Deformations of twisted cscK metrics

A Dissertation presented

by

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to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2016

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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In this dissertation, we describe a new continuity path introduced by X. Chen([10]) aiming to attack the existence problem of constant scalar curvature problem via a direct PDE approach. The path connects the solution of J-equation to the cscK metric. We will present various openness results about the continuity path. The openness at $t = 0$ is in fact a perturbation result from a solution of second order partial differential equation to a solution of a fourth order partial differential equation.

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Acknowledgements

Foremost, I would like to express my sincere gratitude to my advisor Prof. Xiuxiong Chen for the continuous support of my Ph.D study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Ph.D study. I also would like to thank Prof. Mihai Păun for the collaborating on the paper "Deformations of extremal metrics" and clarifying things that I was too young to notice. Thanks Prof. Eric Bedford for the time and patience to listen to my silly ideas which quite often don't make any sense. Thanks Weiyong He for genuine advices and rewarding discussions in the last year of my PhD career.

Besides my advisor, I would like to thank the rest of my thesis committee: Prof. Marcus Khuri, Prof. Song Sun, and Prof. Martin Rocek, for their encouragement, insightful comments, and hard questions.

I also want to thank my PhD fellows for helping me on any kind of math problems and most importantly being valuable friends: Yuanqi Wang, Chengjian Yao, Long Li, Shaosai Huang, Gao Chen, Fangyu Zou, Xuemiao Chen, Selin Taskent etc.

Last but not the least, I would like to thank my family: my wife Ruoxi Wang, for her understanding and unconditionally support; my parents Longxue Zeng and Hua Yu, for giving birth to me and supporting me spiritually throughout my life.

1 Introduction

1.1 Kähler manifolds

A Kähler manifold is a smooth manifold that admits three mutually compatible structures: Riemannian metric, complex structure and symplectic form corresponding to three major fields in differential geometry.

We start with a smooth manifold M. Recall that a Riemannian metric g on M is a positive definite symmetric bilinear form on the tangent bundle TM. In local coordinates $x_1, \cdots, x_n$, one has a natural local basis $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ for TM, then g is locally represented by a smooth matrix-valued function $\{g_{ij}\}$, where the matrix with entries $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is positive definite. The pair (M, g) is usually called a Riemannian manifold.

An almost complex structure J on M is an endomorphism of the tangent bundle TM satisfying $J^2 = -\mathrm{id}$. An almost complex structure J is called integrable if there is a set of charts on M with holomorphic transition functions such that J corresponds to the induced complex multiplication on $TM \otimes_{\mathbb{R}} \mathbb{C}$. An almost complex structure is not necessarily integrable. In fact, we have the following theorem due to Newlander-Nirenberg [38].

Theorem 1.1. *An almost complex structure is integrable if and only if the Nijenhuis tensor* $N_J : TM \times TM \to TM$

$$N_J(u, v) := [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv] \tag{1}$$

is zero.

We say that J is compatible with a Riemannian metric g if $g(u, v) = g(Ju, Jv)$ for any tangent vectors u, v. We can then define

$$\omega_g(\cdot, \cdot) = g(J\cdot, \cdot). \tag{2}$$

One can derive easily that ω_g is in fact a 2-form on M. Usually we call such ω_g the Kähler form of g. For fixed complex structure, we see that g and ω_g are mutually determined by each other, thus often we also call ω_g the Kähler metric.

We denote by ∇ the Levi-Civita connection of the Riemannian metric g, which is the unique torsion free connection such that g is parallel.

Definition 1.2. A Kähler manifold (M, g, J) is a Riemannian manifold (M, g) together with a compatible almost complex structure J such that $\nabla J = 0$.

Remark 1.3. Note that $\nabla J = 0$ implies that $N_J = 0$ and thus J of a Kähler manifold (M, g, J) is automatically integrable.

On a Kähler manifold (M, g, J), we have that $\nabla \omega_g = 0$ and thus $\mathrm{d}\omega_g = 0$. In other words, M admits a symplectic form ω_g such that J is compatible with ω_g. Conversely, we have the following proposition.

Proposition 1.4. *If M admits compatible Riemannian metric g and integrable almost complex structure J, then $\nabla J = 0$ if and only if $\mathrm{d}\omega_g = 0$.*

The proof of this proposition is pure computational and we refer interested readers to [43].

On a Kähler manifold (M, g, J) of dimension $\dim_{\mathbb{C}} M = n$, it is more convenient to work in local holomorphic coordinate $z_i = x_i + \sqrt{-1} y_i$ for $i = 1, \cdots, n$. Besides the obvious basis $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\}$ and $\{\mathrm{d}x_1, \cdots, \mathrm{d}x_n, \mathrm{d}y_1, \cdots, \mathrm{d}y_n\}$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ and complexified cotangent bundle $T^*M \otimes \mathbb{C}$, we have

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}), \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}), \tag{3}$$

for $i = 1, \cdots, n$ of $TM \otimes \mathbb{C}$ corresponding to the $\pm\sqrt{-1}$-eigenspaces $T^{1,0}M$ and $T^{0,1}M$ of the complex structure J and similarly dual basis

$$\mathrm{d}z^i = \mathrm{d}x^i + \sqrt{-1}\mathrm{d}y^i, \mathrm{d}\bar{z}^i = \mathrm{d}x^i - \sqrt{-1}\mathrm{d}y^i, \tag{4}$$

of $T^*M \otimes \mathbb{C}$.

We extend the metric g $\mathbb{C}$-linearly to $TM \otimes \mathbb{C}$ and then we have $g(u, v) = 0$ if $u, v \in T^{1,0}M$ or $u, v \in T^{0,1}M$. Then in local coordinates

$$g = g_{i\bar{j}}(\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j + \mathrm{d}\bar{z}^j \otimes \mathrm{d}z^i), \tag{5}$$

where $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$ and $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$. Thus the Kähler form

$$\omega_g = \sqrt{-1} g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j. \tag{6}$$

The Kähler condition $\mathrm{d}\omega_g = 0$ is then equivalent to

$$\frac{\partial g_{i\bar{p}}}{\partial z_j} = \frac{\partial g_{j\bar{p}}}{\partial z_i}. \tag{7}$$

Furthermore, we extend the Levi-Civita connection ∇ of g in a $\mathbb{C}$-linear way to $\Gamma(TM \otimes \mathbb{C})$. We write the Christoffel symbols as

$$\begin{aligned}\nabla \frac{\partial}{\partial z_j} &= (\Gamma^k_{ij} \mathrm{d}z^i + \Gamma^k_{\bar{i}j} \mathrm{d}\bar{z}^i) \otimes \frac{\partial}{\partial z_k} + (\Gamma^{\bar{k}}_{ij} \mathrm{d}z^i + \Gamma^{\bar{k}}_{\bar{i}j} \mathrm{d}\bar{z}^i) \otimes \frac{\partial}{\partial \bar{z}_k}, \\ \nabla \frac{\partial}{\partial \bar{z}_j} &= (\Gamma^k_{i\bar{j}} \mathrm{d}z^i + \Gamma^k_{\bar{i}\bar{j}} \mathrm{d}\bar{z}^i) \otimes \frac{\partial}{\partial z_k} + (\Gamma^{\bar{k}}_{i\bar{j}} \mathrm{d}z^i + \Gamma^{\bar{k}}_{\bar{i}\bar{j}} \mathrm{d}\bar{z}^i) \otimes \frac{\partial}{\partial \bar{z}_k}.\end{aligned} \tag{8}$$

Using the Kähler condition $\nabla J = 0$, we have that all Christoffel symbols vanish except Γ^k_{ij} and $\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}}$. In fact, we can compute easily that

$$\frac{\partial}{\partial z_i} g_{j\bar{k}} = g(\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}) = \Gamma^l_{ij} g_{l\bar{k}}$$

and thus $\Gamma^k_{ij} = g^{k\bar{p}} \frac{\partial g_{j\bar{p}}}{\partial z_i}$. Given the Levi-Civita connection, the Riemannian curvature tensor $\mathrm{Rm} \in \Gamma(\Lambda^2 T^* M \otimes \mathrm{End}(TM))$ is defined as

$$\mathrm{Rm}(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

Similarly we extend Rm $\mathbb{C}$-linearly to $\Gamma(\Lambda^2 T^* M \otimes \mathrm{End}(TM \otimes \mathbb{C}))$ and under local holomorphic coordinates

$$\mathrm{Rm} = \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \otimes (R^l_{i\bar{j}k} \mathrm{d}z^k \otimes \frac{\partial}{\partial z_l} + R^{\bar{l}}_{i\bar{j}\bar{k}} \mathrm{d}\bar{z}^k \otimes \frac{\partial}{\partial \bar{z}_l}) \tag{9}$$

where $R^l_{i\bar{j}k} = -\frac{\partial}{\partial \bar{z}_j} \Gamma^l_{ik}$ and $R^{\bar{l}}_{i\bar{j}\bar{k}} = -\overline{R^l_{j\bar{i}k}}$. The Ricci tensor $\mathrm{Ric} \in \Gamma(T^* M \otimes T^* M)$ evaluating on $X, Y \in \Gamma(TM)$ is defined as the trace of $\mathrm{Rm}(\cdot, X)Y \in \Gamma(\mathrm{End}(TM))$. Thus in local coordinates, we have

$$\mathrm{Ric} = R_{i\bar{j}} (\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j + \mathrm{d}\bar{z}^j \otimes \mathrm{d}z^i), \tag{10}$$

where $R_{i\bar{j}} = R^l_{l\bar{j}i}$. Define the Ricci form to be $\rho(\cdot, \cdot) = \mathrm{Ric}(J\cdot, \cdot)$ and in local coordinates we have

$$\rho = \sqrt{-1} R_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j. \tag{11}$$

In fact, since

$$R_{i\bar{j}} = -\frac{\partial}{\partial \bar{z}_j} \Gamma^l_{il} = -\frac{\partial}{\partial \bar{z}_j} (g^{l\bar{p}} \frac{\partial g_{l\bar{p}}}{\partial z_i}) = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det g, \tag{12}$$

there is a simple global formula for the Ricci form as the following

$$\rho = -\sqrt{-1} \partial \bar{\partial} \log \det g. \tag{13}$$

Sometimes write as

$$\rho = -\sqrt{-1} \partial \bar{\partial} \log \omega_g^n, \tag{14}$$

since $\mathrm{d}V_g = \omega_g^n = \det(g_{i\bar{j}})\sqrt{-1}\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \cdots \wedge \sqrt{-1}\mathrm{d}z^n \wedge \mathrm{d}\bar{z}^n$.

As a consequence, for fixed complex structure J, if given an another Kähler metric g', the corresponding Ricci form would be

$$\rho' = -\sqrt{-1}\partial\bar{\partial}\log\det g'. \tag{15}$$

and then

$$\rho' - \rho = -\sqrt{-1}\partial\bar{\partial}\log\frac{\det g'}{\det g}, \tag{16}$$

where $\log\frac{\det g'}{\det g}$ is in fact a global function on M. Thus, ρ, ρ' necessarily belongs to the same cohomology class in $H^{1,1}(M,\mathbb{C})\cap H^2(M,\mathbb{R})$, which is in fact 2π multiple of the first chern class of (M, J) denoted as $2\pi c_1(M)$. The converse question, if any representative in $2\pi c_1(M)$ arises as the Ricci form of some Kähler metric on complex manifold (M, J), is indeed much harder. We'll come back to this question in the subsection of Calabi conjecture and Kähler-Einstein problems.

1.2 The space of Kähler metrics

An another advantage of being Kähler is the following lemma.

Lemma 1.5. *($\partial\bar{\partial}$-lemma) Let (M, g, J) be a closed Kähler manifold and $\phi_1, \phi_2 \in H^{1,1}(M,\mathbb{C})$. Suppose that ϕ_1 is cohomologous to ϕ_2. Then there exists a function $f \in C^\infty(M,\mathbb{C})$ such that $\phi_1 - \phi_2 = \partial\bar{\partial}f$.*

Recall that on a complex manifold (M, J), the space of complex valued k-forms on M naturally splits as $\Omega^k(M) = \oplus_{p+q=k}\Omega^{p,q}(M)$, where locally $\Omega^{p,q}(M)$ has basis $\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}\bar{z}^{j_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{j_q}$, for $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$. We have differential operators $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ and $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ defined as the projection of the exterior differential operator d on $\Omega^{p,q+1}(M)$ and $\Omega^{p+1,q}(M)$ components respectively. In fact, $\partial\bar{\partial}$-lemma is also valid for (p, q)-forms with appropriate modifications and the proof requires some ideas from Hodge theory. Since we are only interested in $(1, 1)$-forms on M where the Kähler form lies, in this simple case we provide a quick proof of the $\partial\bar{\partial}$-lemma as below.

Proof. By assumptions, there exists a 1-form α on M such that $\phi_1 - \phi_2 = \mathrm{d}\alpha$. Write $\alpha = \alpha^{1,0} + \alpha^{0,1}$, where $\alpha^{1,0}$ and $\alpha^{0,1}$ denotes the $(1, 0)$ and $(0, 1)$ components of α respectively. Since $\mathrm{d}\alpha = \phi_1 - \phi_2 \in \Omega^{1,1}(M)$ we have that

$\partial\alpha^{1,0} = 0$, $\bar\partial\alpha^{0,1} = 0$ and $\mathrm{d}\alpha = \bar\partial\alpha^{1,0} + \partial\alpha^{0,1}$. Thus to prove the lemma, it suffice to look for functions f, g such that $\partial\alpha^{0,1} = \partial\bar\partial f$ and $\bar\partial\alpha^{1,0} = \bar\partial\partial g$.

It suffices to show that for any $\bar\partial$-closed $(0,1)$-form, say $\alpha^{0,1}$, there exists a function $f \in C^\infty(M, \mathbb{C})$ such that $\alpha^{0,1} + \bar\partial f$ is ∂-closed. We consider differential operator $\bar\partial^* : \Omega^{0,1}(M) \to \Omega^0(M)$ defined as for $\beta = \beta_{\bar j}\mathrm{d}\bar z^j$

$$\bar\partial^*\beta = -g^{i\bar j}\beta_{\bar j,i} \tag{17}$$

where $\beta_{\bar j,i}$ denotes the $(i, \bar j)$ entry of $\nabla\beta$. We set function $f \in C^\infty(M, \mathbb{C})$ to be the solution of the equation

$$-\bar\partial^*\bar\partial f = \bar\partial^*\alpha^{0,1} \tag{18}$$

Since on Kähler manifold, we have $\bar\partial^*\bar\partial f = \frac{1}{2}\Delta_g f$, where Δ_g is the usual Laplacian operator with respect to metric g on M, we know that (18) is solvable if and only if $\bar\partial^*\alpha^{0,1}$ has zero integral on M which follows from its definition by integration by parts directly. Therefore we get that $\theta^{0,1} = \alpha^{0,1} + \bar\partial f$ satisfies both $\bar\partial\theta^{0,1} = 0$ and $\bar\partial^*\theta^{0,1} = 0$. It is left to show that $\partial\theta^{0,1} = 0$. Compute for $\theta^{0,1} = \theta_{\bar j}\mathrm{d}\bar z^j$

$$0 = \partial\bar\partial^*\theta^{0,1} = (-g^{i\bar j}\theta_{\bar j,i})_{,k}\mathrm{d}z^k = -g^{i\bar j}\theta_{\bar j,ik}\mathrm{d}z^k = -g^{i\bar j}\theta_{\bar j,ki}\mathrm{d}z^k. \tag{19}$$

Then consider

$$0 = \int_M \langle\partial\bar\partial^*\theta^{0,1}, \theta^{0,1}\rangle_g \mathrm{d}V_g = \int_M -g^{i\bar j}g^{k\bar l}\theta_{\bar j,ki}\theta_{\bar l}\mathrm{d}V_g = \int_M |\partial\theta^{0,1}|_g^2\mathrm{d}V_g \tag{20}$$

Thus we get $\partial\theta^{0,1} = 0$. In fact, such $\theta^{0,1}$ satisfying both $\bar\partial\theta^{0,1} = 0$ and $\bar\partial^*\theta^{0,1} = 0$ is called $\bar\partial$-harmonic form in Hodge theory. On Kähler manifold, we have that the notion $\bar\partial$-harmonic is equivalent to ∂-harmonic. Thus $\partial\theta^{0,1} = 0$ follows. □

As a corollary to the $\partial\bar\partial$-lemma, if ω is a Kähler form on a closed Kähler manifold, by definition $\omega \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$ and moveover we have that any Kähler form cohomologous to ω will be given by $\omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi$ for some function $\varphi \in C^\infty(M, \mathbb{R})$ such that $\omega_\varphi > 0$. We denote all the Kähler forms cohomologous to ω by $\mathcal{H}_\omega$. By the discussion above, we know that equivalently,

$$\mathcal{H}_\omega = \{\omega_\varphi | \varphi \in C^\infty(M, \mathbb{R}) \text{ such that } \omega_\varphi > 0\}. \tag{21}$$

Such functions φ's are called Kähler potentials and φ is uniquely determined by Kähler form the ω_φ up to constants. Very often, we will drop the subscript "ω" in $\mathcal{H}_\omega$, and call it the space of Kähler metrics whenever it doesn't cause any confusions.

1.3 Calabi conjecture and Kähler-Einstein problem

Let us go back to the question at the end of section 1.1. In 1950's, E. Calabi[4] first raised the question that on a closed Kähler manifold M if any representative in the cohomology class $2\pi c_1(M)$ could be realized as the Ricci form of some Kähler metric, which is the well-known Calabi conjecture. In fact, it suffices to look for the desired Kähler metric within $\mathcal{H}_\omega$, and the calabi conjecture the can be reformulated as given $\alpha \in 2\pi c_1(M)$, there exists a function $\varphi \in C^\infty(M, \mathbb{R})$ such that ω_φ has Ricci form α.

Suppose ω is a Kähler metric on M and thus its Ricci form is given by

$$\rho_\omega = -\sqrt{-1}\partial\bar{\partial}\log\omega^n \in 2\pi c_1(M). \tag{22}$$

Given any $\alpha \in 2\pi c_1(M)$, by the $\partial\bar{\partial}$-lemma, there exists a function $F \in C^\infty(M, \mathbb{R})$ such that $\alpha = \rho_\omega - \sqrt{-1}\partial\bar{\partial}F$. Suppose $\varphi \in C^\infty(M, \mathbb{R})$ such that $\omega_\varphi > 0$ and its correpsonding Ricci form is

$$\rho_\varphi = -\sqrt{-1}\partial\bar{\partial}\log\omega_\varphi^n. \tag{23}$$

Then $\rho_\varphi = \alpha$ is equivalent to

$$\partial\bar{\partial}(\log\frac{\omega_\varphi^n}{\omega^n} - F) = 0, \tag{24}$$

which is equivalent to

$$\log\frac{\omega_\varphi^n}{\omega^n} - F \equiv C, \tag{25}$$

if M is closed. Taking exponential on both hand sides, we have that

$$\det(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}) = e^{F+C}\det(g_{i\bar{j}}), \tag{26}$$

where constant C is obtained by integrating both hand sides on M. Without loss of generality, we can always assume that $F \in C^\infty(M, \mathbb{R})$ satisfies $\int_M e^F\omega^n = \mathrm{Vol}(M)$. Thus, the Calabi conjecture is equivalent to the solvability of the following equation

$$\det(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}) = e^F\det(g_{i\bar{j}}). \tag{27}$$

This equation is called the complex Monge-Ampère equation.

In 1976, S.T. Yau([49]) solved the Calabi conjecture using the continuity method to solve the complex Monge-Ampère. The continuity path he worked on is

$$\det(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}) = \frac{\mathrm{Vol}(M)e^{tF}}{\int_M e^{tF}\omega^n}\det(g_{i\bar{j}}), \tag{28}$$

for parameter $t \in [0,1]$. Set

$$I = \{t \in [0,1] | \text{ Equation (28) with parameter } t \text{ has a smooth solution.}\}. \tag{29}$$

Clearly I is not empty since $0 \in I$. Then one wants to show that $1 \in I$ by proving that I is both open and closed. The openness part is relatively easy. The key ingredient in proving the closedness is the C^2-estimate, which asserts a uniform bound on $(n + \Delta\varphi)$ for all smooth functions φ that solve equation (28) with parameter $t \in [0,1]$. This estimate reflects a unique feature of the Monge-Ampère equations on compact manifolds comparing to the fact that interior C^2-estimate of Monge-Ampère equation on bounded domains is simply not true, see examples by Pogorelov([39]). Once given the C^2-estimate, Yau further showed that $\|\varphi\|_{C^3(M)}$ is uniformly bounded for all smooth functions φ that solve equation (28) with parameter $t \in [0,1]$. Soon after him, by instead using the apriori interior $C^{2,\alpha}$ estimate of Monge-Ampère equation on domains known as the Evans-Krylov theorem([26], [31]), one can skip the C^3-estimate which simplifies his original proof. All the higher order estimates could be obtained via elliptic theory of linear equations by taking differentiation on equation (28). It is important to point out at last that the bound on $(n + \Delta\varphi)$ in fact depends on $\sup_M |\varphi|$, which can be obtained in this case.

In particular, by the assertion of the Calabi conjecture when $c_1(M) = 0$, there exists a Ricci-flat metric on M, called the Calabi-Yau metric.

Definition 1.6. Suppose M is a closed Kähler manifold and $c_1(M)$ denotes its first chern class. We say $c_1(M) > 0$(respectively < 0) if there exists an $\alpha \in c_1(M)$ such that as a $(1,1)$-form on M, $\alpha > 0$(respectively < 0).

When $c_1(M) > 0$, we consider Kähler metrics in the $2\pi c_1(M)$ where the Ricci forms also lie. It is natural to ask among all Kähler metrics in $2\pi c_1(M)$ if there exists one such that its Ricci form equals to itself, namely $\omega \in 2\pi c_1(M)$ such that

$$\rho_\omega = \omega. \tag{30}$$

In the $c_1(M) < 0$ case, similarly $\omega \in -2\pi c_1(M)$ such that

$$\rho_\omega = -\omega. \tag{31}$$

A metric is called Kähler-Einstein(KE) if it is Kähler and its Ricci form is proportional to itself. By rescaling, all Kähler-Einstein metrics are essentially given by one of the following equations: $\rho_\omega = 0$, $\rho_\omega = -\omega$ and $\rho_\omega = \omega$, corresponding to $c_1(M) = 0, < 0, > 0$ respectively.

Existence of KE metrics in the case $c_1(M) = 0$ follows directly from the statement of Calabi conjecture. While for the $c_1(M) < 0$ case, in order to find KE metrics, equivalently one wants to solve

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{F_\omega + \varphi} \det(g_{i\bar{j}}) \tag{32}$$

where $\sqrt{-1}\partial\bar{\partial}F_\omega = \rho_\omega + \omega$ for $\omega \in -2\pi c_1(M)$. Yau's C^2-estimate is still valid for equation above and moreover one can obtain bounds on $\sup_M |\varphi|$ easily using the maximum principle. Thus, when $c_1(M) < 0$, there always exists KE metrics in $-2\pi c_1(M)$. This result is independently proved by Yau([49]) and Aubin([1]) in late 1970s. When $c_1(M) > 0$, the existence of KE metrics is much more subtle. Equivalently, one wants to solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{h_\omega - \varphi} \det(g_{i\bar{j}}) \tag{33}$$

where $\sqrt{-1}\partial\bar{\partial}h_\omega = \rho_\omega - \omega$ for $\omega \in 2\pi c_1(M)$. While Yau's C^2-estimate is still valid, the C^0-estimate on φ doesn't come for free any more due to the negative sign in front of φ.

In fact, there are various obstructions to the existence of KE metrics when $c_1(M) > 0$. Denote the group of all biholomorphisms on complex manifold (M, J) by $\mathrm{Aut}(M)$. In 1957, Matsushima([36]) discovered that if there exists KE metric in $2\pi c_1(M) > 0$, then $\mathrm{Aut}(M)$ is reductive. It follows from this result that there are Kähler manifolds with $c_1(M) > 0$ which don't admit KE metrics. For example, if M is $\mathbb{CP}^2$ blow up one or two points, then one can compute $\mathrm{Aut}(M)$ is not reductive and thus M doesn't admit KE metrics.

In 1983, Futaki([27]) introduced an another obstruction known as the Futaki invariant. Choose $\omega \in 2\pi c_1(M) > 0$ and set $h_\omega \in C^\infty(M, \mathbb{R})$ such that $\sqrt{-1}\partial\bar{\partial}h_\omega = \rho_\omega - \omega$. Then the Futaki invariant is defined as $f_M : \eta(M) \to \mathbb{C}$,

$$f_M(X) = \int_M X(h_\omega)\omega^n, \tag{34}$$

where $\eta(M)$ is the Lie algebra of $\mathrm{Aut}(M)$ that consists of all holomorphic vector fields on M. Futaki showed that f_M is independent of choice of $\omega \in 2\pi c_1(M)$. Therefore, if M with $c_1(M) > 0$ admits KE metrics then necessarily $f_M \equiv 0$. In [27], Futaki also constructed an example of 3-dimensional manifold M with $c_1(M) > 0$ and $\mathrm{Aut}(M)$ reductive but $f_M \neq 0$. Hence such an M doesn't admit KE metrics.

Note that the obstructions above both come from the holomorphic vector fields. In 1990, Tian([44]) has proved that on a complex surface M with $c_1(M) > 0$ it admits KE metrics if and only if $\mathrm{Aut}(M)$ is reducitive. In particular, when $\mathrm{Aut}(M) = \{1\}$, then Kähler surface M with $c_1(M) > 0$ automatically admits KE metric. For a while, people believe that this statement is also true in complex dimension $n \geq 3$. However, this forklore conjecture was disproved by Ding-Tian([25]) on Kähler orbifolds in 1992 where they constructed new obstructions by defining the generalized Futaki invariant on almost Fano varieties(possibly singular). Later, such obstructions are refined and used to define the K-stability condition([45], [19]). An explicit counterexample could be found in [45], which is first contructed in [37].

It is proved by Donaldson-Uhlenbeck-Yau([46],[17]) that the existence of Hermitian-Yang-Mills connection is equivalent to the stability of underlying holomorphic bundle. Inspired by this result, in late 1980's, Yau proposed that the existence of KE metrics should correspond to certain stability of the underlying manifold in the geometric invariant theory. The stability condition is later defined more precicely by Tian([45]) and Donaldson([19]) as the K-stability, which naturally leads to the following famous conjecture.

Conjecture 1.7 (Yau-Tian-Donaldson, [19])**.** *A Fano manifold M admits KE metrics if and only if it is K-stable.*

This conjecture has only recently settled by Chen-Donaldson-Sun([11],[12] and [13]) in 2013.

1.4 Extremal and cscK metrics

In the 1980's, E. Calabi([5]) initiated a broader program aiming to find "the best" canonical metric in each $\mathcal{H}_\omega$. To this end, he considered the L^2-norm of the scalar curvature as a functional on metrics called the Calabi functional, namely we define for any $\varphi \in \mathcal{H}_\omega$,

$$Ca(\varphi) = \int_M R_\varphi^2 \omega_\varphi^n, \tag{35}$$

where $R_\varphi := g_\varphi^{i\bar{j}} \mathrm{Ric}_{\varphi, i\bar{j}} = -g_\varphi^{i\bar{j}} \frac{\partial^2 \log \det g}{\partial z_i \partial \bar{z}_j}$ denotes the scalar curvature of metric ω_φ. Calabi proposed to seek critical points of the Calabi functional in each $\mathcal{H}_\omega$. These metrics are called the extremal metric. By straightforward computation, we have that

$$\delta_u Ca(\varphi) = \int_M \big(2(\delta_u R_\varphi) R_\varphi \omega_\varphi^n + R_\varphi^2 (\delta_u \omega_\varphi^n)\big), \tag{36}$$

where

$$\delta_u R_\varphi = -\Delta_\varphi^2 u - g_\varphi^{i\bar{q}} g_\varphi^{p\bar{j}} (\mathrm{Ric}_\varphi)_{i\bar{j}} u_{,p\bar{q}}, \delta_u \omega_\varphi^n = (\Delta_\varphi u)\omega_\varphi^n. \tag{37}$$

We have the identity by interchanging the order of differentiations

$$u_{,\alpha\bar{p}\beta} = u_{,\alpha\beta\bar{p}} - \sum_l R^l_{\beta\bar{p}\alpha} u_{,l}, \tag{38}$$

where the derivatives are taken in the sense of covariant derivates with respect to the Kähler metric ω_φ and $R^l_{\beta\bar{p}\alpha}$ denotes the Riemannian curvature tensor with respect to metric ω_φ. Thus

$$\begin{aligned}\Delta_\varphi^2 u = g_\varphi^{\alpha\bar{p}} g_\varphi^{\beta\bar{q}} u_{,\alpha\bar{p}\beta\bar{q}} &= g_\varphi^{\alpha\bar{p}} g_\varphi^{\beta\bar{q}} u_{,\alpha\beta\bar{p}\bar{q}} - g_\varphi^{\beta\bar{q}} (\mathrm{Ric}_\varphi)^l_\beta u_{,l\bar{q}} - g_\varphi^{\beta\bar{q}} (\mathrm{Ric}_\varphi)^l_{\beta,\bar{q}} u_{,l}, \\ &= g_\varphi^{\alpha\bar{p}} g_\varphi^{\beta\bar{q}} u_{,\alpha\beta\bar{p}\bar{q}} - g_\varphi^{\beta\bar{q}} (\mathrm{Ric}_\varphi)^l_\beta u_{,l\bar{q}} - R_\varphi^{,l} u_{,l}.\end{aligned} \tag{39}$$

Therefore, we have

$$\begin{aligned}\delta_u Ca(\varphi) &= \int_M \big(2(-g_\varphi^{\alpha\bar{p}} g_\varphi^{\beta\bar{q}} u_{,\alpha\beta\bar{p}\bar{q}} + R_\varphi^{,l} u_{,l}) R_\varphi + R_\varphi^2 (\Delta_\varphi u)\big)\omega_\varphi^n, \\ &= -2\int_M (g_\varphi^{\alpha\bar{p}} g_\varphi^{\beta\bar{q}} R_{\varphi,\bar{p}\bar{q}\alpha\beta}) u \omega_\varphi^n.\end{aligned} \tag{40}$$

Then extremal metrics as the critical points of Calabi functional equivalently are given by the following equation on $\varphi \in \mathcal{H}_\omega$

$$g_\varphi^{\alpha\bar{p}} g_\varphi^{\beta\bar{q}} R_{\varphi,\bar{p}\bar{q}\alpha\beta} = 0. \tag{41}$$

Pairing with R_φ and integrating by parts, we get equivalently φ satisfies that

$$R_{\varphi,\bar{p}\bar{q}} = 0, \tag{42}$$

for any $p, q \in \{1, 2, \cdots, n\}$. We define the $(1,0)$-gradient of R_φ as the $(1,0)$-vector field on M given by $\nabla_\varphi^{1,0} R_\varphi := g_\varphi^{i\bar{j}} \frac{\partial R_\varphi}{\partial \bar{z}_j} \frac{\partial}{\partial z_i}$ and thus $\varphi \in \mathcal{H}_\omega$ being extremal is equivalent to $\nabla_\varphi^{1,0} R_\varphi$ being holomorphic. In particular, when $\mathrm{Aut}(M) = \{1\}$, namely the only global holomorphic vector field on M is given by $X \equiv 0$, φ being extremal is equivalent to $R_\varphi \equiv \underline{R}$ where $\underline{R}$ is a topological constant given by integral average of the scalar curvature. We call Kähler metric ω_φ the constant scalar curvature Kähler(cscK) metric if $R_\varphi \equiv \underline{R}$.

The existence problem of cscK/extremal metrics can be viewed as the continuation of the KE problem for a general Kähler class $\mathcal{H}_\omega$ in the following sense. If we work in cohomology class $2\pi c_1(M) > 0$, then KE $\iff$ cscK. It is easy to see that KE $\Rightarrow$ cscK. The other direction cscK $\Rightarrow$ KE follows from the fact that ρ_ω is harmonic w.r.t. metric $\omega \iff \omega$ is cscK and also the uniqueness of harmonic form in $2\pi c_1(M)$ by Hodge theory. (Note that ω is also harmonic w.r.t metric ω.)

Similar to the KE problem, there are obstructions to the existence of cscK/extremal metrics as well. In [6], Calabi proved the obstruction on the structure of Aut(M) if M admits cscK/extremal metrics as a generalization to the Matsushima's result on KE problems. Moreover, Futaki invariant could be generalized in the cscK/extermal setting as $f_M : \eta(M) \to \mathbb{C}$,

$$f_M(X) = \int_M X(\theta_\omega)\omega^n, \tag{43}$$

where θ_ω is the solution to equation $\Delta_\omega \theta_\omega = R_\omega - \underline{R}$. Again, its definition only depends on the cohomology class $[\omega] \in H^{1,1}(M, \mathbb{R})$ and thus if there exists cscK in $\mathcal{H}_\omega$ then necessarily $f_M \equiv 0$.

In [19], Donaldson presented a precise algebro-geometric condition as an obstruction to the existence of cscK metrics, called the K-stability. Tian([45]) first gave an equivalent definition in the particular case of Fano varieties. Naturally we have the conjecture about the existence of cscK metrics.

Conjecture 1.8 (Yau-Tian-Donaldson, [19])**.** *A smooth polarized manifold (V, L) admits a cscK metric in the class $c_1(L)$ if and only if it is K-stable.*

Besides the Fano case, Donaldson himself proved this conjecture on toric surfaces([19], [21], [22] and [23]). However, in general, the existence problem of cscK metrics is very difficult as explained in an expository article by Donaldson([20]).

2 Continuity path via twisted cscK metrics

Recently, Chen proposed a continuity path in [10] aiming to attack the existence problem of cscK metrics via a direct PDE approach. He considered the path connecting the solution of J-equation to the cscK metric. More precisely, he proposed to solve the equation

$$t(R_\varphi - \underline{R}) - (1-t)(\mathrm{tr}_\varphi \chi - \underline{\chi}) = 0, \tag{44}$$

with parameter $t \in [0,1]$ using continuity method, where $\underline{R}$ is a topological constant given by $\underline{R} = \frac{1}{\int_M \omega^n} \int_M R_\varphi \omega_\varphi^n = [c_1(M)][\omega]^{[n-1]}/[\omega]^{[n]}$. Following the terminology in J. Stoppa([41]), a Kähler metric ω_φ satisfying (44) is called twisted cscK metric.

2.1 The J-equation

On a closed Kähler manifold M with Kähler form ω, given a closed positive $(1,1)$-form χ on M, we define a 1-form on the space of Kähler metrics $\mathcal{H}_\omega$ as

$$\delta_u J_\chi(\varphi) = \int_M u(\chi \wedge \frac{\omega_\varphi^{n-1}}{(n-1)!} - \underline{\chi}\frac{\omega_\varphi^n}{n!}) \tag{45}$$

for $u \in T_\varphi \mathcal{H}_\omega = C^\infty(M)$ where $\underline{\chi}$ is a topological constant given by $\underline{\chi} = \frac{1}{\int_M \omega^n} \int_M \mathrm{tr}_\varphi \chi \omega_\varphi^n = [\chi] \cdot [\omega]^{[n-1]}/[\omega]^{[n]}$. Note that δJ_χ is a closed 1-form on $\mathcal{H}_\omega$ since

$$\begin{aligned} \delta_v \delta_u J_\chi(\varphi) &= \int_M u\sqrt{-1}\partial\bar{\partial} v \wedge (\chi \wedge \frac{\omega_\varphi^{n-2}}{(n-2)!} - \underline{\chi}\frac{\omega_\varphi^{n-1}}{(n-1!)}) \\ &= -\int_M \sqrt{-1}\partial u \wedge \bar{\partial} v \wedge (\chi \wedge \frac{\omega_\varphi^{n-2}}{(n-2)!} - \underline{\chi}\frac{\omega_\varphi^{n-1}}{(n-1)!}), \end{aligned} \tag{46}$$

is symmetric in $u, v \in T_\varphi \mathcal{H}_\omega$. Thus by integration on simply connected space $\mathcal{H}_\omega$, we can define functional J_χ on $\mathcal{H}_\omega$ such that its derivative is given by the 1-form δJ_χ. The functional J_χ on $\mathcal{H}_\omega$ is called J-functional, its critical points by definition satisfies equation

$$\mathrm{tr}_\varphi \chi = \underline{\chi} \iff \chi \wedge \frac{\omega_\varphi^{n-1}}{(n-1)!} = \underline{\chi}\frac{\omega_\varphi^n}{n!} \tag{47}$$

which is the so called J-equation. J-equation was first defined by Donaldson [24] in the setting of moment maps and by Chen[8] in his formula of Mabuchi functional.

Donaldson([24]) first observed that if there exists a smooth solution to J-equation in $\mathcal{H}_\omega$, then necessarily

$$[\underline{\chi}\omega - \chi] > 0.$$

Locally, one can choose holomorphic coordinate such that $\chi_{i\bar{j}} = \delta_{i\bar{j}}$ and $\omega_{i\bar{j}} = \delta_{i\bar{j}}\lambda_i$ at point p. Thus at p the J-equation $\operatorname{tr}_\omega \chi = \underline{\chi}$ can be written as

$$\sum \frac{1}{\lambda_i} = \underline{\chi}. \tag{48}$$

Then necessarily we have $\underline{\chi}\lambda_i > 1$ which implies $\underline{\chi}\omega > \chi$. Donaldson conjectured that condition $[\underline{\chi}\omega - \chi] > 0$ is also sufficient to the existence of solution to the J-equation. This conjecture was confirmed by Chen([8]) in complex dimension 2 when J-equation is equivalent to a complex Monge-Ampère equation that could be solved by Yau's method.

In higher dimension, Weinkove([47], [48]) found a sufficient condition to the existence of solution to J-equation. His condition is

$$[\underline{\chi}\omega - (n-1)\chi] > 0. \tag{49}$$

where n is the complex dimension. In particular, when $n = 2$, it solves Donaldson's conjecture. In [42], Weinkove-Song has found a necessary and sufficient condition: There exists a metric $\omega' \in [\omega]$ such that

$$(\underline{\chi}\omega' - (n-1)\chi) \wedge \omega'^{n-2} > 0. \tag{50}$$

Until now, Donaldson's original conjecture about J-equation is still open in dimension $n \geq 3$.

2.2 Twisted Mabuchi energy and its convexity

As described above solution to J-equation is the critical point of J-functional, while the cscK metric is critical point of the so called Mabuchi functional or K-energy. In 1986, Mabuchi([34]) introduced the Mabuchi functional which has cscK metrics as its critical point. It is defined using its derivative, namely we define a 1-form on $\mathcal{H}_\omega$ as

$$\delta_u \mathcal{M}(\varphi) = -\int_M (R_\varphi - \underline{R}) u \omega_\varphi^n \tag{51}$$

for $u \in T_\varphi\mathcal{H}_\omega$. In fact, $\delta\mathcal{M}$ is a closed on $\mathcal{H}_\omega$ since by direct computation,

$$
\begin{aligned}
\delta_v\delta_u\mathcal{M}(\varphi) &= \int_M \big((\delta_v R_\varphi)u - (R_\varphi - \underline{R})u\Delta_\varphi v\big)\omega_\varphi^n \\
&= \int_M (\Delta_\varphi v)(\Delta_\varphi u)\omega_\varphi^n + (v_{,\bar{\alpha}\beta}\mathrm{Ric}_{\varphi,\alpha\bar{\beta}} - v_{,\alpha\bar{\alpha}}\mathrm{Ric}_{\varphi,\beta\bar{\beta}})u\omega_\varphi^n + \underline{R}u\Delta_\varphi v\omega_\varphi^n \\
&= \int_M (\Delta_\varphi v)(\Delta_\varphi u)\omega_\varphi^n - \sqrt{-1}\partial u \wedge \bar{\partial} v \wedge (\mathrm{Ric}_\varphi - \underline{R}\omega_\varphi) \wedge \omega_\varphi^{n-2},
\end{aligned}
\tag{52}
$$

is symmetric in $u, v \in T_\varphi\mathcal{H}_\omega$. Thus by integrating, we can define the Mabuchi functional $\mathcal{M}$ on $\mathcal{H}_\omega$ where by definition its critical points are cscK metrics.

Mabuchi([35]) in 1987 introduced a Riemannian metric on the infinite dimensional space $\mathcal{H}_\omega$. At point $\varphi \in \mathcal{H}_\omega$, the inner product on the tangent space $T_\varphi\mathcal{H}_\omega$ is given by

$$\langle u, v\rangle = \int_M uv\omega_\varphi^n \tag{53}$$

for $u, v \in C^\infty(M) \cong T_\varphi\mathcal{H}_\omega$. Under this Riemannian metric, it becomes an infinite dimensional symmetric space of nonpositive curvature. Apparently unaware of Mabuchi's work, Semmes-Donaldson [40] re-discover this same metric again from different angles. For a curve $\varphi(t) \in \mathcal{H}_\omega$ $(0 \le t \le 1)$, we define its length by

$$L(\varphi) = \int_0^1 \sqrt{\int_M (\frac{\partial\varphi}{\partial t})^2 \omega_{\varphi(t)}^n}\mathrm{d}t. \tag{54}$$

Then geodesic equation is given by

$$\ddot{\varphi}(t) - g_{\varphi(t)}^{i\bar{j}}\dot{\varphi}(t)_{,i}\dot{\varphi}(t)_{,\bar{j}} = 0 \tag{55}$$

As shown by Mabuchi([34], [35]) the functional $\mathcal{M}$ is convex along smooth geodesics $\varphi(t)$ in $\mathcal{H}_\omega$. In fact one can compute directly that

$$
\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{M}(\varphi(t)) \\
&= \int_M |\dot{\varphi}(t)_{,\alpha\beta}|^2_{\varphi(t)}\omega_{\varphi(t)}^n - \int_M (R_{\varphi(t)} - \underline{R})(\ddot{\varphi}(t) - g_{\varphi(t)}^{i\bar{j}}\dot{\varphi}(t)_{,i}\dot{\varphi}(t)_{,\bar{j}})\omega_{\varphi(t)}^n.
\end{aligned}
\tag{56}
$$

Unfortunately, given two end points $\varphi_1, \varphi_2 \in \mathcal{H}_\omega$, there may be no smooth geodesic $\varphi(t)$ connecting them. In 2000, Chen([9]) showed that there exists a $C^{1,1}$ geodesic(may not be smooth) connecting any given end points

$\varphi_1, \varphi_2 \in \mathcal{H}_\omega$. Moreover, even if the original definition of the Mabuchi functional requires that ω_φ be positive and C^2-smooth (and in particular that φ be C^4-smooth) Chen went on to show([8]) that the Mabuchi functional admits an explicit formula which is well-defined along any $C^{1,1}$-geodesic. It has been conjectured by Chen ([8]) that $\mathcal{M}$ is (weakly) convex along $C^{1,1}$ geodesics. This conjecture has been proved in 2015 by Berndtsson-Berman([2]) and Chen-Li-Păun ([14]). It was shown in [7] by Chen that J-functional is strictly convex along $C^{1,1}$ geodesics.

Let's go back to the continuity path (44). By our discussion above, if $\varphi \in \mathcal{H}_\omega$ solves (44) for $t \in [0, 1]$ then φ is the critical point of functional $t\mathcal{M} + (1 - t)J_\chi$ which is strictly convex along $C^{1,1}$ geodesics. We call $t\mathcal{M} + (1 - t)J_\chi$ the twisted Mabuchi energy. Together with the existence of $C^{1,1}$ geodesic connecting any given $\varphi_1, \varphi_2 \in \mathcal{H}_\omega$ by Chen([9]), one can see that twisted cscK metric($t < 1$) is unique if it exists.

2.3 Uniqueness of cscK/extremal metrics

The uniqueness problem of the cscK/extremal metrics has a long history going back to E. Calabi. We refer to [3], [9], [16], [18], [33] and [2] for the important works generated by this question. In [15], joint with Chen and Păun we gave a new proof of this classical result by studying the deformation of the cscK/extremal metrics. The new deformation results that we will present below are based on the bifurcation technique first introduced by S. Bando and T. Mabuchi in their celebrated work([3]) concerning the uniqueness of Kähler-Einstein metrics modulo holomorphic automorphisms.

Before introducing our work, let us briefly review Bando-Mabuchi's work first. When $c_1(M) > 0$, $\mathrm{Aut}(M)$ is likely nontrivial. Thus in this case, we shouldn't expect that the KE metric is genuinely unique but rather unique up to holomorphic automorphisms. In fact, if ω is KE, then for any $\sigma \in \mathrm{Aut}(M)$, $\sigma^*\omega \in [\omega]$ and it is KE as well. The main idea in Bando-Mabuchi's paper is to solve the Aubin-Yau path backwards from $t = 1$, namely they want to deform a KE metric at $t = 1$ via

$$\rho_{\omega_\varphi} = t\omega_\varphi + (1 - t)\omega \tag{57}$$

backwardly to $t = 0$. If ω_φ varies within the orbit of KE metrics by the group action of $\mathrm{Aut}(M)$, then ω_φ will still be KE. Thus it implies that the linearization of the path equation at $t = 1$ would have nontrivial kernel which locally parametrizes the orbit of KE metrics under $\mathrm{Aut}(M)$ actions. In this case, one can't apply the implicit function theory directly.

To address this difficulty coming from the nontrivial kernel, Bando-Mabuchi introduced a clever trick, called the bifurcation technique based on the geometry of the orbit of KE metrics. They are able to show that on each KE orbit, there exists a unique KE metric from which one can solve the Aubin-Yau path backwardly.

As an analog to their result in the cscK case. we have

Theorem 2.1 (Chen-Păun-Zeng, [15]). *Given an n-dimensional closed Kähler manifold (M,ω) that admits a cscK metric $\omega_{\varphi_0} \in [\omega]$, there exist a constant $\delta > 0$ and a smooth function $\phi : (1-\delta, 1] \times M \to \mathbb{R}$ such that $\varphi_t = \phi(t, \cdot) \in \mathcal{H}^\infty$ verifies the equation*

$$t(R_{\varphi_t} - \underline{R}) - (1-t)(tr_{\varphi_t}\omega - n) = 0. \tag{58}$$

Moreover, there exists a holomorphic automorphism f of M such that $\omega_{\varphi_1} = f^*\omega_{\varphi_0}$.

We also proved an analog in the extremal case as below.

Theorem 2.2 (Chen-Păun-Zeng, [15]). *Given an n-dimensional closed Kähler manifold (M,ω) that admits an extremal metric $\omega_{\varphi_0} \in [\omega]$, there exist a constant $\delta > 0$ and a smooth function $\phi : (1-\delta, 1] \times M \to \mathbb{R}$ such that*

$$\nabla^{1,0}_{\varphi_t}\big(tR_{\varphi_t} - (1-t)tr_{\varphi_t}\omega\big) \tag{59}$$

is a holomorphic vector field on M, where $\varphi_t = \phi(t, \cdot) \in \mathcal{H}^\infty$. Moreover, there exists a holomorphic automorphism f of M such that $\omega_{\varphi_1} = f^*\omega_{\varphi_0}$.

As a consequence of Theorem 2.2, we gave a new proof of the following statement.

Corollary 2.3 ([16], [33], [2] and [15]). *Given an n-dimensional closed Kähler manifold (M,ω) that admits two extremal metrics $\{\omega_j\}_{j=1,2} \subset [\omega]$, there exists a holomorphic automorphism f of M such that $f^*\omega_1 = \omega_2$.*

Our new proof of Corollary 2.3 consists of two main ingredients: the deformation of extremal metrics(Theorem 2.2) and the convexity of twisted Mabuchi funtional introduced above. For simplicity, we will briefly explain how to derive Corollary 2.3 in the cscK case. We add a small strictly convex perturbation J_ω to the Mabuchi functional such that the perturbed functional $t\mathcal{M} + (1-t)J_\omega$ is strictly convex and then its critical point is unique. Thus, if we can deform a given cscK metric via a smooth family of the critical points of the perturbed functionals, then such deformation must be unique. Fortunately, by our Theorem 2.1, one can always find such unique deformation by apriori applying a holomorphic transformation to the given cscK metric. The extremal case follows a similar scheme but with more delicate settings. In [2], they also proved Corollary 2.3, but instead of deriving the deformation theorem above they alternatively deformed the cscK/extremal metric via "approximately critical points".

2.4 Openness results

Set

$$I_\chi = \{t \in [0,1] | \text{ Equation (44) with parameter } t \text{ has a smooth solution.}\}$$

In [10], X. Chen proved the openness of I_χ for $t \in (0,1) \cap I_\chi$ by directly applying the implicit function theorem. However, the openness at $t = 0$ is quite different from the case at $t \in (0,1)$, because equation (44) is a fourth order PDE for all positive t while it reduces to a second order PDE at $t = 0$. In this dissertation, we will first present the proof of Openness of I_χ for $0 < t < 1$ from [10] and then mainly prove the openness of I_χ at $t = 0$ assuming that $0 \in I_\chi$.

Theorem 2.4 ([10])**.** *Given an n-dimensional closed Kähler manifold (M, ω), if $t_0 \in I_\chi \cap (0,1)$ then there exists a constant $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset I_\chi$.*

Theorem 2.5 (Main Theorem)**.** *Given an n-dimensional closed Kähler manifold (M, ω), if $0 \in I_\chi$ then there exists a constant $\delta > 0$ such that $[0, \delta) \subset I_\chi$.*

Similar results to Theorem 2.5 were first proved in [50] for $\chi \in [\omega]$ in which case one automatically has that $0 \in I_\chi$. Later, Hashimoto[30] proved that I_χ is open at $t = 0$ for all smooth $(1,1)$ form $\chi > 0$ with $0 \in I_\chi$. In this dissertation, we will give a proof of the openness of I_χ at $t = 0$ for all all smooth $(1,1)$ form $\chi > 0$ with $0 \in I_\chi$ using the ideas developed in [50].

3 Analytic preparations

We fix a background smooth Kähler metric g on M. For $k \in \mathbb{N}$ and $\alpha \in (0,1)$, we define the function space $C^{k,\alpha}(M)$ to be all functions on M which are continuously differentiable up to kth order with the kth derivatives α-hölder continuous on M. In each local chart $\psi : U \subset M \to \mathbb{R}^{2n}$, we can define norm

$$\|u\|_{C^{k,\alpha}(U,\psi)} = \sum_{|\beta|=0}^{k} \|D^\beta(u\circ\psi^{-1})\|_{L^\infty(\psi(U))} + \sup_{x\neq y\in\psi(U),|\beta|=k} \frac{|D^\beta(u\circ\psi^{-1})(x) - D^\beta(u\circ\psi^{-1})(y)|}{|x-y|^\alpha}.$$

Then by a choice of finite open covers $M = \bigcup_{i=1}^N U_i$ together with local coordinates $\psi_i : U_i \to \mathbb{R}^{2n}$, we can introduce norm $\|\cdot\|_{C^{k,\alpha}(M)} = \sup_i \|\cdot\|_{C^{k,\alpha}(U_i,\psi_i)}$ on $C^{k,\alpha}(M)$. Note that different choices of finite covers and local coordinates may result in different but equivalent norms. In this paper, we fix a finite cover $M = \bigcup_{i=1}^N U_i$ with local coordinates $\psi_i : U_i \to \mathbb{R}^{2n}$.

We introduce the Schauder estimate of Laplacian operator in the following lemma.

Lemma 3.1. *Given a smooth Riemannian metric g on compact manifold M, if $u \in C^\infty(M)$ with $\int_M u\mathrm{d}V_g = 0$ satisfies $\Delta_g u = f$, then for any integer $k \geq 0$ we have*

$$\|u\|_{C^{k+2,\alpha}(M)} \leq C\|f\|_{C^{k,\alpha}(M)}. \tag{60}$$

Proof. It suffices to show the case when $k = 0$. One can obtain higher order estimate via taking differentiations on equation $\Delta_g u = f$.

Using the standard interior schauder estimate on domains, one can obtain that

$$\|u\|_{C^{2,\alpha}(M)} \leq C(\|f\|_{C^\alpha(M)} + \|u\|_{L^2(M)}). \tag{61}$$

To estimate $\|u\|_{L^2(M)}$, we consider

$$\int_M fu\mathrm{d}V_g = \int_M (\Delta_g u)u\mathrm{d}V_g = -\int_M |\nabla u|_g^2 \mathrm{d}V_g \tag{62}$$

By Porncare inequality, we have

$$\int_M u^2\mathrm{d}V_g \leq C\int_M |\nabla u|_g^2 \mathrm{d}V_g \tag{63}$$

Thus, we have estimate

$$\|u\|_{L^2(M)} \leq C\|f\|_{L^2(M)} \leq C\|f\|_{C^\alpha(M)} \tag{64}$$

Then it ends the proof of the lemma. □

Next, we introduce an interpolation equality that we will use in later sections.

Lemma 3.2. *Suppose $u \in C^\infty(M)$ and $\alpha \in (0,1)$. Then for any $\epsilon \ll 1$, we have*

$$\|u\|_{C^{2,\alpha}(M)} \leq \epsilon\|u\|_{C^{4,\alpha}(M)} + C\epsilon^{-\gamma(n,\alpha)}\|u\|_{L^2(M)}, \tag{65}$$

where $\gamma(n,\alpha) = -\frac{2+\alpha}{2} - \frac{4+\alpha}{2}\frac{2n+1}{6}$.

Proof. By Corollary 1.2.19 in [32], we have that

$$\begin{aligned}\|u\|_{C^{2,\alpha}(M)} &\leq C(\|u\|_{L^\infty(M)})^{\frac{2}{4+\alpha}}(\|u\|_{C^{4,\alpha}(M)})^{\frac{2+\alpha}{4+\alpha}}, \\ &\leq \frac{1}{2}\epsilon\|u\|_{C^{4,\alpha}(M)} + C\epsilon^{-\frac{2+\alpha}{2}}\|u\|_{L^\infty(M)}.\end{aligned} \tag{66}$$

For any $p \in (2n, \infty)$, by Sobolev embedding and Theorem 7.28 in [28], we have for $\eta \ll 1$

$$\begin{aligned}\|u\|_{L^\infty(M)} &\leq C_p\|u\|_{W^{1,p}(M)}, \\ &\leq \eta\|u\|_{W^{4,p}(M)} + C_p\eta^{-\frac{1}{3}}\|u\|_{L^p(M)}, \\ &\leq \eta\|u\|_{W^{4,p}(M)} + C_p\eta^{-\frac{1}{3}}\|u\|_{L^2(M)}^{\frac{2}{p}}\|u\|_{L^\infty(M)}^{1-\frac{2}{p}}, \\ &\leq C\eta\|u\|_{C^{4,\alpha}(M)} + \frac{1}{2}\|u\|_{L^\infty(M)} + C_p\eta^{-\frac{p}{6}}\|u\|_{L^2(M)}.\end{aligned} \tag{67}$$

Thus,

$$\|u\|_{L^\infty(M)} \leq C\eta\|u\|_{C^{4,\alpha}(M)} + C_p\eta^{-\frac{p}{6}}\|u\|_{L^2(M)}. \tag{68}$$

Combine inequalities (66) and (68), we get

$$\|u\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}\epsilon\|u\|_{C^{4,\alpha}(M)} + C\epsilon^{-\frac{2+\alpha}{2}}\eta\|u\|_{C^{4,\alpha}(M)} + C_p\epsilon^{-\frac{2+\alpha}{2}}\eta^{-\frac{p}{6}}\|u\|_{L^2(M)}. \tag{69}$$

We can choose $\eta \ll 1$ sufficiently small such that $C\epsilon^{-\frac{2+\alpha}{2}}\eta = \frac{1}{2}\epsilon$ and thus,

$$\|u\|_{C^{2,\alpha}(M)} \leq \epsilon\|u\|_{C^{4,\alpha}(M)} + C_p\epsilon^{-\frac{2+\alpha}{2}-\frac{4+\alpha}{2}\frac{p}{6}}\|u\|_{L^2(M)}. \tag{70}$$

Here the constant $C_p \to \infty$ as $p \to 2n$. We could simply fix $p = 2n+1$ at the beginning and therefore we have

$$\|u\|_{C^{2,\alpha}(M)} \leq \epsilon\|u\|_{C^{4,\alpha}(M)} + C\epsilon^{-\frac{2+\alpha}{2}-\frac{4+\alpha}{2}\frac{2n+1}{6}}\|u\|_{L^2(M)}. \tag{71}$$

□

Next we quote a fredholm theorem for elliptic operators without proof.

Theorem 3.3 ([29])**.** *If M is compact and $P : C^\infty(M) \to C^\infty(M)$ is an elliptic operator on M, then the kernel of P is finite dimensional and $f \in C^\infty(M)$ is in the range of P if and only if*

$$\langle f, v \rangle = 0 \tag{72}$$

for all v in the kernel of the adjoint operator of P, denoted as P^t.

4 Openness for $0 < t < 1$

The proof presented in this section literally follows from [10]. We denote

$$\mathcal{H}^{4,\alpha}(M) = \{\varphi \in C^{4,\alpha}(M) | \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

Define for a closed postive $(1,1)$-form χ,

$$\begin{aligned} F_\chi : \mathcal{H}^{4,\alpha}(M) \times [0,1] &\to C^\alpha(M) \\ (\varphi, t) &\mapsto t(R_\varphi - \underline{R}) - (1-t)(\mathrm{tr}_\varphi \chi - \underline{\chi}) \end{aligned}$$

Theorem 4.1. *Suppose (M, ω) is a closed Kähler manifold and χ is a smooth, closed and positive $(1,1)$-form on M. If there exist $t_0 \in (0,1)$ and $\varphi_0 \in \mathcal{H}^{4,\alpha}(M)$ such that $F_\chi(\varphi_0, t_0) = 0$, then there exists an $\epsilon = \epsilon(t_0) > 0$ such that for any $t \in (t_0 - \epsilon, t_0 + \epsilon)$, there exists $\varphi_t \in \mathcal{H}^{4,\alpha}(M)$ such that $F_\chi(\varphi_t, t) = 0$.*

We'll apply the implicit function theorem on Banach spaces to prove this theorem. First let's consider the linearization of F_χ.

Lemma 4.2. *The linearization of F_χ at $(\varphi, t) \in \mathcal{H}^{4,\alpha}(M) \times [0,1]$ is given by*

$$\begin{aligned} \mathcal{D}F_\chi|_{(\varphi,t)}(u,s) = &- t\Delta_\varphi^2 u - t\langle\sqrt{-1}\partial\bar{\partial}u, \mathrm{Ric}_\varphi\rangle_\varphi + (1-t)\langle\sqrt{-1}\partial\bar{\partial}u, \chi\rangle_\varphi \\ &+ s\big((R_\varphi - \underline{R}) + (\mathrm{tr}_\varphi\,\chi - \underline{\chi})\big), \end{aligned} \tag{73}$$

where $(u,s) \in C^{4,\alpha}(M) \times \mathbb{R}$. Thus it's obvious that $\mathcal{D}F_\mu|_{(\varphi,t)} : C^{4,\alpha}(M) \times \mathbb{R} \to C^\alpha(M)$ is continuous in $(\varphi, t) \in \mathcal{H}^{4,\alpha}(M) \times [0,1]$.

Lemma 4.2 above follows from straightforward computations, so we omit its proof.

Lemma 4.3. *Given any closed $(1,1)$-form μ and function $f \in C^2(M)$, we have the identity*

$$\langle\sqrt{-1}\partial\bar{\partial}f, \mu\rangle_\omega = (f_{,p}\mu_{\alpha\bar{\beta}})_{,\bar{q}}g^{\alpha\bar{q}}g^{p\bar{\beta}} - f_{,p}(\mathrm{tr}_\omega\,\mu)_{,\bar{\beta}}g^{p\bar{\beta}}. \tag{74}$$

Proof. By direct computation, we have

$$\langle\sqrt{-1}\partial\bar{\partial}f, \mu\rangle_\omega = f_{,p\bar{q}}\mu_{\alpha\bar{\beta}}g^{\alpha\bar{q}}g^{p\bar{\beta}} = (f_{,p}\mu_{\alpha\bar{\beta}})_{,\bar{q}}g^{\alpha\bar{q}}g^{p\bar{\beta}} - f_{,p}\mu_{\alpha\bar{\beta},\bar{q}}g^{\alpha\bar{q}}g^{p\bar{\beta}}. \tag{75}$$

Since μ is closed, we have $\mu_{\alpha\bar{\beta},\bar{q}} = \mu_{\alpha\bar{q},\bar{\beta}}$. Thus

$$\begin{aligned}\langle\sqrt{-1}\partial\bar{\partial}f,\mu\rangle_\omega &= (f_{,p}\mu_{\alpha\bar{\beta}})_{,\bar{q}}g^{\alpha\bar{q}}g^{p\bar{\beta}} - f_{,p}\mu_{\alpha\bar{q},\bar{\beta}}g^{\alpha\bar{q}}g^{p\bar{\beta}} \\ &= (f_{,p}\mu_{\alpha\bar{\beta}})_{,\bar{q}}g^{\alpha\bar{q}}g^{p\bar{\beta}} - f_{,p}(\operatorname{tr}_\omega\mu)_{,\bar{\beta}}g^{p\bar{\beta}}\end{aligned} \tag{76}$$

□

Lemma 4.4. *For $(\varphi_0, t_0) \in \mathcal{H}^{4,\alpha}(M) \times (0,1)$ satisfying $F_\chi(\varphi_0, t_0) = 0$, we have that for any $u, v \in C^{4,\alpha}(M)$*

$$\begin{aligned}&\int_M \big(\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(u,0)\big)(v)\omega_{\varphi_0}^n \\ =& -t_0\int_M u_{,\bar{\alpha}\bar{\beta}}v_{,\alpha\beta}\omega_{\varphi_0}^n - (1-t_0)\int_M u_{,\bar{\alpha}}v_{,\beta}\chi_{\alpha\bar{\beta}}\omega_{\varphi_0}^n,\end{aligned} \tag{77}$$

Proof. Proof of Lemma 4.4. By calculations, we have that

$$\begin{aligned}\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(u,0) =& -t_0(u_{,\alpha\beta\bar{p}\bar{q}}g_{\varphi_0}^{\alpha\bar{p}}g_{\varphi_0}^{\beta\bar{q}} - R_{\varphi_0,\bar{p}}u_{,\alpha}g_{\varphi_0}^{\alpha\bar{p}}) \\ &+ (1-t_0)\big((u_{,p}\chi_{\alpha\bar{\beta}})_{,\bar{q}}g_{\varphi_0}^{\alpha\bar{q}}g_{\varphi_0}^{p\bar{\beta}} - u_{,p}(\operatorname{tr}_{\varphi_0}\chi)_{,\bar{\beta}}g_{\varphi_0}^{p\bar{\beta}}\big) \\ =& -t_0u_{,\alpha\beta\bar{p}\bar{q}}g_{\varphi_0}^{\alpha\bar{p}}g_{\varphi_0}^{\beta\bar{q}} + (1-t_0)(u_{,p}\chi_{\alpha\bar{\beta}})_{,\bar{q}}g_{\varphi_0}^{\alpha\bar{q}}g_{\varphi_0}^{p\bar{\beta}} \\ &+ (t_0R_{\varphi_0} - (1-t_0)\operatorname{tr}_{\varphi_0}\chi)_{,\bar{p}}u_{,\alpha}g_{\varphi_0}^{\alpha\bar{p}} \\ =& -t_0u_{,\alpha\beta\bar{p}\bar{q}}g_{\varphi_0}^{\alpha\bar{p}}g_{\varphi_0}^{\beta\bar{q}} + (1-t_0)(u_{,p}\chi_{\alpha\bar{\beta}})_{,\bar{q}}g_{\varphi_0}^{\alpha\bar{q}}g_{\varphi_0}^{p\bar{\beta}}.\end{aligned}$$

Last step is because $F_\chi(\varphi_0, t_0) = 0$. Thus, by integration by parts

$$\begin{aligned}&\int_M \big(\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(u,0)\big)(v)\omega_{\varphi_0}^n \\ =& -t_0\int_M u_{,\alpha\beta}v_{,\bar{p}\bar{q}}g_{\varphi_0}^{\alpha\bar{p}}g_{\varphi_0}^{\beta\bar{q}}\omega_{\varphi_0}^n - (1-t_0)\int_M u_{,p}\chi_{\alpha\bar{\beta}}v_{,\bar{q}}g_{\varphi_0}^{p\bar{\beta}}g_{\varphi_0}^{\alpha\bar{q}}\omega_{\varphi_0}^n.\end{aligned} \tag{78}$$

□

Now with the help of Lemma 4.2 and Lemma 4.4, we are ready to prove Theorem 4.1.

Proof. At given point $\varphi_0 \in \mathcal{H}$ s.t. $F_\chi(\varphi_0, t_0) = 0$, define

$$\begin{aligned}\mathcal{H}_0^{4,\alpha} &= \{\varphi \in \mathcal{H}^{4,\alpha} | \int_M \varphi\omega_{\varphi_0}^n = 0.\}, \\ C_0^{k,\alpha} &= \{f \in C^{k,\alpha}(M) | \int_M f\omega_{\varphi_0}^n = 0.\}, \\ C_0^\infty &= \{f \in C^\infty(M) | \int_M f\omega_{\varphi_0}^n = 0.\}.\end{aligned} \tag{79}$$

Denote π the projection from $C^\alpha(M)$ to its subspace C^α_0. Without loss of generality, we can assume $\varphi_0 \in C^{4,\alpha}_0$. So we can define

$$\begin{aligned} \tilde{F}_\chi : \mathcal{H}^{4,\alpha}_0 \times [0,1] \to C^\alpha_0 \\ (\varphi, t) \mapsto \pi F_\chi(\varphi, t) := F_\chi(\varphi, t) - \int_M F_\chi(\varphi, t)\omega^n_{\varphi_0}. \end{aligned} \tag{80}$$

with $\tilde{F}_\chi(\varphi_0, t_0) = 0$. By Lemma 4.2, we get that the linearization of $\tilde{F}_\chi$ at (φ, t) is continuous for (φ, t) in a small neighborhood of (φ_0, t_0) in $C^{4,\alpha}_0 \times [0,1]$.

By Lemma 4.4, $\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(\cdot, 0) : C^\infty(M) \to C^\infty(M)$ is a self-adjoint elliptic differential operator with kernel and cokernel both equal to subspace of constant functions on M. Thus by the fredholm theorem, we have that $\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(\cdot, 0) : C^\infty_0 \to C^\infty_0$ is a bijection. In fact, we can further prove that $\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(\cdot, 0) : C^{4,\alpha}_0 \to C^\alpha_0$ is an isomorphism between Banach spaces. It suffice to show an apriori estimate of schauder type, namely we will show that $\|u\|_{C^{4,\alpha}(M)} \le C\|\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(u, 0)\|_{C^\alpha(M)}$ for any $u \in C^{4,\alpha}_0$.

Set

$$\begin{aligned} f &= \mathcal{D}F_\chi|_{(\varphi_0,t_0)}(u, 0) \\ &= -t_0\Delta^2_{\varphi_0}u - \langle t_0\mathrm{Ric}_{\varphi_0} - (1-t_0)\chi, \sqrt{-1}\partial\bar{\partial}u\rangle_{\varphi_0}. \end{aligned} \tag{81}$$

Thus by the standard Schauder estimate for Laplacian equations,

$$\|u\|_{C^{4,\alpha}(M)} \le C\|\Delta_{\varphi_0}u\|_{C^{2,\alpha}(M)} \le C'(\|f\|_{C^\alpha(M)} + \|u\|_{C^{2,\alpha}(M)}). \tag{82}$$

By interpolation lemma 3.2, we have

$$\|u\|_{C^{4,\alpha}(M)} \le C(\|f\|_{C^\alpha(M)} + \|u\|_{L^2(M)}). \tag{83}$$

By Lemma 4.4, we have that

$$\begin{aligned} -\int_M fu\omega^n_{\varphi_0} &= (1-t_0)\int_M u_{,\bar{\alpha}}u_{,\beta}\chi_{\alpha\bar{\beta}}\omega^n_{\varphi_0} \\ &\ge C^{-1}\int_M |\nabla u|^2_{\varphi_0}\omega^n_{\varphi_0} \ge C'^{-1}\int_M u^2\omega^n_{\varphi_0} \end{aligned} \tag{84}$$

Last step is because of Porncare lemma and $\int_M u\omega^n_{\varphi_0} = 0$. Thus we get that

$$\|u\|_{L^2(M)} \le C\|f\|_{L^2(M)} \le C\|f\|_{C^\alpha(M)}. \tag{85}$$

Therefore we have

$$\|u\|_{C^{4,\alpha}(M)} \le C\|f\|_{C^\alpha(M)}. \tag{86}$$

Using this apriori estimate together with an approximation argument, we can conclude that $\mathcal{D}F_\chi|_{(\varphi_0,t_0)}(\cdot,0) : C_0^{4,\alpha} \to C_0^\alpha$ is an isomorphism between Banach spaces.

Note that $\mathcal{D}\tilde{F}_\chi|_{(\varphi_0,t_0)}(\cdot,0) = \mathcal{D}F_\chi|_{(\varphi_0,t_0)}(\cdot,0)$. Thus, by the implicit function theorem, there exists $\epsilon = \epsilon(t_0) > 0$ such that for any $|t - t_0| < \epsilon$, we could solve $\varphi_t \in \mathcal{H}_0^{4,\alpha}$ such that

$$\tilde{F}_\chi(\varphi_t, t) = 0. \tag{87}$$

From the definition of π, we could write

$$0 = \tilde{F}_\chi(\varphi_t, t) = F_\chi(\varphi_t, t) - \int_M F_\chi(\varphi_t, t)\omega_{\varphi_0}^n. \tag{88}$$

Thus we get that $F_\chi(\varphi_t, t) \equiv C$. Notice

$$\int_M F_\chi(\varphi_t, t)\omega_{\varphi_t}^n = 0. \tag{89}$$

So $C = 0$ and

$$F_\chi(\varphi_t, t) = 0.$$

Moreover, by the regularity of solutions to the elliptic equations, it follows that $\varphi_t \to \varphi_0$ as $t \to t_0$ in any C^k norms. □

5 Openness at $t = 0$

The main ingredient of the proof is to build an approximated twisted cscK metric for small $t > 0$ via taylor polynomials

$$\varphi_t = \varphi_0 + tu_1 + \frac{t^2}{2}u_2 + \cdots + \frac{t^k}{k!}u_k, \tag{90}$$

where $\varphi_0 \in \mathcal{H}$ satisfies $\mathrm{tr}_{\varphi_0}\chi = \underline{\chi}$ and $u_i's$ are smooth functions to be determined. By a appropriate choice of $u_i's$ we can eliminated the first kth coefficients in the taylor series about t of $t(R_{\varphi_t} - \underline{R}) - (\mathrm{tr}_{\varphi_t}\chi - \underline{\chi})$. The other ingredient is a quantitative inverse function theorem near φ_t so that we can perturb from the approximated twisted cscK metric to a twisted cscK metric.

Without further notice, the "C" in each estimate means a constant depending on the complex dimension n unless specified.

5.1 Approximation of twisted cscK metrics for small $t > 0$

Define

$$\mathcal{H}^{4,\alpha} = \{\varphi \in C^{4,\alpha}(M) | \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}. \tag{91}$$

Then for any $t > 0$, we define the map

$$\begin{aligned} F_t : \mathcal{H}^{4,\alpha} &\to C^\alpha(M), \\ \varphi &\mapsto t(R_\varphi - \underline{R}) - (\mathrm{tr}_\varphi\chi - \underline{\chi}). \end{aligned} \tag{92}$$

To look for twisted cscK metric for parameter $t > 0$ sufficiently small, it suffices to find $\varphi \in \mathcal{H}^{4,\alpha}$ such that $F_t(\varphi) = 0$ for small $t > 0$. We eventually will use a quantitative inverse function theorem to find such φ. But before doing that, let us first look at a good approximation of the twisted cscK metric for small $t > 0$.

Given that $0 \in I_\chi$, then there exists a smooth Kähler potential φ_0 such that $\mathrm{tr}_{\varphi_0}\chi = \underline{\chi}$. Note that $F_t(\varphi_0) = t(R_{\varphi_0} - \underline{R}) \to 0$ in C^∞ sense as $t \to 0$. However, φ_0 is not a good enough approximation for our purpuse due to the possible faster shrinking rate of invertible neighborhood around $F_t(\varphi_0)$ with respect to t as $t \to 0$.

Starting from φ_0, we could build better approximations of twisted cscK metric via taylor polynomials such that $F_t(\cdot)$ is small in terms of powers of t. To be more precise, we introduce the following lemma.

Lemma 5.1. *Suppose $\alpha \in (0,1)$ and $k \in \mathbb{N}$. There exists $\delta_k > 0$ depending on k and φ_0 such that for any $0 < t < \delta_k$, there exists a smooth function $\varphi_t \in C^\infty(M)$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0$ and*

$$\|F_t(\varphi_t)\|_{C^\alpha(M)} \le C_k t^{k+1}, \tag{93}$$

for some constant $C_k > 0$ depending on k and φ_0. Moreover, we have for any $l \in \mathbb{N}$

$$\|\varphi_t - \varphi_0\|_{C^l(M)} \le C_{k,l} t, \tag{94}$$

for some constant $C_{k,l}$ depending on k, l and φ_0.

Proof. We prove this lemma by explicit construction. For $k = 0$, φ_0 would suffice. However we are interested in the case when $k \gg 1$. Consider

$$\varphi_t = \varphi_0 + tu_1 + \frac{t^2}{2}u_2 + \cdots + \frac{t^k}{k!}u_k, \tag{95}$$

where $u_i's$ are smooth functions to be determined. To find these $u_i's$, we consider the taylor expansion of $F_t(\varphi_t)$ with respect to t at $t = 0$,

$$F_t(\varphi_t) = F_0(\varphi_0) + t\big(\frac{\partial}{\partial t}F_t(\varphi_t)\big)|_{t=0} + \cdots + \frac{t^k}{k!}\big(\frac{\partial^k}{\partial t^k}F_t(\varphi_t)\big)|_{t=0} + R, \tag{96}$$

where the remainder is

$$R = \frac{1}{k!}\int_0^t s^k (\frac{\partial}{\partial s})^{k+1} F_s(\varphi_s) ds. \tag{97}$$

The idea is to eliminate lower order term of t by choosing appropriate $u_i's$ such that the first kth coefficients in the taylor expansion vanish. We compute first a few coefficients in the taylor expansion of $F_t(\varphi_t)$. Since $\mathrm{tr}_{\varphi_0}\chi = \underline{\chi}$, we have $F_0(\varphi_0) = 0$. Next compute

$$\frac{\partial}{\partial t}F_t(\varphi_t) = (R_{\varphi_t} - \underline{R}) + t\frac{\partial R_{\varphi_t}}{\partial t} + g_{\varphi_t}^{i\bar{q}} g_{\varphi_t}^{p\bar{j}} \chi_{i\bar{j}} (\frac{\partial \varphi_t}{\partial t})_{,p\bar{q}}. \tag{98}$$

Set $t = 0$ we get

$$\big(\frac{\partial}{\partial t}F_t(\varphi_t)\big)|_{t=0} = R_{\varphi_0} - \underline{R} + g_{\varphi_0}^{i\bar{q}} g_{\varphi_0}^{p\bar{j}} \chi_{i\bar{j}} u_{1,p\bar{q}} \tag{99}$$

Further compute that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}F_t(\varphi_t) =&2(-\Delta^2_{\varphi_t}\frac{\partial\varphi_t}{\partial t} - g^{i\bar{q}}_{\varphi_t}g^{p\bar{j}}_{\varphi_t}(\frac{\partial\varphi_t}{\partial t})_{,i\bar{j}}\mathrm{Ric}_{\varphi_t,p\bar{q}})\\ &+ t\frac{\partial^2 R_{\varphi_t}}{\partial t^2} - g^{i\bar{k}}_{\varphi_t}(\frac{\partial\varphi_t}{\partial t})_{,\bar{k}l}g^{l\bar{q}}_{\varphi_t}g^{p\bar{j}}_{\varphi_t}\chi_{i\bar{j}}(\frac{\partial\varphi_t}{\partial t})_{,p\bar{q}}\\ &- g^{i\bar{q}}_{\varphi_t}g^{p\bar{k}}_{\varphi_t}g^{l\bar{j}}_{\varphi_t}(\frac{\partial\varphi_t}{\partial t})_{,\bar{k}l}\chi_{i\bar{j}}(\frac{\partial\varphi_t}{\partial t})_{,p\bar{q}} + g^{i\bar{q}}_{\varphi_t}g^{p\bar{j}}_{\varphi_t}\chi_{i\bar{j}}(\frac{\partial^2\varphi_t}{\partial t^2})_{,p\bar{q}}.\end{aligned} \tag{100}$$

Set $t = 0$ and we have

$$\begin{aligned}\frac{\partial^2}{\partial t^2}F_t(\varphi_t)|_{t=0} =&2(-\Delta^2_{\varphi_0}u_1 - g^{i\bar{q}}_{\varphi_0}g^{p\bar{j}}_{\varphi_0}(u_1)_{,i\bar{j}}\mathrm{Ric}_{\varphi_0,p\bar{q}})\\ &- g^{i\bar{k}}_{\varphi_0}(u_1)_{,\bar{k}l}g^{l\bar{q}}_{\varphi_0}g^{p\bar{j}}_{\varphi_0}\chi_{i\bar{j}}(u_1)_{,p\bar{q}}\\ &- g^{i\bar{q}}_{\varphi_0}g^{p\bar{k}}_{\varphi_0}g^{l\bar{j}}_{\varphi_0}(u_1)_{,\bar{k}l}\chi_{i\bar{j}}(u_1)_{,p\bar{q}} + g^{i\bar{q}}_{\varphi_0}g^{p\bar{j}}_{\varphi_0}\chi_{i\bar{j}}(u_2)_{,p\bar{q}}.\end{aligned} \tag{101}$$

Compute one more coefficient, we have

$$\begin{aligned}\frac{\partial^3}{\partial t^3}F_t(\varphi_t)|_{t=0} =&3\frac{\partial^2 R_{\varphi_t}}{\partial t^2}|_{t=0} + (t\frac{\partial^3 R_{\varphi_t}}{\partial t^3})|_{t=0} + (\partial\bar{\partial}u_1)^{*3} + \partial\bar{\partial}u_1 * \partial\bar{\partial}u_2\\ &+ + g^{i\bar{q}}_{\varphi_0}g^{p\bar{j}}_{\varphi_0}\chi_{i\bar{j}}(u_3)_{,p\bar{q}},\end{aligned} \tag{102}$$

where $\frac{\partial^2 R_{\varphi_t}}{\partial t^2}|_{t=0}$ is a smooth function M only involving u_1, u_2 and φ_0, and moreover "$*$" abbreviates for multiplying with smooth functions depending on φ_0 and χ as well as proper contractions.

By computations above, we know that each $\frac{\partial^m}{\partial t^m}F_t(\varphi_t)|_{t=0}$ only involves $u_i's$ for $i \leq m$ and the only term involving u_m is $g^{i\bar{p}}_{\varphi_0}g^{q\bar{j}}_{\varphi_0}\chi_{p\bar{q}}u_{m,i\bar{j}}$. This fact suggests that we could define $u_m's$ inductively using previously determined functions $u_i's$ for $i \leq m-1$.

Next, we describe how to define functions $u_i's$. First, let us denote

$$\Delta_{\chi,\varphi_0} := g^{i\bar{p}}_{\varphi_0}g^{q\bar{j}}_{\varphi_0}\chi_{p\bar{q}}\frac{\partial^2}{\partial z_i\partial\bar{z}_j}. \tag{103}$$

Given the fact that $\mathrm{tr}_{\varphi_0}\chi = \underline{\chi}$, Δ_{χ,φ_0} indeed behaves like Laplacian operator on M and $\Delta_{\chi,\varphi_0}u = f$ is solvable if and only if $\int_M f\omega^n_{\varphi_0} = 0$. We know that $F_0(\varphi_0) = 0$ by assumption. Suppose we have smooth functions $u_i's$ defined for all $i \leq m-1$ such that

$$\frac{\partial^i}{\partial t^i}F_t(\varphi_t)|_{t=0} = 0, \tag{104}$$

for all $i \leq m-1$. We hope to find smooth function u_m such that

$$\frac{\partial^m}{\partial t^m} F_t(\varphi_t)|_{t=0} = 0. \tag{105}$$

Following the discussion above we have

$$\frac{\partial^m}{\partial t^m} F_t(\varphi_t)|_{t=0} = \Delta_{\chi,\varphi_0} u_m + S(\varphi_0, u_1, \cdots, u_{m-1}). \tag{106}$$

So to define u_m, it suffices to check if $\int_M S(\varphi_0, u_1, \cdots, u_{m-1})\omega_{\varphi_0}^n = 0$. In fact, when $u_i's$ satisfy (104), it is automatically true. Since for any $0 < t \ll 1$ we have

$$\int_M F_t(\varphi_t)\omega_{\varphi_t}^n = 0, \tag{107}$$

we take derivatives with respect to t of the above identity m times and get

$$\int_M (\frac{\partial}{\partial t})^m F_t(\varphi_t)\omega_{\varphi_t}^n + \sum_{i=1}^m \frac{m!}{i!(m-i)!}\int_M \big((\frac{\partial}{\partial t})^{m-i} F_t(\varphi_t)\big)(\frac{\partial}{\partial t})^i \omega_{\varphi_t}^n = 0. \tag{108}$$

Set $t = 0$ and we have

$$\int_M \big((\frac{\partial}{\partial t})^m F_t(\varphi_t)\big)|_{t=0}\omega_{\varphi_0}^n = 0. \tag{109}$$

Since $\int_M \Delta_{\chi,\varphi_0} u_m \omega_{\varphi_0}^n = 0$ no matter what u_m is, we get that

$$\int_M S(\varphi_0, u_1, \cdots, u_{m-1})\omega_{\varphi_0}^n = 0.$$

Thus we could solve u_m such that (105) holds. Notice in this construction process, all $u_i's$ are smooth functions determined by the previous $u_i's$, and eventually only determined by the initial Kähler metric φ_0. Thus (94) holds.

It remains to check (93) and by our construction it suffices to consider the remainder (97) and to show that

$$\|(\frac{\partial}{\partial s})^{k+1} F_s(\varphi_s)\|_{C^\alpha(M)} \leq C_k, \tag{110}$$

for some constant C_k independent of $s \ll 1$. Choose constant $\delta_k > 0$ such that for any $s < \delta_k$, $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_s > 0$. Denote $\varphi_s^{(i)} = (\frac{\partial}{\partial s})^i \varphi_s$ for $1 \leq i \leq k$ and they are smooth functions with any C^m norm uniformly bounded when s goes to 0. By a direct computation when $s < \delta_k$, we get that

$$(\frac{\partial}{\partial s})^{k+1} F_s(\varphi_s) = F(s, \partial\bar{\partial}\varphi_s, \cdots, \partial\bar{\partial}\varphi_s^{(k)}, \nabla\partial\bar{\partial}\varphi_s, \cdots, \nabla\partial\bar{\partial}\varphi_s^{(k)}, \nabla^4\varphi_s, \cdots, \nabla^4\varphi_s^{(k)}), \tag{111}$$

where F is a smooth functions with respect to its variables, especially smooth up to $s = 0$. Thus, that for $s < \delta_k$, $(\frac{\partial}{\partial s})^{k+1} F_s(\varphi_s)$ is a smooth function on M and its C^α norm is uniformly bounded when $s \to 0$. This ends the proof of the lemma. □

5.2 Quantitative inverse function theorem

In this section, we study the local invertibility of $F_t : \mathcal{H}^{4,\alpha} \to C^\alpha(M)$ at a given point $\psi \in \mathcal{H}$ for fixed $t > 0$ sufficiently small. First of all, we notice that

$$\int_M F_t(\varphi)\omega_\varphi^n = 0, \tag{112}$$

for any $\varphi \in \mathcal{H}^{4,\alpha}$. Given this fact, one could not expect F_t to be surjective on any open set in $C^\alpha(M)$. However, we can modify F_t and the corresponding function spaces to get local invertibility in the modified setting.

At any given point $\psi \in \mathcal{H}$, without loss of generality we could assume that $\int \psi\omega_\psi^n = 0$ and define

$$\begin{aligned} \mathcal{H}_\psi^{4,\alpha} &= \{\varphi \in \mathcal{H}^{4,\alpha} | \int_M \varphi\omega_\psi^n = 0.\}, \\ C_\psi^{k,\alpha} &= \{f \in C^{k,\alpha}(M) | \int_M f\omega_\psi^n = 0.\}, \\ C_\psi^\infty &= \{f \in C^\infty(M) | \int_M f\omega_\psi^n = 0.\}. \end{aligned} \tag{113}$$

Moreover, we could define $F_{t,\psi} : \mathcal{H}_\psi^{4,\alpha} \to C_\psi^\alpha$

$$F_{t,\psi}(\varphi) = F_t(\varphi) - \int_M F_t(\varphi)\omega_\psi^n. \tag{114}$$

Following these definitions, we get that $F_{t,\psi}(\psi) = F_t(\psi)$ since $\int_M F_t(\psi)\omega_\psi^n = 0$ always. Moreover we have that if $F_{t,\psi}(\varphi) = 0$, then $F_t(\varphi) = 0$. It is shown in [10] that for any fixed $t > 0$ if ψ is a twisted cscK metric, namely $F_t(\psi) = F_{t,\psi}(\psi) = 0$, then the map $F_{t,\psi} : \mathcal{H}_\psi^{4,\alpha} \to C_\psi^\alpha$ is locally invertible from $\psi \in \mathcal{H}_\psi^{4,\alpha}$ to $0 \in C_\psi^\alpha$. If we vary ψ a little bit away from the twisted cscK metric, one should expect the local invertibility of $F_{t,\psi}$ still hold.

In the next theorem, we prove that for $t > 0$ sufficiently small, $F_{t,\psi} : \mathcal{H}_\psi^{4,\alpha} \to C_\psi^\alpha$ is still locally invertible from $\psi \in \mathcal{H}_\psi^{4,\alpha}$ to $F_{t,\psi}(\psi) \in C_\psi^\alpha$ if $\psi \in \mathcal{H}$ is close to a twisted cscK metric in the sense that $\|\psi - \varphi_0\|_{C^6(M)} \ll 1$. Moreover, we derive an estimate on the size of invertible neighborhood of $F_{t,\psi} : \mathcal{H}_\psi^{4,\alpha} \to C_\psi^\alpha$ near $F_{t,\psi}(\psi) \in C_\psi^\alpha$.

Theorem 5.2. *Given $\varphi_0 \in \mathcal{H}$ such that $tr_{\varphi_0}\chi = \underline{\chi}$ and $\int_M \varphi_0 \omega_{\varphi_0}^n = 0$, there exists constant $\epsilon = \epsilon(\varphi_0) > 0$ such that for any $0 < t < \epsilon$ and $\psi \in \mathcal{H}$ satisfying $\int_M \psi \omega_\psi^n = 0$ and $\|\psi - \varphi_0\|_{C^6(M)} < \epsilon$, we have that $F_{t,\psi} : \mathcal{H}_\psi^{4,\alpha} \to C_\psi^\alpha$ is locally invertible from ψ to $F_{t,\psi}(\psi)$.*

Moreover, we have that for any $y \in C_\psi^\alpha$ with $\|y - F_{t,\psi}(\psi)\|_{C^\alpha(M)} \leq \epsilon t^{2\gamma+2}$ for γ given in Lemma 3.2, we can find an $x \in \mathcal{H}_\psi^{4,\alpha}$ such that $F_{t,\psi}(x) = y$.

In order to prove Theorem 5.2, first of all one needs to study the linearization of $F_{t,\psi}$ at ψ, denoted by $\mathcal{D}F_{t,\psi}|_\psi$. We compute $\mathcal{D}F_{t,\psi}|_\psi : C_\psi^{4,\alpha} \to C_\psi^\alpha$, for any $u \in C_\psi^{4,\alpha}$

$$\begin{aligned} \mathcal{D}F_{t,\psi}|_\psi(u) &= -t\Delta_\psi^2 u + \langle \chi - t\mathrm{Ric}_\psi, \partial\bar{\partial}u \rangle_\psi \\ &\quad - \int_M \langle \chi - t\mathrm{Ric}_\psi, \partial\bar{\partial}u \rangle_\psi \omega_\psi^n. \end{aligned} \tag{115}$$

Denote $P_{t,\psi}(u) = -t\Delta_\psi^2 u + \langle \chi - t\mathrm{Ric}_\psi, \partial\bar{\partial}u \rangle_\psi$, then

$$\mathcal{D}F_{t,\psi}|_\psi = \pi_\psi \circ P_{t,\psi}, \tag{116}$$

where $\pi_\psi f = f - \int_M f \omega_\psi^n$.

$P = P_{t,\psi} : C^\infty(M) \to C^\infty(M)$ is an elliptic differential operator. By the fredholm theorem, we have the orthogonal decomposition $C^\infty(M) = \mathrm{Im}(P) \oplus \mathrm{Ker}(P^T)$. Denote $P' = \pi_\psi P \pi_\psi$ and $P'^T = \pi_\psi P^T \pi_\psi$. For any $f \in C_\psi^\infty$, we have orthogonal decomposition $f = Pu + v$ for $u \in C^\infty(M)$ and $v \in \mathrm{Ker}(P^T)$. Applying π_ψ, we get $f = \pi_\psi P u + \pi_\psi v$. We have by computation that $\pi_\psi P u = P'u$ and $\pi_\psi v \in \mathrm{Ker}(P'^T)$. Thus $f = P'\pi_\psi u + \pi_\psi v$. Thus we have orthogonal decomposition of $C_\psi^\infty = \mathrm{Im}(P') \oplus \mathrm{Ker}(P'^T)$.

For $t \ll 1$ and $\|\psi - \varphi_0\|_{C^6(M)} \ll 1$ we have that $\mathcal{D}F_{t,\psi}|_\psi = \pi_\psi P_{t,\psi} \pi_\psi : C_\psi^\infty \to C_\psi^\infty$ has kernel and cokernel both equal to zero. More precisely, we introduce the following lemma.

Lemma 5.3. *There exists a constant $\epsilon = \epsilon(\varphi_0) > 0$ such that for any $0 < t < \epsilon$ and $\psi \in \mathcal{H}$ with $\int_M \psi \omega_\psi^n = 0$ and $\|\psi - \varphi_0\|_{C^6(M)} \leq \epsilon$, we have that $\mathcal{D}F_{t,\psi}|_\psi : C_\psi^\infty \to C_\psi^\infty$ is bijective.*

Proof. For any $u \in C^\infty(M)$, we compute

$$\begin{aligned}\int_M (\pi_\psi P_{t,\psi} u) u\omega_\psi^n &= \int_M (P_{t,\psi} u)\pi_\psi u\omega_\psi^n, \\ &= -t\int_M |u_{,\alpha\beta}|_\psi^2 \omega_\psi^n - \int_M u_{,\bar\alpha} u_{,\beta} g_\psi^{p\bar\alpha} g_\psi^{\beta\bar q} \chi_{p\bar q}\omega_\psi^n \\ &\quad - \frac{1}{2}\int_M (\pi_\psi u)^2 \Delta_\psi (tR_\psi - \mathrm{tr}_\psi \chi)\omega_\psi^n, \\ &\le -\int_M u_{,\bar\alpha} u_{,\beta} g_\psi^{p\bar\alpha} g_\psi^{\beta\bar q}\chi_{p\bar q}\omega_\psi^n \\ &\quad + \frac{C_\psi}{2}\sup_M |\Delta_\psi(tR_\psi - \mathrm{tr}_\psi\chi)| \int_M |\nabla u|_\psi^2 \omega_\psi^n,\end{aligned} \tag{117}$$

where C_ψ denotes the Poincaré constant of metric g_ψ. Note that

$$\begin{aligned}&g_{\varphi_0}^{p\bar\alpha} g_{\varphi_0}^{\beta\bar q}\chi_{p\bar q} \ge \delta_{\varphi_0} g_{\varphi_0}^{\beta\bar\alpha}, \text{ for some constant } \delta_{\varphi_0} > 0. \\ &\Delta_{\varphi_0}(tR_{\varphi_0} - \mathrm{tr}_{\varphi_0}\chi) = t\Delta_{\varphi_0} R_{\varphi_0}. \\ &C_\psi \le C_r \text{ for some constant } C_r > 0, \text{ if } r^{-1} g \le g_\psi \le rg.\end{aligned} \tag{118}$$

Thus we can choose $\epsilon > 0$ depending on φ_0 such that for any ψ with $\|\psi - \varphi_0\|_{C^6(M)} \le \epsilon$, we have

$$\begin{aligned}&g_\psi^{p\bar\alpha} g_\psi^{\beta\bar q}\chi_{p\bar q} \ge \frac{1}{2}\delta_{\varphi_0} g_\psi^{\beta\bar\alpha}. \\ &\sup_M |\Delta_\psi(tR_\psi - \mathrm{tr}_\psi\chi)| \le Ct + \frac{1}{4C}\delta_{\varphi_0} \text{ and } C_\psi \le C, \\ &\text{for some constant } C > 0.\end{aligned} \tag{119}$$

We could choose $\epsilon > 0$ even smaller such that for any $0 < t < \epsilon$, we have that $Ct < \frac{1}{4C}\delta_{\varphi_0}$. Thus by (117), we get that

$$\begin{aligned}\int_M (\pi_\psi P_{t,\psi} u) u\omega_\psi^n &\le -\frac{\delta_0}{2}\int_M |\nabla u|_\psi^2 \omega_\psi^n + \frac{\delta_{\varphi_0}}{4}\int_M |\nabla u|_\psi^2\omega_\psi^n, \\ &\le -\frac{\delta_{\varphi_0}}{4}\int_M |\nabla u|_\psi^2 \omega_\psi^n.\end{aligned} \tag{120}$$

Thus $\pi_\psi P_{t,\psi} u = 0$ if and only if u is a constant. The same computation also implies that the adjoint operator $(\pi_\psi P_{t,\psi})^t u = 0$ if and only if u is a constant since by definition,

$$\int_M u(\pi_\psi P_{t,\psi})^t u\omega_\psi^n = \int_M (\pi_\psi P_{t,\psi} u) u\omega_\psi^n. \tag{121}$$

By the fredholm theorem for $\pi_\psi P_{t,\psi}\pi_\psi$, we can get that $\mathcal{D}F_{t,\psi}|_\psi = \pi_\psi P_{t,\psi}\pi_\psi : C_\psi^\infty \to C_\psi^\infty$ is a bijection if $0 < t < \epsilon$ and $\|\psi - \varphi_0\|_{C^6(M)} < \epsilon$. □

We have shown in the last lemma that $\mathcal{D}F_{t,\psi}|_{\psi}$ has an inverse defined on C^{∞}_{ψ}. Next we show a schauder type apriori estimate for operator $\mathcal{D}F_{t,\psi}|_{\psi}$, then by an approximation by smooth functions $f_i \in C^{\infty}_{\psi} \to f \in C^{\alpha}_{\psi}$ in hölder space, we could get that $\mathcal{D}F_{t,\psi}|_{\psi} : C^{4,\alpha}_{\psi} \to C^{\alpha}_{\psi}$ is a surjective. And it's injective because of Lemma 5.3.

Lemma 5.4. *Suppose $\epsilon > 0$ is the constant chosen as in Lemma 5.3. Given $\alpha \in (0,1)$, $0 < t < \epsilon$ and $\psi \in \mathcal{H}$ with $\|\psi - \varphi_0\|_{C^6(M)} < \epsilon$, for any $u \in C^{\infty}_{\psi}$, we have*

$$\|u\|_{C^{4,\alpha}(M)} \leq Ct^{-\gamma-1}\|\mathcal{D}F_{t,\psi}|_{\psi}(u)\|_{C^{\alpha}(M)}, \tag{122}$$

where γ is a constant chosen as in Lemma 3.2 and $C > 0$ is some constant depending on φ_0 only.

Proof. Denote $f = \mathcal{D}F_{t,\psi}|_{\psi}(u)$. Then we get

$$\Delta^2_{\psi} u = -\frac{1}{t} f + \frac{1}{t}\langle \chi - t\mathrm{Ric}_{\psi}, \partial\bar{\partial}u\rangle_{\psi} - \frac{1}{t}\int_M \langle \chi - t\mathrm{Ric}_{\psi}, \partial\bar{\partial}u\rangle_{\psi}\omega^n_{\psi}. \tag{123}$$

Thus by standard schauder estimate, we get

$$\begin{aligned} \|u\|_{C^{4,\alpha}(M)} &\leq C\|\Delta_{\psi} u\|_{C^{2,\alpha}(M)} \\ &\leq \frac{C}{t}\big(\|f\|_{C^{\alpha}(M)} + \|u\|_{C^{2,\alpha}(M)}\big). \end{aligned} \tag{124}$$

By the interpolations in Lemma 3.2,

$$\|u\|_{C^{4,\alpha}(M)} \leq \frac{C}{t}\|f\|_{C^{\alpha}(M)} + \frac{C}{t}\big(\eta\|u\|_{C^{4,\alpha}(M)} + C\eta^{-\gamma}\|u\|_{L^2(M)}\big), \tag{125}$$

where γ is the constant in Lemma 3.2. Choose $\eta = \frac{t}{2C}$, we get

$$\|u\|_{C^{4,\alpha}(M)} \leq \frac{C}{t}\|f\|_{C^{\alpha}(M)} + Ct^{-\gamma-1}\|u\|_{L^2(M)}. \tag{126}$$

Following computations in Lemma 5.3, inequality (120), since $\int_M u\omega^n_{\psi} = 0$, we get

$$\int_M u^2\omega^n_{\psi} \leq C\int_M |\nabla u|^2_{\psi}\omega^n_{\psi} \leq C\int_M fu\omega^n_{\psi} \leq C\|f\|_{L^2(M)}\|u\|_{L^2(M)}. \tag{127}$$

Thus, we have that

$$\|u\|_{L^2(M)} \leq C\|f\|_{L^2(M)}, \tag{128}$$

and then

$$\|u\|_{C^{4,\alpha}(M)} \leq Ct^{-\gamma-1}\|f\|_{C^{\alpha}(M)}. \tag{129}$$

□

Now we're ready to prove Theorem 5.2.

Proof. By Lemma 5.3, Lemma 5.4 and standard inverse function theorem we have already shown the local invertibility of $F_{t,\psi} : \mathcal{H}^{4,\alpha}_{\psi} \to C^{\alpha}_{\psi}$ from ψ to $F_{t,\psi}(\psi)$. Now we estimate the size of invertible neighborhood of $F_{t,\psi}(\psi)$ in C^{α}_{ψ}. Given $y \in C^{\alpha}_{\psi}$, we could define map

$$\begin{aligned}\Psi_y : \mathcal{H}^{4,\alpha}_{\psi} &\to C^{4,\alpha}_{\psi}, \\ x &\mapsto x + (\mathcal{D}F_{t,\psi}|_{\psi})^{-1}(y - F_{t,\psi}(x)).\end{aligned} \tag{130}$$

Note that x is the fixed point of Ψ_y if and only if $y = F_{t,\psi}(x)$. In fact Ψ_y is a contraction map near ψ.

Claim 5.5. *If $t > 0$, $\psi \in \mathcal{H}$ satisfies the assumptions in Theorem 5.2, then there exists a constant $\delta > 0$ depending on φ_0 such that for any $x_0, x_1 \in \mathcal{H}^{4,\alpha}_{\psi}$ with $\|x_0 - \psi\|_{C^{4,\alpha}(M)} \leq \delta t^{\gamma+1}$ and $\|x_1 - \psi\|_{C^{4,\alpha}(M)} \leq \delta t^{\gamma+1}$ for γ given in Lemma 3.2, then*

$$\|\Psi_y(x_1) - \Psi_y(x_0)\|_{C^{4,\alpha}(M)} \leq \frac{1}{2}\|x_1 - x_0\|_{C^{4,\alpha}(M)}, \tag{131}$$

Proof. Proof of claim 5.5. Denote $x_s = sx_1 + (1-s)x_0$ for $s \in [0,1]$ and we compute

$$\begin{aligned}\Psi_y(x_1) - \Psi_y(x_0) &= x_1 - x_0 - (\mathcal{D}F_{t,\psi}|_{\psi})^{-1}(F_{t,\psi}(x_1) - F_{t,\psi}(x_0)), \\ &= \int_0^1 (\mathcal{D}F_{t,\psi}|_{\psi})^{-1}\big((\mathcal{D}F_{t,\psi}|_{\psi} - \mathcal{D}F_{t,\psi}|_{x_s})(x_1 - x_0)\big)ds\end{aligned} \tag{132}$$

where for any $\varphi \in \mathcal{H}^{4,\alpha}_{\psi}$

$$\mathcal{D}F_{t,\psi}|_{\varphi}(u) = -t\Delta^2_{\varphi}u + \langle\chi - t\mathrm{Ric}_{\varphi}, \partial\bar{\partial}u\rangle_{\varphi} - \int_M \big(-t\Delta^2_{\varphi}u + \langle\chi - t\mathrm{Ric}_{\varphi}, \partial\bar{\partial}u\rangle_{\varphi}\big)\omega^n_{\psi}. \tag{133}$$

We first consider

$$\begin{aligned}&(\mathcal{D}F_{t,\psi}|_{\psi} - \mathcal{D}F_{t,\psi}|_{x_s})(x_1 - x_0) \\ &= -t(\Delta^2_{\psi} - \Delta^2_{x_s})(x_1 - x_0) + \big((\chi - t\mathrm{Ric}_{\psi})_{p\bar{q}}g^{\alpha\bar{q}}_{\psi}g^{p\bar{\beta}}_{\psi} \\ &- (\chi - t\mathrm{Ric}_{x_s})_{p\bar{q}}g^{\alpha\bar{q}}_{x_s}g^{p\bar{\beta}}_{x_s}\big)(x_1 - x_0)_{,\alpha\bar{\beta}} \\ &+ \int_M \{t(\Delta^2_{\psi} - \Delta^2_{x_s})(x_1 - x_0) - \big((\chi - t\mathrm{Ric}_{\psi})_{p\bar{q}}g^{\alpha\bar{q}}_{\psi}g^{p\bar{\beta}}_{\psi} \\ &- (\chi - t\mathrm{Ric}_{x_s})_{p\bar{q}}g^{\alpha\bar{q}}_{x_s}g^{p\bar{\beta}}_{x_s}\big)(x_1 - x_0)_{,\alpha\bar{\beta}}\}\omega^n_{\psi}.\end{aligned} \tag{134}$$

Thus we have

$$\|(\mathcal{D}F_{t,\psi}|_{\psi} - \mathcal{D}F_{t,\psi}|_{x_s})(x_1 - x_0)\|_{C^{\alpha}(M)} \leq C\|\psi - x_s\|_{C^{4,\alpha}(M)}\|x_1 - x_0\|_{C^{4,\alpha}(M)}, \tag{135}$$

for some $C > 0$ depending on $\|x_s\|_{C^{4,\alpha}(M)}$ and φ_0. Combining (132) and (135), by Lemma 5.4 we get that

$$\|\Psi_y(x_1) - \Psi_y(x_0)\|_{C^{4,\alpha}(M)} \leq Ct^{-\gamma-1}(\max_{s\in[0,1]} \|\psi - x_s\|_{C^{4,\alpha}(M)})\|x_1 - x_0\|_{C^{4,\alpha}(M)}, \tag{136}$$

where $C > 0$ is a constant depending on $\max_{s\in[0,1]} \|x_s\|_{C^{4,\alpha}(M)}$ and φ_0. If we have $\|x_0 - \psi\|_{C^{4,\alpha}(M)} \leq \delta t^{\gamma+1}$ and $\|x_1 - \psi\|_{C^{4,\alpha}(M)} \leq \delta t^{\gamma+1}$ for some constant $\delta \ll 1$ to be determined, then $\|x_s\|_{C^{4,\alpha}}$ is uniformly bounded on $s \in [0,1]$ and $\|\psi - x_s\|_{C^{4,\alpha}(M)} \leq \delta t^{\gamma+1}$. Therefore, we have

$$\|\Psi_y(x_1) - \Psi_y(x_0)\|_{C^{4,\alpha}(M)} \leq C\delta\|x_1 - x_0\|_{C^{4,\alpha}(M)}. \tag{137}$$

From the last inequality, it's clear that we should choose our $\delta = \frac{1}{2C}$ and then it ends the proof of claim. □

We continue the proof of Theorem 5.2. For $y \in C^{\alpha}_{\psi}$ with $\|y - F_{t,\psi}(\psi)\|_{C^{\alpha}(M)} \leq \epsilon t^{2\gamma+2}$, we have

$$\begin{aligned}\|\Psi_y(\psi) - \psi\|_{C^{4,\alpha}(M)} &= \|(\mathcal{D}F_{t,\psi}|_{\psi})^{-1}(y - F_{t,\psi}(\psi))\|_{C^{4,\alpha}(M)} \\ &\leq Ct^{-\gamma-1}\|y - F_{t,\psi}(\psi)\|_{C^{\alpha}(M)} \leq C\epsilon t^{\gamma+1},\end{aligned} \tag{138}$$

for some constant $C > 0$ just depending on φ_0. We could choose even smaller $\epsilon > 0$ such that $C\epsilon < \frac{1}{2}\delta$ where δ is given in the above claim. Thus $\Psi^i_y(\psi)$ would be a contracting sequence converging to the fixed point of Ψ_y. Following the previous discussion, the fixed point of Ψ_y, say x satisfies that $F_{t,\psi}(x) = y$. □

Combining Theorem 5.2 and Lemma 5.1, we could prove our Main Theorem 2.5 now.

Proof. Let $k = [2\gamma + 2]$ the largest integer smaller than or equal to $2\gamma + 2$. By Lemma 5.1 we could find an approximated twisted cscK metric φ_t for every $t > 0$ with bounds

$$\|F_t(\varphi_t)\|_{C^{\alpha}(M)} \leq Ct^{k+1}, \|\varphi_t - \varphi_0\|_{C^6(M)} \leq Ct. \tag{139}$$

Denote $\psi_t = \varphi_t - \int_M \varphi_t \omega^n_{\varphi_t}$, then

$$\|F_{t,\psi_t}(\psi_t)\|_{C^{\alpha}(M)} \leq C_1 t^{k+1}, \|\psi_t - \varphi_0\|_{C^6(M)} \leq C_2 t. \tag{140}$$

For $t < \min\{(\frac{\epsilon}{C_1})^{k+1-(2\gamma+2)}, \frac{1}{C_2}\epsilon, \epsilon\}$ where ϵ is determined in Theorem 5.2, we have that

$$\|0 - F_{t,\psi_t}(\psi_t)\|_{C^\alpha(M)} \leq \epsilon t^{2\gamma+2}, \|\psi_t - \varphi_0\|_{C^6(M)} < \epsilon. \tag{141}$$

Thus by Theorem 5.2, we get that there exists $\varphi \in C^{4,\alpha}_{\psi_t}$ such that $F_{t,\psi_t}(\varphi) = 0$ and then $F_t(\varphi) = 0$.

$\square$

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