Space of Kähler metrics on singular and non-compact manifolds

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2014

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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Let $\mathscr{H}$ be the space of Kähler metrics in a fixed cohomology class. This space may be endowed with a Weil-Petersson-type metric, referred to as the Mabuchi metric, which allows one to study the geometry of $\mathscr{H}$. It is now well-known that the geometry of the space of Kähler potentials, in particular, the geodesics in $\mathscr{H}$, may be used for studying 'canonical metrics' on the base manifold. In order to be interpreted as the potential of a Kähler metric however, one needs to prove certain regularity for such solutions.

In the first part, I discuss the deriving weighted estimates for the space and time derivatives of solutions in the case of ALE Kähler potentials, and further, prove results regarding the Mabuchi energy and the uniqueness of metrics of constant scalar curvature. In the latter part of the talk I shall discuss certain weighted estimates for the solutions to the geodesic equation when the end points have conical singularities. The results may also be seen as X.-X. Chen's fundamental work on the geodesic convexity of $\mathscr{H}$ in the case of smooth compact manifolds.

Keywords: Space of Kähler potentials, Conical singularity, ALE spaces, c.s.c Kähler metrics

To all who have taught me.

Contents

Acknowledgements

First of all, I should like to thank the committee members, Professors Bedford, Khuri, and Varolin for agreeing to be on the examining committee.

My life, both within mathematics and without it -if one chooses to think in terms of such dichotomy- is marked by my becoming Xiu-Xoing Chen's student. In concrete terms, this thesis reflects his insight. His support, encouragement, and fatherly advice during the two and half years of being under his supervision have had formative impact on my approach to mathematics well beyond this thesis. In the light of this, if one is willing to be a bit literal, I find the word *Doktorvater* more descriptive.

I am indebted to Prof. Khuri for his inexhaustible patience and unwavering support when I was learning riemannian geometry and the analysis of partial differential equations for my oral exam.

During these years in Stony Brook, I have benefited from the lectures by Professors Anderson, Chen, Ebin, Gromov, Grushevsky, Khuri, Lawson, LeBrun, Plamenevskaya, Sullivan, Schul, and Varolin. I would like to thank the latter two particularly for the advice and wisdom I have received from them in the course of many informal discussions. I also have to thank Prof. LeBrun in particular for his support and advice at two critical junctures during my PhD years. I also have to thank Prof. Sormani as I have benefited a lot from attending the seminars she organised on 'metric geometry'.

The material in this thesis reflects the brief and long discussions I have had with people who generously spared time: Prof. Bedford., Prof. Donaldson, Prof. Khuri, Prof. LeBrun, Prof. Şuvaina, and Prof. Varolin. Of course, any shortcomings in this work are due solely to me.

Further, I should like to thank Prof. W. He and Dr. Kai Zheng for carefully reading the drafts of the papers which constitute the content of this dissertation and making useful suggestions and remarks.

Many thanks also go to my colleagues and friends for their friendship and for all I learnt from them during our geometric analysis seminar and outside: Chenjian Yao, Yü Zeng, Cheng Gao, Xiaojie, Li Long, Raquel, Shaosai, Selin, Tsung-Ying, along with Dr. Lorenzo Foscolo, Dr. Chi Li, Dr. Jinzhu Sun, and Dr. Song Sun.

I also thank the many wonderful friends that I made while being at Stony Brook especially Caner, Claudio, Davoud, Dingxin, Dmitry, Gerald, Jan, Jimmy, Panagiotis, Pedro, Rafael, Raquel, Reza, and Shane, with whom I have had many useful discussions and shared a lot of fun time together. Without their company, my stay at Stony Brook would not have been as wonderful and memorable as it has been. Also, thanks go to my dear friends and comrades, Omid and Hassan, for their support and what I learnt from them in physics, mathematics, and a lot of other things.

Above all, I thank my family for all the support and confidence in me, who, though far away, have provided all the support and encouragement that they could.

I hope I am forgiven for my failure in giving a fair and just acknowledgement as that would demand an entire chapter.

Chapter 1

Introduction

1.1 Eine bemerkenswerte Hermite'sche Metrik-1933

'When studying the invariants of a real 2n-dimensional Hermitian metric'

$$ds^2 = \sum \mathfrak{g}_{i\bar{k}} dx_i d\bar{x}_k \tag{1.1}$$

Kähler found it ' ... natural to study, aside from (1.1), the alternating quadratic differential form (forme extérieure)'

$$\omega = \sum \mathfrak{g}_{i\bar{k}} d(x_i, \bar{x}_k) \tag{1.2}$$

'In this approach, the case $\omega' = 0$ appears as a remarkable exception. We find that the metric can be derived in the following way

$$ds^2 = \sum \frac{\partial^2 U}{\partial x_i \partial \bar{x}_k} dx_i d\bar{x}_k \tag{1.3}$$

from a potential U , which evidently is an invariant property equivalent to $\omega' = 0$.' [1]

This apparently casual observation made by young Erich Kähler in the early 1930's was the inception of what is known as Kähler geometry. One of Kähler's motivations seems to have been finding explicit solutions to the static Einstein gravity equation, and it is indeed in that short article where Kähler-Einstein metrics are first considered.

[1] The notation and the usage of indeces in the first three equations might seem a bit out-dated as I am quoting Kähler's original paper and his notation as well. The English translation follows that of Mr. Wolfgang Globke. In the more modern notation, the term $d(x_i, \bar{x}_k)$ for example, may be substituted by $dx^i \wedge dx^{\bar{k}}$, and ω' by $d\omega$.

Regrettably, of all the wonders of this 'remarkable' family of metrics, we shall confine ourselves to stating the properties directly relevant to this work. Any elementary treatment of Kähler geometry can be consulted for their proof, see [27] or [20] for instance.

We shall start with the following simple, nevertheless foundational, lemma which allows us reduce the study of the space of Kähler metrics to the study of potentials.

Lemma 1.1.1. *The dd^c-lemma Let $[\eta] = [\zeta] \in H^{1,1}(M)$, i.e. η and ζ are two cohomologous closed forms. Then, for some function f, unique up to addition of a constant, one has*

$$\eta = \zeta + dd^c f$$

Let us then remind some basic identities that will come to our aid in the calculations in coordinates:

Proposition 1.1.2. We have the following identities about the curvature of a Kähler manifold:

i. $R^{\alpha}_{\beta\bar{\nu}\lambda} = \bar{\partial}_{\nu}\Gamma^{\alpha}_{\lambda\beta} = R_{\nu\bar{\lambda}}\mathfrak{g}^{\alpha\bar{\beta}}R_{\alpha\beta\nu\bar{\lambda}}$,

ii. $R_{\lambda\bar{\nu}} = \mathfrak{g}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}\nu\bar{\lambda}} = R^{\alpha}_{\alpha\lambda\bar{\nu}} = \partial_{\lambda}\,\partial_{\bar{\nu}}\log\det\mathfrak{g}$,

ii'. whereby we have about the Ricci form $\rho = -\mathrm{i}\partial\bar{\partial}\log\det\mathfrak{g}$,

iii. $K = -\Delta\log\det\mathfrak{g} = \rho\wedge\omega^{n-1} = -\mathrm{i}\partial\bar{\partial}\log\det\mathfrak{g}\wedge\omega^{n-1}$ where K denotes the scalar curvature.

The next assertion, which will appear in the calculation in the last chapter, allows us to choose homorphic coordinates in which the curvature has a simple form and the Christoffel symbols vanish.

Proposition 1.1.3. Let $\omega = \mathrm{i}\mathfrak{g}_{\alpha\bar{\beta}}d\mathfrak{z}^{\alpha}\wedge d\mathfrak{z}^{\bar{\beta}}$ be a Kähler form defined on some open set $\mathscr{U}$. Then, for a given point p there is a biholomorphic change of coordinates in which the metric satisfies *at the point p*:

i. $g_{\alpha\bar{\beta}|_p} = \delta_{\alpha\beta}$,

ii. $\partial_{\kappa}\,\mathfrak{g}_{\alpha\bar{\beta}|_p} = \partial_{\bar{\lambda}}\,\mathfrak{g}_{\alpha\bar{\beta}|_p} = 0$, and hence the Christoffel symbols vanish,

iii. $R_{\alpha\bar{\beta}\kappa\bar{\lambda}|_p} = \mathfrak{g}_{\alpha\bar{\beta},\kappa\bar{\lambda}|_p}$.

1.2 The space of Kähler metrics-1954

The systematic study of the space of Kähler potentials seems to have originated by the questions Calabi formulated in the proceedings of the international congress of mathematicians in 1954 [8]. Possibly motivated by the dd^c-lemma, there he defines the space of Kähler metrics on a compact manifold as follows: [2]

Definition 1.2.1. Let M be a Kähler manifold with a fixed Kähler form ω. Then the space of Kähler potentials is given by

$$\mathscr{H} := \{\phi \in C^\infty(X) | \omega + dd^c\phi > 0\} \tag{1.4}$$

As we shall see, various extensions of this definition will be needed which we will introduce in the relevant chapter.

But more geometric approach to understanding this space was yet to develop three decades later. Towards the end of the 1980's, Mabuchi in [30] and Bourguignon in [7] independently started studying the geometric structure of the space of Kähler metrics in a fixed cohomology class over a given manifold. Later, apparently independently, Semmes in [32] and Donaldson in [16] rediscovered this idea from a different viewpoint. The fundamental idea behind these works was the introduction of a Weil-Petersson-type metric on this space that would endow it with a riemannian metric and a compatible connection.

The study proceeded by formal calculations of the connection coefficients, curvature, and the geodesic equation. For example, it was shown that $\mathscr{H}$ is a locally homogeneous space of non-positive sectional curvature. As in the usual case of riemannian manifolds, one can define the length and energy of a curve, and by taking the first variation of the energy functional for curves, one can derive the geodesic equation. It was proved that if the curve $\phi(t)$ is a geodesic in $\mathscr{H}$, it must satisfy the equation [3]

$$\phi'' - \frac{1}{2}|d\phi'|^2_\phi = 0 \tag{1.5}$$

In [16], Donaldson formulated the relation between the geometry of $\mathscr{H}$ and some older problems in few conjectures. This initiated a new programme for studying some classical problems in Kähler geometry. More specifically, these conjectures related problems such as the existence and the uniqueness of extremal Kähler

[2] Henceforth, we shall assume that ω is a fixed smooth positive definite Kähler metric in the background. Also, the norm of various tensors are measured with respect to ω.

[3] Two remarks on notation: Here and hereafter, differential operators without subscript are assumed to be in the space direction. In certain occasions, we may use the subscript X for a space differential operator for emphasis. Also, both ∂_t and t as subscript are used to denote time derivatives.

metrics and Kähler metrics of constant scalar curvature to the geometry of the space $\mathscr{H}$. Of particular importance were the geodesics and the the metric structure of the space $\mathscr{H}$.

Further, it was proved that equation 1.5 can be written as a homogeneous complex Monge-Ampère equation. More precisely, let $\phi_1, \phi_2 \in \mathscr{H}$ be two Kähler potentials on X. Then let τ be the complexification of the time variable $t \in [0,1]$. We complexify the time variable in the following manner: put $\tau := t + \sqrt{-1}\sigma$, where $\sigma \in S^1$ and the boundary data is extended identically along the imaginary direction. Then, τ will belong to Σ, a cylinder of unit height and radius. Assume that $\pi : X \times \Sigma \to X$ is the projection on the first component. We then have the following $(1,1)$- forms on the product: $\Omega := \pi^*\omega$ and $\Omega_\phi = \Omega + dd^c_X\phi + dd^c_\tau\phi$. A curve, therefore, can be thought of as a potential $\phi(x,\tau)$ on $X \times \Sigma$. The equation of the geodesic connecting the two potentials ϕ_1 and ϕ_2 will then be equivalent to the following boundary value problem:

$$\begin{cases} \Omega_\phi^{n+1} = 0 \text{ on } X \times \Sigma \\ \phi|_{\partial\left(X\times S^1\times\{j\}\right)} = \phi_j \ , j = 0,1 \end{cases}$$

Donaldson had conjectured that in $\mathscr{H}$ the geodesics realise the minimum distance, and he further conjectured that the geodesic connecting two smooth potentials is smooth. The X.-X. Chen proved the following theorem in [13]:

Theorem 1.2.2. *[13] Assume that the potentials on the end points, ϕ_0 and ϕ_2, are smooth and $\omega + dd^c\phi_j > 0,\ j = 0,1$. Then, there exists a function with bounded $\partial\bar\partial$-derivatives on $X \times \Sigma$ which solves the geodesic equation weakly. More precisely, there exists a geodesic path $\phi(t) : [0,1] \to \mathscr{H}_{1,1}$ and a uniform constant such that $0 \leq dd^c_{X\times\Sigma}\phi \leq C$, wherein the subscripts for dd^c are to denote that the operators are restricted to the corresponding tangent directions.*

In this work, I shall present two different extensions to this theorem. First, we discuss how this theory can be extended to asymptotically locally euclidean Kähler spaces. In the second part, we shall present some estimates that will allow us to solve the geodesic equation between two potentials with singularities along divisors.

Chapter 2

The space of Kähler metrics over asymptotically locally euclidean manifolds

2.1 Introduction

[1] In this chapter, we present some generalisations of a the result of X-X. Chen reported in [13] for compact Kähler manifolds to the case of asymptotically locally euclidean Kähler manifolds. More specifically, In this chapter, we prove the existence of weak solutions with bounded dd^c-derivatives and derive decay estimates for the potential and the time derivative. This in particular implies that on each time slice the metric is an ALE metric in the extended sense. Further, we show the uniqueness of ALE metrics of constant scalar curvature in each Kähler class under some conditions on the decay rate of curvature and on the anti-canonical bundle.

If we view geodesics as curves with vanishing acceleration, as we shall see, the main rôle will be played by certain curves with preassigned non-zero acceleration which we will refer to as the ε-geodesics.

Theorem 2.1.1. *Let M be an asymptotically locally euclidean Kähler manifold. Assume that ϕ_0 and ϕ_1 are two potentials belonging to $\mathscr{H}^{\mu}_{ALE}$. Then, there is a unique geodesic with spatial laplacian, $\Delta\phi$ satisfying the decay property:*

$$|\Delta\phi| \leq C, \phi = O(r^{-\mu+2}), \partial_t \phi = O(r^{-\mu+2}), |\partial_{tt}\phi| \leq C \tag{2.1}$$

wherein C depends only on the end points and on the lower bound of the curvature

[1]The material for this chapter may be found in [1]

of the reference metric ω. In particular, at each time slice, the potential satisfies

$$\phi(t) = \tilde{\mathscr{H}}^{\mu}_{ALE}$$

One could derive the Euler-Lagrange equation associated to the energy of curves on $\mathscr{H}_{ALE}$ to be the following:

$$\mathscr{G}(\phi) := \phi'' - \mathfrak{g}_{\phi}^{\alpha\bar{\beta}} \phi'_{\alpha}\phi'_{\bar{\beta}} = 0 \tag{2.2}$$

wherein the prime sign denotes time derivative. As in the compact, we interpret the equation in the way that the problem is reduced to solving a degenerate complex Monge-Ampère equation. Namely, let us define $\Sigma = [0,1] \times S^1$, and view it as a Riemann surface with boundary, and extend the potentials on the S^1 factor in the trivial way. Further, let $\pi : M \times \Sigma \to M$ be the obvious projection. By pulling back the metric ω, we shall obtain $\Omega := \pi^*\omega$. Then, on $M \times \Sigma$ we may consider Ω-plurisubharmonic potentials. One may then see by a calculation that

$$\Omega_{\phi}^{m+1} = \mathscr{G}(\phi)\frac{\omega_{\phi}^{m}}{\omega^{m}} \tag{2.3}$$

which allows us to solve the following boundary value problem (2.4) instead.

$$\begin{cases} \Omega_{\phi}^{m+1} = 0 \\ \phi(x,i) = \phi(i), \ i = 0,1. \end{cases} \tag{2.4}$$

The reader is referred to [16] for more on this construction. Notice however that since there is not yet a lower bound on the rank of the complex hessian of the solutions to (2.4), one cannot guarantee the non-degeneracy of the volume form ω_{ϕ}^{m}. As a result, satisfying (2.4), although a necessary condition, is not sufficient for $\mathscr{G}(\phi) = 0$ to hold. One may therefore think of (2.4) as generalised geodesics.

The proof of the Theorem 2.1.1 is based on the resolution of the geodesic equation and proving appropriate asymptotic behaviour as the following theorem states:

Theorem 2.1.2. *Assume that the boundary conditions in the boundary value problem* (2.4) *belong to $\mathscr{H}^{\mu}_{ALE}$. Then, there exists a weak solution in the sense that it is continuous with bounded weak derivative satisfying the decay rates* (2.1).

Proof of Theorem 2.1.2 For the proof, we shall approximate the zero right hand side by strictly positive ones that tend to zero and derive estimates independent of the lower bound of the right hand side, f. This will guarantee the existence of a weak solution by the Arzelà-Ascoli theorem. Namely, we solve the following for positive ε on the right hand side and let ε tend to zero.

$$\begin{cases} \Omega_\phi^{m+1} = \varepsilon \\ \phi(x,i) = \phi(i), \ i = 0,1. \end{cases} \tag{2.5}$$

This is done in the following sections. In Section 2.3, we construct classical solutions for positive right hand side, f, on the strip. In Section 2.4, we derive weighted estimates independent of the lower bound of the right hand side, and thereby guarantee the decay rate of the laplacian of the weak solutions. This will prove that the same bounds hold weakly once one passes to the uniform limit obtained by applying the Arzelà-Ascoli theorem on compact subsets of the strip.

Theorem 2.1.3. *Let M be an ALE Kähler space with $c_1(M) \leq 0$ and $\mu > 2m-2$. Then, there is at most asymptotically locally euclidean Kähler metric of constant scalar curvature in each cohomology class determined by the deacy conditions of ALE potentials. In the particular case when $c_1 = 0$, in each Kähler class there exists one and only one scalar-flat metric which is further Ricci-flat.*

Besides the uniqueness issue, we can further prove the boundedness from below of *Mabuchi's $\mathscr{K}$-energy* as asserted in the following:

Theorem 2.1.4. *Let M be an asymptotically locally euclidean Kähler manifold with $c_1(M) \leq 0$ and $\mu \geq 2m-2$. Then, in each cohomology class, the metric of constant scalar curvature realises the global minimum of the $\mathscr{K}$-energy.*

By Theorem 2.1.3 in the case of vanishing first Chern class, 'Scalar-flat ALE Kähler metrics are Ricci-flat'. This assertion can already be proved using more standard methods as we shall describe in §2.5. For the existence of Ricci-flat metrics in the case of vanishing c_1 we rely on the work of Joyce on the extension of the Calabi conjecture to the ALE Kähler spaces. Along with Theorem 2.1.4, when $c_1(M) = 0$, the $\mathscr{K}$-energy is bounded from below and there always exists a metric of zero scalar curvature in each class which realises the minimum. See also Remark 2.2.2. In the case of $c_1(M) < 0$ however, the uniqueness result does not seem to follow from the methods known before.

2.2 Notation and definitions

In this section, we introduce the basic notations and definitions. The reader can find extensive background material for the subject in §8 of Joyce's book [25].

In what follows, we shall always consider operators such as laplacian and intrinsic derivatives in terms of the reference smooth ALE Kähler metric; the same is the case for constants in the estimates whose dependence is not explicitly stated.

Recall that an asymptotically locally euclidean, abbreviated to *ALE*, is a riemannian manifold that resembles $\mathbb{C}^m/G$ at distant points.

To make this idea more specific, let us fix a finite subgroup of $G \subset SU(m)$ that acts freely on $\mathbb{C}^m - \{0\}$. Then, the euclidean metric h_0 on $\mathbb{C}^m$ descends to a metric on the quotient $\mathbb{C}^m - \{0\}/G$. Let r be the euclidean distance on $\mathbb{C}^m$. Then, we have the following definition:

Definition 2.2.1. Let (M^m, J, g) –which henceforth we shall denote by M for the sake of brevity– be a non-compact Kähler manifold of dimension n. We say that M^n is *asymptotically locally euclidean* of parameter μ asymptotic to $\mathbb{C}^m/G$ provided that there exists a compact set $S \subset\subset M$, a ball of finite radius $B_0(R) \subset\subset \mathbb{C}^m$, and a diffeomorphism, also know as the coordinate system at infinity, $M - S \overset{\pi^{-1}}{\to} (\mathbb{C}^m - B_0(R))/G$. We require the difference of the euclidean metric h_0 and the pull-back of g to satisfy the following decay rates:

$$\nabla^k(\pi_*(\mathfrak{g}) - h_0) = O(r^{-\mu-k}) \text{ for } k \geq 0$$

wherein ∇ is the Levi-Civita connection associated to the flat metric h_0.

Notice that we have not imposed any restrictions on the complex structure J and its pull-back to $(\mathbb{C}^m - B_0(R))/G$, although in the case of spaces obtained as *crepant resolutions* of singular spaces, the coordinate at infinity is indeed a biholomorphism, cf. Chapter 6 in [25].

Remark 2.2.2. We know after the work of Bando, Kasue and Nakajima [3] on the asymptotic behaviour of the Calabi-Yau metrics that the decay rate of such metrics corresponds to the case $\mu = 2m$. In the light of this result, Ricci-flat ALE Kähler spaces satisfy the decay requirements.

Along the same lines, in order to parametrise the space of metrics, let us introduce the space of ALE Kähler potentials. Unlike the case of compact manifolds, there is no ambiguity of adding a constant and to each Kähler metric in the Kähler class there corresponds only one potential. We have the following definition:

Definition 2.2.3. For a given asymptotically locally euclidean Kähler manifold (M, ω, J) we define the space of ALE Kähler potentials to be as follows:

$$\mathscr{H}^{\mu}_{ALE} := \{\phi \in C^\infty | \omega + dd^c\phi > 0, \nabla^k\phi = O(r^{2-\mu-k}), 0 \leq k \leq 3\}$$

Also, we can define a weaker space to which we may refer as the *zero-th order* ALE Kähler potentials:

$$\tilde{\mathscr{H}}^{\mu}_{ALE} = \{\phi | \omega + dd^c\phi \geq 0, \phi = O(r^{2-\mu}), |\Delta\phi| \leq C\}$$

In particular, elements of $\mathscr{H}_{ALE}$ give rise to bounded metrics.

In the rest of this note, we shall refer to the laplacian operators of the metrics ω and ω_ϕ on each time slice by Δ and Δ_ϕ. In order to denote the laplacian on the total space $M \times \Sigma$ with respect to the Kähler forms Ω and Ω_ϕ we shall use $\tilde{\Delta}$ and $\tilde{\Delta}_\phi$.

We also define the following weighted version of Hölder spaces. For some negative real number β, let $\|f\|_{C^k_\beta}$ be defined as

$$\|f\|_{C^k_\beta} := \sum_{j=1}^{k} \sup_M \left| r^{j-\beta} \nabla^j f \right|$$

Let δ be the injectivity radius of the metric ω_0, and let $d(x,y)$ denote the distance between x and y with respect to ω_0. Since the definition is supposed to take farther points into account, one may as well think of the euclidean distance pushed forward via the chart $\pi : \mathbb{C}^m - B(0,R) \to M - K$. Also, let the semi-norm $[.]_{\alpha,\gamma}$ be defined as follows:

$$[f]_{\alpha,\gamma} \sup_{\substack{x \neq y \\ d(x,y) < \delta}} \left(((r(x) \vee r(y))^{-\gamma} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \right) \tag{2.6}$$

wherein $\vee$ denotes the minimum of two numbers. The definitions extend from functions to tensors in the obvious way. We then define the space C^k_β to consist of functions that have finite $\|.\|_{C^{k,\alpha}_\beta}$-norm defined as follows:

$$\|f\|_{C^{k,\alpha}_\beta} := \|f\|_{C^k_\beta} + \left[\nabla^k f \right]_{\alpha,\beta-k-\alpha} \tag{2.7}$$

It is probably the appropriate juncture to clarify the meaning of two key notions we shall use in this context: the ‘first Chern class’ and its sign. In general, notions such as Chern classes do not directly carry over from the framework of compact manifold without boundary to the non-compact case in a straightforward way. Heuristically speaking, we want a notion of the first Chern class that is compatible with the trivial topology of the ALE manifolds outside of some compact set $K \subset\subset M$. Therefore, we define an *admissible* hermitian metric h on the anti-canonical bundle, $-K_M$ as follows.

Let h_0 be the metric induced on $-K_M$ by an element of $\mathscr{H}^\mu_{ALE}$. We say that h is an admissible hermitian metric on $-K_M$ provided that the function $g := \frac{h}{h_0}$ satisfies has finite weighted Hölder norm on its derivatives up to the second order:

$$\|g\|_{C^{2,\alpha}_\beta} < \infty \tag{2.8}$$

for some α and $\beta < 2-2m$.

We have chosen this decay rate since it allows us to integrate the curvature. In particular, the notion of positivity and negativity for a line bundle can be carried over from the compact case and such notion stays well-define. More precisely:

Definition 2.2.4. Let M be an ALE Kähler manifold and $-K_M$ its anti-canonical bundle. We say that $-K_M$ is a negative (respectively positive) line bundle provided that there exists an hermitian metric h on $-K_M$ which satisfies the decay condition (2.8) and, further, its curvature form ρ_h is everywhere a non-positive (respectively non-negative) (1,1)-form and negative (respectively positive) at some point. The notion of zero c_1 can be also extended in the same manner.

We now show that such a notion of *sign* for a line bundle is well-defined provided that we impose the decay estimates on metrics. Let η_1, η_2 be two closed cohomologous forms with decay rates as in (2.8), such that $\eta_2 \geq 0$ whereas $\eta_1 < 0$. Since η_1 and η_2 are required to satisfy the decay conditions and $[\eta_1 - \eta_2] = 0$, the weighted dd^c-lemma, Theorem 8.4.4 in [25], then states that $\eta_2 = \eta_1 + dd^c v$ where $v \in C^{2,\alpha}_{\beta+2}$. Again, the notion of a form being cohomologous to zero should be understood in the category of forms with appropriate decay. Gaffney's extension of Stokes's theorem allows us to integrate by parts and thus observe that $\int_M dd^c v \wedge \omega^{m-1} = 0$. We obtain therefore that

$$0 \leq \int_M \eta_2 \wedge \omega^{m-1} = \int_M (\eta_1 + dd^c v) \wedge \omega^{m-1} = \int_M \eta_1 \wedge \omega^{m-1} < 0$$

which is a contradiction.

2.3 Classical solution of the equation on the the product of the manifold and the compact Riemann surface with positive right hand side

In order to put the geodesic equation that we study here into perspective, we shall here give the basic concepts. More details may be found in the original works which appear with the following historical order in [30], [32], and [16].

As Mabuchi had observed, having fixed a Kähler metric ω_0 as the reference, the space of Kähler potentials over a compact Kähler manifold X, denoted by $\mathscr{H}$, can be endowed with the structure of an infinite dimensional manifold. Formally, the tangent space at any point of this manifold is isomorphic to the space of sufficiently smooth functions on X. It is further endowed with a Weil-Petersson-type metric as follows. At a given potential $\phi \in \mathscr{H}$, let $f, g \in T_\phi \mathscr{H}$ be two vectors tangent to $\mathscr{H}$.

$$\langle \zeta, \xi \rangle_\phi := \int_X \zeta \xi \omega_\phi^m \tag{2.9}$$

With respect to this metric, one may define the notions of length and energy of curves amongst other geometric quantities associated to a riemannian space, at least formally. In analogy to the case of finite dimensional compact manifolds, one define the geodesics to be the stationary points of the energy functional over the space of curves with a fixed end points. The Euler-Lagrange equation corresponding to the energy functional turns out to be equation 2.2.

One may notice that the inner product defined in (2.9) makes sense for potentials belonging to L^2 , in other words, when we have the polynomial decays, when the parameter μ satisfies $\mu > 2m+2$. Nevertheless, as we shall see, for our applications we do not need the Mabuchi metric to be finite and hence this integral to be convergent; we nevertheless are still able to utilise the properties of the geodesic operator.

In this section, we consider the complex Monge-Ampère equation on the product of the asymptotically locally euclidean manifold M and the cylinder Σ -viewed as a Riemann surface. An appropriately chosen sequence of such solutions will then be used to construct a weak solution to the degenerate equation. But the classical solution is important in its own right as we shall see in §2.5 as the ε-geodesics are our tool in proving Theorems 2.1.2 and 2.1.3. The complex Monge-Ampère equation was solved in [25] on ALE manifolds without boundary, but in our case, the presence of the boundary requires a different treatment.

In order to solve the equation with the right hand side f asymptotically equal to a constant, we shall take a sequence of compact domains that expand to the strip. In order to prove the existence of classical solutions on the strip, we shall establish uniform estimates up to order $C^{2,\gamma}$ on compact sets. We will prove uniform laplacian and L^∞ bounds for such solutions. Existence of the laplacian bounds leads to the uniform ellipticity of the linearised operator which will be used in deriving the estimates for the degenerate case.

Theorem 2.3.1. *Consider the boundary value problem*

$$\begin{cases} \Omega_\phi^{m+1} = e^f \Omega^{m+1} \; ; M \times \Sigma \\ \phi = \psi \; ; \; \partial (M \times \Sigma) \end{cases} \tag{2.10}$$

wherein M and Σ are an ALE Kähler manifold and the cylinder respectively, and $f \in C^3(M \times \Sigma)$ satisfies $f = c$, for some positive number c, outside of some set of the form $K \times \Sigma$, where $K \subset\subset M$. Then, this problem has a unique solution in

$C^{2,\gamma}(M \times \overline{\Sigma})$ *for some* γ.

We observe that the estimates we derive for the compact domains are independent of their size and are therefore uniform.

Proof of Theorem 2.10 As mentioned before, we solve the equation on a sequence of compact domains that grow larger and cover the entire strip. Note that the right hand side is kept constant in the process. The boundary condition has be chosen appropriately as we shall explain below. We shall first the detail the technical points that need to be taken into account in the construction of such domain below. Afterwards, we prove the existence of uniform L^∞, and laplacian bounds independent of the size of the sets in this family. In order to make the proof easier to follow, the details of proofs of *a priori* estimates are postponed to Propositions 2.3.2. Having proved the laplacian estimates, one obtains $C^{2,\gamma}$ uniform estimates via the extension of the Evans-Krylov theory to complex hessian equations as is done in [33] for the interior estimates. The boundary $C^{2,\gamma}$ estimates follow from the boundary estimates in the proof Theorem 1 in [11]. Note that since we obtain a uniform bound on the laplacian, the equation becomes uniformly elliptic with uniformly bounded complex hessian. This means that the exponent γ and the $C^{2,\gamma}$ norm are uniformly bounded from above on the entire domain $M \times \Sigma$.

The construction of the compact domains converging to the strip is as follows. Let B_T be the metric ball with respect to the metric ω on the manifold M. Set $\mathscr{G}_T \subset M \times \Sigma$ be the domain obtained by smoothing the corners of the region $B_T \times \Sigma$. Let ψ_T be the function obtained by restricting the function ψ, constructed in §2.13, to $\mathscr{G}_T$.

We solve the problem on each $\mathscr{G}_j$ for $j \in \mathbb{N}$ along with uniform estimates up to $C^{2,\gamma}$, i.e for each $\mathscr{G}_j$ we solve

$$\begin{cases} \Omega_{\phi_T}^{m+1} = e^f \Omega^{m+1} \ ; \mathscr{G}_T \\ \phi_T = \psi_T \ ; \ \partial \mathscr{G}_T \end{cases} \tag{2.11}$$

Since the sets $\mathscr{G}_j$ exhaust the strip, by uniform continuity on the compact sets and by the usual diagonal argument one may obtain a solution saisfying the same $C^{2,\gamma}$ estimates on the entire strip. We then have to guarantee that these estimates remain valid as $T \to \infty$.

Proposition 2.3.2. In the family of boundary value problems 2.11, we have that for all domains $\mathscr{G}_T$, defined in the proof of Theorem 2.10, the quantities $\|\phi\|_{L^\infty}$, $\|\nabla\phi\|_{L^\infty}$, and $\|\Delta\phi\|_{L^\infty}$ are bounded independent of T.

Proof. In order to prove the upper bounds, notice that owing to the fact the boundary data are extended trivially along the S^1-factor, the solution is indeed convexi in the time direction. The convexity in the temporal direction we have that the upper bound in the interior does not exceed that of the boundary. As one may check, for sufficiently large C, the function ψ, which indeed agrees with the solution on the boundary of the domain $\mathscr{G}_T$, clearly serves as a sub-solution and hence provides a lower bound.

In order to prove the laplacian estimates we follow the calculation of Aubin as done in [33], §3 of Chapter 2. Namely, in the normal coordinates at any point one has:

$$\sum_{j=1}^{m+1} \frac{1}{1+\phi_{j\bar{j}}} - B \leq \tilde{\Delta}_\phi \left((m+1+\tilde{\Delta}\phi) + (B+1)\phi \right) \tag{2.12}$$

wherein B is a constant. In the inequality above, all the operators act on both time and space directions. One can now follow the standard line of argument: either the quantity $(n+1+\tilde{\Delta}\phi)+(B+1)\phi$ attains its maximum in the interior, in which case we have an upper bound on $\sum_j \frac{1}{1+\phi_{j\bar{j}}}$ and thereby on $\tilde{\Delta}\phi$, or its maximum is attained on the boundary. Since we have already found a uniform L^∞ bound, finding an estimate on the boundary for the laplacian establishes a uniform estimate.

It is essential to note that the constant C on the right hand side depends only on the curvature properties of the underlying manifold M. In fact, what is needed is a lower bound of the bisectional curvature, $\inf_{\mathscr{G}_T,\alpha,\beta} R_{\alpha\bar{\alpha}\beta\bar{\beta}}$, which is, since M is asymptotically locally euclidean and Σ is flat, bounded independent of T.

In order to prove the boundedness of the quantity $\tilde{\Delta}\phi_T$ at the boundary points we follow Chen's approach in [13], which in turn was inspired by a previous work of B. Guan on the Dirichlet problem for the complex Monge-Ampère equation [22]. Thanks to the behaviour of the boundary conditions, the boundary estimate for the laplacian remains valid independent of T. Hence, the laplacian estimates remain valid independent of T.

Further, we know that for a fixed function f on the strip, boundedness of laplacian leads to strict ellipticity of the operator, which, in turn, combined with a version of the Evans-Krylov theory adapted to the operators of the complex hessian, one obtains $C^{2,\gamma}$ bounds, see §4 [33]. Observe that the exponent γ and the norm $\|\phi\|_{C^{2,\gamma}(\mathscr{G}_T)}$ only depend on the L^∞ and the laplacian estimates, and are, therefore, uniform for all T. This finishes the proof of the existence of classical solutions for the boundary value problem 2.10 on the strip, with globally bounded laplacian.

□

Let us turn to proving the L^∞ bounds. The upper bounds are obtained in this case as in 2.4.1. If the data is not necessarily invariant in the S^1 direction, one can notice that the function is indeed sub-harmonic in the time direction and its maximum therefore appears on the boundary. For the lower barrier, consider the function:

$$\tilde{\psi} = Cr^{-\lambda}t(t-1) + t\phi_0 + (1-t)\phi_1 \tag{2.13}$$

One may verify that this function is a lower barrier for the solutions once $\mu < \lambda < 2\mu + 2$, and the constant C is chosen to be appropriately large.

As for the laplacian bounds, we observe that the arguments for proving the laplacian estimates based on maximum principle are only depend on a lower bound of the curvature of the reference metric ω, which on an ALE space are bounded, and on the L^∞ estimates. In particular, as the domains expand, the estimates are not affected once we can prove uniform estimates on the boundary.

2.3.1 Decay of the higher derivatives of the ε-solutions

As we shall see later in the section on the uniqueness of metrics of constant scalar curvature, for any positive ε on the right hand side, we shall need suitable asymptotics for the curvature that will allow us to integrate by parts the terms that involve the higher derivatives of the ε-*approximate geodesics*. We therefore make the following assertion concerning space derivatives of solutions.

Proposition 2.3.3. In the boundary value problem (2.10), assume that $f > \frac{\varepsilon}{2} > 0$, and that f is equal to ε outside of a compact set, and that the boundary values belong to $\mathscr{H}^{\mu}_{ALE}$. Then, for any $\varepsilon > 0$ we have:

$$\begin{aligned} |\nabla_\alpha \nabla_\beta \nabla_\gamma \phi| &\leq Cr^{-\mu-1} \\ |\nabla_\alpha \nabla_\beta \nabla_\gamma \nabla_\theta \phi| &\leq Cr^{-\mu-2} \ \alpha, \beta, \gamma, \theta \in \{1, ..., m, \bar{1}, ..., \bar{m}\} \end{aligned} \tag{2.14}$$

wherein C depends on $\varepsilon > 0$. As a consequence,

$$|Rm(\omega_\phi)|, |Rc(\omega_\phi)|, |K(\omega_\phi)| \leq Cr^{-\mu-2} \tag{2.15}$$

wherein $Rm(\omega_\phi), Rc(\omega_\phi)$ and $K(\omega_\phi)$ are the curvature tensor, the Ricci tensor, and the scalar curvature.

Further, we have that

$$|\nabla_\alpha \phi'|, |\nabla_\alpha \nabla_\beta \phi'| \leq Cr^{-\mu+1} \tag{2.16}$$

Proof. The proof is an application of Schuader estimates to the space derivatives and a type of boot-strap argument. We can apply the Schauder estimates since we already know membership in $C^{2,\alpha}$ of the potential for the ε-solution.

We shall first derive estimates for the first space derivative. Since at sufficiently far points the right hand side is constant, at those points we obtain the following by differentiating the equation

$$\tilde{\Delta}_\phi \nabla_\xi \phi = 0 \tag{2.17}$$

for some unit spatial direction ξ. We shall now use the fact that the quantity $\nabla_\xi \phi$ using an appropriate barrier. Since the boundary conditions for $\nabla_\xi \phi$ decays like $r^{-\mu+1}$, one can easily verify that a function $v := r^{-\kappa}(t-t^2) + r^{-\mu+1}$ is an upper barrier when $\kappa < \mu+1$. This in particular means that any space derivative decays at least at the rate of r^{-2m+1}, which is the same as the decay rate of the boundary conditions.

Using the decay rate for the first derivatives, we now prove some decay estimate for the space derivatives up to the third order along with their Hölder semi-norms. Consider now the domains $G_R \subset G'_R \subset M \times \Sigma$ defined as follows. Define $G_R := \{(x,t) | R-1 < \rho(r) < R+1\}$, and $G'_R := \{(x,t) | R-2 < \rho(x) < R+2\}$. The decay rate we have obtained guarantees that on the pieces of the boundary $\{\rho(x) = R-2\}$ and $\{\rho(x) = R+2\}$, the quantity $\nabla_\xi \phi$ is bounded by $CR^{-\mu+1}$. Owing to the decay rates of the boundary conditions on the other hand, on the two components $T_{0,1} := \partial(\Omega'_R) \cap \{t = 0,1\}$ we have $\|\nabla_\xi \phi\|_{C^{2,\alpha}(T_{0,1})} \leq CR^{-\mu+1}$. We conclude, by the Schauder estimates, that we have on G'_R:

$$\|\nabla_\xi \phi\|_{C^{2,\alpha}(\Omega_R)} \leq CR^{-\mu+1}$$

As a result, any of the third order derivatives that have at least one spatial direction belong to C^α and their C^α norm is bounded by $CR^{-\mu+1}$. Particularly, this proves (2.16).

We can now differentiate (2.17) in a unit spatial direction ξ' to obtain:

$$\tilde{\Delta}_\phi \left(\nabla_{\xi'} \nabla_\xi \phi \right) = \mathfrak{g}_\phi^{i\bar{n}} \mathfrak{g}_\phi^{m\bar{j}} \phi_{;m\bar{n}\xi'} \phi_{;\xi i\bar{j}} \tag{2.18}$$

wherein the Latin indices vary over both time and space coordinates. Thanks to the decay estimates for the third derivatives of the potential with at least one spatial direction, we observe that on domains G_R and G'_R, the right hand side of (2.18) satisfies:

$$\|\mathfrak{g}_\phi^{\alpha\bar{\nu}} \mathfrak{g}_\phi^{\mu\bar{\beta}} \phi_{\mu\nu\xi'} \phi_{\xi\alpha\bar{\beta}}\|_{C^\alpha(G'_R)} \leq CR^{-2\mu+2}$$

By using an argument similar to the poof of the $C^{3,\alpha}$ estimate in the space direction, we have that

$$\|\phi\|_{C^{4,\alpha}(G'_R)} \leq CR^{-\mu-2}$$

which finishes the proofs of (2.14) and (2.15).

$\square$

2.4 *A priori* estimates for the geodesic equation

As we have seen, the laplacian estimates are reduced to deriving the laplcian estimates at the boundary. Further, the estimates of in [22] work in the case of degenerate right hand side as well and this was further used in [13]. The proof of Proposition 2.3.2 therefore carry over to the case of degenerate right hand side. We only prove the weighted L^∞ estimates.

2.4.1 L^∞-estimates

Similar to the case of compact manifolds, we derive the L^∞ estimates. It will be enough to find a sub- and a super-solution in order to find upper and lower bounds on the function.

For the upper bound, notice that the function ϕ is indeed convex in the time direction, namely $\phi'' \geq 0$. Therefore, the upper bounds can only occur on the boundary points. This, in particular, proves that the upper bound decays at the same rate as the boundary conditions. As for the sub-solution, one may consider any Ω-plurisubharmonic function that restricts to the the boundary conditions. Consider the following function:

$$\psi(\mathfrak{z},t) := t\phi_0 + (1-t)\phi_1 + Cr^{3-2\mu}(t^2-t)$$

wherein γ is the exponent appearing in Theorem 2.10. We show that if the constant C is chosen to be large enough, then one observes that ψ is indeed a sub-solution for the homogeneous problem. To see this, we first see by direct calculation that

$$\begin{aligned} \psi(\mathfrak{z},t)_{t\alpha} &= O(r^{-\mu+1}) \\ \psi(\mathfrak{z},t)_{t\bar{\beta}} &= O(r^{-\mu+1}) \\ \psi(\mathfrak{z},t)_{tt} &= O(r^{-2\mu-3}) \end{aligned} \tag{2.19}$$

Using these rates of decay, by substituting these terms into the following operator

$$\left(\psi_{tt} - \mathfrak{g}_\psi^{\alpha\bar{\beta}}\psi_{t\alpha}\psi_{t\bar{\beta}}\right)\frac{\omega_\psi^m}{\omega^m}$$

we see that for sufficiently large C, this expression is positive, whereby we conclude that ψ is a subsolution.

2.5 $\mathscr{K}$-energy, metrics of constant scalar curvature

Although we shall not explicitly make use of it, one observes that thanks to the asymptotic behaviour of the potentials and their derivatives we can define the Mabuchi L^2-metric on the space of potentials when $m > 2$. What we need is will be the geodesics, the ε-geodesics to be more precise, regardless of their relevance as the extrema of the length functional.

In our calculation in the proof of the next theorem we shall need the following second order operator also referred to as the *Lichnerowicz operator*. It is the complementary part of the -real- hessian to the tensor $dd^c u$. The operator $\mathscr{D}$ applied to a real valued function u, is defined in local coordinates as:

$$\mathscr{D}u := \nabla_\alpha \nabla_\beta u d\mathfrak{z}^\alpha \otimes d\mathfrak{z}^\beta$$

One important property of this operator that we shall use is that the if u lies in the kernel of $\mathscr{D}$, then the vector field

$$\uparrow \bar{\partial} u := \mathfrak{g}^{\alpha\bar{\beta}} \frac{\partial u}{\partial \mathfrak{z}^{\bar{\beta}}} \frac{\partial}{\partial \mathfrak{z}^\alpha}$$

is holomorphic, cf. §1.22 in [20].

In this section we shall always assume that the first Chern class, $c_1(M)$, is non-positive. As introduces in the case of compact manifolds in [29], we define the $\mathscr{K}$-energy by its differential as follows:

$$\delta_\psi \mathscr{K} = -\int_M K_\phi \psi \omega_\phi^n \tag{2.20}$$

wherein K_ϕ is the scalar curvature of the metric ω_ϕ. Notice that by the asymptotics behaviour of the potentials, we know that the integral above is convergent. In this section, we shall extend the proof given in [13] for the uniqueness of metrics of constant scalar curvature in each cohomology class of compact Kähler manifolds to the case ALE Kähler manifolds. It may be seen as a generalisation of the uniqueness theorem for ALE Ricci-flat Kähler metrics proved in [25].

We now turn our attention to the proof of the uniqueness assertion. As we shall see, geodesics are not the only -owing to their lack of regularity, if they are at all- suitable curves for our purpose. The geodesics nevertheless help us find the right curves for our purpose. As we shall see any curve in $\mathscr{H}$ with appropriately assigned, and not necessarily vanishing, acceleration, called ε-geodesics in [13], will do.

Proof of 2.1.3. We follow Chen's approach in [13]. Let us first assume that we have sufficiently regular geodesics as was done in [16] to motivate our choice of curves. Evidently, a metric of constant zero scalar curvature is a stationary point of the functional $\mathscr{K}$. Consider now two potentials ϕ_0 and ϕ_1 that realise two distinct metrics of constant scalar curvature, in particular, two stationary points of the $\mathscr{K}$-energy. By a formal calculation, one obtains

$$\frac{d^2\mathscr{K}}{dt^2} = \int_M |\mathscr{D}\phi'(t)|^2_\phi \omega^n_\phi$$

But since we do not have higher regularity of solutions to the geodesic equation, the term $\mathscr{D}\phi'$, which requires three bounded derivatives, cannot be defined. If the calculation were valid however, one could easily deduce that $\mathscr{K}$ had to be constant along the geodesic connecting ϕ_0 and ϕ_1. Further, one deduces that the term $\mathscr{D}\phi'$ would have to vanish. As a result, $\uparrow\bar{\partial}\phi'$ would have to be the real part of a holomorphic vector field (cf. Lemma 1.22.2 in [20]). Assuming that the same decay rates proved in Proposition 2.3.3 hold for the solutions of the homogeneous complex Monge-Ampère equation, $\uparrow\bar{\partial}\phi'$ would have to be a holomorphic vector field on the entire manifold M which decays at infinity. However, as we shall see in Lemma 2.5.2, there are no such vector fields but the trivial one. Namely, the differential of the function ϕ' would identically vanish . One hence concludes that ϕ' is constant in the space direction. But since the only space-independent solution of the geodesic equation is the linear interpolation in time of the boundary conditions, one has that $\omega_{\phi_0} = \omega_{\phi_1}$.

To overcome the problem of lack of higher regularity of the solutions of the geodesic problem we will use a family of geodesics that approximate the homogeneous problem and follow a similar path of reasoning for proving that $\phi_0 = \phi_1$.

Let ω be the an ALE Kähler metric cohomologous to ω_0 such that $\rho(\omega) \leq 0$, wherein $\rho(\omega)$ is the Ricci form of the Kähler form ω. The existence of such an ALE metric $\omega \in [\omega_0]$ is guaranteed since the first Chern class, c_1, is assumed to be non-positive, and we further know that by the extension of the Calabi conjecture to the ALE space any closed real (1,1)-form $\chi \in [\rho(\omega_0)]$ with appropriate asymptotic behaviour may be realised as the Ricci form of some unique ALE Kähler metric. (see §8.4 and 8.5 of [25] for more details on the de Rham cohomology on ALE spaces and the proof of the Calabi conjecture on ALE spaces). In the case of vanishing first Chern class one could choose ω_0 to be Ricci-flat, and in the case of negative c_1 the form ω_0 could be chosen to be so that the Ricci form $\rho(\omega_0)$ is a negative on some bounded set and zero outside of it. Let us define $\mathscr{G}(\phi) := \phi'' - \frac{1}{2}|d\phi'|^2_\phi$. We now integrate by parts and obtain about the second derivative of the $\mathscr{K}$ along an arbitrary curve, in particular an approximate geodesic, $\phi(t)$:

$$\frac{d^2\mathscr{K}}{dt^2} = \int_M |\mathscr{D}\phi'|^2_\phi \omega^m_\phi - \int_M \mathscr{G}(\phi) K_\phi \omega^m_\phi \tag{2.21}$$

Let us observe that by the decay estimates from Proposition 2.3.3 the integral in the equation above, as well as in the rest of the calculations in this section, are finite. We now prove that the second integral is non-negative as well.

$$\begin{aligned} -\int_M \mathscr{G}(\phi) K_\phi \omega^m_\phi &= -\int_M \mathscr{G}(\phi) Ric(\omega_\phi) \wedge \omega^{m-1}_\phi \\ &= -\int_M \mathscr{G}(\phi) \left(Ric(\omega_\phi) - Ric(\omega)\right) \wedge \omega^{m-1}_\phi \\ &- \int_M \mathscr{G}(\phi) Ric(\omega) \wedge \omega^{m-1}_\phi \\ &= \int_M \mathscr{G}(\phi) dd^c \log \frac{\omega^m_\phi}{\omega^m} \wedge \omega^{m-1}_\phi - \int_M \mathscr{G}(\phi) Ric(\omega) \wedge \omega^{m-1}_\phi \\ &= -\int_M d\mathscr{G}(\phi) \wedge d^c \log \frac{\omega^n_\phi}{\omega^m} \wedge \omega^{m-1}_\phi - \int_M \mathscr{G}(\phi) Ric(\omega) \wedge \omega^{m-1}_\phi \\ &= \int d\mathscr{G}(\phi) \wedge d^c \log \mathscr{G}(\phi) \wedge \omega^{m-1}_\phi - \int_M \mathscr{G}(\phi) Ric(\omega) \wedge \omega^{m-1}_\phi \\ &= \int_M \frac{|\nabla \mathscr{G}(\phi)|^2_\phi}{\mathscr{G}(\phi)} \omega^m_\phi - \int_M \mathscr{G}(\phi) Ric(\omega) \wedge \omega^{m-1}_\phi \end{aligned} \tag{2.22}$$

where we have used the fact that $\log \mathscr{G}(\phi) = -\log \frac{\omega^m_\phi}{\omega^m} + \log \varepsilon$. Notice that the integration by parts carried out above is meaningful by virtue of the asymptotics proved in 2.3.3. More precisely, the asymptotics guarantee the membership in L^1 of the integrands, which, then, by Gaffney's extension of Stokes's theorem on compact riemannian manifold to the case of non-compact manifolds [19], proves the validity of integration by parts.

Now since $Ric(\omega) \leq 0$ and $\mathscr{G} > 0$, the second term is a non-negative finite quantity.

In other words, along the ε-approximate geodesic we have that the $\mathscr{K}$-energy is convex. Further, since the end points are scalar-flat metrics, they are stationary points of the $\mathscr{K}$-energy. Hence, the $\mathscr{K}$-energy is constant along the path, and in particular $\mathscr{D}\phi' = 0$ and we can use 2.5.3 to repeat the argument given in the beginning of the proof with the assumption of the smoothness of geodesics to obtain $\phi_0 = \phi_1$.

□

The following observation can be verified by calculation in local coordinates and we therefore omit its proof.

Lemma 2.5.1. *Let (X,η) be a Kähler manifold. Assume that Ric ≤ 0, and let $\mathfrak{Z}$ be a holomorphic vector field on X. Then $\|\mathfrak{Z}\|_\eta^2$ is a sub-harmonic function, i.e. $\Delta_\eta \|\mathfrak{Z}\|_\eta^2 \geq 0$.*

Lemma 2.5.2. *Let (M,η) be an ALE Kähler space. Then, any non-negative sub-harmonic function u which belongs to L^p for some $p \geq 1$ is identically zero.*

Proof. The proof follows from the mean value inequality for subharmonic functions, along with the observation that if $u \geq 0$ is subharmonic, then so is u^p for any $p \geq 1$. Namely, we have:

$$u(x)^p \leq \frac{1}{Vol(B(x,R))} \int_{B(x,R)} u^p$$

Since the volume of balls grows polynomially whereas for some p the integral of u^p over balls is bounded, u must be zero.

□

Corollary 2.5.3. Let (M,η) be an ALE Kähler space with $c_1(M) \leq 0$ in the sense defined before. Then, any holomorphic vector field, $\mathfrak{Z}$ that satisfies $\|\mathfrak{Z}\|_\eta \in L^p$ for some $p \geq 1$ must be identically zero. In particular, for any solution to the ε-geodesic problem with two scalar-flat end-ponts on a manifold with $c_1(M) \leq 0$, the time derivative vanishes $\phi' = 0$.

Having proved the uniqueness of metrics of constant scalar curvature in each Kähler class, we can now conclude this section by the proof of the boundedness from below of the $\mathscr{K}$-energy on such Kähler manifolds.

Theorem 2.1.4. Let ψ be an arbitrary ALE potential cohomologous to χ, where χ is defined as the proof of 2.1.3. Let $\phi(t)$ be some smooth enough path connecting the two potentials. By the calculations in the the proof of 2.1.3, we have that along the curve $\phi(t)$ the $\mathscr{K}$-energy is convex. Also, since χ realises the minimum of the $\mathscr{K}$-energy, the first derivative of $\mathscr{K}$ along ϕ vanishes at χ. Hence, $\mathscr{K}$ is strictly increasing along $\phi(t)$ which proves the claim.

□

Corollary 2.5.4. Let (M,ω) be an ALE Kähler manifold with $c_1(M) = 0$. Then, $Ric(\omega) = 0$ if and only if it is of constant zero scalar curvature. In other words, any scalar-flat Kähler metric in this case is Ricci-flat. Further, in each Kähler class there exists one and only one metric of constant scalar curvature which is further Ricci-flat.

Proof. Obviously, the fact that being Ricci-flat implies being scalar flat requires no proof. So we only prove the converse. Uniqueness of the scalar-flat metrics in each Kähler class is given by Theorem 2.1.3. By the work of Joyce, Theorem 8.5.1 in [25], there always exists a Ricci-flat metric in each Kähler class when $c_1 = 0$ in the sense of Definition 2.2.4. These two results together prove the claims. $\square$

In the particular case of $c_1 = 0$ we shall give a more direct proof of the uniqueness of ALE metrics of constant scalar curvature. I am not aware of this proof having been adapted to the case of ALE manifolds and I found it worth mentioning here.

Proposition 2.5.5. Let (M, ω) be an ALE Kähler manifold. Then, the Ricci form $\rho(\omega)$ is co-closed, and therefore harmonic, if and only if the scalar curvature s is a constant.

Proof of Proposition 2.5.5. The fact that the Ricci form, ρ, is co-closed, and hence harmonic, is a punctual fact and independent of the global geometry of the Kähler space, cf. Proposition 1.18.2 in [20]. It suffices now to prove that any harmonic form with appropriate asymptotics is indeed co-closed.

$\square$

We can now give the following proof of the uniqueness of Corollary 2.5.4 using the more classical approach.

Alternative proof of Corollary 2.5.4. By Proposition 2.5.5 we know that since the scalar curvature vanishes identically, the Ricci form ρ must be harmonic. We may now evoke the Hodge-de Rham-Kodaira decomposition on ALE manifolds, see Theorem 8.4.1 in [25]. In particular, this means when $c_1 = 0$, the only harmonic form is the trivial one. Noting however that $\rho \in \frac{1}{2\pi} c_1$ yields $\rho = 0$. By the extension of the Calabi conjecture to ALE Kähler spaces detailed in [25], we know that in the same Kähler class one there exists a Ricci-flat, and hence scalar-flat, metric. $\square$

Chapter 3

The space of Kähler potentials with prescribed singularity

3.1 Introduction

This chapter is dedicated to the resolution of the geodesic equation when the boundary potentials are singular. The results of this chapter have been reported in [2].

We start byu noticing that the existence theorem in [13] requires the Kähler metrics on the boundaries to be smooth and strictly positive everywhere. In a more recent work presented in [23], from where our estimates are inspired, W. He has proved that the assumption on the boundary data may be weakened by considering the original geodesic equation, (1.5). Namely, one may still prove regularity of the solution for the boundary conditions whose associated metrics are possibly positive semi-definite and have a bound on their laplacians.

Theorem 3.1.1. *[23] Let ϕ_0 and ϕ_1 be two potentials with bounded laplacian. Then, there exists a generalised solutions, $\phi(t)$, of the geodesic equation, such that*

$$0 \leq n + \Delta\phi(t) \leq C$$

where $C = C(\|\phi_{0,1}\|_\infty, \|\Delta\phi_{0,1}\|_\infty, \omega)$. [1]

A modification of the estimates used to prove Theorem 3.1.1 will be used in the present note in order to derive weighted laplacian estimates. Combined with the observation in [5], it yields the following *a priori* bound, independent of $\varepsilon > 0$.

$$|\partial_t \phi| + |\Delta\phi| < C$$

[1] In this article, the laplacian, Δ, is defined so that it has negative spectrum.

A class of singularities are the so-called *conical* singularities a particular case of which are the *orbifold-type* singularities. The study of such singularities on Riemann surfaces goes back to E. Picard [31]. Recently, after Donaldson's linear theory [17] on conical Kähler metrics, many interesting results have emerged. For instance, cf. [6, 24], [28] and [34]. In [18] this is extended to the Poncaré-type singularities. This generalises the work of Troyanov in [35] on preassigning the Gauß curvature on Riemann surfaces with prescribed conical singularities. Also, in his study of extremal hermitian metrics, the second author considered conical metrics on Riemann surfaces in [12].

Existence of geodesics between conical metrics was studied in [9], where the approaches of [24] and [17] were combined with those in [13] in order to obtain the following.

Theorem 3.1.2. *[9] Let $\mathscr{H}_C \subset \mathscr{H}_{\omega_0} \cap C^{2,\alpha}_{\beta}$ denote the Kähler potentials with bounded Levi-Civita connection and lower bound on Ricci curvature. Then, any two Kähler cone metrics in $\mathscr{H}_C$ can be connected by a unique $C^{1,1}_{\beta}$ cone geodesic.*

In particular, this proves the existence of geodesics once the cone angle is small, $\beta < \frac{1}{2}$. The methods used to prove the theorem above are more intrinsic than the methods we have adopted here in the sense that analysis is done in appropriate function spaces with respect to the cone metric itself as opposed to our approach, which uses a smooth reference metric.

3.2 Main results

Our main focus in the present note will be on proving the a theorem which guarantees existence and uniqueness of geodesics between two metrics with conical singularities along a given divisor such that at a given time slice, the space derivatives are bounded on the points away from the divisor. Our method is a modification of the estimates by W. He and is motivated by the work of Berdtsson in [5], where existence of weak geodesics between bounded potentials is used in order to prove uniqueness of weak Kähler-Einstein metrics with bounded potentials in case of positive Chern class. In light of the crucial rôle this theorem playes in the recent work of Donaldson, Chen, and Sun, [14], it is perhaps important to give a more direct proof of Berdntsson's result on the uniqueness of geodesics between two singular Kähler-Einstein metrics with $C^{1,1}$ bounded potentials, utilizing the main theorem in this paper. We prove the following existence result. For notations and definitions, see §3.3.

Theorem 3.2.1. *Let ϕ_0 and ϕ_1 be two potentials whose corresponding metrics ω_{ϕ_0} and ω_{ϕ_1} have conic singularities of angle β along the smooth divisor V. Then,* *there is a unique weak geodesic, $\phi(t)$, connecting them in the following sense: the solution is everywhere Hölder continuous, and on compact sets away from the singularity, its -spatial- complex hessian is uniformly bounded.*

We can then formulate the more general version of theorem 3.2 the previous theorem as follows.

Theorem 3.2.2. *Assume that ϕ_0, ϕ_1 are two potentials which belong to the space $\mathscr{H}_\xi(X)$ for a singularity $\mathscr{S}$, and an admissible weight function ξ. Then, there is a unique weak geodesic connecting them with bounded laplacian away from the singular set.*

The proof of Theorem which follows depends on the *a priori* bounds that are proved in Theorem 3.2.3.

Proof of Theorem 3.2 We prove Theorem 3.2 by considering the following family of boundary value problems to approximate the degenerate equation 1.5.

$$\begin{cases} \left(\phi_{tt} - |\partial \phi_t|^2_\phi\right) \omega^n_\phi = \frac{\varepsilon e^f}{\left(|s|^2+\eta\right)^p} \omega^n \\ \phi(x,i) = \phi(i),\ i = 0,1. \end{cases} \tag{3.1}$$

and its equivalent form

$$\begin{cases} \frac{\Omega^{n+1}_\phi}{\Omega^{n+1}} = \frac{\varepsilon e^f}{\left(|s|^2+\eta\right)^p} \text{ on } X \times \Sigma \\ \phi|_{\partial(X\times R)} = \phi_j\ , j = 0,1 \end{cases} \tag{3.2}$$

Uniqueness of generalised solutions amongst bounded potentials is already known, see for example [5]. Here and hereafter, we take η to be $\eta(\varepsilon)$ so that $\eta \to 0$ as $\varepsilon \to 0$. We shall, however, drop the explicit dependence. We remark whenever required as to how this dependence may be chosen so that the estimates will hold. In order to prove existence of solutions to this equation, we will need to prove *a priori* estimates for the second derivative of the potential. These estimates are stated in Theorem 3.2.3. Since the second derivative will blow up at a certain rate, depending on the cone angle, close to the divisor, we need to prove that the rate of blowing up is bounded close to the divisor. In particular, we prove that on any compact set not intersecting $\mathscr{S}$, the gradient, $\|\nabla\phi\|$, and the laplacian, $\Delta\phi$, are uniformly bounded.

More precisely, let ϕ_i^k, for $i = 0,1$, be a sequence of smooth potentials that approximate the boundary data in the following sense: on any compact set K that does not intersect the singularity, we let $\phi_i^k \to \phi_i$ in $C^{1,\mu}$ in such a way that $\Delta\phi_i^k$ is preserved uniformly bounded. If $\beta > \frac{1}{2}$, then keep $\|\nabla\phi_i^k\|$ uniformly bounded as well, and if $\beta \leq \frac{1}{2}$, that is, the boundary data is merely Hölder continuous of exponent

2β across the divisor, keep the 2β-Hölder norm bounded across the singular set. Also, choose ε to be $\frac{1}{k}$ and choose η accordingly as it is allowed for the estimates to hold. Thich has been explicitly derived for each estimate. Under these conditions, one also can make the choice so that right hand side of equation 3.1 will tend to zero and $\left\{(\varepsilon_j, \eta_j)\right\}_j \to (0,0)$. The weak solution will be the limit of the smooth solutions thus obtained once we can prove a uniform bound and a uniform modulus of continuity for these solutions. In Section 3.4 we shall derive the C^0 estimates. In Section 3.6, uniform gradient bounds for the case of differentiable boundary data, and a rate of growth for the gradient in the case of smaller angles will be proved. The latter will be used in Section 3.5 to show δ-Hölder norm is bounded across the divisor for some small enough δ.

One will then have a sequence of smooth solutions $\left\{\phi^k\right\}_k$ for the the sequence of boundary value problems with a controlled growth of laplacian close to the divisor and controlled gradient or Hölder norm across the divisor, and with a uniform bound on the time derivative. Therefore, one can extract a subsequence $\left\{\phi^{k_m}\right\}_m$ that will converge in C^γ for some small enough γ, to the generalised solution ϕ. Therefore, in the generalised sense, ϕ will also satisfy the growth conditions on laplacian and will be of class C^γ. In particular, the convergence will be in $C^{1,\mu}$ on compact sets that do not intersect the singularity.

Proof of Theorem 3.2.2 The uniqueness is already known from [13]. In order to prove 3.2.2, we shall need to prove that that the $\mathscr{H}_\xi$-norm of the solutions in the following continuity family are uniformly bounded.

Similar to the case of divisorial conical singularities, we are going to prove existence of solutions to (1.5) in an appropriate sense. We state the following generalisation of Theorem 3.2 for the the following family of boundary value problems, where $\eta > 0$ is chosen depending on $\varepsilon > 0$.

$$\begin{cases} \frac{\omega_\phi^n}{\omega^n}\left(\phi_{tt} - |d\phi_t|_\phi^2\right) = \frac{\varepsilon e^f}{\prod_j \xi_{j,\eta}} \\ \phi(x,i) = \phi(i),\ i = 0,1. \end{cases} \tag{3.3}$$

Here, as in Equation 3.1, η and ε are two parameters, but we shall see how η may be chosen depending on ε in order for the estimates to hold. The content of the following theorem is the required bounds for this equation.

Theorem 3.2.3. *In the family of boundary value problems 3.1, assume that the boundary conditions have conical singularities of angles β_j along V_j. Then, for any $\varepsilon > 0$, the solution ϕ_η of 3.1*

- *if for all β_j we have $\beta_j > \frac{1}{2}$, then $|\partial_t \phi| + |\phi| + |\nabla\phi| + \xi|\Delta\phi| < C$,*
- *if, for some j we have $\beta_j \leq \frac{1}{2}$, then $|\partial_t \phi| + |\phi| + \xi|\Delta\phi| + \|\phi\|_{C^{0,\delta}} < C$ for any $\delta < 2\beta$.*

In the expressions above, C only depends on Δf and the supremum of $|\phi| + \xi|\Delta\phi|$ on the boundary.

The proof of the last theorem in turn will take up §3.4 to §3.7 and Appendix A. The proof of the L^∞-estimate is given first. Then follow the C^α- and C^1-estimates, which in turn rely on the weighted laplacian estimate.

Remark 3.2.4. Note that this theorem does not guarantee that the conical singularity is preserved, but rather, it proves that the growth of the derivatives at any point is not worse than that of the boundary condition, namely the conical case. As a result, away from the singularity of the boundary data we have the usual bounds on the $\partial\bar{\partial}$-derivatives. It also allows a wider class of singularities than conical singularities on the boundary.

Remark 3.2.5. Since C only depends on the $\mathscr{H}_\xi$-norm of boundary data, we may choose the boundary condition to be Kähler metrics that are semi-definite.

Remark 3.2.6. In the the theorem above, we could have stated the theorem in the general case, without differentiating between smaller and larger angles. Namely, we could have used the second estimate for the gradient for both larger and smaller angles.

3.3 Conical Metrics and more general singularities

In this section, basic facts and definitions will be presented. Most of these observations are in some way proved in [18, 24].

Let X^n be a compact Kähler manifold of dimension n and When we talk about a metric with a cone of angle β along a subvariety we mean a metric whose local model is the following metric on $\mathbb{C}^n$ with a cone of angle β along the divisor $[\mathfrak{z}_1 = 0]$.

$$\omega_{model} = \frac{i}{2}|\mathfrak{z}_1|^{2\beta-2} d\mathfrak{z}_1 \wedge d\bar{\mathfrak{z}}_1 + \frac{i}{2}\sum_{j=2} d\mathfrak{z}_j \wedge d\bar{\mathfrak{z}}_j$$

After an appropriate -singular- change of coordinates, one can see that this model metric indeed represents a euclidean cone of total angle $\theta = 2\pi\beta$, whose model on $\mathbb{R}^2$ is the following metric: $d\theta^2 + \beta^2 dr^2$. By the assumption on the asymptotic behaviour we we mean there exists some coordinate chart in which the zero-th order asymptotic of the metric agrees with the model metric. In other words, there is a constant C, such that

$$\frac{1}{C}\omega_{model} \leq \omega \leq C\omega_{model}$$

This asymptotic behaviour of metrics can be translated to the second order asymptotic behaviour of their potentials.

Of particular interest is the case where the conical singularity occurs along a divisor. That is, assume that $V \subset X$ is a smooth complex hypersurface. Also, let (L,h) be the line bundle associated to this hypersurface endowed with some hermitian metric h, and let $s \in \mathscr{O}(L)$ be the defining section of V. We can assume that there are multiple hypersurfaces. Unless stated otherwise, we assume that all the hypersurfaces are smooth and that they do not intersect. In this case, we may observe that in the proofs we can consider the weight functions and hypersurfaces individually. Let us set the following notation for the rest of this article. Let (L_j,h_j) be holomorphic line bundles endowed with hermitian metrics h_j. We then refer by s_j to some -global- holomorphic section of L_j, an element of $H^0(X,\mathscr{O}_X(L_j))$. Also, let $D := \cup V_j$.

Lemma 3.3.1. *Let s_j and (L_j,h_j) be as before. Then, for sufficiently small c, the following $(1,1)$-form*

$$\omega_\beta := \omega + c\sum_j dd^c|s_j|_{h_j}^{2\beta_j} \tag{3.4}$$

defines a Kähler metric with conical singularities of angle β_j along V_j.

Proof. Since the divisors do not intersect, we can consider them individually. We shall therefore drop the subscript in what follows. Adopt a coordinate system in a neighbourhood so that in this coordinate system the divisor corresponds to $[\mathfrak{z}_1 = 0]$. Also, choose unit vector e in that neighbourhood for (L,h). Then, we shall have that $s = \sigma(\mathfrak{z})e$ for some holomorphic function, and further, that $|s|_h = |\sigma|$. Now, by differentiating in local coordinates, one observes that $dd^c|s|_h^{2\beta}$ can be decomposed into a smooth part and a conical part. □

It can be then observed that if we set a smooth metric ω in the background and if we let $\omega_\phi = \omega + dd^c\phi$ be a conical metric, then, close to the singular set, the laplacian of the potential with respect to the reference metric ω, $\Delta\phi$, grows at the rate of $|s|_h^{2\beta-2}$. That is, $\Delta\phi = O(|s|_h^{2\beta-2})$. Similarly, we have about the first derivative that $|\nabla\phi| = O(|s|_h^{2\beta-1})$.

The advantage of using $|s|_h$ is that it is a global function and has an intrinsic geometric meaning. One can, however, observe that as long as the hypersurface is smooth, close to the hypersurface, we could substitute $|s|_h$ with the distance function to the divisor, call it $\rho_D(x)$. Since the distance function is not smooth farther from the support of the divisor, we can define ρ_D to be the distance, with respect to the reference metric ω, to the support of the divisor in the vicinity of $\operatorname{supp} D$ and extend it smoothly to the rest of X. This family of distance functions will be used when we consider more general singularities. We can, therefore, state the growth rate of derivatives in terms of ρ_D as well.

Definition 3.3.2. Assume that $\mathscr{S}$, the singular set, is a subset of the manifold X. A function ξ is called to an *admissible defining function* or *admissible weight function* for the set $\mathscr{S}$ if the following conditions are satisfied:

- the function ξ is an exhaustion function for $\mathscr{S}$, that is it vanishes on, and only on $\mathscr{S}$ with precompact sub-level sets,
- the complex hessian, with respect to the reference metric ω, of $\log \xi$ is uniformly bounded from below on $X - \mathscr{S}$.

In other words, we require that the mixed derivaives of $\log \xi$ be currents bounded from below by some multiple $-C$ of the Kähler form. The following observation provides us with two important families of weight functions.

Lemma 3.3.3. *Let $V \subset X$ be a complex hypersurface.*

1. *Assume that ρ_V is a function equal to the distance to V in a tubular neighbourhood of V and extended smoothly on the rather points. Then, any positive power of the distance function to V, ρ_V^ν for some $\nu > 0$, is an admissible weight function.*

   Further, if $V_j \subset X$ are hypersurfaces and ρ_{V_j}'s are the corresponding weight functions, the product $\prod \rho_{V_j}$ is also an admissible weight function for $\cup V_j$.

2. *Assume that (L, h) is a hermitian holomorphic line bundle and $s \in H^0(X, \mathscr{O}_X(L))$ is the defining section of V. Then, $|s|_h^\nu$ with $\nu > 0$ is an admissible weight function.*

3. *More generally, any analytic set admits admissible weight functions. More specifically, assume that the set $\mathscr{S}$ is the common zero locus of the holomorphic functions $f_1, \ldots, f_N$. Then, ξ may be taken to be any power of $\sum_{j=1}^N |f_j(\mathfrak{z})|^2$ is an admissible weight function.*

Proof. The fact that admissibility of weight functions is preserved under multiplication follows from its definition. The proofs for other claims follow from calculations in local coordinate systems around the submanifolds. □

It will make some of statements clearer if we introduce certain weighted functional spaces. As before, let ξ be a weight function for the gradient and the laplacian of the potential measured with respect to the smooth background metric ω. Since we only use continuous potentials, we have not put a weight on the growth of the C^0. Then, we define the following spaces:

Definition 3.3.4. Let ξ be an admissible weight function. We say that a continuous potential ϕ belongs to the space $\mathscr{H}_\xi$ if $\omega + dd^c\phi \geq 0$ and it further satisfies $\|\phi\|_\xi := |\phi| + |\Delta\phi|\xi < \infty$. We may also speak of an admissible pair $(\mathscr{S}, \xi)$ consisting of the singular set and an admissible weight function.

Theorem 3.3.5. *Assume that in the boundary value problem 3.3, the boundary data, ϕ_0 and ϕ_1 belong to the weighted space $\mathscr{H}_\xi$ for some admissible pair $(\mathscr{S}, \xi)$. Then, for any $\varepsilon > 0$, the $C^{1,1}$ norm of the solution on any compact subset away from $\mathscr{S}$ is bounded independent of η. More precisely, for any $\varepsilon > 0$, we have the following bound independent of η.*

$$|\partial_t \phi| + \xi |\Delta\phi| < C \tag{3.5}$$

for any $\mu \in (0,1)$.

Since the proof of theorem 3.3.5 is *mutatis mutandis* the same as that of 3.2.3, we shall only prove the latter.

3.4 L^∞-estimate

There are various ways to see that the solutions of boundary value problem 3.1 are bounded. One can, for example, generalise the argument [13], where sub- and super-solutions are constructed, to the case of less regular boundary data.

Proposition 3.4.1. In the boundary value problem

$$\begin{cases} \frac{\Omega_\phi^m}{\Omega^m} = \psi \\ \phi_{|\partial(X\times\Sigma)} = \phi_{0,1} \end{cases} \tag{3.6}$$

the L^∞ norm of ϕ in the interior is bounded as soon as the right hand side is square integrable, $\psi \in L^2(X \times \Sigma, \Omega)$.

Proof. Let h be a solution to the following boundary value problem:

$$\begin{cases} \Delta h = n+1 \\ \phi_{|\partial(X\times\Sigma)} = \phi_{0,1} \end{cases}$$

One can verify that h is indeed a supersolution. Also, let ϕ_0 be a an Ω-plurisubharmonic function on $X \times \Sigma$ whose restriction to the boundary agrees with $\phi_{0,1}$. Clearly, ϕ_0 is a subsolution. Therefore, ϕ is bounded from above and below on $X \times \Sigma$. $\square$

Remark 3.4.2. In order for this estimate to hold, one needs to choose $\eta(\varepsilon)$ so that the right hand side in boundary value problem 3.1 stays uniformly bounded, namely, $\varepsilon \leq \eta^p$.

3.5 C^{α}-estimate

In case of certain singularities, including the case of conical metrics, we can prove that the C^{α} norm of the solutions are bounded. In case of conical metrics of angles $\beta > \frac{1}{2}$, since the potential is C^1, this information will be superfluous. However, in the case of smaller angles, $\beta \le \frac{1}{2}$, we shall prove that not only away from the singularity, but also across the divisor Hölder continuity of the solutions is preserved for some appropriate exponent.

Let us first make some observations: [2]

Lemma 3.5.1. *Assume that $g : [0,1] \to \mathbb{R}$ is continuous on $(0,1]$. Further, assume that g is locally Lipschitz of constant $\lambda(\tau)$ on intervals of the form $[\tau,1]$. Then, g is Hölder continuous of exponent μ on $[0,1]$ provided that the following holds:*

$$\overline{\lim_{x\to 0}}\, x^{1-\mu}\lambda(x) < \infty$$

Proof. This can be seen by decomposing the interval $[0,1]$ into subintervals with end points belonging to the sequence $\{2^{-n}\}_n$. $\square$

The previous lemma will allow us to prove μ-Hölder continuity in directions transversal to the divisor once we prove the upper bound on the rate of growth of laplacian. We need to prove that the μ-Hölder modulus is bounded in tangential directions as well. Knowing the rate of growth of the gradient, which is provided in the next section, combined with the following observation, we obtain uniform Hölder continuity.

Lemma 3.5.2. *Assume that $N^n \subset M^m$ is an immersed submanifold and let ρ_N be the distance to N. Let $f \in C^0(M) \cap C^1(M-N)$ be μ-Hölder continuous in directions transversal to N. Further, assume that ∇f, at worst, grows at the rate of $\rho_N^{-\nu}$ for some ν. Then, for $\beta = \frac{\mu}{\mu+\nu}$, one has $f \in C^{\beta}(M)$.*

Proof. Since we already know Hölder continuity in the normal directions, we shall make use of it to prove Hölder continuity in the tangential direction as well. For simplicity, let N and M be $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Since we already assume the control in the normal directions, it will be enough to show that for any two point on the submanifold, $p,q \in \mathbb{R}^n$, $|f(q)-f(p)| \le C\|q-p\|^{\alpha}$. More specifically, let $p = \left(p^1,...,p^n,0,...,0\right), q = \left(q^1,...,q^n,0,...,0\right)$ for simplicity, let us let $r = \|q-p\|$. Choose $\gamma = \frac{1}{\mu+\nu}$. Also, as usual, let e_n denote the n-th element of the standard basis of $\mathbb{R}^m$. Then, by our assumption on the rate of growth of the gradient,

[2] '...It is, of course, a trifle, but there is nothing so important as trifles....' S.H.

$$\begin{aligned}|f(p)-f(q)| &\leq |f(p+r^\gamma e_n)-f(p)|+|f(p+r^\gamma e_n)-f(q+r^\gamma e_n)| \\ &+ |f(q)-f(q+r^\gamma e_n)| \leq C_1 r^{\gamma\mu}+C_2 r^{-\gamma\nu+1}=Cr^{\frac{\mu}{\mu+\nu}} \quad (3.7)\end{aligned}$$

which proves the claim. □

Now, since our estimate on the rate of growth of laplacian close to the divisor only depends on the L^∞ bound, cf. section 3.7, we can estimate the rate of growth of the gradient, as is done in the next section. Combined with the preceding lammata, we may obtain the following proposition which can be interpreted as the continuous embedding $\mathscr{H}_\xi \hookrightarrow C^\delta$.

Proposition 3.5.3. Let ϕ be a solution of the boundary value problem 3.1 for some $\varepsilon > 0$. Then, for $\delta < 2\beta$, $\|\phi\|_{0,\delta} \leq C$ for some uniform C which only depends on the boundary conditions and the geometry of the reference metric ω.

Remark 3.5.4. The C^α estimate is only useful across the divisor. One can easily observe that away from the divisor the solution is indeed C^1.

A more straightforward, nevertheless quite restrictive, approach to proving the Hölder estimates is through the $W^{2,p}$ estimates and the embedding of Sobolev spaces into Hölder spaces which is the content of the following proposition. On complex surfaces, it provides the estimate for $\beta > \frac{1}{2}$.

Proposition 3.5.5. Let the weight function for the laplacian, ξ, satisfy the property that $\frac{1}{\xi} \in L^p$ for some $p > d$, where d denotes the complex dimension of the manifold X. Further, assume that the continuous function ϕ is a solution of boundary value problem 3.1 for some $\varepsilon > 0$. Then, $\phi \in C^\mu$ for $\mu \leq 2 - \frac{2d}{p}$. In particular, in the case of conical singularity along a divisor, it will suffice to have $\beta > 1 - \frac{1}{d}$.

3.6 C^1-estimate

We shall derive two different first order estimates, one for the case of differentiable boundary data, corresponding to the cone angle less than half, and the case of larger cone angle. The distinction, however, is that in the case of smaller cone angle, $\beta > \frac{1}{2}$, we prove that the first space derivatives are bounded. Note that we shall prove the boundedness of the space gradient. In a general context, the boundedness of the temporal derivative was already proved by Berndtsson:

Proposition 3.6.1. [5] Let the $\mathscr{H}^\infty$ be the set of bounded potentials such that $\omega + dd^c\phi \geq 0$, where the inequality is interpreted in the sense of currents. Assume that ϕ is the solution of the following boundary value problem

$$\begin{cases} \frac{\omega_\phi^n}{\omega^n}\left(\phi_{tt} - |d\phi_t|^2_\phi\right) = 0 \\ \phi(x,i) = \phi(i) \in \mathscr{H}^\infty \; i = 0,1. \end{cases} \tag{3.8}$$

Then, $\|\partial_t \phi\|_{L^\infty} \leq C$ for some C which depends on the geometry of the background metric and the boundary conditions.

Proposition 3.6.2. In the boundary value problem 3.1, assume that the boundary conditions have singularities that are no worse that conical singularity of total angle $2\pi\beta$ along the divisor D. Then, if $\beta \geq \frac{1}{2}$, we have

$$|\nabla\phi| \leq C$$

In case of angles strictly larger than $\frac{1}{2}$, we have that for any $\mu \in (0,1)$, the gradient of the solution to 3.1 satisfies the following growth condition close to the set $\operatorname{supp} D$.

$$\|\nabla\phi\| \leq C|s|_h^{\mu-2+2\beta}, \forall\mu \in [0,1)$$

In both cases C is a constant independent of $\varepsilon > 0$ and only dependent on the boundary conditions and the background geometry.

Remark 3.6.3. Applied to our case, we have an *a priori* growth rate for the laplacian $\Delta\phi$ which gives a bound for the rate of growth of μ-Hölder constant of the gradient, $\nabla\phi$, as we approach the singularity. If we consider the conic singularity, when $\beta < \frac{1}{2}$, we have seen in the previous section that $\nabla\phi$ is bounded. This can be retrieved from the lemma above as well by observing that $C(t) \lesssim \frac{1}{\xi} \lesssim O(t^{-2\beta})$ as $t \to 0$ and therefore, since $2\beta < 1$, if we take μ to satisfy $0 < 2\beta < \mu < 1$, the integral in Inequality 3.10 of Corollary 3.6.6 will be finite and therefore $\nabla\phi$ will be bounded everywhere.

Remark 3.6.4. If we had an estimate on the -real- hessian tensor of ϕ, we could have integrated it to obtain the gradient estimate. However, bounding the growth of laplacian only allows us to bound the growth of $C^{1,\mu}$ norm for any $\mu \in [0,1)$.

Remark 3.6.5. Proposition 3.6.2 can be generalised to the case of any admissible pair of singularities, cf. Definition 3.3.2. Indeed, we need the more general form in the proof of Theorem 3.3.5 we need the more general version whose details we have omitted for the sake of simplicity of this exposition.

Proposition 3.6.2 combined with the bound obtain in [5] for $\partial_t \phi$ gives the following:

Corollary 3.6.6. For the boundary value problem 3.1, subject to the same conditions as in proposition 3.6.2, we have the following *a priori* estimate when $\beta < \frac{1}{2}$:

$$\|\nabla\phi\||s|_h^{2-2\beta-\mu} + |\partial_t \phi| < C \tag{3.9}$$

where C is independent of ε, and $\mu \in [0,1)$.

In order to prove the uniform C^1 bounds in the space directions, as in the case of C^α bounds, we shall use the rate of growth of laplacian close to the divisor, which, in turn, as we shall see, only depends on the rate of growth of laplacian on the boundary and the uniform L^∞ estimate. Since the laplacian estimates only depend on the L^∞ estimates of the solution and not the gradient estimates, we can use them to extract information about the rate of growth of the gradient. In particular, in case of a cone singularity along a sub-manifold, we have that the rate of growth of the first derivative close to the divisor is $O(r^{2\beta-1})$ when $\beta \leq \frac{1}{2}$. Moreover, we shall prove that, provided that $\beta > \frac{1}{2}$, the derivative is uniformly bounded as it is the case for the boundary conditions.

In the case where the angle, β, is smaller than $\frac{1}{2}$, however, even the boundary values might not be differentiable, but on the boundary we have a control on the rate of blows-up of the gradient , namely $|\nabla\phi| = O(r^{2\beta-1})$. Let us consider the sub-level sets of the weight function, $\{\xi \leq t\}$. We will give the rate of growth of μ-Hölder constant, $C_\mu(t)$, of $\nabla\phi$ on the set $\{\xi \leq t\}$. Indeed, since we have bounded the rate of growth of laplacian close to the divisor, we know that for any μ such that $0 < \mu < 1$ the μ-Hölder constant, call it $C_\mu(t)$, has, in the worst case, the rate of growth of the laplacian, $\Delta\phi$. That is, $C_\mu(t) \lesssim \xi^{-1} = O(r^{2\beta-2})$. Hence, using 3.6.7 we obtain that for any $0 < \mu < 1$, $|\nabla\phi| \lesssim O(r^{\mu-2+2\beta})$ as $r \to 0$ which is not as strong as the growth rate on the boundary, $O(r^{2\beta-1})$.

Let s be the defining section of the *smooth* divisor V. This means ∇s is no where vanishing. Therefore, there exists some positive number $\delta > 0$, such that $\|\nabla s\| > \delta > 0$ along the divisor. We can therefore state the following lemma. We shall omit the proof since the idea is similar to that of 3.5.1.

Lemma 3.6.7. *Assume $g : [0,1] \to \mathbb{R}$ is continuous on $(0,1]$. Further, assume that g is of Hölder class for some exponent μ on any set of the form $[\tau,1], 0 < \tau$ with constant $C(\tau)$. Then, g is -uniformly- bounded if*

$$\int_0^1 t^{\mu-1} C(t)dt < \infty \tag{3.10}$$

More generally, if the integral above does not converge, the C^0-norm on $[\tau,1]$

will not grow worse than the function $\Theta(\tau)$ defined as follows.

$$\Theta(\tau) := \int_{\tau}^{1} t^{\mu-1} C(t) dt \tag{3.11}$$

In particular, if $C(t) = O(t^{-\gamma})$, we can re-write the expression in 3.11 as follows:

$$\Theta_{\mu}(\tau) := \int_{\tau}^{1} t^{\mu-1} t^{-\gamma} dt = O\left(\tau^{\mu-\gamma}\right)$$

About which we have the following $\Theta(\tau) \lesssim O(\tau^{\mu-2\beta})$.

Proof of Proposition 3.6.2 We know already that for any $\varepsilon > 0$, the gradient is bounded on uniformly on a compact set, call it K, away from the singularity. We can now use lemma 3.6.7 and integrate on a curve connecting some point $q \in K$ to the a point close to the singularity. In the vicinity of the divisor, we can choose this point to be along the shortest path to $\mathscr{S}$. We may now apply the last lemma to the rate of growth of the laplacian and obtain the following

$$\|\nabla\phi\| \leq Cr^{2\beta-2+\mu}$$

3.7 Laplacian estimates

In order to prove the second order estimates, we adopt the approach presented in [23].

3.7.1 Divisorial singularities

As discussed before, consider the divisor V and its defining section s.

Consider the family of equations In this section, we shall prove the laplacian estimate stated in the following proposition:

Proposition 3.7.1. Let (L_j, h_j) and s_j be as before. Assume that in the following family of boundary value problems

$$\left(\phi_{tt} - |d\phi_t|^2_{\phi}\right) \omega^n_{\phi} = \frac{\varepsilon e^f}{(|s|^2 + \eta)^p} \omega^n \tag{3.12}$$

the boundary conditions satisfy the condition $\Delta\phi_k |s|^p_h < C < \infty$, for $k = 0, 1$. Then, the same holds for $\varepsilon \in (0, 1]$, independent of $\varepsilon > 0$ provided that $\varepsilon \leq \eta^p$.

Proof. In order to prove this, we consider, for a fixed ε, the family of equations:

$$\left(\phi_{tt} - |d\phi_t|^2_\phi\right) \omega^n_\phi = \frac{\varepsilon e^f}{(\eta + |s|^2)^p} \omega^n \tag{3.13}$$

With the notation as in [23], we shall prove that the following term

$$w_\eta := \zeta^p_\eta (n + \Delta\phi) := \left(|s|^2 + \eta\right)^p (n + \Delta\phi)$$

can be estimated. The family of functions $\zeta_\eta := \left(|s|^2 + \eta\right)$ are approximations of ζ by functions and all have positive lower bounds. We use the fact that for any $\varepsilon, \eta > 0$, when the right hand side in 3.13 finite, the linearised equation is the laplacian with respect to the metric Ω_ϕ on the manifold with boundary $X \times \Sigma$, and therefore satisfies the maximum principle. Namely, for any positive ε, the linearised operator, which we shall denote by $\mathfrak{D}$, attains its maximum on the boundary. We use this fact in order to prove that the quantity $\log w_\eta$ in the interior, that is for the time $0 < t < 1$, is controlled by its value on the boundary, $t = 0, 1$.

We can rewrite the equation as follows:

$$\log\det\left(\mathfrak{g}_{\alpha\bar\beta} + \phi_{\alpha\bar\beta}\right) + \log\left(\phi_{tt} - |d\phi_t|^2_\phi\right) = \log\varepsilon + f + \log\det\mathfrak{g}_{\alpha\bar\beta} - p\log\zeta_\eta \tag{3.14}$$

Let $\mathfrak{D}$ denote the linearisation of the left hand side in 3.14. One can easily verify that

$$\mathfrak{D}\psi = \Delta_\phi \psi + \frac{\psi_{tt} + \mathfrak{g}^{\alpha\bar\lambda}_\phi \mathfrak{g}^{\kappa\bar\beta}_\phi \phi_{t\alpha}\phi_{t\bar\beta}\psi_{\kappa\bar\lambda} - \mathfrak{g}^{\alpha\bar\beta}_\phi \left(\psi_{t\alpha}\phi_{t\bar\beta} + \psi_{t\bar\beta}\phi_{t\alpha}\right)}{\phi_{tt} - |d\phi_t|^2_\phi} \tag{3.15}$$

In order to keep the expressions shorter, let us choose a shorthand for what we will call the 'geodesic operator' as follows:

$$\mathscr{G}(\phi) := \partial_{tt}\phi - \frac{1}{2}|d\phi_t|^2_\phi \tag{3.16}$$

Of course, by the definition of our continuity family, for any $\varepsilon > 0$ we have $\mathscr{G}(\phi_\varepsilon) > 0$. We will estimate $\mathfrak{D}\log\zeta^p_\eta = p\mathfrak{D}\log\zeta_\eta$ from below in terms of ϕ and $\Delta\phi$. If we apply 3.15 to ζ_η, for which we of course have $\partial_t \zeta_\eta = 0$, we shall obtain the following:

$$\mathfrak{D}\log\zeta_\eta \quad = \quad \Delta_\phi \log\zeta_\eta + \frac{\mathfrak{g}^{\alpha\bar\beta}_\phi \mathfrak{g}^{\kappa\bar\beta}_\phi \phi_{ti}\phi_{t\bar\beta} (\log\zeta_\eta)_{\kappa\bar\beta}}{\mathscr{G}(\phi)} \tag{3.17}$$

We shall also need the following observation, that in a given coordinate system and at a point off the divisor we have

$$\frac{\partial^2}{\partial \mathfrak{z}_l \partial \bar{\mathfrak{z}}_l} \log\left(|s|_h^2 + \eta\right) \geq \frac{|s|_h^2}{|s|_h^2 + \eta} \frac{\partial^2}{\partial \mathfrak{z}_l \partial \bar{\mathfrak{z}}_l} \log |s|_h^2 \geq -C_1 \tag{3.18}$$

for some positive constant C_1. To see why the inequalities of 3.18 hold, let us first recall that on $X - D$, the $(1,1)$-form $dd^c \log |s|_h^2$ represents the first Chern form of the hermitian line bundle (L,h). Therefore, if we take the trace of this form with respect to the Kähler form ω_ϕ, we shall have the lower bound for some constant C_1 which depends only on the geometric properties of (L,h). This also shows that the lower bound holds on the entire manifold X so long as $\eta > 0$. Indeed, one can observe that as currents the following holds on the entire X:

$$dd^c \log\left(|s|_h^2 + \eta\right) \geq dd^c \log |s|_h^2 \geq -C_1 \omega$$

From this point to the end of this section we will postpone the proof of some inequalities to the last chapter, where the calculations of [23] are presented in further details. Also, we will introduce the two quantities $\mathscr{E}_2$ and $\mathscr{A}$ in the following inequality which are clarified in the last chapter.

We now consider the linearisation of the left hand side of 3.14 operator, $\mathfrak{D}$, applied to the function $\log(n + \Delta\phi) - C\phi$ for some constant C whose suitable choice will become clear in the estimates. We shall then have the following inequality which is proved in the last chapter:

$$\begin{aligned} \mathfrak{D}\left(\log(n+\Delta\phi) - C\phi\right) \quad \geq \quad & \frac{\Delta f - B - S}{n + \Delta\phi} - p\frac{\Delta \log \zeta_\eta}{n+\Delta\phi} + \sum_\lambda \frac{C-B}{1+\phi_{\lambda\bar{\lambda}}} - (n+1)C \\ + \quad & (C - 2B)\frac{1}{\mathscr{G}(\phi)} \sum_\lambda \frac{\phi_{t\lambda}\phi_{t\bar{\lambda}}}{(1+\phi_{\lambda\bar{\lambda}})^2} + \frac{\mathscr{E}_2}{(n+\Delta\phi)\mathscr{G}(\phi)} \\ - \quad & \mathscr{A}(n+\Delta\phi) \end{aligned} \tag{3.19}$$

where B is an expression in terms of the $\inf R_{\beta\bar{\beta}\kappa\bar{\kappa}}$ and S is the scalar curvature of g.

We, however, need the combination of this and 3.17 as follows:

$$
\begin{aligned}
\mathfrak{D}(\log w - C\phi) &= \mathfrak{D}(\log\psi + p\log\zeta - C\phi) \geq \frac{\Delta f - B - R}{n+\Delta\phi} - p\frac{\Delta\log\zeta_\eta}{n+\Delta\phi} \\
&- \sum_\lambda \frac{C-B}{1+\phi_{\lambda\bar\lambda}} \\
&- (n+1)C + (C-2B)\frac{1}{\mathscr{G}}\sum_\lambda \frac{\phi_{t\lambda}\phi_{t\bar\lambda}}{(1+\phi_{\lambda\bar\lambda})^2} + \mathscr{E}_2\frac{1}{(n+\Delta\phi)\mathscr{G}(\phi)} \\
&- \mathscr{A}(n+\Delta\phi) + p\Delta_\phi\log\zeta_\eta + p\frac{1}{\mathscr{G}(\phi)}\sum_\lambda \frac{\phi_{t\lambda}\phi_{t\bar\lambda}(\log\zeta)_{\lambda\bar\lambda}}{\left(1+\phi_{\lambda\bar\lambda}\right)^2} \\
&= \frac{\Delta f - B - R}{h} - (n+1)C + \mathscr{E}_2\frac{1}{(n+\Delta\phi)\mathscr{G}(\phi)} - \mathscr{A}(n+\Delta\phi) \\
&+ (C-B)\sum_\kappa \frac{1}{1+\phi_{\kappa\bar\kappa}} + \frac{1}{\mathscr{G}}\sum_\lambda \frac{\phi_{t\lambda}\phi_{t\bar\lambda}}{\left(1+\phi_{\lambda\bar\lambda}\right)^2}\left(p(\log\zeta)_{\lambda\bar\lambda} + C - 2B\right) \\
&+ p\sum_\lambda\left(\frac{1}{1+\phi_{\lambda\bar\lambda}} - \frac{1}{n+\Delta\phi}\right)(\log\zeta_\eta)_{\lambda\bar\lambda} \qquad (3.20)
\end{aligned}
$$

We show that

$$
\frac{1}{\mathscr{G}(\phi)}\sum_\lambda \frac{\phi_{t\lambda}\phi_{t\bar\lambda}}{\left(1+\phi_{\lambda\bar\lambda}\right)^2}\left(p(\log\zeta)_{\lambda\bar\lambda} + C - 2B\right) \geq 0 \qquad (3.21)
$$

for large enough C. To see this, we first observe that for a column vector α, the hermitian matrix obtained by $\alpha\alpha^\dagger$ is non-negative. In particular, the vector can be taken to be $\alpha_j = \phi_{tj}$. Also, from 3.18 we know a lower bound for the mixed derivatives of $\log\zeta$. We can estimate the last term in (3.23) as follows

$$
\sum_\lambda\left(\frac{1}{1+\phi_{\lambda\bar\lambda}} - \frac{1}{n+\Delta\phi}\right)(\log\zeta_\eta)_{\lambda\bar\lambda} \geq -C_1\sum_\lambda \frac{1}{1+\phi_{\lambda\bar\lambda}} \qquad (3.22)
$$

where C_1 is the constant from 3.18. We can then obtain the following:

$$
\begin{aligned}
\mathfrak{D}(\log w - C\phi) &= \frac{\Delta f - B - R}{h} - (n+1)C + \mathscr{E}_2\frac{1}{(n+\Delta\phi)\mathscr{G}(\phi)} - \mathscr{A}(n+\Delta\phi) \\
&+ (C - B - pC_1)\sum_\kappa \frac{1}{1+\phi_{\kappa\bar\kappa}} \qquad (3.23)
\end{aligned}
$$

In the expressions above, if we let C be a large number, the coefficient of the last term, $C - B - pC_1$, will be larger than 1.

Therefore, 3.19 can gives the following:

$$
\begin{aligned}
\mathfrak{D}\left(\log w - C\phi\right) &\geq \frac{\Delta f - B - R}{n+\Delta\phi} \\
&- (n+1)C + \mathscr{E}_2 \frac{1}{(n+\Delta\phi)\mathscr{G}(\phi)} \\
&- \mathscr{A}(n+\Delta\phi) + \sum_{\kappa} \frac{1}{1+\phi_{\kappa\bar{\kappa}}} \qquad (3.24)
\end{aligned}
$$

After some further manipulation of 3.23, the details of which can be found in the last chapter, we obtain:

$$
\mathfrak{D}\left(\log w - C\phi\right) \geq \sum_{\lambda} \frac{1}{1+\phi_{\lambda\bar{\lambda}}} - (n+1)C \qquad (3.25)
$$

Let us observe that, similar to 2.19 in [36], we have

$$
\sum_{\lambda} \frac{1}{1+\phi_{\lambda\bar{\lambda}}} + \frac{1}{\mathscr{G}(\phi)} \geq \left\{ \frac{\sum(1+\phi_{\lambda\bar{\lambda}}) + \mathscr{G}(\phi)}{\left(\prod(1+\phi_{\lambda\bar{\lambda}})\right)\mathscr{G}(\phi)} \right\}^{\frac{1}{n}} \qquad (3.26)
$$

Combining 3.25, 3.34, and the preceding inequality we have the following:

$$
\begin{aligned}
\mathfrak{D}\left(\log w - C\phi + \frac{t^2}{2}\right) &\geq \sum_{\lambda} \frac{1}{1+\phi_{\lambda\bar{\lambda}}} - (n+1)C + \frac{1}{\mathscr{G}(\phi)} \\
&\geq \left\{ \frac{\sum(1+\phi_{\lambda\bar{\lambda}}) + \mathscr{G}(\phi)}{\left(\prod(1+\phi_{\lambda\bar{\lambda}})\right)\mathscr{G}(\phi)} \right\}^{\frac{1}{n}} - (n+1)C \\
&= (n+\Delta\phi+\mathscr{G}(\phi))^{\frac{1}{n}} \left(\frac{(|s|^2+\eta)^p}{\varepsilon e^f}\right)^{\frac{1}{n}} - (n+1)C \\
&= \left\{(n+\Delta\phi)(|s|^2+\eta)^p + (|s|^2+\eta)^p\mathscr{G}(\phi)\right\}^{\frac{1}{n}} \left(\varepsilon s^f\right)^{\frac{-1}{n}} \\
&- (n+1)C \\
&\geq w^{\frac{1}{n}} \left(\varepsilon s^f\right)^{\frac{-1}{n}} - (n+1)C \qquad (3.27)
\end{aligned}
$$

Having this differential inequality, one can argue that either $\log w - C\phi + t^2$ attains its maximum at some interior point P, in which case $\mathfrak{D}\left(\log w - C\phi + t^2\right) \leq$

0, which gives the following upper bound for w:

$$w(P) \leq \varepsilon e^{f}\left((n+1)C\right)^{n}$$

or the maximum of $\log w - C\phi + t^2$ occurs on the boundary. $\square$

The calculations are valid for arbitrary p. In the case of conical singularity, we have seen that $(\Delta\phi)\,|s|_h^{2\beta}$ is bounded on the boundary, therefore, we can choose $p=\beta$. Of course the proposition will still hold for larger p as well, however, that will not be optimal.

3.7.2 Non-divisorial singularities

We may observe that in the proof of Theorem 3.2.3, the only property of $|s|_h^{2\beta}$ we have used is boundedness from below of the mixed derivative of $\log|s|_h$ on $X-D$. We can therefore generalise this result to the case of more general singular sets so long as we can find admissible weight functions.

Proposition 3.7.2. Let W be a smooth embedded complex submanifold of X. Then, the distance function to W, ρ_W, is an admissible weight function.

Similar to what we did in section zero, we consider the family of equations modified as follows:

$$\left(\phi_{tt} - |d\phi_t|_\phi^2\right)\omega_\phi^n = \frac{\varepsilon e^f}{\xi_\eta}\omega^n \tag{3.28}$$

where $\xi_\eta = \left(\rho(\mathfrak{z})^2 + \eta\right)^\alpha$

We claim that the quantity $\xi_\eta\left(\rho(\mathfrak{z})\right)\Delta\phi(t)$ stays bounded independent of η. We shall henceforth denote ξ_η by ξ.

In the end, we can consider the equation modified as follows:

$$\left(\phi_{tt} - |d\phi_t|_\phi^2\right)\omega_\phi^n = \frac{\varepsilon e^f}{\prod_j \xi_j}\omega^n \tag{3.29}$$

where each ξ_j is a weight function with certain properties that vanishes on a set containing the singularity. Since we only need the mixed derivatives of $\log\xi$, we can merely assume ξ is a function whose log is θ-plurisubharmonic for some fixed form θ. This need to hold only in the vicinity of the singular set $\mathscr{S}$ such that on the singularity we have $\xi|_{\mathscr{S}} = 0$. We observe that this holds when $\xi = |s|_h$, where $s \in H^0(L,\mathscr{O})$ and (L,h) is a holomorphic line bundle equipped with a hermitian metric h by inequality 3.18. It also holds if we take ξ to be the distance to a complex submanifold containing the singularity. (cf. lemma 3.3.3).

We also need the following observation.

Lemma 3.7.3. *If ξ is an admissible weight function, then, the elements of the family of function $\xi_\eta := \xi + \eta$, which approximate ξ by strictly positive functions, have a uniform lower bound on $dd^c \log \xi_\eta$, namely, as currents $dd^c \log \xi_\eta \geq -C\omega$ where C is a uniform constant.*

We can repeat

$$\mathfrak{D}\left(\log\left(\xi_\eta \phi\right) - C\phi + t^2\right) \geq \left(\xi_\eta (n + \Delta\phi)\right)^{\frac{1}{n}} \left(\varepsilon s^f\right)^{\frac{-1}{n}} - (n+1)C \tag{3.30}$$

And it is thus proved that

$$\sup_{M\times[0,1]} \{\xi(n+\Delta\phi)\} \leq \sup_M \{\xi(n+\Delta\phi_{0,1})\} \tag{3.31}$$

3.8 Final remarks-Some special cases of singularities

We finish this chapter with some remarks.

Remark 3.8.1. [3] So far we have only considered the case where the singular set is given by the zero locus of some holomorphic section. But thanks to the local nature of the operations, one can merely require that the sigular set be the locally the zero set of a finite number of holomorphic functions. In that case also one may take any power of the modulus those local defining functions to be the 'local' weight function. More specifically, let V be the common zero set of function $f_j, k = 1, ..., k$. Then, the function $\left(\sum_j |f_j|^2\right)^p$, for $p > 0$, is an admissible weight function for the common zero locus of the functions $\{f_j\}_j$.
Further, if the defining functions are defined locally, as in an algebraic variety, one can still construct an admissible weight function for V as follows. Let us observe that if one has a partition of unity $\mu_j(x)$ subordinate to U_j, and if one has admissible weight functions ξ_j on each of the open sets U_j, then the function $\xi := \sum_j \mu_j \xi_j$ is a global admissible weight function. This allows us, in particular, to construct an admissible weight function when the singular set is contained in the common zero set of locally defined holomorphic functions.

Remark 3.8.2. Having obtained an upper bound for the space laplacian of the potential, $\Delta\phi$, we can show that the diameter is uniformly bounded. To see this, let us

[3] I am grateful to Prof. Bedford for pointing this out to me.

note that the set $X-D$ is path connected. Let x be a fixed point outside of the divisor, $x \in X-D$. Then, any point $q \in X-D$ may be connected by a curve $\gamma \subset X-D$. Also, since the divisor is smooth, for any point on the divisor, $p \in D$, there is a curve γ connecting p to the point x, contained in $X-D$ except at p, which is perpendicular to D at p.

Let $d_\phi(p,q)$ denote the distance with respect to the metric ω_ϕ between two points p and q. Fix a point $x \in X-\mathscr{S}$. Then, by the triangle inequality,

$$\operatorname{diam}(M) \leq \sup_{p,q} \big(d_\phi(p,x) + d_\phi(x,q)\big)$$

However, $d_\phi(x,p)$ can be estimated from above by measuring the length of the curve γ, connecting p to x, described in the previous paragraph. The length of any such curve, in turn, can be estimated since we have growth rate of $O(|s|_h^{2\beta-2})$ close to the divisor for the metric.

Remark 3.8.3. Singularities along a totally real submanifold One may observe that the one example of of admissible function is the distance function to a totally real submanifold, $\mathscr{R}$ of X^n once one has that, n, the complex dimension of X, is larger than 1.

3.9 Postponed calculations

In this appendix, we shall provide the details we omitted, including the proof of 3.19, in the proof of laplacian estimates. Much of the calculation is borrowed from [23]. We will henceforth use the normal coordinates, in which at a the given point q, $\mathfrak{g}_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$, $\partial_\kappa \mathfrak{g}_{\alpha\bar{\beta}} = \partial_{\bar{\lambda}} \mathfrak{g}_{\alpha\bar{\beta}} = 0$, and $\mathfrak{g}_{\phi,\alpha\bar{\beta}} = \delta_{\alpha\beta}(1+\phi_{\alpha\bar{\beta}})$, whenever coordinates appear.

We first recall Equation 3.15, the linearisation of the left hand side of (3.14):

$$\mathfrak{D}\psi = \Delta_\phi \psi + \frac{\psi_{tt} + \mathfrak{g}_\phi^{\alpha\bar{\lambda}} \mathfrak{g}_\phi^{\kappa\bar{\beta}} \phi_{t\alpha}\phi_{t\bar{\beta}} \psi_{\kappa\bar{\lambda}} - \mathfrak{g}_\phi^{\alpha\bar{\beta}} \left(\psi_{t\alpha}\phi_{t\bar{\beta}} + \psi_{t\bar{\beta}}\phi_{t\alpha}\right)}{\mathscr{G}(\phi)} \tag{3.32}$$

If we substitute $\psi = \phi$, we obtain the following:

$$\mathfrak{D}\phi = (n+1) - \sum_\beta \frac{1}{1+\phi_{\beta\bar{\beta}}} - \frac{1}{\mathscr{G}(\phi)} \sum_\beta \frac{\phi_{t\beta}\phi_{t\bar{\beta}}}{(1+\phi_{\beta\bar{\beta}})^2} \tag{3.33}$$

And for $\psi = t^2$:

$$\mathfrak{D}t^2 = \frac{2}{\mathscr{G}(\phi)} \tag{3.34}$$

We shall also need the following identity later in calculations:

$$\begin{aligned} \mathfrak{D}\log\psi &= \frac{\mathfrak{D}\psi}{\psi} - \frac{\mathfrak{g}_\phi^{\kappa\bar{\lambda}}\psi_\kappa\psi_{\bar{\lambda}}}{\psi^2} - \frac{\left(\psi_t - \mathfrak{g}_\phi^{\alpha\bar{\lambda}}\phi_{t\alpha}\psi_{\bar{\lambda}}\right)\left(\psi_t - \mathfrak{g}_\phi^{\kappa\bar{\lambda}}\phi_{t\bar{\beta}}\psi_\kappa\right)}{\psi^2\mathscr{G}(\phi)} \\ &=: \frac{\mathfrak{D}\psi}{\psi} - \frac{\mathfrak{g}_\phi^{\kappa\bar{\lambda}}\psi_\kappa\psi_{\bar{\lambda}}}{\psi^2} - \mathscr{A}(\psi) \end{aligned} \tag{3.35}$$

We have implicitly defined $\mathscr{A}(\psi)$ in the identity above. Let us substitute $\psi = \Delta\phi$ and obtain the following:

$$\mathfrak{D}(\Delta\phi) = \Delta_\phi\Delta\phi + \frac{\Delta\partial_{tt}\phi + \mathfrak{g}_\phi^{\alpha\bar{\lambda}}\mathfrak{g}_\phi^{\kappa\bar{\beta}}(\Delta\phi)_{\kappa\bar{\lambda}}\phi_{t\alpha}\phi_{t\bar{\beta}} - \mathfrak{g}_\phi^{\alpha\bar{\beta}}\left((\Delta\phi)_{t\alpha}\phi_{t\bar{\beta}} + (\Delta\phi)_{t\bar{\beta}}\phi_{t\alpha}\right)}{\mathscr{G}(\phi)} \tag{3.36}$$

We can substitute the first two terms, namely $\Delta_\phi\Delta\phi + \frac{\Delta\partial tt\phi}{\mathscr{G}(\phi)}$, following the calculations in section 2 , equations 2.7 and 2.9, of [36], and obtain:

$$
\begin{aligned}
\Delta_\phi \Delta\phi &= \sum_\kappa \mathfrak{g}_\phi^{\alpha\bar\beta} \mathfrak{g}_\phi^{\mu\bar\nu} \phi_{\alpha\bar\nu\kappa}\phi_{\mu\bar\beta\bar\kappa} + \Delta f - \Delta \log \zeta_\eta^p - \Delta \log\left(\partial_{tt}\,\phi - |d\,\partial_t\,\phi|^2_\phi\right) + I \\
&\geq \sum_k \mathfrak{g}_\phi^{\alpha\bar\beta} \mathfrak{g}_\phi^{\mu\bar\nu} \phi_{\alpha\bar\nu\kappa}\phi_{\mu\bar\beta\bar\kappa} + \Delta f - \frac{\Delta\left(\partial_{tt}\,\phi - |d\,\partial_t\,\phi|^2_\phi\right)}{\mathscr{G}(\phi)} + \frac{|d\mathscr{G}(\phi)|^2}{\mathscr{G}(\phi)^2} + I \\
&- \Delta \log \zeta_\eta^p \\
&\geq \sum_\kappa \mathfrak{g}_\phi^{\alpha\bar\beta} \mathfrak{g}_\phi^{\mu\bar\nu} \phi_{\alpha\bar\nu\kappa}\phi_{\mu\bar\beta\bar\kappa} + \Delta f - \frac{\Delta\left(\partial_{tt}\,\phi - |d\,\partial_t\,\phi|^2_\phi\right)}{\mathscr{G}(\phi)} + I - \Delta \log \zeta_\eta^p
\end{aligned}
$$

where C_1 is the same constant that appears in equation 3.18,

$$
I = \sum_{\alpha,\kappa} \frac{1+\phi_{\kappa\bar\kappa}}{1+\phi_{\alpha\bar\alpha}} R_{\alpha\bar\alpha\kappa\bar\kappa} - S
$$

Here S and R denote scalar curvature and curvature tensor respectively. Now if we let B be a positive constant such that $-B \leq \inf R_{\alpha\bar\alpha\kappa\bar\kappa}$, then I satisfies the following inequality:

$$
I \geq -B(n+\Delta\phi)\sum_\alpha \frac{1}{1+\phi_{\alpha\bar\alpha}} - B - S =: -B(n+\Delta\phi)\sum_\alpha \frac{1}{1+\phi_{\alpha\bar\alpha}} - C_2 \tag{3.37}
$$

We would like to bound the terms containing time derivatives from below. Substituting 3.37 into 3.36, leads us to the following:

$$
\begin{aligned}
\mathfrak{D}(\Delta\phi) &\geq \Delta f + \sum_\kappa \mathfrak{g}_\phi^{\alpha\bar\beta} \mathfrak{g}_\phi^{\mu\bar\nu} \phi_{\alpha\bar\mu\kappa}\phi_{\mu\bar\beta\bar\kappa} + I + \mathscr{E}(\phi) - p\Delta\log\zeta_\eta \\
&+ \frac{\mathfrak{g}_\phi^{\alpha\bar\lambda}\mathfrak{g}_\phi^{\kappa\bar\beta}(\Delta\phi)_{\kappa\bar\lambda}\,\phi_{t\alpha}\phi_{t\bar\beta} - \mathfrak{g}_\phi^{\alpha\bar\beta}\left((\Delta\phi)_{t\alpha}\,\phi_{t\bar\beta} + (\Delta\phi)_{t\bar\beta}\,\phi_{t\alpha}\right)}{\mathscr{G}(\phi)}
\end{aligned} \tag{3.38}
$$

where

$$
\mathscr{E}(\phi) = \frac{\Delta|d\phi_t|^2_\phi}{\mathscr{G}(\phi)} \tag{3.39}
$$

We now study the term $\mathscr{E}(\phi)$ as follows. After calculations in normal coordinates, we observe that in the expression above, the numerator, $\Delta|d\phi_t|^2_\phi$, can be written as the sum of three terms which may be analysed separately as follows.

$$\Delta\left|d\phi_t\right|^2_\phi = \mathscr{E}_1 + \mathscr{E}_2 - \mathfrak{g}^{\alpha\bar{\lambda}}_\phi \mathfrak{g}^{\kappa\bar{\beta}}_\phi \left(\Delta\phi\right)_{\kappa\bar{\lambda}} \phi_{t\alpha}\phi_{t\bar{\beta}} + \mathfrak{g}^{\alpha\bar{\beta}}_\phi \left(\left(\Delta\phi\right)_{t\alpha}\phi_{t\bar{\beta}} + \left(\Delta\phi\right)_{t\bar{\beta}}\phi_{t\alpha}\right)$$

where the terms are defined as follows:

$$\begin{aligned}
\mathscr{E}_1 :=& \mathfrak{g}^{\alpha\bar{\nu}}_\phi \mathfrak{g}^{\mu\bar{\beta}}_\phi R_{\mu\bar{\nu}\kappa\bar{\lambda}} \phi_{t\alpha}\phi_{t\bar{\beta}} \left(\mathfrak{g}^{\kappa\bar{\lambda}} + \phi_{\kappa\bar{\lambda}}\right) \\
\mathscr{E}_2 :=& \mathfrak{g}^{\kappa\bar{\lambda}} \mathfrak{g}^{\alpha\bar{\nu}}_\phi \mathfrak{g}^{\mu\bar{\beta}}_\phi \Big\{ \mathfrak{g}^{\rho\bar{\sigma}}_\phi \phi_{t\rho}\phi_{t\bar{\beta}} \left(\phi_{\alpha\bar{\sigma}\kappa}\phi_{\mu\bar{\nu}\bar{\lambda}} + \phi_{\mu\bar{\nu}\kappa}\phi_{\alpha\bar{\sigma}\bar{\lambda}}\right) + \phi_{\mu\bar{\nu}\bar{\lambda}} \left(\phi_{t\alpha\kappa}\phi_{t\bar{\beta}} + \phi_{t\alpha}\phi_{t\kappa\bar{\beta}}\right) \\
&+ \phi_{\mu\bar{\nu}\kappa} \left(\phi_{t\alpha\bar{\lambda}}\phi_{t\bar{\beta}} + \phi_{t\alpha}\phi_{t\bar{\beta}\bar{\lambda}}\right) \Big\} \\
&+ \mathfrak{g}^{\kappa\bar{\lambda}} \mathfrak{g}^{\alpha\bar{\beta}}_\phi \left(\phi_{t\alpha\bar{\lambda}}\phi_{t\bar{\beta}\kappa} + \phi_{t\alpha\kappa}\phi_{t\bar{\beta}\bar{\lambda}}\right)
\end{aligned}$$

The last calculation allows us to cancel the fourth order terms in 3.38 with those of $\mathscr{E}(\phi)$. Since the derivation of the preceding inequality is done by straightforward, nevertheless long, calculations in normal coordinates, we omit the calculation and refer the reader to 2.10 in [23].

We now use 3.38 to obtain

$$\mathfrak{D}(\Delta\phi) \geq \Delta f + \sum_\kappa \mathfrak{g}^{\alpha\bar{\beta}}_\phi \mathfrak{g}^{\mu\bar{\nu}}_\phi \phi_{\alpha\bar{\nu}\kappa}\phi_{\mu\bar{\beta}\bar{\kappa}} + I + \frac{\mathscr{E}_1 + \mathscr{E}_2}{\mathscr{G}(\phi)} - p\Delta\log\zeta_\eta \tag{3.40}$$

By the definition of B, we have $R_{\alpha\bar{\beta}\kappa\bar{\lambda}} \geq -B(\delta_{\alpha\beta}\delta_{\kappa\lambda} + \delta_{\alpha\lambda}\delta_{\kappa\beta})$. We may now combine this piece of information with the obvious inequality $n + \Delta\phi \geq 1 + \phi_{j\bar{j}}$, and obtain:

$$\begin{aligned}
\mathscr{E}_1 &\geq -\mathfrak{g}^{\alpha\bar{\nu}}_\phi \mathfrak{g}^{\mu\bar{\beta}}_\phi B(\delta_{\mu\nu}\delta_{\kappa\lambda} + \delta_{\mu\lambda}\delta_{\kappa\nu})\phi_{t\alpha}\phi_{t\bar{\beta}} \left(\mathfrak{g}^{\kappa\bar{\lambda}} + \phi_{\kappa\bar{\lambda}}\right) \\
&> -2B(n+\Delta\phi)\sum_\beta \frac{\phi_{t\beta}\phi_{t\bar{\beta}}}{(1+\phi_{\beta\bar{\beta}})^2} \qquad (3.41)
\end{aligned}$$

Note that $\mathscr{E}_3$ appears in the numerator of the last term in 3.38. We can, therefore, combine 3.37, 3.38 and 3.41 and get:

$$
\begin{aligned}
\mathfrak{D}(\Delta\phi) &\geq \Delta f + \sum_{\kappa} \mathfrak{g}_{\phi}^{\alpha\bar{\beta}} \mathfrak{g}_{\phi}^{\mu\bar{\nu}} \phi_{\alpha\bar{\nu}\kappa}\phi_{\mu\bar{\beta}\bar{\kappa}} - B(n+\Delta\phi)\sum_{\alpha}\frac{1}{1+\phi_{\alpha\bar{\alpha}}} - C_2 + \frac{\mathscr{E}_1+\mathscr{E}_2}{\mathscr{G}(\phi)} \\
&- p\Delta\log\zeta_{\eta} \\
&\geq \Delta f + \sum_{\kappa} \mathfrak{g}_{\phi}^{\alpha\bar{\beta}} \mathfrak{g}_{\phi}^{\mu\bar{\nu}} \phi_{\alpha\bar{\nu}\kappa}\phi_{\mu\bar{\beta}\bar{\kappa}} - B(n+\Delta\phi)\sum_{\alpha}\frac{1}{1+\phi_{\alpha\bar{\alpha}}} - C_2 \\
&- \frac{2B(n+\Delta\phi)}{\mathscr{G}(\phi)}\sum_{\beta}\frac{\phi_{t\beta}\phi_{t\bar{\beta}}}{(1+\phi_{\beta\bar{\beta}})^2} + \frac{\mathscr{E}_2}{\mathscr{G}(\phi)} - p\Delta\log\zeta_{\eta} \\
&\geq \Delta f - C_2 - B(n+\Delta\phi)\sum_{\alpha}\frac{1}{1+\phi_{\alpha\bar{\alpha}}} + \frac{\mathfrak{g}_{\phi}^{\kappa\bar{\lambda}}(\Delta\phi)_{\kappa}(\Delta\phi)_{\bar{\lambda}}}{n+\Delta\phi} \\
&- \frac{2B(n+\Delta\phi)}{\mathscr{G}(\phi)}\sum_{\beta}\frac{\phi_{t\beta}\phi_{t\bar{\beta}}}{(1+\phi_{\beta\bar{\beta}})^2} - p\Delta\log\zeta_{\eta}
\end{aligned}
$$

In the last inequality, we have used the following consequence of the Schwarz inequality for the third order terms (cf. 2.15 in [36]):

$$\sum_{\kappa} \mathfrak{g}_{\phi}^{\alpha\bar{\beta}} \mathfrak{g}_{\phi}^{\mu\bar{\nu}} \phi_{\alpha\bar{\nu}\kappa}\phi_{\mu\bar{\beta}\bar{\kappa}} \geq \frac{(d\Delta\phi, d\Delta\phi)_{\phi}}{n+\Delta\phi} = \frac{\mathfrak{g}_{\phi}^{\kappa\bar{\lambda}}(\Delta\phi)_{\kappa}(\Delta\phi)_{\bar{\lambda}}}{n+\Delta\phi}$$

We now use the last inequality, 3.35, with $\psi = n+\Delta\phi$, and 3.33 to obtain the following

$$
\begin{aligned}
\mathfrak{D}\left(\log(n+\Delta\phi)-C\phi\right) &= \frac{\mathfrak{D}\left(\Delta\phi\right)}{n+\Delta\phi}+\frac{\sum_\kappa \mathfrak{g}_\phi^{\alpha\bar\beta}\mathfrak{g}_\phi^{\mu\bar\nu}\phi_{\alpha\bar\nu\kappa}\phi_{\mu\bar\beta\bar\kappa}}{n+\Delta\phi}-\mathscr{A}(n+\Delta\phi)\\
&- (n+1)C+C\sum_\beta\frac{1}{1+\phi_{\beta\bar\beta}}+\frac{C}{\mathscr{G}(\phi)}\sum_\beta\frac{\phi_{t\beta}\phi_{t\bar\beta}}{(1+\phi_{\beta\bar\beta})^2}\\
&\geq \frac{\Delta f-C_2}{n+\Delta\phi}-p\frac{\Delta\log\zeta_\eta}{n+\Delta\phi}-\frac{\mathscr{E}_2(\phi)}{\mathscr{G}(\phi)(n+\Delta\phi)}-\mathscr{A}(n+\Delta\phi)\\
&+ \sum_\lambda\frac{(\Delta\phi)_\lambda(\Delta\phi)_{\bar\lambda}}{(1+\phi_{\lambda\bar\lambda})(n+\Delta\phi)^2}-\sum_\lambda\frac{\psi_\lambda\psi_{\bar\lambda}}{\psi^2(1+\phi_{\lambda\bar\lambda})}\\
&- (n+1)C+(C-B)\sum_\beta\frac{1}{1+\phi_{\beta\bar\beta}}+\frac{C-2B}{\mathscr{G}(\phi)}\sum_\beta\frac{\phi_{t\beta}\phi_{t\bar\beta}}{(1+\phi_{\beta\bar\beta})^2}\\
&= \frac{\Delta f-C_2}{n+\Delta\phi}-p\frac{\Delta\log\zeta_\eta}{n+\Delta\phi}-\frac{\mathscr{E}_2}{\mathscr{G}(\phi)(n+\Delta\phi)}-\mathscr{A}(n+\Delta\phi)\\
&- (n+1)C+(C-B)\sum_\beta\frac{1}{1+\phi_{\beta\bar\beta}}+\frac{C-2B}{\mathscr{G}(\phi)}\sum_\beta\frac{\phi_{t\beta}\phi_{t\bar\beta}}{(1+\phi_{\beta\bar\beta})^2}
\end{aligned}
$$

for any constant C, which is 3.19.

We now turn to proving 3.25 based on 3.24. Recall that by 3.24 we had:

$$
\begin{aligned}
\mathfrak{D}\left(\log w-C\phi\right) &\geq \frac{\Delta f-C_2}{n+\Delta\phi}-\mathscr{A}(n+\Delta\phi)+\mathscr{E}_2\frac{1}{(n+\Delta\phi)\mathscr{G}(\phi)}-(n+1)C\\
&+ \sum_\kappa\frac{1}{1+\phi_{\kappa\bar\kappa}}
\end{aligned} \tag{3.42}
$$

It will suffice to prove that

$$
\mathscr{E}_2\geq(n+\Delta\phi)\mathscr{A}(n+\Delta\phi)\mathscr{G}(\phi) \tag{3.43}
$$

which is equivalent to the following:

$$
\mathscr{E}_2(n+\Delta\phi)\geq\left(\psi_t-\mathfrak{g}_\phi^{\alpha\bar\lambda}\phi_{t\alpha}\psi_{\bar\lambda}\right)\left(\psi_t-\mathfrak{g}_\phi^{\kappa\bar\beta}\phi_{t\bar\beta}\psi_\kappa\right)
$$

for $\psi=n+\Delta\phi$. Since this also follows from a straightforward calculation, we refer the reader to (2.21) in [23].

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