On the moduli space of quintic surfaces

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Abstract of the Dissertation

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We describe the GIT compactification for the moduli space of smooth quintic surfaces in projective space. This GIT quotient is used along with the stable replacement for studying the geometry of another special compactification which was developed by Kollár, Shepherd-Barron and Alexeev. In particular, we discuss the interplay between GIT stable quintic surfaces with minimal elliptic singularities and boundary divisors in the KSBA space.

A vosotros, los que me han llevado en el corazon. Gracias.

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Chapter 1

Introduction

To construct a geometric compactification for the moduli space of surfaces of general type is a central problem in algebraic geometry. Work of Gieseker [20] shows that the coarse moduli space $\mathcal{M}^s$ of smooth projective surfaces of general type with fixed numerical invariants p_g, q, and c_1^2 is a quasiprojective variety. A geometric compactification of $\mathcal{M}^s$, called KSBA, was given by the works of Kollár, Shepherd-Barron [39] and Alexeev [1]. However, this compactification is difficult to understand (see Section 4.1), and there is not a description of it for even relatively simple examples such as the quintic surfaces. An alternative approach is to describe a more accessible compactification of $\mathcal{M}^s$, such as the one provided by geometric invariant theory (GIT) and to use this GIT quotient for obtaining information about the KSBA space.

Specifically, here we consider the GIT compactification for quintic surfaces in the projective space, and we discuss the KSBA replacement for some stable GIT surfaces that do not have semi log canonical singularities. Particularly, we related minimal elliptic singularities with boundary divisors in the KSBA space. In our case the GIT compactification $\overline{\mathcal{M}}^{GIT}$ is well suited for being explicitly calculated. It contains an open set, $\mathcal{M}^{stb}$, called the stable locus which has a modular interpretation and it parametrizes a large array of singular surfaces with worse than semi log canonical singularities.

Our geometric quotient is of particular relevance. Indeed, it is a weakly modular compactification of the moduli space of canonical surfaces of general type with invariants $p_g = 4$, $q = 0$ and $c_1^2 = 5$, for which the canonical line bundle is base point free (see Horikawa [27]). The birational geometry associated to singular quintic surfaces is rather rich, and the GIT quotient parametrizes a wealth of surfaces of general type (see Section 4.3). Moreover, It is expected that the GIT compactification for quintic surfaces plays an important role in the birational geometry of the moduli space of surfaces with those invariants. Specifically, we expect a situation similar to the Hassett-Keel program for curves of genus three: *The GIT compactification of plane quartics $\overline{M}^{GIT}_{d=4}$ is the log canonical model of the Deligne-Mumford compactification $\overline{M}_3$* (see [29]).

1.1 Motivation from the Moduli of Curves

The curves with ample canonical bundle are classified by an invariant g called genus of the curve that satisfies the inequality $g \geq 2$. For a fixed value of the genus, there is a quasiprojective variety M_g that parametrizes the smooth curves of genus g. In particular, the curves of genus $g = 3$ are divided into two families.

1. A six dimensional family of curves for which the canonical bundle K_C defines an embedding:

$$\phi_{K_C} : C \hookrightarrow \mathbb{P}(H^0(C, K_C)) \cong \mathbb{P}^2$$

   whose image is a smooth quartic plane curve. These curves are called *non hyperelliptic ones.*

2. A five dimensional family of curves for which the canonical bundle K_C does not induces an embedding into $\mathbb{P}^2$. In this case, the image of C under the associated map ϕ_{K_C} is a nonreduced conic.

In fact, the moduli space of smooth non hyperelliptic curves of genus three M_3^{nh} is identified with the moduli space of smooth quartic plane curves in $\mathbb{P}^2$. This is a six dimensional moduli space which have at least two possible compactifications:

1. The Deligne-Mumford compactification $\overline{M}_3$ (see [9]) which is a modular compactification of M_3^{nh}. This compactification is obtained from considering all curves of genus $g = 3$ with ample canonical bundle, and with at worst log canonical singularities (N.B for curves, they are nodal singularities)

2. A weakly modular compactification obtained from taking the GIT quotient

$$\overline{M}_3^{GIT} = \mathbb{P}\Big(Sym^4\big(H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\right)\big)\Big)^{ss} /\!\!/ SL(3, \mathbb{C})$$

Hyeon and Lee showed that $\overline{M}^{GIT}_{g=3}$ is the log canonical model of $\overline{M}_3$ ([29]), and they described the relationship between these two alternative compactifications.

$$\begin{tikzcd}
\overline{M}_3 \arrow[dr, dotted, "\pi"] & \\
M_3^{nh} \arrow[u, hook] \arrow[r, hook] & \overline{M}_3^{GIT}
\end{tikzcd}$$

We currently understand:

1. The curves of genus three parametrized by $\overline{M}_3$. In particular, there is a boundary divisor $\Gamma \subset \overline{M}_3 \setminus M_3^{nh}$ parametrizing the union of a curve of genus 2 and an elliptic one intersecting at a point. These stable curves are called elliptic tails.

2. The quartic plane curves parametrized by $\overline{M}_3^{GIT}$.

3. The map $\pi : \overline{M}_3 \dashrightarrow \overline{M}_3^{GIT}$. In particular, the generic quartic plane curve parametrized by the loci $\pi(\Gamma)$.

Moreover, we have a procedure, known as stable replacement, that allow us to study the fibers of the $\pi : \overline{M}_3 \dashrightarrow \overline{M}_3^{GIT}$. The stable replacement for toric and quasitoric plane curve singularities was studied by Hassett [26]. A generalization of his work plays a central role in our results.

1.2 Moduli of Surfaces of General Type

Surfaces with ample canonical bundle are classified by three invariants: The geometric genus $p_g = H^0(S, K_S)$, the irregularity $q = H^1(S, K_S)$, and K_S^2. Those invariants satisfy inequalities such as $K_S > 0$ and $K_S^2 \geq 2p_g - 4$. Gieseker [20] showed that if we fixed these invariants, there is a quasiprojective variety parametrizing those surfaces with at worst ordinary double point singularities. In particular, we are interested in the numerical quintic surfaces which have invariants $p_g = 4$, $q = 0$, and $K_S^2 = 5$. These surfaces are, in some way, similar to the curves of genus three (see [27]). Indeed, numerical quintic surfaces are divided into three families:

1. A 40 dimensional family of surfaces for which the canonical bundle K_S defines an embedding

$$\phi_{K_S} : S \hookrightarrow \mathbb{P}(H^0(S, K_S)) \cong \mathbb{P}^3$$

whose image is a quintic surface with at worst ordinary double point singularities. These surfaces are called *type I numerical quintics.*

2. A 39 and a 40 dimensional family of surfaces for which the canonical bundle is ample but with a base point. After solving the base locus of K_S, the induced image of S is a nonreduced quintic surface.

We are interested in the moduli space M_5 of numerical quintic surfaces of type I which we can identify with the moduli space of quintics surfaces in $\mathbb{P}^3$ with at worst DuVal singularities. This is 40 dimensional moduli space with at least two possible compactifications:

1. The Kollar-Shepherd-Barron-Alexeev (KSBA) compactification $\overline{M}_5^{KSBA}$ ([39], [1],[36]) which is a generalization of the Deligne-Mumford construction, but for surfaces of general type. This compactification is obtained by considering numerical quintic surfaces with ample canonical bundle and at worst semi log canonical singularities.

2. A weakly modular compactification obtained from taking the GIT quotient for quintic surfaces

$$\overline{M}_5^{GIT} = \mathbb{P}\Big(Sym^5\big(H^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\right)\big)\Big)^{ss} /\!\!/ SL(4,\mathbb{C})$$

Unfortunately, we do not know if the compactification $\overline{M}_5^{KSBA}$ is divisorial. We do not know which are the surfaces parametrized by $\overline{M}_5^{KSBA}$ or $\overline{M}_5^{GIT}$, and we do not understand the diagram:

$$\begin{tikzcd}
\overline{M}_5^{KSBA} \arrow[dr, dotted, "\pi"] & \\
M_5 \arrow[u, hook] \arrow[r, hook] & \overline{M}_5^{GIT}
\end{tikzcd}$$

In this thesis, we address the following questions.

1. Are there divisors Γ_i on $\overline{M}_5^{KSBA}$ that generalizes the behavior of the elliptic tail divisor on $\overline{M}_3$?

2. What are the GIT stable quintic surfaces?

3. Can we describe the generic surface parametrized by the image $\pi(\Gamma_i)$ in the GIT quotient?

Recent work by J. Rana [58], proves the existence of a divisor on $\overline{M}_5^{KSBA}$ by considering the deformation of surfaces with a $\frac{1}{4}(1,1)$ quotient singularity. Furthermore, J. Rana and J. Tevelev conjectured and partially proved the existence of 22 divisors on $\overline{M}_5^{KSBA}$ associated to Fucshian singularities. We prove the existence 21 of these divisors by different methods.

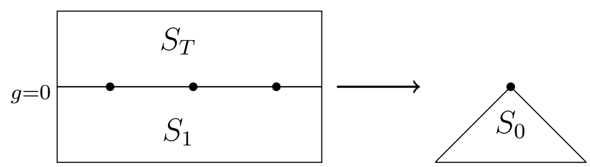
1.3 Main Results

Theorem. *(see Thm 4.2.4) There are at least* 21 *boundary divisors* $\Gamma_i \subset \overline{M}_5^{KSBA}$ *which are contracted by the map:*

$$\pi : \overline{M}_5^{KSBA} \dashrightarrow \overline{M}_5^{GIT}$$

for those divisors it holds:

1. *The generic surface parametrizes by* Γ_i *is the union of a K3 surface* S_T *and a surface of general type* S_1 *with* $p_g = 3$, $q = 0$ *and* $K_X^2 \in \{4, 3, 2\}$.
2. *The surface* S_1 *and* S_T *intersect along a rational curve which support at most five cyclic quotient singularities.*



The generic surface parametrized by $\pi(\Gamma_i)$ *is a quintic surface* S_0 *with one of the following minimal elliptic singularities:*

E_{12}	E_{13}	E_{14}	Z_{11}	Z_{12}	Z_{13}	S_{11}
W_{12}	W_{13}	Q_{10}	Q_{11}	Q_{12}	U_{12}	S_{12}
Z_{15}	Q_{14}	U_{14}	W_{15}	S_{14}	V_{15}	N_{16}

Remark 1.3.1. The previous theorem generalizes behavior observed in the moduli of curves:

1. The minimal elliptic singularities are the two dimensional equivalent of the cusp plane curve singularity.
2. The divisors Γ_i generalize the elliptic tail divisors in $\overline{M}_3$. In our case, a K3 surface plays the role of an elliptic curve.

Sketch of proof: We generalize a construction used by Hassett for the case of plane curves singularities [26]. Let $S_0 := (f(x,y,z) = 0)$ be a surface singularity with a $\mathbb{C}^*$-action. We start with a generic smoothing $(f(x,y,z) + t = 0)$ of S_0. After a base change, $t^a \to t$, induced by the monodromy of the surface singularity, we arrive at a smoothing X of S_0 with a strictly log canonical threefold singularity. The canonical modification $\tilde{X}$ of X is obtained by a weighted blow up. We show that the central fiber $S_1 + S_T$ of $\tilde{X}$ has at worst semi log canonical singularities, and we explicitly check the ampleness of the canonical bundle on our surface $S_1 + S_T$. This procedure generates

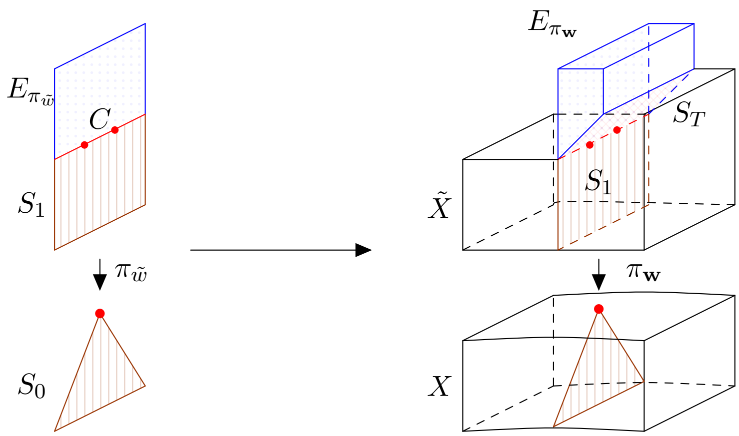


Figure 1.3.1.1: KSBA stable replacement of S_0 into $S_1 + S_T$

KSBA stable surfaces in the central fiber of $\tilde{X} \to \Delta$. Finally, we count the moduli associated to the surface S_1, to the exceptional surface S_T, and to the intersection $S_1 \cap S_T$. □

We also have a good understanding of the quintic surfaces parametrized by the GIT quotient.

Theorem. *(see Section 2) Let S be a quintic surface with at worst one of the following singularities*

1. *Minimal elliptic singularities*
2. *Isolated singularities with Milnor number small than* 21 *or modality smaller than* 5
3. *Isolated double points or isolated triple points with reduced tangent cone*

Then S is stable. On other hand, suppose that the quintic surface has one of the following singularities

1. *a triple line*
2. *a double plane*

3. a point of multiplicity four

Then S is GIT unstable. The GIT boundary has four irreducible disjoint components of dimension 6, 1, 0, *and* 1, *and the maximal stabilizer of a semistable quintic surface is* $SL(2,C)$.

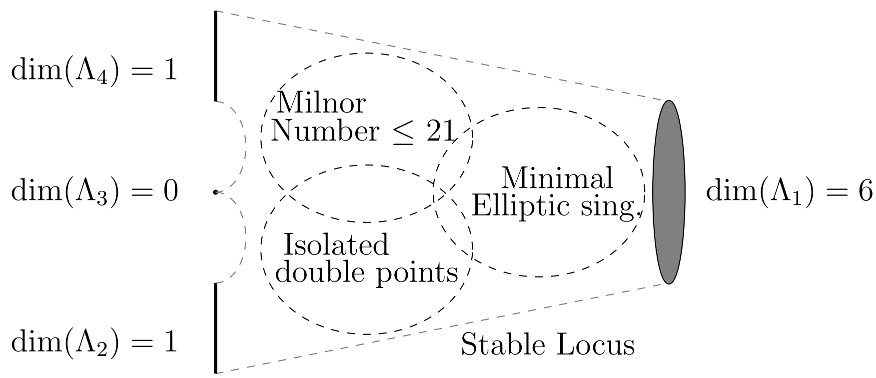


Figure 1.3.1.2: GIT quotient of Quintic Surfaces

Remark 1.3.2. The GIT quotient of quintic surfaces has some similarities with the GIT quotient of quartic plane curves:

1. There is only one non reduced semistable scheme parametrized by the GIT quotient. In the case of quartic plane curves, we have the double conic. In the case of quintic surfaces, we have the union of a double smooth quadratic surface and a transversal hyperplane.

2. All the minimal elliptic singularities are stable. For quartic plane curves the analogous statement is the stability of the cusp singularity.

Sketch of Proof: The GIT construction has three parts: First a combinatorial one based on the Hilbert-Mumford numerical criterion. From this analysis, we obtain a list of critical one parameter subgroups λ_k. Each critical λ_k induces a generic surface $(F_{\lambda_k}(x_0,x_1,x_2,x_3)=0)$ constructed by generic linear combinations of the monomials

$$M^{\oplus}(\lambda_k)=\left\{x_0^{i_0}x_1^{i_1}x_2^{i_2}x_3^{i_3}\mid a_0i_0+...+a_3i_3\geq 0,\quad i_0+i_1+i_2+i_3=5,\;\; i_k\geq 0\right\}$$

The second part of the analysis is to describe the surfaces parametrized by the stable loci. Our approach is to use invariants associated to the singularities and the classifications that they induced. In particular, we use the Milnor number, the modality, the log canonical threshold, and the geometric genus of a singularity because they are semicontinuous. The final part of the analysis is to describe the GIT boundary Λ_i. This description is done by using Luna's Theorem. $\square$

1.4 Future work

There are several questions that we wish to address in our future work:

1. In the case of plane quartic curves, we can blow up the GIT quotient for obtaining a moduli space that includes the hyperelliptic curves. Can we do something similar for the GIT quotient of quintic surfaces? In particular, can we related the Kirwan blow up to the Horikawa divisor parametrizing numerical quintics of type II_b?

2. Quintic surfaces are related to the construction of Enrique and Godeaux surfaces. Can we use the GIT quotient for constructing compactifications of the moduli spaces of those surfaces?

3. Can we generalize the results on quasihomogeneous singularities for constructing KSBA stable surfaces at the boundary of other moduli spaces of surfaces of general type?

1.5 Related work

This work fits in a series of related GIT constructions including Shah [61], Laza [44], Yokoyama [72], Lakhani [41]. For analyzing the singularities, we benefited from the work of Laufer [42], Reid [59], Wall [68] and Prokhorov [57]. The moduli of numerical quintic surfaces was studied first by Horikawa [27], and J. Rana [58] has recently studied its KSBA compactification. On the stable replacement, we follows closely Wahl [67] and Hassett [26]. We use Macaulay [21], Singular [8] for producing examples, and verifying some calculations.

1.6 Notation

The homogeneous coordinates are given by $[x_0 : x_1 : x_2 : x_3]$, we denote as p_i the point $(x_j = x_k = x_l = 0)$ with $i \neq j, k, l$; and we denote as L_{ij} the line $(x_k = x_l = 0)$ with $i, j \neq k, l$. The homogenous polynomials of degree d are denoted $f_d(x_0, x_1, x_2, x_3)$. We work over the complex numbers. Otherwise indicated, whenever a polynomial occurs in an example or calculation, we suppose it has generic coefficients. However, we write it without non-zero coefficients. For example $c_i x_i^2 + c_k x_k^2$ will be written as $x_i^2 + x_k^2$. Moreover, if we work at the completion of the local ring of a singularity, we do not write the coefficients whenever they are invertible elements. For example $u(x,y,z)x^2 + v(x,y,z)y^2$ will be simply written as $x^2 + y^3$ if $u(x,y,z)$ and $v(x,y,z)$ are invertible power series. Finally, let $X = (F_X = 0)$ be a hypersurface with equation $F_X(x_0, x_1, x_2, x_3)$ with respect a given coordinate system, we denote as Ξ_X its set of non zero monomials. Our computational framework follows the one in Mukai [53, sec 7.2].

Chapter 2

GIT for Quintic Surfaces.

2.1 Geometric Invariant Theory Analysis

Geometric invariant theory gives a standard way to compactify some moduli spaces. In particular, the moduli of smooth quintic surfaces, $\mathcal{M}^s$, is an open quasiprojective subset of the GIT compactification $\overline{\mathcal{M}}^{GIT}$ which is given by the quotient

$$\overline{\mathcal{M}}^{GIT} = \mathbb{P}\Big(Sym^5\big(H^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\right)\big)\Big)^{ss} /\!\!/ SL(4, \mathbb{C})$$

As usual, the stability of a given surface X is decided by means of the Hilbert-Mumford Numerical Criterion: *A quintic surface is stable (resp. semistable) if and only if for all the one parameter subgroup* $\lambda(t) : \mathbb{G}_m \to SL(4, \mathbb{C})$, *it holds* $\mu(\lambda, X) < 0$ *(resp.* ≤ 0 *).* We assume that the one parameter subgroups (or 1-PS) are diagonal and their weights are normalized to:

$$\lambda = diag\left(t^{a_0}, ..., t^{a_3}\right) \text{ with } a_0 \geq ... \geq a_3 \quad \text{and} \quad a_0 + ... + a_3 = 0$$

then on our coordinate system, the numerical function can be written as

$$\begin{aligned} \mu(\lambda, X) &= \min\{\lambda.m_k \mid m_k \in \Xi_X\} \\ &= \min\{a_0 i_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 \mid x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} \in \Xi_X\} \end{aligned} \tag{2.1.1}$$

The normalized one parameter subgroups induce *a partial order among the monomials*. Indeed, given two monomials m, m'. Then $m \geq m'$ if and only if $\lambda.m \geq \lambda.m'$ for all normalized 1-PS (see [53, Lemma 7.18]). From the definition of the numerical criterion, the minimal monomials in a configuration Ξ_X are the ones that determine the value of $\mu(\lambda, X)$.

An alternative formulation of the numerical criterion is: X *is not a stable* *surface if and only if there exist a coordinate system and at least one normalized parameter subgroup $\lambda = (a_0, a_1, a_2, a_3)$ such that $X := (F_X = 0)$ and its associated set of monomials Ξ_{F_X} is contained in*

$$M^{\oplus}(\lambda) = \left\{x_0^{i_0}x_1^{i_1}x_2^{i_2}x_3^{i_3} \mid a_0 i_0 + ... + a_3 i_3 \geq 0, \quad i_0 + i_1 + i_2 + i_3 = 5, \ \ i_k \geq 0\right\}$$

For the analysis of the stability, it suffices to consider the maximal sets $M^{\oplus}(\lambda)$ with respect the inclusion. We call them *maximal non stable configurations*; and they are determined by a finite list of 1-PS that we call *critical one parameter subgroups.*

Proposition 2.1.2 (Critical one parameter subgroups)**.** *A quintic surface X is non GIT stable if and only if for a suitable choice of a coordinate system such that $X := (F_X = 0)$, and the monomial configuration Ξ_{F_X} is contained in $M^{\oplus}(\lambda_i)$ for one of the following 1-PS:*

$$\begin{array}{lll}
\lambda_1 = (1, 0, 0, -1) & \lambda_2 = (2, 1, -1, -2) & \lambda_3 = (4, 2, -1, -5) \\
\lambda_4 = (2, 1, 0, -3) & \lambda_5 = (3, 0, -1, -2) & \lambda_6 = (5, 1, -2, -4) \\
\lambda_7 = (2, 1, 1, -4) & \lambda_8 = (2, 2, -1, -3) & \lambda_9 = (7, 1, -4, -4) \\
\lambda_{10} = (8, -1, -2, -5) & &
\end{array}$$

Furthermore, if for a suitable choice of coordinates $\Xi_{F_X} \subseteq M^{\oplus}(\lambda_i)$ for $i > 6$. Then, X is unstable (see Table 2.1.2.1).

Proof. Since there exist finitely many configurations of monomials; only finitely many configurations are relevant for the GIT analysis. To find them, with aid of a computer program, we list all the configurations and we identify the maximal ones. In fact, the computation complexity is greatly reduced by using two basic observations: First, it suffices to consider the configurations associated to $M^{\oplus}(\lambda)$ where λ is such that there exist monomials m_1, m_2 satisfying $\lambda.m_1 = \lambda.m_2 = \lambda.0_d = 0$, where 0_d denotes the centroid in the simplex of monomials. Second, a configuration is characterized by its set of minimal monomials with respect the previously defined partial order.

Moreover, we can also check that our list of critical 1-PS is complete. Indeed, by examining the equations of a hyperplane containing m_1, m_2 and 0_d, it is clear that any $\rho = (a_0, a_1, a_2, a_3)$ satisfies $|a_i| < 3(5)^3$ with $a_i \in \mathbb{Z}$. By using criterion [53, Prop. 7.19], we can confirm that $M^{\rho}(\rho) \subset M^{\oplus}(\lambda_k)$. Our implementation follows similar cases in the literature (e.g. [44],[41], [72]). $\square$

Remark 2.1.3. Let $\overline{F}_{\lambda_i} := \overline{F}_{\lambda_i}(x_0, x_1, x_2, x_3)$ be the equation obtained from a generic linear combination of the monomials stabilized by λ_i. The unstability

Table 2.1.2.1: Singularities associated to Critical 1-PS(See Proposition 2.1.2)

1-PS	Associated Geometric Characteristics
λ_1, λ_3, λ_4	Isolated triple point singularity with non-reduced tangent cone
λ_7	Isolated ordinary quadruple point singularity
λ_2, λ_6, λ_8	Double line of singularities supporting a non isolated triple point
λ_5, λ_9	Double line of singularities with a distinguished double point
λ_{10}	The union of a quartic surface and a hyperplane

of $M^{\oplus}(\lambda_i)$ for $i > 6$ is related to bad geometric properties of $V(\overline{F}_i)$. Indeed, we find that the zero set of $\overline{F}_{\lambda_7} = x_3p_4(x_1, x_2)$ has a quadruple line, the zero set of $\overline{F}_{\lambda_8} = x_3^2p_3(x_0, x_2)$ has a double plane, the zero set of $\overline{F}_{\lambda_9} = x_1^4l(x_2, x_3)$ has a quadruple plane and the zero set of $\overline{F}_{\lambda_{10}} = x_0x_1^3x_3 + x_0x_2^4$ contains a line of multiplicity three and a point of multiplicity four.

To start the GIT analysis, we interpret the geometric characteristics of the zero set associated to the equation $F_{\lambda_i}(x_0, x_1, x_2, x_3)$ which is obtained from a generic linear combination of the monomials on $M^{\oplus}(\lambda_i)$. Our goal is to interpret intrinsically the statement: *There exist a coordinate system such that the surface $X := (F_X = 0)$ satisfies $\Xi_{F_X} \subset M^{\oplus}(\lambda_i)$*

Kempf showed, [34, Thm 3.4], that a non stable point in the GIT quotient always defines a canonical worst one parameter subgroup. From this worst 1-PS, we can relate the failure of stability with geometric properties of our non stable surface. Specifically, each 1-PS λ acts on the vector space $W := H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ determining a weight decomposition $W = \oplus_k W_k$. This decomposition induces a (partial) flag of subspaces $(F_n)_m := \oplus_{k\leq m} W_k \subset W$ which determines a (partial) flag $(F_n)_\lambda := p_\lambda \subset L_\lambda \subset H_\lambda \subset \mathbb{P}^3$. For instance, in a coordinate system such that the normalized λ has different weights a_i. The flag $(F_n)_\lambda$ is:

$$\big(p_\lambda := [0:0:0:1]\big) \in \big(L_\lambda := V(x_0, x_1)\big) \subset \big(H_\lambda := V(x_0)\big)$$

We say that $(F_n)_\lambda$ is a *bad flag* for the surface X, if the associated one parameter subgroup λ satisfies $\mu(\lambda, X) \geq 0$. Typically, the geometric properties of X leading to failure of stability can be expressed in terms of a bad singularity which is singled out by a bad flag in $\mathbb{P}^3$. We call this singularity a destabilizing one, and it is clearly supported at p_λ. Similarly, the other terms of the bad flag impose additional geometric conditions on our surface. For example, the line L_λ usually singles a bad direction or a curve of singularities in X. Given the surfaces $(F_{\lambda_i}(x_0, x_1, x_2, x_3) = 0) \subset \mathbb{P}^3$, our first step is to describe the singularities singled out by their bad flag.

Proposition 2.1.4 (Semistable surfaces)**.** *Let X be a quintic surface, and let Δ be its singular locus. If X is a strictly semistable quintic surface with isolated singularities. Then:*

1. *Δ contains a triple point singularity $p \in X$ whose tangent cone is the union of a double plane H^2 and another one. The intersection multiplicity of the surface with any line in H containing the triple point is five.*

2. *Δ contains a triple point singularity $p \in X$ whose tangent cone is the union of a double plane H^2 and another one intersecting along a line L which is contained in X. The intersection of the hyperplane H with the surface X is the union of a double line L^2 and a nodal cubic plane curve such that the double line is tangent to the cubic curve at the node.*

3. *Δ contains a triple singularity $p \in X$ whose tangent cone is a triple plane H^3. The quintic plane curve obtained from the intersection of the surface X with H has a quadruple point which tangent cone contains a triple line.*

If X is an irreducible strictly semistable quintic surfaces with non isolated singularities, Then:

4. *Δ contains a double line L^2 supporting a special double point whose tangent cone is H^2. At the completion of the local ring, the equation associated to the double point has the form*

$$x^2 + xy^3 + xy^2z + xyz^3 + y^5$$

   The intersection of X with H is a quintuple line supported on L.

5. *Δ contains a double line L^2 supporting a special triple point $p \in X$. The tangent cone of the triple point is the union of three planes intersecting along L. At the completion of the local ring, the equation associated to the triple point has the form:*

$$xf_2(x, y) + y^3z + y^4 + x^2z^2 + xyz^3$$

   The intersection of the surface with one of the hyperplanes H is the union of a conic and a transversal triple line supported on L.

6. *Δ contains a double line L^2 supporting a special triple point whose tangent cone is the union of a double plane H^2 and another one. At the completion of the local ring, the equation associated to the triple point has the form:*

$$x^2y + x^4 + y^4 + x^3z + x^2z^2 + xy^3 + xy^2z + xyz^3$$

The intersection of the surface with the hyperplane H is the union of a quadruple line supported on L and another line.

Remark 2.1.5. A converse result will require an individual analysis which is carried out in the following sections. For an analogous result in quartic surfaces see [61, Thm 2.4]

Proof. We suppose the quintic surface is strictly semistable. Then, by Proposition 2.1.2, we only need to find the geometric characterization of the zero set associated to the equations $F_{\lambda_i}(x_0, x_1, x_2, x_3)$ for $1 \leq k \leq 6$. Our proposition describes the main geometric features of these surfaces. In particular, the intersection of the corresponding generic surface $(F_{\lambda_i}(x_0, x_1, x_2, x_3) = 0)$ with its bad flag $p_{\lambda_k} \in L_{\lambda_k} \subset H_{\lambda_k}$ which singles out the destabilizing singularity. The first case correspond to the quintic surface associated to the equation

$$F_{\lambda_1}(x_0, x_1, x_2, x_3) = x_3^2x_0^2f_1(x_0, x_1, x_2) + x_3x_0f_3(x_0, x_1, x_2) + f_5(x_0, x_1, x_2)$$

The equation associated to the quintic surface described in the second case is:

$$\begin{aligned} F_{\lambda_3}(x_0, x_1, x_2, x_3) = x_3^2x_0^2f_1(x_0, x_1) &+ x_3(x_0^2x_2^2 + x_2f_3(x_0, x_1) + f_4(x_0, x_1)) \\ &+ x_1^2f_3(x_1, x_2) + x_0f_4(x_0, x_1, x_2) \end{aligned}$$

The equation associated to the quintic surface described in the third case is :

$$\begin{aligned} F_{\lambda_4}(x_0, x_1, x_2, x_3) = x_3^2x_0^3 &+ x_3x_1^3f_1(x_1, x_2) + x_3x_0^3h_1(x_0, x_1, x_2) \\ &+ x_3x_0^2f_2(x_1, x_2) + x_3x_0x_1g_2(x_1, x_2) + f_5(x_0, x_1, x_2) \end{aligned}$$

The strictly semistable quintic surfaces with non isolated singularities are destabilized by the one parameter subgroups λ_5, λ_2, λ_6. The fourth case corresponds to the quintic surface defined by the equation

$$\begin{aligned} F_{\lambda_5}(x_0, x_1, x_2, x_3) = x_3^3x_0^2 &+ x_3^2x_0^2f_1(x_0, x_1, x_2) + x_3x_0^2f_2(x_0, x_1, x_2) \\ &+ x_3x_0x_1^2f_1(x_1, x_2) + x_0^2f_3(x_0, x_1, x_2) + x_0x_1g_3(x_1, x_2) + a_1x_1^5 \end{aligned}$$

The equation associated to the fifth case is given by

$$\begin{aligned}F_{\lambda_2}(x_0,x_1,x_2,x_3) &= x_3^2x_0f_2(x_0,x_1) + x_3x_1^3f_1(x_1,x_2) + x_3x_0x_1^2h_1(x_1,x_2)\\ &+ x_3x_0^2g_2(x_0,x_1,x_2) + x_0^2f_3(x_0,x_1,x_2) + x_0x_1g_3(x_1,x_2)\\ &+ x_1^3h_2(x_1,x_2)\end{aligned}$$

The equation associated to the last case is

$$\begin{aligned}F_{\lambda_6}(x_0,x_1,x_2,x_3) &= x_3^2x_0^2f_1(x_0,x_1,x_2) + x_3(x_0^2f_2(x_0,x_1,x_2) + x_0x_1^2h_1(x_1,x_2))\\ &+ x_3a_2x_1^4 + x_1^4g_1(x_1,x_2) + x_0x_1f_3(x_1,x_2) + x_0^2g_3(x_0,x_1,x_2)\end{aligned}$$

To find the local equation of the singularities we use the convention described at Section 1.6. □

Some of the maximal 1-PS induce unstable configurations (Remark 2.1.3). Next, we describe their main geometric characteristics.

Proposition 2.1.6 (Unstable surfaces)**.** *Let X be a quintic surface and let Δ be its singular locus. If for a coordinate system $\Xi_X \subset M^{\oplus}(\lambda_k)$ for $k > 6$. Then, it holds*

1. *Δ contains an ordinary quadruple point.*

2. *Δ contains a double line supporting a special triple point $p \in X$ whose tangent cone is the union of three concurrent hyperplanes intersecting along a line L. At the completion of the local ring, the equation associated to the triple point has the form:*

$$f_3(x,y) + y^2z^3 + xyz^3 + x^2z^3$$

   The intersection of the surface with one of the hyperplanes is the union of a cubic curve and a tangential double line supported at L.

3. *Δ contains a double line supporting a special double point whose tangent cone is H^2. At the completion of the local ring, the equation associated to the double point has the form:*

$$x^2 + xy^2 + y^4 + zy^4 + y^5$$

   The intersection of the surface with H is the union of a quadruple line supported on L and another line.

4. *X is the union of a smooth quartic surface and a hyperplane.*

4.1. The intersection of the hyperplane with the quartic surface is a quartic plane curve with a triple point which tangent cone has a triple line L^3.

4.2. The intersection of the quartic surface with this line L is a quadruple point.

and X is an unstable quintic surface.

Proof. A quintic surface with an isolated quadruple point singularity is destabilized by λ_7. Conversely, the generic equation associated to λ_7 is

$$F_{\lambda_7}(x_0,x_1,x_2,x_3) = x_3 f_3(x_0,x_1,x_2) + f_5(x_0,x_1,x_2,x_3)$$

The equation associated to the second case is given by

$$\begin{aligned} F_{\lambda_8}(x_0,x_1,x_2,x_3) &= x_3^2 f_3(x_0,x_1) + x_3(x_2 f_3(x_0,x_1) + f_4(x_0,x_1)) \\ &\quad + x_2^3 f_2(x_0,x_1) + x_2^2 f_3(x_0,x_1) + x_2 f_4(x_0,x_1) + f_5(x_0,x_1) \end{aligned}$$

The equation associated to the third case is given by

$$\begin{aligned} F_{\lambda_9}(x_0,x_1,x_2,x_3) &= x_3^3 x_0^2 + x_3^2 x_0^2 f_1(x_0,x_1,x_2) + x_3^2 x_0 x_1^2 + x_3 x_2^2 x_0^2 \\ &\quad + x_3 x_2 x_0 f_2(x_0,x_1) + x_3 f_4(x_0,x_1) + x_2^3 x_0^2 + x_2^2 x_0 f_2(x_0,x_1) \\ &\quad + x_2 f_4(x_0,x_1) + f_5(x_0,x_1) \end{aligned}$$

The reducible unstable quintic surface is destabilized by λ_{10}, and its associated equation is given by

$$\begin{aligned} F_{\lambda_{10}} &= x_0\left(x_3^3 x_0 + x_3^2 x_0 l(x_0,x_1,x_2) + x_3 x_2 x_0 f_1(x_0,x_1,x_2) + x_3 f_3(x_0,x_1)\right) \\ &\quad + x_0 f_4(x_1,x_2,x_0) \end{aligned}$$

This proposition follows from computations similar to the previous one, and it is left to the reader. □

Next, we discuss some stability results that appear as consequences of the previous propositions.

Corollary 2.1.7. *Let X be a normal quintic surface with a triple point singularity whose tangent cone contains a double plane. Let $\tilde{X} \to X$ be its monomial transformation with center at the triple point. Then, $\tilde{X}$ is not normal if and only if there is a coordinate system such that $\Xi_{F_X} \subset \Xi_{F_{\lambda_1}}$.*

Proof. It is clear that if $\Xi_{F_X} \subset \Xi_{F_{\lambda_1}}$ then $\tilde{X}$ is not normal. Conversely assume X has the geometric properties stated in the Corollary. Then, we can select a coordinate system such that the triple point is supported at p_3 and the tangent cone is supported at $(x_0 = 0)$. The statement follows from the fact that the singularities of $\tilde{X}$ happens along the intersection of $(x_0 = 0)$ with $(f_4(x_0, x_1, x_2) = 0)$. See [71, Prop. 4.2] or the proof of [71, Prop. 4.5]. $\square$

Corollary 2.1.8. *Let X be a normal quintic surface with at worst an isolated double point singularity or an isolated triple point singularity whose tangent cone is reduced. Then X is stable.*

Proof. Let X be such a quintic surface and assume that X is not stable. Then Ξ_X is contained in one of the $M^{\oplus}(\lambda_k)$ associated to the critical 1-PS of Proposition 2.1.2. It is immediate to check that for every case the corresponding singular point has multiplicity larger than or equal to 3 with non reduced tangent cone, or $M^{\oplus}(\lambda_k)$ has a curve of singularities. $\square$

Corollary 2.1.9. *Let X be a quintic surface containing a line, L, of singularities such that $mult_p(X) = 3$ for all $p \in L \subset X$. Then X is unstable.*

Proof. Let X be such a quintic surface. We can suppose that, in our coordinate system, the triple line is supported on $(x_0 = x_1 = 0)$. Then, the equation associated to X can be written as:

$$x_0^3 f_2(x_0, x_1, x_2, x_3) + x_0^2 x_1 g_2(x_1, x_2, x_3) + x_0 x_1^2 h_2(x_0, x_2, x_3) + x_1^3 p_2(x_0, x_1, x_2, x_3)$$

Now, we apply the numerical criterion with respect λ_8 we find that $\mu(\lambda_8, X) \geq 0$ which implies X is unstable by Proposition 2.1.2. $\square$

Corollary 2.1.10. *Let X be a quintic surface with a singularity of multiplicity larger or equal to four. Then X is unstable.*

Proof. The equation obtained from a generic linear combination of the monomials in $M^{\oplus}(\lambda_7)$ determines a quintic surface with an ordinary quadruple point. Conversely, a quintic surface with a singularity of multiplicity larger or equal to four can be written as $x_3 f_4(x_0, x_1, x_2) + f_5(x_0, x_1, x_2)$ which monomials are contained in $M^{\oplus}(\lambda_7)$. $\square$

Corollary 2.1.11. *Let X be an irreducible quintic surface with a curve of singularities $C \subset Sing(X)$. Suppose the genus of C_{red} is larger than one, and C_{red} does not contain any line. Then X is stable.*

Proof. The Lemma 2.4.2 and our hypothesis on the genus of C imply the quintic surface has not triple point singularities. If X is non stable, there exist a coordinate system such that $\Xi_{F_X} \subset M^{\oplus}(\lambda_i)$. The singularities at X and the geometry associated to the critical 1-PS (see Table 2.1.2.1) imply that $i = 5$

or $i = 9$. Isolated double points are stable by Corollary 2.1.8. Therefore, the singularity singled out by the bad flag must be a non isolated double point. From the GIT analysis, we know that those singularities must be supported on a line. By hypothesis, this is not our case. Then, the surface is stable. $\square$

Corollary 2.1.12. *A nonreduced quintic surface X is semistable if and only if $X = 2Q + H$ where Q is a smooth quadratic surface, and H is a hyperplane intersecting Q along a smooth conic.*

Proof. First, we claim that if X decomposes as the union of a double plane and another cubic surface; then X is unstable. Indeed, we can select a coordinate system such that the double plane is supported on $(x_0 = 0)$. The equation associated to this quintic surface can be written as $(x_0^2 p_3(x_0, x_1, x_2, x_3) = 0)$ which is destabilized by λ_{10}. By degree considerations, it remains to consider whenever X is the union of a double quadratic surface Q^2 and a hyperplane H. If the quadratic surface is singular, it contains a quadruple point which implies it is unstable by Corollary 2.1.10. By Lemma 2.1.15, if the quadratic surface is smooth, the quintic surface is unstable if and only if $Q \cap H$ is singular. $\square$

To decide the semistability of a quintic surface with $SL(2, \mathbb{C})$ stabilizer, we make use of its symmetry to reduce the number of 1-PS for which we have to check the numerical criterion (for a similar argument see [2, Prop. 2.4])

Lemma 2.1.13. *Let X be a quintic surface that decomposes in the union of quartic surface and a hyperplane. Suppose there is a $SL(2, \mathbb{C}) \subset Aut(X)$ action that fixes a smooth conic, C, on X. Then, there is a coordinate system $\{x_i\}$ such that the equation associated to X has the form*

$$x_1\big(f_2(x_0, x_2, x_3)^k + x_1 g_3(x_0, x_1, x_2, x_3)\big)$$

where $(x_1 = f_2(x_0, x_2, x_3) = 0)$ defines the invariant conic. Moreover, the quintic surface X is semistable if and only if it is semistable with respect every 1-PS acting diagonally on $\{x_i\}$ and of the form $\lambda = diag(a_0, a_1, a_2, a_3)$ with $a_1 \geq a_2 \geq a_3$.

Proof. Let $V \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ be the standard two dimensional representation of $SL(2, \mathbb{C})$. The embedding $C \hookrightarrow \mathbb{P}^3$ induces a decomposition

$$W := H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \cong Sym^2(V) \oplus Sym^0(V)$$

where $\mathbb{P}(Sym^2(V))$ is the plane containing the invariant conic C. We say that a basis $\{x_i\}$ of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ is compatible with a reductive group G_x^0 if given a G_x^0-equivariant decomposition of W, the equivariant subspaces are spanned by a subset of the variables $\{x_i\}$. Next, we select a distinguished coordinate system $\{x_i\}$ compatible with our $G_x^0 \cong SL(2,\mathbb{C})$ representation. In particular, $\mathbb{P}(Sym^2(V)) = (x_1 = 0)$.

If X is unstable, there is a destabilizing 1-PS ρ with an associated filtration $(F_\rho)_n$ and a parabolic group P_ρ that preserves it. By Kempf results, [34, Cor 3.5.b], it holds $G_x^0 \subset P_\lambda$ which implies we can write each term of the flag $(F_\rho)_n$ as a direct sum of G_x^0-invariant subspaces induced by $C \hookrightarrow \mathbb{P}^3$. Let T_{max} be a maximal torus compatible with the $\{x_i\}$. By our choice of the coordinate system, it is clear that T_{max} fixes the flag associated to ρ. Therefore, $T_{max} \subset P_\rho$, and the generic equation associated to X is:

$$x_1\big(f_2(x_0,x_2,x_3)^k + x_1 g_3(x_0,x_1,x_2,x_3)\big) \tag{2.1.14}$$

where the invariant conic is given by $(x_1 = f_2(x_0,x_2,x_3) = 0)$ and $1 \leq k \leq 2$. Moreover, in this coordinate system all the one parameter subgroups $\lambda \subset T_{max}$ can be written as $\lambda = diag(a_0,a_1,a_2,a_3)$ with $a_0 \geq a_2 \geq a_3$; where this last condition is achieved by relabelling. Finally, by Kempf [34, Thm 3.4 (c)(4)], every maximal torus T_{max} contains a destabilizing 1-PS. Therefore, the existence of a destabilizing ρ can be decided by the existence of a destabilizing one parameter subgroup $\lambda \subset T_{max}$. □

Proposition 2.1.15. *Let X be a quintic surface that decomposes as the union of a double smooth quadratic surface Q and a hyperplane H. Then X is semistable if and only if $Q \cap H$ is smooth*

Proof. If the intersection $Q \cap H$ is singular, the equation associated to the quintic surface can be written as

$$x_1\big(x_0^2 + x_0x_2 + x_1^2 + x_1x_3\big)^2$$

which is destabilized by $\lambda = (2,2,-1,-3)$. Next, suppose that the conic $Q \cap H$ is smooth and X is unstable. By Lemma 2.1.13, we can suppose our 1-PS have the form $\lambda = (a_0,a_1,a_2,a_3)$ with $a_0 \geq a_2 \geq a_3$. In fact, we can take, by relabelling, λ to be a normalized 1-PS. Although, in that case, the hyperplane's equation may be given by any of the variables $(x_i = 0)$. In this coordinate system the equation of the quintic surface is

$$x_i\big(f_2(x_j,x_k,x_l) + x_i l(x_0,x_1,x_2,x_3)\big)^2$$

The smoothness of the conic implies the equation $f_2(x_k,x_j,x_l)$ has at least one monomial $x_k x_{s_k}$ for each variable x_k. Therefore, the numerical criterion satisfies

$$\mu(X,\lambda) \leq \min\{a_i + 2a_j + 2a_{s_j}, a_i + 2a_k + 2a_{s_k}, a_i + 2a_l + 2a_{s_l}, 3a_i + 2a_k\}$$

for $j, k, l, s_r \neq i$ plus additional conditions that ensure both the conic and the surface Q are smooth. There is a finite number of normalized 1-PS that determine the stability of a quintic surface (see proof of Proposition 2.1.2). With the help of a computer, we checked that the above numerical function is non-negative for all the normalized 1-PS. □

2.2 Minimal Orbits of the GIT compactification

In this section, we describe the boundary components that compactify the stable locus on the GIT quotient. Our main result is Theorem 2.2.1 which describes the semistable quintic surfaces with minimal closed orbits.

Theorem 2.2.1. *The GIT quotient of quintic surfaces is compactified by adding four irreducible boundary components* Λ_1, Λ_2, Λ_3 *and* Λ_4 *of dimensions* 6, 1, 0, 1 *respectively. These boundary components are disjoint (see Figure 1.3.1.2), and the largest stabilizer associated to a closed orbit is* $SL(2,\mathbb{C})$. *Let* X_i *be a generic quintic surface parametrized by the component* Λ_i. *Then,* X_i *has the following geometric properties (see Figure 2.2.1.1)*

1. *The surface* X_1 *is normal; it contains two isolated triple point singularities of geometric genus* 3, *modality* 7, *and Milnor number* 24. *At the completion of the local ring, the equation associated to the singularity is*

$$x^2y + xz^3 + f_5(y,z)$$

which label as V_{24}^* *on [15, Table II] (see also [12, Table 5]).*

2. *The surface* X_2 *is singular along three lines that support two non isolated triple points singularities. At the completion of the local ring, the equation associated to the triple points can be written, for* $t \neq 0$, *as*

$$x^2y + y^2z^2 + xyz^3 + tx^3z$$

3. *The surface* X_3 *has a triple point isolated singularity of geometric genus* 2, *modality* 5, *and Milnor number* 24. *At the completion of the local ring*

the equation associated to the singularity has the form

$$x^2y + y^3z + xz^4$$

Additionally, the surface X_3 is also singular along a line supporting a distinguished triple point of the form

$$x^2y + xy^2z + z^4$$

which after normalization, it becomes a rational triple point of type $C_{1,0}$.

4. *The surface X_4 has an isolated triple point singularity of geometric genus 2, modality 5, and Milnor number 22. At the completion of the local ring the equation associated to the singularity has the form*

$$x^3 + y^3z + xyz^2 + z^5$$

which is label as V'_{22} on [64, pg 244]. Furthermore, the surface X_4 is also singular along a line supporting a distinguished non isolated double point of the form

$$x^2 + xy^3z + txyz^2 + z^5$$

The boundary component Λ_1 is associated to the 1-PS λ_1, $\Lambda_2 \cong \mathbb{P}^1$ is associated to λ_2, Λ_3 is associated to λ_3 and λ_6, $\Lambda_4 \cong \mathbb{P}(3:5)$ is associated to λ_4 and λ_5.

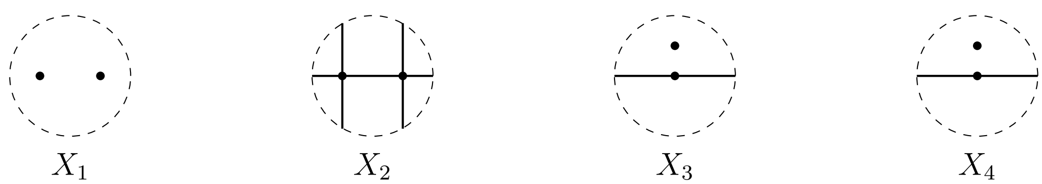


Figure 2.2.1.1: Generic boundary surfaces X_i. The bold lines represent the singular locus of X_i.

Proof. The main theoretical tool for the analysis of the minimal orbits is the Luna's criterion (see [48]): *Let W be an affine variety with an G action and let $x \in W$ be a point stabilized by a reductive subgroup $G_x \subset G$. Then the orbit $G \cdot x$ is closed in W if and only if the orbit $N_G(G_x) \cdot x$ is closed in W^{G_x} where $W^{G_x} \subset W$ denotes the invariant set under the G_x action.* We use the Luna's criterion for the affine space $W = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$, the group $G = SL(4, \mathbb{C})$, and the connected component G_x of the stabilizer of a general quintic surface on the $M^{\oplus}(\lambda_i)$ families. Note also that in our case $N_G(G_x) = G_x \cdot C_G(G_x)$

where $C_G(G_x)$ is the centralizer of G_x in G. Therefore, we can study the quotient of $V^{G_x} /\!\!/ C_G(G_x)$ because G_x acts trivially on V^{G_x} (see also discussion at Section 2.2.1).

We start by describing the first boundary component. The boundary stratum associated to λ_1 parametrizes surfaces which associated equation can be written as

$$x_3^2 x_0^2 x_1 + x_3 x_0 f_3(x_1, x_2) + f_5(x_1, x_2) \tag{2.2.2}$$

and it satisfies conditions described below The generic surface parametrized by this stratum has two isolated singular points analytically isomorphic to the one induced by $x^2y + xz^3 + g_5(y, z)$. The stabilizer is $G_x = (1, 0, 0, -1)$ and the centralizer is given by

$$C_G(G_x) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \frac{1}{a \det A} \end{pmatrix} : A \in GL(2, \mathbb{C}) \right\}$$

We select a coordinate system such that the one parameter subgroups of $C_G(G_x)$ are given by

$$\lambda = diag(1, t^{b+c}, t^{b-c}, t^{-2b})$$

where we can take $c \geq 0$ so our one parameter subgroups are normalized. The Hilbert-Mumford numerical criterion implies we must look configurations of monomials Ξ_{F_X} such that the centroid is inside the convex hull of Ξ_{F_X}. This condition translates into: The term $x_3^2 x_0^2 x_1$ must be different to zero. The term $f_3(x_1, x_2)$ has a root of degree smaller than two at $(x_1 = 0)$. The term $f_5(x_1, x_2)$ has a root of degree smaller than three at $(x_1 = 0)$. Therefore, under an appropriate change of coordinates the equations associated to the unstable degeneration of X_1 can be written as either

$$(x_0 x_3 f_3(x_1, x_2) + f_5(x_1, x_2) = 0)$$

which generically describes a quintic surface with a quadruple point singularity, or

$$\left(x_1 \left(x_0^2 x_3^2 + x_0 x_3 x_1 l(x_1, x_2) + x_1 c(x_1, x_2)\right) = 0\right)$$

which is the equation associated to the first unstable case on Remark 2.2.6, or

$$\left(x_0^2 x_3^2 x_1 + x_0 x_3 x_2^2 l(x_1, x_2) + x_2^4 l_1(x_1, x_2) = 0\right)$$

which also induces a surface with a quadruple point. The dimension of Λ_1 is equal to the dimensions of the quotient

$$(W^{\lambda_1})^{ss} /\!\!/ \big(C_G(\lambda_1)/\lambda_1\big)$$

where the parameter space W^{λ_1} is ten dimensional, and $\dim(C_G(G_x)/G_x) = 4$.

The proofs for the other cases are similar, so we omit some details. The boundary stratum Λ_2 associated to λ_2 is one dimensional and it parametrizes surfaces which associated equation can be taken as:

$$x_3^2x_0x_1^2 + x_3x_0^2x_2^2 + a_1x_3x_2x_1^3 + a_2x_0x_1x_2^3 \quad \text{where } [a_1 : a_2] \in \mathbb{P}^1 \tag{2.2.3}$$

The generic surface parametrized by this stratum is singular along the lines L_{01}, L_{03} and $L_{2,3}$. Those singular curves support two triple points at p_0 and p_3. The quintic surfaces induced by the semistable degeneration associated to $[a_1 : a_2] = [0 : 1]$ and $[a_1 : a_2] = [1 : 0]$ are projectively equivalent. Moreover, if $a_1 = a_2$ the parametrized surface decomposes as

$$(x_3x_0 + x_1x_2)(x_1^3 + x_0x_2^2)$$

The boundary stratum Λ_3 associated associated to λ_3 and λ_6 is zero dimensional and it parametrizes a surface X_3 which associated equation can be taken as:

$$x_3^2x_0^2x_1 + x_3x_1^3x_2 + x_0x_2^4 \tag{2.2.4}$$

The generic surface parametrized by this stratum has an isolated singularity at p_3, and it is singular along the line L_{01} which support a triple point at p_0. We use Singular [8] to find the normalization of it. A study of the syzygies of the normalization implies by Hilbert-Burch theorem than the singularity is determinantal in $\mathbb{C}^4$. The normal form of a rational triple point $C_{1,0}$ can be found after algebraic manipulations. To prove the GIT stability on the associated maximal torus we need to show that

$$0_d \in \mathrm{ConvexHull}\big(x_3^2x_0^2x_1, x_3x_1^3x_2, x_0x_2^4\big)$$

which is deduced from: $2(2,1,0,2) + (0,3,1,1) + (1,0,4,0) = (5,5,5,5)$. The stratum Λ_4 associated to λ_4 and λ_5 is one dimensional and it parametrizes surfaces which associated equation can be taken as:

$$x_0^3x_3^2 + x_3x_2x_1^3 + a_1x_0x_1x_2^2x_3 + a_2x_2^5 \quad \text{where } [a_1 : a_2] \in \mathbb{P}(5 : 3) \cong \mathbb{P}^1. \tag{2.2.5}$$

The generic surface parametrized by this stratum has an isolated singularity at p_3 and it is also singular along the line L_{01} which supports a distinguish double point at p_0. The GIT stability follows from the equalities:

$$(5,5,5,5) = (3,0,0,2) + (0,3,1,1) + 2(1,1,2,1)$$
$$(15,15,15,15) = 5(3,0,0,2) + 5(0,3,1,1) + 2(0,0,5,0)$$

In particular, we require the monomials $x_0^3 x_3^2$ and $x_3 x_2 x_1^3$ plus either $x_0 x_1 x_2^2 x_3$ or x_2^5 to have non zero coefficients. Given $\lambda = (a_0, a_1, a_2, a_3)$ acting on the generic surface X_4, the conditions $3a_0 + 2a_3 = 0$, and $3a_1 + a_2 + a_3 = 0$ imply that the action on the space of parameters is:

$$(a_1, a_2) \to (t^{a_0+a_1+2a_2+a_3} a_1, t^{5a_2} a_2) = (t^{\frac{5}{3}a_2} a_1, t^{a_2} a_2)$$

so $(V^{\lambda_4})^{ss} /\!\!/ (N_G(\lambda_4)) \cong \mathbb{P}(3:5)$. Proposition 2.2.7 determines the dimension of the maximal stabilizer of a semistable quintic surface. $\square$

Remark 2.2.6. We describe non generic surfaces parametrized by Λ_i for $i \neq 1$.

1. The points $[0:1]$ and $[1:0]$ in Λ_2 parametrize a reducible quintic surface as described on the second case of Theorem 2.4.4. The point $[1:1]$ parametrizes a quintic surfaces that decomposes as the union of a non normal cubic surface and a smooth quadratic one.

2. The $[0:1] \in \Lambda_4$ parametrizes a surface as described on the third case of Theorem 2.4.4.

For all surfaces parametrized by $\Lambda_{i\neq 1}$ the points p_0 and p_3 support singularities that are either a triple point or a distinguish double point one. For $i \neq 1$, the point p_1 is always singular as well as the line L_{01} Moreover, a generic point p in a curves of singularities at X_i with $i \neq 1$ is formally of the form $x^2 + y^2$ (see Example 2.3.8) Among the unstable degeneration of a surface parametrized by Λ_1 we find:

1. The union of a hyperplane and a quartic surface Y such that the quartic surface has two $\tilde{E}_7$ singularities, and a line $L \subset Y$ joining them. The intersection of the hyperplane with the tangent cone of one of the $\tilde{E}_7$ singularities is supported on L (compare with the first case on Proposition 2.4.4)

2. The union of a smooth double quadratic surface Q^2, and a hyperplane H such that $Q \cap H$ is a degenerate conic.

Proposition 2.2.7. *Let G_x^0 be the stabilizer associated to the closed orbit of a strictly semistable point. Then $rank(G_x^0) = 1$, and up to isogeny, the largest stabilizer for a semistable quintic surface are $SL(2,\mathbb{C})$ and $\mathbb{G}_m$.*

Proof. By Lemma 2.2.8, we know that $\Lambda_i \cap \Lambda_j = \emptyset$, and that there is not a surface with a $(\mathbb{C}^*)^2$ stabilizer. Then, the statement follows from the Matsushima's criterion: *If G_x^0 is the stabilizer of a closed orbit. Then, G_x^0 is a reductive group*, and the classification of reductive groups of rank one in a field of characteristic zero. An example of a strictly semistable surface with a $SL(2, \mathbb{C})$ stabilizer is given by the union of a double smooth quadratic surface with a transversal hyperplane. □

Lemma 2.2.8. *Let Λ_i and Λ_j be the GIT boundary associated to two different one parameter subgroups. Then $\Lambda_i \cap \Lambda_j = \emptyset$ for every i and j. Moreover, for any semistable surface the largest rank of its stabilizer group G_x^0 is one.*

Proof. Let X_{ij} be a semistable surface parametrized by an open orbit associated to $\Lambda_i \cap \Lambda_j$. Then, its stabilizer $G_{X_{ij}}$ contains two distinct one parameter subgroups λ_i and λ_j which can commute between them or not. We reach a contradiction by studying each case separately.

If λ_i and λ_j commute, they can be simultaneously diagonalized. Then, we can select a coordinate system such that λ_i and λ_j are two of the 1-PS on Proposition 2.1.2 with $i, j \leq 6$. In that case, the configuration of monomials, $\Xi_{X_{ij}}$ is stabilized by λ_i and λ_j. This is impossible because the equation $\lambda_i.m = \lambda_j.m = 0$ is satisfied by only one monomial in all but the two following cases: The equation $a_0x_0x_1^2x_2x_3 + a_1x_1^5$ is invariant under the action of λ_1 and λ_5, but destabilized by λ_4. The equation $a_0x_0x_1x_2^2x_3 + a_1x_2^2$ invariant under the action of λ_1 , λ_4, but destabilized by λ_7. This argument also implies there are not semistable surface with a two dimensional torus stabilizer.

Suppose that λ_i and λ_j do not commute, we will show there is a $g \in P_{\lambda_i}$ such that $g \cdot \lambda_i = \lambda_j$. Therefore, Hilbert-Mumford numerical function is the same for both of them

$$\mu(\cdot, \lambda_i) = \mu(\cdot, g \cdot \lambda_i)$$

This will implies our result by the uniqueness of the worst one parameter subgroup [34, Thm 3.4.c]. By the properties of a good quotient, we can take, without loss of generality, X_{ij} to be parametrized by a closed orbit. We choose $i \neq 1$, and we will work in a coordinate system where λ_i (but not λ_j) is a normalized 1-PS. We claim that the fixed locus of λ_i and λ_j includes most of the coordinate points. Recall that the equivariant decomposition induced by λ_i:

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = \bigoplus_{a_i} V_i \quad \text{where} \quad V_i := \{v \mid \lambda_i \cdot v = t^{a_i} v\} \tag{2.2.9}$$

induces the fixed locus of the action λ_i. For example, in our coordinate system we have: $p_i := \mathbb{P}(V_i)$ and $L_{ij} := \mathbb{P}(V_k \oplus V_l)$. From Remark 2.2.6 , we know every surface parametrized by Λ_i with $i \neq 1$ have distinguished singularities at p_0 and p_3 which are either a triple point or a particularly bad double point. Therefore, those points must be also fixed by the λ_j action. Otherwise, $\lambda_j \subset Aut(X_{ij})$ will generate a curve of bad singularities by taking the λ_j-orbit on X_{ij}. The singularities on that curve are of the same type than our original fix point. This is impossible, because we do not have a curve of singularities with either triple points or bad double points in surfaces parametrized by Λ_i (see Example 2.3.8). Similarly, for all the λ_i there is singular point p_1 and a singular line L_{01} which is fixed by λ_i. By acting with λ_j, we obtain that either L_{01} and p_1 are fixed by the λ_j action, or the λ_j-orbit of L_{01} generates a non reduced surface component of our surface. This last case is impossible, because the only non reduced quintic surface is stabilized solely by λ_1 and $i \neq 1$ (Corollary 2.1.12). The common fix locus of λ_i and λ_j contains the points p_0, p_1, p_3 and the line L_{01}. Next, consider the equivariant decomposition $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = \bigoplus \tilde{V}_i^{n_i}$ induced by λ_j. In our coordinate system and from the discussion on the previous paragraph, each monomial $\{x_i\}$ generates both V_i and $\tilde{V}_i$ for $i \in \{0,1,3\}$. Moreover, $L_{01} := (x_2 = x_3 = 0)$ is fixed by the action of both λ_i and λ_j. Then, $\tilde{V}_2$ is generated by $c_1x_2 + c_3x_3$ and the two coordinate systems that diagonalize λ_i and λ_j are related by a change of coordinates induced by a g that fixes the flag associated to λ_i. So our claim follows. $\square$

2.2.1 Local Analysis of the GIT boundary

Here, we discuss the local structure, in the etale topology, of our GIT quotient

$$\overline{\mathcal{M}}^{GIT} := \left(\mathbb{P}^N\right)^{ss} /\!\!/ G$$

where $G \cong SL(4, \mathbb{C})$ and $N = 55$. The main technical tool is the Luna's slice Theorem [48, App D]. Let $x \in (\mathbb{P}^N)^{ss}$ be a strictly semistable point with stabilizer G_x. There is a G_x-invariant slice V_x to the orbit $G \cdot x$ which can be taken to be a smooth, affine, locally closed subvariety of $(\mathbb{P}^N)^{ss}$ such that $U = G \cdot V_x$ is open in $(\mathbb{P}^N)^{ss}$. Given $(G \times_{G_x} V)/G_x$ where the action on the product is given by $h \cdot (g, v) = (g \cdot h^{-1}, hv)$, and by considering the fiber of the normal bundle $\mathcal{N}_x := \left(\mathcal{N}_{G\cdot x|\mathbb{P}^n}\right)|_x$ we have the following commutative diagram for any $x \in G_x$.

$$\begin{array}{ccccccc}
G \times_{G_x} \mathcal{N}_x & \xleftarrow{\text{étale}} & G \times_{G_x} V & \xrightarrow{\text{étale}} & U & \subset & (\mathbb{P}^N)^{ss} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{N}_x /\!\!/ G_x & \xleftarrow{\text{étale}} & V /\!\!/ G_x & \xrightarrow{\text{étale}} & U /\!\!/ G & \subset & (\mathbb{P}^N)^{ss} /\!\!/ G \cong \overline{\mathcal{M}}^{GIT}
\end{array}$$

From the perspective of the KSBA compactification, $\overline{\mathcal{M}}^{KSBA}$, of the moduli space of surfaces of general type with invariants $p_g = 4$, $q = 0$, and $K_X^2 = 5$. It is of special interest to understand the Kirwan blow up of $\overline{\mathcal{M}}^{GIT}$ at the point ω that parametrizes the union of a double smooth quadratic surface and a transversal hyperplane. Indeed, J. Rana [58, Thm 1.4 and 4.1] prove that $\overline{\mathcal{M}}^{KSBA}$ has a Cartier divisor $\mathcal{D}$ associated to the deformations of the $\frac{1}{4}(1,1)$ singularity. At least one component of this divisor is obtained from taking the stable replacement of the following family of quintic surfaces deforming to ω :

$$X_t = \big(f_2(\mathbf{x})^2 l_1(\mathbf{x}) + t f_2(\mathbf{x}) f_3(\mathbf{x}) + t^2 f_5(\mathbf{x}) = 0\big) \tag{2.2.10}$$

where $f(\mathbf{x}) := f(x_0, x_1, x_2, x_3)$. J. Rana results imply the natural birational map $\overline{\mathcal{M}}^{KSBA} \dashrightarrow \overline{\mathcal{M}}^{GIT}$ contracts the divisor $\mathcal{D}$ to the point ω. It is an open problem to describe all the divisors in the KSBA space that contracts to ω. On other hand, Kirwan constructed a partial desingularization of the GIT quotient by blowing up the loci associated to positive dimensional stabilizers. We expect this blow up to be related to divisors in the KSBA space. In particular, the divisor parametrizing numerical quintic surfaces of type II_b (see [27]) should be related to the exceptional divisor of the Kirwan blow up of $\overline{\mathcal{M}}^{GIT}$ at ω. (for a similar situation in degree four see Shah [61, Sec 4])

Lemma 2.2.11. *Let $\omega \in \overline{\mathcal{M}}^{GIT}$ be the point parametrizing the union of a double smooth quadratic surface Q and a transversal hyperplane H. Let x be a semistable point with closed orbit mapping to the point $\omega \in \overline{\mathcal{M}}^{GIT}$. Then, the natural representation of its stabilizer $G_x^0 \cong SL(2,\mathbb{C})$ on the normal bundle $\mathcal{N}_x$ is isomorphic to*

$$\mathcal{N}_x = \big(Sym^5(V) \otimes Sym^5(V)\big) \oplus Sym^6(V)$$

where $V \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ is the standard three dimensional representation of $SL(2,\mathbb{C})$ induced by the conic $Q \cap H$.

Remark 2.2.12. The exceptional divisor $\mathbb{P}(\mathcal{N}_x)^{ss} /\!\!/ G_x$ associated to the Kirwan blow up often carries itself a modular meaning (for example [45, Cor 4.3], [60, Sec 2], [29, Thm1.3]). In future work, we plan to investigate the modular meaning of the points parametrized by our exceptional divisor by using the geometric interpretation of the $SL(2,\mathbb{C})$ plethysm (for example [18, prop.11.16]).

Proof. The lemma follows from calculating an appropriate $SL(2,\mathbb{C})$ equivariant decomposition of the summands in the normal exact sequence

$$0 \to \mathcal{T}_{G\cdot x} \to \mathcal{T}_{\mathbb{P}^N} \to \mathcal{N}_{G\cdot x|\mathbb{P}^N} \to 0 \tag{2.2.13}$$

which we localize at x. Let $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be an embedding of $\mathbb{P}^1$ into a conic C which lies in the hyperplane H in $\mathbb{P}^3$. The action of $G_x \cong SL(2,\mathbb{C})$ induces an equivariant decomposition

$$W := H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\big) \cong Sym^2(V) \oplus Sym^0(V) \quad \text{where} \quad V \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

To find our equivariant decomposition of $T_{\mathbb{P}^N}$, we use [18, Exer. 11.14] to calculate:

$$\begin{aligned} H^0\big(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)\big) = Sym^{10}(V) \oplus Sym^8(V) \oplus \big(Sym^6(V)\big)^{\oplus 2} \oplus \big(Sym^4(V)\big)^{\oplus 2} \\ \oplus \big(Sym^2(V)\big)^{\oplus 3} \oplus \big(Sym^0(V)\big)^{\oplus 3} \end{aligned}$$

By the Euler sequence, the decomposition of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^3}(1))$ induces one at the tangent space $\mathcal{T}_{\mathbb{P}^N}|_x$. Indeed,

$$0 \to \mathcal{O}_{\mathbb{P}^N}|_x \to \mathcal{O}_{\mathbb{P}^N}(1)|_x \otimes H^0\big(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)\big) \to \mathcal{T}_{\mathbb{P}^N}|_x \to 0 \tag{2.2.14}$$

where $\mathcal{O}_{\mathbb{P}^N}|_x$ correspond to the constants functions $Sym^0(V)$, and $\mathcal{O}_{\mathbb{P}^N}(1)|_x$ is the only one dimensional representation of $SL(2,\mathbb{C})$ which is also $Sym^0(V)$. To calculate the decomposition of the tangent space $\mathcal{T}_{G.x}|_x$ we use the exact sequence

$$0 \to \mathcal{T}_{G_x} \to \mathcal{T}_G \to \mathcal{T}_{G\cdot x} \to 0 \tag{2.2.15}$$

The tangent space $T_{G_x}|_x$ is identified with the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$ which is isomorphic to $Sym^2(V)$. The tangent space of $T_G|_x$ corresponds to the lie algebra $\mathfrak{sl}(4,\mathbb{C})$ which has a 15 dimensional adjoint respresentation. The embedding $C \hookrightarrow \mathbb{P}^3$ induces a decomposition in terms of the $SL(2,C)$ representation V as:

$$T_G|_x \cong Sym^4(V) \oplus \big(Sym^2(V)\big)^{\oplus 3} \oplus Sym^0(V)$$

from which we obtain $T_{G\cdot x}|_x$. Therefore, by comparing irreducible summands on the normal exact sequence 2.2.13, we obtain the following decomposition for $\mathcal{N}_{G\cdot x|\mathbb{P}^n}|_x$:

$$Sym^{10}(V) \oplus Sym^8(V) \oplus \big(Sym^6(V)\big)^{\oplus 2} \oplus Sym^4(V) \oplus Sym^2(V) \oplus Sym^0(V)$$

from which we obtain our statement by [18, Exer. 11.11]. $\square$

The quintic surface $Q^2 + H \subset \mathbb{P}^3$ induces two natural decompositions of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$. Indeed, Let G_Q and G_H be the stabilizer of Q and H respectively. The G_Q-equivariant decomposition induced by the quadratic surface $Q := (F_Q = 0)$ is:

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) \cong W_5 \oplus F_Q \cdot W_3 \oplus F_Q^2 \cdot W_1$$

where $W_5 \cong Sym^5(V) \times Sym^5(V)$ is the space of quintic surfaces intersecting the quadratic surface Q along a $(5,5)$ curve, $W_3 \cong Sym^3(V) \times Sym^3(V)$ corresponds to quintic surfaces decomposing as the union of the quadratic surface Q and a disjoint cubic surface F_3, and $V_1 \cong Sym^1(V) \times Sym^1(V)$ which corresponds to the quintic surfaces that decomposes as a double quadratic surface Q^2 and an arbitrary hyperplane. We decompose W_5 by consider the action of $SL(2, \mathbb{C})$ associated to the conic.

$$Sym^5(V) \times Sym^5(V) \cong \bigoplus_{i=0}^{k=5} Sym^{2k}(V)$$

Similarly, $Sym^6(V) \subset W_3$ parametrizes the six points in C_0 which are marked by the intersection of the a cubic surface $(F_3 = 0)$, the quadratic Q and the hyperplane H. Suppose the quadratic surface, the hyperplane and the invariant conic are given by

$$F_0 := x_1 \left(x_0x_3 - x_2^2 - x_1^2\right)^2; \qquad [s_0 : s_1] \to [s_0^2 : 0 : s_0s_1 : s_1^2]$$

Then, we can write the polynomial parametrized to $(Sym^5(V) \otimes Sym^5(V))$ as

$$\begin{aligned} F(x_0, x_1, x_2, x_3) = x_1(x_0x_3 - x_2^2)^2 &+ (x_0x_3 - x_2^2)^2 l_1(x_0, x_2, x_3) \\ &+ x_1(x_0x_3 - x_2^2)(f_2(x_0, x_2) + g_2(x_2, x_3)) \\ &+ (x_0x_3 - x_2^2)(f_3(x_0, x_2) + g_3(x_2, x_3)) \\ &+ x_1(h_4(x_0, x_2) + p_4(x_2, x_3)) + f_5(x_0, x_2) + g_5(x_2, x_3) \end{aligned}$$

and the term parametrized by $Sym^6(V)$ can be written as:

$$F_2(x_0, x_1, x_2, x_3)G(x_0, x_1, x_2, x_3) = (x_0x_3 - x_2^2 + x_1^2)\left(f_3(x_0, x_2) + g_3(x_2, x_3)\right)$$

The previous discussion, Luna's theorem and Lemma 2.2.11 implies a standardization lemma (for a similar result in quartic surfaces see [61, Lemma 4.2])

Lemma 2.2.16. *We may modify a given family of quintic surfaces specializing to $(F_0 = 0)$, such that the new family is defined by an equation of the form:*

$$P_t = F_0(x_0, x_1, x_2, x_3) + F_2(x_0, x_1, x_2, x_3)G_t(x_0, x_1, x_2, x_3) + F_t(x_0, x_1, x_2, x_3)$$

where

1. $F_t(x_0, x_1, x_2, x_3) \in (Sym^5(V) \otimes Sym^5(V)) \otimes \mathbb{C}[[t]]$ *and* $lim_{t\neq 0} F_t \neq 0$
2. $G_t(x_0, x_1, x_2, x_3) \in Sym^6(V) \otimes \mathbb{C}[[t]]$ *and* $lim_{t\neq 0} G_t \neq 0$

Moreover, the point in $\mathbb{P}(\mathcal{N}_x)$ *corresponding to the limits of* $F_{t=0}$ *and* $G_{t=0}$ *is semistable and belongs to a minimal orbit.*

Next, consider other components of the GIT boundary.

Proposition 2.2.17. *Let* x *the semistable point with closed orbit mapping to the GIT boundary* Λ_2, Λ_3, *or* Λ_4. *Then, the natural representation of its stabilizer* $G_x^0 \cong \mathbb{C}^*$ *on the normal bundle* $\mathcal{N}_x$ *is isomorphic to*

$$\mathcal{N}_x = \bigoplus_{\chi} V_\chi^{\oplus n_\chi}$$

where V_χ *is an irreducible* $\mathbb{G}_m$ *representation with eigenvalue* χ. *Specifically, if* x *is mapped to the boundary* Λ_2, *the values of* χ *and* n_χ *are:*

χ	-10	- 9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
n_χ	1	1	1	2	3	3	2	2	3	2	1	2	3	2	2	3	3	2	1	1	1

If x *is mapped to the boundary* Λ_3, *the values of* χ *and* n_χ *are:*

χ	-25	-21	$-18 \leq n \leq 5, n \neq -15, 0, 1$	1	6	$7 \leq n \leq 9$	10	11	$12 \leq n \leq 16$	18	20
n_χ	1	1	1	2	2	1	2	2	1	1	1

If x *is mapped to the boundary* Λ_4, *the values of* χ *and* n_χ *are:*

χ	-15	$-12 \leq n \leq -8$	-7	-6	-5	-4	-3	-2	-1	0	1	$2 \leq n \leq 6$	7	8	9	10
n_χ	1	1	2	2	1	1	2	2	1	1	2	3	2	2	1	1

Proof. Given the one parameter subgroup $\lambda_k = diag(a_0, a_1, a_2, a_3)$ with $k = \{2, 3, 4\}$, the λ_k equivariant decomposition $W \cong \bigoplus_{a_i} V_{a_i}$ induces a decomposition of the space of monomials

$$Sym^5(W) = \bigoplus_{\alpha} V_\alpha^{\oplus n_\alpha}$$

We can choose the point x so it parametrizes quintic surfaces given by the equations 2.2.3, 2.2.4, and 2.2.5. To calculate $\mathcal{T}_{\mathbb{P}^N}$, we localize the Euler sequence at x. The line bundle $\mathcal{O}_{\mathbb{P}^N}$ has weight zero. We claim that $\mathcal{O}_{\mathbb{P}^N}(1) = V_0$ as well. Indeed, the coordinates in $\mathbb{P}^N$ correspond to the monomials of degree five: $[x_0^5 : \ldots : x_i^2 x_2^3 : \ldots : x_3^5]$. A surface X_k stabilized by λ_k correspond to a point $[0 \ldots : 1 : \ldots : 0]$ with 1 in the places corresponding to monomials m_i that are always part of the equation of X_k. For example, for X_4 we can take $m = x_0^3 x_3^2$ or $m = x_1^2 x_2 x_3$ for λ_4. We choose an affine chart $D_+(m_i)$ by localizing at the coordinate corresponding to m_i. It is clear that the weight of the λ_k-action on $\mathcal{O}_{\mathbb{P}^N}(1)|_{D_+(m_i)}$ is proportional to the weight corresponding the invariant monomial m_i. So, our claim follows. At x, From the Euler sequence 2.2.14 we obtain

$$0 \to V_0 \to V_0 \otimes \bigoplus_{\alpha} V_\alpha^{\oplus n_\alpha} \to \mathcal{T}_{\mathbb{P}^N}|_x \to 0$$

from which we obtain the decomposition of $\mathcal{T}_{\mathbb{P}^N}|_x$. To obtain the decomposition of $\mathcal{T}_{G.x}|_x$, we use the exact sequence 2.2.15. The tangent space to G is the Lie algebra $\mathfrak{sl}(4, \mathbb{C})$, and $\mathcal{T}_{G_x}$ is the adjoint representation of $\lambda_k \cong \mathbb{G}_m$. The one parameter subgroup λ_k acts by conjugation on $\mathfrak{sl}(4, \mathbb{C})$ with eigenvalues of the form $a_i - a_j$ for all i, j. Therefore, the exact sequence 2.2.15 becomes

$$0 \to V_0 \to \bigoplus_{i,j} V_{(a_i - a_j)} \to \mathcal{T}_{G \cdot x}|_x \to 0$$

From the previous discussion and the exact sequence 2.2.13 we find the expression of the normal bundle for each λ_k. $\square$

2.3 Stable Isolated Singularities

In this section, we interpret the failure of stability for a normal irreducible quintic surface in terms of the existence of a bad singularity on it. From the GIT analysis, we know that isolated double points and isolated triple point singularities with reduced tangent cone are stable. Quadruple points are unstable. Therefore, we consider triple points singularities with non reduced tangent cone. We recall that a surface singularity is of type $\tilde{E}_8$ if at the completion of the local ring its equation is equivalent to $z^2+x^3+y^6+tx^2y^2$. This singularity can be recognized by the weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)$. Similarly, a singularity of type Z_{13} is semiquasihomogeneous with respect the weights $\left(\frac{1}{9}, \frac{5}{18}, \frac{1}{6}\right)$ and it is equivalent to $z^2 + x^3y + y^6 + txy^5$.

Proposition 2.3.1. *Let X be a normal quintic surface which has a triple point singularity at $p \in X$ as its unique non-canonical singularity. Let $\tilde{X}$ be the monomial transformation of X with center at p.*

1. *If the tangent cone of $p \in X$ contains a double plane. Then X is stable*

if and only if $\tilde{X}$ does not have a line of singularities or a singularity that deform to Z_{13}.

2. *If the tangent cone of $p \in X$ contains a triple plane. Then, X is stable if and only if $\tilde{X}$ does not have a singularity deforming to a $\tilde{E}_8$ singularity*

Remark 2.3.2. As an immediate corollary, if $\tilde{X}$ has at worst ADE singularities then X are stable. These singularities belong to a larger family called minimal elliptic singularities ([68]). We describe them later in this section 2.3.13.

Proof. We first describe representations of quintic surfaces with a triple point as double covers of $\mathbb{P}^2$. Let $p \in X$ be a triple point on a reduced quintic surface which contains only finitely many lines through p. Let $\tilde{X} \to X$ be the monomial transformation of X from the triple point. We have a natural morphism from $\tilde{X} \to \mathbb{P}^2$. Consider its Stein factorization $\tilde{X} \to X^* \to \mathbb{P}^2$, so the map $X^* \to \mathbb{P}^2$ is finite. Thus, X^* is the double cover of $\mathbb{P}^2$ branching along an octic plane curve $B(X)$. If the equation associated to the quintic surface is

$$F_X(x_0, x_1, x_2, x_3) := x_3^2 f_3(x_0, x_1, x_2) + x_3 f_4(x_0, x_1, x_2) + f_5(x_0, x_1, x_2)$$

then the equation of the branch locus $B(X)$ is

$$f_3(x_0, x_1, x_2) f_5(x_0, x_1, x_2) - f_4(x_0, x_1, x_2)^2 \tag{2.3.3}$$

The map $\tilde{X} \to X^*$ contracts the proper transform of the lines $L \subset X$ through the triple point, and it is an isomorphism everywhere else. In particular, suppose $Sing(X)$ is supported at p. Then, the singularities on $\tilde{X}$ are supported in the exceptional divisor of the monomial transformation. The reduced image of the exceptional divisor of $\tilde{X}$ in $\mathbb{P}^2$ is the curve defined by the equation $f_3(x_0, x_1, x_2) = 0$. By using partial derivatives (see [71, Prop. 4.2]), it is easy to show the singularities of $\tilde{X}$ are supported over the points

$$Sing\big(f_3(x_0, x_1, x_2) = 0\big) \cap \big(f_4(x_0, x_1, x_2) = 0\big)$$

Our surface X is non stable if and only if there is a change of coordinates such that $\Xi_{F_X} \subset M^{\oplus}(\lambda_i)$ for $k \in \{1, 3, 4\}$ (see Table 2.1.2.1). It is clear than on those cases, X satisfies the conditions on the statement. Next, we discuss the converse ones. By Corollary 2.1.7, $\tilde{X}$ is not normal if and only if $\Xi_{F_X} \subset M^{\oplus}(\lambda_1)$. By our hypothesis, the singularities of $\tilde{X}$ are solely induced by the ones at the tangent cone of the triple point. Those are at worst a double line of singularities. On other hand, the singularities of the double cover are induced solely by the branch cover. Therefore, $\tilde{X}$ is not normal if and only if $B(X)$ contains a double line. On other hand, suppose $\tilde{X}$ is normal, the tangent cone of the triple point at X contains a double plane and $B(X)$ has a semiquasihomogeneous singularity of degree 18 with respect the weights $w(x) = 5$, and $w(y) = 3$ at $[0 : 0 : 1]$. We choose a coordinate system such that the triple point is supported at p_3, the double plane in the tangent cone is supported at $(x_0 = 0)$, and the singularity at $B(X)$ is supported at $[0 : 0 : 1]$. The most general equation associated to a branch cover with respect these conditions is:

$$x_0^2 f_1(x_0, x_1)(x_0 f_4(x_0, x_1, x_2) - x_1^2 f_3(x_1, x_2)) - (x_2^2 x_0^2 + x_2 f_3(x_0, x_1) + f_4(x_0, x_1))^2$$

From which, we obtain the quintic surface

$$x_3^2 x_0^2 f_1(x_0, x_1) + x_3 x_2^2 x_0^2 + x_3 x_2 f_3(x_0, x_1) + x_3 f_4(x_0, x_1) + x_1^2 f_3(x_1, x_2) + x_0 f_4(x_0, x_1, x_2)$$

By comparing with the Equation at the proof of Proposition 2.1.4, we can verify that $\Xi_{F_X} \subset M^{\oplus}(\lambda_3)$. Next, suppose that X satisfies the conditions of the second claim in the statement. Select a coordinate system such that the triple point is supported p_3, the tangent cone is supported at $(x_0 = 0)$, and $B(X)$ has a semiquasihomogeneous singularity of degree 6 with respect the weights $w(x) = 3$, and $w(y) = 2$ at $[0 : 0 : 1]$. The most general equation for such an octic plane curve can be written as:

$$F_{B_X} = x_0^3 f_5(x_0, x_1, x_2) - (x_2^2 x_0 f_1(x_0, x_1) + x_2 f_3(x_0, x_1) + f_4(x_0, x_1))^2$$

From which we obtain the quintic surface

$$x_3^2 x_0^3 + x_3(x_2^2 x_0 f_1(x_0, x_1) + x_2 f_3(x_0, x_1) + f_4(x_0, x_1)) + f_5(x_0, x_1, x_2)$$

we can also verify that $\Xi_{F_X} \subset M^{\oplus}(\lambda_4)$.

From our previous discussion, we conclude that if X is a non stable normal quintic surface with a triple point singularity, we can find a general deformation of X that preserves the type of $M^{\oplus}(\lambda_i)$. In particular, the singularities of $\tilde{X}$ deform to either a double line, or to $\tilde{E}_8$, or to Z_{13} depending on the tangent cone associated to the triple point. On other hand, if $\tilde{X}$ has one of those singularities, then the branch locus satisfies Ξ_{B_X} is contained in one of the $\Xi_{F_{\lambda_i}}$ for $\lambda_i = 1, 3, 4$. Therefore, we can find a coordinate system such that $\Xi_{F_X} \subset M^{\oplus}(\lambda_i)$, so X is non stable. $\square$

2.3.1 Invariants of Singularities and GIT Stability

In this section we related the stability of normal quintic surfaces with the study of invariants associated to its singularities. In particular, we use the log canonical threshold, the Milnor number, the geometric genus of the singularity, the classification of singularities due to Arnold [3], Yoshinaga-Suzuki [64, 76], Estrada et. al [76], and the description of minimal elliptic singularities due to Laufer [42], Prokhorov [57], and Reid [59].

First, we recall the definition of the Milnor number and modality of a singularity. Let $(f(x, y, z) = 0)$ be a non degenerated quasihomogeneous singularity with leading term f_w with respect the weights w. The dimension of the formal ring $\mathbb{C}[x, y, z]/(\partial_1 f, \partial_2 f, \partial_3 f)$ is the Milnor number of the singularity. To define modality, we choose a basis of monomial m_i for this formal ring. The number of monomials m_i with weighed degree $w(m_i) \geq w(f)$ is known as the inner modality of $f(x, y, z)$. It is known that for non degenerate quasihomogeneous polynomials the inner modality is also the number of moduli of deformations of $(f_X = 0)$ with the same Milnor number (see [4, pg. 222] [22, pg 118]).

Proposition 2.3.4. *A normal quintic surface having at worst a singularity with either Milnor number small than 22 or modality smaller than 5 is stable.*

Proof. We proved this by contradiction. If the surface X is not stable, the GIT analysis implies there is a coordinate system such that Ξ_{F_X} is contained in one of the $M^{\oplus}(\lambda_i)$ for λ_i as on Proposition 2.1.2. In particular, the destabilizing isolated singularity of X is supported at $p = [0 : 0 : 0 : 1]$; and the singularity $p \in X$ deforms to the singularity of $(F_{\lambda_i} = 0)$ for λ_1, λ_3, λ_4, or λ_7. (see Table 2.1.2.1). We consider a general deformation of X that preserves the type $M^{\oplus}(\lambda_i)$ with respect the given choice of coordinates, By Theorem 2.2.1, the singularities at $(F_{\lambda_i} = 0)$ are either V^*_{24} (notation as [15]), or V^*_{24} (notation as [64]), or V'_{22} (notation as [64, pg 244]) or an ordinary quadruple point. The Milnor number of a quadruple point is at least 27, and its modality is at least 6 (see [64], [15]). Therefore, the statement follows by the upper semicontinuity of both the Milnor number and the modality of the singularity associated to the semistable families. □

Example 2.3.5. The converse statement does not hold. In particular, quintic surfaces can have highly singular isolated double points which are always GIT stable by Corollary 2.1.8. At $(t \neq 1)$ the zero set of the equation

$$F_t(x_0, x_1, x_2, x_3) = x_0^3 x_3^2 + 2x_2 x_1^3 x_3 + x_2^5 + t\big(x_3^3 x_2^2 - 3x_0^2 x_1^2 x_3 + 3x_0 x_1^4 + x_2^3 x_0^2\big)$$

has a weakly elliptic singularity at $[0 : 0 : 0 : 1]$ which is formally equivalent to the singularity induced by the equation $x^2 + y^3 + z^{13}$ (see [71, pg 452]). This singularity has Milnor number equal to 24. The zero set $(F_0(x_0, x_1, x_2, x_3) = 0)$ is a non normal surface parametrized by the boundary Λ_4.

A 1-PS λ defines a natural set of positive rational weights w_λ

Definition 2.3.6. *Given an one parameter subgroup $\lambda = (a_0, a_1, a_2, a_3)$ its associated set of weights is $w_\lambda = (a_0 - a_3, a_1 - a_3, a_2 - a_3)$. Reciprocally, every set of weights w_λ has associated one parameter subgroup λ_w. These correspondences are inverse to each other i.e $\lambda_{w_\lambda} = \lambda$, and we say that w_λ is normalized if λ is a normalized 1-PS. We denote the sum of the weights on w_λ as $w_\lambda(1)$.*

From the perspective of moduli theory, we can classify isolated singularities in two families: Log canonical which are the isolated singularities on the surfaces parametrized by the KSBA compactification (similar to the nodal singularities on the Deligne-Mumford compactification of the moduli of curves); and non log canonical singularities. For a non log canonical singularity $p \in X$, the log canonical threshold $c_p(X)$ is an invariant valued between 0 and 1 such that the smaller its value, the worst is the singularity (see [37, pg. 45] for definitions and details). For example:

$$c_p\left(x^2 + y^{m_1} + z^{m_2}\right) = \frac{1}{2} + \frac{1}{m_1} + \frac{1}{m_2}$$

The relation between the Hilbert-Mumford numerical criterion and the log canonical threshold springs from the fact that, in many cases, the last one can be calculated from a set of weights associated to the variables (see also [40, prop 2.1]).

Lemma 2.3.7. *[37, Prop 8.14] Let $f(x_0, \ldots, x_n)$ be a holomorphic function near $0 \in \mathbb{C}^n$. Assign rational weights $w(x_i)$ to the variables x_i and let $w(f)$ be the weighted multiplicity of f. Then,*

$$c_0(f) \leq \frac{\sum w(x_i)}{w(f)}$$

The equality holds if the log pair

$$\left(\mathbb{C}^n, c_0\left(f_w = 0\right)\right)$$

is log canonical outside of the origin for $c_0 := \frac{\sum w(x_i)}{w(f)}$

Example 2.3.8. The non isolated singularities on irreducible quintic surfaces parametrized by Λ_2, Λ_3 and Λ_4 satisfy the conditions on Lemma 2.3.7. We illustrate the calculation. For the cases, $i = 3, 4$, the surfaces are singular along the line L_{01} with a distinguish singularity at p_0. Consider the completion of the local ring at $p_t := [1 : t : 0 : 0]$ for $t \neq 0$, The distinguish non isolated triple point on the surface parametrized by Λ_3 has the form $x^2y + xzy^2 + z^4$ which at p_t can be simplified, in the completion of the local ring, as:

$$x^2t + xzt^2 + z^4 = \left(x\sqrt{t} + \frac{zt^2}{2\sqrt{t}}\right)^2 - z^2\left(\frac{t^3}{4} - z^2\right) \cong \tilde{x}^2 + \tilde{z}^2 \quad \text{for } t \neq 0$$

Similarly, for the distinguish non isolated double point p_0 parametrized by Λ_4, we have

$$x^2 + xzt^3 + a_1xz^2t + a_2z^5 = \left(x + \frac{zt^3 + a_1tz^2}{2}\right)^2 - t^2z^2\frac{(t^2 + a_1z)^2}{4} + a_2z^5$$

which is formally equivalent to $x^2 + y^2$.

The relationship relation between the log canonical and the GIT stability was noticed by Hacking [23, Prop 10.4], Kim-Lee [35], and Kim [47, Lemma 2.1]. Let $p \in X := (F_X(x_0, x_1, x_2, x_3 = 0)$ be a singular point, by a change of coordinate we can suppose the singular point is supported at p_3. Let $f_p := F_X(x_0, \ldots, x_{n-1}, 1)$ be the localization at p_3. Then, from Equation 2.1.1 and Definition 2.3.6 we obtain $w_\lambda(f_p) = \mu(\lambda, X) - a_3 \deg(X)$. By Lemma 2.3.7 and since the sum of the weights $w_\lambda(1) := -4a_3$, we find that:

$$\begin{aligned} \frac{4}{deg(X)}w_\lambda(f_p) - w_\lambda(1) &= \frac{4}{deg(X)}\mu(\lambda, X) \\ &\leq w_\lambda(1)\left(\frac{4}{\deg(X)c_p(X)} - 1\right) \end{aligned} \tag{2.3.9}$$

This allows us to rewrite the numerical criterion.

Proposition 2.3.10. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Then, X is (semi-stable) stable if for every point $p \in X$, the infimum*

$$inf_{w_\lambda}\left(\frac{4}{deg(X)}w_\lambda(f_p) - w_\lambda(1)\right)$$

is (nonpositive) negative for all the positive rational weights, and for all the linear coordinates which fix the point p. In particular,

1. *A quintic surface having at worst a singularity with log canonical larger than $4/5$ is stable.*

2. *If a stable, isolated singularity has log canonical smaller than $4/deg(X)$. Then, it does not exist a set of weight $\tilde{w}$ that calculate its log canonical threshold (in the sense of Lemma 2.3.7).*

3. *If the leading term of f_p with respect w_λ satisfies the condition on Lemma 2.3.7. Then, $c_p(X) < 4/\deg(X)$ implies X is unstable.*

Proof. The first part follows from Equation 2.3.9 and Lemma 2.3.7. If X is stable, the inequality $\mu(\lambda, X) < 0$ holds for every 1-PS. Therefore, it holds $4/deg(X) < w_\lambda(1)/w_\lambda(f)$ for every set of weights w_λ, and the log canonical threshold cannot be calculate from them. The last statement follows from the second equality at Equation 2.3.9. $\square$

We describe in Proposition 2.3.13 that there is a natural family of singularities, called minimal elliptic with log canonical threshold larger than 4/5. We recall that the geometric genus of a singularity $p \in X$ is, in our case, the higher dimensional analogous of the classical genus drop invariant δ for plane curves singularities (see Proposition 2.3.14).

Definition 2.3.11. *Let X be a normal surface singular at p, the geometric genus of the singularity is $\dim(R^1\pi_*\mathcal{O}_Y)$ where $\pi : Y \to X$ is a resolution of X.*

This invariant induces a well known classification of singularities: *Rational singularities* are those for which the geometric genus is zero. For surfaces, the rational Gorenstein surface singularities are the DuVal ones. After rational surface singularities, we find the family of minimal elliptic ones classified by Laufer [42]. Next, we provide not the original definition of minimal elliptic singularities, but rather a convenient one. Recall that we work with isolated hypersurface singularities which are always Gorenstein.

Definition 2.3.12. *([42, Thm 3.10]) A surface singularity is minimal elliptic if and only if it is Gorenstein and $\dim R^1\pi_*(\mathcal{O}_X) = 1$*

An important application of the log canonical threshold criterion is the GIT stability of the minimal elliptic singularities. In this case, it is possible to show by direct computation that their log canonical threshold is strictly larger than $\frac{4}{5}$. This implies their stability:

Proposition 2.3.13. *Let $X \subset \mathbb{P}^3$ be a surface of degree larger of equal than five with at worst minimal elliptic singularities. Then X is stable. In fact the minimum value reached by the log canonical threshold is $\frac{4}{5} + \frac{1}{180}$.*

Proof. In our case, minimal elliptic singularities are either isolated double points or isolated triple points singularities such that after its blow up the surface has at worst ADE singularities (see [68]). Therefore, our statement follows by Corollary 2.1.8 and Proposition 2.3.1. On another direction, by using the equations of minimal elliptic singularities at [42, pg 1290], it is possible to compute their log canonical threshold. For most of the cases, this can be done with the help of the Lemma 2.3.7. An analysis of the log canonical threshold for minimal elliptic singularities is done by Prokhorov in [57, Table 1-3]. In particular, their log canonical value is larger than or equal to $(4/5 + 1/180)$. Therefore, they are GIT stable by Proposition 2.3.10. $\square$

It is well known that the genus of a singularity $p \in X$ can be interpreted by its effect on the geometric genus, $p_g(X)$, of the variety X. We include a proof for completeness.

Proposition 2.3.14. *Given the minimal resolution $\pi : Y \to X$ of a normal hypersurface of degree d, with an unique non DuVal singularity of genus $R^1(\pi_*\mathcal{O}_Y)$. Then, it holds*

$$\frac{(d-1)(d-2)(d-3)}{6} - p_g(Y) + q(Y) = R^1(\pi_*\mathcal{O}_Y)$$

Furthermore, if X is quintic surface and Y is of general type then $q(Y) = 0$ and we have

$$4 - p_g(Y) = R^1(\pi_*\mathcal{O}_Y)$$

Proof. On a normal hypersurface X of degree d, we have $H^1(X, \mathcal{O}_X) = q(X) = 0$ and

$$H^2(X, \mathcal{O}_X) = p_g(X) = (d-1)(d-2)(d-3)/6$$

From those values and the exact sequence (see [71, pg. 433])

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(Y, \mathcal{O}_Y) \to R^1\pi_*\mathcal{O}_Y \to H^2(X, \mathcal{O}_X) \to H^2(Y, \mathcal{O}_Y) \to 0$$

we obtain $p_g(X) - \dim R^1(\pi_*\mathcal{O}_Y) = p_g(Y) - q(Y) = p_a(Y)$. If X is a quintic surface and Y is of general type, we use the main result of Umezu [65]: Let X be a normal quintic surface and Y denotes its resolution. If Y is of general type then its irregularity, $q(Y)$, vanishes. $\square$

The geometric genus of a quasihomogeneous hypersurface isolated singularity is determined by its weights.

Lemma 2.3.15. *([77, pg 48]) Let f_X be a quasihomogeneous function with an isolated critical point at $(0,0,0)$ and with weights (w_0, w_1, w_2) such that*

$w(f_X)=1$. Denote the triple of rational numbers w_i with common denominator $(n_0/d, n_1/d, n_2/d)$ provided we take that denominator d to be the smallest such on. Then, the geometric genus of the hypersurface isolated singularity $(f_X=0)$ is given by the number of non-negative integer that satisfies

$$\{(i,j,k)\in\mathbb{Z}^3_{\geq 0} \mid d-(n_0+n_1+n_2)\geq n_0 i+n_1 j+n_2 k\} \tag{2.3.16}$$

This lemma allows us to list all quasihomogeneous singularities of a given genus. See [74] [75] for a partial classification of the singularities with geometric genus less than or equal to three. To illustrate the complexity of the surface singularities parametrized by the GIT quotient, we exhibit a lower bound for the geometric genus of the singularities on the semistable surfaces.

Proposition 2.3.17. *There is at least one semistable hypersurface $X\subset\mathbb{P}^3$ of degree $d\geq 4$ with an isolated quasihomogeneous singularity of genus*

$$\frac{d(d-2)(4d-10)}{48} \text{ if } d \text{ is even}$$

$$\frac{(d-1)(d-3)(4d-2)}{48} \text{ if } d \text{ is odd}$$

For quartic surfaces, this value is 1, for quintic surfaces is 3, and for sextic surfaces is 7.

Proof. From the combinatorics of the GIT setting, it is clear that for any degree d the one parameter subgroup $\lambda_1=(1,0,0,-1)$ is always a critical one. From Luna's theorem, (see discussion at proof of Theorem 2.2.1), we can reduce ourselves to study the polynomial $F_{\lambda_1}(x_0,x_1,x_2,x_3)$ that it is stabilized by λ_1. If $d=2m+1$ then

$$F_{\lambda_1}(x_0,x_1,x_2,x_3)=x_3^m x_0^m f_1(x_1,x_2)+x_3^{m-1}x_0^{m-1}f_3(x_1,x_2)+\ldots+f_{2m+1}(x_1,x_2)$$

Also, a similar equation is associated to the case $d=2m$. The quasihomogeneous singularity associated to λ_1 is non degenerate. After localizing, we have a quasihomogeneous polynomial of weights $(2,1,1)$ and weighted multiplicity d. By using Lemma 2.3.15 we find the geometric genus associated to the singularity at the surface $(F_{\lambda_1}(x_0,x_1,x_2,x_3)=0)$. Indeed, Let $\lambda_i=(a_0,a_1,a_2,a_3)$ be a normalized 1-PS with associated weights w_{λ_i}. Suppose the w_{λ_i}-leading term of $F_{\lambda_i}(x_0,x_1,x_2,x_3)$ defines an isolated singularity. We rewrite the Lemma 2.3.15 with the expressions $n_i/d=(a_i-a_3)/w(f)$, and by noting that our configurations are strictly semistable which implies $w_\lambda(f_p)+a_3\deg(X)=0$ (see Equation 2.3.9). Therefore, the geometric genus of the singularity at F_{λ_i}

is given by the number of non-negative integer solutions of the Equations:

$$\begin{aligned} deg(X) &= i_0 + i_1 + i_2 + i_3 \\ |a_3|(\deg(X) - 4) &\geq (a_0 - a_3)i_0 + (a_1 - a_3)i_1 + (a_2 - a_3)i_2 \end{aligned} \tag{2.3.18}$$

In particular if $\lambda_1 = (1, 0, 0, -1)$, and by using [49, Eq. 11], we obtain that the number solutions of the Equations 2.3.18 is

$$\sum_{k=0}^{\left[\frac{\deg(X)-4}{2}\right]} \binom{\deg(X) - 2 - 2k}{2}$$

This formula become the expression of the statement after some algebraic manipulations. □

2.4 Stability and non Isolated Singularities

The GIT semistable locus parametrizes quintic surfaces with non isolated singularities. Then, we need to understand them for completing the description of the GIT quotient. We proved in Corollary 2.1.12 that the only semistable, nonreduced quintic surface is the union of a double quadratic surface and a hyperplane intersecting along a smooth conic. This configuration is a closed orbit, its stabilizer is $SL(2, \mathbb{C})$, and it is parametrized by a point in the GIT quotient. In Corollary 2.1.9, we showed that a quintic surface containing a triple line is unstable. In Corollary 2.1.11, we proved that if an irreducible quintic surface contains a double curve of singularities C with genus larger than one, and such that it does not contain any line. Then, the surface is stable. Therefore, it remains to study the following cases: Reducible reduced quintic surfaces (a quartic surface plus a plane, a cubic surface plus a quadratic one) and irreducible quintic surfaces singular along an elliptic or a rational curve.

Our first step is to bound the degree of the possibles curve of singularities in a quintic surface.

Lemma 2.4.1. *Let $X \subset \mathbb{P}^3$ be an irreducible surface of degree d containing a singular curve C. Then*

$$\deg C \leq \frac{(d-1)(d-2)}{2}$$

Proof. If X is irreducible, then the generic hyperplane section is also irreducible (see [33, Thm 6.10]. The lemma follows from applying the genus formula to the generic hyperplane section

$$g(X \cap H) = \frac{(d-1)(d-2)}{2} - \sum_{p_i \in Sing(C \cap H)} \delta(p_i) \geq 0$$

$\square$

If the quintic surface also contains a triple point. Then, the genus of C can be further bounded.

Lemma 2.4.2. *Let X be an irreducible quintic surface with at least one triple point singularity and a double curve of singularities*

1. *If the triple point is not supported on the curve. Then, the curve of singularities is a twisted cubic or a degeneration of it.*

2. *If the triple point is supported on the curve. Then, the curve of singularities is a twisted cubic, an elliptic quartic curve or a degeneration of those.*

Proof. Let H be a generic hyperplane intersecting X such that $p \in X \cap H$. Then $X \cap H$ is irreducible, and it has a triple point at p. If the triple point singularity is supported on the curve, there are $\deg(C) - 1$ double points on $X \cap H$ which are induced by $C \cap H$. By the genus formula, we have that $7 - \delta_P \geq \deg(C)$ where δ_p is the delta invariant associated to the curve's triple point supported at p. By construction, H is generic, then $\delta_p = 3$ and $4 \geq \deg(C)$. This implies that the genus is strictly smaller than 2 because C is not a quartic plane curve, and there are not curves of degree four and genus two on $\mathbb{P}^3$ (see [25, pg 354]). On the other hand, if the triple point is not supported on the curve the same argument shows $3 \geq \deg(C)$. $\square$

A generic quintic surface that decomposes as the union of a quartic surface and a hyperplane is both GIT and KSBA stable. On our moduli space the locus, called $M(4,1)$, that parametrizes those surfaces is twenty two dimensional: Nineteen dimensions are associated to the moduli of $K3$ surfaces and three dimensions arises from the hyperplane.

Proposition 2.4.3. *Let X be a quintic surface that decomposes as the union of a hyperplane, H, and a quartic normal surface, Y, with isolated singularities such that*

1. *Any isolated singularity satisfies conditions in either Corollary 2.1.8 or Proposition 2.3.1, 2.3.4 or 2.3.13.*

2. *The singular locus of the quartic surface Y is disjoint from the hyperplane, and the quartic plane curve $Y \cap H$ has at worst a triple point which tangent cone has a double line.*

Then X is stable.

Proof. By definition the isolated singularities on our quartic surface cannot destabilize the quintic surface. Therefore, the destabilizing singularity must be supported on the intersection of the hyperplane with the quartic surface. This singularity has at worst multiplicity two, because the singular locus of the quartic surface is disjoint from the hyperplane. Suppose the quintic surface X is non stable and our coordinate system is such that the critical one parameter subgroups are given by Proposition 2.1.2. The set of monomials Ξ_{F_X} is contained in $M^{\oplus}(\lambda_k)$ for $k = 5, 9, 10$ (see Table 2.1.2.1). By the fourth case of Proposition 2.1.6, If $\Xi_X \subset M^{\oplus}(\lambda_{10})$, the intersection of the quartic surface with the hyperplane contains a triple point which tangent cone has a triple line. If $\Xi_X \subset M^{\oplus}(\lambda_k)$ for $k = 5, 9$, the destabilizing singularity is a distinguish double point supported at p_3 whose tangent cone is non reduced and supported at H. Therefore, the tangent cone of the quartic surface Y at p_3 is supported at the hyperplane, as well. By the fourth case of Proposition 2.1.4 and the third case of Proposition 2.1.6, the intersection between the quartic surface Y and the hyperplane H contains at least a quadruple line. The hypotheses imply there is not coordinate system such that $\Xi_{F_X} \subset M^{\oplus}(\lambda_k)$ for $k = 5, 9, 10$. Therefore, X must be stable. $\square$

Next, we describe the intersection between $M(4, 1)$ and our GIT boundary.

Proposition 2.4.4. *Let X be a quintic surface parametrized by a point on the intersection between the locus $M(4, 1)$ and the GIT boundary (as on Theorem 2.2.1). Then, one of the following conditions holds:*

1. *The surface X is parametrized by the first boundary component and it satisfies*

   1.1. *The singular locus of the quartic surface is the union of two double point singularities of type $\tilde{E}_7$.*

   1.2. *The intersection of the hyperplane and the quartic surface is the union of two conics of the form*

$$\big((xy - a_1 z^2)(xy - a_2 z^2) = 0\big)$$

   1.3. *The hyperplane does not intersect the singularities along their tangent cones.*

2. *The surface X is parametrized by the second boundary component and it satisfies*

    2.1. *The singular locus of the quartic surface decomposes as the union of two coplanar double lines L_1 and L_2 intersecting at a non isolated triple point with associated equation of the form*

    $$x^2y + x^3z + y^2z^2$$

    2.2. *The intersection of the hyperplane and the quartic surface decomposes as the union of a cuspidal plane curve, and a line that is contained but not singular, on the quartic surface. The singularity of the cuspidal curve is away from the triple point.*

3. *The surface X is parametrized by the fourth boundary component and it satisfies*

    3.1. *The singular locus of the quartic surface has a double line L and a distinguish triple point given by the equation $x^3 - xyz^2 + zy^3$ which is away from the hyperplane.*

    3.2. *The intersection of the hyperplane and the quartic surface is the union of two lines and a conic tangential to them.*

We represent those geometric characteristics in the following diagram.

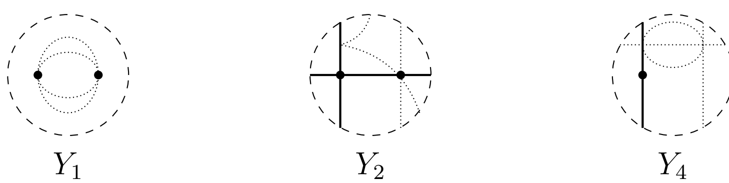


Figure 2.4.4.1: Y_i are our quartic surfaces, the dotted lines represent the intersection $Y_i \cap H$, bold lines represent the singular locus of Y_i.

Proof. Let X be such a quintic surface. Then, there is a one parameter subgroup λ such that X is invariant under the action of it. Let $X = Y \cup H$, then it is easily seen that the hyperplane is also invariant under the action of λ. In particular, this implies that in our coordinate system the equation associated to H must be $(x_i = 0)$. From our results on Section 2.2, and up to a change of coordinates, we have the equations of these surfaces. Indeed, if the surface is a degeneration of X_1, then the associated equation is:

$$x_1\left(x_3^2x_0^2 + x_0x_3f_2(x_1, x_2) + f_4(x_1, x_2)\right)$$

If the surface is a degeneration of X_2, then the associated equation is:

$$x_0(x_3^2x_1^2 + x_3x_0x_2^2 + x_1x_2^3)$$

If the surface is a degeneration of X_3, then the associated equation is:

$$x_3(x_0^3x_3 + x_2x_1^3 + x_0x_1x_2^2)$$

The proposition follows from computations similar to Proposition 2.1.4 and it is left to the reader. $\square$

Remark 2.4.5. The stability for the union of a quartic surface Y and a hyperplane H fits naturally on the VGIT setting for the pair $(Y, \alpha H)$ with $\alpha \geq 0$ (see [46]). Indeed, for $\alpha = 0$ the VGIT-stability is basically the one due to Shah for quartic surfaces [61]. For the case $\alpha = 1$ the pair is stable if and only if the associated quintic surface $Y + H$ is stable. For α large enough the stability reduces to analyzing the quartic plane curves $Y \cap H$ (for a similar example see [43, Thm 2.4]).

We describe a quintic surface with a non-linear curve of singularities of multiplicity of degree three

Proposition 2.4.6. *Let X be a quintic surface with a curve of singularities C such that C does not contain a line and $mult_p(X) = 3$ for every $p \in C$. Then X decomposes as the union of a hyperplane and a quartic surface, and there is a coordinate system such that its associated equation can be written as*

$$x_i\left(f_2(x_j, x_k, x_3)^2 + x_i^2 g_2(x_0, x_1, x_2, x_3) + x_i f_2(x_j, x_k, x_3) l(x_0, x_1, x_2, x_3)\right)$$

Moreover, this surface is generically stable (compare with Corollary 2.1.9).

Remark 2.4.7. We describe an unstable degeneration. The union of two singular quadratic surfaces and an hyperplane such that all of them are intersecting along a conic can be written as

$$\left(x_2\left(x_2x_0 + c_2(x_3x_0 - x_1^2)\right)\left(x_2x_0 + c_1(x_3x_0 - x_1^2)\right) = 0\right)$$

which is unstable with respect $\lambda = (7, 1, -4, -4)$.

Proof. Let C be such a curve. Consider two generic distinct points p and q on it, and let $L_{p,q}$ be the line that join them. Since p and q are triple points, then $L_{p,q}$ intersect X with multiplicity larger or equal than six. However, X is a quintic surface, this implies that the surface contains the line $L_{p,q}$ for every p and q on C. So, X contains the secant variety $Sec(C)$ of C. For a curve

C in $\mathbb{P}^3$, the secant variety of C is either the whole $\mathbb{P}^3$, or it is an hyperplane with the last option only happening if C is a plane curve itself (see [24, pg 144]). Therefore, $Sec(C) \subset X$ which implies that C is a plane curve and X decomposes as a hyperplane H and a quartic surface Y. Moreover, from the hypothesis and by a degree consideration C is a smooth conic.

Let our coordinate system be such that the critical one parameter subgroups are the ones on Proposition 2.1.2. So, we have a partial order among monomials $x_0 \geq x_1 \geq x_2 \geq x_3$ (see discussion after Equation 2.1.1). It is enough to consider the cases when the hyperplane is given by $(x_i = 0)$. Indeed, if the equation of the hyperplane supporting C is $(l(x_0, x_1, x_2, x_3) = 0)$, then we can take the minimal non zero monomial $x_i \in \Xi_l$. By its definition, this change does not affect the value of $\mu(\lambda, X)$. Therefore, the equation associated to the quintic surface can be written as

$$x_i \left(f_4(x_j, x_k, x_l) + x_i g_3(x_0, x_1, x_2, x_3)\right)$$

By hypothesis $m_p(X) = 3$ for every point $p \in C \subset Y \cap H$ and C does not contain a line. This implies $f_4(x_j, x_k, x_l) = (f_2(x_j, x_k, x_l))^2$ and either x_i or $f_2(x_j, x_k, x_l)$ divides $g_3(x_0, x_1, x_2, x_3)$. The equation associated to the quintic surface is obtained by combining both cases.

Given a normalized one parameter subgroup $\lambda = (a_0, a_1, a_2, a_3)$. We have:

$$\mu(\lambda, X) \leq \min\{a_i + 2\mu(\lambda, f_2), 3a_i + \mu(\lambda, g_2), 2a_i + \mu(\lambda, f_2) + \mu(\lambda, l)\}$$

In our coordinate system, the curve cannot be supported at $(x_3 = 0)$ because a triple point is supported at $p_\lambda = [0 : 0 : 0 : 1]$. By the smoothness of the curve, we have

$$f_2(x_j, x_l, x_3) = x_3 l(x_j, x_l) + p_2(x_j, x_l)$$

with the set of monomials Ξ_{f_2} containing at least $\{x_3 x_j, x_l^2\}$ with $j \neq l$ and $j, l \neq i$. Moreover, generically it holds that $\mu(\lambda, g_2) \leq 2a_1$, $\mu(\lambda, l) \leq a_0$. Then, the numerical function with respect λ satisfies:

$$\mu(\lambda, X) \leq \min\{a_i + 2(a_3 + a_j), a_i + 4a_l, 3a_i + 2a_1, 2a_i + (a_3 + a_j) + a_0, 2a_i + 2a_l + a_1\}$$

A direct calculation shows that the numerical criterion is nonpositive for all our critical one parameter subgroups. Then X is semistable. □

A generic quintic surface X that decomposes as the union of a cubic and a quadratic surface is stable. On the moduli space, the locus that parametrizes these surfaces is thirteen dimensional: Nine dimensions arise from the genus 4 curve defined by the intersection of the cubic and the quadratic surface. The other four dimensions arise from the fact that we can add a multiple of the quadratic equation to the cubic surface equation without changing the genus 4 curve.

Proposition 2.4.8. *Let X be the union of a smooth quadratic surface Q and a cubic surface Y with a triple point away from Q. Suppose the surfaces intersect along a smooth curve. Then, X is unstable if and only the tangent cone at the triple point has worst than cuspidal singularities.*

Proof. Suppose that X is not stable, and that our coordinate system is such that the critical 1-PS are the ones in Proposition 2.1.2. In particular, the triple point is supported at p_3. The cubic surface is a cone over a plane cubic curve C; the associated equation to the quintic surface can be written as:

$$f_3(x_0, x_1, x_2)g_2(x_0, x_1, x_2, x_3)$$

By hypothesis, the quadratic surface is away from the triple point. Therefore, the monomial x_3^2 is always present on Ξ_{g_2} which implies $\mu(\lambda, X) = 2a_3 + \mu(\lambda, f_3)$. The following analysis is divided by the singularities on the cubic curve. If C has a triple point, then C is either non reduce or the union of three concurrent lines.Therefore, X is unstable by either Proposition 2.1.12 or Corollary 2.1.9. If C is a conic with a tangent line. We can write its associated equation as $x_0(x_2x_0 - x_1^2)f_2(x_0, x_1, x_2, x_3)$ which is destabilized by λ_9. If C is the union of three non concurrent lines, the associated equation to the quintic surface is:

$$l_1(x_0, x_1, x_2)l_2(x_0, x_1, x_2)l_3(x_0, x_1, x_2)f_2(x_0, x_1, x_2, x_3)$$

In particular, the monomial $x_0x_1x_2x_3^2$ must have coefficient different to zero because the lines are no concurrent. For our critical one parameter subgroups, it holds: $\mu(\lambda_k, X) \leq a_0+a_1+a_2+2a_3 \leq -1$ which implies our surface is stable. The union of a conic and a transversal line deforms to three nonconcurrent lines. Then, it is stable, as well. If C has a cuspidal singularity. We claim the associated surface is stable. The triple point is singled out by p_3. By considering the partial order among monomials (see discussion after Equation 2.1.1) we see from all the possible equations associated to the cusp in our coordinate system, the one that induces the highest value of the numerical criterion is $x_i^2x_j + x_0^3 + p_3(x_0, x_i)$ For the critical 1-PS in Proposition 2.1.2 and $i, j \neq 0$ it holds that:

$$\mu(\lambda_k, X) \leq \min\{2a_i + a_j + 2a_3, 3a_0 + 2a_3\} \leq -2$$

Since, nodal singularities deform to cuspidal ones, the associated surface is stable, as well. $\square$

A quintic surface with at worst double point singularities can be represented as a triple cover of the plane (for example see [71, Thm 10.3]). Next, we explore that representation.

Lemma 2.4.9. *Let $(F_{\lambda_i} = 0)$ be a quintic surface obtained from a generic linear combination of the monomials on $M^{\oplus}(\lambda_i)$ for $i \in 5, 9$. There is an associated surface in $\mathbb{P}(2,1,1,1)$ with equation*

$$G_{\lambda_i}(\psi, x_0, x_1, x_2) := \psi^3 + h_4(x_0, x_1, x_2)\psi + h_6(x_0, x_1, x_3)$$

where $h_4(x_0, x_1, x_2)$ and $h_6(x_0, x_1, x_2)$ are obtained from a generic linear combination of the monomials

$$\Xi_{h_4} = \left\{x_0^{j_0} x_1^{j_1} x_2^{j_2} \mid w_0 j_0 + w_1 j_1 + w_2 j_2 \geq c_1(\lambda_i)\ ;\ j_0 + j_1 + j_2 = 4\right\}$$
$$\Xi_{h_6} = \left\{x_0^{j_0} x_1^{j_1} x_2^{j_2} \mid w_0 j_0 + w_1 j_2 + w_2 j_2 \geq c_2(\lambda_i)\ ;\ j_0 + j_1 + j_2 = 6\right\}$$

with $c_1(\lambda_5) = 10$, $c_2(\lambda_5) = 15$ and $c_1(\lambda_9) = 20$, $c_2(\lambda_9) = 30$. In particular, if the equation associated to $(F_{\lambda_i} = 0)$ is given by

$$x_3^3 x_0^2 + x_3^2 x_0 g_2(x_0, x_1, x_2) + x_3 f_4(x_0, x_1, x_2) + f_5(x_0, x_1, x_2)$$

Then

$$G_{\lambda_i}(\psi, x_0, x_1, x_2) := \psi^3 + \left(f_4 - \frac{g_2^2}{3}\right)\psi + \left(x_0 f_5 + \frac{2}{27} g_2^3 - \frac{g_2 f_4}{3}\right)$$

Conversely, if X is a quintic surface with a double point singularity such that $G_{X,p}(\psi, x_0, x_1, x_2)$ satisfies the above conditions. Then, there is a change of coordinates such that $\Xi_X \subset M^{\oplus}(\lambda_k)$.

Proof. We recall the representation of quintic surfaces with a double point as a finite cover of the plane. Our discussion follows the one at [71, pg 471]. Let X be a, non necessarily normal, quintic surface with at worst double point singularities. We suppose there is at least a double point $p \in X$ with only finitely many lines through it and distinct to a normal crossing or a pinch point. Let $\tilde{X} \to X$ be monomial transformation of X with center p. There is a morphism $\tilde{X} \to \mathbb{P}^2$ induced by the projection from the point $p \in X$ that is generically finite of degree three. The surface $\tilde{X}$ is singular along the exceptional divisor of the monomial transformation. Indeed, if the double point at the quintic surface is supported at p_3 and defined by the equation

$$x_3^3 x_0^2 + x_3^2 x_0 g_2(x_0, x_1, x_2) + x_3 f_4(x_0, x_1, x_2) + f_5(x_0, x_1, x_2)$$

The surface $\tilde{X}$ is given by the equation

$$t^3x_0^2 + t^2sx_0g_2(x_0,x_1,x_2) + ts^2f_4(x_0,x_1,x_2) + s^3f_5(x_0,x_1,x_2)$$

with $[t:s] \in \mathbb{P}^1$. From the equation, we see $\tilde{X}$ is singular along the line $(s = x_0 = 0)$. Blowing up $\tilde{X}$ along this singular line, the total transform X' is given by the set of equations:

$$x_0^2\left(t^3 + t^2sg_2(x_0,x_1,x_2) + ts^2f_4(x_0,x_1,x_2) + x_0s^3f_5(x_0,x_1,x_2)\right)$$
$$s^2\left(t^3x_0^2 + t^2x_0g_2(sx_0,x_1,x_2) + tf_4(sx_0,x_1,x_2) + sf_5(sx_0,x_1,x_2)\right)$$

At the first chart and by taking $\psi = t/s$, we obtain:

$$\psi^3 + \psi^2g_2(x_0,x_1,x_2) + \psi f_4(x_0,x_1,x_2) + x_0f_5(x_0,x_1,x_2)$$

We substitute ψ by $\psi - g_2(x_0,x_1,x_2)$ to find the equation associated to X':

$$G_{X,p} := \psi^3 + \left(f_4 - \frac{g_2^2}{3}\right)\psi + \left(x_0f_5 + \frac{2}{27}g_2^3 - \frac{g_2f_4}{3}\right) \tag{2.4.10}$$

where f_d denotes $f_d(x_0,x_1,x_2)$ and the subindex $p \in X$ denotes the choice of the double point for projecting X to $\mathbb{P}^2$. This equation can be thought as a weighted homogeneous hypersurface of degree 6 in $\mathbb{P}(2,1,1,1)$. For the quintic surfaces $(F_{\lambda_i} = 0)$, we project from the double point singularity singled out by the bad flag at p_3. To characterize the monomials in Ξ_{h_4} and Ξ_{h_6} we notice that for $\lambda = (a_0,a_1,a_2,a_3)$ it holds:

$$\lambda_i.\left(x_0^{i_0}x_1^{i_1}x_2^{i_2}x_3^{i_3}\right) = w_\lambda(x_0^{i_0}x_1^{i_1}x_2^{i_2}) + 5a_3$$

Therefore, for λ_5 we find $w_{\lambda_5} = (5,2,1)$, and a monomial in either Ξ_{f_4}, Ξ_{g_2} or Ξ_{f_3} is induced by one in the maximal configuration if and only if $w_{\lambda_5}(f_4) \geq 10$, $w_{\lambda_5}(g_2) \geq 5$, and $w_{\lambda_5}(f_5) \geq 10$. Similarly, for λ_9, we find that $w_{\lambda_9} = (11,5,0)$, $w_{\lambda_9}(f_5) \geq 30$, $w_{\lambda_9}(f_4) \geq 30$, and $w_{\lambda_9}(g_2) \geq 9$. By the nature of the weights, we can take $w_{\lambda_9}(g_2) \geq 10$. This implies that if $\Xi_{F_X} \subset M^{\oplus}(\lambda_k)$, the Equation G_{X,p_3} will satisfies the conditions of the statement.

Now, suppose $G(\psi,x_0,x_1,x_2)$ satisfies the conditions of the statement and that it arises from a quintic surface. Then, we must prove that every monomial from h_4 and h_6 is induced by one in $M^{\oplus}(\lambda_k)$ which implies the associated quintic surface satisfies $\Xi_{F_X} \subset M^{\oplus}(\lambda_i)$ for $i \in \{5,9\}$. For the monomials in h_4, we must rule out monomials of the form m_1m_2 where $w_\lambda(m_1) < 5$ but $w_\lambda(m_1 + m_2) \geq 10$. This will implies m_1 is not induced by a monomial in a maximal configuration. Indeed, this is not possible because if $m_1 + m_2 \in \Xi_{h_4}$, then also $2m_i$. Similarly, a monomials in h_6 will be from either $x_0 f_5$ or g_2^3 or $g_2 g_4$. In the first case, $w_{\lambda_k}(f_5) \geq c_2(\lambda_k) - w_0$, in the second case, we have $3w_{\lambda_k}(g_2) \geq c_2(\lambda_k)$, in the third case, we have $w_{\lambda_k}(g_2) + w_{\lambda_k}(g_2) \geq c_2(\lambda_k)$. The conclusion follows because there is not a monomial m in a quintic surface such that $w_{\lambda_9}(m) = 19$ which could be the only counterexample. $\square$

Proposition 2.4.11. *A quintic surface X with at worst double point singularities is non stable if and only if there is a coordinate system and a distinguish non double point $p \in X$ with only finitely many lines through it such that the branch locus associated to the morphism*

$$(G_{X,p}(\psi, x_0, x_1, x_2) = 0) \to \mathbb{P}^2$$

can be written as one of the following equations

$$D_{\lambda_5}(x_0, x_1, x_2) = x_0^2 \left(x_2^7 x_0^3 + \sum_{k=1}^{k=2} x_0^k \sum_{i=0}^{i=2} x_2^{3k-i} f_{10-4k+i}(x_0, x_1) + f_{10}(x_0, x_1) \right)$$

$$D_{\lambda_9}(x_0, x_1, x_2) = x_2^6 x_0^5 x_1 + \sum_{i=0}^{i=5} x_2^i x_0^i f_{12-2i} f(x_0, x_1)$$

Proof. Let $p \in X$ be a double point as the one at the statement. We construct its associated surface (see proof of Lemma 2.4.9) which equation is given by:

$$G_{X,p}(\psi, x_0, x_1, x_2) = \psi^3 + h_4(x_0, x_1, x_2)\psi + h_6(x_0, x_1, x_2)$$

The morphism $(G_{X,p} = 0) \to \mathbb{P}^2$ is generically finite of degree three. It associated branch locus is given by the equation:

$$D_{X,p} := 4h_4(x_0, x_1, x_2)^3 + 27h_6(x_0, x_1, x_2)^6$$

The surface X is non stable if and only if there is a coordinate system such that $\Xi_{F_X} \subset M^{\oplus}(\lambda_i)$ for $i \in \{5, 9\}$. By Lemma 2.4.9, it holds that either $w_{\lambda_5}(h_4) \geq 10$ and $w_{\lambda_5}(h_6) \geq 15$ or $w_{\lambda_9}(h_4) \geq 10$ and $w_{\lambda_9}(h_6) \geq 30$. Moreover, those weights determined whenever Ξ_{F_X} is contained in $M^{\oplus}(\lambda_i)$ for $i = \{5, 9\}$. In the first case, the equation associated to the branch locus of $(G_{\lambda_5} = 0) \to \mathbb{P}^2$ is given by D_{λ_5}. In the second case, $\Xi_{F_X} \subset M^{\oplus}(\lambda_9)$ and the equation associated to the branch locus of $(G_{\lambda_9} = 0) \to \mathbb{P}^2$ is D_{λ_9}. $\square$

Chapter 3

KSBA Stable Replacement of Selected Surface Singularities

We are interested in generalizing the procedure for calculating the stable replacement of a plane curve singularity to the case of an isolated surface singularity $S_0 \subset \mathbb{C}^3$ with a good $\mathbb{C}^*$-action. In the one dimensional case, the stable replacement for a generic smoothing involves a base change $t^m \to t$ followed it by a weighted blow up. This construction was studied by Hassett [26] for the case of toric and quasitoric plane curve singularities (see Remark 3.2.5). The purpose of this chapter is to discuss a generalization of that procedure for the 14 exceptional unimodal surface singularities (Theorem 3.5.1), for quasihomogeneous singularities of type I (Theorem 3.4.1), and for certain singularities with a distinguished smoothing (Theorem 3.6.1). We will apply the results of this chapter for describing boundary loci in the moduli space of quintics sufaces in Chapter 4.

Our approach uses the natural grading associated to a surface singularity S_0 with a $\mathbb{C}^*$-action and its weighted blow up, also known as the Steifer partial resolution. Indeed, we consider two perspectives for constructing our one dimensional families of surfaces: First, we impose conditions such that the smoothing $X \to \Delta$ of S_0 is unipotent and that it has a graded isolated hypersurface singularity. The former condition is of particular interest because the monodromy theorem implies semistable families have unipotent monodromy (see [50, pg. 106]). The second type of smoothings are constructed by considering the possible ways the orbifolds singularities associated to the minimal resolution of S_0 can be extended to isolated threefold canonical singularities.

In this chapter we also highlight a relationship between Dolgachev singularities and K3 surfaces which is analogous to the relationship between the cusp singularity on a plane curve and the elliptic tail that appears in the context of moduli of curves. In work in progress, we are generalizing this picture for minimal elliptic singularities which is a significant larger family of non log canonical singularities; these singularities appear naturally on moduli constructions of surface of general type [19, Prop. 4.12]. In Section 3.7, we provide examples illustrating their behaviour which is different to the unimodal case.

The idea of comparing the partial resolution of a smoothing with that of an appropriate hypersurface section has been used recently in a more general setting by Wahl [67]. The relationship between these K3 surfaces, the unimodal singularities and their unipotent smoothings appears in several instances (see [11], [32], [57]). We must warn our readers that developments from the work of Pinkham [56] and Wahl [66] are not included in this note. Moreover, Theorem 3.5.1 is conjectured and partially proved in the recent work of J. Rana [58]; her work partially overlaps with ours but her methods are different.

Remark 3.0.1. In an abuse of notation, we will not distinguish between the germ of a singularity and a generic hypersurface of large degree with that singularity. In general, we denote both cases as S_0.

3.1 Preliminaries on Singularities

Let S_0 be the germ of a surface singularity with a $\mathbb{C}^*$-action defined by $\sigma(t,(x_1,\ldots,x_n)) = (t^{w_0}x_1,\ldots,t^{w_n}x_n)$. If the integers $w_i > 0$, then we say that σ is a good $\mathbb{C}^*$-action. We will always suppose the action is good and our singularities are isolated. In particular, the case of two dimensional hypersurface singularities were classified by Orlik, Wagreich [55, 3.1] and Arnold [4]. Indeed, any of those singularities have a topologically trivial deformation into one of the following polynomials [70, Cor. 3.6]:

$$\text{Class I} = x^{p_0} + y^{p_1} + z^{p_2} \qquad \text{Class II} = x^{p_0} + y^{p_1} + yz^{p_2}$$
$$\text{Class III} = x^{p_0} + y^{p_1}z + yz^{p_2} \qquad \text{Class IV} = x^{p_0} + y^{p_1}z + xz^{p_2}$$
$$\text{Class V} = x^{p_0}y + y^{p_1}z + xz^{p_2}$$
$$\text{Class VI} = x^{p_0} + xy^{p_1} + xz^{p_2} + y^a z^b \qquad (p_0-1)(p_0a+p_2b) = p_0p_1p_2$$
$$\text{Class VII} = x^{p_0}y + xy^{p_1} + xz^{p_2} + xz^a + y^a z^b \quad (p_0-1)(p_0a+p_2b) = b(p_0p_1-1)$$

where $p_i \geq 2$. In this paper, we suppose the equation defining S_0 belongs to one of the previous classes (see Remark 3.1.1) and that S_0 has good properties (see Remark 3.2.3). For us $\tilde{w} = (w_0, w_1, w_2)$ denotes the unique set of integer weights for which the surface singularity is quasihomogeneous, $gcd(w_0, w_1, w_2) = 1$, and the weighted multiplicity $\deg_{\tilde{w}}(S_0)$ reaches its smallest integer value.

Remark 3.1.1. There is not a uniform notation among the references. For example, the class V in [4] is labelled as IV in [70]. We used the notation from [70].

Example 3.1.2. For singularities of class I, the explicit expressions for the weights and their weighed multiplicity are:

$$\begin{aligned} \tilde{w} &= \left(\frac{p_1 p_2}{gcd(p_0p_1, p_1p_2, p_0p_2)}, \frac{p_0 p_2}{gcd(p_0p_1, p_1p_2, p_0p_2)}, \frac{p_0 p_1}{gcd(p_0p_1, p_1p_2, p_0p_2)} \right) \\ deg_w(f) &= \frac{p_0 p_1 p_2}{gcd(p_0p_1, p_1p_2, p_0p_2)}. \end{aligned}$$

The expressions for the other classes are increasingly cumbersome, so we will not display them here.

By the work of Orlik and Wagreich, it is known that the dual graph of the minimal resolution of S_0 is star shaped. The central curve is a curve whose genus is determined by the weights $\tilde{w}$ while the other curves in the branches of the minimal resolution are rational curves (see [55, Thm 2.3.1, Thm 3.5.1]). The contraction of these branches generates a set of cyclic quotient singularities naturally associated to S_0.

Definition 3.1.3. *Let S_0 be a surface singularity with a $\mathbb{C}^*$-action, we denote by $\mathcal{B}_{S_0}$ the set of cyclic quotient singularities supported on the central curve $(S_0 \setminus 0)/\mathbb{C}^*$ and obtained by contracting the branches on the minimal resolution of S_0.*

Remark 3.1.4. Let S_0 be a normal Gorenstein surface singularity with a good $\mathbb{C}^*$- action. Its coordinate ring is a graded algebra $A = \bigoplus_{k=0}^{\infty} A_k$ where $A_0 = \mathbb{C}$ and the singularity is defined by its maximal ideal. Following Dolgachev, there exist a simply connected Riemann surface C, a discrete cocompact group $\Gamma \subset Aut(C)$ and an appropriate line bundle $\mathcal{L}$ such that $A_k = H^0(C, \mathcal{L}^k)^\Gamma$. Moreover, there is a divisor $D_0 \subset C$ and points $p_i \in C$ such that:

$$A_k = L\left(kD_0 + \sum_{i=1}^{r} \left\lfloor k \frac{\alpha_i - \beta_i}{\alpha_i} \right\rfloor \right)$$

where $L(D_R)$ is the space of meromorphic functions with poles bounded by D_R. The set of numbers $b := \deg(D_0) + r$, (α_i, β_i) and $g(C)$ are known as the orbit invariants of S_0. Furthermore, there is a number R, known as the exponent of S_0 that satisfies the relationships:

$$R\left(-b+\sum_{i=1}^{r}\frac{\beta_i}{\alpha_i}\right)=\left(-\deg(K_C)-r+\sum_{i=1}^{r}\frac{1}{\alpha_i}\right)$$

where

$$R\beta_i \equiv 1 \mod (\alpha_i) \ \forall i=1,\dots r$$

with this notation the set of cyclic singularities $\mathcal{B}_{S_0}$ is $\left\{\frac{1}{\alpha_i}(1,\beta_i)\right\}$.

In general there are explicit formulas for finding the singularities in $\mathcal{B}_{S_0}$ (see [55, Sec 3.3]). Next, we describe two examples that will be relevant in the following sections.

Example 3.1.5. Let S_0 be the singularity induced by

$$f(x,y,z)=x^{p_0}+y^{p_1}+z^{p_2}$$

with $p_i \geq 2$. Then, the weights w_i are as on Example 3.1.2, $\alpha_k = gcd(w_i,w_j)$, $R=\deg_{\tilde{w}}(f)-\sum_i w_i$ and $\mathcal{B}_{S_0}$ has cyclic quotient singularities of type $\frac{1}{\alpha_k}(1,\beta_k)$ where $w_k\beta_k \equiv -1 \mod gcd(w_j,w_i)$. In the minimal resolution of S_0, let $\hat{C}$ be the proper transform of $(S_0\setminus 0)/\mathbb{C}^*$. For all the classes of quasihomogeneous singularities, the self intersection of $\hat{C}$ is given by (see [55, Thm 3.6.1])

$$-\hat{C}^2=\frac{\deg_{\tilde{w}}(f)}{w_0w_1w_2}+\sum_{k=1}^{r}\frac{\beta_k}{\alpha_k}$$

although in general α_k may be different to $gcd(w_j,w_i)$. Let E_{20} be a singularity which associated equation is $x^2+y^3+z^{11}$. Then, we have that:

$$\mathcal{B}_{E_{20}}=\left\{\frac{1}{2}(1,1),\frac{1}{3}(1,2),\frac{1}{11}(1,9)\right\}$$

Definition 3.1.6. *(notation as on Remark 3.1.4) A surface singularity S_0 is called Fuchsian if C is the upper half plane and $\mathcal{L}=K_C$. In particular, the 14 exceptional unimodal singularities are Fuchsian ones.*

Lemma 3.1.7. *Let S_0 be a Fuchsian singularity, then*

$$\mathcal{B}_{S_0}=\left\{\frac{1}{\alpha_i}(1,1)\right\}$$

where α_i is an orbit invariant as on Remark 3.1.4 (for a list of the possible α_i see [14, Sec 1, Table 1]).

Definition 3.1.8. *Given a cyclic quotient singularity* $T_1 := \frac{1}{r}(1, d_1)$. *We say that* $\frac{1}{r}(1, d_2)$ *is dual to* T_1 *if the isolated threefold cyclic quotient singularity*

$$\frac{1}{r}(1, d_1, d_2)$$

is canonical. Let $\mathcal{B}_{S_0} = \{T_1, \dots T_n\}$ *be the set of cyclic quotient singularities associated to* S_0. *We say the set of cyclic quotient singularities* $\hat{\mathcal{B}}_{S_0} = \{\hat{T}_1, \dots, \hat{T}_n\}$ *is dual to* $\mathcal{B}_{S_0}$ *if* $\hat{T}_i$ *is dual to* T_i *for every* i.

The dual set can be explicitly founded.

Lemma 3.1.9. *The isolated singularity* $\frac{1}{r}(1, d_2)$ *is the dual* $\frac{1}{r}(1, d_1)$ *if and only if one of the following equations holds (the three first imply the threefold singularity is terminal, the last one implies it is Gorenstein):*

$$1 + d_1 \equiv 0 \mod (r); \qquad 1 + d_2 \equiv 0 \mod (r)$$
$$d_1 + d_2 \equiv 0 \mod (r); \qquad d_1 + d_2 \equiv -1 \mod (r)$$

if $r = 7, 9$ *we also must also consider two, somehow exceptional, canonical cyclic singularities:*

$$\frac{1}{9}(2, 8, 14) \qquad \frac{1}{7}(1, 9, 11)$$

Proof. We use the Reid-Tate criterion [38, pg 105] and the classification due to Morrison and Stevens [52, Thm 2.4], [51, Thm 3] for finding the dual sets $\hat{\mathcal{B}}_{S_0}$. □

Corollary 3.1.10. *Let* S_0 *be a Fuchsian singularity as on Example 3.1.7, then we have the dual sets* $\hat{\mathcal{B}}_{S_0} = \{T_i\}$ *where* T_i *is either* $\frac{1}{\alpha_i}(1, \alpha_i - 1)$ *or* $\frac{1}{\alpha_i}(1, \alpha_i - 2)$.

Example 3.1.11. Let $\mathcal{B}_{E_{20}}$ be as on Example 3.1.5. Then, two possible dual sets are:

$$(\hat{\mathcal{B}}_{E_{20}})_1 = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{11}(1,2)\right\} \quad (\hat{\mathcal{B}}_{E_{20}})_2 = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{11}(1,10)\right\}$$

3.2 On the smoothing of surface singularities

In this section, we construct two types of one dimensional families of surfaces motivated by a similar construction due to Hassett [26] in the context of plane curves singularities (see Remark 3.2.5). Our definitions spring from two questions:

1. What is the base change need it for calculating the stable replacement of a non log canonical surface singularity?
2. Given the union of two surfaces with singularities $\mathcal{B}_{S_0}$ and $\hat{\mathcal{B}}_{S_0}$, is it possible to find a $\mathbb{Q}$-Gorenstein smoothing of them?

Our partial advances in these questions yield the Theorems 3.4.1 and 3.5.1. Next, we describe the weighted blow ups that are used in our work.

Let $X := \operatorname{Spec}(A)$ be an isolated graded normal singularity with a good $\mathbb{C}^*$-action. Then, A can be written as the quotient of a graded polynomial ring $\mathbb{C}[x_1, \ldots x_n]$ where the variables x_i have weights $w_i > 0$ ([55, 1.1]). The $\mathbb{C}^*$-action determines a weighted filtration whose blow up is known as the Steifer partial resolution of X (for more details see [67, Sec. 1]). This resolution depends of the grading, and in our case it is induced by the weighted blow up $\pi_{\mathbf{w}}$ of $\mathbb{C}^n$ with respect the weights $\mathbf{w}$ which we will denoted as $Bl_{\mathbf{w}}\mathbb{C}^n$. In this case, $\pi_{\mathbf{w}}$ amounts to blowing up $\mathbb{C}^n$ along the ideal

$$\left(x_1^{d/w_1}, \ldots, x_n^{d/w_n}\right)$$

where $d = lcm(w_0, \ldots, w_n)$. The exceptional divisor associated to $\pi_{\mathbf{w}}$ is $E_{\mathbf{w}} := \mathbb{P}(w_0, \ldots, w_n)$ and the exceptional divisor on the proper transform $\tilde{X}$ of X is induced by $\tilde{X} \cap E_{\mathbf{w}}$. On our applications, $X \subset \mathbb{C}^4$ is a smoothing of S_0 with an isolated singularity that has a $\mathbb{C}^*$-action. By restricting $\pi_{\mathbf{w}}$ to its central fiber we have a weighted blow up of $S_0 \subset \mathbb{C}^3$ with weights $\tilde{w}$ and exceptional divisor $E_{\tilde{w}} = \mathbb{P}(w_0, w_1, w_2)$. The final configuration is given by (see Figure 3.2.0.1):

$$\begin{tikzcd}
C := S_1 \cap E_{\tilde{w}} \subset S_1 + E_{\tilde{w}} \subset Bl_{\tilde{w}}\hat{\mathbb{C}}^3 \arrow[r] \arrow[d, "\pi_{\tilde{w}}"'] & \tilde{X} + E_{\pi_{\mathbf{w}}} \subset Bl_{\mathbf{w}}\hat{\mathbb{C}}^4 \arrow[d, "\pi_{\mathbf{w}}"] \\
S_0 \subset \hat{\mathbb{C}}^3 \arrow[r] & X \subset \hat{\mathbb{C}}^4
\end{tikzcd}$$

The central fiber $\tilde{X}|_0$ decomposes as the union of two surfaces S_1 and S_T. The surface S_1 is the proper transform of S_0 and $S_T \subset \mathbb{P}(w_0, w_1, w_2, w_3)$ is the exceptional surface contained in the proper transform of $\tilde{X}$. The exceptional divisor $E_{\tilde{w}}$ is not contained in the threefold $\tilde{X}$, but rather it intersects $\tilde{X}$ along the exceptional curve $C \subset S_1$ which satisfies

$$C := S_1 \cap E_{\pi_{\tilde{w}}} = S_1 \cap S_T.$$

The singularities on S_1 are the ones on $\mathcal{B}_{S_0}$. The singularities on S_T depend of the weights $\mathbf{w}$. Note that $\tilde{X}$ may have non isolated singularities. Our applications spring from the fact that, for the appropriate cases, the new one dimensional family of surfaces $\tilde{X} \to \Delta$ have at worst canonical singularities.

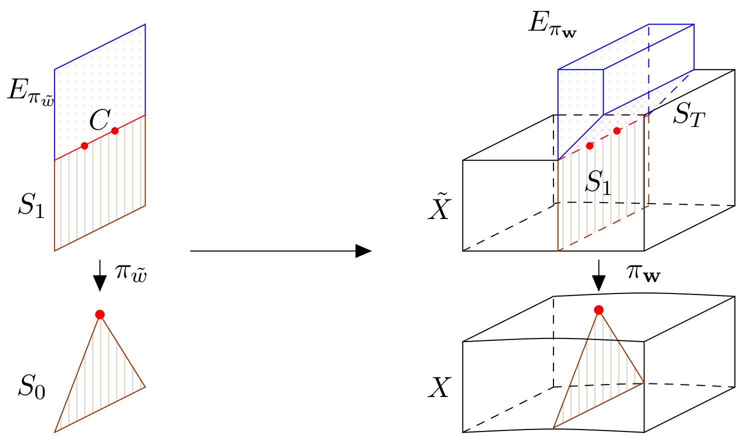


Figure 3.2.0.1: Weighted blow up setting

Lemma 3.2.1. *The pair (S_1, C) has at worst log canonical singularities and $K_{S_1} + C$ is $\pi_{\tilde{w}}$-ample.*

Proof. It follows from either [31, Cor. 3.10]) or [66, 2.3.2]. □

The singularities of S_T are determined by the singularities at $\tilde{E}_{\mathbf{w}}$.

Lemma 3.2.2. *Let $S_T \subset \mathbb{P}(w_0, w_1, w_2, w_3)$ be a generic hypersurface (see Remark 3.2.3), then its singular locus is obtained by intersecting S_T with the edges P_iP_j and with the vertex P_i. In the first case S_T has $\lfloor gcd(w_j, w_i)\deg_{\boldsymbol{w}}(f)/w_jw_i\rfloor$ cyclic quotient singularities of type*

$$\frac{1}{gcd(w_i, w_j)}(1, c_{ij})$$

where c_{ij} is the solution of $w_l - w_kc_{ij} \equiv 0 \mod (gcd(w_i, w_j)),\quad k < l$. The other singularities of S_T are induced by the vertex P_i and they depend on the intersection between S_T and the vertex.

Proof. By hypothesis $E_{\mathbf{w}}$ is well formed, then its singular locus has codimension 2 and it is supported on the edges P_iP_j. By quasismoothness of X, the proof reduces to describe the intersection of S_T with $\mathrm{Sing}(E_{\mathbf{w}})$. This is a standard calculation on weighted projective spaces see for example [73, Lemma 4.1] or [16, I.7.1]; a useful Lemma for counting the points on $S_T \cap P_iP_j$ is [16, I.6.4]. □

Remark 3.2.3. We suppose the associated equation to the singularity is *non degenerated with respect to the Newton polytope*. Therefore, the log canonical, canonical, and minimal model of our singularities can be constructed by a subdivision of the dual fan of their respective Newton polytope (see [31]). We *will suppose the surface S_T is quasismooth*. Then, its singularities are induced only by the cyclic quotient singularities of $\mathbb{P}(w_0, w_1, w_2, w_3)$ (see [16, I.5]). We will also suppose that S_T is well formed, so we can apply the adjunction formula to find its canonical class (for details see [16, I.3.10]). Finally, we *suppose that S_0 and X are generic enough with respect to their weights*. So the singularities of S_T depend only on the weights $\mathbf{w}$. In particular, for each i, the equation contains a term of the form x_i^n or $x_i^m x_k$ with a nonzero coefficient.

Example 3.2.4. (see [69, pg. 72]) Let E_{12} be a minimal elliptic surface singularity with associated equation $(x^2+y^3+z^7=0)$. Consider the smoothing

$$x^2 + y^3 + z^7 + a_1 t^{42} + a_2 xyzt = 0 \ \text{ with } \ [a_1 : a_2] \in \mathbb{P}^1.$$

If $a_1 \neq 0$, then $S_T \subset \mathbb{P}(21, 14, 6, 1)$ is a $K3$ surface of weighted degree 42 with $A_1+A_2+A_6$ singularities. On other hand if $a_1 = 0$, then the exceptional divisor is a rational surface on $\mathbb{P}(21, 14, 6, 1)$ with a $T_{2,3,7}$ singularity. We highlight that this weighted blow up is the simultaneous canonical modification of these singularities (see [30, Ex 3.4])

Remark 3.2.5 (Motivation, see Hassett [26])**.** Let C_0 be the germ of an isolated reduced plane curve singularity with the same topological type as $(f(x,y) = x^p + y^q = 0)$. Let S be the smoothing defined by

$$(f(x,y) + t^{lcm(p,q)} = 0) \to \mathrm{Spec}(\mathbb{C}[[t]]),$$

and let $\pi_{\mathbf{w}}$ be the induced weighted blow up with respect the weights

$$(w_0, w_1, w_3) = \left(\frac{q}{gcd(p,q)}, \frac{p}{gcd(p,q)}, 1\right).$$

a similar setting to ours is (the symbol $\hat{\mathbb{C}}^n$ is to indicate we are working locally).

$$\begin{tikzcd}
C_1 \cap \mathbb{P}(w_0,w_1) \subset C_1 + \mathbb{P}(w_0,w_1) \subset Bl_{\tilde{w}}\hat{\mathbb{C}}^2 \arrow[r] \arrow[d, "\pi_{\tilde{w}}"'] & \tilde{S} + \mathbb{P}(w_0,w_1,1) \subset Bl_{\mathbf{w}}\hat{\mathbb{C}}^3 \arrow[d, "\pi_{\mathbf{w}}"] \\
C_0 \subset \hat{\mathbb{C}}^3 \arrow[r] & S \subset \hat{\mathbb{C}}^3
\end{tikzcd}$$

The central fiber $\tilde{S}|_0$ decomposes as the union of two curves C_1 and C_T. The curve C_1 is the normalization of C_0 and $C_T = \tilde{S} \cap \mathbb{P}(w_0, w_1, 1)$ is the exceptional curve contained on the proper transform of S. The key point is that all the fibers of $\tilde{S}$ are Deligne-Mumford stable [26, Thm 6.2] Therefore, this weighted blow up give us the local stable reduction for those plane curve singularities.

3.3 Smoothing associated to the singularities

Let $X_0 \to \mathrm{Spec}\,(\mathbb{C}[[\tau]])$ be a generic one dimensional smoothing of S_0, then its analytical form is $f(x,y,z) = t$ where $f(x,y,z)$ is an equation defining S_0. We construct another smoothing $X \to \Delta$ of S_0 by taking a base change $t^m \to t$. Our base change is singled out by the monodromy theorem which implies a smoothing $X \to \Delta$ with a semistable family Y dominating it, $Y \to X \to \Delta$, has unipotent local monodromy.

Definition 3.3.1. *Let $X_0 \to \mathrm{Spec}(\mathbb{C}[[\tau]])$ as before. The unipotent base change is the one given by $\tau \to t := \tau^m$ where m is the minimum integer such that:*

$$\Delta := \mathrm{Spec}\,(\mathbb{C}[[t]]) \to \mathrm{Spec}(\mathbb{C}[[\tau]])$$

induces a family $X := X_0 \times_{\Delta_0} \Delta \to \Delta$ with an unique quasihomogeneous singularity and unipotent monodromy.

Varchenko proved that if $S_0 := (f(x,y,z) = 0))$ is non degenerate with respect to its Newton polyope, then the characteristic polynomial of the monodromy of $f(x,y,z)$ at the origin depends only on its associated weights.

Lemma 3.3.2. *Let S_0 be a quasihomogeneous non degenerated singularity, then its unipotent base change is induced by its weighted degree $\deg_{\tilde{w}}(f)$.*

Proof. Let ξ_i be the eigenvalues of the classical monodromy operator associated to a smoothing of S_0; this monodromy is unipotent if and only if $\xi_i^m = 1$ for all i. We must prove that if $m = \deg_{\tilde{w}}(f)$, then $\xi_i^m = 1$, and that $\deg_{\tilde{w}}(f)$ is the smallest possible integer with that property. Since ξ_i are also the roots of the characteristic polynomial $\theta_{S_0}(t)$ of the monodromy, our statement follows directly from an expression, due to Ebeling [13, Thm 1]:

$$\theta_{S_0}(t) = \frac{(1-t^d)^{2g-2+r} \prod_{w_j|d}(1-t^{d/w_j})}{(1-t)\prod_{\alpha_i|d}(1-t^{d/\alpha_i})}$$

where $d = \deg_{\tilde{w}}(f)$, g is the genus of C, α_i and r are as on Remark 3.1.4. Another explicit expression of $\theta_{S_0}(t)$ in terms of the weights is given on [70, Prop. 2.2] □

Let $X \subset \mathbb{C}^4$ be a smoothing of S_0 with an isolated quasihomogeneous singularity, by taking the weighted blow up of X we obtain an one dimensional family $\tilde{X}$ of surfaces degenerating to the union of S_1 and S_T. We are interested whenever the surface S_T has $\hat{\mathcal{B}}_{S_0}$ singularities.

Definition 3.3.3. *Let $X \subset \mathbb{C}^4$ be a smoothing of S_0 with a good $\mathbb{C}^*$-action and let $\hat{\mathcal{B}}_{S_0}$ be one of the dual sets of $\mathcal{B}_{S_0}$. We say that X is a smoothing associated to $\hat{\mathcal{B}}_{S_0}$ if the following conditions holds:*

1. *By taking the associated weighted blow up of X, we obtain a threefold $\tilde{X}$ which central fiber decomposes as $\tilde{X}|_0 = S_1 + S_T$ where S_1 is the proper transform of S_0 and S_T is a well-formed, quasismooth surface in a weighed projective space.*

2. *The singular locus of S_1 is $\mathcal{B}_{S_0}$*

3. *The singular locus of S_T is $\hat{\mathcal{B}}_{S_0}$ and maybe some additional DuVal singularities.*

We denote this smoothing as $X(\hat{\mathcal{B}}_{S_0})$

Example 3.3.4. The set $(\hat{\mathcal{B}}_{S_0})_2$ described on Example 3.1.11 does not have an associated smoothing $X \subset \mathbb{C}^4$ as defined above. Indeed, the possible weights must be of the form $\mathbf{w} = (33, 22, 6, w_3)$. A direct calculation shows that the condition

$$(\hat{\mathcal{B}}_{S_0})_2 \subset Sing(\mathbb{P}(33, 22, 6, w_3)) \cap S_T$$

implies that $w_3 \equiv 49 \mod (66)$. By taking a linear combination of monomials quasihomogeneous with respect those weights, it is not possible to construct a smoothing that satisfies the conditions of Definition 3.3.3.

3.4 Type I Quasihomogeneous Singularities

For this family of surface singularities the setting of Section 3.2 allows us to create a family of surfaces $\tilde{X}$ degenerating to a central fiber with singularities $\mathcal{B}_{S_0}$ and $\hat{\mathcal{B}}_{S_0}$.

Theorem 3.4.1. *Let X be the unipotent smoothing of a quasihomogeneous singularity S_0 of type I; let $\tilde{X}$ be the proper transform of X under the weighted blow up $\pi_{\boldsymbol{w}}$. Then it holds that:*

1. *$\tilde{X}$ has at worst terminal cyclic quotient singularities*

2. *The central fiber $\tilde{X}_0$ has orbifold double normal crossing singularities, and it decomposes in two surfaces $S_1 + S_T$.*

3. *Let $\mathcal{B}_{S_0} = \left\{\frac{1}{r_i}(1, b_i)\right\}$ be the singular locus of S_1. Then, the unipotent smoothing is the one associated to the dual set $\hat{\mathcal{B}}_{S_0} := \left\{\frac{1}{r_i}(1, -b_i)\right\}$*

Remark 3.4.2. An orbifold double normal crossing singularity is locally of the form

$$(xy = 0) \subset \frac{1}{r_i}(1, -1, c_i), \quad (r_i, c_i) = 1$$

Remark 3.4.3. In general for a surface singularity S_0 the set $\mathcal{B}_{S_0}$ have several dual sets $\hat{\mathcal{B}}_{S_0}$ with different associated smoothings (see Section 3.7). Nevertheless, for unimodal singularities a sense of uniqueness is accomplished (see Theorem 3.5.1).

Proof. This result follows from Ishii's characterization of canonical modifications [30]. Next, we describe her approach. The notation is the standard one in toric geometry. Let X be an isolated hypersurface singularity defined by a non degenerated quasihomogeneous polynomial $g := \sum_{\mathbf{a}\in M} c_{\mathbf{a}} x^{\mathbf{a}}$. Oka proved that we can obtain a resolution of X by making a subdivision Σ_0 of the dual fan of the Newton polytope $\Gamma(g) \subset N_{\mathbb{R}}$ (see [54]). This subdivision Σ_0 is induced by primitive vectors $\mathbf{p}_i$ on $N_{\mathbb{R}}$. From the fan associated to the subdivision Σ_0, we obtain a toric variety $T(\Sigma_0)$ such that the proper transform $X(\Sigma_0)$ of X is smooth, intersects transversely each orbit and

$$K_{X(\Sigma_0)} = \varphi^*(K_X) + \sum_{\mathbf{p}_i \in \Sigma_0(1) - \sigma(1)} a(\mathbf{p}_i, X) E_{\mathbf{p}_i}|_{X(\Sigma_0)}$$

where $E_{\mathbf{p}_i}$ are the exceptional divisor associated to the primitive vector $\mathbf{p}_i$, the vectors $\mathbf{p}_i \in \Sigma_0(1) - \sigma(1)$ are the new rays added to the fan, and

$$\begin{aligned} a(\mathbf{p}_i, X) &= \sum_k (\mathbf{p}_i)_k - \min\left\{\mathbf{p}_i(\mathbf{a}) \;\middle|\; \mathbf{a} \in M, g := \sum_{\mathbf{a}\in M} c_{\mathbf{a}} x^{\mathbf{a}}, c_{\mathbf{a}} \neq 0\right\} \\ &= \mathbf{p}_i(\mathbf{1}) - \mathbf{p}_i(g) - 1 \end{aligned}$$

with $\mathbf{p}(\mathbf{a}) = \sum_k p_k a_k$. From our purposes, it is enough to consider the primitive vectors $\mathbf{p}_i$ inside the essential cone of the singularity:

$$C_1(g) := \{\mathbf{s} \in N_{\mathbb{R}} \mid -1 \geq a(\mathbf{s}, X) \quad \text{and} \quad s_i \geq 0\}.$$

because if $\mathbf{p} \notin C_1(g)$, then $a(\mathbf{p}, X) \geq 0$. To prove that $\pi_{\mathbf{w}}$ is the canonical modification of X, we must show that $\mathbf{w} \in C_1(g)$ and that for any $\mathbf{s} \in C_1(g)$,

$\mathbf{s} \neq \mathbf{w}$ the discrepancy associated to $E_{\mathbf{s}}$ is non negative. This translates into a combinatorial condition between $\mathbf{w}$ and $\mathbf{s}$ (see [30, Thm 2.8]); the weighted blow up $\pi_{\mathbf{w}}$ is the canonical modification of X if and only if $\mathbf{w} \in C_1(g)$ and $\mathbf{w}$ is g-minimal in $C_1(g) \cap N \setminus \{0\}$. The g-minimality of $\mathbf{w}$ means that for all $\mathbf{s} \in C_1(g)$ one of the two following inequalities holds for all $i \in \{1 \dots n\}$:

$$\mathbf{w} \leq_g \mathbf{s} := \frac{w_i}{\mathbf{w}(g) - \mathbf{w}(\mathbf{1}) + 1} \leq \frac{s_i}{\mathbf{s}(g) - \mathbf{s}(\mathbf{1}) + 1} \tag{3.4.4}$$

$$\mathbf{w} \preceq_g \mathbf{s} := \frac{w_i}{\mathbf{w}(g)} \leq \frac{s_i}{\mathbf{s}(g)} \tag{3.4.5}$$

and $\mathbf{s}$ belongs to the interior of a $(n+1)$-dimensional cone of $\sigma(\mathbf{w})$. In our case, the equation is given by

$$g(x, y, z, t) := x^{p_0} + y^{p_1} + z^{p_2} + t^{\deg_w(S_0)} + \sum c_{i_0, i_1, i_2, i_3} x^{i_0} y^{i_1} z^{i_2} t^{i_3}$$

with $p_0 \geq p_1 \geq p_2 \geq 2$, $\mathbf{w} = (w_0, w_1, w_3, 1)$ where w_i is as on Example 3.1.2 and $i_0 w_0 + i_1 w_1 + i_2 w_2 + i_3 = \deg_w(S_0)$. The claim that $\mathbf{w} \in C_1(g)$ follows from an inductive argument and by ruling out very low values of the exponents, such as $p_0 = p_1 = p_2 = 2$, which define log canonical surface singularities. By the definition of unipotent smoothing

$$\frac{\mathbf{w}}{\mathbf{w}(g)} = \left(\frac{1}{p_0}, \frac{1}{p_1}, \frac{1}{p_2}, \frac{(p_1 p_2, p_0 p_2, p_0 p_1)}{p_0 p_1 p_2} \right)$$

Therefore, g-minimality follows at once from the definition of weighted degree:

$$\mathbf{s}(g) = \min \left\{ s_0 p_0, s_1 p_1, s_2 p_2, s_3 \frac{p_0 p_1 p_2}{(p_1 p_2, p_0 p_2, p_0 p_1)} \right\}$$

Morever, for $\mathbf{w}$ and any $\mathbf{s} \in C_1(g)$, it holds $\mathbf{w} \preceq \mathbf{s}$. This implies the singularities at $\tilde{X}$ are terminal. Indeed, let Σ be a non singular subdivision of the fan $\Delta(\mathbf{w})$ associated to $Bl_{\mathbf{w}}\mathbb{C}^n$, let ϕ be the proper birational morphism associated to this subdivision.

$$X(\Sigma) \xrightarrow{\phi} \tilde{X} \xrightarrow{\pi_{\mathbf{w}}} X$$

Let $U_i \subset Bl_{\mathbf{w}}\mathbb{C}^n$ be an open set associated to the subfan $\Delta_i \subset \Delta(\mathbf{w})$, and let $\Sigma_i \subset \Sigma$ be the preimage of Δ_i on the non singular subdivision. We can take the restriction $X_i := \tilde{X} \cap U_i$ and its proper transform under the resolution $X(\Sigma_i) := \phi_*^{-1}(X_i)$ to obtain (see [30, Prop. 2.6])

$$K_{X(\Sigma_i)} = \phi^*(K_{U_i} + X_i)\big|_{X(\Sigma_i)} + \sum m_{\mathbf{s}} \left(E_{\mathbf{s}} \cap X(\Sigma_i) \right)_{red}$$

where $\mathbf{s} \neq \mathbf{w}$ and

$$m_{\mathbf{s}} = \frac{s_i}{w_i}\left(\mathbf{w}(f) - \mathbf{w}(\mathbf{1}) + 1\right) - \left(\mathbf{s}(f) - \mathbf{s}(\mathbf{1}) + 1\right)$$

In the proof of Theorem 2.8 at [30] we found that:

1. If $\mathbf{s} \in C_1(g)$ and $\mathbf{w} \preceq_g \mathbf{s}$, then $E_{\mathbf{s}} \cap X(\Sigma_0) = \emptyset$
2. If $\mathbf{s} \notin C_1(g)$ then $m_{\mathbf{s}} > 0$.

In our case $\mathbf{w} \preceq_g \mathbf{s}$ for all $\mathbf{s} \in C_1(g)$, then for any $\mathbf{s}$ we have either $m_s > 0$ or $E_{\mathbf{s}} \cap X(\Sigma_0) = \emptyset$. This implies the singularities on $\tilde{X}$ are terminal, because resolving a canonical singularity will induce an exceptional divisor E_s with $m_s = 0$ and $E_s \cap X(\Sigma_0) \neq \emptyset$. Terminal singularities are of codimension three, so they are isolated on $\tilde{X}$. By construction the singularities on $\tilde{X}$ are cyclic quotient ones and caused solely by the $\mathbb{C}^*$-action. Finally, $\tilde{X}_0$ is reduced and it decomposes into two surfaces S_1 and S_T. The singularities of S_1 are the ones in $\mathcal{B}_{S_0}$ (see Example 3.1.5 for an explicit expression). The singularities of S_T are calculated in Lemma 3.2.2. Our statement follows from those expressions. □

Example 3.4.6. The W_{15} singularity (also known as $A(1, -2, -2, -3, -3)$) is a Fuchsian bimodal singularity and its normal form is $x^2+y^4+z^6$. The unipotent base change $t^{12} \to t$ induces the 8th case on Yonemura's classification [73]. The set of singularities are:

$$\mathcal{B}_{W_{15}} = \left\{2 \times \frac{1}{2}(1,1), 2 \times \frac{1}{3}(1,1)\right\} \qquad \hat{\mathcal{B}}_{W_{15}} = \left\{2 \times \frac{1}{2}(1,1), 2 \times \frac{1}{3}(1,2)\right\}$$

This is the only dual set realized by a smoothing. In this case, the exceptional surface is a $K3$ surface $S_T := S_{12} \subset \mathbb{P}(6,3,2,1)$

3.5 Unimodal Singularities

We focus on unimodal non log canonical singularities. It is well known that there are 14 of those singularities, and that they are all quasihomogeneous. For more details about them, see Arnold [4, pg 247], Laufer [**?**], and Dolgachev [10].

Theorem 3.5.1. *Let S_0 be an unimodal surface singularity, then it holds:*

1. *From all the canonical dual sets associated to $\mathcal{B}_{S_0}$, there is only one dual set $\hat{\mathcal{B}}_{S_0}$ with an associated smoothing $X(\hat{\mathcal{B}}_{S_0}) \to \Delta$ as on Definition 3.3.3.*

2. *The smoothing $X(\hat{\mathcal{B}}_{S_0})$ coincides with the one induced by the unipotent base change. Moreover, this threefold has an unique strictly log canonical singularity.*

3. *The threefold $\tilde{X}$ has isolated terminal cyclic quotient singularities.*

4. *The central fiber $\tilde{X}_0$ has orbifold double normal crossing singularities and it decomposes in two surfaces S_1 and S_T intersecting along a rational curve $C = S_1 \cap S_T$ which satisfies*

$$\left(S_1\big|_{S_T} \right)^2 = \frac{\deg_{\mathbf{w}}(f)}{w_0 w_1 w_2}$$

5. *S_1 is the proper transform of S_0 and it supports the singularities in $\mathcal{B}_{S_0}$.*

6. *S_T is a K3 surface and it supports the singularities in $\hat{\mathcal{B}}_{S_0}$.*

7. *The line bundles $K_{\tilde{X}}|_{S_1}$ and $K_{\tilde{X}}|_{S_T}$ are ample.*

Remark 3.5.2. Yonemura [73] classified all the hypersurface threefold singularities which exceptional surface is a normal $K3$ surface with canonical singularities. There are 95 of those families and they are in bijection with the list of 95 normal K3 surfaces that appear as a hypersurface in a weighted projective space.

Remark 3.5.3. The relationship between monodromy and smoothings of surfaces with semi log canonical (slc) singularities is not straightforward. Indeed, let $S_1 \cup S_T$ be a surface with at worst slc singularities. Then, S_T and S_1 can have cyclic quotient singularities away from their intersection. Any cyclic quotient singularity is log terminal and they can induce arbitrary large monodromy to a generic smoothing of these surfaces. On other hand, the hypothesis that there is a semistable family dominating the smoothing is used in the proof of important theorems related to slc surfaces (For example [39, Thm 5.1]). The monodromy theorem implies that those families have unipotent smoothings.

Proof. Let S_0 be an unimodal singularity with associated weights (w_0, w_1, w_2), the first statement claims that there is only one $1 \leq w_3 \leq \deg_{\tilde{w}}(S_0)$ and one dual set $\hat{\mathcal{B}}_{S_0}$ for which the singularities induced on $S_T \subset \mathbb{P}(w_0, w_1, w_2, w_3)$ are the ones in $\hat{\mathcal{B}}_{S_0}$. The second and third statement means that such unique set of weights is $\mathbf{w} = (w_0, w_1, w_2, 1)$, and that the cyclic quotient singularities on our $\hat{\mathcal{B}}_{S_0}$ are of type $\frac{1}{\alpha_i}(1, \alpha_i - 1)$ (see Example 3.1.10). These statements follows from an individual study of each singularity and their associated weighted projective spaces; this is described on the rest of the section. In fact, we wrote a small computer program in Sage [63] to compute the singularities of S_T, to compare them with the different $\hat{\mathcal{B}}_{S_0}$, and to test the quasismoothness and well formedness of our possible exceptional surfaces. The quasismothness of X implies that $\tilde{X}$ has only cyclic quotient singularities (see [17, Lemma 8]) which are terminal by the nature of $\mathcal{B}_{S_0}$ and $\hat{\mathcal{B}}_{S_0}$. Those terminal singularities are induced by $\tilde{X} \cap Sing\,(\mathbb{P}(w_0, w_1, w_2, 1))$ and they are isolated. We can see that S_T is a K3 surface by the adjunction formula. We remark that the smoothing $X \to \Delta$ and its partial resolution $\tilde{X} \to X \to \Delta$ has been studied by Yonemura [73], Ishii, and Tomari in the context of simple K3 surface singularities. Several of our claims follow by their work. In particular, they show that the weighted blow up is the terminal modification of X, and that in fact it is unique if X is defined by a generic polynomial (see [73, Thm. 3.1]). The relationship between unimodal singularities and Yonemura's classification is described, in another context, by Prokhorov [57].

We find the value of $\left(S_1\big|_{S_T}\right)^2$ by using Lemmas 3.5.14 and 3.5.11. From adjunction formula, the fact that S_T is a normal K3 surface, and that the fibers are numerically equivalent we have:

$$K_{\tilde{X}}\big|_{S_1} = K_{S_1} + S_T\big|_{S_1} \qquad\qquad K_{\tilde{X}}\big|_{S_T} = K_{S_T} + S_1\big|_{S_T} = S_1\big|_{S_T}$$

The surface S_T holds $Pic(S_T) \cong \mathbb{Z}$. Therefore, $K_{\tilde{X}}\big|_{S_T}$ is ample because its degree is positive. The ampleness of $K_{\tilde{X}}\big|_{S_1}$ follows from Lemma 3.2.1 $\square$

Remark 3.5.4. The previous statements are not longer true for all higher modal singularities. See Section 3.7 for details. Moreover, the weighted blow up of an arbitrary quasihomogeneous smoothing does not necessarily yields a threefold with canonical or terminal singularities (see Remark 3.5.6).

Next, we give an explicitly description of the central fiber $\tilde{X}|_0$ for each unimodal singularity. We describe the most details for the E_{12} singularity. The other cases are similar.

3.5.1 The E_{12} singularity

It is also known as $D_{2,3,7}$ or $Cu(-1)$. Its normal form is $x^2+y^3+z^7$, and its unimodal base change $t^{42} \to t$ induces the 20th case of Yonemura's classification. The surface S_1 supports the cyclic quotient singularities:

$$\mathcal{B}_{E_{12}} = \left\{ \frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{7}(1,1) \right\}$$

The possible dual sets are

$$\hat{\mathcal{B}}^1_{E_{12}} = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,2), \frac{1}{7}(1,5)\right\} \qquad \hat{\mathcal{B}}^2_{E_{12}} = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{7}(1,5)\right\}$$
$$\hat{\mathcal{B}}^3_{E_{12}} = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{7}(1,6)\right\} \qquad \hat{\mathcal{B}}^4_{E_{12}} = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,2), \frac{1}{7}(1,6)\right\}$$

The singularities on $\hat{\mathcal{B}}^4_{E_{12}}$ are the only ones that can be realized by a weighed blow up such that the induced surface $S_{42} \subset \mathbb{P}(21, 14, 6, w_3)$ is quasismooth and well-formed as on Definition 3.3.3. In that case $w_3 = 1$, and S_T is a K3 surface. The surfaces in the central fiber intersect along a rational curve $C = S_T \cap S_1$. (see Figure 3.5.4.1)

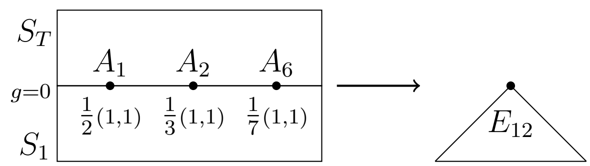


Figure 3.5.4.1: Analysis of the E_{12} singularity

3.5.2 The E_{13} singularity

It is also known as $D_{2,4,5}$ or $Ta(-2,-3)$. Its normal form is $x^2 + y^3 + yz^5$, and its unipotent smoothing $t^{30} \to t$ induces the 50th case on Yonemura's classification. The set of singularities are

$$\mathcal{B}_{E_{13}} = \left\{\frac{1}{2}(1,1), \frac{1}{4}(1,1), \frac{1}{5}(1,1)\right\} \qquad \hat{\mathcal{B}}_{E_{13}} = \left\{\frac{1}{2}(1,1), \frac{1}{4}(1,3), \frac{1}{5}(1,4)\right\}$$

The associated K3 surface is $S_{30} \subset \mathbb{P}(15, 10, 4, 1)$.

3.5.3 The E_{14} singularity

It is also known as $D_{3,3,4}$ or $Tr(-2,-2,-3)$. Its is normal form is $x^2+y^3+z^8$, and its unipotent base change $t^{24} \to t$ induces the 13th case on Yonemura's classification. The set of singularities are:

$$\mathcal{B}_{E_{14}} = \left\{2 \times \frac{1}{3}(1,1), \frac{1}{4}(1,1)\right\} \qquad \hat{\mathcal{B}}_{E_{14}} = \left\{2 \times \frac{1}{3}(1,2), \frac{1}{4}(1,3)\right\}$$

The associated K3 surface is $S_{24} \subset \mathbb{P}(12,8,3,1)$.

Remark 3.5.5. The set of weights $\mathbf{v} = (12,8,3,13)$ seems to induce the singularities of the set $\hat{\mathcal{B}}_{E_{14}}$. However, in this case the surface $S_{24} := (x^2 + y^3+z^8+yzt = 0) \subset \mathbb{P}(12,8,3,13)$ is not quasismooth. In fact, the surface has a A_1 singularity supported on the vertex P_3 which itself supports the singularity $\frac{1}{13}(1,5,10)$.

3.5.4 The U_{12} singularity

It is also known as $D_{4,4,4}$ or $Tr(-3,-3)$. Its normal form is $x^3 + y^3 + z^4$, and its unipotent base change $t^{12} \to t$ induces the 4th case on the Yonemura classification. The set of singularities are:

$$\mathcal{B}_{U_{12}} = \left\{3 \times \frac{1}{4}(1,1)\right\} \qquad\qquad \hat{\mathcal{B}}_{U_{12}} = \left\{3 \times \frac{1}{4}(1,3)\right\}$$

The associated K3 surface is $S_{12} \subset \mathbb{P}(4,4,3,1)$.

Remark 3.5.6. (see Remark 3.5.4) The smoothing induced by the weights $\mathbf{v} = (4,4,3,3)$ induces the exceptional surface

$$S_{12} := (x^3 + y^3 + z^4 + t^4 = 0) \subset \mathbb{P}(4,4,3,3)$$

supporting the singularities

$$\left\{3 \times \frac{1}{4}(1,1), 4 \times \frac{1}{3}(1,1)\right\}$$

The threefold supports the singularities $\frac{1}{4}(1,1,1)$. Next, we apply the Reid-Tai criterion to the associated group generator $\epsilon_4(x,y,z) \to (\epsilon_4 x, \epsilon_4 y, \epsilon_4 z)$. The age of ϵ_4 (see [38, 105]) is $3/4 < 1$ which implies the singularity is not canonical.

Remark 3.5.7. The set of weights $\mathbf{v} = (4,4,3,9)$ induces the quasismooth surface $S_T := (x^3 + y^3 + z^4 + zt = 0) \subset \mathbb{P}(4,4,3,9)$. The induced singularities on S_T are

$$\left\{3 \times \frac{1}{4}(1,3), \frac{1}{3}(1,1), \frac{1}{9}(1,1)\right\}$$

the problem here is that we have additional non DuVal singularities on our exceptional tail.

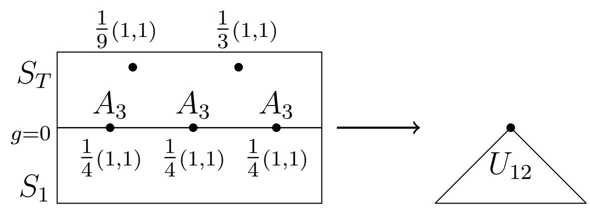


Figure 3.5.7.1: Remark 3.5.7 about the U_{12} singularity

3.5.5 The W_{12} singularity

It is also known as $D_{2,5,5}$ or $Ta(-3,-3)$. Its is normal form is $x^2 + y^4 + z^5$, its unipotent base change $t^{20} \to t$ induces the 9th case on Yonemura's classification. The set of singularities are:

$$\mathcal{B}_{W_{12}} = \left\{ \frac{1}{2}(1,1), 2 \times \frac{1}{5}(1,1) \right\} \qquad \hat{\mathcal{B}}_{W_{12}} = \left\{ \frac{1}{2}(1,1), 2 \times \frac{1}{5}(1,4) \right\}$$

where $\hat{\mathcal{B}}_{W_{12}}$ is supported on the K3 surface $S_{20} \subset \mathbb{P}(10,5,4,1)$.

3.5.6 The W_{13} singularity

It is also known as $D_{3,4,4,}$ or $Tr(-2,-3,-3)$. Its normal form is $x^2 + y^4 + yz^4$, and its unipotent smoothing $t^{16} \to t$ induces the 37th case on Yonemura's classification. The set of singularities are

$$\mathcal{B}_{W_{13}} = \left\{ \frac{1}{3}(1,1), 2 \times \frac{1}{4}(1,1) \right\} \qquad \hat{\mathcal{B}}_{W_{13}} = \left\{ \frac{1}{3}(1,2), 2 \times \frac{1}{4}(1,3) \right\}$$

The associated $K3$ surface is $S_{16} \subset \mathbb{P}(8,4,3,1)$.

Remark 3.5.8. The smoothing obtained by taking linear combination of monomials of degree 16 with respect the weights $(8,4,3,13)$ is associated to the set of singularities

$$\left\{ \frac{1}{3}(1,2), 2 \times \frac{1}{4}(1,3), \frac{1}{13}(1,7) \right\}$$

Then we discard this smoothing because the presence of a non DuVal singularity on the surface S_T

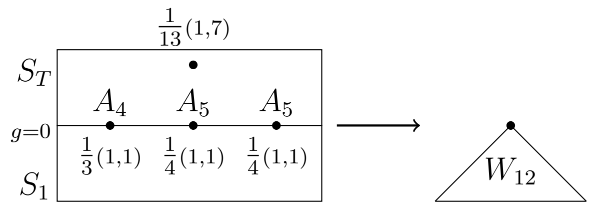


Figure 3.5.8.1: Remarl 3.5.8 about the W_{13} singularity

3.5.7 The Q_{10} singularity

It also known as $D_{2,3,9}$ or $Cu(-3)$. Its normal form is $x^2z+y^3+z^4$, its unipotent base change $t^{24} \to t$ induces the 20th case on Yonemura classification. The set of singularities are

$$\mathcal{B}_{Q_{10}} = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{9}(1,1)\right\} \qquad \hat{\mathcal{B}}_{Q_{10}} = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,2), \frac{1}{9}(1,8)\right\}$$

The associated K3 surface is $S_{24} \subset \mathbb{P}(9,8,6,1)$.

Remark 3.5.9. Consider another smoothing constructed by taking linear combination of monomials of degree 24 with respect the weights $(9,8,6,5)$. The associated exceptional surface has singularities:

$$\left\{\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{9}(1,4), \frac{1}{5}(1,3)\right\}$$

after taking another base change $t^5 \to t$, we will induce the smoothing $(x^2z + y^3+z^4+xt^{15} = 0)$ which a non generic smoothing on the Q_{10} versal deformation space.

3.5.8 The Q_{11} singularity

also known as $D_{2,4,7}$ or $Ta(-2,-5)$. Its associated equation is $x^2z + y^3x + yz^3$, its unipotent base change $t^{18} \to t$ induces the 60th case on Yonemura classification. The sets of singularities are

$$\mathcal{B}_{Q_{11}} = \left\{\frac{1}{2}(1,1), \frac{1}{4}(1,1), \frac{1}{7}(1,1)\right\} \qquad \hat{\mathcal{B}}_{Q_{11}} = \left\{\frac{1}{2}(1,1), \frac{1}{4}(1,3), \frac{1}{7}(1,6)\right\}$$

the associated K3 surface is $S_{18} \subset \mathbb{P}(7,6,4,1)$.

3.5.9 The Q_{12} singularity

It also known as $D_{3,3,6}$ or $Tr(-2,-2,-5)$. Its normal form is $x^2z + y^3 + z^5$, and its unipotent smoothing $t^{15} \to t$ induces the 22th case on Yonemura's classification. The set of singularities are:

$$\mathcal{B}_{Q_{12}} = \left\{2 \times \frac{1}{3}(1,1), \frac{1}{6}(1,1)\right\} \qquad \hat{\mathcal{B}}_{Q_{12}} = \left\{2 \times \frac{1}{3}(1,2), \frac{1}{6}(1,5)\right\}$$

the associated K3 surface is $S_{15} \subset \mathbb{P}(6,5,3,1)$.

Remark 3.5.10. A general smoothing defined by a linear combination of monomials of degree 15 with respect the weights $(6,5,3,2)$ is given by

$$x^2z + y^3 + xz^3 + z^5 + y^2zt + (xy + yz^2)t^2 + (xz + z^3)t^3 + yt^5 + zt^6$$

Therefore, the edge P_0P_3 given by $(y = z = 0)$ is contained on S_T. However, this edge is a line of A_1 singularities so the surface S_T is not normal. Note that the other singularities on S_T are dual to the ones on $\mathcal{B}_{Q_{12}}$

$$\hat{\mathcal{B}}_{Q_{12}} = \left\{2 \times \frac{1}{3}(1,1), \frac{1}{6}(1,4)\right\}$$

3.5.10 The S_{11} singularity

It is also known as $D_{2,5,6}$ or $Ta(-3,-4)$. Its normal form is $x^2z + y^2x + z^4$, and its unipotent base change $t^{16} \to t$ induces the 58th case on Yonemura classification. The set of singularities are:

$$\mathcal{B}_{S_{11}} = \left\{\frac{1}{2}(1,1), \frac{1}{5}(1,1), \frac{1}{6}(1,1)\right\} \qquad \hat{\mathcal{B}}_{S_{11}} = \left\{\frac{1}{2}(1,1), \frac{1}{5}(1,4), \frac{1}{6}(1,5)\right\}$$

The associated K3 surface is given by $S_{16} \subset \mathbb{P}(6,5,4,1)$

3.5.11 The singularity S_{12}

It is also known as $D_{3,4,5}$ or $Tr(-2,-3,-4)$. Its normal form is $x^2z + xy^2 + yz^3$ and its unipotent base change $t^{13} \to t$ induces the 87th case on Yonemura classification. The set of singularities are:

$$\mathcal{B}_{S_{12}} = \left\{\frac{1}{3}(1,1), \frac{1}{4}(1,1), \frac{1}{5}(1,1)\right\} \qquad \hat{\mathcal{B}}_{S_{12}} = \left\{\frac{1}{3}(1,2), \frac{1}{4}(1,3), \frac{1}{5}(1,4)\right\}$$

The associated K3 surface is $S_{13} \subset \mathbb{P}(5,4,3,1)$

3.5.12 The singularity Z_{11}

It is also known as $D_{2,3,8}$ or $Cu(-2)$. Its normal form is $x^2 + y^3z + z^5$ and its unipotent base change $t^{30} \to t$ induces the 38th case on the Yonemura smoothing. The set of singularities are:

$$\mathcal{B}_{Z_{11}} = \left\{ \frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{8}(1,1) \right\} \qquad \hat{\mathcal{B}}_{Z_{11}} = \left\{ \frac{1}{2}(1,1), \frac{1}{3}(1,2), \frac{1}{8}(1,7) \right\}$$

The associated K3 surface is $S_{30} \subset \mathbb{P}(15, 8, 6, 1)$

3.5.13 The singularity Z_{12}

It is also known as $D_{2,4,6}$ or $Ta(-2, -4)$. Its normal form is $x^2 + y^3z + yz^4$, and its unipotent base change $t^{22} \to t$ induces the 78th case of the Yonemura classification. The set of singularities are:

$$\mathcal{B}_{Z_{12}} = \left\{ \frac{1}{2}(1,1), \frac{1}{4}(1,1), \frac{1}{6}(1,1) \right\} \qquad \hat{\mathcal{B}}_{Z_{12}} = \left\{ \frac{1}{2}(1,1), \frac{1}{4}(1,3), \frac{1}{6}(1,5) \right\}$$

The associated K3 surface is $S_{22} \subset \mathbb{P}(11, 6, 4, 1)$

3.5.14 The singularity Z_{13}

It is also known as $D_{3,3,5}$ or $Tr(-2, -2, -4)$. Its normal form is $x^2 + y^3z + z^6$ and its unipotent base change $t^{18} \to t$ induces the 39th case on the Yonemura's classification.

$$\mathcal{B}_{Z_{13}} = \left\{ 2 \times \frac{1}{3}(1,1), \frac{1}{5}(1,1) \right\} \qquad \hat{\mathcal{B}}_{Z_{13}} = \left\{ 2 \times \frac{1}{3}(1,2), \frac{1}{5}(1,4) \right\}$$

The associated K3 surface is $S_{18} \subset \mathbb{P}(9, 5, 3, 1)$,

3.5.15 The line bundle of the exceptional surface

The following lemmas are used to prove the ampleness of the line bundle on Theorems 3.5.1 and 3.6.1.

Lemma 3.5.11. *Let $\tilde{S}_T$ be the smooth model of S_T, let E_k be the exceptional divisors associated to the resolution $\varphi : \tilde{S}_T \to S_T$, and let $\tilde{C}$ be the proper transform of $C = S_1\big|_{S_T}$ on $\tilde{S}_T$. Then, it holds:*

$$C^2 = (\varphi^*(C))^2 = \tilde{C}^2 - \sum_{j,k} (E_j.\tilde{C}) D_{jk} (E_k.\tilde{C})$$

where D_{jk} is the inverse of the intersection matrix E_jE_k. In particular, if all the singularities on $\hat{\mathcal{B}}_{S_0}$ are of type $\{A_{k_1}, \dots A_{k_m}\}$, then

$$C^2 = \tilde{C}^2 + \sum_{j=1}^{j=m} \frac{k_j}{k_j+1} \tag{3.5.12}$$

Proof. By the projection formula, we have $\varphi^*(C).E_k = 0$ for every exceptional divisor E_k. This implies

$$C^2 = \varphi^*(C).\left(\tilde{C} + \sum_j a_j E_j\right) = \tilde{C}^2 + \sum_j a_j(E_j.\tilde{C})$$

On other hand $\varphi^*(C).E_k = 0$ implies $\sum_j a_j E_j.E_k = -\tilde{C}.E_k$ where $E_j.E_k$ is the intersection matrix. Then

$$a_j = \sum_k D_{jk}(-\tilde{C}.E_k)$$

where D_{jk} is the inverse of the intersection matrix E_jE_k. The previous expressions imply

$$C^2 = \tilde{C}^2 + \sum_j \left(\sum_k D_{jk}(-\tilde{C}.E_k)\right) E_j.\tilde{C} = \tilde{C}^2 - \sum_{j,k}(E_j.\tilde{C})D_{jk}(E_k.\tilde{C})$$

Let T_i be the cyclic quotient singularities of S_T supported on the curve C; and let $E(T_i)$ be the intersection matrix of each singularities T_i. Then,

$$D_{j,k} = \begin{pmatrix} E(T_1)^{-1} & \cdots & 0 \\ 0 & E(T_2)^{-1} & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & E(T_m)^{-1} \end{pmatrix}$$

In particular, the exceptional divisors associated to each T_i do not intersect. Therefore, we can consider the contribution of each singularity T_i independently:

$$\sum_{j,k}(E_j.\tilde{C})D_{jk}(E_k.\tilde{C}) = \sum_{E_j,E_k\in R_1}(E_j.\tilde{C})(E(T_1)^{-1})_{jk}(E_k.\tilde{C}) + \dots$$
$$\dots + \sum_{E_j,E_k\in R_m}(E_j.\tilde{C})(E(T_m)^{-1})_{jk}(E_k.\tilde{C})$$

where $E_j \in R_i$ means that E_j is an exceptional divisor associated to T_i. Only the first exceptional divisor of the resolution of T_i intersects the curve $\tilde{C}$ at a point. Therefore,

$$\sum_{E_j,E_k\in R_i}(E_j.\tilde{C})(E(T_i)^{-1})_{jk}(E_k.\tilde{C}) = \left(E(T_i)^{-1}\right)_{1,1}$$

If T_i is an A_k singularity, then by the configuration of its exceptional curves and by writing D_{jk} in terms of the matrix of cofactors, we have:

$$(E(A_k)^{-1})_{1,1} = -\frac{k}{k+1}.$$

From which our statement follows. □

The following is a well known result on degeneration of surfaces.

Lemma 3.5.13. *Let $Y \to \Delta$ be an one dimensional degeneration of surfaces such that Y_t is smooth and $Y_0 = \sum_i n_i V_i$. Denote by C_{ij} the double curve $V_i|_{V_j} \subset V_i$, and the triple point intersection $T_{ijk} = V_i \cap V_j \cap V_k$. Then, we have*

$$N^{n_i}_{V_i|Y} = \mathcal{O}_{V_i}\left(-\sum_{j\neq i} n_j C_{ij}\right)$$
$$V_i^2 V_j = C_{ji}^2$$
$$n_j C_{ij}^2 + n_i C_{ji}^2 = -\sum_{k\neq i,j} n_k |T_{ijk}|$$

On the Expression 3.5.12, we need to find the value of $\tilde{C}^2$. That is the purpose of the following result.

Lemma 3.5.14. *Let Y be a smooth model of the unipotent degeneration $X \to \Delta$, so its central fiber has reduced components $Y|_0 = \sum V_i$. Let $\tilde{S}_T$ be the proper transform of S_T on $Y|_0$, and let $\tilde{C}$ be the proper transform of $C = S_1\big|_{S_T}$ on*

$\tilde{S}_T$. Consider the set of singularities

$$B_{S_0} = \left\{ \frac{1}{\alpha_k}(1, \beta_k) \right\}$$

Then, it holds

$$\tilde{C}^2 = \frac{\deg_{\tilde{w}}(S_0)}{w_0 w_1 w_2} + \sum_k \frac{\beta_k}{\alpha_k} - n_p$$

where n_p is the number of cyclic quotient singularities supported on C. In particular, for Fuchsian hypersurface singularities we have:

$$\tilde{C}^2 = \frac{\deg_{\tilde{w}}(S_0)}{w_0 w_1 w_2} + \sum_k \frac{1}{k+1} - n_p$$

Proof. Let $\tilde{S}_1$ be the proper transform of S_1 on $Y\big|_0$ then it holds that $\tilde{S}_1 \cap \tilde{S}_T$ support n_p triple points where the extra surfaces are exceptional divisors of the cyclic quotient singularities supported on C. Let $\hat{C}$ be the proper transform of C on $\tilde{S}_1$, by Lemma 3.5.13, it holds that:

$$\tilde{C}^2 + \hat{C}^2 = -\sum_{k \neq i,j} |T_{ijk}| = -n_p$$

By [55, Thm 3.6.1] we have that

$$-\hat{C}^2 = \frac{\deg_{\mathbf{w}}(f)}{w_0 w_1 w_2} + \sum_k \frac{\beta_k}{\alpha_k}$$

so the statement follows from it. In the case of Fuchsian hypersurface singularities $\beta_k = 1$ (see Example 3.1.7). □

3.6 Singularities with Higher modality

A common phenomenon in the case of unimodal singularities is that the smoothing X of S_0 has a strict log canonical threefold singularity. Moreover, the singularity at X belongs to a family called simple $K3$ singularities. Yonemura [73] classified those singularities, and he showed that they are in bijection with the list of 95 normal K3 surfaces that appear as a hypersurfaces in a weighted projective space. These surfaces are all the families of weighted projective Gorenstein $K3$ hypersurfaces and they were classifeid by M. Reid in 1979. By exploiting this relationship, we can expand the proofs from previous sections.

Theorem 3.6.1. *Let S_0 be one of the* 53 *quasihomogeneous surface singularities which unipotent smoothing X induces a simple K3 threefold singularity of weight $\boldsymbol{w}$. Let $\tilde{X}$ be the proper transform of X under the weighted blow up $\pi_{\boldsymbol{w}}$. Then it holds that:*

1. *The threefold $\tilde{X}$ has isolated terminal cyclic quotient singularities.*
2. *The central fiber $\tilde{X}_0$ has orbifold double normal crossing singularities and it decomposes in two surfaces S_1 and S_T. The divisor $S_1\big|_{S_T} \subset S_T$ satisfies:*
$$\left(S_1\big|_{S_T}\right)^2 = \frac{\deg_{\mathbf{w}}(f)}{w_0 w_1 w_2}$$
3. *S_1 is the proper transform of S_0 and it supports the singularities in $\mathcal{B}_{S_0}$.*
4. *The line bundles $K_{\tilde{X}}|_{S_1}$ and $K_{\tilde{X}}|_{S_T}$ are ample.*

Proof. Let (w_0, w_1, w_2) be the associated weights of S_0. By construction, the smoothing X has a simple $K3$ singularities with weights

$$\mathbf{w} = (w_0, w_1, w_2, 1)$$

The quasismothness of X implies that $\tilde{X}$ has only cyclic quotient singularities (see [17, Lemma 8]). Moreover, the weighted blow up $\pi_{\mathbf{w}}$ is the terminal modification of X by [73, Thm 3.1].

By definition of a simple $K3$ singularity, S_T is a normal $K3$ surface with at worst A_k singularities [73, Thm 4.2]; and it is realized as a hypersurface of degree $\deg_{\mathbf{w}}(f)$ in $\mathbb{P}(w_0, w_1, w_2, 1)$. From the Table [73, Table 4.6], there are 53 of those singularities. The fact that the central fiber $\tilde{X}|_0$ has double normal crossing singularities follows from the construction of $\tilde{X}$ through a weighted blow up.

We find the value of $\left(S_1\big|_{S_T}\right)^2$ by using Lemmas 3.5.14 and 3.5.11 (using the same argument than on the proof of Theorem 3.5.1). Indeed, from adjunction formula, the fact that S_T is a normal K3 surface, and that the fibers are numerically equivalent we have:

$$K_{\tilde{X}}\big|_{S_1} = K_{S_1} + S_T\big|_{S_1} \qquad\qquad K_{\tilde{X}}\big|_{S_T} = K_{S_T} + S_1\big|_{S_T} = S_1\big|_{S_T}$$

The $K3$ surface holds $Pic(S_T) \cong \mathbb{Z}$. Therefore, $K_{\tilde{X}}\big|_{S_T}$ is ample because its degree is positive. The ampleness of $K_{\tilde{X}}\big|_{S_1}$ follows from Lemma 3.2.1 $\square$

Remark 3.6.2. These 53 quasihomogeneous singularities where first described in another context by Prokhorov [57].

The existence of a semistable resolution is settled for the following results.

Definition 3.6.3. *A germ $x \in X$ (resp. a variety X) admits a semistable resolution if there is a resolution $f : Z \to X \to \Delta$ such that it central fiber $Z|_0$ is a reduced smooth normal crossing divisor (this means $Z \to \Delta$ is semistable)*

Definition 3.6.4. *([7, Def 3.4]) Let X be a threefold, $f : X \to \Delta$ a not necessarily projective morphism. Let $t \in \mathcal{O}$ be a parameter. We say that f has moderate singularities if the analytic germ at every point $x \in X$ is isomorphic to one of the following germs*

1. $(xyz = t) \subset \mathbb{C}^4$
2. $(xy = t) \subset A$ *where* $A = \frac{1}{r}(a, r - a, 1, 0)$ *for some* $gcd(a, r) = 1$.
3. $(xy = z^r + t^n) \subset A$ *for some n, with A as above.*

Lemma 3.6.5. *([7, Lemma 5.2]) Let $x \in X$ be a moderate 3-fold singularity. Then $x \in X$ admits a projective semistable resolution.*

Proposition 3.6.6. *The threefold $\tilde{X}$ obtained in Theorem 3.5.1 and 3.6.1 has moderate 3-fold singularities. Therefore, it admits a projective semistable resolution.*

Proof. The singularities of the threefold $\tilde{X}$ are induced solely by the weighted blow up $\pi_{\mathbf{w}} : \tilde{X} \to X$ and they correspond to the cyclic quotient singularities associated to the set of surface singularities $\mathcal{B}_{S_0}$ and $\hat{\mathcal{B}}_{S_0}$. Our result follows from boting that the singularities at $\tilde{X}$ correspond to the second case of Definition 3.6.4.

□

3.7 Further Examples

Next, we discuss several examples of singularities which behaviour is different to the unimodal ones. We draw our examples from minimal elliptic singularities.

3.7.1 The V'_{18} singularity

It is also known as $4A_{1,-2,o}$; this is a minimal elliptic singularity with Milnor number 18 and modality 4. Its normal form is $x^3+y^4+z^4$, and its set of cyclic quotient singularities is $\mathcal{B}_{V'_{18}} = 4A_2$. There are two dual sets such that can be realized by a weighted blow up:

$$\hat{\mathcal{B}}^1_{V'_{18}} = 4 \times \frac{1}{3}(1,1) \qquad\qquad \hat{\mathcal{B}}^2_{V'_{18}} = 4 \times \frac{1}{3}(1,2)$$

By Theorem 3.4.1, the smoothing associated to the dual set $\hat{\mathcal{B}}^1_{V'_{18}}$ is realized by a linear combination of the monomials of weight 12 with respect the weights $\mathbf{v} = (4,3,3,1)$. This is the unipotent smoothing, the threefold $\tilde{X}$ has singularities $\frac{1}{3}(1,1,2)$, and the exceptional surface is $S_{12} \subset \mathbb{P}(4,3,3,1)$.

On other hand, the smoothing associated to the dual set $\hat{\mathcal{B}}^2_{V'_{18}}$ is realized by a linear combination of the monomials of degree 12 with respect the weights $\mathbf{u} = (4,3,3,2)$. This threefold has a strictly log canonical singularity, and it corresponds to 2nd case on Yonemura's classification. The exceptional surface is the normal $K3$ surface $S_{12} \subset \mathbb{P}(4,3,3,2)$ with singularities $4A_2 + 3A_1$.

3.7.2 The E_{20} singularity.

It is also known as $E_{8,-3}$. It is a minimal elliptic singularity with Milnor number 20 and modality 2. Its normal form is $x^2+y^3+z^{11}$ and its set of cyclic quotient singularities is

$$\mathcal{B}_{E_{20}} = \left\{ \frac{1}{2}(1,1), \frac{1}{3}(1,2), \frac{1}{11}(1,9) \right\}$$

This set has four sets of dual singularities. However, only two of them have an associated smoothing (see Example 3.1.11).

1. The dual set:
$$\hat{\mathcal{B}}^1_{E_{20}} = \left\{ \frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{11}(1,10) \right\}$$
which associated smoothing $X_1 \to \Delta$ is realized by a linear combination of the monomials of degree 66 with respect to the weights $\mathbf{v} = (33,22,6,5)$. The threefold $X_1 \subset \mathbb{C}^4$ has a strictly log canonical singularity and it correspond to 46th case of Yonemura's classification. The exceptional surface is the normal K3 surface $S_{66} := (x^2+y^3+z^{11}+t^{12}z = 0) \subset \mathbb{P}(33,22,11,5)$

2. The smoothing associated to the dual set

$$\hat{\mathcal{B}}_{E_{20}} = \left\{\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{11}(1,2)\right\}$$

is the unipotent one. In that case, the surface $S_{66} \subset \mathbb{P}(33,22,6,1)$ is an non minimal K3 surface (see also [28, Sec 3.3.3])

Chapter 4

Application to the moduli of quintics

4.1 Preliminary

Kollár, Shepherd-Barron and Alexeev developed a geometric compactification of surfaces of general type. They constructed a coarse moduli space for those schemes S satisfying the following properties (see [39, 5.4]):

1. S is a reduced projective surface
2. S is connected with only semi log canonical singularities
3. Let $j : S_0 \to S$ be the inclusion of the locus S_0 of Gorenstein points of S, the sheaf $\omega_S^{[N]}$ defined by $\omega_S^{[N]} = j_* \left(\omega_{S_0}^{\otimes N}\right)$ is an ample line bundle.
4. The self intersection
$$K_S^2 := \frac{\omega_S^{[N]} \cdot \omega_S^{[N]}}{N^2}$$
is constant.
5. $\chi(\mathcal{O}_B) = p_g - q + 1$ is constant.

The moduli functor $\mathcal{M} : (Schemes) \to (Sets)$ such that $\mathcal{M}(T)$ is the set of *isomorphic classes of flat projective morphisms $f : \mathcal{S} \to T$ such that all above properties hold for every geometric fibre of f, and for every geometric point $t \in T$ the natural map*

$$j_* \left(\omega_{S_0/T}^{\otimes N}\right) \otimes k(t) \to \omega_{\mathcal{S}_t}^{[N]}$$

is an isomorphism.

This moduli functor has a coarse moduli space which is a projective scheme of finite type over $\mathbb{C}$ [36]. We highlight that the appropriate definition of a family of stable varieties of surfaces is very a delicate problem and is not completely settled. However, there is an agreement on what sort of surfaces should be allowed in the moduli functor (Definition 4.1.1); the contentious topic is the right notion of families. We do not discuss the definition of families in this thesis, we rather focus in constructing sets of numerical quintic stable surfaces which are parametrized by a codimension one loci in the KSBA space.

Definition 4.1.1. *A stable surface is a connected projective surface S such that S has semi log canonical singularities and the dualizing sheaf ω_X is ample.*

We construct stable surfaces by taking the KSBA stable replacement of quintic surface with distinguished singularities as in Chapter 3. For measuring the codimension of the families of surfaces, we use the following invariants.

Definition 4.1.2. *Let $0 \in S_0$ be an isolated surface singularity analytically equivalent to $(0, f(x,y,z) = 0)$. Then, its Milnor algebra is given by*

$$A(f) = \left(\frac{\mathbb{C}[[x,y,z]]}{Jacobian(f)} \right)$$

The Tjurina number $\tau_0(S_0)$ and the Milnor number $\mu_0(S_0)$ are given by

$$\tau_0(S_0) = \dim_{\mathbb{C}} (A(f)) \qquad\qquad \mu_0(S_0) = \dim_{\mathbb{C}} \left(\frac{A(f)}{(f)} \right)$$

The Milnor and Tjurina number are finite if and only if S_0 has an isolated singularity. For a quasihomogeneous non degenerated isolated singularity $\tau_0(S_0) = \mu_0(S_0)$, but in general $\tau_0(S_0) \leq \mu_0(S_0)$. On other hand, the number of monomials in a monomial base of $A(f)$ with weighted degree larger than or equal to $\mathbf{w}(f)$ is known as the the inner modality of f.

The Milnor number helps us to bound the dimension of the loci associated to a quintic surface with a given singularity.

Proposition 4.1.3. *([22, pg. 373], [4, pg 245]) The modality of the function $f(x,y,z)$ with respect to right-equivalence is equal to the dimension of the μ-constant stratum of $f(x,y,z)$ in the semi-universal deformation of f. Moreover, Kushnirenko and Gabrielov proved that for semi-quasihomogeneous singularities the modality is equal to the inner modality ([4, pg 222]).*

Remark 4.1.4. We are interested in the stratum with constant Milnor Number, because it is known by results of Teissier, Perron, Dung, et al, that for given a family of embedded hypersurfaces a necessary and sufficient condition for them to be all homeomorphic, as embedded varieties, is to have constant Milnor number. Indeed, we say that *a deformation is upper* if it can be written as

$$f(x, y, x) + \sum_{x^i y^j z^k \in A(S_0)} t_{ijk} x^i y^j z^k$$

where $w(x^i y^j z^k) \geq w(f)$. Varchenko proved that for a quasihomogeneous polynomial with an isolated singularity a deformation is μ-constant if and only if it is an upper deformation [5, pg 292].

Example 4.1.5. The V_{15} singularity is a trimodal singularity, and its normal form is $x^2y + y^4 + z^4$. A monomial base of its versal deformation is

$$A(V_{15}) = \{y^3z^2, y^2z^2, yz^2, xz^2, z^2, y^3z, y^2z, yz, xz, z, y^3, y^2, y, x, 1\}_{\mathbb{C}}$$

The singularity V_{15} is quasihomogeneous with respect to the weights $w(x) = 3$, $w(y) = 2$, $w(z) = 2$, and its weighted degree is $w(f) = 8$. We find that $m(f) = 3$. Then, its upper deformations can be written as:

$$x^2y + y^4 + z^4 + c_1y^3z + c_2y^2z^2 + c_3y^3z^2,$$

Remark 4.1.6. The concept of equivalence between singularities is a delicate one. We bypass many of the subtleties because we work with minimal elliptic quasihomogeneous singularities. Indeed, right and contact equivalence are the same in our case [22, Lemma 2.13], and the equations of the minimal elliptic singularities are determined by their resolution graph and their topological type [42]. In particular, we avoid the delicate issue of equisingular deformation for surface singularities, by restricting ourselves to quasihomogeneous singular ones.

Lemma 4.1.7. *Let $\mathcal{U}(S_0) \subset \overline{\mathcal{M}}_5^{GIT}$ be the loci that generically parametrizes a surface with a unique quasihomogeneous singularity S_0. Then it holds that*

$$\dim(\mathcal{U}(S_0)) \leq 40 - \mu_0(S_0) + m(S_0)$$

and the equality holds if there are not local-to-global obstructions to the deformation of S_0.

Proof. We can bound the dimension of $\mathcal{U}(S_0)$ by a local analysis of the singularity. The dimension of the versal deformation of the singularity at S_0 is equal to its Milnor Number $\mu_0(S_0)$. We also take into account the loci where the singularity deforms into itself. By our previous discussion, this loci corresponds to the upper deformations of S_0, and it dimension is the modality of the singularity $m(S_0)$. $\square$

The following result from E. Shustin and I. Tyomkin implies that local deformations of the singularity are realized by global deformations of the hypersurface $S \subset \mathbb{P}^3$ if the Milnor number of the singularity is smaller than 16. This will be a key tool for counting moduli dimensions of our boundary loci.

Proposition 4.1.8. *([62, Thm 1]) Denote $V_d^n(S_1, \dots S_r)$ the set of hypersurface of degree d in $\mathbb{P}^n$ having r isolated singular points of types $S_1, \dots, S_r$ respectively with Tjurina numbers $\tau_0(S_i)$. If $n \geq 2$ and*

$$\sum_{i=1}^{r} \tau_0(S_i) < \max\{9, 4d - 4\}$$

Then $V_d^n(S_1, \dots, S_r)$ is a smooth variety of dimension

$$\binom{d+n}{n} - \sum_i \tau_0(S_i) - 1$$

and the germ of the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$ at any $F \in V_d^n(S_1, \dots S_r)$ is a joint versal deformation of all singular points of F.

Remark 4.1.9. From the classification of quasihomogeneous singularities due to Arnold [3] and Suzuki [76], we find that there are 14 exceptional unimodal, 5 bimodal, and one trimodal quasihomogeneous singularity with Milnor number less than 16. These singularities are not log canonical ones, but they are GIT stable (for a list see Theorem 4.2.4)

Example 4.1.10. The converse does not hold. Indeed, let S_0 be a generic quintic surface with a N_{16} singularity (N.B. the normal form of this singularity is $x^2 + y^5 + z^5$). The general equation of a quintic surface with a non DuVal double point singularity is

$$\begin{aligned} F_{S_0}(x, y, z, 1) := x^2(1 + f_1(x, y, z) + h_2(x, y, z) + h_3(x, y, z)) + \\ + x g_2(y, z) + x f_3(y, z) + x h_4(y, z) + f_4(y, z) + f_5(y, z) \end{aligned}$$

The existence of a N_{16} singularity clearly implies that $f_4(y, z) = x g_2(y, z) = 0$, while there is not restriction over the other terms. Let $g \in Aut(\mathbb{P}^3)$ be an automorphism that fixes the form of the equation. Then $g * p_0 = p_0$ and it acts on the variables as

$$g * [x_0 : x_1 : x_2 : x_3] \to [x_0 : g_1(x_0, x_1, x_2) : h_1(x_0, x_1, x_2) : x_3]$$

The generic form of the associated $g \in SL(4, C)$ is such that:

$$\begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ g_1(x_0, x_1, x_2) \\ h_1(x_0, x_1, x_2) \\ x_3 \end{pmatrix}$$

Counting parameters we find that F_{S_0} is given by 35 parameters (N.B. a general quintic surface depends of 56 parameters). The group of automorphism that fixes the equation F_{S_0} is 7 dimensional. Therefore, the loci $\mathcal{U}_{N_{16}}$ that generically parametrizes quintic surfaces with a N_{16} singularity is 27 dimensional. The Milnor number of this singularity is $\mu_0(N_{16}) = 16$, and its modality is $m(N_{16}) = 3$. Therefore $\mathcal{U}_{N_{16}}$ has the expected dimension by Lemma 4.1.7. We highlight that we can obtain the upper deformations of N_{16}:

$$x^2 + y^5 + z^5 + c_1 y^3 z^2 + c_1 y^2 z^3 + c_3 y^3 z^3$$

by considering the completion of the local ring of a singularity such as

$$x^2 + y^5 + z^5 + c_1 y^3 z^2 + c_1 y^2 z^3 + c_3 x(y^3 + z^3)$$

For the case of curves, it holds the following result from Hassett

Theorem 4.1.11. *([26, Thm 2.11]) Let C_0 be an isolated plane curve singularity. Then the equisingular deformation of C_0 is smooth. For the singularities $x^p = y^q$ it may be represented as*

$$y^p = x^p + \sum t_{ij} x^i y^j$$

where $0 \leq i \leq p-2$, $0 \leq j \leq q-2$, and $qi + pj \geq pq$.

4.2 Boundary divisors

The description of the moduli space of numerical quintic surfaces, due to Horikawa [27], implies the existence of at least one divisor at the boundary of $\overline{\mathcal{M}}_5^{KSBA}$. This divisor parametrizes surfaces that are not realized as quintic surfaces in $\mathbb{P}^3$. If we use our analogy with $\overline{M}_3$, see Section 1, this divisor is similar to the one parametrizing hyperelliptic curves. Our purpose is to describe other divisors that are similar to $\Delta_1 \subset \overline{M}_3$.

Definition 4.2.1. *Let $\mathcal{S} \to \Delta$ be a family of quintic surfaces such that its generic member is a smooth surface, but its central fiber $\mathcal{S}|_0 = S_0$ has a non* *log canonical singularity which is GIT stable. The stable replacement of the family is a procedure for obtaining a family of stable surfaces $\mathcal{S}^c \to \Delta$ such that over the puncture risk, the families $\mathcal{S}^c$ and $\mathcal{S}$ coincide, but the central fiber $\mathcal{S}_0^c$ is a KSBA stable surface. In that case, we say that $\mathcal{S}_0^c$ is the KSBA stable replacement of S_0.*

Suppose that S_0 is a generic surface with a given non log canonical singularity. We denote as

$$D_5(S_0) \subset \overline{\mathcal{M}}_5^{KSBA}$$

the loci parametrizing all the possible stable replacement obtained of S_0.

Lemma 4.2.2. *Suppose the generic stable replacement of S_0 decomposes into the union of two surfaces S_1 and S_T such that:*

1. *$S_1 \cap S_0$ is a curve of genus g supporting $k \geq 3$ cyclic quotient singularities.*
2. *$(S_T, S_1|_{S_T})$ is a marked $K3$ surface with at worst A_k singularities.*
3. *The Picard group of the smooth model $\tilde{S}_T$ of S_T is $Pic(\tilde{S}_T)$.*

Then, it holds

$$D_d(S_0) \leq 63 - \mu_0(S_0) + m(S_0) - rank(Pic(\tilde{S}_T)) - 3g - k$$

Moreover, the equality is achieved whenever the quintic surfaces maps subjectively onto the versal deformation of S_0.

Proof. The moduli contribution from the surface S_1 is described in Lemma 4.1.7. The moduli of marked $K3$ surfaces was calculated by Dolgachev [11, Prop. 2.1], and it is given by:

$$\dim(\mathcal{M}(S_T, C) = 20 - r(Pic(\tilde{S}_T))$$

By construction, our $K3$ surfaces are hypersurfaces in a weighted projective space (see Remark 3.5.2) with at worstk A_k singularities [73, Thm 4.2]. Therefore, the Picard lattice of $\tilde{S}_T$ is generated by the exceptional curves on the resolution of the A_k singularities in S_T. (N.B. The Picard lattice associated to those K3 surfaces were described by Belcastro [6, Table 3]). On other hand, the intersection $S_1 \cap S_T$ is a marked curve of genus g. The k marked points support cyclic quotient singularities on the threefold $\tilde{X}$. From this discussion, we have that

$$\begin{aligned} D_d(S_0) &\leq \dim\left(\mathcal{U}(S_0)\right) + \dim\left(\mathcal{M}(S_T, C)\right) + \dim(M_{g,k}) \\ &\leq (40 - \mu_0(S_0) + m(S_0)) + (20 - r(Pic(\tilde{S}_T)) - (3g - 3 + k) \end{aligned}$$

from which the statement follows. □

Example 4.2.3. We illustrate the previous lemma with an example. Let S_0 be a surface with a N_{16} singularity (N.B its normal form is $z^2+x^5+y^5$). This is a trimodal singularity with Milnor number equal to 16. The surface S_T is a K3 surface whose smooth model $\tilde{S}_T$ has Picard rank equal to six (N.B $Pic(\tilde{S}_T)$ is generated by the -2 curves). The intersection $S_1 \cap S_T$ is a rational curve with 5 marked points (from supporting five A_1 singularities on S_T). We have

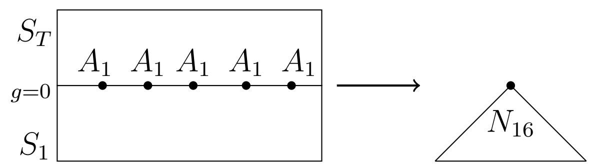


Figure 4.2.3.1: N_{16} singularity (see Example 4.2.3)

$D_5(N_{16})$=39, so it is expected to be a divisor in the KSBA space.

The main application of our result is the study of boundary divisors on the KSBA compactification of surfaces of general type. The following result was conjectured by J. Tevelev and J. Rana.

Theorem 4.2.4. *There are at least* 21 *smooth boundary divisors on the KSBA moduli space of numerical quintic surfaces associated to the stable replacement of quasihomogeneous minimal elliptic singularities with Milnor number less than* 16*. Among those singularities we find the unimodal or Dolgachev ones:*

E_{12}	E_{13}	E_{14}	Z_{11}	Z_{12}	Z_{13}	S_{11}
W_{12}	W_{13}	Q_{10}	Q_{11}	Q_{12}	U_{12}	S_{12}

the following bimodal singularities:

$$Z_{15} \qquad Q_{14} \qquad U_{14} \qquad W_{15} \qquad S_{14}$$

and the trimodal V_{15} *and* N_{16} *singularities.*

Remark 4.2.5. These singularities also belong to the family known as Fuchsian singularities.

Remark 4.2.6. There is a well known relationship between the mirror symmetry for $K3$ surfaces and the Arnold's strange duality. This is reflected here: let T_1 and T_2 be two unimodal exceptional singularities; let S_{T_i} be the associated $K3$ surfaces obtained from the KSBA stable replacement. Then T_1 is dual to T_2 if and only if S_{T_1} is the Mirror partner of S_{T_2}.

Proof. Let S_0 be a generic surface of degree 5 with a unique isolated singularity as the ones in the statement. By Theorems 3.5.1 and 3.6.1, the stable replacement of S_0 is found by taking the appropriate weighted blow up and its unipotent smoothing (see Figure 3.2.0.1). In all our cases, the semistable replacement of S_0 is the union of a surface of general type S_1 and a $K3$ surface S_T. Those surfaces intersect along a rational curve $S_1 \cap S_T$ supporting k cyclic quotient singularities. The number of marked points is $k = 3$ for unimodal singularities, $k = 4$ for bimodal singularities, and $k = 5$ for trimodal singularities. Therefore, by Lemma 4.2.2, the result follows if

$$\mu_0(S_0) - m(S_0) + rank(Pic(\tilde{S}_T)) + k = 24 \tag{4.2.7}$$

because the expected dimension of $D_5(S_0)$ is reached by Proposition 4.1.8 and Example 4.1.10. For the fourteen unimodal singularities, we use their alternative $D_{p,q,r}$ notation (see section 3.5). It is well known that on those cases:

$$\begin{aligned} \mu(D_{p,q,r}) &= 24 - p + q + r \\ rank(Pic(\tilde{S}_T)) &= p + q + r - 2 \\ k &= 3 \end{aligned} \tag{4.2.8}$$

Therefore, if S_0 is a $D_{p,q,r}$ singularity, it holds that $D_d(S_0) = 39$. For the bimodal singularities, it holds $k = 4$ (see [14, Table 2.]. The moduli of the marked $K3$ surfaces follows from Belcastro results [6, Table 3]: For Z_{15} and W_{15} we have that $rank(Pic(\tilde{S}_T)) = 7$. For Q_{14}, S_{14}, and U_{14}, we have that $rank(Pic(\tilde{S}_T)) = 8$. Finally, for trimodal singularities it holds $k = 5$.: For V_{15} we have that $rank(Pic(\tilde{S}_T)) = 7$. The N_{16} calculation is carried in Example 4.2.3. □

Theorem 4.2.9. *In the KSBA space, there are* 31 *other boundary loci parametrizing stable surfaces that decompose as the union of a K3 surface and a surface of general type.*

Proof. Let S_0 be a minimal elliptic singularity for which the unipotent smoothing induces a simple $K3$ threefold singularity. There are 53 of those singularities (see Table 4.6 at [73], and tables at [42]). By Theorem 3.6.1, the stable replacement of S_0 decomposes as the union of a surface of general type S_1 and a $K3$ surface S_T. □

Conjecture 4.2.10. *All the quintic surfaces with minimal elliptic singularities induce boundary loci which parametrizes a surface of general type and a K3 surface.*

Remark 4.2.11. A similar result for curves is part of the folklore: The only plane curve singularities which stable replacement decomposes as the union of the proper transform of the curve and an elliptic curve are the cusp, the tacnode and the ordinary triple point. They have a smoothing with associated unipotent monodromy that induces the only three simple elliptic surface singularities $\tilde{E}_r$. It is well known that the only exceptional curve of their canonical modification is an elliptic curve.

4.3 Birational Geometry of Quintic Surfaces

The Hilbert scheme of all quintic surfaces in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^{55}$, and it contains a Zariski open subset parametrizing surfaces of general type with invariants $p_g = 4$, $q = 0$ and $K_X^2 = 5$. Umezu [65] proved that any surface of general type birational to a quintic surface has irregularity zero. Let $V_{m,n} \subset \mathbb{P}^{55}$ be the set of quintic surfaces birational to a surface of general type with invariants $p_g = m$, $q = 0$ and $K_X^2 = n$. A detailed classification of the possible $V_{m,n}$ is due to Yang [71]. Next, we describe its main aspects:

1. The set $V_{3,2}$ is irreducible, 48 dimensional, and its generic point parametrizes quintic surfaces with a minimal elliptic triple point singularity ([71, Thm 9.4]).

2. The sets $V_{3,4}$ and $V_{3,3}$ are irreducible with 45 and 47 dimensions, respectively. Their generic points parametrize quintic surfaces with a minimal elliptic double point singularity ([71, Thm 10.1 and 10.2] and [78])

3. The sets $V_{2,3}$, $V_{1,2}$, $V_{1,1}$ and $V_{2,2}$ parametrize quintic surfaces which singular locus contains either a weakly elliptic double point singularity or a combination of minimal elliptic double point singularities [71, Table 8.1].

4. The set $V_{2,1}$ has two irreducible components of dimension 39. The generic parametrized quintic surfaces have weakly elliptic singularities of multiplicity two or three ([71, Thm 10.5]).

Our GIT analysis implies that

Proposition 4.3.1. *The loci $V_{3,2}$, $V_{3,4}$, $V_{3,3}$ $V_{2,3}$, $V_{1,2}$, $V_{1,1}$ and $V_{2,2}$ are contained in the stable loci.*

Not every GIT semistable normal quintic surface is birational to a surface of general type. For example the surfaces with a singularity of type V_{24}^* as described in Theorem 2.2.1 are birational to a K3 surface. On other hand, a non normal quintic surface is never birational to a surface of general type. They are birational to either a K3 surfaces, a ruled surface, a fibration of rational or elliptic curves, or a rational surface [71, Sec 7].

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