, y_{\alpha_1}\}$, so it is a dg$C$ algebra map by the same lemma.

We repeat this process inductively over all the groups of free variables. The limit of this process is a dg$C$ algebra $P[x_\alpha, y_\alpha] = P[x_{\alpha_1}, y_{\alpha_1}, x_{\alpha_2}, y_{\alpha_2}, \ldots]$ and a map $P[x_\alpha, y_\alpha] \to P[x_\alpha]$ taking $\{x_\alpha\}$ and $\{y_\alpha\}$ identically to $\{x_\alpha\}$.

The differential is completely determined by what it does on P $\{x_\alpha\}$ and $\{y_\alpha\}$, and it is determined on those by what it does on the subalgebras $P[x_\alpha]$ and $P[y_\alpha]$, so the differential is unique.

Let $f : P[x_\alpha] \to Q$ and $g : P[x_\alpha] \to Q$ be dg$C$ algebra maps that agree on P. Regard their domains as the two dg$C$ subalgebras $P[x_\alpha]$ and $P[y_\alpha]$ of $P[x_\alpha, y_\alpha]$. The maps f and g determine a map on this by what they do on P, $\{x_\alpha\}$ and $\{y_\alpha\}$. This map commutes with the differentials on the generators P, $\{x_\alpha\}$ and $\{y_\alpha\}$, so the map commutes with d on all $P[x_\alpha, y_\alpha]$.

A map $P[\{x_\alpha\} \sqcup \{y_\alpha\}] \to Q$ extending f and g must equal on P, and given by f on $\{x_\alpha\}$ and g on $\{y_\alpha\}$. This data determines a map $P[\{x_\alpha\} \sqcup \{y_\alpha\}] \to Q$, extending both f and g by the properties of free extensions. Since this map commutes with the differentials on P, $\{x_\alpha\}$ and $\{y_\alpha\}$, and the differential is a derivation, it commutes with the differential on the entire algebra. Thus the resulting map $P[\{x_\alpha\} \sqcup \{y_\alpha\}] \to P[\{x_\alpha\}]$ is a map of dg$C$ algebras extending f and g, and the claim follows. ☐

Definition 2.8. *We call the map $m : P[x_\alpha, y_\alpha] \to P[x_\alpha]$ in Proposition 2.9 the* ***product map of*** *$P[x_\alpha]$* ***(relative to the subalgebra*** *P**)**, and we call the map $f \wedge_P g : P[x_\alpha, y_\alpha] \to P[x_\alpha]$ the* ***coproduct of*** *f* ***and*** *g* ***rel*** *P.*

If $P = \mathbb{K}$ is the initial algebra, then $\mathbb{K}[x_\alpha, y_\alpha]$ is canonically isomorphic to the usual coproduct of dgC algebras $\mathbb{K}[x_\alpha] \wedge \mathbb{K}[y_\alpha]$, so we may denote $f \wedge_{\mathbb{K}} g$ using the usual coproduct notation $f \wedge g$.

We will use the product map (relative to a subalgebra) to help us define a notion of homotopy (relative the subalgebra). Before we give this definition, we give imagery to explain it:

One begins with a space $P[x_\alpha]$ and a subspace P, such that $P[x_\alpha]$ is formed by adding cells $\{x_\alpha\}$ to P in groups $\{x_{\alpha_1}\}, \{x_{\alpha_2}\}, \ldots$ such that the boundary of the first group $\{x_{\alpha_1}\}$ is in P, the boundary of the second group $\{x_{\alpha_1}\}$ is in $P[x_{\alpha_1}]$, and so forth.

Suppose we have two maps $f : P[x_\alpha] \to Q$ and $g : P[x_\alpha] \to Q$ of this space to another space Q which agree on the subspace P.

A relative homotopy from f to g means the following: we form a cylinder on the space $P[x_\alpha]$ by taking the Cartesian product of $P[x_\alpha]$ with the closed unit interval. The interval may be regarded as two 0-cells at the ends, and an 1-cell in the middle. The cylinder on $P[x_\alpha]$ consists of a smaller cylinder corresponding to the subspace P, and a copy of the cells $\{x_\alpha\}$ at each end (one copy for each of the two 0-cells in the interval) and for each pair of cells on the end, a cell of one degree higher running between them (corresponding to the product of a cell and the 1-cell of the interval). Call the two copies of the cells on the ends x_α and $\{y_\alpha\}$, and the cells in the middle $\{\bar{\delta}_\alpha\}$. We map the ends to Q using f and g; a homotopy is an extension over the entire cylinder that is constant on the sub-cylinder P.

Alternatively we can pinch the sub-cylinder of P to one copy of P, resulting in a partially pinched cylinder on $P[x_\alpha]$. We can still map the ends of this to Q using f and g, since they agree on P; now, a homotopy relative to P is just a map extending these maps over the entire pinched cylinder.

We could form this partially squished cylinder by building the cells up in layers as follows: start with the subspace P and glue on the cells $\{x_\alpha\}$ and $\{y_\alpha\}$ at the ends in groups starting with x_{α_1} and $\{y_{\alpha_1}\}$, then $\{x_{\alpha_2}\}$ and $\{y_{\alpha_2}\}$ and so on to get two copies of $P[x_\alpha]$ which are glued at P. This is the space $P[x_\alpha, y_\alpha]$. One end is $P[x_\alpha]$, the other is $P[y_\alpha]$ and they are glued at P. Now we add the cells $\{\bar{\delta}_\alpha\}$ in one higher dimension running between $\{x_\alpha\}$ and $\{y_\alpha\}$ that fill in the interior of the cylinder. We add these in groups $\{\bar{\delta}_{\alpha_1}\}, \{\bar{\delta}_{\alpha_2}\}, \ldots$ corresponding to the groups $\{x_{\alpha_1}, y_{\alpha_1}\}, \{x_{\alpha_2}, y_{\alpha_2}\}, \ldots$. The boundary of a cell in $\{\bar{\delta}_{\alpha_1}\}$ has boundary the corresponding cell in $\{x_{\alpha_1}\}$ with one orientation and the corresponding cell in $\{y_{\alpha_1}\}$ with the opposite orientation, along with some possible extra boundary in P. The next cells $\{\bar{\delta}_{\alpha_2}\}$ have boundary corresponding cells in $\{x_{\alpha_2}\}$ and $\{y_{\alpha_2}\}$ with opposite orientations and some extra boundary in $P[x_\alpha, y_\alpha][\bar{\delta}_{\alpha_1}]$. And so on, eventually resulting in the cylinder $P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$.

Or we could have added the cells $\{x_{\alpha_1}\}$, $\{y_{\alpha_1}\}$ followed by $\{z_{\alpha_1}\}$, and then $\{x_{\alpha_2}\}$, $\{y_{\alpha_2}\}$ and then $\{z_{\alpha_2}\}$ to get the cylinder a different way: $P[x_{\alpha_1}, y_{\alpha_1}, z_{\alpha_1}][x_{\alpha_2}, y_{\alpha_2}, z_{\alpha_2}] \ldots \cong P[x_{\alpha_1}, y_{\alpha_1}, z_{\alpha_1}, x_{\alpha_2}, y_{\alpha_2}, z_{\alpha_2}, \ldots] \cong P[x_\alpha, y_\alpha][z_\alpha]$. One might observe this is a very nice way of turning the product map m into a cofibration.

We could also turn the product map into a cofibration recklessly, and show the space $P[x_\alpha, y_\alpha][z_\beta]$ we obtain instead of our nice cyilinder, and then show $P[x_\alpha, y_\alpha][z_\beta]$ is homotopy equivalent to the nice space $P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$ rel $P[x_\alpha, y_\alpha]$, concluding that we can use any of them to define homotopy.

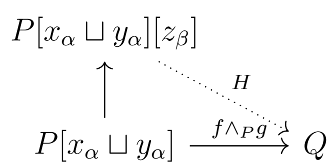
In our algebraic account, we define homotopy using the reckless version, and then use a nicer cylinder version to actually prove things, but it is convenient to have both.

Since algebra maps are backwards compared to topology maps, we call the cylinder-like objects by the name of their dual object objects, path spaces. One observes homotopy in topological spaces may be given using the path space $Y^I := Hom(I, Y)$ using the usual topology on Hom spaces, and the usual homeomorphism:

$$Hom(X \times I, Y) \cong Hom(X, Hom(I, Y))$$

Definition 2.9. *Let $P[x_\alpha]$ be a free triangular extension of a dgC algebra P, and let $m : P[x_\alpha \sqcup y_\alpha] \to P[x_\alpha]$ be the product map (relative to P). We call a free triangular extension $P[x_\alpha \sqcup y_\alpha][z_\beta]$ of $P[x_\alpha \sqcup y_\alpha]$ a* ***path space of*** *$P[x_\alpha]$* ***(relative to*** *P***)** *if there is a quasi-isomorphism $\tilde{m} : P[x_\alpha \sqcup y_\alpha][z_\beta] \to P[x_\alpha]$ extending the product map.*

Let $f : P[x_\alpha] \to Q$ and $g : P[x_\alpha] \to Q$ be a two dgC algebra maps to another dgC algebra Q. Form the coproduct of f and g rel P, $f \wedge_P g : P[x_\alpha \sqcup y_\alpha] \to Q$. We say f ***is homotopic to*** *g* ***rel*** *P if there is an extension of $f \wedge_P g$ over some path space $P[x_\alpha \sqcup y_\alpha][z_\beta]$*



We call such a map $H : P[x_\alpha \sqcup y_\alpha][z_\beta] \to Q$ a ***homotopy from*** *f* ***to*** *g* ***rel*** *P.*

If $P = \mathbb{K}$ is the initial algebra, then we simply say "homotopic" and "homotopy".

Corollary 2.10. *Existence of path spaces follows from the the product map can be turned into a cofibration (Lemma 2.7).*

Remark 2.1. *It is not yet clear that homotopy gives an equivalence relation.*

We construct a nice path space, which we will use to compute, following this lemma:

Lemma 2.11. *Let P be a dgC algebra and let $P[x_\alpha, \partial x_\alpha]$ be a free extension of P by variables $\{x_\alpha, \partial x_\alpha\}$, with $\partial(\partial x_\alpha) := 0$ and $\partial(x_\alpha) := \partial x_\alpha$.*

Then the ideal $\langle x_\alpha, \partial x_\alpha \rangle$ is a differential ideal, and has zero homology

$$H(\langle x_\alpha, dx_\alpha \rangle) = 0$$

Proof. One notes the extension $P[x_\alpha, \partial x_\alpha]$ of P is a two-step free triangular extension using Lemma 2.6.

Since C has zero differerential, the differential ∂ on $P[x_\alpha, \partial x_\alpha]$ is a derivation.

We may define another derivation s on $P[x_\alpha, \partial x_\alpha]$ with the opposite degree by defining it to be zero on P, and by $s(x_\alpha) := 0$ and $s(\partial x_\alpha) := x_\alpha$ (see the appendix).

The commutator of two derivations is a derivation, thus $[\partial, s] := \partial s + s\partial$ is a derivation. One observes that $[\partial, s](P) = 0$, $[\partial, s](x_\alpha) = x_\alpha$ and $[\partial, s](\partial x_\alpha) = \partial x_\alpha$.

We may give a weight grading on $P[x_\alpha, \partial x_\alpha] = P[x_\alpha, \partial x_\alpha]^{(0)} \oplus P[x_\alpha, \partial x_\alpha]^{(1)} \oplus P[x_\alpha, \partial x_\alpha]^{(2)} \oplus \ldots$ by letting all the variables have weight 1 (see appendix). One observes that the weight zero part is $P[x_\alpha, \partial x_\alpha]^{(0)} = P$ and the positive weight parts are the ideal generated by $\{x_\alpha, \partial x_\alpha\}$

$$\langle x_\alpha, \partial x_\alpha\rangle = P[x_\alpha, \partial x_\alpha]^{(1)} \oplus P[x_\alpha, \partial x_\alpha]^{(2)} \oplus \dots$$

One also observes that $[\partial, s]$ of the weight n part is n times the identity map. Thus s scales to a contracting homotopy on the positive weighted parts, thus that the ideal has zero homology as claimed. □

Proposition 2.12. *Let $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \dots]$ be a free extension of a dgC algebra P, and consider the product map $P[x_\alpha, y_\alpha] \to P[x_\alpha]$*

Theres is a free triangular extension $P[x_\alpha \sqcup y_\alpha][z_\alpha] = P[x_\alpha, y_\alpha][\bar{\delta}_{\alpha_1}, z_{\alpha_2}, \dots]$ of $P[x_\alpha, y_\alpha]$ such that

$$\partial\bar{\delta}_\alpha = y_\alpha - x_\alpha + \textit{previous terms}$$

and a quasi-isomorphism

$$\tilde{m} : P[x_\alpha, y_\alpha][\bar{\delta}_\alpha] \xrightarrow{\sim} P[x_\alpha]$$

extending the product map.

It follows that $P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$ with this differential is path space.

Proof. Let $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \dots]$, $P[x_\alpha, y_\alpha]$ and the product map be as above.

Define $p_\alpha(x_\beta) := \partial x_\alpha$, which by assumption lies in earlier terms than x_α.

Define $\delta_\alpha := y_\alpha - x_\alpha$, and define $q_\alpha(y_\beta, x_\beta) := \partial(\delta_\alpha) = \partial y_\alpha - \partial x_\alpha = p_\alpha(y_\beta) - p_\alpha(x_\beta)$. By the proof of Proposition 2.9, $P[x_\alpha, y_\alpha] = P[x_{\alpha_1}, y_{\alpha_1}, x_{\alpha_2}, y_{\alpha_2}, \dots]$ is a free triangular extension with the first sets of variables x_{α_1} and y_{α_1} added first, then the second sets of variables and so forth. Thus $q_\alpha(y_\beta, x_\beta)$ lies in earlier terms terms, too.

The differentials of the first extension $\{x_{\alpha_1}\}$ lands in P, so $\partial x_{\alpha_1} = \partial y_{\alpha_1}$, and so $\partial\delta_{\alpha_1} = \partial(t_{\alpha_1} - x_{\alpha_1}) = 0$. Thus we may form extend the differential over $P[x_{\alpha_1}, y_{\alpha_1}, \bar{\delta}_{\alpha_1}] \cong P[x_{\alpha_1}, \delta_{\alpha_1}, \bar{\delta}_{\alpha_1}] \cong P[x_{\alpha_1}, \bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_1}] \cong P[x_\alpha][\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_1}]$ by defining $\partial\bar{\delta}_{\alpha_1} := \delta_{\alpha_1}$. We may extend the product map on $P[x_{\alpha_1}, y_{\alpha_1}]$ over $\{\bar{\delta}_{\alpha_1}\}$ by zero to a map of graded C algebras. Since the differential commutes with the differential on the generators, it is a map of dg$C$ algebras. And we observes that the ideals $\langle\delta_{\alpha_2}, \bar{\delta}_{\alpha_2}\rangle = \langle\bar{\delta}_{\alpha_2}, \partial\bar{\delta}_{\alpha_2}\rangle$ are equal.

Consider $\partial\delta_{\alpha_2} = q_{\alpha_2}(y_\beta, x_\beta) = p_{\alpha_2}(y_\beta) - p_{\alpha_2}(x_\beta)$. In a similar manner to how one can rewrite a polynomial $aA - bB = (a-b)A + b(A-B)$, one can rewrite $\delta_{\alpha_2} = q_{\alpha_1}(y_\beta, x_\beta)$ as a sum of monomials each involving a term $\delta_{\alpha_1} = y_{\alpha_1} - x_{\alpha_1}$. In other words, $q_{\alpha_2}(y_\beta, x_\beta) = \partial\delta_{\alpha_2}$ is a cycle in the ideal $\langle\delta_{\alpha_1}, \bar{\delta}_{\alpha_1}\rangle = \langle\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_1}\rangle$. By Lemma 2.11, this ideal $\langle\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_1}\rangle$ of $P[x_{\alpha_1}][\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_1}]$ is acyclic, thus $q_{\alpha_2}(y_\beta, x_\beta)$ is a boundary of some $\eta_{\alpha_2} \in \langle\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_1}\rangle \subset P[x_{\alpha_1}][\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_1}] \cong P[x_{\alpha_1}, y_{\alpha_1}, \bar{\delta}_{\alpha_1}]$, that is $q_{\alpha_2}(y_\beta, x_\beta)$ is a boundary in earlier terms than x_{α_2} and y_{α_2}.

We extend the differential over $\{\bar{\delta}_{\alpha_2}\}$ by $\partial\bar{\delta}_{\alpha_2} := y_{\alpha_2} - x_{\alpha_2} - \eta_{\alpha_2}$, which by construction has boundary 0. By our friend, Lemma 2.6, this determines a differential on $P[x_{\alpha_1}, y_{\alpha_1}, \bar{\delta}_{\alpha_1}, x_{\alpha_2}, y_{\alpha_2}, \bar{\delta}_{\alpha_2}] \cong P[x_{\alpha_1}, \delta_{\alpha_1}, x_{\alpha_2}, \delta_{\alpha_2}, \bar{\delta}_{\alpha_1}, \bar{\delta}_{\alpha_2}] \cong P[x_{\alpha_1}, y_{\alpha_1}, \bar{\delta}_{\alpha_2}, \bar{\delta}_{\alpha_2}, \partial\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_2}] \cong P[x_{\alpha_1}, y_{\alpha_1}][\bar{\delta}_{\alpha_2}, \bar{\delta}_{\alpha_2}, \partial\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_2}]$.

Since η_{α_2} is an element of the ideal $\langle\delta_{\alpha_1}, \bar{\delta}_{\alpha_1}$ and that ideal gets mapped to under the extended product map $\tilde{m} : P[x_\alpha, y_\alpha, \bar{\delta}\alpha_1] \to P[x_\alpha]$. One notes that $\delta_{\alpha_2} = y_{\alpha_2} - x_{\alpha_2}$ gets sent to zero under the extend product map, too. Thus $\partial\bar{\delta}_{\alpha_2} = \partial\delta_{\alpha_2} - \eta_{\alpha_2}$ also gets sent to zero under the extended product map. Thus by the same Lemma 2.6, the extension of the product map $P[x_\alpha, y_\alpha][\bar{\delta}_{\alpha_1}, \bar{\delta}_{\alpha_2}] \to P[x_\alpha]$ taking $\bar{\delta}_{\alpha_1}$ and $\bar{\delta}_{\alpha_2}$ to commutes with the differentials, thus is a map of dg$C$ algebras.

We may repeat the process above inductively and take its limit, resulting in a map of dg$C$ algebras

$$\tilde{m} : P[x_\alpha, y_\alpha][\bar{\delta}_\alpha] \cong P[x_\alpha, \delta_\alpha, \bar{\delta}_\alpha] \cong P[x_\alpha, \bar{\delta}_\alpha, \partial\bar{\delta}_\alpha] \cong P[x_\alpha][\bar{\delta}_\alpha, \partial\bar{\delta}_\alpha] \to P[x_\alpha]$$

extending the product map, which sends $\bar{\delta}_\alpha$ to zero, and one computes the kernel of this map is the ideal $\langle\bar{\delta}_\alpha, \partial\bar{\delta}_\alpha\rangle$. This ideal is acyclic by Lemma 2.11, and $\tilde{m}$ is surjective. Thus there is a short exact sequence of linear spaces

$$0 \to \ker(\tilde{m}) \to P[x_\alpha, y_\alpha][\bar{\delta}_\alpha] \xrightarrow{\tilde{m}} P[x_\alpha] \to 0$$

Since the kernel has zero homology, $\tilde{m}$ is a quasi-isomorphism by the usual argument using the induced long exact sequences in homology.

The claim follows. □

2.4.3 The Lifting Lemma

The Lifting Lemma is a relative version of the statement: we can lift a map up to homotopy over a quasi-isomorphism, and any two such lifts are homotopic.

Lemma 2.13 (Lifting lemma)**.** *Let $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$ be a free triangular extension of a dgC algebra P, and let $Q' \xrightarrow{\sim} Q$ be a quasi-isomorphism of dgC algebras.*

Let $f : P[x_\alpha] \to Q$ be a map of dgC algebras. We may extend any lift $\bar{f}$ of f on P over $Q' \xrightarrow{\sim} Q$ to a lift up to homotopy, and any two such lifts are homotopic rel P.

Proof. Assume the hypotheses of the statement, and let $P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$ be as in Proposition 2.12.

Our lift up to homotopy $\bar{f} : P \to Q'$ is already defined on P, and is an actual lift of f on P.

The boundary of x_{α_1} lies in P. Because $\bar{f}(dx_{\alpha_1})$ becomes a boundary downstairs in Q, and because the upstairs Q' is quasi-isomorphic to the downstairs Q, $f(dx_{\alpha_1})$ corresponds to a boundary upstairs of some chain. We can change that chain by a cycle so when pushed downstairs it is homotopic to $f(x_{\alpha_1})$ relative to their common boundary. Define $\bar{f}(y_\alpha)$ to be that upstairs chain. We can fill in the homotopy by extending the maps on the path space over δ_{α_1}. Any other choice of lift would be homotopic relative to the boundary downstairs, thus the difference between choices upstairs is a boundary of some chain; if we change that chain by a cycle so when mapped downstairs it is homotopic to the downstairs homotopy relative to their common boundary. To make our computation later easier, we can change the homotopies between the two different images of y_{α_1} downstairs so that the homotopy upstairs maps exactly to their sum. Thus we can lift over the first layer $\{\bar{\delta}_{\alpha_1}\}$

Now we lift the second layer $\{\bar{\delta}_{\alpha_2}\}$. Recall that $\partial\bar{\delta}\alpha_2 = y_{\alpha_2} - x_{\alpha_2} - \eta_{\alpha_2}$ and that dx_{α_2}, ∂y_{α_2} and $\partial\eta_{\alpha_2}$ lie in terms that have already been lifted. Applying ∂ to both sides of this equality reveals that $\partial\eta_{\alpha_2} = \partial y_{\alpha_2} - \partial x_{\alpha_2}$. We already have maps taking $x_{\alpha_2} - y_{\alpha_2}$ to downstairs Q, thus the image of $\partial y_{\alpha_2} = \partial(x_{\alpha_2} - \eta_{\alpha_2})$ is a boundary. Since upstairs Q' and downstairs Q are quasi-isomorphic, ∂y_{α_2} is a boundary upstairs, too, of some chain. We can change that chain by a cycle so when mapped downstairs it is homotopic to $x_{\alpha_2} - \eta_{\alpha_2}$ relative to their common boundary. Define $\bar{f}(y_{\alpha_2})$ to be this upstairs chain. Since the downstairs image of y_{α_2} is homotopic to the image of $x_{\alpha_2} - \eta_{\alpha_2}$, we can extend the homotopy over δ_α.

Now we fill in the homotopy between the two different lifts upstairs. We already filled in the homotopy between the lifts at their first step, and we did it so the $\bar{\delta}_{\alpha_1}$ for the homotopy between the lifts upstairs maps to the sum of the homotopies of the lifts downstairs with the original map. Upstairs we want to solve $\partial\bar{\delta}_{\alpha_2} = y_{\alpha_2} - y'_{\alpha_2} - \tilde{\eta}_{\alpha_2}$. Downstairs this maps to the difference of the images of $y_{\alpha_2} = \partial\bar{\delta}_{\alpha_2} - x_{\alpha_2} - \eta_{\alpha_2}$ and $y'_{\alpha_2} = \partial\bar{\delta}'_{\alpha_2} - \eta'_{\alpha_2}$ and also $\tilde{\eta}_{\alpha_2}$. The boundary of this difference is zero, and the difference lies in the image of the homotopies assembled into one map from $P[x_{\alpha_1}, x_{\alpha_2}, y_{\alpha_1}, y_{\alpha_2}, y'_{\alpha_1}, y'_{\alpha_2}, \bar{\delta}_{\alpha_1}, \bar{\delta}_{\alpha_2}, \bar{\delta}'_{\alpha_1}, \bar{\delta}'_{\alpha_2}] \cong P[x_{\alpha_1}, x_{\alpha_2}, \bar{\delta}_{\alpha_1}, \bar{\delta}_{\alpha_2}, \bar{\delta}'_{\alpha_1}, \bar{\delta}'_{\alpha_2}, \partial\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_2}, \partial\bar{\delta}'_{\alpha_1}, \partial\bar{\delta}'_{\alpha_2}]$. In fact, the difference lives in the image of the ideal $\langle \bar{\delta}_{\alpha_1}, \bar{\delta}_{\alpha_2}, \bar{\delta}'_{\alpha_1}, \bar{\delta}'_{\alpha_2}, \partial\bar{\delta}_{\alpha_1}, \partial\bar{\delta}_{\alpha_2}, d\bar{\delta}'_{\alpha_1}, \partial\bar{\delta}'_{\alpha_2} \rangle$. Since that ideal is contractible, the difference is a boundary of some ζ in the image of of the this ideal. Since upstairs Q' and downstairs Q are quasi-isomorphic, we can find a homotopy upstairs between the two different lifts. We change this homotopy by a cycle so that downstairs it becomes homotopic to $\bar{\delta}_{\alpha_1} - \bar{\delta}'_{\alpha_2} - \zeta$ relative to their common boundary. To make the computation in the next step easier, we can change the homotopy on $\bar{\delta}_{\alpha_2}$ so the upstairs $\bar{\delta}_{\alpha_2}$ maps exactly down to $\bar{\delta}_{\alpha_2} - \bar{\delta}'_{\alpha_2} - \zeta$ (thus putting it in the image of the next level of homotopies assemble into a single map from $P[\ldots]$).

We can repeat this process inductively. The limit is a lift up to homotopy of the original map. And since any other lift up to homotopy gives alternative choices in each step, and we can build a homotopy between it and our choice of lift, any two lifts up to homotopy are homotopic (all relative to P). The claim follows. □

Proposition 2.14. *If the map $Q \overset{\sim}{\twoheadrightarrow} Q'$ in the Lifting Lemma is surjective, then the lift can be chosen strictly.*

Proof. Since $Q \to Q'$ is a quasi-isomorphis, its kernel has zero homology.

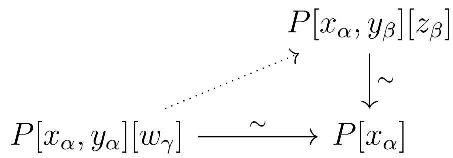
Suppose we've chosen an strict lift on the first several layers of free generators. Let x_α be a generator in the next layer. Then our map lifts $p_\alpha(x_\beta) := \partial x_\alpha$ strictly. Pick any lift of x_α upstairs. Its boundary differs from the lift of $p(x_\alpha)$ by an element of the kernel of $Q \to Q'$, which one checks is a cycle. Since the kernel has zero homology, this cycle is a boundary of some element which is also in the kernel. We change our lift of x_α by this element to get an strict lift of x_α whose boundary is $p(x_\alpha)$. Thus we can strictly lift the next layer. The limit is a strict lift.

We will soon see that homotopy is reflexive (without using this bonus proposition), so the lift is a lift up to homotopy rel P, and the claim follows. □

The lifting lemma has a number of consequences.

Lemma 2.15. *If two maps are homotopic, then there is a homotopy from any path space.*

Proof. Let $P[x_\alpha]$ be a free triangular extension of a dg$C$ algebra P, and let $P[x_\alpha, y_\alpha][z_\beta]$ and $P[x_\alpha, y_\alpha][w_\gamma]$ be two path spaces. They have quasi-isomorphisms $P[x_\alpha]$ extending the product map $P[x_\alpha, y_\alpha] \to P$. The identity map on $P[x_\alpha, y_\alpha]$ supplies a lift of one over the other on its subspace $P[x_\alpha, y_\alpha]$. By the lifting lemma, we can extend this to a lift up over the entire domain:



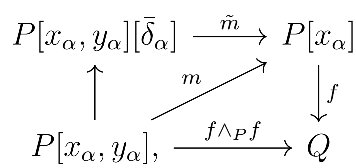
Suppose there is homotopy out of $P[x_\alpha, y_\alpha][z_\beta]$. Then precomposing with the dotted map provides a homotopy out of $P[x_\alpha, y_\alpha][w_\gamma]$. □

Proposition 2.16. *Homotopy (relative to subalgebra) gives an equivalence relation.*

Proof. Let $P[x_\alpha]$ be a free triangular extension of a dg$C$ algebra P.

Denote the product map $m : P[x_\alpha, y_\alpha] \to P[x_\alpha]$, and let $P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$ be path space as in Proposition 2.12, with its quasi-isomorphism $\tilde{m} : P[x_\alpha, y_\alpha][\bar{\delta}_\alpha] \to P[x_\alpha]$ extending the product map.

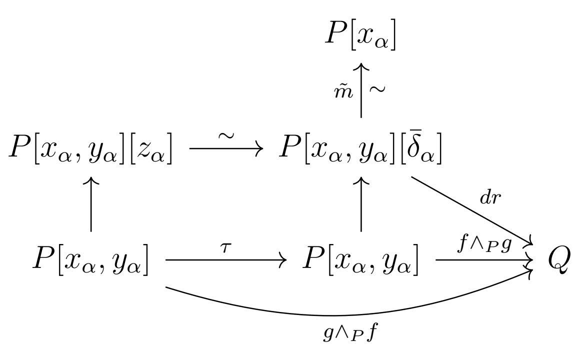
1. Reflexivity. Let $f : P[x_\alpha] \to Q$ be a dg$C$ algebra map. Form $f \wedge_P f : P[x_\alpha, y_\alpha] \to Q$, and observe that $f \circ m = f \wedge_P f$, thus the diagram commutes

   The map $H_{ff} := f \circ \tilde{m}$ gives a homotopy rel P from f to itself. It follows that that homotopy is reflexive.

2. Symmetry. Suppose f is homotopic to g rel P. By Lemma 2.15, there is a homotopy $H_{fg} : P[x_\alpha, y_\alpha][\bar{\delta}_\alpha] \to Q$, ie a dg$C$ algebra map extending $f \wedge_P g$.

   Define the twist map $\tau : P[x_\alpha, y_\alpha] \to P[x_\alpha, y_\alpha]$ to be identity P, and extended over x_α and y_α by $\tau(x_\alpha) := y_\alpha$ and $\tau(y_\alpha) := x_\alpha$. This commutes with the differential, thus it is a dg$C$ algebra map by Lemma 2.6. Consider the composite $P[x_\alpha, y_\alpha] \xrightarrow{\tau} P[x_\alpha, y_\alpha] \to P[x_\alpha, y_\alpha][\bar{\delta}]$. By Lemma 2.7 we can turn this map into a cofibration, thus there is a space $P[x_\alpha, y_\alpha][z_\beta]$ and maps so the square in the diagram commutes

   The upward maps in the bottom row are inclusion maps. The composite map $P[x_\alpha, y_\alpha][z_\beta] \to P[x_\alpha, y_\alpha][\bar{\delta}_\alpha] \to P[x_\alpha]$ is a quasi-isomorphism extending the product map, so $P[x_\alpha, y_\alpha][z_\alpha]$

is a path space. The map $f \wedge_P g \circ \tau = g \wedge_P f$. Thus the map from $P[x_\alpha, y_\alpha]$ to Q extends $g \wedge_P f$ over a path space, that is, its a homotopy from g to f rel P.

It follows that homotopy is symmetric.

3. Transitivity. Let $P[x_\alpha, y_\alpha][\bar{\delta}(x_\alpha, y_\alpha)]$ and $P[y_\alpha, z_\alpha][\bar{\delta}(y_\alpha, z_\alpha)]$ denote two copies of our path space.

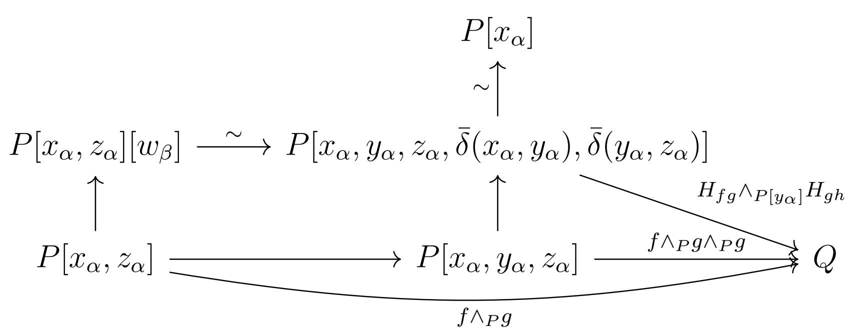
   In a similar manner to how we formed the product map (Proposition 2.9), we may form a dg$C$ algebra $P[x_\alpha, y_\alpha, z_\alpha]$ extending three copies of $P[x_\alpha]$ with a differential extending the differentials on those three copies. Form a path space $P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$ as in Proposition 2.12. We may adjoin to $P[x_\alpha, y_\alpha, z_\alpha]$ two copies of the generators, $\{\bar{\delta}(x_\alpha, y_\alpha)\}$ and $\{\bar{\delta}(y_\alpha, z_\alpha)\}$ with their differentials, to form a free triangular extension $P[x_\alpha, y_\alpha, z_\alpha, \bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha)]$ with differentials extending the differentials on $P[x_\alpha, y_\alpha, \bar{\delta}(x_\alpha, y_\alpha)] \cong P[x_\alpha, y_\alpha][\bar{\delta}(x_\alpha, y_\alpha)]$ and $P[y_\alpha, z_\alpha, \bar{\delta}(y_\alpha, z_\alpha)] \cong P[y_\alpha, z_\alpha][\bar{\delta}(y_\alpha, z_\alpha)]$, along with a dg$C$ algebra map

$$P[x_\alpha, y_\alpha, z_\alpha, \bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha)] \to P[x_\alpha]$$

   extending the maps $P[x_\alpha, y_\alpha, \bar{\delta}(x_\alpha, y_\alpha)] \to P[x_\alpha]$ and $P[y_\alpha, z_\alpha, \bar{\delta}(y_\alpha, z_\alpha)]$. Like in Proposition 2.12, one observes $P[x_\alpha, y_\alpha, z_\alpha, \bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha)] \cong P[x_\alpha, \delta(x_\alpha, y_\alpha), \delta(y_\alpha, z_\alpha), \bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha$ $P[x_\alpha][\bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha), d\bar{\delta}(x_\alpha, y_\alpha), d\bar{\delta}(y_\alpha, z_\alpha)]$, and that the kernel of the above map is the ideal $\langle \bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha), d\bar{\delta}(x_\alpha, y_\alpha), d\bar{\delta}(y_\alpha, z_\alpha) \rangle$. This ideal is acyclic by Lemma 2.11, thus the above map is a quasi-isomorphism.

   Now form the inclusion $P[x_\alpha, z_\alpha] \to P[x_\alpha, y_\alpha, \bar{\delta}(x_\alpha, y_\alpha)]$. Turn this into a cofibration by Lemma 2.6 to obtain a free triangular extension $P[x_\alpha, y_\alpha][w_\beta]$ and quasi-isomorphism $P[x_\alpha, y_\alpha][w_\beta] \to P[x_\alpha, y_\alpha, \bar{\delta}(x_\alpha, y_\alpha)]$ extending the old map.

   Thus we have most of the commutative diagram:



   We get the rest as follows: let f, g and h be dg$C$ algebra maps $P[x_\alpha]$ to Q that agree on P, and assume is f homotopic to g and g is homotopic to h, relative to P. One defines maps $f \wedge_P g \wedge_P h$ and $H_{fg} \wedge_{P[y_\alpha]} H_{gh}$ much as we defined the coproduct of two maps along a subspace, in Proposition 2.9. One notes that $P[x_\alpha, z_\alpha][w_\beta]$ is a path space, because it maps to $P[x_\alpha]$ by a quasi-isomorphism extending the product map.

   The map along the bottom is $f \wedge_P h$ and the map from $P[x_\alpha, y_\alpha][w_\beta] \to Q$ along the middle extends this over a path path space, this provides a homotopy from f to h relative to P. Thus homotopy is transitive.

It follows that homotopy rel a subalgebra gives an equivalence relation. □

Lemma 2.17. *Algebraic homotopy gives a finer equivalence relation than chain homotopy, that is if two maps are algebraicly homotopic, then they're chain homotopic.*

Proof. Let $P[x_\alpha]$ be a free triangular extension of P. Form a path space $P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$ as in 2.12. Recall that $P[x_\alpha]$ is a subalgebra of this in two different ways, by sending x_α to x_α or sending x_α to y_α. This gives two inclusion maps i_1 and i_2, which are both chain maps. First we show these are chain homotopic.

Let $p(x_\alpha)$ be an element $P[x_\alpha]$. Observe that

$$i_1(p(x_\alpha)) - i_2(p(x_\alpha)) = p(x_\alpha) - p(y_\alpha)$$

In the proof of Lemma 2.12 we showed that such elements lie in the contractible ideal $\langle \bar{\delta}, d\bar{\delta} \rangle$. Thus $i_1 - i_2$ is a chain map that factors through the contractible ideal. Let s be a contracting homotopy on the ideal, and $i \langle \bar{\delta}_\alpha, \partial \bar{\delta}_\alpha \rangle \hookrightarrow P[x_\alpha, y_\alpha][\bar{\delta}_\alpha]$ the inclusion map (which is a chain map).

Define $s' := i \circ s \circ (i_1 - i_2)$ we observe that

$$\partial s' + s' \partial = i \circ (\partial s + s \partial) \circ (i_1 - i_2) = i_1 - i_2$$

Thus i_1 is chain homotopic to i_2.

Suppose $f : P[x_\alpha] \to Q$ and $g : P[x_\alpha] \to Q$ be dg$C$ algebra maps, which are homotopic rel P. Then there is a homotopy $H_{fg} : P[x_\alpha, y_\alpha][\bar{\delta}_\alpha] \to Q$. One observes that the $f = H_{fg} \circ i_1$ and $g = H_{fg} \circ i_2$. Define $s'' := H_{fg} \circ s'$. We check

$$\partial s'' + s'' \partial = \partial s' H_{fg} + s' H_{fg} d = (\partial s' + \partial s') H_{fg} = (i_1 - i_2) H_{fg} = f - g$$

Thus f and g are chain homotopic. □

2.5 Homotopies and Homotopical Relations

We make some observations about homotopies and certain relations that use them.

Definition 2.10. *If f is homotopic to g rel P, we write $f \sim_P g$. If they are homotopic (ie homotopic rel the initial algebra) then we write $f \sim g$.*

We note the following basic properties of homotopies:

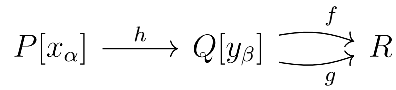
Proposition 2.18. *Consider maps of dg$C$ algebras*

$$P[x_\alpha] \underset{g}{\overset{f}{\rightrightarrows}} Q \xrightarrow{h} R$$

If $f \sim_P g$ then $hf \sim_P hg$.

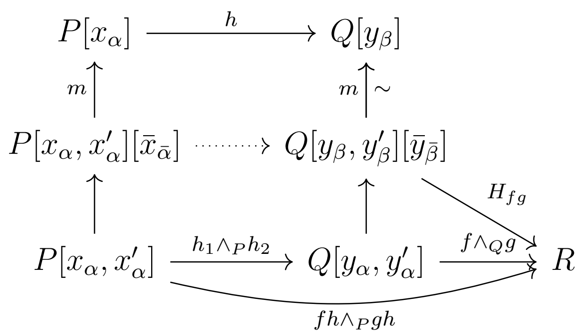
Proof. If $H_{fg} : P[x_\alpha, y_\alpha][z_\alpha] \to Q$ is a homotopy from f to g rel P, then $h \circ H_{fg}$ is a homotopy from hf to hg rel P. □

Proposition 2.19. *Consider maps of dgC algebras*



If $fh = gh$ on P (for example, if $g(P) = f(P) \subset Q$) and $f \sim_P g$, then $fh \sim_Q$.

Proof. Form path spaces $\tilde{m} : P[x_\alpha, x'_\alpha][\bar{x}_{\bar{\alpha}}] \xrightarrow{\sim} P[x_\alpha]$ and $\tilde{m} : Q[y_\beta, y'_\beta][\bar{y}_{\bar{\beta}}] \xrightarrow{\sim} Q[y_\alpha]$ and consider the diagram



where H_{fg} is a homotopy from f to g. Using the two inclusions of $Q[y_\beta]$ into $Q[y_\beta, y'_\beta]$ we can map $P[x_\alpha]$ to $Q[y_\beta, y'_\beta]$ using h in two different ways, which agree on P. Thus there is a map $h_1 \wedge_P h_2$. One observes that $f \wedge_Q g \circ h_1 = fh$ and $f \wedge_Q g \circ h_2 = gh$. It follows that $f \wedge_Q g \circ h_1 \wedge_P h_2 = fh \wedge_P gh$.

The outermost rectangle on the left found by ignoring the dotted arrow commutes, thus $h_1 \wedge_P h_2$ provides a lift of $P[x_\alpha, x'_\alpha][\bar{x}_{\bar{\alpha}}]$ to $Q[y_\beta, y'_\beta][\bar{y}_{\bar{\beta}}]$ on the subspace $P[x_\alpha, x'_\alpha]$ of the composite map $P[x_\alpha, x'_\alpha][x_{\bar{\alpha}}] \xrightarrow{m} P[x_\alpha] \xrightarrow{h} Q[y_\beta]$. Thus by the Lifting Lemma (Lemma 2.13), this lift can be extended to a lift up homotopy, thus the square below the dotted line commutes. Thus the middle path from $P[x_\alpha, x'_\alpha][\bar{x}_\alpha]$ to R provides a homotopy from fh to gh rel P, and the claim follows. □

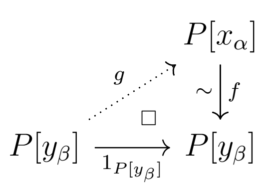
Definition 2.11. *Consider a map $f : P[x_\alpha] \to P[y_\beta]$ extending the identity map on P. We say f is a* ***homotopy equivalence rel*** *P if there is a map the other way $g : P[y_\beta] \to P[x_\alpha]$ also extending the identity map such that $gf \sim_P 1_{P[x_\alpha]}$ and $gf \sim_P 1_{P[y_\beta]}$.*

If $P = \mathbb{K}$ is the initial algebra, then we say, simply, ***homotopy equivalent****.*

One should think of a homotopy equivalence as a map that is invertible up to homotopy.

Proposition 2.20. *An algebra map $f : P[x_\alpha] \to P[y_\beta]$ extending the identity map on P is a homotopy equivalence if an only if it is a quasi-isomorphism.*

Proof. Consider the diagram



We can lift the identity map on $P[y_\beta]$ over f on P by identity on P. By the Lifting Lemma (Lemma 2.13), there is a lift g making the diagram commute up to homotopy rel P.

Thus we have found g such that $fg \sim_P 1_{P[y_\beta]}$.

Since homotopic maps are chain homotopic (by Lemma 2.17), fg is chain homotopic to the identity map on $P[y_\alpha]$, thus it induces isomorphism on homology. Since f is a quasi-isomorphism, it also induces an isomorphism on homology. Thus g, too, induces isomorphism on homology. Thus, as before, we may find an $f_2 : P[x_\alpha] \to P[y_\beta]$ extending the identity on P such that $gf_2 \sim_P 1_{P[x_\alpha]}$.

Using the properties of homotopies, above, we compute

$$gf = gf1_{P[x_\alpha]} \sim_P gfgf_2 \sim_P g1_{P[y_\beta]}f_2 \sim_P 1_{P[x_\alpha]}$$

Thus $gf \sim_P 1_{P[x_\alpha]}$.

Thus if f is a quasi-isomorphism, then it is a homotopy equivalence.

If f is a homotopy equivalence (rel a subalgebra), then there is a a map g the other way such that fg and gf are homotopic–thus chain homotopic–to their respective identity maps. Thus their induced maps on homology are isomorphisms. Thus f is a quasi-isomorphism. Thus homotopy equivalence implies quasi-isomorphism, and the claim follows. □

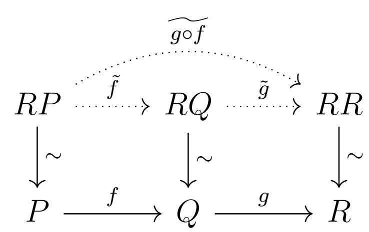
Corollary 2.21. *Consider a map $P \to Q$. Suppose we have two free triangular extensions $P[x_\alpha]$ and $P[y_\beta]$ and quasi-isomorphisms $P[x_\alpha] \xrightarrow{\sim} Q$ and $P[y_\beta] \xrightarrow{\sim} Q$ extending the original. Then $P[x_\alpha]$ is homotopy equivalent to $P[y_\alpha]$ rel P.*

Proof. This is an immediate consequence of the Lifting Lemma (Lemma 2.13) and Proposition 2.20. □

Proposition 2.22. *Any two resolutions are homotopy equivalence. Maps of algebras lift to a unique homotopy class of maps of resolutions. Lifts of composite maps are homotopic to composites of the lifted maps.*

Proof. That any two resolutions are homotopy equivalent follows from Corollary 2.21, and that maps lift to unique homotopy classes of maps follows immediately from the Lifting Lemma.

Consider algebra maps $P \xrightarrow{f} Q \xrightarrow{g} R$. Take resolutions $RP \xrightarrow{\sim} P$ $RQ \xrightarrow{\sim} Q$ and $RR \xrightarrow{\sim} R$ and consider lifts $\tilde{f}$, $\tilde{g}$ and $\widetilde{g \circ f}$ of f, g and $f \circ g$ (which exist by the Lifting Lemma.



One checks that $\widetilde{\tilde{g} \circ \tilde{f}}$ is a lift of $g \circ f$ up to homotopy using the properties of homotopies. By construction $g \circ f$ is also a lift of $\widetilde{g \circ f}$ up to homotopy. Since lifts up to homotopy are unique up to homotopy, $\tilde{g} \circ \tilde{f}$ and $\widetilde{g \circ f}$ are homotopic, and the claim follows. □

Definition 2.12. *We say a map of resolved algebras $f : P_1 \to Q_1$ is* ***homotopy equivalent*** *to another map of resolved algebras $g : P_2 \to Q_2$ if there are homotopy equivalences denoted by the dotted arrows such that the diagram commutes up to homotopy:*

$$\begin{array}{ccc} P_1 & \overset{\sim}{\dashrightarrow} & P_2 \\ \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle g} \\ Q_1 & \overset{\sim}{\dashrightarrow} & Q_2 \end{array}$$

We say a general algebra map is ***homotopy equivalent*** *to another general algebra map if a lift of the first to resolutions is homotopy equivalent to a lift of the second to resolutions.*

Lemma 2.23. *Homotopy equivalence is reflexive, symmetric, and transitive, thus we can form classes (though not sets) of homotopy equivalent maps.*

Proof. First we check that homotopy equivalence on resolved algebras is reflexive, symmetric and transitive.

On resolved algebras homotopy equivalence is obviously reflexive.

Suppose f and g are homotopy equivalent maps of resolved algebras. Since quasi-isomorphisms of resolved algebras are homotopy equivalences, there is a diagram commuting up to homotopy, such that the top two and bottom two maps are inverse homotopy equivalences

$$\begin{array}{ccc} & \overset{\sim}{\curvearrowleft} & \\ & \square & \\ RP_1 & \xrightarrow{\sim} & RP_2 \\ \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle g} \\ RQ_1 & \xrightarrow{\sim} & RQ_2 \\ & \square & \\ & \underset{\sim}{\curvearrowleft} & \end{array}$$

One checks, using the properties of homotopies, that the outermost paths from RP_2 to RQ_1 are homotopic, thus homotopy equivalence is reflexive.

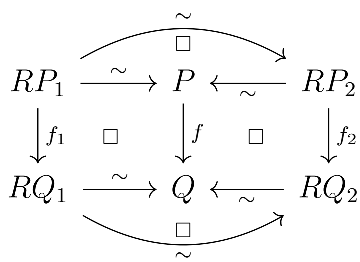
Suppose f, g and h are maps between resolved algebras. Suppose f is homotopy equivalent to g and g is homotopy equivalent to h. Then there is a diagram that commutes up to homotopy

$$\begin{array}{ccccc} RP_1 & \xrightarrow{\sim} & RP_2 & \xrightarrow{\sim} & RP_3 \\ \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle g} & \square & \downarrow{\scriptstyle h} \\ RQ_1 & \xrightarrow{\sim} & RQ_2 & \xrightarrow{\sim} & RQ_3 \end{array}$$

The composite of two quasi-isomorphisms is a quasi-isomorphisms, and and one uses the properties of homotopies to show the outermost routes from RP_1 to RQ_2 are homotopic, thus f is homotopy equivalent to h.

The claim follows for maps of resolved algebras. Before we prove the claim for general maps, we check that any two lifts of a map to (possibly different) resolutions are homotopy equivalent.

Let $f : P \to Q$ and suppose we have resolutions $RP_1 \xrightarrow{\sim} P \xleftarrow{\sim} RP_2$ and $RQ_1 \xrightarrow{\sim} P \xleftarrow{\sim} RQ_2$. And consider the diagram where f_1 and f_2 are lifts of f up to homotopy to homotopy, and the curved maps at the top and bottom are lifts up to homotopy between resolutions.

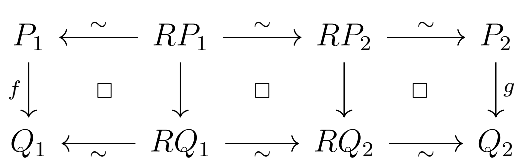


Using the properties of homotopies, one easily checks that any two ways of getting from RP_1 to Q are homotopic. Thus the two ways of getting to from RP_1 to RQ_2 and then to Q are homotopic. Thus both routes from RP_1 to RQ_2 are lifts up to homotopy of any of the routes from RP_1 to Q. Thus by the Lifting Lemma (Lemma 2.13), the two routes from RP_1 to RQ_2 are homotopic, that is, the outermost square in the diagram commutes up to homotopy. Thus f_1 and f_2 are homotopy equivalent.

Now we prove the claim for general algebra maps. Homotopy equivalence is obviously reflexive for general maps. It's symmetric, because we can find an inverse up to homotopy of the homotopy equivalence between the resolutions.

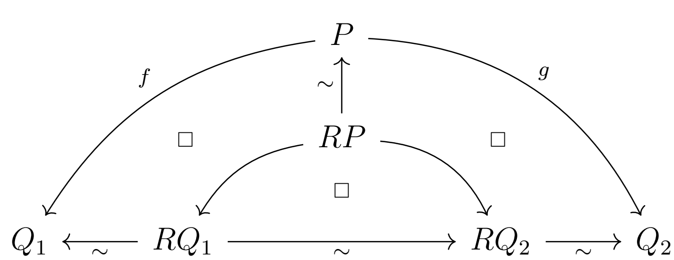
Suppose f, g and h are general algebra maps, and f is homotopy equivalent to g and g is homotopy equivalent to h. Since any two lifts of g to resolutions are homotopy equivalence, we may assume the homotopy equivalences of f to g and g to h use the same resolutions for g. Now we can combine the diagrams of the homotopy equivalences at the resolutions of g. Since homotopy equivalence is transitive on maps of resolved algebras, we get a homotopy equivalence from f to h. Thus homotopy equivalence is transitive on general maps, and the claim follows. □

One observes that a homotopy equivalence from $f : P_1 \to Q_1$ to $g : P_2 \to Q_2$ amounts to the existence of a resolutions and maps making the diagram commute up to homotopy



If we fix the domain, $P_1 = P_2 = P$ then we can use the the same resolution for P_1 and P_2 and the identity quasi-isomorphism on these resolutions. This gives the following relation:

Definition 2.13. *Let $f : P \to Q_1$ and $g : P \to Q_2$ be algebra maps with the same domain. We say f is* ***homotopy equivalent with restricted domain*** *to g if there are resolutions $RP \to P$, $RQ_1 \to Q_1$ and $RQ_2 \to Q_2$ and quasi-isomorphisms making the following diagram commute.*



Proposition 2.24. *Homotopy equivalence with restricted domain is reflexive, symmetric and transitive, thus we can form classes of maps which are homotopy equivalent with restricted domain.*

Proof. One easily adapts the proposition for general homotopy equivalence of maps to this situation. □

Similarly, can consider homotopy equivalence with restricted co-domain, or restricted domain and co-domain. The latter amounts maps on the same spaces which are homotopic after lifting to resolutions.

2.6 Algebraic Structures

Let V be a chain complex. The multilinear operations on V with k inputs and l outputs form a chain complex $Hom(V^{\otimes k}, V^{\otimes l})$.

Definition 2.14. *Let $\mathcal{O}$ be a set of operations types ranging most generally over all pairs $\binom{k}{l}$ (written vertically) for $k = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$*

Let V be a chain complex. In the presence of such a set $\mathcal{O}$, the space of all operations on V is $End(V) := \{Hom(V^{\otimes k}, V^{\otimes l}) | \binom{k}{l} \in \mathcal{O}\}$. We may denote the k-to-l operations by $End(V)\binom{k}{l} := Hom(V^{\otimes k}, V^{\otimes l})$.

Though later in this thesis we only consider combination algebras that act by certain distinguished combination operations on the space of operations of a chain complex V, for example, by composing operations, this need not be the case: we might wish to include combination operations that rely on, say, a choice of pairing on the chain complex. This leads us to make the following, modest definition:

Definition 2.15. *Let C be a combination algebra with zero differential, and let P be a dg$C$ algebra.*

*A **dg$P$ algebra structure on** V is a choice of a dg$C$ algebra structures on $End(V)$ and a dg$C$ algebra map $P \to End(V)$. We call this map the algebra structure's **structure map**. Since a map of dg$C$ algebras includes the information of the the dg$C$ algebra structure on the domain and range, we may refer to an algebra structure by its structure map.*

*We call operations in $End(V)$ with zero inputs and zero outputs **constant operations**. Since $Hom(\mathbb{K}, \mathbb{K})$ is canonically isomorphic to the the ground fields $\mathbb{K}$, we may identify the constant constant operations with the constants $\mathbb{K}$.*

If $P \to End(V)$ is an algebra structure, we call it's structure constants the induced map on the constant operations on homology, that is

$$\mathbf{H}(P\tbinom{0}{0}) \to \mathbb{K}$$

Recall that there are several notions of homotopy-theoretic equivalences on maps of dg$C$ algebras. (See Section 2.5). Since algebra structures are given by maps of dg$C$ algebras, we may apply these equivalences to algebra structures.

Definition 2.16. *We say two algebraic structures are* ***homotopy equivalent*** *if their structure maps are homotopy equivalent. We say they're* ***homotopy equivalent with restricted domain (or codomain)*** *if their structures maps are. We say two algebra structures with the same domain and range are* ***homotopic*** *if they're homotopic (after lifting to a resolution).*

We note how structure constants behave on classes of equivalent algebra structures for various equivalences.

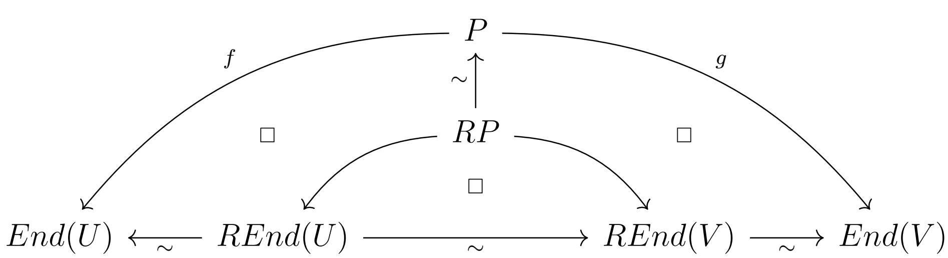
Theorem 2.25. *Homotopic algebra structures have equal structure constants. Algebra structures which are homotopy equivalent with restricted domain have equal structure constants up to scaling.*

Proof. To prove homotopic algebra structures have equal structure constants, its enough to resolve the domain. Let $f : P \to End(V)$ and $g : P \to End(V)$ be two dg$P$ algebra structures on V and $RP \xrightarrow{\sim} P$ a free resolution of P. If f and g are homotopic (after composing with the resolution) then the top and bottom path induce the same maps on homology

$$RP \xrightarrow[\sim]{h} P \underset{g}{\overset{f}{\rightrightarrows}} End(V)$$

Since the induced map of h is an isomorphism on homology, f and g induce the same maps on homology, thus they give equal structure constants $f_0 = g_0 : \mathbf{H}(P_0) \to \mathbb{K}$.

Let $P \to End(U)$ and $P \to End(V)$ be two dg$P$ algebra structures which are homotopy equivalent with restricted domain. Then there are resolutions and maps making the following diagram commute up to homotopy



Since this diagram commutes on homology, there is a commutative diagram

$$\begin{array}{ccc} & \mathbf{H}(P_0) & \\ {}^{f_0}\swarrow & & \searrow^{g_0} \\ \mathbb{K} & \xrightarrow{\cong} & \mathbb{K} \end{array}$$

The isomorphism on the bottom is a choice of scalar. Thus their structure constants are equal up to scaling, and the claim follows.

□

We observe the following basic, but key fact:

Proposition 2.26. *Fix a combination algebra C with zero differential. Let U and V be chain complexes, and choose a dgC algebra structure on $End(U)$ an $End(V)$.*

If $End(U)$ and $End(V)$ have homotopy equivalent resolutions as dgC algebras, then for every algebra structure $P \to End(U)$ there is a homotopy equivalent algebra structure $Q \to End(V)$. If P is resolved, then we can pick an algebra structure on V which is is homotopy equivalent with restricted domain.

Proof. Let $P \to End(V)$ be an algebra structure on V, and suppose $End(U)$ and $End(V)$ have homotopy equivalent resolutions. Then there are resolutions and diagram without the dotted arrow

$$\begin{array}{ccccccc} P & \xleftarrow[\sim]{} & RP & & & & \\ \downarrow & \square & \vdots & & & & \\ End(U) & \xleftarrow[\sim]{} & REnd(U) & \xrightarrow{\sim} & REnd(V) & \xrightarrow{\sim} & End(V) \end{array}$$

By the Lifting Lemma (Lemma 2.13), we can lift the map from RP to $End(U)$ to $REnd(U)$ up to homotopy. We now get a map $RP \to End(V)$ by compositing. One easily checks $RP \to End(V)$ is homotopy equivalent to $P \to End(V)$.

If P is resolved, then we can simply lift starting from P, so the claim follows. □

In the next chapter we show that certain class of combination algebras behaves very well.

Chapter 3

Algebraic Structures with Natural Action of Combination Algebras on Operations on Chain Complexes

In this chapter, we give conditions on the actions of a combination algebra, which insure that algebra structures are particularly well-behaved.

We find it convenient to first discuss algebra structures over a fixed algebra of operations.

3.1 The Category Algebra Structures Over a Fixed Algebra Of Operations

Fix a combination algebra C and a dg$C$ algebra P.

It is convenient to refer to chain complex with a distinguished dg$P$ algebra structures as a **dg$P$ algebra**. Recall that this action includes an action of C on $End(V)$ and that there may be different such actions. We may consider the category of dg$P$ algebras with maps in the usual sense. Namely, a map $\psi : U \to V$ is an dg$P$algebra map if for every operation p in P with any number of inputs and outputs k and l, the following diagram commutes

$$\begin{array}{ccc} U^{\otimes k} & \xrightarrow{\psi^{\otimes k}} & V^{\otimes k} \\ \downarrow f & & \downarrow f \\ U^{\otimes l} & \xrightarrow{\psi^{\otimes l}} & V^{\otimes l} \end{array}$$

Recall that the zero tensor power of a chain complex is the chain complex $\mathbb{K}$ consisting of the ground field in degree zero and zero differential, and the zero tensor power of a chain map is the identity map $1_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$, thus the diagram makes sense for k or l zero, too.

Remark 3.1. *It follows that algebra maps exactly preserve constant operations*

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{1_{\mathbb{K}}} & \mathbb{K} \\ \downarrow f & & \downarrow f \\ \mathbb{K} & \xrightarrow{1_{\mathbb{K}}} & \mathbb{K} \end{array}$$

We say a subcomplex W of a dp$P$ algebra V is a **subalgebra** if the operations on V restrict to W, and we say a subcomplex I of V is an **ideal** if the operations on V induce maps on the quotient. We say an algebra map is an **isomorphism** if it is an isomorphism as a linear space.

One observes the inverse of an isomorphism is again an isomorphism, because the diagrams above commute, using the inverse map. One checks the image of an algebra map is a subalgebra of the image, and that kernels are ideals. One also checks the usual isomorphism theorem holds.

3.1.1 Restriction, Induction, and Transportation

Let V be a chain complex, and $f : V^{\otimes k} \to V^{\otimes l}$ an operation on V. Let W be a subcomplex of V with inclusion map $i : W \to V$. We say that f restricts to W if there is a map $f|_W$ making the diagram commute

$$\begin{array}{ccc} W^{\otimes}k & \xrightarrow{i^{\otimes k}} & V^{\otimes k} \\ \downarrow{\scriptstyle f|_W} & & \downarrow{\scriptstyle f} \\ W^{\otimes k} & \xrightarrow{i^{\otimes l}} & V^{\otimes l} \end{array}$$

One notes $f|_W$ exists if and only if $f(W^{\otimes k}) \subset W^{\otimes l}$. We call $f|_W$ the **restriction** of f to W, and note that the restriction $f|_W = 0$ if and only if $f(W^{\otimes k}) = 0$.

Let I be a subcomplex of V and $p : V \to V/I$ the projection map. We say an operation $f : V^{\otimes k} \to V^{\otimes l}$ induces an operation on V/I if there is a map $\bar{f}$ making the diagram commute

$$\begin{array}{ccccc} \sum_{i+1+j=k} V^{\otimes i} \otimes I \otimes V^{\otimes j} & \hookrightarrow & V^{\otimes k} & \xrightarrow{p^{\otimes k}} & V/I^{\otimes k} \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ \sum_{i+1+j=l} V^{\otimes i} \otimes I \otimes V^{\otimes j} & \hookrightarrow & V^{\otimes l} & \xrightarrow{p^{\otimes l}} & V/I^{\otimes l} \end{array}$$

One notes such an $\bar{f}$ exists if and only if $f(\sum_{i+1+j=k} V^{\otimes i} \otimes I \otimes V^{\otimes j}) \subset \sum_{i+1+j=l} V^{\otimes i} \otimes I \otimes V^{\otimes j}$, and that the restriction $\bar{f} = 0$ if and only if $f(V^{\otimes k}) \subset \sum_{i+1+j=l} V^{\otimes i} \otimes I \otimes V^{\otimes j}$. We call $\bar{f}$ the **induction** of f to V/I.

Let V be a chain complex, and consider the space of operations $End(V)$ on V of some types $\mathcal{O}$. Given a subspace W of V denote $Res(W, V)$ the space of operations in $End(V)$ that restrict to operation on W, and given a subspace I if V, denote $Ind(V, V/I)$ the space of operation in $End(V)$ that induce operations on V/I.

Lemma 3.1. *The spaces $Res(W, V)$ and $Ind(V, V/I)$ are subcomplexes of $End(V)$, and the maps given by restriction and induction are surjective chain maps*

$$\begin{array}{rcl} Res(W, V) & \twoheadrightarrow & End(W) \\ f & \mapsto & f|_W \end{array}$$

$$\begin{array}{rcl} Ind(V, V/I) & \twoheadrightarrow & End(V/I) \\ f & \mapsto & \bar{f} \end{array}$$

Proof. Let V be a chain complex and W and I subcomplexes. An operation $f : V^{\otimes k} \to V^{\otimes l}$ in $End(V)$ restricts to a W if and only if $f(W^{\otimes k}) \subset W^{\otimes l}$, and induces an map on V/I if and only if $f(\sum_{i+1+j=k} V^{\otimes i} \otimes I \otimes V^{\otimes j}) \subset \sum_{i+1+j=l} V^{\otimes i} \otimes I \otimes V^{\otimes j}$. If two operations satisfy one of these, then so do their sums and boundaries, thus $Res(W, V)$ and $Ind(V, V/I)$ are subcomplexes of $End(V)$.

Suppose f and g are two k-to-l operation in $Res(W, V)$. Then $(f+g)|_W(w) = (f+g)(w) = f(w) + g(w) = (f|_W + g_W)(w)$, moreover $\partial(f)|_W(w) = (df \pm fd)(w) = df(w) \pm fdw = df|_W(w) \pm f|_W(dw) = \partial(f|_W)(w)$, thus one sees restriction is a chain map. If f and g are in $Ind(V, V/I)$, then we observe that $\overline{(f+g)}(\bar{v}) = \overline{(f+g)(v)} = \overline{f(v)} + \overline{g(v)} = \bar{f}(\bar{v}) + \bar{g}(\bar{v})$, moreover $\overline{\partial(f)}(\bar{v}) = \overline{\partial(f)(v)} = \overline{df(v) \pm fd(v)} = d\bar{f}(\bar{v}) \pm \bar{f}(d(\bar{v})) = \partial(\bar{f})(\bar{v})$, thus one sees induction is a chain map.

Exact sequences of vector spaces always split, thus so do exact sequences of graded vector spaces. Thus one can extend any operation on W and V/I to operations on V. Thus the restriction and induction maps are surjective, and the claim follows. $\square$

Define the kernel of restriction to be $Ann(W, V)$ and the kernel of induction to be $CoAnn(V, V/I)$. We call these spaces the **annihilator** of W in V and the **co-annihilator** of V on V/I. These contain the operations, which restrict to zero on W and the operations which induce zero on V/I, respectively.

Proposition 3.2. *Given a space $End(V)$ of operation on V, and subspaces I and W of V there are diagrams with exact rows and columns*

$$\begin{array}{ccccccccc} & & & & End(V) & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & Ann(W,V) & \longrightarrow & Res(W,V) & \longrightarrow & End(W) & \longrightarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

$$\begin{array}{ccccccccc} & & & & End(V) & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & CoAnn(V,V/I) & \longrightarrow & Ind(V,V/I) & \longrightarrow & End(V/I) & \longrightarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

We prove a crucial lemma after after recalling a few facts about chain complexes: a chain complex is called **contractible** if the identity map is chain homotopic to zero, and such a chain homotopy is called a **contracting homotopy**. A chain complex is called **acyclic** if it has zero homology. One checks that a chain complex (over a field) is acyclic if and only if it is contractible. Let V and W be chain complexes, and consider the chain complex $Hom(V, W)$ with differential $\partial(f) := [d, f] := df - (-1)^{|f|} fd$. Suppose V is acyclic, thus contractible with contracting homotopy s. Then $S(f) := sf$ is a contracting homotopy for $Hom(V, W)$, thus $Hom(V,W)$ is acyclic, ie has zero homology. If W is acyclic, a similar trick also shows $Hom(V,W)$ is acyclic. Finally Let $U \to V \to V/U \to 0$ be a short exact sequence of chain complexes. This induces a long exact sequence in homology, which by inspections shows that U is acyclic if and only if $V \xrightarrow{\sim} V/U$ is a quasi-isomorphism, and V/U is acyclic if and only if $U \xrightarrow{\sim} V$ is a quasi-isomorphism. If I is an acyclic complex with contracting homotopy s, and V is another chain complex, then $S = 1^{\otimes n} \otimes s \otimes 1^{\otimes m}$ is a contracting homotopy on $V^{\otimes n} \otimes I \otimes V^{\otimes m}$, thus it, too, is acyclic. One observes that direct sums of acyclic complexes are also acyclic.

Lemma 3.3. *If the inclusion $i : W \xrightarrow{\sim} V$ is a quasi-isomorphism then so are the maps*

$$\begin{array}{ccc} End(V) & & \\ \sim\uparrow & & \\ Res(W,V) & \xrightarrow{\sim} & End(W) \end{array}$$

and $Ann(W,V)$ is acyclic.

If the projection map $p : V \xrightarrow{\sim} V/I$ is a quasi-isomorphism (or equivalently, if I is acyclic), then so are the maps

$$\begin{array}{ccc} End(V) & & \\ \sim\uparrow & & \\ Ind(V,V/I) & \xrightarrow{\sim} & End(V/I) \end{array}$$

and $CoAnn(V,V/I)$ is acyclic.

Proof. Suppose W is a quasi-isomorphic subspace of V, meaning the inclusion map $W \overset{\sim}{\hookrightarrow} V$ is a quasi-isomorphism. Denote the inclusion map $i : W \to V$ and the quotient map $p : V \to V/W$.

Let $f : V^{\otimes k} \to V^{\otimes l}$ be a k-to-l operation in $End(V)$. Using the familiar universal property of quotients, one checks that f restricts an operation $f|_W$ on W if and only if $p^{\otimes l} \circ f \circ i^{\otimes k} = 0$, and it induces the zero operation $f|_W = 0$ if an only if it induces a map $\bar{f} : V^{\otimes k}/W^{\otimes k} \to V^{\otimes l}/W^{\otimes l}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^{\otimes k} & \xrightarrow{i^{\otimes k}} & V^{\otimes k} & \longrightarrow & V^{\otimes k}/W^{\otimes k} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f|_W} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \\ 0 & \longrightarrow & W^{\otimes l} & \xrightarrow{i^{\otimes l}} & V^{\otimes l} & \xrightarrow{p^{\otimes l}} & V^{\otimes l}/W^{\otimes l} & \longrightarrow & 0 \end{array}$$

This gives a map $Ann(W,V)\binom{k}{l} \to Hom(V^{\otimes k}/W^{\otimes k}, V^{\otimes l}/W^{\otimes l})$ that one checks is an isomorphism of chain complexes. These two facts give a diagram with and exact row and exact column:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & Hom(W^{\otimes k}, V^{\otimes l}/W^{\otimes l}) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & End(V)\binom{k}{l} & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & Hom(V^{\otimes k}/V^{\otimes k}, V^{\otimes l}/W^{\otimes l}) & \longrightarrow & Res(W,V)\binom{k}{l} & \longrightarrow & End(W)\binom{k}{l} & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Since the inclusion $W \overset{\sim}{\hookrightarrow} V$ is a quasi-isomorphism, so is the n-fold tensor product of the inclusion $W^{\otimes n} \overset{\sim}{\hookrightarrow} V^{\otimes n}$. Thus the quotients $V^k/W^{\otimes n}$ are all acyclic, and thus so are the various Hom spaces involving them. Recall that when n is zero, the zeroth tensor power of a chain complex is $\mathbb{K}$ and the zeroth tensor power of a chain map is the identity map on $\mathbb{K}$, so the diagram makes sense for any k and l.

It follows that $Ann(W,V)$ is acyclic, and the maps from $Res(W,V)$ to $End(V)$ and $End(W)$ are quasi-isomorphisms.

For the other half the claim, we observe that an operation $f : V^{\otimes k} \to V^{\otimes l}$ on V induces an operation $\bar{f}$ on V/I if and only if the composition $q_l \circ f \circ j_k = 0$, and induces the zero operation if and only if it restricts to a map on the kernels $ker(j_k) \to ker(j_l)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ker(j_k) & \xrightarrow{j} & V^{\otimes k} & \longrightarrow & (V/W)^{\otimes k} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f} & & \vdots{\scriptstyle \bar{f}} & & \\
0 & \longrightarrow & ker(j_l) & \longrightarrow & V^{\otimes l} & \xrightarrow{q} & (V/I)^{\otimes l} & \longrightarrow & 0
\end{array}$$

This gives a map from $CoAnn(V,V/I) \to Hom(ker(j_k), ker(j_l))$, which one checks is an isomorphism of chain complexes. Thus there is an isomorphism $CoAnn(V,V/I)\binom{k}{l} \cong Hom(V^{\otimes k}, (V/I)^{\otimes l})$, and a diagram with an exact row and and exact column:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & Hom(ker(j_k),(V/I)^{\otimes l}) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & End(V)\binom{k}{l} & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & Hom(ker(j_k),ker(j_l)) & \longrightarrow & Ind(V,V/I)\binom{k}{l} & \longrightarrow & End(V/I)\binom{k}{l} & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

One observes that $ker(j_k) = \sum_{m+1+n=k} V^{\otimes m} \otimes I \otimes V^{\otimes n}$, which is zero when $k = 0$. Since I is acyclic, so are tensor products involving I, and thus so is the sum of several of those, $ker(j_k)$. Thus the various Hom complexes involving $ker(j_k)$ are acyclic.

It follows that $CoAnn(V, V/I)$ is acyclic, and the maps from $Ind(V, V/I)$ to $End(V)$ and $End(V/I)$ are quasi-isomorphisms. The claim follows. □

Aside from restriction and induction, we also consider transportation of operations on chain complexes via isomorphisms of the chain complex:

Definition 3.1. *Let $\phi : U \to V$ be an isomorphism of chain complexes. Form $End(U)$. Given an operation $f : U^{\otimes k} \to U^{\otimes l}$ we define an operation $f^{\phi} : V^k \to V^{\otimes l}$ by the composition*

$$\begin{array}{ccc}
U^{\otimes k} & \xleftarrow{(\phi^{-1})^{\otimes k}} & V^{\otimes k} \\
f\downarrow & & \downarrow f^{\phi} \\
U^{\otimes l} & \xrightarrow{\phi^{\otimes l}} & V^{\otimes l}
\end{array}$$

*We call this the **transportation of f via ϕ**. Transportation of operations on U to operations on V via an isomorphism ϕ gives a map*

$$\begin{array}{ccc}
End(U) & \overset{\cong}{\to} & End(V) \\
f & \mapsto & f^{\phi}
\end{array}$$

*which we call the **transportation map (of ϕ)***

Proposition 3.4. *If $\phi : U \to V$ is an isomorphism, then the transportation map $End(U) \to End(V)$ is an isomorphism that commutes with the differential.*

Proof. One easily checks that the differential commutes with transportation, because ϕ, and hence its inverse, is a chain map. The inverse map of ϕ is also a chain map, and transportation by it provides the inverse map. □

One observes that if two operations restrict to a subspace, then so do composition, moreover their composition commutes with restriction. Similar facts hold for operations, which induce operations on quotients. One also notes transportation commutes with composition. We will see that a combination algebra built from such combination operations (as composition) have particularly nice properties, as the following tautology foreshadows:

Proposition 3.5. *Let C be a combination algebra and P a dg$C$ algebra. Consider the category of dg$P$ algebras.*

W is a dg$P$ subalgebra of V if and only if the structure maps lift to a commutative diagram (of dg linear spaces):

$$\begin{array}{ccc} End(V) & \longleftarrow & P \\ \uparrow & \swarrow & \downarrow \\ Res(W,V) & \longrightarrow & End(W) \end{array}$$

V/I is a dg$P$ quotient algebra of V by an ideal I if and only if their structure maps lift to a commutative diagram

$$\begin{array}{ccc} End(V) & \longleftarrow & P \\ \uparrow & \swarrow & \downarrow \\ Ind(V,V/I) & \longrightarrow & End(V/I) \end{array}$$

And $\phi : U \to V$ is an isomorphism of dg$P$ algebras if and only if their structure maps are related by the map the transportation map of ϕ:

$$\begin{array}{ccc} End(U) & \longleftarrow & P \\ & \searrow & \downarrow \\ & & End(V) \end{array}$$

Remark 3.2. *One notes that if the diagrams above were maps of dg$C$ algebra structures, then the algebra structures related by quasi-isomorphic inclusions, quasi-isomorphic projections and isomorphisms would be homotopy equivalent with restricted domain.*

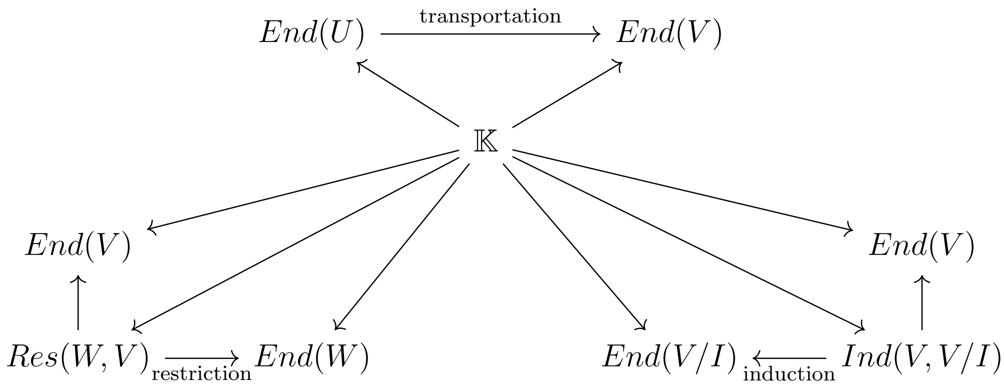
3.2 Natural Actions of Combination Algebras

3.2.1 Compatibility of Action

Fix a combination algebra C with zero differential. Recall that there is an initial algebra $\mathbb{K}$ over C, which may be found by taking the free algebra on the empty set. One looks at the construction of a free algebra to discover that $\mathbb{K}$ is simply the space of 0-to-1 operations in C.

Let V be a chain complex, W and I subcomplexes of V and $U \to V$ an isomorphism. Suppose we also have some choice of dg$C$ algebra structures on $End(V)$, $End(W)$, $End(V/I)$ and $End(U)$, such that $Res(W,V)$ and $Ind(V,V/I)$ are dg$C$ subalgebras of $End(V)$, and such that restriction, induction and transportation are dg$C$ algebra maps.

Since $\mathbb{K}$ is initial, there would be a commutative diagram as follows.



This not only means that W, V, V/I and U are bestowed upon a choice of dg$\mathbb{K}$ algebra structures, but that the inclusion $W \to V$, quotient $V \to V/I$ and isomorphism $U \to V$ are maps of dg$\mathbb{K}$ algebras (by Proposition 3.5).

If a dg$\mathbb{K}$ algebra structure on a chain complex V gives V any nontrivial structure (like a pairing), thus we cannot do better (ie require less structure) than inclusions, quotients and isomorphisms for some choices of dg$\mathbb{K}$ algebra structures on V, W, I, and U.

Proposition 3.6. *Let C be a combination algebra with zero differential. Fix a class of dg$C$ algebra structures on operations $End(V)$ on chain complexes V. Assume that transportation is a dg$C$ algebra map for any choice of dg$C$ algebra structures on operations on chain complexes. Then for every chain complex, V the dg$C$ algebra structure on $End(V)$ is unique.*

Proof. Let V be any chain complex. The identity map is a chain map, and so for any two choices of dg$C$ algebra structures on $End(V)$, transportation $End(V) \to End(V)$ (which is also the identity map) is a dg$C$ algebra map, thus C acts the same way on each. Thus the action on $End(V)$ is unique. □

Thus to do better, we need to have a distinguished dg$C$ algebra structures of C on spaces $End(V)$.

We note however, that we can't do any worse than have the diagrams for dg$\mathbb{K}$, in the following sense:

Proposition 3.7. *Let C be a combination algebra. Fix a class of dg$C$ algebra structures on spaces of operations $End(V)$ for chain complexes V. Let $\mathbb{K}$ be the initial dg$C$ algebra and P be a dg$C$ algebra.*

If dg$\mathbb{K}$ subalgebra, quotient algebra, and algebra isomorphism the diagrams in 3.5 are diagrams of dg$C$ algebras, then the same is true with P in place of $\mathbb{K}$.

Proof. A dg$P$ algebra structure on a chain complex V includes a choice of action of C on $End(V)$, thus it induces a dg$\mathbb{K}$ algebra structure on V by the map $\mathbb{K} \to End(V)$. One observes that if $U \to V$ is dg$P$ algebra map, it is automatically a dg$\mathbb{K}$ algebra map with the U and V regarded as dg$\mathbb{K}$ algebras. The claim follows. □

Finally, given distinguished dg$C$ algebra structures on spaces $End(V)$, it would also be convenient to ask they not depend on differentials, and that their induce dg$C$ algebra structure on homology match their distinguished dg$C$ algebra structure on homology .

We find these conveniences adequate, thus we make the following definition:

Definition 3.2. *Let C be a combination algebra with zero differential. We say C acts **naturally** on operations if for every chain complex V if there is a distinguished action of C on $End(V)$, which is defined independently of the differential, commutes with taking homology, and if for all subcomplexes W and I of V and any isomorphism of chain complexes $U \to V$, the restriction and induction spaces, $Res(W, V)$ and $Ind(V, V/I)$, are dgC subalgebras of $End(V)$, and restriction, induction and transportation are dgC algebra maps.*

Thus we observe:

Corollary 3.8. *If a combination algebra acts naturally on operations, then any chain complex has a distinguished dg$\mathbb{K}$ algebra structure, and any chain map is an algebra map of dg$\mathbb{K}$ algebras, thus diagrams in Proposition 3.5 are diagrams of dgC algebras for any P. Additionally, dgP algebra $U \to V$ maps induce dg$H(P)$ algebra maps $H(U) \to H(V)$ with the distinguished dgC algebra structures on $End((H(U)))$ and $End(H(V))$.*

Proposition 3.9. *Let C be a combination algebra with zero differential, with a distinguished action on $End(V)$ on every chain complex V.*

If C is generated by operations whose actions on operations only depend on the the underlying graded vector space, commute with taking homology, preserve restriction and induction spaces, and commute with restriction, induction and transportation, then C acts naturally.

Proof. If two operations only depend only on their underlying graded vector space of a chain complex, then so do operations obtained by composing them and permuting their inputs. The units of a combination algebra don't depend on the differential. Thus the algebra generated by operations that don't depend on the differential doesn't depend on the differential. Thus the action of C on operations on a chain complex only depends on its underlying graded vector space.

Let c_1 and c_2 be operations in C acting on $End(V)$ and $End(H(V))$. Let $[c_1]$ and $[c_2]$ denote the operations on $End(H(V))$ induced by c_1 and c_2 on $End(V)$ by taking homology. One recalls that, $[c_1]^\sigma = [c_1^\sigma]$ and $[c_1 \,{}_i\!\circ\, c_2] = [c_1] \,{}_i\!\circ\, [c_2]$, and that $[1_a] = 1_a$. If $[c_1] = c_1$ and $[c_2] = c_2$, then $[c_1^\sigma] = [c_1]^\sigma = c_1^\sigma$, and any composition $[c_1 \,{}_i\!\circ\, c_2] = [c_1] \,{}_i\!\circ\, [c_2] = c_1 \,{}_i\!\circ\, c_2$. It follows that the subalgebra of operation in C generated by operations whose actions commute with homology also acts by operations, which commute with homology. Thus the actions of C commute with homology.

Let V be a chain complex. Form the space $End(End(V))$ of all combination operations. Recall that this is an algebra under units, permutation of inputs and compositions. Consider the space $U(V)$ of all combination operations in $End(End(V))$ that preserve all restriction spaces, induction spaces, and which commute with restriction, induction and transportation (and have zero differential). We check below that $U(V)$ is a subalgebra of $End(End(V))$ (with zero differential), and that restriction, induction and transportation commute composition, units and permutations of inputs of $U(V)$.

Since generators of C land in the subalgebra $U(V)$, so does C. Since composition, units, and permutation of inputs commute with restriction, induction and transportation of these elements the restriction induction and transportation maps are dg$C$ algebra maps, so the claim follows once we've checked $U(V)$ is a subalgebra.

Let W and I be any subcomplexes of V, and $\phi : U \to V$ be an isomorphism. Consider the unit combination operation $1_{\binom{k}{l}}$. This fixes $End(V)\binom{k}{l}$, so it fixes $Res(W,V)\binom{k}{l}$ and $Ind(V,V/I)\binom{k}{l}$, moreover $\big(1_{\binom{k}{l}}(f)\big)|_W = f|_W = 1_{\binom{k}{l}}(f|_W)$ when f restricts, $\overline{1_{\binom{k}{l}}(f)} = \bar{f} = 1_{\binom{k}{l}}(\bar{f})$ when g induces, and $\big(1_{\binom{k}{l}}(h)\big)^{\phi} = h^{\phi} = 1_{\binom{k}{l}}(h^{\phi})$ for any h. The boundary of $1_{\binom{k}{l}}$ is zero. Thus $1_{\binom{k}{l}} \in U(V)$.

If c_1 and c_2 are combination operations in $U(V)$ with the same kinds of inputs and outputs, then linear combinations also preserve subspaces, induce operations on the subspaces and the induced operations commute with restriction, induction and transportation. Thus $U(V)$ is a linear space.

If c is in $U(V)$ and $f_1, \ldots, f_n$ restrict, then $c^{\sigma}(f_1 \otimes \ldots \otimes f_n)|_W = \pm c(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)})|_W = \pm c(f_{\sigma(1)}|_W \otimes \ldots \otimes f_{\sigma(n)}|_W) = c^{\sigma}(f_{\sigma(1)}|_W \otimes \ldots \otimes f_{\sigma(n)}|_W)$, so c^{σ} preserves restriction spaces and commutes with restriction. Similar processions show c^{σ} induces and σ commutes with induction, and that transportation commutes with permutation. Thus $c \in U(V)$.

If c_1 and c_2 are in $U(V)$ and $f_1, \ldots, f_m, g_1, \ldots g_n$ restrict ,then $\big((c_1\ {}_i\circ\ c_2)(f_1 \otimes \ldots \otimes g_1 \otimes \ldots \otimes g_n \otimes \ldots \otimes f_m)\big) = \big(c_1(f_1 \otimes \ldots \otimes c_2(g_1 \otimes \ldots \otimes g_n) \otimes \ldots \otimes f_m)\big)|_W = c_1(f_1|_W \otimes \ldots \otimes c_2(g_1 \otimes \ldots \otimes g_n)|_W \otimes \ldots \otimes f_m|_W) = c_1(f_1|_W \otimes \ldots \otimes c_2(g_1|_W \otimes \ldots \otimes g_n|_W) \otimes \ldots \otimes f_m|_W) = (c_1\ {}_i\circ\ c_n)(f_1|_W \otimes \ldots \otimes g_1|_W \otimes \ldots \otimes g_n|_W \otimes \ldots \otimes f_m|_W)$, so $c_1\ {}_i\circ\ c_2$ preserves restriction spaces and commutes with restriction. Similar processions show compositions preserves induction spaces and commutes with induction and transportation.

Thus $U(V)$ is closed under all the operations on $End(End(V))$, thus its a subspace and the claim follows. □

Recall that an combination operation in a combination algebra acts by a chain map if and only if ∂ is a derivation of it, and if and only if it is a boundary.

Definition 3.3. *Let c be a combination operation with a distinguished action on operations $End(V)$ of each chain complex V.*

We say c is a ***natural combination operation*** *if c actions preserves all restriction and induction spaces, c commutes with restriction, induction and transportation, the action of c only depends on the underlying graded vector space of V, its induced action on homology is the same as its distinguished action on homology, and the differential on any space of operations acts by a derivation of c (that is, its distinguished operations are cycles). The claim follows by the proposition above.*

Corollary 3.10. *If C is generated by natural operations, then it acts naturally.*

Proof. Since a generating set has a distinguished action on operation on each chain complex, so has the entire combination algebra. Since a generating set acts by derivations, its differential is zero on a generating set, so it is zero on all of C. □

Theorem 3.11. *Combination operations given by various compositions of operations, tensor product of operations, permutations of inputs and outputs and the identity map, are all natural operations.*

Proof. A 0-to-1 combination preserves restriction and induction spaces by simply mapping into them. The identity map restricts to the identity map on subspaces, and induces the identity map on quotient spaces, and transports to the identity map on isomorphic spaces. Thus it restricts, induces and commutes with restriction, induction and transportation. It obviously induces the identity map on homology, thus commutes with taking homology. Thus the identity map gives a natural operation.

Let f and g be two operations in $End(V)$ with k_1 and k_2 inputs, and l_1 and l_2 outputs, respectively. We note that the definition of $f \otimes g$ means the composition

$$V^{\otimes k_1+k_2} \stackrel{\cong}{\to} V^{\otimes k_1} \otimes V^{\otimes k_2} \otimes \stackrel{f \otimes g}{\to} V^{\otimes l_1} \otimes V^{\otimes l_2} \stackrel{\cong}{\to} V^{\otimes l_1+l_2}$$

which involves two natural isomorphisms.

We observe that $[f \otimes g]([v_1 \otimes \ldots \otimes v_{k_1+k_2}]) = [(f \otimes g)(v_1 \otimes \ldots \otimes v_{k_1+k_2})] = [f(v_1 \otimes \ldots \otimes v_{k_1}) \otimes g(v_{k_1+1} \otimes \ldots \otimes v_{k_1+k_2})]$ Thus after applying the natural isomorphism $H(V^{\otimes l_1}) \otimes H(V^{\otimes l_2}) \cong H(v^{\otimes l_1+l_2})$, thus the induced operation in $End(H(V))$ gives $[f \otimes g]([v_1] \otimes \ldots \otimes [v_{k_1+k_2}]) = [f(v_1 \otimes \ldots \otimes v_{k_1}] \otimes [g(v_{k_1+1} \otimes \ldots \otimes v_{k_1+k_2})] = ([f] \otimes [g])([v_1] \otimes \ldots \otimes [v_{k_1+k_2}])$, so $[f \otimes g] = [f] \otimes [g]$. Thus the action of tensoring operations commutes with taking homology.

Recall that an operation restricts to a subspace if and only it preserves the subspace. If f and g preserve a subspace, then clearly so does $f \otimes g$, thus tensor product preserves restriction spaces. It is trivial to check it commutes with restriction. Recall that an operation induces an operation on the quotient by an ideal I if given at least one input in I at least one output is in I. If f and g restrict, and $f \otimes g$ is given at least one input in I, then one of f or g in $f \otimes g$ gets I as an input, thus given an output in I, and thus $f \otimes g$ has an output in I. Thus $f \otimes g$ preserves induction spaces. One uses that canonical isomorphism $(V \otimes V)/(V \otimes I + I \otimes V) \cong V/I \otimes V/I$ to check that tensor product commutes with restriction. An easy computation also shows that tensor product commutes with transportation. Thus the tensoring operation is natural.

One may define various kinds of compositions, which can be combined from the following composition and permutations. Let $f : V^{\otimes k} \to V^{\otimes m+l}$ and $g :^{\otimes m} \to V^{\otimes n}$ be operations on V and l at least 1. We define $f \circ_L g := f \circ (g \otimes 1_V^{\otimes l})$, which is independent of the differential. This clearly preserves restriction and induction spaces and commutes with restriction, induction and transportation, and one checks it commutes with the boundary map, that is $\partial(f \circ_L g) = \partial(f) \circ_L g + (-1)^{|f|} f \circ_L \partial(g)$. We check that this action of composition commutes with taking homology: $[f \circ_L g]([v_1] \otimes \ldots \otimes [v_{n+m}]) = [(f \circ_L g)(v_1 \otimes \ldots \otimes v_{n+m})] = [f(g(v_1 \otimes \ldots \otimes v_n) \otimes v_{m+1} \otimes \ldots \otimes v_{n+m})] = [f]([g(v_1 \otimes \ldots \otimes v_m)] \otimes [v_{m+1} \otimes \ldots \otimes [v_n]]) = ([f] \circ_L [g])([v_1] \otimes \ldots \otimes [v_{n+m}])$. Thus this composition operation is natural.

It is straight forward to check that permutations of inputs and outputs give natural operations. □

This makes it rather easy to build natural combination algebras out of familiar operations.

Let N be a set of natural operations. Then there is distinguished map to the combination algebra of endomorphisms $End(End(V))$ on operations of any chain complex V. Thus there is a unique map extending this over the free combination algebra, $\mathbb{K}[N] \to End(End(V))$. Since N acts by chain maps, they may be given the zero differential. We may extend the differential over $\mathbb{K}[N]$ by zero.

Proposition 3.12. *The space U of all combination operations in $\mathbb{K}[N]$ that vanish for all V is an ideal of $\mathbb{K}[N]$.*

Proof. Suppose not. U is certainly a linear space, and is closed under permutation of inputs. If it is also closed under composition, then it is an ideal. So it isn't closed under composition. Then there is a combination operation u in $\mathbb{K}[N]$ that acts by zero on all operation on all chain complexes, but when composed with some other c in $\mathbb{K}[N]$ doesn't act by zero on operations on some chain complex V. The kernel of $\mathbb{K}[N] \to End(End(V))$ is an ideal, thus the composition actually does act by zero on $End(V)$, contradicting that it acts in by something other than zero. Thus U is an ideal. □

It follows that the distinguished maps $\mathbb{K}[N] \to End(End(V))$ factor through the quotient $C(N) := \mathbb{K}[N]/U$.

Theorem 3.13. *Let $\mathcal{O}$ be a set of operation types, and $\mathcal{C}(\mathcal{O})$ the set of combination operation types on $\mathcal{O}$.*

Let N be a set of natural relations. The combination algebra $C(N)$, above, factors through any other combination algebra extending the action of N, and it acts naturally on operations.

Proof. The free property for combination algebras ensures there is a map $\mathbb{K}[N] \to D$ to any other combination algebra D extending the action of N, and it commutes with the differential. Since U is the the kernel of the map to D, by commutativity, the map to D factors through $C(N) := \mathbb{K}[N]/U$, and the claim follows. □

Definition 3.4. *We call $C(N)$ the* ***universal combination algebra generated by*** *N.*

Thus:

Corollary 3.14. *The universal combination algebra generated by various compositions of operations, tensor product of operations, permutations of inputs and outputs and the identity map give natural actions on chain complexes.*

Perhaps the most common example, though usually not presented in this manner is the following: let $\mathcal{O}$ be the set of k-to-1 operation types for $k = 0, 1, 2, \ldots$, and let N contain the identity map, maps permuting the inputs of operations by any permutation and composition of one output into any one input. We define the universal combination algebra generated by these operations $Operad := C(N)$. Algebras over $Operad$ are called dg operads, and algebra structures over those are called algebras over operads.

3.3 Algebraic Structures with a Natural Action on Operations

We will see in the presence of a natural action of a combination algebra on operations that certain pleasant facts hold true. First, if U and V are chain homotopy equivalent chain complexes, then $End(U)$ and $End(V)$ have homotopy equivalent resolutions. Recall that this implies we can transfer structures between homotopy equivalent chain complexes up homotopy equivalence. Second, we can often tell that two dg$P$ algebra structures are homotopy equivalent with restricted domain without having to pass to resolutions. And third, we given a bit of extra structure relating a complex to a subcomplex or quotient complex, we give explicit formulas for transfered structures. We can always generate such data relating a chain complex to its homology, and use it to get an explicit formula for of an algebra structure transfered to its homology.

3.3.1 Zig-Zags of Algebras

Fix a combination algebra C with zero differential that acts naturally on operations on chain complexes.

Definition 3.5. *Fix a dg$C$ algebra P, and let U and V be dg$P$ algebras.*

*A **zig-zag (of dg$P$) algebras from** U **to** V is a finite list of dg$P$ algebras $U = W_1, W_2, \dots W_n = V$ with dg$P$ algebra maps going either left or right between adjacent pairs, which are either quasi-isomorphic inclusions, quasi-isomorphic projections, or (quasi-isomorphic) isomorphisms. We denote a particular choice of zig-zag from U to V with an arrow $U \rightsquigarrow V$.*

*We say U is **zig-zag equivalent** to V if there is a zig-zag from U to V.*

It is trivial to check:

Proposition 3.15. *Zig-zag equivalence is reflexive, symmetric and transitive, thus we can form classes of zig-zag equivalent algebras.*

Theorem 3.16. *If two algebras are zig-zag equivalent, then their structure maps are homotopy equivalent with restricted domain.*

Proof. Suppose $W \to V$ is a quasi-isomorphic inclusion of dg$P$ algebras, then there is a commutative diagram of dg$C$ algebras (by Proposition 3.5):

$$\begin{array}{ccc} End(V) & \longleftarrow & P \\ {\scriptstyle\sim}\uparrow & \swarrow & \downarrow \\ Res(W,V) & \xrightarrow{\sim} & End(W,) \end{array}$$

One easily checks this implies $P \to End(W)$ and $P \to End(V)$ are homotopy equivalent (by taking resolutions, using the lifting lemma, and the properties of homotopies). The other diagrams in Proposition 3.5 take care of quasi-isomorphic projections, and isomorphisms (which are also quasi-isomorphisms).

Now consider a general zig-zag from U to V. Then the structure maps of adjacent algebras are homotopy equivalent with restricted domain. Since homotopy equivalent with restricted domain is an equivalence relation, the structure maps of U and V are homotopy equivalent, as desired. $\square$

One recalls the following facts:

Lemma 3.17. *Any chain map can be factored into an inclusion, followed by an isomorphism, followed by a projection (using the chain complex version of a mapping cylinder). If the chain map is a quasi-isomorphism, then these can be chosen to be quasi-isomorphisms. Quasi-isomorphic chain maps are chain homotopy equivalences.*

Proposition 3.18. *The category of dg$\mathbb{K}$ algebras is isomorphic to the category of chain complexes, and chain homotopy equivalence and zig-zag equivalence give the same relation.*

Proof. Every chain complex has a unique dg$\mathbb{K}$ algebra structure, because there is a unique dg$C$ algebra structure on $End(V)$ and there is a unique map $\mathbb{K} \to End(V)$.

Since the action of C is natural, restriction and induction spaces $Res(W, V)$ and $Ind(V, V/I)$ are dg$C$ subalgebras of $End(V)$ for any subcomplexes W and I of V, and the restriction, induction and transportation maps are dg$C$ algebra maps. Since $\mathbb{K}$ is initial, there are commutative diagrams as in Proposition 3.5 with $P = \mathbb{K}$. The same proposition implies that inclusions, quotients and isomorphisms are dg$\mathbb{K}$ algebra maps.

Since every chain map can be factored into inclusion maps, isomorphisms and quotient maps, and compositions of algebra maps are algebra maps, every chain map is a dg$\mathbb{K}$ algebra map.

It follows that the category of chain complexes and dg$\mathbb{K}$ algebras are isomorphic.

Chain homotopy equivalences are quasi-isomorphisms and vice versa, and they can be factored into quasi-isomorphic inclusions, isomorphisms, and quasi-isomorphic quotient maps. It follows that homotopy equivalent chain complexes and zig-zag equivalent dg$\mathbb{K}$ algebras are the same thing, as desired. □

Proposition 3.19. *In the presence of a natural action of a combination algebra, if chain complexes U and V are homotopy equivalent then their algebras of endomorphisms $End(U)$ and $End(V)$ have homotopy equivalent resolutions.*

Proof. If U and V are homotopy equivalent, then U and V are zig-zag equivalent as dg$\mathbb{K}$ algebras. It is quickest, though perhaps obfuscatory, to note that by Proposition 3.16, the structure maps $\mathbb{K} \to End(U)$ and $\mathbb{K} \to End(V)$ are homotopy equivalent with restricted domain, thus there is a diagram with resolutions of $End(U)$ and $End(V)$ and a homotopy equivalence between them. However, one should really note that there is a zig-zag of dg$C$ algebras between $End(U)$ and $End(V)$ built from restriction, induction and endomorphism spaces. Thus a resolution of $End(U)$ lifts up to homotopy over this whole zig-zag to a resolution of $End(V)$. □

3.3.2 Existence of Transfered Structures

Fix a combination algebra C that acts naturally on operations on chain complexes.

As a consequence of the Proposition 3.19 above and 2.26, if two chain complexes are homotopy equivalent, then we can transfer structures from one to the other up to homotopy equivalent, or homotopy equivalence with restricted domain, given their domain is resolved. One recalls that if two algebra structures are homotopy equivalent with restriced domain, then their structure constants are equal up to overal scaling (Theorem 2.25), but in fact, we can do slightly better:

Theorem 3.20. *Let* $\mathbb{K}[x_\alpha]$ *be a resolved dg*C *algebra. If* U *and* V *are homotopy equivalent, then we can transfer any algebra structure on* U *to an algebra structure on* V*, which is homotopy equivalent with restricted domain and has equal structure constants. Any two such transfered structures are homotopic.*

Proof. Pick a homotopy equivalence from U to V. Since the category of chain complexes is isomorphic to the category of dg$\mathbb{K}$ algebras (Proposition 3.18), there is a zig-zag of dg$\mathbb{K}$ algebras from U to V. Thus there is a zig-zag of dg$C$ algebras from $End(U)$ to $End(V)$. Thus the ground fields of $End(U)$ and $End(V)$ are canonically identified. The claim follows. □

If we have a partially resolved dg$C$ aalgebra $P[x_\beta]$, then we can transfer dg$P[x_\alpha]$ alebras, given we have a way to transfer dg$P$ algebra structures

Theorem 3.21. *Let* P *be a dg*C *algebra, and* $P[x_\alpha]$ *a free triangular extension of* P*, and let* U *and* V *be dg*P *algebras.*

If U *and* V *are zig-zag equivalent then we can transfer any dg*$P[x_\alpha]$ *algebra structure on* U *(extending its dg*P *algebra structure) to a dg*$P[x_\alpha]$ *algebra structure on* V *(extending its dg*P *algebra structure) which is homotopy equivalent with restricted domain and equal structure constants. Any two such transfered structures are homotopic rel* P*.*

Proof. A zig-zag from U to V gives a lift of P through a zig-zag from $End(U)$ to $End(V)$. We just lift up to homotopy over this zig-zag to get a dg$P[x_\alpha]$ algebra structure on $End(U)$. Any two such transfered structures with structure maps f and g are homotopy equivalent with restricted domain, and agree on P, thus using a resolution $RP[x_\alpha]$ of $P[x_\alpha]$ one checks f and g are both lifts up to homotopy rel P of some map $RP \to End(V)$. Thus they are homotopic rel P by the Lifting Lemma. □

3.3.3 Explicit Formulas For Transferred Maps

One recalls that a Hodge decomposition on a chain complex in the usual sense gives a splitting of a chain complex into homology, plus the image of d plus a space which is isomorphic to the image of d by d. This splitting gives standard representatives for homology classes, and a standard way to kill boundaries.

We give a more general notion of Hodge decomposition, which has harmonic part a subcomplex containing all the homology. This gives us a way of compressing an algebra structure to a subcomplex, or off of a contractible ideal.

The idea for this in inspired by [1, **?**]

Hodge Decompositions

Fix a set $\mathcal{O}$ of operation types. This is independent of an action of a combination algebra.

Be warned, the following definition is not the usual definition, but one finds it in the literature [1]:

Definition 3.6. *Let* V *be a chain complex with differential* d*. A* ***Hodge decomposition*** *on* V *with maps* $s : V \to V$ *and* $t : V \to V$ *such that:*

1. *t is a degree zero projection that commutes with the boudnary, ie* $t^2 = t$ *and* $dt = td$*,*

2. *s is a chain homotopy of t to the identity map, which squares to zero, ie: $ds+sd = 1-t$ and $s^2 = 0$*

3. *s and t are "orthogonal", ie $st = 0$ and $ts = 0$.*

We call the map t of a Hodge decomposition the **harmonic projection** and s the **contracting homotopy**. Since t is a projection, so is $1 - t$. We call t the **harmonic projection** and $1 - t$ the **projection onto the contractible part**.

We denote the inclusion map $i : tV \to V$ and the projection, given $p : V \to tV$, which is just the restriction of the codomain of t. One observes that tV and $(1-t)V$ are subcomplexes and that $V \cong tV \oplus (1-t)V$. One also observes that s acts by zero on tV and is a contracting homotopy on $(1 - t)V$, thus $(1 - t)V$ has zero homology. Additionally, i and p are inverse chain homotopy equivalences, with $ip = t$ and $pi = id_{tV}$, and because t is chain homotopic to the identity (by definition). Thus i and p induce isomorphisms on homology. We call tV the **harmonic subcomplex**, and $(1 - t)V$ the **contractible subcomplex**.

Now consider the chain complex $Hom(V^{\otimes k}, V^{\otimes l})$. Given an operation $f : V^{\otimes k} \to V^{\otimes l}$ we define $T(f) := t^{\otimes l} \circ f \circ t^{\otimes k}$ and call it the **compression of f to the harmonic subcomplex**. One notes that $T(f)$ restricts to tV in the sense that if $f \in End(V)$, then $T(f) \in Restrict(tV, V)$. One recalls that $V^{\otimes 0} = \mathbb{K}$ and $t^{\otimes 0} = id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$, thus T acts by identity on the constant operations.

We define an operator S on $Hom(V^{\otimes k}, V^{\otimes l})$ as follows:

$$S(f) := \sum_{i+j+1=l} (t^{\otimes i} \otimes s \otimes 1^{\otimes j}) \circ f + (-1)^{|f|} \sum_{i+j+1=k} t^{\otimes l} \circ f \circ (t^{\otimes i} \otimes s \otimes 1^{\otimes j})$$

Proposition 3.22. *Let V be a chain complex with a hodge decomposition. Then $Hom(V^{\otimes k}, V^{\otimes l})$ has a Hodge decomposition with the operators S and T as above. We denote the inclusion and projection operators*

$$I : T(Hom(V^{\otimes k}, V^{\otimes l}) \to Hom(V^{\otimes k}, V^{\otimes l})$$

$$P : Hom(V^{\otimes k}, V^{\otimes l}) \to T(Hom(V^{\otimes k}, V^{\otimes l}))$$

Proof. One checks $T^2 = T$, $\partial T = T\partial$, $S^2 = 0$ and $TS = 0 = ST$.

To check $\partial S + S\partial = 1 - T$, one first notes that

$$(1-T)(f) = \sum_{i+j+1=l} (t^{\otimes i} \otimes (1-t) \otimes 1^{\otimes j}) \circ f + \sum_{i+j+1=k} t^{\otimes l} \circ f \circ (t^{\otimes i} \otimes (1-t) \otimes 1^{\otimes j})$$

□

The notion of Hodge decomposition extends in an obvious way to sets of chain complexes, thus

Corollary 3.23. *Let V be a chain complex with differential d, and Hodge decomposition s and t. Then $End(V)$ with differential ∂ has a Hodge decomposition given by S and T on each component. We denote the projection and inclusion maps*

$$I : T(End(V)) \to End(V)$$

$$P : End(V) \to T(End(V))$$

We call S and T the induced Hodge decomposition on $End(V)$.

Lemma 3.24. *If V is a chain complex with a Hodge decomposition, then the induced Hodge decomposition on $End(V)$ restricts to Hodge decompositions on $Res(tV, V)$ and $Ind(V, V/(1-t)V)$, which split into harmonic and contractible parts as*

$$Res(tV, V) = T(End(V)) \oplus Ann(tV, V)$$

$$Ind(V, V/(1-t)V) = T(End(V)) \oplus CoAnn(V, V/(1-t)V)$$

Thus restriction and induction give isomorphisms

$$T(End(V)) \xrightarrow{\cong} End(tV)$$

$$T(End(V)) \xrightarrow{\cong} End(V/(1-t)V)$$

Proof. First, one notes that operations of $T(End(V))$ given inputs in tV give outputs in tV. Thus they restrict to maps on tV. Thus $T(End(V)) \subset Res(tV, V))$. Operations in $T(End(V))$ given any input in $(1-t)V$ give zero, thus they induce maps on V/I, thus $T(End(V)) \subset Ind(V, V/(1-t)V)$. Since they contain the entire image of T, it follows that the harmonic pars of $Res(tV, V)$ and $Ind(V, V/(1-t)V)$ are both $T(End(V))$.

If an operation f restricts to tV then some inputs or output of $S(f)$ always has an s next to it, thus maps $S(f)((tV)^{\otimes k}) = 0$. Thus $S(f)$ also restricts to tV (to zero, in fact).

If an operation f induces a map on $V/(1-t)V$, then

$$f\left(\sum_{i+1+j=k} V^{\otimes i} \otimes (1-t)V \otimes V^{j}\right) \subset \sum_{i+1+j=l} V^{\otimes i} \otimes (1-t)V \otimes V^{j}$$

Some of the summands of of $S(f)$ have are f with an s and some output, and and some have f with s at some input and t at all outputs, so $S(f)\left(\sum_{i+1+j=k} V^{\otimes i} \otimes (1-t)V \otimes V^{\otimes j}\right) = 0$. So $S(f)$ restricts to $V/(1-t)V$ (to zero, in fact).

Thus S and T both restrict to $Res(tV, V)$ and $Ind(V, V/(1-t)V)$. One observes, thus that they give hodge decompositions.

Suppose f restricts to zero on tV, one checks that $T(f) = 0$, so $1 - T(Ann(tV, V)) = Ann(tV, V)$. One also checks if g restricts to any map on tV that $(1-T)(g)$ restricts to zero on $Ann(V)$. Thus $(1-T)(Res(tV, V)) = Ann(tV, V)$, that is the annihilator is the contractible part of the Hodge decomposition.

Suppose f induces the zero map on $V/(1-t)V$. Then $f(V^{\otimes k}) \subset \sum_{i+1+j=l} V^{\otimes i} \otimes (1-t)V \otimes V^{j}$, so $T(f) = 0$. So $(1-T)(CoAnn(V, V/(1-t)V)) = CoAnn(V, V/(1-t)V)$. One also checks if g induces any map on $V/(1-t)V$, then $(1-T)(g)$ induces the zero map on $V/(1-t)V$, becuase one may write $(1-T)(g)$ as a sum of gs with a $(1-t)$ at at least one input or output (along with some ts at other inputs and outputs) and so $(1-T)(g)(V^{\otimes}k) \subset \sum_{i+1+j=l} V^{\otimes i} \otimes (1-t)V \otimes V^{j}$. Thus $(1-T)(Ind(V, V/(1-t)V)) = CoAnn(V, V/(1-t)V)$.

It follows that $Res(tV, V)$ and $Ind(V, V/(1-t)V)$ split as in the statement.

Since $Ann(tV, V)$ and $CoAnn(V, V/(1-t)V)$ are the kernels of restriction and induction, and restriction and induction are surjective, restriction and induction are isomorphisms, and the claim follows. ☐

There is always a Hodge decomposition whose harmonic part is the homology.

Proposition 3.25. *Let V be a chain complex with differential d. There is Hodge decomposition on V whose harmonic part has zero differential, thus is isomorphic to the homology $H(V)$.*

Proof. Let V be a chain complex with zero differential. Choose a complement H of $im(d)$ in $ker(d)$. Choose a complement K of $ker(d) = H \oplus im(d)$ in V. Observe that $d(K) = d(V) \subset im(d)$. Thus $d : K \overset{\cong}{\to} im(d)$ is an isomorphism. Let s be the inverse isomorphism extended over $H \oplus im(d)$ by zero, thus $im(d) = K$ and $V = H \oplus im(d) \oplus im(s)$. Define t to be the projection on to the component H of this decomposition. One checks s and t give a Hodge decomposition on V with harmonic part $tV = H$. Observe that $dH = 0$. Thus the homology of H is just H itself. Recall that the inclusion $H = tV \to V$ is a quasi-isomorphism, thus the inclusion induces an isomorphism $H \cong H(V)$.

The claim follows. □

Explicit Transfer Formula Using a Hodge Decomposition

Fix a combination algebra C with zero differential, which acts naturally on operations.

Given an algebra structure, a Hodge decomposition picks out a new algebra structure and a homotopy to it:

Lemma 3.26. *Let P be a dgC algebra and $P[x_\alpha]$ a free triangular extension.*

Let V be a chain complex with differential d, and a Hodge decomposition given by s and t, and let $End(V)$ be the operations on V with differential ∂ and induced Hodge decomposition given by S and T.

Form a path space $P[x_\alpha, y_\alpha, \bar{\delta}_\alpha]$ as in Proposition 2.12.

Given any algebra structure $f : P[x_\alpha] \to End(V)$, there is a new algebra structure $g : P[x_\alpha] \to End(V)$ and a homotopy H from f to g rel P, which are defined recursively by the formulas

$$g(x_\alpha) := T(f(x_\alpha) + H(\eta_\alpha)) + S(g(\partial x_\alpha))$$

$$H(\bar{\delta}_\alpha) := -S(H(\eta_\alpha) + f(x_\alpha))$$

If the harmonic part $W = tV$ is a dgP subalgebra of V, then g maps into $Res(W, V)$, and if W is a dg$P[x_\alpha]$ subalgebra of V, then $T(H(\eta_\alpha)) = 0$.

If the contractible part $I = (1 - t)V$ is a dgP ideal of V, g maps into $Ind(I, V/I)$, and if I is a dg$P[x_\alpha]$ algebra ideal of V, then $T(H(\eta_\alpha)) = 0$.

Proof. The free triangular extension $P[x_\alpha] = P[x_{\alpha_1}, x_{\alpha_2}, \ldots]$ consists of a sequence of extensions with the differential landing in earlier terms.

Recall that η_α is chosen to lie in the ideal $\langle \bar{\delta}_\beta, d\bar{\delta}_\beta \rangle$ of previous terms so that $\partial\eta_\alpha = \partial y_\alpha - \partial x_\alpha = p_\alpha(y_\beta) - p_\alpha(x_\beta)$.

We check the formula, below, but note that one may arrive to it by attempting to define $g(x_\alpha) := T(x_\alpha)$ and then correcting. One way to do this is by first correcting by $T(H(\eta_\alpha))$ and then by a term that kills $(1 - T)(g(\partial x_\alpha))$, which one can do by applying S. Then one has to fill in the homotopy to kill $-(1 - T)(H(\eta_\alpha) + f(x_\alpha))$, which one can do by applying S.

Suppose we have defined g and the homotopy H over

$$g : P[x_{\alpha_1}, \dots x_{\alpha_{n-1}}] \to End(V)$$

$$H : P[x_{\alpha_1}, y_{\alpha_1}, \bar{\delta}_{\alpha_1}, \dots, x_{\alpha_{n-1}}, y_{\alpha_{n-1}}, \bar{\delta}_{\alpha_{n-1}}] \to End(V)$$

to dg$C$ algebra maps by the formula in the hypothesis.

Let x_{α_n} be a variable in the next extension, and define $g(x_{\alpha_n}) := T(f(x_{\alpha_n}))+T(H(\eta_{\alpha_n}))+S(g(\partial x_{\alpha_n}))$ as in the hypothesis. By Lemma 2.6, we can extend g over this next extension to a dg$C$ algebra map, if $\partial g(x_{\alpha_n}) = g(\partial x_{\alpha_n})$ (since $\partial g(\partial(x_{\alpha_n} = g(\partial^2 x_{\alpha_n} = 0)$. We check this identity, noting that $\partial S = (1-T)+S\partial$ and $H(\partial \eta_{\alpha_n}) = H(\partial y_{\alpha_n} - \partial x_{\alpha_n}) = g(\partial x_{\alpha_n}) - f(\partial x_{\alpha_n})$

$$\begin{aligned}
\partial g(x_{\alpha_n}) &:= \partial T(f(x_{\alpha_n})) + \partial T(H(\eta_{\alpha_n})) + \partial S(g(\partial x_{\alpha_n})) \\
&= T(f(\partial x_{\alpha_n})) + T(H(\partial \eta_{\alpha_n})) + (1-T)(g(\partial x_{\alpha_n})) - S\partial(g(\partial x_{\alpha})) \\
&= T(f(\partial x_{\alpha_n})) + T(g(\partial x_{\alpha_n})) - T(f(\partial x_{\alpha_n})) + (1-T)(g(\partial x_{\alpha_n})) - S(g(\partial^2(x_{\alpha_n}))) \\
&= g(\partial x_{\alpha_n})
\end{aligned}$$

Thus we can extend over the next layer of variables by the formula to get a dg$C$ algebra map, and by the same lemma, we can extend the homotopy over the next layer of variables x_{α_n} and y_{α_n} by f and g, respectively. We may also extend it over the next layer of variables $\bar{\delta}_{\alpha_n}$ by the formula in the hypothesis $H(\bar{\delta}_{\alpha_n}) := -S(H(\eta_{\alpha_n})) - S(f(x_{\alpha_n}))$, and this extension will be a dg$C$ algebra map, if $\partial H(\bar{\delta}_{\alpha_n}) = H(\partial \eta_{\alpha_n})$, by the same lemma. We check this, noting that $S(g(x_{\alpha_n})) = 0$ and $T(H(\bar{\delta}_{\alpha_n})) = 0$

$$\begin{aligned}
\partial H(\bar{\delta}_{\alpha_n}) &= -\partial S(H(\eta_{\alpha_n}) + f(x_{\alpha})) \\
&= S\partial(H(\eta_{\alpha_n})) + f(x_{\alpha})) - (1-T)(H(\eta_{\alpha_n}) + f(x_{\alpha_n})) \\
&= S(g(\partial(x_{\alpha_n}) - f(\partial x_{\alpha_n}) + f(x_{\alpha_n})) - (1-T)(H(\eta_{\alpha_n}) + f(x_{\alpha_n})) \\
&= S\partial(g(x_{\alpha_n})) - (1-T)(H(\eta_{\alpha_n}) + f(x_{\alpha_n})) \\
&= \partial S(g(x_{\alpha_n})) + (1-T)(g(x_{\alpha_n})) - (1-T)(H(\eta_{\alpha_n}) + f(x_{\alpha_n})) \\
&= (1-T)(g(x_{\alpha_n}) - f(x_{\alpha_n}) - \eta_{\alpha_n}) \\
&= (1-T)(H(\partial \bar{\delta}_{\alpha_n})) \\
&= H(\partial \bar{\delta}_{\alpha_n}) - \partial T(H(\partial)) \\
&= H(\partial \bar{\delta}_{\alpha_n})
\end{aligned}$$

The first part of the claim follows.

Let $p_\alpha(x_\beta) := \partial x_\alpha$. Recall, because $P[x_\alpha]$ is a free triangular extension, the $p(x_\beta)$ lie in earlier terms.

Suppose that the harmonic part $W = tV$ is a dg$P$ subalgebra of V. Then we show by induction that $g(x_\alpha)$ lands in $Res(W,V)$. One first notes, that T of any operation lands in $Res(W,V)$ by Lemma 3.24, moreover $Res(W,V)$ is closed under $\mathcal{S}$. Suppose some operations x_β map to $Res(W,V)$, Then $p_\alpha(x_\beta)$ is some combination of operations which all land in $Res(W,V)$. Since C acts naturally, $Res(W,V)$ is a dg$C$ subalgebra of $End(V)$, so $p(x_\alpha)$ also lands in $Res(W,V)$. It follows that $g(x_\alpha) = T(f(x_\alpha) + H(\eta_\alpha)) + S(p_\alpha(g(x_\beta)))$ is in $Res(W,V)$ if early $g(x_\beta)$ are. One notes the first terms p_{α_1} are just elements of P, thus $g(x_{\alpha_1}) = T(f(x_{\alpha_1}) + H(\eta_{\alpha_1})) + S(p_{\alpha_1}))$ is in $Res(W,V)$. By induction, $g(x_\alpha)$ is in $Res(W,V)$ for all α.

If W is a dg$P[x_\alpha]$ subalgebra of V, then the original map $P[x_\alpha]$ lifts to $Res(W,V)$ so one observes the entire computation takes place in $Res(W,V)$. Since $H(\bar{\delta}_\alpha)$ is always in the image of S, and the image of S is $Ann(W, V)$, $H(\bar{\delta}_\alpha)$ is in the annihilator, and so is its boundary. Since $H(\eta_{alpha})$ is in the ideal of $H(\bar{\delta}_\alpha)$ and its boundary, which are in the annihilator, and the annihilator is an ideal of $Res(W, V)$, it follows that $H(\eta_\alpha)$ is in the annihilator. The annihilator is annihilated by the operator T, thus $T(H(\eta_\alpha)) = 0$, as derived.

The second part of the claim follows. The third part follows by virtually the same argument as for the second part. □

It follows that we can transfer algebra structures to "harmonic subalgebras" and "harmonic quotient algebras" with the same recursive formulas, and because the transfer is through a zig-zag from $End(V)$ to $End(W)$ or $End(V/I)$, the structure constants are equal.

Theorem 3.27. *If W is a dgP subalgebra of V, and there is a Hodge decomposition on V whose harmonic part is W, then a dg$P[x_\alpha]$ algebra structure on V determined by operations $f(x_\alpha)$ in $End(V)$ is transfered up to homotopy equivalence with restricted domain (and equal structure constants) to a dg$P[_\alpha]$ algebra structure on W determined operations $g(x_\alpha) := T(f(x_\alpha) + H(\eta_\alpha))$.*

If V/I is a dgP quotient algebra of V and there is Hodge decomposition on V whose contractible part is I, then a dg$P[x_\alpha]$ algebra structure on V determined by operations $f(x_\alpha)$ in $End(V)$ is transfered up to homotopy equivalence with restricted domain (and equal structure constants) to a dg$P[x_\alpha]$ algebra structure on V/I determined by the same formula, and regarding $V/I = tV$.

If $f(x_\alpha)$ preserves W or restricts to V/I, then the term involving $H(\eta_\alpha)$ can be zeroed out, and one can simply compress $f(x_\alpha)$ to the harmonic subspace.

We can go the other way: given an dg$P[x_\alpha]$ algebra structure on a dg$P$ quotient algebra or subalgebra on an algebra, we can extend over the algebra up to homotopy equivalence.

Suppose V is a dg$P$ algebra and that the harmonic part $W = tV$ is a dg$P$ subalgebra. The subspace $T(End(V))$ of $Res(W, V)$ maps isomorphically onto $End(W)$ via the restriction map. Let $J : End(W) \to Res(W, V)$ be the inverse extended to $Res(W, V)$ by inclusion.

Suppose V is a dg$P$ algebra with contractible part I which is an ideal so that V/I is a dg$P$ quotient algebra. The subspace $T(End(V))$ of $Ind(V, V/I)$ maps isomorphically to $End(V/I)$ via the induction map. We let $J : End(W) \to Ind(V, V/I)$ be the inverse map, extended to $Ind(V, V/I)$ by inclusion.

Theorem 3.28. *Let V be a dgC algebra with a Hodge decomposition, and give $End(V)$ the induce hodge decomposition with S and T.*

Suppose V is a dgP algebra whose harmonic $W = tV$ is a subalgebra. Then a dg$P[x_\alpha]$ algebra structure on W determined by $f(x_\alpha)$ is extend to a dg$P[x_\alpha]$ algebra structure on V determined by the recursive formula

$$g(x_\alpha) := J(f(x_\alpha)) + S(g(\partial x_\alpha))$$

which is homotopy equivalence with restricted domain (and equal structure constants).

Suppose V is a dgP algebra whose contractible part $I = (1-t)V$ is an ideal of V. Then a dg$P[x_\alpha]$ structure on V/I determined by $f(x_\alpha)$ is extends to a dg$P[x_\alpha]$ algebra structure on V by the same recursive formula to an algebra structure which is homotopy equivalent with restricted domain (and equal structure constants).

Proof. Fist consider the case W is a dg$P$ subalgebra of V. Since the restriction map $Res(W,V) \to End(W)$ is surjective, and the structure map $P \to End(W)$ lifts to $Res(W,V)$ by Proposition 3.5, we can find a strict lift of $P[x_\alpha]$, by Proposition 2.14.

Let $f : P[x_\alpha] \to End(W)$ be the structure map on W, and $g : P[x_\alpha] \to Res(W,V)$ be its strict lift. Recall that the Hodge decomposition on $End(V)$ restricts to a Hodge decomposition on $Res(W,V)$. To prove the formula, we simply follow the procedure for lifting a surjectively, as in 2.14.

On observes that $Res \circ J$ is the identity on $End(W)$, and $im(J) = im(T)$ in $Res(W,V)$. Recall also that the kernel of restriction is $Ann(W,V)$.

Since W is a dg$P$ subalgebra and $p_{\alpha_1} := \partial x_{\alpha_1} \in P$, $g(\partial x_{\alpha_1})$ is a strict lift of $f(\partial x_{\alpha_1})$. Since $Res \circ J$ is identity, $J(f(x_{\alpha_1}))$ provides a strict lift of x_1, though $\partial(J(f(x_{\alpha_1})))$ isn't necessarily equal to $g(p_{\alpha_1})$. The difference $\partial(J(f_{\alpha_1})) - g(p_{\alpha_1})$, however, restricts to zero on $End(W)$, and thus is in the annihilator $Ann(W,V)$. It's a cycle, too, so its the boundary of $S(g(p_{\alpha_1}) - \partial J(f(x_{\alpha_1})))$, since ∂ commutes with J and $SJ = 0$, the second term drops off, leaving $S(g(p_{\alpha_1}))$ Thus $g(x_\alpha) := J(f(x_{\alpha_1})) + S(g(p_{\alpha_1})) = J(f(x_{\alpha_1})) + S(g(\partial x_{\alpha_1}))$.

Subsequent terms follow the same pattern, thus the formula is a strict lift, thus also a lift up to homotopy. This formula gives a dg$P[x_\alpha]$ algebra structure on V, which is homotopy equivalent with restricted domain. Thus the inclusion map is a quasi-isomorphism. Thus by Proposition 3.16, they are homotopy equivalent algebra structures with restricted domain.

The proof for the second part is nearly identical. □

3.4 A Simple Example

We consider an example of an algebra structure whose homotopy classes are completely determined by constant operations, and which we can compute everything completely.

Fix the set of operation types $\mathcal{O}$ to contain only the three following types: 1-to-0 operations, 0-to-0 operations, and 0-to-1 operations.

Given a chain complex V, the space of operations $End(V)$ on V consists of $Hom(\mathbb{K}, V)$, $Hom(\mathbb{K}, \mathbb{K})$ and $Hom(V, \mathbb{K})$, which we may regard as V, $\mathbb{K}$ and V^* respectively, that is

$$End(V) = \{V, \mathbb{K}, V^*\}$$

The evaluation map $V^* \otimes V \to \mathbb{K}$ is a composition, and we denote it $[\binom{1}{0} \circ \binom{0}{1}]$. With permuted inputs, we write it $[\binom{0}{1} \circ \binom{1}{0}]$.

Let $N := \{\circ\}$ be the set containing the $[\binom{1}{0} \circ \binom{0}{1}]$. Given any chain complex V, there is a distinguished map $N \to End(V)$ taking $[\binom{1}{0} \circ \binom{0}{1}]$ to the evaluation map. By Theorem 3.11, $[\binom{1}{0} \circ \binom{0}{1}]$ is a natural operations, thus the universal combination algebra $C(N)$ generated by N acts naturally on operations on chain complexes (and has the zero differential). One observes that $C(N)$ is simply

$$C(N) = \mathbb{K}\langle 1_V, 1_{\mathbb{K}}, 1_{V^*}, [\binom{1}{0} \circ \binom{0}{1}], [\binom{0}{1} \circ \binom{1}{0}]\rangle$$

One notes

$$v^* \circ w = (-1)^{|w||v|} w \circ v^*$$

Let M be a smooth oriented manifold of dimension n, with differential forms $\Omega(M)$ and exterior derivative d. Integrating over the fundamental class gives a chain map, by Stoke's Theorem,which is of degree minus n:

$$\int : \Omega(M) \to \mathbb{K}$$

Let ω be a choice of volume form

$$\omega \in \Omega(M)^n$$

The volume form is a cycle, which we may regard as a 0-to-1 operation which is a chain map. It gives the manifold some volume

$$Vol := \int \omega$$

which we may regard as a 0-to-0 operation.

We study algebra structure given by these operations which have a fixed integration map.

The integration alone forms an dg$C(N)$ algebra

$$Int := \mathbb{K}[\textstyle\int] = \mathbb{K}\langle \textstyle\int \rangle$$

with zero differential. All three operations together (integration, volume form and volume) form a dg$C(N)$ algebra of operations

$$IntVol := \mathbb{K}[\textstyle\int, \omega, Vol]/\langle Vol - \textstyle\int \circ \omega \rangle = \mathbb{K}\langle \textstyle\int, \omega, Vol \rangle$$

with zero differential.

Int is a dg$C(N)$ subalgebra of $IntVol$, so we may resolve $IntVol$ relative to this subalgebra by turning the inclusion map into a cofibration. One observes that we simply have to append a single new operation ω to Int with zero boundary and map it to ω in $IntVol$ to get our quasi-isomorphism

$$Int[\omega] \xrightarrow{\sim} IntVol$$

Let $Int[\omega']$ be another copy. We obtain a path space of $Int[\omega]$ rel Int by forming the free extension

$$Int[\omega, \omega', \bar{\omega}]$$

and extending the differential over $\bar{\omega}$ by

$$\partial \bar{\omega} := \omega' - \omega$$

Suppose we have two dg$IntVol$ algebra structures on $\Omega(M)$ with the same integration map. We may view them as $Int[\omega]$ algebras with structure maps $Int[\omega] \to End(\Omega(M))$.

Thus we observe they are homotopic rel Int if and only if their volume forms differ by a boundary, this implies that their volumes are equal (which we already know to be true, by Theorem 2.25). One notes the converse: if two volume forms give the same volume, then they differ by a boundary. Thus

Proposition 3.29. *Two dg$Int[\omega]$ algebra structures are homotopic rel Int if and only if they have the same volume.*

One knows given a Riemannian metric on a manifold, we get a co-differential d^*, which determines a Hodge decomposition in the classical sense. We get a splitting of differential forms into three pieces $\Omega(M) = \mathcal{H} \oplus im(d) \oplus im(d^*)$ such that $\mathcal{H}$ has zero differential, and d and d^* are isomorphisms between $im(d)$ and $im(d^*)$. One observes that since $(d^*)^2 = 0$, Δ is an isomorphism on $im(d) \oplus im(d^*)$ (thus this sum is acyclic). The inverse map G is called a Green's operator [1]. One checks that since Δ commutes with the exterior derivative, so does $\mathcal{G}$, and since the output of d^* is in the domain of $\mathcal{G}$, the map $s := Gd^*$ makes sense. One defines $t := 1 - (ds + sd)$ and observes that s and t give a Hodge decomposition on $\Omega(M)$ in the sense of Definition 3.6, and that $\mathcal{H} = t\Omega(M)$ is the harmonic part, and $im(d) \oplus im(d^*) = (1-t)\Omega(M)$ is the contractible part.

Given a dg$Int[\omega]$ algebra structure on $\Omega(M)$, we note that $\mathcal{H}$ is automatically a dgInt subalgebra, but not necessarily a dg$Int[\omega]$ subalgebra. We compute the formula from Lemma 3.26, which allows us to compress the algebra structure onto the harmonic part $\mathcal{H} = t\Omega(M)$ to an algebra structure which is homotopy equivalent with restricted domain.

We denote our starting structure map $f[\omega] : Int \to \Omega(M)$, and compute the transfered one g, which is equal to f on Int, which turns out to be, simply, $g(\omega) := T(f(\omega)) = tf(\omega)$. We can compute the homotopy, too; it's $H(\bar{\omega}) := -S(f(\omega)) = -sf(\omega) = -sf(\omega) = -Gd^* f(\omega)$.

This new structure map restricts to a structure map $g : Int[\omega] \to End(\mathcal{H})$ which is homotopy equivalent with restricted domain and equal structure constants, by Theorem 3.27. In fact, one checks $tf(\omega)$ is just the old harmonic projection of $f(\omega)$ to $\mathcal{H}$, and $H(\bar{\omega}) = -sf(\omega)$ is the co-exact form (ie form in $im(d^*)$) whose boundary is the difference between $f(\omega)$ and its harmonic representative $tf(\omega)$. Thus

Proposition 3.30. *An orientation and a choice of volume form on M determines a dg$Int[\omega]$ algebra structure on $\Omega(M)$. Given a Riemannian metric on M, the dg$Int[\omega]$ algebra structure on the space of harmonic forms $\mathcal{H}(\Omega(M))$ given by restricting integration and sending ω to the harmonic representative the volume form in $\Omega(M)$ is homotopy equivalent with restricted domain and has the same volume.*

Chapter 4

Appendix: Combination Algebras and Algebras of Over Combination Algebras

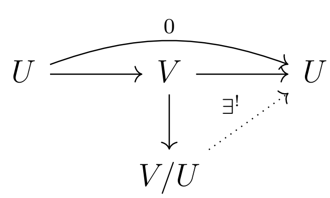
4.1 Linear Spaces

We study algebraic structures on chain complexes. A space of operations on a chain complex is a set of chain complexes indexed by the various kinds of operations one wishes to consider. The space of operation on a space of operations is another set of chain complexes indexed by the various kinds of operation on chain complexes one wishes to consider. Sets of chain complexes have similar structure to chain complexes, just componentwise. The purpose of this section is mainly to set conventions.

4.1.1 Vector Spaces

We fix a ground field $\mathbb{K}$ of characteristic zero. Tensor product $\otimes$ mean over tensor product over the ground field $\otimes_{\mathbb{K}}$.

Quotients One recalls that quotients satisfy a universal property, namely, given a subspace U of V, and a map $V \to W$ whose kernel contains U, the factors uniquely through the quotient map



moreover this map is injective if and only if W is equal to kernel of $V \to W$.

Let U and V be vector spaces with subspaces $U' \subset U$ and $V' \subset V$. One shows that any map $U \otimes V \to W$ with $U' \otimes V + U \otimes V'$ in its kernel factors uniquely through the map $U \otimes V \to (U/U') \otimes (V/V')$. Thus there is a canonical identification

$$(U \otimes V)/(U' \otimes V + U \otimes V') \cong (U/U') \otimes (V/V')$$

Coproducts and Direct Limits Let $\{V_i\}$ be a set of vector spaces. We call a vector space C with maps $V_i \to C$ a **coproduct** of $\{V_i\}$ if it satisfies the following universal property: Given any maps $f_i : V_i \to W$ to a vector space W, these maps factor uniquely through C

$$\begin{array}{ccc} C & \overset{\exists!}{\dashrightarrow} & W \\ \uparrow & \nearrow & \\ V_i & & \end{array}$$

This defines coproducts uniquely up to canonical isomorphism. One observes the vector space $\sum_i V_i$ with inclusion maps $V_i \to \sum_i V_i$ is a coproduct.

We use direct limits in the most basic sense. Let

$$V_0 \subset V_1 \subset V_2 \subset \dots$$

be an increasing sequence of spaces. We say a sequence of maps $V_i \to W$ is **coherent** if two agree where they're both defined. We way V together with a coherent sequence of maps $V_i \to V$ satisfies the universal property of a directed limit if for any coherent sequence of maps $V_i \to W$ there is a unique map $V \to W$ making the diagrams commute

$$\begin{array}{ccc} V & \overset{\exists!}{\dashrightarrow} & W \\ \uparrow & \nearrow & \\ V_i & & \end{array}$$

and we call V a **directed limit**). This defines filtered limits uniquely up to unique isomorphism.

One observes that the union $V = \cup_i V_i$ of an increasing sequence of spaces with the inclusion maps $V_i \to V$ is a direct limit.

Given two sequences of increasing spaces U_i and V_i we may view their tensor products $U_i \otimes V_i$ as an increasing sequence of subspaces of $U \otimes V$. One checks that $\cup_i U_i \otimes V_i = U \otimes V$, thus $U \otimes V$ is a direct limit of $U_0 \otimes V_0 \subset U_1 \otimes V_1 \subset U_2 \otimes V_2 \subset \dots$.

Free Objects Given a set X, we define $\mathbb{K}\langle X\rangle$ to be the vector space with basis X. This space comes with an obvious inclusion map $X \to \mathbb{K}\langle X\rangle$.

One observes this construction is functorial, and that $K\langle X\rangle$ with the inclusion map satisfy the following universal property: given a map of sets $X \to V$ to a vector space V, there is a unique linear map $\mathbb{K}\langle X\rangle \to W$ extending this map

$$\begin{array}{ccc} \mathbb{K}\langle X\rangle & \dashrightarrow & V \\ \uparrow & \nearrow & \\ X & & \end{array}$$

One observes that $\mathbb{K}\langle X\rangle \otimes \mathbb{K}\langle Y\rangle$ with the obvious map $X \times Y \to \mathbb{K}\langle X\rangle \otimes \mathbb{K}\langle Y\rangle$ satisfies the same property, thus it is canonically identified with $\mathbb{K}\langle X \times Y\rangle$.

There is a similar notion of increasing sequences and direct limits for sets. Let X_i be an increasing sequence of sets with direct limit X with distinguished maps $X_i \to X$. One observes that the spaces $\mathbb{K}\langle X_0\rangle \subset \mathbb{K}\langle X_1\rangle \subset \mathbb{K}\langle X_2\rangle \subset \dots$ for an increasing sequence of vector spaces and that the space $\mathbb{K}\langle X\rangle$ together with the induced maps $\mathbb{K}\langle X_i\rangle \to \mathbb{K}\langle X\rangle$ is a direct limit.

One notes further that given a sequence of coherent maps $X_i \to V$ to a vector space V, with limit $X \to V$, the induced maps $\mathbb{K}\langle X_i\rangle \to V$ have direct limit $\mathbb{K}\langle X\rangle \to V$ which is the induced map of $X \to V$.

4.1.2 Graded Vector Spaces and Chain Complexes

By a **graded vector space**, denoted V, we mean a sequence of vector spaces with k-th component V^k, one for each $k \in \mathbb{Z}$. We say a vector v in the k-th component has degree k and denote this $|v|$.

By a **map of graded vector spaces of degree** k, denoted $f : V \to W$, we mean a sequence of maps $f^n : V^n \to W^{n+k}$. Thus a degree k map has degree k either sense, and we denote this $|f| = k$.

We denote the vector space of degree k maps from V to W by $Hom^k(V,W)$. The maps of all degrees from a graded vector space $Hom(V,W)$ called the Hom **space of** V **and** W, and composition is performed componentwise $(f \circ g)^k = f^k \circ g^k$ and is thus associative. 1_V means the degree 0 map in $Hom(V,V)$ with identity map in each degree, and as the identity for composition is called the **unit** (or **identity map**) on V. One observes that $|f(v)| = |f| + |v|$ and $|f \circ g| = |f| + |g|$.

Sums, quotients, products, kernels, cokernels, subspaces, unions and so forth, are all defined componentwise

$$(\oplus_\alpha V_\alpha)^k = \oplus_\alpha (V\alpha)^k \qquad (\oplus_\alpha f_\alpha)^k = \oplus_\alpha (f_\alpha)^k$$

Tensor products and products, on the other hand, are convolved

$$(V \otimes W)^k = \oplus_{i+j=k} V^i \otimes W^j$$

and maps of tensor products are designed with the Koszul sign .convention built in [see: **??**], that is,

$$(f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w)$$

It follows that

$$(f \otimes g) \circ (f' \otimes g') = (-1)^{|g||f'|} (f \circ f') \otimes (g \circ g')$$

One observes that that $|v \otimes w| = |v| + |w|$ and $|f \otimes g| = |f| + |g|$

One observes that quotient, tensor products and direct limits of graded vector spaces satisfy similar universal property as for non-graded vector spaces.

Thus, since composition is bilinear, we may regard it as a map

$$- \circ - : Hom(V,W) \otimes Hom(U,V) \to Hom(U,W)$$

which one computes is degree zero.

One checks, $V' \subset V$ and $W' \subset W$, there is a canonical isomorphism

$$(V \otimes W)/(V' \otimes W + V \otimes W') \cong (V/V') \otimes (W/W')$$

We take $\mathbb{K}$ to denote the graded vector space with the ground field in degree zero and zeros elsewhere.

There are obvious isomorphisms

$$\alpha_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$$

$$\lambda_V : \mathbb{K} \otimes V \to V$$

$$\rho_V : V \otimes \mathbb{K} \to V$$

along with a twisting map

$$\tau_{V,W} : V \otimes W \to V \otimes W$$

$$\tau_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

which one checks are natural in each slot

Given finitely many graded chain complexes $V_1, \ldots, V_n$, and any two ways of tensoring them together (in pairs, in any order) X_1 and X_2, any two isomorphism $X_1 \to X_2$ built out $\otimes, \alpha, \lambda, \rho$ and τ are equal. By the argument of Saunders Mac Lane [5], it is enough to check this on a few diagrams. Thus we may mostly dispense with parenthesis, keeping in mind natural isomorphisms are involved. This is called being a symmetric monoidal category.

There is an evident notion of higher tensor products, which for n multiples of a single graded vector space V we denote $V^{\otimes n}$. The zero tensor power and empty tensor product are taken to mean the graded vector space $\mathbb{K}$. The zero tensor power of a map is the identity map on $\mathbb{K}$. One observes that

$$|v_1 \otimes \ldots \otimes v_n| = |v_1| + \ldots + |v_n|$$

Proposition 4.1. *There is a left action of the symmetric group S_n on the n-fold tensor product V^n defined by*

$$\sigma(v_1 \otimes \ldots \otimes v_n) = \pm v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

whose sign is determined by the Koszul sign convention.

Proof. Let the transpositions $(i, i+1)$ act by

$$1 \otimes \ldots \otimes \tau_{V,V} \otimes \ldots \otimes 1$$

with $\tau_{V,V}$ in the i-th slot and apply the remark above. The Koszul sign convention is encoded by τ. □

This left action turns into a right action on $Hom(V^n, V)$ by letting

$$f^\sigma = f \circ \sigma$$

There are similar notions of graded sets, with component-wise unions, intersections, and so forth, and convolved products.

We may restrict to the category of chain complexes with degree zero maps. These have coproducts, which for a set of graded vector spaces V_i is given by the sum $\sum_i V_i$ with inclusion maps $V_i \to \sum_i V_i$.

Given a graded set X, we define $\mathbb{K}\langle X\rangle$ to be the graded vector space generated by X^k in its k-th component $\mathbb{K}\langle X\rangle^k = \mathbb{K}\langle X^k\rangle$. One observes that it satisfies an analogous free property as for vector spaces with the obvious map $X \to \mathbb{K}\langle X\rangle$. One observes that $\mathbb{K}\langle X\rangle \otimes \mathbb{K}\langle X\rangle$ with the obvious choice of map $X \times Y \to \mathbb{K}\langle X\rangle$ satisfies the free property, thus $\mathbb{K}\langle X\rangle \otimes \mathbb{K}\langle X\rangle$ is canonically identified with $\mathbb{K}\langle X \times Y\rangle$. We call $\mathbb{K}\langle X\rangle$ the **free graded vector space generated by** X

One notes there are analogous notions of limits for graded sets and graded vector space and so the analogous facts hold. In particular, one notes that for increasing sequences U_i and V_i of graded vector spaces, that $U_i \otimes V_i$ can be viewed as an increasing sequence of subspaces of $U \otimes V$ with union all of $U \otimes V$. Hence $U \otimes V$, with inclusion maps $U_i \to V_i \to U \otimes V$, is a direct limit.

Chain Complexes A **differential** d on a graded vector space is a map of degree ± 1 of square zero. Those of positive degree are called cohomologically graded and those of negative degree are called homologically graded. By default, our differentials are cohomologically graded.

A **chain complex** is a graded vector space with a differential. Maps, sums, kernels, unions, and so forth for chain complexes are the same as for graded vector spaces, only with the addition of a differential. By **chain map** one means a degree zero map that commutes with the differentiation.

The differential on the Hom space $H(V, W)$ is given by

$$\partial_{Hom(V,W)}(f) := d_W \circ f - (-1)^{|f|} f \circ d_V$$

which we often write forgetting subscripts. One notes that a chain map is a cycle in the corresponding Hom complex. One computes that ∂ is a derivation of composition, that is

$$\partial(f \circ g) = \partial(f) \circ g + (-1)^{|f|} f \circ \partial(g)$$

This implies that composition is a chain map, thus a cycle in

$$Hom\Big(Hom(V, W) \otimes Hom(U, V), Hom(U, W)\Big)$$

One notes that the usual notion of a derivation is the same as a multilinear chain map. Thus we often call multilinear chain maps **derivations**.

The differential on sums, products, unions, subspaces, and so forth is componentwise

$$(\textstyle\bigoplus_\alpha d_\alpha)^k = \bigoplus_\alpha (d_\alpha)^k$$

The differential the tensor products is more interesting; it's given by

$$d_{V \otimes W} = d_V \otimes 1 + 1 \otimes d_W$$

with the Koszul signs spilling out of the tensor product

$$d_{V\otimes W}(v\otimes w)=d_V(v)\otimes w+(-1)^{|w|}v\otimes d_W(w)$$

To show this squares to zero, one notes

$$(1\otimes d_W)\circ(d_V\otimes d_V)=-(d_V\otimes d_W)=-(d_V\otimes 1)\circ(1\otimes d_W)$$

The differential is defined analogously on higher tensor products.

We denote the chain complex $\mathbb{K}$ to be the graded vector space $\mathbb{K}$ with zero differential.

One observes that differentials commute with the natural isomorphisms α,λ,ρ, and τ. We compute this for τ, since it isn't obvious:

$$\begin{aligned}\tau_{V,W}\circ d_{V\otimes W}(v\otimes w) &= (-1)^{(|v|+1)|w|}w\otimes d_V(v)+(-1)^{|v|(|w|+1)+|v|}d_W(w)\otimes v\\ &= (-1)^{|v||w|}(d_{V\otimes W})w\otimes v\\ &= d_{W\otimes V}\circ\tau_{V,W}(v\otimes w)\end{aligned}$$

It follows that the differential commutes with the action of the permutation group on S_n on $V^{\otimes n}$ and $Hom(V^{\otimes n},V)$

Thus just as for chain complexes, we can essentially dispense with parenthesis.

One checks various notions of quotients, unions, kernels, cokernels and direct limits extend to chain complexes. If we restrict to using only degree zero maps of chain complexes, then there are coproducts, which is the same as the coproduct for graded vector spaces, with the addition of the sum differential.

We may regard graded vector spaces as chain complexes with zero differential. One calls vectors in a chain complex **chains** or **cochains** depending on the degree of the differential. We simply call everything chains, unless the context demands otherwise. Given a chain $v\in V$ with differential d we call dv the **boundary** of v. If the boundary of a chain is zero, then we call the chain a **cycle**. Since differentials square to zero, boundaries are cycles.

One recalls that given a chain complex V we may form the graded vector space $H(V):=ker(d)/im(d)$, called the **homology** which we may regard as a chain complex with zero differential. Given a cycle v, we denote its homology class $[v]$. One observes that maps $f:V\to W$ which are cycles in the Hom complex induce maps on homology, which only depend on their homology class. If two maps $f\in Hom(W,V)$ with zero boundary differ by a boundary of some map s, that is $\partial(s)=g-f$, we say they're chain homotopic. If f and g are degree zero, ie chain maps, this implies the familiar $ds+sd=g-f$.

$$\begin{aligned}[f]:H(V) &\to H(W)\\ [v] &\to [f(v)]\end{aligned}$$

Since we're working over a field, one recalls that there are natural isomorphisms

$$\begin{aligned}H(V)\otimes H(W) &\to H(V\otimes W)\\ [v]\otimes[w] &\mapsto [v\otimes w]\end{aligned}$$

$$\begin{aligned}H(Hom(W,V)) &\to Hom(H(W),H(V))\\ [f] &\mapsto \big\{[w]\mapsto[f(v)]\big\}\end{aligned}$$

(by the Kunneth Theorem). We say a chain map is a quasi-isomorphism if it induces an isomorphism on homology. We say a chain complex V s contractible if its identity map is homotopic to the identity map, meaning its identity map has the zero homology class of $Hom(V, V)$, meaning it induces the zero map on homology, thus the space has zero homology. We call a space with zero homology acyclic. One checks that a chain complex is acyclic if and only if it is contractible.

There is an obvious notion of short exact sequence for chain complexes and maps of short exact sequences. One recalls that given a short exact sequence of chain complexes

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

induces a long exact sequence of homology groups, which we don't reproduce here, but we note that if f (or g) is a quasi-isomorphism, then by the long exact sequence the U (or W) is acyclic.

4.1.3 Sets of Graded Objects

We find it convenient to organize several graded vector spaces or chain complexes into a sets indexed by some fixed set. In the presence of a set, a linear space means a set of chain complexes or graded vector spaces indexed by that set. These form a category with sums, unions, subspaces, kernels, images, quotients, Hom spaces, homology and so forth taken componentwise. There is an analogous notion of sets of graded sets. We generally refer to these simply as sets when the context is clear. We may form free linear spaces on them componentwise. One checks there are direct limits, coproducts, and so forth.

4.2 Combination Magmas

Definition 4.1. *We let $\mathcal{O}$ denote a set; we call it the set of* ***operation types****. In the presence of a set of operation types, we define*

$$\mathcal{C}(\mathcal{O}) := \{\left(\begin{smallmatrix} a_1 & \cdots & a_n \\ & a_0 & \end{smallmatrix}\right) | a_1, \ldots, a_n, a_0 \in \mathcal{O}, n = 0, 1, 2, \ldots\}$$

to be the set of ***combination operation types*** *on operation types $\mathcal{O}$.*

Let $\mathrm{F} = \left(\begin{smallmatrix} a_1 & \cdots & a_n \\ & a_0 & \end{smallmatrix}\right)$ be a type of combination operation on $\mathcal{O}$. We call $a_1, \ldots, a_n$ the ***inputs*** *or* ***input types*** *of F, and a_0 the* ***output*** *or* ***output type*** *of F. We call the number of input types of F its* ***arity****.*

Let $P = \{P(a) | a \in \mathcal{O}\}$ be a set of chain complexes indexed by $\mathcal{O}$. We define the space of ***combination operations on*** *P to be the set of vector spaces $End(P) := \{End(P)(\mathrm{F}) | \mathrm{F} \in \mathcal{C}(\mathcal{O})\}$ indexed by $\mathcal{C}(\mathcal{O})$ where $End(P)\left(\begin{smallmatrix} a_1 & \cdots & a_n \\ & a_0 & \end{smallmatrix}\right) := Hom(P(a_1) \otimes \ldots \otimes P(a_n), P(a_0))$. If $n = 0$, then this is simply $End(P)(_{a_0}) := Hom(\mathbb{K}, P(a_0))$. We denote its differential ∂. We can make the same definition forgetting differentials.*

Such spaces P will be spaces of operations later on. In the presence of a set $\mathcal{O}$, we refer to a chain complex P like above, as a space of operations (of types $\mathcal{O}$).

Let f and g be two combination operations of types $\mathrm{F} = \binom{a_1 \ldots a_m}{a_0}$ and $\mathrm{G} = \binom{b_1 \ldots b_n}{b_0}$, respectively. If the output b_0 of g and the i-th input a_i of f matched, then we could compose g into the i-th slot of g to get a new operations

$$(f \ {}_i\circ \ g) := f \circ (1 \otimes \ldots \otimes g \otimes \ldots \otimes 1)$$

of type

$$F \ {}_i\circ \ G := \binom{a_1 \ldots \otimes a_{i-1} b_1 \ldots b_n a_{a+1} \ldots a_m}{a_0}$$

or we could permute the m inputs of f with a permutation σ in the m-th permutation group S_m to get a new operation

$$f^{\sigma}$$

of type

$$F^{\sigma} := \binom{a_{\sigma^{-1}(1)} \ldots a_{\sigma^{-1}(m)}}{a_0}$$

For each operation type a, there is also an identity operation

$$1_a$$

of type

$$\binom{a}{a}$$

Thus, given $\mathcal{O}$ we may consider all the expressions $\mathrm{F} \ {}_i\circ \ \mathrm{G}$, F^{σ} and 1_a that make sense.

We consider linear spaces with these operations.

Definition 4.2. *In the presence of a set $\mathcal{O}$, a **combination magma** is a set of chain complexes $M := \{M(\mathrm{F}) | \mathrm{F} \in \mathcal{C}(\mathcal{O})\}$ with distinguished chain maps, called composition, permutation of inputs, and units, respectively,*

$$[\mathrm{F} \ {}_i\circ \ \mathrm{G}] : M(\mathrm{F}) \otimes M(\mathrm{G}) \to M(\mathrm{F} \ {}_i\circ \ \mathrm{G})$$

$$[\mathrm{F}]^{\sigma} : M(\mathrm{F}) \to M(\mathrm{F}^{\sigma})$$

$$1_a : \mathbb{K} \to M\binom{a}{a}$$

for all expressions that make sense, and all $a \in \mathcal{O}$, with no relations assumed. We denote its differential $\mathscr{D}$. There is an analogous definition without differentials. We say dg combination magma to emphasize that there are differentials, or graded combination magma to emphasize that there aren't.

One observes that the operations being chain maps is equivalent to $\mathscr{D}$ being a derivation of composition, commuting with permutation of inputs. We may treat 1_a as a map, as above, or as the element of $M\binom{a}{a}$ picked out by the unit in $\mathbb{K}$. The map 1_a is a chain map if and only if the element 1_a is a boundary.

Proposition 4.2. *For any set P chain complexes indexed by $\mathcal{O}$, the space of combination operations on P with the distinguished operations of composition, permutation of inputs, and units give it the structure of a combination magma.*

We may consider the category of combination magmas, with maps that preserve structure. There are evident notions of subalgebras, kernels, ideals (closed under permutations and compositions), differential ideals (ideals also closed under the differential), and quotients. There is an evident notion of the ideal generated by a subset R, which we denote $\langle R \rangle$.

Definition 4.3. *Let M is a combination magma with a direct sum decomposition $M = M^{(0)} \oplus M^{(1)} \oplus M^{(2)} \oplus \dots$. If an element m is in some component, let $w(m)$ denote the number of that component.*

We say the direct sum decomposition is a ***weight grading*** *on M if*

$$w(f \,{}_i\!\circ\, g) = w(f) + w(g)$$

$$w(f^\sigma) = w(f)$$

$$w(1_a) = 0$$

when the expressions make sense. If M has a distinguished weight grading, we say it is ***weight graded****, and we call $M^{(n)}$ the weight n component. If M has a differential, then we also assume that M preserves components, that is,*

$$w(\mathscr{D} f) = w(f)$$

If I is an ideal of a weight graded combination magma M, we say I ***respects the weight grading on*** *M if $I^{(n)} := I \cap M^{(n)}$ gives a direct sum decomposition $I = I^{(0)} \oplus I^{(1)} \oplus I^{(2)} \oplus \dots$. If I is a differential ideal, then we assume the differential preserves the components of I.*

If R is a graded set of a weight graded combination magma M, we say R respects the weight graded on M if $R^{(n)} := R \cap M^{(n)}$ gives a partition $R = R^{(0)} \sqcup R^{(1)} \sqcup R^{(2)} \sqcup \dots$. If we with to consider the differential, then we assume it preserves the components of R.

Definition 4.4. *Let $s : M \to M$ be a map of linear spaces with degree k. We say s is a derivation (not necessarily of square zero) if it is a derivation of the operations on M, that is*

$$s(f \,{}_i\!\circ\, g) = s(f) \,{}_i\!\circ\, g + (-1)^{|s||f|} f \,{}_i\!\circ\, s(g)$$

$$s(f^\sigma) = s(f)^\sigma$$

$$s(1_a) = 0$$

We say s ***respects the ideal*** *I of M if I is closed under s, and we say s* ***respects a subset*** *R of M if R is closed under s.*

Proposition 4.3. *Quotients by ideals satisfy the universal property of quotients. Kernels are ideals. Analogous statements hold when there is a differential.*

If a subset of a weight graded combination magma respects its weight grading, then so does the ideal it generates. If an ideal of a weight graded combination magma respects its weight grading, then the quotient has an induced weight grading.

If a subset is respects a derivation, then so does the ideal it generates. If an ideal respects a derivation, then the derivation passes to the quotient.

Proof. The quotient M/I of M by an ideal I satisfies the universal property of quotients for linear spaces. The unit operations automatically pass to the quotient. The unary permutation operations pass to the quotient, because the ideal is preserved by permutation of inputs. The binary operations pass to the quotient, because the space $I \otimes M + M \otimes I$ maps into the quotient and because $M/I \otimes M/I$ is canonically isomorphic to $(M \otimes M)/(I \otimes M + M \otimes I)$. Kernels are ideals, because algebra maps commute with the operations. If there are differentials, the kernel is a differential ideal, because it the algebra map is a chain map, and thus commutes with the differential.

Suppose a subset R of a C respects the weight grading, then so does the linear space R_1 it generates, in the obvious sense a linear subspace can respect the grading. Suppose we've defined R_n that sits in $\langle R \rangle$ and respects the grading (as a linear subspace). Define R_{n+1} to be the linear space generated by R_n, composites of R_n and C, composites of C and R_n and the elements of R_n with their inputs permuted in all ways. This is clearly still in the ideal $\langle R \rangle$. Since the operations preserve weight, the spaces from which R_{n+1} is composed also preserve weight. Thus so does R_{n+1}. We obtain an increasing sequence $R_1 \subset R_2 \subset \dots$ contained in $\langle R \rangle$. One observes that the direct limit of this sequence is an ideal in $\langle R \rangle$, thus equal to $\langle R \rangle$. Since every element of $\langle R \rangle$ lies in some finite step R_n it can be written as a sum of elements in $\langle R \rangle$ which lie in some weighted components of M. Thus $\langle R \rangle$ splits by weight, and thus respects the weight grading on M.

If an ideal I respects the weight grading, then the quotient of M has an induced decomposition $(M/I)^{(k)} := M^{(k)}/I^{(k)}$. Since weight is determined by the weight of any representative, and the operations are computed on representatives, the operations preserve sum weight. Thus the induced decomposition of M/I is a weight grading.

Suppose a subset R respects derivations. Then so does its the linear subspace R_1 it generates (in the obvious sense that a subspace can preserves a derivation). One constructs the same increasing sequence as before, noting that each additional step is closed under derivation. Since every element of the limit lies in a finite step, and the finite steps are are closed under the derivation and contained in the limit, the limit is also closed under derivations. Thus $\langle R \rangle$ respects derivations.

Suppose an ideal respects derivations. Then it induces a map on the quotient. One checks the map on the quotient is a derivation, because we can compute on representatives.

The claim follows. □

Definition 4.5. *In the presence of a set $\mathcal{O}$, we say a set of graded sets $Y = \{Y(\text{F}) | \text{F} \in \mathcal{C}(\mathcal{O})\}$ indexed by $\mathcal{C}(\mathcal{O})$ is a combination magma if it has analogous operations to a combination magma on a linear space, that is maps of sets, called composition, permutation of inputs and units, respectively,*

$$[\text{F} \;{}_i{\circ}\; \text{G}] : Y(\text{F}) \times Y(\text{G}) \to Y(\text{F} \;{}_i{\circ}\; \text{G})$$

$$[\text{F}]^\sigma : Y(\text{F}) \to Y(\text{F}^\sigma)$$

$$1_a : \{\star\} \to Y\tbinom{a}{a}$$

We may treat 1_a as a map or element.

Definition 4.6. *Let Y be a combination magma with partition $Y = Y^{(0)} \sqcup Y^{(1)} \sqcup Y^{(2)} \sqcup \dots$. Denote the partition number of an element y by $w(y)$. We say the partition is a weight* *grading on Y if*

$$w(x\ {}_i\circ\ y) = w(x) + w(y)$$
$$w(y^\sigma) = w(y)$$
$$w(1_a) = 0$$

*One observes that a partition on Y is equivalent data to a function $Y \to \mathbb{N}$ (on the elements of the elements of Y). We say a function $Y \to \mathbb{N}$ is a **weight function** if its corresponding partition is a weight grading.*

*We say Y is **weight graded** if it has a distinguished weight grading or weight function.*

Definition 4.7. *Given a set $X = \{X(\mathrm{F})|\mathrm{F} \in \mathcal{C}(\mathcal{O})\}$ of graded sets indexed by $\mathcal{C}(\mathcal{O})$ we can form the set of all magmatic words in X by taking X and inductively adding in units, composition, and permutations of current elements (putting them in appropriate components and degrees), and then taking the direct limit, ie union W_X. One observes that W_X is closed under composition, permutation, and units, thus is a combination magma. We call W_X the **combination magma of words on** X (and we give a more specific account of the construction in a proof, below). There is a distinguished map $X \to W_X$ given by the inclusion.*

*The operations on W_X pass to operations on the linear space $\mathbb{K}\langle W_X\rangle$ generated by W_X, giving it the structure of a combination magma. There is a distinguished map $X \to \mathbb{K}\langle W_X\rangle$ given by including. We will call $\mathbb{K}\rangle W_X\rangle$ the **free combination magma generated by** X once we show it satisfies the free property.*

Proposition 4.4. *The combination magma W_X of words on a set X (with its distinguished map $X \to W_X$) satisfies the free property in the category of combination magmas on sets indexed by $\mathcal{C}(\mathcal{O})$, and $\mathbb{K}\langle W_X\rangle$ (with its distinguished map $X \to \mathbb{K}\langle W_X\rangle$) satisfies the free property on X in the category of graded combination magmas.*

Any map $s : X \to \mathbb{K}\langle W_X\rangle$ (possibly of non-zero degree) extends uniquely to a derivation on $\mathbb{K}\langle W_X\rangle$ (which doesn't necessarily square to zero).

Any map $w : X \to \mathbb{N}$ on extends uniquely to a weight grading on W_X, and any weigh grading on W_X extends uniquely to a weight grading on $\mathbb{K}\langle W_X\rangle$ in the sense that there is a unique weigh grading on $\mathbb{K}\langle W_X\rangle$ such that the weight of $x \in X$ viewed as an element of $\mathbb{K}\langle W_X\rangle$ is determined by w.

Proof. Given a set X (of graded sets indexed by $\mathcal{C}(\mathcal{O})$) and $f \in X$, let $[\![f]\!]$ denote the type of f in $\mathcal{C}(\mathcal{O})$. Recall that $|f|$ denotes the degree of f. We define a formal composition to be a triple (f, i, g) if there is an operation $\mathrm{F}\ {}_i\circ\ \mathrm{G}$, and we denote the triple $f\ {}_i\circ\ g$. We give it degree $|f\ {}_i\circ\ g| := |f| + |g|$ and type $[\![f\ {}_i\circ\ g]\!] := [\![f]\!]\ {}_i\circ\ [\![g]\!]$. There are similar definitions of formal permutations and formal units.

Let $X_1 := X$ be given. Given X_n we define X_{n+1} to be the union of all formal compositions and permutations of elements in X_n along with formal units. Thus we get an increasing sequence $X = X_1 \subset X_2 \subset \ldots$ of sets. Let W_X be the direct limit. Define compositions, permutations and units on W_X by formal composition, formal permutation of inputs and formal units. This gives W_X the structure of a magma, with inclusion map $X \to W_X$.

We show W_X is free. Suppose Y is a combination magma, and $\phi : X = X_1 \to W_X$ is a map preserving degree and type. If we have f defined on X_n so it commutes with operations, then there is a unique way to extend it over X_{n+1} to commute with operations. The direct limit $\phi : W_X \to Y$ is map that commutes with operations on any finite step and it's clearly unique. Thus W_X is free.

Form the space $\mathbb{K}\langle W_X\rangle$ with its distinguished map $X \to \mathbb{K}\langle W_X\rangle$. Suppose M is a combination magma (now a linear space). If we forget its linear structure, it is a combination magma as sets. Thus there is unique an extension $W_X \to M$ as combination magmas of sets. This extends uniquely over $\mathbb{K}\langle W_X\rangle \to M$ to a linear map. This map commutes with the combination operations on generators W_X, thus it commutes overall, thus it is a map of combination algebras. Any other map would have to agree on W_X, thus would have to agree. Thus the map $W_X \to M$ is unique. And so $\mathbb{K}\langle W_X\rangle$ with its distinguished map is free.

Suppose $s : X \to \mathbb{K}\langle W_X\rangle$ is a map of sets, which possibly changes degree. Supposing we've extended s over X_n so it satisfies the identities of a derivation on the subspace $\mathbb{K}\langle X_n\rangle$. Then there is a unique way to define it on X_{n+1} so it satisfies the required identities on the next subspace. Taking a direct limit, we get a derivation on $\mathbb{K}\langle W_X\rangle$. Since its unique on every extension, its unique.

Suppose $w : X = X_1 \to \mathbb{N}$ is given. Suppose we've extended it over X_n so it satisfies the required identities on X_{n-1} There is a unique way to extend w over X_{n+1} so w satisfies the required identities on elements in X_n. Taking the direct limit we get a function on W_X that satisfies the required identities on all finite steps, hence overall. Hence it is a weight function, and it is obviously unique. This gives a weight decomposition on $\mathbb{K}\langle W_X\rangle$ by $\mathbb{K}\langle W_X\rangle^{(k)} := \mathbb{K}\langle W_X^{(k)}\rangle$. Any weight decomposition on $\mathbb{K}\langle W_X\rangle$ with the weight of elements of X given by w, would induce a weight function on W_X, thus would have to match the old one.

The claim follows. □

4.3 Combination Algebras

Let $\mathcal{O}$ be a set (which we call a set of operation types).

If P be a space of operations (of types $\mathcal{O}$), its space of combination operations $End(P)$ is a space of operations of type $\mathcal{C}(\mathcal{O})$ and it has distinguished operations called compositions, permutations of inputs and units. These operations live in the space of combination operations on $End(End(P))$, where, if there is a differential on P, they are cycles. Let $\mathcal{G} := \{[\mathrm{F}\ {}_i\!\circ\ \mathrm{G}], [\mathrm{F}]^\sigma, 1_a\}$ denote the set of these operations. We can form the free operadic magma $\mathbb{K}\langle W_{\mathcal{G}}\rangle$ on $\mathcal{G}$. Then for any space of operations P there is a distinguished map

$$\mathbb{K}\langle W_{\mathcal{G}}\rangle \to End(End(P))$$

The operations on $End(P)$ satisfy relations, for example, composition is associative. If one draws operations as rooted trees, one finds there are several relations, which we detail below. These relations can all be expressed in terms of the operations on combination operations, that is, they are in the kernel of the maps $\mathbb{K}\langle W_{\mathcal{G}}\rangle \to End(End(P)$ above.

In stead of working out what the relations are, it would be convenient to form the ideal $U(\mathcal{G})$ of elements which are relations for all choices of P, and define a combination algebra to be a space with composition, permutation and unit operations which satisfy the relations in $U(\mathcal{G})$. We don't really need to know what they are. However, at a certain point below, we will need a fact about this space of relations, which isn't obvious at the time of writing.

The fact is roughly as follows. We would like to form magmas with certain weight gradings and derivations, and we would like these to pass to the quotient by the ideal of relations (determined by $U(\mathcal{G})$), which make the magma into a combination algebra. This works if the ideal respects the weight grading or derivation on the magma, bu it is unknown to the author that this is the case. So in lieu of forming $U(\mathcal{G})$, we take a certain subset $\mathcal{R} \subset U(\mathcal{G})$, which not only generates all apparent relations, but for which weight gradings and derivations pass to the quotient.

Lemma 4.5. *Let P be a space of operations, and $End(P)$ the magma of combination operations on P. Recall that this has compositions, permutations of inputs, and units*

$$[\mathrm{F}\ {}_i\circ\ \mathrm{G}](f\otimes g)(v_1\otimes\ldots\otimes v_{n+m-1}) := (f\ {}_i\circ\ g)(v_1\otimes\ldots\otimes v_{n+m-1}) := \pm f(v_1\otimes\ldots\otimes v_{i-1}\otimes g(v_i\otimes\ldots\otimes v_{i+m-1})\otimes v_{i+m}$$

$$[\mathrm{F}]^\sigma(f)(v_1\otimes v_n) := f^{\otimes\sigma}(v_1\otimes\ldots\otimes v_n) := \pm f(v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)})$$

$$1_{\mathrm{F}}(f) := f$$

where f and g have n and m inputs, respectively, $f \in P(\mathrm{F})$ and $g \in P(\mathrm{G})$.

These operations satisfy relations $\mathcal{R}$ come from associativity of composition, group action given by permutation of inputs, units of composition, and equivariance of composition with respect to the group action.

If M is a combination magma with a weight grading grading (or derivation), then the ideal $I(\mathcal{R})$ of relations generated by the relations of $\mathcal{R}$ applied in all ways to elements of M respects all weight gradings and derivations. If there is a differential, $I(\mathcal{R})$ is a differential ideal, and respects them weight gradings and derivations as a differential ideal.

Proof. The relations as are follows, where f g and h are combination operations in $End(P)$ of types F, G and H, respectively.

Associativity of Composition One observes there are two general of ways to compose three combination operations: in serial and in parallel:

$$f\Big(\ldots g\big(\ldots h(\ldots)\ldots\big)\ldots\Big) \qquad f\big(\ldots g(\ldots)\ldots h(\ldots)\ldots\big)$$

Each of these compositions can be done in two different orders, and the result obviously doesn't depend on the order.

Leaving the subscripts off the units 1_a, because there is no ambiguity, we write associativity for serial composition

$$\begin{aligned} & f\ {}_i\circ\ (g\ {}_j\circ\ h) = \\ = \;& f\circ(1\otimes\ldots\otimes \overset{i}{(g\ {}_j\circ\ h)}\otimes\ldots\otimes 1) \\ = \;& f\Big(1\otimes\ldots\otimes \overset{i}{g}\ \big(1\otimes\ldots\otimes \overset{j}{h}\otimes\ldots\otimes 1\big)\otimes 1\otimes\ldots\otimes 1\Big) = \\ = \;& f\circ(1\otimes\ldots\otimes \overset{i}{g}\otimes\ldots\otimes)\circ(1\otimes\ldots\otimes \overset{i+j-1}{h}\otimes\ldots\otimes 1) \\ = \;& (f\ {}_i\circ\ g)\ {}_{i+j-1}\circ\ h \end{aligned}$$

and letting m be the arity of g, we write associativity for parallel composition

$$\begin{aligned} & (f\ {}_i\circ\ g)\ {}_{i+m-1}\circ\ h = \\ = \ & f \circ (1 \otimes \ldots \otimes \overset{i}{g} \otimes \ldots \otimes 1) \circ (1 \otimes \ldots \otimes \overset{i+m-1}{h} \otimes \ldots \otimes 1) \\ = \ & (-1)^{|g||h|} f \circ (1 \otimes \ldots \otimes \overset{j}{h} \otimes \ldots \otimes 1) \circ (1 \otimes \ldots \otimes \overset{i}{g} \otimes \ldots \otimes 1) \\ = \ & (-1)^{|g||h|} (f\ {}_j\circ\ h)\ {}_i\circ\ g \end{aligned}$$

The relations can be expressed in $U(\mathcal{G})$, as follows:

$$[\mathrm{F}\ {}_i\circ\ (\mathrm{G}\ {}_j\circ\ \mathrm{H})] \circ \Big(1_{\mathrm{F}} \otimes [\mathrm{G}\ {}_j\circ\ \mathrm{H}]\Big) = [(\mathrm{F}\ {}_i\circ\ \mathrm{G})\ {}_{i+j-1}\circ\ \mathrm{H}] \circ \Big([\mathrm{F}\ {}_j\circ\ \mathrm{G}] \otimes 1_{\mathrm{H}}\Big)$$

$$[(\mathrm{F}\ {}_i\circ\ \mathrm{G})\ {}_{i+m-1}\circ\ \mathrm{H}] \circ \Big([\mathrm{F}\ {}_i\circ\ \mathrm{G}] \otimes 1_{\mathrm{H}}\Big) = [(\mathrm{F}\ {}_i\circ\ \mathrm{H})\ {}_j\circ\ \mathrm{G}] \circ \Big([\mathrm{F}\ {}_i\circ\ \mathrm{H}] \otimes 1_{\mathrm{G}}\Big) \circ \Big(1_{\mathrm{F}} \otimes \tau_{\mathrm{G},\mathrm{H}}\Big)$$

when the compositions make sense.

Permutations Give Group Actions Permutation of inputs is a group action, that is

$$(f^\sigma)^\tau = f^{\sigma\tau}$$

$$f^{id} = f$$

Thus the operations satisfy the following relations in $U(\mathcal{G})$:

$$[\mathrm{F}^\sigma]^\tau \circ [\mathrm{F}]^\sigma = [\mathrm{F}]^{\sigma\tau}$$

$$[\mathrm{F}]^{id_n} = 1_F$$

the permutations make sense.

Units are Units of Composition Because composing with the identity in a particular slot doesn't change a map, any map f of type $\mathrm{F} = \binom{a_1 \ldots a_n}{a_0}$, satisfies

$$1_{a_0\ 1}\circ\ f = f$$

$$f\ {}_i\circ\ 1_{a_i} = f$$

when the compositions make sense. We may write these relations as elements of $U(\mathcal{G})$ as follows

$$[\mathrm{F}\ {}_i\circ\ 1_a] = 1_{\mathrm{F}}$$

$$[1_a\ {}_i\circ\ \mathrm{F}] = 1_{\mathrm{F}}$$

Permutations Act Equivariantly With Respect to Composition Suppose we permute the inputs of the combination operations f and g with σ and τ and then compose them

$$f^{\sigma}\ {}_i\circ\ g^{\tau}$$

One observes the same map may be achieved by plugging g into the $\sigma(i)$-th slot of f, and then permuting the variables in some way determined by σ, τ and i. We call this permutation $\sigma\ {}_i\circ\ \tau$ and observe that is given by permuting the block of inputs corresponding to the inputs of g in the $\sigma(i)$-th slot by τ, and then using σ to permute all of the inputs, with the block of inputs of g treated as a single unit (one should draw trees to see what they are; it is useful to let τ or σ be the identity permutation). Thus

$$(f\ {}_{\sigma(i)}\circ\ g)^{\sigma\ {}_i\circ\ \tau} = f^{\sigma}\ {}_i\circ\ g^{\tau}$$

This, too, gives a relation on the operations in $U(\mathcal{G})$:

$$[\mathrm{F}^{\sigma}\ {}_i\circ\ \mathrm{G}^{\tau}] \circ \Big([\mathrm{F}]^{\sigma} \otimes [\mathrm{G}]^{\tau}\Big) = [\mathrm{F}\ {}_{\sigma(i)}\circ\ \mathrm{G}]^{\sigma\ {}_i\circ\ \tau} \circ [\mathrm{F}\ {}_{\sigma(i)}\circ\ \mathrm{G}]$$

Compatibility With the Differential The unit elements $1_a \in End(V)\binom{a}{a}$ are identity maps, thus obviously chain maps, that is

$$\mathcal{D}(1_a) = 0$$

One observes this is equivalent to the assignment

$$\mathbb{K} \to End\big(P\big)\tbinom{a}{a}$$

beging a chain map. We note

$$f\ {}_i\circ\ g = f \circ (1_{a_1} \otimes \ldots \otimes \overset{i}{g} \otimes \ldots \otimes 1_{a_n})$$

for some $a_k \in A$. One uses that ∂ commutes with composition of functions and $\partial(1_a) = 0$ to show that

$$\mathcal{D}(f\ {}_i\circ\ g) = \mathcal{D}(f)\ {}_i\circ\ g + (-1)^{|f|} f\ {}_i\circ\ \mathcal{D}(g)$$

This is equivalent to saying the composition maps $[\mathrm{F}\ {}_i\circ\ \mathrm{G}]$ are chain maps.

Since the differential commutes with the action of permutation, the maps $[\mathrm{F}]^{\sigma}$ are also chain mappings.

For a more detailed discussion of these relations one can find them under the name "colored operads".

We let $\mathcal{R}$ be the set of all the relations above in $U(\mathcal{G})$.

Given a magma M, we may form the ideal $I(\mathcal{R})$ as follows: take all elements of M and plug them in all possible ways that make sense to get a set of relators, that is all the elements for all combination operations f, g and h that make sense:

$$f\ {}_i\circ\ (g\ {}_h\circ\ ) - (f\ {}_i\circ\ g)\ {}_{i+j-1}\circ\ h$$

$$(f\ {}_i\circ\ g)\ {}_{i+m-1}\circ\ h - \pm(f\ {}_j\circ\ h)\ {}_i\circ\ g$$

$$(m^\sigma)^\tau - m^{\sigma\tau}$$
$$f - 1_a \,{}_1{\circ}\, f$$
$$f - f \,{}_i{\circ}\, 1_a$$
$$f^\sigma \,{}_i{\circ}\, g^\tau - (f \,{}_{\sigma(i)}{\circ}\, g)^{\sigma \,{}_i{\circ}\, \tau}$$

and if there is a differential

$$\mathcal{D}(1_a)$$
$$\mathcal{D}(f \,{}_i{\circ}\, g) - (\mathcal{D}f \,{}_i{\circ}\, g \pm f \,{}_i{\circ}\, \mathcal{D}g)$$
$$\mathcal{D}(f^\sigma) - (\mathcal{D}f)^\sigma$$

One observes that all these relations respect weight gradings, and ignoring the relations for a differential, that they respect derivations. The claim follows. □

Definition 4.8. *In the presence of a set $\mathcal{O}$, a **combination algebra** is a combination magma that satisfies the relations of Lemma 4.5, which we call the **relations of a combination algebra**.*

We can make differential or non-differential versions, thus we say differential graded combination algebras to emphasize there is a differential, or graded combination algebra to emphasize there isn't.

Thus by construction:

Corollary 4.6. *If P is a space of operations, then the endomorphisms $End(P)$ with the canonical operations form a combination algebra, which we call the **combination algebra of endomorphisms on** P.*

The category of combination algebras is simply the subcategory of combination magmas which satisfy the relations of a combination algebra, with maps that preserves their structure, in other words, the full subcategory. This category has the same notion of subalgebras and ideals. One notes that quotients by an ideals automatically satisfy the relations of a combination algebra, because the induced operations are computed on representatives, which are assumed to to satisfy relations of a combination algebra. Weight gradings and derivations on combination algebras are defined the same way as for magmas. One observes that direct limits of combination algebras as combination magmas are again combination algebras, thus combination algebras inherit direct limits.

Proposition 4.7. *The facts in Proposition 4.3, which hold for combination magmas also hold for combination magmas, as follows:*

Quotients by ideals satisfy the universal property of quotients. Kernels are ideals. Analogous statements hold when there is a differential.

If a subset of a weight graded combination magma respects its weight grading, then so does the ideal it generates. If an ideal of a weight graded combination magma respects its weight grading, then the quotient has an induced weight grading.

If a subset is respects a derivation, then so does the ideal it generates. If an ideal respects a derivation, then the derivation passes to the quotient.

Proof. The proofs are virtually the same as for combination magmas. □

Obviously, not ever thing passes perfectly. Free magmas definitely aren't free as combination algebras.

Definition 4.9. *Let M be a magma. Let $I(\mathcal{R})$ be the ideal generated by relations of a combination algebra applied in all ways to M, as in Lemma 4.5. We call $I(\mathcal{R})$ the* ***ideal of relations of a combination algebra in*** *M.*

Given a set $X = \{X(\mathrm{F})|\mathrm{F} \in \mathcal{C}(\mathcal{O})\}$ of graded sets indexed by $\mathcal{C}(\mathcal{O})$, we may form the combination algebra $\mathbb{K}[X] := \mathbb{K}\langle W_X\rangle/I(\mathcal{R})$. This has a obvious distinguished map $X \to \mathbb{K}[X]$. We will call $\mathbb{K}[X]$ the ***free combination algebra on the set*** *X once we have shown it satisfies the free property in the category of graded combination algebras. We may call X a* ***set of free generators*** *or* ***free variables****.*

Proposition 4.8. *The combination algebra $\mathbb{K}[X]$ with distinguished map $X \to \mathbb{K}[X]$ satisfies the free property in the category of graded combination algebras.*

Any map $s : X \to \mathbb{K}[X]$ of underlying sets of graded sets, possibly of non-zero degree extends uniquely to a derivation on $\mathbb{K}[X]$ (which doesn't necessarily square to zero).

If s_1 and s_2 are derivations, then their graded commutator

$$[s_1, s_2] := s_1s_2 - (-1)^{|s_1||s_2|}s_2s_1$$

is a derivation. Thus if s is an derivation of odd degree, which is zero on generators, then $s^2 = 0$. If s has degree plus or minus one, then it is a differential.

Let $f : C \to C'$ be a map of graded combination algebras. Suppose C and C' have derivations s and s', respectively, which are of the same degree. If f commutes with the derivations on a generating set, then it commutes with the derivations.

Any map $w : X \to \mathbb{N}$ extends uniquely to a weight grading on $\mathbb{K}[X]$ in the sense that there is a unique weight grading on $\mathbb{K}[X]$ such that the weight of the equivalence class of $x \in X$ is determined by W.

Proof. Given a map $X \to C$ combination algebra, there is unique map extending it to a map of magmas $\mathbb{K}\langle W_X\rangle \to C$. The ideal $I(\mathcal{R})$ is automatically in the kernel, so the map factors uniquely through $\mathbb{K}[X]$. Another such map would give the same map on $\mathbb{K}\langle W_X\rangle$ thus induce the same map on $\mathbb{K}[X]$, thus the map is unique. Thus $\mathbb{K}[X]$ is free.

Let $s : X \to \mathbb{K}[X]$ be a map of sets of graded sets of some, possibly nonzero, degree. If it extends to a derivation, it extends uniquely by considering the images of spaces $\mathbb{K}\langle X_n\rangle$ in $\mathbb{K}[X]$ and exhausting $\mathbb{K}[X]$ much as we did in Proposition 4.8. For existence, lift s to a map $\tilde{s} : X \to \mathbb{K}\langle W_X\rangle$. By the analogous proposition for combination magmas, $\tilde{s}$ extends uniquely to a derivation on $\mathbb{K}\langle W_X\rangle$, and since the ideal $I(\mathcal{R})$ respects all derivations it induces a derivation on the quotient $\mathbb{K}[X]$ which extends s by construction.

Suppose s_1 and s_2 are derivations of the same degree. Let $[s_1, s_2] : s_1s_2 - (-1)^{|s_1||s_2|}s_2s_1$. Clearly $[s_1, s_2](1_a) = 0$ and $[s_1, s_2](f^\sigma) = ([s_1, s_2](f))^\sigma$. We compute it is a derivation of composition in the case both s_1 and s_2 have odd degree, because that is the case we care about, and the signs are few. The general case is similar

$$\begin{aligned}
[s_1, s_2](f \ {}_i\circ \ g) &= s_1 s_2(f \ {}_i\circ \ g) + s_2 s_1(f \ {}_i\circ \ g) \\
&= s_1\Big((s_2 f) \ {}_i\circ \ g + (-1)^{|f|} f \ {}_i\circ \ (s_2 g)\Big) \\
&\quad +s_2\Big((s_1 f) \ {}_i\circ \ g + (-1)^{|f|} f \ {}_i\circ \ (s_1 g)\Big) \\
&= (s_1 s_2 f) \ {}_i\circ \ g - (-1)^{|f|}(s_2 f) \ {}_i\circ \ (s_1 g) \\
&\quad (-1)^{|f|}(s_1 f) \ {}_i\circ \ (s_2 g) + (f \ {}_i\circ \ (s_1 s_2 g) \\
&\quad (s_2 s_1 f) \ {}_i\circ \ g - (-1)^{|f|}(s_1 f) \ {}_i\circ \ (s_2 g) \\
&\quad (-1)^{|f|}(s_2 f) \ {}_i\circ \ (s_1 g) + f \ {}_i\circ \ (s_2 s_1 g) \\
&= (s_1 s_2 + s_2 s_1)(f) \ {}_i\circ \ g + f \ {}_i\circ \ (s_1 s_2 + s_2 s_1)(g) \\
&= [s_1, s_2](f) \ {}_i\circ \ g + f \ {}_i\circ \ [s_1, s_2](g)
\end{aligned}$$

The general case is similar. Thus commutators of derivations are derivations.

If s is an odd derivation, then $s^2 = \frac{1}{2}[s, s]$ is a derivation, too. If a derivation is zero on generators, then it is zero, thus if $s^2 = 0$ on its entire domain.

Let C and C' be graded combination algebras with derivations s and s' of the same degree. Suppose $f : C \to C'$ is a map of graded combination algebras, and that $fs = s'f$ on a generating set. One observes that it also commutes on products, units and compositions. So it commutes on the space of compositions of generators, generators with permuted inputs, and units, too. Doing this repeatedly generates in increasing sequence of subspace on which it commutes. Thus it commutes on the limit, which is the entire algebra.

Let $w : X \to \mathbb{N}$ be a function. If there is a weight grading on $\mathbb{K}[X]$ such that the weight of classes of elements of X is determined by w, then it is unique, because then the weights of classes of elements in W_X are completely determined, and $\mathbb{K}[X]$ is generated by the classes of W_X.

By Proposition 4.4 this extends to a unique weight grading on $\mathbb{K}\langle X\rangle$ such that the weight of elements of X is determined by w. The kernel $I(\mathcal{R})$ respects weigh gradings, thus the weight grading induces a weight grading on the quotient $\mathbb{K}[X]$. Weight is computed by any member of any class, thus the weight of classes of X in $\mathbb{K}[X]$ are given by w. Thus the desired weight grading exists and is unique.

The claims follows. □

We wish to define coproducts of combination algebras. The following notion is convenient

Definition 4.10. *Let C be a combination algebra. We call the map*

$$\mathbb{K}[C] \to C$$

*induced by the identity map on C the **tautological map**, and we denote the kernel $I(C)$, which is an ideal, and we call it the **tautological ideal**. Thus is a canonical isomorphism $\mathbb{K}[C]/I(C) \cong C$.*

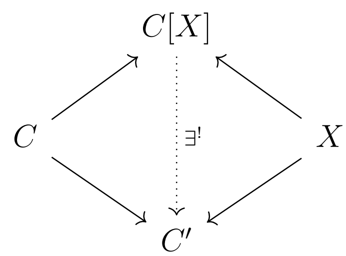
If C has a differential, then, the differential extends to a differential on $\mathbb{K}[C]$ (by Proposition 4.8), thus $\mathbb{K}[C]$ is a differential graded combination algebra map. Since d is a derivation on each and the map $\mathbb{K}[C] \to C$ commutes with the differentials on generators, it commutes with the differential on all of $\mathbb{K}[C]$. Thus the map is a chain map. Thus the ideal is a differential ideal, and the isomorphism is an isomorphism of dg combination algebras.

Let C_1 and C_2 be combination algebras Then there are canonical maps $\mathbb{K}[C_2] \to \mathbb{K}[C_1 \sqcup C_2] \leftarrow \mathbb{K}[C_2]$. One observes there induced maps $\mathbb{K}[C_2]/I(C_1) \to \mathbb{K}[C_1 \sqcup C_2]/\langle I(C_1) \sqcup I(C_2)\rangle \leftarrow \mathbb{K}[C_2]/I(C_2)$, and that if there are differentials, then the maps commute with the differentials ($\langle I(C_1) \sqcup I(C_2)\rangle$ is a differential ideal, because its generators are closed under d). We denote $C_1 \wedge C_2 := \mathbb{K}[C_1 \sqcup C_2]/\langle I(C_1 \sqcup I(C_2))\rangle$. Thus there are distinguished maps $C_1 \to C_1 \wedge C_2 \leftarrow C_2$. We will call $C_1 \wedge C_2$ the ***coproduct*** *of C_1 and C_2, once we show it satisfies the appropriate universal property.*

Let C be a combination algebra and X a set of free generators. We denote $C[X] := C \wedge \mathbb{K}[X]$ and call it the free extension of C by X. There are distinguished maps from C, X and $\mathbb{K}[X]$ to $C[X]$.

Proposition 4.9. *Let C_1 and C_2 be combination algebras (with or without differentials). Then $C_1 \wedge C_2$ with the distinguished maps $C_1 \to C_1 \wedge C_2 \leftarrow C_2$ satisfies the universal property of coproducts. The distinguished maps are inclusions, so we may treat C_1 and C_2 as subalgebras.*

Let C be a combination algebra and X a set of free generators. Let C' be a combination algebra. Given any map of sets (preserving degree) $X \to C'$ and map of combination algebras $C \to C'$, there is a unique extension over $C[X]$, that is, there is unique map $C[X] \to C'$ such that the diagram commutes



Proof. That $C_1 \wedge C_2$ satisfies the universal property of coproducts, follows from applying the universal property of quotients. One observe the distinguished maps are inclusions by using properties of linear spaces. One uses the universal free property and universal property of coproducts to show that free extensions satisfy their universal property. □

Lemma 4.10. *Let C be a dg combination algebra and X a set of generators. A map $d : X \to C$ (of the same degree as the differential on C) determines a differential on $C[X]$ extending the differential on C if $d^2 = 0$ on X.*

Let $C \to C'$ be a map of dg combination algebras, assume $C[X]$ has differential extending the differential on C. Suppose further that we have a map $f : C[X] \to C'$ of graded combination algebras algebras (ignoring differentials). Then f is a map of dg combination algebras (ie commutes with the differential) if and only if $fd = df$ on X.

Given a function $w : X \to \mathbb{N}$ there is a unique weight grading om $C[X]$ such that C has weight zero, and elements of X have weight determined by w.

Proof. We can assemble the map d and the differential on C into a single map $d : C \sqcup X \to C \sqcup X$, which one observes has square zero. This induces a differential on $\mathbb{K}[C \sqcup X]$ by proposition 4.8. The differential on C induces a differential on $K[C]$. There is an evident map $\mathbb{K}[C] \to \mathbb{K}[C \sqcup X]$, which one notes commutes with the differential. Quotienting the by the differential ideals $I(C)$ and $\langle I(C)\rangle$, respectively, we get a map of dg combination algebras $C \to C[X]$, which is the distinguished maps $C \to C[X]$. Thinking of the isomorphic image of C as a subalgebra of C, it follows that the differential on $C[X]$ extends the differential on C.

If a map commutes with the differential on generators (which ours does), then it commutes with the differential. The claim follows.

Suppose $w : X \to \mathbb{N}$ is a weight function. extend it over C to be zero. Then $\mathbb{K}[C \sqcup X]$ has a unique weight grading satisfying the desired properties. The image of $\mathbb{K}[C]$ has weight zero, thus $I(C)$ respects the weight grading on $\mathbb{K}[C \sqcup X]$, thus the grading passes to $C[X]$. The weight of any element is given by the weigh on representatives, whose weights are completely determined by the weights on X and C. Thus the weight grading is unique.

The claim follows. □

Proposition 4.11. *The homology $H(C)$ of a dg combination algebra C has an induced combination algebra structure. A map of dg combination algebras $C \to C'$ induces a map of combination algebras $H(C) \to H(C')$.*

Proof. Let C be a combination algebra with homology $H(C)$. Since the operations on C (composition, permutation of inputs, units) are chain maps, they induce maps on homology. Since homology commutes with tensor products (up to unique isomorphism), we get operations

$$[\mathrm{F}\ {}_i\circ\ \mathrm{G}] : H(C(\mathrm{F})) \otimes H(C(\mathrm{G})) \cong H(C(\mathrm{F}) \otimes C(\mathrm{G})) \to H(C(\mathrm{F}\ {}_i\circ\ \mathrm{G}))$$

$$[\mathrm{F}]^\sigma : H(C(\mathrm{F})) \to H(C(\mathrm{F}^\sigma))$$

$$1_a : \mathbb{K} \to H(C)\tbinom{a}{a} = H(C\tbinom{a}{a})$$

If we think of 1_a as an element of $H(C^a_a)$, then 1_a is a cycle, thus $1_a \in H(C\binom{a}{a})$. One observes that the conversion between the element/map notions of a unit commutes with homology.

Using the natural isomorphisms, one sees that these operations satisfy the relations of a combination algebra, because they do on homological representatives.

Let $f : C \to C'$ be a map of combination algebras, with induced map $[\phi] : H(C) \to H(C')$ on homology. The induced map $[\phi]$ commutes with the induced operations on $H(C)$ and $H(C')$, because it is computed on representatives, and it commutes on those. □

4.4 Algebras of Operations over A Combination Algebra

Fix a combination algebra C (on some set of operation types $\mathcal{O}$).

Let P be a space of operations. Recall that the endomorphisms $End(P)$ of combination operations on P is a combination algebra.

Definition 4.11. *Let C be a combination algebra and P a space of operations. An* ***action of*** *C* ***on*** *P means a map $C \to End(P)$ of combination algebras. We say P* ***is an algebra***

over *C if there is a distinguished action of C on P. We may make this definition with or without differentials. We emphasize there is or isn't a differential by saying graded C algebra or differential graded C algebra.*

If we fix a combination algebra C, we may consider the category of algebras over C, with morphisms given by maps of degree zero (or chain maps) which commute with action of C. These have evident notions of subalgebras and ideals (one need only check I is closed under combination operations with at least one input), differential ideals, ideals generated by a subset, and quotient algebras. One observes that kernels are ideals, and that the universal property of quotients holds.

There are analogous notions of weight grading as splittings such that operations preserve the total weight sum, that is

$$w(c(p_1 \otimes \ldots \otimes c_n)) = w(p_1) + \ldots + w(p_n)$$

for any operations $p_1, \ldots, p_n$ in P an algebra of operations over a combination algebra C, and c an operation in C. If c has no inputs, then it is tantamount to an operation in P, and we require if have weight zero, $w(c) = 0$. If there is a differential, we require it preserve weight, too.

We say s is a derivation on an graded algebra of operations P over a combination algebra C if the graded commutator $[s, c]$, defined as follows, vanishes for every operation C,

$$[s, c](p_1 \otimes \ldots \otimes p_n) := sc(p_1 \otimes \ldots \otimes p_n) - (-1)^{|s||c|} \sum_i \pm c(p_1 \otimes \ldots \otimes sp_i \otimes \ldots \otimes p_n)$$

If a combination operation c has not inputs, this means that $s(c) = 0$. This is the same as how we required that a derivation on a combination algebra satisfy $s(1_a) = 0$.

Remark 4.1. *One notes that if the differential on C is non-zero, then the differential on a dgC algebra P need not act by derivation of the combination operations, since, for example*

$$\partial c(p_1 \otimes p_2) - (-1)^{|c|}(c(\partial p_1 \otimes p_2) \pm c(p_1 \otimes \partial p_2)) = \mathcal{D}(c)(p_1 \otimes p_2)$$

One observes that the boundary $\mathcal{D}(c)$ of a combination operation measures, exactly, the failure of d to be a derivation. Because of this, we concern ourselves mainly with combination algebras with zero differential.

In other words:

Proposition 4.12. *Let C be a dg combination algebra and P a dgC algebra. The differential on P is a derivation of all combination operations if and only if the differential on C is zero.*

There are analogous notions of what it means for an ideal to respect a derivation or weight grading.

Proposition 4.13. *The same kinds kinds of facts which hold for combination magmas and combination algebras in Propositions 4.3 and 4.7 also hold for algebras over combination algebras, namely:*

Quotients by ideals satisfy the universal property of quotients. Kernels are ideals. Analogous statements hold when there is a differential.

If a subset of a weight graded combination magma repsects its weight grading, then so does the ideal it generates. If an ideal of a weight graded combination magma repsects its weight grading, then the quotient has an induced weight grading.

If a subset is respects a derivation, then so does the ideal it generates. If an ideal respects a derivation, then the derivation passes to the quotient.

Proof. The proofs are virtually the same. □

As one expects, we have to work some to get free algebras, derivations and induced weight gradings, but due to the following proposition, we don't have to work very hard.

Definition 4.12. *Let C be a combination algebra. One notes that the zero-to-one operations on C form a space of operations. We denote this space $C_0 := \{C_0(a) := C\binom{}{a} | a \in \mathcal{O}\}$, and we call it the combination algebra's* ***(underlying) space of operations****.*

One observes that an n-to-1 combination operation $c \in C\binom{a_1 \dots a_n}{a_0}$ determines an n-to-1 operation on C_0 as follows:

$$\mu : C \quad \to \quad End(C_0)$$

where given $p_1, \dots, p_n$ in $C_0(a_1), \dots, C_0(a_n)$, respectively,

$$\mu_c(p_1 \otimes \dots \otimes p_n) := (\dots(c \ {}_1\!\circ \ p_1) \ {}_1\!\circ \ p_2 \dots) \ {}_1\!\circ \ p_n$$

If P is a space of operations, then we may regard $End(P)_0 = Hom(\mathbb{K}, P)$ simply as the space P, to which it is canonically isomorphic.

Proposition 4.14. *Let C be a combination algebra. The map $\mu : C \to End(C_0)$ gives C_0 a C algebra structure, that is μ is a map of combination algebras. If there is a differential on C, then it gives C_0 a dgC algebra structure.*

Proof. One should draw the operations as rooted trees to come up with the following computations.

Let c be in C and $p_1, \dots p_n$ in C_0. We compute:

$$\begin{aligned}
& \Big(\Big[\mathcal{D}, \mu_c\Big] - \mu_{\mathcal{D}c}\Big)(p_1 \otimes \dots \otimes p_n) = \\
& \Big(\mathcal{D}\mu_c - \pm\mu_C\mathcal{D} - \mu_{\mathcal{D}c}\Big)(p_1 \otimes \dots \otimes p_n) = \\
= \ & \mathcal{D}\mu_c(p_1 \otimes \dots \otimes p_n) - \pm\mu_c\mathcal{D}(p_1 \otimes \dots \otimes p_n) - \mu_{\mathcal{D}c}(p_1 \otimes \dots \otimes p_n) \\
= \ & \mathcal{D}\Big(\mu_c(p_1 \otimes \dots \otimes p_n)\Big) - \textstyle\sum_i \Big(\mu_c(p_1 \otimes \dots \otimes \mathcal{D}p_i \otimes \dots \otimes p_n\Big) - \mu_{\mathcal{D}c}(p_1 \otimes \dots \otimes p_n) \\
= \ & \mathcal{D}\Big((\dots(c \ {}_1\!\circ \ p_1) \ {}_1\!\circ \ p_2 \dots) \ {}_1\!\circ \ p_n\Big) \\
& -\Big((\dots(\mathcal{D}c \ {}_1\!\circ \ p_1) \ {}_1\!\circ \ p_2 \dots) \ {}_1\!\circ \ p_n \pm (\dots(c \ {}_1\!\circ \ \mathcal{D}p_1) \ {}_1\!\circ \ p_2 \dots) \ {}_1\!\circ \ p_n \pm \dots\Big) \\
= \ & 0
\end{aligned}$$

Thus μ is a chain map if there is a differential on C.

Let c and c' be elements of C and $p_1, \ldots, p_n$ be elements of C_0. We compute:

$$
\begin{aligned}
& \mu_{c \ {}_i\circ \ c'}(c_1 \otimes \ldots \otimes c_n) = \\
= \ & \Big(\Big(\Big(c \ {}_i\circ \ c'\Big) \ {}_1\circ \ p_1\Big) \ldots\Big) \ {}_1\circ \ p_n = \\
= \ & \pm\Big(\Big(\Big(\Big(c \ {}_1\circ \ p_1\Big) \ {}_{i-1}\circ \ c'\Big) \ {}_1\circ \ p_2\Big) \ldots\Big) \ {}_1\circ \ p_n \\
\vdots \ & \text{ move first i-1 inputs before q} \\
= \ & \pm\Big(\Big(\Big(\Big(\Big(c \ {}_1\circ \ p_1\Big) \ldots\Big) \ {}_1\circ \ p_{i-1}\Big) \ {}_1\circ \ c'\Big) \ {}_1\circ \ p_i\Big) \ {}_1\circ \ \ldots\Big) \ {}_1\circ \ p_n \\
= \ & \pm\Big(\Big(\Big(\Big(\Big(c \ {}_1\circ \ p_1\Big) \ldots\Big) \ {}_1\circ \ p_{i-1}\Big) \ {}_1\circ \ \Big(c' \ {}_1\circ \ p_i\Big)\Big) \ {}_1\circ \ \ldots\Big) \ {}_1\circ \ p_n \\
\vdots \ & \text{ use associativity to compose the } k \text{ inputs of } q \text{ first} \\
= \ & \pm\Big(\Big(\Big(\Big(\Big(c \ {}_1\circ \ p_1\Big) \ldots\Big) \ {}_1\circ \ p_{i-1}\Big) \ {}_1\circ \ \Big(\Big(\Big(c' \ {}_1\circ \ p_i\Big) \ldots\Big) \ {}_1\circ \ p_{i-1+k}\Big) \ {}_1\circ \ \ldots\Big) \ {}_1\circ \ p_n \\
= \ & \pm\Big(\Big(\Big(\Big(\Big(c \ {}_1\circ \ p_1\Big) \ldots\Big) \ {}_1\circ \ p_{i-1}\Big) \ {}_1\circ \ \mu_{c'}(p_i \otimes \ldots \otimes p_{i-1+k})\Big) \ {}_1\circ \ \ldots\Big) \ {}_1\circ \ p_n \\
= \ & \pm\mu_c\Big(p_1 \otimes \ldots \otimes p_{i-1} \otimes \mu_{c'}(p_i \otimes \ldots \otimes p_{i-1+k}) \otimes p_{i+k} \otimes \ldots \otimes p_n\Big) \\
= \ & \Big(\mu_c \ {}_i\circ \ \mu_{c'}\Big)(p_1 \otimes \ldots \otimes p_n)
\end{aligned}
$$

Thus $\mu_{c \ {}_i\circ \ c'} = \mu_c \ {}_i\circ \ \mu_{c'}$.

It's enough to check μ commutes with permutations for the permutations $(1k)$, since these generate all permutations.

First we note that if c in C p_1 and q and p are in C_0, then

$$\Big(c \ {}_1\circ \ p\Big) \ {}_k\circ \ q = \pm(c \ {}_{k+1}\circ \ q) \ {}_1\circ \ p$$

and

$$
\begin{aligned}
\Big(c^{(1k)} \ {}_k\circ \ p\Big) \ {}_1\circ \ q \ &= \ \pm\Big(c \ {}_1\circ \ p\Big)^{(1k) \ {}_1\circ \ id} \ {}_1\circ \ q \\
&= \ \pm\Big(c \ {}_1\circ \ p\Big)^{((k-1)12\ldots(k-2))} \ {}_1\circ \ q \\
&= \ \pm\Big(c \ {}_1\circ \ p\Big) \ {}_{k-1}\circ \ q\Big)^{id} \\
&= \ \Big(\pm \Big(c \ {}_1\circ \ p\Big) \ {}_{k-1}\circ \ q \\
&= \ \pm\Big(c \ {}_k\circ \ q\Big) \ {}_1\circ \ p
\end{aligned}
$$

Now we compute:

$$\begin{aligned}
& \mu_{c^{(1k)}}(p_1 \otimes \ldots \otimes p_n) = \\
= \;& \Bigg(\bigg(\Big(\Big(\Big(c^{(1k)}\ {}_1\circ\ p_1\Big)\ldots\Big)\ {}_1\circ\ p_{k-1}\Big)\ {}_1\circ\ p_k\bigg)\ldots\Bigg)\ {}_1\circ\ p_n \\
= \;& \pm\Bigg(\bigg(\Big(\Big(\Big(c^{(1k)}\ {}_1\circ\ p_1\Big)\ldots\Big)\ {}_2\circ\ p_k\Big)\ {}_1\circ\ p_{k-1}\bigg)\ldots\Bigg)\ {}_1\circ\ p_n \\
\vdots\;& \text{push } p_k \text{ all the way to the beginning using the first computation above} \\
= \;& \pm\bigg(\Big(\Big(c^{(1k)}\ {}_k\circ\ p_k\Big)\ {}_1\circ\ p_1\Big)\ldots\bigg)\ {}_1\circ\ p_n \\
& \text{using the second computation above} \\
= \;& \pm\bigg(\Big(\Big(c\ {}_k\circ\ p_1\Big)\ {}_1\circ\ p_k\Big)\ldots\bigg)\ {}_1\circ\ p_n \\
\vdots\;& \text{push } p_1 \text{ all the way to the k-th using first computation above} \\
= \;& \pm\Bigg(\bigg(\Big(\Big(\Big(c\ {}_1\circ\ p_k\Big)\ldots\Big)\ {}_2\circ\ p_1\Big)\ {}_1\circ\ p_{k-1}\bigg)\ldots\Bigg)\ {}_1\circ\ p_n \\
= \;& \pm\Bigg(\bigg(\Big(\Big(\Big(c\ {}_1\circ\ p_k\Big)\ldots\Big)\ {}_1\circ\ p_{k-1}\Big)\ {}_1\circ\ p_1\bigg)\ldots\Bigg)\ {}_1\circ\ p_n \\
= \;& \pm\mu_c(p_k \otimes p_2 \otimes \ldots \otimes p_{k-1} \otimes p_1 \otimes p_{k+1} \otimes \ldots \otimes p_n) \\
= \;& {\mu_c}^{(1k)}(p_1 \otimes \ldots \otimes p_n)
\end{aligned}$$

Thus $\mu_{c^{(1k)}} = {\mu_c}^{(1k)}$. It follows that μ commutes with all permutations.

Finally, we check that μ preserves the units. Let p be in $C_0(a)$. We compute:

$$\mu_{1_a}(p) = 1_a\ {}_1\circ\ p = 1_a(p)$$

Thus μ sends units to units.

The claim follows. □

Proposition 4.14 allows us to transplant certain tools from combination algebras into algebras of operations.

Definition 4.13. *In the presence of a set $\mathcal{O}$, we refer to the a set of graded sets indexed by $\mathcal{O}$ as a* ***set of operations****,* ***free generators****, etc.*

We may regard a set or space of operations as a set or space of combination operations by treating them as the 0*-to-*1 *operations (opposite of how we did it in Definition 4.12).*

In the presence of a combination algebra C, and given a set X of operations, we define $\mathbb{K}[X]$ to be the space of zero-to-one operations of $C[X]$, that is, $\mathbb{K}[X] := C[X]_0$. We will call $\mathbb{K}[X]$ the ***free graded C algebra on X*** *once we know it satisfies the free property. There is an evident distinguished map $X \to \mathbb{K}[X]$.*

Proposition 4.15. *The following facts hold for algebras over a combination algebra, which are analogous to those in Proposition 4.8:*

Fix a combination algebra C.

Then the graded C algebra $\mathbb{K}[X]$ with its distinguished map $X \to \mathbb{K}[X]$ satisfies the free property in the category of graded C algebras.

Let $\mathbb{K}[X]$ be a free graded C algebra. A map $s : X \to \mathbb{K}[X]$ (of any degree) extends uniquely to a derivation on $\mathbb{K}[X]$.

Any map $s : X \to \mathbb{K}[X]$ of underlying sets (of any degree) extends uniquely to a derivation on $\mathbb{K}[X]$ (which doesn't necessarily square to zero).

If s_1 and s_2 are derivations, then their graded commutator

$$[s_1, s_2] := s_1 s_2 - (-1)^{|s_1||s_2|} s_2 s_1$$

is a derivation. Thus if s is an derivation of odd degree, which is zero on generators, then $s^2 = 0$. If s has degree plus or minus one, then it is a differential.

Let $f : P \to P'$ be a map of graded combination algebras. Suppose P and P' have derivations s and s', respectively, which are of the same degree. If f commutes with the derivations on a generating set, then it commutes with the derivations.

Any map $w : X \to \mathbb{N}$ extends uniquely to a weight grading on $\mathbb{K}[X]$ in the sense that there is a unique weight grading on $\mathbb{K}[X]$ such that the weight of the equivalence class of $x \in X$ is determined by W.

Proof. One recalls that $C[X]$ with the distinguished map $X \to C[X]$ satisfies the free property in the category of graded combination algebras.

Suppose P is an algebra over C, and and we're given a map $X \to P$ of sets of operations. Regarding P as the space of operations underlying $End(P)$, we get a map $X \to End(P)$, and since P is C algebra, we have a map $C \to End(P)$. Thus by universal property of $C[X]$, we get a map of combination algebras $C[X] \to End(P)$. Restricting this to their underlying spaces of operations gives our map $\mathbb{K}[X] \to P$m where $\mathbb{K}[X]$ is given the action of $C[X]$ as its space of operations, restricted to an action of C. We check it is a map of C algebras, and the unique one extending the original map $X \to P$. Denote the map $C[X] \to End(P)$ by f. We simply have to check that $f(c(c_1 \otimes \ldots \otimes c_n)) = c(f(c_1) \otimes \ldots \otimes f(c_n))$. But this follows immediately from the the definition of $c(c_1 \otimes \ldots \otimes c_n)$ in definition 4.12 and the fact f is a map of combination algebras.

Suppose there were another map, $\mathbb{K}[X] \to P$ extending the map $X \to P$. This extends uniquely to a map $C[X] \to End(P)$ of combination algebras which extend the action of C and the original map $X \to P$. Thus the map is f. Thus its restriction to a map $\mathbb{K}[X] \to P$ is f. Thus $f = g$.

One proves the remaining claims using the same trick. □

We define the tautological map and tautological ideal just as we did for combination algebras. One recalls to show the tautological map is a chain map in the context of combination algebras, we used that the differentials are derivations. In the context of an algebra of operations P over a dg combination algebra, the differentials are derivations if and only if the combination algebra has zero differential (see Lemma 4.12 and preceding remark).

Definition 4.14. *Let C be a combination algebra.*

Given a C algebra P, we define the tautological map and tautological ideal analogously to Definition 4.10. If C has the zero differential, and P is a dgP algebra, the tautological ideal is a differential ideal.

We define coproducts and free extensions of C algebras analogously. If C has zero differential, then the ideals defining them are differential ideals, thus the coproducts have differentials.

Proposition 4.16. *Ignoring the differential, coproducts satisfy the property of coproducts, and free extensions satisfy an analogous property to free extensions of combination algebras, as in Proposition 4.9.*

If the combination algebra has zero differential, then there are coproducts of differential algebras of operations.

Proof. The same proof works. □

The following is a key lemma, which we use repeatedly in the other chapters.

Lemma 4.17. *Let C be a dg combination algebra with zero differential.*

Let P be a dg$C$ algebra, and let and X a set of generators. A map $d : X \to P$ (of the same degree as the differential on P) determines a differential on $P[X]$ extending the differential on P if $d^2 = 0$ on X.

Let $P \to Q$ be a map of dg combination algebras, and assume $P[X]$ has differential extending the differential on P. Suppose we have a map $f : P[X] \to Q$ of graded C algebras (ignoring differentials). Then f is a map of dg$C$ algebras (ie commutes with the differential) if and only if $fd = df$ on X.

Given a function $w : X \to \mathbb{N}$ there is a unique weight grading on $P[X]$ such that P has weight zero, and elements of X have weight determined by w.

Proof. The proof is virtually identical to the proof of the analogous lemma for combination algebras (Lemma 4.10) □

Proposition 4.18. *Let C be a combination dg combination algebra, and P a dg$C$ algebra.*

There is an induced algebra structure on the homology $H(P)$ over the induced combination algebra $H(C)$.

A map $P \to Q$ of dg$C$ algebras, induces a map $H(P) \to H(Q)$ of $H(C)$ algebras.

Proof. One observes there is a canonical isomorphism $H(End(P)) \cong End(H(P))$. By the same analogous proposition for combination algebras (Proposition 4.11), There map $C \to End(P)$ induces a combination algebra map on homology $H(C) \to H(End(P))) \cong End(H(P))$, thus inducing an $H(C)$ algebra structure on $H(P)$.

If $P \to Q$ is a map of dg$P$ algebras, then there is an induced map of linear spaces $H(P) \to H(Q)$. This map commutes with the operations induced by $H(C)$, because they are commuted on representatives, and they commute on representatives. □

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