The Ricci flow on manifolds with boundary

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Abstract of the Dissertation

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In this thesis, we investigate issues related to boundary value problems for the Ricci flow.

First, we focus on a compact manifold with boundary and show the short-time existence, regularity and uniqueness of the flow. To obtain these results we impose the boundary conditions proposed by Anderson for the Einstein equations, namely the mean curvature and the conformal class of the boundary. We also show that a certain continuation principle holds. Our methods still apply when the manifold is not compact, as long as it has compact boundary and an appropriate control of the geometry at infinity.

Secondly, motivated by the static extension conjecture in Mathematical General Relativity, we study a boundary value problem for the Ricci flow on warped products. We impose the boundary data proposed by Bartnik for the static vacuum equations, which are the mean curvature and the induced metric of the boundary of the base manifold.

We conclude the thesis applying the results above to study the flow on a 3-manifold with symmetry. We show the long time existence of the flow and study its behavior in different situations.

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Chapter 1

Introduction

Ricci flow has been proven to be a powerful tool of geometric analysis in the study of the relation between the geometry and topology of Riemannian manifolds. In this thesis, motivated by certain problems in geometry and mathematical general relativity on manifolds with boundary, we develop a general existence theory for the Ricci flow on such manifolds. This chapter begins with a brief overview of the Ricci flow, continues with a description of prior work on the Ricci flow on manifolds with boundary and concludes with the main results of the thesis.

1.1 A brief overview of Ricci flow.

Let (M, g_0) be a complete, compact Riemannian manifold. We say that a one-parameter family of complete Riemannian metrics $g(t)$ with $g(0) = g_0$ evolves by the Ricci flow, if it satisfies the partial differential equation

$$\partial_t g = -2\,\mathrm{Ric}(g).$$

The Ricci flow was introduced by Hamilton in 1982 in [Ham82], where he showed that for every intial metric g_0 the flow admits a unique solution for small time. The intuition is that the Ricci flow should behave as the geometric analogue of the heat diffusion equation

$$\partial_t u = \Delta u.$$

Heat diffusion tends to smooth out uneven heat distributions and flow towards harmonic functions (i.e. constants when M is closed). In a similar manner it is expected that Ricci flow should inherit some of this nice behavior, although it is a much more complicated, nonlinear system of equations.

Due to the diffeomorphism invariance of the Ricci tensor it is preferable to understand Ricci flow as a dynamical system in the quotient space of Riemannian metrics modulo diffeomorphisms and rescalings. The steady state solutions in this picture will be Einstein metrics, i.e. metrics that satisfy $\operatorname{Ric}(g) = \lambda g$, or more generally metrics which satisfy an equation of the form $\operatorname{Ric} + \mathcal{L}_X g = \lambda g$ for some constant λ and a vector field X. Such metrics are called Ricci solitons in the literature.

The normalized Ricci flow, defined as

$$\partial_t g = -2 \operatorname{Ric} + \frac{2}{n} r g,$$

r being the average of the scalar curvature, is equivalent to the Ricci flow up to rescalings and preserves the volume. The following Theorem by Hamilton, justifies the analogy with the heat equation.

Theorem 1.1. *Let (M, g_0) be a three dimensional closed Riemannian manifold with positive Ricci curvature. Then, the solution to the normalized Ricci flow exists for all time and converges exponentialy to a metric with constant positive curvature.*

The Ricci flow has had many successes since then. We will only mention and give references to a few of them here. On surfaces, in [Ham88] and [Cho91] it is shown that the normalized Ricci flow always converges to a metric of constant curvature. The normalized Ricci flow on manifolds of dimension great or equal to four and positive curvature operator converges to a metric with constant positive curvature. See for instance [Ham86],[BW08]. Also, in [Ham86] Ricci flow is used to classify all 4-manifolds that admit metrics with nonnegative curvature operator. Moreover, Brendle and Schoen used Ricci flow to prove the differentiable sphere Theorem in [BS09].

The analogy with the heat equation, taken literally turns out not to be accurate since the flow will in general develop singularities. It is a major area of active research on Ricci flow to understand in depth how these singularities form. Perelman, introduced new tools in [Per02], [Per03a], [Per03b] that allowed him to understand singularities in dimension three and realized Hamilton's program towards the proof of Thurston's Geometrization conjecture using Ricci flow with surgery.

In the realm of complete noncompact manifolds, Shi in [Shi89b] proved that if the initial metric has uniformly bounded curvature, then Ricci flow admits a solution with uniformly bounded curvature for small time. In particular he shows the following Theorem.

Theorem 1.2. *Let (M, g_0) be an n-dimensional complete noncompact Riemannian manifold with Riemannian curvature tensor satisfying*

$$|\operatorname{Rm}|^2 \leq k_0, \quad \textit{on } M.$$

Then, there exists $T(n, k_0) > 0$ such that the Ricci flow equation has a smooth solution $g(t)$ with $g(0) = g_0$ for $0 \leq t \leq T(n, k_0)$. Moreover it satisfies estimates

$$\sup_M |\nabla^m \operatorname{Rm}|^2 \leq \frac{C_m}{t^m}, \quad 0 \leq t \leq T(n, k_0).$$

He also proceeds to investigate the behaviour of the flow on noncompact three-manifolds with bounded nonnegative Ricci curvature obtaining the full classification of such manifolds in [Shi89a].

1.2 Prior work on manifolds with boundary and incomplete manifolds.

In light of the results described in the previous section, it is a natural question whether it is possible to use Ricci flow to deform metrics which are incomplete, and what kind of conditions would be required in order to obtain reasonable existence and uniqueness statements. In particular, if M is a manifold with boundary, what would be the appropriate boundary data?

Proving an existence theorem for the Ricci flow, apart from the nonlinearity of the equation, a serious obstacle comes from the diffeomorphism invariance of the Ricci tensor. This phenomenon had been observed long time ago in the Einstein equations in General Relativity and other PDEs that have a gauge invariance. For the Ricci flow, Hamilton first overcame the difficulty using the implicit function theorem of Nash-Moser. However, DeTurck later in [DeT83] discovered a simpler proof for the existence of Ricci flow on closed manifolds. He showed that, up to diffeomorphisms, it is equivalent to a modified parabolic equation, where standard parabolic theory applies.

On manifolds with boundary the situation is more complicated. First, due to the diffeomorphism invariance of Ricci flow, one would like to impose boundary conditions which are geometric, in the sense that they are preserved under the action of diffeomorphisms that fix the boundary. Since Ricci flow is a second order equation, this hints that the boundary conditions should involve the induced metric and the second fundamental form. Secondly, the boundary conditions should provide a well posed parabolic boundary value problem for the modified equation.

The first work on the topic was by Shen, in [She96], where he studied a natural Neumann-type boundary value problem for the Ricci flow, imposing conditions on the second fundamental form of the boundary. He proves the following Theorem.

Theorem 1.3. *For any given compact Riemannian manifold with boundary (M, g_0) with $\mathcal{A}(g_0) = \lambda g_0$, there is a short time solution to the Ricci flow satisfying $g(0) = g_0$ and*

$$A_t = \lambda g_t$$

on ∂M for all t, where $\mathcal{A}$ is the second fundamental form of ∂M and λ a constant.

However, one would like a more general existence theory which will allow the deformation of arbitrary metrics. In this direction, Pulemotov in [Pul12] shows that one can deform metrics with boundary of merely constant mean curvature.

Theorem 1.4. *Let (M, g_0) be a compact Riemanninan manifold with boundary. Assume that the boundary has constant mean curvature $\mathcal{H}_0$. Then, given any smooth function $\mu(t)$ with $\mu(0) = 1$ there exists a solution to the Ricci flow satisfying on ∂M:*

$$\mathcal{H}_t = \mu(t)\mathcal{H}_0,$$

where $\mathcal{H}_t$ denotes the mean curvature of ∂M.

Unfortunately there is only control of the mean curvature along this flow and we should expect that a uniqueness statement doesn't hold in this case. More boundary conditions need to be imposed to achieve uniqueness.

Regarding the long time behavior of Ricci flow, Shen and Cortissoz in [She96] and [Cor09] extend Theorem 1.1 of Hamilton, under the assumption that the boundary is totally geodesic or convex umbilic.

On surfaces there has been more progress. One should mention the work of Brendle in [Bre02a], [Bre02b], Tong Li in [Li93], Cortissoz in [Kor07] and Cortissoz and Murcia in [CM12]. They study the Ricci flow under Neumann-type boundary conditions. Related is also the work of Brendle on the closely related Yamabe flow in [Bre02c].

To conclude this section, and put the discussion above in the broader context of the Ricci flow on general incomplete manifolds, we mention a different point of view considered by Topping in [Top10]. Using the pseudolocality of the Ricci flow and exploiting its special properties in dimension 2 he shows the existence of "instantaneously complete" Ricci flow solutions initiating from possibly incomplete surfaces. Later, with

Giesen in [GT09] they show the uniqueness of in this class of solutions, in the case where the initial metric has negative Gauss curvature.

1.3 Motivation and main results.

The aim of this section is to motivate the work presented in this thesis. It becomes clear from the preceding discussion that we would like to supplement the Ricci flow with boundary conditions which will allow the flow to start from arbitrary metrics on manifolds with boundary. Moreover, these boundary data should be able to determine the flow uniquely and also be invariant under diffeomorphisms that fix the boundary.

In the following, we will describe some other geometric problems that this work is motivated and inspired from.

1.3.1 Boundary value problems for Einstein metrics.

The first motivation comes from the work of Anderson on boundary value problems for the Einstein equations in [And08]. Theorem 1.1 of [And08], states that the moduli space $\mathcal{E}_\lambda^{k,\alpha}$ of $C^{k,\alpha}$ Einstein metrics with Einstein constant λ, assuming a certain topological condition, is a Banach manifold ($\mathcal{E}$ is the space of Einstein metrics modulo the action of diffeomorphisms that fix the boundary).

Theorem 1.5 (Theorem 1.1 in [And08])**.** *Suppose $\pi_1(M, \partial M) = 0$. Then for any $\lambda \in \mathbb{R}$, the moduli space $\mathcal{E}_\lambda^{k,\alpha}$, if nonempty, is an infinite dimensional C^∞ smooth Banach manifold.*

We can view this as a "local" solvability result, since the existence of an Einstein metric $\tilde{g}$ immediately implies that any infinitesimal Einstein deformation integrates to a path of Einstein metrics.

However, to understand the global problem, it is necessary to impose appropriate boundary conditions. The geometric nature of the Einstein equations hints that they should involve the induced metric and the second fundamental form of the boundary. However, as we see below, the obvious choice of boundary data is not the correct one.

The Dirichlet and Neumann problems are not elliptic.

Naturaly one would like to investigate the Dirichlet and Neumann boundary value problems for the Einstein equations.

- *Dirichlet problem.* Given a $C^{k,\alpha}$ Riemannian metric γ on ∂M, find a $C^{k,\alpha}$ Einstein metric g on M such that the induced metric g^T on ∂M is γ.
- *Neumann problem.* Given a $C^{k-1,\alpha}$ symmetric 2-tensor h on ∂M, find a $C^{k,\alpha}$ Riemannian Einstein metric g on M such that the second fundamental form $\mathcal{A}$ of ∂M with respect to g is h.

However, a consequence of the Gauss equation is that the Dirichlet condition does not yield a well posed elliptic boundary value problem.

Let g be an Einstein metric satisfying $\mathrm{Ric} = \lambda g$. The Gauss equation for ∂M becomes

$$|\mathcal{A}|^2 - \mathcal{H}^2 + s_\gamma - (n-1)\lambda = 0,$$

where s_γ is the scalar curvature of γ. The ellipticity of the Dirichlet problem would imply that if Π is the boundary map

$$\begin{array}{rcl} \Pi : \mathcal{E}^{k,\alpha} & \to & Met^{k,\alpha}(\partial M) \\ g & \mapsto & g^T \end{array} \tag{1.1}$$

then its linearization $D\Pi_g$ at any Einstein metric g should be a Fredholm operator.

However, if g is a $C^{k,\alpha}$ Einstein metric the Gauss equation implies that s_γ should be $C^{k-1,\alpha}$. For an arbitrary metric γ though, s_γ will only be $C^{k-2,\alpha}$, which is an infinite codimension subspace of $C^{k-1,\alpha}$.

It turns out that neither Neumann boundary conditions give an elliptic problem. In this case the boundary map is

$$\begin{array}{rcl} \Pi' : \mathcal{E} & \to & S^2(T^*\partial M) \\ g & \mapsto & \mathcal{A}(g) \end{array}$$

Its linearization $D\Pi'_g$ at an Einstein metric g with totally geodesic boundary (or with second fundamental form vanishing at an open set of the boundary) will have an infinite dimensional kernel, due to the infinitesimal deformations of g by diffeomorphisms of M which do not fix the boundary.

An elliptic boundary value problem for the Einstein equations.

In Theorem 1.2 of [And08] Anderson proposes a boundary value problem for the Einstein equations which turns out to be elliptic. In particular, he proves the following Theorem.

Theorem 1.6. *The boundary map*

$$\begin{array}{rcl} \tilde{\Pi} : \mathcal{E}^{k,\alpha} & \rightarrow & \mathcal{C}^{k,\alpha}(\partial M) \times C^{k-1,\alpha}(\partial M) \\ g & \mapsto & ([g^T], \mathcal{H}(g)) \end{array}$$

is C^∞ smooth and Fredholm of index 0.

Here, $\mathcal{C}^{k,\alpha}$ denotes the pointwise conformal classes of $C^{k,\alpha}$ metrics on ∂M, and g^T, $\mathcal{H}(g)$ the induced metric and the mean curvature of the boundary respectively.

The natural question that arises is whether the boundary conditions described above yield a good existence and uniqueness theory for the Ricci flow. This would lead to many new interesting questions regarding the behaviour of the solutions and should also lead to a better understanding of the corresponding elliptic problem.

1.3.2 A Boundary value problem for the static vacuum equations.

The second motivation of this work comes from General Relativity. The simplest possible solutions to the Einstein vacuum equations, i.e. $\text{Ric} = 0$, are static metrics. These are Lorenzian metrics on $\mathbb{R} \times M$ of the form

$$-V^2 dt^2 + g_M,$$

where g_M is a Riemannian metric on a 3-manifold M and V is a smooth function on M. In this setting, the Einstein equations become the following elliptic system on g_M and V

$$\text{Ric}(g_M) = V^{-1} D^2_{g_M} V \qquad (1.2)$$

$$\Delta_{g_M} V = 0. \qquad (1.3)$$

The equations above are called *static vacuum equations*. The Riemannian metric $V^2 dt^2 + g_M$ induced by a solution (g, V) of (1.2),(1.3) is also Ricci flat.

Clearly, on closed 3-manifolds solutions g_M of the static vaccum equations are flat and V is constant. However, Anderson in [And99], generalizing a result of Lichnerowicz, proves that the same is true for noncompact complete solutions. Therefore, the natural setting to seek nontrivial solutions to these equations is a manifold with boundary.

Bartnik in [Bar89],[Bar02], motivated by his definition of quasilocal mass in general relativity, poses the following conjecture.

Conjecture. Given a Riemannian metric g in $B^3 \subset \mathbb{R}^3$, there is a unique static asymptotically flat Riemannian metric $\bar{g}$ in $\mathbb{R}^3 \setminus B^3$ which satisfies

$$\begin{aligned} \bar{g}^T &= g^T, \\ \mathcal{H}(\bar{g}) &= \mathcal{H}(g), \end{aligned}$$

where B^3 denotes the unit ball in $\mathbb{R}^3$ and $\mathcal{H}$, g^T the mean curvature and the induced metric respectively of the boundary ∂B^3.

We will refer to the boundary data above, i.e. the induced metric and mean curvature, as *Bartnik data*. From an analytic point of view, the static vacuum equations with the Bartnik boundary data form an elliptic boundary value problem. For work on this conjecture we refer the reader to the work of Miao in [Mia03], and Anderson and Khuri in [AK11].

1.3.3 Main results.

The conformal class and mean curvature as boundary data for the Ricci flow.

Let g_0 be a Riemannian metric on M^{n+1} and $\gamma(x,t)$ arbitrary smooth Riemannian metrics on ∂M and η an arbitrary smooth function on $\partial M \times [0,\infty)$. Assume the compatibility conditions $[\gamma(0)] = [g_0^T]$ and $\mathcal{H}(g_0) = \eta|_{t=0}$. In Chapter 4 we prove the following Theorem.

Theorem 1.7 (Theorem 4.9)**.** *There exists a $T > 0$ and a smooth solution to the Ricci flow defined away from $\partial M \times 0$ which satisfies on ∂M the boundary conditions*

$$\mathcal{H}(g(t)) = \eta_t, \tag{1.4}$$

$$\left[g(t)^T\right] = [\gamma(t)], \tag{1.5}$$

for $t > 0$. As $t \to 0$, $g(t)$ converges in the Cheeger-Gromov $C^{1,\alpha}$ sense (i.e. up to diffeomorphisms that fix the boundary) to g_0 and C^∞ away from the boundary.

Moreover, higher order compatibility conditions on the data (g^0, γ, η) can improve this convergence to $C^{k,\alpha}$ and the regularity of g up to $\partial M \times 0$ (see Theorem 4.9).

We note that a version of this Theorem which satisfies the initial data in the usual sense $g(0) = g^0$ does hold. However, such a solution will generally not be C^∞ smooth up to the boundary even for positive time. This issue is related to the invariance of the equation under diffeomorphisms and is discussed in remark 4.4.2.

Moreover, the Theorem also holds when the manifold is noncompact (complete) with compact boundary, since our techniques still apply once we assume that g_0 has geometry bounded in $W^{2,p}$ (see Definition 1) and $\mathrm{Rm}(g_0) \in L^p$.

It is well known that incomplete solutions of the Ricci flow are in general not unique. On a manifold with boundary though, the boundary data (1.4),(1.5) allow us to obtain the following uniqueness result.

Theorem 1.8 (Theorem 4.10). *A C^3 (up to $\partial M \times 0$) solution to the Ricci flow satisfying the boundary conditions (1.4),(1.5) is uniquely determined by the initial data g_0 and the boundary data $([\gamma], \eta)$.*

Finally, in chapter 6 we obtain an extension condition for the flow. On closed manifolds, a Ricci flow defined for $t < T$ with uniformly bounded curvature tensor can be extended past time T. On compact manifolds with boundary, we observe that appropriate control of the data $([\gamma], \eta)$ and uniform bounds on the ambient curvature and the second fundamental form $\mathcal{A}$ of the boundary suffice for the extension of the flow. This is the following Theorem.

Theorem 1.9. *[Theorem 6.1] Let $g(t)$ be a smooth Ricci flow with maximal time of existence $T < \infty$ and smooth boundary data $([\gamma(x,t)], \eta(x,t))$ defined for all $0 \le t < \infty$. Then*

$$\sup_{0 \le t < T} \left(\sup_{x \in M} |\mathrm{Rm}(g(t))|_{g(t)} + \sup_{x \in \partial M} |\mathcal{A}(g(t))|_{g(t)} \right) = +\infty.$$

Bartnik data for the Ricci flow on warped products.

Let $N = M^{n+1} \times F^m$ and $\hat{g}$ be an Einstein metric on F, with Einstein constant λ. Given a Riemannian metric g_0 and a function f_0 on M, one can define the warped product metric $h_0 = g_0 + f_0^2 \hat{g}$ on N. If M is noncompact (complete), assume that $\hat{g}$ is Ricci flat, $\mathrm{Rm}(g_0) \in L^p$, $f_0 \in W^{2,p}$, that g_0 has geometry bounded in $W^{2,p}$ (again, see Definition 1) and ∂M is compact.

The Ricci flow equation on N, assuming the solution has the form

$h(t) = g(t) + f(t)^2\hat{g}$, becomes

$$\partial_t g = -2\,\mathrm{Ric}_g + 2mf^{-1}D^2_g f, \tag{1.6}$$

$$\partial_t f = \Delta_g f + (m-1)\frac{|df|^2_g}{f} - \frac{\lambda}{f}. \tag{1.7}$$

For $F = \mathbb{R}$ and $m = 1$ this becomes a Ricci flow on static metrics.

Let $\gamma(t)$, $\eta(t)$ be arbitrary, smooth, one parameter families of Riemannian metrics and functions on ∂M, satisfying the zeroth order compatibility conditions $\gamma(0) = g_0^T$ and $\eta(0) = \mathcal{H}(g_0)$. Imposing the Bartnik data, and applying the techniques of Chapter 4 we obtain the following existence result for (1.6), (1.7) in Chapter 5.

Theorem 1.10 (Theorem 5.2). *Given h_0 as above, there exists $T > 0$ and solutions g, f of (1.6),(1.6), smooth and defined away from the corner $\partial M \times 0$, such that $h(t) = g(t) + f(t)^2\hat{g}$ is a Ricci flow for $t \leq T$, satisfying*

$$g(t)^T = \gamma(t),$$
$$\mathcal{H}(g(t)) = \eta(t),$$

for all $t > 0$. In addition, there exists a a family of diffeomorphisms ϕ_t of M such that $\phi_t^* g(t), \phi_t^* f(t)$ convege to g_0, f_0 respectively, as $t \to 0$ in $C^{1,\alpha}(\overline{M_T})$ (uniformly). Moreover, the convergence is also C^k up to $\partial M \times 0$ if $\mathrm{Ric}(h)$ satisfies higher order compatibility conditions.

In addition, if M is noncompact and g_0 is controlled in C^2, then $h(t)$ has uniformly bounded curvature for all $t \leq T$.

Chapter 2

Background material

2.1 Function spaces

In this section we define the function spaces we will need. Let M^{n+1} be a compact or noncompact n+1-dimensional manifold with compact boundary ∂M and interior M^o. We denote $M_T = M^o \times (0,T)$, $\partial M_T = \partial M \times (0,T)$.

Fix a smooth Riemannian metric $\bar{g}$ on M and its Levi-Civita connection $\overline{\nabla}$. Let $\{U_s\}$ be an open cover of M, and ϕ_s a collection of charts such that

$$\begin{aligned} \phi_s : U_s \to B(0,\bar{r}) \subset \mathbb{R}^{n+1} &\qquad \text{, if } U_s \text{ does not intersect the boundary} \\ \phi_s : U_s \to B(0,\bar{r})^+ \subset \mathbb{R}^{n+1} &\qquad \text{, if } U_s \text{ intersects the boundary.} \end{aligned}$$

for some $\bar{r} > 0$ (where $\bar{r}$ is uniform over all charts). In the last case assume that

$$\phi_s|_{\partial M \cap U_s} : \partial M \cap U_s \to V := B^n(0,\bar{r}) \subset \mathbb{R}^n.$$

Let also ρ_s be a partition of unity subordinate to that open cover.

We will use the convention that Greek indices correspond to directions tangent to the boundary, ranging from 1 to n, while the 0 index to the inward transversal direction.

Definition 1. We say that $\bar{g}$ has uniformly bounded geometry in $W^{2,p}$, if there exists a uniform constant $C > 1$ so that the components of $\bar{g}$ in each of the coordinate systems defined above satisfy

$$\begin{aligned} C^{-1}\delta_{ij} \leq \ & \bar{g}_{ij} \ \leq C\delta_{ij} \quad \text{(as bilinear forms)} \\ ||\bar{g}_{ij}||_{W^{2,p}} \ & \leq \ C \end{aligned}$$

Here the $W^{2,p}$ norm is defined with respect to the flat metric induced in each coordinate chart.

When M is non-compact we will always assume that $\bar{g}$ has uniformly bounded geometry in $W^{2,p}$.

Consider any tensor bundle E of rank k over M, with projection map π, equipped with the connection inherited by $\overline{\nabla}$. The completion of the space of the time dependent $C^\infty(M_T)$ compactly supported sections of E, with respect to the norm

$$\|u\|_{W_p^{2,1}(M_T)} = \|u\|_{L_p(M_T)} + \|\overline{\nabla}u\|_{L_p(M_T)} + \|\overline{\nabla}^2 u\|_{L_p(M_T)} + \|\partial_t u\|_{L_p(M_T)}$$

will be denoted by $W_p^{2,1}(M_T)$. Let also

$$|u|_{L_p^{2,1}(M_T)} = \|\partial_t u\|_{L_p(M_T)} + \|\overline{\nabla}^2 u\|_{L_p(M_T)}$$

be the principal part of the norm $\|u\|_{W_p^{2,1}(M_T)}$. If τ is a section of E, we will denote by ${}^s\tau^{ijk\ldots}_{\mu\nu\ldots}$ the coordinates of this tensor with respect to the trivialization based at U_s.

We define the following norm for time dependent $L_p(\partial M_T)$, and $W^{1,p}(\partial M_T)$ sections of $E_{\partial M} = \{v \in E \,|\, \pi \circ v \in \partial M\}$, for $\lambda = 1 - \frac{1}{p}$.

$$\|v\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} = \|v\|_{L_p(\partial M_T)} + |v|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)},$$

$$\|v\|_{W_p^{1+\lambda,\frac{1+\lambda}{2}}(\partial M_T)} = \|v\|_{L_p(\partial M_T)} + \|\overline{\nabla}^T v\|_{L_p(\partial M_T)} + |\overline{\nabla}^T v|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)},$$

where $\overline{\nabla}^T$ denotes the connection induced on $E_{\partial M}$. Setting $\hat{\rho}_s = \rho_s \circ \phi_s^{-1}$, we define

$$|v|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} = \sum_s \max_{i_1,\ldots,i_k} |\hat{\rho}_s {}^s v^{i_1,\ldots,i_l}{}_{i_{l+1},\ldots,i_k}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)}$$

where, for every function $f \in L_p(V_T)$

$$\begin{aligned}
|f|^p_{\mathcal{L}_p^{\alpha,\beta}(V_T)} &= |f|^p_{\mathcal{L}_p^{\alpha,0}(V_T)} + |f|^p_{\mathcal{L}_p^{0,\beta}(V_T)} \\
|f|^p_{\mathcal{L}_p^{\alpha,0}(V_T)} &= \sum_{\mu=1}^{n} \int_0^{+\infty} h^{-(1+p\alpha)} \|\Delta_{\mu,h} f\|^p_{L_p(V_{i,h,T})} dh \\
|f|^p_{\mathcal{L}_p^{0,\beta}(V_T)} &= \int_0^{+\infty} h^{-(1+p\beta)} \|\Delta_{t,h} f\|^p_{L_p(V_{T-h})} dh.
\end{aligned}$$

In the above,

$$
\begin{aligned}
\Delta_{\mu,h} f(y,t) &= f(y+he_\mu,t) - f(y,t)\\
\Delta_{t,h} f(y,t) &= f(y,t+h) - f(y,t)\\
V_{\mu,h,T} &= \left\{(y,t) \in V_T | y + he_\mu \in V\right\}.
\end{aligned}
$$

Analogous spaces exist also in the elliptic setting, see for instance [Sol67].

For $l > 0$ nonintegral, we will denote by $C^{l,l/2}(M \times [0,\tau], E)$ the Banach space of time dependent sections u of E having continuous up to the boundary derivatives $\partial_t^r \overline{\nabla}^q u$ for all r, q satisfying $2r + q < l$, satisfying appropriate Hölder conditions in the time and space directions. More precisely, the norm is given by

$$
|u|_{l,l/2} = \sup_s \max_I |{}^s u_I|_{[l],B(0,1)} + \sup_s \max_I \langle {}^s u_I \rangle_{l,l/2,B(0,1)},
$$

where ${}^s u_I$ are the coordinate functions of u in the coordinate system U_s and

$$
\begin{aligned}
|f|_{k,B(0,1)} &= \sum_{0 \le 2r+q \le k} ||\partial_t^r \partial_x^q f||_\infty\\
\langle f \rangle_{l,l/2,B(0,1)} &= \sum_{2r+q=[l]} \left\langle \partial_t^r \partial_x^q f \right\rangle_{l-[l],x} + \sum_{0<l-2r-q<2} \left\langle \partial_t^r \partial_x^q f \right\rangle_{\frac{l-2r-q}{2},t}.
\end{aligned}
$$

Here, for $0 < \rho < 1$

$$
\begin{aligned}
\langle f \rangle_{\rho,x} &= \sup_{x \neq y,\, t} \frac{|f(x,t) - f(y,t)|}{|x-y|^\rho}\\
\langle f \rangle_{\rho,t} &= \sup_{t \neq t',\, x} \frac{|f(x,t) - f(x,t')|}{|t-t'|^\rho}.
\end{aligned}
$$

We will also denote by $|u|_k$ and $\langle u \rangle_{l,l/2}$ the norms

$$
\begin{aligned}
\langle u \rangle_{l,l/2} &= \sup_s \max_I \langle {}^s u_I \rangle_{l,l/2,B(0,1)}\\
|u|_k &= \sup_s \max_I |{}^s u_I|_{k,B(0,1)}.
\end{aligned}
$$

For any integer $k \ge 0$ we will denote by $C^k(M \times [0,\tau])$ the space of sections with all the derivatives $\partial_t^r \overline{\nabla}^q u$ for $2r + q \le k$ continuous, equipped with the norm $|\cdot|_k$.

It is not hard to see that $C^{\epsilon,\frac{\epsilon}{2}}(\partial M_T)$ and $C^{1+\epsilon,\frac{1+\epsilon}{2}}(\partial M_T)$ embed in $W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)$ and $W_p^{1+\lambda,\frac{1+\lambda}{2}}(\partial M_T)$ respectively, provided that $\epsilon > \lambda$. We will also need the following embedding theorems.

Lemma 2.1.

1. *For* $1 < p < \infty$, *and* $u \in W_p^{2,1}(M_T)$,

$$||u||_{W_p^{1+\lambda,(1+\lambda)/2}} + ||\overline{\nabla}\, u||_{W_p^{\lambda,\lambda/2}(\partial M_T)} \le C_1 ||u||_{W_p^{2,1}(M_T)}.$$

2. *If* $\frac{n+3}{2} < p < \infty$ *and* $0 < \alpha < \min(1, 2 - (\frac{n+3}{p}))$, *then*

$$\langle u \rangle_{\alpha,\alpha/2} \le C_2 \left(\delta^{2-\frac{n+3}{p}-\alpha} |u|_{L_p^{2,1}(M_T)} + \delta^{-\frac{n+3}{p}-\alpha} \|u\|_{L^p(M_T)} \right).$$

3. *If* $n+3 < p < \infty$ *and* $0 < \alpha \le 1 - \frac{n+3}{p}$, *then*

$$\langle \overline{\nabla}\, u \rangle_{\alpha,\alpha/2} \le C_3 \left(\delta^{1-\frac{n+3}{p}-\alpha} |u|_{L_p^{2,1}(M_T)} + \delta^{-(1+\frac{n+3}{p}+\alpha)} \|u\|_{L^p(M_T)} \right).$$

In the above, the constants do not depent on $T > 0$ *and* $0 < \delta \le \min(d, T^{1/2})$, *where* d *is a constant depending on the chosen atlas* $\{U_s\}$.

Estimates (2), (3) still hold in the case M *is noncompact, under the assumption that the geometry is bounded in* $W^{2,p}$ *and* $p > n+2$.

Proof. By Lemma 3.3 in Chapter II of [LSU67] or Lemma A.1 in [Wei91] the estimates hold for domains in $\mathbb{R}^{n+1}$ or $\mathbb{R}^{n+1}_+$ satisfying the cone condition. We will show below that the global estimates (2),(3) hold in the case that M is noncompact and $\bar{g}$ has geometry bounded in $W^{2,p}$.

First, by the Sobolev embedding (since $p > n+2$) there is a constant $C > 0$, uniform in s, such that $|u_{ij}|_{0,B(0,\bar{r})\times[0,T]} \le C||u_{ij}||_{W_e^{1,p}(B(0,\bar{r})\times[0,T])}$, where u_{ij} are the components of the 2-tensor u in a given coordinate chart (U_s, ϕ_s) and $W_e^{1,p}$ denotes the Sobolev space with respect to the standard Euclidean metric and connection.

However, the uniform control of the geometry in $W^{2,p}$ provides uniform bounds on $|\bar{g}_{ij}|_{C^1}$ (since $p > n+1$), which imply the estimate

$$||u_{ij}||_{W_e^{1,p}} \le C||u||_{W_{\bar{g}}^{1,p}(B(0,\bar{r})\times(0,T))} \le C||u||_{W_p^{2,1}(M_T)},$$

where $W^{1,p}_{\bar g}$ is defined using $\bar g$ and its connection. Here too, the constant is independent of the choice of the chart. Therefore the estimates above provide uniform C^0 bounds on $u_{ij}(x,t)$, in any coordinate chart.

Now, we estimate

$$\begin{aligned} ||u_{ij}||^p_{W^{2,1}_{p,e}(B(0,\bar r)\times[0,T])} &= \int |u|^p_e dx + \sum_k \int |\partial_k u_{ij}|^p_e dx + \sum_{k,l} \int |\partial_k \partial_l u_{ij}|^p_e dx \\ &\le C\left(\int |u|^p_{\bar g} dvol_{\bar g} + \int |\overline\nabla u|^p_{\bar g} dvol_{\bar g} + \int |\overline\nabla^2 u|^p_{\bar g} dvol_{\bar g} \right) \\ &\le C||u||^p_{W^{2,1}_p(M_T)}. \end{aligned}$$

Where $C > 0$ does not depend on the choice of the particular chart. Note that the only terms inside $\overline\nabla^2 u_{ij}$ which contain second derivatives of $\bar g$ are of the form $u * \partial^2 \bar g$. However, since there is a uniform C^0 bound on u_{ij}, the uniform $W^{2,p}$ bound on $\bar g_{ij}$ is enough to estimate such terms. Then estimates (2), (3) directly follow from the corresponding estimates in each coordinate chart, since the constants above maybe chosen the same for all charts. $\square$

From now on we fix some $p > n+3$ and some $\alpha = 1 - \frac{n+3}{p}$ and $\epsilon > \lambda$. Then, as the previous Lemma implies, the Sobolev space $W^{2,1}_p(M_T)$ embeds in the Hölder space $C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{M_T})$. Moreover, we get the following estimates (see Corollary A.2 in [Wei91]).

Lemma 2.2. *For all $u \in W^{2,1}_p(M_T)$, with $u(.,0) \equiv 0$, $n+3 < p < \infty$, $0 < \gamma = (1 - \frac{n+3}{p})/2$ and all sufficiently small $T > 0$*

1. $|u|_1 \le C_4 T^\gamma |u|_{L^{2,1}_p(M_T)}$.
2. $|u|_{\mathcal{L}^{\beta,\beta/2}_p(\partial M_T)} \le C_5 T^\gamma |u|_{L^{2,1}_p(M_T)}$, *for all $\beta \in (0,1)$.*

Finally, we will often use the following product estimate.

Lemma 2.3. *If $f_1, f_2 \in \mathcal{L}^{\alpha,\beta}_p(V_T) \cap L^\infty(V_T)$ and $\hat\rho = \rho \circ \phi^{-1}$, then*

$$|\hat\rho f_1 f_2|_{\mathcal{L}^{\alpha,\beta}_p(V_T)} \le C_6 ||f_1 f_2||_\infty + ||f_1||_\infty |\hat\rho f_2|_{\mathcal{L}^{\alpha,\beta}_p(V_T)} + ||f_2||_\infty |\hat\rho f_1|_{\mathcal{L}^{\alpha,\beta}_p(V_T)}.$$

2.2 The second fundamental form and mean curvature.

Let g be a Riemannian metric on M and N the ourward unit normal to ∂M with respect to g. The second fundamental form $\mathcal{A}$ of the boundary is defined by

$$\mathcal{A} = \frac{1}{2} \left(\mathcal{L}_N g\right)^T .$$

Its trace is the mean curvature, given by

$$2\mathcal{H}(g) = \mathrm{tr}_{g^T} \mathcal{L}_N g.$$

First variation of the mean curvature.

Lemma 2.4. *If g_t is a smooth one-parameter family of metrics, such that $g_0 = g$, and $\partial_t g|_{t=0} \equiv h$, the variation of the mean curvature of the boundary is given by the formula:*

$$2\mathcal{H}'_g(h) = \mathrm{tr}_{g^T} \nabla_N h + 2\delta_{\partial M}\left(h(N)^T\right) - h(N,N)\mathcal{H}(g). \tag{2.1}$$

Proof.

$$\begin{aligned}
2\mathcal{H}'_g &= -h^{\alpha\beta}(\mathcal{L}_N g)_{\alpha\beta} + \mathrm{tr}_\gamma(\mathcal{L}_N g)' = -2\langle A, h\rangle + \mathrm{tr}_\gamma \mathcal{L}_{N'} g + \mathrm{tr}_\gamma \mathcal{L}_N h \\
&= -2\langle A, h\rangle + \mathrm{tr}_\gamma \mathcal{L}_{\frac{-h_{00}N}{2}} g + \mathrm{tr}_\gamma \mathcal{L}_{-h(N)^T} g + \mathrm{tr}_\gamma \mathcal{L}_N h \\
&= -2\langle A, h\rangle - h(N,N)H + \delta_{\partial M}\left(h(N)^T\right) + \mathrm{tr}_\gamma \nabla_N h + 2\langle A, h\rangle \\
&= \mathrm{tr}_\gamma \nabla_N h + 2\delta_{\partial M}\left(h(N)^T\right) - h(N,N)H.
\end{aligned}$$

□

The mean curvature in local coordinates.

Lemma 2.5. *In local coordinates the mean curvature of the boundary of M is*

$$2\mathcal{H} = g^{T,\alpha\beta}\nu^i\partial_i(g_{\alpha\beta}) - \left(\frac{2g^{0l}g^{\alpha k}}{\sqrt{g^{00}}} + \frac{g^{0l}g^{0k}g^{0\alpha}}{(\sqrt{g^{00}})^3} - \frac{g^{T,\alpha\beta}g_{0\beta}g^{0l}g^{0k}}{\sqrt{g^{00}}}\right)\partial_\alpha(g_{kl}).$$

Proof.

$$\begin{aligned}
2\mathcal{H}(g) &= \operatorname{tr}_{g^T}(\mathcal{L}_N g) = g^{T,\alpha\beta}(\mathcal{L}_N g)_{\alpha\beta} \\
&= g^{T,\alpha\beta}\left(N(g_{\alpha\beta}) - g([N,\partial_\alpha],\partial_\beta) - g(\partial_\alpha,[N,\partial_\beta])\right) \\
&= g^{T,\alpha\beta}\left(\nu^i\partial_i(g_{\alpha\beta}) + 2\partial_\alpha(\nu^i)g_{i\beta}\right) \\
&= g^{T,\alpha\beta}\nu^i\partial_i(g_{\alpha\beta}) + 2g^{T,\alpha\beta}\partial_\alpha(\nu^\epsilon)g_{\epsilon\beta} + 2g^{T,\alpha\beta}\partial_\alpha(\nu^0)g_{0\beta} \\
&= g^{T,\alpha\beta}\nu^i\partial_i(g_{\alpha\beta}) + 2\partial_\alpha(\nu^\alpha) + 2g^{T,\alpha\beta}g_{0\beta}\partial_\alpha(\nu^0).
\end{aligned}$$

Since $\nu^i = -\frac{g^{0i}}{\sqrt{g^{00}}}$ we compute

$$\partial_\alpha(\nu^i) = \left(\frac{g^{0l}g^{ik}}{\sqrt{g^{00}}} - \frac{g^{0l}g^{0k}g^{0i}}{2(\sqrt{g^{00}})^3}\right)\partial_\alpha(g_{kl}).$$

Therefore

$$2\mathcal{H} = g^{T,\alpha\beta}\nu^i\partial_i(g_{\alpha\beta}) + \left(\frac{2g^{0l}g^{\alpha k}}{\sqrt{g^{00}}} - \frac{g^{0l}g^{0k}g^{0\alpha}}{(\sqrt{g^{00}})^3} + \frac{g^{T,\alpha\beta}g_{0\beta}g^{0l}g^{0k}}{\sqrt{g^{00}}}\right)\partial_\alpha(g_{kl}). \quad \square$$

Chapter 3

Two linear parabolic boundary value problems.

In this chapter we discuss two linear parabolic boundary value problems on M which will later arise as linearizations of the Ricci-DeTurck equation. Theorems 3.1 and 3.2 are applications of the work of Solonnikov on the solvability in L^p spaces of parabolic boundary value problems satisfying general boundary conditions (see[Sol65]). The adaptation of his results in our case does not require any essential modification if M is compact or noncompact with geometry bounded in $W^{2,p}$.

Let g be a $C^{1+\epsilon}$ Riemannian metric on M, for some $\epsilon > 1 - \frac{1}{p}$ and $p > n+3$, and γ the induced metric on the boundary. Then,

$$\beta_g(u) = \operatorname{div}_g u - \frac{1}{2} d \operatorname{tr}_g u$$

is the Bianchi operator and $\mathcal{H}'_g$ the linearization of the mean curvature at g (see 2.1).

Theorem 3.1. *Consider the following linear parabolic initial-boundary value problem on symmetric 2-tensors on M*

$$\partial_t u - \operatorname{tr}_g \overline{\nabla}^2 u = F(x,t)$$

$$\left.\begin{array}{rcl} \beta_g(u) & = & G(x,t) \\ \mathcal{H}'_g(u) & = & D(x,t) \\ u^T - \frac{\operatorname{tr}_\gamma u^T}{n}\gamma & = & 0 \end{array}\right\} \text{on } \partial M \tag{3.1}$$

$$u|_{t=0} = u_0$$

for $F \in L^p(M_T)$, G, D in the corresponding $W_p^{\lambda,\lambda/2}(\partial M_T)$ space and $u_0 \in W^{2,p}(M^o)$. Assuming that the zeroth order compatibility conditions

$$\begin{aligned}\beta_g(u_0) &= G(x,0)\\ \mathcal{H}'_g(u_0) &= D(x,0)\\ u_0^T - \frac{\operatorname{tr}_\gamma u_0^T}{n}\gamma &= 0\end{aligned}$$

hold, problem (3.1) has a unique solution $u \in W_p^{2,1}(M_T)$ which satisfies the estimate

$$\|u\|_{W_p^{2,1}(M_T)} \leq C_8 \left(\|F\|_{L^p(M_T)} + \|G\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} \right. \\ \left. + \|D\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} + \|u_0\|_{W^{2,p}(M^o)} \right). \quad (3.2)$$

Moreover the constant C_8 stays bounded as $T \to 0$ and depends on the $C^{1+\epsilon}$ norms of g and g^{-1}.

Proof. The method followed in Chapter IV of [LSU67] and Theorem 5.4 of [Sol65] carries over to the manifold setting, after the necessary adaptation to the realm of manifolds and vector bundles (see [Pul12]). We only need to show that the following boundary value problem on $\mathbb{R}^{n+1}_+ = \{x^0 \geq 0\} \subset \mathbb{R}^{n+1}$ satisfies the complementing condition (see [LSU67],[Sol65] and [Èĭd69]).

$$\begin{aligned}\partial_t u_{kl} - \Delta_{eucl} u_{kl} &= \widehat{F}_{kl}\\ \delta^{ij}\partial_i(u_{jk}) - \frac{1}{2}\delta^{ij}\partial_k u_{ij} &= \widehat{G}_k\\ \delta^{\alpha\beta}\partial_o u_{\alpha\beta} - 2\delta^{\alpha\beta}\partial_\alpha u_{\beta 0} &= \widehat{D}\\ u_{\alpha\beta} - \frac{\delta^{\epsilon\zeta}u_{\epsilon\zeta}}{n}\delta_{\alpha\beta} &= 0\\ u|_{t=0} &= 0.\end{aligned} \quad (3.3)$$

Here, $\widehat{F}_{kl} \in L_p(\mathbb{R}^{n+1}_+)$ and $\widehat{G}_k, \widehat{D} \in W_{p,0}^{\lambda,\lambda/2}(\partial\mathbb{R}^{n+1}_+)$ (by $W_{p,0}^{\lambda,\lambda/2}$ we denote the subspace with initial condition $u|_{t=0} = 0$). One obtains (3.3) by expressing (3.1) in local coordinates around a point x of the boundary, with

$g_{ij}(x) = \delta_{ij}$, freezing the coefficients at $(x, 0)$ and keeping the higher order terms. The principal symbols of the boundary operators are:

$$i \sum_l \xi_l h_{lk} - \frac{i}{2} \sum_l \xi_k h_{ll} \tag{3.4}$$

$$i\xi_0 \sum_\alpha h_{\alpha\alpha} - 2i \sum_\alpha \xi_\alpha h_{0\alpha} \tag{3.5}$$

and the principal symbol of the parabolic operator $\partial_t - \Delta_{eucl}$, is $(p + |\zeta|^2 + \tau^2)h_{ij}$, where $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$ and $|\zeta|$ its Euclidean norm. We obtain the following positive root $\hat{\tau} = i\sqrt{p + |\zeta|^2}$. Setting equations (3.4), (3.5) to zero and letting $\xi_0 = \hat{\tau}$, $\xi_\alpha = \zeta_\alpha$, we get the following system:

$$\begin{aligned}
i\hat{\tau}h_{00} + i\sum_\alpha \zeta_\alpha h_{\alpha 0} - \frac{i}{2}\hat{\tau}\sum_l h_{ll} &= 0 && (3.6)\\
i\hat{\tau}h_{0\mu} + i\sum_\alpha \zeta_\alpha h_{\alpha\mu} - \frac{i}{2}\zeta_\mu \sum_l h_{ll} &= 0 && (3.7)\\
i\hat{\tau}\sum_\alpha h_{\alpha\alpha} - 2i\sum_\alpha \zeta_\alpha h_{0\alpha} &= 0 && (3.8)\\
h_{\alpha\beta} &= \phi\delta_{\alpha\beta}. && (3.9)
\end{aligned}$$

Since the principal symbol of the equation is in diagonal form, the complementing condition is equivalent to proving that system (3.6)-(3.9) has only the zero solution when (p, ζ) satisfy

$$\mathrm{Re}p \geq -\delta_1|\zeta|^2 \tag{3.10}$$

for some $0 < \delta_1 < 1$.

From equation (3.6) we have

$$2i\sum_\alpha \zeta_\alpha h_{\alpha 0} = i\hat{\tau}\sum_l h_{ll} - 2i\hat{\tau}h_{00} = i\hat{\tau}(\operatorname{tr} h - 2h_{00}), \tag{3.11}$$

while multiplying equation (3.7) by $2\zeta_\mu$ and then adding over μ we find:

$$\sum_\mu 2i\hat{\tau}\zeta_\mu h_{0\mu} + 2i\sum_{\alpha,\mu} \zeta_\alpha\zeta_\mu h_{\alpha\mu} - i\sum_\mu \zeta_\mu^2 \operatorname{tr} h = 0. \tag{3.12}$$

This gives, taking (3.11) and $h_{\alpha\mu} = \phi\delta_{\alpha\mu}$ into account:

$$i\hat{\tau}^2(\operatorname{tr} h - 2h_{00}) + 2i|\zeta|^2\phi - i|\zeta|^2 \operatorname{tr} h = 0 \tag{3.13}$$

which, after substituting for $\hat{\tau}$, leads to the equation:

$$ph_{00} = pn\phi + 2(n-1)|\zeta|^2\phi \tag{3.14}$$

Now, by equation (3.8) we have:

$$2i\sum_{\alpha}\zeta_\alpha h_{0\alpha} = i\hat{\tau}\sum_{\alpha}h_{\alpha\alpha} = i\hat{\tau}\phi n \tag{3.15}$$

which combined with (3.6) gives:

$$2i\hat{\tau}h_{00} + i\hat{\tau}\phi n - i\hat{\tau}\,\mathrm{tr}\,h = 0 \tag{3.16}$$

and therefore $i\hat{\tau}h_{00} = 0$. Now, (3.10) implies that $p \neq -|\zeta|^2$, which gives $\hat{\tau} \neq 0$ and thus $h_{00} = 0$.

Now, by (3.14) we have that

$$\phi\left(pn + 2|\zeta|^2(n-1)\right) = 0. \tag{3.17}$$

However, assumption (3.10) implies that $\left(pn + 2|\zeta|^2(n-1)\right) \neq 0$, since $\frac{2(n-1)}{n} > 1$ for $n \geq 2$. This gives that $\phi = 0$.

Now we have established that $\phi = h_{00} = 0$ it is easy to see that $h_{0\mu} = 0$, by (3.8). This proves the complementing condition for system (3.3). $\square$

Remark 3.0.1. Theorem 3.1 is still valid if we consider γ_t and g_t evolving such that $\gamma_0 = g^T$. Note that the complementing condition is satisfied if γ_t and g_t^T are in the same conformal class. If not, the openness of this condition implies that it holds at least for some short time $\hat{\tau} > 0$ depending on $C^{\epsilon,\epsilon/2}$ bounds of γ_t and g_t. Thus, we either get local (in time) existence or a global solution and the constant C_8 depends on the $C^{1+\epsilon,\frac{1+\epsilon}{2}}$ norms of g_t and γ_t and the C^0 norms of g^{-1} and γ^{-1}.

Before we state Theorem 3.2 a few definitions are in order. Given a metric g and a positive function f on M, we can define the following first order differential operator

$$P_{(g,f)}(u,\phi)^r = g^{rl}g^{pq}\,\overline{\nabla}_p\,u_{ql} - \frac{1}{2}g^{rl}g^{pq}\,\overline{\nabla}_l\,u_{pq} - mf^{-1}g^{rk}\,\overline{\nabla}_k\,\phi$$

acting on a symmetric 2-tensor u and a function ϕ on M.

In the following, u and F_1 are symmetric 2-tensors, β_1 a section of the restricton to ∂M of the tangent bundle of M, F_2 and β_2 functions on B and ∂M respectively, and β_3 a symmetric 2-tensor on ∂M. Abusing slightly notation we will not use different symbols to distinguish among function spaces containing sections of different bundles but of the same nature.

Moreover, for the coefficients of the boundary operator $P_{(g,f)}$ to have the right regularity so that the following Theorem holds, we note that the background metric $\bar{g}$ has to be at least $C^{1+\epsilon}$ (with respect to the atlas defined in Chapter 2).

Theorem 3.2. *Let $F_1, F_2 \in L^p$, boundary data $\beta_1, \beta_2 \in W_p^{\lambda,\lambda/2}$ and $\beta_3 \in W_p^{\lambda+1,\frac{\lambda+1}{2}}$. Then, given initial data $u_0 \in W^{2,p}$, $\phi_0 \in W^{2,p}$, and assuming that the zeroth order compatibility conditions hold, there exists a unique solution $(u,\phi) \in W_p^{2,1}$ to the following parabolic system*

$$\partial_t u_{kl} - \mathrm{tr}_g \overline{\nabla}^2 u_{kl} = F^1_{kl} \tag{3.18}$$

$$\partial_t \phi - \mathrm{tr}_g \overline{\nabla}^2 \phi = F^2 \tag{3.19}$$

satisfying on ∂M the conditions

$$P_{(g,f)}(u,\phi)^r = \beta_1^r \tag{3.20}$$

$$\mathcal{H}'_g(u) = \beta_2 \tag{3.21}$$

$$g^T_{\epsilon\zeta} = \beta^3_{\epsilon\zeta} \tag{3.22}$$

Moreover, u and ϕ satisfy the following estimate

$$\begin{aligned} ||u||_{W_p^{2,1}} + ||\phi||_{W_p^{2,1}} \leq C \Big(& ||F_1||_{L^p} + ||F_2||_{L^p} + ||\beta_1||_{W_p^{\lambda,\lambda/2}} + ||\beta_2||_{W_p^{\lambda,\lambda/2}} \\ & + ||\beta_3||_{W_p^{\lambda+1,\frac{\lambda+1}{2}}} + ||u_0||_{W^{2,p}} + ||\phi_0||_{W^{2,p}} \Big). \end{aligned} \tag{3.23}$$

Proof. The symbols of the boundary operators, setting $\xi = (\tau,\zeta)$ are:

$$\sum_k i\xi_k \sigma_{kl} - \frac{i}{2}\xi_l \sum_k \sigma_{kk} - imf^{-1}\xi_l \varphi, \tag{3.24}$$

$$i\tau \sum_\alpha \sigma_{\alpha\alpha} - 2i \sum_\alpha \zeta_\alpha \sigma_{\alpha 0}, \tag{3.25}$$

$$\sigma_{\alpha\beta}. \tag{3.26}$$

For $l = 0$, setting (3.24) equal to zero , we obtain

$$i\tau\sigma_{00} + i\sum_\alpha \zeta_\alpha \sigma_{\alpha 0} - \frac{i}{2}\tau \sum_k \sigma_{kk} - i\mu\tau\varphi = 0, \tag{3.27}$$

where $\mu = mf^{-1}$. Similarly, for $l = \beta$ in (3.24) we get the equation

$$i\tau\sigma_{\beta 0} + i\sum_{\alpha}\zeta_\alpha\sigma_{\alpha\beta} - \frac{i}{2}\zeta_\beta\sum_{k}\sigma_{kk} - i\mu\zeta_\beta\varphi = 0, \tag{3.28}$$

while setting (3.25) and (3.26) equal to zero we obtain

$$i\tau\sum_{\alpha}\sigma_{\alpha\alpha} - 2i\sum_{\alpha}\zeta_\alpha\sigma_{\alpha 0} = 0, \tag{3.29}$$

$$\sigma_{\alpha\beta} = 0. \tag{3.30}$$

In particular, combining (3.29) with (3.30) we get

$$\sum_{\alpha}\zeta_\alpha\sigma_{\alpha 0} = 0. \tag{3.31}$$

Equations (3.27) and (3.30) now give

$$\frac{i}{2}\tau\sigma_{00} - i\mu\tau\varphi = 0. \tag{3.32}$$

Moreover, multiplying (3.28) with ζ_β, summing over β and combining with (3.31) we obtain

$$i\tau\sum_{\beta}\zeta_\beta\sigma_{0\beta} - \frac{i}{2}|\zeta|^2\sigma_{00} - i\mu|\zeta|^2\varphi = \frac{i}{2}|\zeta|^2\sigma_{00} + i\mu|\zeta|^2\varphi = 0. \tag{3.33}$$

This, together with (3.32) imply that $\sigma_{00} = \varphi = 0$, since $|\zeta|^2 \neq 0$, $\mu \neq 0$ and $\tau \neq 0$.

Finally, with the aid of equation (3.28) we obtain that $\sigma_{0\alpha} = 0$. Therefore, the boundary value problem satisfies the complementing condition. □

Chapter 4

A general boundary value problem for the Ricci flow.

In this chapter we describe a boundary value problem for the Ricci flow and study issues of short-time existence, regularity and uniqueness. We will focus in the case that the manifold M is compact, describing the required estimates in detail.

The results of this chapter can be extended to non-compact manifolds, as long as the boundary is compact, and the initial metric has geometry bounded in $W^{2,p}$ and curvature in L^p. The modifications needed to obtain the result for the noncompact case are treated in chapter 5.

Let g^0 be a $W^{2,p} \cap C^{1+\epsilon}$ Riemannian metric on M^{n+1}. Consider also $\gamma(x,t) \in C^{1+\epsilon,\frac{1+\epsilon}{2}}(\partial M_T)$, a family of boundary metrics and a function $\eta(x,t) \in C^{\epsilon,\frac{\epsilon}{2}}(\partial M_T)$, where ϵ is always $1-\frac{1}{p} < \epsilon < 1$ and $p > n+3$. Moreover, assume that the following zeroth order compatibility conditions hold.

$$\begin{aligned} [\gamma_0] &= \left[\tilde{g}^T\right] \\ \eta(\cdot,0) &= \mathcal{H}(\tilde{g}) \end{aligned} \tag{4.1}$$

We supplement the Ricci flow equation

$$\partial_t g = -2\,\mathrm{Ric}(g) \tag{4.2}$$

with the boundary conditions

$$\begin{aligned} \left[g^T\right] &= [\gamma_t] \\ \mathcal{H}(g) &= \eta(x,t) \end{aligned} \tag{4.3}$$

and the initial condition

$$g(0) = g^0, \tag{4.4}$$

and aim to study the existence and regularity of solutions.

As is well known, the Ricci flow equation is not strongly parabolic, so we will first study the Ricci-DeTurck equation

$$\partial_t g = -2\,\mathrm{Ric}(g) + \mathcal{L}_{\mathcal{W}(g,\tilde{g})} g, \tag{4.5}$$

with the boundary conditions

$$\begin{array}{rcl} \mathcal{W}(g,\tilde{g}) & = & 0 \\ \left[g^T\right] & = & [\gamma_t] \\ \mathcal{H}(g) & = & \eta(x,t). \end{array} \tag{4.6}$$

Here, $\mathcal{W}(g,\tilde{g})_l = g_{lr} g^{pq} \left(\Gamma^r_{pq}(g) - \tilde{\Gamma}^r_{t,pq}\right)$, $\Gamma(g)$ being the Christoffel symbols of g, and $\tilde{\Gamma}_t$ the Christoffel symbols of a family of metrics $\tilde{g}_t$ with $\tilde{g}|_{t=0} = g^0$. We will assume $\tilde{g}$ is at least $C^2(\overline{M}_T)$, unless g^0 itself doesn't belong to C^2 in which case we chose $\tilde{g} \equiv g^0$.

Finally, define

$$\kappa = \max\left\{ \|g^0\|_{W^{2,p}(M^o)}, |g^0|_{1+\epsilon}, |(g^0)^{-1}|_0, |\gamma|_{1+\epsilon,\frac{1+\epsilon}{2}}, |\eta - \eta_0|_{\epsilon,\frac{\epsilon}{2}} \right\}.$$

Remark 4.0.2. The geometric nature of Ricci flow requires the boundary data to be geometric, namely invariant under diffeomorphisms that fix the boundary. The data (4.3) have this property. However, passing to the DeTurck equation we need to impose the additional, gauge-dependent boundary condition $\mathcal{W}(g,\tilde{g}) = 0$.

Remark 4.0.3. We allow the background metric $\tilde{g}_t$ to vary and define a time dependent reference gauge. This, as will be discussed in section 4.2, allows higher regularity of the solution on $\partial M \times 0$.

4.1 Short-time existence of the Ricci-DeTurck flow.

We can now state and prove the main short time existence Theorem

Theorem 4.1. *Consider the boundary value problem* (4.5),(4.6) *with initial con-*

dition $g(0) = g^0$. For the data $(g^0, \tilde{g}, \eta, \gamma)$ define

$$\Lambda = \max\left\{\kappa, \sup_t \left\{\|\tilde{g} - g^0\|_{W^{2,p}(M^o)} + \|\partial_t \tilde{g}(t)\|_{L_p(M^o)}\right\}\right\}.$$

For any $K > 0$ there exists a $T = T(\Lambda, K) > 0$ and a solution $g(t) \in W^{2,1}_p(M_T)$ of this initial-boundary value problem which satisfies $\|g - g^0\|_{W^{2,1}_p(M_T)} \leq K$.

Proof. Using the background connection $\overline{\nabla}$ the Ricci-DeTurck equation (4.5) can be expressed as

$$\partial_t g - \operatorname{tr}_g \overline{\nabla}^2_{\cdot,\cdot}\, g = \mathcal{R}(g(x,t), \overline{\nabla}\, g(x,t)) - \mathcal{L}_{V(g)} g, \tag{4.7}$$

where $V(g) = g_{ir} g^{pq} (\widetilde{\Gamma}^r_{t,pq} - \overline{\Gamma}^r_{pq})$, while in local coordinates we get (see [CLN06])

$$\begin{aligned}
\mathcal{R}(g, \overline{\nabla}\, g)_{ij} \;\; = \;\; & g^{pq} \bar{g}^{kl} \left(g_{ik} \overline{R}_{jplq} + g_{jk} \overline{R}_{iplq}\right) \\
& - g^{pq} g^{kl} \left(\frac{1}{2}\, \overline{\nabla}_i\, g_{kp}\, \overline{\nabla}_j\, g_{lq} + \overline{\nabla}_p\, g_{jk}\, \overline{\nabla}_l\, g_{iq} - \overline{\nabla}_p\, g_{jq}\, \overline{\nabla}_q\, g_{il}\right) \\
& + g^{pq} g^{kl} \left(\overline{\nabla}_j\, g_{kp}\, \overline{\nabla}_q\, g_{il} + \overline{\nabla}_i\, g_{kp}\, \overline{\nabla}_q\, g_{jl}\right).
\end{aligned}$$

Moreover, we will express the boundary condition for the conformal class in the form:

$$g^T_t - \frac{\operatorname{tr}_{\gamma_t} g^T_t}{n} \gamma_t = 0.$$

Following [Wei91], for $K, T > 0$ we define the following subset of $W^{2,1}_p(M_T)$:

$$M^T_K(g^0) = \left\{ u \in W^{2,1}_p(M_T) \,\middle|\, u|_{t=0} = g^0,\ \|u - g^0\|_{W^{2,1}_p(M_T)} \leq K \right\}.$$

Choose $\delta > 0$ such that $(g^0)^{ij} \xi_i \xi_j \geq \delta |\xi|^2_{eucl}$ in every coordinate system of the fixed atlas. Note that δ is controlled from below in terms of κ. Lemma 2.2 implies that for every $K > 0$, there exists $0 < T_o(K, g^0) \leq 1$ such that $\det(u_{ij}) \geq \delta/2$ and $(u^{-1})^{ii} \geq \delta/2$ for every $u \in M^{T_o}_K(g^0)$. In particular, $u(x,t)$ is a metric for all $t \in [0, T_o]$.

Now, let $T \leq T_o$. For every $w \in M^T_K(g^0)$ the following linear parabolic boundary value problem is well defined:

$$
\begin{aligned}
\partial_t u - \operatorname{tr}_{g^0} \overline{\nabla}^2 u &= \mathcal{R}(w(x,t), \overline{\nabla}\, w(x,t)) - \mathcal{L}_{V(w)} w - \operatorname{tr}_{g^0} \overline{\nabla}^2 w + \operatorname{tr}_w \overline{\nabla}^2 w \equiv F_w \\
\beta_{g^0}(u) &= \beta_{g^0}(w) - \mathcal{W}(w) \equiv D_w \\
\mathcal{H}'_{g^0}(u) &= \mathcal{H}'_{g^0}(w) - \mathcal{H}(w) + \eta(x,t) \equiv G_w \\
u^T - \frac{\operatorname{tr}_{\gamma_t} u^T}{n}\gamma_t &= 0 \\
u|_{t=0} &= g^0
\end{aligned}
$$

and has a unique solution $u \in W^{2,1}_p(M_T)$, by Theorem 3.1. This defines a map

$$S : M^T_K(g^0) \to W^{2,1}_p(M_T)$$

where $S(w)$ is this solution.

Notice that a fixed point of S solves the nonlinear boundary value problem. Therefore, it suffices to prove that S is a map from $M^T_K(g^0)$ to itself and also a contraction, as long as T is small enough. The existence of the fixed point will follow, since $M^T_K(g^0)$ is a complete metric space.

It is easy to see that $\sigma = S(w) - g^0$ satisfies

$$
\begin{aligned}
\partial_t \sigma - \operatorname{tr}_{g^0} \overline{\nabla}^2 \sigma &= F_w + \operatorname{tr}_{g^0} \overline{\nabla}^2 g^0 \equiv \widehat{F}_w \\
\beta_{g^0}(\sigma) &= D_w \\
\mathcal{H}'_{g^0}(\sigma) &= \mathcal{H}'_{g^0}(w - g^0) - (\mathcal{H}(w) - \mathcal{H}(g^0)) + \eta(x,t) - \eta(x,0) \equiv \widehat{G}_w \\
\sigma^T - \frac{\operatorname{tr}_{\gamma_t} \sigma^T}{n}\gamma_t &= 0 \\
\sigma|_{t=0} &= 0
\end{aligned}
$$

Here we used that $\beta_{g^0}(g^0) = 0$ and the compatibility condition $\mathcal{H}(g^0) = \eta|_{t=0}$. Lemma 4.2 below and the parabolic estimate of Theorem 3.1 show that for any K, S maps $M^T_K(g^0)$ to itself, if T is small enough.

Finally, for any $w_1, w_2 \in M^T_K(g^0)$, $S(w_1) - S(w_2)$ similarly satisfies a linear initial-boundary value problem of the form (3.1). Then, the estimate of Lemma 4.3 below shows that S is a contraction for small $T > 0$.

The uniform bound of existence time follows from the fact that a uniform bound of κ implies uniform bounds of the constants of Lemmata 4.2, 4.3, and the constant C_8 of the parabolic estimate of Theorem 3.1. $\square$

4.1.1 The key estimates: Lemmata 4.2 and 4.3

Lemma 4.2. *Let $w \in M_K^T(g^0)$ for some $K > 0$ and $T \leq T_o(K, g^0)$. Then, there exists a constant $C(K, \tilde{g}, \eta)$ and a function $\zeta : [0, +\infty) \to [0, +\infty)$ with $\zeta(T) \to 0$ as $T \to 0$, such that the following estimate holds:*

$$\|\widehat{F}_w\|_{L_p(M_T)} + \|D_w\|_{W_p^{\lambda, \frac{\lambda}{2}}(\partial M_T)} + \|\widehat{G}_w\|_{W_p^{\lambda, \frac{\lambda}{2}}(\partial M_T)} \leq C(K, \tilde{g}, \eta)\zeta(T),$$

where

$$C(K, \tilde{g}, \eta) = C\left(K, \sup_t \left\{\|\tilde{g}_t - g^0\|_{W^{2,p}(M^o)} + \|\partial_t \tilde{g}_t\|_{L_p(M^o)}\right\},\right.$$
$$\left. \|g^0\|_{W^{2,p}(M^o)}, |\eta - \eta_0|_{\epsilon, \frac{\epsilon}{2}}\right)$$

Proof. Since $w \in M_K^T(g^0)$, Lemma 4.5 below implies that $w \in C^1(\overline{M_T})$ and therefore
$$|\mathcal{R}(w, \overline{\nabla}\, w)|_h \leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}\right).$$

This gives

$$\|\mathcal{R}(w, \overline{\nabla}\, w)\|_{L_p(M_T)} \leq C\left(K, \|g^0\|_{W^{2,p}(M^o)}\right)\zeta(T).$$

Next, we estimate

$$\begin{aligned}
\|\mathcal{L}_{V(w)} w\|_{L_p(M_T)} &\leq T^{1/p} \sup_t \|\mathcal{L}_{V(w)} w|_t\|_{L_p(M^o)} \\
&\leq C\left(K, \sup_t \|\tilde{g}_t\|_{W^{2,p}(M^o)}\right)\zeta(T).
\end{aligned}$$

We also have

$$\|\operatorname{tr}_{g^0} \overline{\nabla}^2 g^0\|_{L_p(M_T)} \leq C\left(\|g^0\|_{W^{2,p}(M^o)}\right) T^{1/p}.$$

Combining these estimates we obtain

$$\|\mathcal{R}(w, \overline{\nabla}\, w) - \mathcal{L}_{V(w)} w + \operatorname{tr}_{g^0} \overline{\nabla}^2 g^0\|_{L_p(M_T)} \leq C\left(K, \sup_t \|\tilde{g}(t)\|_{W^{2,p}(M^o)}\right)\zeta(T).$$

To estimate the rest of F_w we estimate using Lemma 4.5:

$$\begin{aligned}
|((g^0)^{ij} - w^{ij})\,\overline{\nabla}^2_{i,j}\,w|_{\bar{g}} &\leq C \max_{k,l} |((g^0)^{ij} - w^{ij})\,\overline{\nabla}^2_{i,j}\,w_{k,l}| \\
&\leq C\left(K, \|g^0\|_{W^{2,1}_p(M_T)}\right) \max_{i,j,k,l} |\,\overline{\nabla}^2_{i,j}\,w_{k,l}| \\
&\leq C\left(K, \|g^0\|_{W^{2,1}_p(M_T)}\right) \zeta(T) |\,\overline{\nabla}^2\,w|_{\bar{g}}.
\end{aligned}$$

This gives the estimate

$$\|\operatorname{tr}_{g^0}\overline{\nabla}^2\,w - \operatorname{tr}_w\overline{\nabla}^2\,w\|_{L_p(M_T)} \leq C\left(K, \|g^0\|_{W^{2,p}(M^o)}\right)\zeta(T)$$

and proves

$$\|\widehat{F}_w\|_{L_p(M_T)} \leq C\left(K, \sup_t \|\tilde{g}(t)\|_{W^{2,p}(M^o)}\right)\zeta(T).$$

It now remains to control the norms of $\widehat{G}_w$ and D_w. Given any $w \in M^T_K(g^0)$ (we assume that $T < T_o(K, g^0)$), define $h = w - g^0$, and for every $0 \leq s \leq 1$

$$g_s(x,t) \;=\; g^0(x) + s \cdot h(x,t)$$

Then by the fundamental theorem of calculus we get that

$$2\mathcal{H}(w) - 2\mathcal{H}(g^0) = \int_0^1 2\mathcal{H}'_{g_s}(h)ds$$

and therefore

$$\begin{aligned}
\widehat{G}_w &:= 2\mathcal{H}'_{g^0}(h) - \left(2\mathcal{H}(w) - 2\mathcal{H}(g^0)\right) + 2(\eta(x,t) - \eta(x,0)) \\
&= \int_0^1 (2\mathcal{H}'_{g_0}(h) - 2\mathcal{H}'_{g_s}(h))ds + 2(\eta(x,t) - \eta(x,0)).
\end{aligned}$$

Now, denoting $A_s := 2\mathcal{H}'_{g_0}(h) - 2\mathcal{H}'_{g_s}(h)$ we calculate

$$A_s = \underbrace{tr_{g_0^T}(\nabla_{0,N_0}h) - tr_{g_s^T}(\nabla_{s,N_s}h)}_{\alpha_1^s}$$
$$+ \underbrace{2\delta_{0,\partial M}(h(N_0)^T) - 2\delta_{s,\partial M}(h(N_s)^T)}_{\alpha_2^s} + \underbrace{h(N_s,N_s)\mathcal{H}(g_s) - h(N_0,N_0)\mathcal{H}(g_0)}_{\alpha_3^s} \tag{4.8}$$

which, in the coordinates of the fixed atlas, are

$$\begin{aligned}
\mathcal{H}(g) &= \frac{1}{2}\left(g^{\alpha\beta}\nu^i\partial_i(g_{\alpha\beta}) + 2\partial_\alpha(\nu^\alpha) + 2g^{\alpha\beta}g_{0\beta}\partial_\alpha(\nu^0)\right) \\
\alpha_1^s &= (g_0^{\alpha\beta}\nu_0^i - g_s^{\alpha\beta}\nu_s^i)\partial_i h_{\alpha\beta} - 2(g_0^{\alpha\beta}\nu_0^i\Gamma^l_{o,i\alpha} - g_s^{\alpha\beta}\nu_s^i\Gamma^l_{s,i\alpha})h_{l\beta} \\
\alpha_2^s &= \left(g_s^{\alpha\beta}\nu_s^i - g_o^{\alpha\beta}\nu_o^i\right)\partial_\alpha h_{i\beta} + \left(g_s^{\alpha\beta}\partial_\alpha(\nu_s^i) - g_o^{\alpha\beta}\partial_\alpha(\nu_o^i)\right)h_{i\beta} \\
&\quad + \left(g_o^{\alpha\beta}\nu_o^i\hat{\Gamma}^j_{o,\alpha\beta} - g_s^{\alpha\beta}\nu_s^i\hat{\Gamma}^j_{s,\alpha\beta}\right)h_{ij} \\
\alpha_3^s &= \frac{1}{2}\left\{\left(\nu_s^i\nu_s^j\nu_s^k g_s^{\alpha\beta}\partial_k(g_{s,\alpha\beta}) - \nu_0^i\nu_0^j\nu_0^k g_0^{\alpha\beta}\partial_k(g_{0,\alpha\beta})\right)h_{ij}\right. \\
&\quad +2\left(\nu_s^i\nu_s^j\partial_\alpha(\nu_s^\alpha) - \nu_0^i\nu_0^j\partial_\alpha(\nu_0^\alpha)\right)h_{ij} \\
&\quad +2\left(\nu_s^i\nu_s^j g_s^{\alpha\beta}g_{s,0\beta}\partial_\alpha(\nu_s^0) - \nu_0^i\nu_0^j g_0^{\alpha\beta}g_{0,0\beta}\partial_\alpha(\nu_0^0)\right)h_{ij},
\end{aligned}$$

where $N_s = \nu_s^i\partial_i = -\frac{g_s^{0i}}{(g_s^{00})^{1/2}}\partial_i$ is the outward unit normal, $\hat{\Gamma}$ the Christoffel symbols of the connection induced on ∂M, and g_s^{ij} represents the inverse of the matrix $g_{s,ij}$(i.e the induced metric on the cotangent bundle). To simplify notation, $g_s^{\alpha\beta}$ denotes the inverse of the matrix $\{g_{\alpha\beta}\}_{\alpha,\beta=1,\ldots,n}$.

Now, to indicate how the estimates of this lemma are established, we show how the term

$$(g_0^{\alpha\beta}\nu_0^i - g_s^{\alpha\beta}\nu_s^i)\partial_i h_{\alpha\beta} = \left((g_0^{\alpha\beta} - g_s^{\alpha\beta})\nu_0^i + (\nu_0^i - \nu_s^i)g_s^{\alpha\beta}\right)\partial_i h_{\alpha\beta}$$

is estimated. We have

$$
\begin{aligned}
|\hat{\rho}(g_0^{\alpha\beta} - g_s^{\alpha\beta})\nu_0^i \partial_i h_{\alpha\beta}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} &\leq C|g_0^{\alpha\beta} - g_s^{\alpha\beta}|_0 |\partial_i h_{\alpha\beta}|_0 |\nu_0^i|_0 \\
&\quad + |g_0^{\alpha\beta} - g_s^{\alpha\beta}|_0 |\nu_0^i|_0 |\hat{\rho}\partial_i h_{\alpha\beta}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&\quad + |g_0^{\alpha\beta} - g_s^{\alpha\beta}|_0 |\partial_i h_{\alpha\beta}|_0 |\hat{\rho}\nu_0^i|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&\quad + |\nu_0^i|_0 |\partial_i h_{\alpha\beta}|_0 |\hat{\rho}(g_0^{\alpha\beta} - g_s^{\alpha\beta})|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&\leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}\right) \zeta(T) \|h\|_{W_p^{2,1}(M_T)},
\end{aligned}
$$

where the last inequality follows from Lemma 4.5.

The terms that are of zeroth order in h, for example $2(g_0^{\alpha\beta}\nu_0^i\Gamma^l_{o,i\alpha} - g_s^{\alpha\beta}\nu_s^i\Gamma^l_{s,i\alpha})h_{l\beta}$, are of first order in g_0 and g_s, but they are estimated in a similar way:

$$
\begin{aligned}
&2(g_0^{\alpha\beta}\nu_0^i\Gamma^l_{o,i\alpha} - g_s^{\alpha\beta}\nu_s^i\Gamma^l_{s,i\alpha})h_{l\beta} = \\
&\qquad 2\left((g_0^{\alpha\beta} - g_s^{\alpha\beta})\nu_0^i\Gamma^l_{0,i\alpha} + g_s^{\alpha\beta}(\nu_0^i - \nu_s^i)\Gamma^l_{0,i\alpha} + g_s^{\alpha\beta}\nu_s^i(\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha})\right) h_{l\beta}.
\end{aligned}
$$

For example, the term $g_s^{\alpha\beta}\nu_s^i(\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha})h_{l\beta}$ can be estimated again using Lemma 4.5;

$$
\begin{aligned}
|\hat{\rho} g_s^{\alpha\beta}\nu_s^i(\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha})h_{l\beta}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} &\leq C|g_s^{\alpha\beta}\nu_s^i|_0 |\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha}|_0 |h_{l\beta}|_0 \\
&+ |g_s^{\alpha\beta}\nu_s^i|_0 |h_{l\beta}|_0 |\hat{\rho}(\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha})|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&+ |g_s^{\alpha\beta}|_0 |h_{l\beta}|_0 |\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha}|_0 |\hat{\rho}\nu_s^i|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&+ |\nu_s^i|_0 |h_{l\beta}|_0 |\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha}|_0 |\hat{\rho} g_s^{\alpha\beta}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&+ |\nu_s^i g_s^{\alpha\beta}|_0 |\Gamma^l_{0,i\alpha} - \Gamma^l_{s,i\alpha}|_0 |\hat{\rho} h_{l\beta}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&\leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}\right) \zeta(T) \|h\|_{W_p^{2,1}(M_T)}.
\end{aligned}
$$

The procedure indicated above carries over to estimate all the terms of

A_s, providing us with the estimate

$$\begin{aligned} |A_s|_{0,V_T} + |\hat{\rho} A_s|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} &\leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}\right) \zeta(T) \|h\|_{W_p^{2,1}(M_T)} \\ &\leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}\right) \zeta(T), \end{aligned}$$

since $\|h\|_{W_p^{2,1}(M_T)} \leq K$. Now, under the assumptions for η, this proves that

$$\|\widehat{G}_w\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} \leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}, |\eta - \eta_0|_{\epsilon,\frac{\epsilon}{2}}\right) \zeta(T).$$

Similarly, the linearization of the map $w \mapsto \mathcal{W}(w, \tilde{g})$ at $u(x,t)$ is given by

$$\mathcal{W}'_u(\tau)_l = \beta_u(\tau)_l + (\tau_{lr} u^{pq} - u_{lr} \tau_{ij} u^{ip} u^{jq})(\Gamma(u)^r_{pq} - \tilde{\Gamma}^r_{t,pq}). \tag{4.9}$$

So, given any $w \in M_K^T(g^0)$, $T < T_0$, since $\beta_{g^0}(g^0) = 0$ we have

$$\begin{aligned} (D_w)_l &= \beta_{g^0}(w)_l - (\mathcal{W}(w,\tilde{g}) - \mathcal{W}(g^0,\tilde{g}))_l - \mathcal{W}(g^0,\tilde{g})_l \\ &= \beta_{g^0}(h)_l - \int_0^1 \mathcal{W}'_{g_s}(h)_l ds - \mathcal{W}(g^0,\tilde{g})_l \\ &= \int_0^1 (\beta_{g^0}(h) - \beta_{g_s}(h))_l ds + \\ &\quad \int_0^1 (h_{lr} g_s^{pq} - g_{s,lr} h_{ij} g_s^{ip} g_s^{jq})(\Gamma^r_{s,pq} - \tilde{\Gamma}^r_{t,pq}) ds - \mathcal{W}(g^0,\tilde{g})_l. \end{aligned}$$

Again, using a coordinate system intersecting the boundary, we have:

$$\begin{aligned} \beta(h) &= g^{ij}\left(\partial_i h_{jl} - h_{rl}\Gamma^r_{ij} - h_{jr}\Gamma^r_{il}\right) - \frac{1}{2}\partial_l(g^{ij}h_{ij}) \\ \beta_{g^0}(h)_l - \beta_{g_s}(h)_l &= (g^{0,ij} - g_s^{ij})\partial_i h_{jl} - (g^{0,ij}\Gamma^r_{0,ij} - g_s^{ij}\Gamma^r_{s,ij})h_{rl} \\ &\quad -(g^{0,ij}\Gamma^r_{0,il} - g_s^{ij}\Gamma_{s,il})h_{jr} \\ &\quad -\frac{1}{2}\left((\partial_l g^{0,ij} - \partial_l g_s^{ij})h_{ij} + (g^{0,ij} - g_s^{ij})\partial_l h_{ij}\right). \end{aligned}$$

Finally, a series of estimates of the same form as those used for the mean curvature part of the boundary conditions gives the required estimate:

$$\begin{aligned} \|D_w\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} &\leq C\left(K, \|\tilde{g} - g^0\|_{W_p^{2,1}(M_T)}, \|g^0\|_{W^{2,p}(M^o)}\right) \zeta(T) \\ &\leq C(K, \tilde{g}, \eta)\zeta(T). \end{aligned}$$

□

Lemma 4.3. *Let $K > 0$ and $T \leq T_o(K, g^0)$. Then, there exists a constant $C(K, \tilde{g})$ such that for every $w_1, w_2 \in M_K^T(g^0)$ the following estimate holds:*

$$\|F_{w_1} - F_{w_2}\|_{L_p(M_T)} + \|D_{w_1} - D_{w_2}\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} + \|G_{w_1} - G_{w_2}\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} \leq C(K, \tilde{g})\zeta(T)\|w_1 - w_2\|_{W_p^{2,1}(M_T)}, \quad (4.10)$$

where

$$C(K, \tilde{g}) = C\left(K, \sup_t \left\{\|\tilde{g} - g^0\|_{W^{2,p}(M^o)} + \|\partial_t \tilde{g}\|_{L_p(M^o)}\right\}, \|g^0\|_{W^{2,p}(M^o)}\right).$$

Proof. If $w_1, w_2 \in M_K^T(g^0)$ and w_1^{-1}, w_2^{-1} are the metrics induced on the cotangent bundle, Lemma 2.2 gives that

$$|w_1 - w_2|_1 + |w_1^{-1} - w_2^{-1}|_0 \leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}\right)\zeta(T)\|w_1 - w_2\|_{W_p^{2,1}(M_T)}.$$

which is enough to prove in a similar manner with Lemma 4.2 that

$$\|\mathcal{R}(w_1, \overline{\nabla}\, w_1) - \mathcal{R}(w_2, \overline{\nabla}\, w_2) - (\mathcal{L}_{V(w_1)}w_1 - \mathcal{L}_{V(w_2)}w_2)\|_{L_p(M_T)} \leq C\left(K, \sup_t \|\tilde{g}(t)\|_{W^{2,p}(M^o)}\right)\zeta(T)\|w_1 - w_2\|_{W_p^{2,1}(M_T)}. \quad (4.11)$$

Then, again as in the proof of Lemma 4.2, we have

$$\begin{aligned}
&|\operatorname{tr}_{g^0} \overline{\nabla}^2 w_1 - \operatorname{tr}_{w_1} \overline{\nabla}^2 w_1 - (\operatorname{tr}_{g^0} \overline{\nabla}^2 w_2 - \operatorname{tr}_{w_2} \overline{\nabla}^2 w_2)|_h \leq \\
&\qquad |(g^{0,ij} - w_1^{ij})\,\overline{\nabla}^2_{i,j}(w_1 - w_2)|_h \quad + |(w_2^{ij} - w_1^{ij})\,\overline{\nabla}^2_{i,j}\, w_2|_h \\
&\qquad \leq C(K, g^0)\zeta(T)|\,\overline{\nabla}^2(w_1 - w_2)|_h + C(K, g^0)|w_2 - w_1|_0\,|\,\overline{\nabla}^2\, w_2|_h
\end{aligned}$$

which, by Lemma 2.2, gives

$$\|\operatorname{tr}_{g^0} \overline{\nabla}^2 w_1 - \operatorname{tr}_{w_1} \overline{\nabla}^2 w_1 - (\operatorname{tr}_{g^0} \overline{\nabla}^2 w_2 - \operatorname{tr}_{w_2} \overline{\nabla}^2 w_2)\|_{L_p(M_T)} \leq C\left(K, \|g^0\|_{W_p^{2,1}(M_T)}\right)\zeta(T)|w_1 - w_2|_{L_p^{2,1}(M_T)} \quad (4.12)$$

and proves

$$\|F_{w_1} - F_{w_2}\|_{L_p(M_T)} \leq C\left(K, \sup_t \|\tilde{g}\|_{W^{2,p}(M^o)}\right)\zeta(T)\|w_1 - w_2\|_{W_p^{2,1}(M_T)}.$$

To establish the estimates for the boundary data, we define

$$\begin{aligned} h &= w_1 - w_2 \\ g_s &= w_1 + s \cdot h. \end{aligned}$$

Then, again as in the proof of Lemma 4.2, we get:

$$\begin{aligned} G_{w_1} - G_{w_2} &= \int_0^1 (2\mathcal{H}'_{g^0}(h) - 2\mathcal{H}'_{g_s}(h))ds \\ D_{w_1} - D_{w_2} &= \int_0^1 (\beta_{g^0}(h) - \beta_{g_s}(h))_l ds + \\ &\quad \int_0^1 (h_{lr} g_s^{pq} - g_{s,lr} h_{ij} g_s^{ip} g_s^{jq})(\Gamma^r_{s,pq} - \tilde{\Gamma}^r_{t,pq})ds \end{aligned}$$

and the same computation gives the estimate

$$\|G_{w_1} - G_{w_2}\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} + \|D_{w_1} - D_{w_2}\|_{W_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} \leq C(K, \tilde{g})\zeta(T)\|w_1 - w_2\|_{W_p^{2,1}(M_T)}.$$

□

4.1.2 A few technical Lemmata.

Lemma 4.4. *Let $\delta_0 > 0$. There exists a positive constant C, such that for matrix valued functions $g, g_l \in L^\infty(V_T) \cap \mathcal{L}_p^{\alpha,\beta}(V_T)$, $l = 1, 2$ for which $det(g_{ij}), det(g_{l,ij}) \geq \delta_0$ and $g^{ii}, g_l^{ii} \geq \delta_0$ holds:*

1. $|\hat{\rho} g^{ij}|_{\mathcal{L}_p^{\alpha,\beta}(V_T)} \leq C|g|_0 \left(|\hat{\rho} g|_{\mathcal{L}_p^{\alpha,\beta}(V_T)} + 1\right)$
2. $|\hat{\rho}\left(g_1^{ij} - g_2^{ij}\right)|_{\mathcal{L}_p^{\alpha,\beta}(V_T)} \leq C \cdot B_1 \cdot B_2 \left(|\hat{\rho}(g_1 - g_2)|_{\mathcal{L}_p^{\alpha,\beta}(V_T)} + |g_1 - g_2|_0\right)$
3. $|\hat{\rho}(g^{00})^{-1/2}|_{\mathcal{L}_p^{\alpha,\beta}(V_T)} \leq C \cdot |g|_0 \left(|\hat{\rho} g|_{\mathcal{L}_p^{\alpha,\beta}} + 1\right)$

4. $|\hat{\rho}\left((g_1^{00})^{-1/2}-(g_2^{00})^{-1/2}\right)|_{\mathcal{L}_p^{\alpha,\beta}(V_T)} \leq C\cdot B_1\cdot B_2.\left(|\hat{\rho}\left(g_1-g_2\right)|_{\mathcal{L}_p^{\alpha,\beta}(V_T)}+|g_1-g_2|_0\right)$

where $|g_l|_0 \leq B_1$ *and* $|\hat{\rho}g_l|_{\mathcal{L}_p^{\alpha,\beta}(V_T)} \leq B_2$ *and the constants depend on* δ *and the cutoff function* $\hat{\rho}$.

Proof. The result follows from Weidemaier, Corollary A.3. □

Lemma 4.5. *Let* $g_0, g_1 \in M_K^T(g^0)$, $T \leq T_0(K,g^0)$, $g_s = g_0 + s(g_1-g_0)$ *and* (U,ϕ,ρ) *a chart whose domain intersects the boundary, with the corresponding cutoff function* ρ *and* $\hat{\rho} = \rho\circ\phi^{-1}$. *Let also* $\nu_s^i = -\frac{g_s^{0i}}{(g_s^{00})^{-1/2}}$ *be the components of the outward unit normal to the boundary with respent to* g_s. *Then:*

1. $|g_{s,ij}|_{1,V_T} + |\hat{\rho}g_{s,ij}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} + |\hat{\rho}\partial_k g_{s,ij}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K,g^0)$

2. $|g_0^{ij} - g_s^{ij}|_1 + |g_{0,ij} - g_{s,ij}|_1 \leq C(K,g^0,\delta)\zeta(T)\|h\|_{W_p^{2,1}(M_T)}$

3. $|\hat{\rho}(g_0^{ij} - g_s^{ij})|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} + |\hat{\rho}(g_{0,ij} - g_{s,ij})|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K,g^0,\delta)\zeta(T)\|h\|_{W_p^{2,1}(M_T)}$

4. $|\hat{\rho}\partial_k g_s^{ij}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K,g^0,\delta)$

5. $|\nu_s^i|_0 + |\hat{\rho}\nu_s^i|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} + |\hat{\rho}\partial_\alpha \nu_s^i|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K,g^0,\delta)$

6. $|\nu_0^i - \nu_s^i|_1 + |\hat{\rho}(\nu_0^i - \nu_s^i)|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K,g^0,\delta)\zeta(T)\|h\|_{W_p^{2,1}(M_T)}$

7. $|\hat{\rho}\partial_\alpha \nu_s^i|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K,g^0,\delta)$

where $\lim\limits_{T\to 0^+}\zeta(T) = 0$.

Proof. Since g_0, g_1 are in $M_K^T(g^0)$ we get that $g_s \in M_K^T(g^0)$. Therefore the estimate

$$\|g_s - g^0\|_{W_p^{2,1}(M_T)} \leq K$$

holds. Using Lemma 2.2, and assuming $T \leq 1$ we compute:

$$\begin{aligned}
|g_s|_1 &\leq |g_s - g^0|_1 + |g^0|_1 \leq C_4 K + C_4^*(g^0) \\
|g_s|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} &\leq |g_s - g^0|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} + |g^0|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} \leq C_5 K + C_5^*(g^0)
\end{aligned}$$

where $C_4^* = |g^0|_1$ and $C_5^* = |g^0|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)}$.

We also have that $\|g_s\|_{W_p^{2,1}(M_T)} \leq \|g_s - g^0\|_{W_p^{2,1}(M_T)} + \|g^0\|_{W_p^{2,1}(M_T)}$. This, using Lemma 2.1, gives:

$$|\overline{\nabla}\, g_s|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(\partial M_T)} \leq C_1 \|g_s\|_{W_p^{2,1}(M_T)} \leq C_1 K + C_1^*(g^0)$$

where $C_1^*(g^0) = \|g^0\|_{W_p^{2,1}(M_T)}$. Now, the estimate

$$|\hat{\rho}\partial_k g_{s,ij}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C\left(\bar{g}, \{U_s\}, \phi_s\right) |\hat{\rho}\,\overline{\nabla}_k\, g_{s,ij}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)}$$

completes the proof of (1).

Estimate (2) follows directly from (1), Lemma 2.2 and the fact that $\det(g_{s,ij}) \geq \delta/2$, since the inverse is given in terms of the determinant and the cofactor matrix.

Lemma 4.4, proves (3), since

$$\begin{aligned}
|g_{s,ij}|_0 + |\hat{\rho} g_{s,ij}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} &\leq C(K, g^0) \\
|\hat{\rho}(g_{0,ij} - g_{s,ij})| &\leq C_5 T^\gamma \|g_0 - g_s\|_{W_p^{2,1}(M_T)} \leq C_5 T^\gamma \|h\|_{W_p^{2,1}(M_T)} \\
|g_0 - g_s|_0 &\leq C_4 T^\gamma \|h\|_{W_p^{2,1}(M_T)}
\end{aligned}$$

where the first inequality follows from (1) while the second and third inequalities by Lemma 2.2.

For estimate (4), note that $\partial_k(g_s^{ij}) = -g_s^{qi}\partial_k g_{s,pq} g_s^{pj}$. Then, by (1) we get the estimate

$$|\hat{\rho}\partial_k(g_s^{ij})|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K, g^0, \delta)$$

The estimate of $|\nu_0^i - \nu_s^i|_0 + |\partial_\alpha(\nu_0^i - \nu_s^i)|_0$ follows from (1) and (2) of this Lemma. For the rest we have:

$$\begin{aligned}
|\hat{\rho}(\nu_0^i - \nu_s^i)|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} &= \left|\hat{\rho}\left(\frac{g_0^{0i}}{\sqrt{g_0^{00}}} - \frac{g_s^{0i}}{\sqrt{g_s^{00}}}\right)\right|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&\leq \left|\hat{\rho}\left((g_0^{00})^{-\frac{1}{2}} - (g_s^{00})^{-\frac{1}{2}}\right) g_0^{0i}\right|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} + |\hat{\rho}(g_0^{0i} - g_s^{0i})(g_s^{00})^{-\frac{1}{2}}|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \\
&\leq C(K, g^0, \delta)\zeta(T)\|h\|_{W_p^{2,1}(M_T)}
\end{aligned}$$

where the last inequality follows from Lemma 4.4, Lemma 2.2 and the fact that $T \leq T_0$.

Lemma 4.4, since $g_s^{00} \geq \delta/2$, also proves that

$$|\hat{\rho}\partial_\alpha(\nu_s^i)|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} = \left|\partial_\alpha\left(\frac{g_s^{0i}}{\sqrt{g_s^{00}}}\right)\right|_{\mathcal{L}_p^{\lambda,\frac{\lambda}{2}}(V_T)} \leq C(K, g^0, \delta)$$

and this finishes the proof of this Lemma. □

4.2 Regularity of the Ricci-DeTurck flow.

The solution of the Ricci-DeTurck boundary value problem obtained in the previous section is in the Sobolev space $W_p^{2,1}(M_T)$ for $p > n+3$, and therefore has only $C^{1+\alpha,\frac{1+\alpha}{2}}$ regularity in $\overline{M_T}$. In this section we show that certain higher order compatibility conditions on ∂M are necessary and sufficient to obtain higher regularity on $\partial M \times 0$. We also obtain an automatic smoothing effect of the flow for positive time (up to the boundary).

4.2.1 Higher order compatibility conditions

Assuming that $g(x,t)$ is a $C^{l+2,\frac{l}{2}+1}(\overline{M_T})$ solution to the Ricci-DeTurck flow

$$\partial_t g = -2\,\mathrm{Ric}(g) + \mathcal{L}_{\mathcal{W}(g,\tilde{g})} g$$

we easily see that all the derivatives $h_k \equiv \partial_t^k g|_{t=0} \in C^{l+2-2k}(M)$, $0 \leq k \leq \left[\frac{l}{2}\right] + 1$ are determined by the initial data $g|_{t=0} = g^0 \in C^{l+2}(\bar{M})$, by differentiating the equation with respect to t, and then commuting ∂_t^k with $\partial_i\partial_j$, which is possible as long as $2k+2 < l+2$, i.e. $0 \leq k \leq \left[\frac{l}{2}\right]$.

Moreover, if $g(x,t)$ satisfies the boundary conditions (4.6), differentiating with respect to t we get:

$$\partial_t^k \mathcal{W}(g_t, \tilde{g}_t)|_{t=0} = 0 \tag{4.13}$$

$$\partial_t^k \mathcal{H}(g_t)|_{t=0} = \partial_t^k \eta|_{t=0} \tag{4.14}$$

$$\partial_t^k \left(g_t^T - \frac{\mathrm{tr}_{\gamma_t}(g_t^T)}{n}\gamma_t\right)\Bigg|_{t=0} = 0 \tag{4.15}$$

So, from (4.15) for $k \leq \left[\frac{l}{2}\right] + 1$, we see that additional conditions need to be satisfied by h_k, and hence by g^0, on ∂M. Similarly, for $k \leq \left[\frac{l+1}{2}\right]$ (so as $2k+1 < l+2$), the ∂_t^k derivatives commute with the space derivatives of the first order operators $\mathcal{W}, \mathcal{H}$ on (4.13), (4.14), and give additional restrictions on the initial data on the boundary.

In particular, if $l > 0$, since $(g^0)^T = \gamma|_{t=0}$ we see that $\dot{\gamma}\ |_{t=0}$ is specified, up to a conformal factor by h_1:

$$\dot{\gamma}\ |_{t=0} = h_1^T - \frac{\mathrm{tr}_{\gamma_0} h_1^T}{n}\gamma_0 + f\gamma_0 \tag{4.16}$$

where f is an arbitrary function. Moreover, if $l > 1$, $\dot{\eta}\ |_{t=0}$ is also specified by the initial data:

$$\dot{\eta}\ |_{t=0} = \mathcal{H}'_{g^0}(h_1) \tag{4.17}$$

In the section below, we show how parabolic regularity implies that these conditions are also sufficient to obtain higher regularity of a solution to the Ricci-DeTurck boundary value problem.

4.2.2 Regularity up to $t = 0$

Theorem 4.6. *Let $g(x,t) \in W_p^{2,1}(M_T)$ a solution of the Ricci-DeTurck boundary value problem (4.21),(4.22). For $l = k + \alpha$, $\alpha \leq 1 - \frac{n+3}{p}$, if $\tilde{g} \in C^{l+2,\frac{l+2}{2}}(M^o \times [0,T])$ then $g \in C^{l+2,\frac{l+2}{2}}(M^o \times [0,T])$. Moreover, if $\eta \in C^{l+1,\frac{l+1}{2}}(\partial M_T)$, $\gamma \in C^{l+2,\frac{l+2}{2}}(\partial M_T)$ and $\tilde{g} \in C^{l+2,\frac{l+2}{2}}(\overline{M_T})$ and the data g^0,η,γ,$\tilde{g}$ satisfy the necessary compatibility conditions, there exists a τ, $0 < \tau \leq T$, such that $g \in C^{l+2,\frac{l+2}{2}}(\overline{M_\tau})$.*

Proof. First we prove interior regularity. Let $p \in M^o$, any point in the interior of M, U a neighbourhood of p not intersecting the boundary, a smooth chart $\phi : U \to B(0,1)$ and a smooth cutoff function ζ defined on $\mathbb{R}^{n+1}$ such that $\zeta|_{B(0,1/2)} \equiv 1$ and $\zeta|_{B(0,1)-B(0,3/4)} \equiv 0$.

Given a Riemannian metric g_{ij} on $B(0,1)$ we can define the differential operator

$$\mathcal{L}(\partial_t, \partial_x, g_{ij})(u)_{kl} = \partial_t(u_{kl}) - g^{ij}\partial_i\partial_j(u_{kl})$$

acting on symmetric 2-tensors on $B(0,1)$.

Now, since $g(x,t)$ solves the Ricci-DeTurck flow, in these coordinates it satisfies the parabolic equation

$$\mathcal{L}(\partial_t, \partial_x, g_{ij})(g)_{kl} = \mathcal{S}(g, \partial g, \tilde{g}, \partial \tilde{g}, \partial^2 \tilde{g})_{kl} \tag{4.18}$$

Define $v_{kl} = \zeta g_{kl}$. It is easy to see that it satisfies the equation

$$\partial_t v_{kl} - g^{ij}\partial_i\partial_j(v_{kl}) = \zeta \mathcal{S}_{kl} - g^{ij}\partial_i\partial_j(\zeta)g_{kl} - 2g^{ij}\partial_i(\zeta)\partial_j(g_{kl}) \tag{4.19}$$

on $B(0,1)$, with initial data $v_{kl}|_{t=0} = \zeta g^0_{kl} \in C^{2+\alpha}$ and Dirichlet boundary data $v|_{\partial B(0,1)} \equiv 0$. Since the compatibility conditions of any order hold (ζ is zero on a neighbourhood of $\partial B(0,1)$), and the coefficients of the equation are in $C^{\alpha,\frac{\alpha}{2}}(\overline{B(0,1)_T})$ by Lemma 2.2, it follows from standard parabolic theory (see [LSU67]) that this boundary value problem has a unique solution in $C^{2+\alpha,(2+\alpha)/2}(\overline{B(0,1)_T})$. Since $p \in M^o$ was arbitrary, we obtain that $g \in C^{2+\alpha,\frac{2+\alpha}{2}}(M^o \times [0,T])$.

The argument presented above, since the coefficients and the right hand side of (4.19) are in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{B(0,1)} \times [0,T])$, shows that if $\tilde{g}$ is in $C^{3+\alpha,\frac{3+\alpha}{2}}(M^o \times [0,T])$, so is g. Thus, a standard bootstrapping argument shows that $g \in C^{k+\alpha,\frac{k+\alpha}{2}}(M^o \times [0,T])$ as long as $\tilde{g}$ is.

It remains to prove the regularity of the solution of the Ricci-DeTurck equation in a neighbourhood of the boundary, under the assumptions of the Theorem.

We need to establish some notation first. Let $\phi : U \to \phi(U) \subset \mathbb{R}^{n+1}$ be any smooth chart on a domain U intersecting the boundary of M, such that $\phi(U \cap \partial M) = \phi(U) \cap \{x^0 = 0\}$, and let g_{ab}, $\gamma_{\varepsilon\sigma}$ be symmetric positive definite $(n+1) \times (n+1)$ and $n \times n$ matrices respectively. Define the following differential operators

$$\begin{aligned}
B(\partial_x, g_{ab})(u)_i &= g^{pq}\partial_p(u_{qi}) - \frac{1}{2}g^{pq}\partial_i(u_{pq}) \\
H(\partial_x, g_{ab})(u) &= g^{T,\alpha\beta}\nu^i\partial_i(u_{\alpha\beta}) + \\
&\quad \left(\frac{2g^{0l}g^{\alpha k}}{\sqrt{g^{00}}} - \frac{g^{0l}g^{0k}g^{0\alpha}}{(\sqrt{g^{00}})^3} + \frac{g^{T,\alpha\beta}g_{0\beta}g^{0l}g^{0k}}{\sqrt{g^{00}}}\right)\partial_\alpha(u_{kl}) \\
C(\gamma_{\varepsilon\sigma})(u)_{\alpha\beta} &= u_{\alpha\beta} - \frac{\gamma^{\mu\nu}\gamma_{\alpha\beta}}{n}u_{\mu\nu}
\end{aligned}$$

Now, take any $p \in \partial M$, and consider a smooth coordinate system as the above with $\phi(p) = 0$ and $g_{ij}(0)|_{t=0} = \delta_{ij}$.

By Lemma 2.5, $2\mathcal{H}(g) = H(\partial_x, g_{ab}(x,t))(g)$. Thus, in addition to (4.18), $g_{ij}(x,t)$ satisfy the following conditions on $\phi(U) \cap \{x^0 = 0\}$.

$$\begin{aligned} B(\partial_x, g_{ab}(x,t))(g)_i &= g^{pq} g_{ri} \tilde{\Gamma}^r_{pq} \\ H(\partial_x, g_{ab}(x,t))(g) &= 2\eta(x,t) \\ C(\gamma_{\varepsilon\sigma})(g)_{\alpha\beta} &= 0 \end{aligned} \tag{4.20}$$

and the initial condition $g_{ij}|_{t=0} = g^0_{ij}$.

Notice that after "freezing" the coefficients at $x = 0, t = 0$, the operators $\mathcal{L}(\delta_{ab})$, $B(\delta_{ab})$, $H(\delta_{ab})$, $C(\delta_{\varepsilon\sigma})$, satisfy the complementing condition, as the computation in Theorem 3.1 shows. The openness of this condition implies that the same is true for the operators $\mathcal{L}(g_{ab})$, $B(g_{ab})$, $H(g_{ab})$, $C(\gamma_{\varepsilon\sigma})$, as long as g_{ab} and $\gamma_{\varepsilon\sigma}$ are close to δ_{ab}.

We will extend (4.19), (4.20) to a parabolic boundary value problem on $\mathbb{R}^{n+1}_+$ using a smooth cutoff function $0 \leq \eta \leq 1$ on $\mathbb{R}^{n+1}_+$ supported in a ball $B^+(0,r)$ such that $\eta|_{B^+(0,r/2)} \equiv 1$. For this, define the metrics

$$a_{ab}(x,t) = \eta g_{ab}(x,t) - (1-\eta)\delta_{ab}$$

on $\mathbb{R}^{n+1}_+$, and

$$\alpha_{\varepsilon\sigma} = \eta\gamma_{t,\varepsilon\sigma} + (1-\eta)\delta_{\varepsilon\sigma}$$

on $\mathbb{R}^n$. Then, choosing $r, \tau > 0$ small enough, the operators $\mathcal{L}(a_{ab}(x,t))$, $B(a_{ab}(x,t))$, $H(a_{ab}(x,t))$, $C(\alpha_{\varepsilon\sigma}(x,t))$ will satisfy the complementing condition, defining a parabolic boundary value problem on $\mathbb{R}^{n+1}_+ \times [0,\tau]$.

Now, let $0 \leq \zeta \leq 1$ be a smooth cutoff function supported in $B^+(0,r/2)$, $\zeta|_{B^+(0,r/3)} \equiv 1$, and set $v = \zeta g$.

Then v satisfies the equation

$$\mathcal{L}(\partial_t, \partial_x, a_{ij})(v)_{kl} = \zeta\mathcal{S}_{kl} - g^{ij}\partial_i\partial_j(\zeta)g_{kl} - 2g^{ij}\partial_i(\zeta)\partial_j(g_{kl})$$

and the boundary conditions

$$\begin{aligned} B(\partial_x, a_{ab}(x,t))(v)_i &= \left(1 - \frac{n}{2}\right)\partial_i(\zeta) + \zeta g^{pq}\tilde{\Gamma}^r_{t,pq} g_{ri} \\ H(\partial_x, a_{ab}(x,t))(v) &= 2\zeta\eta + n\nu^i\partial_i(\zeta) + \nu^\alpha\partial_\alpha(\zeta) + \frac{g_T^{\alpha\beta} g_{0\beta} g^{0l} g^{0k} g_{kl}}{\sqrt{g^{00}}}\partial_\alpha(\zeta) \\ C(\alpha_{\varepsilon\sigma})(v)_{\alpha\beta} &= 0 \end{aligned}$$

on $\mathbb{R}^{n+1}_+$. Therefore, by Theorem 5.4 of [Sol65], and the same bootstrapping argument we used for the interior regularity, we can prove that $g \in C^{l+2,\frac{l+2}{2}}(\overline{M} \times [0,\tau])$ for some small $\tau > 0$, since ζg^0_{kl} satisfies the necessary higher order compatibility conditions as long as g^0_{kl} does. $\square$

Remark 4.2.1. For $g \in W^{2,1}_p(M_T)$, the metrics $g(t)$ are uniformly equivalent and satisfy a uniform Hölder condition in the t direction. Therefore, one can iterate the argument above to show boundary regularity up to time T.

4.2.3 Boundary regularity for $t > 0$

Theorem 4.7. *Let g be a solution in $W^{2,1}_p(M_T)$ of the Ricci-DeTurck boundary value problem with respect to a C^∞ family of background metrc is $\tilde{g}$ and C^∞ initial and boundary data. Then g is in $C^\infty(M \times (0,\tau])$. In particular the compatibility conditions of any order are satisfied by $g(t)$ for any $t > 0$.*

Proof. We already know that $g \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{M_T})$, for $\alpha \leq 1 - \frac{n+3}{p}$. Setting $l = k + \alpha$, we are going to do induction on the order k of the regularity. Suppose that $g \in C^{l+1,\frac{l+1}{2}}(M \times (0,\tau])$.

Set $d = \left[\frac{l}{2}\right] + 1$, and choose $0 < \tau_o < \tau$. We will use the notation

$$v^{\tau_o}(x,t) \equiv v(x,t+\tau_o).$$

The idea is to use the arguments in the proof of Theorem 4.6 for the quantity

$$w_{kl} = t^d \zeta g^{\tau_o}_{kl}$$

For the boundary regularity (interior regularity can be proven in the same way), w will satisfy the equation

$$\mathcal{L}(\partial_t, \partial_x, a^{\tau_o}_{ab})(w)_{kl} = t^d \left\{ \zeta \mathcal{S}_{kl} - g^{ij}\partial_i\partial_j(\zeta) g_{kl} - 2g^{ij}\partial_i(\zeta)\partial_j(g_{kl}) \right\}^{\tau_o} + dt^{d-1}\zeta g^{\tau_o}_{kl}$$

on $\mathbb{R}^{n+1}_+$, the conditions

$$\begin{aligned}
B(\partial_x, a^{\tau_o}_{ab})(w)_i &= t^d \left\{ \left(1 - \frac{n}{2}\right) \partial_i(\zeta) + \zeta g^{pq} \tilde{\Gamma}^r_{pq} g_{ri} \right\}^{\tau_o} \\
H(\partial_x, a^{\tau_o}_{ab})(w) &= t^d \left\{ 2\zeta\eta + n\nu^i \partial_i(\zeta) + \nu^\alpha \partial_\alpha(\zeta) + \frac{g_T^{\alpha\beta} g_{0\beta} g^{0l} g^{0k} g_{kl}}{\sqrt{g^{00}}} \partial_\alpha(\zeta) \right\}^{\tau_o} \\
C(\alpha^{\tau_o}_{\varepsilon\sigma})(w)_{\alpha\beta} &= 0
\end{aligned}$$

on $\partial\mathbb{R}^{n+1}_+$ and the initial condition $w|_{t=0} = 0$.

The coefficients of $\mathcal{L}$, B, H and C are in $C^{l+1,\frac{l+1}{2}}(M \times [0, \tau - \tau_o])$, while the right hand side of the equation is in $C^{l,l/2}(M \times [0, \tau - \tau_o])$. Then w will be in $C^{l+2,\frac{l+2}{2}}(M \times [0, \tau - \tau_o])$ as long as the necessary compatibility conditions hold.

To prove this, first note that assuming that $w \in C^{l+2,\frac{l+2}{2}}(M \times [0, \tau - \tau_o])$, we have that

$$\partial_t^i w|_{t=0} = \begin{cases} 0 & i < d \\ d!\zeta g^{\tau_o} & i = d \end{cases}$$

Using this, and commuting derivatives, we compute for $0 \le i \le d-1$

$$\begin{aligned}
\partial_t^i (B(a_{ab}^{\tau_o})(w)|_{t=0} &= \partial_t^i \left(t^d \left\{ \left(1 - \frac{n}{2}\right) \partial_i(\zeta) + \zeta g^{pq} \tilde{\Gamma}^r_{pq} g_{ri} \right\}^{\tau_o} \right)\Big|_{t=0} = 0 \\
\partial_t^i (H(a_{ab}^{\tau_o})(w)|_{t=0} &=
\end{aligned}$$

$$\partial_t^i \left(t^d \left\{ 2\zeta\eta + n\nu^i \partial_i(\zeta) + \nu^\alpha \partial_\alpha(\zeta) + \frac{g_T^{\alpha\beta} g_{0\beta} g^{0l} g^{0k} g_{kl}}{\sqrt{g^{00}}} \partial_\alpha(\zeta) \right\}^{\tau_o} \right)\Bigg|_{t=0} = 0$$

and $\partial_t^i (C(\alpha_{\varepsilon\sigma}^{\tau_o})(w))|_{t=0} = 0$ for $0 \le i \le d$. This shows that the initial and boundary data satisfy the compatibility conditions.

Hence, $w \in C^{l+2,\frac{l+2}{2}}(M \times [0, \tau - \tau_o])$, and since τ_o was arbitrary $g \in C^{l+2,\frac{l+2}{2}}(M \times (0, \tau])$. □

Remark 4.2.2. The assumption that the data are smooth is not necessary. Given an initial metric in $C^{k,\epsilon}(M) \cap W^{2,p}(M^o)$, $k \ge 1$, boundary data $\gamma \in C^{l,l/2}(\partial M \times (0, \tau])$, $\eta \in C^{l-1,\frac{l-1}{2}}(\partial M \times (0, \tau])$ and $\tilde{\Gamma}_t \in C^{l-1,\frac{l-1}{2}}(\partial M \times (0, \tau])$, for $l = k' + \alpha$, $k' \ge k$, the solution will be in $C^{l,l/2}(M \times (0, \tau])$.

4.2.4 Regularity of the DeTurck vector field

According to the following Proposition, the DeTurck vector field $\mathcal{W}$ can gain one derivative, without requiring all the compatibility conditions needed to increase the regularity of g. Only higher order compatibility of the initial data with the reference metrics is needed. Thus, it can be assumed to be as smooth as the solution to the Ricci-DeTurck flow.

Proposition 4.2.1. *Let $g \in C^{l,l/2}(\overline{M_T})$ be a solution of the Ricci-DeTurck equation, $l > 3$. Assume further that $\tilde{g}$ is in $C^{l+1,\frac{l+1}{2}}(\overline{M_T})$ and that the compatibility* *condition $h_k = \partial_t^k \tilde{g}_t|_{t=0}$ is satisfied for $k \leq \left[\frac{l+1}{2}\right]$. Then, the DeTurck vector field $\mathcal{W}$ is in $C^{l,l/2}(\overline{M_T})$.*

Proof. Applying the Bianchi operator $\beta_g = \operatorname{div}_g - \frac{1}{2} d \operatorname{tr}_g$ in both sides of the Ricci-DeTurck equation we get

$$\beta_g(\partial_t g) = \beta_g(\mathcal{L}_{\mathcal{W}} g).$$

Commuting derivatives we obtain

$$\beta_g(\mathcal{L}_{\mathcal{W}} g) = \Delta \mathcal{W} + \operatorname{Ric}(\mathcal{W}),$$

where $\Delta = \operatorname{tr}_g \nabla^2$ and $\operatorname{Ric}(\mathcal{W}) = \operatorname{Ric}(\mathcal{W}, \cdot)$.

By the linearization formula of $\mathcal{W}$ (4.9) we get that

$$\partial_t \mathcal{W}_b = \beta_g(\partial_t g)_b - g_{br} U_{ij} g^{ip} g^{jq} \left(\Gamma^r_{pq} - \tilde{\Gamma}^r_{t,pg} \right) - g^{pq} \partial_t (\tilde{\Gamma}^r_{t,pq}),$$

with $U_{ij} = -2 \operatorname{Ric}_{ij} + \mathcal{L}_{\mathcal{W}} g_{ij}$.

Combining the above we get the following evolution equation for $\mathcal{W}$

$$\partial_t \mathcal{W} = \Delta \mathcal{W} + \operatorname{Ric}(\mathcal{W}) + Q,$$

where Q is an expression involving at most two derivatives of the metric. By parabolic regularity, given the Dirichlet boundary condition $\mathcal{W}|_{\partial M} = 0$ and the validity of the compatibility conditions at $t = 0$ it follows that $\mathcal{W} \in C^{l,l/2}(M \times [0, T])$ as long as $g \in C^{l,l/2}(M \times [0, T])$. □

4.3 Uniqueness of the Ricci-DeTurck flow

Let $g_1, g_2 \in W^{2,1}_p(M_T)$ be two solutions to the Ricci-DeTurck boundary value problem (4.21),(4.22) satisfying the same initial and boundary data. Choosing $K > 0$ such that $g_i \in M^T_K(g^0)$, there is a $\hat{\tau} > 0$ such that the map S defined in the proof of the existence Theorem is a contraction map of $M^{\hat{\tau}}_K(g^0)$ to itself, and therefore has a unique fixed point. Since g_1, g_2 are both fixed points, they have to agree on $[0, \hat{\tau}]$. Assuming the data are smooth enough to guarantee that $g_i(t)$ are C^2 for $t > 0$, one can apply the same argument regarding t_0 as initial time. Then, an open-closed argument concludes that $g_1 \equiv g_2$ on $[0, T]$.

In particular, assuming that the data $(g_0, \tilde{g}, [\gamma], \eta)$ are smooth, the results above prove the following Theorem.

Theorem 4.8. *For every $K > 0$ and*

$$\Lambda = \max\left\{\kappa, \sup_t \left\{\|\tilde{g}(t) - g^0\|_{W^{2,p}(M^o)} + \|\partial_t \tilde{g}(t)\|_{L_p(M^o)}\right\}\right\},$$

there exists a unique solution $g(t)$ for some short time $T = T(\Lambda, K) > 0$ of the Ricci-DeTurck equation

$$\partial_t g = -2\,\mathrm{Ric}(g) + \mathcal{L}_{\mathcal{W}(g,\tilde{g})} g, \tag{4.21}$$

where $\mathcal{W}(g,\tilde{g})_l = g_{lr} g^{pq}(\Gamma(g)^r_{pq} - \Gamma(\tilde{g}_t)^r_{pq})$, satisfying on ∂M the boundary conditions:

$$\begin{aligned} \mathcal{W}(g,\tilde{g}) &= 0 \\ \mathcal{H}(g) &= \eta \\ \left[g^T\right] &= [\gamma]. \end{aligned} \tag{4.22}$$

and the estimate $\|g - g^0\|_{W^{2,1}_p(M_T)} \leq K$. The solution is C^∞ away from the corner $\partial M \times 0$, and extends on $M \times [0, T]$ as a $C^{1+\alpha, \frac{1+\alpha}{2}}$ family of metrics. Moreover, if the data g^0, γ, η and $\tilde{g}$ satisfy the necessary higher order compatibility conditions, then g is $C^{k+\alpha, \frac{k+\alpha}{2}}$ up to $\partial M \times 0$.

4.4 The boundary value problem for the Ricci flow

Let g^0 be a smooth Riemannian metric on a compact Riemannian manifold with boundary M, γ_t be a smooth family of metrics of the boundary, and η a smooth function on $\partial M \times [0, +\infty)$. We assume that they satisfy the zero order compatibility condition (4.1). The aim is to study the existence and regularity of a Ricci flow evolution of g^0 on M, such that the conformal class of the boundary metric is $[\gamma_t]$ and the mean curvature of the boundary is η. The existence will follow by the standard argument of pulling back a solution of the Ricci-DeTurck flow by a family of diffeomorphisms. However the issue of how smooth this family is at the corner $\partial M \times 0$ of the parabolic domain will become relevant, as it may be only C^0 despite being smooth everywhere else. Theorem 4.9 describes how this phenomenon affects the existence and regularity. Before discussing the proof we make some remarks on the regularity of a solution to the Ricci flow $g(t)$ with the boundary conditions under consideration.

As it was shown in subsection (4.2), certain higher order compatibility conditions among the initial and boundary data are necessary for the regularity of the Ricci-DeTurck flow on the corner $\partial M \times 0$. Naturally, such obstruction to regularity appears in any evolution initial-boundary value problem, and so does for the Ricci flow.

For instance, for the boundary value problem (4.2), (4.3), (4.4), the compatibility conditions (4.16), (4.17), with $h_1 = -2\,\mathrm{Ric}(g^0)$, are needed for a C^2 or C^3 solution to exist. Notice that these compatibility conditions are exclusively formulated in terms of the Ricci tensor of the initial metric. More generally, differentiating the boundary conditions with respect to time, we get

$$\mathrm{Ric}^T - \frac{\mathrm{tr}_\gamma \mathrm{Ric}^T}{n}\gamma = f\gamma + \frac{\mathrm{tr}_\gamma g^T}{n}\dot{\gamma},$$

so that since $[\gamma] = [g^T]$,

$$\mathrm{Ric}^T - \frac{\mathrm{tr}_{g^T} \mathrm{Ric}^T}{n}g^T = f\gamma + \frac{\mathrm{tr}_\gamma g^T}{n}\dot{\gamma}.$$

Also, the mean curvature condition gives,

$$\mathcal{H}'_g(\mathrm{Ric}) = -\frac{1}{2}\dot{\eta}.$$

Using the evolution equation of the Ricci tensor under Ricci flow (see for instance [CLN06]), and the contracted second Bianchi identity we observe that Ric satisfies the following boundary value problem

$$\partial_t \mathrm{Ric} = \Delta_L \mathrm{Ric}, \qquad \text{on } M \tag{4.23}$$

where Δ_L is the Lichnerowicz Laplacian, and on ∂M

$$\begin{aligned} \mathrm{Ric}^T - \frac{\mathrm{tr}_{g^T} \mathrm{Ric}^T}{n}g^T &= f\gamma + \frac{\mathrm{tr}_{\gamma^T} g^T}{n}\dot{\gamma} \\ \mathcal{H}'_g(\mathrm{Ric}) &= -\frac{1}{2}\dot{\eta} \\ \beta_g(\mathrm{Ric}) &= 0. \end{aligned} \tag{4.24}$$

Notice that the computation in Theorem 3.1 shows that it satisfies the complementing condition and is parabolic.

Now, since $\partial_t^k g = -2\partial_t^{k-1}\mathrm{Ric}$, it follows that the compatibility conditions on $g(0)$ needed for $g(t) \in C^k(\overline{M_T})$, with $k > 3$, are the same as those for $\operatorname{Ric} \in C^{k-2}$, satisfying (4.23), (4.24). Note also that by the contracted second Bianchi identity the compatibility conditions of any order hold for the last boundary condition.

For example $\operatorname{Ric} \in C^2(\overline{M_T})$ and $g \in C^4(\overline{M_T})$ require an additional compatibility condition between $(\Delta_L \operatorname{Ric})^T$ and $\ddot{\gamma}|_{t=0}$, which, in the simple case that the conformal class stays fixed along the flow, will be

$$\left(\Delta_L \operatorname{Ric}(g^0)\right)^T = \rho g^{0,T}$$

for some function ρ on ∂M.

Theorem 4.9. *Let g^0, γ, η as in Theorem 4.8. There exists a $T > 0$ and a smooth family of metrics $g(t)$ for $0 < t \leq T$ that solves the Ricci flow equation*

$$\partial_t g = -2 \operatorname{Ric}(g) \tag{4.25}$$

and satisfies on ∂M the boundary conditions

$$\begin{aligned} \mathcal{H}(g) &= \eta \\ \left[g^T\right] &= [\gamma]. \end{aligned} \tag{4.26}$$

In addition, $g(t)$ converges in the Cheeger-Gromov $C^{1,\alpha}$ sense (i.e. up to diffeomorphisms that fix the boundary) to g^0 and C^∞ away from the boundary, as $t \to 0$.

Moreover, if g^0, γ, η satisfy the necessary higher order compatibility conditions for the Ricci tensor Ric *to be in the class $C^k(\overline{M_T})$, then*

1. *$g(t)$ converges to g^0 in the geometric $C^{k+2,\alpha}$ sense.*

2. *$g \in C^k(\overline{M_T}) \cap C^\infty(M^o \times [0,T])$, and there exists a C^{k+1} diffeomorphism ϕ of M which fixes the boundary and is C^∞ in the interior such that $g(0) = \phi^* g^0$. Also, if $k \geq 1$, ϕ is C^{k+2} and $g \in C^{k+1}(\overline{M_T})$.*

3. *The Riemann tensor* Rm *is in $C^k(\overline{M_T})$ and $\operatorname{Rm}(g(0)) = \phi^* \operatorname{Rm}(g^0)$.*

Also, T is controlled from below in terms of κ.

Proof. By Theorem 4.8, choosing a family of smooth background metrics $\tilde{g}$, there exists a solution $\hat{g}(t)$ to the Ricci-DeTurck boundary value problem (4.21),(4.22), which is in $C^\infty\left(\overline{M_T} - (\partial M \times 0)\right)$ and in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{M_T})$ if no other higher order compatibility conditions hold.

The DeTurck vector field $\mathcal{W}(\hat{g}(t), \tilde{g})$ is also in $C^\infty(\overline{M_T} - (\partial M \times 0))$.Then, for some $\varepsilon > 0$, the ODE

$$\begin{aligned} \frac{d}{dt}\psi &= -\mathcal{W} \circ \psi \\ \psi_\varepsilon &= id_M \end{aligned} \tag{4.27}$$

defines a unique smooth flow ψ_t for $t > 0$, which extends at $t = 0$ continuously up to the boundary, and smoothly in the interior.

Then, $g(t) = \psi_t^* \hat{g}(t)$ solves the Ricci flow equation (see for instance [CLN06]). Moreover, since the diffeomorphisms ψ_t fix the boundary, and the mean curvature and conformal class are invariant under such diffeomorphisms, it follows that $g(t)$ satisfies the boundary conditions (4.26).

Since $\left(\psi_t^{-1}\right)^* g(t) = \hat{g}(t)$ and $\hat{g}(t) \to g^0$ in the $C^{1,\alpha}$ sense as $t \to 0$, we get that $g(t) \to g^0$ in the geometric $C^{1,\alpha}$ sense.

Now, assume that g^0, γ, η satisfy the higher order compatibility conditions necessary for the Ricci tensor to be in $C^k(\overline{M_T})$ and the metric g in $C^{k+2}(\overline{M_T})$ under the Ricci flow. We need similar compatibility conditions to hold for the Ricci-DeTurck flow, in order to improve the regularity of $\hat{g}$. In general we don't expect them to hold for an arbitrary choice of background metrics $\tilde{g}$, so we have to choose them carefully.

As the discusion in subsection (4.2) shows, the time derivatives at $t = 0$ of solutions $g, \hat{g}$ to the Ricci flow and Ricci-DeTurck flow respectively

$$h_k = \partial_t^k g|_{t=0}$$

$$\hat{h}_k = \partial_t^k \hat{g}|_{t=0}$$

are completely specified by the initial data g^0 and the background metrics $\tilde{g}$ in the case of $\hat{h}_k$. Observe that if $\tilde{g}_t$ is chosen so that $\partial_t \tilde{g}|_{t=0} = 0$ and $\partial_t^k \tilde{g}|_{t=0} = h_k$ for $k > 1$, we get

$$\hat{h}_k = h_k. \tag{4.28}$$

To see this, note that $\hat{h}_l$ is determined, through the equation, by $\hat{h}_0, \ldots, \hat{h}_{l-1}$ and $\partial_t^k \tilde{g}|_{t=0}$ for $k < l$. Thus, assuming that (4.28) holds for $k < l$ we get

$$\partial_t^k (\mathcal{L}_{\mathcal{W}(\hat{g}, \tilde{g})} \hat{g})|_{t=0} = 0, \qquad \text{for } k < l$$

since $\partial_t^k \mathcal{W}|_{t=0} = 0$ for $k < l$. Note that $\partial_t \mathcal{W}|_{t=0} = 0$, by the contracted second Bianchi identity. Then, we compute

$$\hat{h}_l = \partial_t^{l-1}(-2\,\mathrm{Ric}(\hat{g}) + \mathcal{L}_{\mathcal{W}(\hat{g},\tilde{g})}\hat{g})|_{t=0} = \partial_t^{l-1}(-2\,\mathrm{Ric}(\hat{g}))|_{t=0} = F(\hat{h}_1, \ldots, \hat{h}_{l-1})$$

for some expression F. On the other hand,

$$h_l = \partial_t^{l-1}(-2\,\mathrm{Ric}(g))|_{t=0} = F(h_1, \ldots, h_{l-1})$$

for the same expression F. Hence, (4.28) follows by induction, since $\hat{h}_0 = h_0 = g^0$.

Now, (4.28) implies that higher order compatibility of the data of the Ricci flow boundary value problem imply higher order compatibility of the same order for the Ricci-DeTurck flow.

Theorem 4.8 shows that $\hat{g}(t)$ is actually in $C^{k+2+\alpha,\frac{k+2+\alpha}{2}}(\overline{M_T})$, which immediately implies that $g(t)$ converges to g^0 in the geometric $C^{k+2,\alpha}$ sense.

Moreover, the regularity of ψ in $M \times [0, T]$ is at least C^{k+1}, i.e. it has t time and s space derivatives for $2t + s \leq k + 1$, since $\mathcal{W}$ is of first order on the metric. It follows that $\mathrm{Rm}(g(t)) = \psi_t^*(\mathrm{Rm}(\hat{g}(t)))$ is in $C^k(\overline{M_T})$, and $\mathrm{Rm}(g(0)) = \psi_0^*\,\mathrm{Rm}(\hat{g}(0))$.

By Proposition (4.2.1), if $k \geq 1$, the DeTurck field and also ψ is in C^{k+2}, therefore $g(t) \in C^{k+1}(\overline{M_T})$. Otherwise, if $k = 0$, $g(t) \in C^0(\overline{M_T})$ (up to the boundary, at $t = 0$).

The lower bound of the existence time $T > 0$ is a consequence of the corresponding estimate for the Ricci-DeTurck flow, after the observation that the background metrics $\tilde{g}$ can be chosen so that

$$\sup_t \{\|\tilde{g}(t) - g^0\|_{W^{2,p}(M^o)} + \|\partial_t \tilde{g}(t)\|_{L_p(M^o)}\} \leq 1.$$

□

Remark 4.4.1. By parabolic theory, necessary compatibility conditions are also sufficient to get higher regularity of a solution. However, the Ricci flow is not parabolic, and this is manifested by loss of derivatives. On the other hand, the Ricci tensor satisfies a parabolic boundary value problem and, as predicted, the compatibility conditions give the expected smoothness.

Remark 4.4.2. Setting the initial condition $\psi|_{t=0} = id_M$ in (4.27), we obtain a solution to the Ricci flow satisfying $g(0) = g^0$. However, the diffeomorphisms ψ will have finite degree of regularity up to the boundary, even for $t > 0$, depending on the compatibility of the data. Thus, $g(t)$ will also have finite regularity along $\partial M \times [0,T]$. This is in contrast to the behaviour of solutions to parabolic boundary value problems, which become immediately smooth for $t > 0$, as long as the boundary data are smooth.

The simple example of a rotationaly symmetric Ricci flow on the $n+1$ dimensional ball illustrates the situation. Consider metrics of the form

$$g = \phi^2(r)dr^2 + \psi^2(r)ds_n^2$$

where $0 < r \leq 1$ and ds_n^2 is the standard metric on S^n. Notice that the symmetries imposed fix most of the gauge freedom, allowing only reparametrizations of the radial variable r. Under Ricci flow the evolution equations of ϕ and ψ are (see [CK04])

$$\partial_t\phi = n\frac{\partial_s^2\psi}{\psi}\phi \tag{4.29}$$

$$\partial_t\psi = \partial_s^2\psi - (n-1)\frac{1-(\partial_s\psi)^2}{\psi} \tag{4.30}$$

where $\partial_s = \phi^{-1}\partial_r$.

The diffeomorphism freedom in the r direction is the reason that ϕ does not satisfy a parabolic equation and satisfies a transport-type equation instead. In case the initial and boundary data don't satisfy the first compatibility condition for the mean curvature, ψ will be worse than $C^3(M \times [0,T])$ and the right hand side of (4.29) will be worse than $C^1(M \times [0,T])$. Equation (4.29) doesn't enjoy the smoothing properties of a parabolic equation and, as an ODE in t, we can't expect smooth dependence on the initial data. Thus, low regularity of $g(t)$ at $\partial M \times 0$ can propagate in $\partial M \times \{t > 0\}$.

4.5 Uniqueness of the Ricci flow.

We can now use the harmonic map heat flow for manifolds with boundary (see [Ham75]) to establish the uniqueness of $C^3(\overline{M}_T)$ solutions to the boundary value problem for the Ricci flow obtained in Theorem 4.9 . We obtain the following Theorem.

Theorem 4.10. *A solution to the boundary value problem (4.25),(4.26) in $C^3(\overline{M}_T)$ is uniquely determined by the initial data g^0 and the boundary data $([\gamma], \eta)$.*

Proof. Let $g_1(t), g_2(t)$ be two $C^3(\overline{M}_T)$ solutions to the Ricci flow satisfying the same initial and boundary conditions. Consider the following heat equations for maps $\phi_i : (M, g_i) \to (M, g^0)$:

$$\begin{aligned} \frac{d\phi_i}{dt} &= \Delta_{g_i(t), g^0}\phi_i \quad \text{in } M, & (4.31) \\ \phi|_{\partial M} &= id_{\partial M} \quad \text{on } \partial M, & (4.32) \end{aligned}$$

with initial condition

$$\phi|_{t=0} = id_M. \tag{4.33}$$

For integral $m > 0$ and $p > n+3$ we can define the Sobolev spaces $W_p^{2m,m}(M_\varepsilon, M)$ of maps $f : M \to M$, by requiring the coordinate representations of f with respect to an atlas of M to be in $W_p^{2m,m}(M_\varepsilon)$. This space consists of the L_p functions on $M_\varepsilon = M \times (0, \varepsilon)$ with the derivatives $\partial_t^r \hat{\nabla}_s$ in $L_p(M_\varepsilon)$ for $2r + s \leq 2m$. The space $W_p^{2m,m}(M_\varepsilon, M)$ does not depend on the atlas used for its definition, as long as $p > n+3$.

The results in Part IV, section 11 of [Ham75] show that there exist solutions $\phi_i \in W_p^{2,1}(M_\varepsilon, M)$, for small $\varepsilon > 0$. The convexity assumption of this result for the target (M, g^0) is not needed here, since $\phi|_{t=0} = id_M$ and thus $\phi_i(t)$ remain diffeomorphisms of M for small t. Also, by the theory in [LSU67] and [Sol65] these results hold under the current assumption for the regularity of $g_i(t)$.

Moreover, the first order compatibility condition for the boundary value problem (4.31)-(4.33) holds since

$$\frac{d\phi_i}{dt} = \Delta_{g_i(0), g^0}\phi_i(0) = \Delta_{g^0, g^0} id_M = 0.$$

Thus, the diffeomorphisms $\phi_i(t)$ are in $W_p^{4,2}(M_\varepsilon, M)$.

Given the regularity of ϕ_i and g_i we know that $\hat{g}_i = (\phi_i(t))_* g_i(t) \in W_p^{2,1}(M_\varepsilon)$. Then, $\hat{g}_i$ satisfies the Ricci-DeTurck equation with background metric g^0 and the geometric boundary data are still satisfied by $\hat{g}_i$ since ϕ_i fix the boundary. Also notice that the gauge condition

$$\mathcal{W}(\hat{g}_i, g^0)|_{\partial M_T} = 0$$

holds, since

$$\Delta_{g_i(t), g^0}\phi_i(t) = -\mathcal{W}(\hat{g}_i(t), g^0) \circ \phi_i(t) \tag{4.34}$$

and

$$\Delta_{g_i(t), g^0}\phi_i(t)|_{\partial M_T} = 0.$$

By the uniqueness of $W_p^{2,1}$ solutions of the Ricci-DeTurck boundary value problem, we have that $\hat{g}_1(t) = \hat{g}_2(t)$ and $\mathcal{W}(\hat{g}_1(t), g^0) = \mathcal{W}(\hat{g}_2(t), g^0)$ for $0 \leq t \leq \varepsilon$. Now (4.34) and (4.31) imply that $\phi_1 = \phi_2$, thus $g_1 = \phi_1^* \hat{g}_1 = \phi_2^* \hat{g}_2 = g_2$. $\square$

Theorem 4.10 has the following corollary:

Corollary 4.11. *If ϕ is an isometry of g^0 which preserves the boundary data, namely*

$$\begin{aligned} \phi^* \eta(x,t) &= \eta(x,t) \\ [\phi^* \gamma(x,t)] &= [\gamma(x,t)] \,, \end{aligned}$$

and $g(t)$ is a solution to the corresponding Ricci flow boundary value problem then ϕ is an isometry of $g(t)$ for all t.

Proof. It is a consequence of the diffeomorphism invariance of the Ricci flow equation and the uniqueness. If ϕ is an isometry of g^0 which preserves the boundary data and $g(t)$ is a solution of the Ricci flow boundary value problem then $\phi^* g(t)$ is also a solution with the same initial and boundary data. By uniqueness we obtain that $\phi^* g(t) = g(t)$. $\square$

4.6 A more general boundary value problem.

The methods used in the preceding sections can be applied to prove the following generalization of Theorem 4.9, in which the mean curvature at any time t depends on the induced metric on ∂M via a given smooth function $\eta(x, t, g^T, (g^T)^{-1})$.

Theorem 4.12. *Theorem 4.9 holds if we replace the boundary condtion for the mean curvature with*

$$\mathcal{H}(g) = \eta\left(x, t, g^T, (g^T)^{-1}\right) . \tag{4.35}$$

The existence time T is controlled from below in terms of

$$\max\left\{ \|g^0\|_{W^{2,p}(M^o)}, |g^0|_{1+\epsilon}, |(g^0)^{-1}|_0, |\gamma|_{1+\epsilon, \frac{1+\epsilon}{2}}, |\eta|_{C^2} \right\},$$

where $|\eta|_{C^2}$ is a C^2 norm in all the arguments of η.

Proof. Regarding the short-time existence of the Ricci-DeTurck flow, estimates on the line of Corollary A.3 of [Wei91] establish that the estimates of

Lemmata 4.2 and 4.3 remain valid when

$$\begin{aligned}
\widehat{G}_w &= \mathcal{H}'_{g^0}(w-g^0) - (\mathcal{H}(w) - \mathcal{H}(g^0)) + \\
&\quad \eta(x,t,w^T,(w^T)^{-1}) - \eta(x,0,g^{0,T},(g^{0,T})^{-1}) \\
G_{w_1} - G_{w_2} &= \mathcal{H}'_{g^0}(w_1-w_2) - (\mathcal{H}(w_1) - \mathcal{H}(w_2)) + \\
&\quad \eta(x,t,w_1^T,(w_1^T)^{-1}) - \eta(x,0,w_2^{0,T},(w_2^{0,T})^{-1})
\end{aligned}$$

with the corresponding constants now controlled by the norm of $\eta(x,t,\cdot,\cdot)$.

The regularity theorems are still valid since the dependence of η on g^T is of zero order. Now, pulling back by the DeTurck diffeomorphisms we obtain a solution to the Ricci flow satisfying (4.35). Finally, the arguments in section 4.3 and Theorem 4.10 establish the uniqueness of solutions of the boundary value problems for the Ricci-DeTurck and the Ricci flow equations. $\square$

Remark 4.6.1. Ricci flow is typically thought as a flow on the space of metrics modulo diffeomorphisms and rescalings. However, prescribing the mean curvature does not fit well in this picture, unless it vanishes. We point out that the generalization above gives the possibility to impose scale invariant boundary conditions on the mean curvature.

For instance, we may require $\mathcal{H}(g(t)) = \frac{\bar{\eta}}{\phi}$, where $\bar{\eta}$ and ϕ are smooth functions on $\partial M \times [0,+\infty)$, ϕ defined by $g^T = \phi^2\gamma$ and η arbitrary (but such that the zeroth order compatibility conditions hold).

Chapter 5

Ricci flow on warped products with Bartnik's data.

5.1 Some background on the geometry of warped products.

Let (M^{n+1}, g) and $(F^m, \hat{g})$ be smooth Riemannian manifolds. Given a smooth function f on M, we can define the Riemannian metric $h = g + f^2\hat{g}$ on $N = M \times F$. In this section we will describe the connection and the basic curvature quantities of h in terms of g and f.

We will write ∇^h, ∇ and $\widehat{\nabla}$ for the connections of h, g and $\hat{g}$ respectively. Let $p_1 : N \to M$, $p_2 : N \to F$ be the projections to each of the factors of N. Given vector fields X, Y, Z on M and U, V, W on F, we will also use X, Y, Z, U, V, W for the pullbacks under p_1, p_2 of these vector fields on N. We will also assume that the vector fields above commute. The following Propositions are standard formulae, which can be found in the literature, see for instance [Bes08]. We state them here without proof.

Proposition 5.1.1. *The connection ∇^h of h on the vector fields X, Y, U, V as above is given by*

$$\begin{aligned}
\nabla^h_Y X &= \nabla_Y X \\
\nabla^h_Y U &= \frac{Y(f)}{f} U \\
\nabla^h_U X &= \frac{X(f)}{f} U \\
\nabla^h_V U &= -\frac{\nabla f}{f} h(U, V) + \widehat{\nabla}_V U
\end{aligned}$$

Proposition 5.1.2. *The curvature tensor* Rm_h *of* h *is given in terms of the curvature* Rm *of* g *and* f *by*

$$\begin{aligned}
\mathrm{Rm}_h(X,Y)Z &= \mathrm{Rm}(X,Y)Z\\
\mathrm{Rm}_h(X,Y)U &= 0\\
\mathrm{Rm}_h(X,U)Z &= \frac{D^2_g f(X,Z)}{f}U\\
\mathrm{Rm}_h(X,U)V &= -\frac{D^2_g f(X,\cdot)}{f}h(U,V)\\
\mathrm{Rm}_h(U,V)Z &= 0\\
\mathrm{Rm}_h(U,V)W &= \widehat{\mathrm{Rm}}(U,V)W - \frac{|df|^2_g}{f^2}h(V,W)U + \frac{|df|^2_g}{f^2}h(U,W)V
\end{aligned}$$

Proposition 5.1.3. *The Ricci tensor of* h *is given by*

$$\begin{aligned}
\mathrm{Ric}_h(V,W) &= \widehat{\mathrm{Ric}}(V,W) - \left(\frac{\Delta_g f}{f} + (m-1)\frac{|df|^2_g}{f^2}\right)h(V,W)\\
\mathrm{Ric}_h(X,Y) &= \mathrm{Ric}_g(X,Y) - \frac{m}{f}D^2_g f(X,Y)\\
\mathrm{Ric}_h(X,V) &= 0
\end{aligned}$$

Moreover, if $\hat{g}$ *is Einstein, i.e.* $\widehat{Ric} = \lambda\hat{g}$ *we have*

$$\mathrm{Ric}_h(V,W) = -\left(\frac{\Delta_g f}{f} + (m-1)\frac{|df|^2_g}{f^2} - \frac{\lambda}{f^2}\right)h(V,W)$$

5.2 The Ricci flow on warped products with Einstein fibres

A direct consequence of Proposition 5.1.3 is the following.

Proposition 5.2.1. *The Ricci flow on* $(N, h(t) = g(t) + f(t)^2\hat{g})$ *is equivalent to the following system of equations on* g *and* f

$$\partial_t g = -2\,\mathrm{Ric}_g + 2mf^{-1}D^2_g f \tag{5.1}$$

$$\partial_t f = \Delta_g f + (m-1)\frac{|df|^2_g}{f} - \frac{\lambda}{f} \tag{5.2}$$

Next, we express the Ricci-DeTurck equation for $h(t)$ in terms of g and f. Using orthonormal frames $\{X_i\}$ of g and $\{U_j\}$ around points in M and F respectively, we can compute the DeTurck vector field with respect to a background metric $\tilde{h} = \tilde{g} + \tilde{f}^2\hat{g}$ as:

$$\begin{aligned}
\mathcal{W}(h,\tilde{h}) &= \sum_{i=1}^{n}(\nabla^h_{X_i}X_i - \nabla^{\tilde{h}}_{X_i}X_i) + \sum_{j=1}^{p}(\nabla^h_{U_j}U_j - \nabla^{\tilde{h}}_{U_j}U_j) \\
&= \mathcal{W}(g,\tilde{g}) - \frac{m\nabla^g f}{f} + \frac{m\nabla^{\tilde{g}}\tilde{f}}{\tilde{f}}\left(\frac{\tilde{f}}{f}\right)^2,
\end{aligned}$$

and compute

$$\begin{aligned}
\mathcal{L}_{\mathcal{W}(h,\tilde{h})}h &= \mathcal{L}_{W(h,\tilde{h})}g + 2f\mathcal{W}(h,\tilde{h})(f)\hat{g} \\
&= \mathcal{L}_{W(g,\tilde{g})}g - p\mathcal{L}_{f^{-1}\nabla^g f}g + \mathcal{L}_{\mathcal{V}}g + 2f\mathcal{W}(h,\tilde{h})(f)\hat{g} \\
&= \mathcal{L}_{W(g,\tilde{g})}g - 2mf^{-1}D^2_g f + 2mf^{-2}df\otimes df + \mathcal{L}_{\mathcal{V}}g + 2f\mathcal{W}(h,\tilde{h})(f)\hat{g},
\end{aligned}$$

where $\mathcal{V}(f,\tilde{f},\tilde{g}) = \frac{m\nabla^{\tilde{g}}\tilde{f}}{\tilde{f}}\left(\frac{\tilde{f}}{f}\right)^2 = m\tilde{f}f^{-2}\nabla^{\tilde{g}}\tilde{f}$. Putting everything together we obtain the following Proposition.

Proposition 5.2.2. *The Ricci-DeTurck equation on h*

$$\partial_t h = -2Ric_h + \mathcal{L}_{\mathcal{W}(h,\tilde{h})}h$$

is equivalent to the system

$$\partial_t g = -2\,\mathrm{Ric}_g + \mathcal{L}_{\mathcal{W}(g,\tilde{g})}g + 2mf^{-2}df\otimes df + \mathcal{L}_{\mathcal{V}}g \tag{5.3}$$

$$\partial_t f = \Delta_g f + (m-1)\frac{|df|^2_g}{f} - \frac{\lambda}{f} + \mathcal{W}(h,\tilde{h})(f). \tag{5.4}$$

Following Shi [Shi89b] we express the flow using the background connection $\overline{\nabla}$ of a Riemannian metric $\bar{g}$, which we will choose later to be the initial metric. Since we want to allow for varying background metrics $\tilde{g}$, we write

$$\mathcal{W}(g,\tilde{g}) = \mathcal{W}(g,\bar{g}) + \Phi(g),$$

where $\Phi(g)_r = g_{rl}g^{pq}(\overline{\Gamma}^l_{pq} - \widetilde{\Gamma}^l_{pq})$. After a direct computation we obtain

$$\begin{aligned}
\partial_t g_{kl} - g^{ij}\,\overline{\nabla}_i\,\overline{\nabla}_j\, g_{kl} &= \mathcal{A}(g,f;\tilde{g},\tilde{f}), \\
\partial_t f - g^{ij}\,\overline{\nabla}_i\,\overline{\nabla}_j\, f &= \mathcal{B}(g,f;\tilde{g},\tilde{f}),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}(g,f;\tilde g,\tilde f) &= \mathcal{R}(g,\overline{\nabla}\, g)_{kl} + \mathcal{L}_\Phi g_{kl} + 2mf^{-2}\,\overline{\nabla}_k f\,\overline{\nabla}_l f + \mathcal{L}_{\mathcal{V}} g_{kl},\\
\mathcal{B}(g,f;\tilde g,\tilde f) &= -f^{-1}g^{ij}\,\overline{\nabla}_i f\,\overline{\nabla}_j f - \lambda f^{-1} - g^{ij}g^{ab}\left(\overline{\nabla}_i\,\tilde g_{ja} - \frac{1}{2}\,\overline{\nabla}_a\,\tilde g_{ij}\right)\overline{\nabla}_b f\\
&\quad + m\tilde f f^{-2}\tilde g^{ij}\,\overline{\nabla}_i \tilde f\,\overline{\nabla}_j f,
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{R}_{kl} &= g^{pq}\bar g^{rs}\left(g_{kr}\bar R_{lpsq} + g_{lr}\bar R_{kpsq}\right)\\
&\quad - g^{pq}g^{rs}\left(\frac{1}{2}\,\overline{\nabla}_k\, g_{rp}\,\overline{\nabla}_l\, g_{sq} + \overline{\nabla}_p\, g_{lr}\,\overline{\nabla}_s\, g_{kq} - \overline{\nabla}_p\, g_{lq}\,\overline{\nabla}_q\, g_{ks}\right)\\
&\quad + g^{pq}g^{rs}\left(\overline{\nabla}_l\, g_{rp}\,\overline{\nabla}_q\, g_{ks} + \overline{\nabla}_k\, g_{rp}\,\overline{\nabla}_q\, g_{ls}\right),\\
\mathcal{L}_{\mathcal{V}} g_{kl} &= m\left(\overline{\nabla}_k(\tilde f f^{-2} g_{lb}\tilde g^{ba}\,\overline{\nabla}_a \tilde f) + \overline{\nabla}_l(\tilde f f^{-2} g_{kb}\tilde g^{ba}\,\overline{\nabla}_a \tilde f)\right)\\
&\quad - \mathcal{V}_c g^{cd}(\overline{\nabla}_k\, g_{ld} + \overline{\nabla}_l\, g_{kd} - \overline{\nabla}_d\, g_{kl})\\
&= m\Big(\overline{\nabla}_k(\tilde f f^{-2} g_{lb}\tilde g^{ba}\,\overline{\nabla}_a \tilde f) + \overline{\nabla}_l(\tilde f f^{-2} g_{kb}\tilde g^{ba}\,\overline{\nabla}_a \tilde f)\\
&\quad - \tilde f f^{-2}\tilde g^{ba}\,\overline{\nabla}_a \tilde f(\overline{\nabla}_k\, g_{lb} + \overline{\nabla}_l\, g_{kb} - \overline{\nabla}_b\, g_{kl})\Big).
\end{aligned}$$

5.3 A boundary value problem for the Ricci-DeTurck equation.

Now let M^{n+1} be a manifold with compact boundary ∂M, $(F,\hat g)$ be a complete Riemannian manifold and $h_0 = g_0 + f_0^2\hat g$ a warped product metric on $N = M\times F$.

If M is noncompact, we assume that $\hat g$ is Ricci flat, $\mathrm{Rm}(g_0)\in L^p$, $f_0 \in W^{2,p}$ $(p > n+3)$ and bounded away from zero, and that g_0 has geometry bounded in $W^{2,p}$. Note that we allow $\mathrm{Rm}(g_0)$ and $D^2_{g_0} f_0$ to be unbounded in L^∞.

Moreover, let $\gamma(t)$ be $C^{1+\epsilon,\frac{1+\epsilon}{2}}$ family of Riemannian metrics on ∂M, and $\eta(t)$ a $C^{\epsilon,\frac{\epsilon}{2}}$ family of functions on ∂M. We assume the zeroth order compatibility conditions

$$\begin{aligned}
g_0^T &= \gamma(0),\\
\mathcal{H}(g_0) &= \eta(0).
\end{aligned}$$

As in chapter 4, aiming to prove the short-time existence of the Ricci flow with Bartnik's boundary conditions, the mean curvature $\mathcal{H}(g)$ and the induced metric g^T, we first focus on the Ricci-DeTurck equation

$$\partial_t h = -2Ric_h + \mathcal{L}_{\mathcal{W}(h,\tilde{h})} h. \tag{5.5}$$

It is defined with the aid of a time dependent family of background metrics $\tilde{h}$. To obtain a well posed parabolic boundary value problem, we need to impose that a solution $h(t)$ will satisfy the following conditions on ∂M, for every t.

$$\begin{aligned} \mathcal{W}(h,\tilde{h})(t) &= 0, && (5.6)\\ \mathcal{H}(g(t)) &= \eta(t), && (5.7)\\ g(t)^T &= \gamma(t). && (5.8) \end{aligned}$$

Now we can prove the following existence theorem.

Theorem 5.1. *There exists a $T > 0$ and a unique solution $h(t) = g(t) + f(t)^2\hat{g}$ of (5.5)-(5.8), with $g(0) = g_0$, $g - g_0 \in W^{2,1}_p(M_T)$ and $f \in W^{2,1}_p(M_T)$. Moreover, f, g are smooth (C^∞) solutions of (5.3)-(5.4) away from the corner $\partial M \times 0$ and they belong to $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{M_T})$. In particular, if M is not compact we have, for all $t \in [0,T]$,*

$$\sup_M (|\overline{\nabla}\, g(t)|_{\bar{g}} + |\overline{\nabla}\, f(t)|_{\bar{g}}) < \infty. \tag{5.9}$$

If the data $(g_0, \tilde{g}, f_0, \tilde{f}, \gamma, \eta)$ satisfy the necessary compatibility conditions then g and f belong to $C^{k+\alpha,\frac{k+\alpha}{2}}(\overline{U}_T)$, for any precompact neighbourhood $U \subset M$ intersecting ∂M.

Moreover, if $(g_0, f_0) \in C^{k,\alpha}(\overline{M})$ for $k \geq 2$ then $(g(t), f(t)) \in C^{k,\alpha}(\overline{M})$ for all t.

Proof. The proof is via a fixed point argument, following the method in [Wei91], where a $W^{2,1}_p$ solution to the nonlinear boundary value problem is obtained as a fixed point of a suitable map.

We will assume that $\bar{g} = g_0$. For $K, T > 0$, we may define the following set of pairs $(w, \sigma) \in W^{2,1}_p$, where w are 2-tensors and σ functions:

$$M^T_K = \left\{ (w,\sigma) \in W^{2,1}_p \;\middle|\; w|_{t=0} = 0,\ \sigma|_{t=0} = 0,\ ||w||_{W^{2,1}_p} \leq K, ||\sigma||_{W^{2,1}_p} \leq K \right\}.$$

For any $(w,\sigma) \in M^T_K$, define $g_w = g_0 + w$, $f_\sigma = f_0 + \sigma$ and $h_{(w,\sigma)} = g_w + f_\sigma^2\hat{g}$. Note that for $T > 0$ small enough, depending on K, g_w and $h_{(w,\sigma)}$ will define Riemannian metrics on M and N respectively, as $W^{2,1}_p$ embeds to $C^{1+\alpha,\frac{1+\alpha}{2}}$ for $p > n+3$ and $\alpha = 1 - \frac{n+3}{p}$. Then, the following boundary value problem is well defined.

$$\begin{aligned}
\partial_t u_{kl} - \bar{g}^{ij}\,\overline{\nabla}_i\,\overline{\nabla}_j\,u_{kl} &= \mathcal{A}(g_w, f_\sigma; \tilde{g}, \tilde{f}) - (\bar{g}^{ij} - g_w^{ij})\,\overline{\nabla}_i\,\overline{\nabla}_j\,w_{kl} \quad (5.10)\\
\partial_t \phi - \bar{g}^{ij}\,\overline{\nabla}_i\,\overline{\nabla}_j\,\phi &= \mathcal{B}(g_w, f_\sigma; \tilde{g}, \tilde{f}) - (\bar{g}^{ij} - g_w^{ij})\,\overline{\nabla}_i\,\overline{\nabla}_j\,\sigma\\
&\quad + \bar{g}^{ij}\,\overline{\nabla}_i\,\overline{\nabla}_j\,f_0 \quad (5.11)
\end{aligned}$$

$$\begin{aligned}
P_{h_0}(u,\phi) &= P_{h_0}(w,\sigma) - \mathcal{W}(h_{(w,\sigma)}, \tilde{h})\\
\mathcal{H}'_{g_0}(u) &= \mathcal{H}'_{g_0}(w) - \mathcal{H}(g_w) + \eta \quad (5.12)\\
u(t)^T &= \gamma(t) - \gamma(0)
\end{aligned}$$

Note that for any $(w,\sigma) \in M_K^T$ the right-hand sides of equations (5.10),(5.11) belong to L^p. This is a consequence of the embedding of $W_p^{2,1}$ to $C^{1+\alpha,\frac{1+\alpha}{2}}$, and the assumption that the curvature of $\bar{g}$ is in L^p. Therefore, by Theorem 3.2, the boundary value problem (5.10), (5.11), (5.12) has a unique solution $(u,\phi) = S(w,\sigma)$ satisfying $u|_{t=0} = 0, \phi|_{t=0} = 0$. In other words, it defines a map

$$S : M_K^T \to W_p^{2,1}.$$

Now, a series of estimates similar to those in section 4.1 show that for $T > 0$ small enough S maps M_K^T to itself and it is also a contraction. Hence, it has a fixed point (w', σ') and the corresponding Riemannian metric $h_{(w',\sigma')}$ solves (5.3), (5.4) with the boundary conditions (5.6)-(5.8). It is important to note that these estimates depend heavily on the embedding of $W_p^{2,1}$ in $C^{1+\alpha,\frac{1+\alpha}{2}}$ mentioned above, which still holds in the noncompact case, under the additional assumption of geometry bounded in $W^{2,p}$.

The regularity statement of the Theorem can be proven using the linear parabolic theory for boundary value problems developed in [Sol65] to bootstrap the regularity, after appropriately localizing the equation as in section 4.2. The overall argument however is similar to that of the analogous situation is 4.2, so we omit it here.

Estimate (5.9) follows simply from the fact that g, f are in $C^{1+\alpha,\frac{1+\alpha}{2}}$. Now we proceed to show that if $g_0 \in C^{k+\alpha}$ then $g(t) \in C^{k+\alpha}$. Is enough to show that uniform estimates hold for all charts not intersecting the boundary.

Let $B(0,\bar{r}) \subset \mathbb{R}^n$ be the Euclidean ball of radius $\bar{r}$ and fix a cutoff function ζ supported in $B(0,\bar{r})$. If g and f denote the solutions of (5.3), (5.4), consider the quantities

$$\begin{aligned} v_{kl} &= \zeta g_{kl}, \\ \psi &= \zeta f, \end{aligned}$$

expressed in the coordinates described in chapter 2. In $B(0,\bar{r})$ they satisfy equations of the form

$$\begin{aligned} \partial_t v_{kl} - g^{ij}\partial_i\partial_j v_{kl} &= \mathcal{S}_1(g,\partial g,f,\partial f;\tilde{g},\tilde{f},\partial\tilde{g},\partial\tilde{f},\bar{g},\partial\bar{g},\partial^2\bar{g}), \\ \partial_t \psi - g^{ij}\partial_i\partial_j \psi &= \mathcal{S}_2(g,\partial g,f,\partial f;\tilde{g},\tilde{f},\partial\tilde{g},\partial\tilde{f},\bar{g},\partial\bar{g}), \end{aligned}$$

zero boundary conditions, and initial conditions controlled in $C^{k+\alpha}$. By standard parabolic estimates we obtain

$$\begin{aligned} |v_{kl}|^{B(0,\rho)_T}_{2+\alpha,\frac{2+\alpha}{2}} + |\psi|^{B(0,\rho)_T}_{2+\alpha,\frac{2+\alpha}{2}} &\leq c(|\mathcal{S}_1|^{B(0,\rho)_T}_{\alpha,\frac{\alpha}{2}} + |\mathcal{S}_2|^{B(0,\rho)_T}_{\alpha,\frac{\alpha}{2}} + |g_0|_{2,\alpha} + |f_0|_{2,\alpha}) \\ &\leq c\left(|g_{kl}|^{B(0,\rho)_T}_{1+\alpha,\frac{1+\alpha}{2}} + |f|^{B(0,\rho)_T}_{1+\alpha,\frac{1+\alpha}{2}} + |g_0|_{2,\alpha} + |f_0|_{2,\alpha}\right), \end{aligned}$$

where $c > 0$ does not depend on the particular coordinate chart, but depends on $|\bar{g}|_{C^{2,\alpha}}$ and T (Note that we can choose $\tilde{g} = \bar{g}$ away from ∂M).

However, the fixed point (w', σ') satisfies the bound

$$||w'||_{W^{2,1}_p} + ||\sigma'||_{W^{2,1}_p} \leq 2K.$$

By the embedding of $W^{2,1}_p(M_T)$ in $C^{1+\alpha,\frac{1+\alpha}{2}}$ we obtain uniform control of $|v_{kl}|^{B(0,\rho)_T}_{2+\alpha,\frac{2+\alpha}{2}}$ and $|\psi|^{B(0,\rho)_T}_{2+\alpha,\frac{2+\alpha}{2}}$ in terms of K.

Bootstrapping the above estimate gives uniform control in $C^{k,\alpha}$. □

5.4 The Ricci flow.

As in chapter 4 we can now use Theorem 5.1 to prove the short-time existence of Ricci flow on warped products with Bartnik's data. For simplicity we assume that all the data (g_0, γ, η), defined at the begining of section 5.3 are smooth (C^∞). The result is described in the following Theorem.

Theorem 5.2. *There exists $T > 0$ and solutions g, f of (5.1)-(5.2), smooth and defined away from the corner $\partial M \times 0$, such that $h(t) = g(t) + f(t)^2\hat{g}$ is a Ricci* *flow, satisfying*

$$\begin{aligned} g(t)^T &= \gamma(t), \\ \mathcal{H}(g(t)) &= \eta(t), \end{aligned} \tag{5.13}$$

for all $t > 0$. In addition, there exists a a family of diffeomorphisms ϕ_t of M such that $\phi_t^* g(t), \phi_t^* f(t)$ convege to g_0, f_0 respectively, as $t \to 0$ in $C^{1,\alpha}(\overline{M_T})$ (uniformly).

Moreover, the regularity up to $\partial M \times 0$, and the convergence of $\phi_t^* g(t), \phi_t^* f(t)$ to the initial data is improved, if $\mathrm{Ric}(h)$ *satisfies higher order compatibility conditions.*

In addition, if M is noncompact and g_0 is controlled in C^2, then $h(t)$ has uniformly bounded curvature for all $t \leq T$.

Proof. Given the data g_0, f_0, γ, η, we first need to choose $\tilde{h} = \tilde{g} + \tilde{f}^2 \hat{g}$ appropriately so that compatibility conditions that are satisfied for the boundary value problem (5.1)-(5.2), (5.13)-(5.14) will still hold for the Ricci-DeTruck boundary value problem (5.3)-(5.4), (5.6)-(5.7), as in Theorem 4.9. Note that in the non-compact case, $\tilde{g}, \tilde{f}$ may coincide with g_0, f_0 outside a compact set. Pulling back by the diffeomorphisms ϕ_t generated by $\mathcal{W}(h, \tilde{h})$, setting an initial condition $\phi_\varepsilon = id_M$, we obtain the solution to the Ricci flow. In the non-compact case, the flow of $\mathcal{W}(h, \tilde{h})$ is well defined, since it is uniformly bounded by 5.9. $\square$

Remark 5.4.1. Assuming higher order control of the geometry at infinity on can obtain uniform convergence of $\phi_t^* h(t)$ to h_0 in the appropriate topology.

Chapter 6

A continuation principle.

As Ricci flow is a nonlinear system of equations, its solutions are not expected to exist for all time. For that reason, a central issue in the study of Ricci flow is understanding what is the nature of the singularities that appear and also what kind of conditions suffice for the continuation of the flow. However, Ricci flow is a PDE with geometric character due to the invariance of the equation under the action of diffeomorphisms. Thus, it is expected and desirable to have a geometric characterization of the developing singularities, as a first step towards understanding how the flow could cease to exist at a finite time.

On closed manifolds, it is a well known result of Hamilton [Ham82] that the flow exists as long as the norm of the curvature tensor stays bounded. The following theorem is the appropriate generalization on compact manifolds with boundary.

Theorem 6.1. *Let $g(t)$ be a smooth Ricci flow with maximal time of existence $T < \infty$ and smooth boundary data $([\gamma(x,t)], \eta(x,t))$ (as in Theorem 4.9) defined for all $0 \leq t < \infty$. Then*

$$\sup_{0\leq t<T} \left(\sup_{x\in M} |\operatorname{Rm}(g(t))|_{g(t)} + \sup_{x\in\partial M} |\mathcal{A}(g(t))|_{g(t)} \right) = +\infty.$$

Proof. Assume that $T < \infty$ and for some $K > 0$

$$\sup_{x\in M} |\operatorname{Rm}(g(t))|_{g(t)} + \sup_{x\in\partial M} |\mathcal{A}(g(t))|_{g(t)} \leq K \tag{6.1}$$

for all $t < T$.

The bound (6.1) implies that $g(t)$ are uniformly equivalent and in addition that g^T have bounded curvature for $t < T$. We show that g^T are actually controlled in $C^{1,\epsilon}$ as $t \to T$. Let $u(x,t)$ be a function such that $g^T = u^{\frac{4}{n-2}}\gamma$ (when $n \geq 3$). If $R(\gamma)$ and $R(g^T)$ are the scalar curvatures of γ and g^T respectively, it is known that u satisfies an elliptic equation of the form

$$a\Delta u + R(\gamma)u - R(g^T)u^{\frac{n+2}{n-2}} = 0,$$

where $a = a(n)$ and Δ is the Laplacian with respect to the uniformly equivalent and controlled in $C^{1,\epsilon}$ metrics $\gamma(t)$. Now, elliptic regularity and the uniform bounds on $u, R(\gamma), R(g^T)$ imply that u, and hence g^T, is controlled in $C^{1,\epsilon}$. For $n = 2$ the situation is similar.

Next, we observe that the interior injectivity radius i_M, the injectivity radius of the boundary $i_{\partial M}$ and the "boundary injectivity radius" i_b are uniformly bounded below for $t < T$. Here we need to clarify that $i_M \geq i_o$ means that for any $p \in M^o$ the exponential map exp_p restricted to a ball of radius $\rho < \min\{i_o, dist(p, \partial M)\}$ is a diffeomorphism onto its image while the boundary injectivity radius determines the size of the collar neigbourhood of the boundary in which the normal exponential map is a diffeomorphism.

Since $g(t)$ are uniformly equivalent, for any $p \in M^o$ there exists a $r_o > 0$ such that $dist_t(p, \partial M) \geq r_o$ for all $t < T$. This also shows that the volume ratio

$$\frac{Vol_t(B_t(p,r))}{r^{n+1}} \geq c$$

for all $r \leq r_o$ and $t < T$, which together with the curvature bound gives that $inj_M(p)$ is bounded below. A similar argument controls the injectivity radius $i_{\partial M}$ of the boundary.

Moreover, by comparison geometry the bounds on the curvature and the second fundamental form control the "focal" distance of the boundary. Then, since the metrics are uniformly equivalent, the boundary cannot form "self-intersections", hence the boundary injectivity radius i_b is also bounded below.

Now, (6.1) and the discussion above implies that there exist positive constants R_0, i_0, d_0, S_0 such that for all $t \in [0, T)$

$$|\operatorname{Ric}(M,g)|_g \leq R_0, \quad |\operatorname{Ric}(\partial M, g^T)|_{g^T} \leq R_0 \tag{6.2}$$

$$i_M \geq i_0, \quad i_{\partial M} \geq i_0, \quad i_b \geq 2i_0 \tag{6.3}$$

$$|\mathcal{H}(g)|_{Lip(\partial M)} \leq S_0, \quad \operatorname{diam}(M,g) \leq d_0. \tag{6.4}$$

Theorem 3.1 of [AKK$^+$04] states that the class of Riemannian manifolds satisfying the bounds above is $C^{1,\epsilon}$-precompact. Thus, there is a sequence

$t_j \to T$ and $C^{2,\epsilon}$ diffeomorphisms ϕ_j of M such that the metrics $h_j = \phi_j^* g(t_j)$ converge in $C^{1,\epsilon}$. In addition, the fact that both h_j^T and $g(t_j)^T$ are uniformly bounded in $C^{1,\epsilon}$ implies that $\phi_j|_{\partial M}$ are actually controlled in $C^{2,\epsilon}$.

Next, we show that h_j are uniformly bounded in $W^{2,p}(M^o)$. The bounds above imply that the "harmonic radius" of (M, h_j) is uniformly bounded below. It is also known that in harmonic coordinates the Ricci curvature becomes elliptic. Moreover, in boundary harmonic coordinates (see [AKK$^+$04]) the metric components satisfy a boundary value problem of the form (4.20), which is elliptic in the sense of [ADN64]. The mean curvature and conformal class are C^ϵ and $C^{1,\epsilon}$ controlled respectively and the harmonic coordinate functions are controlled in $C^{2,\epsilon}$. Thus, the L^p estimates in Solonnikov [Sol67] provide uniform $W^{2,p}$ control of h_j in these coordinates (in a ball or "half ball" of smaller size). Note that the $W^{2,p}$ estimates up to the boundary of [Sol67] hold under the current regularity assumptions on h_j, γ and η. Finally, since the harmonic coordinate functions $C^{2,\epsilon}$ controlled we obtain the uniform estimate of h_j in $W^{2,p}(M^o)$.

Now, $g(t)$ is a smooth Ricci flow for $t < T$, and therefore $g(t_j)$ satisfy the necessary compatibility conditions of any order. The same is true for h_j since for the Ricci flow these conditions are imposed on the Ricci tensor and are diffeomorphism invariant. Thus, by the short time existence result and the uniform control of h_j, $[\phi_j^* \gamma]$ and $\phi_j^* \eta$, there exist smooth solutions $h_j(t)$ to the Ricci flow boundary value problem with $h_j(t_j) = h_j$, for a uniform amount of time.

By uniqueness, $g(t) = (\phi_j)_* h_j(t)$ for all j and $t \geq t_j$. Therefore, taking j large, the argument above shows that the solution $g(t)$ can be extended past time T. □

Remark 6.0.2. An examination of the proof of Theorem 3.1 in [AKK$^+$04] indicates that Hölder control on the mean curvature is probably enough to obtain precompactness in Hölder spaces. The assumption on the Lipschitz control of the mean curvature could then be removed. However, we will avoid the technical details of this improvement here, since in general the mean curvature η can be assumed to have high degree of regularity.

Chapter 7

An example.

In this chapter study the Ricci flow on $M = [0,1] \times S^1 \times S^1$ equipped with metrics of the form

$$g = p^2(r)dr^2 + q^2(r)d\theta^2 + u^2(r)d\phi^2. \tag{7.1}$$

We obtain long time existence if the mean curvature of the boundary is zero and its the conformal class remains fixed along the flow. Moreover, we show that the flow may converge to a product metric or diverge, depending on the conformal class of the boundary.

We begin by introducting the notation and some basic facts on the geometry of metrics of the form 7.1. The distance parameter s is defined as

$$s = \int_0^r p(r)dr.$$

We will also write $\partial_s := \frac{1}{p}\partial_r$ and indicate differentiation with respect to ∂_s with $'$.

A direct computation shows that the sectional curvatures of tangent planes containing ∂_s are

$$sec(s,\theta) = -\frac{q''}{q} = -Q'' - (Q')^2, \tag{7.2}$$

$$sec(s,\phi) = -\frac{u''}{u} = -U'' - (U')^2, \tag{7.3}$$

where $P = \log p$, $Q = \log q$, $U = \log u$. We also compute

$$sec(\theta,\phi) = Q'U'.,$$

Letting $A(r_0)$ be the second fundamental form of the slice $r = r_0$, with respect to ∂_s , we compute that

$$\begin{aligned} A(r) &= \partial_s Q q^2 d\theta^2 + \partial_s U u^2 d\phi^2 \\ |A|^2 &= (Q')^2 + (U')^2 \end{aligned}$$

Thus the mean curvature $\eta(r)$ of each slice becomes

$$\eta(r) = (Q+U)'.$$

The components of the Ricci tensor are

$$\begin{aligned} \mathrm{Ric}_{rr} &= -p^2 \left(Q'' + U'' + (Q')^2 + (U'^2) \right) \\ \mathrm{Ric}_{\theta\theta} &= -q^2 \left(Q'' + (Q')^2 + Q'U' \right) \\ \mathrm{Ric}_{\phi\phi} &= -u^2 \left(U'' + (U')^2 + Q'U' \right) \end{aligned}$$

If $g(t)$ solves by the Ricci flow, the evolution equations of P, Q and U are

$$\frac{dP}{dt} = Q'' + U'' + (Q')^2 + (U')^2 \tag{7.4}$$

$$\frac{dQ}{dt} = Q'' + (Q')^2 + Q'U' \tag{7.5}$$

$$\frac{dU}{dt} = U'' + (U')^2 + Q'U' \tag{7.6}$$

Setting $V = Q + U$ and $C = Q - U$, the obtain the following equivalent evolution equations.

$$\begin{aligned} \frac{dV}{dt} &= V'' + (Q')^2 + (U')^2 + 2Q'U' = V'' + (V')^2 \\ \frac{dC}{dt} &= C'' + (Q')^2 - (U')^2 = C'' + C'V' \end{aligned}$$

Now, let g_0 be an initial metric of the form 7.1, and assume that the mean curvature of the two boundary components is α_0, α_1 respectively (with respect to the outward pointing unit normal vectors).

Suppose that $g(t)$ is the Ricci flow provided by Theorem 4.9 preserving the mean curvature and the conformal class of the (disconnected) boundary. By Corollary 4.11, the form 7.1 of the metric is preserved, since it is uniquely determined by its symmetries.

The boundary conditions, in terms of V and C, become

$$\begin{aligned} V'|_{r=0} &= -\alpha_0, & V'|_{r=1} &= \alpha_1 \\ C|_{r=0} &= c_0, & C|_{r=1} &= c_1 \end{aligned}$$

where c_0 and c_1 represent the corresponding values for g_0. The pair (c_0, c_1) parametrizes the conformal classes of a metric of the form 7.1 on ∂M.

In order to compute the evolution equations of quantities involving derivatives of V and C we will need the commutator $[\partial_t, \partial_s]$, which is given by

$$\begin{aligned} [\partial_t, \partial_s](u) &= -\dot{P}\partial_s u \\ &= -\left(V'' + (Q')^2 + (U')^2\right) u' \\ &= -\left(V'' + |A|^2\right) u'. \end{aligned} \tag{7.7}$$

Using (7.8) we compute the evolution equation of $\eta(r) = V'$

$$\begin{aligned} \frac{d\eta}{dt} &= \partial_s \dot{V} - \left(V'' + (Q')^2 + (U')^2\right)\eta \\ &= \left(V'' + (V')^2\right)' - \left(V'' + (Q')^2 + (U')^2\right)\eta \\ &= \eta'' + \eta\eta' - \left((Q')^2 + (U')^2\right)\eta \\ &= \eta'' + \eta\eta' - |A|^2\eta. \end{aligned}$$

Similarly, we compute the evolution equation of $w = C'$

$$\begin{aligned} \frac{dw}{dt} &= \partial_s \dot{C} - \left(V'' + |A|^2\right) w \\ &= \left(C'' + w\eta\right)' - \eta' w - |A|^2 w \\ &= w'' + \eta w' - |A|^2 w. \end{aligned}$$

The following lemmata describe the evolution of the distance between the two boundary components and the volume of the manifold.

Lemma 7.1. *If the sum of the mean curvatures of the two boundary components is non-negative, i.e.* $\alpha_0 + \alpha_1 \geq 0$*, their distance* l *is nondecreasing.*

Proof.

$$\begin{aligned}\frac{dl}{dt}=\int_0^1\frac{dp(r)}{dt}dr &= \int_0^1\left(V''+|A|^2\right)pdr=\int_0^l(V''+|A|^2)ds\\ &\geq V'(l)-V'(0)=\alpha_1+\alpha_0\geq 0\end{aligned}$$

□

Lemma 7.2. *The evolution of the volume is given by*

$$\frac{dVol}{dt}=2\alpha_1e^{V(l)}+2\alpha_0e^{V(0)}+\int_0^l\left(\frac{(C')^2-(V')^2}{2}\right)ds. \tag{7.8}$$

Proof.

$$\begin{aligned}\frac{dVol}{dt} &= \frac{d}{dt}\int_0^1 e^V pdr=\int_0^1(\dot V+\dot P)e^V pdr\\ &= \int_0^l\left(V''+(V')^2+V''+|A|^2\right)e^Vds\\ &= 2\int_0^l V''e^Vds+\int_0^l((V')^2+|A|^2)e^Vds\\ &= 2V'e^V|_0^l+\int_0^l\left(|A|^2-(V')^2\right)e^Vds\\ &= 2\alpha_1e^{V(l)}+2\alpha_0e^{V(0)}+\int_0^l\left(\frac{(C')^2-(V')^2}{2}\right)ds\\ &= 2\alpha_1e^{V(l)}+2\alpha_0e^{V(0)}+\int_0^l sec(\theta,\phi)ds\end{aligned}$$

□

Remark 7.0.3. Note that if C is constant at $t=0$, it will remain constant along the flow. In this case, if the boundary components are minimal, Lemma 7.2 implies that the volume is nonincreasing along the flow. This, however, is not in general the case.

From now on we assume that both boundary components are minimal, namely $\alpha_0=\alpha_1=0$. A direct application of the maximum principle yields the following preliminary estimates.

Lemma 7.3. *As long as the flow exists, V and C are bounded from above and below.*

Proof. Since $V' = 0$ on the boundary, at any maximum (r,t) of V we have $V'' \leq 0$ so

$$\dot{V}(r,t) = V'' + (V')^2 \leq 0$$

If we define

$$V_{max}(t) = \max_r V(r,t)$$

then

$$\dot{V}_{max} \leq 0$$

which implies that the maximum V is nonincreasing. The same argument shows that the minimum of V is nondecreasing, hence V is bounded above and below along the flow.

On the other hand, C satisfies fixed Dirichlet data so we need to focus on the behaviour in the interior. At an interior maximum (r,t) of C, $C' = 0$ and the evolution equation gives $\dot{C} \leq 0$ there, so C remains bounded above. Like before, the same argument can show that C is also bounded below. $\square$

Next, we show that the Ricci flow has the expected uniformizing effect on C and V.

Lemma 7.4. *As long as the flow exists the estimate $|A|^2 \leq \frac{1}{2t}$ holds.*

Proof. A computation shows that $|A|^2 = \frac{\eta^2+w^2}{2}$, which corresponds to the decomposition of the second fundamental form to the trace and a trace free parts.

Set $\rho = \frac{\eta^2+w^2}{2}$. A direct computation shows that

$$\begin{aligned} \dot{\rho} &= \rho'' + \eta\rho' - \Big((\eta')^2 + (w')^2\Big) - \frac{(\eta^2+w^2)^2}{2}, \\ &= \rho'' + \eta\rho' - \Big((\eta')^2 + (w')^2\Big) - 2\rho^2. \end{aligned}$$

Now, $\eta = V'$ satisfies homogeneous Dirichlet boundary conditions, i.e. $\eta|_{r=0} = \eta|_{r=1} = 0$, and since the conformal classes of the boundary components are fixed along the flow

$$\dot{C}|_{r=0,1} = C''|_{r=0,1} + C'V'|_{r=0,1} = C''|_{r=0,1} = w'|_{r=0,1} = 0.$$

Therefore, $\rho'|_{r=0} = \rho'|_{r=1} = 0$ and the maximum principle implies that $\rho_{max}(t) = \max_r \rho(r,t)$ satisfies

$$\dot{\rho}_{max} \leq -2\rho^2.$$

Hence, $\rho_{max}(t) \le \frac{1}{2t+\frac{1}{\rho_{max}(0)}} \le \frac{1}{2t}$, which proves the lemma. $\square$

Lemma 7.5. *The length l satisfies the estimate $l(t) \le l(0)(2\rho_{max}(0)t+1)^{1/2}$.*

Proof. By the evolution equation for l we obtain

$$\dot{l} = \int_0^l \rho ds \le l\rho_{max}(t) \le \frac{l}{2t+\frac{1}{\rho_{max}(0)}}, \tag{7.9}$$

hence $\frac{\dot{l}}{l} \le \frac{1}{2t+\frac{1}{\rho_{max}(0)}}$, which gives

$$l \le l_0(2\rho_{max}(0)t+1)^{1/2}.$$

$\square$

The next step is to show that the flow exists for all time. By Theorem 6.1, it suffices to show that the ambient curvature and the second fundamental form of the boundary remain bounded. The arguments above show that the required control on $|A|^2$ holds, so it remains to control the curvature.

Lemma 7.6. *The curvature remains bounded along the flow.*

Proof. Notice that

$$|\operatorname{Ric}|^2 = (V''+|A|^2)^2 + (Q''+(Q')^2+Q'U')^2 + (U''+(U')^2+Q'U')^2.$$

Given that $|A|^2 = (Q')^2+(U')^2$ is controlled, it is enough to control $s_1 := V''$ and $s_2 := C''$. A computation shows that their evolution equations are

$$\begin{aligned} \dot{s}_1 &= s_1''+\eta s_1' - (2\eta^2+w^2)s_1 - \eta w s_2 && (7.10)\\ \dot{s}_2 &= s_2''+\eta s_2' - (2w^2+\eta^2)s_2 - \eta w s_1 && (7.11) \end{aligned}$$

Setting $\sigma = \frac{1}{2}(s_1^2+s_2^2)$ we compute

$$\begin{aligned} \dot{\sigma} &= \sigma'' - (s_1)^2 - (s_2)^2 + \eta\sigma' - (2\eta^2+w^2)s_1^2 - (2w^2+\eta^2)s_2^2 - \eta w s_1 s_2 - \eta w s_1 s_2\\ &= \sigma'' + \eta\sigma' - (\eta^2+w^2)\sigma - [(s_1')^2 + (s_2')^2 + (\eta s_1 + w s_2)^2]\\ &\le \sigma'' + \eta\sigma' - (\eta^2+w^2)\sigma. && (7.12) \end{aligned}$$

On the boundary we have $\dot{\eta} = \eta = w' = 0$, hence

$$\sigma' = s_1 s_1' + s_2 s_2' = \eta'\eta'' + w'w'' = \eta'(\dot{\eta} - \eta\eta' + |A|^2\eta) = 0.$$

Combining this with the evolution equation of σ we obtain

$$\dot{\sigma}_{max} \leq 0,$$

which shows that σ remains bounded above. $\square$

In the next Lemma we show that σ behaves like $\frac{1}{t}$, as $t \to \infty$.

Lemma 7.7. *σ satisfies an estimate of the form $\sigma \leq \frac{f}{t}$, for an approrpiate constant $f > 0$.*

Proof. The evolution equation of $F = t\sigma + \rho$ is

$$\begin{aligned}
\dot{F} &= t\dot{\sigma} + \sigma + \dot{\rho} \\
&= t\left(\sigma'' + \eta\sigma - (\eta^2 + w^2)\sigma - \left((s_1')^2 + (s_2')^2 + (\eta s_1 + w s_2)^2\right)\right) \\
&= (t\sigma)'' + \eta(t\sigma)' - (\eta^2 + w^2)(t\sigma) - t\left((s_1')^2 + (s_2')^2 + (\eta s_1 + w s_2)^2\right) + \sigma \\
&\quad + \rho'' + \eta\rho' - \frac{(\eta^2 + w^2)^2}{2} - 2\sigma \\
&\leq F'' + \eta F'.
\end{aligned}$$

On the boundary $F' = t\sigma' + \rho' = \eta\eta' + ww' = 0$. Therefore, by the maximum principle F is bounded. This implies that

$$\sigma \leq \frac{f}{t}$$

for some constant f, which proves the lemma. In particular, since ρ_{max} is decreasing and $F|_{t=0} = \rho$, it follows that we can choose $f = \rho_{max}(0)$. $\square$

Now we state and finish the proof of the result.

Theorem 7.8. *Let g_0 be a Riemannian metric on $M = S^1 \times S^1 \times [0, 1]$ of the form 7.1, with minimal boundary. The Ricci flow solution which keeps the boundary minimal and fixes its conformal class exists for all time. Moreover,*

1. *$|\operatorname{Rm}| \leq \frac{f}{t}$ and $|A|^2 \leq \frac{1}{2t}$ for some constant $f > 0$.*
2. *There exists a constant $\Lambda > 0$ such that if $c_0 = c_1$ and $l(0)|A_0| \leq \frac{1}{\sqrt{\Lambda}}$, then l remains bounded and the flow converges in C^2 to a flat product metric. The same holds if $C|_{t=0}$ is constant.*
3. *If $c_0 \neq c_1$, the distance l of the two boundary components satisfies $l \geq a\sqrt{t + b}$ and the flow diverges.*

Proof. By Lemmata 7.4, 7.7 we know that the flow exists for all time, and the curvature decays to zero. Moreover, the flow will converge in C^2 to a flat product metric, provided that the distance of the two boundary components l remains bounded.

We first demonstrate that this is not the case if $c_0 \neq c_1$. By Lemma 7.4

$$a := |c_0 - c_1| = \left| \int_0^l C' ds \right| \leq \int_0^l \frac{1}{\sqrt{t+b}} ds = \frac{l}{\sqrt{t+b}}$$

Therefore, we obtain $l \geq a\sqrt{t+b}$.

By Lemma 7.3, $V \geq v_*$ for a uniform constant v_*. Hence, we have the estimate

$$Vol = \int_0^l e^V ds \geq l v_*.$$

Therefore, the convergence statement (1) will follow once we show that the volume is uniformly bounded. By 7.8 we have:

$$\frac{dVol}{dt} = \int_0^l \frac{w^2 - \eta^2}{2} ds \leq \int_0^l \frac{w^2}{2} ds \tag{7.13}$$

We have to estimate $\int_0^l w^2 ds$. For this, we compute

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^l w^2 ds\right) &= \int_0^l 2w\dot{w} ds + \int_0^l w^2 \dot{P} ds \\
&= \int_0^l 2w(w'' + \eta w' - |A|^2 w) ds + \int_0^l w^2 (V'' + |A|^2) ds \\
&= 2\int_0^l w w'' ds + 2\int_0^l w w' \eta ds - 2\int_0^l |A|^2 w^2 ds \\
&\quad + \int_0^l w^2 \eta' ds + \int_0^l w^2 |A|^2 ds \\
&\leq -2\int_0^l (w')^2 ds.
\end{aligned} \tag{7.14}$$

In the last step we integrate by parts and use that $w' = \eta = 0$ on the boundary (since the mean curvature and $\dot{C}$ are there).

Since $\int_0^l w ds = \int_0^l C' ds = c_1 - c_0 = 0$, there exists a constant Λ, independent of l, such that

$$\int_0^l w^2 ds \leq \Lambda l^2 \int_0^l (w')^2 ds. \tag{7.15}$$

Combining (7.14) and (7.15) we obtain

$$\frac{d}{dt}\left(\int_0^l \frac{w^2}{2} ds\right) \leq -\frac{2}{\Lambda l^2} \int_0^l \frac{w^2}{2} ds. \tag{7.16}$$

Setting $Y(t) = \int_0^l \frac{w^2}{2} ds$ we obtain $\frac{d}{dt} \log Y \leq -\frac{2}{\Lambda l^2}$. Then, the estimate of l in Lemma 7.5 imples that

$$\frac{d}{dt} \log Y \leq -\frac{1}{\Lambda l(0)^2 \rho_{max}(0)(t + \frac{1}{2\rho_{max}(0)})} = -\frac{\kappa}{t + \frac{1}{2\rho_{max}(0)}},$$

where $\kappa = \frac{1}{\Lambda l(0)^2 \rho_{max}(0)}$. Therefore Y satisfies the estimate

$$Y \leq \frac{Y_0}{(2\rho_{max} t + 1)^\kappa}.$$

Now, by 7.13, for the volume of M to remain bounded, it suffices to have $\int_0^\infty Y(t) dt < \infty$. This holds if $\kappa > 1$ or equivalently $l(0)^2 \rho_{max}(0) < 1/\Lambda$. If instead $C|_{t=0}$ is constant, then $Y(0) = 0$. Thus $Y \equiv 0$ and the statement again follows.

□

Remark 7.0.4. It is not hard to show that the scale invariant quantity $l\rho_{max}$ is nonincreasing along the flow. Indeed, we compute

$$\begin{aligned} \frac{d(l^2 \rho_{max})}{dt} = 2l\dot{l}\rho_{max} + l^2 \dot{\rho}_{max} &\leq 2l\dot{l} - 2l^2 \rho_{max}^2 \\ &= 2(l^2 \rho_{max}) \left(\frac{\dot{l}}{l} - \rho_{max}\right) \leq 0, \end{aligned}$$

where the last inequality follows from inequality (7.9).

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