Deformations of Axially Symmetric Initial Data and the Angular Momentum-Mass Inequality

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Abstract of the Dissertation

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In this dissertation, we study geometric inequalities for black holes, mainly the angular momentum-mass inequality and the angular momentum-mass-charge inequality.

Firstly, we show how to reduce the general formulation of the angular momentum-mass inequality, for (non-maximal) axially symmetric initial data of the Einstein equations, to the known maximal case. This procedure is based on a certain deformation of the initial data which preserves the relevant geometry, while achieving the maximal condition. More importantly, we compute the scalar curvature formula for the deformation of initial data, which shows that the dominant energy condition holds in a weak sense.

Through this procedure, we develop a geometrically motivated system of quasi-linear elliptic equations which is conjectured to admit a solution. The primary equation bears a strong resemblance to the Jang-type equations studied in the context of the positive mass theorem and the Penrose inequality.

Secondly, in a similar sense, we show how to reduce the general formulation of the angular momentum-mass-charge inequality, for (non-maximal) axially symmetric initial data of the Einstein-Maxwell equations with zero magnetic field, to the known maximal case, whenever there exists a solution for the system of quasi-linear elliptic equations.

Lastly, we combine these two results and the area-angular momentum inequality to show the lower bound of the area in terms of ADM mass, angular momentum, and charge for black holes under the same assumptions.

Contents

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Chapter 1

Introduction

The standard picture of gravitational collapse [6], [11] asserts that generically, an asymptotically flat spacetime should eventually settle down to a stationary final state, consisting of (possibly multiple) disconnected black hole spacetimes. The black hole uniqueness theorem implies that, in vacuum, each of these solutions must be the Kerr spacetime; note that there are still important unresolved technical aspects associated with this uniqueness result [9]. It is also conceivable that these black holes are coupled to matter fields. In any event, as in Kerr, the following inequality holds between mass and angular momentum $m_f \geq \sqrt{|\mathcal{J}_f|}$ for each of the connected components of the final state, and hence for the final state itself. Moreover, as gravitational radiation carries positive energy, the mass of any initial state should not be smaller than that of the final state $m \geq m_f$. If auxiliary conditions are imposed, one of which usually includes axisymmetry, in order to ensure the conservation of angular momentum, then $\mathcal{J} = \mathcal{J}_f$ where $\mathcal{J}$, $\mathcal{J}_f$ denote the (ADM) angular momentums of the initial and final state. This leads to the angular momentum-mass inequality

$$m \geq \sqrt{|\mathcal{J}|} \tag{1.0.1}$$

for any initial state. A counterexample to (1.0.1) would pose a serious challenge to this standard picture of collapse, whereas a verification of (1.0.1) would only lend credence to this model.

Consider an initial data set (M, g, k) for the Einstein equations. This consists of a 3-manifold M, Riemannian metric g, and symmetric 2-tensor k representing the extrinsic curvature (second fundamental form) of the embedding into spacetime, which satisfy the constraint equations

$$\begin{aligned} 16\pi\mu &= R + (Tr_g k)^2 - |k|_g^2, \\ 8\pi J &= div_g(k - (Tr_g k)g). \end{aligned} \tag{1.0.2}$$

Here μ and J are the energy and momentum densities of the matter fields, respectively, and R is the scalar curvature of g. The following inequality will be referred to as the dominant energy condition

$$\mu \geq |J|_g. \tag{1.0.3}$$

Suppose that M has at least two ends, with one designated end being asymptotically flat, and the remainder being either asymptotically flat or asymptotically cylindrical. Recall that a domain $M_{\text{end}} \subset M$ is an asymptotically flat end if it is diffeomorphic to $\mathbb{R}^3 \setminus$ Ball, and in the coordinates given by the asymptotic diffeomorphism the following fall-off conditions hold

$$g_{ij} = \delta_{ij} + o_l(r^{-\frac{1}{2}}), \quad \partial g_{ij} \in L^2(M_{\text{end}}), \quad k_{ij} = O_{l-1}(r^{-\lambda}), \quad \lambda > \frac{5}{2}, \tag{1.0.4}$$

for some $l \geq 6$[1]. In the context of the angular momentum-mass inequality,

[1]The notation $f = o_l(r^{-a})$ asserts that $\lim_{r\to\infty} r^{a+n}\partial^n f = 0$ for all $n \leq l$, and $f = O_l(r^{-a})$ asserts that $r^{a+n}|\partial^n f| \leq C$ for all $n \leq l$. The assumption $l \geq 6$ is needed for the results in [8].

these asymptotics may be weakened, see for example [32]. The asymptotics for cylindrical ends is most easily described in Brill coordinates, to be given in the next section.

We say that the initial data are axially symmetric if the group of isometries of the Riemannian manifold (M, g) has a subgroup isomorphic to $U(1)$, and that the remaining quantities defining the initial data are invariant under the $U(1)$ action. In particular, if η denotes the Killing field associated with this symmetry, then

$$\mathfrak{L}_\eta g = \mathfrak{L}_\eta k = 0, \tag{1.0.5}$$

where $\mathfrak{L}_\eta$ denotes Lie differentiation. If M is simply connected and the data are axially symmetric, it is shown in [8] that the analysis reduces to the study of manifolds diffeomorphic to $\mathbb{R}^3$ minus a finite number of points. Each point represents a black hole, and has the geometry of an asymptotically flat or cylindrical end. The fall-off conditions in the designated asymptotically flat end guarantee that the ADM mass and angular momentum are well-defined by the following limits

$$m = \frac{1}{16\pi} \int_{S_\infty} (g_{ij,i} - g_{ii,j})\nu^j, \tag{1.0.6}$$

$$\mathcal{J} = \frac{1}{8\pi} \int_{S_\infty} (k_{ij} - (Tr_g k) g_{ij})\nu^i \eta^j, \tag{1.0.7}$$

where S_∞ indicates the limit as $r \to \infty$ of integrals over coordinate spheres S_r, with unit outer normal ν. Note that (1.0.4) implies that the ADM linear momentum vanishes.

Angular momentum is conserved [21] if

$$J_i \eta^i = 0. \tag{1.0.8}$$

Moreover, when M is simply connected, this is a necessary and sufficient condition [21] for the existence of a twist potential ω:

$$\epsilon_{ijl}(k^{jn} - (Tr_g k)g^{jn}))\eta^l \eta_n dx^i = d\omega \tag{1.0.9}$$

where ϵ_{ijl} is the volume form for g.

In [16] Dain has confirmed (1.0.1) under the hypotheses that the initial data have two ends, are maximal ($Tr_g k = 0$), vacuum ($\mu = |J|_g = 0$), and admit a global Brill coordinate system. He also established the rigidity statement, which asserts that equality occurs in (1.0.1) if and only if the initial data arise as the $t = 0$ slice of the extreme Kerr spacetime. Chrusciel, Li, and Weinstein [8], [12] improved these results by showing that global Brill coordinates exist under general conditions, and by replacing the vacuum assumption with the hypotheses that $\mu \geq 0$ and a twist potential exists; they also studied the case of multiple black holes. Later Schoen and Zhou [32] gave a simplified proof for more general asymptotics, still assuming the maximal condition, and Zhou [36] treated the near maximal case. It should be noted that such results are false without the assumption of axial symmetry [25].

The focus of this paper is on the general case without the maximal or near maximal hypothesis. We will exhibit a reduction argument by which the general case is reduced to the maximal case, assuming that a canonical system of elliptic PDE possesses a solution. The procedure is motivated by, and bears a resemblance to, previous reduction arguments that have been applied to other geometric inequalities such as the positive mass theorem and the Penrose inequality [2], [3], [22], [27], [28], [33]. Moreover, the primary equation is related to the Jang-type equations that appear in each of these procedures. The end result yields a natural deformation of the initial data, in which the geometry relevant to the angular momentum-mass inequality is preserved, while achieving the maximal condition. In particular, this answers a question posed by R. Schoen [36]:

Question 1.0.1. *Is there a canonical way to deform a non-maximal, axisymmetric, vacuum data to a unique maximal, vacuum data with the same physical quantities, i.e. the mass and angular momentum, which also preserves the axial symmetry?*

This thesis is organized as follows. In the next chapter we describe the deformation in detail, while the reduction argument is established and the case of equality is treated. We will leave some remarks on the analysis of the canonical system of PDEs, which will be fully provided in the joint work with Marcus Khuri. In Chapter 3, we describe the deformation of initial data for Einstein-Maxwell equation in special case, which will provide how to treat the non-maximal case for the angular momentum-mass-charge inequality. Direct application follows in Chapter 4. And finally the derivation of the curvature formula will be provided in Chapter 5.

Chapter 2

Angular Momentum-Mass Inequality

2.1 Brill's Initial Data Set

(M, g, k) is a simply connected, axially symmetric initial data set with multiple ends. Simple connectedness and axial symmetry imply [8] that $M \cong \mathbb{R}^3 \setminus \sum_{n=1}^{N} i_n$, where i_n are points in $\mathbb{R}^3$ and represent asymptotic ends (in total there are $N+1$ ends). Moreover there exists a global (cylindrical) Brill coordinate system (ρ, ϕ, z) on M, where the points i_n all lie on the z-axis, and in which the Killing field is given by $\eta = \partial_\phi$. In these coordinates the metric takes a simple form

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)^2, \tag{2.1.1}$$

where $\rho e^{-U}(d\phi + A_\rho d\rho + A_z dz)$ is the dual 1-form to $|\eta|^{-1}\eta$ and all coefficient functions are independent of ϕ. Let M^0_{end} denote the end associated with limit

$r = \sqrt{\rho^2 + z^2} \to \infty$. The asymptotically flat fall-off conditions (1.0.4) will be satisfied if

$$U = o_{l-3}(r^{-\frac{1}{2}}), \qquad \alpha = o_{l-4}(r^{-\frac{1}{2}}), \qquad A_\rho, A_z = o_{l-3}(r^{-\frac{3}{2}}). \tag{2.1.2}$$

The remaining ends associated with the points i_n will be denoted by M^n_{end}, and are associated with the limit $r_n \to 0$, where r_n is the Euclidean distance to i_n. The asymptotics for asymptotically flat and cylindrical ends are given, respectively, by

$$U = 2\log r_n + o_{l-4}(r_n^{\frac{1}{2}}), \qquad \alpha = o_{l-4}(r_n^{\frac{1}{2}}), \qquad A_\rho, A_z = o_{l-3}(r_n^{\frac{3}{2}}), \tag{2.1.3}$$

$$U = \log r_n + o_{l-4}(r_n^{\frac{1}{2}}), \qquad \alpha = o_{l-4}(r_n^{\frac{1}{2}}), \qquad A_\rho, A_z = o_{l-3}(r_n^{\frac{3}{2}}). \tag{2.1.4}$$

It will also be assumed that the dominant energy condition (1.0.3) is satisfied, and that

$$div_g k(\eta) = 0, \tag{2.1.5}$$

which is equivalent to (1.0.8). Equation (2.1.5) gives rise to a twist potential ω (1.0.9) that is constant on each connected component of the axis of rotation. Let I_n denote the interval of the z-axis between i_{n+1} and i_n, where $i_0 = -\infty$ and $i_{N+1} = \infty$. Then a standard formula [16] yields the angular momentum for each black hole

$$\mathcal{J}_n = \frac{1}{8}(\omega|_{I_n} - \omega|_{I_{n-1}}). \tag{2.1.6}$$

According to (1.0.7) and (2.1.5), the total angular momentum is given by

$$\mathcal{J} = \sum_{n=1}^{N} \mathcal{J}_n. \tag{2.1.7}$$

2.2 Deformation of Initial Data for Angular Momentum-Mass inequality

In this section we will describe the deformation procedure which leads to the reduction argument for the angular momentum-mass inequality.

We seek a deformation of the initial data $(M, g, k) \to (\overline{M}, \overline{g}, \overline{k})$ such that the manifolds are diffeomorphic $M \cong \overline{M}$, the geometry of the ends is preserved, and

$$\overline{m} = m, \quad \overline{\mathcal{J}} = \mathcal{J}, \quad Tr_{\overline{g}}\overline{k} = 0, \quad \overline{R} \geq |\overline{k}|^2_{\overline{g}} \text{ weakly,} \tag{2.2.1}$$

where $\overline{m}$, $\overline{\mathcal{J}}$, and $\overline{R}$ are the mass, angular momentum, and scalar curvature of the new data. The inequality in (2.2.1) is said to hold 'weakly' if it is valid when integrated against an appropriate test function. The validity of this inequality plays a central role in the proof of the angular momentum-mass inequality in the maximal case, and it is precisely the lack of this inequality in the non-maximal case which prevents the proof from generalizing. Thus, the primary goal of the deformation is to obtain such a lower bound for the scalar curvature, while preserving all other aspects of the geometry.

With intuition from the previous work [2], [3], [33] we search for the deformation in the form of a graph inside a stationary 4-manifold

$$\overline{M} = \{t = f(x)\} \subset (\mathbb{R} \times M, \varphi dt^2 + 2Y_i dx^i dt + g), \tag{2.2.2}$$

where the 1-form $Y = Y_i dx^i$ and functions φ and f are defined on M and satisfy

$$\mathfrak{L}_\eta f = \mathfrak{L}_\eta \varphi = \mathfrak{L}_\eta Y = 0. \tag{2.2.3}$$

Define

$$\overline{g}_{ij} = g_{ij} + f_i Y_j + f_j Y_i + \varphi f_i f_j, \qquad \overline{k}_{ij} = \frac{1}{2u}\left(\overline{\nabla}_i Y_j + \overline{\nabla}_j Y_i\right), \tag{2.2.4}$$

where $f_i = \partial_i f$, $\overline{\nabla}$ is the Levi-Civita connection with respect to $\overline{g}$, and

$$u^2 = \varphi + |Y|^2_{\overline{g}}. \tag{2.2.5}$$

In the 'Riemannian' setting (2.2.2), $\overline{g}$ arises as the induced metric on the graph $\overline{M}$. However in the 'Lorentzian' setting

$$\overline{M} = \{t = f(x)\} \subset (\mathbb{R} \times M, -\varphi dt^2 - 2Y_i dx^i dt + \overline{g}), \tag{2.2.6}$$

the deformed data arise as the induced metric and second fundamental form of the $t = 0$ slice. Notice that

$$\partial_t = un - \overline{Y}, \tag{2.2.7}$$

where n is the unit normal to the $t = 0$ slice and $\overline{Y}$ is the vector field dual to Y with respect to $\overline{g}$. Thus $(u, -\overline{Y})$ comprise the lapse and shift of this stationary spacetime. Based on the structure of the Kerr spacetime, we make the following simplifying assumption that Y has only one component

$$\overline{Y}^i \partial_i := \overline{g}^{ij} Y_j \partial_i = Y^\phi \partial_\phi. \tag{2.2.8}$$

Lemma 2.2.1. *Under the hypothesis* (2.2.8)*, $\overline{g}$ is a Riemannian metric, $Tr_{\overline{g}}\overline{k} = 0$, and $\varphi = u^2 - g_{\phi\phi}(Y^\phi)^2$. Moreover if $\{e_i\}_{i=1}^3$ is an orthonormal frame for $\overline{g}$ with $e_3 = |\eta|^{-1}\eta$, then*

$$\overline{k}(e_i, e_j) = \overline{k}(e_3, e_3) = 0, \qquad \overline{k}(e_i, e_3) = \frac{|\eta|}{2u} e_i(Y^\phi), \qquad i, j \neq 3. \tag{2.2.9}$$

Proof. From (2.2.3) it follows that $\overline{g}_{\phi\phi} = g_{\phi\phi}$, and so $|Y|^2_{\overline{g}} = g_{\phi\phi}(Y^\phi)^2$. This yields the formula for φ. Next observe that

$$\begin{aligned} uTr_{\overline{g}}\overline{k} &= \overline{\nabla}_i Y^i \\ &= \partial_i Y^i - \overline{\Gamma}^i_{ij} Y^j \\ &= -\overline{\Gamma}^i_{i\phi} Y^\phi \\ &= \left(\frac{1}{\sqrt{\det \overline{g}}} \partial_\phi \sqrt{\det \overline{g}}\right) Y^\phi \\ &= 0, \end{aligned} \tag{2.2.10}$$

where $\overline{\Gamma}^l_{ij}$ are Christoffel symbols.

We now show that $\overline{g}$ is Riemannian. Equations (2.2.3) and (2.2.8) imply that

$$Y_\phi = g_{\phi\phi} Y^\phi, \quad Y_i = \overline{g}_{ij} Y^j = \overline{g}_{i\phi} Y^\phi = (g_{i\phi} + f_i Y_\phi) Y^\phi = (g_{i\phi} + f_i g_{\phi\phi} Y^\phi) Y^\phi. \tag{2.2.11}$$

Inserting this into (2.2.4) produces

$$\overline{g}_{ij} = g_{ij} + (f_i g_{j\phi} + f_j g_{i\phi}) Y^\phi + (u^2 + g_{\phi\phi}(Y^\phi)^2) f_i f_j. \tag{2.2.12}$$

Take a g-orthonormal frame $(d_1, d_2, d_3 = |\eta|^{-1}\eta)$ at a point, and express $\overline{g}$ as a matrix with respect to this frame

$$\overline{g} = \begin{pmatrix} 1 + (u^2 + g_{\phi\phi}(Y^\phi)^2) f_1^2 & (u^2 + g_{\phi\phi}(Y^\phi)^2) f_1 f_2 & \sqrt{g_{\phi\phi}} Y^\phi f_1 \\ & 1 + (u^2 + g_{\phi\phi}(Y^\phi)^2) f_2^2 & \sqrt{g_{\phi\phi}} Y^\phi f_2 \\ & & 1 \end{pmatrix}. \tag{2.2.13}$$

The determinant of the lower 2×2 minor is $1 + u^2 f_2^2 > 0$, and the full determinant is given by

$$\det \overline{g} = (1 + u^2 |\nabla f|^2_g) \det g > 0. \tag{2.2.14}$$

It follows that $\overline{g}$ is positive definite.

In order to establish (2.2.9), observe that

$$2u\overline{k}_{ij} = \overline{\nabla}_i Y_j + \overline{\nabla}_j Y_i = \partial_i Y_j + \partial_j Y_i - 2\overline{\Gamma}^a_{ij} Y_a, \tag{2.2.15}$$

and

$$\partial_i Y_j = \partial_i(\overline{g}_{\phi j} Y^\phi) = (\partial_i \overline{g}_{\phi j}) Y^\phi + \overline{g}_{\phi j} \partial_i Y^\phi, \tag{2.2.16}$$

$$\begin{aligned} 2\overline{\Gamma}^a_{ij} Y_a &= \overline{g}^{al}(\partial_i \overline{g}_{jl} + \partial_j \overline{g}_{il} - \partial_l \overline{g}_{ij}) Y_a \\ &= (\partial_i \overline{g}_{j\phi} + \partial_j \overline{g}_{i\phi}) Y^\phi. \end{aligned} \tag{2.2.17}$$

Therefore

$$2u\overline{k}_{ij} = \overline{g}_{\phi i} \partial_j Y^\phi + \overline{g}_{\phi j} \partial_i Y^\phi. \tag{2.2.18}$$

Clearly $\overline{k}(e_3, e_3) = 0$, and if we express e_i, $i = 1, 2$ in coordinates (2.3.4), then for $i, j = 1, 2$

$$\begin{aligned} 2u\overline{k}(e_i, e_j) &= e^{2\overline{U} - 2\overline{\alpha}}(\overline{k}_{ij} - A_i \overline{k}_{j\phi} - A_j \overline{k}_{i\phi} + A_i A_j \overline{k}_{\phi\phi}) \\ &= e^{2\overline{U} - 2\overline{\alpha}}(\overline{g}_{\phi i} \partial_j Y^\phi + \overline{g}_{\phi j} \partial_i Y^\phi - A_i \overline{g}_{\phi\phi} \partial_j Y^\phi - A_j \overline{g}_{\phi\phi} \partial_i Y^\phi) \\ &= 0, \end{aligned} \tag{2.2.19}$$

since $\overline{g}_{\phi i} = A_i \overline{g}_{\phi\phi}$ from (2.3.1). Also

$$2u\overline{k}(e_i, e_3) = \frac{\overline{g}_{\phi\phi}}{|\eta|} e_i(Y^\phi) = |\eta| e_i(Y^\phi). \tag{2.2.20}$$

□

This lemma shows that the deformed data set is maximal, satisfying one requirement of (2.2.1). Furthermore, it shows that φ is determined by the functions u and Y^ϕ. Thus, the three functions (u, Y^ϕ, f) completely determine the new data, and will be chosen to satisfy the remaining statements in (2.2.1), so as to yield a reduction argument for the angular momentum-mass inequality.

The next task is to show how to choose the three functions (u, Y^{ϕ}, f). In order to apply the techniques from the maximal case, the existence of a twist potential for $(\overline{M}, \overline{g}, \overline{k})$ is needed. Therefore we require

$$div_{\overline{g}}\overline{k}(\eta) = 0. \tag{2.2.21}$$

This turns out to be a linear elliptic equation for Y^{ϕ} (if u is independent of Y^{ϕ}), as is shown in the following subsection. The function Y^{ϕ} is uniquely determined among bounded solutions of (2.2.21), if the r^{-3}-fall-off rate is prescribed at M^0_{end}. In particular, we will choose the following boundary condition

$$Y^{\phi} = -\frac{2\mathcal{J}}{r^3} + o_2\left(\frac{1}{r^{\frac{7}{2}}}\right) \quad \text{as} \quad r \to \infty. \tag{2.2.22}$$

Lemma 2.2.2. *If $\overline{g}$ is asymptotically flat and $u \to 1$ as $r \to \infty$, then the boundary condition* (2.2.22) *guarantees that $\overline{\mathcal{J}} = \mathcal{J}$.*

Proof. Observe that since $g_{\phi\phi} \sim r^2 \sin^2\theta$ as $r \to \infty$, where $\rho = r\sin\theta$ and $z = r\cos\theta$, we have

$$\begin{aligned}
\overline{\mathcal{J}} &= \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} \overline{k}(\partial_\phi, \partial_r) \\
&= \lim_{r\to\infty} \frac{1}{16\pi} \int_0^{\pi} \int_0^{2\pi} g_{\phi\phi}\partial_r Y^{\phi} r^2 \sin\theta d\phi d\theta \\
&= \frac{3\mathcal{J}}{4} \int_0^{\pi} \sin^3\theta d\theta \\
&= \mathcal{J}.
\end{aligned} \tag{2.2.23}$$

□

Let us now show how to choose f. As with previous deformations arising from the positive mass theorem and Penrose inequality, f is chosen to impart positivity properties to the scalar curvature. With this in mind, it is instructive to calculate the scalar curvature for an arbitrary f. The following result requires a long and detailed computation, and is therefore relegated to the Chapter 5.

Theorem 2.2.3. *Suppose that* (1.0.5), (2.1.5), (2.2.3), (2.2.8), *and* (2.2.21) *are satisfied, then the scalar curvature of* $\overline{g}$ *is given by*

$$\begin{aligned}\overline{R} - |\overline{k}|^2_{\overline{g}} =& 16\pi(\mu - J(v)) + |k-\pi|^2_g + 2u^{-1} div_{\overline{g}}(uQ) \\ &+ (Tr_g\pi)^2 - (Tr_g k)^2 + 2v(Tr_g\pi - Tr_g k),\end{aligned} \tag{2.2.24}$$

where

$$\pi_{ij} = \frac{u\nabla_{ij}f + u_i f_j + u_j f_i + \frac{1}{2u}(g_{i\phi}Y^{\phi}_{,j} + g_{j\phi}Y^{\phi}_{,i})}{\sqrt{1+u^2|\nabla f|^2_g}} \tag{2.2.25}$$

is the second fundamental form of the graph $\overline{M}$ *in the Lorentzian setting,*

$$v^i = \frac{uf^i}{\sqrt{1+u^2|\nabla f|^2_g}}, \qquad w^i = \frac{uf^i + u^{-1}\overline{Y}^i}{\sqrt{1+u^2|\nabla f|^2_g}}, \tag{2.2.26}$$

and

$$Q_i = \overline{Y}^j \overline{\nabla}_{ij} f - u\overline{g}^{jl} f_l \overline{k}_{ij} + w^j(k-\pi)_{ij} + uf_i w^l w^j (k-\pi)_{lj}\sqrt{1+u^2|\nabla f|^2_g}. \tag{2.2.27}$$

Furthermore, if $Y \equiv 0$ *then the same conclusion holds without any of the listed hypotheses.*

This theorem, together with the dominant energy condition (1.0.3), make it clear that in order to obtain the inequality $\overline{R} \geq |\overline{k}|^2_{\overline{g}}$ at least weakly, f should be chosen to solve the equation

$$Tr_g(\pi - k) = 0. \tag{2.2.28}$$

It follows that

$$\overline{R} - |\overline{k}|^2_{\overline{g}} = 16\pi(\mu - J(v)) + |k - \pi|^2_g + 2u^{-1}div_{\overline{g}}(uQ), \tag{2.2.29}$$

which yields the inequality in (2.2.1) after multiplying by u and applying the divergence theorem; it is assumed that appropriate asymptotic conditions are imposed (see below) in order to ensure that the boundary integrals vanish in each of the ends. Equation (2.2.28) is similar to previous Jang-type equations that have been used in connection with deformations of initial data, in particular for the positive mass theorem [33] and the Penrose inequality [3]. These previous equations have the form

$$Tr_{\overline{g}}(\pi - k) = 0, \tag{2.2.30}$$

where it is assumed that $u = Y = 0$ [33], and $Y = 0$ [3]. Note that (2.2.30) does not reduce to (2.2.28) even in the setting of [33] or [3]. This suggests that there is a significant difference between these two equations. In fact, solutions of (2.2.28) do not blow-up, while solutions of (2.2.30) typically blow-up at apparent horizons or can be prescribed to blow-up at these surfaces. This separate behavior arises from the fact that the trace in (2.2.28) is taken with respect to g, whereas the trace in (2.2.30) is taken with respect to $\overline{g}$. As a result, the analysis of (2.2.28) is much more simple than that of (2.2.30). Lastly, in order to ensure that $\overline{m} = m$, we will impose the following asymptotics

$$|f| + r|\nabla f|_g + r^2|\nabla^2 f|_g \leq cr^{-\varepsilon} \quad \text{in} \quad M^0_{end}, \tag{2.2.31}$$

for some $0 < \varepsilon < 1$. A bounded solution may be obtained by prescribing the following asymptotics at the remaining ends

$$r_n^{-1}|\nabla f|_g + r_n^{-2}|\nabla^2 f|_g \leq c \quad \text{in asymptotically flat} \quad M^n_{end}, \tag{2.2.32}$$

$$|\nabla f|_g + |\nabla^2 f|_g \leq c r_n^{\frac{1}{5}} \quad \text{in asymptotically cylindrical} \quad M_{end}^n. \tag{2.2.33}$$

At this point we have shown how to choose f and Y, in order to produce a deformation of the initial data which satisfies (2.2.1). It remains to choose u, in such a way as to facilitate a proof of the angular momentum-mass inequality. This shall be accomplished in the next section.

2.3 The Reduction Argument

In this section, we will follow the maximal case proof of the angular momentum-mass inequality, within the setting of the deformed initial data $(\overline{M}, \overline{g}, \overline{k})$. The primary difficulty arises from a lack of the pointwise scalar curvature inequality as appearing in (2.2.1). However a choice of u will overcome this difficulty.

Assuming that the functions (u, Y^ϕ, f) are chosen to possess the appropriate asymptotics, the geometry of the ends will be preserved in the deformation. Since the deformed data are also simply connected and axially symmetric, the results of [8] apply to yield a global Brill coordinate system $(\overline{\rho}, \phi, \overline{z})$ such that

$$\overline{g} = e^{-2\overline{U}+2\overline{\alpha}}(d\overline{\rho}^2 + d\overline{z}^2) + \overline{\rho}^2 e^{-2\overline{U}}(d\phi + A_{\overline{\rho}} d\overline{\rho} + A_{\overline{z}} d\overline{z})^2. \tag{2.3.1}$$

Next, recall that (2.2.21) implies the existence of a twist potential $\overline{\omega}$. An important property of the Brill coordinates is that they yield a simple formula for the mass [4], [16]

$$\overline{m} - \mathcal{M}(\overline{U}, \overline{\omega}) = \frac{1}{32\pi}\int_{\mathbb{R}^3} \left(2e^{-2\overline{U}+2\overline{\alpha}}\,\overline{R} + \overline{\rho}^2 e^{-2\overline{\alpha}}(A_{\overline{\rho},\overline{z}} - A_{\overline{z},\overline{\rho}})^2 - \overline{g}_{\phi\phi}^{-2}|\partial\overline{\omega}|^2\right) dx, \tag{2.3.2}$$

where $|\partial\overline{\omega}|$ and dx denote the Euclidean norm and volume element, and

$$\mathcal{M}(\overline{U}, \overline{\omega}) = \frac{1}{32\pi}\int_{\mathbb{R}^3} \left(4|\partial\overline{U}|^2 + \overline{g}_{\phi\phi}^{-2}|\partial\overline{\omega}|^2\right) dx. \tag{2.3.3}$$

Let

$$e_{\overline{\rho}} = e^{\overline{U}-\overline{\alpha}}(\partial_{\overline{\rho}} - A_{\overline{\rho}}\partial_\phi), \quad e_{\overline{z}} = e^{\overline{U}-\overline{\alpha}}(\partial_{\overline{z}} - A_{\overline{z}}\partial_\phi), \quad e_\phi = \frac{1}{\sqrt{\overline{g}_{\phi\phi}}}\partial_\phi, \tag{2.3.4}$$

be an orthonormal frame. Then according to (1.0.9) and $\overline{g}_{\phi\phi} = g_{\phi\phi}$,

$$\overline{k}(e_{\overline{\rho}}, e_\phi) = \frac{1}{2|\eta|^2_{\overline{g}}} e_{\overline{z}}(\overline{\omega}) = \frac{e^{\overline{U}-\overline{\alpha}}}{2g_{\phi\phi}}\partial_{\overline{z}}\overline{\omega}, \quad \overline{k}(e_{\overline{z}}, e_\phi) = -\frac{1}{2|\eta|^2_{\overline{g}}} e_{\overline{\rho}}(\overline{\omega}) = \frac{e^{\overline{U}-\overline{\alpha}}}{2g_{\phi\phi}}\partial_{\overline{\rho}}\overline{\omega}. \tag{2.3.5}$$

In light of Lemma 2.2.1 it follows that

$$|\overline{k}|^2_{\overline{g}} = 2(\overline{k}(e_{\overline{\rho}}, e_\phi)^2 + \overline{k}(e_{\overline{z}}, e_\phi)^2) = \frac{e^{2\overline{U}-2\overline{\alpha}}}{2g^2_{\phi\phi}}|\partial\overline{\omega}|^2, \tag{2.3.6}$$

and hence with the help of Theorem 2.2.3 and the dominant energy condition

$$\begin{aligned} \overline{m} - \mathcal{M}(\overline{U}, \overline{\omega}) &\geq \frac{1}{32\pi}\int_{\mathbb{R}^3} 2e^{-2\overline{U}+2\overline{\alpha}}(\overline{R} - |\overline{k}|^2_{\overline{g}})dx \\ &\geq \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{e^{-2\overline{U}+2\overline{\alpha}}}{u} div_{\overline{g}}(uQ)dx \\ &\geq \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{e^{\overline{U}}}{u} div_{\overline{g}}(uQ)dx_{\overline{g}}, \end{aligned} \tag{2.3.7}$$

where the volume element for $\overline{g}$ is given by $dx_{\overline{g}} = e^{-3\overline{U}+2\overline{\alpha}}dx$.

Inequality (2.3.7) suggests that we choose

$$u = e^{\overline{U}} = \frac{\overline{\rho}}{\sqrt{\overline{g}_{\phi\phi}}} = \frac{\overline{\rho}}{\sqrt{g_{\phi\phi}}}. \tag{2.3.8}$$

If $\overline{g}$ preserves the asymptotic geometry of g, then based on (2.1.2), (2.1.3), (2.1.4)

$$u = 1 + o_{l-3}(r^{-\frac{1}{2}}) \quad \text{as} \quad r \to \infty \quad \text{in} \quad M^0_{end}, \tag{2.3.9}$$

$$u = r_n^2 + o_{l-4}(r_n^{\frac{5}{2}}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically flat} \quad M^n_{end}, \tag{2.3.10}$$

$$u = r_n + o_{l-4}(r_n^{\frac{3}{2}}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically cylindrical} \quad M^n_{end}, \tag{2.3.11}$$

where r_n is the Euclidean distance to the point i_n defining the end. Therefore, with the help of the asymptotics for f (2.2.31), (2.2.32) and Y^ϕ (2.2.22), the asymptotic boundary integrals arising from the right-hand side of (2.3.7) all vanish. We will therefore close this section by showing the following lemma. Note that it is crucial to choose $\mathcal{J} = \overline{\mathcal{J}}$ in the proof.

Lemma 2.3.1. *When $u = e^{\overline{U}}$, the following holds.*

$$\int_M \frac{1}{u} div_{\overline{g}}(uQ(\cdot))e^{\overline{U}} dv_{\overline{g}} = 0, \tag{2.3.12}$$

Proof. Recall that $Q(\cdot)$ is the one form on $(M, \overline{g})$ as following.

$$Q(\cdot) = (Hessf)(\overline{Y}, \cdot) - \overline{k}(u\overline{\nabla}f, \cdot) + (k-\pi)(\overline{w}, \cdot) + (k-\pi)(\overline{w}, \overline{w})\frac{u \cdot df}{\sqrt{1 - u^2|\overline{\nabla}f|^2_{\overline{g}}}} \tag{2.3.13}$$

We will use divergence theorem to prove (2.3.12). The limit of the boundary integration at spatial infinity will be clearly 0 by (2.1.2), (2.2.31), and $Y^\phi \to \frac{C}{r^3}$. Therefore we will concentrate on analyzing the boundary behavior of $Q(\cdot)$ near the origin.

As we have studied in the previous sections, the asymptotic conditions for f, Y^ϕ, u vary upto the geometry of (M, g), i.e. (2.1.2), (2.2.31), (2.2.32), and (2.2.33). For spatial infinity, regardless the geometry of (M, g), (2.2.31) holds. Firstly for the asymptotic behavior near the origin, we will only assume the following, provided that $|k - \pi|_g, |k(\partial_\phi, \cdot)|_g$ and $|\pi(\partial_\phi, \cdot)|_g$ are bounded near the origin.

$$u \to 0, \quad |\nabla f|_g \to 0 \quad Y^\phi \to \mu \quad \text{uniformly near the origin} \tag{2.3.14}$$

μ is a constant depending on the choice of $\overline{\mathcal{J}}$. Note that we can easily see that $u|\partial f|_\delta \to 0$ by our choice of u and (2.3.14), where δ is the euclidean metric.

We will not assume any particular fall off rate for (2.3.14) yet. Regardless, (2.3.14) implies that the fall off rate for $u|\overline{\nabla} f|_{\overline{g}} = \frac{u|\nabla f|_g}{\sqrt{1+u^2|\nabla f|_g^2}}$ same as the fall off rate for $u|\nabla f|_g$ near the origin.

Let us start to analyze asymptotic behavior of $Q(\cdot)$. We will later show that (2.3.12) holds in each case, whenever (M, g) has two asymptotically flat ends or an asymptotically flat end and an asymptotically cylindrical end.

According to (2.3.14), $\overline{w}$ and $\pi(\overline{Y}, \cdot)$ behave as following near the origin. For $\pi(\overline{Y}, \cdot)$, we will apply the computations in Chapter 5. Second fundamental form π with respect to $\overline{g}$ metric is computed in (5.2.6). By substituting $\pi(\overline{w}, \overline{Y})$ in (5.3.16) to (5.2.6), we can easily verify the first line of (2.3.16).

$$\overline{w} = \frac{u\nabla_g f + \frac{\overline{Y}}{u}}{\sqrt{1 + u^2|\nabla f|_g^2}} \to \frac{\overline{Y}}{u} + O(|u\nabla f|_g) \tag{2.3.15}$$

and

$$\begin{aligned}\pi(\overline{Y}, \cdot) &= \frac{u}{\sqrt{1 - u^2|\overline{\nabla} f|_{\overline{g}}^2}} \left((Hess_{\overline{g}} f)(\overline{Y}, \cdot) + \frac{\overline{k}(\overline{Y}, \cdot)}{u} - \pi(\overline{w}, \overline{Y}) df \right) \\ &\to u(Hess f)(\overline{Y}, \cdot) + \overline{k}(\overline{Y}, \cdot) - \pi(\overline{Y}, \overline{Y}) df + O(|u\nabla f|_g)\end{aligned} \tag{2.3.16}$$

We will apply (2.3.14), (2.3.15) and (2.3.16) to (2.3.13). Therefore the following holds near the origin.

$$\begin{aligned}Q(\cdot) &\to Hess_{\overline{g}} f(\overline{Y}, \cdot) - \overline{k}(u\overline{\nabla} f, \cdot) + (k - \pi)\left(\frac{\overline{Y}}{u}, \cdot\right) + (k - \pi)\left(\frac{\overline{Y}}{u}, \frac{\overline{Y}}{u}\right) udf + O(|\nabla f|_g) \\ &\to k\left(\frac{\overline{Y}}{u}, \cdot\right) + k\left(\frac{\overline{Y}}{u}, \frac{\overline{Y}}{u}\right) udf - \overline{k}(u\overline{\nabla} f, \cdot) - \overline{k}\left(\frac{\overline{Y}}{u}, \cdot\right) + O(|\nabla f|_g)\end{aligned} \tag{2.3.17}$$

Let us assume that $S = \{\overline{r} = \epsilon\}$ is a coordinate sphere around the origin in $(M, \overline{g})$. In Brill's coordinate system, the normal vector $\overline{n}_S$ for S is as following.

$$\overline{n}_S = e_{\overline{r}} = e^{\overline{U} - \overline{\alpha}}(\partial_{\overline{r}} - A_{\overline{r}}\partial_\phi) \tag{2.3.18}$$

Note that for the normal vector $\overline{n}_S$ for any axially symmetric closed surface S near the origin, $\overline{k}(\nabla f, \overline{n}_S) = 0$. We will analyze the boundary behavior of $Q(\overline{n}_S)$ as $\epsilon \to 0$ as follows.

$$Q(\overline{n}_S) \to k\left(\frac{\overline{Y}}{u}, \overline{n}_S\right) + k\left(\frac{\overline{Y}}{u}, \frac{\overline{Y}}{u}\right) u(n_S(f)) - \overline{k}\left(\frac{\overline{Y}}{u}, \overline{n}_S\right) + O(|\nabla f|_g) \quad (2.3.19)$$

In sum, the following holds regardless the geometry of (M, g). Note that $dA_{\overline{g}} = \overline{r}^2 e^{-2\overline{U}+\overline{\alpha}} d\overline{\theta} d\phi$ is the volume form of $S = \{r = \epsilon\}$ in Brill's coordinate system.

$$\begin{aligned}
&\lim_{\epsilon \to 0} \int_{S=\{\overline{r}=\epsilon\}} uQ(\overline{n}_S) \quad dA_{\overline{g}} \\
&= \lim_{\epsilon \to 0} \int_{S=\{\overline{r}=\epsilon\}} \left(k(\overline{Y}, \overline{n}_S) + k(\overline{Y}, \overline{n}_S)\overline{n}_S(f) - \overline{k}(\overline{Y}, \overline{n}_S)\right) + O(u|\nabla f|_g) \quad dA_{\overline{g}} \\
&= \lim_{\epsilon \to 0} \int_{S=\{\overline{r}=\epsilon\}} (k(\overline{Y}, \overline{n}_S) + k(\overline{Y}, \overline{Y})\overline{n}_S(f) + O(u|\nabla f|_g)) \quad dA_{\overline{g}} - 8\pi\mu\overline{\mathcal{J}}
\end{aligned} \quad (2.3.20)$$

The last line is from applying the divergence theorem for $div_{\overline{g}}(\overline{k}(\partial_\phi, \cdot)) = 0$. Note that $\overline{\mathcal{J}}$ is the angular momentum of $(M, \overline{g}, \overline{k})$, and the constant μ is the limit of Y^ϕ at the origin, i.e. $Y^\phi \to \mu$, depending on $\overline{\mathcal{J}}$.

Before we compute (2.3.20) further, we will analyze $k(\overline{Y}, \overline{n}_S) + k(\overline{Y}, \overline{Y})\overline{n}_S(f)$ with respect to g metric and show the following lemma.

Lemma 2.3.2. *Consider the 2-dimensional hyper surface $S_g = \{\overline{r} = constant, t = f\}$ in $(\mathbb{R} \times M, \widetilde{g} = -\phi dt^2 - 2Y_i dx^i \cdot dt + \overline{g})$. Then the following holds.*

$$\lim_{\epsilon \to 0} \int_{S=\{\overline{r}=\epsilon\}} (k(\overline{Y}, \overline{n}_S) + k(\overline{Y}, \overline{Y})\overline{n}_S(f)) dA_{\overline{g}} = \lim_{\epsilon \to 0} \int_{S_g} \left(Y^\phi k(\partial_\phi, n_S) + O(u|\nabla f|_g)\right) dA_g \quad (2.3.21)$$

where n_S is the outward unit normal vector of S_g embedded in (Σ, g, π) and dA_g is the volume form of the S_g.

Proof. We will compute the outward unit normal vector n_S and the volume form of the $S_g = \{\overline{r} = constant, t = f\}$, which is a 2 - dimensional axially symmetric closed surface embedded in $(\Sigma = \{t = f\}, g)$. Recall that $\{\overline{n}_S = e_{\overline{r}}, e_{\overline{\theta}}, e_\phi\}$ forms the orthonormal basis on $(M, \overline{g})$. $\{e_{\overline{\theta}}, e_\phi\}$ forms orthonormal basis on $S = \{\overline{r} = constant\}$ in $(M, \overline{g})$ with the outward normal vector $\overline{n}_S = e_{\overline{r}}$. Also we assume that $\{X_r, X_\theta, X_\phi\}$ forms the basis for the tangent space of (Σ, g).

$$X_i = e_i + e_i(f)\partial_t \quad i = \overline{r}, \overline{\theta}, \phi \tag{2.3.22}$$

Note that $\{X_\theta, X_\phi\}$ forms the basis for the tangent space of S_g. We will firstly compute v_S, which is the outward normal vector to the S_g in (Σ, g) of the following form, for some B_θ, B_ϕ.

$$v_S = X_r + B_\theta X_\theta + B_\phi X_\phi \tag{2.3.23}$$

Let us compute B_θ, B_ϕ as follows.

$$\begin{aligned}
0 &= \widetilde{g}\langle v_S, X_\phi\rangle \\
&= \widetilde{g}\langle X_r + B_\theta X_\theta + B_\phi X_\phi, e_\phi\rangle \\
&= \widetilde{g}\langle e_{\overline{r}}(f)\partial_t, e_\phi\rangle + B_{\overline{\theta}}\widetilde{g}\langle e_{\overline{\theta}}(f)\partial_t, e_\phi\rangle + B_\phi \\
&= -e_{\overline{r}}(f)Y(e_\phi) - B_\theta e_{\overline{\theta}}(f)Y(e_\phi) + B_\phi
\end{aligned} \tag{2.3.24}$$

$$
\begin{aligned}
0 &= \widetilde{g}\langle v_S, X_{\overline{\theta}}\rangle \\
&= \widetilde{g}\langle X_r + B_\theta X_\theta + B_\phi X_\phi, e_{\overline{\theta}} + e_{\overline{\theta}}(f)\partial_t\rangle \\
&= \widetilde{g}\langle e_{\overline{r}} + e_{\overline{r}}(f)\partial_t, e_{\overline{\theta}} + e_{\overline{\theta}}(f)\partial_t\rangle + B_\theta \widetilde{g}\langle e_{\overline{\theta}} + e_{\overline{\theta}}(f)\partial_t, e_{\overline{\theta}} + e_{\overline{\theta}}(f)\partial_t\rangle + B_\phi e_{\overline{\theta}}(f)\widetilde{g}\langle e_\phi, \partial_t\rangle \\
&= -e_{\overline{\theta}}(f)Y(e_{\overline{r}}) - e_{\overline{r}}(f)Y(e_{\overline{\theta}}) - \phi e_{\overline{r}}(f)e_{\overline{\theta}}(f) + B_\theta(1 - 2e_{\overline{\theta}}(f)Y(e_{\overline{\theta}}) - \phi e_{\overline{\theta}}(f)^2) \\
&\quad - B_\phi e_{\overline{\theta}}(f)Y(e_\phi) \\
&= -\phi e_{\overline{r}}(f)e_{\overline{\theta}}(f) + B_\theta(1 - \phi e_{\overline{\theta}}(f)^2) - B_\phi e_{\overline{\theta}}(f)Y(e_\phi)
\end{aligned} \tag{2.3.25}
$$

We used the identity $Y(e_{\overline{r}}) = Y(e_{\overline{\theta}}) = 0$ in the last line of (2.3.25). We can easily solve (2.3.24) and (2.3.25) for B_θ, B_ϕ. Therefore v_S is as following.

$$
v_S = X_r + \frac{u^2 e_{\overline{r}}(f)e_{\overline{\theta}}(f)}{1 - u^2 e_{\overline{\theta}}(f)^2}X_\theta + \frac{e_{\overline{r}}(f)Y(e_\phi)}{1 - u^2 e_{\overline{\theta}}(f)^2}X_\phi \tag{2.3.26}
$$

Therefore the outward unit normal vector n_S to the S_g is easily derived as following.

$$
n_S = \sqrt{\frac{1 - u^2 e_{\overline{\theta}}(f)^2}{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}\left(X_r + \frac{u^2 e_{\overline{r}}(f)e_{\overline{\theta}}(f)}{1 - u^2 e_{\overline{\theta}}(f)^2}X_\theta + \frac{e_{\overline{r}}(f)Y(e_\phi)}{1 - u^2 e_{\overline{\theta}}(f)^2}X_\phi\right) \tag{2.3.27}
$$

Since k and π are trivially extended, i.e. $k(\partial_t, \cdot) = \pi(\partial_t, \cdot) = 0$, we will only consider the spatial component of n_S from now on. We will also define this spatial component as n_S. Recall that $e_{\overline{r}} = \overline{n}_S$, the outward unit normal vector to $S = \{\overline{r} = constant\}$ on $(M, \overline{g})$. Then n_S is as following.

$$
n_S = \sqrt{\frac{1 - u^2 e_{\overline{\theta}}(f)^2}{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}(\overline{n}_S + \frac{u^2 e_{\overline{r}}(f)e_{\overline{\theta}}(f)}{1 - u^2 e_{\overline{\theta}}(f)^2}e_{\overline{\theta}} + \frac{e_{\overline{r}}(f)Y(e_\phi)}{1 - u^2 e_{\overline{\theta}}(f)^2}e_\phi) \tag{2.3.28}
$$

Also the volume form dA_g of S_g is as following by the direct computation. $dA_{\overline{g}}$ is a volume form of $S = \{\overline{r} = constant\}$ in $(M, \overline{g})$.

$$
dA_g = \sqrt{1 - u^2 e_{\overline{\theta}}(f)^2}dA_{\overline{g}} \tag{2.3.29}
$$

Next, we will analyze the behavior of n_S near the origin. By (2.3.28), the following holds.

$$
\begin{aligned}
n_S &= \sqrt{\frac{1-u^2 e_{\overline{\theta}}(f)^2}{1-u^2|\overline{\nabla} f|_{\overline{g}}^2}}\left(\overline{n}_S + \frac{u^2 e_{\overline{r}}(f) e_{\overline{\theta}}(f)}{1-u^2 e_{\overline{\theta}}(f)^2} e_{\overline{\theta}} + \frac{e_{\overline{r}}(f) Y(e_\phi)}{1-u^2 e_{\overline{\theta}}(f)^2} e_\phi\right) \\
&\to \overline{n}_S + \overline{n}_S(f) Y(e_\phi) e_\phi + O(u^2 |\nabla f|_g^2) e_\theta
\end{aligned}
\tag{2.3.30}
$$

Also the volume form of S_g becomes as following near the origin, i.e. (2.3.29).

$$
dA_g \to dA_{\overline{g}} \tag{2.3.31}
$$

Therefore

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{S=\{\overline{r}=\epsilon\}} uQ(\overline{n}_S) \quad dA_{\overline{g}} \\
&= \lim_{\epsilon\to 0}\int_{S=\{\overline{r}=\epsilon\}} \left(k(\overline{Y},\overline{n}_S) + k(\overline{Y},\overline{Y})\overline{n}_S(f) + O(u|\nabla f|_g)\right) \quad dA_{\overline{g}} - 8\pi\mu\overline{\mathcal{J}} \\
&= \lim_{\epsilon\to 0}\int_{S_g} (k(\overline{Y}, n_S) + O(u|\nabla f|_g)) \quad dA_g - 8\pi\mu\overline{\mathcal{J}}
\end{aligned}
\tag{2.3.32}
$$

□

We will now finish computing (2.3.20) as well as (2.3.12). Let us first assume that (M, g) has two asymptotically flat ends. In this case, $u = e^{\overline{U}} = O(r^2)$ and $|\nabla f|_g = O(r^1)$ near the origin by (2.1.2) and (2.2.31). Also note that $dA_{\overline{g}} = O(r^{-2})$. Therefore the following holds.

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{S=\{\overline{r}=\epsilon\}} uQ(\overline{n}_S) \quad dA_{\overline{g}} \\
&= \lim_{\epsilon\to 0}\int_{S_g} (k(\overline{Y}, n_S) + O(u|\nabla f|_g)) \quad dA_g - 8\pi\mu\overline{\mathcal{J}} \\
&= 8\pi\mu\cdot(\mathcal{J} - \overline{\mathcal{J}})
\end{aligned}
\tag{2.3.33}
$$

In the last two lines in (2.3.33) we used (2.3.30), (2.3.31), fall-off conditions for k, f, dA_g and $div_g k(\partial_\phi) = 0$. Also recall that $Y^\phi \to \mu$ uniformly near the origin, as discussed in previous sections, where μ is a constant depending on the choice of $\overline{\mathcal{J}}$. Therefore, by choosing $\overline{\mathcal{J}} = \mathcal{J}$,

$$\int_M \frac{1}{u} div_{\overline{g}}(uQ(\cdot)) \quad e^U dv_{\overline{g}} = 0 \tag{2.3.34}$$

Let us consider the case when (M, g) has an asymptotically flat end and an asymptotically cylindrical end. Recall that $u = e^U = O(r)$, $|\nabla f|_g = O(r^\delta)$ for some $\delta > 0$ and $dA_{\overline{g}} = O(1)$. The result directly follows from applying similar arguments as in (2.3.30), (2.3.31), and (2.3.33). $\square$

2.4 The Proof of the Main Theorem and Case of Equality

In this section, we will prove the angular momentum-mass inequality whenever the system of the elliptic equation admits a solution.

Theorem 2.4.1. *Let* (M, g, k) *be a smooth, simply connected, axially symmetric initial data set satisfying the dominant energy condition* (1.0.3) *and condition* (1.0.8)*, and with two ends, one designated asymptotically flat and the other either asymptotically flat or asymptotically cylindrical. If the system of equations* (2.2.21)*,* (2.2.28)*,* (2.3.8) *admits a smooth solution* (u, Y^ϕ, f) *satisfying the asymptotics* (2.2.22)*,* (2.2.31)*,* (2.2.32)*,* (2.3.9)*-*(2.3.11)*, then*

$$m \geq \sqrt{|\mathcal{J}|} \tag{2.4.1}$$

and equality holds if and only if (M, g, k) *arises from an embedding into the extreme Kerr spacetime.*

Proof. The existence of a solution (u, Y^{ϕ}, f) ensures that we may apply the maximal case proof to the deformed initial data $(\overline{M}, \overline{g}, \overline{k})$ as above, arriving at the inequality (2.3.12). The results of [12], [16], [32] then imply that

$$\mathcal{M}(\overline{U}, \overline{\omega}) \geq \sqrt{|\overline{\mathcal{J}}|}. \tag{2.4.2}$$

Moreover, according to (2.2.1) $\overline{m} = m$ and $\overline{\mathcal{J}} = \mathcal{J}$, and hence (2.3.12) yields the desired inequality (2.4.1).

Consider now the case of equality in (2.4.1). In the process of deriving (2.3.12), several positive terms were left out from the right-hand side. These terms arise from (2.2.24) and (2.3.2). In the current situation, they must all vanish

$$|\mu - J(v)| = |k - \pi|_g = |A_{\overline{\rho},\overline{z}} - A_{\overline{z},\overline{\rho}}| = 0. \tag{2.4.3}$$

Furthermore, in light of the dominant energy condition, the fact that $|v|_g < 1$, and the identity

$$\mu - J(v) = (\mu - |J|_g) + (1 - |v|_g)|J|_g + (|J|_g|v|_g - J(v)), \tag{2.4.4}$$

it follows that

$$\mu = |J|_g = 0. \tag{2.4.5}$$

We claim that $(\overline{M}, \overline{g}, \overline{k})$ is now a vacuum initial data set. By Lemma 2.2.1 $Tr_{\overline{g}}\overline{k} = 0$, so that the momentum density is given by

$$8\pi\overline{J} = div_{\overline{g}}\overline{k}. \tag{2.4.6}$$

Let $\{e_i\}_{i=1}^3$ denote the orthonormal basis (2.3.4) with $e_3 = e_\phi = \frac{\partial_\phi}{|\partial_\phi|}$, then

$$\begin{aligned}(div_{\overline{g}}\overline{k})(e_i) &= \sum_{i=1}^{3}(\overline{\nabla}_{e_j}\overline{k})(e_i, e_j)\\ &= \sum_{j=1}^{3}\left[e_j(\overline{k}(e_i, e_j)) - \sum_{a=1}^{3}\langle\overline{\nabla}_{e_j}e_i, e_a\rangle\overline{k}(e_a, e_j) - \sum_{a=1}^{3}\langle\overline{\nabla}_{e_j}e_j, e_a\rangle\overline{k}(e_i, e_a)\right].\end{aligned} \tag{2.4.7}$$

Assume now that $i \neq 3$, then by Lemma 2.2.1

$$\sum_{j=1}^{3} e_j(\overline{k}(e_i, e_j)) = 0 \tag{2.4.8}$$

and

$$(div_{\overline{g}}\overline{k})(e_i) = -\sum_{j=1}^{2}\langle\overline{\nabla}_{e_j}e_i, e_3\rangle\overline{k}(e_3, e_j) - \sum_{a=1}^{2}\langle\overline{\nabla}_{e_3}e_i, e_a\rangle\overline{k}(e_a, e_3) - \sum_{j=1}^{3}\langle\overline{\nabla}_{e_j}e_j, e_3\rangle\overline{k}(e_i, e_3). \tag{2.4.9}$$

The last sum is zero since ∂_ϕ is a Killing field. Moreover

$$\langle\overline{\nabla}_{e_3}e_i, e_a\rangle = -\langle e_i, \overline{\nabla}_{e_3}e_a\rangle = -\frac{1}{|\partial_\phi|}\langle e_i, \overline{\nabla}_{e_a}\partial_\phi\rangle = \langle\overline{\nabla}_{e_a}e_i, e_3\rangle, \tag{2.4.10}$$

since

$$[\partial_\phi, e_a] = \mathcal{L}_{\partial_\phi}e_a = 0. \tag{2.4.11}$$

Thus, we need only show that the first sum in (2.4.9) vanishes. To accomplish this, observe that

$$\langle\overline{\nabla}_{e_j}e_i, e_3\rangle = -\langle\overline{\nabla}_{e_i}e_j, e_3\rangle \tag{2.4.12}$$

as ∂_ϕ is Killing. Furthermore a direct computation shows that

$$\langle[e_{\overline{\rho}}, e_{\overline{z}}], e_3\rangle = e^{2\overline{U}-2\overline{\alpha}}(A_{\overline{\rho},\overline{z}} - A_{\overline{z},\overline{\rho}})|\partial_\phi| = 0, \tag{2.4.13}$$

where (2.4.3) was used. Therefore

$$\langle\overline{\nabla}_{e_j}e_i, e_3\rangle = \langle\overline{\nabla}_{e_i}e_j, e_3\rangle, \tag{2.4.14}$$

and it follows that the first sum in (2.4.9) vanishes. Hence $\overline{J} = 0$.

Consider now the energy density

$$16\pi\overline{\mu} = \overline{R} + (Tr_{\overline{g}}\overline{k})^2 - |\overline{k}|^2_{\overline{g}} = \overline{R} - |\overline{k}|^2_{\overline{g}}. \tag{2.4.15}$$

A lengthy computation in Chapter 5 shows that

$$\overline{R}-|\overline{k}|^2_{\overline{g}} = -2(div_{\overline{g}}\overline{k})(u\overline{\nabla}f)+16\pi(\mu-J(v))+|k|^2_g-|\pi|^2_g+2(div_g k)(v)-2(div_g\pi)(v), \tag{2.4.16}$$

when equation (2.2.28) is satisfied. However, $\overline{J} = 0$ and (2.4.3) imply that the right-hand side vanishes. Thus $\overline{\mu} = 0$, and $(\overline{M}, \overline{g}, \overline{k})$ is a vacuum initial data set.

Next, since $\overline{m} = \overline{\mathcal{J}}$ we may now apply the results of [16] and [32] to conclude that $(\overline{M}, \overline{g}, \overline{k})$ is isometric to the $t = 0$ slice $(\mathbb{R}^3 - \{0\}, g_{EK}, k_{EK})$ of the extreme Kerr spacetime $\mathbb{EK}^4$. Consider the map $M \to \mathbb{EK}^4$ given by $x \mapsto (x, f(x))$. The induced metric on the graph is given by

$$(g_{EK})_{ij} - f_i(Y_{EK})_j - f_j(Y_{EK})_i - (u^2_{EK} - |Y_{EK}|^2_{g_{EK}})f_i f_j, \tag{2.4.17}$$

where

$$(k_{EK})_{ij} = \frac{1}{2u_{EK}}\left(\nabla^{EK}_i(Y_{EK})_j + \nabla^{EK}_j(Y_{EK})_i\right), \tag{2.4.18}$$

and $(u_{EK}, -Y_{EK})$ are the lapse and shift. If ∂_ϕ denotes the spacelike Killing field in this spacetime, then $g^{ij}_{EK}(Y_{EK})_j\partial_i = Y^\phi_{EK}\partial_\phi$ and Y^ϕ_{EK} satisfies equation (2.2.21) with $\overline{g}$ replaced by g_{EK}, as well as boundary condition (2.2.22). Since there is a unique solution to (2.2.21), (2.2.22), and $\overline{g} \cong g_{EK}$, we have that $Y = Y_{EK}$. Moreover it is a direct calculation to find that $u_{EK} = e^{U_{EK}} = e^{\overline{U}} = u$, where U_{EK} arises from the Brill coordinate expression for g_{EK}. It now follows from (2.2.4) and (2.2.5) that g agrees with the induced metric (2.4.17). Furthermore, from (2.4.3) $\pi = k$, showing that the second fundamental form of the embedding $(M, g) \hookrightarrow \mathbb{EK}^4$ is given by k. Therefore the initial data (M, g, k) arise from the extreme Kerr spacetime.

Lastly, if (M, g, k) arises from extreme Kerr, then by the properties of this spacetime equality in (2.4.1) holds. □

Theorem 2.4.1 reduces the proof of the angular momentum-mass inequality, in the general non-maximal case, to the existence of a solution (u, Y^{ϕ}, f) to the system of equations (2.2.21), (2.2.28), and (2.3.8). Notice that this is in fact a coupled system, as the definition of u depends on $\overline{g}$. The first task, which is addressed in the next section, is to analyze the given asymptotic boundary value problems associated with each equation (2.2.21) and (2.2.28). Before doing so, however, we record the reduction statement for multiple black holes. Let

$$\mathcal{F}(\mathcal{J}_1, \ldots, \mathcal{J}_N) \tag{2.4.19}$$

denote the numerical value of the action functional (2.3.3) evaluated at the harmonic map, from $\mathbb{R}^3 - \{\overline{\rho} = 0\}$ to the two-dimensional hyperbolic space, constructed in Proposition 2.1 of [12]. Whether the square of this value agrees with

$$\mathcal{J} = \sum_{n=1}^{N} \mathcal{J}_n \tag{2.4.20}$$

is an important open problem. The proof of the following theorem is analogous to that of Theorem 2.4.1.

Theorem 2.4.2. *Let* (M, g, k) *be a smooth, simply connected, axially symmetric initial data set satisfying the dominant energy condition* (1.0.3) *and* *condition* (1.0.8)*, and with* $N+1$ *ends, one designated asymptotically flat and the others either asymptotically flat or asymptotically cylindrical. If the system of equations* (2.2.21)*,* (2.2.28)*,* (2.3.8) *admits a smooth solution* (u, Y^{ϕ}, f) *satisfying the asymptotics* (2.2.22)*,* (2.2.31)*,* (2.2.32)*,* (2.3.9)-(2.3.11)*, then*

$$m \geq \mathcal{F}(\mathcal{J}_1, \ldots, \mathcal{J}_N). \tag{2.4.21}$$

2.5 Remarks on the Solvability of the Coupled System

In this section, we will provide some remarks on the solvability of the coupled quasi-local elliptic equations for f, Y^{ϕ}, u. In [5] it is shown that the each equation admits a solution with the proper asymptotic conditions, provided that the system is not coupled. We will briefly describe the main theorem as well as the asymptotic fall off conditions. Let (M, g, k) be a simply connected, axisymmetric initial data set with two ends denoted $M^{\pm}_{end}$, such that M^{+}_{end} is asymptotically flat and M^{-}_{end} is either asymptotically flat or asymptotically cylindrical. Then there is a global Brill coordinate system (ρ, ϕ, z) in which the metric takes the form (2.1.1). Here we make a change of coordinates to (r, ϕ, θ), where $\rho = r\sin\theta$ and $z = r\cos\theta$. The metric may then be expressed by

$$g = e^{-2U+2\alpha}(dr^2 + r^2 d\theta^2) + e^{-2U} r^2 \sin^2\theta (d\phi + A_r dr + A_\theta d\theta)^2. \tag{2.5.1}$$

In addition to (2.1.2)-(2.1.4), it is assumed that the initial data and u satisfy the following asymptotics

$$u = 1 + o_2(r^{-\frac{1}{2}}), \quad Tr_g k = O_2(r^{-2-\varepsilon}), \quad \text{in} \quad M^{+}_{end}, \tag{2.5.2}$$

for some $\varepsilon \in (0,1)$, and

$$u = r^2 + o_2(r^{\frac{5}{2}}), \quad Tr_g k = O_2(r^4), \quad \text{in asymptotically flat} \quad M^-_{end}, \quad (2.5.3)$$

$$u = r + o_2(r^{\frac{3}{2}}), \quad Tr_g k = O_2(r^{\frac{3}{2}}), \quad \text{in asymptotically cylindrical} \quad M^-_{end}. \quad (2.5.4)$$

Note that the asymptotics for u are consistent with the choice (2.3.8) and the asymptotics (2.1.2)-(2.1.4), while the asymptotics for $Tr_g k$ are weaker in M^+_{end}, and stronger in asymptotically flat M^-_{end}, as compared with (1.0.4).

First, we will describe the quasi-local elliptic equation for f, provided that a smooth function u is given. In local coordinates, with the help of (2.2.25), equation (2.2.28) is given by

$$g^{ij}\left(\frac{u\nabla_{ij}f + u_i f_j + u_j f_i}{\sqrt{1+u^2|\nabla f|^2}} - k_{ij}\right) = 0. \quad (2.5.5)$$

Observe that this equation may also be expressed in divergence form

$$div_g(u^2\nabla f) = u(Tr_g k)\sqrt{1+u^2|\nabla f|^2_g}. \quad (2.5.6)$$

The desired asymptotics are

$$|f| + r|\nabla f|_g + r^2|\nabla^2 f|_g \leq cr^{-\varepsilon} \quad \text{in} \quad M^+_{end}, \quad (2.5.7)$$

$$r^{-1}|\nabla f|_g + r^{-2}|\nabla^2 f|_g \leq c \quad \text{in asymptotically flat} \quad M^-_{end}, \quad (2.5.8)$$

$$|\nabla f|_g + |\nabla^2 f|_g \leq cr^{\frac{1}{2}} \quad \text{in asymptotically cylindrical} \quad M^-_{end}, \quad (2.5.9)$$

where c is a constant.

In [5], we prove the following theorem.

Theorem 2.5.1. *Given initial data* (M, g, k) *and a smooth positive function* u *satisfying* (2.1.2)-(2.1.4) *and* (2.5.2)-(2.5.4), *there exists a smooth uniformly bounded solution* f *of* (2.5.5) *satisfying the asymptotics* (2.5.7)-(2.5.9).

For the proof of Theorem 2.5.1, we mainly utilize the continuity method as in [33]. We also provide the (uniform) radial sub and super solution to show that f satisfies the proposed asymptotic conditions.

Let us now consider the equation for Y^ϕ, provided that the smooth fuctions u and f are given, although f is not required to satisfy an equation here. In particular, u and f satisfy the asymptotics (2.5.2)-(2.5.4) and (2.5.7)-(2.5.9). As described in the previous section, Y^ϕ should satisfy the following differential equation.

$$div_{\overline{g}}\,\overline{k}(\eta) = 0 \quad \text{on} \quad M, \tag{2.5.10}$$

with solutions satisfying the following asymptotics

$$Y^\phi = -\frac{2\mathcal{J}}{r^3} + o_2\left(r^{-\frac{7}{2}}\right) \quad \text{in} \quad M^+_{end}, \tag{2.5.11}$$

$$Y^\phi = \mu + O_2(r^5) \quad \text{in asymptotically flat} \quad M^-_{end}, \tag{2.5.12}$$

$$Y^\phi = \mu + O_2(r) \quad \text{in asymptotically cylindrical} \quad M^-_{end}, \tag{2.5.13}$$

where $\mathcal{J}$ and μ are constants. In order to obtain a unique solution the value of $\mathcal{J}$ will be prescribed, and in this case the value of μ is determined by $\mathcal{J}$ and the initial data. Note that these asymptotics are consistent with those of the (sole component of the) shift vector field Y_{EK} in the extreme Kerr spacetime.

The equation (2.5.10) may be expressed in a more revealing way as follows

$$\Delta_{\overline{g}} Y^\phi + \overline{\nabla}\log(u^{-1}g_{\phi\phi}) \cdot \overline{\nabla} Y^\phi = 0. \tag{2.5.14}$$

which is equivalent to the following.

$$\begin{aligned} 0 = &\left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|^2_g}\right)\left(\nabla_{ij}Y^\phi - \frac{u\pi_{ij}f^l}{\sqrt{1+u^2|\nabla f|^2_g}}\partial_l Y^\phi\right) \\ &+ \left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|^2}\right)\left(\partial_i \log g_{\phi\phi} - \frac{\partial_i \log u}{1+u^2|\nabla f|^2}\right)\partial_j Y^\phi, \end{aligned} \tag{2.5.15}$$

where

$$\pi_{ij} = \frac{u}{\sqrt{1+u^2|\nabla f|^2}} \left(\nabla_{ij} f + (\log u)_i f_j + (\log u)_j f_i + \frac{g_{i\phi} Y^{\phi}_{,j} + g_{j\phi} Y^{\phi}_{,i}}{2u^2} \right) \tag{2.5.16}$$

is the second fundamental form of the graph $\overline{M} = \{t = f(x)\}$ in the Lorentzian setting. It is important to note that the linear character of the equation, arises from the fact that

$$\left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|^2} \right) \pi_{ij} \tag{2.5.17}$$

does not depend on Y^{ϕ}. The equivalence of the three equations (2.5.10), (2.5.14), and (2.5.15) will be proved in the following subsection.

In [5], we show the existence of the solution for (2.5.15) with appropriate asymptotic condition.

Theorem 2.5.2. *Given initial data* (M, g, k) *and smooth functions* $u > 0$, f *satisfying* (2.1.2)-(2.1.4), (2.5.2)-(2.5.4), *and* (2.5.7)-(2.5.9), *there exists a unique, smooth, uniformly bounded solution* Y^{ϕ} *of* (2.5.15) *satisfying the asymptotics* (2.5.11)-(2.5.13).

The main idea of the proof is similar to the 1.0.7. We mainly utilize the continuity method, along with constructing (uniform) radial sub and super solution, which describe the desired asymptotic conditions for Y^{ϕ}. The derivative estimates for Y^{ϕ} will be given separately upto the geometry of the manifold M.

2.5.1 The equation for Y

In this subsection, we will compute the quasi-local elliptic equation for Y^{ϕ} with respect to the $\overline{g}$, and g. This leads the equivalence between the three equations (2.5.10), (2.5.14), and (2.5.15). In [21], it is shown that there exists a twist potential on $(M, \overline{g}, \overline{k})$ if and only if the following holds.

$$div_{\overline{g}}\overline{k}(\partial_\phi, \cdot) = 0 \tag{2.5.18}$$

Due to our choice of Y, $\overline{g}^{ij}Y_i = Y^\phi \partial_\phi$, (2.5.18) gives a second order (quasi) linear elliptic equation on Y^ϕ. Note that we will adopt a global coordinate system, Brill's coordinate for $(M, \overline{g}, \overline{k})$. This let us treat Y^ϕ as merely a function on M. In this section we will compute this differential equation on Y^ϕ with respect to both $\overline{g}$ and g. We will show that it will depend on f and the choice of u.

First, let us compute (2.5.18) in terms of Y^ϕ, u with respect to $\overline{g}$. Here $\{i, j, k\}$ represents $\{\rho, z, \phi\}$ components while $\{p, q\}$ represents $\{\rho, z\}$ components. Here, ∂_i represents a derivative, not a covariant derivative with respect to $\overline{g}$. Recall that $\overline{k}_{i\phi} = \frac{Y_{\bar{i};\phi} + Y_{\phi;\bar{i}}}{2u} = \frac{\overline{g}_{\phi\phi} Y^\phi_{,i}}{2u}$.

$$\begin{aligned}
\overline{g}^{ij}\overline{k}_{\phi\bar{i};\bar{j}} &= \overline{g}^{ij}\left(\partial_j(\overline{k}_{\phi i}) - \overline{\Gamma}^l_{ij}\overline{k}_{l\phi} - \overline{\Gamma}^l_{j\phi}\overline{k}_{li}\right) \\
&\text{where} \quad \overline{g}^{ij}\overline{\Gamma}^l_{j\phi}\overline{k}_{li} = \frac{1}{2}\overline{k}^{jl}(\overline{g}_{l\phi,j} - \overline{g}_{j\phi,l}) = 0 \\
&= \overline{g}^{ij}(\partial_j\left(\frac{\overline{g}_{\phi\phi}Y^\phi_{,i}}{2u}) - \overline{\Gamma}^l_{ij}\overline{k}_{l\phi}\right) \\
&= -\frac{1}{u}\overline{k}(\partial_\phi, \overline{\nabla}u) + \frac{1}{2u}\overline{g}^{ij}(\partial_j(\overline{g}_{\phi\phi}\partial_i Y^\phi) - \overline{\Gamma}^l_{ij}\overline{g}_{\phi\phi}\partial_l(Y^\phi)) \\
&= -\frac{1}{u}\overline{k}(\partial_\phi, \overline{\nabla}u) + \frac{\overline{g}_{\phi\phi}}{2u}(\overline{\Delta}Y^\phi + \overline{g}\langle\overline{\nabla}log(\overline{g}_{\phi\phi}), \overline{\nabla}Y^\phi\rangle) \\
&= \frac{\overline{g}_{\phi\phi}}{2u}\left(\overline{\Delta}Y^\phi + \overline{g}\langle\overline{\nabla}log(\frac{\overline{g}_{\phi\phi}}{u}), \overline{\nabla}Y^\phi\rangle\right)
\end{aligned} \tag{2.5.19}$$

Second, we will derive (2.5.19) in Brill's coordinate system,

$$\overline{g} = e^{-2\overline{U}+2\overline{\alpha}}(d\overline{\rho}^2 + d\overline{z}^2) + \overline{\rho}^2 e^{-2\overline{U}}(d\phi + A_{\overline{\rho}}d\overline{\rho} + A_{\overline{z}}d\overline{z})^2 \tag{2.5.20}$$

The following features will be useful for the further computation.

$$\begin{aligned}
&\overline{g}^{pq} = e^{2\overline{U}-2\overline{\alpha}}\delta^{pq} \quad \overline{g}^{p\phi} = -A_p e^{2\overline{U}-2\overline{\alpha}} \\
&\overline{g}^{\phi\phi} = \overline{\rho}^{-2}e^{2\overline{U}} + e^{2\overline{U}-2\overline{\alpha}}(A_{\overline{\rho}}^2 + A_{\overline{z}}^2) \\
&e_p = e^{\overline{U}-\overline{\alpha}}(\partial_p - A_p\partial_\phi), \quad e_\phi = \overline{\rho}^{-1}e^{\overline{U}}\partial_\phi
\end{aligned} \tag{2.5.21}$$

$$\begin{aligned}
&\overline{g}\langle\overline{\nabla}_{e_\rho}e_\rho, e_z\rangle \\
&= e^{\overline{U}-\overline{\alpha}}(\overline{\Gamma}^z_{\rho\rho} - 2A_{\overline{\rho}}\overline{\Gamma}^z_{\rho\phi} + A_{\overline{\rho}}^2\overline{\Gamma}^z_{\phi\phi}) \\
&= e^{\overline{U}-\overline{\alpha}}\left(\frac{\overline{g}^{zz}}{2}(2\overline{g}_{\rho z,\rho} - \overline{g}_{\rho\rho,z}) + \overline{g}^{z\phi}\overline{g}_{\rho\phi,\rho}\right) \\
&- e^{\overline{U}-\overline{\alpha}}A_{\overline{\rho}}(\overline{g}^{zz}(\overline{g}_{\phi z,\rho} - \overline{g}_{\rho\phi,z}) + \overline{g}^{z\phi}\overline{g}_{\phi\phi,\rho}) - \frac{e^{\overline{U}-\overline{\alpha}}A_{\overline{\rho}}^2\overline{g}^{zz}\overline{g}_{\phi\phi,z}}{2} \\
&= \partial_z(\overline{U} - \overline{\alpha})e^{\overline{U}-\overline{\alpha}}
\end{aligned} \tag{2.5.22}$$

likewise $\quad \overline{g}\langle\overline{\nabla}_{e_z}e_z, e_\rho\rangle = \partial_\rho(\overline{U} - \overline{\alpha})e^{\overline{U}-\overline{\alpha}}$

$$\overline{g}\langle\overline{\nabla}_{e_\phi}e_\phi, e_p\rangle = \frac{e^{-\overline{U}+\overline{\alpha}}}{\overline{g}_{\phi\phi}}\overline{\Gamma}^p_{\phi\phi} = -\frac{e^{\overline{U}-\overline{\alpha}}\partial_p(\log(\overline{g}_{\phi\phi}))}{2}$$

Therefore, each terms in (2.5.22) with respect to Brill's coordinate system will be as follows :

$$
\begin{aligned}
&\overline{\Delta} Y^\phi \\
&= \sum_{p=\rho,z} \left(e_p(e_p Y^\phi) - \overline{\nabla}_{e_p} e_p(Y^\phi) - \overline{\nabla}_{e_\phi} e_\phi(Y^\phi) \right) \\
&= \sum_{p=\rho,z} \left(e^{\overline{U}-\overline{\alpha}} \partial_p (e^{\overline{U}-\overline{\alpha}} \partial_p Y^\phi) - e^{2\overline{U}-2\overline{\alpha}} (\partial_p(\overline{U}-\overline{\alpha}) \partial_p Y^\phi) + \frac{e^{2\overline{U}-2\overline{\alpha}} (\partial_p log(\overline{g}_{\phi\phi}))}{2} \partial_p Y^\phi \right) \\
&= \sum_{p=\rho,z} \left(e^{2\overline{U}-2\overline{\alpha}} \partial_p^2 (Y^\phi) + \frac{e^{2\overline{U}-2\overline{\alpha}} \cdot (\partial_p log(\overline{g}_{\phi\phi}))}{2} \partial_p Y^\phi \right)
\end{aligned} \tag{2.5.23}
$$

and

$$
\begin{aligned}
&\overline{g} \langle \overline{\nabla} log(\frac{\overline{g}_{\phi\phi}}{u}), \overline{\nabla} Y^\phi \rangle \\
&= \sum_{p=\rho,z} e^{2\overline{U}-2\overline{\alpha}} (\partial_p Y^\phi) \left(\frac{\overline{g}_{\phi\phi,p}}{\overline{g}_{\phi\phi}} - \frac{u_{,p}}{u} \right)
\end{aligned} \tag{2.5.24}
$$

In sum (2.5.19) becomes as follows.

$$
\begin{aligned}
div_{\overline{g}} \overline{k}(\partial_\phi) &= \frac{\overline{g}_{\phi\phi}}{2u} (\overline{\Delta} Y^\phi + \overline{g} \langle \overline{\nabla} log(\frac{\overline{g}_{\phi\phi}}{u}), \overline{\nabla} Y^\phi \rangle) \\
&= \frac{\overline{g}_{\phi\phi} e^{2\overline{U}-2\overline{\alpha}}}{2u} \sum_{p=\rho,z} \left(\partial_p^2 Y^\phi + (\partial_p Y^\phi)(\frac{3}{2} \frac{\overline{g}_{\phi\phi,p}}{\overline{g}_{\phi\phi}} - \frac{u_{,p}}{u}) \right) \\
&= \sum_{p=\rho,z} \frac{e^{2\overline{U}-2\overline{\alpha}}}{\sqrt{\overline{g}_{\phi\phi}}} \left(\partial_p \left(\frac{\overline{g}_{\phi\phi}^{\frac{3}{2}} \partial_p Y^\phi}{2u} \right) \right)
\end{aligned} \tag{2.5.25}
$$

We will summarize computation results so far as a remark below.

Remark 2.5.3. $(M, \bar{g}, \bar{k})$ has twist potential if and only if Y^ϕ is a solution of following differential equations depending on f and our choice of u.

$$
\begin{aligned}
&\frac{\overline{g}_{\phi\phi}}{2u} \left(\overline{\Delta} Y^\phi + \overline{g} \langle \overline{\nabla} log(\frac{\overline{g}_{\phi\phi}}{u}), \overline{\nabla} Y^\phi \rangle \right) = 0 \\
&\frac{e^{2\overline{U}-2\overline{\alpha}}}{\sqrt{\overline{g}_{\phi\phi}}} \sum_{p=\rho,z} \left(\partial_p \left(\frac{\overline{g}_{\phi\phi}^{\frac{3}{2}} \partial_p Y^\phi}{2u} \right) \right) = 0 \quad \text{in Brill's coordinate}
\end{aligned} \tag{2.5.26}
$$

Lastly we will compute (2.5.26) with respect to g. Recall that $\overline{g}^{ij}$ in terms of g^{ij}, Y^ϕ, u is as follows, i.e. (5.1.5).

$$Y^i = g^{ij}Y_j = g^{ij}(Y^\phi g_{i\phi} + f, i|\overline{Y}|^2) = Y^\phi \delta^i_\phi + f^i|\overline{Y}|^2 \tag{2.5.27}$$

$$\overline{g}^{ij} = g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|^2_g} - \frac{Y^\phi(\delta^i_\phi f^j + f^i \delta^j_\phi)}{1+u^2|\nabla f|^2_g} + \frac{|\nabla f|^2 (Y^\phi)^2 \delta^i_\phi \delta^j_\phi}{1+u^2|\nabla f|^2_g} \tag{2.5.28}$$

We will start to compute each terms in (2.5.26) as follows.

$$\begin{aligned} \overline{\Delta} Y^\phi &= \overline{g}^{ij}(\partial_i \partial_j Y^\phi - \Gamma^k_{ij} Y^\phi_{,k}) + \overline{g}^{ij}(\Gamma^k_{ij} - \overline{\Gamma}^k_{ij}) Y^\phi_{,k} \\ \overline{g}\langle \overline{\nabla} log\left(\frac{\overline{g}_{\phi\phi}}{u}\right), \overline{\nabla} Y^\phi \rangle &= \overline{g}^{ij} log\left(\frac{\overline{g}_{\phi\phi}}{u}\right)_{,i} Y^\phi_{,j} \end{aligned} \tag{2.5.29}$$

We will first compute $\overline{g}^{ij}(\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})Y^\phi_{,k}$. The explicit formula for $\Gamma^k_{ij} - \overline{\Gamma}^k_{ij}$ can be found in **Identity 1** from the previous section.

$$\begin{aligned} &\overline{g}^{ij}(\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})Y^\phi_{,k} \\ &= -\overline{w}(Y^\phi) tr_{\overline{g}}\pi + |\overline{\nabla} f|^2_{\overline{g}}\, \overline{g}\langle \frac{\overline{\nabla}\phi}{2}, \overline{\nabla} Y^\phi \rangle - \overline{g}^{kl} f^j (Y_{l;\overline{j}} - Y_{\overline{j};l}) Y^\phi_{,k} \end{aligned} \tag{2.5.30}$$

Now let us compute each terms in (2.5.29) with respect to g using (2.5.27) and (2.5.28).

$$
\begin{aligned}
&\overline{g}^{ij}(\partial_i\partial_j Y^\phi - \Gamma^k_{ij}Y^\phi_{,k}) \\
&= \left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|^2_g}\right)(Hess_g Y^\phi(\partial_i,\partial_j)) + \frac{2Y^\phi f^j \Gamma^k_{\phi j}Y^\phi_{,k}}{1+u^2|\nabla f|^2_g} - \frac{|\nabla f|^2 (Y^\phi)^2 \Gamma^k_{\phi\phi}Y^\phi_{,k}}{1+u^2|\nabla f|^2_g} \\
&\overline{g}^{ij}(\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})Y^\phi_{,k} \\
&= -\overline{w}(Y^\phi)Tr_{\overline{g}}\pi + |\overline{\nabla} f|^2_{\overline{g}}\overline{g}\langle\frac{\overline{\nabla}\phi}{2}, \overline{\nabla}Y^\phi\rangle - \overline{g}^{kl}\overline{f}^j(Y_{l;j} - Y_{j;l})Y^\phi_{,k} \\
&= -\frac{u(\nabla f)Y^\phi\cdot(Tr_{\overline{g}}\pi)}{\sqrt{1+u^2|\nabla f|^2_g}} + \frac{|\nabla f|^2}{1+u^2|\nabla f|^2_g}g\langle\nabla\frac{\phi}{2}, \nabla Y^\phi\rangle - \frac{u^2|\nabla f|^2 f^l\phi_{,l}\cdot f^j Y^\phi_{,j}}{2(1+u^2|\nabla f|^2_g)^2} \\
&+ \frac{2Y^\phi\cdot(Y^\phi)^l\Gamma^m_{l\phi}f_{,m}}{1+u^2|\nabla f|^2_g} + \frac{|\nabla f|^2 Y_\phi|\nabla Y^\phi|^2}{1+u^2|\nabla f|^2_g} - \frac{Y_\phi(\nabla f(Y^\phi))^2}{1+u^2|\nabla f|^2_g}
\end{aligned}
\tag{2.5.31}
$$

and

$$
\begin{aligned}
&\overline{g}\langle\overline{\nabla}log(\frac{\overline{g}_{\phi\phi}}{u}), \overline{\nabla}Y^\phi\rangle \\
&= \overline{g}^{ij}log\left(\frac{\overline{g}_{\phi\phi}}{u}\right)_{,i}Y^\phi_{,j} \\
&= \left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|^2_g}\right)log\left(\frac{g_{\phi\phi}}{u}\right)_{,i}Y^\phi_{,j} \\
&= g\langle\nabla log\left(\frac{g_{\phi\phi}}{u}\right), \nabla Y^\phi\rangle - \frac{u^2 f^l log\left(\frac{g_{\phi\phi}}{u}\right)_{,l}\cdot f^j Y^\phi_{,j}}{1+u^2|\nabla f|^2_g}
\end{aligned}
\tag{2.5.32}
$$

where

$$Tr_{\overline{g}}\pi = \left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|_g^2} - \frac{Y^\phi(\delta^i_\phi f^j + f^i \delta^j_\phi)}{1+u^2|\nabla f|_g^2} + \frac{|\nabla f|^2 (Y^\phi)^2 \delta^i_\phi \delta^j_\phi}{1+u^2|\nabla f|_g^2}\right)\pi_{ij}$$

$$\text{where} \quad \pi = \frac{u}{\sqrt{1+u^2|\nabla f|_g^2}}\left(f_{;ij} + \frac{1}{2u^2}(g_{i\phi}Y^\phi_{,j} + g_{j\phi}Y^\phi_{,i}) + \frac{u_{,j}f_{,i}}{u} + \frac{u_{,i}f_{,j}}{u}\right)$$

$$\begin{aligned} &= (g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|_g^2})\pi_{ij} \\ &- \frac{Y^\phi g_{\phi\phi}}{u^2(1+u^2|\nabla f|_g^2)}\left(\frac{u\nabla f(Y^\phi)}{\sqrt{1+u^2|\nabla f|_g^2}}\right) - \frac{(Y^\phi)^2|\nabla f|^2}{1+u^2|\nabla f|_g^2}\left(\frac{u\Gamma^l_{\phi\phi}f_{,l}}{\sqrt{1+u^2|\nabla f|_g^2}}\right) \end{aligned} \tag{2.5.33}$$

Also use the following for simplifying $\overline{Y}$ equation :

$$\left(\frac{\nabla\phi}{2}\right)^l - (Y^\phi)^2\Gamma^l_{\phi\phi} + (Y_\phi \nabla Y^\phi)^l = (u\nabla u)^l \tag{2.5.34}$$

In conclusion, if we compute the Y^ϕ equation in (2.5.29) with respect to g using (2.5.31), (2.5.32), (2.5.33), and (2.5.34) it will be as follows :

$$\begin{aligned} &\overline{\Delta}Y^\phi + \overline{g}\langle\overline{\nabla}log\left(\frac{\overline{g}_{\phi\phi}}{u}\right), \overline{\nabla}Y^\phi\rangle \\ &= \left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|_g^2}\right)(HessY^\phi(\partial_i, \partial_j)) \\ &- \frac{u\nabla f(Y^\phi)}{\sqrt{1+u^2|\nabla f|_g^2}}\left(\left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|_g^2}\right)\pi_{ij}\right) \\ &+ \left(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|_g^2}\right)\left(\log(g_{\phi\phi})_{,i}Y^\phi_{,j} - \frac{u_{,i}Y^\phi_{,j}}{u(1+u^2|\nabla f|_g^2)}\right) \end{aligned} \tag{2.5.35}$$

Remark 2.5.4. In (2.5.35), $(g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|_g^2})\pi_{ij}$ does not explicitly depend on Y^ϕ. It depends on f, u and g.

Chapter 3

Angular Momentum-Mass-Charge inequality

3.1 Deformation of Initial Data for Angular Momentum-Mass-Charge inequality

In this chapter, we will observe that the reduction argument given previous chapter, is applicable to another geometric inequality, the angular momentum-mass-charge inequality for the axisymmetric black holes. Let (M, g, k, E) be the axisymmetric initial data for the Einstein-Maxwell equations with vanishing magnetic field. M is a 3 dimensional Riemannian manifold which is simply conected, and possesses only two ends denoted $M^{\pm}_{end}$, such that M^{+}_{end} is asymptotically flat and M^{-}_{end} is either asymptotically flat or asymptotically cylindrical. Also we will assume that (M, g, k, E) satisfies the charged dominant energy condition if the following constraints hold.

$$\mu_{EM} \geq |J_{EM}| \tag{3.1.1}$$

where

$$16\pi\mu_{EM} = 16\pi\mu - 2|E|^2, \quad J_{EM} = J, \quad div_g E = 0 \tag{3.1.2}$$

Such data are said to be strongly asymptotically flat, if in addition to the standard definition $E \sim |x|^{-2}$ at spatial infinity. The total charge is given by

$$q = q_E = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r} E \cdot \nu$$

In order to obtain a twist potential, it is necessary to assume that $J_{EM}(\eta) = 0$ as in the previous chapter, where η is the killing vector generating the axial symmetry. Based on the heuristic physical arguments leading to (1.0.1), the charge may be included in the angular momentum-mass inequality as follows.

Conjecture 3.1.1. *Let (M, g, k, E) be a simply connected, axially symmetric initial data set with two ends, one strongly asymptotically flat and the other either strongly asymptotically flat or asymptotically cylindrical. If $J_{EM}(\eta) = 0$, $div_g E = 0$ and the charged dominant energy condition is satisfied, then*

$$m^2 \geq \frac{q^2 + \sqrt{q^4 + 4\mathcal{J}^2}}{2}. \tag{3.1.3}$$

Moreover, equality holds if and only if (M, g, k, E) arises from the extreme Kerr-Newman spacetime.

This result has been established in the maximal case, where $Tr_g k = 0$ by Chrusciel and Costa [15], [10], and Schoen and Zhou [32]. By utilizing a deformation argument similar to the previous chapter, we are able to establish this result modulo the existence of a solution to the coupled system. The main difference is that the electric field must also be deformed as $\overline{E}$ so that

$$|E|_g^2 \geq |\overline{E}|_{\overline{g}}^2, \tag{3.1.4}$$

$$div_{\overline{g}}\,\overline{E} = 0 \tag{3.1.5}$$

$$\overline{q} = q \tag{3.1.6}$$

There is a natural way to accomplish this, by taking $\overline{E}$ to be the induced electric field on the graph Σ. We will show the deformation process in this section and prove the main theorem in the following section.

Consider the deformation of initial data set $(M, \overline{g}, \overline{k})$ as in the previous chapter. For the deformation of the electric field, as mentioned above, we will first consider the 4-momentum $\overline{F} = \frac{1}{2}\overline{F}_{ab}dx^a \wedge dx^b$ on $(\mathbb{R} \times M, \widetilde{g} = -\phi dt^2 - 2Y_i dx^i dt + \overline{g})$ as following, where $\overline{F}_{0i} = u\overline{E}_i$ and $\overline{F}_{ij} = 0$ for $i, j, k = 1, 2, 3$ with x^i coordinate in $(M, \overline{g})$.

$$\overline{F} = \begin{pmatrix} 0 & u\overline{E}_1 & u\overline{E}_2 & u\overline{E}_3 \\ -u\overline{E}_1 & 0 & 0 & 0 \\ -u\overline{E}_2 & 0 & 0 & 0 \\ -u\overline{E}_3 & 0 & 0 & 0 \end{pmatrix} \tag{3.1.7}$$

Notice that $\overline{E} = \overline{F}(\overline{n}, \cdot)$ with the unit normal vector $\overline{n} = \frac{\partial_t + \overline{Y}}{u}$ to the $M = \{t = 0\}$. The formulation of $\overline{E}$ in terms of the initial data (M, g, k, E) will be followed by the relation between $\overline{F}(N, \cdot)$ and the electric field E. Since (M, g) is embedded in $(\mathbb{R} \times M, \widetilde{g})$ as $\{t = f\}$ graph, the following holds.

$$\overline{F}(N, \cdot) = E(\cdot) \tag{3.1.8}$$

where N is the outward unit normal vector to $(\Sigma = \{t = f\}, g, \pi)$. We will first derive the exact formulation of $\overline{E}$ with respect to E by utilizing (3.1.8).

Lemma 3.1.2.

$$\overline{E}_i = \frac{E_i + E(u^2\nabla f + Y^\phi \eta) f_i}{\sqrt{1 + u^2|\nabla f|_g^2}}$$

Proof. We will define $\overline{E}$ in the following way that $E = \overline{F}(N, \cdot)$, where $N = \overline{w}^i\partial_i + N^t\partial_t = \frac{u^2\overline{\nabla}f+\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} + \frac{\partial_t}{u\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}$ is the unit normal vector to the graph $\{t = f\}$. As in the previous chapter, $\{X_i = \partial_i + f_{,i}\partial_t\}$ are tangent vectors of (Σ, g, π).

$$E_i = F(N, X_i) = uN^t\overline{E}_i - u\overline{E}(\overline{w})f_i \tag{3.1.9}$$

We will solve (3.1.9) for $\overline{E}$. Note that the spatial component of N can be written with respect to $\overline{g}$ and g as following.

$$\overline{w} = \frac{u^2\overline{\nabla}f + \overline{Y}}{u\sqrt{1 - u^2|\overline{\nabla}f|_{\overline{g}}^2}} = \frac{u^2\nabla f + Y^\phi\eta}{u\sqrt{1 + u^2|\nabla f|_g^2}} \tag{3.1.10}$$

From (3.1.9),

$$w^iE_i = u\overline{E}(\overline{w})(N^t - w^if_i) = \frac{\overline{E}(\overline{w})}{\sqrt{1 + u^2|\nabla f|_g^2}} \tag{3.1.11}$$

We will substitute (3.1.11) to (3.1.9). Also note that $N^t = \frac{\sqrt{1+u^2|\nabla f|_g^2}}{u}$. Therefore Lemma 3.1.2 easily follows by solving (3.1.9) for $\overline{E}_i$. □

Remark 3.1.3. Lemma 3.1.2 implies that the total charge $\overline{q}$ induced by $\overline{E}$ is same as q, assuming the asymptotic condition for f, Y^ϕ at spatial infinity as (2.2.22), (2.2.31).

Next, We will show that $|\overline{E}|_{\overline{g}} \leq |E|_g$ as expected. We will prove the following lemma by direct computation of $|E|_g$ with respect to $\overline{g}$ metric.

Lemma 3.1.4.

$$|\overline{E}|_{\overline{g}} \leq |E|_g$$

Proof.

$$\begin{aligned}
|E|_g^2 &= g^{ij}E_iE_j \\
&= u^2\left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)(N^t\overline{E}_i - \overline{E}(\overline{w})f_i)(N^t\overline{E}_j - \overline{E}(\overline{w})f_j) \\
&= u^2(N^t)^2|\overline{E}|_{\overline{g}}^2 - (N^t)^2\overline{E}(\overline{Y})^2 - 2u^2N^t\overline{E}(\overline{\nabla}f)\overline{E}(\overline{w}) \\
&+ u^2\overline{E}(\overline{w})^2\left((N^t)^2 - 2\overline{w}^lf_lN^t + |\overline{\nabla}f|_{\overline{g}}^2 + (\overline{w}^lf_l)^2\right)
\end{aligned} \tag{3.1.12}$$

where

$$u^2N^t\overline{E}(\overline{\nabla}f) = \overline{E}(\overline{w}) - N^t\overline{E}(\overline{Y}) \tag{3.1.13}$$

and

$$\begin{aligned}
&u^2((N^t)^2 - 2\overline{w}^lf_lN^t + |\overline{\nabla}f|_{\overline{g}}^2 + (\overline{w}^lf_l)^2) \\
&= u^2(N^t - \overline{w}^lf_l)^2 + u^2|\overline{\nabla}f|_{\overline{g}}^2 = 1
\end{aligned} \tag{3.1.14}$$

By substituting (3.1.13) and (3.1.14), (3.1.12) becomes as following:

$$\begin{aligned}
|E|_g^2 &= u^2(N^t)^2|\overline{E}|_{\overline{g}}^2 - (N^t)^2\overline{E}(\overline{Y})^2 + 2\overline{E}(\overline{w})N^t\overline{E}(\overline{Y}) - \overline{E}(\overline{w})^2 \\
&= u^2(N^t)^2|\overline{E}|_{\overline{g}}^2 - (\overline{E}(\overline{w}) - N^t\overline{E}(\overline{Y}))^2 \\
&= u^2(N^t)^2(|\overline{E}|_{\overline{g}}^2 - \overline{E}(u\overline{\nabla}f)^2) \\
&\geq u^2(N^t)^2(1 - u^2|\overline{\nabla}f|_{\overline{g}}^2)|\overline{E}|_{\overline{g}}^2 = |\overline{E}|_{\overline{g}}^2
\end{aligned} \tag{3.1.15}$$

□

Lastly, we will prove the following lemma to show that $div_{\overline{g}}\overline{E} = 0$ if and only if $div_gE = 0$.

Lemma 3.1.5.

$$div_{\overline{g}}\overline{E} = \frac{div_g E}{\sqrt{1+u^2|\nabla f|_g^2}}$$

Proof. We will compute the divergence of $\overline{E}$ with respect to divergence of E in two different ways. First proof is based on conceptual understanding whereas the second proof is based on the direct computations, utilizing the computation in Chapter 5.

First Proof Let us define 4-current $\overline{J}$ for $\overline{F}$ as following.

$$\overline{J}_b = \widetilde{\nabla}\overline{F}_{ab} \quad a,b = 0...3 \tag{3.1.16}$$

where $\widetilde{\nabla}$ is the covariant derivative with respect to the 4-metric $\widetilde{g} = -\phi dt^2 - Y_i dx^i \cdot dt + \overline{g}$ on $\mathbb{R}\times M$. The relation between the 4-current and $E, \overline{E}$ are as follows.

$$div_{\overline{g}}\overline{E} = -\overline{J}(\overline{n}), \qquad div_g E = -\overline{J}(N) \tag{3.1.17}$$

Therefore

$$\begin{aligned} div_{\overline{g}}\overline{E} &= -\overline{J}(\overline{n}) = -\overline{J}(\sqrt{1-u^2|\overline{\nabla} f|_{\overline{g}}^2}N) + \overline{J}(u\overline{\nabla} f) \\ &= \sqrt{1-u^2|\overline{\nabla} f|_{\overline{g}}^2} div_g E + \overline{J}(u\overline{\nabla} f) \end{aligned} \tag{3.1.18}$$

We will show that $\overline{J}(u\overline{\nabla} f) = 0$ by directly computing $\overline{J}_i$ with $i = 1...3$. Recall that the Christoffel symbols for $(\mathbb{R}\times M, \widetilde{g})$ are as following.

$$\begin{aligned} &\widetilde{g}^{tt} = -\frac{1}{u^2}, \qquad \widetilde{g}^{ti} = -\frac{\overline{Y}^i}{u^2}, \qquad \widetilde{g}^{ij} = \overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} \\ &\widetilde{\Gamma}^t_{tt} = 0, \qquad \widetilde{\Gamma}^k_{tt} = \frac{1}{2}\phi^k, \qquad \widetilde{\Gamma}^t_{ij} = \frac{\overline{k}_{ij}}{u} \\ &\widetilde{\Gamma}^t_{it} = \frac{\phi_i + \overline{Y}^l(Y_{\overline{l};i} - Y_{\overline{i};l})}{2u^2} \\ &\widetilde{\Gamma}^k_{it} = -\frac{1}{2}\overline{g}^{kl}(Y_{\overline{l};i} - Y_{\overline{i};l}) + \frac{\overline{Y}^k}{2u^2}(\phi_i + \overline{Y}^l(Y_{\overline{l};i} - Y_{\overline{i};l})) \\ &\widetilde{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} \end{aligned} \tag{3.1.19}$$

We will compute the spatial component of $\overline{J}_i$ and show that $\overline{J}(u\overline{\nabla}f) = 0$ with any differentiable function f which is axially symmetric, i.e. $f_{,\phi} = 0$.

$$\begin{aligned} \overline{J}_i &= \widetilde{\nabla}^a \overline{F}_{ai} \\ &= \widetilde{\nabla}^t \overline{F}_{ti} + \widetilde{\nabla}^l \overline{F}_{li} \\ &= \widetilde{g}^{tb} \overline{F}_{ti\tilde{;}b} + \widetilde{g}^{lb} \overline{F}_{li\tilde{;}b} \\ &= \widetilde{g}^{tt} \overline{F}_{ti\tilde{;}t} + \widetilde{g}^{tl} \overline{F}_{ti\tilde{;}l} + \widetilde{g}^{jl} \overline{F}_{li\tilde{;}j} + \widetilde{g}^{lt} \overline{F}_{li\tilde{;}t} \end{aligned} \tag{3.1.20}$$

Let us compute the last four terms in (3.1.20) by substituting (3.1.19) and (3.1.7). Recall that with the choice of $Y_i = \overline{g}_{\phi i} Y^{\phi}$, $Tr_{\overline{g}} \overline{k} = \overline{k}(\overline{Y}, \overline{Y}) = 0$.

$$\begin{aligned} \widetilde{g}^{tt} \overline{F}_{ti\tilde{;}t} &= -\frac{1}{u^2} \left(\partial_t F_{ti} - \widetilde{\Gamma}^a_{tt} \overline{F}_{ai} - \widetilde{\Gamma}^a_{ti} \overline{F}_{ta} \right) = \frac{\widetilde{\Gamma}^l_{ti} \overline{F}_{tl}}{u^2} \\ &= -\frac{1}{2u} \overline{g}^{jl} (Y_{\bar{j};\bar{i}} - Y_{\bar{i};\bar{j}}) \overline{E}_l + \frac{\widetilde{\Gamma}^t_{ti} \overline{E}(\overline{Y})}{u} \end{aligned} \tag{3.1.21}$$

$$\tilde{g}^{tl} \overline{F}_{ti\tilde{;}l} = -\frac{\overline{Y}^l}{u^2} \left(\partial_l \overline{F}_{ti} - \widetilde{\Gamma}^a_{lt} \overline{F}_{ai} - \widetilde{\Gamma}^a_{il} \overline{F}_{ta} \right) = \frac{\overline{Y}^l \widetilde{\Gamma}^j_{il} \overline{F}_{tj}}{u^2} = \frac{\overline{Y}^l \widetilde{\Gamma}^j_{il} \overline{E}_j}{u} \tag{3.1.22}$$

$$\begin{aligned} \widetilde{g}^{jl} \overline{F}_{li\tilde{;}j} &= \left(\overline{g}^{jl} - \frac{\overline{Y}^j \overline{Y}^l}{u^2} \right) \left(\partial_j \overline{F}_{li} - \widetilde{\Gamma}^a_{lj} \overline{F}_{ai} - \widetilde{\Gamma}^a_{ij} \overline{F}_{la} \right) \\ &= -\left(\overline{g}^{jl} - \frac{\overline{Y}^j \overline{Y}^l}{u^2} \right) \widetilde{\Gamma}^t_{ij} \overline{F}_{lt} = \left(\overline{g}^{jl} - \frac{\overline{Y}^j \overline{Y}^l}{u^2} \right) \overline{k}_{ij} \overline{E}_l \end{aligned} \tag{3.1.23}$$

and

$$\widetilde{g}^{tl} \overline{F}_{li\tilde{;}t} = -\frac{\overline{Y}^l}{u^2} \left(\partial t \overline{F}_{li} - \widetilde{\Gamma}^a_{tl} \overline{F}_{ai} - \widetilde{\Gamma}^a_{ti} \overline{F}_{la} \right) = \frac{\widetilde{\Gamma}^t_{ti} \overline{Y}^l \overline{F}_{lt}}{u^2} = -\frac{\widetilde{\Gamma}^t_{ti} \overline{E}(\overline{Y})}{u} \tag{3.1.24}$$

By substituting (3.1.21), (3.1.22), (3.1.23), (3.1.24) and (3.1.19) to (3.1.20), J_i

is as following.

$$
\begin{aligned}
\overline{J}_i &= -\frac{1}{2u}\overline{g}^{jl}(Y_{\bar{j};\bar{i}} - Y_{\bar{i};\bar{j}})\overline{E}_l + \frac{\overline{Y}^l\widetilde{\Gamma}^j_{il}\overline{E}_j}{u} + \left(\overline{g}^{jl} - \frac{\overline{Y}^j\overline{Y}^l}{u^2}\right)\overline{k}_{ij}\overline{E}_l \\
&= \frac{1}{u}\overline{g}^{jl}\overline{E}_l(Y_{\bar{i};\bar{j}}) + \frac{\overline{Y}^l\overline{\Gamma}^j_{il}\overline{E}_j}{u} \\
&= \frac{1}{u}\overline{g}^{jl}\overline{E}_l\left(\partial_j(\overline{g}_{i\phi}Y^{\phi}) - \overline{\Gamma}^k_{ij}\overline{Y}_k\right) + \frac{Y^{\phi}\overline{g}^{jl}\overline{E}_l(\overline{g}_{j\phi,i} - \overline{g}_{i\phi,j})}{2u} \\
&= \frac{1}{u}\overline{g}^{jl}\overline{E}_l(\partial_j(\overline{g}_{i\phi}Y^{\phi}) - \frac{Y^{\phi}}{2}(\overline{g}_{\phi i,j} + \overline{g}_{\phi j,i}) + \frac{Y^{\phi}}{2}(\overline{g}_{j\phi,i} - \overline{g}_{i\phi,j})) \\
&= \frac{\overline{g}_{i\phi}}{u}\overline{g}^{jl}\overline{E}_l(\partial_j(Y^{\phi}))
\end{aligned}
\tag{3.1.25}
$$

From (3.1.25), $\overline{J}(u\overline{\nabla}f) = f_{,\phi}\overline{g}^{jl}\overline{E}_l(\partial_j(Y^{\phi})) = 0$. Therefore (3.1.18) is now as following.

$$
div_{\overline{g}}\overline{E} = \sqrt{1 - u^2|\overline{\nabla}f|^2_{\overline{g}}}\,div_g E = \frac{div_g E}{\sqrt{1 + u^2|\nabla f|^2_g}} \tag{3.1.26}
$$

Second Proof We will now verify Lemma 3.1.5 by direct computation. Recall that the Christoffel symbol of (Σ, g, π) and the second fundamental form π with respect to $\overline{g}$ are as follows.

$$
\begin{aligned}
g^{ij} &= \overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j \\
\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} &= -\overline{w}^k\pi_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} + f_i\widetilde{\Gamma}^k_{jt} + f_j\widetilde{\Gamma}^k_{it} + f_if_j\widetilde{\Gamma}^k_{tt} \\
\pi_{ij} &= \frac{u\left(f_{\bar{;}ij} + \frac{\overline{k}_{ij}}{u} + f_i(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt}f_k) + f_j(\widetilde{\Gamma}^t_{it} - \widetilde{\Gamma}^k_{it}f_k) - f_if_j\widetilde{\Gamma}^k_{tt}f_k\right)}{\sqrt{1 - u^2|\overline{\nabla}f|^2_{\overline{g}}}}
\end{aligned}
\tag{3.1.27}
$$

We will compute $div_g E$ with respect to $\overline{g}$ and $\overline{E}$. Recall that $E_i = uN^t\overline{E}_i -$

$uf_i\overline{E}(\overline{w})$ by (3.1.9).

$$\begin{aligned}
div_g E &= g^{ij} E_{i;j} \\
&= \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\left(uN^t\overline{E}_i - uf_i\overline{E}(\overline{w})\right)_{\overline{;}j} \\
&- \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\left(\Gamma^k_{ij} - \bar{\Gamma}^k_{ij}\right)\left(uN^t\overline{E}_k - uf_k\overline{E}(\overline{w})\right) \\
&= uN^t div_{\overline{g}}\overline{E} + uN^t\left(-\frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\overline{E}_{i\overline{;}j} \\
&+ \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\left(\partial_j(uN^t)\overline{E}_i - \partial_j(u\overline{E}(\overline{w}))f_i - u\overline{E}(\overline{w})f_{\overline{;}ij}\right) \\
&- \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)(\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})(uN^t\bar{E}_k - uf_k\bar{E}(\bar{w}))
\end{aligned} \tag{3.1.28}$$

We will compute each terms in (3.1.28). The second and the third terms in the last part of (3.1.28) are as following.

$$uN^t\left(-\frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\overline{E}_{i\overline{;}j} = uN^t\left(\frac{\overline{E}(\overline{\nabla}_{\overline{Y}}\overline{Y})}{u^2} + \overline{w}(\overline{E}(\overline{w})) - \overline{E}(\overline{\nabla}_{\overline{w}}\overline{w})\right) \tag{3.1.29}$$

and

$$\begin{aligned}
&\left(\overline{g}^{ij} + \overline{w}^i\overline{w}^j\right)\left(\partial_j(uN^t)\overline{E}_i - \partial_j(u\overline{E}(\overline{w}))f_i\right) \\
&= \overline{E}(\overline{\nabla}(uN^t)) + \overline{w}(uN^t)\overline{E}(\overline{w}) - N^t\overline{w}(u\overline{E}(\overline{w}))
\end{aligned} \tag{3.1.30}$$

Note that (3.1.29) and (3.1.30) can be simplified by utilizing $uN^t\overline{\nabla}_{\overline{w}}\overline{w} =$

$u\overline{w}(N^t)\overline{w} + u(N^t)^2\overline{\nabla}_{\overline{w}}(u^2\overline{\nabla}f + \overline{Y})$.

$$
\begin{aligned}
&uN^t\left(-\frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\overline{E}_{\bar{i};j} + \left(\overline{g}^{ij} + \overline{w}^i\overline{w}^j\right)\left(\partial_j(uN^t)\overline{E}_i - \partial_j(u\overline{E}(\overline{w}))f_i\right)\\
&= uN^t\left(\frac{\overline{E}\left(\overline{\nabla}_{\overline{Y}}\overline{Y}\right)}{u^2} - \overline{E}\left(\overline{\nabla}_{\overline{w}}\overline{w}\right)\right) + \overline{E}\left(\overline{\nabla}(uN^t)\right) + u\overline{w}(N^t)\overline{E}(\overline{w})\\
&= \overline{E}(\overline{\nabla}(uN^t)) + \frac{N^t\overline{E}\left(\overline{\nabla}_{\overline{Y}}\overline{Y}\right)}{u} - u(N^t)^2\overline{E}\left(\overline{\nabla}_{\overline{w}}(u^2\overline{\nabla}f + \overline{Y})\right)\\
&= u^3(N^t)^3\overline{E}\left(u|\overline{\nabla}f|^2\overline{\nabla}u + u^2\overline{\nabla}_{\overline{\nabla}f}\overline{\nabla}f\right) + \frac{N^t\overline{E}\left(\overline{\nabla}_{\overline{Y}}\overline{Y}\right)}{u}\\
&- u(N^t)^3\overline{E}\left(\overline{\nabla}_{u^2\overline{\nabla}f+\overline{Y}}(u^2\overline{\nabla}f + \overline{Y})\right)\\
&= u^3(N^t)^3\overline{E}\left(u|\overline{\nabla}f|^2\overline{\nabla}u - \overline{\nabla}_{\overline{\nabla}f}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{\nabla}f - |\overline{\nabla}f|^2\overline{\nabla}_{\overline{Y}}\overline{Y} - 2uf^lu_l\overline{\nabla}f\right)
\end{aligned}
\tag{3.1.31}
$$

Lastly, We will compute the rest in (3.1.28). Note that we will rewrite $f_{\bar{;}ij}$ in terms of $\pi, \overline{\nabla}f, \overline{Y}$ by using the formulation of π with respect to $\overline{g}$ in (3.1.27). For computing the last term in (3.1.28), we will also use $\overline{Y}^l\widetilde{\Gamma}^t_{lt} = 0$, $\widetilde{\Gamma}^k_{jt} = \widetilde{\Gamma}^t_{jt}\overline{Y}^k - \frac{1}{2}\overline{g}^{kl}(Y_{l\bar{;}j} - Y_{j\bar{;}l})$.

$$
\begin{aligned}
&\left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)u\overline{E}(\overline{w})f_{\bar{;}ij}\\
&= \overline{E}(\overline{w})\left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\left(\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\pi_{ij} - \overline{k}_{ij} - 2uf_i(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt}f_k) + uf_if_j\widetilde{\Gamma}^k_{tt}f_k\right)\\
&= \sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\overline{E}(\overline{w})Tr_g\pi - \overline{E}(\overline{w})\left(\overline{k}(\overline{w},\overline{w}) + \frac{2uf^j(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt}f_k)}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\right)\\
&+ \overline{E}(\overline{w})\left(2uN^t(\overline{w}^lf_l)\overline{Y}^j\widetilde{\Gamma}^k_{jt}f_k + \frac{u|\overline{\nabla}f|^2\widetilde{\Gamma}^k_{tt}f_k}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\right)
\end{aligned}
\tag{3.1.32}
$$

and

$$
\begin{aligned}
&\left(\overline{g}^{ij}-\frac{\overline{Y}^i\overline{Y}^j}{u^2}+\overline{w}^i\overline{w}^j\right)\left(\Gamma^k_{ij}-\overline{\Gamma}^k_{ij}\right)\\
&=\left(\overline{g}^{ij}-\frac{\overline{Y}^i\overline{Y}^j}{u^2}+\overline{w}^i\overline{w}^j\right)\left(-\overline{w}^k\pi_{ij}+\frac{\overline{Y}^k}{u}\overline{k}_{ij}+f_i\widetilde{\Gamma}^k_{jt}+f_j\widetilde{\Gamma}^k_{it}+f_if_j\widetilde{\Gamma}^k_{tt}\right)\\
&=-(Tr_g\pi)\overline{w}^k+\frac{\overline{k}(\overline{w},\overline{w})\overline{Y}^k}{u}+\frac{2f^j\widetilde{\Gamma}^t_{jt}\overline{Y}^k-\overline{g}^{kl}f^j(Y_{\bar{l};j}-Y_{\bar{j};l})}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\\
&-(\overline{w}^lf_l)N^t\overline{Y}^j(\overline{g}^{kl}(Y_{\bar{l};j}-Y_{\bar{j};l}))+\frac{|\overline{\nabla}f|^2\widetilde{\Gamma}^k_{tt}}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}
\end{aligned}
\tag{3.1.33}
$$

We will simplify (3.1.32) and the last term of (3.1.28) together, using (3.1.33), in that both terms contain $\overline{E}(\overline{w})$. Next, we will substitute $\widetilde{\Gamma}$ in (3.1.19) and $\overline{k}_{ij}=\frac{Y_{\bar{i};j}+Y_{\bar{j};i}}{2u}$ to evaluate it further.

$$
\begin{aligned}
&-\left(\overline{g}^{ij}-\frac{\overline{Y}^i\overline{Y}^j}{u^2}+\overline{w}^i\overline{w}^j\right)u\overline{E}(\overline{w})f_{\bar{;}ij}+\left(\overline{g}^{ij}-\frac{\overline{Y}^i\overline{Y}^j}{u^2}+\overline{w}^i\overline{w}^j\right)\left(\Gamma^k_{ij}-\overline{\Gamma}^k_{ij}\right)(uf_k\overline{E}(\overline{w}))\\
&=\overline{E}(\overline{w})\left(-uN^tTr_g\pi+\overline{k}(\overline{w},\overline{w})+\frac{2uf^j(\widetilde{\Gamma}^t_{jt}-\widetilde{\Gamma}^k_{jt}f_k)}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\right)\\
&+\overline{E}\overline{w}\left(-2uN^t(\overline{w}^lf_l)\overline{Y}^j\widetilde{\Gamma}^k_{jt}f_k-uN^t(\overline{w}^lf_l)\overline{Y}^j(f^l(Y_{\bar{l};j}-Y_{\bar{j};l}))\right)\\
&=\overline{E}(\overline{w})\left(-uN^tTr_g\pi+\frac{\overline{Y}^lf^j(Y_{\bar{l};j}+Y_{\bar{j};l})}{u(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}+\frac{\overline{\nabla}f(\phi)+\overline{Y}^lf^j(Y_{\bar{l};j}-Y_{\bar{j};l})}{u(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}\right)\\
&=\overline{E}(\overline{w})\left(-uN^tTr_g\pi+\frac{2f^lu_l}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\right)
\end{aligned}
\tag{3.1.34}
$$

Let us further simplify the second last term of (3.1.28) by (3.1.33). We will evaluate it by substituting $\widetilde{\Gamma}$ in (3.1.19) as well. Notice that $\overline{g}^{kl}f^jY_{\bar{j};l}=-\overline{Y}^j\overline{g}^{kl}f_{\bar{;}jl}=-(\overline{\nabla}_{\overline{Y}}\overline{\nabla}f)^k$, and $\overline{g}^{kl}\overline{Y}^jY_{\bar{j};l}=\frac{(\overline{\nabla}|Y|^2)^k}{2}$. Therefore the following holds.

$$
\begin{aligned}
&-\left(\overline{g}^{ij}-\frac{\overline{Y}^i\overline{Y}^j}{u^2}+\overline{w}^i\overline{w}^j\right)\left(\Gamma_{ij}^k-\overline{\Gamma}_{ij}^k\right)uN^t\overline{E}_k \\
&=uN^t\left((Tr_g\pi)\overline{E}(\overline{w})-\overline{E}(\overline{Y})\left(\frac{\overline{k}(\overline{w},\overline{w})}{u}+\frac{2f^j\widetilde{\Gamma}_{jt}^t}{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}\right)\right) \\
&+uN^t\left(\frac{\overline{E}\left(\overline{\nabla}_{\overline{\nabla}f}\overline{Y}+\overline{\nabla}_{\overline{Y}}\overline{\nabla}f\right)}{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}+\frac{|\overline{\nabla}f|^2\overline{E}\left(\overline{\nabla}_{\overline{Y}}\overline{Y}-\frac{\overline{\nabla}|\overline{Y}|^2}{2}\right)}{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}-\frac{|\overline{\nabla}f|^2}{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}\overline{E}\left(\frac{\bar{\nabla}\phi}{2}\right)\right) \\
&=uN^t(Tr_g\pi)\overline{E}(\overline{w})-2u^2(N^t)^3f^lu_l\overline{E}(\overline{Y}) \\
&+u^3(N^t)^3\overline{E}\left(\overline{\nabla}_{\overline{\nabla}f}\overline{Y}+\overline{\nabla}_{\overline{Y}}\overline{\nabla}f+|\overline{\nabla}f|^2\overline{\nabla}_{\overline{Y}}\overline{Y}-u|\overline{\nabla}f|^2\overline{\nabla}u\right)
\end{aligned}
\tag{3.1.35}
$$

In conclusion, the sum of (3.1.31), (3.1.34) and (3.1.35) is 0. This leads the desired result from (3.1.28) as following.

$$
div_gE=uN^tdiv_{\overline{g}}\overline{E}=\frac{div_{\overline{g}}\overline{E}}{\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} \tag{3.1.36}
$$

which is equivalent to (3.1.26). □

So far we show how the deformation of initial data $(M,\overline{g},\overline{k},\overline{E})$ satisfies all the desired conditions, especially (3.1.4), (3.1.5), and (3.1.6). In the following section, we will prove the angular momentum-mass-charge inequality, with the similar reduction argument utilized in the previous chapter.

3.2 The Reduction Argument

Let (M,g,k,E) be a simply connected, axially symmetric initial data set with two ends, one strongly asymptotically flat and the other either strongly asymptotically flat or asymptotically cylindrical with the dominant energy condition

$\mu_{EM} \geq |J_{EM}|$ and $J_{EM}(\eta) = 0$. In the previous section, we derive the deformation of initial data $(M, \overline{g}, \overline{k}, \overline{E})$ such that

$$Tr_{\overline{g}}\overline{k} = 0, \quad div_{\overline{g}}\overline{k}(\eta) = 0$$

and

$$|E|_g \geq |\overline{E}|_{\overline{g}}, \quad div_{\overline{g}}\overline{E} = 0, \quad q = \overline{q} \tag{3.2.1}$$

We will combine the reduction argument in section 2.3 and the standard argument in [15], [10]. First, recall that the following scalar curvature formula (2.2.29) holds for $(M, \overline{g}, \overline{k})$.

$$\overline{R} - |\overline{k}|^2_{\overline{g}} = 16\pi(\mu - J(v)) + |k - \pi|^2_g + 2u^{-1}div_{\overline{g}}(uQ), \tag{3.2.2}$$

We will substitute $16\pi\mu = 16\pi\mu_{EM} + 2|E|^2$, and $J_{EM} = J$ to (3.2.2) so that

$$\overline{R} - |\overline{k}|^2_{\overline{g}} - 2|\overline{E}|^2_{\overline{g}} = 16\pi(\mu_{EM} - J_{EM}(v)) + 2(|E|^2_g - |\overline{E}|^2_{\overline{g}}) + |k - \pi|^2_g + 2u^{-1}div_{\overline{g}}(uQ) \tag{3.2.3}$$

Provided that there exist smooth solutions (f, Y^ϕ, u) for (2.2.28), (2.3.8), (2.4.2) with appropriate asymptotic conditions described in section 2.3, $(M, \overline{g}, \overline{k})$ has a global Brill's coordinate system,

$$\overline{g} = e^{-2\overline{U}+2\overline{\alpha}}(d\overline{\rho}^2 + d\overline{z}^2) + \overline{\rho}^2 e^{-2\overline{U}}(d\phi + A_{\overline{\rho}}d\overline{\rho} + A_{\overline{z}}d\overline{z})^2 \tag{3.2.4}$$

Also, there exists a twist potential $\overline{\omega}$ for $\eta = \partial_\phi$ in $(M, \overline{g}, \overline{k})$ as well. In this coordinate system, it is shown in (2.3.7) that

$$\overline{m} - \frac{1}{32\pi}\int_{\mathbb{R}^3} \left(4|\partial\overline{U}|^2 + \overline{g}_{\phi\phi}^{-2}|\partial\overline{\omega}|^2\right) dx \geq \frac{1}{32\pi}\int_{\mathbb{R}^3} 2e^{-2\overline{U}+2\overline{\alpha}}(\overline{R} - |\overline{k}|^2_{\overline{g}})dx \tag{3.2.5}$$

We will substitute (3.2.3) to (3.2.5). In addition, we will directly apply Lemma 2.3.1 to make the boundary integration zero in the following computation. Then,

$$\begin{aligned}
&\overline{m} - \frac{1}{32\pi}\int_{\mathbb{R}^3} \left(4|\partial\overline{U}|^2 + \overline{g}_{\phi\phi}^{-2}|\partial\overline{\omega}|^2 + 4e^{-2\overline{U}+2\overline{\alpha}}|\overline{E}|_{\overline{g}}^2\right) dx \\
&\geq \frac{1}{32\pi}\int_{\mathbb{R}^3} 2e^{-2\overline{U}+2\overline{\alpha}}(16\pi(\mu_{EM} - J_{EM}(v)) + 2(|E|_g^2 - |\overline{E}|_{\overline{g}}^2) + |k-\pi|_g^2)dx \\
&\geq 0
\end{aligned} \tag{3.2.6}$$

The last inequality holds due to the dominant energy condition $\mu_{EM} \geq |J_{EM}|$ and (3.2.1).

From now on, we will follow the standard argument in [10] and [15]. Let us adopt the Brill's coordinate system for (M, g), $g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)^2$. Recall that $\{e_i = e^{U-\alpha}(\partial_i - A_i\partial_\phi), e_\phi = \frac{\partial_\phi}{|\partial_\phi|}\}$ $i = \rho, z$ forms the orthonormal basis of the Brill's coordinate system. Note that throughout the process, we will make a deformation on E so that $E(e_\phi) = 0$ whereas $E(e_i)$ remains same. In particular,

$$E(\partial_\phi) = 0$$

This deformation preserves the total electric charge. It is straightforward to check that the dominant energy condition still holds. More importantly, we will show that $div_g E = 0$ in the following computation. In general, for any vector E with $\mathfrak{L}_\eta E = 0$, the following holds.

$$
\begin{aligned}
& div_g E \\
&= \sum_{i,j=\rho,z} \big(e_i(E(e_i)) - g\langle \nabla_{e_i} e_i, e_j \rangle E(e_j) - g\langle \nabla_{e_i} e_i, e_\phi \rangle E(e_\phi) - g\langle \nabla_{e_\phi} e_\phi, e_j \rangle E(e_j) \big) \\
&= \sum_{i,j=\rho,z} \big(e_i(E(e_i)) - g\langle \nabla_{e_i} e_i, e_j \rangle E(e_j) - g\langle \nabla_{e_\phi} e_\phi, e_j \rangle E(e_j) \big)
\end{aligned} \tag{3.2.7}
$$

The last line holds since $\sum_i (g\langle \nabla_{e_i} e_i, e_\phi \rangle) = -\frac{1}{|\partial_\phi|} \sum_i (g\langle \nabla_{e_i} \partial_\phi, e_i \rangle) = 0$. (3.2.7) shows that $div_g E$ is independent from the value of $E(e_\phi)$ whenever $E(e_i)$ is fixed. Therefore under this deformation,

$$div_g E = 0$$

Without loss of generality, we will simply assume that $E(\partial_\phi) = 0$ from now on. In this case, simple computation same as in (2.5.22) shows that in Brill's coordinate system,

$$div_g E = \frac{e^{2U-2\alpha}}{|\partial_\phi|} \sum_{i=\rho,z} \left(\partial_i (|\partial_\phi| E_i) \right). \tag{3.2.8}$$

Here, $E_i = E(\partial_i)$. If we compute $\overline{E}$ accordingly as in the previous section, Lemma 3.1.2 shows that

$$\overline{E}(\eta) = 0 \tag{3.2.9}$$

In addition, $\overline{E}$ also satisfies (3.2.1) and therefore, (3.2.6) still holds. Note that Lemma 3.1.5 implies that

$$div_{\overline{g}} \overline{E} = 0. \tag{3.2.10}$$

The main purpose of the new deformation is to show the following lemma.

Lemma 3.2.1. *For 4-momentum $\overline{F}$ in* (3.1.7)*, the following holds with $\eta = \partial_\phi$.*

$$d(\overline{F}(\eta, \cdot)) = d(*\overline{F}(\eta, \cdot)) = 0 \tag{3.2.11}$$

$*$ is the Hodge star operator in $(\mathbb{R} \times M, \widetilde{g})$.

Proof. Let us take Brill's coordinate system for $(M, \overline{g}, \overline{k})$ with orthonormal frame $\{e_{\overline{\rho}}, e_{\overline{z}}, e_\phi\}$ as usual. Then $\{\overline{n}, e_{\overline{\rho}}, e_{\overline{z}}, e_\phi\}$ forms the orthonormal frame for $(\mathbb{R}\times M, \widetilde{g} = -\phi dt^2 - 2Y_i dt dx^i + \overline{g})$, with $\overline{n} = \frac{\partial_t + \overline{Y}}{u}$. Let us take $\{\theta^n, \theta^{\overline{\rho}}, \theta^{\overline{z}}, \theta^\phi\}$ as dual one forms accordingly. In this frame, the following holds.

$$\overline{F}(\eta, \cdot) = -|\eta|\overline{E}(e_\phi)\theta^n, \quad *\overline{F}(\eta, \cdot) = -|\eta|\overline{E}(e_{\overline{z}})\theta^\rho + |\eta|\overline{E}(e_{\overline{\rho}})\theta^z \tag{3.2.12}$$

By (3.2.9), $\overline{F}(\eta, \cdot) = 0$. Next, let us compute the exterior derivative for $*\overline{F}(\eta, \cdot)$ as follows. Note that $\theta^{\overline{\rho}} = e^{-\overline{U}+\overline{\alpha}} d\overline{\rho}, \theta^{\overline{z}} = e^{-bu+\overline{\alpha}} d\overline{z}$.

$$\begin{aligned} d(*\overline{F}(\eta, \cdot)) &= d(-|\eta|\overline{E}(e_{\overline{z}})\theta^{\overline{\rho}} + |\eta|\overline{E}(e_{\overline{\rho}})\theta^{\overline{z}}) \\ &= d(-|\eta|\overline{E}_{\overline{z}} d\overline{\rho} + |\eta|\overline{E}_{\overline{\rho}} d\overline{z}) \\ &= \left(\partial_{\overline{z}}(|\eta|\overline{E}_{\overline{z}}) + \partial_{\overline{\rho}}(|\eta|\overline{E}_{\overline{\rho}})\right) d\overline{\rho} \wedge d\overline{z} \end{aligned} \tag{3.2.13}$$

The analog of (3.2.8) with respect to $\overline{g}$ metric and (3.2.10) shows that $*\overline{F}(\eta, \cdot)$ is indeed closed. □

Since M is simply connected, Lemma 3.2.1 implies that there exist functions $\overline{\chi}, \overline{\psi}$ such that

$$\overline{F}(\eta, \cdot) = d\overline{\chi} \quad * \overline{F}(\eta, \cdot) = d\overline{\psi} \tag{3.2.14}$$

By comparing (3.2.12) and (3.2.14), we deduce the following. Note also that $\overline{\chi}$ is trivial.

$$\begin{aligned} |\overline{E}|^2_{\overline{g}} &\geq \frac{|\overline{\nabla}\overline{\chi}|^2_{\overline{g}} + |\overline{\nabla}\overline{\psi}|^2_{\overline{g}}}{|\eta|^2_{\overline{g}}} \\ &\geq \frac{e^{4\overline{U}-2\overline{\alpha}}}{\overline{\rho}^2} \left(|\partial\overline{\chi}|^2_\delta + |\partial\overline{\psi}|^2_\delta\right) \end{aligned} \tag{3.2.15}$$

where δ is a flat metric. Now we are ready to gain the standard mass inequality as (2.16) in [10]. (3.2.6) and (3.2.15) deduce the following inequality.

$$\overline{m} \geq \frac{1}{32\pi} \int_{\mathbb{R}^3} \left(4|\partial \overline{U}|^2 + \overline{g}_{\phi\phi}^{-2} |\partial \overline{\omega}|^2 + 4\frac{e^{2\overline{U}}}{\overline{\rho}^2} \left(|\partial \overline{\chi}|_\delta^2 + |\partial \overline{\psi}|_\delta^2 \right) \right) dx \tag{3.2.16}$$

Let us simply take $2\overline{v} = \overline{\omega}$. This is valid since $\overline{\chi}$ is trivial in our case. Therefore, the following holds.

$$\overline{m} \geq \mathcal{I}(\overline{U}, \overline{v}, \overline{\chi}, \overline{\psi}) \tag{3.2.17}$$

Where $\mathcal{I}$ is the standard mass functional given by

$$\mathcal{I}(U, v, \chi, \psi) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \left(|\partial U|^2 + \frac{e^{4U}}{\rho^2} |\partial v + \chi \partial \psi - \psi \partial \chi|^2 + \frac{e^{2U}}{\rho^2} \left(|\partial \chi|^2 + |\partial \psi|^2 \right) \right) dx. \tag{3.2.18}$$

In [15] and [32], it is shown that $\mathcal{I}$ is minimized by the Extreme Kerr Newman data with the fixed total electric charge, total magnetic charge and the angular momentum. Therefore $\mathcal{I}(\overline{U}, \overline{v}, \overline{\chi}, \overline{\psi})$ is bounded below by the energy from Extreme Kerr-Newman data $(U_{EKN}, V_{EKN}, \chi_{EKN}, \psi_{EKN})$ with zero total magnetic charge, total electric charge $q_{EKN} = \overline{q}$ and angular momentum $\mathcal{J}_{EKN} = \overline{\mathcal{J}}$. Therefore the following holds.

$$\begin{aligned} m = \overline{m} &\geq \mathcal{I}(U_{EKN}, V_{EKN}, \chi_{EKN}, \psi_{EKN}) \\ &= \frac{q_{EKN}^2 + \sqrt{q_{EKN}^4 + 4\mathcal{J}_{EKN}^2}}{2} \\ &= \frac{\overline{q}^2 + \sqrt{\overline{q}^4 + 4\overline{\mathcal{J}}^2}}{2} = \frac{q^2 + \sqrt{q^4 + 4\mathcal{J}^2}}{2} \end{aligned} \tag{3.2.19}$$

In conclusion, we state the main theorem of the chapter.

Theorem 3.2.2. *Let (M, g, k, E) be a simply connected, axially symmetric initial data set with two ends, one strongly asymptotically flat and the other either strongly asymptotically flat or asymptotically cylindrical. If $J_{EM}(\eta) = 0$,* *the charged dominant energy condition is satisfied, and the system of equations* (2.2.21), (2.2.28), (2.3.8) *admits a smooth solution* (u, Y^{ϕ}, f) *satisfying the asymptotics* (2.2.22), (2.2.31), (2.2.32), (2.3.9)-(2.3.11), *then*

$$m^2 \geq \frac{q^2 + \sqrt{q^4 + 4\mathcal{J}^2}}{2}. \tag{3.2.20}$$

Chapter 4

A Lower Bound for Area in Terms of Mass, Angular Momentum and Charge

In this chapter we observe that the reduction argument given above, immediately applies to another geometric inequality for axisymmetric black holes. Let (M, g, k, E) be as in the previous chapter, with the restriction that it possesses only two ends denoted $M_{end}^{\pm}$, such that M_{end}^{+} is asymptotically flat and M_{end}^{-} is either asymptotically flat or asymptotically cylindrical. Based on the heuristic arguments of Chapter 2 leading to the angular momentum-mass inequality (1.0.1), combined with the Hawking area theorem [24], with admitting no charged matter, the following upper and lower bounds are derived [21]

$$m^2-\frac{q^2}{2}-\sqrt{\left(m^2-\frac{q^2}{2}\right)^2-\frac{q^4}{4}-J^2}\leq\frac{A_{min}}{8\pi}\leq m^2-\frac{q^2}{2}+\sqrt{\left(m^2-\frac{q^2}{2}\right)^2-\frac{q^4}{4}-J^2}, \tag{4.0.1}$$

where A_{min} is the minimum area required to enclose M_{end}^{-}. In [21] the lower bound is established in the maximal case. The proof relies upon the angular momentum-mass-charge inequality and the area-angular momentum inequality $A_{min} \geq 8\pi|\mathcal{J}|$ ([14], [20], [10]). In the non-maximal case, the area-angular momentum inequality has also been established when A_{min} is replaced by the area of a stable, axisymmetric, marginally outer trapped surface ([14], [18]). Thus, since we have shown in the previous chapters how to reduce the non-maximal case of the angular momentum-mass inequality to the problem of solving a coupled system of elliptic equations, an analogous lower bound for area may also be reduced to the same problem. More precisely, Theorems 2.4.1, 3.2.2 combined with Theorem 1.1 in [14] and the proof of a Theorem 2.5 in [21], produces the following result.

Theorem 4.0.3. *Let (M, g, k, E) be a simply connected, axially symmetric initial data set with two ends, one (M_{end}^{1}) asymptotically flat and the other (M_{end}^{2}) either asymptotically flat or asymptotically cylindrical. We assume that the dominant condition is satisfied, $J_i\eta^i = 0$, and there is no charged matter. In addition, if the data possesses a stable axisymmetric marginally outer trapped surface with area $\mathcal{A}$, and the system of equations* (2.2.21), (2.2.28), (2.3.8) *admits a smooth solution (u, Y^{ϕ}, f) satisfying the asymptotics* (2.2.22), (2.2.31), (2.2.32), (2.3.9)-(2.3.11), *then*

$$\frac{A_{min}}{8\pi} \geq m^2 - \frac{q^2}{2} - \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}, \tag{4.0.2}$$

where A_{min} is the minimum area required to enclose M_{end}^{2}.

Chapter 5

The Scalar Curvature Formula

In this chapter, we will derive a formula for scalar curvature $\overline{R}$ of $\overline{g} = \phi df^2 + Y \otimes df + df \otimes Y + g$ in terms of the energy density μ, the momentum density J, ϕ, Y^{ϕ}, and f under the appropriate assumption on Y we discussed before. We will follow the notation in [3] from now on. In general, barred quantities will be associated with $(M, \bar{g})$ and unbarred quantities will be associated with $(\Sigma = \{t = f\}, g)$.

(Σ, g) is considered as an image of the graph $t = f$ embedded in the stationary spacetime $(\mathbb{R} \times M, \widetilde{g} = -\phi dt^2 - Y \otimes dt - dt \otimes Y + \bar{g})$, where $(M, \bar{g}, \bar{k})$ is $t = 0$ slice of the constructed spacetime.

Let $\overline{\partial_0} = \partial_t$ and $\overline{\partial_i}$ be tangent vectors to $(M, \overline{g})$. Define

$$\partial_i = X_i = \overline{\partial_i} + f_i \overline{\partial_0} \tag{5.0.1}$$

to be corresponding tangent vectors to (Σ, g). In this coordinate system,

$$\begin{aligned} g_{ij} &= \overline{g}_{ij} - f_i Y_j - f_j Y_i - \phi f_i f_j \\ g^{ij} &= \overline{g}^{ij} - \frac{\overline{Y}^i \overline{Y}^j}{u^2} + \overline{w}^i \overline{w}^j \end{aligned} \tag{5.0.2}$$

where the lapse function u and $\overline{w}$, the spatial component of the unit normal vector to (Σ, g) are as following.

$$\begin{aligned} u &= \sqrt{\phi + |\overline{Y}|_{\overline{g}}^2} \\ \overline{w}^i &= \frac{u^2 \overline{f}^i + \overline{Y}^i}{u\sqrt{1 - u^2|\overline{\nabla} f|_{\overline{g}}^2}} \end{aligned} \tag{5.0.3}$$

The second fundamental form of $(M, \overline{g})$ in spacetime is

$$\overline{k}_{ij} = \frac{\overline{\nabla}_j Y_i + \overline{\nabla}_i Y_j}{2u} \tag{5.0.4}$$

Observe the useful identity shown in section 5.1

$$(1 + u^2|\nabla f|_g^2)(1 - u^2|\overline{\nabla} f|_{\overline{g}}^2) = 1 \tag{5.0.5}$$

under the assumption on the Y. This is from direct comparison for the volume form of (Σ, g) and $(M, \overline{g})$.

In section 5.2, we will compute the second fundamental form π of (Σ, g) in our constructed stationary spacetime as follows :

$$\begin{aligned} \pi_{ij} &= \frac{u\nabla_{ij} f + \frac{1}{2u}(g_{i\phi} Y^{\phi}_{,j} + g_{i\phi} Y^{\phi}_{,j}) + f_i u_j + f_j u_i}{\sqrt{1 + u^2|\nabla f|_g^2}} \\ &= \frac{u\overline{\nabla}_{ij} f + \overline{k}_{ij} + \frac{f_i \phi_j}{2u} + \frac{f_j \phi_i}{2u} - \frac{\overline{\nabla} f(\phi)}{2} f_i f_j}{\sqrt{1 - u^2|\overline{\nabla} f|_{\overline{g}}^2}} \\ &+ \frac{f_i}{2}\overline{w}^l (Y_{l,j} - Y_{j,l}) + \frac{f_j}{2}\overline{w}^l (Y_{l,i} - Y_{i,l}) \end{aligned} \tag{5.0.6}$$

We will extend π and k trivially in the constructed spacetime so that

$$\pi(\partial_t, \cdot) = k(\partial_t, \cdot) = 0 \quad \rightarrow \quad \pi(\partial_i, \partial_j) = \pi(\overline{\partial}_i, \overline{\partial}_j) \tag{5.0.7}$$

Let us compute a formula for $\overline{R}$. In $(\mathbb{R} \times M, \widetilde{g})$, the future pointing normal vector N of the graph (M, g, π) and the future pointing normal vector n of

$t = 0$ slice $(M, \overline{g}, \overline{k})$ are as following :

$$n = \frac{\partial_t + \overline{Y}}{u}, \quad N = \frac{u\overline{\nabla} f + n}{\sqrt{1 - u^2 |\overline{\nabla} f|^2_{\overline{g}}}} \tag{5.0.8}$$

Note that ∂_t is a killing vector on $(\mathbb{R} \times M, \widetilde{g})$. Therefore there is an obvious one to one correspondence between $\Sigma = \{t = f\}$ and $M = \{t = 0\}$. In that sense, we will decompose n on $\Sigma = \{t = f\}$ as a normal component and a tangential component $tan_\Sigma(n)$.

$$n = \sqrt{1 + u^2 |\nabla f|^2_g} N + tan_\Sigma(n) \tag{5.0.9}$$

$$tan_\Sigma(n) = g^{ij} \langle n, \partial_i \rangle \partial_j = -u\nabla f \tag{5.0.10}$$

As in [3], we will compute $G(N, n)$ in two different ways using the Gauss-Codazzi equations. First we will compute $G(N, n)$ on $(M, \overline{g}, \overline{k})$. By applying (5.0.8) and (5.0.5),

$$\begin{aligned} G(N, n) &= \sqrt{1 + u^2 |\nabla f|^2_g} (G(n, n) + G(u\overline{\nabla} f, n)) \\ &= \sqrt{1 + u^2 |\nabla f|^2_g} (\overline{R} + (Tr_{\overline{g}} \overline{k})^2 - |\overline{k}|^2_{\overline{g}})/2 \\ &+ \sqrt{1 + u^2 |\nabla f|^2_g} div_{\overline{g}} (\overline{k} - (Tr_{\overline{g}} \overline{k}) \overline{g})(u\overline{\nabla} f) \end{aligned} \tag{5.0.11}$$

Second, compute $G(N, n)$ on (Σ, g, π). Recall that $n = \sqrt{1 + u^2 |\nabla f|^2_g} N - u\nabla f$ from (5.0.9) and (5.0.10).

$$\begin{aligned} G(N, n) &= \sqrt{1 + u^2 |\nabla f|^2_g} G(N, N) + G(N, tan_\Sigma(n)) \\ &= \sqrt{1 + u^2 |\nabla f|^2_g} (R + (Tr_g \pi)^2 - |\pi|^2_g)/2 \\ &+ div_g (\pi - (Tr_g \pi) g)(-u\nabla f) \end{aligned} \tag{5.0.12}$$

Comparing (5.0.11) and (5.0.12) gives the following.

$$\begin{aligned} &\overline{R} + (Tr_{\overline{g}} \overline{k})^2 - |\overline{k}|^2_{\overline{g}} + 2 div_{\overline{g}} (\overline{k} - (Tr_{\overline{g}} \overline{k}) \overline{g})(u\overline{\nabla} f) \\ &= R + (Tr_g \pi)^2 - |\pi|^2_g - 2 div_g (\pi - (Tr_g \pi) g)(v) \end{aligned} \tag{5.0.13}$$

where

$$v = \frac{u\nabla f}{\sqrt{1+u^2|\nabla f|_g^2}}, \quad |v|_g \leq 1 \tag{5.0.14}$$

Lastly, recall the definition of the energy density μ and momentum density J for (M, g, k) as following.

$$\begin{aligned} 8\pi\mu &= G(N,N) = (R + (Tr_g k)^2 - |k|_g^2)/2 \\ 8\pi J(\cdot) &= G(N,\cdot) = div_g(k - (Tr_g k)g)(\cdot) \end{aligned} \tag{5.0.15}$$

By (5.0.15), $R = 16\pi\mu - (Tr_g k)^2 + |k|_g^2$. Recall that our choice of Y makes $Tr_{\overline{g}}\overline{k} = 0$. Therefore we can rewrite (5.0.13) in terms of the energy and momentum density as follows :

$$\begin{aligned} &\overline{R} - |\overline{k}|_{\overline{g}}^2 + 2div_{\overline{g}}\overline{k}(u\overline{\nabla} f) \\ &= 16\pi(\mu - J(v)) - |\pi|_g^2 + |k|_g^2 - 2div_g(\pi)(v) + 2div_g(k)(v) \\ &+ (Tr_g\pi)^2 - (Tr_g k)^2 + 2v(Tr_g\pi - Tr_g k) \end{aligned} \tag{5.0.16}$$

Notice that by the dominant energy condition, $\mu - J(v) \geq 0$. Therefore $\overline{R} \geq 0$ if $\pi = k, Y = 0$. In [3], Bray and Khuri computed (5.0.16) with respect to $\overline{g}$ in the case that $Y = 0 \to \overline{k} = 0$. In addition, they showed that $\overline{R}$ is weakly nonnegative if f is a solution of the generalized Jang equation $Tr_\Sigma(k - \pi) = 0$.

We will compute the divergence terms (5.0.16) with respect to $\overline{g}$. Under the appropriate assumption on k and Y^ϕ, i.e. $div_g k(\partial_\phi) = 0$ and $div_{\overline{g}}\overline{k}(\partial_\phi) = 0$, we can also show that $\overline{R}$ is weakly nonnegative as well if f is a solution of a quasi-local elliptic equation $Tr_g(k - \pi) = 0$. The detailed computation is provided in section 5.3. We will refer each identities in section 5.3 here and briefly discuss how to compute **Identity 7**, the revised Schoen-Yau Identity.

Identity 1

$$\begin{aligned}\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} &= -\overline{w}^k \pi_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} + f_i f_j (\frac{\phi^k}{2}) \\ &+ f_i \left(-\frac{1}{2}\overline{g}^{kl}(Y_{l,j} - Y_{j,l}) + \frac{\overline{Y}^k}{2u^2}(\phi_j + \overline{Y}^l (Y_{l,j} - Y_{j,l})) \right) \\ &+ f_j \left(-\frac{1}{2}\overline{g}^{kl}(Y_{l,i} - Y_{i,l}) + \frac{\overline{Y}^k}{2u^2}(\phi_i + \overline{Y}^l (Y_{l,i} - Y_{i,l})) \right)\end{aligned} \tag{5.0.17}$$

Identity 2

$$\begin{aligned}div_g k(w) &= \frac{1}{u} div_{\overline{g}}(uk(\overline{w}, \cdot)) + \overline{w}(k(\overline{w}, \overline{w})) \\ &- \widetilde{g}\langle k, \pi\rangle - 2\widetilde{g}\langle k(\overline{w}, \cdot), \pi(\overline{w}, \cdot)\rangle + (Tr_{\widetilde{g}}\pi) k(\overline{w}, \overline{w})\end{aligned} \tag{5.0.18}$$

Identity 3

$$\begin{aligned}&div_g k(\partial_\phi) \\ &= div_{\overline{g}}(k(\partial_\phi, \cdot)) + \frac{1}{u} div_{\overline{g}}(uk(\partial\phi, \overline{w})\overline{w}) \\ &- g\langle k(\partial_\phi, \cdot), \pi(v, \cdot)\rangle + \overline{g}\langle k(\partial_\phi, \cdot), \overline{k}(u\overline{\nabla} f, \cdot)\rangle - \frac{\overline{\nabla} f(u)}{u} k(\partial_\phi, \overline{Y})\end{aligned} \tag{5.0.19}$$

Identity 4

$$\begin{aligned}&div_{\overline{g}}(\pi(\partial_\phi, \cdot)) \\ &= \frac{1}{\sqrt{1 - u^2 |\overline{\nabla} f|^2_{\overline{g}}}} div_{\overline{g}}(uHess_{\overline{g}} f(\partial_\phi, .)) - \frac{1}{u} div_{\overline{g}}(u\pi(\partial\phi, \overline{w})\overline{w}) \\ &+ g\langle \pi(\partial_\phi, \cdot), \pi(v, \cdot)\rangle - \overline{g}\langle \pi(\partial_\phi, \cdot), \overline{k}(u\overline{\nabla} f, \cdot)\rangle + \frac{\overline{\nabla} f(u)}{u} \pi(\partial_\phi, \overline{Y})\end{aligned} \tag{5.0.20}$$

Identity 5

$$div_g\pi(\partial_\phi) = \frac{1}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} div_{\overline{g}}(uHess_{\overline{g}} f(\partial_{\phi,\cdot})) \tag{5.0.21}$$

Identity 6

$$\begin{aligned} &div_g(k-\pi)(v) \\ &= |\pi|^2 - g\langle \pi, k\rangle - u\overline{k}^{ij}(\overline{\nabla}_{ij} f) \\ &+ \frac{1}{u} div_{\overline{g}}(u(Hess_{\overline{g}}(\overline{Y},\cdot) + (k-\pi)(\overline{w},\cdot) + (k-\pi)(\overline{w},\overline{w})\frac{u\cdot df}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}})) \end{aligned} \tag{5.0.22}$$

Our main goal is to compute $div_g(k-\pi)(v)$ with respect to $\overline{g}$. In **Identity 1** we discuss the difference between Christoffel symbols with respect to g and $\overline{g}$. Second, we will compute **Identity 2** and **3** separately using **Identity 1**. Note that these two results give $div_g k(v)$ in that $\overline{v} = \overline{w} - \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}\overline{Y}}{u}$. The reason why we will compute it separately is $div_g k(\partial_\phi) = 0$. Next, we will compute $div_g\pi(\partial_\phi)$ with respect to $\overline{g}$ in **Identity 4**. On the other hand, **Identity 3** gives explicit formula for $div_g\pi(\partial_\phi)$ by substituting $\pi \to k$. Therefore by substituting **Identity 4** into **Identity 3** with π, we get **Identity 5**. We will use $div_{\overline{g}}\overline{k}(\partial_\phi) = 0$ in this process as well. We get **Identity 6** by combining **Identity 2** for $(k-\pi)$ and **Identity 5**.

Finally, by substituting **Identity 6** into (5.0.16) we get the following revised Schoen-Yau Identity.

Identity 7 (Revised Schoen-Yau Identity)

$$\begin{aligned} \overline{R} - |\overline{k}|^2_{\overline{g}} &= 16\pi(\mu - J(v)) + |k-\pi|^2_g + \frac{2}{u} div_{\overline{g}}(uQ(\cdot)) \\ &\quad + (Tr_g\pi)^2 - (Tr_g k)^2 + 2v(Tr_g\pi - Tr_g k) \end{aligned} \tag{5.0.23}$$

where Q is one form on $(M, \overline{g})$ such that

$$Q(\cdot) = (Hess_{\overline{g}} f)(\overline{Y}, \cdot) - \overline{k}(u\overline{\nabla} f, \cdot) + (k-\pi)(\overline{w}, \cdot) + (k-\pi)(\overline{w}, \overline{w})(\frac{udf}{\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}, \cdot) \tag{5.0.24}$$

In the special case where $Y = 0$, similar Identity is derived by Bray and Khuri as Identity 9 in [3]. The main difference lies in that the generalized Scheon-Yau identity represents $\overline{R}$ with respect to $\overline{g}$, which is a metric for Jang surface satisfying $Tr_\Sigma(k - \pi) = 0$. But this new revised version of Schoen-Yau Identity is represented with respect to g and $\overline{g}$, and it naturally proposes the following quasi-local elliptic equation for f,

$$Tr_g(k - \pi) = 0 \tag{5.0.25}$$

which will easily imply that $\overline{R} - |\overline{k}|^2_{\overline{g}} + \frac{2}{u} div_{\overline{g}}(uQ(\cdot)) \quad \geq 0$. This is our desired result for proving the angular momentum-mass inequality in that we need

$$\int_M u(\overline{R} - |\overline{k}|^2) dv_{\overline{g}} \geq 0 \tag{5.0.26}$$

But this may not be directly applicable to prove the positive mass theorem or the Penrose inequality due to the divergence term.

5.1 Volume form

In this section, we will compute the $\overline{g}$ metric with respect to g metric, Y^ϕ and f. We will also compute the volume form of $(M, \overline{g})$ accordingly, which will be useful in section 5.3. We note that $dvol_{\overline{g}}$ is the volume form of $(M, \overline{g})$ whereas $dvol_g$ is the volume form of (Σ, g). Recall that the $\overline{g}$ metric can be written by g, f, Y^ϕ as follows.

$$\overline{g}_{ij} = g_{ij} + Y_i f_j + Y_j f_i + \phi f_i f_j \tag{5.1.1}$$

Also recall that Y^ϕ is a function from M to $\mathbb{R}$ as $(M, \overline{g})$ allows the global coordinate system with the killing vector as a coordinate vector ∂_ϕ. Therefore the following holds by our choice for Y_i.

$$Y_i = \overline{g}_{i\phi} Y^\phi = (g_{i\phi} + Y_\phi f_i) Y^\phi = g_{i\phi} Y^\phi + |\overline{Y}|^2_{\overline{g}} f_i \tag{5.1.2}$$

By substituting (5.1.2) to (5.1.1), the following holds.

$$\overline{g}_{ij} = g_{ij} + Y^\phi g_{j\phi} f_i + Y^\phi g_{i\phi} f_j + \left(u^2 + |\overline{Y}|^2_{\overline{g}}\right) f_i f_j \tag{5.1.3}$$

Therefore, $(\overline{\Sigma}, \overline{g})$ can be referred as the $\{t = f\}$ graph embedded in the Riemannian 4 dimensional manifold $(\mathbb{R} \times M, \widehat{g})$, where $\widehat{g}$ is as follows. (M, g) is $\{t = 0\}$ slice of $(\mathbb{R} \times M, \widehat{g})$ under the Brill's coordinate system.

$$\widehat{g} = \left(u^2 + (Y^\phi)^2 g_{\phi\phi}\right) dt^2 + 2Y^\phi g_{\phi i} dx^i dt + g \tag{5.1.4}$$

Note that the lapse function in $(\mathbb{R} \times M, \widehat{g})$ is u, while the shift vector is $Y^\phi \partial_\phi$ as in the Lorentzian setting. Therefore $\overline{g}^{ij}$ can be computed as following, within similar sense.

$$\begin{aligned}\overline{g}^{ij} &= g^{ij} + \frac{(Y^\phi)^2\delta^i_\phi\delta^j_\phi}{u^2} - \left(\frac{u^2 f^i + Y^\phi\delta^i_\phi}{u\sqrt{1+u^2|\nabla f|^2_g}}\right)\left(\frac{u^2 f^j + Y^\phi\delta^j_\phi}{u\sqrt{1+u^2|\nabla f|^2_g}}\right) \\ &= g^{ij} - \frac{u^2 f^i f^j}{1+u^2|\nabla f|^2_g} - \frac{Y^\phi(\delta^i_\phi f^j + f^i\delta^j_\phi)}{1+u^2|\nabla f|^2_g} + \frac{|\nabla f|^2(Y^\phi)^2\delta^i_\phi\delta^j_\phi}{1+u^2|\nabla f|^2_g}\end{aligned} \tag{5.1.5}$$

It is already computed in the previous chapter, section 2 (2.2.14) that the volume form of $(M, \overline{g})$ is as following by the direct computation.

$$dvol_{\overline{g}} = \sqrt{1+u^2|\nabla f|^2_g}\, dvol_g \tag{5.1.6}$$

We can also compute the volume form of (Σ, g) embedded in $(\mathbb{R}\times M, \widetilde{g})$ with respect to $\overline{g}$ in similar sense, especially since the lapse and shift formulations for Riemannian and Lorentzian settings are now identical. Therefore, the following holds.

$$dvol_g = \sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}\, dvol_{\overline{g}} \tag{5.1.7}$$

Therefore we get following identity by comparing the two volume forms, which will be very useful in the following sections.

$$\left(1+u^2|\nabla f|^2_g\right)\left(1-u^2|\overline{\nabla} f|^2_{\overline{g}}\right) = 1 \tag{5.1.8}$$

5.2 Second Fundamental Form

In this section, we will compute the Second Fundamental Form π of (Σ, g) with respect to $\overline{g}$ and also g. In addition we will show that $Tr_g(k-\pi) = 0$

with respect to g is a quasi-linear elliptic equation depending on the choice of u. Interestingly it depends on $\overline{Y}$ only if our choice of u depends on $\overline{Y}$.

First we will compute π with respect to $\overline{g}$.

The Christoffel symbols for $\widetilde{g} = -\phi dt^2 - 2Y_i dt dx^i + \overline{g}_{ij}$ are as follows : index i,j,k represents spatial component whereas index t represents t component. Barred quantities are with respect to $\overline{g}$ metric whereas unbarred quantities are with respect to g.

$$\begin{aligned} &\widetilde{g}^{tt} = -\frac{1}{u^2} \quad \widetilde{g}^{ti} = -\frac{\overline{Y}^i}{u^2} \quad \widetilde{g}^{ij} = \overline{g}^{ij} - \frac{\overline{Y}^i \overline{Y}^j}{u^2} \\ &u^2 = \phi + |\overline{Y}|^2_{\overline{g}} \quad \overline{k}_{ij} = \frac{1}{2u^2}(\overline{\nabla}_j Y_i + \overline{\nabla}_i Y_j) \end{aligned} \tag{5.2.1}$$

$$\begin{aligned} &\widetilde{\Gamma}^t_{tt} = 0 \qquad \widetilde{\Gamma}^k_{tt} = \frac{1}{2}\overline{g}^{kl}\phi_l = \frac{\phi^k}{2} \qquad \widetilde{\Gamma}^t_{ij} = \frac{1}{u}\overline{k}_{ij} \\ &\widetilde{\Gamma}^t_{it} = \frac{1}{2u^2}(\phi_{,i} + \overline{Y}^l (Y_{l,i} - Y_{i,l})) \\ &\widetilde{\Gamma}^k_{it} = -\frac{1}{2}\overline{g}^{kl}(Y_{l,i} - Y_{i,l}) + \frac{\overline{Y}^k}{2u^2}(\phi_{,i} + \overline{Y}^l (Y_{l,i} - Y_{i,l})) \\ &\widetilde{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} \end{aligned} \tag{5.2.2}$$

For (Σ, g) embedded in $(\mathbb{R} \times M, \widetilde{g})$, recall that

$$\begin{aligned} g_{ij} &= \overline{g}_{ij} - f_i Y_j - f_j Y_i - \phi f_i f_j \\ g^{ij} &= \overline{g}^{ij} - \frac{\overline{Y}^i \overline{Y}^i}{u^2} + \overline{w}^i \overline{w}^j \end{aligned} \tag{5.2.3}$$

where spatial component $\overline{w}$ of normal vector N of (Σ, g) is as follows.

$$\begin{aligned} N &= \frac{u\overline{\nabla} f + n}{\sqrt{1 - u^2 |\overline{\nabla} f|^2_{\overline{g}}}} = \overline{w} + \frac{\partial_t}{u\sqrt{1 - u^2 |\overline{\nabla} f|^2_{\overline{g}}}} \\ \overline{w}^i &= \frac{u^2 \bar{f}^i + \overline{Y}}{u\sqrt{1 - u^2 |\overline{\nabla} f|^2_{\overline{g}}}} \end{aligned} \tag{5.2.4}$$

We will compute the second fundamental form π of (Σ, g) as follows.

$$
\begin{aligned}
&\pi_{ij}\\
&= -\langle \widetilde{\nabla}_{X_i} X_j, N \rangle\\
&= -\langle \widetilde{\nabla}_{\partial_i + f_{,i}\partial_t} \partial_j + f_{,j}\partial_t, \frac{u\overline{\nabla} f}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} + \frac{n}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} \rangle\\
&= -(\widetilde{\Gamma}^k_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt})\langle \partial_k, \frac{u\overline{\nabla} f}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} \rangle\\
&- (f_{,ij} + \widetilde{\Gamma}^t_{ij} + f_{,i}\widetilde{\Gamma}^t_{jt} + f_{,j}\widetilde{\Gamma}^t_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^t_{tt})\langle \partial_t, N \rangle\\
&= \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} \Big(f_{,ij} + \widetilde{\Gamma}^t_{ij} + f_{,i}\widetilde{\Gamma}^t_{jt} + f_{,j}\widetilde{\Gamma}^t_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^t_{tt} - (\widetilde{\Gamma}^k_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt}) f_{,k} \Big)\\
&= \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} \Big(\overline{\nabla}_{ij} f + \frac{1}{u}\overline{k}_{ij} + f_i(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt} f_{,k}) + f_j(\widetilde{\Gamma}^t_{it} - \widetilde{\Gamma}^k_{it} f_{,k}) + f_{,i}f_{,j}(\widetilde{\Gamma}^t_{tt} - \widetilde{\Gamma}^k_{tt} f_{,k}) \Big)
\end{aligned}
\tag{5.2.5}
$$

Therefore we get π with respect to $\overline{g}$ by substituting $\widetilde{\Gamma}$ from (5.2.2).

$$
\begin{aligned}
\pi_{ij} = &\frac{u\overline{\nabla}_{ij} f + \overline{k}_{ij} + \frac{f_i \phi_j}{2u} + \frac{f_j \phi_i}{2u} - f_i f_j \frac{\overline{\nabla} f(\phi)}{2}}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\\
&+ \frac{f_i}{2}\overline{w}^l(Y_{l,j} - Y_{j,l}) + \frac{f_j}{2}\overline{w}^l(Y_{l,i} - Y_{i,l})
\end{aligned}
\tag{5.2.6}
$$

Second, we will compute π with respect to g. It is crucial to apply the difference between covariant derivatives. But as we can see below, the difference between Christoffel symbols with respect to $\overline{g}$ itself contains $\overline{k}$, which also has covariant derivative within $\overline{g}$. This is the main reason why computation can be complicated. To simplify the process, we will convert $\overline{k}$ with respect to g first. And then we will compute $\overline{\nabla}_{ij} f$ with regard to g. The difference between Christoffel symbols is shown as **Identity 1** in section 5.2. Here we will refer the result and compute π further.

Identity 1

$$
\begin{aligned}
\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} &= -\overline{w}^k \pi_{ij} + \frac{\overline{Y}^k}{}\overline{k}_{ij} + f_{,i} f_{,j} \widetilde{\Gamma}^k_{tt} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} \\
&= -\overline{w}^k \pi_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} + f_i f_j (\frac{\phi^k}{2}) \\
&+ f_i \left(-\frac{1}{2}\overline{g}^{kl}(Y_{l,j} - Y_{j,l}) + \frac{\overline{Y}^k}{2u^2}(\phi_j + \overline{Y}^l (Y_{l,j} - Y_{j,l}))\right) \\
&+ f_j \left(-\frac{1}{2}\overline{g}^{kl}(Y_{l,i} - Y_{i,l}) + \frac{\overline{Y}^k}{2u^2}(\phi_i + \overline{Y}^l (Y_{l,i} - Y_{i,l}))\right)
\end{aligned}
\tag{5.2.7}
$$

As mentioned, we will compute $\overline{k}$ with respect to g as follows :

$$
\begin{aligned}
&\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} \\
&= \frac{\overline{Y}^k}{2u^2}(\overline{\nabla}_j Y_i + \overline{\nabla}_i Y_j) - \overline{w}^k \pi_{ij} + f_{,i} f_{,j}\widetilde{\Gamma}^k_{tt} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} \\
&= \frac{\overline{Y}^k}{2u^2}(Y_{i;j} + Y_{j;i}) + \frac{\overline{Y}^k}{u^2}(\Gamma^l_{ij} - \overline{\Gamma}^l_{ij})Y_l - \overline{w}^k \pi_{ij} + f_{,i} f_{,j}\widetilde{\Gamma}^k_{tt} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it}
\end{aligned}
\tag{5.2.8}
$$

We will apply (5.2.8) to Y_k and solve for $(\Gamma^l_{ij} - \overline{\Gamma}^l_{ij})Y_l$:

$$
\begin{aligned}
&\frac{\phi}{u^2}(\Gamma^l_{ij} - \overline{\Gamma}^l_{ij})Y_l \\
&= \frac{|\overline{Y}|^2_{\overline{g}}}{2u^2}(Y_{i;j} + Y_{j;i}) - \pi_{ij}\overline{w}^l Y_l + f_{,i} f_{,j}\widetilde{\Gamma}^l_{tt} Y_l + f_{,i}\widetilde{\Gamma}^l_{jt} Y_l + f_{,j}\widetilde{\Gamma}^l_{it} Y_l
\end{aligned}
\tag{5.2.9}
$$

Now we will substitute (5.2.9) to the last line of (5.2.8).

$$
\begin{aligned}
&\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} \\
&= \frac{\overline{Y}^k}{2\phi}(Y_{i;j} + Y_{j;i}) - \pi_{ij}(\overline{w}^k + \frac{\overline{w}^l Y_l}{\phi}\overline{Y}^k) \\
&+ f_{,i}(\widetilde{\Gamma}^k_{jt} + \frac{\widetilde{\Gamma}^l_{jt} Y_l}{\phi}\overline{Y}^k) + f_{,j}(\widetilde{\Gamma}^k_{it} + \frac{\widetilde{\Gamma}^l_{it} Y_l}{\phi}\overline{Y}^k) + f_{,i} f_{,j}(\widetilde{\Gamma}^k_{tt} + \frac{\widetilde{\Gamma}^l_{tt} Y_l}{\phi}\overline{Y}^k)
\end{aligned}
\tag{5.2.10}
$$

(5.2.6) shows that π with respect to $\overline{g}$ contains $Hess_{\overline{g}} f$, $\overline{k}$. We will compute $f_{\overline{;}ij} + \frac{\overline{k}}{u}$ with respect to g by substituting (5.2.10). Note that $\overline{Y}^k f_k = f^k Y_k = 0$

with respect to $\overline{g}$.

$$\begin{aligned}
&f_{\overline{;}ij} + \frac{1}{u}\overline{k}_{ij} \\
&= f_{;ij} + \frac{1}{2u^2}(Y_{i;j} + Y_{j;i}) + (\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})(f_{,k} + \frac{1}{u^2}Y_k) \\
&= -\pi_{ij}(\overline{w}^l f_{,l} + \frac{\overline{w}^l Y_l}{\phi}) + f_{;ij} + \frac{1}{2\phi}(Y_{i;j} + Y_{j;i}) \\
&+ f_{,i}(\widetilde{\Gamma}^l_{jt} f_{,l} + \frac{\widetilde{\Gamma}^l_{jt} Y_l}{\phi}) + f_{,j}(\widetilde{\Gamma}^l_{it} f_{,l} + \frac{\widetilde{\Gamma}^l_{it} Y_l}{\phi}) + f_{,i} f_{,j}(\widetilde{\Gamma}^l_{tt} f_{,l} + \frac{\widetilde{\Gamma}^l_{tt} Y_l}{\phi})
\end{aligned} \tag{5.2.11}$$

Also, from (5.2.5),

$$\begin{aligned}
&f_{\overline{;}ij} + \frac{1}{u}\overline{k}_{ij} \\
&= \frac{\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\pi_{ij} - f_{,i}(\widetilde{\Gamma}^t_{it} - \widetilde{\Gamma}^l_{jt} f_{,l}) - f_{,j}(\widetilde{\Gamma}^t_{it} - \widetilde{\Gamma}^l_{jt} f_{,l}) + f_{,i} f_{,j}(\widetilde{\Gamma}^l_{tt} f_{,l})
\end{aligned} \tag{5.2.12}$$

We will compare (5.2.11) and (5.2.12). Solve for π to derive the desired result as follows.

$$\begin{aligned}
&\left(\frac{\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u} + \overline{w}^l f_{,l} + \frac{\overline{w}^l Y_l}{\phi}\right)\pi_{ij} \\
&= \frac{u}{\phi\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}\pi_{ij} \\
&= f_{;ij} + \frac{1}{2\phi}(Y_{i;j} + Y_{j;i}) + f_{,i}(\widetilde{\Gamma}^t_{jt} + \frac{\widetilde{\Gamma}^l_{jt} Y_l}{\phi}) + f_{,i}(\widetilde{\Gamma}^t_{it} + \frac{\widetilde{\Gamma}^l_{it} Y_l}{\phi}) \\
&= f_{;ij} + \frac{1}{2\phi}(Y_{i;j} + Y_{j;i}) + \frac{f_{,i}\phi_{,j}}{2\phi} + \frac{f_{,j}\phi_i}{2\phi}
\end{aligned} \tag{5.2.13}$$

Therefore

$$\pi_{ij} = \frac{1}{u\sqrt{1 + u^2|\nabla f|^2_g}}\left(\phi f_{;ij} + \frac{Y_{i;j} + Y_{j;i}}{2} + \frac{f_{,i}\phi_j}{2} + \frac{f_{,j}\phi_i}{2}\right) \tag{5.2.14}$$

But the coefficient for $f_{;ij}$ is not exact, since $Y_i \quad i \neq \phi$ depends on f under our assumption on Y, i.e. $\overline{Y} = Y^\phi \partial_\phi$. To get the correct formula, we need to substitute

$$Y_i = \overline{Y}^{\phi}\overline{g}_{\phi i} = \overline{Y}^{\phi}(g_{\phi i} + f_{,i}Y_{\phi}) = \overline{Y}^{\phi}g_{\phi i} + f_{,i}|\overline{Y}|^2_{\overline{g}} \tag{5.2.15}$$

Since we are on a global coordinate, Y^{ϕ} is actually a function on M which we can choose in general. Note that as shown in the previous sections, we chose Y^{ϕ} so that $div_{\overline{g}}(\overline{k})(\partial_{\phi}) = 0$.

$$Y_{i;j} + Y_{j;i} = 2|\overline{Y}|^2_{\overline{g}}f_{;ij} + f_{,i}(|\overline{Y}|^2_{\overline{g}})_{,j} + f_{,j}(|\overline{Y}|^2_{\overline{g}})_{,i} + Y^{\phi}_{,i}g_{\phi j} + Y^{\phi}_{,j}g_{\phi i} \tag{5.2.16}$$

We will substitute (5.2.16) to (5.2.14),

$$\pi_{ij} = \frac{u}{\sqrt{1 + u^2|\nabla f|^2_g}}\left(f_{;ij} + \frac{Y^{\phi}_{,i}g_{\phi j} + Y^{\phi}_{,j}g_{\phi i}}{2u^2} + \frac{f_{,i}u_{,j}}{u} + \frac{f_{,j}u_{,i}}{u}\right) \tag{5.2.17}$$

Therefore the $Tr_g(\pi - k) = 0$ is as follows.

Remark 5.2.1.

$$Tr_g(\pi - k) = \frac{u}{\sqrt{1 + u^2|\nabla f|^2_g}}\left(\Delta f + \frac{2\nabla f(u)}{u}\right) - Tr_g k \tag{5.2.18}$$

5.3 Derivation of the Scalar Curvature Formula : Identity 1-7

In this section, we will compute all **Identity** from previous sections and will verify **Identity 7**.

Identity 1

$$\begin{aligned}\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} &= -\overline{w}^k \pi_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} + f_i f_j (\frac{\phi^k}{2}) \\ &+ f_i \left(-\frac{1}{2}\overline{g}^{kl}(Y_{l\bar{;}j} - Y_{j\bar{;}l}) + \frac{\overline{Y}^k}{2u^2}(\phi_j + \overline{Y}^l (Y_{l\bar{;}j} - Y_{j\bar{;}l})) \right) \\ &+ f_j \left(-\frac{1}{2}\overline{g}^{kl}(Y_{l\bar{;}i} - Y_{i\bar{;}l}) + \frac{\overline{Y}^k}{2u^2}(\phi_i + \overline{Y}^l (Y_{l\bar{;}i} - Y_{\bar{;}l})) \right)\end{aligned} \tag{5.3.1}$$

Proof.

$$\begin{aligned}&\widetilde{\nabla}_{X_i} X_j \\ &= \nabla_{X_i} X_j - \langle \widetilde{\nabla}_{X_i} X_j, N \rangle N \\ &= \Gamma^k_{ij} X_k + \pi_{ij} N \\ &= \Gamma^k_{ij} X_k + \pi_{ij}(\overline{w}^k X_k) + \frac{\sqrt{1 - u^2 |\overline{\nabla} f|^2_{\overline{g}}}}{u} \pi_{ij} \partial_t\end{aligned} \tag{5.3.2}$$

where the last line is from (5.2.4).

$$\begin{aligned}\widetilde{\nabla}_{X_i} X_j &= \widetilde{\nabla}_{\partial_i + f_{,i}\partial_t} \partial_j + f_{,j}\partial_t \\ &= (\widetilde{\Gamma}^k_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt})\partial_k \\ &+ (f_{,ij} + \widetilde{\Gamma}^t_{ij} + f_{,i}\widetilde{\Gamma}^t_{jt} + f_{,j}\widetilde{\Gamma}^t_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^t_{tt})\partial_t \\ &= (\widetilde{\Gamma}^k_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt})X_k \\ &+ (f_{,ij} + \widetilde{\Gamma}^t_{ij} + f_{,i}\widetilde{\Gamma}^t_{jt} + f_{,j}\widetilde{\Gamma}^t_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^t_{tt})\partial_t \\ &- (\widetilde{\Gamma}^k_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt})f_{,k}\partial_t \\ &= (\widetilde{\Gamma}^k_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt})X_k + \frac{\sqrt{1 - u^2 |\overline{\nabla} f|^2_{\overline{g}}}}{u} \pi_{ij} \partial_t\end{aligned} \tag{5.3.3}$$

Last line of (5.3.3) is from π in (5.2.5). Therefore by comparing (5.3.2) and (5.3.3),

$$\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} = -\overline{w}^k \pi_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_i f_j \widetilde{\Gamma}^k_{tt} \tag{5.3.4}$$

By substituting $\widetilde{\Gamma}$ from (5.2.2) we get (5.3.1). □

Identity 2

$$\begin{aligned} div_g k(w) = \frac{1}{u} div_{\overline{g}}(uk(\overline{w},\cdot)) + \overline{w}(k(\overline{w},\overline{w})) \\ - \widetilde{g}\langle k,\pi\rangle - 2\widetilde{g}\langle k(\overline{w},\cdot),\pi(\overline{w},\cdot)\rangle + (Tr_{\widetilde{g}}\pi)k(\overline{w},\overline{w}) \end{aligned} \tag{5.3.5}$$

Proof.

$$\begin{aligned} & div_g(k)(w) \\ &= \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\overline{w}^l k_{il;j} \\ &= \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\overline{w}^l (k_{il\overline{;}j} - (\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})k_{kl} - (\Gamma^k_{jl} - \overline{\Gamma}^k_{jl})k_{ik}) \\ &= div_{\overline{g}}(k)(\overline{w}) - \frac{1}{u^2}(\overline{\nabla}_{\overline{Y}}k(\overline{Y},\overline{w})) + \overline{\nabla}_{\overline{w}}k(\overline{w},\overline{w}) \\ &- \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\overline{w}^l \left((\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})k_{kl} + (\Gamma^k_{jl} - \overline{\Gamma}^k_{jl})k_{ik}\right) \end{aligned} \tag{5.3.6}$$

We will explicitly compute the each terms in (5.3.6) as follows. We will compute covariant derivative of $\overline{w}$ and k in **Identity 2-1** to **Identity 2-4**. We will combine and simplify them in (5.3.21), (5.3.25), (5.3.26) as follows. The result will be in (5.3.27). Lastly, we will explicitly compute the last line of (5.3.6) in **Identity 2-5**. (5.3.27) and **Identity 2-5** will result in **Identity 2**.

Identity 2-1

$$\overline{w}_{\bar{i};j} = \pi_{ij} + (\pi(\overline{w}, \partial_j) - \pi\left(\frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}, \partial_j)\right)\overline{w}_i$$
$$- \frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}\left(A_{ij} - \frac{Y_{\bar{i};\bar{j}}}{u^2}\right) - \frac{2u_{,j}}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}Y_i \qquad (5.3.7)$$
$$+ \frac{1}{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}\left(\frac{u_{,i}}{u} - u^2 A(\overline{\nabla}f, \partial_j)\right)\overline{w}_i$$

where $A_{ij} = \frac{1}{u}\overline{k}_{ij} + f_{,i}(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt}f_{,k}) + f_{,j}(\widetilde{\Gamma}^t_{it} - \widetilde{\Gamma}^k_{it}f_{,k}) - f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt}f_{,k}$

Proof.

$$\overline{w}_{\bar{i};j} = \left(\frac{u\overline{\nabla}f + \frac{1}{u}\overline{Y}}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}\right)_{\bar{i};j}$$
$$= \frac{u\left(f_{\bar{;}ij} + \frac{1}{u^2}\overline{Y}_{\bar{i};j} + \partial_j(\frac{1}{u^2})\overline{Y}_i\right)}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}} + \partial_j\left(log\frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}\right)\overline{w}_i$$
$$= \frac{u(f_{\bar{;}ij} + \frac{1}{u^2}\overline{Y}_{\bar{i};j} + \partial_j(\frac{1}{u^2})\overline{Y}_i)}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}} + \frac{1}{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}\left(\frac{u_{,j}}{u} + u^2 f^l f_{\bar{;}lj}\right)\overline{w}_i \qquad (5.3.8)$$

Substitute $f_{\bar{;}ij} = \frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}{u}\pi_{ij} - A_{ij},$

$$f^l f_{\bar{;}lj} = \left(\frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}{u}\overline{w}^l - \frac{\overline{Y}^l}{u^2}\right)\left(\frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\bar{g}}}}{u}\pi_{ij}\right) - f^l A_{lj}$$

□

Identity 2-2

$$
\begin{aligned}
div_{\overline{g}}(k)(\overline{w}) &= div_{\overline{g}}k(\overline{w},\cdot)) - \overline{g}\langle k,\pi\rangle \\
&- \overline{g}\langle k(\overline{w},\cdot),\pi\left(\overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}},\cdot\right)\rangle \\
&+ \frac{uk^{ij}\left(A_{ij} - \frac{Y_{\overline{i};\overline{j}}}{u^2}\right)}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} + \frac{u^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}\overline{g}\rangle k(\overline{w},\cdot), A(\overline{\nabla} f,\cdot)\langle \\
&+ 2k\left(\overline{Y}, \frac{\overline{\nabla} u}{u^2\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right) - k\left(\overline{w}, \frac{\overline{\nabla} u}{u(1-u^2|\overline{\nabla} f|^2_{\overline{g}})}\right)
\end{aligned} \tag{5.3.9}
$$

Proof.

$$
div_{\overline{g}}(k)(\overline{w}) = div_{\overline{g}}(k(\overline{w},\cdot)) - k^{ij}\overline{w}_{\overline{i};\overline{j}}
$$

Substitute **Identity 2-1**. □

Identity 2-3

$$
\begin{aligned}
\overline{\nabla}_{\overline{Y}} k(\overline{Y},\overline{w}) &= -k(\overline{\nabla}_{\overline{Y}}\overline{Y},\overline{w}) - \overline{g}\langle k(\overline{Y},\cdot),\pi(\overline{Y},\cdot)\rangle \\
&+ \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\overline{g}\langle k(\overline{Y},\cdot), A(\overline{Y},\cdot)\rangle - \frac{k\left(\overline{Y},\overline{\nabla}_{\overline{Y}}\overline{Y}\right)}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}
\end{aligned} \tag{5.3.10}
$$

Proof.

$$
\begin{aligned}
\overline{\nabla}_{\overline{Y}} k(\overline{Y},\overline{w}) &= \overline{Y}(k(\overline{Y},\overline{w})) - k\left(\overline{\nabla}_{\overline{Y}}\overline{Y},\overline{w}\right) - k\left(\overline{\nabla}_{\overline{Y}}\overline{w},\overline{Y}\right) \\
&= -k\left(\overline{\nabla}_{\overline{Y}}\overline{Y},\overline{w}\right) - k\left(\overline{\nabla}_{\overline{Y}}\overline{w},\overline{Y}\right)
\end{aligned} \tag{5.3.11}
$$

Since $\overline{Y} = \overline{Y}^{\phi}\partial_\phi$, the following holds.

$$
\begin{aligned}
\left(\overline{\nabla}_{\overline{Y}}\overline{w}\right)_i = \overline{Y}^l\overline{w}_{\overline{i};l} = \overline{Y}^l\left(\frac{u\overline{\nabla} f + \frac{1}{u}\overline{Y}}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)_{\overline{i};l} &= \frac{u\overline{Y}^l f_{\overline{;}il} + \frac{1}{u}\overline{Y}^l\overline{Y}_{\overline{i};l}}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} \\
= \pi(\partial_i,\overline{Y}) - \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\left(A(\partial_i,\overline{Y}) - \frac{(\overline{\nabla}_{\overline{Y}}\overline{Y})_i}{u^2}\right)
\end{aligned} \tag{5.3.12}
$$

By substituting (5.3.12) to (5.3.11), we get (5.3.10). □

Identity 2-4

$$\begin{aligned}\overline{\nabla}_{\overline{w}}k(\overline{w},\overline{w}) &= \overline{w}(k(\overline{w},\overline{w})) - 2\overline{g}\langle k(\overline{w},\cdot),\pi(\overline{w},\cdot)\rangle - 2k(\overline{w},\overline{w})\pi(\overline{w},\overline{w}) \\ &- 2k(\overline{w},\overline{w})\left(\frac{(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}{2u^2}\overline{w}(\phi)\right) \\ &+ 4k(\overline{w},\overline{Y})\left(\frac{\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right) \\ &+ \frac{2u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\left(\overline{g}\langle k(\overline{w},\cdot),A(\overline{w},\cdot)\rangle - \frac{1}{u^2}k(\overline{w},\overline{\nabla}_{\overline{w}}\overline{Y})\right)\end{aligned} \tag{5.3.13}$$

Proof.

$$\overline{\nabla}_{\overline{w}}k(\overline{w},\overline{w}) = \overline{w}(k(\overline{w},\overline{w})) - 2k\left(\overline{\nabla}_{\overline{w}}\overline{w},\overline{w}\right) \tag{5.3.14}$$

By substituting **Identity 2-1**,

$$\begin{aligned}&\left(\overline{\nabla}_{\overline{w}}\overline{w}\right)_i = \overline{w}^l\overline{w}_{i\bar{;}l} \\ &= \pi(\partial_i,\overline{w}) + \pi(\overline{w},\overline{w})\overline{w}_i \\ &- \frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\left(A(\partial_i,\overline{w}) - \frac{\left(\overline{\nabla}_{\overline{w}}\overline{Y}\right)_i}{u^2}\right) - \frac{2\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}Y_i \\ &+ \left(\frac{1}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\left(\frac{\overline{w}(u)}{u} - u^2A(\overline{\nabla}f,\overline{w})\right) - \pi\left(\overline{w},\frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)\right)\overline{w}_i\end{aligned} \tag{5.3.15}$$

We will compute the last line of (5.3.15) by the direct computation using definition of $A, u, \overline{k}$ along with $\overline{Y}^l f_{\bar{;}il} = -f^l Y_{l\bar{;}i}$, $f^i f^j Y_{i\bar{;}j} = 0$ as follows.

$$\begin{aligned}
&\pi(\overline{w}, \overline{Y}) \\
&= \frac{u}{\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}} \left(\overline{w}^i \overline{Y}^j f_{\bar{;}ij} + \frac{\overline{k}(\overline{w}, \overline{Y})}{u} + (\overline{w}^i f_i) \frac{\overline{Y}^i f^j (Y_{j\bar{;}i} - Y_{i\bar{;}j})}{2} \right) \\
&= -\frac{\overline{Y}^i f^j Y_{j\bar{;}i}}{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}} + \frac{\overline{Y}^i f^j (Y_{i\bar{;}j} + Y_{j\bar{;}i})}{2(1 - u^2|\overline{\nabla} f|^2_{\overline{g}})} + \frac{u^2|\overline{\nabla} f|^2 \overline{Y}^i f^j (Y_{j\bar{;}i} - Y_{i\bar{;}j})}{2(1 - u^2|\overline{\nabla} f|^2_{\overline{g}})} \\
&= \frac{1}{2} \overline{Y}^i f^j (Y_{i\bar{;}j} - Y_{j\bar{;}i})
\end{aligned} \tag{5.3.16}$$

and

$$\frac{\overline{w}(u)}{u} = \frac{\overline{w}\left(\phi + |\overline{Y}|^2_{\overline{g}}\right)}{2u^2} \tag{5.3.17}$$

and

$$\begin{aligned}
&A(\overline{\nabla} f, \overline{w}) \\
&= \frac{\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u} A(\overline{w}, \overline{w}) - \frac{1}{u^2} A(\overline{Y}, \overline{w}) \\
&= \frac{\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u} \left(\frac{1}{u} \overline{k}(\overline{w}, \overline{w}) + (\overline{w}^i f_i) \frac{\overline{w}(\phi)}{u^2} - (\overline{w}^i f_i)^2 \frac{\overline{\nabla} f(\phi)}{2} \right) \\
&- \frac{1}{u^2} \left(\overline{k}\left(\frac{\overline{\nabla} f}{\sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}, \overline{Y}\right) + (\overline{w}^l f_l) \frac{1}{2} \overline{Y}^i f^j (Y_{j\bar{;}i} - Y_{i\bar{;}j}) \right) \\
&= \frac{\overline{w}(\phi)}{2} \left(\frac{2|\overline{\nabla} f|^2}{u^2} - |\overline{\nabla} f|^4 \right) + \frac{\overline{Y}^i f^j (Y_{i\bar{;}j} + Y_{j\bar{;}i})}{2u^3 \sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}} + \frac{|\overline{\nabla} f|^2 \overline{Y}^i f^j (Y_{i\bar{;}j} - Y_{j\bar{;}i})}{2u \sqrt{1 - u^2|\overline{\nabla} f|^2_{\overline{g}}}}
\end{aligned} \tag{5.3.18}$$

We will simplify the last line of (5.3.15) by substituting (5.3.16), (5.3.17), and (5.3.18) as follows.

$$\begin{aligned}&\frac{\pi(\overline{w},\overline{Y})}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}-\frac{\overline{w}(u)}{u(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}+\frac{u^2A(\overline{\nabla}f,\overline{w})}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\\&=-\frac{(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}{2u^2}\overline{w}(\phi)\end{aligned}\tag{5.3.19}$$

We get (5.3.13) by substituting (5.3.15) and (5.3.19) to (5.3.14). □

Now we will simplify $div_{\overline{g}}(k)(\overline{w})-\frac{1}{u^2}(\overline{\nabla}_{\overline{Y}}k(\overline{Y},\overline{w}))+\overline{\nabla}_{\overline{w}}k(\overline{w},\overline{w})$ in (5.3.6) by combining (5.3.9)(5.3.10) and (5.3.13).

$$\begin{aligned}&div_{\overline{g}}(k)(\overline{w})-\frac{1}{u^2}\left(\overline{\nabla}_{\overline{Y}}k(\overline{Y},\overline{w})\right)+\overline{\nabla}_{\overline{w}}k(\overline{w},\overline{w})\\&=div_{\overline{g}}(k(\overline{w},\cdot))+\overline{w}(k(\overline{w},\overline{w}))-3\overline{g}\langle k(\overline{w},\cdot),\pi(\overline{w},\cdot)\rangle-2k(\overline{w},\overline{w})\pi(\overline{w},\overline{w})\\&-\overline{g}\langle k,\pi\rangle+\frac{1}{u^2}\overline{g}\langle k(\overline{Y},\cdot),\pi(\overline{Y},\cdot)\rangle+\frac{1}{u^2}k\left(\overline{\nabla}_{\overline{Y}}\overline{Y},\overline{w}\right)\\&-2k(\overline{w},\overline{w})\left(\frac{(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}{2u^2}\overline{w}(\phi)\right)+4k(\overline{w},\overline{Y})\left(\frac{\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)\\&-\frac{2}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}k\left(\overline{w},\overline{\nabla}_{\overline{w}}\overline{Y}\right)+\frac{k\left(\overline{Y},\overline{\nabla}_{\overline{Y}}\overline{Y}\right)}{u^3\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\\&+2k\left(\overline{Y},\frac{\overline{\nabla}u}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)-k\left(\overline{w},\frac{\overline{\nabla}u}{u(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}\right)\\&+\frac{uk^{ij}\left(A_{ij}-\frac{Y_{\bar{i};j}}{u^2}\right)}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}+\frac{3u\overline{g}\langle k(\overline{w},\cdot),A(\overline{w},\cdot)\rangle}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\\&+\frac{\overline{g}\langle k(\overline{w},\cdot),\pi(\overline{Y},\cdot)\rangle}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}-\frac{\overline{g}\langle k(\overline{w},\cdot),A(\overline{Y},\cdot)\rangle}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}-\frac{\overline{g}\langle k(\overline{Y},\cdot),A(\overline{Y},\cdot)\rangle}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\end{aligned}\tag{5.3.20}$$

We will simplify each terms involving A in last three lines in (5.3.20). Evaluate

A by $\widetilde{\Gamma}$ with $\overline{\nabla} f = \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{w} - \frac{\overline{Y}}{u^2}$.

$$
\begin{aligned}
&\frac{uk^{ij}\left(A_{ij} - \frac{Y_{\bar{i};j}}{u^2}\right)}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} \\
&= 2k\left(\overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}, \overline{g}^{jl}(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt} f_k)\partial_l\right) \\
&- \left(\widetilde{\Gamma}^k_{tt} f_k\right) k\left(\overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}, \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{w} - \frac{\overline{Y}}{u^2}\right) \\
&= 2k\left(\overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}, \frac{\overline{\nabla}\phi}{2u^2} + \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} - Y_{\bar{j};i})\partial_l\right) \\
&- \frac{\overline{\nabla} f(\phi)}{2} k\left(\overline{w} - \frac{\overline{Y}}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}, \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{w} - \frac{\overline{Y}}{u^2}\right)
\end{aligned}
\tag{5.3.21}
$$

and

$$
\begin{aligned}
&\frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\overline{g}\langle k(\overline{w},\cdot), A(\overline{w},\cdot)\rangle \\
&= \frac{\overline{g}\langle k(\overline{w},\cdot), \overline{k}(\overline{w},\cdot)\rangle}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} + k\left(\overline{w}, \overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)\left(\overline{w}^j(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt} f_k)\right) \\
&+ \frac{u^2|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} k\left(\overline{w}, \overline{g}^{jl}(\widetilde{\Gamma}^t_{jt} - \widetilde{\Gamma}^k_{jt} f_k)\partial_l\right) - k\left(\overline{w}, \overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)(\widetilde{\Gamma}^k_{tt} f_k) \\
&= k\left(\overline{w}, \frac{\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} + Y_{\bar{j};i})\partial_l}{2u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right) + \frac{u^2|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} k\left(\overline{w}, \frac{\overline{\nabla}\phi}{2u^2} + \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} - Y_{\bar{j};i})\partial_l\right) \\
&+ k\left(\overline{w}, \overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)\frac{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})\overline{w}(\phi)}{2u^2}
\end{aligned}
\tag{5.3.22}
$$

and

$$
\begin{aligned}
&\overline{g}\langle k(\overline{w},\cdot),\pi\left(\frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}},\cdot\right)\rangle-\frac{1}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\overline{g}\langle k(\overline{w},\cdot),A(\overline{Y},\cdot)\rangle\\
&=-\frac{k\left(\overline{w},\overline{g}^{jl}f^iY_{\bar{i};\bar{j}}\partial_l\right)}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}
\end{aligned}
\tag{5.3.23}
$$

and lastly,

$$
\begin{aligned}
&\frac{1}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\overline{g}\langle k(\overline{Y},\cdot),A(\overline{Y},\cdot)\rangle\\
&=\frac{k\left(\overline{Y},\overline{g}^{jl}\overline{Y}^i(Y_{\bar{i};\bar{j}}+Y_{\bar{j};\bar{i}})\partial_l\right)}{2u^3\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}-k\left(\overline{Y},\frac{\overline{w}}{u^2}-\frac{\overline{Y}}{u^3\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)\pi(\overline{w},\overline{Y})
\end{aligned}
\tag{5.3.24}
$$

We will simplify (5.3.20) by (5.3.21), (5.3.22), (5.3.23), and (5.3.24). We will write $(u\overline{\nabla}u)^l = \left(\frac{\overline{\nabla}\phi}{2}\right)^l + \overline{g}^{jl}\overline{Y}^i Y_{\bar{i};\bar{j}}$.

$$
\begin{aligned}
&-2k(\overline{w},\overline{w})\left(\frac{(1-u^2|\overline{\nabla}f|_{\overline{g}}^2)}{2u^2}\overline{w}(\phi)\right) + 4k(\overline{w},\overline{Y})\left(\frac{\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}\right) \\
&-\frac{2}{u\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}k\left(\overline{w},\overline{\nabla}_{\overline{w}}\overline{Y}\right) + \frac{k\left(\overline{Y},\overline{\nabla}_{\overline{Y}}\overline{Y}\right)}{u^3\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} \\
&+2k\left(\overline{Y},\frac{\overline{\nabla}u}{u^2\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}\right) - k\left(\overline{w},\frac{\overline{\nabla}u}{u(1-u^2|\overline{\nabla}f|_{\overline{g}}^2)}\right) \\
&+\frac{uk^{ij}\left(A_{ij}-\frac{Y_{\bar{i};\bar{j}}}{u^2}\right)}{\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} + \frac{3u\overline{g}\langle k(\overline{w},\cdot),A(\overline{w},\cdot)\rangle}{\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} \\
&+\frac{\overline{g}\langle k(\overline{w},\cdot),\pi(\overline{Y},\cdot)\rangle}{u\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} - \frac{\overline{g}\langle k(\overline{w},\cdot),A(\overline{Y},\cdot)\rangle}{1-u^2|\overline{\nabla}f|_{\overline{g}}^2} - \frac{\overline{g}\langle k(\overline{Y},\cdot),A(\overline{Y},\cdot)\rangle}{u\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} \\
&= -k\left(\overline{Y},\frac{\overline{g}^{jl}f^iY_{\bar{i};\bar{j}}\partial_l}{u\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}\right) + k\left(\overline{Y},\frac{\overline{g}^{jl}\overline{Y}^i(Y_{\bar{i};\bar{j}}+Y_{\bar{j};\bar{i}})\partial_l}{2u^3\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}\right) + \frac{1}{u^2}k\left(\overline{Y},\overline{\nabla}_{\overline{w}}\overline{Y}\right) \\
&-k(\overline{Y},\overline{Y})\left(\frac{\overline{w}(\phi)}{2u^4} + \frac{\pi(\overline{w},\overline{Y})}{u^3\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}\right) \\
&+k(\overline{w},\overline{Y})\left(\frac{4\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}} - \frac{\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}\,\overline{w}(\phi)}{2u^3} + \frac{1}{u^2}\pi(\overline{w},\overline{Y})\right) \\
&+k\left(\overline{w},\frac{1}{2}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};\bar{j}}-Y_{\bar{j};\bar{i}})\partial_l\right)\left(\frac{3+u^2|\overline{\nabla}f|^2}{u\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}\right) \\
&+k\left(\overline{w},\frac{1}{2}\overline{\nabla}\phi\right)\left(\frac{1+u^2|\overline{\nabla}f|^2}{u^2(1-u^2|\overline{\nabla}f|_{\overline{g}}^2)}\right)
\end{aligned}
\tag{5.3.25}
$$

We will convert $f^i\overline{Y}_{\bar{i};\bar{j}} = -\overline{Y}^i f_{\bar{;}ij} = -\frac{\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}{u}\pi(\overline{Y},\partial_j)+A(\overline{Y},\partial_j) = -\frac{\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}}{u}\pi(\overline{Y},\partial_j)+\frac{\overline{Y}^i(Y_{\bar{i};\bar{j}}+Y_{\bar{j};\bar{i}})}{2u^2}-\pi(\overline{w},\overline{Y})(\frac{\overline{w}_j}{u^2}-\frac{\overline{Y}_j}{u^3\sqrt{1-u^2|\overline{\nabla}f|_{\overline{g}}^2}})$. Also note that $f^lY^jY_{\bar{i};\bar{j}} = -\overline{Y}^l\overline{Y}^j f_{\bar{;}jl} =$

$-\frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\pi(\overline{Y},\overline{Y})$. Then we will easily get the following.

$$
\begin{aligned}
& div_{\overline{g}}(k)(\overline{w}) - \frac{1}{u^2}\left(\overline{\nabla}_{\overline{Y}} k(\overline{Y},\overline{w})\right) + \overline{\nabla}_{\overline{w}} k(\overline{w},\overline{w}) \\
&= div_{\overline{g}}(k(\overline{w},\cdot)) + \overline{w}(k(\overline{w},\overline{w})) - 3\overline{g}\langle k(\overline{w},\cdot),\pi(\overline{w},\cdot)\rangle - 2k(\overline{w},\overline{w})\pi(\overline{w},\overline{w}) \\
&- \overline{g}\langle k,\pi\rangle + \frac{2}{u^2}\overline{g}\langle k(\overline{Y},\cdot),\pi(\overline{Y},\cdot)\rangle - \frac{1}{u^4}k(\overline{Y},\overline{Y})\pi(\overline{Y},\overline{Y}) \\
&+ \frac{1}{u^2}k\left(\overline{w},\overline{\nabla}_{\overline{Y}}\overline{Y}\right) + \frac{1}{u^2}k\left(\overline{Y},\overline{\nabla}_{\overline{w}}\overline{Y}\right) - k(\overline{Y},\overline{Y})\left(\frac{\overline{w}(u)}{u^3}\right) \\
&+ k(\overline{w},\overline{Y})\left(\frac{4\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} - \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u^3}\overline{w}(\phi) + \frac{2}{u^2}\pi(\overline{w},\overline{Y})\right) \\
&+ k\left(\overline{w},\frac{1}{2}\overline{w}^l(Y_{\bar{l};i} - Y_{\bar{i};l})\right)\left(\frac{3+u^2|\overline{\nabla} f|^2}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right) \\
&+ k\left(\overline{w},\frac{\overline{\nabla}\phi}{2})(\frac{1+u^2|\overline{\nabla} f|^2}{u^2(1-u^2|\overline{\nabla} f|^2_{\overline{g}})}\right)
\end{aligned}
\tag{5.3.26}
$$

From now on, we will compute the last line in (5.3.6) to complete **Identity 2**.

Identity 2-5

$$\left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\overline{w}^l\left((\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})k_{kl} + (\Gamma^k_{jl} - \overline{\Gamma}^k_{jl})k_{ik}\right)$$
$$= -(Tr_{\widetilde{g}}\pi)k(\overline{w},\overline{w}) - 2k(\overline{w},\overline{w})\pi(\overline{w},\overline{w}) - \overline{g}\langle k(\overline{w},\cdot),\pi(\overline{w},\cdot)\rangle$$
$$- \frac{1}{u^2}k\left(\overline{w},\frac{1}{2}\overline{\nabla}|\overline{Y}|^2_{\overline{g}}\right) + \frac{1}{u^2}k\left(\overline{w},\overline{\nabla}_{\overline{Y}}\overline{Y}\right) + \frac{1}{u^2}k\left(\overline{Y},\overline{\nabla}_{\overline{w}}\overline{Y}\right) - k(\overline{Y},\overline{Y})\left(\frac{\overline{w}(u)}{u^3}\right)$$
$$+ k(\overline{w},\overline{Y})\left(\frac{4\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}} - \frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{2u^3}\overline{w}(\phi)\right)$$
$$+ k\left(\overline{w},\frac{1}{2}\overline{w}^l(Y_{\overline{l};i} - Y_{\overline{i};l})\right)\left(\frac{3+u^2|\overline{\nabla}f|^2}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)$$
$$+ k\left(\overline{w},\frac{\overline{\nabla}\phi}{2}\right)\left(\frac{2|\overline{\nabla}f|^2}{(1-u^2|\overline{\nabla}f|^2_{\overline{g}})}\right) \tag{5.3.27}$$

Proof. Recall that $\Gamma^k_{ij} - \overline{\Gamma}^k_{ij} = -\overline{w}^k\pi_{ij} + \frac{\overline{Y}^k}{u}\overline{k}_{ij} + f_{,i}\widetilde{\Gamma}^k_{jt} + f_{,j}\widetilde{\Gamma}^k_{it} + f_{,i}f_{,j}\widetilde{\Gamma}^k_{tt}$.

$$\widetilde{g}^{ij}(\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})$$
$$= -\overline{w}^kTr_{\widetilde{g}}\pi + 2f^j\widetilde{\Gamma}^k_{jt} + |\overline{\nabla}f|^2\widetilde{\Gamma}^k_{tt}$$
$$\overline{w}^i\overline{w}^j(\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})$$
$$= -\overline{w}^k\pi(\overline{w},\overline{w}) + \frac{\overline{Y}^k}{u}\overline{k}(\overline{w},\overline{w}) + 2(\overline{w}^lf_l)\overline{w}^j\widetilde{\Gamma}^k_{jt} + (\overline{w}^lf_l)^2\widetilde{\Gamma}^k_{tt}$$
$$\overline{g}^{ij}\overline{w}^l(\Gamma^k_{jl} - \overline{\Gamma}^k_{jl})k_{ik} \tag{5.3.28}$$
$$= -\overline{g}\langle k(\overline{w},\cdot),\pi(\overline{w},\cdot)\rangle + \frac{1}{u}\overline{g}\langle k(\overline{Y},\cdot),\overline{k}(\overline{w},\cdot)\rangle$$
$$+ (\overline{w}^lf_l)\overline{g}^{ij}\widetilde{\Gamma}^k_{jt}k_{ik} + (\overline{w}^lf_l)k(\overline{\nabla}f,\widetilde{\Gamma}^k_{tt}\partial_k) + k(\overline{\nabla}f,\overline{w}^l\widetilde{\Gamma}^k_{lt}\partial_k)$$
$$\overline{Y}^i\overline{Y}^j\overline{w}^l(\Gamma^k_{jl} - \overline{\Gamma}^k_{jl})k_{ik}$$
$$= -k(\overline{w},\overline{Y})\pi(\overline{w},\overline{Y}) + \frac{1}{u}k(\overline{Y},\overline{Y})\overline{k}(\overline{w},\overline{Y}) + (\overline{w}^lf_l)k(\overline{Y},\overline{Y}^l\widetilde{\Gamma}^k_{lt}\partial_k)$$

Evaluate (5.3.28) by substituting $\widetilde{\Gamma}$ It is useful to notice that $\widetilde{\Gamma}^k_{it} = \widetilde{\Gamma}^t_{it}\overline{Y}^k - \frac{1}{2}\overline{g}^{kl}(Y_{\overline{l};i} - Y_{\overline{l};i})$. Also convert $\overline{\nabla} f = \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{w} - \frac{1}{u^2}$ as before. Simple computation leads (5.3.27). □

In conclusion, we get **Identity 2** by combining **Identity 2-1** to **Identity 2-5**, i.e. subtract **Identity 2-5** from (5.3.26) as follows.

$$
\begin{aligned}
& div_g(k)(w) \\
& = div_{\overline{g}}(k(\overline{w}, \cdot)) + k\left(\overline{w}, \frac{\overline{\nabla} u}{u}\right) + \overline{w}(k(\overline{w}, \overline{w})) + (Tr_{\widetilde{g}}\pi)k(\overline{w}, \overline{w}) \\
& - \overline{g}\langle k, \pi\rangle + \frac{2}{u^2}\overline{g}\langle k(\overline{Y}, \cdot), \pi(\overline{Y}, \cdot)\rangle - \frac{1}{u^4}k(\overline{Y}, \overline{Y})\pi(\overline{Y}, \overline{Y}) \\
& - 2\overline{g}\langle k(\overline{w}, \cdot), \pi(\overline{w}, \cdot)\rangle + \frac{2}{u^2}k(\overline{w}, \overline{Y})\pi(\overline{w}, \overline{Y}) \\
& = \frac{1}{u}div_{\overline{g}}(uk(\overline{w}, \cdot)) + \overline{w}(k(\overline{w}, \overline{w})) \\
& - \widetilde{g}\langle k, \pi\rangle - 2\widetilde{g}\langle k(\overline{w}, \cdot), \pi(\overline{w}, \cdot)\rangle + (Tr_{\widetilde{g}}\pi)k(\overline{w}, \overline{w})
\end{aligned}
\tag{5.3.29}
$$

□

Identity 3

$$
\begin{aligned}
& div_g k(\partial_\phi) \\
& = div_{\overline{g}}(k(\partial_\phi, \cdot)) + \frac{1}{u}div_{\overline{g}}(uk(\partial\phi, \overline{w})\overline{w}) \\
& - g\langle k(\partial_\phi, \cdot), \pi(v, \cdot)\rangle + \overline{g}\langle k(\partial_\phi, \cdot), \overline{k}(u\overline{\nabla} f, \cdot)\rangle - \frac{\overline{\nabla} f(u)}{u}k(\partial_\phi, \overline{Y})
\end{aligned}
\tag{5.3.30}
$$

Proof.

$$
\begin{aligned}
& div_g(k)(\partial_\phi) \\
& = g^{ij} k_{i\phi;j} \\
& = g^{ij}(\partial_j(k_{i\phi}) - \Gamma^k_{ij} k_{k\phi} - \Gamma^k_{j\phi} k_{ki}) \\
& \text{where} \quad g^{ij}\Gamma^k_{j\phi}k_{ki} = \frac{1}{2}k^{lj}(g_{l\phi,j} - g_{j\phi,l}) = 0 \\
& = \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\left(\overline{\nabla}_{\partial_j}(k(\partial_\phi,\cdot))(\partial_i) - (\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})k_{k\phi}\right) \\
& = div_{\overline{g}}(k(\partial_\phi,\cdot)) + \frac{1}{u^2}\left(k\left(\overline{\nabla}_{\overline{Y}}\overline{Y},\partial_\phi\right)\right) + \overline{\nabla}_{\overline{w}}(k(\partial_\phi,\cdot))(\overline{w}) \\
& - \left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\left((\Gamma^k_{ij} - \overline{\Gamma}^k_{ij})k_{k\phi}\right)
\end{aligned}
\tag{5.3.31}
$$

Identity 3-1

$$
\begin{aligned}
& \overline{\nabla}_{\overline{w}}(k(\partial_\phi,\cdot))(\overline{w}) \\
& = \overline{w}(k(\overline{w},\partial_\phi)) - g\langle k(\partial_\phi,\cdot),\pi(v,\cdot)\rangle + \overline{g}\langle k(\partial_\phi,\cdot),\overline{k}(u\overline{\nabla}f,\cdot)\rangle \\
& - \frac{\overline{\nabla}f(u)}{u}k(\partial_\phi,\overline{Y}) - \frac{k\left(\partial_\phi,\overline{\nabla}_{\overline{w}}\overline{Y}\right)}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}} \\
& + 2k(\partial_\phi,\overline{Y})\left(\frac{\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right) + \frac{|\overline{\nabla}f|^2 k\left(\partial_\phi,u\overline{\nabla}u\right)}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}} + \frac{k\left(\partial_\phi,f^lY_{\overline{l;i}}\right)}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}
\end{aligned}
\tag{5.3.32}
$$

Proof.

$$
\overline{\nabla}_{\overline{w}}(k(\partial_\phi,\cdot))(\overline{w}) = \overline{w}(k(\overline{w},\partial_\phi)) - k\left(\overline{\nabla}_{\overline{w}}\overline{w},\partial_\phi\right) \tag{5.3.33}
$$

where $(\overline{\nabla}_{\overline{w}}\overline{w})_i = \overline{w}^l \overline{w}_{\bar{i};l}$. Recall that the following holds by (5.3.19).

$$\begin{aligned}
&\frac{1}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\pi(\overline{w},\overline{Y}) - \frac{\overline{w}(u)}{u(1-u^2|\overline{\nabla} f|^2_{\overline{g}})} + \frac{u^2 A(\overline{\nabla} f,\overline{w})}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} \\
&= -\frac{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})}{2u^2}\overline{w}(\phi)
\end{aligned} \tag{5.3.34}$$

Substitute (5.3.34) and (5.3.15) to evaluate $\overline{w}^l \overline{w}_{\bar{i};l}$ as in **Identity 2-4**.

$$\begin{aligned}
&\overline{\nabla}_{\overline{w}}(k(\partial_\phi,\cdot))(\overline{w}) \\
&= \overline{w}(k(\overline{w},\partial_\phi)) - \overline{g}\langle k(\partial_\phi,\cdot),\pi(\overline{w},\cdot)\rangle - k(\partial_\phi,\overline{w})\pi(\overline{w},\overline{w}) \\
&- k(\overline{w},\partial_\phi)(\frac{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})}{2u^2}\overline{w}(\phi)) \\
&+ 2k(\partial_\phi,\overline{Y})\left(\frac{\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right) \\
&+ \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}(\overline{g}\langle k(\partial_\phi,\cdot),A(\overline{w},\cdot)\rangle - \frac{1}{u^2}k(\partial_\phi,\overline{\nabla}_{\overline{w}}\overline{Y}))
\end{aligned} \tag{5.3.35}$$

We will evaluate (5.3.35). Recall that from (5.3.21), the following holds.

$$
\begin{aligned}
&\frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\overline{g}\langle k(\partial_\phi,\cdot), A(\overline{w},\cdot)\rangle \\
&= k\left(\partial_\phi, \frac{\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j}+Y_{\bar{j};i})\partial_l}{2u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right) \\
&+ \frac{u^2|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} k\left(\partial_\phi, \frac{\overline{\nabla}\phi}{2u^2} + \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j}-Y_{\bar{j};i})\partial_l\right) \\
&+ k\left(\partial_\phi, \overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)\frac{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})\overline{w}(\phi)}{2u^2}
\end{aligned}
\tag{5.3.36}
$$

substitute $\overline{w}^i(Y_{\bar{i};j}-Y_{\bar{j};i}) = -\overline{w}^i(Y_{\bar{i};j}+Y_{\bar{j};i}) + 2\overline{w}^i Y_{\bar{i};j}$

$$
\begin{aligned}
&= \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u} k\left(\partial_\phi, \bar{g}^{jl}\overline{w}^i(Y_{\bar{i};j}+Y_{\bar{j};i})\partial_l\right) \\
&+ \frac{|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} k(\partial_\phi, u\overline{\nabla} u) + \frac{u^2|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} k\left(\partial_\phi, \overline{g}^{jl} f^i Y_{\bar{i};j}\partial_l\right) \\
&+ k\left(\partial_\phi, \overline{w} - \frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)\frac{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})\overline{w}(\phi)}{2u^2}
\end{aligned}
$$

In addition we will use the following identity. Recall that $\overline{v} = \overline{w} - \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{Y}$, $\pi(\overline{w},\overline{Y}) = \frac{1}{2}\overline{Y}^i f^j(\overline{Y}_{\bar{i};j} - \overline{Y}_{\bar{j};i})$ from previous sections. We will also substitute

$\frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\pi(\overline{Y},\partial_j) = -f^l\overline{Y}_{\overline{l};j} + \frac{\overline{k}(\overline{Y},\partial_j)}{u} - f_j\pi(\overline{w},\overline{Y})$.

$$\begin{aligned}
&\overline{g}\langle k(\partial_\phi,\cdot),\pi(\overline{w},\cdot)\rangle + k(\partial_\phi,\overline{w})\pi(\overline{w},\overline{w})\\
&= g\langle k(\partial_\phi,\cdot),\pi(v,\cdot)\rangle + \frac{1}{u^2}k(\partial_\phi,\overline{Y})\pi(\overline{v},\overline{Y})\\
&+ \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}(k(\partial_\phi,\overline{w})\pi(\overline{w},\overline{Y}) + \overline{g}\langle k(\partial_\phi,\cdot),\pi(\overline{Y},\cdot)\rangle)\\
&\text{substitute}\pi(\overline{Y},\partial_j)\\
&= g\langle k(\partial_\phi,\cdot),\pi(v,\cdot)\rangle + \frac{1}{u}\overline{g}\langle k(\partial_\phi,\cdot),\overline{k}(\overline{Y},\cdot)\rangle\\
&- k\left(\partial_\phi,\overline{g}^{jl}f^iY_{\overline{i};j}\partial_l\right) + \frac{1}{u^2}k(\partial_\phi,\overline{Y})\left(\frac{1}{2}\overline{\nabla} f(|\overline{Y}|^2_{\overline{g}})\right)
\end{aligned} \tag{5.3.37}$$

We will simplify (5.3.35) by substituting (5.3.36) and (5.3.37).

$$\begin{aligned}
&- \overline{g}\langle k(\partial_\phi,\cdot),\pi(\overline{w},\cdot)\rangle - k(\partial_\phi,\overline{w})\pi(\overline{w},\overline{w})\\
&- k(\overline{w},\partial_\phi)\left(\frac{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})}{2u^2}\overline{w}(\phi)\right)\\
&+ \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\overline{g}\langle k(\partial_\phi,\cdot),A(\overline{w},\cdot)\rangle\\
&= -g\langle k(\partial_\phi,\cdot),\pi(v,\cdot)\rangle - \frac{1}{u}\overline{g}\langle k(\partial_\phi,\cdot),\overline{k}(\overline{Y},\cdot)\rangle\\
&+ \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}k\left(\partial_\phi,\overline{g}^{jl}\overline{w}^i(Y_{\overline{i};j}+Y_{\overline{j};i})\partial_l\right)\\
&- \frac{\overline{\nabla} f(u)}{u}k(\partial_\phi,\overline{Y}) + \frac{|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}k(\partial_\phi,u\overline{\nabla} u) + \frac{k\left(\partial_\phi,\overline{g}^{jl}f^iY_{\overline{i};j}\partial_l\right)}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}
\end{aligned} \tag{5.3.38}$$

where $\frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{w} = \overline{\nabla} f + \frac{\overline{Y}}{u^2}$. Therefore with (5.3.35) and (5.3.38), we get (5.3.32). □

Identity 3-2

$$\left(\overline{g}^{ij} - \frac{\overline{Y}^i\overline{Y}^j}{u^2} + \overline{w}^i\overline{w}^j\right)\left(\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k\right)k_{k\phi}$$
$$= -k(\overline{w}, \partial_\phi)(Tr_{\widetilde{g}}\pi + \pi(\overline{w}, \overline{w})) + \frac{1}{u^2}k\left(\partial_\phi, \overline{\nabla}_{\overline{Y}}\overline{Y}\right) - \frac{1}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}k(\partial_\phi, \overline{\nabla}_{\overline{w}}\overline{Y})$$
$$+ 2k(\partial_\phi, \overline{Y})\left(\frac{\overline{w}(u)}{u^2\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right) + k(\partial_\phi, u\overline{\nabla}u)\frac{|\overline{\nabla}f|^2}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}} + \frac{k\left(\partial_\phi, \overline{g}^{ij}f^lY_{\overline{l};i}\partial_j\right)}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}} \tag{5.3.39}$$

Proof. We do the same as **Identity 2-5**. Recall that $\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k = -\overline{w}^k\pi_{ij} + \frac{\overline{Y}^k}{\overline{u}}\overline{k}_{ij} + f_{,i}\widetilde{\Gamma}_{jt}^k + f_{,j}\widetilde{\Gamma}_{it}^k + f_{,i}f_{,j}\widetilde{\Gamma}_{tt}^k$.

$$\widetilde{g}^{ij}(\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k) = -\overline{w}^kTr_{\widetilde{g}}\pi + 2f^j\widetilde{\Gamma}_{jt}^k + |\overline{\nabla}f|^2\widetilde{\Gamma}_{tt}^k$$
$$\overline{w}^i\overline{w}^j(\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k) = -\overline{w}^k\pi(\overline{w}, \overline{w}) + \frac{\overline{Y}^k}{u}\overline{k}(\overline{w}, \overline{w}) + 2(\overline{w}^lf_{,l})\overline{w}^j\widetilde{\Gamma}_{jt}^k + (\overline{w}^lf_{,l})^2\widetilde{\Gamma}_{tt}^k \tag{5.3.40}$$

Evaluate above by directly substituting $\widetilde{\Gamma}$. It is useful to notice that $\widetilde{\Gamma}_{it}^k = \widetilde{\Gamma}_{it}^t\overline{Y}^k - \frac{1}{2}\overline{g}^{kl}(Y_{l;i} - Y_{\overline{l};i})$. □

Therefore, if we combine **Identity 3-1** and **Identity 3-2**, we get the following.

$$div_gk(\partial_\phi)$$
$$= div_{\overline{g}}(k(\partial_\phi, \cdot)) + \overline{w}(k(\overline{w}, \partial_\phi)) + k(\overline{w}, \partial_\phi)(Tr_{\widetilde{g}}\pi + \pi(\overline{w}, \overline{w})) \tag{5.3.41}$$
$$- g\langle k(\partial_\phi, \cdot), \pi(v, \cdot)\rangle + \overline{g}\langle k(\partial_\phi, \cdot), \overline{k}(u\overline{\nabla}f, \cdot)\rangle - \frac{\overline{\nabla}f(u)}{u}k(\partial_\phi, \overline{Y})$$

In addition we will use the following important identity to get .

Identity 3-3

$$\overline{g}^{ij}\overline{w}_{\bar{i};\bar{j}} = Tr_{\widetilde{g}}\pi + \pi(\overline{w},\overline{w}) - \frac{\overline{w}(u)}{u} \tag{5.3.42}$$

Proof. By **Identity 2-1**,

$$\begin{aligned}
&\overline{g}^{ij}\overline{w}_{\bar{i};\bar{j}} \\
&= Tr_{\overline{g}}\pi + \pi(\overline{w},\overline{w}) - \pi\left(\frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}},\overline{w}\right) + \frac{\overline{w}(u)}{u(1-u^2|\overline{\nabla} f|^2_{\overline{g}})} \\
&- \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\overline{g}^{ij}\left(A_{ij} - \frac{Y_{\bar{i};\bar{j}}}{u^2}\right) - \frac{u^2 A(\overline{\nabla} f,\overline{w})}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} \\
&= Tr_{\overline{g}}\pi + \pi(\overline{w},\overline{w}) - \pi(\overline{Y},\overline{w})\left(\frac{u|\overline{\nabla} f|^2}{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})^{\frac{3}{2}}} + \frac{3}{u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right) - \frac{u\overline{k}(\overline{w},\overline{\nabla} f)}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}} \\
&+ \frac{\overline{w}(u)}{u(1-u^2|\overline{\nabla} f|^2_{\overline{g}})} - \frac{\overline{w}(\phi)}{u^2(1-u^2|\overline{\nabla} f|^2_{\overline{g}})} + \frac{|\overline{\nabla} f|^2\overline{w}(\phi)}{2(1-u^2|\overline{\nabla} f|^2_{\overline{g}})} \\
&= Tr_{\overline{g}}\pi + \pi(\overline{w},\overline{w}) - \frac{1}{u^2}\pi(\overline{Y},\overline{Y}) - \frac{\overline{w}(u)}{u} \\
&= Tr_{\widetilde{g}}\pi + \pi(\overline{w},\overline{w}) - \frac{\overline{w}(u)}{u}
\end{aligned} \tag{5.3.43}$$

The last line holds since $\pi(\overline{Y},\overline{Y}) = -\frac{uf^l\overline{Y}^j Y_{\bar{l};\bar{j}}}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}$. □

In sum, by (5.3.42), the first line of (5.3.41) will be replaced by the following

:

$$\begin{aligned}
&\overline{w}(k(\overline{w},\partial_\phi)) + k(\overline{w},\partial_\phi)(Tr_{\widetilde{g}}\pi + \pi(\overline{w},\overline{w})) \\
&= \overline{w}(k(\overline{w},\partial_\phi)) + k(\overline{w},\partial_\phi)(\overline{g}^{ij}\overline{w}_{\bar{i};\bar{j}} + \frac{\overline{w}(u)}{u}) = \frac{1}{u}div_{\overline{g}}(uk(\partial\phi,\overline{w})\overline{w})
\end{aligned} \tag{5.3.44}$$

(5.3.42) and (5.3.44) result in **Identity 3**. □

Identity 4

$$div_{\overline{g}}(\pi(\partial_\phi,\cdot))$$
$$= \frac{1}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}} div_{\overline{g}}(uHess_{\overline{g}}f(\partial_{\phi,}.)) - \frac{1}{u} div_{\overline{g}}(u\pi(\partial\phi,\overline{w})\overline{w}) \tag{5.3.45}$$
$$+ g\langle \pi(\partial_\phi,\cdot),\pi(v,\cdot)\rangle - \overline{g}\langle \pi(\partial_\phi,\cdot),\overline{k}(u\overline{\nabla}f,\cdot)\rangle + \frac{\overline{\nabla}f(u)}{u}\pi(\partial_\phi,\overline{Y})$$

Proof. Recall that

$$\pi(\partial_\phi,\cdot) = \frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\left(Hess_{\overline{g}}f(\partial_\phi,\cdot) + \frac{1}{u}\overline{k}(\partial_\phi,\cdot) + df\cdot\frac{1}{2}f^l(Y_{l\bar{;}\phi} - Y_{\phi\bar{;}l})\right) \tag{5.3.46}$$

where $\pi(\overline{w},\partial_\phi) = -\frac{1}{2}f^l(Y_{l\bar{;}\phi} - Y_{\phi\bar{;}l})$.

The following is the necessary condition for the existence of twist potential for $(M,\overline{g},\overline{k})$. We will assume that (5.3.47) holds throughout all computation in this section.

$$(div_{\overline{g}}\overline{k})(\partial_\phi) = 0 \tag{5.3.47}$$

We will directly compute $div_{\overline{g}}\pi$ from (5.3.46).

$$div_{\overline{g}}(\pi(\partial_\phi,\cdot))$$
$$= \frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}(div_{\overline{g}}(Hess_{\overline{g}}f(\partial_\phi,\cdot)) - \overline{k}\left(\partial_\phi,\frac{\overline{\nabla}u}{u^2}\right) - \overline{\nabla}f(\pi(\overline{w},\partial_\phi)) - \overline{g}^{ij}f_{\bar{;}ij}\pi(\overline{w},\partial_\phi))$$
$$+\pi\left(\partial_\phi,\overline{\nabla}log\left(\frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)\right) \tag{5.3.48}$$

Identity 4-1

$$\pi\left(\partial_\phi, \overline{\nabla}\log\left(\frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)\right)$$
$$= g\langle \pi(\partial_\phi,\cdot), \pi(v,\cdot)\rangle - \overline{g}\langle \pi(\partial_\phi,\cdot), \overline{k}(u\overline{\nabla}f,\cdot)\rangle + \pi\left(\partial_\phi, \frac{\overline{\nabla}u}{u}\right)$$
$$- \pi(\partial_\phi,\overline{w})\pi(\overline{w},\overline{w}) - \frac{\overline{\nabla}f(\phi)\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{2u}\pi(\partial_\phi,\overline{w}) + \frac{\overline{\nabla}f(u)}{u}\pi(\partial_\phi,\overline{Y}) \tag{5.3.49}$$

Proof.

$$\pi(\partial_\phi, \overline{\nabla}\log(\frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}))$$
$$= \pi(\partial_\phi, \frac{\overline{\nabla}u}{u}) + \frac{|\overline{\nabla}f|^2}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\pi(\partial_\phi, u\overline{\nabla}u) + \frac{u^2}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\pi(\partial_\phi, \frac{\overline{\nabla}|\overline{\nabla}f|^2}{2})$$
$$= \frac{\pi(\partial_\phi,\overline{\nabla}u)}{u(1-u^2|\overline{\nabla}f|^2_{\overline{g}})} + \overline{g}\langle\pi(\frac{u\overline{\nabla}f}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}},\cdot), \pi(\partial_\phi,\cdot)\rangle$$
$$- \frac{u^2}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\overline{g}\langle A(\overline{\nabla}f,\cdot), \pi(\partial_\phi,\cdot)\rangle \tag{5.3.50}$$

First we will evaluate the last term in (5.3.50). We will follow computations in (5.3.21) for computing $A(\overline{\nabla}f,\cdot)$. If we convert k into $\pi(\partial_\phi,\cdot)$ in (5.3.21), we get as follows.

$$
\begin{aligned}
&\frac{u^2}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\overline{g}\langle A(\overline{\nabla}f,\cdot),\pi(\partial_\phi,\cdot)\rangle \\
&=\frac{u}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\overline{g}\langle A(\overline{w},\cdot),\pi(\partial_\phi,\cdot)\rangle-\frac{\overline{g}\langle A(\overline{Y},\cdot),\pi(\partial_\phi,\cdot)\rangle}{(1-u^2|\overline{\nabla}f|^2_{\overline{g}})} \\
&=\pi\left(\partial_\phi,\frac{\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j}+Y_{\bar{j};i})\partial_l}{2u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)+\frac{u^2|\overline{\nabla}f|^2}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\pi\left(\partial_\phi,\frac{\overline{\nabla}\phi}{2u^2}+\frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{2u}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j}-Y_{\bar{j};i})\partial_l\right) \\
&+\pi\left(\partial_\phi,\overline{w}-\frac{\overline{Y}}{u\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}\right)\frac{(1-u^2|\overline{\nabla}f|^2_{\overline{g}})\overline{w}(\phi)}{2u^2}-\frac{\overline{g}\langle A(\overline{Y},\cdot),\pi(\partial_\phi,\cdot)\rangle}{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}
\end{aligned}
\tag{5.3.51}
$$

Second, we will evaluate $\overline{g}\langle\pi\left(\frac{u\overline{\nabla}f}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}},\cdot\right),\pi(\partial_\phi,\cdot)\rangle$ in (5.3.50) for the simplicity of further computation.

$$
\begin{aligned}
&\overline{g}\langle\pi\left(\frac{u\overline{\nabla}f}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}},\cdot\right),\pi(\partial_\phi,\cdot)\rangle \\
&=\overline{g}\langle\pi\left(\overline{v}-\frac{u|\overline{\nabla}f|^2\overline{Y}}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}},\cdot\right),\pi(\partial_\phi,\cdot)\rangle \\
&=g\langle\pi(v,\cdot),\pi(\partial_\phi,\cdot)\rangle+\frac{1}{u^2}\pi(\partial_\phi,\overline{Y})\pi(\overline{v},\overline{Y}) \\
&-\pi(\partial_\phi,\overline{w})\pi\left(\overline{w}-\frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}\overline{Y}}{u},\overline{w}\right)-\overline{g}\langle\pi\left(\frac{u|\overline{\nabla}f|^2\overline{Y}}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}},\cdot\right),\pi(\partial_\phi,\cdot)\rangle
\end{aligned}
\tag{5.3.52}
$$

Substitute (5.3.51), (5.3.52) to (5.3.50). We will substitute $\pi\left(\frac{u|\overline{\nabla}f|^2\overline{Y}}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}},\partial_i\right)=$

$\frac{u^2|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}(-f^l Y_{\bar{l};i} + A(\overline{Y}, \partial_i))$ to simplify (5.3.50) further as follows.

$$\pi\left(\partial_\phi, \overline{\nabla}\log\left(\frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)\right)$$
$$= g\langle \pi(v,\cdot), \pi(\partial_\phi,\cdot)\rangle - \pi(\partial_\phi,\overline{w})\pi(\overline{w},\overline{w}) - \pi(\partial_\phi,\overline{w})\frac{(1-u^2|\overline{\nabla} f|^2_{\overline{g}})\overline{w}(\phi)}{2u^2}$$
$$+ \overline{g}\langle A(\overline{Y},\cdot), \pi(\partial_\phi,\cdot)\rangle - \pi\left(\partial_\phi, \frac{\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} + Y_{\bar{j};i})\partial_l}{2u\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\right)$$
$$- \frac{u^2|\overline{\nabla} f|^2}{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}\pi\left(\partial_\phi, \frac{\overline{\nabla}\phi}{2u^2} + \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} - Y_{\bar{j};i})\partial_l - \overline{g}^{jl}f^iY_{\bar{i};j}\partial_l\right)$$
$$+ \frac{\pi(\partial_\phi,\overline{\nabla} u)}{u(1-u^2|\overline{\nabla} f|^2_{\overline{g}})} + \pi(\partial_\phi,\overline{Y})\left(\frac{\overline{\nabla} f(\phi)}{2u^2} + \frac{\pi(\overline{v},\overline{Y})}{u^2}\right) + \pi(\partial_\phi,\overline{w})\pi\left(\frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}\overline{Y}}{u}, \overline{w}\right)$$
$$(5.3.53)$$

Where we will use the following to simplify (5.3.53).

$$\frac{\overline{\nabla}\phi}{2u^2} + \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} - Y_{\bar{j};i})\partial_l - \overline{g}^{jl}f^iY_{\bar{i};j}\partial_l$$
$$= \frac{\overline{\nabla} u}{u} - \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}\overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} + Y_{\bar{j};i})\partial_l \tag{5.3.54}$$

and

$$\overline{g}\langle A(\overline{Y},\cdot), \pi(\partial_\phi,\cdot)\rangle$$
$$= \pi\left(\partial_\phi, \frac{\overline{g}^{jl}\overline{Y}^i(Y_{\bar{i};j} + Y_{\bar{j};i}\partial_l)}{2u^2}\right) - \pi\left(\partial_\phi, \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{w} - \frac{\overline{Y}}{u^2}\right)\pi(\overline{w},\overline{Y})$$
$$= -\overline{g}\langle \pi(\partial_\phi,\cdot), \overline{k}(u\overline{\nabla} f,\cdot)\rangle + \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{2u}\pi\left(\partial_\phi, \overline{g}^{jl}\overline{w}^i(Y_{\bar{i};j} + Y_{\bar{j};i})\partial_l\right)$$
$$- \pi\left(\partial_\phi, \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u}\overline{w} - \frac{\overline{Y}}{u^2}\right)\pi(\overline{w},\overline{Y})$$
$$(5.3.55)$$

and lastly,

$$\pi(\overline{v},\overline{Y}) = \frac{f^l\overline{Y}^j(Y_{\overline{i};j}+Y_{\overline{j};i)})}{2u} = -\pi(\overline{w},\overline{Y}) + \frac{\overline{\nabla}f(|\overline{Y}|^2)}{2u^2} \tag{5.3.56}$$

In conclusion, we get (5.3.49) by substituting (5.3.54), (5.3.55) and (5.3.56) to (5.3.53). □

We will finish computing **Identity 4**. Brief computation using **Identity 2** shows the following :

$$\begin{aligned}
\overline{g}^{ij}f_{\overline{;}ij} &= \frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{u}Tr_{\overline{g}}(\pi) - \overline{g}^{ij}A_{ij} \\
&= \frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{u}Tr_{\overline{g}}(\pi) - \frac{2}{u^2}\pi(\overline{w},\overline{Y}) - \left(\frac{2}{u^2}-|\nabla f|^2\right)\frac{\overline{\nabla}f(\phi)}{2} \\
&= \frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{u}Tr_{\widetilde{g}}(\pi) + \frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{u^3}\pi(\overline{Y},\overline{Y}) \\
&- \frac{2}{u^2}\pi(\overline{w},\overline{Y}) - \left(\frac{2}{u^2}-|\overline{\nabla}f|^2\right)\frac{\overline{\nabla}f(\phi)}{2} \\
&= \frac{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}{u}Tr_{\widetilde{g}}(\pi) - \frac{\overline{\nabla}f(u)}{u} - \frac{(1-u^2|\overline{\nabla}f|^2_{\overline{g}})\overline{\nabla}f(\phi)}{2u^2}
\end{aligned} \tag{5.3.57}$$

Note that we get the last line of (5.3.57) by substituting $\pi(\overline{w},\overline{Y}) = \frac{\overline{Y}^i f^j(Y_{\overline{i};j}-Y_{\overline{j};i})}{2}$, $\pi(\overline{Y},\overline{Y}) = \frac{u(\overline{Y}^i\overline{Y}^j f_{\overline{;}ij})}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}} = -\frac{u(\overline{Y}^i f^j Y_{\overline{j};i})}{\sqrt{1-u^2|\overline{\nabla}f|^2_{\overline{g}}}}$. Now we will substitute (5.3.49) and

(5.3.57) to (5.3.48). Therefore $div_{\overline{g}}(\pi(\partial_\phi,\cdot))$ becomes as follows.

$$
\begin{aligned}
& div_{\overline{g}}(\pi(\partial_\phi,\cdot)) \\
&= \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}(div_{\overline{g}}(Hess_{\overline{g}} f(\partial_\phi,\cdot)) - \overline{k}\left(\partial_\phi, \frac{\overline{\nabla} u}{u^2})\right) + \pi\left(\partial_\phi, \frac{\overline{\nabla} u}{u}\right) \\
&+ g\langle \pi(\partial_\phi,\cdot), \pi(v,\cdot)\rangle - \overline{g}\langle \pi(\partial_\phi,\cdot), \overline{k}(u\overline{\nabla} f,\cdot)\rangle + \pi(\partial_\phi, \overline{Y})\left(\frac{\overline{\nabla} f(u)}{u}\right) \\
&- \overline{w}(\pi(\partial_\phi,\overline{w})) - \pi(\partial_\phi,\overline{w})\left(Tr_{\widetilde{g}}\pi + \pi(\overline{w},\overline{w}) - \frac{\overline{w}(u)}{u}\right)
\end{aligned} \tag{5.3.58}
$$

Last line of (5.3.58) is equal to $div_{\overline{g}}(\pi(\partial_\phi,\cdot)\overline{w})$ by (5.3.42). Lastly, we will substitute $\pi\left(\partial_\phi, \frac{\overline{\nabla} u}{u}\right) = \frac{u}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}\left(Hess_{\overline{g}} f(\frac{\overline{\nabla} u}{u}, \partial_\phi) + \frac{\overline{k}(\overline{\nabla} u, \partial_\phi)}{u^2} - \frac{\overline{\nabla} f(u)}{u}\pi(\overline{w}, \partial_\phi)\right)$ to achieve (5.3.58). □

Identity 5

$$
div_g \pi(\partial_\phi) = \frac{1}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} div_{\overline{g}}\left(u Hess_{\overline{g}} f(\partial_{\phi,}.)\right) \tag{5.3.59}
$$

Proof. Substitute $k \to \pi$ in **Identity 3**.

$$
\begin{aligned}
& div_g \pi(\partial_\phi) \\
&= div_{\overline{g}}(\pi(\partial_\phi,\cdot)) + \frac{1}{u} div_{\overline{g}}(u\pi(\partial\phi,\overline{w})\overline{w}) \\
&- g\langle \pi(\partial_\phi,\cdot), \pi(v,\cdot)\rangle + \overline{g}\langle \pi(\partial_\phi,\cdot), \overline{k}(u\overline{\nabla} f,\cdot)\rangle - \frac{\overline{\nabla} f(u)}{u}\pi(\partial_\phi,\overline{Y})
\end{aligned} \tag{5.3.60}
$$

Substitute $div_{\overline{g}}(\pi(\partial_\phi,\cdot))$ from **Identity 4**. □

Identity 6

$$
\begin{aligned}
& div_g(k-\pi)(v) \\
& = |\pi|^2 - g\langle \pi, k\rangle - u\overline{k}^{ij}(f_{\bar{;}ij}) \\
& + \frac{1}{u} div_{\overline{g}} \left(u \left(Hess_{\overline{g}}(\overline{Y}, \cdot) + (k-\pi)(\overline{w}, \cdot) + (k-\pi)(\overline{w}, \overline{w}) \frac{u \cdot df}{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}} \right) \right)
\end{aligned} \tag{5.3.61}
$$

Proof.

$$
div_g(k-\pi)(v) = div_g(k-\pi)(w) - \frac{\sqrt{1-u^2|\overline{\nabla} f|^2_{\overline{g}}}}{u} div_g(k-\pi)(\overline{Y}) \tag{5.3.62}
$$

We will substitute $(k-\pi)$ to **Identity 2** for $div_g(k-\pi)(w)$. Next, substitute **Identity 5** to evaluate $Y^{\phi} div_g \pi(\partial_{\phi})$. Note that $Y^{\phi} div_g k(\partial_{\phi}) = 0$.

$$
\begin{aligned}
& div_g(k-\pi)(v) \\
& = |\pi|^2_{\widetilde{g}} - \widetilde{g}\langle \pi, k\rangle \\
& + 2|\pi(\overline{w}, \cdot)|^2_{\widetilde{g}} - 2\widetilde{g}\langle \pi(\overline{w}, \cdot), k(\overline{w}, \cdot)\rangle + \frac{1}{u} div_{\overline{g}}(u(k-\pi)(\overline{w}, \cdot)) \\
& + \overline{w}((k-\pi)(\overline{w}, \overline{w})) + (k-\pi)(\overline{w}, \overline{w})(Tr_{\widetilde{g}}\pi) \\
& + \frac{1}{u} div_{\overline{g}}(u Hess_{\overline{g}} f(\overline{Y}, \cdot)) - u\overline{k}^{ij}(f_{\bar{;}ij})
\end{aligned} \tag{5.3.63}
$$

Last line is valid since $\overline{k} = \frac{(Y_{\bar{i};j} + Y_{\bar{j};i})}{2u}$, $\overline{k}(\overline{\nabla} f, \overline{\nabla} u) = 0$. Now let us simplify first four terms in (5.3.63) as follows.

$$
\begin{aligned}
& |\pi|^2_{\widetilde{g}} + 2|\pi(\overline{g}, \cdot)|^2_{\widetilde{g}} = |\pi|^2_g - \pi(\overline{w}, \overline{w})^2 \\
& \widetilde{g}\langle \pi, k\rangle + 2\widetilde{g}\langle \pi(\overline{w}, \cdot), k(\overline{w}, \cdot)\rangle = g\langle \pi, k\rangle - \pi(\overline{w}, \overline{w}) k(\overline{w}, \overline{w})
\end{aligned} \tag{5.3.64}
$$

Therefore the following holds.

$$
\begin{aligned}
div_g(k-\pi)(v) = |\pi|_g^2 - g\langle \pi, k\rangle + \frac{1}{u} div_{\overline{g}}(u(k-\pi)(\overline{w},\cdot)) \\
+ \frac{1}{u} div_{\overline{g}}(uHess_{\overline{g}} f(\overline{Y},\cdot)) - u\overline{k}^{ij}(f_{\bar{;}ij}) \\
+ \overline{w}((k-\pi)(\overline{w},\overline{w})) + (k-\pi)(\overline{w},\overline{w})(Tr_{\widetilde{g}}\pi + \pi(\overline{w},\overline{w}))
\end{aligned}
\tag{5.3.65}
$$

Recall that $\overline{g}^{ij}\overline{w}_{i\bar{;}j} = Tr_{\widetilde{g}}\pi + \pi(\overline{w},\overline{w}) - \frac{\overline{w}(u)}{u}$. Therefore last two terms form the following which result in **Identity 6**.

$$
\begin{aligned}
&\overline{w}((k-\pi)(\overline{w},\overline{w})) + (k-\pi)(\overline{w},\overline{w})(Tr_{\widetilde{g}}\pi + \pi(\overline{w},\overline{w})) \\
&= \frac{1}{u} div_{\overline{g}}(u(k-\pi)(\overline{w},\overline{w})\overline{w}) \\
&\text{since} \quad \overline{g}^{ij} Y_{i\bar{;}j} = 0, \overline{Y} = Y^{\phi}\partial_{\phi}, \\
&= \frac{1}{u} div_{\overline{g}} \left(u(k-\pi)(\overline{w},\overline{w}) \frac{u \cdot df}{\sqrt{1-u^2|\overline{\nabla} f|_{\overline{g}}^2}} \right)
\end{aligned}
\tag{5.3.66}
$$

□

Identity 7

$$
\begin{aligned}
\overline{R} - |\overline{k}|_{\overline{g}}^2 \quad = \quad & 16\pi(\mu - J(v)) + |k-\pi|_g^2 + \frac{2}{u} div_{\overline{g}}(uQ(\cdot)) \\
& + (Tr_g \pi)^2 - (Tr_g k)^2 + 2v(Tr_g\pi - Tr_g k)
\end{aligned}
\tag{5.3.67}
$$

where Q is one form on $(M, \overline{g})$ such that

$$
Q(\cdot) = (Hess_{\overline{g}} f)(\overline{Y},\cdot) - \overline{k}(u\overline{\nabla} f,\cdot) + (k-\pi)(\overline{w},\cdot) + (k-\pi)(\overline{w},\overline{w}) \frac{udf}{\sqrt{1-u^2|\overline{\nabla} f|_{\overline{g}}^2}}
\tag{5.3.68}
$$

Proof. Recall the $\overline{R}$ formula (5.0.16).

$$\begin{aligned}
&\overline{R} - |\overline{k}|^2_{\overline{g}} + 2div_{\overline{g}}\overline{k}(u\overline{\nabla}f) \\
&= 16\pi(\mu - J(v)) - |\pi|^2_g + |k|^2_g - 2div_g(\pi)(v) + 2div_g(k)(v) \\
&+ (Tr_g\pi)^2 - (Tr_gk)^2 + 2v(Tr_g\pi - Tr_gk)
\end{aligned} \tag{5.3.69}$$

Where $div_{\overline{g}}\overline{k}(u\overline{\nabla}f) = \frac{1}{u}div_{\overline{g}}(u\overline{k}(u\overline{\nabla}f, \cdot)) - u\overline{k}^{ij}f_{\bar{;}ij}$. Substitute **Identity 6** and solve for $\overline{R} - |\overline{k}|^2_{\overline{g}}$. □

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