Affine Stratifications and equivariant vector bundles
on the moduli of principally polarized abelian varieties

A Dissertation presented

by

Anant Atyam

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2014

Stony Brook University

The Graduate School

Anant Atyam

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Samuel Grushevsky - Dissertation Advisor

Professor, Department of Mathematics

Robert Lazarsfeld - Chairperson of Defense

Professor, Department of Mathematics

Jason Starr

Professor, Department of Mathematics

Gavril Farkas

Professor, Department of Mathematics, Humboldt Universität, Berlin

This dissertation is accepted by the Graduate School

Charles Taber

Dean of the Graduate School

Abstract of the Dissertation

Affine Stratifications and equivariant vector bundles on the moduli of principally polarized abelian varieties

by

Anant Atyam

Doctor of Philosophy

in

Mathematics

Stony Brook University

2014

We explicitly construct a locally closed affine stratification of the coarse moduli space of principally polarized complex abelian four folds $\mathcal{A}_4$ and using [32], produce an upper bound for the coherent cohomological dimension and the constructible cohomological dimension of $\mathcal{A}_4$, and make a conjecture on the cohomological dimension of $\mathcal{A}_g$ for any g. In the second part of the thesis we study a subring of $K_0(\mathcal{A}_g)$ and study the Chern character map from this ring to the tautological ring of $\mathcal{A}_g$ which leads us to give a conjectural representation theoretic interpretation of the tautological ring of $\mathcal{A}_g$.

Dedication Page

I dedicate this Thesis to my late grandmother Leela Devi, your selflessness and silent presence has contributed more to my life than I can possibly understand. I would also like to dedicate this work to my girlfriend and my best friend Sanja Nedić, it's been a beautiful journey with you. To Dad, the best role model I could ever have. To Mom, Thank you for having faith in me at times when most people would have given up. To Akka, Thank you for rooting for me at times when I needed it the most. And finally to Aniket, my friend, my Guru, thank you for reigniting the spark in me when it was all but dead.

Acknowledgements

I would like to thank my advisor Sam Grushevsky for being excessively patient and understanding of my deficiencies as a budding mathematician, and for making my foray into algebraic geometry a joyous experience. I would also like to thank Jimmy Mathews for patiently giving me a crash course in representation theory which made a part of this thesis possible and for also inspiring me with his single minded devotion to understand nature via mathematics. I would like to also thank Dennis Sullivan for giving me an opportunity to come to Stony Brook to do Mathematics.

Contents

Chapter 1

Summary of main results

We study the moduli space of principally polarized abelian varieties $\mathcal{A}_g$ over the field of complex numbers $\mathbb{C}$ and discuss the question of the minimum number of affine open subsets of $\mathcal{A}_g$ required to cover $\mathcal{A}_g$ as a coarse moduli space and the related question of the minimum number of locally closed and disjoint affine strata whose union gives us $\mathcal{A}_g$.

If one manages to provide an upper bound for the minimum number of affine open subsets or the minimal affine stratification number, as defined by Roth and Vakil [32] required to cover a quasi projective variety X, the relation between sheaf cohomology and Čech cohomology gives us a concrete upper bound for the coherent cohomological dimension and the constructible cohomological dimension of the variety X.

We now state our main results.

Theorem 1.0.1. *There exists an affine stratification of $\mathcal{A}_4$ consisting of seven strata.*

As a corollary of the above theorem and the results of Roth and Vakil

Corollary 1.0.2. *The cohomological dimension* $\mathrm{cd}(\mathcal{A}_4(\mathbb{C})) \leq 6$*, while the constructible cohomological dimension* $\mathrm{ccd}(\mathcal{A}_4(\mathbb{C})) \leq 16$*.*

In another direction based on the study of the Tautological ring of $\mathcal{A}_g$ and the work of Oort and van der Geer, we prove the following theorem.

Theorem 1.0.3. *The minimum number of affine open sets required to cover* $\mathcal{A}_g$ *is greater than or equal to* $\frac{g(g-1)}{2} + 1$

Based on the above Theorem and the study of the tautological ring of $\mathcal{A}_g$ we conjecture the following statement.

Conjecture 1.0.4. *The cohomological dimension of* $\mathcal{A}_g$ *in all characteristics is equal to* $\frac{g(g-1)}{2}$

We make a departure from the study of the cohomological dimension of $\mathcal{A}_g$ in the second part of the thesis and attempt to give an alternative conjectural description of the tautological ring. In the second part of the thesis we relate the ring of representations of the groups $\mathrm{GL}(g, \mathbb{C})$ and the groups $\mathrm{Sp}(g, \mathbb{C})$ and make the following conjecture

Conjecture 1.0.5. *There is a natural monomorphism from the representation ring* $i : R[\mathrm{Sp}(g, \mathbb{C})] \to R[\mathrm{GL}(g, \mathbb{C})]$ *and the tautological ring* $\mathcal{R}^*(\mathcal{A}_g)$ *can be described as a shifted quotient of* $R[\mathrm{GL}(g, \mathbb{C})]$ *by the representation ring* $R[\mathrm{Sp}(g, \mathbb{C})]$ *and an extra explicit relation in the representation ring* $R[\mathrm{GL}(g, \mathbb{C})]$*.*

We now proceed to give a brief exposition of the theory of principally polarized abelian varieties and introduce the necessary prerequisites for this thesis.

Chapter 2

Introduction: abelian varieties and their moduli

We provide a brief survey and introduction the the general theory of principally polarized abelian varieties and recall necessary classical facts that will be necessary for the remainder of the thesis

2.1 Principally polarized abelian varieties and their moduli

The primary object of study in this dissertation is the moduli space of principally polarized abelian varieties (abbreviated ppav) $\mathcal{A}_g$ and the closely related moduli space of smooth curves of genus g, $\mathcal{M}_g$.

Definition 2.1.1. A complex torus of dimension g is a group quotient $\mathbb{C}^g/\Lambda$ where $\Lambda \subset \mathbb{C}^g$ is a lattice of full rank: that is a subgroup $\Lambda \simeq \mathbb{Z}^{2g}$ such that

$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^g$.

Definition 2.1.2. An abelian variety A is a projective algebraic group, i.e. A is a projective variety that has the structure of a group such that the composition map and inverse map are algebraic morphisms.

Remark 2.1.3. It can be proven that over the complex numbers an abelian variety is a complex torus and in fact conversely any complex torus that is a projective variety is an abelian variety. It is a non-trivial fact that not every complex torus is projective, we will shortly discuss the criteria for a complex torus to be projective which is related to the notion of a polarization of an abelian variety. The primary references for the material in this chapter are [16] and [4].

We now state the definition of a principally polarized abelian variety.

Definition 2.1.4. A polarized abelian variety is a pair $(A, c_1(L))$ with L being an ample line bundle on the abelian variety A.

Definition 2.1.5. A principally polarized abelian variety (ppav) is a polarized abelian variety $(A, c_1(L))$ with $h^0(A, L) = 1$.

Remark 2.1.6. It is a non trivial fact that the dimension of the space of global sections of an ample line bundle on an abelian variety is dependent just on the Chern class of the corresponding ample line bundle.

Definition 2.1.7. We define the Siegel upper half space to be the set of complex symmetric matrices with positive definite imaginary part

$$\mathcal{H}_g := \left\{\tau \in Mat_{g\times g}(\mathbb{C}) | \tau = \tau^t,\ \mathrm{Im}(\tau) > 0\right\}.$$

We now state a result that allows us to describe ppav explicitly as complex tori satisfying some conditions on the lattice defining the torus.

Theorem 2.1.8. *A complex torus A is an abelian variety if and only if it is biholomorphic to the complex torus $\mathbb{C}^g/\Delta\mathbb{Z}^g + \tau\mathbb{Z}^g$ for some $\tau \in \mathcal{H}_g$ and some diagonal $g \times g$ matrix Δ with positive integral diagonal entries δ_i such that $\delta_i | \delta_{i+1}$. This is to say that we think of the lattice Λ as generated by the g columns of Δ (which are thus integer multiples of coordinate vectors) and the columns of τ.[16]*

Remark 2.1.9. The Δ indicates a choice of a polarization $c_1(L)$ on our abelian variety and moreover $h^0(A, L) = \det(\Delta)$, thus $(A, c_1(L))$ is a ppav if and only if A can be described as $A \simeq \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, i.e $\Delta = Id$.

To prove this theorem, known as Riemann's bilinear relations, we will need the fundamental results of Hodge theory for Kähler manifolds, as well as Kodaira embedding and Lefschetz (1,1)-theorems, which we now recall in general, and then apply to complex tori. [16]

Theorem 2.1.10. *[16] The cohomology of a compact Kähler manifold X admits a decomposition $H^r(X, \mathbb{C}) \simeq \oplus_{p+q=r} H^{p,q}(X, \mathbb{C})$ where $H^{p,q}(X, \mathbb{C}) \simeq H^q(X, \Omega^p)$ is the q'th cohomology of the bundle of holomorphic p-forms on X. Moreover, this decomposition satisfies $H^{p,q}(X, \mathbb{C}) \simeq \overline{H^{q,p}(X, \mathbb{C})}$ and in particular for $q = 0$, we have $H^{0,q}(X, \mathbb{C}) = H^0(X, \Omega^q)$.*

Theorem 2.1.11 (Kodaira's embedding theorem)**.** *[16] A compact Kähler manifold X is a projective variety — that is, can be embedded in a complex projective space — if and only if there exists a non zero positive closed two form $\omega \in H^2(X, \mathbb{Q}) \cap H^{1,1}(X, \mathbb{C})$ i.e. $\omega(w, \overline{w}) > 0$ for any $w \in T_X$.*

Theorem 2.1.12 (Lefschetz theorem on (1,1) classes)**.** *[16] Let X be a compact Kähler manifold and let $\omega \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Q})$, then $\omega = c_1(L)$ for some holomorphic line bundle L on X.*

We now state the Künneth Formula which allows us to compute the cohomology of the cartesian product of two spaces

Theorem 2.1.13. *If X and Y are two finite CW complexes, then the cohomology $H^k(X \times Y, R) \simeq \bigoplus_{p+q=k} H^p(X, R) \otimes H^q(Y, R)$, if the cohomology groups of X and Y are free R modules.*

Since a complex torus $X = \mathbb{C}^g/\Lambda$ is just homeomorphic to $(S^1)^{2n}$ one obtain the following result as an application of Kunneth formula and the universal coefficient theorem

Theorem 2.1.14. *The cohomology ring $H^*(X, R)$ of a complex torus $X = \mathbb{C}^g/\Lambda$ is isomorphic to the al $\wedge^* H^1(X, R)$ for any ring R and $H^1(X, \mathbb{Z}) \simeq \Lambda^*$.*

We now apply this general machinery of Kähler geometry to the case of complex tori. We first note that we can define a Kähler structure on complex torus $X = \mathbb{C}^g/\Lambda$ by using the form $\omega = \sum_{i=1}^{g} dz_i \wedge \overline{dz_i}$, we see that this form is defined on the universal cover of X i.e. $\mathbb{C}^g$ and it is invariant under translations by elements of Λ and is hence well defined on X.

Since $X = \mathbb{C}^g/\Lambda$ is a Kähler manifold and a complex torus, we obtain the following result as a consequence of the Hodge decomposition

Proposition 2.1.15. *The cohomology ring of the complex torus X admits the* *following decomposition.*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}) = \bigoplus_{p+q=k} \wedge^p H^{1,0}(X,\mathbb{C}) \otimes \wedge^q H^{0,1}(X,\mathbb{C}).$$

Remark 2.1.16. If X is a complex torus given as a quotient $X = \mathbb{C}^g/\Lambda$ and $z_1, \ldots, z_g$ are coordinates on $\mathbb{C}^g$ then $\{dz_1, dz_2, \ldots. dz_g\}$ gives a natural basis for $H^{1,0}(X,\mathbb{C})$ and similarly $\left\{\overline{dz_1}, \overline{dz_2}, \ldots.\overline{dz_g}\right\}$ gives a basis of the conjugate space $H^{0,1}(X,\mathbb{C})$ respectively.

We now use this general machinery to give a proof of Riemann's bilinear relations

Proof of theorem 2.1.8. Let X be a complex torus given as $X = \mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a full rank lattice. Then by theorem 2.1.14, $H^1(X,\mathbb{Z}) = \Lambda^*$ and $H^2(X,\mathbb{Z}) = \wedge^2\Lambda^*$. By Kodaira's embedding theorem X is an abelian variety if and only if there exists a closed positive two form $\omega \in H^2(X,\mathbb{Z}) \cap H^{1,1}(A,\mathbb{C})$. We first prove that if such ω exists, then $X \simeq \mathbb{C}^g/\Delta\mathbb{Z}^g + \tau\mathbb{Z}^g$ for some diagonal Δ, and $\tau \in \mathcal{H}_g$.

Since by theorem 2.1.14 we know that $H^2(A,\mathbb{Z}) = \wedge^2\Lambda^*$, and that ω is positive. Now using a standard fact about non degenerate quadratic forms over $\mathbb{Z}$ we can represent ω as a skew symmetric matrix of the form $\left(\begin{smallmatrix} 0 & \Delta \\ -\Delta & 0 \end{smallmatrix}\right)$ with respect to some basis $\{\lambda_1, \ldots \lambda_{2g}\}$ of Λ where Δ is a diagonal matrix with $\delta_i|\delta_{i+1}$. Let $\{x_1, \ldots x_{2g}\}$ be the dual integral basis to $\{\lambda_1, \ldots \lambda_{2g}\}$. Now since $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^g$, we know that $\{x_1, \ldots x_{2g}\}$ are real linear functionals on $\mathbb{C}^g$, thus we can choose linear complex coordinates $\{z_1, \ldots z_g\}$ on $\mathbb{C}^g$ such that $x_i = \frac{1}{2\delta_i}(z_i + \overline{z_i})$, for $i = 1 \ldots g$ and with respect to these coordinates

$x_{g+i} = \sum_{j=1}^{g} \tau_{ij} z_j + \sum_{j=1}^{g} \overline{\tau_{ij} z_j}$. Now our two form ω can be rewritten as $\omega = \sum_{i=1}^{g} \delta_i dx_i \wedge dx_{g+i}$. Applying a change of basis from $\{dx_1, dx_2, \dots dx_{2g}\}$ to $\left\{dz_1, \dots dz_g, \overline{dz_1}, \dots \overline{dz_g}\right\}$ on $(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})^*$ we can rewrite ω as

$$\begin{aligned}
\omega &= \sum_{i=1}^{g} \delta_i \delta_i^{-1} \frac{1}{2}(dz_i + \overline{dz_i}) \wedge (\sum_{j=1}^{g} \tau_{ij} dz_j + \overline{\tau_{ij}} \overline{dz_j}). \\
&= \sum_{i,j} (\tau_{ij} - \tau_{ji}) dz_i \wedge dz_j + \sum_{i,j} (\overline{\tau_{ij}} - \overline{\tau_{ji}}) \overline{dz_i} \wedge \overline{dz_j} + \sum_{i,j} (\tau_{ij} - \overline{\tau_{ji}}) dz_i \wedge \overline{dz_j}).
\end{aligned}$$

Recall now that our positive form ω was supposed to lie in $H^{1,1}(A, \mathbb{C})$, while the first summand in the formula above lies in $H^{2,0}(A, \mathbb{C})$, and the second lies in $H^{0,2}(A, \mathbb{C})$ — thus these two summands must vanish, which is to say that for any i, j we must have $\tau_{ij} = \tau_{ji}$. This simply says that τ is symmetric. Since ω has to be positive, the last summand (where we note that $\tau_{ij} - \overline{\tau_{ji}} = 2i \operatorname{Im} \tau_{ij}$) must give a positive-definite quadratic form, which is precisely to say that $\operatorname{Im}(\tau)$ is a positive definite matrix.

We have thus proven that $X \simeq \mathbb{C}^g / \Delta\mathbb{Z}^g + \tau\mathbb{Z}^g$ with $\tau \in \mathcal{H}_g$ for the choice of some complex basis on $\mathbb{C}^g$.

We now prove the reverse implication, that is, Suppose $X = \mathbb{C}^g / \Delta\mathbb{Z}^g + \tau\mathbb{Z}^g$ for $\tau \in \mathcal{H}_g$ then X is an abelian variety. By Kodaira's embedding theorem, it is enough to construct a form $\omega \in H^2(A, \mathbb{Z}) \cap H^{1,1}(A, \mathbb{C}) > 0$. We choose coordinates $\{z_1, \dots, z_g\}$ on $\mathbb{C}^g$ so that $\left\{dz_1, dz_2, \dots .dz_g, \overline{dz_1}, \overline{dz_2}, \dots .\overline{dz_g}\right\}$ gives a basis for $H^1(A, \mathbb{C})$. Since $H^1(A, \mathbb{Z}) \simeq (\Delta\mathbb{Z}^g + \tau\mathbb{Z}^g)^*$ we see that the basis $\{dx_1, dx_2, ...dx_{2g}\}$ of $H^1(X, \mathbb{C})$ defined by $dx_i = \frac{1}{\delta_i} \frac{1}{2}(dz_i + \overline{dz_i})$ for $i = 1....g$ and $dx_{g+i} = \sum_{j=1}^{g} \tau_{ij} dz_j + \overline{\tau_{ij}} \overline{dz_j}$ is also an integral basis for $H^1(X, \mathbb{Z})$.

A simple computation using the above relation between the rational basis given by the $\{dx_i\}_{i=1}^{2g}$ and the complex basis given by $\left\{dz_i, \overline{dz_i}\right\}_{i=1}^{g}$ tells us that the form $\omega = \sum_{i=1}^{g} \delta_i dx_i \wedge dx_{g+i} \in H^2(A, \mathbb{Z}) \cap H^{1,1}(A, \mathbb{C})$ is positive. The positivity of the form follows from the positive definiteness of $\mathrm{Im}(\tau) = -i/2(\tau_{ij} - \overline{\tau_{ij}})$. and the fact that it lies in $H^{1,1}(A, \mathbb{C})$ follows from the fact that $\tau = \tau^t$. □

Since our interest is describing the moduli space of abelian varieties, and by Riemann bilinear relations any principally polarized abelian variety (which means $\Delta = Id$) is isomorphic to $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ for some $\tau \in \mathcal{H}_g$, we will now need to determine when two such abelian varieties are biholomorphic. Before doing so we need the following preliminary definition.

Definition 2.1.17. There is a group action of $\mathrm{Sp}(g, \mathbb{Z})$ on $\mathcal{H}_g$ given by $M \circ \tau = (A\tau + B)(C\tau + D)^{-1}$ for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\tau \in \mathcal{H}_g$.

Remark 2.1.18. The fact that the above action is a group action is not immediately obvious, but follows from a strightforward computation

Theorem 2.1.19. *Two ppav* $A_1 = \mathbb{C}^g/\mathbb{Z}^g + \tau_1\mathbb{Z}^g$ *and* $A_2 = \mathbb{C}^g/\mathbb{Z}^g + \tau_2\mathbb{Z}^g$ *are biholomorphic as ppav, i.e. by a biholomorphism that sends the polarization on* A_1 *to the polarization on* A_2 *if and only if* $\tau_2 = M \circ \tau_1$ *for some* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(g, \mathbb{Z})$.

Before proving the statement, we note that together with 2.1.8, 2.1.19 it implies the following corollary

Corollary 2.1.20. *The moduli space of ppav over* $\mathbb{C}$ *can be analytically described as* $\mathcal{H}_g/\,\mathrm{Sp}(g, \mathbb{Z})$.

Remark 2.1.21. The case of $g = 1$ corresponds to the case of the description of the moduli space of pointed elliptic curves as $\mathcal{H}/\operatorname{SL}(2,\mathbb{Z})$. Every point on this quotient is an orbifold point with a finite stabilizer.

Proof of theorem 2.1.19. Let $f : \mathbb{C}^g/\mathbb{Z}^g + \tau_1\mathbb{Z}^g \to \mathbb{C}^g/\mathbb{Z}^g + \tau_2\mathbb{Z}^g$ be a biholomorphic map preserving polarization, which is to say that we must have $f^*(c_1(L_2)) = c_1(L_1)$. Now consider the map $\tilde{f} : \mathbb{C}^g \to \mathbb{C}^g$ denoting the associated holomorphic map between the universal covers. It can be shown that this map is linear[4], for which we must have Now since $\tilde{f}(\mathbb{Z}^g + \tau_1\mathbb{Z}^g) \subset \mathbb{Z}^g + \tau_2\mathbb{Z}^g$ and $f^*(c_1(L_2)) = c_1(L_1)$. The map $\tilde{f}$ restricted to the lattices is a symplectic linear transformation with integer coefficients as it has to preserve the symplectic pairing on the lattice as it preserves the lattices., i.e. $\tilde{f} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with respect to the standard basis of the lattices, coming from the columns of the matrix $\begin{bmatrix} Id & \tau_i \end{bmatrix}$ for $i = 1, 2$. Now the map $\tilde{f} : \mathbb{C}^g \to \mathbb{C}^g$ is a complex linear transformation as $\tilde{f}$ can be naturally identified as the map between the associated lie algebra of A_1 and A_2. Thus $\tilde{f}(x) = Mx$ for some $M \in \operatorname{GL}(g,\mathbb{C})$. By our earlier discussion on $\tilde{f}$ restricted to the lattice $\mathbb{Z}^g + \tau_1\mathbb{Z}^g$, we can say that the image of the standard basis of $\mathbb{C}^g$ under $\tilde{f}$ is $C\tau + D$ and $\tilde{f}(\tau_1) = A\tau_1 + B$, since $\tilde{f}$ can be represented in a matrix form by $C\tau_2 + D$, we see that $\tilde{f}^{-1} = (C\tau + D)^{-1}$. Therefore $\tilde{f}^{-1} \circ \tilde{f}(\tau_1) = (A\tau_2 + B)(C\tau_2 + D)^{-1} = \tau_1$. □

Remark 2.1.22. We would like to point out that the moduli space of principally polarized abelian varieties is an orbifold or algebraically speaking has the structure of a stack. This is because an abelian variety with a polarization has a natural automorphism of order two given by $x \mapsto -x$ for $x \in A$ and because could also have other automorphisms. It is not too hard to see that every principally polarized abelian variety has a finite stabilizer. This can be seen because any element of the stabilizer of a point $[\tau] \in \mathcal{H}_g / \operatorname{Sp}(g,\mathbb{Z})$ is an automorphism of the associated ppav $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. This automorphism can be lifted to a linear automorphism of $\mathbb{C}^g$ to itself preserving the lattice and having determinant one, because having a determinant larger than one wouldn't make it an automorphism of the torus, it would be a many to one map of the torus to itself.

The universal family of ppav, i.e. a fibration over $\mathcal{H}_g / \operatorname{Sp}(g,\mathbb{Z})$ where the fibers over $[\tau]$ can be identified with the quotient of the abelian variety $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ by its automorphisms can be described analytically as $\mathcal{H}_g \times \mathbb{C}^g / \operatorname{Sp}(g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ where the action is given by

$$(\tau, z) \to ((A\tau + B)(C\tau + D)^{-1}, (C\tau + D)^{-1}(z + \gamma\tau + \delta)),$$

for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \operatorname{Sp}(g,\mathbb{Z})$ and $\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \in \mathbb{Z}^{2g}$.

Remark 2.1.23. While the discussion above is only applicable over the field of complex numbers $\mathbb{C}$, we can define the notion of moduli space of principally polarized abelian varieties over a field of characteristic p. We won't be making use of this moduli space explicitly, except to point out the differences in the geometry of the moduli spaces in different characteristics.

2.2 Theta functions and line bundles on abelian varieties

One natural question that arises is to describe a line bundle on a ppav $X = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ whose Chern class is the principal polarization. We do so by defining a holomorphic function, the theta function, on $\mathbb{C}^g$ that is quasi periodic with respect to the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$ and hence its zero locus descends as a divisor to X and thus describes a line bundle whose Chern class turns out to be the principal polarization on X.

Definition 2.2.1. Let $\tau \in \mathcal{H}_g$ and $m = \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \in (\mathbb{Z}/2\mathbb{Z})^{2g}$, we define the theta function $\theta_m : \mathbb{C}^g \to \mathbb{C}$ with characteristic m as

$$\theta_m(z,\tau) := \sum_{n \in \mathbb{Z}^g} e^{(\pi i (n+\varepsilon/2)^t \tau (n+\varepsilon/2) + 2\pi i (n+\varepsilon/2)^t (z+\delta/2))}.$$

Remark 2.2.2. The positive definiteness of $\operatorname{Im}\tau$ shows that $\theta_m(z,\tau)$ converges for all z and is hence a holomorphic function on $\mathbb{C}^g$. By a direct calculation one sees that for any $a, b \in \mathbb{Z}^g$

$$\theta_m(z + a + b\tau, \tau) = e^{\pi i b^t \tau b - 2\pi i b^t z}\theta_m(z,\tau). \tag{2.2.1}$$

Since the factor $e^{\pi i b^t \tau - 2\pi i b^t z} \neq 0$ the zero locus of $\theta_m(z,\tau)$ is well defined as a subset of on $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$.

Remark 2.2.3. We denote $\theta(z,\tau)$ the theta function with characteristic $m = 0 \in (\mathbb{Z}/2\mathbb{Z})^{2g}$.

Definition 2.2.4. The zero locus of $\theta(z,\tau)$ is called the theta divisor Θ on $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ and similarly the zero locus of $\theta_m(z,\tau)$ is called the Θ_m divisor on A_τ for $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$.

We now state a theorem which relates the theta divisor to the principal polarization.

Theorem 2.2.5. *For any $\tau \in \mathcal{H}_g$ the zero locus of $\theta(z,\tau)$ defines on $A_\tau := \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ an ample line bundle Θ with one section, whose c_1 defines a principal polarization.*

Remark 2.2.6. The above theorem really states two theorems, the first one being that $\mathcal{O}_{A_\tau}(\Theta)$ is an ample line bundle since a polarization is the chern class of an ample line bundle and ampleness of a line bundle is a numerical property as the second one that states $h^0(A_\tau, \mathcal{O}_{A_\tau}(\Theta)) = 1$. The proofs of both these theorems are non trivial and we do not indicate their proofs.

A natural question that arises is the following: given a line bundle L on a ppav A_τ and if $c_1(L) = c_1(\mathcal{O}_{A_\tau}(\Theta))$, what can we say about the divisor defined by the non zero section of L. Recall that the dimension of the space of sections of an ample bundle only depends on c_1 in this case, so is equal to one

Theorem 2.2.7. *For a ppav $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, let L be a line bundle such that $c_1(L) = c_1(\mathcal{O}_{A_\tau}(\Theta))$, then $h^0(A_\tau, L) = 1$ and if $s \in H^0(A_\tau, L) \neq 0$ then the zero locus $Z(s) = Z(\theta(z+b,\tau))$ for some $b \in \mathbb{C}^g$.*

We now restate the above theorem as a consequence of the following theorem

Theorem 2.2.8. *Let* $\mathrm{Pic}^0(A_\tau)$ *denote the group of invertible sheaves of degree* 0 *on a ppav* $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$*, i.e. the group of invertible sheaves numerically equivalent to the structure sheaf of* A_τ*. Then* $\mathrm{Pic}^0(A_\tau)$ *is a ppav that is naturally biholomorphic to* A_τ *as a ppav.*

We will be using the above theorem and the definition below, to classify all line bundles that give us a principal polarization.

Definition 2.2.9. Let $x \in A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ we define the translation map $\tau_x : A_\tau \to A_\tau$ by $\tau_x(z) := x + z$

Corollary 2.2.10. *Let* L *be a line bundle on* $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ *such that* $c_1(L) = c_1(\mathcal{O}_{A_\tau}(\Theta))$ *then* $L = \tau_b^*(\mathcal{O}_{A_\tau}(\Theta))$ *for some* $b \in A_\tau$*, thus the zero locus of a non zero section of* L *is the zero locus of* $\theta(z+b,\tau)$ *which descends to* A_τ*. Moreover the map* $b \to \tau_b^*(\mathcal{O}_{A_\tau})$ *is an isomorphism between* A_τ *and* $\mathrm{Pic}^0(A_\tau)$*.*

Remark 2.2.11. The zero locus of $\theta(z+b,\tau)$ is a priori defined on $\mathbb{C}^g$ but due to 2.2.1, its zero locus descends to a well defined codimension one subvariety of A_τ.

We indicated earlier that $c_1(\mathcal{O}_{A_\tau}(\Theta_m))$ is the principal polarization on A_τ, but we also stated a theorem that said that every principal polarization is the chern class of the translate of the $\mathcal{O}_{A_\tau}(\Theta)$ line bundle. We therefore now state a relation between $\theta(z + \frac{\varepsilon\tau+\delta}{2}, \tau)$ and $\theta_m(z,\tau)$ for $m = \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \in (\mathbb{Z}/2\mathbb{Z})^{2g}$, the relation is as follows.

$$\theta(z + \frac{\varepsilon\tau + \delta}{2}, \tau) = e^{\pi i(-\varepsilon^t\tau\varepsilon/4 - \varepsilon^t\delta/2 - \varepsilon^t z)}\theta_m(z,\tau) \tag{2.2.2}$$

We now define the notion of the parity of an element $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ and then relate it to properties of the corresponding theta function with characteristic m.

Definition 2.2.12. For $m = \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ we define the parity of m to be $e(m) := \varepsilon^t \delta \in \mathbb{Z}/2\mathbb{Z}$. We say that m is even if $e(m) = 0$ and odd otherwise

The following theorem relates the parity of a theta characteristic $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ with the corresponding theta function.

Proposition 2.2.13. *The theta function with characteristic m, $\theta_m(z, \tau)$ is an even (resp. odd) function of z if and only if m is even (resp. odd).*

The evenness or oddness of theta functions with characteristic indicates that the standard involution on the abelian variety leaves the Θ_m divisor fixed. We rephrase this as follows.

Consider the involution $i : A_\tau \to A_\tau$ defined by $i(z) := -z$, then $i^*(\mathcal{O}_{A_\tau}(\Theta_m)) = \mathcal{O}_{A_\tau}(\Theta_m)$, in fact 2.2.8 tells us that the only line bundles with the same chern class as the principal polarization that will be invariant under the involution will be $\mathcal{O}_{A_\tau}(\Theta_m)$ for $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$, we thus state the following result.

Proposition 2.2.14. *Let L be a line bundle on a ppav $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ such that $i^*(L) = L$ for the standard involution i on A_τ and such that $c_1(L) = c_1(\Theta)$. Then $L = \mathcal{O}_{A_\tau}(\Theta_m)$ for some $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$.*

We recall that $\theta_m(z, \tau)$ can be identified with the section of a line bundle on $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, now 2.1.19 tells us that that two ppav $A_{\tau 1} = \mathbb{C}^g/\mathbb{Z}^g + \tau_1\mathbb{Z}^g$ and $A_{\tau 2} = \mathbb{C}^g/\mathbb{Z}^g + \tau_2\mathbb{Z}^g$ are isomorphic if and only if $\tau_2 = (A\tau_1 + B)(C\tau_1 + D)^{-1}$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(g, \mathbb{Z})$

This leads to the following question: if $f : A_{\tau 1} \to A_{\tau 2}$ is an isomorphism of ppav then how does $f^*(\mathcal{O}_{A_{\tau 2}}(\Theta_m))$ behave or how does the theta function $\theta_m(z, \tau_1)$ transform with respect to f?

To answer this question we first introduce an affine action of $\mathrm{Sp}(g, \mathbb{Z})$ on the set of characteristics $(\mathbb{Z}/2\mathbb{Z})^{2g}$ and some subgroups of $\mathrm{Sp}(g, \mathbb{Z})$. We first define two subgroups $\Gamma(n) \subset \mathrm{Sp}(g, \mathbb{Z})$ and $\Gamma(n, 2n) \subset \mathrm{Sp}(g, \mathbb{Z})$

Definition 2.2.15. The subgroup $\Gamma(n) \subset \mathrm{Sp}(g, \mathbb{Z})$ is defined by

$$\Gamma(n) := \{\gamma \in \mathrm{Sp}(g, \mathbb{Z}) | \gamma \equiv Id_{2g} \mod n\}$$

The subgroup $\Gamma(n, 2n) \subset \mathrm{Sp}(g, \mathbb{Z})$ is defined by

$$\Gamma(n, 2n) := \left\{\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(g, \mathbb{Z}) | \gamma \equiv Id_{2g} \bmod n, A^t B = C^t D = 0 \mod 2n\right\}.$$

We denote $\mathcal{H}_g/\Gamma(n)$ by $\mathcal{A}_g(n)$ and $\mathcal{H}_g/\Gamma(n, 2n)$ by $\mathcal{A}_g(n, 2n)$ — these are finite (stack) covers of $\mathcal{A}_g$.

We now define an affine action of $\mathrm{Sp}(g, \mathbb{Z})$ on $(\mathbb{Z}/2\mathbb{Z})^{2g}$, this action is not the usual linear action of $\mathrm{Sp}(g, \mathbb{Z})$.

Definition 2.2.16. Let $m = \left(\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right) \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ and $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(g, \mathbb{Z})$, then $\gamma.m := \left(\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right)\left(\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right) + \left(\begin{smallmatrix} \operatorname{diag} A^t B \\ \operatorname{diag} C^t D \end{smallmatrix}\right) \bmod 2$.

We denote by $\circ$ the group action of $\mathrm{Sp}(g, \mathbb{Z})$ on $\mathcal{H}_g$ and now state the transformation formula of theta functions w.r.t. the group $\mathrm{Sp}(g, \mathbb{Z})$

Theorem 2.2.17. *Let* $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *and let* $(m \in \mathbb{Z}/2\mathbb{Z})^{2g}$ *be a theta characteristic, then the following transformation formula holds for all theta functions* *with characteristics*

$$\theta_m((C\tau + D)^{-1}z, \gamma \circ \tau) = \phi(\varepsilon, \delta, \gamma, z, \tau) \det(C\tau + D)^{1/2}\theta_{\gamma.m}(z, \tau)$$

Remark 2.2.18. The function ϕ is a complicated function of $z, \varepsilon, \delta, \tau, \gamma$ with the property that $\phi|_{z=0}$ is independent of τ and is an eigth root of unity and $\gamma.m = m$ if $\gamma \in \Gamma(2)$ and $\phi|_{z=0} = 1$ for $\gamma \in \Gamma(4, 8)$

One can easily see that the affine action of $\mathrm{Sp}(g, \mathbb{Z})$ is transitive on the set of even characteristics, and in fact it is also doubly transitive. More specifically we state a theorem of Igusa, that allows us to understand the orbits of sets of theta characteristic under the affine action of $\mathrm{Sp}(g, \mathbb{Z})$.

Theorem 2.2.19 ([21], see also [33])**.** *Given two p-tuples of* even *theta characteristics $I = (m_1, m_2, ...m_p)$ and $J = (n_1, n_2....n_p)$, they are in the same orbit of the action of the deck group* $\mathrm{Sp}(2g, \mathbb{Z}\backslash 2\mathbb{Z})$ *Use* $\mathrm{Sp}(g, \mathbb{Z})$ *for consistency on the set of p-tuples of theta characteristics, then there exists an element mapping one to the other, preserving the numbering if and only if all of the following conditions hold:*

1. *Linear relations among even number of terms in I translate to corresponding linear relations in J, i.e., $m_{i_1} + m_{i_2} .... + m_{i_{2l}} = 0$ if and only if $n_{i_1} + n_{i_2} .... + n_{i_{2l}} = 0$.*

2. *$e(m_i + m_j + m_k) = e(n_i + n_j + n_k)\ \forall i, j, k$.*

2.3 Modular Forms and equivariant vector bundles

We define the notion of Siegel modular forms as certain holomorphic functions on $\mathcal{H}_g$ satisfying some properties and interpret them as sections of vector bundles on level covers of $\mathcal{A}_g$.

Definition 2.3.1. A rational representation of an algebraic group, i.e. a group that is also an algebraic variety is a representation $\rho : G \to \mathrm{GL}(g, \mathbb{C})$ that is also a rational map of algebraic varieties.

Definition 2.3.2. Let $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ be a subgroup and let $\rho : \mathrm{GL}(g, \mathbb{C}) \to \mathrm{GL}(k, \mathbb{C})$ be a rational representation, we say that $F : \mathcal{H}_g \to \mathbb{C}^k$ is a ρ-valued modular form with respect of $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ if F is a holomorphic function such that $F((A\tau + B)(C\tau + D)^{-1}) = \rho(C\tau + D)F(\tau)$ for all $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma$. If $\rho(M) = \det(M)^n$ for $n \in \mathbb{Q}$ we say that F is a scalar modular form of weight n with respect to Γ.

Remark 2.3.3. In general for a scalar modular form of weight $n \in \mathbb{Q}$ with respect to a certain subgroup Γ, the n'th power is well defined, without any discrepancy in the choice of a root of unity if the subgroup Γ is appropriately chosen.

Remark 2.3.4. We observe that theorem 2.2.17, Remark 2.2.18 and Proposition 2.2.13 tell us that for $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ with m even, $\theta_m(0, \tau)$ is a scalar modular form of weight $\frac{1}{2}$ with respect to $\Gamma(4, 8)$ and $\theta_m(0, \tau)^8$ is a scalar modular form of weight 4 with respect to $\Gamma(2)$ or this can be restated by saying that the theta constants with characteristic m, $\theta_m(0,\tau)$ for m even are elements of $H^0(\mathcal{A}_g(4,8), \det(\mathbb{E})^{1/2})$

The algebro geometric interpretation of Siegel modular forms is that they represent sections of some vector bundles on level covers of $\mathcal{A}_g$. We describe these vector bundles on $\mathcal{A}_g$, but before doing so we describe the Hodge bundle on $\mathcal{A}_g$ from which many of the vector bundles associated to siegel modular forms can be defined.

Definition 2.3.5. Consider the universal family $\pi : \mathcal{X}_g \to \mathcal{A}_g$, recall this was constructed at least analytically, and consider the sheaf of relative differentials, $\Omega_{\mathcal{X}_g/\mathcal{A}_g}$. The Hodge bundle $\mathbb{E}$ is the locally free sheaf of rank g on $\mathcal{A}_g$ defined by $\pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$

Remark 2.3.6. We should mention that the fine moduli space of principally polarized abelian varieties is not a scheme but a Deligne-Mumford stack because ppav's admit automorphisms, the generic ppav $[A_\tau] \in \mathcal{A}_g$ has the standard involution as an automorphism, but if we pass to an appropriate level cover, e.g. to $\mathcal{A}_g(3)$, then this space is the moduli space of ppav's with a level structure and it turns out [] that this space is a smooth complex manifold or, there are no automorphisms of ppav's with a level 3 structure. Thus we see in particular that $\mathcal{A}_g$ is a global quotient of a manifold by a non free finite group action.

Remark 2.3.7. The definition of $\mathbb{E}$ indicates that $\mathbb{E}|_{[A_\tau]}$ for a ppav is isomorphic to $H^{1,0}(A_\tau)$, thus $(dz_1, dz_2, dz_3 ... dz_g)$ is in the sheaf of sections of $\mathbb{E}$ and it can be shown by writing down the action of $\mathrm{Sp}(g,\mathbb{Z})$ on $\mathcal{H}_g$ explicitly that the Hodge bundle corresponds to the identity representation of $GL(g,\mathbb{C})$, i.e. if

$f : A_{\tau_1} \to A_{\tau_2}$ is an isomorphism of ppav with $\tau_2 = (A\tau_1 + B)(C\tau_1 + D)^{-1}$, then consider the morphism from $f^* H^{1,0}(A_{\tau_2}) \to H^{1,0}(A_{\tau_1})$ is given by $(C\tau_1 + D)^{-1}$ with respect to the standard dual basis of $\mathbb{C}^g$. In this case it means that the sections of $\mathbb{E}$ are Siegel modular forms for the identity representation of $\mathrm{GL}(g, \mathbb{C})$

We now state the following theorem of Borel which classify line bundles on $\mathcal{A}_g$ up to their pullbacks on level covers.

Theorem 2.3.8. *[5] The rational Picard group of the moduli space $\mathcal{A}_g$ is $\mathrm{Pic}(\mathcal{A}_g) \otimes \mathbb{Q} = \mathbb{Q}\lambda_1$ where we denote $\lambda_1 := c_1(\mathbb{E}) \neq 0$.*

2.3.1 Satake Compactification

A natural question that arises is whether the determinant of the Hodge bundle $\det \mathbb{E}$ is ample on $\mathcal{A}_g$ and its level covers. Note that ampleness of pull backs of line bundles is preserved under finite morphisms. Indeed this turns out to be the case and can be seen from the following theorem

Theorem 2.3.9. *[21] The map $Th_m : \mathcal{A}_g(4, 8) \to \mathbb{P}^{2^{g-1}(2^g-1)-1}$ given by $Th_m(\tau) := [\theta_m(0, \tau)]_{m \in (\mathbb{Z}/2\mathbb{Z})^{2g}_{even}}$ is an embedding*

Corollary 2.3.10. *The line bundle $\det \mathbb{E}$ is very ample on $\mathcal{A}_g(4, 8)$ and thus $\det \mathbb{E}$ is ample on $\mathcal{A}_g$*

The above statement is a corollary because the positivity of curvature of a line bundle is preserved under pullbacks of finite maps, hence ampleness is preserved

Definition 2.3.11. The compactification of $\mathcal{A}_g$ induced from the linear system $|\det \mathbb{E}|^k$ is called the Satake-Baily-Borel compactification, $\mathcal{A}_g^{sat}$.

The set theoretic description of $\mathcal{A}_g^{sat}$ is given by

$$\mathcal{A}_g^{sat} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \mathcal{A}_{g-2} \ldots \ldots \mathcal{A}_0 = pt$$

The boundary of $\mathcal{A}_g^{sat}$ i.e. $\mathcal{A}_g^{sat} \setminus \mathcal{A}_g$ is contained inside the singular locus of $\mathcal{A}_g^{sat}$ and is highly singular and is of codimension g.

We describe a partial compactification of $\mathcal{A}_g$, i.e. $\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1}$

2.4 Jacobians and Riemann singularity theorem

One of the earliest studied examples of principally polarized abelian varieties were the Jacobians of smooth curves. We discuss this example and the associated principal polarizations and their connection to the geometry of the underlying curve. The material here is by and large obtained from [16] and [28].

Definition 2.4.1. Given a smooth curve C of genus g, the Jacobian of a curve is the group of invertible sheaves of degree 0 on the curve.

Definition 2.4.2. Given a smooth curve C of genus g, a divisor D on C is an element of the free abelian group generated by the points of X, thus a divisor D can be represented as $D = \sum_{p \in C} n_p p$ where $n_p \in \mathbb{Z}$ and $n_p = 0$ for all but finitely many points $p \in C$. In particular a divisor of degree k on C is a divisor

$D = \sum_{i=1}^{l} n_i p_i$ with $\sum n_i = k$. If $n_i \geq 0$ we say that the divisor D is effective on C. The support of a divisor $D = \sum_{p \in C} n_p p$ on C is defined to be the set of points $p \in C$ where $n_p \neq 0$.

Definition 2.4.3. Let $f : C_1 \to C_2$ be a finite morphism between smooth curves C_1 and C_2, if D is a divisor of the form $D = \sum_{i=1}^{k} n_i p_i$ then $f^*(D) = \sum_{i=1}^{k} n_i f^*(p)$, where $f^*(p) = \sum_{q \in f^{-1}(\{p\})} mult_f(q)$. where $mult_f(q)$ is the degree of the ramification of f at q.

Definition 2.4.4. Two effective divisors D_1 and D_2 are called linearly equivalent if there exists a morphism $f : C \to \mathbb{P}^1$ such that $f^*(0) + S = D_1$ and $f^*(\infty) + S = D_2$ for some effective divisor S in C.

The major theorem we wish to prove is the following

Theorem 2.4.5. *The Jacobian of a smooth curve C of genus g, denoted* $\mathrm{Pic}^0(C)$ *can be endowed with the structure of a principally polarized abelian variety.*

Remark 2.4.6. We will denote the set of invertible sheaves of degree k by $\mathrm{Pic}^k(C)$. The endowment of the structure of a ppav above arises as there is a natural polarization on the torsor $\mathrm{Pic}^{g-1}(C)$ of $\mathrm{Pic}^0(C)$ and as they are isomorphic as varieties, the chosen isomorphism gives a polarization on $\mathrm{Pic}^0(C)$ as well.

To prove the above theorem we first give a well known analytic description of the Jacobian by recalling some standard facts regarding curves.

Fact 2.4.7. *Given a closed oriented surface of genus g (C, p) we can view it as a closed orientable surface and we can choose a standard set of representatives* *of $\pi_1(C,p)$ given by $\{a_1,\dots a_g, b_1 \dots b_g\}$, i.e. closed loops around p such that we can cut C along the a_i and b_j to construct a $4g$ sided polygon $\mathcal{P}$ with sides being identified with a_i and b_j and a_i^{-1} and b_j^{-1}.*

Fact 2.4.8. *Since C is an oriented closed surface of genus g the homology $H_1(C,\mathbb{Z})$ has a symplectic basis $\{a_1, a_2, \dots a_g, b_1, b_2, \dots b_g\}$ such that $a_i \cdot a_j = b_i \cdot b_j = 0$ and $a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij}$, where δ_{ij} is the Krönecker delta. The symplectic basis can be thought of as the representatives of the loops $\{a_i, b_i\}_{i=1}^g$ given above as standard generators of $\pi_1(C,p)$ viewed in $H_1(C,\mathbb{Z})$*

Fact 2.4.9. *The following duality $H^{1,0}(C,\mathbb{C}) = H^{0,1}(C,\mathbb{C})^*$ holds and it arises from the non degeneracy of the mapping $H^{1,0}(C,\mathbb{C}) \times H^{0,1}(C,\mathbb{C}) \to \mathbb{C}$ given by $\int_C \alpha \wedge \beta$.*

Fact 2.4.10. *We also recall that the space of holomorphic one forms on C is a g-dimensional vector space $H^0(C,\Omega_C^1)$ that can be identified with $H^{1,0}(C,\mathbb{C})$, while for the de Rham cohomology we have $H^1(C,\mathbb{C}) = H^{1,0}(C,\mathbb{C}) \oplus H^{0,1}(C,\mathbb{C})$ with $H^{0,1}(C,\mathbb{C}) = \overline{H^{1,0}(C,\mathbb{C})} = H^{0,1}(C,\mathbb{C})^*$.*

The long exact sequence associated to the exponential sheaf sequence $0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$ leads to the natural inclusion $H^1(C,\mathbb{Z}) \hookrightarrow H^1(C,\mathcal{O}_C)$, since Serre duality tells us that $H^1(C,\mathcal{O}_C) = H^0(C,\Omega_C^1)^*$, we can naturally identify $H^1(C,\mathcal{O}_C)$ with $H^{0,1}(C,\mathbb{C})$. We recall that $H^1(C,\mathbb{Z}) = H_1(C,\mathbb{Z})^*$ if C is a smooth curve of genus g. Furthermore We can say that given an integral symplectic basis $\{a_1, a_2 \dots a_g, b_1, \dots b_g\}$ of $H_1(C,\mathbb{Z})$, the inclusion $H^1(C,\mathbb{Z}) \hookrightarrow H^1(C,\mathcal{O}_C)$ tells us that we can choose a basis $\{\overline{\omega_1} \dots \overline{\omega_g}\}$ of $H^{0,1}(C,\mathbb{C})$ such that $\int_{a_i} \overline{\omega_j} = \delta_{ij}$. This is seen to be the case because the dual integral basis to the linearly independent set $\{a_i\}_{i=1}^g \subset H_1(C,\mathbb{Z})$ given by $\{\delta_i\}_{i=1}^g \subset H^1(C,\mathbb{Z})$

can be interpreted as complex vector space basis for $H^{0,1}(C,\mathbb{C})$, due to the inclusion $H^1(C,\mathbb{Z}) \hookrightarrow H^{0,1}(C,\mathbb{C})$ as a rank $2g$ lattice inside a g dimensional complex vector space. Thus we can identify $\delta_i \in H^1(C,\mathbb{Z}) = H_1(C,\mathbb{Z})^*$ as one forms $\overline{\omega_i} \in H^{0,1}(C,\mathbb{C})$ such that $\int_{a_i} \overline{\omega_j} = \delta_{ij}$. Now if $\{\overline{\omega_i}\}_{i=1}^g$ is a complex vector basis for $H^{0,1}(C,\mathbb{C})$ then it's conjugate elements $\omega_i \in H^{1,0}(C,\mathbb{C})$ give us a basis for $H^{1,0}(C,\mathbb{C})$, such that $\int_{a_i} \omega_j = \delta_{ij}$.

We call the matrix $\tau = [\int_{b_i} \omega_j]$ for the choice of ω_j dual to the $a_i's$ as above the period matrix of C.

Remark 2.4.11. Observe that the period matrix τ for the curve C depends upon the choice of symplectic basis for $H_1(C,\mathbb{Z})$

Theorem 2.4.12. *Given $p_1, p_2 \in C$ the g tuple $(\int_{p_1}^{p_2} \omega_1, \dots \int_{p_1}^{p_2} \omega_g)$ is well defined on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ for our choice of ω_i's as chosen above and where τ is the period matrix of C.*

Proof. The statement of the theorem is true because if we choose two paths $\gamma_i : [0,1] \to C$, $i = 1,2$ such that $\gamma_i(0) = p_1$ and $\gamma_i(1) = p_2$, then the concatenation of $\gamma_1(t)$ and $\gamma_2(1-t)$ given by γ is a closed loop in C hence $\int_\gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega$ for any $\omega \in H^{1,0}(C,\mathbb{C})$, but since γ is a closed path in C it can be identified with an element of $H_1(C,\mathbb{Z})$ and thus $\gamma = \sum_{i=1}^g (\alpha_i a_i + \beta_i b_i)$, therefore $\int_\gamma \omega = \sum_{i=1}^g (\alpha_i \int_{a_i} \omega + \beta_i \int_{b_i} \omega)$ for $\alpha_i, \beta_i \in \mathbb{Z}$, since $\int_{a_i} \omega_j = \delta_{ij}$ and $\int_{b_i} \omega_j = \tau_{ij}$, we see that $(\int_{\gamma_1} \omega_1, \dots \int_{\gamma_1} \omega_g) - (\int_{\gamma_2} \omega_1, \dots \int_{\gamma_2} \omega_g) = (\sum_{j=1}^g (\alpha_j \int_{a_j} \omega_1 + \beta_j \int_{b_j} \omega_1), \dots \sum_{j=1}^g (\alpha_j \int_{a_j} \omega_g + \beta_j \int_{b_j} \omega_g) \in \mathbb{Z}^g + \tau\mathbb{Z}^g$ □

Definition 2.4.13. Let $\mathrm{Div}^k(C)$ denote the set of effective divisors of degree k on C. Given $p \in C$ and denoting τ the period matrix constructed above, we define $\mu_k : \mathrm{Div}^k(C) \to \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ by $\mu_k(\sum_{i=1}^k p_i) = (\sum_{i=1}^k \int_p^{p_i} \omega_1, \dots \sum_{i=1}^k \int_p^{p_i} \omega_g)$, and denote $W_k := \mu_k(D_k)$ its image.

We state Abel's Theorem which will allow us to prove Theorem 2.4.5.

Theorem 2.4.14. *(Abel's Theorem) Two effective divisors $D_1, D_2 \in \mathrm{Div}^k(C)$, they are linearly equivalent if and only if $\mu_k(D_1) = \mu_k(D_2)$.*

We also state Jacobi Inversion theorem that states that μ_k is surjective for $k \geq g$.

Theorem 2.4.15. *(Jacobi Inversion) The map $\mu_k : \mathrm{Div}^k(C) \to \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ is surjective for $k \geq g$.*

Riemann Roch tells us that for $k >> 0$ and a divisor D of degree k(not necessarily effective), $h^0(C, \mathcal{O}_C(D)) > 0.$, thus every element of $\mathrm{Pic}^k(C)$ can be expressed as $\mathcal{O}_C(D)$ for D being an effective divisor, thus the above discussion gives us the following theorem.

Theorem 2.4.16. *There is an isomorphism between $f : \mathrm{Pic}^k(C) \to \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ where the isomorphism is given by $f(\mathcal{O}_C(D)) = \mu_k(D)$ for an effective divisor D of degree $k >> 0$*

There is an isomorphism between $\mathrm{Pic}^k(C)$ and $\mathrm{Pic}^0(C)$, this can be seen by tensoring any invertible sheaf $\mathcal{L} \in \mathrm{Pic}^0(C)$ with $\mathcal{O}_C(k.p)$ for $p \in C$.

Thus if we manage to prove that $\tau = \tau^t$ and $Im(\tau) > 0$ we will have managed to prove Theorem 2.4.5. Before proving Theorem 2.4.5 we recall a standard construction from Surface topology and state a lemma that will be useful to us in our proof of Theorem 2.4.5

Remark 2.4.17. Given a closed $\mathcal{C}^\infty$ 1-form α on our smooth curve C we will abuse notation and call $q^*(\alpha)$ as α on $\mathcal{P}$ for the natural quotient $q : \mathcal{P} \to C$ and since α is exact on $\mathcal{P}$ we will denote by $f_\alpha(x)$ the integral $\int_p^x \alpha$ for a choice of a base point p in the interior of $\mathcal{P}$.

Lemma 2.4.18. *Let α and β be closed $\mathcal{C}^\infty$ one forms on C, then $\int_{\partial\mathcal{P}} f_\alpha\beta = \sum_{i=1}^g \int_{a_i}\alpha\int_{b_i}\beta - \int_{a_i}\beta\int_{b_i}\alpha$ where ∂P can be viewed as a cycle of the form $\sum_{i=1}^g a_i + b_i + a_i^{-1} + b_i^{-1}$*

Proof. Let $x \in a_i$ be identified with $x' \in a_i^{-1}$, and let σ_x be a path between x and x' in $\mathcal{P}$, then $f_\alpha(x) - f_\alpha(x') = \int_p^x \alpha - \int_p^{x'} \alpha = -\int_{\sigma_x} \alpha$, since integrals of α are zero around closed paths. Now on C the path σ_x is homotopic to b_i, therefore since α is a closed form we end up with $f_\alpha(x) - f_\alpha(x') = -\int_{b_i} \alpha$. We can similarly show that for a point $y \in b_i$ and the corresponding $y' \in b_i^{-1}$, $f_\alpha(y) - f_\alpha(y') = \int_{a_i} \alpha$. Note that for any one form α we have $\int_{a_i^{-1}} \alpha = -\int_{a_i}\alpha$ and $\int_{b_i^{-1}} \alpha = -\int_{b_i}\alpha$, finally note that β is a one form on C thus, the values of β along a_i and $a_i^{-1^{-1}}$ are the same. Based on this we see that $\int_{\partial P} f_\alpha\beta = \sum_{i=1}^g (\int_{a_i} + \int_{a_i^{-1}} + \int_{b_i} + \int_{b_i^{-1}}) f_\alpha\beta = \sum_{i=1}^g (\int_{x\in a_i} [f_\alpha(x) - f_\alpha(x')]\beta + \int_{y\in b_i} [f_\alpha(y) - f_\alpha(y)]\beta = \sum_{i=1}^g (\int_{x\in a_i} [\int_{b_i} \alpha]\beta = \sum_{i=1}^g \int_{a_i}\alpha\int_{b_i}\beta - \int_{a_i}\beta\int_{b_i}\alpha$.

□

Proof. (Proof of Theorem 2.4.5) We will view our smooth curve C of genus g as a quotient of a polygon $\mathcal{P}$ as above and choose a basis of $H^0(C, \Omega_C^1)$ as above. Thus we have $\mathrm{Pic}^0(C) \simeq \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ and if we manage to prove $\tau = \tau^t$ and $\mathrm{Im}(\tau) > 0$, then we are done.With the established notation Lemma 2.4.18 gives us the following

$$\sum_{i=1}^{g} \int_{a_i} \omega_k \int_{b_i} \omega_j - \int_{a_i} \omega_j \int_{b_i} \omega_k = \int_{\mathcal{P}} \omega_k \wedge \omega_j = \int_C \omega_k \wedge \omega_j = 0. \tag{2.4.1}$$

Plugging in $\tau_{ij} = \int_{b_i} \omega_j$ and $\int_{a_i} \omega_j = \delta_{ij}$ we see that $\tau = \tau^t$.

Similarly if we choose $\omega \in H^0(C, \Omega_C^1)$ as $\omega = \sum_{i=1}^{g} c_i \omega_i \neq 0$. Then we know that $-i \int_C \omega \wedge \overline{\omega} > 0$. Thus using Lemma 2.4.18 again we end up with the identity $\int_C \omega \wedge \overline{\omega} = \sum_{i,j} c_i c_j (\int_{b_i} \omega_j - \overline{\int_{b_i} \omega_j})$. This is precisely saying that $\mathrm{Im}(\tau) > 0$.

□

2.4.1 Riemann Singularity Theorem

We wish to relate the subvariety $W_{g-1} \subset \mathrm{Pic}^0(C) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ with τ defined as above, with the theta divisor $\Theta_\tau \subset \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ thought of explicitly as the zero locus of the theta function. Before doing so, let us denote by $\mathcal{O}_C(D)$ the invertible sheaf associated to a divisor D on C, we now state a theorem that tells us something about $h^0(C, \mathcal{O}_C(D))$ for a generic effective divisor D of degree k.

We first state a result that is a simple corollary of the Riemann Roch Theorem

Fact 2.4.19. *Let D be a generic effective divisor of degree $k \leq g-1$ on C then $h^0(C, \mathcal{O}_C(D)) = 1$.*

Corollary 2.4.20. *The subvariety $W_k \subset \mathrm{Pic}^0(C)$ is a subvariety of dimension k for $k \leq g-1$ in particular W_{g-1} is a divisor of $\mathrm{Pic}^0(C)$.*

It turns out that W_{g-1} is the divisor in $\mathrm{Pic}^0(C)$ whose associate Chern class is the principal polarization:

Theorem 2.4.21. *(Riemann's theorem) If* $\mathrm{Pic}^0(C) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ *then there exists* κ *such that* $W_{g-1} = \Theta + \kappa$*, where* $\kappa = \mu(L)$ *such that* $L^{\otimes 2} = K_C$*.*

Remark 2.4.22. The half canonical line bundles L obtained above depends on our choice of p

Riemann's theorem and the map $\mu_{g-1} : \mathrm{Div}^{g-1}(C) \to W_{g-1}$ allow us to specifically relate the singularities of the Theta function $\theta_m(z, \tau)$ to the dimension of the space of sections of line bundles.

Theorem 2.4.23. *The multiplicity of a point* $\mu_{g-1}(D)$ *on* W_{g-1} *is* $h^0(C, \mathcal{O}_C(D))$*.*

Recall that $\theta_m(z, \tau)$ is even if and only if m is even(2.2.13) then using 2.4.21 and 2.4.23 we obtain the following corollary

Corollary 2.4.24. *If* $\mathrm{Pic}^0(C) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ *then* $\theta_m(0, \tau) = 0$ *for even* m *implies that there an invertible sheaf* L *on* C *such that* $L^{\otimes 2} = K_C$ *and* $h^0(C, L) \geq 2$ *and* $h^0(C, L) \equiv 0 \mod 2$*.*

2.5 Torelli theorem and the moduli of curves

In the previous section we observed that to any smooth curve C of genus g we associated the principally polarized abelian variety $(\mathrm{Pic}^0(C), W_{g-1})$. A fundamental question that arises is the following, if the jacobians of two curves are isomorphic as ppav, is it necessarily true that the curves are isomorphic.

This is indeed the case and is shown in

Theorem 2.5.1 (Torelli theorem)**.** *If* $(\mathrm{Pic}^0(C), [W_{g-1}]) \simeq (\mathrm{Pic}^0(C'), [W'_{g-1}])$, *as ppav then* $C \simeq C'$.

Based on the above theorem we can define the coarse moduli space of curves $\mathcal{M}_g$ as follows

Definition 2.5.2. The coarse moduli space of smooth curves of genus g $\mathcal{M}_g$ can be defined as the subset $\mathcal{M}_g \subset \mathcal{A}_g$ whose points correspond to principally polarized abelian varieties that are jacobians of curves

We denote the moduli space of smooth curves of genus g with n ordered marked distinct points by $\mathcal{M}_{g,n}$

Remark 2.5.3. The moduli space of curves can be constructed independently using Geometric Invariant Theory or via Teichmuller theory. Just like the moduli space of ppav the moduli of smooth curves of genus g, $\mathcal{M}_g$ when viewed as a fine moduli space is not a scheme but a Deligne Mumford stack, but the associated coarse moduli space is a scheme. We won't distinguish between them unless we explicitly need to. In reality 2.5.1 gives a $2:1$ morphism $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ of fine moduli stacks ramified along the hyperelliptic locus

A well known modular compactification that was constructed for $\mathcal{M}_g$ is the Deligne Mumford compactification of $\mathcal{M}_g$, denoted by $\overline{\mathcal{M}_g}$. The advantage of the Deligne Mumford Compactification is that it is a divisorial compactification of $\mathcal{M}_g$, i.e. it's boundary components are of codimension one and it has an interpretation as a moduli space of stable curves, more precisely the points of $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ correspond parameterize isomorphism classes of stable nodal curves.

Remark 2.5.4. The word stable nodal curves means curves that have finitely many automorphisms and the only singularities they can have are nodes, i.e. singularities of the form $xy = 0$.

In fact $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ is a reducible divisor with $\lfloor \frac{g}{2} \rfloor + 1$ divisorial components, denoted by δ_0, δ_2, ... $\delta_{\lfloor \frac{g}{2} \rfloor}$. The generic point of δ_0 corresponds to an irreducible nodal curve of arithmetic genus g with a single node as a singularity. While as the generic point of δ_i corresponds to a reducible curve of arithmetic genus g with the irreducible components of the curve being smooth of genus g and genus $g - i$ and the intersection point of the two curves corresponds to a nodal singularity.

More precisely a generic point $[C] \in \delta_i$ has a description given by $[C] = C_i \sqcup C_{g-i}/p \sim q$ with $p \in C_i$ and $q \in C_{g-i}$ and p or equivalently q is a nodal singularity on C where $g(C_i) = g$ and $g(C_{g-i}) = g - i$.

For our purposes we will be working with a Zariski open subset of $\overline{\mathcal{M}_g}$ given by $\mathcal{M}_g^{ct} = \overline{\mathcal{M}_g} \setminus \delta_0$ called the moduli space of curves of compact type.

The reason we will be working with this space is because the Torelli embedding $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ extends as a morphism $\overline{\mathcal{J}} : \mathcal{M}_g^{ct} \to \mathcal{A}_g$. Observe that we specifically mention the word morphism, because the Torelli embedding is no longer an embedding once extended to $\mathcal{M}_g^{ct}$.

To understand how the Torelli morphism extends to $\mathcal{M}_g^{ct}$ consider a generic point of δ_i given by $[C]$ where $C = C_i \sqcup C_{g-i}/p \sim q$, with C_i being a curve of genus i and C_{g-i} being a curve of genus $g - i$. It can be shown [] this is easy, maybe explain why in the analytic picture where the holomorphic differentials live on one of the two components that the image of $[C]$ under the extended

Torelli morphism is $\mathrm{Pic}^0(C_i) \times \mathrm{Pic}^0(C_{g-i})$.

Observe that under the image of the Torelli morphism the point of attachment $p \in C_i$ is forgotten, indeed the fiber $J^{-1}([C]) = C_i \times C_{g-i}$. On a global level, the statement changes to $J^{-1}(\mathcal{M}_i \times \mathcal{M}_{g-i}) = \mathcal{M}_{i,1} \times \mathcal{M}_{g-i,1}$.

2.6 Tautological Ring and Grothendieck Riemann Roch Theorem

We now study a subring of the rational chow ring $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ called the tautological ring and a corresponding subring of the rational chow ring of the partial compactification $\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \subset \mathcal{A}_g^{sat}$.

Definition 2.6.1. The tautological ring $\mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g) \subset CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ is the ring generated by the Chern classes $\lambda_k := c_k(\mathbb{E})$ of the Hodge bundle

We denote by $\mathcal{R}H^*(\mathcal{A}_g, \mathbb{Q})$ the image of the tautological ring in the cohomology ring $H^*(\mathcal{A}_g, \mathbb{Q})$ and indicate the relations that the λ classes satisfy. These relations were proven in [34] and [8]

Theorem 2.6.2. *The image of the tautological ring $\mathcal{R}H^*(\mathcal{A}_g, \mathbb{Q})$ in the cohomology ring of $\mathcal{A}_g$ is described as follows*

$$\mathcal{R}H^*(\mathcal{A}_g, \mathbb{Q}) = \mathbb{Q}[\lambda_1, \lambda_2, \dots \lambda_g]/((\star), \lambda_g = 0),$$

where the relation $(\star)$ is

$$(\star)(1 + \lambda_1 + \lambda_2 \dots + \lambda_g)(1 - \lambda_1 + \lambda_2 \dots + (-1)^g \lambda_g) = 1.$$

Remark 2.6.3. The above discussion just gives us with the relation $c(\mathbb{E})c(\mathbb{E}^*) = 1 \in H^*(\mathcal{A}_g, \mathbb{Q})$ The relation $c(\mathbb{E})c(\mathbb{E}^*) = 1$ holds in the chow ring as well, we will discuss this relation in the chow ring below in a later subsection.

We indicate an outline of the proof of relation $(\star)$ in cohomology first and then later in Chow after introducing the Grothendieck Riemann Roch theorem.

Indeed, the tautological relation can be interpreted as the following relation between the Chern classes of vector bundles

$$c(\mathbb{E})c(\mathbb{E}^*) = 1$$

where $c(V)$ denotes the total Chern class of a vector bundle V. To prove this relation, we notice that $\mathbb{E} \oplus \mathbb{E}^*$ is a local system, i.e a vector bundle with a flat connection

The Hodge bundle can stalk wise be naturally identified as $H^{0,1}(A_\tau, \mathbb{C})$ for $[\tau] \in \mathcal{A}_g$ and the existence of a principal polarization $\omega \in H^{1,1}(A_\tau, \tau) \cap H^2(A_\tau, \mathbb{Z})$ on our abelian variety A_τ allows us to identify $H^{1,0}(A_\tau, \mathbb{C})$ with $H^{0,1}(A_\tau, \mathbb{C})^*$. This is the case because the pairing defined by $Q : H^{1,0}(A_\tau, \mathbb{C}) \otimes H^{0,1}(A_\tau, \mathbb{C}) \to \mathbb{C}$ by $Q(\alpha, \beta) = \int_{A_\tau} \alpha \wedge \beta \wedge \omega^{g-2}$ is non degenerate []. The non degeneracy follows from numerical criteria of ampleness of ω.

We know that $H^{1,0}(A_\tau, \mathbb{C}) \oplus H^{0,1}(A_\tau, \mathbb{C}) \simeq H^1(A_\tau, \mathbb{C})$ by the Hodge decomposition. At a global level this just says that $\mathbb{E} \oplus \mathbb{E}^*$ is isomorphic to the local system $R^1_{\pi_*}(\underline{\mathbb{C}})$ where π denotes the universal family $\pi : \mathcal{X}_g \to \mathcal{A}_g$. This local system in particular is a flat vector bundle corresponding to the tautological representation of the orbifold fundamental group of $\mathcal{A}_g$ given by $\mathrm{Sp}(2g, \mathbb{Z})$. Thus we see that $c(\mathbb{E})c(\mathbb{E}^*) = 1 \in H^*(\mathcal{A}_g, \mathbb{Q})$.

Theorem 2.6.4. *[34] The tautological ting of $\mathcal{A}_g$, $\mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g) \subset CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ is described as* $\mathbb{Q}[\lambda_1, \lambda_1, ...\lambda_g]/((1+\lambda_1+\lambda_2....\lambda_g)(1-\lambda_1+\lambda_2-\lambda_3...(-1)^g\lambda_g) = 1, \lambda_g = 0)$.

If one works over the partial compactification $\mathcal{A}^*_g$, the Hodge bundle extends as a coherent sheaf on $\mathcal{A}^*_g$ and the following result is known.

Theorem 2.6.5. *[34] The Tautological ring of $\mathcal{A}^*_g$ described by the subring of the chow ring of $\mathcal{A}_g$ is generated by the classes λ_i is described as* $\mathbb{Q}[\lambda_1, \lambda_2, ...\lambda_g]/((1+\lambda_1+\lambda_2....\lambda_g)(1-\lambda_1+\lambda_2-\lambda_3...(-1)^g\lambda_g) = 1)$

We now indicate the main ideas of the proof of relation $(\star)$ in the chow ring which is really the Grothendieck Riemann Roch theorem, We recall this theorem now and give some introductory definitions before stating the theorem and reproducing the proof of $(\star)$ from [34]

2.6.1 Grothendieck Riemann Roch Theorem

Definition 2.6.6. [14] Let X denote a smooth quasi projective variety and let $\mathrm{Coh}(X)/\sim$ denote the monoid of coherent sheaves on X with the following relation $[A] + [C] \sim [B]$ where A, B, C are coherent sheaves on X and $0 \to A \to B \to C \to 0$ is an exact sequence of $\mathcal{O}_X$ modules. The Grothendieck group of bounded complexes of coherent sheaves on X, $K_0(X)$ is the group associated to the monoid given by $\mathrm{Coh}(X)/\sim$

Definition 2.6.7. Let $f : X^{[n]} \to Y^{[m]}$ denote a proper morphism between smooth quasi projective varieties X and Y. We define a group homomorphism $f_! : K_0(X) \to K_0(Y)$ as $f_! = \sum(-1)^i R^i f_*$.

Definition 2.6.8. The Chern character on a quasi projective variety X is a homomorphism $Ch : K_0(X) \to A(X)$ satisfying the following conditions

- For a line bundle L on X, $Ch(L) = e^{c_1(L)}$
- For a vector bundle $V = \bigoplus_{i=1}^{n} L_i$ written as a direct sum of line bundles, with $Ch(\bigoplus_{i=1}^{n} L_i) = \sum_{i=1}^{n} Ch(L_i)$
- For two vector bundles V and W on X, $Ch(V \otimes W) = Ch(V)Ch(W)$
- for a morphism $f : Y \to X$ of quasi projective varieties X and Y, the following functorial property holds $f^*(Ch(V)) = Ch(f^*(V))$

Definition 2.6.9. The Todd class Td on a quasi projective variety X is a multiplicative homomorphism $Td : K_0(X) \to A(X)$ characterized by the following properties

- For a line bundle L, with $c_1(L) = x$ the Todd class $Td(L) = \frac{x}{1-e^{-x}}$
- For a vector bundle $V = \bigoplus_{i=1}^{n} L_i$, the Todd class of V given by $Td(V) = \prod_{i=1}^{n} Td(L_i)$

We now state the Grothendieck-Riemann-Roch theorem

Theorem 2.6.10. *[14] Let $f : X \to Y$ denote a proper morphism between smooth quasi projective schemes and let T_f denote the relative tangent sheaf of f, then for an element $\mathcal{F} \in K_0(X)$ then the following formula holds in $K_0(Y)$*

$$Ch(f_!\mathcal{F}) = f_*(Ch(\mathcal{F})Td(T_f))$$

Remark 2.6.11. It should be mentioned that the universal family $\mathcal{X}_g$ and $\mathcal{A}_g$ are Deligne Mumford stacks and their associated coarse moduli spaces are singular but by the work of Mumford et. al [] we can treat them as smooth varieties and can apply GRR to $\pi : \mathcal{X}_g \to \mathcal{A}_g$, without worrying about the stackiness. For the purposes of the proof of theorem 2.6.4 we will view $\mathcal{A}_g$ as a stack defined over $\operatorname{Spec}\mathbb{Z}$, the advantage of this viewpoint for our purposes is that $CH^*_{\mathbb{Q}}(\mathcal{A}_g \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p) \simeq CH^*_{\mathbb{Q}}(\mathcal{A}_g) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$. This allows us to move back and forth between sub varieties of $\mathcal{A}_g$ defined over positive characteristics and characteristic 0.

2.6.2 Outline of Proof of Theorem 6.4

Recall that we have to prove relations $(\star)$ and $\lambda_g = 0$ in the Chow ring, and then establish that there are no further relations.

Proof of relation $(\star)$ in the Chow ring, following van der Geer's original work [34]

Let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ denote the universal family over $\mathcal{A}_g$. We Recall that $\Omega_{\mathcal{X}_g/\mathcal{A}_g} \simeq \pi^*(\mathbb{E})$ and consider the line bundle L corresponding to the universal Θ divisor on $\mathcal{X}_g$.

We apply Grothendieck Riemann Roch to $\pi : \mathcal{X}_g \to \mathcal{A}_g$ and $L = \mathcal{O}(\Theta)$ which gives us

$$Ch(\pi_! L) = \pi_*(Ch(L)Td((\pi^*\mathbb{E})^*))$$

Now it is known that $R^i_{\pi_*}(L) = 0$ []. Thus $\pi_!(L)$ is a line bundle as L

represents the principal polarization on the universal abelian variety. If we denote $c_1(\pi_!(L)) = \theta$. The above expression by the defining property of Ch of a line bundle equates to.

$$\sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \pi_*(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!})Td(\mathbb{E}^*)$$

Comparison of the terms of degree 1 give us that

$$\theta = -\lambda_1/2 + \pi_*(\frac{\Theta^{g+1}}{(g+1)!})$$

Now if instead of using L on $\mathcal{X}_g$ we use L^n. It can be shown in [] that $Ch(\pi_!(L^n)) = n^g Ch(\pi_!(L))$[]. This can be seen because of the fact that $\pi_!(L^n)$ is a numerical function of degree n^g, at the level of stalks, for a ppav A_τ, $H^0(A_\tau, L^n) = n^g$, this follows from the fact that if a polarization $c_1(M)$ is of the form $\begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix}$ then $H^0(A_\tau, M) = \det(\Delta)$. Applying GRR we end up with the identity

$$n^g \sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \pi_*(\sum_{k=0}^{\infty} \frac{n^{g+k}\Theta^{g+k}}{(g+k)!}).Td(\mathbb{E}^*)$$

Comparing coefficients of powers of n we see $\pi_*(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!}) = 1$. This implies that 2.6.2 gives us the relation

$$2\theta = -\lambda_1$$

Further if we simplify 2.6.2 by rewriting $Td(\mathbb{E}^*)$ in terms of λ classes, we end up with $Td(\mathbb{E} \oplus \mathbb{E}^*) = 1$ **??**. which by the general theory of characteristic classes [34] imply that $c(\mathbb{E})c(\mathbb{E}^*) = 1$ thus proving relation $(\star)$.

Proof of the relation $\lambda_g = 0$

We now indicate the proof of the relation $\lambda_g = 0$, which in fact is also obtained by applying GRR, this time to the structure sheaf $\mathcal{O}_{\mathcal{X}_g}$ obtaining

$$Ch(\pi_!(\mathcal{O}_{\mathcal{X}_g})) = \pi_*(Ch(\mathcal{O}_{\mathcal{X}_g}).Td(\Omega_1^*)) = \pi_*(1)Td(\mathbb{E}^*) \tag{2.6.1}$$

Now for a vector bundle B of rank n on a variety X we have the following relation in general $\sum_{i=0}^{n}(-1)^i \wedge^i B^* = c_n(B)Td(B)^{-1}$ [34]. as follows easily from some of the defining properties of Td.

This gives the relation $Ch(\sum_{i=0}^{g}(-1)^i \wedge^i \mathbb{E}^*) = c_g(\mathbb{E})Td(\mathbb{E})^{-1} = 0$. Since $c_g(\mathbb{E}) = \lambda_g \in CH^*_{\mathbb{Q}}(\mathbb{E})$ and since $Td(\mathbb{E}^*) = Td(\mathbb{E})^{-1}$, we see that $c_g(\mathbb{E}) = \pi_*(1) = 0$.

Proof that there are no further relations in $\mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g)$

We first introduce the graded ring $R_g := \mathbb{Q}[u_1, u_2, u_3 ... u_g]/((1 + u_1 + u_2 ... + u_g)(1 - u_1 + u_2 + ...(-1)^g u_g) = 1)$, with the grading defined by $\deg u_k = k$. It can be shown [8][36] that R_g is a gorenstein ring with socle in degree $g(g+1)/2$. This is the case because $\prod u_i^{e_i}$ is an additive basis for R_g.

It was shown in Ekedahl and Oort [] that $\mathcal{A}_g \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$ has a complete subvariety of dimension $\frac{g(g-1)}{2}$. This results in the fact that $\lambda_1^{g(g-1)/2} \neq 0$. Now the fact that $\lambda_g = 0 \in CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ along with relation $(\star)$ tells us that there is a natural homomorphism from $R_{g-1} \to \mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g)$ and the fact that $\lambda_1^{g/(g-1)/2}$ coupled with the fact that R_{g-1} is gorenstein tells us that the $R_{g-1} \simeq \mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g)$.

Chapter 3

Affine Covers and Stratifications

We study the notion of affine covers and stratifications on $\mathcal{A}_g$ and $\mathcal{M}_g$ and recall some constructions of stratifications on these spaces and prove Theorem 1.0.1. We also discuss how the study of the tautological ring of $\mathcal{A}_g$ and $\mathcal{M}_g$ are related to these stratifications in general.

3.1 Stratifications and Complete Subvarieties of $\mathcal{M}_g$

Before discussing stratifications and the cohomological dimension of $\mathcal{A}_g$ we give a brief survey on the affine covers and affine stratifications of $\mathcal{M}_g$ and the tautological ring that serves as motivation for our work.

It can be shown that $\mathcal{M}_g$ for $g \geq 3$ is not affine, as we can construct a complete curve inside it. To construct such a curve, denote $\mathcal{M}_g^{sat}$ the closure of $\mathcal{M}_g \subset \mathcal{A}_g^{sat}$, which is a projective variety. Further note that $codim(\mathcal{M}_g^{sat} \setminus \mathcal{M}_g) \subset \mathcal{M}_g^{sat} = 2$. This is seen to be the case because there is a morphism $\mathcal{J}^{sat} : \overline{\mathcal{M}_g} \to \mathcal{M}_g^{sat}$. For a generic point of δ_i, $i > 0$ given by $[C] = [C_i \sqcup C_{g-i}/p \sim q]$ where C_i and C_{g-i} are smooth curves of genus i and $g-i$ respectively, we have $\mathcal{J}^{sat}[C] = \mathrm{Pic}^0(C_i) \times \mathrm{Pic}^0(C_{g-i})$. Thus the fibers of the morphism of $\mathcal{J}^{sat}$ restricted to the generic point of δ_i are of dimension 2. For $i = 0$, the generic point of δ_0 is identified with $[C, p, q]/$ where the geometric genus $g(C) = g-1$ and there is a nodal singularity at $p \sim q$ on the quotient. In this case the generic point $[C, p, q]$ is mapped to $\mathrm{Pic}^0(C) \in \mathcal{A}_g^{sat} \setminus \mathcal{A}_g$. Thus the fibers of the morphism of $\mathcal{J}^{sat}$ restricted to δ_0 also has fibers of dimension 2.

Thus if we choose sufficiently general very ample divisors on $\mathcal{M}_g^{sat}$, we can construct a complete curve inside $\mathcal{M}_g$ that does not intersect the boundary. More precisely we choose general very ample divisors D_i on $\mathcal{M}_g^{sat}$ for $i = 1 \ldots 3g-4$ such that $\cap_{i=1}^{k} D_i$ intersects $\mathcal{M}_g^{sat} \setminus \mathcal{M}_g$ in a subscheme of codimension k. Thus we end up with $\cap_{i=1}^{3g-4} D_i \cap \mathcal{M}_g^{sat} \setminus \mathcal{M}_g = \emptyset$. But since $\dim(\mathcal{M}_g) = 3g-3$ we get that $\cap_{i=1}^{3g-4} D_i \cap \mathcal{M}_g^{sat} = \cap_{i=1}^{3g-4} D_i \cap \mathcal{M}_g \neq \emptyset$. Thus we can find a complete curve inside $\cap_{i=1}^{3g-4} D_i \cap \mathcal{M}_g$.

We note that $\mathcal{M}_g$ is known not to be projective, as it is not compact, and it is natural to ask what is the maximal dimension of a complete subvariety, we mention Diaz's Theorem in this context.

Theorem 3.1.1. *The dimension of any complete subvariety in $\mathcal{M}_g$ is less than or equal to $g-2$.*

The motivation for Diaz's theorem arises from the Weierstrass stratification on $\mathcal{M}_g$ indicated by Arbarello[2]. Arbarello had conjectured that the stratification was affine. It was recently proven by Arbarello and Mondello that the stratification is almost never affine [2], while Krichever recently proved that the strata in fact do not contain complete curves [26] which implies the Diaz theorem above.

Definition 3.1.2. The Weierstrass stratification on $\mathcal{M}_g$ is defined as

$$\mathcal{M}_g = W_g \supset W_{g-1} .... \supset W_k ... \supset W_2$$

where $W_\alpha := \{[C] \in \mathcal{M}_g \mid \exists p \in C, h^0(C, \alpha p) \geq 2\}$.

The idea was to prove that $W_k \setminus W_{k-1}$ could not contain any complete curves. If one exhibits such a stratification then any complete subvariety of $\mathcal{M}_g$ of dimension d would intersect W_k in a complete subvariety of dimension $d-k+1$ and thus intersect W_2 in a subvariety of dimension at most $d-g+2$, if we prove that W_2 does not contain any complete curves, we would be done.

A simpler way of understanding the strata is by looking at curves that admit a map to $\mathbb{P}^1$ that is completely ramified at p. One sees from the definition of the Weierstrass stratification that W_2 is just the hyperelliptic locus in $\mathcal{M}_g$. It was further known that W_k is of codimension $g-k$ in $\mathcal{M}_g$ and moreover that W_k is irreducible and closed.

Before the work of Krichever on Arbarello's conjecture, Diaz constructed an alternative stratification of $\mathcal{M}_g$, and showed that its strata did not contain complete curves. Diaz considered the stratification of $\mathcal{M}_g$ by $\mathcal{M}_g = D_g \supset D_{g-1} \supset .... \supset D_2$ where D_k consists of those curves C of genus g such that there is a rational morphism $\pi : C \to \mathbb{P}^1$ with $\#\pi^{-1}\{0, \infty\} \leq k$. Diaz then showed that for any k $D_k \setminus D_{k-1}$ contains no complete curves It is then shown that the stratification $D_k \setminus D_{k-1}$ does not contain complete curves. It is not known if $D_k \setminus D_{k-1}$ are affine, but it is known that D_k is of pure codimension k. Using this stratification Diaz proved his theorem

One does however know that W_2 is affine, i.e. that the hyperelliptic locus $Hyp_g \subset \mathcal{M}_g$ is affine. We indicate the proof below, as it will be needed for some of the things we do in what follows.

Theorem 3.1.3 ([25])**.** *The hyperelliptic locus or the moduli of hyperelliptic curves $Hyp_g \subset \mathcal{M}_g$ is affine.*

Proof. Indeed, hyperelliptic curves of genus g are double covers of $\mathbb{P}^1$ branched along $2g+2$ distinct points on $\mathbb{P}^1$. Thus a hyperelliptic curve is determined by a choice of $2g+2$ points on $\mathbb{P}^1$ up to an action of $\mathrm{Aut}(\mathbb{P}^1)$. Thus Hyp_g can be naturally identified with a finite group quotient of $\mathcal{M}_{0,2g+2}$, here $\mathcal{M}_{0,2g+2}$ denotes the moduli space of ordered $2g+2$ points on $\mathbb{P}^1$. The space $\mathcal{M}_{0,2g+2}$ is known to be affine. This can be seen to be the case because it is a finite quotient of $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{2g-1} \setminus \Delta$ where Δ is the union of all the big diagonals, i.e. where two or more coordinates coincide in $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{2g-1}$. □

3.2 Tautological Ring of $\mathcal{M}_g$

One of the active areas of research has been understanding the cohomology and Chow rings of $\mathcal{M}_g$. The first result in this direction is

Theorem 3.2.1. *[18] The rational Picard group* $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_g) = \mathbb{Q}\lambda$.

The Chow ring and the cohomology of $\mathcal{M}_g$ is not known for all g and its theory is quite complicated in general. That is why we study a subring of the Chow ring of $\mathcal{M}_g$ called the tautological ring. The tautological ring of $\mathcal{M}_g$ is related to the tautological ring of $\mathcal{A}_g$ by the Torelli morphism, i.e. the pullbacks of tautological classes by the Torelli morphism end up giving tautological classes in the new ring, but the pullback homomorphism is not surjective for $g \geq 6$.

Definition 3.2.2. Let $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$, denote the universal family over $\mathcal{M}_g$, i.e. that sends $\pi([C,p]) \to [C]$ for $C \in \mathcal{M}_g$ and $p \in C$ and now consider the relative sheaf of differentials given by $\omega_\pi = \Omega^1_{\mathcal{M}_{g,1}/\mathcal{M}_g}$. Let $\psi := c_1(\omega_\pi)$. The kappa classes are defined as $\kappa_{n-1} := \pi_*(\psi^n) \in CH^{n-1}(\mathcal{M}_g)$.

Definition 3.2.3. The tautological ring of $\mathcal{M}_g$, denoted $\mathcal{R}^*_{\mathbb{Q}}(\mathcal{M}_g)$, is defined to be the subring of the Chow ring generated by the kappa classes, i.e. $R^* := \mathbb{Q}[\kappa_1, \kappa_2, ...\kappa_{3g-3}]$.

Recall that the tautological ring $R^*(\mathcal{A}_g)$ was defined as the ring generated by the Chern classes of the Hodge bundle $\mathbb{E}$. Also recall that the Hodge bundle was stalk wise seen to be $H^0(A_\tau, \Omega^1)$ on $[A_\tau] \in \mathcal{A}_g$. If A_τ is the jacobian of a smooth curve C of genus g then construction of $\mathrm{Pic}^0(C)$ allows us to view $H^0(A_\tau, \Omega^1) \simeq H^0(C, \omega_C)$. Now we consider the pullback $\mathcal{J} : \mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g) \to CH^*_{\mathbb{Q}}(\mathcal{M}_g)$ with respect to the Torelli map. Thus we can consider the lambda classes, i.e. the Chern classes of the Hodge bundle, in the Chow ring of $\mathcal{M}_g$. We also observe that if $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ is the universal family, then $\pi_*(\omega_\pi) = \mathbb{E}|_{\mathcal{M}_g}$. This suggests that we could apply the Grothendieck Riemann Roch Theorem for π, and indeed this yields the following result of Mumford.

Theorem 3.2.4. *If $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ denotes the Torelli embedding, then $\mathcal{J}^*(\mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g)) \subset \mathcal{R}^*_{\mathbb{Q}}(\mathcal{M}_g)$.*

Proof. If we manage to prove that the restriction of λ classes to $\mathcal{M}_g$ can be rewritten as polynomials in kappa classes we are done. For this purpose we apply the Grothendieck Riemann Roch Theorem to $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ and the invertible sheaf ω_π i.e. sheaf of relative differentials with respect to π. We observe that $R^0\pi_*(\omega_\pi) = \mathbb{E}$ and $R^1\pi_*(\omega_\pi) = \mathcal{O}_{\mathcal{M}_g}$. Thus Grothendieck Riemann Roch gives us the following equality

$$ch(\mathbb{E} - \mathcal{O}_{\mathcal{M}_g}) = \pi_*(e^\psi \frac{-\psi}{1 - e^\psi}).$$

On the left hand side we end up with $g - 1 + p(\lambda_1, \lambda_2, \ldots \lambda_g)$, on the right hand side we end up with a polynomial in the kappa classes. and moreover one can see that the k'th graded piece of $p(\lambda_1 \ldots \lambda_g)$ will be a polynomial of the form $c\lambda_k + q_k(\lambda_1 \ldots \lambda_{g-1})$ where $q_k(\lambda_1, \ldots \lambda_{g-1}) \in \mathcal{R}^k_{\mathbb{Q}}(\mathcal{M}_g)$ and $c \in \mathbb{Q}$ is non-zero. Thus by induction one can see that the λ classes can be expressed as polynomials in the kappa classes. $\square$

The inclusion is in fact an equality for $g \leq 5$, for $g \geq 6$, it is a proper inclusion, i.e. all kappa classes cannot be expressed as lambda classes [11]. In fact the recent work of Vakil-Penev [31] shows that $CH^*_{\mathbb{Q}}(\mathcal{M}_g) = \mathcal{R}^*_{\mathbb{Q}}(\mathcal{M}_g)$ for $g \leq 6$.

Unlike the case of the tautological ring of the moduli space of ppav, the relations defining $\mathcal{R}^*_{\mathbb{Q}}(\mathcal{M}_g)$ for all g are the current subject of intensive research. There is a conjectural description of the tautological ring due to Faber [10], where some of the conjectures have been proven. We state two of these conjectures.

Conjecture 3.2.5. *a. $\mathcal{R}^*(\mathcal{M}_g)$ is a Gorenstein ring with socle in dimen-*

sion $g-2$. i.e. $\mathcal{R}_{\mathbb{Q}}^{g-2}(\mathcal{M}_g) \simeq \mathbb{Q}$ and there is a perfect pairing $\mathcal{R}_{\mathbb{Q}}^{k}(\mathcal{M}_g) \times \mathcal{R}_{\mathbb{Q}}^{g-2-k}(\mathcal{M}_g) \to \mathcal{R}_{\mathbb{Q}}^{g-2}(\mathcal{M}_g)$.

b. The first $\left\lfloor \frac{g}{3} \right\rfloor$ kappa classes $\left\{\kappa_1, \kappa_2 .... \kappa_{\left\lfloor \frac{g}{3} \right\rfloor}\right\}$ generate the ring with no relations in degrees $\leq \left\lfloor \frac{g}{3} \right\rfloor$

Theorem 3.2.6. *1. (Looijenga) It is known that $\mathcal{R}^{g-2}(\mathcal{M}_g) = \mathbb{Q}\lambda_1^{g-2}$ with $\lambda_1^{g-2} \neq 0$ and $\mathcal{R}^i(\mathcal{M}_g) = 0$ for $i > g-2$*

2. Part b of the above conjecture is true when the tautological classes are viewed in cohomology instead of the Chow ring.

The one-dimensionality is a combination of the work of Looijenga[27], who showed $\dim_{\mathbb{Q}}(\mathcal{R}_{\mathbb{Q}}^{g-2}(\mathcal{M}_g)) \leq 1$, and the work of Faber, who showed non-vanishing [9]. The vanishing in degree higher than $g-2$ is due to Looijenga [27].

The perfect pairing part of the conjecture is still open, in fact it is doubtful if the conjecture holds because of the failure of the modifications of the conjecture for compactifications of $\mathcal{M}_g$. The second part of the Theorem is the culmination of the work of various topologists [30]

The theorem of Diaz follows easily from the vanishing statement above. Indeed, if X is a complete subvariety of $\mathcal{M}_g$ of dimension k, then since λ_1 is ample on $\mathcal{M}_g$ (as the pullback of an ample line bundle from $\mathcal{A}_g$), we must have $\lambda_1^k \cdot [X] > 0$. But $\lambda_1^k = 0$ for $k > g-2$ by Loojienga's Theorem, and thus we must have $k \leq g-2$.

One of the primary reasons for people's interest in the tautological ring is that geometrically meaningful loci in $\mathcal{M}_g$ such as Brill Noether loci are tautological [11].

3.3 Affine covers, affine stratifications and cohomological dimension

Given a quasi projective variety X, a fundamental question which one wishes to ask is how far it is from being projective. To make the statement precise we introduce the notions of affine covering number acn, affine stratification number asn and the cohomological dimension cd of a variety X. The results below are due to Vakil and Roth and are discussed in detail in [32].

Definition 3.3.1. [32] The affine covering number (acn) of a scheme X is defined to be one less than the smallest number of open affine sets required to cover X.

Definition 3.3.2. An affine stratification of a scheme X is a finite decomposition $X = \sqcup_{k\in\mathbb{Z}\geq 0,i} Y_{k,i}$ into locally closed affine subschemes $Y_{k,i}$, where for each $Y_{k,i}$,

$$\overline{Y}_{k,i} \setminus Y_{k,i} \subseteq \cup_{k'>k,j} Y_{k',j}.$$

The length of an affine stratification is the largest k such that $\cup_j Y_{k,j}$ is nonempty.

The affine stratification number $\operatorname{asn} X$ of a scheme X is the minimum of the lengths of all possible affine stratifications of X.

Definition 3.3.3. The coherent cohomological dimension $\mathrm{cd}(X)$ of a variety X is the smallest i such that $H^j(X, \mathcal{F}) = 0$, for all $j > i$ and for any quasicoherent sheaf $\mathcal{F}$ on X.

Definition 3.3.4. The constructible cohomological dimension $\mathrm{ccd}(X)$ of a variety X is the smallest i such that $H^j(X,\mathcal{F})=0$, for all $j>i$ and for any constructible sheaf $\mathcal{F}$ on X

The reason the above definitions measure how far a variety is away from being projective is Grothendieck's theorem.

Theorem 3.3.5. *[20] A quasi projective variety X is projective if and only if* $\mathrm{cd}(X)=\mathrm{acn}(X)=\mathrm{asn}(X)=dim(X)$. *Similarly X is affine if and only if* $\mathrm{cd}(X)=\mathrm{acn}(X)=\mathrm{asn}(X)=0$.

We now state a result of Roth and Vakil, which we will use to state some of the main results in this Thesis.

Theorem 3.3.6. *[32] The coherent cohomological dimension of a scheme X is bounded above by the affine stratification number of the scheme X which in turn is bounded above by the affine covering number of X:*

$$\mathrm{cd}(X)\leq \mathrm{asn}(X)\leq \mathrm{acn}(X).$$

Similarly the constructible cohomological dimension is bounded above by $\dim X+\mathrm{asn}\,X$*, i.e.*

$$\mathrm{ccd}\,X\leq \dim X+\mathrm{asn}\,X.$$

3.4 Looijenga's conjecture and affine covers and stratification of $\mathcal{M}_g$

Looijenga has made the following conjecture

Conjecture 3.4.1. *The moduli space of smooth curves of genus g, $\mathcal{M}_g$ has an open covering by $g-1$ open affine sets i.e. $acn(\mathcal{M}_g) \leq g-1$.*

We will now show that the bound $g-1$ is definitely the optimal one, and will also present a proof of the conjecture for $g \leq 5$ due to Fontanari and Pascolutti [13].

Lemma 3.4.2. *If U is an open affine subset of X with X being a normal quasi projective variety with $\mathrm{Pic}_{\mathbb{Q}}(X) \simeq \mathbb{Q}$ then $U \subset X \setminus D$, where D is a divisor.*

Proof. Let $i : U \to X$ be the inclusion map. Since X is normal, $\mathrm{Pic}_{\mathbb{Q}}(U) = 0$. The only way for $i^* : Pic_{\mathbb{Q}}(X) \to Pic_{\mathbb{Q}}(U)$ to be the zero map is if the λ class goes to zero and that can happen only if U is contained in the complement of the divisor, as removing a higher codimension locus of $\mathcal{M}_g$ does not affect the image of $\mathrm{Pic}_{\mathbb{Q}}(X)$. □

We apply the above lemma to the case when $X = \mathcal{M}_g$ and provide a proof for the following result.

Theorem 3.4.3. *We have the following inequality* $\mathrm{acn}(\mathcal{M}_g) \geq g-1$.

Remark 3.4.4. As a consequence of this theorem Looijenga's conjecture can be restated as the equality $\mathrm{acn}(\mathcal{M}_g) = g-1$.

Proof. Suppose that $\mathcal{M}_g = \cup_{i=1}^n U_i$, where U_i are affine open subsets of $\mathcal{M}_g$. Then by lemma 3.4.2 we see that $U_i \subset \mathcal{M}_g \setminus D_i$ where D_i are divisors, this implies that $\cap_{i=1}^n D_i = \emptyset$. Theorem 3.2.1 tells us that D_i is represented by a non zero multiple of λ_1 in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_g)$, this implies that $\lambda_1^n = 0$ on $\mathcal{M}_g$. Theorem 3.2.6 says that the smallest n for which $\lambda_1^n = 0$ is $n = g-1$. Thus we see that $\mathrm{acn}(\mathcal{M}_g) \geq g-1$. □

Looijenga's conjecture has been proven by Fontanari and Pascolutti [13] for $g \leq 5$ and we indicate the outline of the proof.

Fontanari and Pascolutti constructed $g-1$ modular forms for $3 \leq g \leq 5$ such that the intersection of the zero loci of these modular forms on $\mathcal{M}_g \subset \mathcal{A}_g$ is empty. We will indicate their construction below, but before doing so we introduce some preliminary results on modular forms and their loci necessary for the construction.

Theorem 3.4.5. *[29] Let $\tau \in \mathcal{H}_g$; then τ is the period matrix of a hyperelliptic curve if and only if exactly $2^{g-1}(2^g+1) - \frac{1}{2}\binom{2g+2}{g+1}$ suitable even theta constants vanish at the point τ. Each suitable sequence of theta constants defines an irreducible component of $Hyp_g(2) = p^{-1}(Hyp_g)$ for $p : \mathcal{A}_g(2) \to \mathcal{A}_g$, and the group $\mathrm{Sp}(2g, \mathbb{Z})$ acts transitively on the respective sequences of theta constants. More specifically 2.2.19 tells us that each irreducible component of $Hyp_g(2)$ corresponds to the orbits of a particular $2^{g-1}(2^g+1) - \frac{1}{2}\binom{2g+2}{g+1}$ tuple of theta characteristics satisfying suitable azygetic/syzygetic properties.*

Let us denote by $\mathcal{E}_g$ the set of all even theta characteristics. Let us denote by $\mathcal{B}_g$ a sequence of even theta constants such that for $[\tau] \in Hyp_g(2)$, $\theta_m(0, \tau) = 0$ if and only if $m \in \mathcal{B}_g$. Of course the number of such sequences is non unique, but they are all orbits of each other under the affine action of $\mathrm{Sp}(g, \mathbb{Z})$ on tuples. In the case above we just pick and fix a particular sequence.

For $S \subset \mathcal{E}_g$ we denote $P(S) := \prod_{m \in S} \theta_m$.

Remark 3.4.6. We have $\#\mathcal{B}_3 = 1$, $\#\mathcal{B}_4 = 10$, and $\#\mathcal{B}_5 = 66$, i.e. the hyperelliptic locus Hyp_g is described by the vanishing of 1 theta constant, 10 theta constants and 66 theta constants in genera 3, 4 and 5 respectively. In the case of genus 3 if more than one theta constant vanishes, then the zero locus described is the decomposable locus of $\mathcal{A}_3$.

The idea behind the proof of Fontanari and Pascolutti is to construct $g-1$ ample divisors $\{D_i\}_{i=1}^{g-1}$ on $\mathcal{M}_g$ that are described as zero loci of scalar modular forms on $\mathcal{M}_g$ such that $\cap D_i = \emptyset$ so that the complements of this divisors cover $\mathcal{M}_g$.

Remark 3.4.7. A quick word on notation, if F is a scalar modular form of weight k with respect to $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ we denote the zero locus of F on $\mathcal{H}_g/\Gamma$ by $Z(F)$.

Before reproducing the proof we state a lemma that will help us prove the conjecture for $\mathcal{M}_5$.

Remark 3.4.8. We recall that a curve C carrying a g_d^r means that there exists a divisor D on C such that $deg(D) = d$ and $h^0(C, \mathcal{O}_C(D)) = r + 1$. In particular a smooth curve C carrying a g_2^1 is necessarily hyperelliptic.

Lemma 3.4.9. *If a curve of genus* 5 *with a base point free* g_3^1 *carries two half canonical linear* g_4^1 *then* C *is hyperelliptic.*

The above lemma can be shown using standard algebro geometric techniques from [1]

We now reproduce Fontanari and Pascolutti's [12] proof of Looijenga's conjecture for $2 \leq g \leq 5$:

Proof. **g =2.**

The moduli space of smooth curves of genus 2 is affine as every smooth curve of genus 2 is hyperelliptic, this can be seen because of the following, if C

is a curve of genus 2 then by Riemann Roch, $h^0(C, K_C) = 2$, but $\deg(K_C) = 2(2) - 2 = 2$, hence the canonical map is a $2:1$ map of the curve to $\mathbb{P}^1$. Thus we see that $\mathcal{M}_2 = Hyp_2$ and by Theorem 10.2, $\mathcal{M}_2$ is affine.

g=3.

1. *First Divisor* D_1. Let $\Theta_{null} := \prod_{m\in\mathcal{E}_g} \theta_m$, this is a modular form of weight $2^{g-1}(2^g - 1)$ on $\mathcal{A}_g$, and moreover corollary 2.4.24 tells us that if $A_\tau = \mathrm{Pic}^0(C)$ for a smooth curve of genus g such that $\Theta_{null}(\tau) = 0$ then there exists a g_2^1 on the curve C. Therefore C is hyperelliptic.

2. *Second Divisor* D_2. Let

$$F_H := \sum_{\gamma.\mathcal{B}_g, \gamma\in\mathrm{Sp}(2g,\mathbb{Z})} P(\mathcal{E}_g \setminus \gamma.\mathcal{B}_g),$$

where we recall that P(S) denotes the product $\prod_{m\in S} \theta_m(0,\tau)$ for $m \in S$ By Theorem 3.4.5 we see that $Z(F_1) \cap Hyp_3 = \emptyset$ thus we let $D_2 = Z(F_1)$ therefore $U_2 = Z(F_1)$ and since $D_1 \cap D_2 = \emptyset$ we see that $U_1 \cup U_2 = \mathcal{M}_3$.

g=4.

1. *First Divisor* D_1. We choose our first divisor to be $D_1 := Z(\Theta_{null})$, Corollary 2.4.24 tells us that if τ is the period matrix for C such that $\theta_m(0,\tau) = 0$ then $\exists L$ on C such that $h^0(C, L) \geq 2$ and $L^{\otimes 2} = K_C$, in this case if C was a canonical curve then let $D_1 \neq D_2 \in |L|$, now since $2(D_1 + D_2) = 2D_1 + 2D_2$ and since $D_1 + D_2 \in |K_C|$ and $2D_1$ and $2D_2 \in |K_C|$ tells us that C lies on a quadric of the form $x^2 - yz$ in $\mathbb{P}^{g-1}$ or a quadric of rank 3 in general. This is seen to be the case because if $i : C \to \mathbb{P}^3$ is the embedding given by the canonical linear system, i.e. $i^*(\mathcal{O}(1)) = K_C$, and if $y, z, x \in H^0(\mathbb{P}^3, \mathcal{O}(1))$ such that the divisors of y, z, x on C are $2D_1$ and $2D_2$ and $D_1 + D_2$ thus since the

divisor associated to y^2 is the same as the divisor associated to xz, hence by an appropriate change of coordinates we end up with the fact that the curve passes through a quadric of the form $y^2 - xz$.

2. *Second Divisor* D_2. Let

$$F_1 := \sum_{m \in \mathcal{E}_g} \frac{\Theta_{null}^8}{\theta_m^8}$$

and let $D_2 := Z(F_1)$. The zero locus $D_1 \cap D_2$ gives us the locus of curves where at least two Theta constants vanish. A result of Igusa [22][21] tells us that if C is a smooth genus 4 curve with period matrix τ such that $\theta_{m_1}(0,\tau) = \theta_{m_2}(0,\tau) = 0$ for $m_1 \neq m_2$ then C is hyperelliptic.

3. *Third Divisor* D_3. We choose our third divisor $D_3 = Z(F_H)$ by our definition of F_H we can see that $Z(F_H) \cap Hyp_3 = \emptyset$, since $D_1 \cap D_2 = Hyp_3$ we end up with $D_1 \cap D_2 \cap D_3 = \emptyset$, thus $\cup_{i=1}^3 U_i = \cup_{i=1}^3 D_i^c = \mathcal{M}_4$.

g=5.

1. *First Divisor* D_1. In a paper of Grushevsky and Salvati Manni [17] it was shown that the closure of the open locus of trigonal curves of genus 5 could be described as the zero locus of the Schottky-Igusa modular form

$$F_T := 2^g \sum_{m \in \mathcal{E}_g} \theta_m^{16} - (\sum_{m \in \mathcal{E}_g} \theta_m^8)^2,$$

We thus choose $D_1 := Z(F_T)$, which is thus the locus of trigonal curves.

2. *Second Divisor* D_2. We choose our second divisor $D_2 := Z(\Theta_{null})$; by our earlier discussion the generic point of this divisor represents canonical curves that lie in a singular quadric of rank 3. Thus the generic point of $D_1 \cap D_2$

represents canonical curves that are trigonal and lie on a quadric of rank 3.

3. *Third Divisor* D_3. The third divisor D_3 is given by $D_3 = Z(F_1)$, The generic point of the zero locus $D_1 \cap D_2 \cap D_3$ represents curves that have a base point free g^3_1 and two pencils of half canonical linear systems. We wish to prove that if $D_1 \cap D_2 \cap D_3 = Hyp_5$. But this follows from Lemma 3.4.9.

4. *Fourth Divisor* D_4. The fourth divisor D_4 is given by $D_4 = Z(F_H)$, by the construction of F_H, we see that $D_4 \cap Hyp_5 = \emptyset$, therefore $D_1 \cap D_2 \cap D_3 \cap D_4$ is empty hence $\mathcal{M}_5 = \cup_{i=1}^4 U_i = \cup_{i=1}^4 D_i^c$ $\square$

3.5 Proof of Main Theorem

This result has appeared in our paper [3], that is slated to appear in the Proceedings of the AMS and we reproduce the argument from the paper.

Before proceeding with the proof of the main theorem we would like to make a few comments about notations,

The subvariety $\mathcal{A}_k \times \mathcal{A}_{g-k} \subset \mathcal{A}_g$ denotes the subvariety whose points correspond to products of k dimensional ppav's and $g-k$ dimensional ppav. Strictly speaking this is an abuse of notation as $\mathcal{A}_k \times \mathcal{A}_{g-k} = \mathcal{A}_{g-k} \times \mathcal{A}_k \subset \mathcal{A}_g$, thus $\mathcal{A}_2 \times \mathcal{A}_2 = \mathrm{Sym}^2(\mathcal{A}_2) \subset \mathcal{A}_4$ and $\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 = \mathrm{Sym}^4(\mathcal{A}_1) \subset \mathcal{A}_4$. We denote the loci of ppav in $\mathcal{A}_g$ that are products of lower dimensional ppav by $\mathcal{A}_g^{dec}$

Salvati Manni in [33] characterized the loci $\mathcal{A}_k \times \mathcal{A}_{g-k}$ by the vanishing of theta constants, more precisely

For a given k, let n_k be the number of theta characteristics in $\mathcal{E}$ of the form $\begin{bmatrix} \varepsilon \\ \\ \delta \end{bmatrix} = [\begin{smallmatrix} \varepsilon_1 & \varepsilon_2 \\ \delta_1 & \delta_2 \end{smallmatrix}]$ with $\begin{bmatrix} \varepsilon_1 \\ \\ \delta_1 \end{bmatrix} \in (\mathbb{Z}/2\mathbb{Z})^{2k}_{odd}$ and $\begin{bmatrix} \varepsilon_2 \\ \\ \delta_2 \end{bmatrix} \in (\mathbb{Z}/2\mathbb{Z})^{2(g-k)}_{odd}$, i.e. $\varepsilon_1^t\delta_1 = \varepsilon_2^t\delta_2 = 1 \in \mathbb{Z}\ /2\mathbb{Z}$.

Let $I \in \mathcal{E}^{n_k}$ be a n_k tuple all of whose characteristics are of the form described above. Then we have the following theorem.

Theorem 3.5.1 ([33])**.** *Given $\tau \in \mathcal{H}_g$ then $[\tau]$ corresponds to a ppav that is a product of a ppav of dimension k and $g-k$ if and only iff $\exists J = (m_1, m_2 \dots m_{n_k}) \in \mathcal{E}^{n_k}$ in the* $\mathrm{Sp}(g, \mathbb{Z}\ /2\mathbb{Z})$ *orbit of I with $\theta_{m_1}(0,\tau) = \theta_{m_2}(0,\tau) = \dots\dots = \theta_{m_{n_k}}(0,\tau) = 0$ under the affine action of* $\mathrm{Sp}(g, \mathbb{Z}/2\mathbb{Z})$.

We will prove Theorem 1.0.1 by two steps, the first step will involve the construction of an affine stratification by considering three divisors on $\mathcal{A}_4$ and taking their intersections to define a stratification. The second step will involve constructing four additional strata without the use of modular forms and by geometric considerations and proving that the stratification is affine.

Step 1 (Strata induced by zero loci of modular forms):

1st Stratum: Consider the Schottky form defined by the following formula, using the notation in [13]

$$F_T := 16 \sum_{m \in \mathcal{E}} \theta_m^{16} - (\sum_{m \in \mathcal{E}} \theta_m^8)^2. \tag{3.5.1}$$

Igusa [23] showed that the zero locus $Z(F_T)$ of F_T is equal to $\mathcal{J}_4 \subset \mathcal{A}_4$ here $\mathcal{J}_4 = \overline{\mathcal{M}_4} \subset \mathcal{A}_4$, We define our first stratum to be

$$X_0 := \mathcal{A}_4 \setminus Z(F_T).$$

2nd Stratum: Consider the Theta null form

$$\Theta_{null} := \prod_{m \in \mathcal{E}} \theta_m. \tag{3.5.2}$$

From Riemann's theta singularity theorem it easily follows that the modular form Θ_{null} does not vanish identically on $\mathcal{M}_g$ in any genus. We thus define our second stratum to be

$$X_1 := Z(F_T) \setminus Z(\Theta_{null}).$$

This stratum corresponds to Jacobians of canonical curves of genus four that are given by a complete intersection of a smooth cubic and a smooth quadric [6].

3rd Stratum: Consider the modular form F_1 given by

$$F_1 := \sum_{m \in \mathcal{E}_3} \frac{\Theta_{null}^8}{\theta_m^8}. \tag{3.5.3}$$

The Zariski closed subset given by $Z(\Theta_{null}) \cap Z(F_1) \cap Z(F_T)$ can be interpreted as the closed subset of $\mathcal{J}_4$ where some two theta constants vanish. Igusa [22] proved that a curve $[C] \in \mathcal{M}_4 \subset \mathcal{A}_4$ is hyperelliptic if and only if two theta constants vanish on the Jacobian of C. On the other hand Theorems 2.2.19 and and our interpretation of $Z(F_T)$ tells us that $\mathcal{A}_4^{dec} \subset Z(F_T) \cap Z(\Theta_{null}) \cap Z(F_1)$. Hence we see $Z(F_T) \cap Z(\Theta_{null}) \cap Z(F_1) = Hyp_4 \cup \mathcal{A}_4^{dec}$. We thus define our third stratum to be

$$X_2 := (Z(F_T) \cap Z(\Theta_{null})) \setminus Z(F_1).$$

This stratum corresponds to Jacobians of canonical curves of genus four that are a complete intersection of a smooth cubic and a singular quadric of rank three [6].

Remark 3.5.2. The group action of $\mathrm{Sp}(g, \mathbb{Z}\backslash 2\mathbb{Z})$ on theta characteristics becomes important from the third stratum onwards. The reason for its importance is because we are unable to continue the strategy of allowing a successive number of theta constants to vanish on $\mathcal{A}_g$ or on a level cover $\mathcal{A}_g(2)$ i

In general, it follows from the transformation formula (**??**) of theta constants, that if m_i's and n_i's satisfy the conditions of Theorem 2.2.19, then the Galois group $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ of the cover $\mathcal{A}_g(2) \to \mathcal{A}_g$ conjugates $Z(\theta^8_{m_1}, \theta^8_{m_2}, ...\theta^8_{m_k}) \subset \mathcal{A}_g(2)$ to $Z(\theta^8_{n_1}, \theta^8_{n_2}....\theta^8_{n_k}) \subset \mathcal{A}_g(2)$.

A concrete example of different orbits of 10 theta characteristics describing different loci is as follows: R. Varley constructed an example of a set of 10 theta constants vanishing on a level cover of $\mathcal{A}_4$ that describes a single ppav, which corresponds to the Segre cubic threefold with an even point of order two; i.e., a cubic threefold with 10 nodes [37]. The correspondence between cubic threefolds and $\mathcal{A}_4$ arises due to the work of Donagi and Smith [7], further explored by Izadi [24], which establishes a birational map between $\mathcal{A}_4$ and a level cover of the moduli of cubic three folds $\mathcal{C}$. By contrast, the vanishing of a different 10-tuple of theta constants describes the hyperelliptic locus $Hyp_4 \subset \mathcal{A}_4$.

Remark 3.5.3. The two modular forms given by F_1 and Θ_{null} were used by Fontanari and Pascolutti in proving Looijenga's conjecture for $\mathcal{M}_4$ [13]. They then use another modular form F_H again constructed as a sum of products of theta constants. They then go on to show that this modular form does not intersect $Hyp_4 \subset \mathcal{M}_4$. *We will not be able to use F_H to construct our 4th stratum as* $Z(F_T, F_1, \Theta_{null}, F_H) = \mathcal{A}_4^{dec}$ (see Theorems 2.2.19 and 3.5.1), since the codimension of $\mathcal{A}_4^{dec} \subset \mathcal{A}_4$ equals to 3, we see that the stratification using F_H is not optimal as we will be using four forms to describe a geometric locus of codimension 3.

Step 2 (Affine stratification constructed by identifying appropriate closed subvarieties)

4th Stratum: Consider the boundary of the previous stratum, which is $\overline{X_2} \setminus X_2 = Hyp_4 \cup \mathcal{A}_4^{dec}$. We know that Hyp_g is affine for all g. The boundary of $\overline{Hyp_4}$ is equal to $(\mathcal{A}_1 \times \overline{Hyp_3}) \cup (\mathcal{A}_2 \times \mathcal{A}_2)$ [19]. It is known classically that in $\mathcal{A}_3$, Jacobians of hyperelliptic curves are characterized by the vanishing of a single theta constant. More precisely $\overline{Hyp_3} = Z(\Theta_{null})$. Since moreover $\mathcal{A}_1$ is affine, the product $\mathcal{A}_1 \times (\mathcal{A}_3 \setminus \overline{Hyp_3})$ is affine, as a product of affine varieties. We thus define our fourth stratum to be

$$X_3 := Hyp_4 \ \sqcup \ \left(\mathcal{A}_1 \times (\mathcal{A}_3 \setminus \overline{Hyp_3})\right).$$

Since X_3 is a disjoint union of two affine varieties of the same dimension such that Hyp_4 and $\mathcal{A}_1 \times (\mathcal{A}_3 \setminus \overline{Hyp_3})$ are closed in the subspace Zariski topology on X_3 we see that X_3 is affine.

5th Stratum: Consider the boundary of the previous stratum

$$\overline{X_3} \setminus X_3 = (\mathcal{A}_2 \times \mathcal{A}_2) \cup (\mathcal{A}_1 \times \overline{Hyp_3}).$$

Now recall that that any indecomposable abelian threefold is a Jacobian of a smooth curve [19]; it thus follows that $\overline{Hyp_3} \setminus Hyp_3 = \mathcal{A}_1 \times \overline{Hyp_2}$, while $\mathcal{A}_2 \times \mathcal{A}_2 = \mathcal{M}_2^{ct} \times \mathcal{M}_2^{ct}$. Thus we see that

$$(\mathcal{A}_2 \times \mathcal{A}_2) \setminus (\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2) = \mathcal{M}_2 \times \mathcal{M}_2.$$

We know $\mathcal{M}_2$ to be affine as $\mathcal{M}_2 = Hyp_2$. On the other hand, we have

$$(\mathcal{A}_2 \times \mathcal{A}_2) \cap \left(\mathcal{A}_1 \times \overline{Hyp_3}\right) = \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2.$$

We thus define our fifth stratum to be

$$X_4 := (\mathcal{A}_1 \times Hyp_3) \sqcup (\mathcal{M}_2 \times \mathcal{M}_2).$$

This stratum is seen to be affine as $\mathcal{M}_2 \times \mathcal{M}_2$ and $\mathcal{A}_1 \times Hyp_3$ are affine, disjoint and closed in the subspace Zariski topology of X_4.

6th Stratum: Consider the boundary of the previous stratum $\overline{X_4} \setminus X_4 = \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2$. Since $\mathcal{A}_1$ is known to be affine we just need to stratify $\mathcal{A}_2$. But $\mathcal{A}_2 = \mathcal{M}_2^{ct}$, and $\mathcal{M}_2^{ct} = \mathcal{M}_2 \sqcup \mathcal{A}_1 \times \mathcal{A}_1$. Thus we define our sixth stratum to be

$$X_5 := (\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2) \setminus (\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1).$$

This stratum is affine as it is simply equal to $\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{M}_2$.

7th Stratum: We finally define our seventh stratum to be

$$X_6 := \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1.$$

We then see that $\mathcal{A}_1$ is affine as $\mathcal{A}_1 = Hyp_1$, and the theorem is thus proven.

3.6 Cohomological dimension of $\mathcal{A}_g$

Looijenga's conjecture rests upon two observations, the first one being that $\lambda_1^{g-2} \neq 0 \in \mathcal{R}^*_{\mathbb{Q}}(\mathcal{M}_g)$ and $\lambda_1^{g-1} = 0$, this piece of information along with Theorem 3.2.6 and Lemma 3.4.2 tells us that there does not exist a collection of open affine sets of $\mathcal{M}_g$, $\{U_i\}_{i=1}^{g-2}$ such that $\cup_{i=1}^{g-2} U_i = \mathcal{M}_g$.

We make analogous observations for $\mathcal{A}_g$, the first one being that $\lambda_1^{g(g-1)/2} \neq 0 \in \mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g)$ while $\lambda_1^{g(g-1)/2+1} = 0$. These can be seen due to the fact that $\mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g)$ is a Gorenstein ring with socle in degree $\frac{g(g-1)}{2}$, thus due to Theorem 2.6.4 and Lemma 3.4.2 we arrive at the conclusion that there does not exist a collection of open affine sets of $\mathcal{A}_g$, $\{U_i\}_{i=1}^{g(g-1)/2}$ such that $\cup_{i=1}^{g(g-1)/2} U_i = \mathcal{A}_g$.

We have thus proven the following

Theorem 3.6.1. *The affine covering number* $\operatorname{acn}(\mathcal{A}_g) \geq g(g-1)/2$*.*

Theorem 1.0.1 and Theorem 3.6.1 lead us to make the following conjecture

Conjecture 3.6.2. *The cohomological dimension* $\operatorname{cd}(\mathcal{A}_g) = g(g-1)/2$ *in all characteristics.*

As a corollary of Theorem 1.0.1 and Theorem 3.3.6 we have the following

Corollary 3.6.3. *The cohomological dimension* $\operatorname{cd}(\mathcal{A}_4(\mathbb{C})) \leq \operatorname{asn}(\mathcal{A}_4(\mathbb{C})) \leq 6$*, while the constructible cohomological dimension* $\operatorname{ccd}(\mathcal{A}_4(\mathbb{C})) \leq 16$*.*

We now state an important theorem of Keel and Sadun that highlights the differences between the geometry of $\mathcal{A}_g$ in finite characteristic and over $\mathbb{C}$.

Theorem 3.6.4. *Let X be a compact subvariety of $\mathcal{A}_g \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C}$ such that $c_i(\mathbb{E})|_X = 0$ in $H^{2i}(X, \mathbb{R})$ for $g \geq i \geq 0$ then $dim(X) \leq \frac{i(i-1)}{2}$ with strict inequality if $i \geq 3$.*

As a corollary they obtained Oort's conjecture.

Corollary 3.6.5. *There is no compact codimension g subvariety of $\mathcal{A}_g \times_{\mathrm{Spec}\,\mathbb{Z}}$ $\mathrm{Spec}\,\mathbb{C}$ for $g \geq 3$.*

Remark 3.6.6. Unlike Diaz's Theorem, Oort's conjecture does not follow from the study of the tautological ring of $\mathcal{A}_g$ because by theorem 6.4 $\mathcal{R}^*(\mathcal{A}_g)$ is isomorphic to R_{g-1} with socle in degree $\frac{g(g-1)}{2}$ which is codimension g in $\mathcal{A}_g$. In fact Oort's conjecture fails in characteristic p, as there exists a complete codimension g subvariety Y_o of $\mathcal{A}_g \times_{Spec\,\mathbb{Z}} Spec\,\mathbb{F}_p$ of codimension g, the locus of ppav the scheme of whose p-torsion points is supported at the origin (completeness follows from the fact that $\mathbb{G}_m$ always has non-trivial p-torsion, so that no semiabelic variety can lie in the closure). Since Y_p is complete, we have $\mathrm{cd}(Y_p) = g(g-1)/2$, and thus since Y_p is a closed subvariety of $\mathcal{A}_g(\mathbb{F}_p)$, it follows that $\mathrm{cd}(\mathcal{A}_g(\mathbb{F}_p)) \geq \mathrm{cd}(Y_p) = g(g-1)/2$. In particular we have $\mathrm{cd}(\mathcal{A}_4(\mathbb{F}_p)) \geq 6$.

Chapter 4

Representation Theory and the tautological ring of $\mathcal{A}_g$

We introduce the necessary background in representation theory to state Conjecture 1.0.5 in a precise manner and prove it for $g \leq 5$ and give empirical evidence for the conjecture in general.

4.1 Representation Theory

We observed earlier that the equivariant vector bundles on $\mathcal{A}_g$ corresponded to the representations of $\mathrm{GL}(g, \mathbb{C})$. We will recall some of the basics of representation theory of $\mathrm{GL}(g, \mathbb{C})$ and $\mathrm{Sp}(2g, \mathbb{Z})$ and $\mathrm{Sp}(2g, \mathbb{C})$ in order to study equivariant vector bundles on $\mathcal{A}_g$ in further detail. The material in this section is mostly reproduced from [15] and [35]

We state a Theorem that classifies irreducible representations of $\mathrm{GL}(g, \mathbb{C})$:

Theorem 4.1.1. *Let V be a g-dimensional complex vector space; then the* *irreducible representations of* $\mathrm{GL}(V)$ *are in one-to-one correspondence with non-decreasing g-tuples of integers* $\mu = (\mu_1, \mu_2, \dots \mu_g)$, *i.e.* $\mu_1 \geq \mu_2 \dots \geq \mu_g$. *We denote this irreducible representation by* $\mathbb{S}_\mu(V)$.

Remark 4.1.2. The representation of $\mathrm{GL}(V)$ on $\mathrm{Sym}^k(V)$ is given by the tuple $(k, 0, 0, \dots 0)$, while the representation on $\wedge^k(V)$ is given by the tuple $(\underbrace{1, 1, \dots 1}_{k}, 0, 0 \dots 0)$. We should mention that the representation associated to $\wedge^g(V)^*$ is given by the tuple $(-1, -1, -1 \dots -1)$ and that $\wedge^k(V)^* \simeq \wedge^{g-k} V \otimes \wedge^g(V)^*$. Note also that the representations $\mathbb{S}_\mu(V) \otimes (\det(V))^{\otimes k}$ is given by the tuple $(\mu_1 + k, \mu_2 + k, \mu_3 + k, \dots, \mu_g + k)$.

We recall that every finite dimensional representation of $\mathrm{GL}(V)$ is determined up to conjugacy by its character. Based on this we define the Schur functions associated to irreducible representations as their characters as follows:

Definition 4.1.3. Let $x \in GL(V)$ and let $x_1, x_2 \dots x_g$ be its eigenvalues x acting on $\mathbb{S}_\mu$. Then the Schur function corresponding to x and $\mathbb{S}_\mu$ is defined to be $S_\mu(x_1, x_2 \dots x_g) := \mathrm{tr}_{\mathbb{S}_\mu}(x)$, i.e. the trace of the image of the element in the appropriate representation. This function $S_\mu(x)$ is invariant under conjugation of x hence just depends on the eigenvalues of x counted with multiplicities occuring in the characteristic polynomial of x.

Theorem 4.1.4. *[15]*

For a g-tuple $\mu = (\mu_1, \mu_2, \mu_3, \dots \mu_g)$ *with all* μ_i *non-negative and* $\mu_1 \geq \mu_2 \geq \dots \geq \mu_g$ *the associated Schur function* $S_\mu(x_1, x_2, \dots x_g)$ *is a symmetric polynomial of degree* $\sum_{i=1}^{g} \mu_i$.

We won't prove the above theorem because it will lead us too far afield into the combinatorics of young diagrams and partitions. But it is an important statement for our purposes to understand the structure of the representation ring of $\mathrm{GL}(V)$

Remark 4.1.5. The fact that the Schur function is a symmetric function can be easily seen as the character of a representation is invariant under conjugation, hence under permutation of eigenvalues in particular. The fact that the Schur function is a polynomial for g tuples with non negative entries uses the machinery of Schur Weyl Duality. For the representation $\wedge^k(V)$ the associated Schur polynomial is $S_{(\underbrace{1,1,\ldots 1}_{k\ times},0,0\ldots 0)} \sum_{i_1<i_2<\ldots<i_k} x_{i_1}x_{i_2}\ldots x_{i_k}$ is simply the k'th elementary symmetric polynomials in g variables.

Now Remark 4.1.2 tells us that every irreducible representation $\mathbb{S}_\mu(V)$ can be tensored with a power of $\det(V)$ so that its associated Schur function is a polynomial.

We recall that if ρ_1 and ρ_2 are representations associated to a group G then the character functions satisfy $\chi_{\rho_1\otimes\rho_2} = \chi_{\rho_1}\chi_{\rho_2}$ and $\chi_{\rho_1\oplus\rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.

Based on the above we make a definition

Definition 4.1.6. The representation ring of $\mathrm{GL}(V)$ denoted $R[\mathrm{GL(V)}]$ is the ring generated as a $\mathbb{Z}$ algebra by the characters of finite dimensional complex representations of $\mathrm{GL}(V)$.

We then have the following

Theorem 4.1.7. *The ring $R[\mathrm{GL}(V)]$ is a finitely generated algebra $\mathbb{Z}[\chi_V, \ldots \chi_{\wedge^g(V)}, \chi_{\wedge^g(V)^{-1}}]$ and thus can be identified with $\mathbb{Z}[e_1, e_2, \ldots e_g, e_g^{-1}]$where e_k is the k'th elemen-*

tary symmetric polynomial in g variables. In particular, it is a free polynomial ring generated by the e_k and the only relation is $e_g e_g^{-1} = 1$.

Proof. Every finite dimensional representation ρ of $\mathrm{GL}(V)$ is a direct sum of irreducible representations $\rho \simeq \oplus_{j=1}^{n} \mathbb{S}_{\mu_j}(V)$. For sufficiently large n the weights of the tensor product $\rho \otimes \det V^{\otimes n}$ are all non-negative, and thus the corresponding Schur functions are symmetric polynomials. The character of $\rho \otimes \det(V)^{\otimes n}$ is χ_ρ times the n'th power of e_g, and is a symmetric polynomial. Since the elementary symmetric polynomials form a polynomial basis for the ring of symmetric polynomials and moreover since $\mathbb{Z}[e_1, e_2, \dots e_g] \simeq \mathbb{Z}[y_1 y_2, \dots y_g]$ i.e. the ring of symmetric polynomials is isomorphic to the standard polynomial ring in g variables.. Thus we have $\chi_{\rho(V)\otimes\det(V)^k} = \chi_{\rho(V)}\chi_{\det(V)^k} = \chi_{\rho(V)} e_g^k = p(e_1, e_2, \dots e_g)$, so that $\chi_{\rho(V)} = p(e_1, e_2, \dots e_g) e_g^{-k}$. $\square$

Remark 4.1.8. As a matter of convention we set e_0 to be equal to 1, which will simplify some formulas.

Theorem 4.1.9. *The representation ring of* $\mathrm{Sp}(2g, \mathbb{C})$*,* $R[\mathrm{Sp}(g, \mathbb{C})]$ *is isomorphic to* $\mathbb{Z}[y_1, y_2, \dots y_g]$ *where* y_k *is the character associated to the representation* $\wedge^k \mathbb{C}^{2g}$ *with the standard symplectic pairing on* $\mathbb{C}^{2g}$*.*

We will not provide the proof of the above theorem, it can be found in [15] and uses many of the same tools as the proof for $\mathrm{GL}(g, \mathbb{C})$.

We will need the description of these rings to motivate and provide evidence for Conjecture 1.0.5 and state it more precisely.

4.2 Chern Character map

Definition 4.2.1. A fundamental domain of $\mathcal{A}_g$ is a connected open set $U \subset \mathcal{H}_g$, i.e. the Siegel upper half space with non empty interior, such that the quotient map $p : \mathcal{H}_g \to \mathcal{A}_g$ is injective on U and surjective on $\overline{U}$

Remark 4.2.2. $\mathcal{A}_g$ can be geometrically constructed by identifying the faces of $\overline{U}$ and therefore is a quotient $\overline{U}/\sim$

Recall that given a vector bundle V on $\mathcal{A}_g$ the pullback of the vector bundle $\pi^*(V)$ where $\pi : \mathcal{H}_g \to \mathcal{A}_g$ is the quotient map is a trivial vector bundle as $\mathcal{H}_g$ is contractible. Thus the above definition tells us that if we pick an element τ of $\overline{U} \setminus U^\circ$ if we can write down the gluing maps of the vector bundle in a neighborhood of τ we will have characterized the vector bundle on $\mathcal{A}_g$ completely

Definition 4.2.3. Given a representation $\rho : \mathrm{GL}(g, \mathbb{C}) \to \mathrm{GL}(k, \mathbb{C})$, the vector bundle $\rho(\mathbb{E})$ is the rank k vector bundle on $\mathcal{A}_g$ such that if U and V are two fundamental domains of $\rho(\mathbb{E})$ such that $\tau \in \overline{U} \cap \overline{V}$, recall there exists $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that $\gamma(\overline{U}) = \overline{V}$, then the transition map of $\pi^*(\rho(\mathbb{E}))$ for the induced trivializations on the fundamental domains under the action of $\mathrm{Sp}(g, \mathbb{Z})$ is given by $\gamma(\tau, v) = ((A\tau + B)(C\tau + D)^{-1}, \rho(C\tau + D)v)$.

Definition 4.2.4. The homomorphism $j : R[GL(g, \mathbb{C})] \to K_0(\mathcal{A}_g, \mathbb{Q})$ is defined by applying the above construction to any representation, and then sending addition to direct sums and product to tensor products, more specifically $j(\chi_{\rho_1} + \chi_{\rho_2}) := \rho_1(\mathbb{E}) \oplus \rho_2(\mathbb{E})$ and $j(\chi_{\rho_1}\chi_{\rho_2}) := \rho_1(\mathbb{E}) \otimes \rho_2(\mathbb{E})$.

Recall that we also have the Chern character morphism $Ch : K_0(\mathcal{A}_g, \mathbb{Q}) \to CH^*_{\mathbb{Q}}(\mathcal{A}_g)$, and we will now consider the composition

$$ch := Ch \circ j : R[\mathrm{GL}(g, \mathbb{C})] \otimes \mathbb{Q} \to CH^*_{\mathbb{Q}}(\mathcal{A}_g).$$

We now state another result which allows us to give a conjectural representation theoretic interpretation of the kernel of ch.

Proposition 4.2.5. *There is a natural ring homomorphism $i : R[\mathrm{Sp}(2g, \mathbb{C})] \to R[\mathrm{GL}(g, \mathbb{C})]$ given by $i(y_k) := \sum_{p+q=k} e_p e_{g-q} e_g^{-1}$.*

Proof. Strictly speaking this is not a proposition, because we can define an arbitrary homomorphism from a free polynomial ring in n variables to another polynomial ring by defining the image of the generators. We want to justify the usage of the word natural. In this case, we recall that every representation of $Sp(g, \mathbb{C})$ can be completely determined by its restriction to the maximal torus and hence the block diagonal matrices of the form $\left[\begin{smallmatrix} A & 0 \\ 0 & A^{t-1} \end{smallmatrix}\right]$. Thus the standard or tautological representation of $\mathrm{Sp}(g, \mathbb{C})$ naturally corresponds to the direct sum of the standard representation of $\mathrm{GL}(g, \mathbb{C})$ and its dual representation. Since the dual representation of $\mathrm{GL}(g, \mathbb{C})$ can be identified with $e_{g-1}e_g^{-1}$, we thus see that there is a natural correspondence between y_1 and $e_1 + e_{g-1}e_g^{-1}$. Similarly taking exterior products of the standard representation of $\mathrm{Sp}(g, \mathbb{C})$ we see that the correspondence in general can be seen to be between y_k and $\sum_{p+q=k} e_p e_{g-q} e_g^{-1}$. Thus we can define a 'natural' homomorphism i as above. □

We now state a result that gives us a conjectural description of the tautological ring

Theorem 4.2.6. *For $g \leq 5$ the homomorphism $ch : R[\mathrm{GL}(g, \mathbb{C})] \otimes \mathbb{Q} \to R_{\mathbb{Q}}(\mathcal{A}_g)$ is surjective and its kernel is the ideal generated by*

$$(i(y_k) - \binom{2g}{k}, \sum_{k=0}^{g} (-1)^k e_{g-k} e_g^{-1})$$

where $i : R[\mathrm{Sp}(2g, \mathbb{C})] \to R[\mathrm{GL}(g, \mathbb{C})]$ is the monomorphism defined above.

Conjecture 4.2.7. *The above theorem holds true for all g.*

Part of the Theorem states that the image of ch is contained in the tautological ring of $\mathcal{A}_g$. Indeed, if one views $\lambda_k = c_k(\mathbb{E})$ as the k'th elementary symmetric polynomial in the g virtual chern roots, then using the fact that the k'th graded piece $ch_k(\wedge^n \mathbb{E})$ is a symmetric polynomial in the g virtual chern roots of $\mathbb{E}$ and also using Theorem 4.1.7 gives $ch(R[\mathrm{GL}(g, \mathbb{C})]) \subset \mathcal{R}^*_{\mathbb{Q}}(\mathcal{A}_g)$. As of now we cannot prove the conjecture in its entirety, but it may be approachable via appropriate invariant theory and commutative algebra. We can prove one direction of the statement for arbitrary g.

Theorem 4.2.8. *The ideal generated by $(i(y_k) - \binom{2g}{k}, \sum_{k=0}^{g} (-1)^k e_{g-k} e_g^{-1})$ is contained in the kernel of ch for all g.*

Proof. We prove that the polynomial $i(y_k) - \binom{2g}{k}$ is in the kernel of ch. Recall that y_k is the representation associated to $\wedge^k \mathbb{C}^{2g}$ for a $2g$ dimensional vector space, it can also be identified with the flat vector bundle on $\mathcal{A}_g$ whose stalks are identified with $H^k(A_\tau, \mathbb{C})$ on $[A_\tau] \in \mathcal{A}_g$. Since $Ch(W) = rank(W)$ for a vector bundle W with a flat connection, we can say that $ch(i(y_k)) = \binom{2g}{k}$. We recall from 2.6.2 that $ch(\sum_{k=0}^{g}(-1)^k e_{g-k}e_g^{-1}) = Td(\mathbb{E})^{-1}\cdot\lambda_g$. Since 2.6.2 tells us that $\lambda_g = 0$, we end up with our desired result. $\square$

We now give the proof of Theorem 4.2.6

Proof. We have proven the above theorem by programming it in macaulay by working with Chern classes in terms of virtual Chern roots. More specifically, we identified the representation ring of $\mathrm{GL}(g,\mathbb{C})$ with $\mathbb{Q}[x_1, x_2, \ldots, x_g, x_g^{-1}]$ and defined $ch : \mathbb{Q}[x_1, \ldots, x_g, x_g^{-1}] \to \mathbb{Q}[X_1, X_2, \ldots, X_g]^{\mathcal{S}_g}$ by the expression $\sum_{i_1<i_2<i_3<\ldots<i_k}^{\frac{g(g+1)}{2}} {}_{l=1} \frac{(X_{i_1}+X_{i_2}\ldots+X_{i_k})^l}{l!}$.

After defining the above homomorphism we define a monomorphism $h : \mathbb{Q}[y_1, y_2, \ldots, y_g] \to \mathbb{Q}[X_1, X_2, \ldots, X_g]$ by $h(y_k) := e_k(X_1, \ldots, X_g)$ where e_k is the k'th elementary symmetric polynomial and verify that $im(ch) \subset im(h)$ and we consider the map $h^{-1}\circ ch$. Once we define this map we take the quotient of $\mathbb{Q}[y_1, \ldots, y_g]$ by the relations obtained in each graded piece by the tautological relation, i.e. $(1 + y_1 + \ldots + y_g)(1 - y_1 + \ldots + (-1)^g y_g) = 1$. Thus we just have to check for the surjectiveness of the map $h^{-1}\circ ch : \mathbb{Q}[x_1, \ldots, x_g, x_g^{-1}] \to \mathbb{Q}[y_1, \ldots, y_g]/T$ where T is the ideal generated by the tautological relations and check if the kernel of this homomorphism is what we defined it to be. $\square$

Bibliography

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1985.

[2] Enrico Arbarello and Gabriel Mondello. *Two remarks on the Weierstrass flag*. American Mathematical Society, 2012.

[3] Anant Atyam. Affine stratification of $\dashv_4$. 12 2013.

[4] C. Birkenhake and H. Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2004.

[5] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[6] I. Dolgachev. *Classical algebraic geometry: a modern view*. Cambridge University Press, Cambridge, 2012.

[7] R. Donagi and R. Smith. The structure of the Prym map. *Acta Math.*, 146(1-2):25–102, 1981.

[8] H. Esnault and E. Viehweg. Chern classes of Gauss-Manin bundles of weight 1 vanish. *K-Theory*, 26(3):287–305, 2002.

[9] C. Faber. A non-vanishing result for the tautological ring of $\mathcal{M}_g$. 1997. preprint arXiv:math/9711219.

[10] C. Faber. A conjectural description of the tautological ring of the moduli space of curves. In *Moduli of curves and abelian varieties*, Aspects Math., E33, pages 109–129. Vieweg, Braunschweig, 1999.

[11] C. Faber and R. Pandharipande. Relative maps and tautological classes. *J. Eur. Math. Soc. (JEMS)*, 7(1):13–49, 2005.

[12] G. Farkas and M. Popa. Effective divisors on $\overline{\mathcal{M}}_g$, curves on $K3$ surfaces, and the slope conjecture. *J. Algebraic Geom.*, 14(2):241–267, 2005.

[13] Claudio Fontanari and Stefano Pascolutti. An affine open covering of $\mathcal{M}_g$ for $g \leq 5$. 03 2011.

[14] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 1998.

[15] William Fulton and Joe Harris. *Representation theory: a first course*, volume 129. Springer, 1991.

[16] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[17] S. Grushevsky and R. Salvati Manni. The superstring cosmological constant and the schottky form in genus 5. *Amer. J. Math.*, 133(4):1007–1027, 2011.

[18] J. Harer. The second homology group of the mapping class group of an orientable surface. *Invent. Math.*, 72(2):221–239, 1983.

[19] J. Harris and I. Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[20] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[21] J.-I. Igusa. *Theta functions*, volume 194 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1972.

[22] J.-I. Igusa. On the irreducibility of Schottky's divisor. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):531–545 (1982), 1981.

[23] J.-I. Igusa. Schottky's invariant and quadratic forms. In *E. B. Christoffel (Aachen/Monschau, 1979)*, pages 352–362. Birkhäuser, Basel, 1981.

[24] E. Izadi. The geometric structure of $\mathcal{A}_4$, the structure of the Prym map, double solids and Γ_{00}-divisors. *J. Reine Angew. Math.*, 462:93–158, 1995.

[25] P. Katsylo and F Bogomolov. Rationality of some quotient varieties. *Mat. Sb.*, 126(168):584–589, 1985.

[26] I. Krichever. Real normalized differentials and Arbarello's conjecture. 2011. preprint arXiv:1112.6427.

[27] E. Looijenga. On the tautological ring of $\mathcal{M}_g$. *Invent. Math.*, 121(2):411–419, 1995.

[28] R. Miranda. *Algebraic Curves and Riemann Surfaces.* Dimacs Series in Discrete Mathematics and Theoretical Comput. American Mathematical Society, 1995.

[29] D. Mumford. *Tata lectures on theta. II.* Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2007. Jacobian theta functions and differential equations, With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura, Reprint of the 1984 original.

[30] R. Pandharipande. The kappa classes on the moduli spaces of curves. 2011. preprint arXiv:1108.5984.

[31] Nikola Penev and Ravi Vakil. The chow ring of the moduli space of curves of genus 6. 07 2013.

[32] M. Roth and R. Vakil. *The affine stratification number and the moduli space of curves*, volume 38 of *CRM Proc. Lecture Notes*, pages 213–227. Amer. Math. Soc., Providence, RI, 2004.

[33] R. Salvati Manni. Modular varieties with level 2 theta structure. *Amer. J. Math.*, 116(6):1489–1511, 1994.

[34] G. van der Geer. *Cycles on the moduli space of abelian varieties*, pages 65–89. Aspects Math., E33. Vieweg, Braunschweig, 1999.

[35] G. van der Geer. Siegel modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 181–245. Springer, Berlin, 2008.

[36] G. van der Geer. The cohomology of the moduli space of abelian varieties. 2011. preprint arXiv:1112.2294; Handbook of Moduli, to appear.

[37] Robert Varley. Weddle's surfaces, humbert's curves, and a certain 4 dimensional abelian variety. *Amer. J. Math.*, (108):931–952, 1986.