On the route to chaos for two-dimensional modestly area-contracting analytic maps

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Abstract of the Dissertation

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It has long been conjectured that the two-dimensional dissipative maps, like the one-dimensional maps, experience the period-doubling cascade to chaos. In [CEK], Collet, Eckmann and Koch proved this conjecture for highly dissipative families. In this dissertation, we introduce the notion of nested systems that generalizes Hénon maps, and construct two operators acting on the space of nested systems based on the idea of the return maps. We use the two operators to show that if a dissipative nested system satisfies some apriori bounds, then they can be replaced with simpler but dynamically equivalent nested systems. We prove that the total number of applications of the operators is bounded by the number of periodic points. When the procedure of the applications stops, we obtain "little Hénon" maps whose dynamics are well-understood. We then show that if the first nested system of a family contains finitely many periodic points, then the family only experiences either saddle-node or periodic-doubling bifurcations. We conclude that if there are sufficiently many periodic points, then moderate dissipative nested systems can be transformed into highly dissipative ones so that the results of [CEK] applies.

To my mother

Contents

List of Figures

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Chapter 1

Introduction

In Subsection 1.1, we introduce some general history about chaos theory. In Subsection 1.2 and 1.3, we provide some background and a brief review of important works related to the problem that we are interested in. We formulate our main results and outline the structure of this dissertation in Subsection 1.4.

1.1 The birth and rebirth of dynamical system theory

The central theme of dynamical system theory is the notion of *chaos*, and it was first observed by H.J. Poincaré [P] in 1889. At the time, the King Oscar II of Sweden sponsored a mathematical contest. One of the four questions in the contest was the famous n-body problem, and Poincaré chose to work on a simplified version of it. The simplified problem assumes that one celestial body is almost massless and the plane in which it moves is entirely defined by the other two bodies. The simplified problem is to determine the trajectory of the almost massless body.

Poincaré [P] realized that this simplified problem has a special class of solutions, now called *transverse homoclinic trajectories*. The mere existence of such solutions would imply that the series solutions of the equations of motion would diverge. In other words, the trajectory will be somewhat unpredictable. This unpredictability was later understood and described as the *sensitivity with respect to initial condition*, i.e. chaos. This paper [P] was the very first work of the dynamical system theory. See [Ho] for more details.

In the next half century, the dynamical system theory gradually progressed, but in a rather slow pace. It was after Edward Lorenz rediscovered the unpredictable behavior in [Lo] in 1963 that the dynamical system theory started to draw a great deal of attention from many scientific fields, including mathematics.

At the time, Lorenz was trying to mathematically identify the problems which prevent an accurate long-term weather prediction. He chose to model the conviction motion of fluid as his starting point. Heuristically speaking, a convection occurs when the fluid is heated from below and cooled from above: the fluid at the bottom becomes warmer and more buoyant, and then rises; meanwhile the fluid on the top becomes colder and denser, and then descends. The model that he worked on assumes a very idealized boundary condition.

Lorenz was able to approximate the PDEs of the simplified convection motion by a three-order system of ODEs in time, and then he used a rather rudimentary computer to stimulate the solution. One day he repeated the simulation that he had done earlier. In order to save time, he entered the numbers that were in the middle of the printout of his last stimulation. However, the simulated solution that he received this time was entirely different from that he did earlier. Lorenz was able to figure out the origin of the difference: while the computer internally processed numbers to 6 decimal places, the numbers on the printout were rounded off to 3 decimal places.

The story highlights the key characteristic of the chaos: small changes in initial conditions are able to create disproportionately large changes in the long-term behavior.

It is far beyond our capacity and our intention to give a comprehensive account of the dynamical ideas. We will hereafter restrict our attention to those ideas that are closely related to our own interest. We will discuss how chaos is defined and explain why the presence of homoclinic points implies chaos in the Subsection 1.2. In the following Subsection 1.3 we will discuss the *route to chaos* which means intuitively how a family of dynamical systems becomes chaotic. Both will be discussed in the context of discrete systems.

1.2 The notion of chaos

We work with the map $\varphi : M \to M$ of a smooth compact Riemannian manifold M to itself; the differentiability will be specified each time since the requirements vary.

It was until 1975 that T. Li and J. A. Yorke defined the term chaos mathematically in *Period three implies chaos* ([LiY]). In their definition, a continuous map $\varphi : M \to M$ is chaotic if there exists an uncountable subset $N \subset M$ with the following properties:

1. $\limsup_{n\to\infty} d(\varphi^n(x), \varphi^n(y)) > 0$ for every $x, y \in N$ and $x \neq y$,
2. $\liminf_{n\to\infty} d(\varphi^n(x), \varphi^n(y)) = 0$ for every $x, y \in N$ and $x \neq y$,
3. $\limsup_{n\to\infty} d(\varphi^n(x), \varphi^n(p)) > 0$ for every $x \in N$ and periodic point $p \in M$,
4. there is a periodic point in M of period n, for any $n \in \mathbb{Z}^+$.

Li and Yorke [LiY] proved that if a continuous interval map φ has a periodic point with period 3 then φ is chaotic in the above sense.

It is worth mentioning that the condition 4 is a special case of Sarkovskii's theorem discovered by A.N. Sarkovskii [Sa] in 1964. Sarkovskii's theorem has extremely weak hypotheses, and a remarkably elegant conclusion. The statement of Sarkovskii's theorem is as follows:

If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function that has a periodic point of period n, then φ also has periodic orbits of any period l with $n \rhd l$ in the Sarkovskii-order.

The Sarkovskii-order is an order on the set of positive integers $\mathbb{Z}^+$:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \dots$$
$$\rhd 2^3\cdot 3 \rhd 2^3\cdot 5 \rhd 2^3\cdot 7 \rhd \dots 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

There are different definitions of the notion of chaos and a universally accepted definition has not yet fully emerged (Li and Yorke's work was one of the many attempts). However, another notion, called *sensitive dependence on initial conditions*, is the fundamental characteristic of almost, if not all, definitions of chaos, and its definition is universally defined.

Let $\varphi : M \to M$ be of C^0 class. The map φ is said to have *sensitive dependence on initial conditions* if there exists an $\epsilon > 0$ such that for any $x \in M$ and any $\delta > 0$ there exists a number $y \in \{z \in M | \mathrm{dist}(x,z) < \delta\}$ and a positive integer $n \in \mathbb{Z}^+$ with $\mathrm{dist}(\varphi^n(x), \varphi^n(y)) > \epsilon$.

The definition of sensitivity again reflects the idea of unpredictability: no matter how close two points initially are, there is always a chance that their orbits get apart by at least ϵ. Sensitive dependence on initial conditions is such a fundamental concept that it is often used interchangeably with chaos. Throughout this dissertation, we will use them interchangeably when the context is clear.

Various indicators are conceived to detect the sensitivity dependence on initial conditions. One of them is the notion of *topological entropy*. We assume that M is a non-empty compact Hausdorff space and $\varphi : M \to M$ a continuous map. The topological entropy is often used to measure "the degree" of the dynamical system's sensitivity. The definition was first introduced by Adler, Konheim and McAndrew in 1965 as follows: for an open cover $\mathcal{U}$ of M, let $N(\mathcal{U})$ denote the smallest cardinality of a sub-cover of $\mathcal{U}$ (i.e., a sub-family of $\mathcal{U}$ whose union still covers X). By compactness, $N(\mathcal{U})$ is always finite. If $\mathcal{U}$ and $\mathcal{V}$ are open covers of M, then

$$\mathcal{U} \vee \mathcal{V} \equiv \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$$

is called their common refinement. Let $\mathcal{U}^n \equiv \mathcal{U} \vee \varphi^{-1}(\mathcal{U}) \vee \cdots \vee \varphi^{-(n-1)}(\mathcal{U})$, where $\varphi^{-k}(\mathcal{U}) \equiv \{\varphi^{-k}(U) : U \in \mathcal{U}\}$. Using the sub-additivity argument, one can show that the limit

$$\lim h(\mathcal{U}, \varphi) \equiv \lim_{n\to\infty} \frac{\log N(\mathcal{U}^n)}{n}$$

exists for any open cover $\mathcal{U}$ and equals $\inf_{n\in\mathbb{N}} \frac{log N(\mathcal{U}^n)}{n}$. The *topological entropy* $h(\varphi)$ is defined as follows:

$$h(\varphi) \equiv \sup_{\mathcal{U}} h(\mathcal{U}, \varphi),$$

where the supremum ranges over all open covers $\mathcal{U}$ of M.

The topological entropy is an important indicator of chaos because of the following fact: *the positive topological entropy implies sensitivity dependence* (see [BGKM]). Although the converse need not hold (see [Sj]), the topological entropy will serve as a primary criterion that distinguishes non-chaotic maps from chaotic ones.

Another relevant indicator is the *non-wandering* set. A point $x \in M$ is a non-wandering point if for any neighborhood U of x, there is an integer n such that $\varphi^n(U) \cap U \neq \emptyset$. The union of all non-wandering points, denoted by $\Omega(\varphi)$, is called the non-wandering set. The non-wandering set captures all the interesting dynamics of φ. The following fact will be used in this dissertation: *if $\Omega(\varphi)$ consists of finitely many periodic points, then the map φ cannot be sensitive dependent on initial conditions*. In other words, finiteness and periodicity of all non-wandering points together is a sufficient condition of being non-chaotic.

In many definitions of chaos, *recurrence* and *dense periodic orbits* are also considered as the characteristics of chaos. Recurrence heuristically means that the system will, in finite time, return to where is very close to the initial state. However, unlike the sensitive dependence, whether the recurrence and dense periodic points are considered as indispensable properties is under debate.

Having discussed characteristics of chaos, we now turn to examples that exhibit sensitive dependence on initial conditions. The first example is the dynamical system that has homoclinic points: the mere presence of a transverse homoclinic point implies chaos.

In the remaining part of this subection, we assume φ is a diffeomorphism for simplicity. A point $p \in M$ is a hyperbolic fixed point of φ if $\varphi(p) = p$ and $(d\varphi)_p$ has no eigenvalue of norm 1. For a hyperbolic fixed point, the *stable* and *unstable manifolds* are

$$W^s(p) = \{x \in M | \varphi^n(x) \to p \text{ for } n \to +\infty\},$$

and

$$W^u(p) = \{x \in M | \varphi^n(x) \to p \text{ for } n \to -\infty\},$$

respectively.

The Stable Manifold Theorem (also known as the Hadamard-Perron Theorem since it was discovered by Hadamard and Perron independently in the 1920s) proved that both $W^s(p)$ and $W^u(p)$ are injectively immersed submanifolds of M which are as differentiable as φ. Moreover, the dimension of $W^s(p)$ (resp., $W^u(p)$) is equal to the number of eigenvalues of $(\mathrm{d}\varphi)_p$ with norms smaller (resp., greater) than 1. Suppose p is a hyperbolic fixed point of φ, q is *homoclinic* to p if

$$p \neq q, \quad \text{and} \quad q \in W^s(p) \cap W^u(p).$$

In other words, $q \neq p$ and $\lim_{n\to\pm\infty} \varphi^n(q) = p$. The second condition reflects why Poincaré called them "bi-asymptotic". We say that q is a *transverse homoclinic point* if

$$T_q M = T_q W^s(p) \oplus T_q W^u(p).$$

It is clear that if p has a homoclinic point q, then not all the eigenvalues of $(\mathrm{d}\varphi)_p$ lie inside or outside the unit disk. Such a fixed point is called of *saddle type* (if all the eigenvalues lie inside the unit disk, p is called a *sink*; if all lie outside the unit disk, p is a *source*).

We hereafter assume that M is a Riemann surface and $\varphi : M \to M$ is of class C^2. Let p be a saddle fixed point of φ. It has two real eigenvalues λ and σ and we can assume $0 < \lambda < 1 < \sigma$ without loss of generality. P. Hartman [Hp] showed that φ admits C^1 coordinates (x, y) in a neighborhood U of p such that $p = (0, 0)$ and $\varphi(x, y) = (\lambda \cdot x, \sigma \cdot y)$.

We can extend the domain of the C^1 coordinates (x, y) to a somewhat larger domain along $W^s(p)$. Without loss of generality, we assume that $\mathrm{cl}(U)$ is the unit square $B \equiv [-1, 1] \times [-1, 1]$ in the C^1 linearizing coordinates. Suppose U is a small neighborhood so that $\varphi^{-1}((\lambda, 1] \times [-1, 1]) \cap B = \emptyset$. Then we can extend the (x, y) coordinates to $\varphi^{-1}((\lambda, 1] \times [-1, 1])$ as follows: let a be a point in $a \in \varphi^{-1}((\lambda, 1] \times [-1, 1])$ so that $\varphi(a) = (x, y)$, then the point a has coordinate $(\lambda^{-1} \cdot x, \sigma^{-1} \cdot y)$.

Since the stable manifold $W^s(p)$ is not self-intersecting, the extension can be repeated along $W^s(p)$ as many as possible provided that the original domain is small enough. Similarly, the domain can be extended along $W^u(p)$. However, homoclinic points are obstacles: these two linearizing coordinates do not coincide with each other near the homoclinic point q.

In the domain of the extended coordinates, we can construct a rectangle $R \equiv \{x_1 \le x \le x_2, y_1 \le y \le y_2\}$ along l^s, which is an arc in $W^s(p)$ containing p and q such that for some $N \in \mathbb{Z}^+$

1. $R \cap \varphi^i(R)$ consists of one rectangle containing p for $0 \le i < N$,
2. $R \cap \varphi^N(R)$ consists of two rectangles, one containing p and the other containing q,
3. $\{x = x_2, y_1 \le y \le y_2\} \cap \varphi^N(R) = \emptyset$,
4. $\varphi(\{x_1 \le x \le x_2, y = y_2\}) \cap R = \emptyset$.

If the rectangle R is thin enough, the sides of R and $\varphi^N(R)$ intersect transversely because $W^s(p)$ and $W^u(p)$ intersect transversely at q. See [PT, Ch 2] for details.

In order to explain the complicated behavior associated with the homoclinic points, S. Smale invented the Smale horseshoe map in [Ss] in 1965. The horseshoe map is a diffeomorphism $Q : B \to \mathbb{R}^2$ where $B = [0,1] \times [0,1] \subset \mathbb{R}^2$ is the unit square; it maps B in such a way that

1. Q is linear on both components of $B \cap Q^{-1}(B)$,
2. $1, 2, 3, 4$ are mapped to $1', 2', 3', 4'$.

The map is shown in Figure 1.2.1.

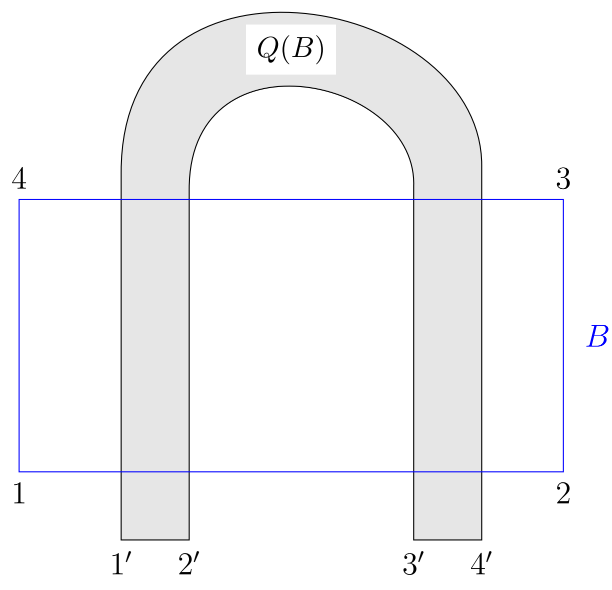
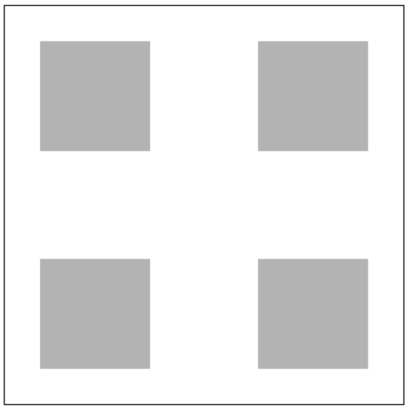


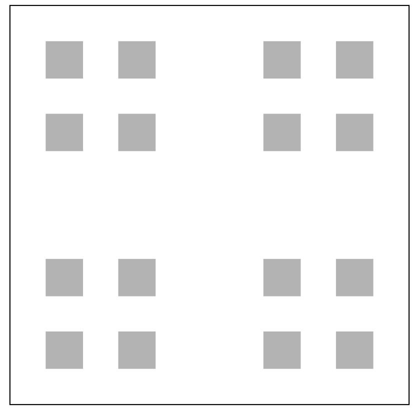
Figure 1.2.1: Horseshoe Map

The Horseshoe map is an abstraction of $\varphi^N|_R$. Indeed, it is easy to observe that the map φ^N is almost a horseshoe map on rectangle R in the linearizing coordinates, except that φ^N is not necessarily linear on the component of $R \cap \varphi^N(R)$ containing q. However, this difference can be taken care of by assuming N is sufficiently large (R is sufficiently thin) so that the restriction of φ^N to the component is roughly $(\mathrm{d}\varphi^N)_q$.

For simplicity, we assume that $N = 1$, i.e. φ is a Smale horseshoe map on R. Moreover, φ preserves both vertical and horizontal directions on both components of $\varphi(R) \cap R$ (as shown in Figure 1.2.1). The *maximal invariant subset* of R under φ, denoted by $\Lambda \equiv \bigcap_{n\in\mathbb{Z}} \varphi^n(R)$, is the set of points $s \in R$ so that $\varphi^n(s) \in R$ for all $n \in \mathbb{Z}$; see Figure 1.2.2.



(a) The shaded area is invariant under one iteration of both φ and φ^{-1}.



(b) The shaded area is invariant under two iterations of both φ and φ^{-1}.

Figure 1.2.2: The limit set, $\Lambda \equiv \bigcap_{n\in\mathbb{Z}} \varphi^n(R)$, is an invariant Cantor set under all forward- and backward- iterations of φ.

Denote by $\bar{0}$ the component of $\varphi(R) \cap R$ which contains the fixed point p and by $\bar{1}$ the other component which contains the homoclinic point q. Therefore we can assign to each point in Λ a symbolic representation. Put

$$\Sigma \equiv \{a = (\dots, a_{-1}, a_0, a_1, \dots) | a_i \in \{0, 1\}\}.$$

A map $h : \Lambda \to \Sigma$ can be defined as follows: for $s \in \Lambda$, $h(s) = (\dots, s_{-1}, s_0, s_1, \dots)$ so that

$$s_n = \begin{cases} 0 & \text{if} \quad \varphi^n(s) \in \bar{0}, \\ 1 & \text{if} \quad \varphi^n(s) \in \bar{1}. \end{cases}$$

It is obvious that the map $h : \Lambda \to \Sigma$ is surjective. Smale proved in [Ss] that $h : \Lambda \to \Sigma$ is a homeomorphism and the two sets, Λ and Σ, are *topologically conjugate* by h. More precisely,

$$h \circ \varphi = \sigma \circ h,$$

where $\sigma : \Sigma \to \Sigma$ is the left shift operation (i.e. $\sigma(a_i)_{i\in\mathbb{Z}} = (a'_i)_{i\in\mathbb{Z}}$ with $a'_i = a_{i+1}$). The commutation diagram is shown in Figure 1.2.3.

The dynamics of two conjugated maps are the same from the topological point of view. The chaotic behavior of φ when a transverse homoclinic point is present (which was also what Poincare

$$\begin{tikzcd}
\Lambda \arrow[r, "\varphi"] \arrow[d, "h"] & \Lambda \arrow[d, "h"] \\
\Sigma \arrow[r, "\sigma"'] & \Sigma
\end{tikzcd}$$

Figure 1.2.3: The map φ restricted to the invariant set Λ is conjugate to the left shift map on the space of bi-infinite sequences of two symbols Σ.

observed) can now be fully explained by the behavior of the left shift map on the space of bi-infinite sequences of two symbols. By studying all possible sequences of symbols, it can be concluded that $\varphi : \Lambda \to \Lambda$ has

1. periodic cycles of any periods,
2. a dense subset of periodic cycles,
3. a dense orbit,

and more importantly,

4. sensitive dependence on initial conditions.

It is worth mentioning that the topological entropy of the horseshoe map is $\log 2$. See [De2, Ch 2] for more detailed analysis. We will use the fact that the existence of homoclinic point implies chaos many times in our proof.

There are other chaotic dynamical systems. One example is the bizarre figure that Lorenz [Lo] observed. It was later named as the *strange attractor* by Belgian physicist D. Ruelle in 1971.

The rigorous definition of an attractor is given by J. Milnor in [Mj]: a set $A \subset M$ is an *attractor* of a continuous map $\varphi : M \to M$ if

1. A is forward-invariant under φ,
2. the *basin of attraction*, which consists of points $x \in M$ whose limit set of forward orbits - denoted by $\omega(x)$ - is a subset in A, must have strictly positive measure, and
3. no strictly smaller forward-invariant closed set $A' \subset A$ has the same basin of attraction as does A.

In [Mj], Milnor gives a nice explanation of the last two requirements: the second condition "says that there is some positive possibility that a randomly chosen point will be attracted to A", and the third "says that every part of A plays an essential role".

Attractors themselves do not necessarily imply sensitivity dependence on initial conditions. For example, a sink is an attractor and all the points in the basin of attraction move towards the sink.

Regarding the strange attractor, there are also a number of different definitions. Some definitions do not imply a strange attractor being chaotic either; see [GOPY]. However, we choose such a definition that requires sensitivity dependence on initial conditions (see [PT, Ch 7]): an invariant set $A \subset M$ under map $\varphi : M \to M$ is a *strange attractor* if

1. A is an attractor with basin of attraction $B(A)$,
2. with total probability on $B(A) \times B(A)$, φ has sensitive dependence on initial conditions *or* almost all orbits (total probability) on $B(A)$ are sensitive.

The difference between the invariant chaotic set Λ given by the transverse homoclinic point and the strange attractor A is clear. Although φ is sensitive dependent on initial conditions on both Λ and A themselves, no knowledge at all for the orbit of any single point in $M \backslash \Lambda$ can be inferred by studying Λ while one can conclude that the (forward) orbits of the points in $B(A)$ is around the set A.

The numerical simulation of Lorenz's original three-dimensional differential equations suggests that there is a strange attractor. Based on the numerical data, Guckenheimer and Williams [GW] built a geometrical model describing the dynamics of the flow and proved the existence of strange attractors under the conditions of this geometrical model. W. Tucker [W] proved that this geometrical model is indeed an accurate description of Lorenz's original equation. Therefore the Lorenz attractor is a strange attractor.

Another famous example of strange attractor comes from the famous *Hénon family of maps*:

$$\varphi_{a,b}(x, y) = (1 - ax^2 + y, bx).$$

Hénon [Hm] conjectured, based on his computational simulation, that for $a = 1.4$ and $b = 0.3$ the corresponding map exhibits a strange attractor. Benedicks and Carleson [BC] investigate the case where the parameter a is near 2 and $b > 0$ is small. For each given b, there is a positive Lebesgue measure set E_b such that if $a \in E_b$, then $\varphi_{a,b}$ has a strange attractor $A_{a,b}$ which is the closure of the unstable manifold of some saddle point $p_{a,b}$.

There is no short of strange attractors despite their strangeness. In fact, Palis conjectured and L. Mora and M. Viana later [MV] proved that the Benedicks-Carleson's result can be generalized as follows: *Let $\{\varphi_t\}_t$ be a C^∞ one-parameter family of surface diffeomorphism and suppose that φ_0 has a homoclinic tangency associated to some periodic point p_0. Then under generic (even open and dense) assumptions, there is a positive Lebesgue measure set E of parameter values near $t = 0$, such that for $t \in E$, the diffeomorphism φ_t exhibits a strange attractor, or repeller (the attractor of φ^{-1}) near the orbit of tangency.* The similar results - the abundant existence of strange attractors - were proved by Mora and Viana [MV] for one-dimensional maps and by M. Viana [Vm] for higher-dimensional maps. See [PT, Ch 7] for more discussions about strange attractors.

We close this subsection by one remark: although our attention is on the maps of a compact manifold M, this condition is not essential. If M is non-compact, all the discussions still are valid by restricting all the notions and theorem to points in M whose positive orbits have compact closure. In this case, ∞ can be conventionally considered as an attracting fixed point.

1.3 The route to the chaos

Given a family of maps $\{\varphi_t : M \to M\}_{t\in[-1,1]}$ where φ_{-1} has a simple dynamics (for example, a single fixed point that attracts all the orbits) and φ_1 has a chaotic dynamics, it is natural to ask how the family evolves from a simple dynamics system to a chaotic one as the index parameter t varies. The *route to chaotic dynamics* refers to such a process, and it has been a subject of much research in the past several decades.

Experiments and theoretical deductions have provided many various routes to chaos. Here are a few: (a) period-doubling route to chaos, (b) homoclinic bifurcation route to chaos, (c) period-adding route to chaos, and (d) quasi-periodic (torus) route.

A family of dynamical systems becomes chaotic via the period-doubling route to chaos means that it undergoes infinitely many period-doubling bifurcations. A period bifurcation happens when the parameter t passes some critical moment λ_n in which one eigenvalue of a previously attracting cycle of period-n becomes -1 so that the cycle of period-n loses stability and gives birth to an attracting periodic cycle of period-$2n$. The previously attracting cycle of period-n later becomes a saddle periodic cycle of period-n, and remains there independent of the change of the parameter t.

The period-doubling route is of special interest because of a remarkable discovery made by Feigenbaum [Fe] and independently by P. Coullet and C. Tresser [CT2] in mid-1970's. They analyzed the "typical" one-parameter families of *unimodal* interval maps which are at the transition from simple to chaotic dynamics. An interval map $f : I \to I$ is unimodal if it has one and only one critical point $c \in I$; for example, if $I = [-1, 1]$, the quadratic family $f_\lambda(x) \to \lambda x(1 - x)$ is such a family. The parameter values λ_n's at which those successive period-doubling bifurcations occurred in the route were recorded. They both observed that independent of the choice of the families (as long as they are unimodal with enough smoothness, say C^3), a universal scaling ratio exists:

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \to 4.669201609101990\ldots.$$

Furthermore, the limit map of the sequence of maps f_{λ_n}, denoted by f_{λ_∞} (where $\lambda_\infty = \lim_n \lambda_n$), has an invariant Cantor set Λ as its attractor. P. Coullet and C. Tresser [CT2] also found another "universal scaling" property within the the small scale geometry of the invariant Cantor set of f_{λ_∞}.

To explain these phenomena, a non-linear operator, called the *period-doubling operator* $\mathcal{R}$, has been invented; this operator acts on the space of unimodal maps. A unimodal map $f : I \to I$ is said to be *renormalizable* if there is a number $p > 1$ and a family of $I_0, I_1, \ldots, I_p = I_0$ of compact disjoint intervals in I such that I_0 contains the critical point c in its interior, and $f(I_{j-1}) \subset I_j$ for all $0 \le j < p$. Then the renormalization $\mathcal{R}$ of f is defined as an affinely rescaled version of the first return map to I_0, $f^p : I_0 \to I_0$; specifically

$$\mathcal{R}(f) = \phi \circ f^p \circ \phi^{-1},$$

where $\phi : I_0 \to I$ is an affine rescaling of I_0 to the original inteval I. It can be easily shown that the $\mathcal{R}(f)$ is still a unimodal map. A unimodal map f is said to be *infinitely renormalizable* if $\mathcal{R}^n(f) \equiv (\mathcal{R}\circ\mathcal{R}\circ\cdots\circ\mathcal{R})(f)$ is defined for all $n \in \mathbb{Z}^+$. They then proposed the following conjecture.

The Renormalization Conjecture: *In the proper class of infinitely renormalizable maps, the period-doubling renormalization operator has a unique fixed point* f_* *that is hyperbolic with a one-dimensional unstable manifold and a co-dimension one stable manifold consisting of the systems on the route to chaos.*

This conjecture would imply both of the above universalities: the universal scaling ratio $4.669\ldots$ is in fact the unstable eigenvalue of the derivative of the operator at the fixed, i.e. the unstable eigenvalue of $(D\mathcal{R})_{f_*}$; the universal geometry of the attractor of $\Lambda_{f_{\lambda_\infty}}$ of f_{λ_∞} given by all proper families can be explained by the attractor Λ_* of f_*. In this language, the notion of universality can be rephrased as follows:

Universality: *There exists a positive number* $\rho < 1$ *so that for any two infinitely renormalizable maps* f *and* g*,*

$$\mathrm{dist}(\mathcal{R}^n f, \mathcal{R}^n g) = O(\mathrm{dist}(f, g)\rho^n).$$

Given the universality, two infinitely renormalizable maps f and g are conjugate with a conjugation map h; that is, there exists a homeomorphism h between the domains of the two maps so that

$$h \circ f = g \circ h.$$

A priori the conjugation map h may not be smooth. A stronger property called *rigidity* is defined as follows.

Rigidity: *The conjugation between two infinitely renormalizable maps is differentiable on the attractors.*

A rigorous proof of this conjecture turned out to be very difficult. In about thirty years, many had contributed towards the final proof. Among them are D. Sullivan (see [S]), C. McMullen (see [Mc]) and M. Lyubich (see [Ly]). They all considered the holomorphic extensions of the analytic maps on the interval and exploited the idea of *quadratic-like* maps defined by Douady and Hubbard (see [DH]). A quadratic-like map is a holomorphic branched covering $F : U \to U'$ of degree 2 between two topological disks U and U' so that $\mathrm{cl}(U) \subset U'$. A standard example is $F_c(z) = z^2 + c$. M. Lyubich completed the proof of the Renormalization Conjecture in the space of analytic interval maps in [Ly].

Later, with the techniques that were developed for the analytic maps, many people investigate the conjecture in much larger spaces of unimodal maps. M. Martens [M] has the most general results about the existence of fixed points of the Renormalization operator. He showed that the renormalization operator, acting on the space of unimodal maps with critical exponent greater than one, has periodic points of any combinatorial type.

However, both the hyperbolicity of the Renormalization fixed point and the rigidity are very sensitive to the differentiability of maps. A. Dave proved that the the fixed point is hyperbolic in the $C^{2+\alpha}$ family of maps for $0 < \alpha < 1$; see [Da]. Quite recently, V. Chandramouli, M. Martens, W. de Melo and C. Tresser in [CMMT] showed that (a) in the space of C^2 unimodal maps the fixed point is not hyperbolic, (b) in a smoother space called $C^{2+|\cdot|}$ (the map in which is less smooth than the usual $C^{2+\alpha}$ for $0 < \alpha < 1$) the failure of hyperbolicity is tamer than that in C^2, and (c) when the smoothness is less than (but can be very close to) C^2, the fixed point is not even unique, therefore universality and rigidity do not exist.

Nevertheless, it is somewhat safe to assure ourselves that we have a quite clear knowledge of what happens for a generic one-dimensional dynamic system which is on the route to chaos and what its boundary of chaos looks like.

The study of route to chaos in higher-dimensional spaces is, however, far more difficult and far from complete. Many new phenomena can happen and one cannot expect the period-doubling route to chaos being the generic route as in the one-dimensional case.

Some of them have been confirmed theoretically. For example, we might meet the homoclinic bifurcation in area-preserving systems. A homoclinic bifurcation is a process of unfolding a homoclinic tangency which leads to the appearance or disappearance of the transverse homoclinic points. Clearly, the homoclinic bifurcation can create chaos in a sudden burst.

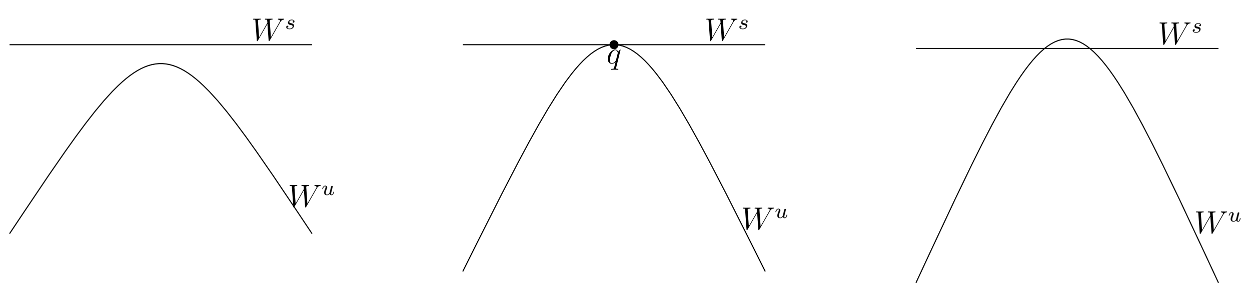


Figure 1.3.1: Unfolding a homoclinic tangency

The family of Hénon maps provides an example of homoclinic bifurcation. Hénon maps are planar maps of the form:

$$(x, y) \mapsto (-x^2 - by + c, x)$$

where b and c are real numbers (this form is slightly different from the one presented in Subsection 1.2, but they are equivalent). When $b = 1$, the Hénon maps are area-preserving. R. Devaney showed that (a) when $c < -1$, Hénon map does not have any periodic point, (b) when $c = -1$, there exists a fixed point in $[-1, 1]$ but no other periodic point and (c) for all $c > -1$, the Hénon maps admit transverse homoclinic points; see [De]. Therefore, this conservative Hénon family shows that the chaotic behavior in a conservative system need not arise via a sequence of period doubling bifurcations as that in the one-dimensional case.

One of the other possibilities is the quasi-periodic route to chaos, which is also called Ruelle-Takens-Newhouse route as they discovered this phenomenon (see [RT] and [NRT]). They showed that by two Hopf bifurcations (a Hopf bifurcation occurs when a pair of conjugate eigenvalues of the periodic points are on the unit circle in complex plane), the system develops a strange attractor. The quasi-periodic route to chaos is also observed experimentally in chemical systems (see [Sc]). Another route to chaos is via consecutive period-adding bifurcations. In contrast to period-doubling bifurcation by which a period-n cycle directly bifurcates to a period-$2n$ cycle, each period-adding bifurcation increases the period by a constant $d \in \mathbb{Z}^+$. This route is also observed in many biological processes. To summarize, the route to chaos in higher dimensional dynamical systems is far more complicated than that in the one-dimensional case; this has been confirmed theoretically as well as experimentally.

However, the period-doubling route to chaos is still of special interest in the high-dimensional systems for two reasons. The first reason relates to the fact that all systems can be divided in two categories: conservative and dissipative ones. The conservative systems are friction-free whereas the dissipative ones are not. As a consequence, the asymptotic behavior of dissipative systems should resemble one-dimensional case more than that of conservative ones does. J.M. Gambaudo and C. Tresser in [GT] has conjectured that *the dissipative* $\mathbb{R}^2$ *maps on the boundary of positive topological entropy exhibit a period-doubling cascade*; therefore, even though other routes to chaos might exist in the conservative systems (for example, the homoclinic bifurcation route to chaos, illustrated in the example of the conservative Hénon family), they should not exist in dissipative systems. Although this is still an open problem, it seems very plausible. S. Crovisier, E. Pujals and C. Tresser very recently prove the conjecture assuming some extra conditions on the dissipative maps. There are similar conjectures for higher-dimensional dynamical systems.

Secondly, the period-doubling phenomenon is still abundantly observed in high-dimensional (both conservative and dissipative) dynamical systems. Furthermore, the universality is always observed as well when the family does go through the period-doubling route to chaos. However, it is worth mentioning that the universalities observed are not necessarily the same universality that exists in one-dimensional unimodal maps. For example, for the conservative $\mathbb{R}^2$ systems in which the period-doubling route occurs, the universal scaling ratio is about 8.72 other than 4.66. We will explain such differences in details in the later discussion of conservative $\mathbb{R}^2$ maps in this subsection.

The idea of Feigenbaum-Coullet-Tresser period-doubling renormalization operator is again applied to prove the universality for the two-dimensional systems. In short, one can introduce a *renormalization operator* acting on the class of maps of interest, and an analogue of Renormalization Conjecture can be formulated as follows: *in the proper class of maps, the specifically-defined renormalization operator has a unique fixed point that is hyperbolic with a one-dimensional unstable manifold and a co-dimension one stable manifold consisting of the systems on the route to chaos; moreover the universal scaling ratio is the unstable eigenvalue of the derivative of the operator at the fixed point.*

Although the underlying classes of maps are various, the universalities are not the same, and the renormalization operators are case-by-case defined to suit their own underlying class, it is worth noting that the wording of the analogue is always a verbatim rephrase of the original Renormalization Conjecture. More importantly, the analogue and the original Renormalization Conjecture share the same essence: if the analogue of Renomralization Conjecture of a particular class of maps can be demonstrated, then the universality of this class of maps follows as that does in the one-dimensional case.

The $\mathbb{R}^2$ maps are the simplest among higher-dimensional maps. In the following part of this subsection, we discuss the renormalizations (therefore the universality and rigidity behaviors) of dissipative and conservative $\mathbb{R}^2$ maps. When the context is clear, we will speak of renormalization operators without explicitly mentioning the underlying classes of maps.

Collet, Eckmann and Koch started the investigation into the dissipative systems. In [CEK], they defined a renormalization operator for highly dissipative systems and showed that the one-dimensional period-doubling renormalization fixed point is also a hyperbolic fixed point for highly dissipative two-dimensional maps of the renormalization operator. Later on, Gambaudo, van Strien and Tresser [GST] proved that the highly dissipative infinitely renormalizable two-dimensional maps also have an attracting Cantor set on which the maps act as the adding machine.

However, even for those highly dissipative maps, the geometry is not the same as that of their one-dimensional counterparts. A. De Carvalho, M. Lyubich, and M. Martens showed in [CLM] that the geometry of infinitely renormalizable dissipative two-dimensional maps has the universality property with two-dimensional characteristics and, unlike the one-dimensional case, does not have the rigidity property. Because of the surprising contrast, here we give a little more description about these discoveries.

In [CLM], they consider a class of two-dimensional maps that are given by small perturbations of unimodal maps

$$F(x,y) \mapsto (f(x) - \epsilon(x,y), x) \tag{1.3.1}$$

where $f(x)$ is a unimodal map satisfying some regularity assumptions. They also assume that the C^1 norm of the perturbation ϵ is small (therefore the map is highly dissipative).

In [CLM], the renormalization is defined in such a way that if f is period-doubling renormalizable, F is CLM-renormalizable. The renormalization operator $\mathcal{R}_{CLM}$ is defined by

$$\mathcal{R}_{CLM} F = H^{-1} \circ F^2|_U \circ H,$$

where U is a neighborhood of the "tip" point $v = (f(0), 0)$ and H is an explicit non-linear change of variables.

This renormalization operator has three unique characteristics. First, the "tip" point v can be viewed as a "critical value". Put in another way, this renormalization region U is near the "critical value" rather than near the "critical point" around which other renormalization operators are defined. The subtle choices like this are crucial to their proof. Second, the renormalization domain U is defined by its geometric properties. Finally, the rescaling maps are specifically defined diffeomorphisms based on the renormalization domain instead of universally defined affine maps: it is shown that one particular change of coordinates can "rescale" F^2 back to the Hénon-like map.

They first showed that the degenerate map $F_*(x,y) \equiv (f_*(x), x)$, where f_* is the fixed point of the one-dimensional renormalization operator, is a hyperbolic fixed point for the operator $\mathcal{R}_{CLM}$ with a one-dimensional unstable manifold. They then showed that, for any infinitely renormalizable Hénon-like map F, there exists a hierarchical family of "pieces" $\{B^n_\sigma\}$, 2^n on each level n, so that the set

$$\mathcal{O} = \mathcal{O}_F \equiv \bigcap_n \bigcup_\sigma B^n_\sigma$$

is an attracting Cantor set on which F acts like the dyadic adding machine. Moreover, the diameters of the pieces B^n_σ shrink exponentially at the rate of $O(\lambda^{-n})$, where λ is about 2.6 and is the universal scaling ratio of one-dimensional renormalization. This gives the the explicit formula of $\mathcal{R}_{CLM}$:

$$\mathcal{R}^n_{CLM}F(x,y) = (f_n(x) - b^{2^n}a(x)y(1+O(\rho^n)), x)$$

where f_n is the n-th renormalization of f such that $f_n \to f_*$ exponentially fast, b is the average Jacobian of F, $0 < \rho < 1$, and $a(x)$ is a function. That the renormalizations $\mathcal{R}^n_{CLM}F$ of infinitely renormalizable maps converge at an exponential rate follows directly from this formula. The universality thus resembles the one-dimensional case.

However, this Cantor set $\mathcal{O}$ is not rigid. Indeed, if F and G are two infinitely renormalizable Hénon-like maps with different average Jacobians b_F and b_G, then *the conjugation h between the attractor of F and that of G does not admit a differentiable extension to R^2*. Similar theory has also been generalized to three dimensional dissipative Henon-like maps by Y. Nam in [Ny].

Finally, the authors of [CLM] show that the Cantor sets of generic infinitely renormalizable Hénon-like maps, unlike their one-dimensional counterparts, do not have a universal bounded geometry.

A heuristic explanation of these new features is as follows: near the "tip"of Hénon-like maps, the renormalization regions are slightly tilted and bent parallelograms instead of rectangles so that non-linear part plays a role in the two-dimensional case. The geometry of the Cantor sets therefore is distorted.

There are also many studies about the area-preserving maps as well. Universality has also been observed in families of area-preserving maps by several authors in the early 1980s (see [Bo], [EKW] and [CEK2]). For example, it is shown in [Bo] that the area-preserving Hénon family

$$F_a(x,y) = (-ax^2 - y + 1, x)$$

exhibits the period-doubling cascade to chaos. Maps in this family possess a fixed point

$$\Big(\frac{-1+\sqrt{1+a}}{a}, \frac{-1+\sqrt{1+a}}{a}\Big)$$

which is stable when $1 < a < 3$. When $a_1 = 3$ this fixed point becomes unstable, a cycle of period 2 is born. The period-2 periodic points of F_a are $(x_\pm, x_\mp)$, where $x_\pm = \frac{(1\pm\sqrt{a-3})}{a}$. This cycle is stable until $a_2 = 4$, and then a cycle of period 4 is born. In general, there exists a sequence of parametric values a_k, at which the cycle of period 2^{k-1} turns unstable and a cycle of period 2^k is born. The sequence of parameter values a_k accumulates to a number a_∞. The crucial observation is that the scaling ratio

$$\lim_{k\to\infty} \frac{a_k - a_{k-1}}{a_{k+1} - a_k} = 8.721\ldots$$

given by this family is indeed universal for a larger class of families of area-preserving maps.

This scaling ratio sharply distinguishes itself from the Feigenbaum-Coullet-Tresser scaling ratio that exists in one-dimensional and dissipative two-dimensional systems; the universality in conservative systems are therefore conjectured and later proved very different from that in dissipative systems.

The authors of [EKW] propose the renormalization operator for a proper class of area-preserving maps:

$$\mathcal{R}_{EKW}(F) = \Lambda_F^{-1} \circ F^2 \circ \Lambda_F,$$

where Λ_F is some F-dependent change of coordinates. It has been showed in [CEK2] that Λ_F is a diagonal linear transformation, that is, $\Lambda_F(x, y) = (\lambda_F x, \mu_F y)$.

The authors of [EKW] and [EKW2] also proved the existence of a hyperbolic fixed point for such a period-doubling renormalization operator with the help of the computer.

In stark contrast to dissipative case, the rigidity of such a renormalization operator in area-preserving systems has been established by D. Gaidashev, T. Johnson, and M. Martens in [GJM]: *the period doubling Cantor sets of area-preserving maps in the universality class of the Eckmann-Koch-Wittwer renormalization fixed point are smoothly conjugate (more precisely, they are conjugated by a $C^{1+\alpha}$ map).*

More detailed discussions about the period-doubling route to chaos and renormalizations of both one-dimensional and two-dimensional maps can be found in [LM].

Nevertheless, as we mentioned earlier, the study of the route to chaos in higher-dimensional cases is far from complete. First of all, most recent works are about the two dimensional-maps in Euclidean space other than higher dimensions and other manifolds. With the growth of dimensions, the complexity of chaos grows as well; the properties of manifolds may also lead to different behaviors.

Secondly, even the study of two-dimensional maps is nowhere near complete. Take the dissipative maps for example: few of the works are on the general dissipative families of maps (other than highly dissipative families). A fundamental question, Gambaudo and Tresser's Conjecture that *the dissipative $\mathbb{R}^2$ maps on the boundary of positive topological entropy exhibit a period-doubling cascade*, remains open; see [GT].

While believing that Gambaudo and Tresser's Conjecture is true for generic maps, our work aims at a special (also very large) family of analytic and modestly dissipative (say, the Jacobian of F is less $\frac{1}{2}$) maps and shows they experience the period-doubling route to chaos under certain conditions. The frame of our work is discussed in the next subsection.

1.4 Statement of results

The purpose of this dissertation is to present a partial answer to the long-standing conjecture propose by Gambaudo and Tresser:

The orientation-preserving area-contracting embeddings between two disks in $\mathbb{R}^2$ experience the period-doubling route to chaos.

In this dissertation, we provide a large class of area-contracting maps and show that the route to chaos of this class of maps is via the period-doubling cascade.

Throughout this dissertation, we assume the maps are real analytic. It seems natural that the stronger differentiability the maps have, the more likely the conjecture holds. The analytic maps therefore serve a good starting point. More importantly, the fact that analytic $\mathbb{R}^2$ maps have holomorphic extensions to an open neighborhood in $\mathbb{C}^2$ allows us to take advantage of complex analysis tools. Indeed, in this dissertation we have our discussion in the complex ($\mathbb{C}^2$, or an open subset of $\mathbb{C}^2$) setting rather than in the real ($\mathbb{R}^2$, or an open subset of $\mathbb{R}^2$) setting. Unless otherwise specified, all the objects are assumed to be complex in the following discussion.

In addition, we hope that the our analysis in the complex setting sheds light on analytic maps. Therefore, we assume that all the subsets in $\mathbb{C}^2$ are symmetrical about $\mathbb{R}^2$ and all the $\mathbb{C}^2$ maps commute with conjugation of complex points. This real symmetry requirement simplifies our analysis and fits the holomorphic extensions of analytic maps.

The outline of this dissertation is as follows. We start by introducing the notion of a *branch*, which is a pair consisting of a specially-structured domain $D \subset \mathbb{C}^2$, called a *strip*, and a map $F : D \to \mathbb{C}^2$. Branches are generalized Hénon maps. Indeed, they are essentially "little Hénons" in many ways. Figure 1.4.1 is the projection over $\mathbb{R}^2$ of a branch; it highly resembles the Hénon-maps.

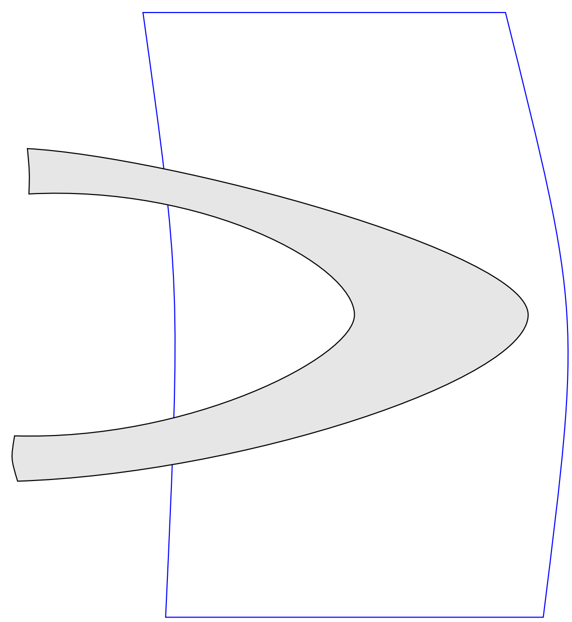


Figure 1.4.1: The projection of a branch onto $\mathbb{R}^2$. The white rectangular area bounded by blue lines is its domain and the shaded area is its image.

The merit of branches is that their compositions are still branches under certain topological restraints, therefore the class of branches is inclusive enough in such a sense. The preliminary properties of branches are discussed in Chapter 2.

The main part of this dissertation considers clusters of finite branches whose domains and ranges are both orderly displayed. Roughly speaking, the member branches have the domains pairwise inclusive, and their real images mutually "envelop" each other. Moreover, the two inclusivenesses are consistent; see Figure 1.4.2. Such clusters are called *nested systems*. As branches are generalizations of the Hénon maps, nested systems are generalizations of the return maps of Hénon maps. A single branch is the simplest form of a nested system.

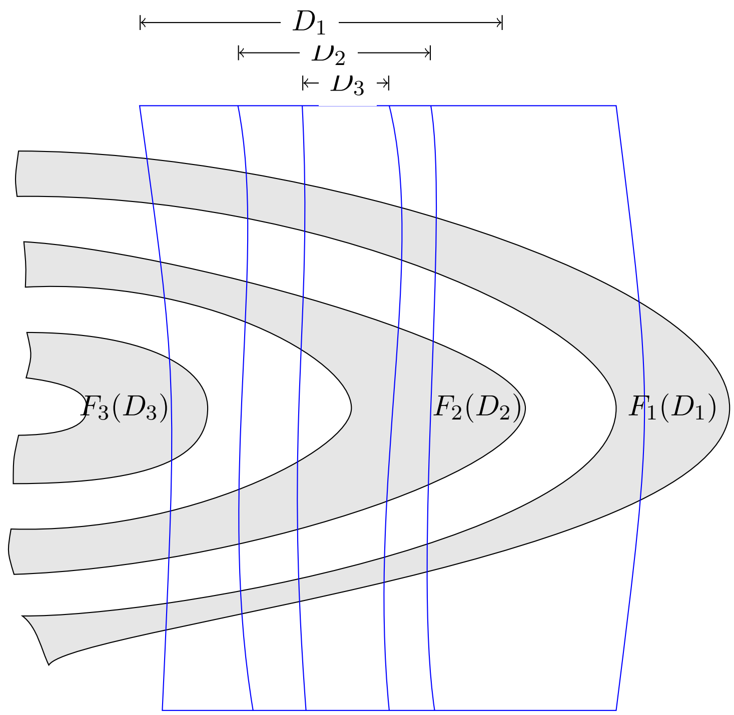


Figure 1.4.2: A nested system consists of three branches. The domains are pairwise inclusive - $D_1 \supset D_2 \supset D_3$, and the real ranges “envelop” each otherz; $F(D_1)$ is “higher” than $F(D_2)$ and $F(D_2)$ is “higher ”than $F(D_3)$. Two orders are consistent.

Because it is the route to chaos that we are interested in, it is natural to assume that all the nested systems in this dissertation have zero topological entropy and finite periodic points (except the one on the boundary of chaos).

We denote the nested systems containing one branch by *simple nested systems*, and those containing two branches by *basic nested systems*. They are not only the simplest nested systems, but also the only forms of nested systems that we need to investigate into. Indeed, in Subection 4.2 we show that, *under zero topological entropy assumption, a nested systems which contains n members is equivalent to a family of n either simple or basic nested systems in terms of non-wandering points.* More specifically, we partition the non-wandering set in such a way that each subset involves at most two consecutive member branches of the original nested system. The three-member nested system in Figure 1.4.2 is equivalent to a family of one basic nested system and one simple nested system illustrated in Figure 1.4.3. Since $D_3 \cap F_3(D_3) = \emptyset$, the branch (D_3, F_3) does not show at all in the equivalence family. The family contains only one basic nested system consisting of (D_1, F_1) and (D_2, F_2), and one simple nested system consisting of (D_2, F_2).

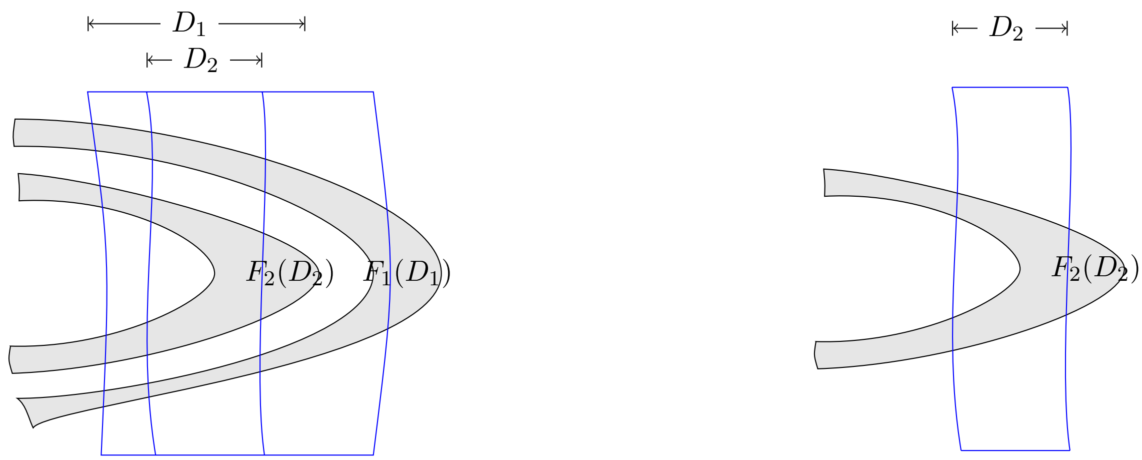


Figure 1.4.3: The equivalence family of Figure 1.4.2

In Chapter 5, we define the *renormalization operator* in the family of basic nested systems, and the *renewal operator* in the family of simple nested systems. Our renormalization operator is based on the first return maps of some *relevant* points and the maximal connected components on which those first return maps are continuous. The relevant point is chosen based on the following idea. Suppose there exists a non-wandering point q in $D_1 \backslash D_2$. As a non-wandering point, it must return to $D_1 \backslash D_2$. We further observe that q must be in $F^{-1}(D_1)$ and return to $F^{-1}(D_1)$ (otherwise, $F_1^2(q) \notin F(D_1) = V$, therefore it cannot be a non-wandering point). We then apply this idea inductively to find the most "relevant" region containing all non-wandering points and use other criteria to refine the "relevant" points. By carefully defining relevant points, we prove that this operator produces a new "smaller" nested system that still describes all the dynamics that are of interest. This new nested system is called the *renormalization* of the original basic nested system. For simplicity, the renormalization of a simple nested system is defined to be the simple nested system itself.

Since a general nested system is equivalent to a family of some either basic or simple nested systems, its renormalization is naturally defined as the family of renormalizations of those either basic or simple nested systems. The new family is called the *first generation (equivalent renormalized) family*. The fact that all the members of the first generation family are nested systems allows an inductive definition of *n-th generation (equivalent renormalized) family*.

On the other hand, if for some generation family, all of its members are simple nested systems, then the renormalization simply replace the family by itself without rendering anything new. Therefore, the inductive process stops.

Generations of nested systems form a *(renormalization) tree*. A priori, a tree can have finitely or infinitely many generations. Figure 1.4.4 shows a two-generation tree starting from a three-member nested system.

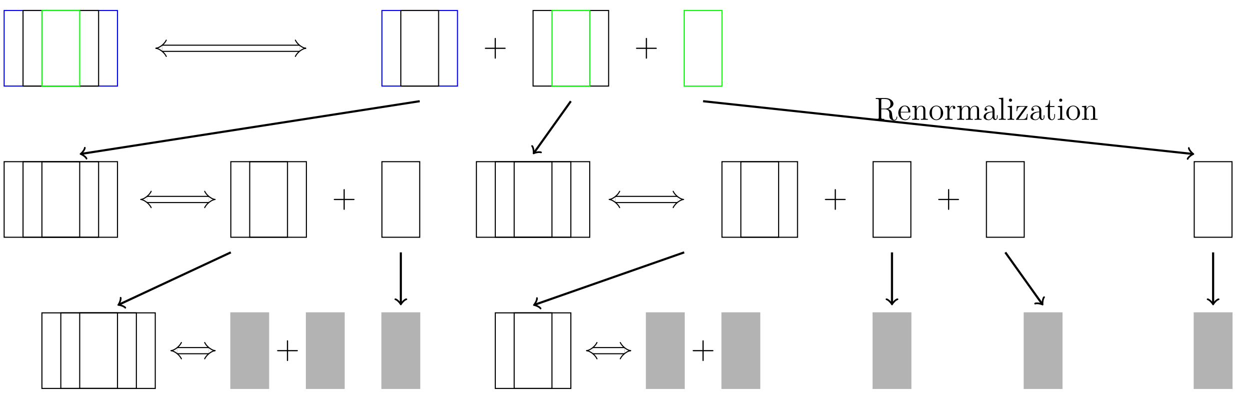


Figure 1.4.4: Tree of generations of nested systems. Each rectangle represents a branch. After two generations of renormalizations, the original family is equivalent to a family of simple nested systems.

However, in Subsection 5.5 we show that the finiteness of generations of a tree is guaranteed by the finiteness of periodic points. Therefore, any nested system is equivalent to finitely many simple nested systems. The proof relies on the fact that each renormalization shorts the periods of periodic points.

If all the periodic points are hyperbolic, the simple nested systems split into two categories: the ones containing a saddle and an attractor, and the ones containing more than one saddle points. If a simple nested system is of the first type, it has been shown ([LM2]) that the attractor must be a global attractor; we consequently understand its dynamics.

The nested systems of the second type have far more mixed dynamics. They are dealt with by the renewal operator which creates a new and "finer" (non-simple) nested system from the old one. Roughly speaking, if there are more than one saddle, then at least one of them, denoted by p_1, has sufficiently "bent" stable manifold that does not go straightly into infinity. The foliations of p_1 provide the natural structure that defines a strip. Depending on the return time of the points of this strip to itself, we can divide this strip into mutually inclusive strips, each of which has its own return map. Each strip and its associated return map together form a branch. Moreover, the real images of those branches "envelop" each other. In this way we obtain a new nested system.

In order to pull off the renewal operator, we need the nested system to be moderately dissipative (its Jacobians are uniformly less than $\frac{1}{2}$) with an a priori bound. The new nested system is conveniently called the *renewal of the simple nested system* (or simply the renewal).

The renewal invokes another round of renormalizations which in turn calls for another renewal. This is a "cycle" of operations. We show that there are only finitely many such cycles by exploiting the finiteness of periodic points in Subsection 6.2.

When the cycles stop, all the member branches are "Little Henons" (simple nested systems) each of which has a global attracting fixed point (and a saddle). To summarize, we have the following Theorem which we prove in Subsection 6.2.

Theorem 1.4.1. *If a nested system has zero entropy and finitely many hyperbolic periodic points, then all the points are either attracted to these periodic points or escape to infinity.*

The case that some of the periodic points are non-hyperbolic has a similar conclusion which follows the hyperbolic case easily and is shown in Chapter 6. In either case, periodic points control all dynamics of the nested system that is away from chaos.

Each cycle has its own renormalization tree. In Chapter 7, by putting trees in sequence, we have a *total tree* of the nested system. Following the shape of a total tree, we therefore obtain a "combinatorial" description of a nested system. For a family of nested systems, the route to chaos is then revealed by its trajectory of combinatorials. Indeed, for a family of nested systems $\{NS_t\}_{t\in[a,b]}$, if the combinatorial does not change for all $t \in [a, b]$, the dynamics does not change either. Conversely, if the combinatorial changes at some point t_1, one of three things can happen at NS_{t_1}: saddle node bifurcation, periodic-doubling bifurcation, and a formal change due to the definition of the renormalization operators.

Among the three possibilities, the formal change is innocent: finite such changes do not impact the dynamics at all, while infinite such changes only accumulate to a saddle node bifurcation. In short, they alone cannot create a route to chaos.

The saddle node bifurcations present a possible route to chaos, but it can be solved. With the growth of the number of periodic points, the total tree also grows in both cycles and generations in each cycles. Both renormalization and renewal operators shrink the Jacobian; more notably, the renormalization operator shrink the Jacobian at a exponential rate. In other words, the modest dissipative nested systems are transformed into highly dissipative simple nested systems. Simple nested systems are a generalization of Hénon maps. Similar to the family of highly dissipative Hénon maps, the family of highly dissipative simple nested systems must experience period-doubling cascade to chaos.

We therefore conclude the main theorem in this dissertation:

Theorem 1.4.2. *A modest dissipative family of nested systems satisfying an apriori bound experiences period-doubling route to chaos.*

Theorem 1.4.2 is proved in Chapter 7. By Theorem 1.4.2, the moderate dissipative family of Henon maps experiences period-doubling route to chaos.

We here highlight two assumptions in our proof. The first is concerned with a topological property of nested systems, see Assumption 5.1.4. The second one deals with the renewal operator. In order to start a renewal operation, the family needs an a priori bound. Although the renormalization operator preserves (even improves) the bound, we have not completely shown that the renewal operator preserves it as well. This is illustrated in the Assumption 6.2.7.

1.5 Notations

1. Throughout the paper we use z and w to denote points in $\mathbb{C}$, and use x and y to denote points in $\mathbb{R}$.

2. Throughout the paper, all maps are holomorphic unless otherwise stated.

3. The coordinate projections are denoted by $\pi_{1,2} : \mathbb{C}^2 \to \mathbb{C}$.

4. A *disc* is a homeomorphic copy of the closed unit disc in $\mathbb{C}$.

5. Put
$$X \equiv \{(z,0) : |z| \leq 1\} \qquad \text{and} \qquad Y \equiv \{(0,w) : |w| \leq 1\}.$$
When there is no confusion, we consider X and Y as discs in $\mathbb{C}$. For example, for $g : X \to \mathbb{C}^2$ we define $g(z) \equiv g(z,0)$.

6. For each $p = (z,w) \in \mathbb{C}^2$, define its conjugate as $\overline{p} = (\overline{z}, \overline{w})$.

7. For each subset $K \subset \mathbb{C}^2$, put
$$KR \equiv K \cap \mathbb{R} \quad \text{or} \quad K \cap \mathbb{R}^2$$
when there is no confusion.

8. Throughout this paper we use I to denote a finite index set and $\mathcal{I}$ to denote either a finite or a countably infinite index set.

9. Put $\mathcal{T} \equiv [-1,1]$.

Chapter 2

Basic Notions

This dissertation is built on a class of objects called branches. They are introduced by M. Lyubich and M. Martens in their recent unpublished manuscript [LM2]. In order to provide a complete picture of our proof, in Subsections 2.1 and 2.2 we recall the definitions, lemmas, and propositions from [LM2] that are necessary for this dissertation. More precisely, we introduce the notion of strips and branches in Subsection 2.1, and the notion of scaling ratio which measures the space between two strips in Subsection 2.2. We then start to discuss real symmetrical strips and branches and introduce an order $\leq_R$ to real symmetrical branches in Subsection 2.3. We will work with the space of the real symmetrical strips and branches in this dissertation.

2.1 Strips and Branches

Let $K \subset \mathbb{C}$ be a disc and $S \subset \mathbb{C}^2$ be a compact set. If $u : K \to S$ is an embedding and $u(K) \subset \partial S$, then u is called a *proper embedding* and $u(K)$ is a *properly embedded disc*.

Put $\gamma \equiv \overline{\cup_j^n u_j(K_j)}$ where $u_j(K_j)$ is a properly embedded disc for every $j = 1, \ldots, n$. In other words, γ is the union of finite properly embedded discs. If the sets $u_j(\mathrm{int}(K_j))$ are pairwise disjoint, then the sets $\overline{u_j(K_j)}$ are called *interior connected components*.

Definition 2.1.1 ([LM2])**.** The connected compact set $S \subset \mathbb{C}^2$ is a *fibration over the disc* $X_S \subset \mathbb{C}$ *with the structure map* $s : U \to \mathbb{C}$, where U is an open neighborhood of $S \subset U$, if

1. $s(S) = X_S$,
2. Rank $\mathrm{Ja}(s(z)) = 1$ for every $z \in U$, where $\mathrm{Ja}(s(z))$ is the Jacobian Matrix of the structure map s at z,
3. each fibre $\gamma_z = s^{-1}(z) \cap S$ is a union of finitely many properly embedded discs, each of which is called a component of the fibre.

Moreover, the *s-boundary* of S is $\partial^s S \equiv s^{-1}(\partial X_S) \cap S \subset \partial S$. An embedding $u : K \to S$ is *properly embedded* relative to the fibration on S if $u(\partial K) \subset \partial^s(S)$.

Definition 2.1.2 ([LM2])**.** A connected compact set $S \subset \mathbb{C}^2$ is called a *strip* with the *base* X_S and the *height* Y_S if

1. S is fibered over Y_S by
$$\pi_2 : \mathbb{C}^2 \to \mathbb{C},$$
2. S is fibered over X_S by
$$s : U \to \mathbb{C},$$
3. for each $w \in Y_S$, the horizontal slice $S_w = \pi_2^{-1}(y) \cap S$, called a π_2-fibre, is properly embedded relative to the s-fibration
$$\partial S_w \subset \partial^s S,$$
4. for each $z \in X_S$, the fibre $\gamma_z = s^{-1}(z) \cap S$ is properly embedded relative to the π_2-foliation
$$\partial \gamma_z \subset \partial^{\text{hor}} S = \partial^{\pi_2} S,$$
5. $$S = s^{-1}(X_S) \cap \pi_2^{-1}(Y_S).$$

Strictly speaking, given a strip S, there are two structure maps: $s : U \to \mathbb{C}^2$ fibers S over the base X_S and $\pi_2 : U \to \mathbb{C}^2$ fibers over the height Y_S. However, π_2 is a universal structure map independent of strips. We will hereafter call the map s to be the structure map of S for simplicity. We will also denote by X and Y the base and the height of any general strip, should no confusion arise. The *horizontal boundary* of the strip S given in Definition 2.1.2 is the π_2-boundary $\partial^{\text{hor}} S = \partial^{\pi_2} S$. The boundary of a strip S satisfies $\partial S = \partial^s S \cup \partial^{\text{hor}} S$.

If the two strips $S_0 \subset S_1$ have the same height, then S_0 is called a *sub-strip* of S_1.

Let $u : K \to S$ be a properly embedded disc. It is called a *vertical disc* if
$$u(\partial K) \subset \partial^{\text{hor}} S.$$
The topological degree $\deg_0(\pi_2 \circ u : K \to Y)$ is properly defined. The *degree* of the vertical disc u is its topological degree. It is called a *horizontal disc* if
$$u(\partial K) \subset \partial^s S.$$
The topological degree $\deg_0(s \circ u : K \to X)$ is properly defined. The degree of the horizontal disc u will be defined later.

Lemma 2.1.3 ([LM2])**.** *Let S be a strip with the base X and the height Y. Then for $z \in X$ and $w \in Y$*
$$\deg_0(s : S_w \to X) = \deg_0(\pi_2 : \gamma_z \to Y),$$
and independent of z and w. The degree of a strip is defined as
$$\deg(S) = \deg_0(s : S_w \to X).$$

Lemma 2.1.4 ([LM2])**.** *Let $u : K \to S$ be a horizontal disc and $\gamma : K_\gamma :\to S$ a vertical disc in S. Then*

$$\#(\gamma(K_\gamma) \cap u(K)) \leq \deg(S) \cdot \deg_0(\gamma) \cdot \deg_0(u).$$

Definition 2.1.5 ([LM2])**.** Let $u : K \to S$ be a horizontal disc map in S. For every sub-strip $D \subset S$ let $K_D = u^{-1}(D)$. The degree of a horizontal disc $u : K \to S$ is defined as

$$\deg(u) = \sup_{D \subset S} \frac{\deg_0(s_D \circ u : K_D \to X_D)}{\deg(D)}.$$

Let $D \subset V$ be a substrip and $u : K \to V$ a horizontal disc. Define

$$\deg_D(u) = \frac{\deg_0(s_D \circ u : K_D \to X_D)}{\deg(D)}.$$

Lemma 2.1.6 ([LM2])**.** *Let w be in Y which is the height of the strip V, and let V_w^j be an interior connected components of V_w. Each V_w^j is a horizontal disc $u_w : V_w^j \to V$. For every substrip $D \subset V$*

$$\sum_j \deg_D(u_j) = 1.$$

In particular, $\deg(u_j) \leq 1$*.*

Lemma 2.1.7 ([LM2])**.** *Let $W \subset \mathbb{C}^2$ be a connected compact one-dimensional manifold and $Y \subset \mathbb{C}$ be a disc such that*

1) $\partial W \subset \pi_2^{-1}(Y)$,

2) $\pi_2(W) = Y$.

Then W is a disc.

Definition 2.1.8 ([LM2])**.** A *branch* is a pair (D, F) consisting of a strip D whose height is Y and the structure map is $s_D : D \to X_D$, and a map $F : D \to \mathbb{C}^2$ that is defined on an open neighborhood of D and is diffeomorphic onto its image. The strip D is called the *domain* of the branch.

Definition 2.1.9 ([LM2])**.** A branch (D, F) *wraps* around a strip V with height Y if

1. $\pi_2 \circ F(D) \subset \mathrm{int}(Y)$,
2. $F(\partial^s D) \cap \mathrm{int}(V) = \emptyset$,
3. $F(D) \cap \mathrm{int}(V) \neq \emptyset$.

The map F is called the *wrapping map*. The branch *wraps strictly* around V, if $F^{-1}(V) = D$, and $s_D = s_V \circ F$.

Lemma 2.1.10 ([LM2]). *Let $S \subset V$ be a sub-strip of V. If a branch $F : D \to \mathbb{C}^2$ wraps around V and $F(D) \cap \operatorname{int}(S) \neq \emptyset$, then it also wraps around S.*

Lemma 2.1.11 ([LM2]). *Let S be a strip with the height Y and $u : K \to \mathbb{C}^2$ be a disc with*

1. *$u(K) \subset \pi_2^{-1}(Y)$,*
2. *$u(\partial K) \cap \operatorname{int}(S) = \emptyset$,*
3. *$u(K) \cap \operatorname{int}(S) \neq \emptyset$.*

Then

$$u^{-1}(S) = \bigcup K_i,$$

where K_i are discs with pairwise disjoint interiors.

Lemma 2.1.12 ([LM2]). *Consider a branch $F : D \to \mathbb{C}^2$ which wraps around V. If $\gamma : K \to V$ is a vertical disc, then $F^{-1}(\gamma(K))$ is a finite union of vertical discs in D:*

$$F^{-1}(\gamma(K)) = \bigcup \gamma_i(K_i),$$

where $\gamma_i : K_i \to D$ are vertical discs in D with pairwise disjoint interiors.

Proposition 2.1.13 ([LM2]). *Let $D_1 \subset V$ be a sub-strip of V with the structure map $s_1 : D_1 \to \mathbb{C}$. If a branch (D, F) wraps around V and $D_0 \subset D$ is a non-empty connected component of $F^{-1}(D_1) \subset D$, then $D_0 \subset D$ is a sub-strip with the induced structure map*

$$s_0 = s_1 \circ F : D_0 \to \mathbb{C}.$$

Moreover, the branch (D_0, F) wraps strictly around D_1.

The Proposition 2.1.13 illustrates the main idea behind the notion of strips and branches: the pre-images of strips are also strips. Therefore, the wrapping map can be assigned a degree as follows. Let (D, F) be a branch wrapping around V. By Proposition 2.1.13, for any sub-strip $D' \subset V$, each connected component D'_j of $F^{-1}(D') \subset D$ is a sub-strip of D.

Definition 2.1.14 ([LM2]). The degree of F is

$$\deg(F) = \sup_{D' \subset V} \frac{\sum_j \deg(D'_j)}{\deg(D')}.$$

Lemma 2.1.15 ([LM2]). *The degree of a branch is a finite positive number.*

Lemma 2.1.16 ([LM2]). *Let $u : K \to V_0$ be a horizontal disc and $D \subset V_0$ be a sub-strip. The branch (D, F) wraps strictly around V_1. Let K_0 be the closure of a connected component of $\operatorname{int}(u^{-1}(D))$, then K_0 is a disc. This defines a horizontal disc $F \circ u : K_0 \to V_1$ in V_1 satisfying*

$$\deg(F \circ u) \leq \deg(F) \cdot \deg(u).$$

2.2 Scaling Ratios

In this section we introduce the notion of scaling ratio of a sub-strip D of V. The scaling ratio is an adaptation of the notion of the extremal length to the context of strips.

Let $K \subset \mathbb{C}$ be a disc and $B \subset \mathrm{int}(K)$ be a compact subset. Let $\Gamma(B, K)$ be the collection of curves *connecting the boundary* of K to B, i.e. curves $\gamma : [t_K, t_B] \to K$ with $\gamma(t_K) \in \partial K$ and $\gamma([t_K, t_B)) \subset K \setminus B$, and $\gamma(t_B) \in B$. A non-negative function $\rho \in L^2(K)$ is K-*allowable* if

$$A(\rho) = \int_K \rho^2 dxdy \in (0, \infty).$$

If ρ is measurable on $\gamma \in \Gamma(B, K)$ then

$$L_\gamma(\rho) = \int_\gamma \rho |dz|,$$

otherwise $L_\gamma(\rho) = \infty$. Furthermore, let

$$L(\rho) = \inf_{\gamma \in \Gamma(B,K)} L_\gamma(\rho).$$

Definition 2.2.1 ([LM2])**.** The *extremal length* of the pair $B \subset K$ is

$$\lambda(B, K) = \sup_\rho \frac{L(\rho)^2}{A(\rho)}.$$

In [LM2], they use the discs to illustrate the calculation of extremal length. Let $K_{r_1} \subset \mathbb{C}$ and $K_{r_2} \subset \mathbb{C}$ be two Euclidean discs centered at the origin with radius r_1 and r_2 respectively. If $r_1 < r_2$, then

$$\lambda(K_{r_1}, K_{r_2}) = \frac{1}{2\pi} \ln \frac{r_2}{r_1}.$$

Lemma 2.2.2 ([LM2])**.** *If $u : K \to \mathbb{C}$ is a univalent map, then*

$$\lambda(B, K) = \lambda(u(B), u(K)).$$

Lemma 2.2.3 ([LM2])**.** *Let $f : K_0 \to K_1$ with $f(\partial K_0) = \partial K_1$ and $B_0 = f^{-1}(B_1)$ with $B_1 \subset \mathrm{int}(K_1)$. Then*

$$\lambda(B_0, K_0) \geq \frac{1}{\deg(f)} \cdot \lambda(B_1, K_1).$$

Lemma 2.2.4 ([LM2])**.** *If $B_1 \subset B_2 \subset \mathrm{int}(K)$, then*

$$\lambda(B_1, K) \geq \lambda(B_2, K).$$

Lemma 2.2.5 ([LM2])**.** *Let $D_i \subset K$ be a collection of pairwise disjoint discs contained in the disc K. Each D_i contains a compact set $E_i \subset D_i$. Let $D = \cup D_i$ and $E = \cup E_i$. Then*

$$\lambda(E, K) \geq \lambda(D, K) + \inf \lambda(E_i, D_i).$$

Lemma 2.2.6 ([LM2])**.** *If* $\mathrm{int}(B) \neq \emptyset$*, then* $\lambda(B, K) < \infty$*. Moreover,* $\lambda(\emptyset, K) = \infty$*.*

Although scaling ratios will only be used for sub-strips, the notion is defined for any compact subset $D \subset \mathrm{int}(V)$, where V is a strip and $\mathrm{int}(D) \neq \emptyset$. Let $u : K \to V$ be a horizontal disc in V and

$$B_u = u^{-1}(D) \subset K.$$

Definition 2.2.7 ([LM2])**.** Let $D \subset V$ be a compact set with $\mathrm{int}(D) \neq \emptyset$ and $D \subset \mathrm{int}(V)$. The space around D with respect to v is

$$[D, V] = \inf_u \deg(u) \cdot \lambda(B_u, K).$$

An interior connected component V_y^j of a slice V_y of V is a disc. Let D be a sub-strip of V and $D_y^j = D \cap V_y^j$. Then, by Lemma 2.1.6,

$$\lambda(D_y^j, V_y^j) \geq [D, V].$$

In particular, $[D, V]$ measures the worst extremal length in each interior component of slices.

Lemma 2.2.8. *For every pair* $D \subset \mathrm{int}(V)$

$$0 < [D, V] < \infty.$$

Definition 2.2.9 ([LM2])**.** The *scaling-ratio* of $D \subset \mathrm{int}(V)$, where D is compact with $\mathrm{int}(D) \neq \emptyset$ and $D \subset \mathrm{int}(V)$, is

$$\sigma(D, V) = e^{-2\pi \cdot [D,V]}.$$

Lemma 2.2.10 ([LM2])**.** *Let* $D \subset V$ *be a sub-strip and* E *be a compact subset of* D *with* $\mathrm{int}(E) \neq \emptyset$ *and* $E \subset \mathrm{int}(D)$*. Then*

$$[E, V] \geq [E, D] + [D, V],$$

or equivalently

$$\sigma(E, V) \leq \sigma(E, D) \cdot \sigma(D, V).$$

Lemma 2.2.11 ([LM2])**.** *Let* $K_1 \supset K_2 \supset K_3 \supset \cdots$ *be a sequence of discs with*

$$\sum_n \lambda(K_{n+1}, K_n) = \infty.$$

Then

$$\bigcap K_n = \{p\}.$$

Lemma 2.2.12 ([LM2])**.** *Let* $V \supset D_1 \supset D_2 \supset \cdots \supset D_n \supset D_{n+1} \supset \cdots$ *be a sequence of sub-strips with*

$$\sum_n [D_{n+1}, D_n] = \infty.$$

Then for every $y \in Y$

$$\dim(V_y \cap \bigcap D_n) = 0.$$

In particular,

$$\mathrm{int}(\bigcap D_n) = \emptyset.$$

Lemma 2.2.13 ([LM2]). *Let $D_1 \subset V$ be a sub-strip and (D, F) be a branch which stritcly wraps around V and $D_0 = F^{-1}(D_1) \subset D$.*

$$[D_0, D] \geq \frac{1}{\deg(F)} \cdot [D_1, V],$$

or equivalently

$$\sigma(D_0, D) \leq \sigma(D_1, V)^{\frac{1}{\deg(F)}}.$$

2.3 Real Symmetrical Strips and Branches

A compact set $U \subset \mathbb{C}^2$ is called a *real symmetrical* set if for each $p \in U$ its conjugate $\overline{p}$ is also in U.

Let $U \subset \mathbb{C}^2$ be a real symmetrical set. If a map $F : U \to \mathbb{C}^2$ commutes with the standard conjugation, it is called a *real symmetrical map* on U. For example, The Henon maps

$$F : \mathbb{C}^2 \to \mathbb{C}^2, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x^2 - by - c \\ x \end{pmatrix}$$

are real symmetrical on $\mathbb{C}^2$ provided that b and c are both real.

Definition 2.3.1. Let $S \subset \mathbb{C}^2$ be a strip with the base X, the height Y, the structure map $s : S \to X$, and the projection $\pi_2 : S \to Y$. S is a *real symmetrical strip* if

1. S is a real symmetrical set and
2. the structure map s is real symmetrical .

Definition 2.3.2. Let (D, F) be a branch that warps around a real symmetrical strip V. (D, F) is called a *real symmetrical branch* if

1. D is a real symmetrical strip and
2. F is a real symmetrical map.

Lemma 2.3.3. *If (D, F) wraps around a real symmetrical strip V and $W \subset V$ is a real symmetrical substrip with the structure map s_W, then $F^{-1}(W) \subset D$ is a real symmetrical strip with the structure map $s_W \circ F$.*

Proof. For each $p \in D$ and $q = F(p) \in W$, we have $\overline{q} \in W$ because W is real symmetrical. Since (D, F) is real symmetrical , it follows that $\overline{p} \in D$ and $F(\overline{p}) = \overline{q} \in W$. Hence $\overline{p} \in F^{-1}(W)$. The structure map of $F^{-1}(W)$ is $s_W \circ F$ by Proposition 2.1.13 satisfying

$$s_W \circ F(\overline{p}) = s_W(\overline{F(p)}) = \overline{s_W(F(p))}. \qquad \square$$

Given a real symmetrical map F, its restriction to $\mathbb{R}^2$, denoted by $F|_{\mathbb{R}^2}$, is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. The Jacobian matrix of $F|_{\mathbb{R}^2}$ is denoted by $\mathrm{Jac}(F|^2_{\mathbb{R}})$. Let (D, F) be a real symmetrical branch that strictly wraps around a real symmetrical strip V. (D, F) is called a *positive* branch

$$\mathrm{Jac}(F|_{\mathbb{R}^2}) \geq 0;$$

(D, F) is called a *dissipative* branch if there exists some $0 \leq d < 1$ such that

$$\mathrm{Jac}(F|_{\mathbb{R}^2}) \leq d.$$

If F has degree n, (D, F) is called a *degree-n branch* for short.

Assume the strip V has degree one. The branch (D, F) that strictly wraps around V is called a *left-end branch* if

$$F\big(s_D^{-1}(\pm 1)\big) \cap \mathbb{R}^2 \;\subset\; s_V^{-1}(-1) \cap \mathbb{R}^2;$$

it is called a *right-end branch* if

$$F\big(s_D^{-1}(\pm 1)\big) \cap \mathbb{R}^2 \;\subset\; s_V^{-1}(+1) \cap \mathbb{R}^2.$$

For simplicity, this paper assumes that all the branches, if applicable, are left-end. All the arguments are the same if branches are right-end.

Remark 2.3.4. Let (D, F) be a real symmetrical degree-2 left-end branch that wraps around a real symmetrical strip V with $\deg(V) = 1$. Let $K \subset D$ be a horizontal disc with $\deg(K) = 1$. Then $F(K)$ is a horizontal disc in V and $\deg(F(K)) \leq \deg(F) \cdot \deg(K) = 2$ because of Lemma 2.1.16. For any degree 1 vertical disc K_γ in V,

$$\begin{aligned}\#\big\{F(K) \cap K_\gamma\big\} &\leq \deg(V) \cdot \deg(K_\gamma) \cdot \deg_0(F(K)) \\ &\leq \deg(V) \cdot \deg(K_\gamma) \cdot (\deg(F(K)) \cdot \deg(V)) \leq 2.\end{aligned}$$

The first inequality follows from Lemma 2.1.4, and the second inequality follows from the definition of the degree of a horizontal disc.

The vertical slice $s_V^{-1}(z)$ is a degree 1 vertical disc in V for any $z \in X$, hence we have

$$\#\big\{F(K) \cap s_V^{-1}(z)\big\} \leq 2.$$

On the other hand, if K is further assumed to be real symmetrical, then it follows the fact that F is left end, we have

$$\#\big\{F(K) \cap s_V^{-1}(-1) \cap \mathbb{R}^2\big\} \geq 2.$$

Thus,

$$\#\big\{F(K) \cap s_V^{-1}(-1)\big\} = 2.$$

The two intersection points are in $s_V^{-1}(-1) \cap \mathbb{R}^2$, therefore they determine a unique path in $s_V^{-1}(-1) \cap \mathbb{R}^2$ with which $F(KR) \equiv F(K \cap \mathbb{R}^2)$ forms a Jordan curve. This type of Jordan curves is called *a left attached Jordan curve associated with $F(KR)$*.

In particular, if $\deg(D) = 1$, then for any $y \in Y$, the horizontal slice $\pi_2^{-1}(y) \cap D$ is a degree 1 horizontal real symmetrical disc in D. In this case, we introduce the following notations for later convenience. For every $x \in XR$ and $y \in YR$, put

$$\begin{aligned} &l_y \equiv \pi_2^{-1}(y) \cap D \cap \mathbb{R}^2, \qquad && l_y(x) \equiv l_y \cap s_D^{-1}(x) \\ &\gamma_y \equiv F(l_y), && \gamma_y(x) \equiv F(l_y(x)). \end{aligned} \tag{2.3.1}$$

Clearly, for any $y \in YR$,

$$\gamma_y(\pm 1) = \left\{ F(\pi_2^{-1}(y) \cap D) \cap s_V^{-1}(-1) \right\}.$$

The left attached Jordan curve associated with γ_y is denoted by J_y.

Definition 2.3.5. Let (D, F) be a real symmetrical degree-2 left-end branch wrapping around a real symmetrical strip V such that $\deg(D) = \deg(V) = 1$.

1. For a point $p \in VR$ (resp. a set $K \subset VR$), write

   $$p \leq_R D \text{ (resp. } K \leq_R D)$$

   if p (resp. K) lies inside J_y for all $y \in YR$.

2. For a real symmetrical branch $(\widetilde{D}, \widetilde{F})$ that strictly wraps around V, write

   $$\widetilde{D} \leq_R D$$

   if $\widetilde{F}(\widetilde{D}R)$ lies inside J_y for all $y \in YR$.

Lemma 2.3.6. *Let (D, F) and V be as in definition 2.3.5, and $K \subset D$ be a real symmetrical horizontal disc with $\deg(K) = 1$. If $p \leq_R D$, then p lies inside the left attached Jordan curve associated with $F(KR)$.*

Proof. Let $y_1 = \inf \pi_2(KR)$ and $y_2 = \sup \pi_2(KR)$. The left attached Jordan curve associated with $F(KR)$ must lie either outside of J_{y_1} or J_{y_2}. The Lemma follows. ☐

For later convenience, we introduce the following notations: If in a setting we have $(n+1)$ real symmetrical strips $D_i, i = 1, ..., n$ and V,

1. their structure maps will be denoted by $s_i, i = 1, ..., n$ and s, respectively, and

2. for $i = 1, ..., n$, $x \in X$ and $y \in Y$

$$\begin{aligned} &b^{\pm c} \equiv s^{-1}(\pm 1), \quad b^{\pm} \equiv b^{\pm c}|_{\mathbb{R}^2}, \quad b^{uc/dc} \equiv \pi_2^{-1}(\pm 1) \cap V, \quad b^{u/d} \equiv b^{uc/dc}|_{\mathbb{R}^2}, \\ &b_i^{\pm c} = s_i^{-1}(\pm 1), \quad b_i^{\pm} = b_i^{\pm c}|_{\mathbb{R}^2}, \quad b_i^{uc/dc} \equiv \pi_2^{-1}(\pm 1) \cap D_i, \quad b_i^{u/c} \equiv b_i^{uc/dc}|_{\mathbb{R}^2}, \\ &b^{\pm}(y) \equiv b^{\pm} \cap \pi_2^{-1}(y), \quad b^{u/d}(x) \equiv b^{u/d} \cap s^{-1}(x), \\ &b_i^{\pm}(y) \equiv b_i^{\pm} \cap \pi_2^{-1}(y), \quad b_i^{u/d}(x) \equiv b_i^{u/d} \cap s^{-1}(x). \end{aligned} \tag{2.3.2}$$

If there are only two strips D and V, the above notations are modified modestly as follows:

1. their structure maps will be denoted by s_D and s_V, respectively, and
2. b_1 is replaced with Db. For example, $Db^{\pm c} \equiv b_1^{\pm c}$.

Using this set of notations, for example, we can rephrase the definition of left-end as follows: Let (D, F) strictly wraps around a real symmetrical V which has degree 1. The branch (D, F) is a *left-end* branch if

$$F\big(Db^{\pm}\big) \subset b^{-}.$$

Definition 2.3.7 (Vertical Curves)**.** Let V be a strip. $l \subset VR$ is called a *vertical curve* if l is the graph of a function over YR.

Example 2.3.8. If V is a degree one real symmetrical strip, $b^{\pm}$ are vertical curves.

Every vertical curve in V intersects with b^u and b^d once, respectively. Given two disjoint vertical curves l_1 and l_2, we therefore have four points p_u, p_d, q_u and q_d where

$$\{p_u\} = l_1 \cap b^u, \quad \{p_d\} = l_1 \cap b^d, \quad \{q_u\} = l_2 \cap b^u \quad \text{and} \quad \{q_d\} = l_2 \cap b^d.$$

Suppose the strip V has degree 1. Then p_u and q_u (resp. p_d and q_d) determine a unique path in b^u (resp. b^d) connecting them. The two paths and the two disjoint vertical curves together are form a Jordan curve. This Jordan curve is said to be *prescribed by the two vertical curves l_1 and l_2.*

Both the Jordan curve and the region (resp. its closure) enclosed by the Jordan curve are sometimes referred to as a rectangle. Conversely, a region $R \subset V \cap \mathbb{R}^2$ (resp. its closure $\overline{R}$) is called a *rectangle* if ∂R is a Jordan curve prescribed by two vertical curves in V.

Definition 2.3.9. Let V be a degree one real symmetrical strip and $l \subset V$ be a vertical curve. A point $p \in V$ or a subset $K \subset V$ is said on the *left (resp. right) side* of l with respect to V if it lies inside the Jordan curve prescribed by l and b^-(resp. b^+).

Remark 2.3.10. When $\pi_2(b^-(y)) < \pi_2(b^+(y))$, the definition 2.3.9 is consistent with our intuition. Without loss of generality, we hereafter assume $\pi_2(b^-(y)) < \pi_2(b^+(y))$.

Definition 2.3.11 (Holomorphic Vertical Curves)**.** A curve $l \subset V$ is called a *holomorphic vertical curve* if l is the restriction to $\mathbb{R}^2$ of a vertical disc $K_\gamma \subset V$, where $\deg(K_\gamma) = 1$. K_γ is called the *containing (vertical) disc* of l.

A holomorphic vertical curve is a vertical curve.

Definition 2.3.12. Let V be a strip. A compact set $l \subset VR$ is called a *horizontal curve* if $s(l)$ is the graph of a function over XR. The curve l is called a *holomorphic horizontal curve* if l is the restriction to $\mathbb{R}^2$ of a horizontal disc $K \subset V$, where $\deg(K) = 1$. K is called the *containing (horizontal) disc* of l.

Similar to the vertical curve case, if $\deg(V) = 1$, two disjoint horizontal curves c_1 and $c_2 \subset V$ with part of $b^{\pm}$ form a Jordan Curve. This Jordan curve is said *to be prescribed by two horizontal curves c_1 and c_2.*

Chapter 3

The linkage between positive entropy and orbits of certain points

In this chapter, we show that the existence of certain orbits implies the existence of transverse homoclinic points; see Figure 3.2.1 and Proposition 3.2.1. This property plays a crucial role in the following chapters.

3.1 Preliminary Lemmas

Lemma 3.1.1. *Let (D, F) be a branch wrapping around a strip V and $K_\gamma \subset V$ be a vertical disc. Put $K_0 \equiv F^{-1}(K_\gamma)$. Then K_0 is a union of vertical discs in D. Moreover,*

$$\deg(F^{-1}(K_\gamma)) \leq \deg(V) \cdot \deg(F) \cdot \deg(K_\gamma).$$

Proof. Lemma 2.1.12 concludes that K_0 is a union of vertical discs in D. For every $y \in Y$, $\pi_2^{-1}(y) \cap D$ is a union of horizontal discs in D, denoted by $K_y^j, j = 1, 2, ..., n$, where

$$\sum_j \deg_D(K_y^j) = 1.$$

It follows Lemma 2.1.16 that $\{F(K_y^j), j = 1, 2, ..., n\}$ is a union of horizontal discs in V, and

$$\sum_j \deg_V F(K_y^j) \leq \sum_j \deg_D(K_y^j) \cdot \deg(F) = \deg(F) \cdot \sum_j \deg_D(K_y^j) = \deg(F).$$

By the definition of degree of horizontal discs, we therefore have

$$\sum_j \deg_0 F((K_y^j)) = \sum_j \deg_V F((K_y^j)) \cdot \deg(V) \leq \deg(F) \cdot \deg(V).$$

The number of intersection points of a vertical disc and a horizontal disc has an upper bound given by Lemma 2.1.4, that is

$$\#((K_\gamma) \cap \bigcup_j F(K_y^j)) \leq \deg(K_\gamma) \cdot \sum_j \deg_0 F(K_y^j) \leq \deg(K_\gamma) \cdot \deg(F) \cdot \deg(V).$$

Since F is a diffeomorphism,

$$\#(F^{-1}(K_\gamma) \cap \pi_2^{-1}(y)) \leq \deg(K_\gamma) \cdot \deg(F) \cdot \deg(V),$$

the Lemma follows. $\square$

Corollary 3.1.2. *Let (D, F) be a branch wrapping around a strip V. Let $l \subset V$ be a holomorphic vertical curve and $l_0 \equiv F^{-1}(l)$. For every $y \in YR$, l_y and γ_y are defined as in Equation 2.3.1. Then for every $y \in YR$,*

$$\#(l_y \cap l_0) = \#(\gamma_y \cap l) \leq \deg(F) \cdot \deg(V).$$

In particular, if $\deg(V) = 1$, *then*

$$\#(l_y \cap l_0) = \#(\gamma_y \cap l) \leq \deg(F).$$

Proof. By definition, the containing disc of l, denoted by K_γ, must be a degree one vertical disc in V. The corollary follows Lemma 3.1.1 immediately. $\square$

Lemma 3.1.3. *Let (D, F) be a real symmetrical degree-2 left-end branch that wraps around a real symmetrical strip V such that* $\deg(V) = 1$. *Let $K_\gamma \subset V$ be a real symmetrical vertical disc with degree one, and K_0 be $F^{-1}(K_\gamma)$. If K_0R consists of two disconnected vertical curves, denoted by l_1 and l_2, then K_0 consists of two degree one vertical discs in D.*

Proof. We have $\deg(K_0) \leq 2$ by Lemma 3.1.1, and $\deg(K_0) \geq 2$ by observing the existence of two vertical curves. Therefore, K_0 either consists of two vertcial discs in D, each of which has degree one, or consists of a single vertical disc in D with degree two.

Presume K_0 is a single vertical disc in D with degree two. Then $F(K_0) = F(F^{-1}(K_\gamma)) \subset K_\gamma$ is a real symmetrical disc. Since $\deg(K_\gamma) = 1$, K_γ can be considered as a graph over Y. It follows that $F(K_0)$ can be considered as a graph over a real symmetrical disc $W \subset Y$. Therefore, $F(K_0) \cap \mathbb{R}^2$ is a nonempty connected component. However,

$$F(K_0) \cap \mathbb{R}^2 = F(l_1) \cup F(l_2)$$

is disconnected since l_1 and l_2 are disconnected. This contradiction indicates that our presumption is impossible, i.e. K_0 must consist of two topological degree one vertical discs in D. $\square$

Lemma 3.1.4. *Let (D, F) be a real symmetrical degree-2 left-end branch that wraps around a real symmetrical strip V. Let $S \in V$ be a real symmetrical strip. Assume* $\deg(D) = \deg(V) = \deg(S) = 1$. *Let $K \subset D$ be a real symmetrical horizontal disc with degree one, and $K_1 \equiv F(K) \cap S$. If K_1R consists of two disconnected holomorphic horizontal curves (denoted by c_1 and c_2) in S, then K_1 consists of two degree one horizontal discs in S.*

Proof. The proof is almost identical to the proof of Lemma 3.1.3, hence it is omitted. $\square$

Lemma 3.1.5. *Let (D, F) be a real symmetrical branch that wraps around a real symmetrical strip V and $S_\gamma \subset V$ be a real symmetrical strip. Put $S_0 \equiv F^{-1}(S_\gamma)$. If S_0R is connected, so is S_0.*

Proof. If there were more than one connected componentof S_0, then S_0R would be disconnected as well. $\square$

Lemma 3.1.6. *Let (D, F) be a real symmetrical degree-2 left-end branch wrapping around a real symmetrical strip V such that $\deg(D) = \deg(V) = 1$. Let $l \subset V$ be a holomorphic vertical curve. If there exists a point $p \in V$ on the right side of l such that $p \leq_R D$, then $l_0 \equiv F^{-1}(l)$ consists of two disconnected vertical curves in D.*

Proof. Let J_L denote the Jordan curve prescribed by the b^- and l. For every $y \in YR$, let l_y, γ_y, and J_y be as in Equation 2.3.1. The construction of J_y indicates that either J_y is inside of J_L or $J_y \cap J_L$ (especially, $\gamma_y \cap Db^-$)is nonempty. Since l is on the right side of b^-, $p \in DR$ implies that p lies outside of J_L. However, by definition $p \leq_R D$ is equivalent to p lying inside J_y. It follows that $\gamma_y \cap l \neq \emptyset$. In particular, $\#(\gamma_y \cap l) \geq 2$. On the other hand, Corollary 3.1.2 implies $\#(\gamma_y \cap l) \leq 2$.

Therefore, for every $y \in YR$, there are two and only two points p_y and $q_y \in l_y$ such that $F(p_y)$ and $F(q_y)$ belong to l. By continuity, $l_p \equiv \{p_y : y \in YR\}$ and $l_q \equiv \{q_y : y \in YR\}$ each forms a vertical curve in D. It is obvious they are disconnect. $\square$

Let $C \subset V$ be consisting of only two disjoint horizontal curves c_1 and c_2. For simplicity, the Jordan Curve horizontally prescribed by c_1 and c_2 is called the Jordan Curve prescribed by C.

Lemma 3.1.7. *Let (D, F) be a real symmetrical degree-2 left-end branch wrapping around a real symmetrical strip V and $S \subset V$ be a real symmetrical strip with the structure map $\hat{s}$. Assume $\deg(S) = \deg(D) = \deg(V) = 1$. Put $Sb^\pm \equiv \hat{s}^{-1}(\pm 1)$. Assume Db^- is on the left side of Db^+ with respect to V. Let α and $\beta \subset D$ be two holomorphic horizontal curves, and $p \in D$ be a point inside the Jordan Curve horizontally prescribed by α and β. If $F(p)$ is on the right side of Sb^+, then either $\alpha_1 \equiv F(\alpha) \cap S$ or $\beta_1 \equiv F(\beta) \cap S$ consists of two disconnected holomorphic horizontal curves in S. More precisely, let J_α and J_β be the left attached Jordan curve associated to $F(\alpha)$ and $F(\beta)$, respectively. Either J_α is inside J_β or J_β is inside J_α. Suppose J_α is inside J_β, then α_1 must consist of two disconnected horizontal curves in S.*

Proof. The proof is almost identical to the proof of Lemma 3.1.6, thus it is omitted. $\square$

In Lemma 3.1.7, both α and $\beta \subset D$ are two holomorphic horizontal curves. By definition, α (resp. β) has a containing horizontal disc K_α (resp. K_β) in D. Suppose it is α_1 that consists of two disconnected horizontal curves c_1 and c_2 in S. By Lemma 3.1.4, $F(K_\alpha) \cap S$ contains two disconnected horizontal discs in S, each of which has degree one. Moreover, they must be the containing discs of c_1 and c_2, respectively.

Lemma 3.1.8. *Let (D, F) be the same as in Lemma 3.1.6. Let $\widetilde{V} \subset V$ be a real symmetrical strip. Assume $l \subset \widetilde{V}R$ is a holomorphic vertical curve so that $\widetilde{V}R$ is on the right side of l. If $F^{-1}(l)$ consists of two disconnected (holomorphic) vertical curves, denoted by l_L and l_R, then $F^{-1}(\widetilde{V}R) \subset D$ is inside the Jordan curve prescribed by l_L and l_R. Moreover, the left side region in DR of l_L, denoted by R_L, and the right side region in DR, denoted by R_R, are mapped to the left side of l in VR.*

Proof. For every $y \in YR$, we parametrize l_y by an orientation preserving diffeomorphism $t_y : [0,1] \to l_y$. (D,F) is a left-end branch, hence $F(t_y(0)$ and $F(t_y(1) \in b^-$. Since $\widetilde{V}R$ lies on the right side of l, by definition, it lies outside the Jordan Curve prescribed by l and b^-, denoted by J. For any point $p \in l_{y'}$ so that $F(p) \in \widetilde{V}R$, $F(p)$ lies outside of J. Say $p = t_{y'}(t')$ for some t'. Paths $t_{y'}([0,t'])$ and $t_{y'}([t',1])$ each has one end point inside J and the other endpoint outside of J. Hence, there exist $t'_1 \in (0,t')$ and $t'_2 \in (t',1)$ so that $F(t'_y(t'_i) \in J,\ for\ i = 1,2$. Moreover, $F(D) \cup b^{u/d} = \emptyset$, therefore $F(t_{y'}(t'_i)) \in l$, for $i = 1,2$. In other words, $t_{y'}^{-1}(F^{-1}(p)) \in (t'_1, t'_2)$. The lemma follows. □

3.2 Existence of homoclinic points

Let $\{A_n\}_{n=1}^{\infty}$ be a family of subsets of $\mathbb{C}^2$. A set A is called the *limit* of A_n, denoted by

$$A = \lim_n A_n,$$

if A consists of the limit points of all sequence of points $\{(z_n, w_n) \in A_n\}_{n=1}^{\infty}$.

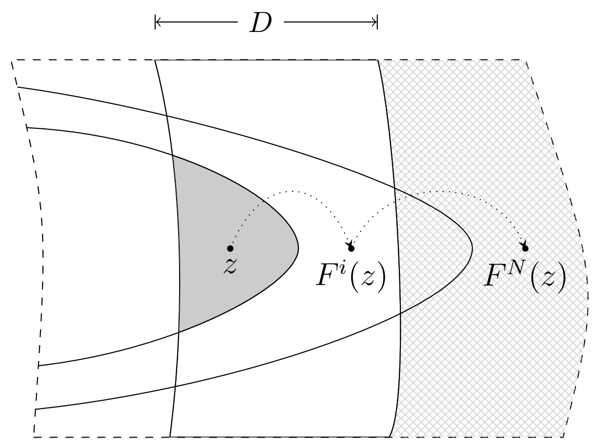


Figure 3.2.1: The real point $z \in D$ lies in the shaded areas (i.e. $z \leq_R D$). If there exists a point $N \in \mathbb{Z}^+$ so that $F^N(z)$ lies in the crosshatched area (i.e. on the right hand side of D), then there exists a homoclinic point.

Proposition 3.2.1. *Let (D,F) be a real symmetrical degree-2 left-end branch that wraps around a real symmetrical strip V such that $D \subset V$ and $\deg(D) = \deg(V) = 1$. Assume Db^- is on the left side of Db^+ with respect to V, $z \in DR$, and $z \leq_R D$. If there exists an $N \in \mathbb{Z}^+ \cup \{0\}$ such that $F^N(z)$ lies on the right side of Db^+, then F has a transverse homoclinic point.*

Proof. 1) The first step is to prove that there exists a fixed point $P_\gamma \in DR$ that has a stable manifold. Put

$$D_m \equiv F^{-m}(D).$$

Let $b_m^{\pm}$ be as define in Equation 2.3.2. We prove by induction that for every $m \in \mathbb{Z}^+$, there exists a connected component of b_m^-, denoted by l_m, such that

1. l_m is a holomorphic vertical curve of which $D_m R$ is on the right side, and

2. z is on the right side of l_m.

For $m = 0$, put $l_0 \equiv b_0^-$, the above claims are true. Suppose they are true for $m < M$. Since z is on the right side of l_{M-1}, the preimage of l_{M-1} consists of two disconnected (holomorphic) vertical curves by Lemma 3.1.6. Since l_{M-1} is the leftmost holomorphic vertical curve in D_{M-1}, by Lemma 3.1.8, $D_M R$ is on the right side of one of them, denoted by l_M. On the other hand, if z were on the left side of l_M, by Lemma 3.1.8, $F(z)$ would be on the left side of $D_{M-1}R$. So by applying Lemma 3.1.8 M-times, $F^M(z)$ would be on the left side of DR. However, this contradicts the assumption of Theorem 3.2.1 that there exists an $N \in \mathbb{Z}^+$ such that $F^N(p)$ lies on the right side of Db^+. So the claims are true for $m = M$. This completes the proof of the claim.

The containing disc of l_0 (i.e b_0^-) is b_0^{-c}. For every $m \in \mathbb{Z}^+$, the holomorphic vertical curve $l_m \subset F^{-1}(l_{m-1})$. It follows that the containing disc of l_m must be a connected component of $F^{-m}(b_0^{-c})$, denoted by K_γ^m. Especially,

$$K_\gamma^m \subset F^{-1}(K_\gamma^{m-1}).$$

For every $m \in \mathbb{Z}^+$, K_γ^m is the graph of a bounded function $\psi_m : Y \longrightarrow V$. Since $\{\psi_m, m \in \mathbb{Z}^+\}$ is a normal family, it uniformly converges to a biholomorphic fuction

$$\psi_\infty : Y \longrightarrow V.$$

Put

$$K_\infty^\gamma \equiv \{(z, w) | w \in Y, z = \psi_\infty(w)\}.$$

It is clear that $K_\infty^\gamma = \lim_m K_k^\gamma$.

For every (z, w), there exists a sequence of points $\{(z_m, w_m) \in K_k^\gamma\}_k$ such that

$$\lim_k (z_m, w_m) = (z, w).$$

For every $m \in \mathbb{Z}^+$ and every pair (z_m, w_m), there exists $(z'_{m-1}, w'_{m-1}) \in K_{m-1}^\gamma$ such that

$$(z_m, w_m) = F^{-1}((z'_{m-1}, w'_{m-1})).$$

Therefore,

$$\begin{aligned} F((z, w)) =& F(\lim(z_m, w_m)) = \lim F((z_m, w_m)) \\ =& \lim F(F^{-1}((z'_{m-1}, w'_{m-1})) = \lim(z'_{m-1}, w'_{m-1}) \in \lim K_{m-1}^\gamma = K_\infty^\gamma. \end{aligned}$$

That is,

$$F(K_\infty^\gamma) \subset K_\infty^\gamma.$$

Since K_∞^γ is a holomorphic disc, by Schwartz' Lemma, there is a fixed point $P_\gamma \in K_\infty^\gamma$ whose stable manifold is K_∞^γ.

2) The next goal is to show that there exists a fixed point $P \in DR$ with an unstable manifold. We prove by induction that for every $\in \mathbb{N}$, there exists a pair of holomorphic horizontal curves, denoted by $\alpha_m \subset V$ and $\beta_m \subset V$. Let J^m be the Jordan curve horizontally prescribed by α_m and β_m, and R^m be the closure of the region inside J^m. The following properties are true:

1. z lies inside J^m and
2. $F(R^m) \cap D$ lies inside J^m (i.e. $F(R^m) \cap D \subset R^m$).

Put

$$\alpha_0 \equiv b^d \qquad \text{and} \qquad \beta_0 \equiv b^u.$$

The containing horizontal disc of α_0 is b^{dc} and that of β_0 is b^{uc}. Naturally, we have

1. z lies inside J^0;
2. $F(R^0) \cap D$ lies inside J^0.

Suppose the claim is true for every $m < M$. Because $F^N(z)$ is properly defined, $F^{N-1}(z) \in D$. Moreover, for $m < M$, $F(R^m) \cap D \subset R^m$, we have

$$F^{N-1}(z) \in \Big(F^{N-1}(R^{M-1}) \cap D\Big) \subset \Big(F^{N-2}(R^{M-1}) \cap D\Big) \subset ... \subset \Big(F(R^{M-1}) \cap D\Big) \subset R^{M-1}.$$

That is, $F^{N-1}(z)$ lies inside the Jordan curve J^{M-1}. By Lemma 3.1.7, either $F(\alpha_{M-1}) \cap D$ or $F(\beta_{M-1}) \cap D$ has two disconnected holomorphic horizontal curves. Without loss of generality, we can assume the left attached Jordan curve associated with $F(\beta_{M-1})$ lies inside that associated with $F(\alpha_{M-1})$. Lemma 3.1.7 further assures that $F(\alpha_{M-1}) \cap D$ must have two holomorphic horizontal curves, α_M and β_M. That is, the second claim is true for $m = M$, i.e. $F(R^M) \cap D$ lies inside J^M. Moreover, Lemma 2.3.6 shows that z is inside the left attached Jordan curve associated $F(\alpha_{M-1})$. Combined with the fact that z is in DR, we conclude that z lies inside J^M. This completes the proof of the claim.

For every m, since the containing disc of α_0 is $b^{d/c}$, the containing horizontal disc of holomorphic horizontal curve α_m is a connected component of $F^m(b^{d/c})$, denoted by K_m. Then

$$\deg(K_m) = 1 \qquad \text{and} \qquad F(K_m) \supset K_{m+1}.$$

Since $\deg(D) = 1$, the map H given by

$$H : D \longrightarrow \mathbb{C}^2, \qquad (z, w) \mapsto \big(S_D(z), \pi(w)\big)$$

is biholomorphic to its image. $H(K_m)$ therefore is the graph of a bounded holomorphic function ϕ_m over X. Since $\{\phi_m\}$ is a normal family, it uniformly converges to a map

$$\phi_\infty : X \longrightarrow \mathbb{C}^2.$$

Put

$$K_\infty \equiv \{H^{-1}(z,w) | z \in X, w = \phi_\infty(z)\}.$$

Since

$$F(H^{-1}(\phi_m(X))) \supset H^{-1}(\phi_{m+1}(X)),$$

it follows that

$$F(H^{-1}(\phi_\infty(X))) \supset H^{-1}(\phi_\infty(X)).$$

Therefore, $\phi_\infty(X)$ has a fixed point P' under the map

$$H \circ F \circ H^{-1} : H(D) \longrightarrow H(D),$$

whose unstable manifold is $\phi_\infty(X)$. Since $H \circ F \circ H^{-1}$ is a conjugation map of F, F has a fixed point $P = H^{-1}(P')$ whose unstable manifold is $K_\infty \equiv H^{-1}(\phi_\infty(X))$.

3) For each pair of positive integers (n, m), K_n^γ is a degree one vertical disc and K is a degree one horizontal disc. Say

$$P_{n,m} \equiv K_n^\gamma \cap K_m.$$

$\{P_{n,m}\}$ is monotonically increasing with respect to n and m in the following sense: there exists a homotopy $H : D \times I \longrightarrow D$ such that for all (n, m) and all $q' \in K_n^\gamma \cup K_m$,

$$H(q', 0) = q'$$

and

$$H(q',1) \subset s^{-1}(K_n^\gamma \cap b^d) \quad \text{if} \quad q' \in K_n^\gamma \qquad \text{and} \qquad H(q',1) \subset \pi_2^{-1}(K_m \cap b^-) \quad \text{if} \quad q' \in K_m.$$

Thus,

$$\pi_2\big(H(P_{n,m},1)\big) \leq \pi_2\big(H(P_{n,m+1},1)\big) \qquad \text{and} \qquad s\big(H(P_{n,m},1)\big) \leq s\big(H(P_{n+1,m},1)\big).$$

Therefore,

$$\lim_m \lim_n P_{n,m} = \lim_m P_{\infty,m} = \lim_n P_{n,\infty} = P_{\infty,\infty}.$$

Observe that $P_{\infty,\infty}$ is a fixed point of F, because

$$\lim_{n,m} F(P_{n,m}) = \lim_{n,m} P_{n-1,m+1} = P_{\infty,\infty}.$$

On the other hand,

$$P_{\infty,\infty} \subset (K\gamma_\infty \cap K_\infty),$$

so $P_{\infty,\infty}$ is in the stable manifold of P_γ and the unstable manifold of P. Hence

$$P_{\infty,\infty} = P_\gamma = P.$$

For every k, $\#(F(K_k) \cap b_0^+) = 2$. So by continuity

$$1 \leq \#(F(K_\infty) \cap b_0^+) \leq 2.$$

Since K_∞^γ lies on the left side of b_0^+,

$$\#(F(K_\infty) \cap K_\infty^\gamma) = 2,$$

i.e. P has a transverse homoclinic point. □

Chapter 4

Nested Systems

In this chapter, we introduce the notion of nested systems, which are clusters of nicely ordered branches, and discuss the equivalence in terms of non-wandering points between different nested systems. We show that every nested system with zero entropy is equivalent to a family of one or two-branch nested systems; see Proposition 4.2.1.

4.1 Definition of nested systems and equivalence between nested systems

Definition 4.1.1. Let (D_i, F_i) be a real symmetrical degree-2 left-end positive dissipative branch strictly wrapping around a real symmetrical strip V for $i = 1, ..n$. $\{(D_i, F_i)_{i=1,\ldots,n}, V\}$ is a *pre-system* if

1. $D_i \subset \text{int}(D_{i-1})$ for any $i = 2, \ldots, n$ and $D_1 \subset \text{int}(V)$,
2. $\deg(D_i) = \deg(V) = 1$ for $i = 1, ..., n$,
3. $F_i(D_i) \cap F_j(D_j) = \emptyset$ for $i \neq j$,
4. $D_i \leq_R D_{i-1}$ for $i = 2, ..., n$,
5. $\sigma(D_1, V) < 1$ where $\sigma(D_1, V)$ is the scaling-ratio of D_1 in V,
6. for $i = 1, \ldots, n$, the left attached Jordan Curve associated with $F_i(b_i^-)$ lies outside the left attached Jordan Curve associated with $F_i(b_i^+)$.

The assumption 6 of Definition 4.1.1 is only needed for simplicity. Without it, all statements will remain true but the proofs will involve more cases whose arguments will be identical to the current ones.

Definition 4.1.2. Let $\{(D_i, F_i)_{i=1}^n, V\}$ be a pre-system. Define

$$F : \bigcup_i D_i \longrightarrow V$$

by

$$F(p) = F_i(p) \qquad \text{if } p \in D_i \backslash D_{i+1} \quad \text{for} \quad i = 1, ..., n-1.$$

Put

$$NS \equiv \{(D_i)_{i=1,\dots,n}, V, F\}. \tag{4.1.1}$$

NS is called a *system.*

Example 4.1.3. If (D, F) is a real symmetrical degree-2 left-end positive branch wrapping around a real symmetrical strip V where $\deg(D) = \deg(V) = 1$, then (D, V, F) is a system. On the other hand, if (D, V, F) is a system, then

1. V must be a real symmetrical strip,
2. (D, F) must be a real symmetrical degree-2 left-end positive branch wrapping around V, and
3. $\deg(D) = \deg(V) = 1$.

We may introduce a system $NS = \{(D_i)_{i=1,\dots,n}, V, F\}$ without explaining its pre-system explicitly. When no confusion arises, the map that wraps D_i around V is always denoted by F_i.

Notation 4.1.4. For later convenience, if $NS = \{(D_i)_{i=1}^n, V, F\}$ be a system, put

$$\text{Jac}(NS) \equiv \text{Jac}(F) \equiv \sup_i \text{Jac} F_i.$$

If $\mathcal{NS} = \{NS_j\}_{j\in\mathcal{I}}$ is a family of systems where NS_j is a systems for every $j \in \mathcal{I}$, put

$$\text{Jac}(\mathcal{NS}) \equiv \sup_j \text{Jac}(NS_j).$$

Lemma 4.1.5. *Let $I' \subset I$ be an index set and $NS \equiv \{(D_i)_{i\in I}, V, F\}$ be a system. Then the pre-system of SS induces a pre-system*

$$\{(D_j, F_j)_{j\in I'}, V\}.$$

It in turn gives a system

$$NS' \equiv \{(D_j)_{j\in I'}, V, F_{I'}\}.$$

We say NS' is a sub-system of NS indexed by I'.

Proof. The tuple $\{(D_j, F_j)_{j\in I'}, V\}$ naturally satisfies all the properties except property 5 in the Definition 4.1.1. If the sub-index set I' contains 1, property 5 holds. If $1 \notin I'$, by Lemma 2.2.10 property 5 holds. $\square$

Definition 4.1.6. Given a system $NS \equiv \{(D_i)_{i=1,\dots,n}, V, F\}$, put

$$\Omega^c(NS) \equiv \{q \in \cup D_i, \text{ s.t. } q \text{ is a non-wandering point under } F\},$$
$$\Omega^c_{NS}(D_i) \equiv \{q \in \Omega^c(NS) \cap (D_i \backslash D_{i+1}), \text{ s.t. } \mathcal{O}(q) \cap (\bigcup_{j=1}^{i-1} D_j \backslash D_i) = \emptyset\},$$
$$\Omega(NS) \equiv \{q \in \cup D_i R, \text{ s.t. } q \text{ is a non-wandering point under } F\}, \text{ and}$$
$$\Omega_{NS}(D_i) \equiv \{q \in \Omega(NS) \cap (D_i \backslash D_{i+1}), \text{ s.t. } \mathcal{O}(q) \cap (\bigcup_{j=1}^{i-1} D_j \backslash D_i) = \emptyset\}.$$

By definition

$$\Omega(NS) = \Omega^c(NS) \cap \mathbb{R}^2 \qquad \text{and} \qquad \Omega_{NS}(D_i) = \Omega^c_{NS}(D_i) \cap \mathbb{R}^2.$$

In other words, $\Omega(NS)$ and $\Omega_{NS}(D_i)$ are about the real non-wandering points. Since we are interested in real dynamics, they are the focus of this paper.

Notation 4.1.7. When no confusion arises, for example only one system is involved, we denote $\Omega^c_{NS}(D_i)$ by $\Omega^c(D_i)$ and $\Omega_{NS}(D_i)$ by $\Omega(D_i)$.

Let $\mathcal{NS} = \{NS_j\}_{j \in \mathcal{I}}$ be a family of systems. Put

$$\Omega^c(\mathcal{NS}) \equiv \bigcup_j \Omega^c(NS_j) \qquad \text{and} \qquad \Omega(\mathcal{NS}) \equiv \bigcup_j \Omega(NS_j).$$

Definition 4.1.8. Let U_1 and U_2 be two subset of $\mathbb{R}^2$. If we have two maps $f_A : U_1 \to \mathbb{R}^2$ and $f_B : U_2 \to \mathbb{R}^2$, let $\Omega(A)$ and $\Omega(B)$ be the non-wandering set of f_A and f_B, respectively. Given a point $p \in U_1$ (resp. U_2), denote the orbit of p under f_A (resp. f_B) by $\mathcal{O}_A(q)$ (resp. $\mathcal{O}_B(q)$). We say $\Omega(A)$ and $\Omega(B)$ have *the same orbit space* if either of the following is true:

1. $\Omega(B) \subset \Omega(A)$. Moreover, for every $p \in \Omega(A)$, there is some $q \in \Omega(B)$ such that $\mathcal{O}_B(q) \subset \mathcal{O}_A(p)$.

2. $\Omega(A) \subset \Omega(B)$. Moreover, for every $q \in \Omega(B)$, there is some $p \in \Omega(A)$ such that $\mathcal{O}_A(p) \subset \mathcal{O}_B(q)$.

In either cases, p and q are called mirror points.

Definition 4.1.9. Let f_A and f_B be two maps whose non-wondering sets $\Omega(A)$ and $\Omega(B)$ respectively have the same orbit space. We say $\Omega(A)$ and $\Omega(B)$ are *essentially the same* if for every pair of mirror points $p \in \Omega(A)$ and $q \in \Omega(B)$,

1. if they are periodic points, then they have the same hyperbolicity;

2. there exists some point $P \in U_1$ whose α-limit (resp. ω-limit) set under f_A contains $\mathcal{O}(p)$, if and only if, there exists some point $Q \in U_2 \cap \mathcal{O}_A(p)$ whose α-limit (resp. ω-limit) set under f_B contains $\mathcal{O}_B(q)$.

The second property of Definition 4.1.9 shows that if $p \in \Omega^c(A)$ and $q \in \Omega^c(B)$ are periodic mirror points, then if one of them is an attractor (resp. a saddle point or a repeller) then the other must as well be an attractor (resp. a saddle point or a repeller).

Remark 4.1.10. The Definitions 4.1.8 and 4.1.9 can be generalized so that every condition is defined up to conjugation. However, for our purpose, the conjugation is not necessary.

If two maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ and $g : \mathbb{C}^2 \to \mathbb{C}^2$ are real symmetrical, then the Definition 4.1.8 and 4.1.9 can apply onto its restriction over $\mathbb{R}^2$.

Example 4.1.11. In general $\Omega(NS) \neq \bigcup_i \Omega(D_i)$. But $\Omega(NS)$ is essentially the same as $\bigcup_i \Omega(D_i)$.

Definition 4.1.12. Let NS and NS' be two systems, and $\mathcal{NS}$ and $\mathcal{NS}'$ two families of systems. NS is said to be *equivalent* to NS' if $\Omega(NS)$ and $\Omega(NS')$ are essentially the same. $\mathcal{NS}$ is said to be *equivalent* to $\mathcal{NS}'$ if $\Omega(\mathcal{NS})$ and $\Omega(\mathcal{NS}')$ are essentially the same. NS is said to be *equivalent* to $\mathcal{NS}$ if $\Omega(NS)$ and $\Omega(\mathcal{NS})$ are essentially the same. They are denoted by $\Omega(NS) \sim \Omega(NS')$, $\Omega(\mathcal{NS}) \sim \Omega(\mathcal{NS}')$, and $\Omega(NS) \sim \Omega(\mathcal{NS})$, respectively.

Every strip D_i has degree one, thus $b_i^{\pm}$ and $b^{\pm}$ are all vertical curves. Given a set $K \in VR$, if K lies inside the Jordan Curve prescribed by b_i^- and b^-, then $K \in VR$ is said lying on the left side of D_i; if K lies inside the Jordan Curve prescribed by b_i^+ and b^+, then $K \in VR$ is said lying on the right side of D_i.

Definition 4.1.13. Let $NS = \{(D_i)_{i=1}^n, V, F\}$ be a system. NS is an *l-order nested system*, if

1. $F_1^{-(l-1)}(D_1) \supset D_2$,
2. $F_1^{-l}(D_1)$ lies on the right side of D_2, and
3. $F_i^{-1}(D_i)$ lies on the right side of D_{i+1} for $i \geq 2$.

The number l is called the *order* of the nested system.

Definition 4.1.14. Let $\mathcal{NS} = \{NS_j\}_{j \in \mathcal{I}}$ be a family of systems. The order of each NS_j is l_j. If

$$\sup_j l_j = l < \infty,$$

then we say the *order* of $\mathcal{NS}$ is l; otherwise, we say the order of $\mathcal{NS}$ is $+\infty$.

Notation 4.1.15. Let $NS = \{(D_i)_{i \in I}, V, F\}$ be a system. NS is called a *nested system* if there exists a positive integer l so that NS is an l-order nested system.

Lemma 4.1.16. *Let $NS = \{(D_i)_{i=1}^n, V, F\}$ be an l-order nested system. We have*

$$\Omega^c(D_1) \subset F_1^{-(l-1)}(D_1) \backslash D_2 \quad \text{and} \quad \Omega(D_1) \subset F_1^{-(l-1)}(D_1 R) \backslash D_2 R.$$

Proof. By definition, $\Omega^c(D_1) \not\subset D_2$ and $\Omega(D_1) \not\subset D_2R$. Assume $p \in D_1 \backslash F_1^{-(l-1)}(D_1)$ is a non-wandering point. Since $F_1^{-(l-1)}(D_1) \supset D_2$,

$$F(p) = F_1(p) \in F_1(D_1)\backslash F_1^{-(l-2)}(D_1).$$

If $F(p) \notin D_1$, p would not be a non-wandering point. That is to say

$$F(p) \in D_1\backslash F_1^{-(l-2)}(D_1) \subset D_1\backslash F_1^{-(l-1)}(D_1).$$

By induction, we have that for any $1 \leq j < l-1$,

$$F^j(p) = F_1^j(p) \in D_1\backslash F_1^{-(l-1)+j}(D_1).$$

Especially, for $j = l-2$,

$$F^{l-2} \in D_1\backslash F_1^{-(l-1)+(l-2)}(D_1) = D_1\backslash F_1^{-1}(D_1).$$

Therefore, $F^{l-1}(p) = F(F^{l-2}(p)) \in F(D_1)\backslash D_1$, which contradicts the assumption that p is a non-wandering point. Hence $\Omega^c(D_1) \subset F_1^{-(l-1)}(D_1)\backslash D_2$. Furthermore, since D_1, D_2, F_1, F_2 and V are all real symmetrical, $\Omega(D_1) \subset F_1^{-(l-1)}(D_1R)\backslash D_2R$. □

Corollary 4.1.17. *If NS is an l-order nested system, we have*

$$\Omega^c(D_1) \subset F_1^{-l}(D_1)\backslash D_2, \quad \text{and} \quad \Omega(D_1) \subset F_1^{-(l)}(D_1R)\backslash D_2R.$$

Proof. Suppose $q \in \Omega^c(D_1)$. By definition, $q \notin D_2$, thus $F(q) = F_1(q)$. It follows from Lemma 4.1.16 that $F(q) \in F_1^{-(l-1)}(D_1)$, i.e. $q \in F_1^{-l}(D_1)$. □

Lemma 4.1.18. *Let $NS = \{(D_i)_{i=1}^n, V, F\}$ be an l-order nested system. For every $i = 1, 2, ..., n-1$, $\Omega(D_i)$ must not lie on the left side of D_{i+1}.*

Proof. Since $F_1^{-l}(D_1)$ lies on the right side of D_2, if $q \in \Omega(D_1)$ lies on the left side of D_2, then $q \notin D_2 \cup F_1^{-l}(D_1)$. Thus,

$$F(p) = F_1(p) \notin F_1^{-(l-1)}(D_1).$$

Since $F_1^{-(l-1)}(D_1) \supset D_2$, $F(p) \in \Omega(D_1)$. However, this contradicts Lemma 4.1.16. The Lemma thus is true when $i = 1$. Suppose $q \in \Omega(D_i)$ for any $1 < i < n$ so that q lies on the left side of D_{i+1}. Since $F_i^{-1}(D_i)$ lies on the right side of D_{i+1}, Lemma 3.1.8 shows that there exist some $1 \leq j < i$ so that $F(q) \in D_j$ and $F(q)$ lies on the left side of D_{j+1}. The lemma then follows by induction. □

Definition 4.1.19. Let $NS = \{(D_i)_{i\in I}, V, F\}$ be a nested system. If $|I| = 1$, NS is called a *simple nested system*. If $|I| = 2$, NS is called a *basic nested system*. If NS is a simple nested system, by convention, it is a *zero-order nested system*.

Lemma 4.1.20. *Let $NS = \{(D_i)_{i\in I}, V, F\}$ be an l-order nested system. If $NS' \equiv \{(D_j)_{j\in I'}, V, F_{I'}\}$ is a sub-system of NS indexed by $I' \subset I$, then NS' is also a nested-system. More precisely, if*

1. *if $|I'| = 1$, NS' is a zero-order nested system;*
2. *if $|I'| \neq 1$ and $1 \in I'$, NS' is an l-order nested system;*
3. *if $|I'| \neq 1$ and $1 \notin I'$, NS' is a one-order nested system.*

The proof of Lemma 4.1.20 is straightforward, hence it is omitted.

4.2 Zeroth equivalent family of zero entropy nested systems

Proposition 4.2.1. *Given an l-order nested system $NS = \{(D_i)_{i\in I}, V, F\}$, suppose $q \in \Omega(D_i)$ for $i \in I$. Then either $\mathcal{O}(q) \cap D_{i+2} = \emptyset$, or there exists a homoclinic point.*

Proof. Suppose $q \in \Omega(D_i)$ so that there exists a positive integer N_1 such that $F^{N_1}(q) \in \mathcal{O}(q) \cap D_{i+2}$. Since $D_i \leq_R D_{i-1}$ for every $i = 2, ..., n$, $F^{N_1+1}(q) \leq_R D_{i+1}$. On the other hand, $q \in \Omega(D_i)$ implies that there exists another positive integer $N_2 \geq N_1 + 1$ such that $F^{N_2}(q) \in D_i \backslash D_{i+1}$. That is, $F^{N_2}(q) \in \Omega(D_i)$. Lemma 4.1.18 assures that $F^{N_2}(q)$ lied on the right side of D_{i+1}. Therefore there is a homoclinic point by Theorem 3.2.1. □

Corollary 4.2.2. *Let $NS \equiv \{(D_i)_{i\in I}, V, F\}$ be an l-th nested system. If NS has zero entropy, then for every $q \in \Omega(D_i)$, $\mathcal{O}(q)$ depends only on (D_i, F_i) and (D_{i+1}, F_{i+1}).*

Corollary 4.2.3. *Let $NS \equiv \{(D_i)_{i\in I}, V, F\}$ be an l-th nested system with zero entropy. Put*

$$I_i \equiv \{i, i+1\} \subset I,$$

for every $i = 1, \ldots, n-1$, and

$$I_n \equiv \{n\} \subset I.$$

Denote by NS_i the sub-nested system of NS indexed by I_i. Then

$$\Omega_{NS_i}(D_i) = \Omega_{NS}(D_i).$$

Therefore, if we put

$$\mathcal{CS} \equiv \{NS_i\}_{i\in I},$$

then $\mathcal{CS}$ is equivalent to NS (i.e. $\Omega(\mathcal{CS})$ is essentially the same as $\Omega(NS)$.

Corollary 4.2.2 and 4.2.3 follow from Proposition 4.2.1 immediately.

It is clear that NS_i is a basic nested system for any $i \neq n$ and NS_n a simple nested system. We can therefore studying simple and basic nested systems instead of complicated nested system.

Given a basic nested system $BS = \{D_i, D_{i+1}, V, F\}$, we can further decompose $\Omega(D_i)$ into two smaller sets $\Omega(D_{i,i})$ and $\Omega(D_{i,i+1})$ as follows: for every $q \in \Omega(D_i)$,

$$q \in \begin{cases} \Omega(D_{i,i}) & \text{if } \mathcal{O}(q) \cap D_{i+1} = \emptyset, \\ \Omega(D_{i,i+1}) & \text{if } \mathcal{O}(q) \cap D_{i+1} \neq \emptyset. \end{cases}$$

If $\Omega(D_{i,i+1}) = \emptyset$, $\Omega(D_i)$ depends only on one branch (D_i, F_i). More precisely, say the pre-system of BS is

$$\big\{(D_i, F_i)\, , \, (D_{i+1}, F_{i+1})\, , \, V\big\}.$$

Instead of dealing with BS, we only need to work with the two simple sub-nested system $SS'_i = \{D_i, V, F_i\}$ and $SS'_{i+1} = \{D_{i+1}, V, F_{i+1}\}$ so as to study $\Omega(D_i)$.

Definition 4.2.4. Let $BS = \{D_i, D_{i+1}, V, F\}$ be a basic nested system. If $\Omega(D_{i,i+1}) \neq \emptyset$, BS is called a *non-reducible* basic nested system; otherwise it is called a *reducible* basic nested system.

If $BS = \{D_i, D_{i+1}, V, F\}$ is a reducible nested system, put

$$\mathcal{CS} \equiv \big\{\{D_i, V, F_i\}, \{D_{i+1}, V, F_{i+1}\}\big\}.$$

$\mathcal{CS}$ is equivalent to BS.

Definition 4.2.5. Let $SS_i = \{D_i, V, F_i\}$ be a simple nested system. If $\Omega(D_i) \neq \emptyset$, SS is called a *essential* simple nested system; otherwise it is called a *auxiliary* simple nested system.

Definition 4.2.6. Let $NS = \{(D_i)_{i\in I}, V, F\}$ be a nested system. Put

$$I_B \equiv \{i \in I | BS_i \text{ is non-reducible basic sub-nested system of } NS \text{ indexed by } I_i\}.$$

I_B is called the *(non-reducible) basic index set* of NS. Put

$$I_S \equiv I \backslash I_B.$$

I_S is called the *simple index set* of NS.

Definition 4.2.7. Let $NS = \{(D_i)_{i\in I}, V, F\}$ be a nested system with zero entropy. Let I_B and I_S be the basic and simple index set of NS, respectively. For any $i \in I$, the basic sub-nested system of NS indexed by I_i is denoted by BS_i and the simple sub-nested system of NS indexed by $\{i\}$ is denoted by SS_i. Put

$$\mathcal{CS} \equiv \{BS_i | i \in I_B\} \cup \{SS_j | j \in I_S\}.$$

$\mathcal{CS}$ is called the (zero-th) *equivalent family* of NS. Since NS has zero entropy, $\mathcal{CS}$ is equivalent to NS.

Suppose $\mathcal{CS}$ is the equivalent family of some nested system. By the definition not all elements of $\mathcal{CS}$ contains interesting dynamics. Precisely, $\mathcal{CS}$ contains some auxiliary simple sub-nested system of NS. This inclusion seems redundant here, however, it is very crucial for us to study a family of nested systems in Chapter 7.

Definition 4.2.8. Let $\mathcal{NS} \equiv \{NS_j | j \in I\}$ be a family of nested systems. If for all $j \in I$, NS_j has zero entropy, then $\mathcal{NS}$ is called a *zero entropy family.*

Definition 4.2.9. Let $\mathcal{NS} \equiv \{NS_j | j \in I\}$ be a zero entropy family of nested systems, where NS_j is an l_j-th nested system for $j \in I$. We denote the (zero-th) equivalent family of NS_j by $\mathcal{CS}_j$. Then

$$\mathcal{CNS} \equiv \bigcup_j \mathcal{CS}_j$$

is called the (zero-th) *equivalent family* of $\mathcal{NS}$.

The equivalent family of either a nested system or a family of nested systems is always a family of simple and basic nested systems. Moreover, the equivalent family is only defined for nested systems or families with zero entropy.

For a simple nested system, depending on how many saddle nodes the system contains, we either conclude that there exist a saddle node and a global attractor (thus no other nonwandering point exists) or need to renew the simple system. Both cases will be discussed in the following chapter.

Chapter 5

Renormalizaiton

In this chapter, we construct the renormalization operator acting on basic nested systems. We sequentially construct three operators (the pre-renormalization in Subsection 5.1, the quasi-renormalization operator in Subsection 5.2, and the basic-renormalization operator in Subsection 5.3). Each of them refines the previous operator so that the basic-renormalization of a basic nested system is indeed a nested system.

Given the basic-renormalization operator acting on non-reducible nested systems, we then define the renormalizable general nested systems and the corresponding renormalization operator in Subsection 5.4. In the same subsection, we also define the K-th renormalizable nested systems for $K \in \mathbb{Z}^+$ by induction. We prove that a nested system with finite periodic points cannot be infinitely renormalizable in the Subsection 5.5.

5.1 Pre-renormalization acting on basic nested systems

Given a basic nested system $BS = \{D_i, D_{i+1}, V, F\}$, there is a simple sub-nested system SS of BS indexed by $i+1$. We notice that

$$\Omega^c(BS) = \Omega^c_{BS}(D_i) \cup \Omega^c(SS).$$

So only $\Omega^c_{BS}(D_i)$ needs to be studied. From Corollary 4.1.17 we know $\Omega^c_{BS}(D_i) \subset F_i^{-l}(D_i)$.

Definition 5.1.1. Let BS be a basic l-th nested system whose pre-system is

$$\big\{(D_i, F_i)\,,\,(D_{i+1}, F_{i+1})\,,\,V\big\}.$$

For every $q \in F_i^{-l}(D_i)$, if a positive integer $N(q)$ is the least positive integer such that $F^N(q) \in F_i^{-l}(D_i)$, then we call $N(q)$ the *return time* of q. We denote by $E(q)$ the maximal connected component of D_i such that $q \in E(q)$ and F^N is continuous on $E(q)$. The procedure of finding out all such $E(q)$'s and the associated return maps is called the (first) *pre-renormalization* of BS. The map F^N is called the *return map* of q (or $E(q)$) with respect to the first pre-renormalization.

For simplicity, when the return time is not explicitly required, we denote the map associated to $E(q)$ by F_q. If a point $q \in F_i^{-l}(D_i)$ whose return time is finite, then the point q is called a *pre-relevant point* with respect to pre-renormalization.

Lemma 5.1.2. *If $q \notin F_i^{-1}\big(F_i^{-l}(D_i) \cup D_{i+1}\big)$, q is not a pre-relevant point.*

Proof. For every pre-relevant point q, $q \in F_i^{-l}(D_i)$. Then $F(q) = F_i(q)$. If

$$q \notin F_i^{-1}(F_i^{-l}(D_i) \cup D_{i+1}),$$

then

$$F(q) \notin F_i^{-l}(D_i) \cup D_{i+1}.$$

In particular, $F^2(q) = F_i^2(q) \notin F_i^{-(l-1)}(D_i)$. By induction, we have

$$F^{l+1}(q) = F^l(F_i(q)) = F^{l-1}(F_i^2(q)) = \cdots = F_i^{l+1}(q) \notin D_i,$$

where we used the fact $D_{i+1} \subset F_i^{-(l-1)}(D_i) \subset \cdots \subset D_i$. □

Lemma 5.1.3. *Let $BS = \{D_i, D_{i+1}, V, F\}$ be a basic nested system. Put*

$$A \equiv \{q \mid q \text{ is a pre-relevant point with respect to pre-renormalization of } BS\}.$$

Then

$$\Omega_{BS}^c(D_i) \subset \mathrm{cl}(A) \qquad \text{and} \qquad \Omega_{BS}^c(D_i) \subset \mathrm{cl}(A) \cap \mathbb{R}^2.$$

Proof. Let q' be a point in $F_i^{-l}(D_i) \backslash \mathrm{cl}(A)$. There exists a neighborhood $U \subset F_i^{-l}(D_i)$ of q' so that for any $n \in \mathbb{Z}^+$, $U^n \cap F_i^{-l}(D_i) = \emptyset$. Thus $q' \notin \Omega_{BS}^c(D_i)$. □

Lemma 5.1.3 implies that the pre-renormalization captures all the real important dynamics.

Let BS be as in Definition 5.1.1. For points q_1 and $q_2 \in F_i^{-l}(D_i)$, q_1 is said to be *equivalent* to q_2 with respect to pre-renormalization, denoted by $q_1 \sim q_2$, if

$$E(q_1) = E(q_2).$$

If for any $n \in \mathbb{Z}^+$, $F^n(q) \notin \Omega_{BS}^c(D_i)$, by convention, we say the return time of q is $+\infty$. q might belong to some $E(q')$, but q is not equivalent to q'.

Assumption 5.1.4. *Let (D, F) be a real symmetrical degree-2 left-end branch that wraps around a real symmetrical strip V and $S_\gamma \subset V$ be a real symmetrical strip. Put $S_0 \equiv F^{-1}(S_\gamma)$. Assume* $\deg(S_\gamma) = \deg(D) = 1$. *If S_0R consists of two interior disconnected components, then S_0 consists of two degree one strips in D.*

Proposition 5.1.5. *Let BS be as in Definition 5.1.1. Then*

1. *the maximal components $E(q)$'s defined in Definition 5.1.1 are real symmetrical strips,*
2. $\deg(E(q)) = 1$ *and the associated return map F_q is a degree-2 real symmetrical map, that wraps $E(q)$ around V, and*
3. *$(E(q), F_q)$ are positive, dissipative and left-end.*

Proof. Say q has return time $N(q) < \infty$. Suppose there is some $j < N(q)$ so that $F^j(q) \notin D_{i+1}$. Since $N(q)$ is the least return time, $F^j(q) \notin F_i^{-l}(D_i)$ either. Then we have $F(F^j(q)) \notin F_i^{-(l-1)}(D_i)$. It follows induction that $F^{N(q)-j}(F^j(q)) \notin D_i$. This however contradicts the assumption that $N(q) < \infty$. Hence for any q, its return map must have the form

$$\underbrace{F_{i+1} \circ F_{i+1} \circ \dots F_{i+1}}_{N(q)-1} \circ F_i. \tag{5.1.1}$$

1. Since the return map is in form of Equation 5.1.1, the maximal connected component on which $F^N(q)$ is connected has the form

$$\begin{cases} (F_{i+1}^{N(q)-2} \circ F_i)^{-1}(D_{i+1}) & \text{if} \quad N(q) > 1 \\ D_i & \text{if} \quad N(q) = 1 \end{cases}$$

   In either case, it is real symmetrical.

2. If $N(q) = 1$, $\deg(D_i) = 1$. If $N(q) = 2$, $F^2(q) = F(F(q)) \in F^{-l}(D_i)$. $F(q)$ lies on the right side of D_{i+1}. Then Lemma 3.1.6 implies that for any holomorphic vertical curve $l \in D_{i+1}$, $F_i^{-1}(l)$ contains two disconnected holomorphic vertical curves. That is, $F_i^{-1}(D_{i+1}R)$ contains two disconnected components. Then it follows Assumption 5.1.4 that $F_{i+1}^{-1}(D_{i+1})$ has two degree one strips. Therefore $\deg E(q) = 1$ and $\deg(F_{i+1} \circ F_i) = 2$.

   A similar argument also holds for any $N(q) = n > 2$ by noticing that

   (a) The connected component of $(F_{i+1}^{n-2} \circ F_i)^{-1}(D_{i+1})$ that contains q, say $\widetilde{E}(q)$ has degree one;

   (b) $F_{i+1}^{n-1} \circ F_i$ has degree two on $\widetilde{E}(q)$,

   where we used induction. Hence $\deg(E(q)) = 1$ and the associated return map $\deg(F_q) = 2$.

3. It follows immediately from Equation 5.1.1, that all F_q's are positive, dissipative, and left-end.

□

For a relevant point q, we sometimes call the set $E(q)$ the associated strip of q.

Let $BS = \{D_i, D_{i+1}, V, F\}$ be a basic nested system. Propositon 5.1.5 shows if there is some q whose return time is $N(q) \geq 2$, then $F_i^{-1}(D_{i+1})$ contains two disconnected strip, denoted by S_1 and S_2. Since $\deg(S_1) = \deg(S_2) = 1$, without loss of generality, we assume S_1 lies on the left side of S_2. Because the left attached Jordan Curve associated with b_1^- lies outside the left attached Jordan Curve associated with b_1^+, $S_2 \leq_R S_1$, for all the branches $(E(q), F_q)$ except where $E(q) = D_i$, we have $E(q) \leq_R S_1$.

Lemma 5.1.6. *Let $BS = \{D_i, D_{i+1}, V, F\}$ be a basic nested system with zero entropy. Then*

$$S_2 \cap \Omega(D_1) = \emptyset.$$

Proof. Suppose there exists some $q \in S_2 \cap \Omega(D_i)$. Hence, there exists some N so that $F^N(q) \in S_2$. By the construction of return maps, there exist some pre-relevant points $q_1, \ldots, q_n \in F_i^{-l}(D_i)$ so that

$$q_1 = q, \qquad \text{and} \qquad q_k = F_{q_{k-1}} \circ \cdots \circ F_{q_1}(q)$$

for any $2 \le k \le n$, and

$$F^N = F_{q_n} \circ \cdots \circ F_{q_1}.$$

Lemma 5.1.2 implies all the $q_k \in F_i^{-1}\big(F_i^{-l}(D_i) \cup D_{i+1}\big)$. More precisely, for any $k > 2$, $q_k \in F_i^{-1}(F_i^{-l}(D_i)) \cup S_1$ and $q_1 \in S_2$. Suppose $1 \le K \le n$ is the largest integer that $q_k \in F_i^{-1}(D_{i+1})$.

$$q_{K+1} = F_{q_K} \circ \cdots \circ F_{q_1}(q) \le_R S_1.$$

On the other hand, $q_{K+1} \in F_i^{-1}\big(F_i^{-l}(D_i) \cup D_{i+1}\big)$. Hence $q_{K+1} \in F_i^{-1}(F_i^{-l}(D_i)) \cup S_2$. In either case q_{K+1} lies on the right side of S_1. Theorem 3.2.1 shows there exists a transverse homoclinic point which contradicts that BS has zero-entropy. □

Corollary 5.1.7. *Suppose BS is a basic nested system with zero entropy. For every $j \in \mathcal{I}$, if there exists some pre-relevant point q_j whose return time is j, then $(F_{i+1}^{j-2} \circ F_i)^{-1}(D_{i+1})$ has only one connected component, denoted by E_j, so that*

1. *E_j is a strip,*
2. *$\deg(E_j) = 1$ and*
3. *$\big((F_{i+1}^{j-2} \circ F_i)^{-1}(D_{i+1}) \backslash E_j\big) \bigcap \Omega(D_i) = \emptyset$.*

Moreover, given branches $(E_j, F_{i+1}^{j-1} \circ F_i)$ and $(E_k, F_{i+1}^{k-1} \circ F_i)$, if $k \le j$

$$E_j \le_R E_k.$$

Proof. The corollary follows Lemma 5.1.6 by induction.

□

Corollary 5.1.7 does not promise that $E_j \cap \Omega(D_i) \neq \emptyset$. On the contrary, this Corollary implies E_j is the only connected component that we are interested in.

5.2 Quasi-renormalization action on basic nested systems

A nested system by definition consist of only finitely many strips. In this and next subsection, we attempt to reduce the number of the strips E_q created by the pre-renormalization.

Definition 5.2.1. Let BS be a basic nested system with zero entropy. The *quasi-renormalization* of BS is a process that finds all the E_j defined in Corollary 5.1.7. A pre-relevant point q_j is called a *quasi-relevant point* if its return time is j and $q_j \in E_j$. The map associated to each E_j is renamed as eF_j for all $j \in \mathbb{Z}^+$.

We are slightly abusing notations here: q, $E(q)$ and F_q are used in Definitions 5.1.1, 5.2.1 and 5.3.1. However, in different subsections, it is clear which definition we are referring to.

The collection $\{(E_j, eF_j)_{j=1,2\ldots}, V\}$ is almost a pre-system by Propostion 5.1.5 and Corollary 5.1.7. However, it lacks a crucial property: there might exist infinitely many branches. This problem can be solved. First, we have the following Lemma.

Lemma 5.2.2. *If there exists some $M \in \mathbb{N}$ so that*

$$eF_M(\mathrm{int}(E_M)) \cap \mathrm{int}(E_M) \cap \mathbb{R}^2 = \emptyset,$$

then for any $j \geq M$, $E_j \cap \Omega(D_i) = \emptyset$.

Proof. If $i < j$, by definition, $E_j \subset D_i$. On the other hand, Corollary 5.1.7 shows that $E_j \leq_R E_i$. Thus the lemma follows. $\square$

5.3 Basic-renormalization acting on basic nested systems

Intuitively speaking, Lemma 5.2.2 implys that many E_j is irrelevant to the real dynamics. Thus, we modify quasi-renormalization in the following way in order to obtain basic-renormalization.

Definition 5.3.1. Let BS be an l-order basic nested system. For every $q \in F^{-l}(D_i)$ if

1. q is a quasi-relevant point and
2. $F_q(E(q)) \cap \mathrm{int}(E(q)) \neq \emptyset$,

it is called a *relevant point*. If there exists some relevant point, we say basic-renormalization exists and the basic-renormalization of BS is a process that identifies all such relevant points q's, the associated $E(q)$'s and return map F_q's.

Lemma 5.3.2. *Given a basic nested system $BS = \{D_i, D_{i+1}, V, F\}$, if the basic-renormalization does not exist, then $\Omega(D_i) = \emptyset$.*

Proof. If $q \in \Omega(D_i) \neq \emptyset$, then q must have finite return time $N(q)$, $q \in E(q)$ contains at least a non-wandering point q and $F_q(E(q) \cap \mathrm{int}(E(q)) \neq \emptyset$ by the definition of a non-wandering point. Hence, q is a relevant point. $\square$

Basic-renormalization differs from pre-renormalization by eliminating certain branches. So Theorem 5.1.5 holds for basic-renormalization as well. Therefore, $\{(E_j, eF_j)_{j\in\mathcal{I}}, V\}$ is properly defined as well. Although there appear to be infinitely many E_j, the scaling ratio help us handle this issue.

Lemma 5.3.3. *Suppose $\sigma \equiv \sigma(BS)$. Then for every $j \in \mathbb{N}$,*

$$\sigma(E_j, E_{j+1}) \leq \sigma^{\frac{1}{2}}.$$

Proof. For every $j \in \mathbb{N}$, eF_j strictly wraps around V and $\deg(eF_j) = 2$. By Lemma 2.2.13,

$$\sigma(E_j, E_{j+1}) \leq \sigma(D_{i+1}, V)^{\frac{1}{2}} \leq \sigma^{\frac{1}{2}},$$

where we used the fact that $eF_{j+1} \subset eF_j^{-1}(D_{i+1})$. □

Lemma 5.3.4. *If there are infinitely many branches $\{(E_j, eF_j)_{j\in\mathbb{Z}^+}$ strictly wrapping around V, then*

$$\mathrm{int}(\bigcap_j E_j) = \emptyset.$$

Proof. For every $j \in \mathbb{Z}^+$, $\sigma(E_j, E_{j+1}) < \sigma^{\frac{1}{2}} < \rho < 1$, hence

$$\prod_j \sigma(E_j, E_{j+1}) \leq \lim \rho^{j+1} = 0.$$

Then this lemma is a direct application of Lemma 2.2.12 □

Lemma 5.3.5. *If there are infinitely many branches $\{(E_j, eF_j)_{j\in\mathbb{Z}^+}\}$, then BS has either a transverse or a tangent homoclinic point.*

Proof. 1) The first step is to show that there exists a fixed point $P \in D_{i+1}R$ with an unstable manifold. We claim that for every $j \in \mathbb{N}$ where E_{j+1} exists, there exist two holomorphic horizontal curves in D_{i+1}, denoted by α_j and β_j so that

1. $\alpha_j \cup \beta_j \subset F_{i+1}(\beta_{j-1})$,
2. $eF_j(E_j) \cap D_{i+1}R$ is inside the Jordan Curve horizontally prescribed by α_j and β_j.

Put $\beta_0 \equiv b_i^d$. Assume $j = 1$, $eF_1 = F$. Since E_2 exists, there exists some $q \in E_2$ so that $F^2(q) \in F^{-l}(D_i)$. That is $F^2(q)$ lies on the right side of D_{i+1}. On the other hand, $F(q)$ lies inside the left attached Jordan Curve associated with $F_i(b_i^-)$. Then $F_i(b_i^-) \cap D_{i+1}$ contains two disconnected holomorphic horizontal curves by Lemma 3.1.7, denoted by α_1 and β_1. Since

$$eF_j(E_j) \cap D_{i+1}R = F_i(D_i) \cap D_{i+1}R,$$

it is clearly that $eF_j(E_j)\cap D_{i+1}R$ lies inside the Jordan Curve horizontally prescribed by α_1 and β_1.

Assume the claims are true for all $j < j'$. Denoted by J^j the Jordan Curve horizontally prescribed by α_j and β_j. Similarly, since $E_{j'+1}$ exists, there exists some $q \in E_{j'+1}$ so that $F^{j'+1}(q) \in F^{-l}(D_i)$. That is, $F^{j'+1}(q)$ lies on the right side of D_{i+1}. On the other hand

$$F^{j'}(q) \in F_{i+1}(F^{j'-1}(q)) \subset F_{i+1}(F^{j'-1}(E_{j'-1})).$$

Since $F^{j'-1}(E_{j'-1})$ lies inside $J^{j'-1}$, $F^{j'}(q)$ lies inside the left attached Jordan Curve associated with $F_{i+1}(\beta_{j'-1})$. Lemma 3.1.7 shows $F_{i+1}(\beta_{j'-1}) \cap D_{i+1}$ contains two disconnected holomorphic horizontal curves, denoted by α'_j and β'_j. Moreover, $eF'_j(E'_j) \cap D_{i+1}R$ is inside the Jordan Curve horizontally prescribed by $\alpha_{j'}$ and β'_j.

Since there exists infinitely many branches (E_j, eF_j), we then have an infinite sequence of horizontal holomorphic curves, $\{\beta_j\}$. Besides, $F_{i+1}(\beta_j) \supset \beta_{j+1}$. Then the same argument which proved existence of a fixed point with an unstable manifold in the proof of Theorem 3.2.1 can be applied. Hence, there exists a fixed point $P \in D_{i+1}R$ with an unstable manifold L.

2) The second step is to prove if a point $p \in \bigcap_j E_j$,

$$\lim_n F^n(p) = P.$$

In other words, $\bigcap_j E_j$ is part of the stable manifold of P.

Put $b^u_j \equiv b^u \cap E_j$ and $b^d_j \equiv b^d \cap E_j$. For each j, (E_j, eF_{j-1}) is a degree-1 branch that strictly wraps around D_{i+1}. Therefore, $L^u_j \equiv eF_{j-1}(b^u_j)$ and $L^d_j \equiv eF_{j-1}(b^d_j)$ are horizontal holomorphic curves, and $eF_{j-1}(E_j)$ lies inside the Jordan Curve horizontal prescribed by L^u_j and L^d_j. Put $L^u \equiv \lim_j L^u_j$ and $L^d \equiv \lim_j L^d_j$. As the limit of bounded horizontal holomorphic curves, L^u and L^d are both horizontal holomorphic curves in D_{i+1}. Since L^u_j lies inside the Jordan Curve horizontal prescribed by L^d_j and L^d_{j+1} for every j, $L^u = L^d$. Observe $L^d_j = \beta_j$, so $L^u = L^d = L$. Hence if $p \in \bigcap_j E_j$,

$$\lim_k F^k_{i+1}(F(p)) \in L.$$

Since L is an unstable manifold, $\lim_k F^k_{i+1}(F(p)) = P$.

We can now prove there is a (transverse or tangent) homoclinic point. By Lemma 2.2.12, $\mathrm{m}(\bigcap_j E_j) = 0$. On the other hand, $\bigcap_j E_j$ and $\bigcap_j E_jR$ are connected. $\bigcap_j E_j$ is a vertical curve in D_i, denoted by L_γ. For every j, $eF_j(E_jR) \cap \mathrm{int}(E_jR) = \emptyset$. In particular, $eF_j(b^d_j) \cap \mathrm{int}(E_jR) \neq \emptyset$. Since $eF_j(b^d_j) = F_{i+1}(K^d_j)$, $F_2(L) \cap \mathrm{int}(E_jR) \neq \emptyset$. Hence, $F_2(L) \cap L_\gamma \neq \emptyset$. In other wordss, the stable manifold and unstable manifold of P have an intersection point other than P. The Lemma follows. $\square$

Notation 5.3.6. Let $\mathcal{W}$ denote the space of nested systems which have positive entropy.

Proposition 5.3.7. *Let BS be a basic nested system whose basic-renormalization exists. If $BS \notin$ $\mathrm{cl}(\mathcal{W})$, $\{(E_j, eF_j)_{j=\mathcal{I}}, V\}$ is a pre-system, i.e. $|\mathcal{I}| < \infty$.*

Proof. The proposition follows from the Lemma 5.3.5. $\square$

Definition 5.3.8. Let BS be a basic nested system whose basic-renormalization exists. If $BS \notin$ $\mathrm{cl}(\mathcal{W})$, we have a pre-system $\{(E_j, eF_j)_{j=I}, V\}$. We define the map

$$eF : \bigcup_{j\in I} E_j \to V$$

by

$$eF(p) = eF_j(p), \qquad \text{if } p \in E_j \backslash E_{j+1} \; for \; j \geq 1.$$

Put

$$ES \equiv \{(E_j)_{j\geq 1}, V, eF\}. \tag{5.3.1}$$

ES is called the *basic-renormalized nested system* of BS.

Remark 5.3.9. Hereafter, if a basic nested system BS is said to have its basic-renormalized nested system, it is implicitly assumed that BS is basic-renormalizable and $BS \notin \mathrm{cl}(\mathcal{W})$.

Lemma 5.3.10. *If $ES = \{(E_j)_{j\in I}, V, eF\}$ is the basic renormalized nested system of an l-order basic nested system $BS = \{D_i, D_{i+1}, V, F\}$, then $\Omega(ES)$ is essentially the same as $\Omega_{BS}(D_i)$. In particular,*

$$\Omega_{BS}(D_i) = \Omega(ES).$$

Proof. The renormalization process implies that $\Omega_{BS}(D_i) \subset \bigcup_{j\in I} E_j \bigcap F_i^{-l}(D_i)$. Since each eF_j is some iterates of F, if $P \in \Omega(ES)$, then $\mathcal{O}_{eF}(P) \subset \mathcal{O}_F(P)$. Moreover, since eF_j is the return map to $F_i^{-l}(D_i)$,

$$\mathcal{O}_{eF}(P) = \mathcal{O}_F(P) \cap F_i^{-l}(D_i).$$

The lemma follows. □

Proposition 5.3.11. *Let BS be a basic l-order nested system and ES be the basic-renormalized nested system of BS. Then ES is a $(l+1)$-order nested system.*

Proof. The proposition follows from Corollary 5.1.7 and Propostion 5.3.7. □

The Proposition 5.3.11 therefore justifies the Definitionn 5.3.8 where ES has already been assumed "nested".

Corollary 5.3.12. *Let BS be a reducible nested system and $ES = \{(E_j)_{j\in I'}, V, eF\}$ be its basic-renormalized nested system. Then for any $j \geq 2$, E_j does not contain any nonwandering point.*

Proof. Let $q \in BS$ be a non-wandering point. If $q \in E_j$ for some $j \geq 2$, then its return map is of the form $F_{i+1} \circ \cdots \circ F_{i+1} \circ F_i$. That contradicts the assumption that BS is reducible in which $\mathcal{O}(q) \cap D_{i+1} = \emptyset$. □

The above corollary implies that the basic-renormalization of reducible basic nested systems provides no new information about non-wandering points. Indeed, a reducible basic nested system is equivalent to two simple nested systems, and the zero-th equivalent family of any nested system does not contain reducible basic nested systems.

5.4 Renormalization on general nested systems

Let $I \neq \emptyset$ and $\widetilde{I}$ be two index sets. For each $j \in I$, let BS_j be a non-reducible basic nested system whose basic-renormalized nested system exists, denoted by ES_j. For each $k \in \widetilde{I}$, let SS'_k be a simple nested system. Put

$$\mathcal{CS} \equiv \{BS_j \,|\, j \in I\} \cup \{SS'_k \,|\, k \in \widetilde{I}\}.$$

If $I \neq \emptyset$, then $\mathcal{CS}$ is said to be basic-renormalizable and

$$e\mathcal{CS} \equiv \{ES_j \,|\, j \in I\} \cup \{SS'_k \,|\, k \in \widetilde{I}\}$$

is called the *basic-renormalized family* of $\mathcal{CS}$. Moreover, let $\mathcal{ES}_j$ be the zero-th equivalent family of ES_j for each $j \in I$. Put

$$\mathcal{ES} \equiv \{\mathcal{ES}_j \,|\, j \in I\} \cup \{SS'_k \,|\, k \in \widetilde{I}\}.$$

$\mathcal{ES}$ is the zero-th equivalent family of $e\mathcal{CS}$. In other wordss, $\mathcal{ES}$ is *zero-th equivalent family of basic-renormalized family* of $\mathcal{CS}$. If $I = \emptyset$, then $\mathcal{CS}$ is said to be non-renormalizable.

Definition 5.4.1. Let NS be a nested system and $\mathcal{CS}$ be its (zero-th) equivalent family. NS is said to be *renormalizable* if $\mathcal{CS}$ is basic-renormalizable; otherwise NS is said to be non-renormalizable. If NS is renormalizable, then the *first equivalent family* of NS is the zero-th equivalent family of basic-renormalized family of $\mathcal{CS}$. The procedure of finding the first equivalent family of NS is called the *first renormalization* of a nested system NS.

There are differences between basic-renormalizablity and renormalizablity. If a basic nested system BS is renormalizable, BS must be a non-reducible basic nested system. On the other hand, even a reducible basic nested system can have basic-renormalization.

Definition 5.4.2. Let $\mathcal{NS} = \{NS_j\}_{j\in\mathcal{I}}$ be a family of nested systems. If there exists some $j \in \mathcal{I}$ where NS_j is renormalizable, then $\mathcal{NS}$ is said to be renormalizable; otherwise $\mathcal{NS}$ is said to be non-renormalizable. Given $\mathcal{NS}$ is renormalizable, there exists an index set $\mathcal{I}' \subset \mathcal{I}$ so that for any $j \in \mathcal{I}'$, NS_j is renormalizable. The first equivalent family of NS_j for any $j \in \mathcal{I}'$ is denoted by $\mathcal{ES}_j$. The zero-th equivalent family of NS_k for any $k \in \mathcal{I}\backslash\mathcal{I}'$ is denoted by $\mathcal{CS}_k$. Put

$$\mathcal{ES} \equiv \{\mathcal{ES}_j \,|\, j \in \mathcal{I}'\} \cup \{\mathcal{CS}_k \,|\, k \in \mathcal{I}\backslash\mathcal{I}'\}.$$

$\mathcal{ES}$ is called the *first equivalent family* of $\mathcal{NS}$. The procedure of finding $\mathcal{ES}$ is called the *first renormalization* of $\mathcal{NS}$.

Notation 5.4.3. Suppose a family of nested system $\mathcal{NS} = \{NS_j\}_{j\in\mathcal{I}}$ is renormalizable. It might exist some $k \in \mathcal{I}$ so that NS_k is non-renormalizable itself. However, for later convenience, should no confusion arises, we may say as if every element in $\mathcal{NS}$ were renormalizable. For example, we may say that the first equivalent family of NS_j for any $j \in \mathcal{I}$ is denoted by $\mathcal{ES}_j$ and the first equivalent family of $\mathcal{NS}$ is simply $\{\mathcal{ES}_j \,|\, j \in \mathcal{I}\}$.

Lemma 5.4.4. *Let NS be a nested system. If NS is non-renormalizable then its zero-th equivalent family $\mathcal{CS}$ is a zero-order family.*

Proof. The zero-th equivalent family consists of simple and basic nested systems. If some basic nested systems, say $BS = \{D_1, D_2, V, F\}$ were non-renormalizable, then $\Omega(D_1) = \emptyset$ by Lemma 5.3.2. However, if BS is in the zero-th equivalent family of NS, by definition, $\Omega(D_1) \neq \emptyset$. $\square$

Definition 5.4.5. Let NS be a nested system. If the m-th equivalent family of NS is renormalizable, the $(m+1)$*-th equivalent family* of NS is the first equivalent family of m-th equivalent family of NS. Let $\mathcal{NS}$ be a family of nested system. If the m-th equivalent family of $\mathcal{NS}$ is renormalizable, the $(m+1)$*-th renormalized family* of $\mathcal{NS}$ is the first renormalized family of m-th renormalized family of $\mathcal{NS}$. The procedure of finding the m-th equivalent family is called m*-th renormalization* of NS (resp. $\mathcal{NS}$).

Definition 5.4.6. Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system). If there exists some $M < \infty$ such that the M-th equivalent family of NS (resp. $\mathcal{NS}$) is non-renormalizable, then we say the renormalization of NS (resp. $\mathcal{NS}$) *stops in M steps* and M is the *renormalization stop number* of NS (resp. $\mathcal{NS}$).

We say the renormalization of a nested system (resp. a family of nested systems) *stops in finite steps* if there exists some $M < \infty$ such that the M-th equivalent family is non-renormalizable.

Corollary 5.4.7. *Let NS be a nested system with order l. If m-th equivalent family of NS exists, then its order is less than $(m + l)$. Similarly, let $\mathcal{NS}$ be a family of nested system with order l. If the m-th equivalent family of NS exists, then its order is less than $(m + l)$.*

Proof. Say for any $j \leq m$, the j-th equivalent family is denoted by $k\mathcal{NS}$. Proposition 5.3.11 shows j-th equivalent family has order l_j, then $j + 1$-th equivalent family has at most order $l_j + 1$. $\square$

5.5 Finiteness of Renormalizations

A priori the renormalizations of a nested system do not necessarily stop in finitely many steps. In this subsection, we will show two Propositions 5.5.1 and 5.5.2. They link the existence of any non-wandering point with that of a fixed point. Since the renormalization reduces the period of periodic points, the two positions imply that the renormalization stops in finite steps provided finiteness of periodic points. Moreover, if a family of nested systems has only finitely many periodic points, the cardinality of index set of the family must be finite.

Proposition 5.5.1. *Let $SS = (D, V, F)$ be a simple nested system. If SS has no fixed point, then $\Omega(SS) = \emptyset$.*

Proof. The Jordan curve J prescribed by Db^- and Db^+ has four corners P_1, P_2, P_3, and P_4. Then DR is the closure of the region inside J. In this proof, if the context is clear, we may denote a curve by its two endpoints. For example, P_1P_2 refers to the curve Db^-. Since (D, F) is a real symmetrical degree-2 left-end positive branch wrapping around the real symmetrical strip V, $F(P_1P_2)$, $F(P_2P_3)$, $F(P_3P_4)$ and $F(P_4P_1)$ are four curves each of which can be thought as a graph over a connected component of YR. Especially $F(P_1P_2) \cap F(P_3P_4) \in b^-$, and $F(P_2P_3)$ and $F(P_4P_1)$ are parabola-like in the sense that they have only one critical point. Put

$$P_i' \equiv F(P_i)$$

for $i = 1, \ldots, 4$.

Let $l_i : [0, 1) \longrightarrow \mathbb{R}^2 \backslash \{DR \backslash \{\cup_{i=1}^4 P_i\}\}$, for $i = 1, \ldots, 4$ be four pairwise disjoint curves such that

1. $l_i(0) = P_i$, for $i = 1, \ldots, 4$,
2. putting

$$P_{i,\infty} \equiv l_i(1) \equiv \lim_{t \to 1} l_i(t)$$

for $i = 1, \ldots, 4$, then l_1 and l_4 extend upwards to infinity in the sense that

$$\pi_2(l_1(1)) = \pi_2(l_4(1)) = +\infty,$$

while l_2 and l_3 extend downwards to infinity in the sense that

$$\pi_2(l_2(1)) = \pi_2(l_3(1)) = -\infty,$$

and

3. for all $i = 1, \ldots, 4$,

$$\pi_1(l_i(1)) > -\infty.$$

We construct a map $\mathcal{F} : \big(\cup_{\omega \in \{A,B,C,D\}} l_\omega \big) \cup DR \longrightarrow \mathbb{R}^2$ such that

1. $\mathcal{F}(DR) = F(DR)$ (In particular, $\mathcal{F}(l_i(0)) = P'_i$ for each i),
2. for $i = 1, \ldots, 4$, l_i is homeomorphic to $\mathcal{F}(l_i)$,
3. $\mathcal{F}(l_i)$'s are pairwise disjoint,
4. $\big(\cup_{i=1}^4 \mathcal{F}(l_i) \big) \cap \big(\cup_{i=1}^4 l_i \cup DR \big) = \emptyset$, and
5. for $i = 1, \ldots, 4$, $\mathcal{F}(l_i)$ extends leftwards to infinity in the sense that

$$\pi_1(\mathcal{F}(l_i(1))) = -\infty.$$

Put

$$P'_{i,\infty} \equiv \mathcal{F}(l_i(1))$$

for $i = 1, \ldots, 4$.

Figure 5.5.1 is an illustration of the construction. By the hypothesis of the theorem, $\mathcal{F}|_{DR} = F$ has no fixed point. Furthermore, the construction of $\mathcal{F}$ shows that there exists some $d > 0$ such that for all $i = 1, \ldots, 4$ and all point $p \in l_i$,

$$\operatorname{dist}(\mathcal{F}(p), p) > d.$$

If all l_i and $\mathcal{F}(l_i)$ are defined carefully, we can construct an orientation preserving homeomorphism $\mathcal{F}_L$ between the region R_L bounded by curves $P_{1,\infty}P_1$, P_1P_2 with $P_2P_{2,\infty}$ and the region R'_L bounded by $P'_{1,\infty}P'_1$, $P'_1P'_2$ with $P'_2P'_{2,\infty}$ so that

1. $\mathcal{F}_L|_{P_{1,\infty}P_1 \cup P_1P_2 \cup P_2P_{2,\infty}} = \mathcal{F}(p)|_{P_{1,\infty}P_1 \cup P_1P_2 \cup P_2P_{2,\infty}}$,
2. $\mathcal{F}_L$ does not have a fixed point.

Similarly, there is an orientation-preserving homeomorphism $\mathcal{F}_R$ between the region R_D bounded by curve $P_{2,\infty}P_2$, P_2P_3 with $P_3P_{3,\infty}$ and the region R'_D bounded by $P'_{2,\infty}P'_2$, $P'_2P'_3$ with $P'_3P'_{3,\infty}$ so that

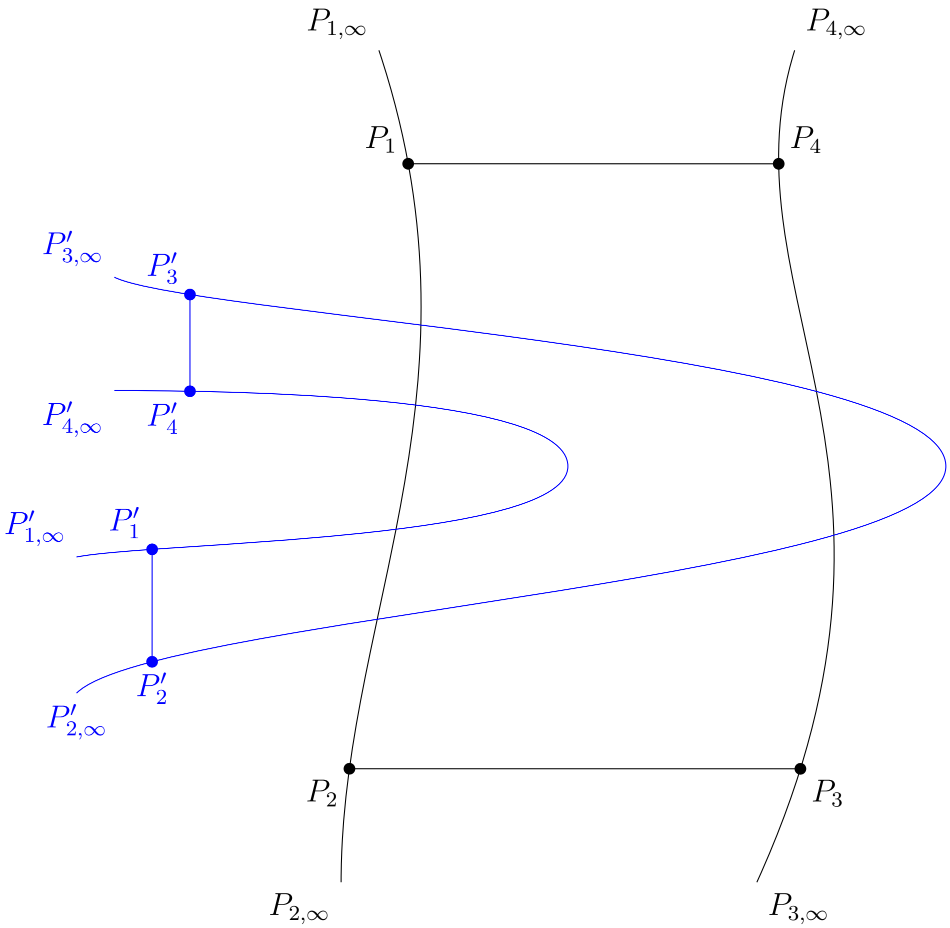


Figure 5.5.1: Extension of a simple nested system

1. $\mathcal{F}_R|_{P_{2,\infty}P_2 \cup P_2P_3 \cup P_3P_{3,\infty}} = \mathcal{F}(p)|_{P_{2,\infty}P_2 \cup P_2P_3 \cup P_3P_{3,\infty}}$,
2. $\mathcal{F}_R$ does not have a fixed point.

In other words, $\mathcal{F}$ can be extended to the larger domain $DR \cup R_L \cup R_D$ which is homeomorphic to its image $\mathcal{F}(DR) \cup R'_L \cup R'_D$ under the extended map; such extension has no fixed point. It is obvious that any map whose domain is $\mathbb{R}^2 \backslash \{DR \cup R_l \cup R_D\}$ and whose range is $\mathbb{R}^2 \backslash \{F(DR) \cup R'_l \cup R'_D\}$ does not have any fixed point. So we can extend $\mathcal{F}$ to $\mathbb{R}^2$ such that $\mathcal{F}$ is an orientation preserving planar homeomorphism without fixed point.

The Brouwer Translation Theorem states that if an orientation preserving planar homeomorphism has no fixed point, then it is conjugate to a planar translation. Therefore, $\mathcal{F}$ has no non-wandering point. Since $\mathcal{F}|_{DR} = F$, F cannot have any non-wandering point either. $\square$

Proposition 5.5.2. *Let $BS = \{D_1, D_2, V, F\}$ be a basic nested system whose pre-nested system is*

$$\{(D_1, F_1), (D_2, F_2), V\}.$$

If D_1 has no fixed point under F_1 and D_2 has no fixed point under F_2, then $\Omega(BS) = \emptyset$.

Proof. We extend F as Figure 5.5.2 illustrates. For each $i = 1, 2, 3, 4$, the extension maps $P_{i,\infty}$ to $P'_{i,\infty}$, and $Q_{i,\infty}$ to $Q'_{i,\infty}$. The regions bounded by those curves are extended accordingly. Similar argument to the proof of Proposition 5.5.1 shows that the extension F is an orientation preserving homeomorphism from $\mathbb{R}^2$ into $\mathbb{R}^2$ without fixed points. The Brouwer Translation Theorem concludes the Proposition 5.5.2. $\square$

Proposition 5.5.1 and 5.5.2 highlight the importance of fixed points. For later use, we introduce the following notations.

Notation 5.5.3. Let $NS = \{D_i, V, F\}_{i \in I}$ be a nested system. Put

$$\text{Fix}(NS) \equiv \{\text{fixed points of F}\} \qquad \text{and} \qquad P(NS) \equiv \{\text{periodic points of F}\}.$$

For any $i \in I$,put

$$\text{Fix}_{NS}(D_i) \equiv \text{Fix}(NS) \cap (D_i \backslash D_{i+1}) \quad \text{and} \quad P_{NS}(D_i) \equiv \{q \in P(NS) | \mathcal{O}(q) \cap (\cup_{j<i} D_j) = \emptyset\}.$$

When no confusion arises, we may use $\text{Fix}(D_i)$ for $\text{Fix}_{NS}(D_i)$ and $P(D_i)$ for $P_{NS}(D_i)$. Let $\mathcal{NS} = \{NS_i\}$ be a family of nested systems. Put

$$\text{Fix}(\mathcal{NS}) \equiv \bigcup_i \text{Fix}(NS_i) \qquad \text{and} \qquad P(\mathcal{NS}) \equiv \bigcup_i P(NS_i).$$

Lemma 5.5.4. *Let NS be a nested system with zero entropy. If* $\text{Fix}(NS) = \emptyset$*, then* $\Omega(NS) = \emptyset$.

Proof. Since NS has zero entropy, its zero-th equivalent family exists and is denoted by $\mathcal{CS} = \{CS_j\}$. For every j, CS_j is either a simple nested system or a basic nested system. Since $\text{Fix}(NS) = \emptyset$, $\text{Fix}(CS_j) = \emptyset$. By Propositions 5.5.1 and 5.5.2, $\Omega(CS_j) = \emptyset$. $\square$

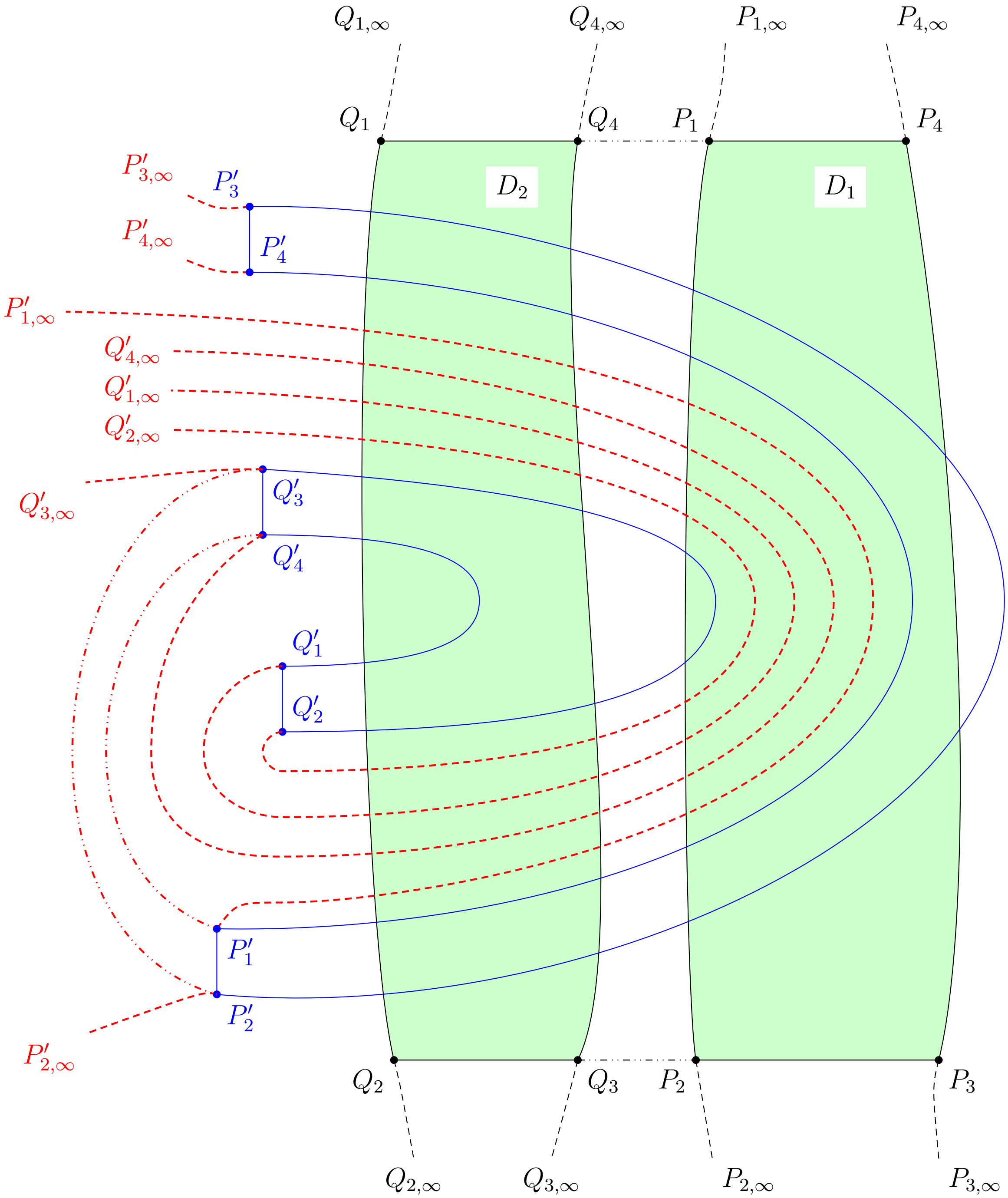


Figure 5.5.2: Extension of a basic nested system

Corollary 5.5.5. *Let $BS = \{D_1, D_2, V, F\}$ be a basic nested system. $BS \neq \mathrm{cl}(\mathcal{W})$. If $\Omega(D_1)$ does not contain any periodic point, then $\Omega(D_1) = \emptyset$.*

Proof. If BS is reducible, then $P_B S(D_1) = \emptyset$ implies $\Omega(D_1) = \emptyset$. Therefore, if $\Omega(D_1) \neq \emptyset$, then BS must be non-reducible. Moreover, $BS \neq \mathrm{cl}(\mathcal{W})$, its basic-renormalized nested system exists and is denoted by ES. By Lemma 5.3.10, $\Omega(ES)$ is essentially the same as $\Omega(D_1)$. Therefore, $P(ES) = P_{BS}(D_1) = \emptyset$. In particular, ES does not contain any fixed point. The corollary thus follows from Lemma 5.5.4. □

Lemma 5.5.6. *Let $BS = \{D_1, D_2, V, F\}$ be a basic nested system so that $BS \notin \mathrm{cl}(\mathcal{W})$. If D_2 does not contain periodic points under F, then BS is a reducible basic nested system.*

Proof. 1) The first step is to prove that the renormalization of BS must stop in finite steps assuming that the renormalization did exist.

Suppose the renormalization of BS does not stop in finite steps. Denote the first renormalized nested system by $ES = \{(E_j)_{j\in I}, V, eF\}$. If D_2 does not contain any periodic point under F, then E_j does not contain any periodic point under eF_j for $j \geq 2$. Hence the zero-th equivalent family of ES has only one basic nested system, whose pre-system is $\{(E_1, eF_1), (E_2, eF_2), V\}$. For the same reason, for every $m \in \mathbb{N}$, the m-th equivalent family contains only one basic family, say they are ${}_mES = \{{}_mE_1, {}_mE_2, V, {}_mF\}$. Since ${}_mE_1 = D_1$, put ${}_mE = {}_mE_2$. If $q \in {}_mE$, ${}_mF(q) = F_2 \circ F_1^m(q)$. The basic renormalization construction implies

$${}_mE_2 \subset F_1^{-1}({}_{m-1}E_2) \subset \cdots \subset F_1^{-m}(D_2),$$

for every m. In particular, for every m, there is a connected component of $F_1^{-m}(b_2^-)$, denoted by ${}_mb^-$ so that ${}_mb^- \subset F_1^{-1}({}_{m-1}b^-)$. Because $\deg({}_mE_2) = 1$, ${}_mb^-$ is a vertical holomorphic curve for every m. Using the same argument as in the proof of Theorem 3.2.1, we obtain a fixed point P whose real stable manifold is $L_\gamma = \lim_m ({}_mb^-)$.

For each $k \in \mathbb{Z}^+$, F_1^k strictly wraps ${}_{m+k}E$ around ${}_mE$ and the branch $({}_{m+k}E, F_1^k)$ has degree one. Put ${}_mb_d = b_d \cap {}_mE$. In particular, $F_1^k({}_{m+k}b_d)$ is a degree one horizontal holomorphic curve in ${}_mE$ for any m and k. If we fix m and let $j \to \infty$, we then have a horizontal holomorphic curve $L_m = \lim_j F_1^k({}_{m+k}b_d)$.

Since ${}_mE$ lies on the left side of ${}_{m+1}E$, $\lim_m L_m = Q \in L_\gamma$. We now prove $P = Q$. Observe

$$F_1(L_{m+1}) = F_1(\lim F_1^{k-1}({}_{m+k}b_d)) = \lim F_1(F_1^{k-1}({}_{m+k}b_d)) = L_m.$$

Thus we have

$$F_1(Q) = F_1(\lim_m L_m) = \lim_m F_1(L_m) = \lim_m L_{m-1} = Q.$$

Moreover, for all m, $\lim_k F_1^{-k}(L_m) = \lim_k L_{m+k} = Q = P$ implies that L_m are part of the unstable manifold of P

On the other hand, ${}_mF({}_mE) \cap_m E \neq \emptyset$ by the definition of basic-renormalization. Since

$${}_mF({}_mE) = F_2(F_1^{m-1}({}_mE)) \supset F_2(F_1^{m-1}(F_1({}_{m+1}E))),$$

then ${}_mF({}_mE) \cap_{m+k} E \neq \emptyset$ for any $k \in \mathbb{N}$. Therefore ${}_mF({}_mE) \cap L_\gamma \neq \emptyset$. In particular,

$$ {}_mF({}_mb_d) \cap L_\gamma = F_2(F_1^m({}_mb_d)) \cap L_\gamma \neq \emptyset, $$

where we used the fact that ${}_mF({}_mE)$ lies inside the left attached Jordan Curve associated with ${}_mF({}_mb_d)$. Since $L_1 = \lim_m F_1^{m-1}({}_mb_d)$, $F_2(F_1(L_1)) \cap L_\gamma \neq \emptyset$. This implies there exists either a transverse homoclinic point or a homoclinic tangency. This contradicts $BS \notin \mathrm{cl}(\mathcal{W})$.
2) Assume BS is renormalizable. We now have that the renormalization of BS must stop in finite steps. Say the first renormalized nested system of BS is $ES = \{E_j, V, eF\}$. Since for any $j \geq 2$, $\Omega(E_j)$ does not contain periodic points, $\Omega(E_j) = \emptyset$ by Proposition 5.5.1 and 5.5.2. Then we have

$$ \Omega(D_1) = \Omega(ES) = \bigcup \Omega_{ES}(E_j) = \Omega_{ES}(E_1) \subset F_1^{-(l+2)}(D_1). $$

Assume the M-th renormalized nested system of BS is no longer renormalizable. By induction, for any $m < M$, the m-th equivalent family of BS contains only one basic nested system, denote by $mES = \{{}_mE_1, {}_mE_2, V, mF\}$, which has non-wandering points. We have

$$ \Omega(D_1) = \Omega_{ES}(E_1) = \cdots = \Omega_{{}_mES}({}_mE_1) \subset F_1^{-(l+m+1)}(D_1). $$

When $m = M$, the basic nested system ${}_MES = \{{}_ME_1, {}_ME_2, V, MF\}$ is reducible. If $p \in \Omega_{{}_MES}({}_ME_1) = \Omega(D_1)$, then $\mathcal{O}_{{}_MF}(p) \cap {}_ME_2 = \emptyset$. Hence,

$$ \mathcal{O}_{{}_MF}(p) = \mathcal{O}_{F_1}(p) = \mathcal{O}_F(p). \qquad \square $$

Corollary 5.5.7. *Let $BS = \{D_1, D_2, V, F\}$ be a basic nested system. If for any priodic point $p \in D_2$, $\mathcal{O}_F(p) \in D_2$, then BS is a reducible basic nested system.*

Proof. Suppose BS is non-reducible. Denote the first renormalized nested system by $ES = \{(E_j)_{j\in I}, V, eF\}$. Since $\mathcal{O}_F(p) \in D_2$, the strip E_j does not contain any periodic point under eF for any $j \geq 2$. The nested system ES is reducible by Lemma 5.5.6. The corollary follows from the fact that $\Omega(E_2) = \emptyset$ because of E_2 has no periodic points. $\square$

Lemma 5.5.8. *Let $BS = \{(D_1, D_2), V, F\}$ be an l-order basic nested system whose pre-system is $\{(D_1, F_1), (D_2, F_2), V\}$. Let $eBS = \{(E_j)_{j\geq 1}, V, eF\}$ be the first renormalized nested system. $p \in \Omega(D_1)$ is a periodic point of BS under F. Then either the period of p under eF is strictly less than that under F or $\mathcal{O}_F(p) \subset F^{-(l+1)}(D_1)$.*

Proof. Since all the relevant points $q \notin F^{-(l+1)}(D_1)$ have return time longer than 1, the proposition follows. $\square$

Lemma 5.5.9. *Let $BS = \{(D_1, D_2), V, F\}$ be an l-order basic nested system so that $BS \notin \mathrm{cl}(\mathcal{W})$. It has only finitely many fixed point $p_1, \ldots, p_n$. The BS is a reducible basic nested system.*

Proof. Since all p_i have period one, their periods cannot be lowered. It follows Proposition 5.5.8 that for every j, either $p_i \in D_2$ or $p_i \in F^{-(l+1)}(D_1)$. Then Corollary 5.5.7 implies BS is a reducible basic nested system, which is non-renormalizable. $\square$

Proposition 5.5.10. *Let $BS = \{D_1, D_2, V, F\}$ be a basic nested system so that $BS \notin \mathrm{cl}(\mathcal{W})$. If BS contains only finite periodic points, then exists some M so that the M-th equivalent family of BS consists of only finitely many simple nested systems (i.e. the renormalization of BS stops in finite steps).*

Proof. If the periodic points have orbits that do not involve D_2. The Lemma follows by definition. Otherwise, the Lemma 5.5.8 shows that the renormalization will lower the periods. Therefore, after finitely many renormalizations, periods will be lowered to one. This corollary follows from Lemma 5.5.9. □

Proposition 5.5.11. *Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system) with only finitely many periodic points. There exists a positive integer M so that the M-th equivalent family of NS(or $\mathcal{NS}$) contains only finitely many simple nested systems. (i.e. Its renormalization will stop in finite steps).*

Proof. The zero-th equivalent family consists of only simple and basic nested system. Hence, this proposition follows from Proposition 5.5.10. □

Definition 5.5.12. Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system). If there exists a positive integer m so that the m-th equivalent family of NS (resp. $\mathcal{NS}$) consists of only simple nested system, then the m-th equivalent family is said to be the *the zero-th equivalent simple family of NS*.

Before proceeding to the next chapter, we discuss two more things. The first is how their determinants behave under renormalization.

Lemma 5.5.13. *Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system). If the k-th equivalent family of NS (resp. $\mathcal{NS}$) exists, denoted it by kNS (resp. $k\mathcal{NS}$), then*

$$\mathrm{Jac}(kNS) \leq \mathrm{Jac}(NS) \qquad (\textit{resp.}\ \mathrm{Jac}(k\mathcal{NS}) \leq \mathrm{Jac}(\mathcal{NS})).$$

Proof. The renormalization map of F is in the form of $F_{i+1}^{n-1} \circ F_i$ where n is the return time of the corresponding relevant points. Since $\mathrm{Jac}(NS) \leq d < 1$, it is obvious that each renormalization shrinks the Jacobian by the factor of at least d. The lemma follows. □

The second is that we discuss how scaling ratio behaves under renormalization.

Definition 5.5.14. Let $NS = \{(D_i)_{i \in I}, V, F\}$ be a nested system. The *scaling ratio* of NS is

$$\sigma(NS) = \sigma(D_1, V).$$

Clearly $\sigma(D_i, V) \leq \sigma(NS)$ for every $i \in I$.

Definition 5.5.15. Let $\mathcal{NS} = \{NS_j\}_{j \in I}$ be a family of nested systems. The *scaling ratio* of NS is

$$\sigma(\mathcal{NS}) = \sup_{j \in I} \sigma(NS).$$

Remark 5.5.16. Let $\mathcal{NS}$ be a family of nested system. Then $\sigma(\mathcal{NS}) > 0$.

Lemma 5.5.17. *Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system). If the k-th equivalent family of NS (resp. $\mathcal{NS}$) exists, we denote it by kNS (resp. $k\mathcal{NS}$). We have $\sigma(kNS) \leq \sigma(NS)$ (resp. $\sigma(k\mathcal{NS}) \leq \sigma(\mathcal{NS})$).*

Proof. Let $BS = \{D_1, D_2, V, F\}$ be a basic nested system and its first renormalized nested family $ES = \{E_j, V, eF\}$. The scaling ratio of the first equivalent family of BS is

$$\sup_j \sigma(E_j, V) = \sigma(D_1, V).$$

For arbitrary nested system NS, their zero-th equivalent family consists of either basic or simple nested systems, therefore $\sigma(1NS) \leq \sigma(NS)$. For arbitrary k-th equivalent family, the lemma follows by induction. $\square$

Chapter 6

Renewal

Simple nested systems can be classified into two sub-types based on the numbers of saddle points that they contain. If a simple nested system contains no more than one saddle, its dynamics is simple and well-understood; we study such simple nested systems in Subsection 6.1. On the other hand, if a simple nested system contains more than one saddle and satisfies an apriori bound, we can construct a refiner nested system on which the renormalization operator can act on (See Subsection 6.2). In Subsection 6.1, we also define the differentiable one-parameter families of simple nested systems and study the saddle-node bifurcations in such families.

6.1 Saddle node bifurcation in Simple nested systems

Put

$$\Delta \equiv \{(x,x) : x \in \mathbb{R}\}.$$

Δ is therefore the diagonal of $\mathbb{R}^2$.

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a planar homeomorphism such that $F(x,y) \mapsto (f(x,y),x)$. Then $F^{-1}(\Delta)$ is can be viewed as an graph of a function over x-axis. In other words, there exists a function $\delta : \mathbb{R} \longrightarrow \mathbb{R}$ so that $F^{-1}(\Delta) = \{x, \delta(x)\}$. For later use, we say such $\delta : \mathbb{R} \longrightarrow \mathbb{R}$ is the *diaganol function* of F.

Lemma 6.1.1. *Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an orientation preserving diffeomorphism such that $F(x,y) \mapsto (f(x,y),x)$. If F does not have any fixed point then F does not contain any nonwandering point.*

Proof. There are two cases based on whether the diagonal function of F is above or below the diagonal; see Figure 6.1.1. Without loss of generality, we assume the diagonal function of F lies below the diagonal as in the part (a) of Figure 6.1.1.

Let $p = (x,y)$ be a point in $\mathbb{R}^2$, and $F(p) = (x_1,y_1)$ be its image.

1) Assume p is above the diagonal, i.e. $x < y$. Since F is orientation preserving, then p' lies on the diagonal, i.e. $x_1 < y_1$. Therefore, $x_1 < y_1 = x < y$. Denote $F^n(p) = (x_n, y_n)$. By induction, we have

$$x_n < y_n = x_{n-1} < y_{n-1} < \cdots < x_1 < y_x = x < y.$$

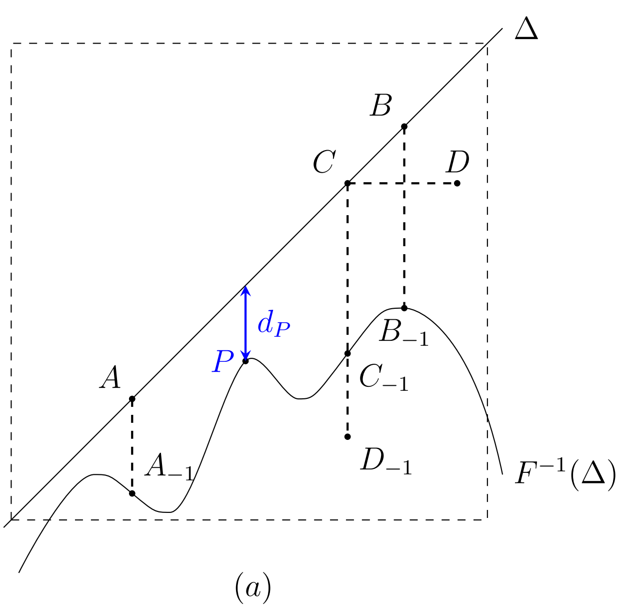


Figure 6.1.1: Because there is a positive distance between the diagonal function of F and the diagonal, the dynamics is clear when F has no fixed point.

Since there is no fixed point, $\lim x_n = -\infty$ and $\lim y_n = -\infty$. The point p cannot be a non-wandering point.

2) Assume p is below the diagonal but above $F^{-1}(\Delta)$. Since F is orientation preserving, then $F(p)$ lies above the diagonal.

3) Assume p is below $F^{-1}(\Delta)$. If there exists an $N \in \mathbb{Z}^+$ so that $F^N(p)$ is above the diagonal, then we are done. Otherwise, we have

$$x_n > y_n = x_{n-1} > y_{n-1} > \cdots > x_1 > y_1 = x > y.$$

Since there is no fixed point, $\lim x_n = \infty$ and $\lim y_n = \infty$. The point p cannot be a non-wandering point.

$\square$

The Lemma 6.1.1 is a direct application of Brouwer fixed-point Theorem. However, the technique we use in the proof is more relevant in our setting.

Lemma 6.1.2. *Let F be as in Lemma 6.1.1. If Δ and $F^{-1}(\Delta)$ only intersect tangentially, then $\Omega(F)$ consists of only non-hyperbolic fixed points.*

Proof. Suppose p is above the diagonal Δ. the identical argument of the proof of Lemma 6.1.1 shows that

$$x_n < y_n = x_{n-1} < y_{n-1} < \cdots < x_1 < y_x = x < y.$$

Hence, we have $\lim x_n = x'$ and $\lim y_n = y'$. Moreover

$$F(x', y') = F(\lim x_n, \lim y_n) = \lim F(x_n, y_n) = \lim(x_{n+1}, y_{n+1}) = (x', y').$$

The point p is attracted to a fixed point, therefore it cannot be non-wandering point. $\square$

Definition 6.1.3. Let $\{F_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2\}_{t\in\mathcal{T}}$ be a family of maps. Let $\mathcal{F} : \mathcal{T} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map where $\mathcal{F}(t, x, y) = F_t(x, y)$. If $\mathcal{F}$ is differentiable with respect to t, then the family is said *differentiable with respect to* t.

Lemma 6.1.4. *Let $\{F_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2\}_t$ be a family of orientation preserving diffeomorphisms such that $F_t(x, y) \mapsto (f_t(x, y), x)$. The family is differentiable with respect to t. Let $\Omega(F_t)$ be the nonwandering set of F_t for any t. If for $t < 0$ $\Omega(F_t) = \emptyset$ and $\Omega(F_0) \neq \emptyset$, then $\Omega(F_0)$ consists of fixed points and all the fixed points are non-hyperbolic.*

Proof. Since the family $\mathcal{F}$ is differential with respect to t, the diagonal Δ and $F_0^{-1}(\Delta)$ interest tangentially. The lemma therefore follows from Lemma 6.1.1 and 6.1.2. $\square$

If the family $\{F_t\}$ in Lemma 6.1.4 is of homeomorphisms other than diffeomorphisms, the first part of conclusion that $\Omega(F_0)$ only contains fixed points is still valid. However, the second part that all the fixed points are non-hyperbolic is no longer valid since hyperbolicity is not properly defined when maps are merely homeomorphisms.

Let $\{F_t\}_t$ be as in Lemma 6.1.4. For any t, let δ_t be the diagonal function of F_t. Let $\bar{\delta}$ be a map whose domain is (t, x) plane where $\bar{\delta}(t, x) = \delta_t(x)$. For a generic family $\{F_t\}$, at any fixed point p of F_0,

1. $\bar{\delta}_0''(p) \neq 0$
2. $\left.\frac{\partial\bar{\delta}}{\partial t}\right|_{t=0}(p) \neq 0.$

Therefore, for every $p \in \Omega(F_0)$, the family F_t experiences a saddle node bifurcation at p when $t = 0$. In other words, there exists an $\epsilon_p > 0$ so that for any $\Omega(F_t)$ where $t < \epsilon_p$ contains one saddle fixed point p_t and one sink fixed point q_t where

$$\lim_{t\to 0} p_t = \lim_{t\to 0} q_t = p.$$

If we further assume that $\Omega(F_0)$ consists of only n finitely many fixed points, then there exists an $\epsilon > 0$ so that $\Omega(F_t)$ contains only $2n$ fixed points for all $t < \epsilon$.

Remark 6.1.5. Lemma 6.1.1, 6.1.4 and the following remarks are still valid if the map (resp. the family of maps) is defined only on a subset of $\mathbb{R}^2$.

Definition 6.1.6. Let $SS_t = \{D_t, V, F_t\}_{t\in\mathcal{T}}$ be a family of simple nested systems. The structure map for each D_t is s_t. The family is said differentiable with respect to t, if

1. Let $\Phi : \mathcal{I} \times X \times Y \longrightarrow \mathbb{C}^2$ be a map where $\Phi(t, z, w) = (s_t^{-1}(z, w), w)$. Φ is differentiable with respect to t.
2. Let $\mathcal{F} : \mathcal{I} \times X \times Y \longrightarrow V$ be a map where $\mathcal{F}(t, z, w) = F_t \circ \Phi(t, z, w)$. $\mathcal{F}$ is differentiable with respect to t.

Remark 6.1.7. Let $SS_t = \{D_t, V, F_t\}_{t\in\mathcal{T}}$ be a family of simple nested systems. By continuity, the followings are clear:

1. If SS_0 has a hyperbolic periodic point, then there exists an $\epsilon > 0$ so that for any $t < \epsilon$, SS_t has a hyperbolic periodic point with same period.

2. If for any $\epsilon > 0$ there exists an $t < \epsilon$ so that SS_t has a periodic point with period T, then SS_0 has a periodic point with period T.

6.2 Renewal

Definition 6.2.1. Let $SS = \{D, V, F\}$ be a simple nested system. If there exists a (non-simple) nested system $rSS = \{rD_i, rV, rF\}_{i\in I}$ so that

1. $\Omega(SS) \backsimeq \Omega(rSS)$,

2. for any $i \in I$, $\Omega(rD_i) \neq \Omega(SS)$,

then rSS is said to be the *renewal* of SS. The zero-th equivalent family of rSS, denoted by $r\mathcal{SS}$, is called the *first renewal equivalent family* of SS.

The definition shows the renewal of SS must not be SS itself, therefore it refines the nonwandering set $\Omega(SS)$. More precisely, we have the following Lemma.

Lemma 6.2.2. *Let SS be a renewable simple nested system and $r\mathcal{SS} = \{BS_i\}_{i\in I}$ be its first renewal equivalent family. For any $i \in I$, $Fix(BS_i) \subsetneq Fix(SS)$.*

Proof. If there existed an $i \in I$ so that $Fix(BS_i) = Fix(SS)$, then BS_j does not have any fixed point for any $j \neq i$. Thus BS_j does not contain any non-wandering point by Proposition 5.5.1 and 5.5.2. ☐

Definition 6.2.3. Let $\mathcal{SS} = \{SS_i\}_{i\in I}$ be a family of simple nested system. If

$$I' = \{i \in I | SS_i \text{ is renewable}\} \neq \emptyset,$$

then $\mathcal{SS}$ is said to be renewable. Let $r\mathcal{S}_i$ denote the first renewal equivalent family by SS_i for any $i \in I'$ and put

$$r\mathcal{SS} \equiv (\cup_{i\in I'} r\mathcal{S}_i) \bigcup (\cup_{j\in I\setminus I'} SS_j).$$

$r\mathcal{SS}$ is the called first renewal equivalent family of $\mathcal{SS}$.

Definition 6.2.4. Let NS (resp. $\mathcal{NS}$) be a nested system (a family of nested system). If its zero-th equivalent simple family, denoted by $\mathcal{SS}$, exists and is renewable, then NS (resp. $\mathcal{NS}$) is said to be renewable and its first renewal equivalent family is the first renewal equivalent family of $\mathcal{SS}$.

If a nested system is non-renewable, by definition there are two possible reasons: First, its zero-th equivalent simple family does not exist. Second its zero-th equivalent simple family is not renewable. When there is no confusion, we will hereafter say a nested system (a family of nested systems) renewable without explicitly mentioning the existence of its zero-th equivalent simple family.

Definition 6.2.5. Let σ be a positive number so that $0 < \sigma \leq 1$. A positive number $b \equiv b(\sigma)$ is called a *renewable determinant of* σ if for any simple nested system SS with scaling ratio $\sigma \equiv \sigma(SS)$, if $\mathrm{Jac}(SS) < b$ then one of the followings happens:

1. $\Omega(SS) = p$ where p is a non-hyperbolic fixed point;
2. $\Omega(SS) = \{p, q\}$ where p is a saddle and q a attracting fixed point;
3. $\Omega(SS)$ has at least two saddle nodes and SS therefore has a renewal rSS.

Definition 6.2.6. Define a function $\mathfrak{b} : (0, 1) \longrightarrow (0, \infty)$ so that for any $\sigma \in (0, 1)$

$$\mathfrak{b}(\sigma) \equiv \sup\{b | b \text{ is a renewable derminant of } \sigma\}.$$

This function $\mathfrak{b}$ is called the *renewable determinant function*.

Let SS be a simple nested system. Suppose its determinant is $\mathrm{Jac}(SS)$ and its scaling ratio is $\sigma(SS)$. If $\mathrm{Jac}(SS) < \mathfrak{b}(\sigma(SS))$, then we say that SS has *a renewal compatible pair of determinant and scaling ratio*.

Assumption 6.2.7. *Let σ be a positive number so that $0 < \sigma \leq 1$. There exists a renewable determinant $b(\sigma)$ of σ. That is, the renewable determinant function exists. Moreover, if a simple nested system SS has a renewable compatible pair of determinant* $\mathrm{Jac}(SS)$ *and* $b(\sigma)$ *and has at least two saddle nodes, then its renewal rSS has a renewable compatible pair of determinant* $\mathrm{Jac}(rSS)$ *and scaling ratio $\sigma(rSS)$. More precisely, we have*

$$\sigma(rSS) \leq \sigma \qquad \text{and} \qquad \mathrm{Jac}(rSS) \leq \mathrm{Jac}(SS).$$

The function which relates the scaling ratio σ and the determinant b given in above Assumption 6.2.7 is inversely proportional, i.e., the larger the σ is the smaller the b is. From the proof of above proposition, we have $\lim_{\sigma \to 0} \mathfrak{b}(\sigma) < 1/2$.

Although the renewal of a nested system is not necessarily unique, if a nested system whose scaling ratio and determinant satisfy the relation given in Assumption 6.2.7 is renewable, we will always assume the renewal is that given in the proof of Assumption 6.2.7.

Corollary 6.2.8. *Let $\mathcal{SS} = \{SS_i \equiv \{D_i, V_i, F_i\}\}$ be a family of simple nested systems. The number $\sigma \equiv \sigma(\mathcal{SS})$ is the scaling ratio of $\mathcal{SS}$. Suppose for any i,* $\mathrm{Jac}(F_i)$ *is less than $b(\sigma)$ given in Assumption 6.2.7. Then either of the following happens:*

1. *For any i, $\Omega(SS_i)$ either consists of only one non-hyperbolic fixed point or consists of one saddle and one attracting fixed point.*

2. $\mathcal{SS}$ is renewable.

Proof. Because $\sigma(\mathcal{SS}) = \sup \sigma(SS_i)$, the corollary follows the Assumption 6.2.7 immediately. $\square$

Definition 6.2.9. Let NS (resp. $\mathcal{NS}$) be a nested system (a family of nested system). If the k-th renewal equivalent family of NS (resp. $\mathcal{NS}$), then the $(k+1)$-th renewal equivalent family of NS (resp. $\mathcal{NS}$) is the first renewal equivalent family of its k-th renewal equivalent family.

Let NS (resp. $\mathcal{NS}$) be a nested system (a family of nested system). If there exists an $K \in \mathbb{Z}^+$ so that the K-th renewal equivalent family of NS (resp. $\mathcal{NS}$) exists but is not renewable, then we say the renewal process of NS (resp. $\mathcal{NS}$) *stops* at K steps.

Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system). Given k and $m \in \mathbb{Z}^+$, we hereafter denote the m-th equivalent family of its k-the renewal equivalent family of NS (resp. $\mathcal{NS}$) by ${}_{k,m}NS$ (resp. ${}_{k,m}\mathcal{NS}$)provided it exists.

Given a nested system (or a family of nested system), if it is either renormalizable or renewable then we say it is actionable.

Definition 6.2.10. Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system). If there exist an K and $M \in \mathbb{Z}^+$ so that ${}_{K,M}NS$ (resp. ${}_{K,M}\mathcal{NS}$) exists but ${}_{K,M}NS$ is not actionable, then (K, M) is said to be the *stop number* of NS (resp. $\mathcal{NS}$). Moreover, if ${}_{(K,M)}NS$ (resp. ${}_{K,M}\mathcal{NS}$) is the simple equivalent family of the K-th renewal of NS, then we say NS is fully actionable and acted at (K, M).

Notation 6.2.11. Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system). Suppose (K, M) is its stop number. For any $i < K$, we know the k-th renewal equivalent family has its own simple equivalent family through finitely many renormalizations. Therefore for later use, we denote the stop number of renormalization for the k-th renewal equivalent family by $m(k)$ for $k < K$. If NS (resp. $\mathcal{NS}$) is fully acted, then we can assign $m(K)$ as well. It is clear $m(K) = K$.

Lemma 6.2.12. *Let SS be a simple nested system whose scaling ratio and determinant satisfy the relation given in Assumption 6.2.7. Suppose SS is renewable and rS is its first renewal. Suppose the zero-th equivalent simple family of rS exists, denoted by $r\mathcal{S}' = \{rS_i'\}$. Then one of the following happens: 1) $r\mathcal{S}'$ is renewable, and 2) $\Omega(rS_i')$ either contains a non hyperbolic fixed point or contains one saddle and one sink fixed point for any $i \in I$.*

Proof. Assumption 6.2.7 shows that the first renewal rS has a compatible pair of determinant and scaling ratio. Renormalization operator preserves such compatibility by Lemma 5.5.17. Then either the zero-th equivalent simple family $r\mathcal{S}'$ is renewable, or rS_i' satisfies the first two properties of Definition 6.2.5. $\square$

Theorem 6.2.13. *Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system) with zero entropy and finitely many periodic points. Suppose its scaling ratio σ and determinant $b \equiv \mathrm{Jac}(NS)$ are compatible. Then there exist K, $M \in \mathbb{Z}^+$ so that NS is fully acted at the actionable number (K, M). Moreover, suppose ${}_{(K,M)}NS = \{KS_i\}_{i \in I}$, then all KS_i are simple nested systems, with each $\Omega(KS_i)$ containing either a non-hyperbolic fixed point or one saddle and one sink fixed point for any $i \in I$.*

Proof. Let $rS = \{BS_i\}$ be the first renewal equivalent family of NS. Lemma 6.2.2 shows that each BS_i inherits strictly less fixed point from NS. Especially, there cannot be more than n times renewal.

On the other hand, suppose the k-th renewal of NS exists. Proposition 5.5.11 implies (by finitely many renormalizations) the existence of the k-th equivalent simple family, denoted by ${}_{(k,m(k))}NS = \{kS_i\}_{i \in I}$. Lemma 6.2.12 shows that either the $k+1$-th renewal exists or each $\Omega(S_i)$ contains either a non-hyperbolic fixed point or one saddle and one sink fixed point. □

The above theorem has a straightforward but important corollary as follows:

Corollary 6.2.14. *Let NS (resp. $\mathcal{NS}$) be a nested system (resp. a family of nested system) as in Theorem 6.2.13. Then $\Omega(NS)$ (resp. $\Omega(\mathcal{NS})$) contains and only contains periodic points.*

Proof. For every pair of positive integers $k \leq K$ and $m \leq M$, the (k,m)-th equivalent family of NS has the same real non-wandering set as that of NS. The corollary follows from Theorem 6.2.13. □

Chapter 7

A family of nested systems on the route to chaos

In this chapter, we study the route to chaos of nested systems. We first construct a combinatorial representation of nested systems based on the renormalization and the renewal operation in Subsection 7.1. Then we study what dynamical change of a one-parameter family of nested systems we can infer from the change of the combinatorial representations. More precisely, we show that 1) if the dynamics does not change, the combinatorial representation does not change (except a possible formal change; see Definition 7.2.3), and 2) if the dynamics does change, the combinatorial must change correspondingly.

With the the combinatorial representation, we conclude that a nested system can only experience saddle-node bifurcations and period-doubling bifurcations (see Propositions 7.2.12 and 7.2.15). We then conclude our main theorem that a nested system has the period-doubling cascade as its route to chaos.

7.1 Combinatorial representation of a nested system

Notation 7.1.1. Hereafter, we denote

1. $\mathcal{L} \equiv \{B, S, O, \bullet\}$.
2. $Map(\mathbb{Z}^+, \mathcal{L})$ is the space of all maps $\Psi : \mathbb{Z}^+ \longrightarrow \mathcal{L}$.

Definition 7.1.2. Let $NS = \{D_i, V, F\}_{i \in I}$ be a nested system and $\mathcal{CS}$ be its zero-th equivalent family. Define a map ${}_{(0,0)}\Psi : \mathbb{Z}^+ \longrightarrow \mathcal{L}$ by

$$
{}_{(0,0)}\Psi(i) = \begin{cases} B, & \text{if } \mathcal{CS} \text{ contains the basic sub-nested systems of } NS \text{ indexed by } \{i, i+1\} \\ S, & \text{if } \mathcal{CS} \text{ contains the essential simple sub-nested systems of } NS \text{ indexed by } \{i\} \\ O, & \text{if } \mathcal{CS} \text{ contains the auxiliary simple sub-nested systems of } NS \text{ indexed by } \{i\} \\ \bullet, & \text{otherwise} \end{cases}
$$

${}_{0,0}\Psi$ is called the *the* $(0,0)$*-th representation map of* NS.

A nested system must consist of finitely many branches. In other wordss, there must exist a positive integer $k \leq |I|$ so that ${}_{(0,0)}\Psi(k) \neq \bullet$ and for all $j > k$, ${}_{(0,0)}\Psi(j) = \bullet$. The number k is called the *cardinality* of ${}_{(0,0)}\Psi$.

Example 7.1.3. If $NS = \{D, V, F\}$ is a simple nested system with a periodic point, then

$$ {}_{(0,0)}\Psi(i) = \begin{cases} S, & \text{if } i = 1, \\ \bullet, & i > 1. \end{cases} $$

The cardinality of ${}_{(0,0)}\Psi(i)$ is 1.

Lemma 7.1.4. *The map ${}_{(0,0)}\Psi$ as given in Definition 7.1.2 is well-defined.*

Proof. Since $\mathcal{CS}$ is the zero-th equivalent family of NS, it cannot contain both the basic sub-nested systems of NS indexed by $\{i, i+1\}$ and the simple sub-nested systems of NS indexed by $\{i\}$. Moreover, the simple nested system cannot be both essential and auxiliary at the same time by definition. $\square$

Definition 7.1.5. Let BS be a basic nested system and ES be its first renormalized nested system. A map $\mu : \mathbb{Z}^+ \longrightarrow \mathcal{L}$ is called the *modeling map* of BS if it is the the $(0,0)$-th representation map of ES. The cardinality of μ is the cardinality of the $(0,0)$-th representation map of ES.

Let SS be a simple nested system. The modeling map $\mu : \mathbb{Z}^+ \longrightarrow L$ is the the $(0,0)$-th representation map of SS itself. The cardinality of μ is the cardinality of the $(0,0)$-th representation map of SS.

Definition 7.1.6. Let NS, $\mathcal{CS}$, and ${}_{(0,0)}\Psi(i)$ be as in Definition 7.1.2. If $CS' \in \mathcal{CS}$ is the sub-nested system of NS indexed by i or $\{i, i+1\}$, then it is called *the* (i)*-position* nested system with respect to NS. The modeling map of CS', denoted by $\mu_{(i)}$, is called the (i)*-position modeling map*.

Definition 7.1.7. Let SS be a renewable simple nested system and the nested system RS be its renewal. The *remodeling map of SS*, denoted by $\rho : \mathbb{Z}^+ \longrightarrow \mathcal{L}$, is the $(0,0)$-th representation map of RS. The cardinality of ρ is the cardinality of the $(0,0)$-th representation map of RS.

Let SS' be a non-renewable simpled nested system. The *remodeling map of SS'*, denoted by $\rho : \mathbb{Z}^+ \longrightarrow \mathcal{L}$, is the $(0,0)$-th representation map of SS' itself. The cardinality of ρ is the cardinality of the $(0,0)$-th representation map of SS'.

Definition 7.1.8. The bullet map $\mu_\bullet : \mathbb{Z}^+ \longrightarrow \mathcal{L}$ is defined by

$$ \mu_\bullet(i) = \bullet \qquad \text{for all } i \in \mathbb{Z}^+. $$

Definition 7.1.9. Let NS be a nested system. For a pair of numbers $(k, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, if NS is (k, m)-th actionable, then we define (k, m)*-th location number of NS*, denoted by $p(k, m)$, as follows:

$$ p(k, m) \equiv \Sigma_{i=0}^{k-1} M(i) + k + m, \tag{7.1.1} $$

where $M(i)$ is the renormalization stop number of the i-th renewal equivalent family of NS for each $0 \leq i < k$.

Definition 7.1.10. Let NS be a nested system. For any pair $(k,m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that NS is (k,m)-th actionable, we then have the (k,m)-th location number of NS, denoted by $p(k,m)$. Given the $(0,0)$-th representation of NS as in Definition 7.1.2, we define the (k,m)-th representation map of NS

$$_{(k,m)}\Psi : \mathbb{N}^{p(k,m)} \longrightarrow \mathrm{Map}(\mathbb{Z}^+, \mathcal{L})$$

inductively on $p(k,m)$ as follows:

$$_{(k,m)}\Psi(a_1,\dots,a_{p(k,m)}) = \begin{cases} \rho_{(a_1,\dots,a_{p(k,m)})}, & \text{if } m = 0 \text{ and } (a_1,\dots,a_{p(k,m)})\text{-position} \\ & \quad \text{nested system exists} \\ \mu_{(a_1,\dots,a_{p(k,m)})}, & \text{if } m \neq 0 \text{ and } (a_1,\dots,a_{p(k,m)})\text{-position} \\ & \quad \text{nested system exists} \\ \mu_\bullet, & \text{otherwise} \end{cases}$$

If a nested system NS' is the $a_{p(k,m)+1}$-th position nested system with respect to $(a_1,\dots,a_{p(k,m)})$-th position nested system with respect to NS, NS' is called the $(a_1,\dots,a_{p(k,m)+1})$-position nested system with respect to NS. Furthermore, the modeling map of NS', denoted by $\mu_{(a_1,\dots,a_{p(k,m)+1})}$, is called the $(a_1,\dots,a_{p(k,m)+1})$-position modeling map; if $m = M(k)$ the remodeling map of NS', denoted by $\rho_{(a_1,\dots,a_{p(k,m)+1})}$, is called the $(a_1,\dots,a_{p(k,m)+1})$-position remodeling map.

7.2 Dynamical equivalent windows, Bifurcation moment and Route to chaos

Let $\{NS_t\}_{t\in\mathcal{T}} = \{(D_{ti})_{i\in I}, V, F_t\}_{t\in\mathcal{T}}$ be a family of nested systems. Put

$$SS_{tj} \equiv \{D_{tj}, V, F_t\}_{t\in\mathcal{T}}$$

for every $j \in I$. SS_{tj} is called the sub-family of NS_t indexed by j.

Definition 7.2.1. Let $\{NS_t\} = \{(D_{ti})_{i\in I}, V, F_t\}_{t\in\mathcal{T}}$ be a family of nested systems and SS_{tj} be the sub-family of NS_t indexed by j. The family is said to be differentiable with respect to t, if SS_{tj} is differentiable with respect to t for any $j \in I$.

Let $\{NS_t\} = \{(D_{ti})_{i\in I}, V, F_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. We may say $\{NS_t\}$ is differentiable for $t \in \mathcal{T}$ for simplicity.

Moreover, assume the $(a_1,\dots,a_m)$-th position nested system of NS_t, denoted by NS'_t, exists for any $t \in \mathcal{T}'$ where $\mathcal{T}' \subset \mathcal{T}$ is an interval. If there exists a $\mathcal{T}'' \subset \mathcal{T}'$ so that the family $\{NS'_t\}_{t\in\mathcal{T}''}$ is differentiable with respect to t, we say for simplicity the family of $(a_1,\dots,a_m)$-th position nested system of NS_t is differentiable for $t \in \mathcal{T}''$.

Lemma 7.2.2. *Let $\{BS_t\} = \{(D_{ti})_{i=1,2}, V, F_t\}_{t\in\mathcal{T}}$ be a family of basic nested systems with respect to t. Denote the basic-renormalization of BS_t by $\{(E_{ti}, eF_{ti})\}_{i\in\mathcal{I}_t}$. Given a positive integer $j \leq |\mathcal{I}_0|$, there exists some $\epsilon > 0$ so that for all $|t| < \epsilon$ and for all $i = 1,\dots,j$, (E_{ti}, eF_{ti}) exist. Moreover, for every $i < j$, the family of $\{E_{ti}, V, eF_{ti}\}$ is differentiable with respect to t when $|t| < \epsilon$.*

Proof. Let $q \in D_{01}$ be a relevant point, $E(q)$ be its associated strip, and F_{0q} be its associated return map. By definition of basic-renormalization, $F_{0q}(E(q)) \cap \mathrm{int}(E(q)) \neq \emptyset$. Also $F_{0q} = F^i$ for some $i \in \mathbb{Z}^+$. The lemma holds by from continuity. $\square$

If $BS_0 \notin (cl)(\mathcal{W})$, then its renormalized nested family has only finite branches. Lemma 5.2.2 shows for all $i = 1, \ldots, n$,

$$eF_{0i}(E_{0i}R) \cap \mathrm{int}(E_{0i}R) \neq \emptyset.$$

On the other hand, the quasi-renormalization of BS_0 might have more than n elements, i.e. the $(n+1)$-th element $(E_{0(n+1)}, eF_{0(n+1)})$ exists. Especially, there might exist a boundary case:

$$eF_{0(n+1)}(E_{0(n+1)}R) \cap E_{0(n+1)}R \neq \emptyset \quad \text{but} \quad eF_{0(n+1)}(E_{0(n+1)}R) \cap \mathrm{int}(E_{0(n+1)}R) = \emptyset.$$

Definition 7.2.3. Let $\{BS_t\} = \{(D_{ti})_{i=1,2}, V, F_t\}_{t \in \mathcal{T}}$ be a family of basic nested systems and $\{(E_{ti}, eF_{ti})\}_{i \in \mathcal{I}_t}$ be their respective quasi-renormalization. Suppose the renormalized nested system of BS_0 is $ES_0 = \{(E_{0i})_{i=1,\ldots,n}, V, eF_0\}$. The family is said to have a *renormalization strip transformation at* $t = 0$ if there exists an $\epsilon > 0$, so that

1. for all $i \leq n+1$, the family of $\{E_{ti}, V, eF_{ti}\}$ is differentiable with respect to t for all $t \in (-\epsilon, 0)$ (or for all $t \in (0, \epsilon)$), and
2. $eF_{0(n+1)}(E_{0(n+1)}R) \cap E_{0(n+1)}R \neq \emptyset$.

Renormalization strip transformation is a formal change which does not imply any dynamical change. In fact, Lemma 5.2.2 shows that unless $eF_{0(n+1)}(E_{0(n+1)}R) \cap \mathrm{int}(E_{0(n+1)}R) \neq \emptyset$, the strip $E_{0(n+1)}R$ does not contain any non-wandering point. Lemma 5.2.2 further guarantees that $eF_{0j}(E_{0j}R) \cap \mathrm{int}(E_{0j}R) = \emptyset$ for any positive integer $j > n+1$. Nevertheless, such transformations can affect the combinatorial representations as is explained in the following corollary.

Corollary 7.2.4. *Let $\{BS_t\} = \{(D_{ti})_{i=1,2}, V, F_t\}_{t \in \mathcal{T}}$ be a family of basic nested systems. If BS_0 is renormalizable, then there exists an $\epsilon > 0$ so that BS_t is also renormalizable for all $|t| < \epsilon$. Moreover, suppose ES_t is the renormalized nested system of BS_t for respective t, then either there exists a renormalization strip transformation at $t = 0$ or $\{ES_t\}_{|t|<\epsilon}$ is differentiable with respect to t.*

Proof. By the definition of renormalization strip transformation, this corollary holds. $\square$

On the other hand, the renormalization strip transformation is also the only non-dynamical change that affects the combinatorial presentation. In the following part of this subsection, we will show that all other changes that affect the combinatorial representation are dynamical changes. More precisely, combinatorial representation changes if and only if there exists either a saddle-node bifurcation or period-doubling bifurcation or renormalization strip transformation; see Proposition 7.2.19.

Let $\{NS_t\}_{t \in \mathcal{T}}$ be a family of nested systems. We assume that for all $t \in [-1, 1)$, $NS_t \notin \mathrm{cl}(\mathcal{W})$ hereafter. We denote the zeroth-equivalent family of NS_t by $\mathcal{CS}_t = \{CS_{ti}\}$ (for respective t). Without loss of generality, we can also assume the i-position nested system of NS_t is CS_{ti} which is either $CS_{ti} = \{D_{ti}, D_{t(i+1)}, V, F\}$ or $CS_{ti} = \{D_{ti}, V, F\}$.

Lemma 7.2.5. *Let $NS_t = \{(D_{ti})_{i\in I}, V, F_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. Denote the $(0,0)$-th representation map of NS_t by $\Psi_t : \mathbb{N} \longrightarrow \mathcal{L}$. Then for all $t \in \mathcal{T}$,*

$$\Psi_t(i) = \bullet \iff \Psi_0(i) = \bullet.$$

Proof. For all $t \in \mathcal{T}$, NS_t has $|I|$ many branches. □

Lemma 7.2.6. *Let NS_t and Ψ_t be as in Lemma 7.2.5. Assume all the periodic points of NS_0 are hyperbolic. Then*

$$\Psi_t(i) = O, \qquad \text{for all } t \in (-\epsilon, \epsilon)$$

for some $\epsilon > 0$ if and only if $\Psi_0(i) = O$.

Proof. 1) Assume there exists an $\epsilon > 0$ so that for any $|t| < \epsilon$, $\Psi_t(i) = O$. If $\Psi_0(i) \neq O$, then $\Psi_0(i) = S/B$ ($\Psi_0(i) \neq \bullet$ because of Lemma 7.2.5). Proposition 5.5.1 and 5.5.2 conclude that CS_{0i} must have hyperbolic fixed points in either case (hyperbolicity follows from the assumption of the lemma). The fact that hyperbolic fixed points continue to exist in a short period of time contradicts $\Psi_0(i) = \bullet$ for all $t < \epsilon$.

2) Assume $\Psi_0(i) = O$. If there exists a sequence $\{t_j\}$ so that $t_j \to 0$ and $\Psi_{t_j}(i) = S/B$, then for each j, $CS_{t_j i}$ has at leas a fixed point, denoted by P_j. The accumulation point of $\{P_j\}_{i\to\infty}$ must be a fixed point of CS_{0i}, i.e. $\Psi_0(i) \neq O$. □

Corollary 7.2.7. *Let NS_t and Ψ_t be as in Lemma 7.2.5. Suppose $\Psi_0(i) \neq O$ and there exists an $\epsilon > 0$ so that $\Psi_t(i) = O$ for any $0 < t < \epsilon$. Then $\{NS_t\}$ goes through saddle-node bifurcations when $t = \epsilon$ in $D_{\epsilon i}$.*

Proof. The proof of Lemma 7.2.6 shows that if for some i $\Psi_0(i) \neq O$ and there exists an $\epsilon > 0$ so that $\Psi_t(i) = O$ for any $0 < t < \epsilon$, then the fixed points in D_i must be non-hyperbolic. Therefore, $\{NS_t\}$ goes through saddle-node bifurcations when $t = \epsilon$ in $D_{\epsilon i}$. □

Generally speaking, $\{NS_t\}$ goes through only one saddle-node bifurcation.

Lemma 7.2.8. *Let NS_t and Ψ_t be as in Lemma 7.2.6. Then there exists an $\epsilon > 0$ so that for any $t < \epsilon$*

$$\Psi_t(i) = \begin{cases} S, & \text{if } \Psi_0(i) = S, \\ B, & \text{if } \Psi_0(i) = B. \end{cases}$$

In other wordss, The existence of an $\epsilon > 0$ so that $\Psi_t(i) = S$ (resp. $\Psi_t(i) = B$) for any $|t| < \epsilon$ is equivalent to that $\Psi_0(i) = S$ (resp. $\Psi_t(i) = B$).

Proof. Suppose $\Psi_0(i) = S$. If $i \neq \max_{j\in I} j$, let BS_t be the basic sub-nested system of NS_t indexed by $\{i, i+1\}$. It is clear that the family $\{BS_t\}_{t\in\mathcal{T}}$ is differentiable with respect to t. Since $NS_0 \notin \mathrm{cl}(\mathcal{W})$, $BS_0 \notin \mathrm{cl}(\mathcal{W})$ either. Corollary 7.2.4 shows there exists an ϵ' so that for any $0 \leq t < \epsilon'$, the first renormalized nested system of BS_t, denoted by $ES_t = \{E_{tj}, V, eF_t\}_{j\in I'}$, exists and $\{ES_t\}_{0\leq t<\epsilon'}$ is differentiable with respect to t. Since $\Psi_0(i) = S$, BS_0 is a reducible basic nested system. By Corollary 5.3.12, E_{0j} does not contain any nonwandering point, especially any periodic points for any $j > 1$. Remark 6.1.7 indicates that there exists an $\epsilon < \epsilon'$ so that for any

$0 \leq t < \epsilon$, E_{tj} does not contain any periodic point for any $j > 2$. Corollary 5.5.7 therefore implies that BS_t is a reducible basic nested system for any $0 \leq t < \epsilon$. In other words, the basic sub-nested sytem of NS_t indexed by $\{i, i+1\}$ is not in the zero-th equivalent family of NS_t for any $0 \leq t < \epsilon$. Hence $\Psi_t(i) \neq B$ for any $0 \leq t < \epsilon$. If we further make ϵ small enough so that Lemma 7.2.6 applies, then $\Psi_t(i) = S$ for any $0 \leq t < \epsilon$. If however, $i = \max_{j \in I} j$, then $\Psi_t(i) \neq B$ naturally. Thus if the ϵ satisfies Lemma 7.2.6, we have $\Psi_t(i) = S$ for $0 \leq t < \epsilon$.

Similarly, suppose $\Psi_0(i) = B$. Let BS_t be the basic sub-nested system of NS_t indexed by $\{i, i+1\}$. There exists an ϵ' so that for any $0 \leq t < \epsilon'$, BS_t has its first renormalized nested system $ES_t = \{E_{tj}, V, eF_t\}_{j \in I'}$, where $\{ES_t\}_{0 \leq t < \epsilon'}$ is differentiable with respect to t. Moreover, there must exist an $j \in I'$ so that E_{0j} contains an periodic point P. Since P is hyperbolic, there exists an $\epsilon < \epsilon'$ so that E_{tj} has a periodic point as well. By Lemma 5.3.12, BS_t is a non-reducible basic nested system. Thus $\Psi_t(i) = B$ for any $0 < t < \epsilon$. $\square$

Corollary 7.2.9. *Let $\{NS_t\}$, Ψ_t and ϵ be as in Lemma 7.2.8. Suppose there exists some $i \in I_{t_\epsilon}$ so that*

$$\text{either} \quad \Psi_\epsilon(i) \neq S \quad \text{where} \quad \Psi_0(i) = S, \qquad \text{or} \qquad \Psi_\epsilon(i) \neq B \quad \text{where} \quad \Psi_0(i) = B,$$

then the family $\{NS_t\}$ goes through a saddle-node bifurcation at $t = \epsilon$.

Proof. If $\Psi_\epsilon(i) = O$ when $\Psi_0(i) = S/B$, then the corollary follows from Corollary 7.2.7.

Suppose $\Psi_\epsilon(i) = B$ when $\Psi_0(i) = S$. Let BS_t and $ES_t = \{E_{tj}, V, eF_t\}_{j \in I'}$ be as in the proof of Lemma 7.2.8. The proof of Lemma 7.2.8 shows that for all $0 \leq t < \epsilon$, E_{tj} does not contain any periodic point for any $j > 2$. On the other hand, Lemma 5.5.6 concludes that if $E_{\epsilon j}$ did not contain any periodic point for any $j > 2$, then BS_ϵ is reducible basic nested system. Therefore, there exists an integer $j > 2$, so that ES_t goes through a saddle-node bifurcation in E_{tj} at $t = \epsilon$.

A similar argument applies if we assume $\Psi_\epsilon(i) = S$ when $\Psi_0(i) = B$. $\square$

Lemma 7.2.10. *Let $NS_t = \{(D_{ti})_{i \in I}, V, F_t\}_{t \in \mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. Assume all the periodic points of NS_0 are hyperbolic and $NS_0 \notin \mathrm{cl}(\mathcal{W})$. Then there exists an $\epsilon > 0$ so that*

1. *for any $t < \epsilon$, NS_t has the same $(0,0)$-th representation map as NS_0 does, and*
2. *for any $i \in I$, the family of i-th nested system of NS_t is differentiable in $0 \leq t < \epsilon$.*

Proof. Proposition follows from Lemma 7.2.5, 7.2.6 and 7.2.8. $\square$

Lemma 7.2.11. *Let $\{NS_t\}_{t \in \mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. Assume all the periodic points of NS_0 are hyperbolic and NS_0 is renormalizable. Then there exists an $\epsilon > 0$ so that for any $t < \epsilon$ and $i \in I$*

1. *the i-th postition modeling map of NS_t, denoted by μ_i^t, exists,*
2. *$\mu_i^t = \mu_i^0$, denote the cardinality of μ_i^t by I_i, and*

3. for any $0 < j \le I_i$, the family of (i,j)-th position nested systems is differentiable for $0 \le t < \epsilon$.

Proof. Corollary 7.2.7 implies there exists an $\epsilon' > 0$ so that for any $t < \epsilon'$ NS_t is renormalizable. Therefore, for any $t < \epsilon'$ and $i \in I$, the modeling map of i-th position nested system of NS_t exists. Lemma 7.2.10 shows there exists an $0 < \epsilon'' < \epsilon'$ so that for any $t < \epsilon''$ NS_t has the same $(0,0)$-th representation map. Denote the representation map of NS_t by Ψ_t. If $\Psi_0(i) = S$, then $\Psi_t(i) = S$ too. Since the modeling map of a simple nested system is the $(0,0)$-th representation of itself, for any $t < \epsilon''$ we have

$$\mu_i^t(j) \equiv \mu_i^0(j) = \begin{cases} S, & j = 1, \\ \bullet, & j > 1. \end{cases}$$

Thus, the proposition follows. Similarly, if $\Psi_0(i) = O$, we also have $\mu_i^t = \mu_i^0$ for for any $t < \epsilon''$. Suppose $\Psi_0(i) = \Psi_t(i) = B$ for $t < \epsilon''$. Denote the i-th position nested system of NS_t by CS_{ti} for any $t \in \mathcal{T}$ and $i \in I$. Lemma 7.2.8 implies if we take ϵ'' to be small enough, we have the family of basic nested systems, $\{CS_{ti}\}_{0 \le t < \epsilon''}$, is differentiable with respect to t. By Corollary 7.2.7, we obtain an $\bar{\epsilon} < \epsilon''$ so that for $0 \le t < \bar{\epsilon}$ the first renormalized nested system of CS_{ti} exists, denoted by ES_{ti}. Moreover, $\{ES_{ti}\}_{0 \le t < \epsilon}$ is a differentiable family of nested system. By definition, μ_i^t is the $(0,0)$-th representation map of ES_{ti}, hence it follows from Lemma 7.2.10 that there exists an $0 < \epsilon < \bar{\epsilon}$ so that

$$\mu_i^t = \mu_i^0$$

for any $0 \le t < \epsilon$ and $i \in I$. □

If a number ϵ is the largest number that satisfies the three properties in Lemma 7.2.11, then there exists some i, so that either

$$\mu_i^\epsilon \ne \mu_i^0,$$

or

$$\mu_i^\epsilon = \mu_i^0,$$

but there exists some $\delta > 0$ so that for any $0 < t < \delta$.

$$\mu_i^{\epsilon+\delta} \ne \mu_i^0.$$

In either case, we will have the following corollary.

Corollary 7.2.12. *Let $\{NS_t\}$, μ_i^t and ϵ be as in Lemma 7.2.11. If we further assume that ϵ is the largest number that satisfies the three properties in Lemma 7.2.11, then there exists some i, so that $\{NS_t\}$ either goes through a saddle-node bifurcation at $t = \epsilon$ in D_i or has a renormalization strip transformation at $t = \epsilon$ in D_i.*

Proof. If $\mu_i^\epsilon \ne \mu_i^0$, then by Lemma 7.2.5, 7.2.6 and 7.2.8, there exists some j so that one of the following happens:

1. $\mu_i^0(j) = O$ and $\mu_i^\epsilon(j) = S$,

2. $\mu_i^0(j) = O$ and $\mu_i^\epsilon(j) = B$,

3. $\mu_i^0(j) = B$ and $\mu_i^\epsilon(j) = o$

4. $\mu_i^0(j) = S$ and $\mu_i^\epsilon(j) = B$,

5. $\mu_i^0(j) = B$ and $\mu_i^\epsilon(j) = S$.

When the first case happens, there exists a non-hyperbolic periodic point $D_{\epsilon i}$ which does not exist in D_{ti} for all $t < \epsilon$. Therefore NS_t experiences a saddle-node bifurcation at $t = \epsilon$. The rest cases follows from the definition of renormalization strip transformation. If $\mu_i^\epsilon = \mu_i^0$ but there exists some $\delta > 0$ so that for any $0 < t < \delta$ $\mu_i^{\epsilon+\delta} \neq \mu_i^0$, the corollary holds by similar arguments. □

Proposition 7.2.13. *Let $\{NS_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. Assume all the periodic points of NS_0 are hyperbolic. If there exists an integer $M < \infty$ so that NS_0 is M-th renormalizable, then there exists an $\epsilon > 0$, so that for any $0 \leq m \leq M$ and any $t < \epsilon$ the $(0, m)$-th representation map of NS_t is the same as the $(0, m)$-th representation map of NS_0.*

Proof. This proposition follows from Lemma 7.2.11 by applying it inductively on m. □

Lemma 7.2.14. *Let $\{SS_t\}_{t\in\mathcal{T}}$ be a family of simple nested systems which is differentiable with respect to t. Denote the remodeling map of SS_t by ρ^t and its cardinality of ρ^t by I. Assume all the periodic points of SS_0 are hyperbolic. Then there exists an $\epsilon > 0$ so that for any $t < \epsilon$*

1. $\rho^t = \rho^0$, and

2. for any $0 < j \leq I$, the family of $(1, j)$-th position nested systems of SS_t is differentiable for $0 \leq t < \epsilon$.

Proof. The construction of the renewal operator on the space of simple nested systems implies that the renewal of $SS_{t'}$ is differentiable with respect to t in a neighborhood of t' when all the periodic points of $SS_{t'}$ are hyperbolic. □

Corollary 7.2.15. *Let $\{SS_t\}$ and ρ^t be as in Lemma 7.2.14. We further assume that for any $t \in \mathcal{T}$, SS_t has a renewal compatible pair of determinant and scaling ratio. If the ϵ given in 7.2.14 is the largest number that satisfies the two conditions in Lemma 7.2.14, then $\{SS_t\}$ either goes through a period doubling bifurcation or a saddle-node bifurcation at $t = \epsilon$.*

Proof. The corollary follows from the construction of the renewal operator. □

Proposition 7.2.16. *Let $\{NS_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. Assume all the periodic points of NS_0 are hyperbolic. Moreover there exist a $K \in \mathbb{Z}^+$ and an $M \in \mathbb{Z}^+$ so that NS_0 is (K, M)-th actionable. Then there exists an $\epsilon > 0$, so that for any $0 \leq k \leq K$, $0 \leq m \leq M$ and $0 \leq t < \epsilon$, the (k, m)-th representation map of NS_t is the same as the (k, m)-th representation map of NS_0.*

Proof. It follows from Propostion 7.2.13 and Lemma 7.2.14. □

Let $\{NS_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. Denote the (k,m)-th representation map of NS_t by ${}_{(k,m)}\Psi_t$. Suppose the stop number for NS_{t_0} is $(K_{t_0}, M_{t_0}) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Let $\mathcal{U} \subset \mathcal{T}$ be the maximal open connected component containing t_0 so that

$$ {}_{(K_{t_0},M_{t_0})}\Psi_t =_{(K_{t_0},M_{t_0})} \Psi_{t_0} $$

for any $t \in \mathcal{U}$. Then $\mathcal{U}$ is called the *dynamical equivalent window* containing t_0 in $\mathcal{T}$.

Corollary 7.2.17. *Let $\{NS_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. If $\mathcal{U} \equiv (a', b') \subset \mathcal{T}$ is a dynamical equivalent window, then the family $\{NS_t\}$ goes through one of the followings at $t = a'$ and $t = b'$, respectively:*

1. *a saddle-node bifurcation,*
2. *a period-doubling bifurcation, or*
3. *a renormalization strip transformation.*

Proof. It follows from the Corollary 7.2.12 and 7.2.15. □

Proposition 7.2.18. *Let $\{NS_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. Suppose the stop number for NS_0 is $(K, M) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Furthermore, we assume for all $t \in [-1, 1)$,*

1. *NS_t has a renewal compatible pair of determinant and scaling ratio, and*
2. *${}_{(K,M)}\Psi_t =_{(K,M)} \Psi_0$.*

Then for all $t \in [-1, 1)$,

$$ \Omega(NS_t) = P(NS_t). $$

Moreover, for any t_1 and $t_2 \in \mathcal{T}$,

$$ P(NS_{t_1}) = P(NS_{t_2}) $$

up to a set of degenerate non-hyperbolic periodic points. In other words, if

$$ p \in P(NS_{t_1})\backslash P(NS_{t_2}) \bigcup P(NS_{t_2})\backslash P(NS_{t_1}), $$

then p must be a degenerate non-hyperbolic periodic point.

Proof. Because NS_t cannot contain any non-wandering points which are not periodic by Corollary 6.2.14, if a point p is in $P(NS_{t_1})$ or $P(NS_{t_2})$, then p must be a periodic point. Moreover, the periodic point p cannot be non-degenerate non-hyperbolic by the definition of dynamical equivalent window.

Suppose the periodic point $p \in P(NS_{t_1})\backslash P(NS_{t_2})$ is hyperbolic. Then there exists a interval $\mathcal{T}'$ about t_1 and a neighborhood U about p and a smooth function $\mathfrak{p} : \mathcal{T} \to \mathbb{C}^2$ such that $\mathfrak{p}(t_1) = p$ and $F^n(p_t) = p_{t'} = p_t$ where n is the period of p under F_t. Without loss of generality, we can assume $t_1 < t_2$. Put $t' \equiv \sup \mathcal{T}'$. By continuity, $t' \in \mathcal{T}'$ and $p_{t'}$ must be a non-hyperbolic periodic point. If $p_{t'}$ is non-degenerate, then either $\{NS_t\}$ has a saddle-node bifurcation or a periodic-bifurcation at t'. If $NS_{t'}$ has a saddle-node bifurcation, then either there exists a pair of numbers (k, m) so that two possibilities can happen in (k, m)-th renormalized equivalent family of $NS_{t'}$:

- an auxiliary simple nested system becomes an essential nested system, or vice versa,
- a reducible basic nested system becomes a non-reducible nested system, or vice versa,

or there exists a number k so that two possibilities can happen in the k-th equivalent simple family of $NS_{t'}$:

- a non-renewable auxiliary simple nested system becomes an essential non-renewable simple nested syste, or vice versa,
- a non-renewable simple nested system becomes renewable, or vice versa.

Each of the first two possibilities will affect ${}_{(k,m)}\Psi_{t'}$ and each of the last two will affect ${}_{(k,m(k))}\Psi_{t'}$ where $m(k)$ is the stop number of renormalization for the k-th renewal equivalent family. The argument is similar if the family $\{NS_t\}$ has a period-doubling bifurcation at time t'. Therefore, $\mathcal{T}$ cannot be a dynamical equivalent window. □

Corollary 7.2.17 explains why a dynamical equivalent window must be open and Proposition 7.2.18 further justifies what the "equivalent" means.

Proposition 7.2.19. *Let $\{NS_t\}_{t\in\mathcal{T}}$ be a family of nested systems which is differentiable with respect to t. The family $\{NS_t\}$ satisfies the following conditions:*

1. *for any $t \in [-1, 1)$, $NS_t \notin cl(\mathcal{W})$,*
2. *NS_a has only finitely many hyperbolic periodic points, and*
3. *for any $t \in \mathcal{T}$, NS_t has a renewal compatible pair of determinant and scaling ratio.*

Then

1. *the family $\{NS_t\}$ can only go through three types of dynamical changes: saddle-node bifurcation, period-doubling bifurcation, and renormalization strip transformation; moreover*
2. *those bifurcations/transformations are isolated.*

Proof. Dynamical changes at the boundary of a dynamical equivalent window are either a saddle-node bifurcation or a period-doubling bifurcation which only increases or decreases the number of periodic points by finitely many, therefore the proposition follows from the Corollary 7.2.17 and Proposition 7.2.18. □

Renormalization strip transformations do not change the dynamics.

Lemma 7.2.20. *Let $\{NS_t\}$ be a family of nested systems as in Proposition 7.2.19. If there does not exist any saddle-node or period-doubling bifurcations, then $NS_b \notin \mathrm{cl}(\mathcal{W})$.*

Proof. If there are only finitely many renormalization strip transformations, the lemma holds by definition.

Suppose infinitely many renormalization strip transformations occur at $t = t_i$ consecutively and $\lim t_i \to b$. For each i, we assume that the $(a_1^i, \ldots, a_{n(i)}^i)$-th position nested system of NS_{t_i} has different modeling map from that of $NS_{t'}$ for $t_{i-1} < t' < t_i$. If for any integer i, there exists an integer $j > i$, so that $(a_1^i, \ldots, a_{n(i)}^i)$ is not the same as the first $a_{n(i)}^i$ elements in $(a_1^j, \ldots, a_{n(j)}^j)$, then there is no dynamical change at $t = b$. Otherwise, NS_t has a saddle-node bifurcation at $t = b$. That is, $NS_b \notin \mathrm{cl}(\mathcal{W})$. □

Theorem 7.2.21. *Let $\{NS_t\}$ be the family of nested system as in Proposition 7.2.19. If there exists a number $\rho \in (0, 1)$ so that $\mathrm{Jac}(NS_t) \leq \rho$ for all $t \in \mathcal{T}$, then the family $\{NS_t\}$ experiences period-doubling cascade to chaos.*

Proof. We first observe that the family NS_t must have infinitely many saddle-node or period-doubling bifurcation moments in $[-1, 1)$ by applying Lemma 7.2.20.

A saddle-node bifurcation or period-doubling bifurcation increases the number of periodic points. With the increase of the number of periodic points, the number of renormalizations also increases. Equation 5.1.1 implies that renormalizations reduce the Jacobian exponentially. In other wordss, if the Jacobian of the (k_t, m_t)-th equivalent family equivalent family of NS_t exists, denoted by ${}_{(k,m)}NS_t$, then

$$\mathrm{Jac}({}_{(k,m)}NS_t) \leq \mathrm{Jac}(NS - t))^{p(k,m)},$$

where $p(k, m)$ is as in Equation 7.1.1. Assume NS_t is fully actioned at (K_t, M_t). Since there are infinitely saddle-node or period-doubling bifurcations in $[-1, 1)$, there must exist a number $t' < b$ so that NS_t has sufficiently many periodic points and ${}_{(K_t,M_t)}NS_t$ is a highly dissipative family of nested systems for all $t > t'$. Our family experiences period-doubling cascade to chaos because highly dissipative two-dimensional maps must experience period-doubling cascade to chaos. □

Chapter 8

Appendix

8.1 One-dimensional piecewise continuous functions

In this subsection, we consider a special type of piecewise continuous functions. We show that if such a function does not possess a stable fixed point, then the only non-wandering points are period-two periodic points.

Throughout the argument, for all interval A we denote by l_A and r_A the left and right end points of $\mathrm{Cl}(A)$, respectively.

Definition 8.1.1. A map f from a subset of $I = [0, 1]$ to $\mathrm{Int}(I) = (0, 1)$ is of *MCV type* if it satisfies all of the following conditions.

1. f is defined on the union of a collection of open intervals $\{\widetilde{B}_i\}_{i=0}^{\infty}$ of $(0, 1)$ such $m(\cup_i \widetilde{B}_i) = 1$ and
$$\widetilde{B}_i \cap \widetilde{B}_j = \emptyset \qquad \forall i \neq j.$$
Such a $\widetilde{B}_i$ is called a *branch*.

2. $f|_{\widetilde{B}_i}$ is continuous and maps intervals to intervals.

3. For each i, $f|_{\widetilde{B}_i}$ extends continuously to $\mathrm{Bd}(\widetilde{B}_i)$ such that
$$f(l_{\widetilde{B}_i}) \equiv \lim_{x \to l_{\widetilde{B}_i}+} f|_{\widetilde{B}_i}(x) \quad \text{and} \quad f(r_{\widetilde{B}_i}) \equiv \lim_{x \to r_{\widetilde{B}_i}-} f|_{\widetilde{B}_i}(x) \quad \in \quad \{0, 1\}.$$

Remark 8.1.2. Condition 2 in Definition 8.1.1 implies that no interval is mapped to a single point. Condition 3 in general does not guarantee that f extends to $\mathrm{Cl}(\cup_i \widetilde{B}_i) = I$ continuously.

Lemma 8.1.3. *If f is of MCV type, so is f^n for every $n > 0$.*

Proof. Put $B_{ij}^n \equiv f^{-n}(B_i) \cap B_j$. For each i and j, the set B_{ij}^n is a branch of f^n which satisfies the Definition 8.1.1. □

Definition 8.1.4. A family of functions $\{f_t\}_{-1\leq t\leq 1}$ of MCV type is said to be *continuous* with respect to t, if for every t_0, $\widetilde{B}_j(t_0)$, and $\epsilon > 0$, there exists some $\delta > 0$ such that for all $t \in (t_0-\delta, t_0+\delta)$, there is a branch $\widetilde{B}_j(t)$ of f_t such that

$$\left|\chi_{\widetilde{B}_j(t_0)} - \chi_{\widetilde{B}_j(t)}\right| < \epsilon \qquad \text{and} \qquad \left|f_{t_0}\circ\chi_{\widetilde{B}_j(t_0)} - f_t\circ\chi_{\widetilde{B}_j(t)}\right| < \epsilon,$$

where χ_Ω is the characteristic function of each set Ω.

Definition 8.1.5. Let f_0 be a function of MCV type. A periodic point p of f_0 is said to be *stable* if there is some $\epsilon_0 > 0$ so that for all positive $\epsilon < \epsilon_0$, there exists a δ such that for all $t\in(-\delta,\delta)$, f_t has a periodic point $p_t\in(p-\epsilon, p+\epsilon)$ of f_t with the same period.

Lemma 8.1.6. *Let $\{f_t\}_{-1\leq t\leq 1}$ be a family of MCV type, continuous with respect to t. If A is a closed interval such that $A\subset \mathrm{Cl}(\widetilde{B}_0)$, where $\widetilde{B}_0$ is a branch of f_0^k for some $k>0$, and*

$$f_0^k(\mathrm{Cl}A)\supset \mathrm{Cl}A \qquad \text{and} \qquad \mathrm{Bd}(f_0^k(\mathrm{Cl}A))\cap \mathrm{Bd}A = \emptyset, \tag{8.1.1}$$

then there exists a stable periodic point p of f_0^k in $\mathrm{Int}A$.

Proof. Put $N_x(r)\equiv[x-r,x+r]$. Without loss of generality, we may assume $k=1$. By the boundary condition (8.1.1), every periodic point of f_0 must be in $\mathrm{Int}A$. In particular, there exists a $p\in\mathrm{Int}A$ and a closed neighborhood $W\subset A$ such that either

$$\begin{aligned} f(x) &< x \quad \text{if } x\in[l_W, p] \\ f(x) &> x \quad \text{if } x\in[p, r_W], \end{aligned} \tag{8.1.2}$$

or

$$\begin{aligned} f(x) &> x \quad \text{if } x\in[l_W, p] \\ f(x) &< x \quad \text{if } x\in[p, r_W], \end{aligned}$$

The above inequalities are all strict, because f_0 maps intervals to intervals. Without loss of generality, we assume (8.1.2) is the case. Put

$$d = \min\{p-l_W, r_W-p\}.$$

For all $\epsilon\in(0,d/2)$, there exists δ_1 such that for all $t\in N_0(\delta_1)$, there exists $\widetilde{B}(t)$ so that

$$\left|\chi_{\widetilde{B}(t)} - \chi_{\widetilde{B}(0)}\right| < \epsilon.$$

Since

$$l_{N_p(\epsilon)} - l_{\widetilde{B}(t)} > (l_{N_p(\epsilon)} - l_{\widetilde{B}(0)}) - |l_{\widetilde{B}(0)} - l_{\widetilde{B}(t)}| \geq d-\epsilon > \epsilon$$

and $r_{\widetilde{B}(t)} - r_{N_p(\epsilon)} > \epsilon$ likewise, we have

$$N_p(\epsilon) \subset \bigcap_{t \in N_0(\delta_1)} \widetilde{B}(t).$$

Hence $f_t|_{N_p(\epsilon)}$ is continuous for all $t \in N_0(\delta_1)$. Now fix $N_p(\epsilon)$ and put

$$\lambda = \frac{1}{2} \min \big\{ f_0(r_{N_p(\epsilon)}) - r_{N_p(\epsilon)}, l_{N_p(\epsilon)} - f_0(l_{N_p(\epsilon)}) \big\}.$$

For the given λ, there exists $\delta_2 \in (0, \delta_1)$ such that for all $t \in N_0(\delta_2)$,

$$\big| f_t \circ \chi_{N_p(\epsilon)} - f_0 \circ \chi_{N_p(\epsilon)} \big| < \lambda.$$

Hence

$$f_t(r_{N_p(\epsilon)}) - r_{N_p(\epsilon)} \geq \big(f_0(r_{N_p(\epsilon)}) - r_{N_p(\epsilon)} \big) - \big| f_0(r_{N_p(\epsilon)}) - f_t(r_{N_p(\epsilon)}) \big| \geq 2\lambda - \lambda = \lambda.$$

Similarly $l_{N_p(\epsilon)} - f_t(l_{N_p(\epsilon)}) \geq \lambda$. Therefore,

$$f_t(N_p(\epsilon)) \supset N_p(\epsilon),$$

which implies that f_t has a fixed point $p_t \in N_p(\epsilon)$. In other wordss, $\forall \epsilon$ small enough, we find $\delta = \delta_2$ such that for each $t \in N_0(\delta_2)$, f_t has a fixed point $p_t \in N_p(\epsilon)$. The periods are obviously all 1. $\square$

Definition 8.1.7. Let f be of MCV type such that f has no periodic points. A branch A is a *high branch* of f if $f(l_A) = f(r_A) = 1$, while a *low branch* of f if $f(l_A) = f(r_A) = 0$.

Remark 8.1.8. If f is of MCV type such that f has no periodic points, every branch of f is either a high branch or a low branch.

Theorem 8.1.9. *If f is of MCV type such that there are no (stable) periodic points, then there are no other non-wandering points either.*

Corollary 8.1.10. *Let $\{f_t\}_{t \in (-1,1)}$ be a family of MCV functions such that f_t has no non-wandering points for all $t < 0$. Then all of the non-wandering points of f_0, if there is any, are (non-stable) periodic points.*

Proof of Theorem 8.1.9. Since f has no fixed points,

$$f(l_{\widetilde{B}_i}) = f(r_{\widetilde{B}_i}) \qquad \forall i.$$

Hence each $\widetilde{B}_i$ is either a high branch or a low branch of f. Without loss of generality, suppose

$$\widetilde{B}_0 \text{ is a low branch} \tag{8.1.3}$$

and $q \in \widetilde{B}_0$ is a non-wandering point. If $q \in \mathrm{Int}(\widetilde{B}_0)$, there exist a neighborhood $W \subset \widetilde{B}_0$ of q and $T \gg 1$ such that

$$f^T(W) \cap W \neq \emptyset \qquad \text{and} \qquad f^r(W) \cap W = \emptyset \quad \forall\, 0 < r < T. \tag{8.1.4}$$

Claim 8.1.11. *For the neighborhood W given by (8.1.4), there exists a branch $B_0 \subset \widetilde{B}_0$ such that*

1. *B_0 is a branch of f^T,*
2. *$f^T(B_0 \cap W) \cap W \neq \emptyset$, and*
3. *$f^T|_{B_0}^{-1}\{0,1\} = \{l_{B_0}, r_{B_0}\}$.*

Proof of Claim 8.1.11. By (8.1.4), there exists some $p_0 \in W$ such that $f^T(p_0) \in W$. Hence $f^r(p_0) \notin \cup_i \{l_{\widetilde{B}_i}, r_{\widetilde{B}_i}\}$ for all $0 \leq r \leq T$. The claim then follows by taking the branch B_0 of f^T that contains p_0. □

For B_0 given by Claim 8.1.11, put

$$B_k \equiv f^k(B_0) \qquad \forall k \geq 0. \tag{8.1.5}$$

Claim 8.1.12. *As in (8.1.3), if $\widetilde{B}_0$ is a low branch of f, the branch B_0 defined in Claim 8.1.11 is then a high branch of f^T for some $T \gg 1$.*

Proof of Claim 8.1.12. For T given by (8.1.4), denote by C_1 the branch containing B_1,on which f^{T-1} is continuous. If B_0 were a low branch of f^T, then C_1 would also be a low branch of f^{T-1}. Therefore,

$$\sup f^{T-1}\big(f(B_0 \cap W)\big) \leq \sup f(B_0 \cap W). \tag{8.1.6}$$

Since $\widetilde{B}_0$ is a branch of f, $\sup f(B_0 \cap W) \leq \sup W$. So the second part of (8.1.4) implies $\sup f(B_0 \cap W) \leq \inf W$. By (8.1.6),

$$\sup f^T(B_0 \cap W) \cap W \leq \inf W.$$

Since W is open,

$$f^T(B_0 \cap W) \cap W = \emptyset,$$

which contradicts the second condition of Claim 8.1.11. Therefore, B_0 is a high branch of f^T. □

Claim 8.1.13. *For T defined in (8.1.4) and $\{B_k\}_{k \geq 0}$ defined in (8.1.5),*

$$\mathrm{Int}(B_k) \cap \mathrm{Int}(B_{k+1}) = \emptyset \qquad \text{and} \qquad \sup(B_{k+1}) \leq \inf(B_k)$$

for all $0 \leq k \leq T-2$.

Proof of Claim 8.1.13. Claim 8.1.13 is proved by induction.
(1) We show the claim for $k = 0$. Presume

$$\mathrm{Int}(B_0) \cap \mathrm{Int}(B_1) = \emptyset \tag{8.1.7}$$

were not true. Note that $f^{T-1}|_{B_1}$ being continuous implies that $f|_{B_1}$ is continuous. Were (8.1.7) not true, $B_0 \cup B_1$ is an interval such that $f|_{B_0 \cup B_1}$ is continuous. So $B_0 \cup B_1 \subset \widetilde{B}$ by the definition of $\widetilde{B}_0$. Moreover, (8.1.7) being not true implies that

$$\mathrm{Int}(B_k) \cap \mathrm{Int}(B_{k+1}) \neq \emptyset \qquad \forall 0 \leq k \leq T-1.$$

Hence by the same argument, we obtain

$$\bigcup_{k=0}^{T-1} B_k \subset \widetilde{B}.$$

Since $\widetilde{B}_0$ is a low branch,

$$f(x) < x \qquad \forall x \in \widetilde{B}.$$

Besides, since $f - \mathrm{id}|_{\widetilde{B}_0}$ is continuous and $\widetilde{B}_0$ is precompact,

$$\inf \left| f(x) - x \right| > \epsilon > 0 \tag{8.1.8}$$

for some ϵ. Thus,

$$\sup f^T(B_0 \cap W) \;<\; \sup f^{T-1}(B_0 \cap W) \;<\; \cdots \;<\; \sup f(B_0 \cap W). \tag{8.1.9}$$

By (8.1.4), $f(B_0 \cap W) \cap W = \emptyset$. Hence $\sup f(B_0 \cap W) \le \sup W$ implies $\sup f(B_0 \cap W) \le \inf W$. By (8.1.9),

$$\sup f^T(B_0 \cap W) < \inf W. \tag{8.1.10}$$

Since W is open, (8.1.10) implies that $f^T(B_0 \cap W) \cap W = \emptyset$, which contradicts Claim 8.1.11. Hence (8.1.7) holds. Since $\widetilde{B}_0$ is a low branch, $\sup B_1 \le \sup B_0$. By (8.1.7), we obtain $\sup B_1 \le \inf B_0$, which completes the $k = 0$ case.

(2) Now assume Claim 8.1.13 is true up to $k-1$. First we show

$$\mathrm{Int}(B_k) \cap \mathrm{Int}(B_{k+1}) = \emptyset. \tag{8.1.11}$$

By the same reasoning as in part (1), (8.1.11) is true for all $k < T-1$. Presume (8.1.11) were not true for $k = T-1$. If B_k is in a low branch of f, by induction $\{B_l\}_{l=0}^{k-1}$ are all in the low branches of f. Thus, by (8.1.8) and the second part of (8.1.4),

$$\sup f^T(B_0 \cap W) \;<\; \sup f^{T-1}(B_0 \cap W) \;<\; \cdots \;<\; \sup f^k(B_0 \cap W) \;<\; \sup f(B_0 \cap W) \;\le\; \inf W.$$

Since W is open, once again we obtain $f^T(B_0 \cap W) \cap W = \emptyset$, which contradicts Claim 8.1.11.

If B_k is in a high branch of f, say, $\widetilde{B}_k$, then $\widetilde{B}_k \neq \widetilde{B}_0$. Since $\sup B_k \le \inf B_0$, consider the branch C_0 of f^{k+1} such that $B_0 \subset C_0 \subset \widetilde{B}_0$. The same argument as in the proof of Claim 8.1.11 implies

$$f^{k+1}(l_{C_0}) = f^{k+1}(r_{C_0}) \in \{0, 1\}.$$

However, $f^k(l_{C_0})$ and $f^k(r_{C_0})$ are either $l_{\widetilde{B}_k}$ or $r_{\widetilde{B}_k}$. Hence

$$f^{k+1}(l_{C_0}) = f^{k+1}(r_{C_0}) = 1.$$

Therefore,

$$f^{k+1}(C_0) \supset [r_{B_k}, 1] \supset [r_{\widetilde{B}_k}, 1] \supset [l_{\widetilde{B}_0}, r_{\widetilde{B}_0}] = \mathrm{Cl}\widetilde{B}_0 \supset C_0. \tag{8.1.12}$$

It is clear that $l_{C_0} \neq 1$. Indeed, $r_{C_0} \neq 1$, for otherwise $\widetilde{B}$ would then be a high branch. By (8.1.12), there exists a stable fixed point, which contradicts our assumption (8.1.3). This shows that (8.1.11) is also true for $k = T - 1$.

(3) Next we prove

$$\sup B_{k+1} \leq \inf B_k. \tag{8.1.13}$$

Suppose otherwise. By (8.1.11), (8.1.13) being not true implies that

$$\sup B_k \leq \inf B_{k+1}.$$

So the branch $\widetilde{B}_k$ of f that contains B_k is a high branch of f. Hence there exists a branch C_0 of f^{k+1} such that

$$B_0 \cap W \subset B_0 \subset C_0 \subset \widetilde{B}_0.$$

Since $f^{k+1}(C_0) = f(f^k(B_0)) \subset f(\widetilde{B}_0)$, C_0 is a high branch of f^{k+1}. Note that $B_{k+1} \backslash W$ has at most two connected components. Denote by L the component with $\sup L \leq \inf W$ and by R the component with $\inf R \geq \sup W$ (one of L or R may not exists). Then either

1. $f^{k+1}(B_0 \cap W) \subset L$ or
2. $f^{k+1}(B_0 \cap W) \subset R$.

In Case 1, C_0 being a high branch of f^{k+1} implies that $f^{k+1}(r_{C_0}) = 1$. Since there exists some $x_0 \in B_0 \cap W$ such that $f^{k+1}(x_0) < \inf W$,

$$f^{k+1}[x_0, r_{C_0}] \supset [f^{k+1}(x_0), 1] \supsetneq [\inf W, 1] \supset [x_0, r_{C_0}].$$

If $r_{C_0} = 1$, the fact that f is fixed-point-free implies $\widetilde{B}_0$ is a high branch of f, which contradicts the assumption (8.1.3). Hence

$$r_{C_0} < 1. \tag{8.1.14}$$

Moreover,

$$f^{k+1}(x_0) \neq x_0. \tag{8.1.15}$$

Hence by Lemma 8.1.6 there exists a stable periodic point Q, which is a contradiction.

In Case 2 (i.e. $f^{k+1}(B_0 \cap W) \subset R$), there exists $y_0 \in f^{k+1}(B_0 \cap W) \subset B_{k+1}$ such that

$$f^{k+1}(y_0) < \sup W \leq \inf f^{k+1}(B_0 \cap W) < y_0.$$

Consider the branch $D_{T-(k+1)}$ of $f^{T-(k+1)}$ that contains B_{k+1}. By Claim 8.1.12, B_0 is a high branch of f^T. Consequently

$$\begin{aligned} 1 &= f^{T-(k+1)} \circ f^{k+1}(l_{B_0}) = f^{T-(k+1)} \circ f^{k+1}(r_{B_0}) \\ &\in \mathrm{Cl}\big(f^{T-(k+1)} \circ f^{k+1}(B_0)\big) \subset \mathrm{Cl}\big(f^{T-(k+1)}(D_{T-(k+1)})\big). \end{aligned}$$

This implies that $D_{T-(k+1)}$ is a high branch of $f^{T-(k+1)}$. Thus,

$$f^{T-(k+1)}[y_0, r_{D_{T-(k+1)}}] \supsetneq [\sup W, 1] \supsetneq [y_0, r_{D_{T-(k+1)}}],$$

As in the proof of Case 1, there exists a stable periodic point $Q' \in [y_0, D_{T-(k+1)}]$ by Lemma 8.1.6, which is a contradiction. In sum, neither Case 1 nor Case 2 can happen, so (8.1.13) holds. This completes the induction of the proof. □

By Claim 8.1.13, each B_i lies on the left of B_{i-1}, $1 \le i \le T_1$. By Claim 8.1.12, B_0 is a high branch of f^T, hence $1 \in B_T$. Note that $T \gg 1$, so we may assume $T > 2$. Since $f|_{\widetilde{B}_0}$ is continuous and $f(r_{\widetilde{B}}) = 0$, there exists an interval $K \subset \widetilde{B}_0$ such that

$$\inf K \ge \sup B_0 \qquad \text{and} \qquad f(K) = B_2.$$

Since $W \cap B - 0 \ne \emptyset$, $\inf W < \inf K$. Put

$$K_1 \equiv K \cap W \qquad \text{and} \qquad K_2 \equiv K \cap W^c.$$

By (8.1.4)

$$f^{T-1}(K_1) \cap W = f^{T_1}(K \cap W) \cap W = \emptyset.$$

Thus,

$$\sup W < \sup K \qquad \text{and} \qquad f^{T-1}(K_2) \cap W \ne \emptyset. \tag{8.1.16}$$

By (8.1.16), there exists $z_0 \in K_2$ such that

$$\sup W \le z_0 \qquad \text{and} \qquad f^{T-1}(z_0) \le \sup W. \tag{8.1.17}$$

Denote by E the branch of f^{T-1} such that $K \subset E \subset \widetilde{B}_0$. Then by (8.1.17),

$$f^{T-1}[z_0, r_E] \supsetneq [\sup W, 1] \supsetneq [z_0, r_E],$$

Since z_0 and r_E are not periodic points, by Lemma 8.1.6, there exists a stable periodic point $Q'' \in [z_0, r_E]$, which is a contradiction. This completes the proof of Theorem 8.1.9. □

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