Holomorphic Twistor Spaces and Bihermitian Geometry

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Abstract of the Dissertation

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Ever since the 1970's, holomorphic twistor spaces have been used to study the geometry and analysis of their base manifolds. In this dissertation, I will introduce integrable complex structures on twistor spaces fibered over complex manifolds that are equipped with certain geometrical data. The importance of these spaces will be shown to lie, for instance, in their applications to bihermitian geometry, also known as generalized Kahler geometry. (This is part of the generalized geometry program initiated by Nigel Hitchin.) By analyzing their twistor spaces, I will develop a new approach to studying bihermitian manifolds. In fact, I will demonstrate that the twistor space of a bihermitian manifold is equipped with two complex structures and natural holomorphic sections as well. This will allow me to construct tools from the twistor space that will lead, in particular, to new insights into the real and holomorphic Poisson structures on the manifold.

Contents

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Chapter 1

Introduction

In the 1970's, Atiyah, Hitchin and Singer introduced a tautological complex structure on a certain twistor space fibered over an anti-selfdual four manifold that has had, until today, a major impact on differential and complex geometry [3, 22]. It was generalized in the 1980's to the almost complex structure J^{∇}_{taut} on the twistor space of any even dimensional manifold [4, 21] (see Section 2.3 for the definition). Although J^{∇}_{taut} has led to many advances in geometry, it is nonetheless almost never integrable and its applications have consequently been limited to results about four dimensional manifolds or those of special type.

The purpose of this dissertation is to introduce other almost complex structures on twistor spaces that are not only widely integrable but also rich in their applications to various fields of geometry. The twistor spaces that we consider are the bundles of complex structures associated to even rank real vector bundles that are fibered over complex manifolds (Section 2.1). The complex structures that we build depend on connections that satisfy certain curvature conditions (Theorem 2.4.1). As these conditions are in fact easy to satisfy, our holomorphic twistor spaces exist in abundance. For instance, we demonstrate in Section 3.3 that the twistor spaces of well known Hermitian manifolds—SKT, bihermitian, as well as Calabi-Yau manifolds [7, 9, 1]—all admit complex structures that depend on the given geometrical data. More examples of holomorphic twistor spaces are given throughout Part 1 (Chapters 2-4) of this dissertation.

The focus of Part 2 (Chapters 5-8) is to demonstrate how to use these twistor spaces to derive results about the complex geometries of the base manifolds. In particular, we develop a new way of thinking about bihermitian manifolds—also known as generalized Kahler manifolds—that leads to insights into their real and holomorphic Poisson structures.

To give more details, our first application of holomorphic twistor spaces is given in Chapter 5. There we use their holomorphic sections to decompose the base manifold into different types of holomorphic subvarieties, denoted by M^{δ} (Theorem 5.1.1). The main idea behind this construction is that the holomorphic sections of twistor space not only induce holomorphic bundles over the base manifold but different types of holomorphic bundle maps as well. Some of the M^{δ} then correspond to the degeneracy loci of these maps while others refine their structure.

We then establish in Section 7.1 a twistor point of view of the M^{δ} by realizing them as intersections of different complex submanifolds and holomorphic subvarieties in twistor space. This allows us to develop tools to study the M^{δ} inside this space and leads us to derive a number of results about them. Our first result is given in Section 7.2 where we establish lower bounds on their dimensions. Secondly we determine necessary conditions for there to exist curves in the base manifold that lie in certain M^{δ} (Propositions 7.3.1 and 7.3.3). As described in Section 7.3, these conditions lead to *upper bounds* on the dimensions of the subvarieties.

To demonstrate the importance of these results, we will now describe two of our major classes of examples of a holomorphic twistor space equipped with holomorphic sections—when the base manifold is a bihermitian manifold and holomorphic twistor space. As part of the second example, we will provide more details as to how we used twistor spaces to derive the above results.

1.1 Bihermitian and Generalized Kahler Manifolds

A bihermitian manifold is a Riemannian manifold equipped with a pair of complex structures that satisfy certain relations (Section 3.3.2). These manifolds were first introduced by physicists in [9], as the target spaces of supersymmetric sigma models, and were later found to be equivalent to (twisted) generalized Kahler manifolds [13, 16] (see also [1]). Consequently, there are several approaches in the literature that are used to study these manifolds; and in this paper, we introduce yet another—we study them via their twistor spaces.

Indeed, one of our major results of Sections 3.3 and 3.4 is that the twistor space of a bihermitian manifold not only admits two complex structures but natural holomorphic sections as well. By then applying Theorem 5.1.1 to this case, we decompose the bihermitian manifold in Chapter 8 into holomorphic subvarieties that are new to the literature.

The importance of these subvarieties lies in their connection to known Poisson structures on the manifold: Some of the subvarieties are the degeneracy loci of a holomorphic Poisson structure while the ones that are new to the literature are surprisingly the loci of *real* Poisson structures. At the same time, there are others that refine the structure of both of these loci. As a consequence, we can now study the Poisson structures on a bihermitian manifold by using the new tools from twistor spaces that were described above.

For instance, by applying the general bounds of Theorem 7.2.6, we derive in Section 8.3 an existence result about the subvarieties in bihermitian manifolds. More precisely, we demonstrate that there are classes of bihermitian structures on $\mathbb{CP}^3$ that cannot admit certain M^{δ}. As these subvarieties refine the degeneracy loci of the corresponding holomorphic Poisson structures on $\mathbb{CP}^3$, our result provides new information about the structure of these loci.

Along with these results, we derive in Section 8.2 additional bounds on the dimensions of the subvarieties in a bihermitian manifold by using Poisson geometry. Before we do so, we apply the general existence conditions of Proposition 7.3.3, to show that it would be natural to expect these bounds just by knowing a few simple relations.

1.2 Stratifications of Twistor Spaces

Our second major class of examples of a twistor space that is equipped with holomorphic sections is when the base manifold is itself a holomorphic twistor space (Chapter 6). In this case, we not only produce different stratifications of twistor spaces, whose strata are complex submanifolds and holomorphic subvarieties, but also use these structures to derive the results about the general M^δ of Chapter 7.

As we show in Chapter 6, some of the complex submanifolds that we produce in twistor space can be viewed as Schubert cells in a certain Grassmannian space. By using this correspondence and defining special charts for the twistor space, we determine the dimensions of these submanifolds (as well as the dimensions of the other subvarieties) and describe their tangent bundles.

These properties are in fact important in our derivation of the results about the M^δ given in Chapter 7. The way that we derive them is to first holomorphically embed the base manifold into its twistor space and then, as mentioned above, to realize the M^δ as intersections of the different complex submanifolds and holomorphic subvarieties (Section 7.1). One advantage of this point of view is that the codimension of the intersection of any two holomorphic subvarieties is always bounded from above by the sum of their codimensions. Being that we have already determined the dimensions of the subvarieties in twistor space in Section 6.2, we arrive at the bounds on the M^δ given in Section 7.2. Moreover, we also apply the description of the tangent bundles given in Section 6.2 to derive the necessary conditions for there to exist curves that lie in certain M^δ as specified in Propositions 7.3.1 and 7.3.3.

Having explored some applications of holomorphic twistor spaces to bihermitian manifolds as well as twistor spaces, in our forthcoming paper [12] we will use them to derive results about Lie groups that admit, what we call, twistor holomorphic representations.

Let us now turn to the task of defining integrable complex structures on twistor spaces. We begin with the following preliminaries.

Part I

Holomorphic Twistor Spaces

Chapter 2

Complex Structures on Twistor Spaces

2.1 Preliminaries

Let V be a $2n$ dimensional real vector space and let $\mathcal{C}(V) = \{J \in EndV |\ J^2 = -1\}$ be one of its twistor spaces. To describe some of the properties of $\mathcal{C}(V)$, consider the action of $GL(V)$ on $EndV$ via conjugation: $B \cdot A = BAB^{-1}$. As $\mathcal{C}(V)$ is a particular orbit of this action, it is isomorphic to

$$GL(V)/GL(V, I),$$

where $I \in \mathcal{C}(V)$ and $GL(V, I) = \{B \in GL(V)|\ [B, I] = 0\} \cong GL(n, \mathbb{C})$. It then follows that the dimension of $\mathcal{C}(V)$ is $2n^2$ and that if we consider $\mathcal{C}(V)$ as a submanifold of $EndV$ then

$$T_J\mathcal{C} = [EndV, J] = \{A \in EndV|\ \{A, J\} = 0\}.$$

With this, we may define a natural almost complex structure on $\mathcal{C}(V)$ that is well known to be integrable:

$$I_{\mathcal{C}}A = JA, \text{ for } A \in T_J\mathcal{C}.$$

If we now equip V with a positive definite metric g then another twistor space that we will consider is $\mathcal{T}(V, g) = \{J \in \mathcal{C}(V)|\ g(J\cdot, J\cdot) = g(\cdot, \cdot)\}$. In this case, $\mathcal{T}$ is an orbit of the action of $O(V, g)$ on $EndV$ by conjugation, and is thus isomorphic to the Hermitian symmetric space $O(V, g)/U(I)$, where $I \in \mathcal{T}$ and $U(I) \cong U(n)$. It then follows that the dimension of $\mathcal{T}$ is $n(n-1)$ and that if we consider $\mathcal{T}$ as a submanifold of $EndV$ then

$$T_J\mathcal{T} = [\mathfrak{o}(V, g), J] = \{A \in \mathfrak{o}(V, g)|\ \{A, J\} = 0\}.$$

As $I_{\mathcal{C}}$ naturally restricts to $T_J\mathcal{T}$, $\mathcal{T}$ is a complex submanifold of $\mathcal{C}$.

2.1.1 Twistors of Bundles

Let now $E \longrightarrow M$ be an even dimensional vector bundle fibered over an even dimensional manifold. Generalizing the previous discussion to vector bundles, we will define $\mathcal{C}(E) =$

$\{J \in EndE|\ J^2 = -1\}$, which is a fiber subbundle of the total space of $\pi : EndE \longrightarrow M$ with general fiber $\mathcal{C}(E_x)$, for $x \in M$. Since the fibers of $\pi_{\mathcal{C}} : \mathcal{C}(E) \longrightarrow M$ are complex manifolds, $\mathcal{C}(E)$ naturally admits the complex vertical distribution $V\mathcal{C} \subset T\mathcal{C}(E)$, where $V_J\mathcal{C} = T_J\mathcal{C}(E_{\pi(J)}) \cong [EndE|_{\pi(J)}, J]$. Using the section $\phi \in \Gamma(\pi_{\mathcal{C}}^* EndE)$ defined by $\phi|_J = J$, we will then identify $V\mathcal{C}$ with the subbundle $[\pi_{\mathcal{C}}^* EndE, \phi]$ of $\pi_{\mathcal{C}}^* EndE$.

Letting g be a positive definite metric on E, we will also consider $\mathcal{T}(E, g) = \{J \in \mathcal{C}(E)|\ g(J\cdot, J\cdot) = g(\cdot, \cdot)\}$. Similar to the case of $\mathcal{C}(E)$, $\mathcal{T}(E, g)$ naturally admits the complex vertical distribution $V\mathcal{T}$, defined by $V_J\mathcal{T} = T_J\mathcal{T}(E_{\pi(J)}) \cong [\mathfrak{o}(E_{\pi(J)}, g), J]$. If we denote the projection map from $\mathcal{T}(E, g)$ to M by $\pi_{\mathcal{T}}$ then we will identify $V\mathcal{T}$ with the subbundle $[\pi_{\mathcal{T}}^* \mathfrak{o}(E, g), \phi]$ of $\pi_{\mathcal{T}}^* EndE$, where now $\phi \in \Gamma(\pi_{\mathcal{T}}^* EndE)$.

Notation 2.1.1. *As was done above and will be continued below, we will at times denote $\mathcal{C}(E)$ by $\mathcal{C}$ and $\mathcal{T}(E, g)$ by $\mathcal{T}(E)$, $\mathcal{T}(g)$ or just $\mathcal{T}$. Moreover, there are also times when we will denote $\pi_{\mathcal{C}}$ or $\pi_{\mathcal{T}}$ by just π.*

2.2 Horizontal Distributions and Splittings

With this background at hand, we will now take the first steps in defining integrable complex structures on $\mathcal{C}(E)$ and $\mathcal{T}(E, g)$ in the case when M is a complex manifold. Given a connection ∇ on E we will define the horizontal distribution $H^\nabla\mathcal{C}$ in $T\mathcal{C}$, so that this latter bundle splits into $V\mathcal{C} \oplus H^\nabla\mathcal{C}$. Similarly, in the case when g is a metric on E and ∇ is a metric connection, we will describe how to split $T\mathcal{T}$ into $V\mathcal{T} \oplus H^\nabla\mathcal{T}$. Once we have described these splittings we will define the desired complex structures on the above twistor spaces in Section 2.3.

To begin, let, as above, $E \longrightarrow M$ be a vector bundle, though the base manifold is not yet assumed to be a complex manifold, and let ∇ be any connection. As $\mathcal{C}$ is a fiber subbundle of the total space of $\pi : EndE \longrightarrow M$, we will find it convenient to split its tangent bundle by first splitting $TEndE$.

Although there are other ways to define this splitting the basic idea here is to use parallel translation with respect to ∇. First, if $A \in EndE$ and $\gamma : \mathbb{R} \longrightarrow M$ satisfies $\gamma(0) = \pi(A)$ then the parallel translate of A along γ will be denoted by $A(t)$. The horizontal distribution $H^\nabla EndE$ in $TEndE$ is then defined as follows.

Definition 2.2.1. Let $H_A^\nabla EndE = \{\frac{dA(t)}{dt}|_{t=0}|$ for all $\gamma, \gamma(0) = \pi(A)\}$.

It is straightforward to show that $H^\nabla EndE$ is a complement to the vertical distribution:

Lemma 2.2.2. *$TEndE = VEndE \oplus H^\nabla EndE$.*

Remark 2.2.3. *The above procedure can actually be used to split the tangent bundle of any vector bundle with a connection. Another way to define such a splitting is to consider the bundle as associated to its frame bundle and then use the standard theory of connections. These two methods yield the same splittings and are essentially equivalent.*

Now if $J \in \mathcal{C} \subset EndE$ and $\gamma : \mathbb{R} \longrightarrow M$ is a curve that satisfies $\gamma(0) = \pi(J)$ then it is clear that the associated parallel translate $J(t)$ lies in $\mathcal{C}$ for all relevant $t \in \mathbb{R}$. It then follows that $H_J^\nabla EndE$ lies in $T_J\mathcal{C}$, so that we have:

Lemma 2.2.4. *$T_J\mathcal{C} = V_J\mathcal{C} \oplus H^\nabla_J\mathcal{C}$, where $H^\nabla_J\mathcal{C} = H^\nabla_J EndE$.*

Similarly, if g is a metric on E and $\nabla g = 0$ then the parallel translate of $J \in \mathcal{T}$ along γ lies in $\mathcal{T}$. We thus have

Lemma 2.2.5. *$T_J\mathcal{T} = V_J\mathcal{T} \oplus H^\nabla_J\mathcal{T}$, where $H^\nabla_J\mathcal{T} = H^\nabla_J EndE$.*

With the above splittings, it will be useful for later calculations to derive a certain formula for the vertical projection operator $P^\nabla : TEndE \longrightarrow VEndE \cong \pi^*EndE$, which, upon suitable restriction, will also be valid for the corresponding projection operators for $T\mathcal{C}$ and $T\mathcal{T}$. The formula will depend on the tautological section ϕ of π^*EndE that is defined by $\phi|_A = A$:

Proposition 2.2.6. *Let $X \in T_A EndE$, then*

$$P^\nabla(X) = (\pi^*\nabla)_X\phi,$$

where we are considering P^∇ to be a section of $T^*EndE \otimes \pi^*EndE$.

Proof of Proposition 2.2.6. Let $\{e_i\}$ be a local frame for E over some open set $U \subset M$ about the point $\pi(A)$, where $A \in EndE$, and let $\{e_i \otimes e^j\}$ be the corresponding frame for $EndE$. Then for $X \in T_A EndE$,

$$(\pi^*\nabla)_X\phi = (\pi^*\nabla)_X\phi^i_j\pi^*(e_i \otimes e^j) \tag{2.2.1}$$
$$= d\phi^i_j(X)e_i \otimes e^j|_{\pi(A)} + A^i_j\nabla_{\pi_*X}e_i \otimes e^j. \tag{2.2.2}$$

Let us now consider the following two cases.

A) Let X be an element of $V_A EndE$, which for the moment is not identified with $EndE|_{\pi(A)}$, so that $\pi_*X = 0$. Also let A(t) be a curve in $EndE|_{\pi(A)}$ such that $A(0) = A$ and $\frac{dA(t)}{dt}|_{t=0} = X$. Then by Equation 2.2.2, $(\pi^*\nabla)_X\phi = \frac{dA(t)^i_j}{dt}|_{t=0}e_i \otimes e^j|_{\pi(A)} = P^\nabla(X) \in EndE|_{\pi(A)}$.

B) Let $X \in H^\nabla_A EndE$ so that it equals $\frac{d}{dt}A(t)|_{t=0}$, where $A(t)$ is the parallel translate of A along some curve $\gamma : \mathbb{R} \longrightarrow M$ that satisfies $\gamma(0) = \pi(A)$. As $d\phi^i_j(X) = \frac{d}{dt}A(t)^i_j|_{t=0}$, Equation 2.2.2 becomes $\frac{d}{dt}A(t)^i_j|_{t=0}e_i \otimes e^j|_{\pi(A)} + A^i_j\nabla_{\frac{d\gamma}{dt}|_{t=0}}e_i \otimes e^j$, which is zero since $A(t)$ is parallel. □

If we consider the corresponding projection operator $P^\nabla : T\mathcal{C} \longrightarrow V\mathcal{C}$ then it follows from the above proposition that $P^\nabla(X) = (\pi^*_\mathcal{C}\nabla)_X\phi$, where ϕ is now a section of $\pi^*_\mathcal{C}EndE \longrightarrow \mathcal{C}$. Note that since $\phi^2 = -1$, $(\pi^*_\mathcal{C}\nabla)_X\phi$, for $X \in T_J\mathcal{C}$, is indeed contained in $V_J\mathcal{C} = \{A \in EndE|_{\pi(J)}|\ \{A, J\} = 0\}$. In the case when g is a metric on E and $\nabla g = 0$, an analogous formula holds for $P^\nabla : T\mathcal{T} \longrightarrow V\mathcal{T}$.

Remark 2.2.7. *We respectfully report that similar formulas for the projection operators for $T\mathcal{C}$ and $T\mathcal{T}$ were derived in [21] but with a small error.*

2.3 The Complex Structures

Now let $E \longrightarrow (M, I)$ be an even dimensional bundle that is fibered over an almost complex manifold and let ∇ be a connection on E. We will define the following almost complex structure on the total space of $\pi : \mathcal{C}(E) \longrightarrow M$ and explore its integrability conditions in the next section.

Definition 2.3.1. $\boldsymbol{\mathcal{J}^{(\nabla,I)}}$ **:** First use ∇ to split

$$T\mathcal{C} = V\mathcal{C} \oplus H^{\nabla}\mathcal{C},$$

and then let

$$\begin{aligned}&(1)\ \mathcal{J}^{(\nabla,I)}A = JA\\&(2)\ \mathcal{J}^{(\nabla,I)}v^{\nabla} = (Iv)^{\nabla},\end{aligned}$$

where $A \in V_J\mathcal{C} \subset EndE|_{\pi(J)}$ and $v^{\nabla} \in H^{\nabla}_J\mathcal{C}$ is the horizontal lift of $v \in T_{\pi(J)}M$.

In other words, $\mathcal{J}^{(\nabla,I)}$ on $V\mathcal{C} \oplus H^{\nabla}\mathcal{C}$ equals $\phi \oplus \pi^*I$, where we have identified $V\mathcal{C}$ with $[\pi^*EndE, \phi]$ and $H^{\nabla}\mathcal{C}$ with π^*TM.

It then follows from the definition of $\mathcal{J}^{(\nabla,I)}$ that π is pseudoholomorphic:

Proposition 2.3.2. $\pi : (\mathcal{C}, \mathcal{J}^{(\nabla,I)}) \longrightarrow (M, I)$ *is a pseudoholomorphic submersion.*

In the case when g is a metric on E and ∇ is a metric connection, the claim is that $\mathcal{J}^{(\nabla,I)}$ on $\mathcal{C}$ restricts to $\mathcal{T}$, so that $\mathcal{T} \subset (\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ is an almost complex submanifold. The reason is that $T_J\mathcal{T}$ splits into $V_J\mathcal{T} \oplus H^{\nabla}_J\mathcal{T}$, where $H^{\nabla}_J\mathcal{T} = H^{\nabla}_J\mathcal{C} = H^{\nabla}_J EndE$, as explained in the previous section.

Remark 2.3.3. *It should be noted that $\mathcal{J}^{(\nabla,I)}$ has not yet been studied in this generality in the literature. In [24], Vaisman did study $\mathcal{J}^{(\nabla,I)}$ only in the special case when $E = TM$ and $\nabla I = 0$ and only on certain submanifolds of $\mathcal{C}(TM)$. However, for our applications we do not want to restrict ourselves to $E = TM$ and we especially do not want to require $\nabla I = 0$.*

With $\mathcal{J}^{(\nabla,I)}$ defined, let us now compare it to the tautological almost complex structures on twistor spaces that are usually considered in the literature [3, 21, 4]. If ∇' is a connection on $TM \longrightarrow M$, where here M is any even dimensional manifold, then based on the splitting of $T\mathcal{C}$ into $V\mathcal{C} \oplus H^{\nabla'}\mathcal{C}$, we define $\mathcal{J}^{\nabla'}_{taut}$ on $\mathcal{C}(TM)$ as follows.

Definition 2.3.4. Let $\mathcal{J}^{\nabla'}_{taut} = \phi \oplus \phi$, where we have identified $V\mathcal{C}$ with $[\pi^*EndTM, \phi]$ and $H^{\nabla'}\mathcal{C}$ with π^*TM, and where the first ϕ factor acts by left multiplication.

To compare it to $\mathcal{J}^{(\nabla,I)}$, note that $\mathcal{J}^{\nabla'}_{taut}$ does not require M to admit an almost complex structure, while the former one does. On the other hand, $\mathcal{J}^{\nabla'}_{taut}$ is only defined for the bundle $E = TM$ whereas $\mathcal{J}^{(\nabla,I)}$ is defined for any even dimensional vector bundle. Also, given (M, I), the projection map $(\mathcal{C}(TM), \mathcal{J}^{\nabla'}_{taut}) \longrightarrow (M, I)$ is never pseudoholomorphic, whereas $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)}) \longrightarrow (M, I)$ is always so. Lastly, $\mathcal{J}^{\nabla'}_{taut}$ is rarely integrable– except in special cases such as when M is an anti-selfdual four manifold [3]– whereas the integrability conditions of $\mathcal{J}^{(\nabla,I)}$ are very natural to be fulfilled, as we will show below.

Although $\mathcal{J}^{(\nabla,I)}$ and $\mathcal{J}_{taut}^{\nabla'}$ are defined quite differently and have opposite properties, they are still related holomorphically. Indeed, given (M, I) and a connection ∇ on TM and letting $\mathcal{C} = \mathcal{C}(TM)$, we will show in Section 3.4.2 that $\mathcal{J}_{taut}^{\nabla}$, which is almost never integrable, is always a pseudoholomorphic section of $(\mathcal{C}(T\mathcal{C}), \mathcal{J}) \longrightarrow (\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ for some appropriately chosen $\mathcal{J}$.

Having compared the above almost complex structures, let us now return to the general setup of a vector bundle $E \longrightarrow (M, I)$ that is fibered over an almost complex manifold and that is equipped with a connection ∇. The goal is to determine the conditions on I and the curvature of ∇, R^∇, that are equivalent to the integrability of $\mathcal{J}^{(\nabla,I)}$ not just on $\mathcal{C}$ but on other almost complex submanifolds $\mathcal{C}'$ as well. Although these conditions can be worked out for any $\mathcal{C}'$, we will focus on the case when the corresponding projection map $\pi_{\mathcal{C}'} : \mathcal{C}' \longrightarrow M$ is a surjective submersion. If g is a metric on E and $\nabla g = 0$ then as an example we can take $\mathcal{C}' = \mathcal{T}(g)$. We will describe other examples below in Section 2.5.

The method that we will use to explore the integrability conditions of $\mathcal{J}^{(\nabla,I)}$ on $\mathcal{C}'$ is to calculate its Nijenhuis tensor on $\mathcal{C}$.

2.3.1 Nijenhuis Tensor

In this section, let $\pi : \mathcal{C}(E) \longrightarrow M$ be the projection map and define $\mathcal{J} := \mathcal{J}^{(\nabla,I)}$ and $P := P^\nabla : T\mathcal{C} \longrightarrow V\mathcal{C} \subset \pi^*EndE$, as in Section 2.2. We will presently compute the Nijenhuis tensor, $N^{\mathcal{J}}$, of $\mathcal{J}$ that is given by

$$N^{\mathcal{J}}(X, Y) = [\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}[\mathcal{J}X, Y] - \mathcal{J}[X, \mathcal{J}Y] - [X, Y],$$

in terms of the Nijenhuis tensor of I and the curvature of ∇, R^∇.

Proposition 2.3.5. *Let $X, Y \in T_J\mathcal{C}$ and let $v = \pi_*X$ and $w = \pi_*Y$. Then*

1) $\pi_*N^{\mathcal{J}}(X, Y) = N^I(v, w)$
2) $PN^{\mathcal{J}}(X, Y) = [R^\nabla(v, w) - R^\nabla(Iv, Iw), J] + J[R^\nabla(Iv, w) + R^\nabla(v, Iw), J].$

Proof of Proposition 2.3.5, Part 1). This easily follows from the fact that if $X \in \Gamma(T\mathcal{C})$ is π-related to $v \in \Gamma(TM)$ then $\mathcal{J}X$ is π-related to Iv. □

Letting, as above, $\phi \in \Gamma(\pi^*EndE)$ be defined by $\phi|_J = J$, the proof of Part 2 of the proposition, will be based on the following lemma.

Lemma 2.3.6. *Let $X, Y \in \Gamma(T\mathcal{C})$. Then*

$$P^\nabla([X, Y]) = -[R^{\pi^*\nabla}(X, Y), \phi] + \pi^*\nabla_X P(Y) - \pi^*\nabla_Y P(X).$$

Proof. Consider

$$\begin{aligned} P^\nabla([X, Y]) &= \pi^*\nabla_{[X,Y]}\phi \\ &= -R^{(\pi^*\nabla,\pi^*EndE)}(X, Y)\phi + \pi^*\nabla_X\pi^*\nabla_Y\phi - \pi^*\nabla_Y\pi^*\nabla_X\phi, \end{aligned}$$

where $R^{(\pi^*\nabla,\pi^*EndE)}$ is the curvature of $\pi^*\nabla$, which is considered as a connection on π^*EndE. The lemma then follows from the identity: $R^{(\pi^*\nabla,\pi^*EndE)}(X,Y)\phi = [R^{\pi^*\nabla}(X,Y),\phi]$.

$\square$

Proof of Proposition 2.3.5, Part 2). Let $X,Y \in \Gamma(T\mathcal{C})$ and consider $PN^{\mathcal{J}}(X,Y) = P([\mathcal{J}X,\mathcal{J}Y] - \mathcal{J}[\mathcal{J}X,Y] - \mathcal{J}[X,\mathcal{J}Y] - [X,Y])$. By using the previous lemma as well as the fact that $P\mathcal{J} = \phi P$, we can express $PN^{\mathcal{J}}(X,Y)$ as the sum of two sets of terms. The first set involves the curvature of $\pi^*\nabla$:

$$[R^{\pi^*\nabla}(X,Y) - R^{\pi^*\nabla}(\mathcal{J}X,\mathcal{J}Y),\phi] + \phi[R^{\pi^*\nabla}(\mathcal{J}X,Y) + R^{\pi^*\nabla}(X,\mathcal{J}Y),\phi].$$

When restricted to $J \in \mathcal{C}$ this gives the expression for $PN^{\mathcal{J}}(X,Y)$ that is contained in Part 2 of the proposition.

The second set of terms is

$$\pi^*\nabla_{\mathcal{J}X}P(\mathcal{J}Y) - \phi\pi^*\nabla_{\mathcal{J}X}P(Y) - \phi\pi^*\nabla_X P(\mathcal{J}Y) - \pi^*\nabla_X P(Y) - (X \leftrightarrow Y).$$

Using $P\mathcal{J} = \phi P$, it easily follows that the first four terms and the last four, which are represented by $(X \leftrightarrow Y)$, separately add to zero. $\square$

2.3.2 Integrability Conditions

We are now prepared to explore the integrability conditions of $\mathcal{J}^{(\nabla,I)}$ on $\mathcal{C}'$, where, as above, $\mathcal{C}'$ is any almost complex submanifold of $(\mathcal{C}(E),\mathcal{J}^{(\nabla,I)})$ such that $\pi_{\mathcal{C}'} : \mathcal{C}' \longrightarrow M$ is a surjective submersion. As is well known, $\mathcal{J}^{(\nabla,I)}$ on $\mathcal{C}'$ will be integrable if and only if $\pi_* N^{\mathcal{J}}(X,Y)$ and $PN^{\mathcal{J}}(X,Y)$ are both zero $\forall X,Y \in T_J\mathcal{C}'$ and $\forall J \in \mathcal{C}'$. By Proposition 2.3.5, the first condition is equivalent to the vanishing of the Nijenhuis tensor of I, while the second is equivalent to

$$[R^{\nabla}(v,w) - R^{\nabla}(Iv,Iw),J] + J[R^{\nabla}(Iv,w) + R^{\nabla}(v,Iw),J] = 0$$

$\forall v,w \in T_{\pi(J)}M$ and $\forall J \in \mathcal{C}'$. To analyze this condition, we will express it in terms of $R^{0,2}$, the (0,2)-form part of the curvature R^{∇}:

Lemma 2.3.7. *The condition*

$$[R^{\nabla}(v,w) - R^{\nabla}(Iv,Iw),J] + J[R^{\nabla}(Iv,w) + R^{\nabla}(v,Iw),J] = 0$$

$\forall v,w \in T_{\pi(J)}M$ *holds true if and only if*

$$[R^{0,2},J]E_J^{0,1} = 0.$$

We thus have:

Theorem 2.3.8. $(\mathcal{C}',\mathcal{J}^{(\nabla,I)})$ *is a complex manifold if and only if*

1) I *is integrable*
2) $[R^{0,2},J]E_J^{0,1} = 0, \quad \forall J \in \mathcal{C}'.$

Note that the second condition in the above theorem is equivalent to $R^{0,2} : E_J^{0,1} \longrightarrow E_J^{0,1}, \forall J \in \mathcal{C}'$.

2.4 (1,1) Curvature

Assuming henceforth that I is integrable, an important case of Part 2 of the above theorem that guarantees that $(\mathcal{C}', \mathcal{J}^{(\nabla,I)})$ is a complex manifold is when $R^{(0,2)} = 0$, or equivalently, when R^∇ is (1,1) with respect to I. In particular, we have:

Theorem 2.4.1. *Let $E \longrightarrow (M, I)$ be fibered over a complex manifold and let ∇ be a connection on E that has (1,1) curvature. Then $\mathcal{J}^{(\nabla,I)}$ is an integrable complex structure on $\mathcal{C}(E)$. In addition, if g is a metric on E and $\nabla g = 0$ then $\mathcal{T}(E, g)$ is a complex submanifold of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$.*

If we $\mathbb{C}$-linearly extend ∇ to a complex connection on $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ then the condition that R^∇ is (1,1) can also be expressed as $(\nabla^{0,1})^2 = 0$. We thus have:

Lemma 2.4.2. *Let ∇ be a connection on $E \longrightarrow (M, I)$. Then R^∇ is (1,1) if and only if $\nabla^{0,1}$ is a $\overline{\partial}-$operator on $E_{\mathbb{C}}$.*

In Section 4.2 we will use the fact that $\nabla^{0,1}$ is a $\overline{\partial}-$operator to holomorphically embed $(\mathcal{C}', \mathcal{J}^{(\nabla,I)})$ into a more familiar complex manifold that is associated to the holomorphic bundle $E_{\mathbb{C}}$– the Grassmannian bundle, $Gr_n(E_{\mathbb{C}})$.

Example 2.4.3 (Pseudoholomorphic Curves)**.** Let $E \longrightarrow (M, I)$ be an even dimensional vector bundle fibered over a complex curve. If ∇ is any connection on E then $R^{0,2}$ is automatically zero and hence $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$ is a complex manifold. Moreover if g is a metric on E and ∇ is a metric connection then $\mathcal{T}(E, g)$ is a complex submanifold of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$.

As an application, let $E \longrightarrow (N, J)$ be an even dimensional vector bundle that is fibered over an almost complex manifold and let ∇ be any connection on E. The goal is to show that although $(\mathcal{C}(E), \mathcal{J}^{(\nabla,J)})$ is only an almost complex manifold, it always contains many pseudoholomorphic submanifolds that are in fact complex manifolds. The idea is to use the well known existence of a plethora of pseudoholomorphic curves in N. Indeed, if we let $i : (S, I) \longrightarrow (N, J)$ be a pseudoholomorphic embedding of a complex curve into N then the curvature of $i^*\nabla$ on i^*E is (1,1) and thus $(\mathcal{C}(i^*E), \mathcal{J}^{(i^*\nabla,I)})$ is a complex manifold. As it is straightforward to show that i induces a pseudoholomorphic embedding of $\mathcal{C}(i^*E)$ into $\mathcal{C}(E)$, $\mathcal{C}(i^*E)$ is one of many examples of pseudoholomorphic submanifolds of $\mathcal{C}(E)$ that are themselves complex manifolds.

Further connections between twistors and pseudoholomorphic curves will be explored in the near future. $\square$

Another example of a vector bundle that naturally admits connections with (1,1) curvature is a holomorphic Hermitian bundle. We will describe this case in detail in Sections 3.1-3.3, but in the following we present examples where the base manifold M is a twistor space itself as well as an anti-selfdual four manifold.

Example 2.4.4. (Twistors) Let V be an even dimensional real vector space and, as in Section 2.1, let $\mathcal{C}(V)$ be its twistor space with complex structure $I_{\mathcal{C}}$. The bundle E that we will consider is the trivial bundle $\mathcal{C}(V) \times V \longrightarrow \mathcal{C}(V)$. We could then choose the trivial connection d on E to define a complex structure on $\mathcal{C}(E)$ but let us modify it by using a certain section ϕ of $EndE \longrightarrow \mathcal{C}(V)$ that is defined by $\phi|_J = J$. (ϕ has appeared before in Sections 2.1.1 and 2.2 where we were discussing the twistor spaces associated to bundles and not just vector spaces.) The connection that we will then choose is $\nabla := d + \frac{1}{2}(d\phi)\phi$ since its curvature has the desired property:

Proposition 2.4.5. *∇ is a connection on E with (1,1) curvature.*

To prove this, we need the following lemma, which is a special case of Proposition 2.2.6.

Lemma 2.4.6. *Let $A \in T_J\mathcal{C} = \{B \in EndV | \ \{B, J\} = 0\}$. Then $d_A\phi = A$.*

Proof of Proposition 2.4.5. The curvature R^∇ is given by $-\frac{1}{4}(d\phi \wedge d\phi)$. To show that it is (1,1) first note that if $A \in T_J\mathcal{C} \subset EndV$ then $I_{\mathcal{C}}A = JA$ and by the above lemma, $d_A\phi = A$. It then follows that for A and $B \in T_J\mathcal{C}$, $R^\nabla(JA, JB) = -\frac{1}{4}[JA, JB]$ and since A and B anticommute with J, this equals $-\frac{1}{4}[A, B] = R^\nabla(A, B)$. Thus R^∇ is (1,1). □

It then follows from the above proposition as well as Theorem 2.4.1 that $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I_{\mathcal{C}})})$ is a complex manifold.

Of course, we could have replaced $\frac{1}{2}(d\phi)\phi$ in the definition of ∇ with, for example, just $d\phi$. The reason that we chose this specific term is that ∇ will then satisfy $\nabla\phi = 0$, which will especially be used in the forthcoming chapters.

In Section 3.4 we will show how this example can be understood as part of a general procedure that produces new connections with (1,1) curvature from holomorphic sections of twistor space. □

Example 2.4.7 (Anti-selfdual Curvatures)**.** Consider an even dimensional vector bundle $E \longrightarrow (M, g)$ that is fibered over a four dimensional oriented Riemannian manifold. Since the manifold is oriented, the bundle of 2-forms, $\wedge^2 T^*$, splits into a direct sum of $\wedge^+$ and $\wedge^-$, the $+1$ and -1 eigenbundles of the Hodge star operator. To obtain a complex structure on $\mathcal{C}(E)$, suppose ∇ is a connection on E with anti-selfdual curvature, i.e. $R^\nabla \in \Gamma(\wedge^- \otimes EndE)$. Moreover, suppose I is a complex structure on M that is compatible with g and that also induces the same orientation as that of the given one. The claim then is that R^∇ is automatically (1,1) with respect to I. The reason is that it is well known that $\wedge^+ =< w > \oplus(\wedge^{2,0} \oplus \wedge^{0,2})$ and $\wedge^- = \wedge^{1,1}_0$, where $w(\cdot, \cdot) = g(I\cdot, \cdot)$ and $\wedge^{1,1}_0$ is the orthogonal complement to $< w >$ in $\wedge^{1,1}$. We thus have:

Corollary 2.4.8. *$(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ is a complex manifold.*

Let us now consider the following special case, which is more familiar in the literature. Let $E = TM$ and let (M, g, I) be an anti-selfdual Hermitian four manifold whose orientation is determined by I. As the curvature of the Levi Civita connection, R^∇, lies in $\Gamma(\wedge^- \otimes \mathfrak{o}(TM, g))$, there are at least two integrable complex structures on $\mathcal{T}^+$, which is the subbundle of $\mathcal{T}$ whose elements induce the same orientation as that of I. The first of these complex structures is $\mathcal{J}^{(\nabla,I)}$, as it is integrable on all of $\mathcal{T}$, and the second is the tautological complex structure $\mathcal{J}^{\nabla}_{taut}$ as defined in Section 2.3 [3].

Given that $\mathcal{T}^+$ admits these two complex structures, it is natural to explore some bundles over this twistor space that admit connections with (1,1) curvature, for they in turn can be used to define complex structures on other associated twistor spaces. Letting $\pi : \mathcal{T}^+ \longrightarrow M$ be the projection map, the first bundle that we will consider is π^*TM. The claim then is that there are at least two connections on this bundle that have (1,1) curvatures with respect to both $\mathcal{J}^{(\nabla,I)}$ and $\mathcal{J}^{\nabla}_{taut}$. The first of these connections is simply $\pi^*\nabla$– its curvature is (1,1) because $R^\nabla \in \Gamma(\wedge^- \otimes \mathfrak{o}(TM,g))$– while the other connection is $\pi^*\nabla' = \pi^*\nabla + \frac{1}{2}(\pi^*\nabla\phi)\phi$, where $\phi \in \Gamma(\pi^*EndTM)$ is defined by $\phi|_J = J$. One way to prove that this latter connection has the desired (1,1) curvature property is to generalize the proof of Proposition 2.4.5. Although this is straightforward to carry out, we will prove it instead in Section 3.4 by first showing that ϕ is a holomorphic section of $(\mathcal{C}(\pi^*TM), \mathcal{J}^{(\pi^*\nabla,\mathcal{I})}) \longrightarrow (\mathcal{T}^+,\mathcal{I})$ for $\mathcal{I} \in \{\mathcal{J}^{(\nabla,I)}, \mathcal{J}^{\nabla}_{taut}\}$. Note that, similar to the discussion in the previous example, the connection $\pi^*\nabla'$ satisfies $\pi^*\nabla'\phi = 0$, which will have several applications later on.

Now we can use these connections on π^*TM to produce connections on the tangent bundle of $\mathcal{T}^+$, $T\mathcal{T}^+$, that also have (1,1) curvatures with respect to both $\mathcal{J}^{(\nabla,I)}$ and $\mathcal{J}^{\nabla}_{taut}$. For this, split $T\mathcal{T}^+ = V\mathcal{T}^+ \oplus H^\nabla\mathcal{T}^+$, which is a special case of Lemma 2.2.5, and identify $V\mathcal{T}^+$ with $[\pi^*(\mathfrak{o}(TM,g)),\phi]$ and $H^\nabla\mathcal{T}^+$ with π^*TM. As one may check, $\pi^*\nabla + \frac{1}{2}[(\pi^*\nabla\phi)\phi,\cdot] \oplus D$, where D is either $\pi^*\nabla$ or $\pi^*\nabla'$, defines a connection on $T\mathcal{T}^+$ with (1,1) curvature and thus defines different complex structures on $\mathcal{C}(T\mathcal{T}^+)$.

We will consider more properties of $\mathcal{J}^{(\nabla,I)}$ and $\mathcal{J}^{\nabla}_{taut}$ as well as their interaction in Section 3.4.2. $\square$

In Chapter 3, we will give more examples of bundles that admit connections with (1,1) curvature and show, in particular, that SKT, bihermitian as well as Calabi-Yau manifolds naturally admit complex structures on their twistor spaces.

2.5 Other Curvature Conditions

Although, by Theorem 2.3.8, the condition $R^{(0,2)} = 0$ guarantees the integrability of $\mathcal{J}^{(\nabla,I)}$ on $\mathcal{C}' \subset \mathcal{C}(E)$, it is not the most general one. The present goal is to demonstrate some of these more general conditions for certain $\mathcal{C}'$.

As a first example, consider a $\mathcal{C}'$ that satisfies the following condition: given any $J \in \mathcal{C}'$, $-J$ is also in $\mathcal{C}'$.

Proposition 2.5.1. *If $\mathcal{C}'$ satisfies the above condition then $(\mathcal{C}', \mathcal{J}^{(\nabla,I)})$ is a complex manifold if and only if $[R^{0,2}, J] = 0$ for all $J \in \mathcal{C}'$.*

Proof. If $(\mathcal{C}', \mathcal{J}^{(\nabla,I)})$ is a complex manifold then given $J \in \mathcal{C}'$, it follows from Theorem 2.3.8 that $[R^{0,2}, J]E^{0,1}_J$ and $[R^{0,2}, J]E^{0,1}_{-J}$ are both zero. Hence $[R^{0,2}, J] = 0$ for all $J \in \mathcal{C}'$. As I is already assumed to be integrable, the converse also follows from Theorem 2.3.8. $\square$

In the case when $\mathcal{C}' = \mathcal{C}$, it is straightforward to show that the condition $[R^{0,2}, J] = 0$ for all $J \in \mathcal{C}$ is equivalent to the endomorphism part of $R^{0,2}$ being pointwise constant. We thus have:

Proposition 2.5.2. *$(\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ is a complex manifold if and only if $R^{(0,2)} = \lambda \otimes \mathbf{1}$, where λ is a (0,2) form on M and $\mathbf{1}$ is the identity endomorphism on $E_{\mathbb{C}}$.*

Example 2.5.3. To take a simple example, let ∇' be a connection on $E \longrightarrow (M, I)$ that has (1,1) curvature and let $\nabla = \nabla' + (w \otimes \mathbf{1})$ for some 1-form w. Then $(R^{\nabla})^{0,2} = (\nabla^{0,1})^2$ on $E_{\mathbb{C}}$ equals $\overline{\partial} w^{0,1} \otimes \mathbf{1}$ and hence $\mathcal{J}^{(\nabla,I)}$ is a complex structure on $\mathcal{C}$. This complex structure, however, is not new since $\mathcal{J}^{(\nabla,I)}$ is actually equal to $\mathcal{J}^{(\nabla',I)}$. The reason is that although the connections ∇ and ∇' are not equal on E they are in fact the same on $EndE$.

More interesting examples will be the subject of future work. □

For another example of a $\mathcal{C}'$ of the above type, let g be a metric on $E \longrightarrow (M, I)$ and let ∇ be a metric connection. As in the case for $\mathcal{C}$, it follows from Proposition 2.5.1 that $\mathcal{J}^{(\nabla,I)}$ is integrable on $\mathcal{C}' = \mathcal{T}(g)$ if and only if the endomorphism part of $R^{0,2}$ is pointwise constant. However, in this case $R^{0,2}$ is a (0,2) form that takes values in the skew endomorphism bundle $\mathfrak{o}(E_{\mathbb{C}}, g)$, so that its trace is zero. We thus have:

Proposition 2.5.4. *$(\mathcal{T}, \mathcal{J}^{(\nabla,I)})$ is a complex manifold if and only if $R^{(0,2)} = 0$.*

Example 2.5.5. In Chapter 6 we will be considering other types of $\mathcal{C}'$. For example, let $E \longrightarrow (M, I)$ be equipped with a metric g and a metric connection ∇. If $J \in \Gamma(\mathcal{T})$ satisfies $\nabla J = 0$ then we will show, in particular, that the following are almost complex submanifolds of $(\mathcal{T}, \mathcal{J}^{(\nabla,I)})$:

$$1)\mathcal{T}^{(m_1,*)}(J) = \{K \in \mathcal{T} |\ dimKer(K + J) = 2m_1\}$$
$$2)\mathcal{T}^{(*,m_{-1})}(J) = \{K \in \mathcal{T} |\ dimKer(K - J) = 2m_{-1}\}.$$

In addition, if R^{∇} is (1,1) then the above are complex submanifolds that form the strata of some of the stratifications of $\mathcal{T}$ that were mentioned in the introduction. □

Chapter 3

More Examples

The goal of this chapter is to describe various connections with (1,1) curvature on holomorphic Hermitian bundles and the resulting complex structures on the twistor spaces $\mathcal{C}$ and $\mathcal{T}$. In particular, we will demonstrate that the twistor spaces of SKT, bihermitian and Calabi-Yau manifolds naturally admit complex structures. In Section 3.4, we will explore the holomorphic sections of these twistor spaces and show how they can be used to construct other connections that have (1,1) curvature.

We will now begin by considering the Chern connections of Hermitian bundles.

3.1 Chern Connections

Let $E \longrightarrow (M, I)$ be a holomorphic bundle fibered over a complex manifold. Here, we will view it as a real bundle equipped with a fiberwise complex structure, J. If g is any fiberwise metric on E that is compatible with J then, as is well known, the associated Chern connection ∇^{Ch} (considered as a real connection on E) has (1,1) curvature. We thus have

Corollary 3.1.1. *$(\mathcal{C}, \mathcal{J}^{(\nabla^{Ch}, I)})$ is a complex manifold and $\mathcal{T}$ is a complex submanifold.*

Example 3.1.2. As a simple example, let (M, I) be any complex manifold that admits a Kahler metric g. Then the Chern connection, ∇^{Ch}, on TM equals the Levi Civita connection, ∇^{Levi}. Thus $\mathcal{J}^{(\nabla^{Levi}, I)}$ is an integrable complex structure on $\mathcal{C}$ and $\mathcal{T}$. $\square$

If we now $\mathbb{C}$-linearly extend ∇^{Ch} to $E_{\mathbb{C}}$ then, as a particular case of Lemma 2.4.2, $\nabla^{Ch(0,1)}$ is a $\bar{\partial}$-operator for this bundle. To describe this $\bar{\partial}$-operator in more familiar terms, let us consider the holomorphic bundle $E^{1,0} \oplus E^{*1,0}$, where $E^{1,0}$ is the $+i$ eigenbundle of J. The claim then is that the map

$$1 \oplus g : E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1} \longrightarrow E^{1,0} \oplus E^{*1,0}$$

is an isomorphism of holomorphic vector bundles. If we denote the Chern connection on $E^{1,0}$ by $\tilde{\nabla}^{Ch}$ then this follows from the following proposition, whose proof is straightforward.

Proposition 3.1.3. $\nabla^{Ch} = \tilde{\nabla}^{Ch} \oplus g^{-1}\tilde{\nabla}^{Ch} g$, *as complex connections on $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$.*

Thus in particular if $\{e_i\}$ is a local holomorphic trivialization of $E^{1,0}$ then $\{e_i, g^{-1}(e^i)\}$ is a holomorphic trivialization of $E_{\mathbb{C}}$.

Now if g' is another metric on E that is compatible with J then in Chapter 4 we will address the question of whether $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'}, I)})$ is biholomorphic to $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch}, I)})$ by holomorphically embedding twistor spaces into Grassmannian bundles.

3.2 $\bar{\partial}$-operators

In the previous section, we found it useful to describe $\nabla^{Ch(0,1)}$ on $E_{\mathbb{C}}$ by considering the natural $\bar{\partial}$-operator $\bar{\partial}$ on $E^{1,0} \oplus E^{*1,0}$ and the isomorphism

$$1 \oplus g : E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1} \longrightarrow E^{1,0} \oplus E^{*1,0}.$$

In this section, we will give more examples of $\bar{\partial}-$operators on $E^{1,0} \oplus E^{*1,0}$ and use this same isomorphism to transfer them to ones on $E_{\mathbb{C}}$. These in turn will give metric connections on E with (1,1) curvature that can be used to define complex structures on $\mathcal{T}$.

To begin, let $(E, g, J) \longrightarrow (M, I)$ be, as above, a holomorphic Hermitian vector bundle and consider the following natural symmetric bilinear form $<,>$ on $E^{1,0} \oplus E^{*1,0}$: $< X + \mu, Y + \nu >= \frac{1}{2}(\mu(Y) + \nu(X))$. A general $\bar{\partial}$-operator that preserves this metric is of the form $\bar{\partial} + \mathcal{D}'^{0,1}$, where $\mathcal{D}'^{0,1} \in \Gamma(T^{*0,1} \otimes \mathfrak{so}(E^{1,0} \oplus E^{*1,0}))$. If we now consider the splitting of $\mathfrak{so}(E^{1,0} \oplus E^{*1,0}) = EndE^{1,0} \oplus \wedge^2 E^{*1,0} \oplus \wedge^2 E^{1,0}$ then we may decompose

$$\mathcal{D}'^{0,1} = \begin{pmatrix} A & \alpha \\ D & -A^t \end{pmatrix},$$

where A, D and α are (0,1) forms with values in $EndE^{1,0}, \wedge^2 E^{*1,0}$ and $\wedge^2 E^{1,0}$, respectively.

Since $\bar{\partial} + \mathcal{D}'^{0,1}$ squares to zero, there are differential conditions on these sections. If we take, for example, the case when $\mathcal{D}'^{0,1} = D$ then these conditions are equivalent to $\overline{\partial} D = 0$; a similar statement holds for the case when $\mathcal{D}'^{0,1} = \alpha$.

To obtain $\overline{\partial}$-operators on $E_{\mathbb{C}}$, consider, as above, the isomorphism,

$$1 \oplus g : (E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}, \frac{g}{2}) \longrightarrow (E^{1,0} \oplus E^{*1,0}, <,>).$$

$\overline{\partial} + \mathcal{D}'^{0,1}$ on $E^{1,0} \oplus E^{*1,0}$ then corresponds to $\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$ on $E_{\mathbb{C}}$, where

$$\mathcal{D}_g^{0,1} = \begin{pmatrix} A & \alpha g \\ g^{-1} D & -g^{-1} A^t g \end{pmatrix}.$$

As we are interested in real connections on E, note that $\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$ is the (0,1) part of the real connection $\nabla^{Ch} + \mathcal{D}_g := \nabla^{Ch} + \mathcal{D}_g^{0,1} + \overline{\mathcal{D}_g^{0,1}}$, whose curvature is (1,1).

Corollary 3.2.1. *$\mathcal{J}^{(\nabla^{Ch} + \mathcal{D}_g, I)}$ is a complex structure on $\mathcal{C}$ and $\mathcal{T}$.*

For convenience, we summarize the $\bar{\partial}$-operators and connections that we have discussed so far in the following table.

E	$E_{\mathbb{C}}$	$E^{1,0} \oplus E^{*1,0}$
$\nabla^{Ch} + \mathcal{D}_g$	$\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$	$\overline{\partial} + \mathcal{D}'^{0,1}$

If we now take the case when $\mathcal{D}'^{0,1} = D$ then in Section 4.4 we will explore how $\mathcal{J}^{(\nabla^{Ch}+\mathcal{D}_g,I)}$ on $\mathcal{T}$ depends on the Dolbeault cohomology class of D in $H^{0,1}(\wedge^2 E^{*1,0})$, i.e. if $B \in \Gamma(\wedge^2 E^{*1,0})$ then we will determine whether $\overline{\partial} + D$ and $\overline{\partial} + D + \overline{\partial} B$ give isomorphic complex structures on $\mathcal{T}$.

Moreover we will also address a question that is a generalization of the one raised in the previous section: if g' were another metric on E that is compatible with J then given $\mathcal{D}'^{0,1} \in \Gamma(T^{*0,1} \otimes \mathfrak{so}(E^{1,0} \oplus E^{*1,0}))$, is it true that $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'}+\mathcal{D}_{g'},I)})$ is biholomorphic to $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch}+\mathcal{D}_g,I)})$?

3.3 Three Forms

An important case of the above discussion is when $E = TM$ is fibered over a Hermitian manifold (M, g, I) that is equipped with a real three form $H = \overline{H^{2,1}} + H^{2,1}$ of type (1,2) + (2,1), such that $\overline{\partial} H^{2,1} = 0$. In this case, we will let $\mathcal{D}'^{0,1} = H^{2,1}$, which is defined to be a section of $T^{*0,1} \otimes \mathfrak{so}(T^{1,0} \oplus T^{*1,0})$ by setting $H^{2,1}_v w = H^{2,1}(v, w, \cdot)$, for $v \in T^{0,1}$ and $w \in T^{1,0}$. It then follows that $\nabla^{Ch(0,1)} + g^{-1}H^{2,1}$, where here $g^{-1}H^{2,1} = g^{-1}H^{2,1}_{\frac{1}{2}(1+iI)}$, is a $\overline{\partial}$-operator on $TM_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$. As the corresponding $\mathcal{D}_g$ in the above table is $\frac{1}{2}I[g^{-1}H, I]$, we have

Proposition 3.3.1. *$\nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I]$ is a metric connection on TM with (1,1) curvature.*

Hence $\mathcal{J}^{(\nabla^{Ch}+\frac{1}{2}I[g^{-1}H,I],I)}$ is a complex structure on $\mathcal{C}$ and $\mathcal{T}$.

As we will now show, natural examples of the above three form H can be found on SKT manifolds, bihermitian manifolds and Calabi-Yau threefolds.

3.3.1 SKT Manifolds

A natural example of a real three form on any Hermitian manifold, (M, g, I), is $H = -d^c w = i(\partial - \overline{\partial})w$, where $w(\cdot, \cdot) = g(I\cdot, \cdot)$. If we take its (2,1) part, $H^{2,1}$, then it is straightforward to check that it is $\overline{\partial}$ closed if and only if $dH = 0$. Manifolds whose H satisfy this condition are known in the literature as strong Kahler with torsion (SKT) manifolds and have recently become very popular in the mathematics and physics communities [7, 6]. One of the associated $\overline{\partial}$-operators on $TM_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$ is $\nabla^{Ch(0,1)} - g^{-1}H^{2,1}$ and was actually introduced in a paper of Bismut in his study of Dirac operators [5]. The main point that we would like to stress here is that, as a corollary of the above discussion, this $\overline{\partial}$-operator leads to complex structures on the twistor spaces $\mathcal{C}$ and $\mathcal{T}$ that can be described as follows. First note that $\nabla^{Ch(0,1)} - g^{-1}H^{2,1}$ is the (0,1) part of the real connection $\nabla^{Ch} - \frac{1}{2}I[g^{-1}H, I]$ which can be shown to be equal to $\nabla^- := \nabla^{Levi} - \frac{1}{2}g^{-1}H$, where ∇^{Levi} is the Levi Civita connection. The connection ∇^- is closely related to the Bismut connection, $\nabla^+ := \nabla^{Levi} + \frac{1}{2}g^{-1}H$ (see below for a general definition as well as [5, 10]).

Corollary 3.3.2. *If (M, g, I) is SKT then $(\mathcal{C}, \mathcal{J}^{(\nabla^-,I)})$ is a complex manifold and $\mathcal{T}$ is a complex submanifold.*

The Bismut connection that was mentioned above is actually defined for any almost Hermitian manifold, (M, g, I):

Definition 3.3.3. The Bismut connection is the unique connection, ∇^+, that satisfies

1. $\nabla^+ = \nabla^{Levi} + \frac{1}{2}g^{-1}H$, where H is a 3-form
2. $\nabla^+ I = 0$.

It can be shown that H is (1,2) +(2,1) if and only if I is integrable and in this case it equals $-d^c w$ [10, 13].

3.3.2 Bihermitian Manifolds

A source of SKT manifolds is bihermitian manifolds [9, 1, 13, 16]. A bihermitian manifold is by definition a Riemannian manifold (M, g) that is equipped with two metric compatible complex structures J_+ and J_- that satisfy the following conditions

$$\nabla^+ J_+ = 0 \quad \text{and} \quad \nabla^- J_- = 0,$$

where $\nabla^\pm = \nabla^{Levi} \pm \frac{1}{2}g^{-1}H$, for a closed three form H.

It then follows from Definition 3.3.3 that ∇^+ and ∇^- are the respective Bismut connections for J_+ and J_-. Thus an equivalent way to express the above bihermitian conditions is

$$H = -d^c_+ w_+ = d^c_- w_- \quad \text{and} \quad dH = 0.$$

Since dH is assumed to be zero, (g, J_+) and (g, J_-) are two SKT structures for M and hence by Corollary 3.3.2 the associated twistor space $\mathcal{T}$ admits the following two complex structures that depend on the three form H:

Corollary 3.3.4. *$\mathcal{J}^{(\nabla^-, J_+)}$ and $\mathcal{J}^{(\nabla^+, J_-)}$ are two complex structures for $\mathcal{C}$ and $\mathcal{T}$.*

We will derive more results about bihermitian manifolds in Sections 3.4.2 and 8. As for some examples, Kahler and hyperkahler manifolds are bihermitian. Other examples of bihermitian structures have been found, in particular, on compact even dimensional Lie groups, Del Pezzo surfaces and more generally on Fano manifolds [13, 17, 15].

3.3.3 Calabi-Yau Threefolds

Another class of Hermitian manifolds that admit $\overline{\partial}$ closed (2,1) forms, which by Proposition 3.3.1 can be used to define complex structures on $\mathcal{C}$ and $\mathcal{T}$, are Calabi-Yau threefolds. Indeed, $H^{2,1}_{Dolbeault}$ parametrizes the deformations of the complex structure on the threefold. It is interesting to note that at the same time there is a well defined map from $H^{2,1}_{Dolbeault}$ to the space of complex structures on the twistor space (modulo biholomorphisms) as will be described for a more general setup in Section 4.4.1. We are currently investigating the connection between deformations of complex structures on Calabi-Yau threefolds (as well as on general complex manifolds) and the complex geometries of twistor space.

3.4 Holomorphic Sections of Twistor space

In the previous sections we gave a number of examples of the general setup of a bundle $E \longrightarrow (M, I)$ that is equipped with a connection, ∇, that has (1,1) curvature. Given the associated complex manifold $(\mathcal{C}, \mathcal{J}^{(\nabla,I)})$, which holomorphically fibers over M, it is natural to consider its holomorphic sections. While we will use these sections in Chapter 5 to produce holomorphic subvarieties in M and in particular stratifications of $\mathcal{C}$, as was mentioned in the introduction, our focus here is to use them to construct more connections with (1,1) curvature– and thus more complex structures on twistor spaces. To begin, let us characterize the holomorphic sections of $\mathcal{C}$.

3.4.1 Holomorphic Sections and (1,1) Curvature

As above, let $E \longrightarrow (M, I)$ be equipped with a connection ∇ that has (1,1) curvature.

Proposition 3.4.1. *$J : M \longrightarrow (\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ is a holomorphic section if and only if $J\nabla_v J = \nabla_{Iv} J$, for all $v \in TM$.*

Proof. Letting $P^\nabla : T\mathcal{C} \longrightarrow V\mathcal{C}$, as in Section 2.2, be the projection operator that is based on the splitting of $T\mathcal{C}$ into $V\mathcal{C} \oplus H^\nabla\mathcal{C}$, let us consider the holomorphicity condition of J: $\mathcal{J}^{(\nabla,I)} J_* = J_* I$. If $v \in T_x M$, we then have:

1) $\mathcal{J}^{(\nabla,I)} J_* v = \mathcal{J}^{(\nabla,I)}(P^\nabla(J_* v) + v^\nabla)$, where $v^\nabla \in H^\nabla_{J(x)}\mathcal{C}$ is the horizontal lift of $v \in T_x M$. This then equals $JP^\nabla(J_* v) + (Iv)^\nabla$.

2) $J_*(Iv) = P^\nabla(J_* Iv) + (Iv)^\nabla$.

Hence J is holomorphic if and only if

$$JP^\nabla(J_* v) = P^\nabla(J_* Iv). \tag{3.4.1}$$

Using the formula $P^\nabla = \pi^*\nabla\phi$, as given in Proposition 2.2.6, it is straightforward to show that $P^\nabla(J_* v) = \nabla_v J$. Plugging this into Equation 3.4.1 proves Proposition 3.4.1. $\square$

If we consider the $\overline{\partial}-$operator $\nabla^{0,1}$ on $E_\mathbb{C}$ then the above holomorphicity condition is equivalent to $(\nabla^{0,1} J)E^{0,1}_J = 0$. This in turn is equivalent to $J\nabla^{0,1} e = -i\nabla^{0,1} e$, for all $e \in \Gamma(E^{0,1}_J)$. We thus have:

Proposition 3.4.2. *$J : M \longrightarrow (\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ is a holomorphic section if and only if $E^{0,1}_J$ is a holomorphic subbundle of $(E_\mathbb{C}, \nabla^{0,1})$.*

Having described the holomorphic sections of $\mathcal{C}$, let us now use them to build other connections on E with (1,1) curvature.

Proposition 3.4.3. *Let $J : M \longrightarrow (\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ be a holomorphic section. Then $\nabla + \nabla J(a + bJ)$, where $a, b \in \mathbb{R}$, is a connection with (1,1) curvature.*

Proof. We will show that $\nabla^{0,1} + \nabla^{0,1} J(a + bJ)$ is a $\overline{\partial}$-operator on $E_\mathbb{C}$. By using Proposition 3.4.2, it is straightforward to show that this (0,1) connection is of the form $\nabla^{0,1} + A$, where $A \in \Gamma(T^{*0,1} \otimes End E_\mathbb{C})$ satisfies $\nabla^{0,1} A = 0$, $AE^{1,0}_J \subset E^{0,1}_J$ and $AE^{0,1}_J = 0$. It then follows that $\nabla^{0,1} + A$ squares to zero. $\square$

These new connections then lead to other complex structures on the twistor space:

Corollary 3.4.4. *If J is a holomorphic section of $(\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ then $\mathcal{J}^{(\nabla+\nabla J(a+bJ),I)}$ is a complex structure on $\mathcal{C}$. In addition, if (E,g,J) is a Hermitian bundle and ∇ is a metric connection then this complex structure restricts to $\mathcal{T}$.*

Among the above connections, there is a particular one that we wish to focus on:

Proposition 3.4.5. $\nabla' := \nabla + \frac{1}{2}(\nabla J)J$ *is a connection with (1,1) curvature that satisfies* $\nabla' J = 0$.

Thus J, originally chosen to be a holomorphic section of $(\mathcal{C}, \mathcal{J}^{(\nabla,I)})$, is also, by definition, a parallel section of $(\mathcal{C}, \mathcal{J}^{(\nabla',I)})$. Moreover, if we consider the holomorphic bundle $(E_{\mathbb{C}}, \nabla'^{0,1})$, then the fact that $\nabla' J = 0$ implies that $E_J^{1,0}$ and $E_J^{0,1}$ are holomorphic subbundles. We will explore the consequences of this in Chapter 5.

3.4.2 Examples of Holomorphic Sections

In searching for examples of holomorphic sections of $(\mathcal{C}, \mathcal{J}^{(\nabla,I)})$, it should first be noted that since $\pi : \mathcal{C} \longrightarrow (M, I)$ is a holomorphic submersion, there are plenty of local ones. As for global holomorphic sections, in the case when $\mathcal{C} = \mathcal{C}(TM) \longrightarrow (M, I)$, there is a natural candidate– namely I itself. While we will describe other examples of holomorphic sections later on, our first goal is to describe a certain class of twistor spaces, which include those associated to SKT and bihermitian manifolds, where I is holomorphic. Yet in fact we will find it natural to begin with a more general situation where I is not necessarily integrable and explore the condition that guarantees that I is pseudoholomorphic. We will use this, in particular, to show that the (pseudo)holomorphicity condition on sections of twistor space is a generalization of the integrability condition.

To begin, let (M, g, I) be an almost Hermitian manifold that is equipped with a real three form H. Using the natural Chern connection, ∇^{Ch}, on TM (see for example [10]) we will consider $\nabla = \nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I]$ and the corresponding twistor space $(\mathcal{T}, \mathcal{J}^{(\nabla,I)})$, which is only an almost complex manifold. We will now explore the conditions on H so that I and $-I$ are pseudoholomorphic sections:

Proposition 3.4.6.

$$1) I : M \longrightarrow \mathcal{T} \text{ is pseudoholomorphic if and only if } H \text{ is } (1,2) + (2,1).$$
$$2) - I : M \longrightarrow \mathcal{T} \text{ is pseudoholomorphic if and only if } H \text{ is } (3,0) + (0,3).$$

Proof. As Proposition 3.4.1 is true regardless of whether I is integrable, it follows that I is pseudoholomorphic if and only if $I[g^{-1}H, I] = [g^{-1}H_I, I]$, which is equivalent to H being (1,2) + (2,1). The proof of 2) is similar. □

If we now choose H to be the three form that is contained in the Bismut connection $\nabla^+ = \nabla^{Levi} + \frac{1}{2}g^{-1}H$, which was defined in Definition 3.3.3, then the pseudoholomorphicity of I is equivalent to its integrability:

Proposition 3.4.7. *$I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla,I)})$ is pseudoholomorphic if and only if I is integrable.*

Proof. This follows from Proposition 3.4.6 and the fact that for the three form H in the Bismut connection, I is integrable if and only if H is (1,2)+(2,1). □

It is in this sense, as we remarked above, that the pseudoholomorphic condition on sections of twistor space generalizes the integrability condition on almost complex structures.

Using Proposition 3.4.6, we can now describe a class of holomorphic twistor spaces that always admit I as a holomorphic section. Indeed, let (M, g, I) be a Hermitian manifold that is equipped with a $\overline{\partial}$ closed (2,1) form, $H^{2,1}$, and let $H = \overline{H^{2,1}} + H^{2,1}$. By Proposition 3.3.1, the connection $\nabla = \nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I]$ on TM has (1,1) curvature and thus $(\mathcal{T}, \mathcal{J}^{(\nabla,I)})$ is a complex manifold. Since H is (1,2)+(2,1), we have:

Corollary 3.4.8. *$I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla,I)})$ is holomorphic.*

As was described in the previous section, whenever we have a holomorphic section of a twistor space we automatically obtain other connections with (1,1) curvature. For the above case, one such connection is $\nabla' = \nabla + \frac{1}{2}(\nabla I)I$, which satisfies $\nabla' I = 0$. This particular connection is in fact a familiar one:

Proposition 3.4.9. $\nabla' = \nabla^{Ch}$

Proof. Since $\nabla I = [g^{-1}H, I]$, $\nabla' = \nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I] + \frac{1}{2}([g^{-1}H, I])I = \nabla^{Ch}$. □

To take an example of the above setup, let (M, g, I) be an SKT manifold, as defined in Section 3.3.1, so that $H = -d^c w = i(\partial - \overline{\partial})w$ satisfies $dH = 0$. Since this latter condition is equivalent to $\overline{\partial}H^{2,1} = 0$, the connection $\nabla^{Ch} - \frac{1}{2}I[g^{-1}H, I]$, which equals $\nabla^- = \nabla^{Levi} - \frac{1}{2}g^{-1}H$, has (1,1) curvature. Taking Proposition 3.4.6 into account, we have:

Proposition 3.4.10. *Let (M, g, I) be an SKT manifold.*

> 1)*$I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^-,I)})$ is holomorphic.*
>
> 2) *$-I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^-,I)})$ is holomorphic if and only if (M, g, I) is Kahler.*

Proof. To prove 2), note that by Proposition 3.4.6, $-I$ is holomorphic if and only if H is (3,0)+(0,3). As H is already (1,2)+(2,1), this is true if and only if $H = 0$, or equivalently, (M, g, I) is Kahler. □

Since a bihermitian manifold admits the two SKT structures (g, J_+) and (g, J_-) as explained in Section 3.3.2, we have

Proposition 3.4.11. *Let (M, g, J_+, J_-) be a bihermitian manifold. The following are holomorphic sections:*

$$1) J_+ : (M, J_+) \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^-,J_+)})$$
$$2) J_- : (M, J_-) \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^+,J_-)}).$$

Since $\nabla^{\pm} J_{\pm} = 0$, J_+ and J_- are also, by definition, parallel sections of $(\mathcal{T}, \mathcal{J}^{(\nabla^+, J_-)})$ and $(\mathcal{T}, \mathcal{J}^{(\nabla^-, J_+)})$ respectively. In Chapter 8 we will use these parallel and holomorphic sections of $\mathcal{T}$ to produce holomorphic subvarieties in (M, J_+) and (M, J_-) that are new to the literature.

Having given some examples where the section $I : (M, g, I) \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla, I)})$ is holomorphic, we will now consider other examples of holomorphic sections of twistor space. The following demonstrates that there are twistor spaces that are fibered over other twistor spaces that naturally admit holomorphic sections.

Example 3.4.12. Let $E \longrightarrow (M, I)$ be equipped with a connection ∇ that has (1,1) curvature. Denoting the projection map from $\mathcal{C}(E)$ to M by π, in this example we will be focusing on the complex manifold $(\mathcal{C}(E), \mathcal{I})$, where $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$, along with its pullback bundle $\pi^* E$. Since π is holomorphic, the connection $\pi^*\nabla$ on $\pi^* E$ has (1,1) curvature so that the total space of $(\mathcal{C}(\pi^* E), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})}) \longrightarrow \mathcal{C}(E)$ is a complex manifold.

As we discussed before, ϕ is a natural section of $\mathcal{C}(\pi^* E)$, defined by $\phi|_J = J$, and the claim is that it is holomorphic:

Proposition 3.4.13. *ϕ is a holomorphic section of $(\mathcal{C}(\pi^* E), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})}) \longrightarrow \mathcal{C}(E)$.*

Proof. It follows from Proposition 3.4.1 that ϕ is holomorphic if and only if $\phi(\pi^*\nabla)_X \phi = \pi^*\nabla_{\mathcal{I}X}\phi$, for all $X \in T\mathcal{C}(E)$. By Proposition 2.2.6, this is equivalent to $\phi P^{\nabla}(X) = P^{\nabla}(\mathcal{I}X)$, where $P^{\nabla} : T\mathcal{C} \longrightarrow V\mathcal{C}$ is the vertical projection operator that is induced by ∇. This last expression follows directly from the definition of $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$. □

Since ϕ is holomorphic, by Proposition 3.4.5, the connection $\pi^*\nabla' = \pi^*\nabla + \frac{1}{2}(\pi^*\nabla\phi)\phi$ on $\pi^* E \longrightarrow (\mathcal{C}(E), \mathcal{I})$ also has (1,1) curvature and satisfies $\pi^*\nabla'\phi = 0$. We thus have:

Corollary 3.4.14. *$\mathcal{J}^{(\pi^*\nabla', \mathcal{I})}$ is another complex structure on $\mathcal{C}(\pi^* E)$.*

The fact that $R^{\pi^*\nabla'}$ is (1,1) implies that $(\pi^* E_{\mathbb{C}}, \pi^*\nabla'^{0,1})$ is a holomorphic bundle over $(\mathcal{C}(E), \mathcal{I})$ and since $\pi^*\nabla'\phi = 0$, $\pi^* E^{1,0}_{\phi}$ and $\pi^* E^{0,1}_{\phi}$ are holomorphic subbundles. We will use this result in Chapter 6 to produce complex submanifolds that form the strata of several stratifications of $(\mathcal{C}(E), \mathcal{I})$.

If we now restrict $(\pi^* E, \pi^*\nabla)$ to a particular fiber of $\pi : \mathcal{C}(E) \longrightarrow M$ then we obtain the setup of Example 2.4.4: an even dimensional real vector space, V, and the trivial bundle $E' = \mathcal{C}(V) \times V \longrightarrow \mathcal{C}(V)$ that is equipped with its trivial connection d. It follows from Proposition 3.4.13 that $\phi \in \Gamma(End E')$, defined by $\phi|_J = J$, is a holomorphic section of $(\mathcal{C}(E'), \mathcal{J}^{(d, I_{\mathcal{C}})}) \longrightarrow \mathcal{C}(V)$, where $I_{\mathcal{C}}$ is the standard complex structure on $\mathcal{C}(V)$. Another way to show the holomorphicity of ϕ is to first note that the map

$$\begin{aligned} (\mathcal{C}(E'), \mathcal{J}^{(d, I_{\mathcal{C}})}) &\longrightarrow \mathcal{C}(V) \times \mathcal{C}(V) \\ J &\longrightarrow (K, J), \end{aligned}$$

where $J \in \mathcal{C}(E')|_K$, is a biholomorphism. The section ϕ of $\mathcal{C}(E')$ then corresponds to the diagonal map from $\mathcal{C}(V)$ into $\mathcal{C}(V) \times \mathcal{C}(V)$, which is holomorphic.

We can now rederive the results of Example 2.4.4. (Note in that example E' was denoted by E.)

Corollary 3.4.15.

$$1)\nabla' = d + \frac{1}{2}(d\phi)\phi \textit{ has (1,1) curvature.}$$
$$2)(\mathcal{C}(E'), \mathcal{J}^{(\nabla', \mathcal{I}_{\mathcal{C}})}) \textit{ is a complex manifold.}$$

As an application of the above discussion, let $TM \longrightarrow (M, I)$ be equipped with a connection ∇ of (1,1) curvature and let $\pi : \mathcal{C} = \mathcal{C}(TM) \longrightarrow M$ be the projection map. The goal is to prove a result that was stated in Section 2.3: that the almost complex structure $\mathcal{J}^{\nabla}_{taut}$ on $\mathcal{C}$, as defined in Definition 2.3.4, is a holomorphic section of $(\mathcal{C}(T\mathcal{C}), \mathcal{J}) \longrightarrow (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, for some appropriately chosen $\mathcal{J}$. To define $\mathcal{J}$, use ∇ to split $T\mathcal{C} = V\mathcal{C} \oplus H^{\nabla}\mathcal{C}$ and identify $V\mathcal{C}$ with $[End(\pi^*TM), \phi]$ and $H^{\nabla}\mathcal{C}$ with π^*TM. It then follows from the above discussion that $\tilde{\nabla} = \pi^*\nabla + \frac{1}{2}[(\pi^*\nabla\phi)\phi, \cdot] \oplus \pi^*\nabla$ is a connection on $T\mathcal{C}$ that has (1,1) curvature with respect to $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$. Hence $\mathcal{J} = \mathcal{J}^{(\tilde{\nabla}, \mathcal{I})}$ is a complex structure on $\mathcal{C}(T\mathcal{C})$. Using Proposition 3.4.13, it is then straightforward to show:

Proposition 3.4.16. *$\mathcal{J}^{\nabla}_{taut}$ is a holomorphic section of $(\mathcal{C}(T\mathcal{C}), \mathcal{J}^{(\tilde{\nabla}, \mathcal{I})}) \longrightarrow (\mathcal{C}, \mathcal{I})$.*

As another application, consider the setup in Example 2.4.7: a Hermitian anti-selfdual four manifold (M, g, I) whose orientation is determined by I, and $\pi : \mathcal{T}^+ \longrightarrow M$, the subbundle of $\mathcal{T}(TM)$ whose elements induce the same orientation as the given one. If we let ∇ be the Levi Civita connection then as discussed in that example, $\pi^*\nabla$ is a connection on π^*TM that has (1,1) curvature with respect to both of the integrable complex structures $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}^{\nabla}_{taut}$ on $\mathcal{T}^+$. Based on the discussion surrounding Proposition 3.4.13, it is straightforward to show that $\phi : (\mathcal{T}^+, \mathcal{I}) \longrightarrow (\mathcal{C}(\pi^*TM), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})})$ is not only holomorphic for $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$ but for $\mathcal{I} = \mathcal{J}^{\nabla}_{taut}$ as well. Hence $\pi^*\nabla' = \pi^*\nabla + \frac{1}{2}(\pi^*\nabla\phi)\phi$ is a connection on π^*TM that has (1,1) curvature with respect to both $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}^{\nabla}_{taut}$, as was claimed in Example 2.4.7. As $\pi^*\nabla'\phi = 0$, we obtain several holomorphic structures on the bundles $\pi^*TM^{1,0}_{\phi}$ and $\pi^*TM^{0,1}_{\phi}$ that are fibered over $(\mathcal{T}^+, \mathcal{I})$. □

More examples and applications of holomorphic sections of twistor spaces will be given in the upcoming chapters.

Chapter 4

Twistors and Grassmannians

In the previous chapter, we have not only given examples of a bundle $E \longrightarrow (M, I)$ with a connection ∇ that has (1,1) curvature but have also raised various questions about the complex manifold structure of $(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, especially in Section 3.2. In this chapter we will address these questions by holomorphically embedding $\mathcal{C}$ into a more familiar complex manifold– a certain Grassmannian bundle. Indeed, as we noted previously, the condition that R^{∇} is (1,1) is equivalent to $\nabla^{0,1}$ being a $\overline{\partial}$-operator on $E_{\mathbb{C}}$, and if we let $dim_{\mathbb{R}} E = 2n$ then the Grassmannian bundle that we will take will be the holomorphic bundle $Gr_n(E_{\mathbb{C}})$.

To define the embedding, we will first show how to holomorphically embed the fibers of $\mathcal{C}$ into those of $Gr_n(E_{\mathbb{C}})$.

4.1 Embedding the Fibers

Let V be a $2n$ dimensional real vector space and let $Gr_n(V_{\mathbb{C}})$ be the Grassmannians of complex n planes. The map that we will consider is

$$\begin{aligned} \psi : \mathcal{C}(V) &\longrightarrow Gr_n(V_{\mathbb{C}}) \\ J &\longrightarrow V_J^{0,1}; \end{aligned}$$

it has the following properties:

Proposition 4.1.1.

1. *The map $\psi : \mathcal{C}(V) \longrightarrow Gr_n(V_{\mathbb{C}})$ is a holomorphic embedding.*
2. *The image of this embedding is $\{P \in Gr_n(V_{\mathbb{C}}) | P \oplus \overline{P} = V_{\mathbb{C}}\}$, which is an open submanifold of the Grassmannians.*

Proof. Consider $\psi_* : T_J\mathcal{C}(V) \longrightarrow T_{V_J^{0,1}} Gr_n(V_{\mathbb{C}})$ and choose the holomorphic chart

$$\begin{aligned} End(V_J^{0,1}, V_J^{1,0}) &\longrightarrow Gr_n(V_{\mathbb{C}}) \\ B &\longrightarrow Graph(B), \end{aligned}$$

where $Graph(B) = \{v^{0,1} + Bv^{0,1} | v^{0,1} \in V_J^{0,1}\}$. If we let A be a general element in $T_J\mathcal{C}(V) \cong \{D \in EndV | \{D, J\} = 0\}$ then we need to show that $\psi_*(JA) = \mathcal{I}\psi_*(A)$, where $\mathcal{I}$ is the complex structure on the Grassmannians.

First consider,

$$\begin{aligned}\psi_*(JA) &= \frac{d}{dt}|_{t=0}\psi(exp(\frac{-tA}{2})Jexp(\frac{tA}{2})) \\ &= \frac{d}{dt}|_{t=0}exp(\frac{-tA}{2})(V_J^{0,1}).\end{aligned}$$

Using the above chart, $\psi_*(JA)$ then corresponds to $-\frac{A}{2}$, as an element of $End(V_J^{0,1}, V_J^{1,0})$.

Similarly we have $\psi_*(A) = \frac{d}{dt}|_{t=0}exp(-\frac{tAJ}{2})(V_J^{0,1})$, so that under the above chart, $\mathcal{I}\psi_*(A)$ corresponds to $-\frac{iAJ}{2}$, which as an element of $End(V_J^{0,1}, V_J^{1,0})$ equals $-\frac{A}{2}$.

The proof of the other parts of the proposition are straightforward. □

If we now choose a positive definite metric, g, on V then by restriction, the above map, ψ, gives a holomorphic embedding of $\mathcal{T}(V)$ into $Gr_n(V_{\mathbb{C}})$. Since the metric is positive definite, the image of this map is precisely $MI(V_{\mathbb{C}}) = \{P \in Gr_n(V_{\mathbb{C}}) | g(v, w) = 0, \forall v, w \in P\}$, the space of maximal isotropics of $V_{\mathbb{C}}$ defined by using the $\mathbb{C}$-bilinearly extended metric. For convenience we state this as a proposition.

Proposition 4.1.2.

$$\begin{aligned}\mathcal{T}(V) &\longrightarrow Gr_n(V_{\mathbb{C}}) \\ J &\longrightarrow V_J^{0,1}\end{aligned}$$

is a holomorphic embedding with image $MI(V_{\mathbb{C}})$.

4.2 The Holomorphic Embedding

Let us now consider a $2n$ dimensional real vector bundle $E \longrightarrow (M, I)$ that is fibered over a complex manifold. As discussed above, a connection ∇ on E with (1,1) curvature gives rise to two complex analytic manifolds: the twistor space $(\mathcal{C}, \mathcal{J}^{(\nabla,I)})$ and the holomorphic fiber bundle $\pi_{Gr} : Gr_n(E_{\mathbb{C}}) \longrightarrow M$. To holomorphically embed $\mathcal{C}$ into $Gr_n(E_{\mathbb{C}})$, we will generalize the map ψ that was defined in the previous section:

Theorem 4.2.1. *The map*

$$\begin{aligned}\psi : (\mathcal{C}, \mathcal{J}^{(\nabla,I)}) &\longrightarrow Gr_n(E_{\mathbb{C}}) \\ J &\longrightarrow E_J^{0,1}\end{aligned}$$

is a holomorphic embedding.

In the case when E is equipped with a metric g and ∇ is a metric connection, we will define $MI(E_{\mathbb{C}})$ to be the space of maximal isotropics in $Gr_n(E_{\mathbb{C}})$; we then have:

Proposition 4.2.2.

$$\begin{aligned}(\mathcal{T}, \mathcal{J}^{(\nabla,I)}) &\longrightarrow Gr_n(E_{\mathbb{C}})\\ J &\longrightarrow E_J^{0,1}\end{aligned}$$

is a holomorphic embedding with image $MI(E_{\mathbb{C}})$.

To prove Theorem 4.2.1, we will need to describe the complex structure on the Grassmannians similarly to how we defined $\mathcal{J}^{(\nabla,I)}$ on $\mathcal{C}$. The first step will be to define the horizontal distribution $H^{\nabla}Gr_n$ on $Gr_n(E_{\mathbb{C}})$. But before giving the definition, let us first recall that if $P \in Gr_n(E_{\mathbb{C}})$ and $\gamma : \mathbb{R} \longrightarrow M$ satisfies $\gamma(0) = \pi_{Gr}(P)$ then we can use ∇, considered as a complex connection on $E_{\mathbb{C}}$, to parallel transport P along γ as follows. If we set $P =< e_1, ..., e_n >_{\mathbb{C}}$, so that $\{e_i\}$ is a basis for P, then define $P(t) =< e_1(t), ..., e_n(t) >_{\mathbb{C}}$, where $\gamma^*\nabla e_i(t) = 0$ and $e_i(0) = e_i$. Since ∇ is a complex connection on $E_{\mathbb{C}}$, $P(t)$ does not depend on the basis $\{e_i\}$ for P that was chosen.

With this, let us define the desired horizontal distribution on $Gr_n(E_{\mathbb{C}})$.

Definition 4.2.3. Let $H_P^{\nabla}Gr_n = \{\frac{dP(t)}{dt}|_{t=0} | P(t)$ is the parallel translate of P along $\gamma, \gamma(0) = \pi_{Gr}(P)\}$.

Along with $H^{\nabla}Gr_n$, there is also the natural vertical distribution VGr_n; as it is defined by the fibers of $Gr_n(E_{\mathbb{C}})$, it is a complex vector bundle and satisfies $\pi_{Gr*}(V_PGr_n) = 0$, for all $P \in Gr_n(E_{\mathbb{C}})$. It is straightforward to prove that these two distributions are complements to each other:

Lemma 4.2.4. $T_PGr_n = V_PGr_n \oplus H_P^{\nabla}Gr_n$.

We may now use the above lemma to define an almost complex structure on $Gr_n(E_{\mathbb{C}})$, which we will show in Proposition 4.2.6 to be the complex structure that is induced by $\nabla^{0,1}$ and which we will use to prove Theorem 4.2.1. As the definition of this almost complex structure is similar to that of $\mathcal{J}^{(\nabla,I)}$ on $\mathcal{C}$, we will denote it by the same symbol:

Definition 4.2.5. Let $\mathcal{J}^{(\nabla,I)}$ on $Gr_n(E_{\mathbb{C}})$ be defined as follows. First split

$$TGr_n = VGr_n \oplus H^{\nabla}Gr_n$$

and then let

$$\mathcal{J}^{(\nabla,I)} = \mathcal{J}^V \oplus \pi_{Gr}^* I,$$

where $\mathcal{J}^V$ is the standard fiberwise complex structure on VGr_n and where we have used the natural identification of $H^{\nabla}Gr_n$ with π_{Gr}^*TM.

If we consider the complex manifold structure of $Gr_n(E_{\mathbb{C}})$ that is induced by the $\overline{\partial}$-operator $\nabla^{0,1}$ on $E_{\mathbb{C}}$, we then have:

Proposition 4.2.6. *The complex structure on $Gr_n(E_{\mathbb{C}})$ is $\mathcal{J}^{(\nabla,I)}$.*

We will prove the above proposition for a more general setup in the next section; here we will use it to prove Theorem 4.2.1 by showing that the map $\psi : (\mathcal{C}, \mathcal{J}^{(\nabla,I)}) \longrightarrow (Gr_n(E_{\mathbb{C}}), \mathcal{J}^{(\nabla,I)})$, which is given by $\psi(J) = E_J^{0,1}$, is holomorphic. Recalling the splitting of $T\mathcal{C} = V\mathcal{C} \oplus H^{\nabla}\mathcal{C}$, as given in Lemma 2.2.4, let us first consider the following:

Lemma 4.2.7. *The map ψ_* preserves horizontals: $\psi_* : H^{\nabla}_J\mathcal{C} \longrightarrow H^{\nabla}_{E^{0,1}_J}Gr_n$. In fact, $\psi_*(v^{\nabla}) = v^{(\nabla,Gr)}$, where v^{∇} and $v^{(\nabla,Gr)}$ are the appropriate horizontal lifts of $v \in T_xM$.*

Proof. Let $\gamma(t)$ be a curve in M such that $\gamma(0) = x$ and $\gamma'(0) = v$. Also let $J(t)$ be the parallel translate of $J \in \mathcal{C}(E_x)$ along γ (by using ∇), so that

$$\psi_*(v^{\nabla}) = \frac{d}{dt}|_{t=0}\psi(J(t)).$$

The claim then is that $\psi(J(t))$, which is by definition $E^{0,1}_{J(t)}$, equals $E^{0,1}_J(t)$, the parallel translate of $E^{0,1}_J$ along γ. To show this just note that if $e(t)$ is the parallel translate of $e \in E^{0,1}_J$ then $J(t)e(t)$ is also parallel and since $Je = -ie$, it follows that $J(t)e(t) = -ie(t)$ for all relevant $t \in \mathbb{R}$. Hence $\frac{d}{dt}|_{t=0}\psi(J(t)) = \frac{d}{dt}|_{t=0}E^{0,1}_J(t) = v^{(\nabla,Gr)}$. □

Assuming Proposition 4.2.6, we can now prove that ψ is holomorphic:

Proof of Theorem 4.2.1. Consider $\psi_* : T_J\mathcal{C} \longrightarrow T_{E^{0,1}_J}Gr_n$. By Proposition 4.2.6, we need to show that $\psi_*\mathcal{J}^{(\nabla,I)} = \mathcal{J}^{(\nabla,I)}\psi_*$.

A) If $A \in V_J\mathcal{C}$, the vertical tangent space to J, then it follows from Proposition 4.1.1 that

$$\psi_*(JA) = \mathcal{J}^{(\nabla,I)}\psi_*(A),$$

so that ψ_* is holomorphic in the vertical directions.

B) As for the horizontal directions, let $v^{\nabla} \in H^{\nabla}_J\mathcal{C}$ be the horizontal lift of $v \in T_xM$. Then $\psi_*(\mathcal{J}^{(\nabla,I)}v^{\nabla}) = \psi_*((Iv)^{\nabla})$, which by Lemma 4.2.7 equals $(Iv)^{(\nabla,Gr)}$. This in turn equals $\mathcal{J}^{(\nabla,I)}v^{(\nabla,Gr)} = \mathcal{J}^{(\nabla,I)}\psi_*(v^{\nabla})$. □

4.3 Proof of Proposition 4.2.6

In this section, we will prove a slightly more general version of Proposition 4.2.6; this will then complete the proof of Theorem 4.2.1. To begin, we will find it useful to describe the complex structures on holomorphic vector bundles:

Let $\pi_F : F \longrightarrow (M, I)$ be a complex vector bundle that is equipped with a $\overline{\partial}$-operator, $\overline{\partial}$, and let ∇ be a complex connection on F such that $\nabla^{0,1} = \overline{\partial}$. Below we will let $\mathcal{J}^{(\nabla,I)}$ be the almost complex structure on either F or $Gr_k(F)$ that is defined in a by now familiar way: use ∇ to split the appropriate tangent bundle into vertical and horizontal distributions, and define $\mathcal{J}^{(\nabla,I)}$ to be the direct sum of the given fiberwise complex structure on the verticals and the lift of I on the horizontals.

Proposition 4.3.1. *Let ∇ be a complex connection on F such that $\nabla^{0,1} = \overline{\partial}$. Then the associated complex structure on F is $\mathcal{J}^{(\nabla,I)}$.*

Proof. Let $\{f_i\}$ $(1 \le i \le dim_{\mathbb{C}}F)$ be a holomorphic frame for F over $U \subset M$ and let W be the complex vector space that is generated by $\{w_i\}$ over $\mathbb{C}$. To prove the proposition, we need to show that the map

$$\begin{aligned}\sigma : (F|_U, \mathcal{J}^{(\nabla,I)}) &\longrightarrow U \times W \\ a_i f_i|_x &\longrightarrow (x, a_i w_i)\end{aligned}$$

is holomorphic. For this, consider $\sigma_* : T_f F \longrightarrow T_{\sigma(f)}(U \times W)$, where $\pi_F(f) = x$.

1) Since $\sigma|_x$ is a complex linear isomorphism from $F|_x$ to W, σ is holomorphic in the vertical directions, i.e., $\sigma_*(if') = i\sigma_*(f')$, where $f' \in V_f F = F|_x$.

2) As for the horizontal directions, we need to show that $\sigma_*(\mathcal{J}^{(\nabla,I)} v^\nabla) = \mathcal{I}\sigma_*(v^\nabla)$, where v^∇ is the horizontal lift of $v \in T_x M$ to $H_f^\nabla F \subset T_f F$ and $\mathcal{I}$ is the complex structure on $U \times W$. Let us first consider,

$$\begin{aligned}\sigma_*(\mathcal{J}^{(\nabla,I)} v^\nabla) &= \sigma_*((Iv)^\nabla) \\ &= \frac{d}{dt}|_{t=0}\sigma(f(t)),\end{aligned}$$

where $f(t)$ is the parallel translate of f along the curve $\gamma : \mathbb{R} \longrightarrow M$ that satisfies $\gamma(0) = x$ and $\gamma'(0) = Iv$. If we let $f(t) = a_j(t) f_j|_{\gamma(t)}$ then the above equals

$$(Iv, \frac{da_j(t)}{dt}|_{t=0} w_j).$$

Similarly, $\sigma_*(v^\nabla) = (v, \frac{d\tilde{a}_j(t)}{dt}|_{t=0} w_j)$, where $\tilde{f}(t) = \tilde{a}_j(t) f_j|_{\tilde{\gamma}(t)}$ is the parallel translate of f along the curve $\tilde{\gamma} : \mathbb{R} \longrightarrow M$ that satisfies $\tilde{\gamma}(0) = x$, $\tilde{\gamma}'(0) = v$. Now since $\mathcal{I}\sigma_*(v^\nabla) = (Iv, i\frac{d\tilde{a}_j(t)}{dt}|_{t=0} w_j)$, σ is holomorphic if and only if $\frac{da_j(t)}{dt}|_{t=0} = i\frac{d\tilde{a}_j(t)}{dt}|_{t=0}$.

To show this equality, note that the condition $\tilde{\gamma}^* \nabla \tilde{f(t)} = 0$ together with $a_j := \tilde{a}_j(0) = a_j(0)$ imply that $i\frac{d\tilde{a}_j(t)}{dt}|_{t=0} f_j = -i a_j \nabla_v f_j$. This then equals $-a_j \nabla_{Iv} f_j$ because $\nabla^{0,1} f_j = 0$, which in turn equals $\frac{da_j(t)}{dt}|_{t=0} f_j$ since $\gamma^* \nabla f(t) = 0$. Hence σ is holomorphic. □

As for the Grassmannians, we have:

Proposition 4.3.2. *The complex structure on $Gr_k(F)$ that is induced from $(F, \overline{\partial})$ is $\mathcal{J}^{(\nabla,I)}$.*

The proof of the above proposition and hence of Proposition 4.2.6 is just a straightforward generalization of the previous proof. This then completes the proof of Theorem 4.2.1 as well.

4.4 Corollaries of the Embedding

We will now demonstrate some of the corollaries of the holomorphic embedding $\psi : (\mathcal{C}, \mathcal{J}^{(\nabla,I)}) \longrightarrow Gr_n(E_{\mathbb{C}})$, as given in Theorem 4.2.1. In particular, we will address certain issues regarding the holomorphic structure of twistor spaces that were raised in Section 3.2.

Let E and E' be two real vector bundles of even dimension that are fibered over (M, I) and that are respectively equipped with connections ∇ and ∇' of (1,1) curvature.

Proposition 4.4.1. *Let $A : E \longrightarrow E'$ be a bundle map such that its $\mathbb{C}$-extension, $A : (E_{\mathbb{C}}, \nabla^{0,1}) \longrightarrow (E'_{\mathbb{C}}, \nabla'^{0,1})$ is an isomorphism of holomorphic vector bundles. Then this map induces a fiber preserving biholomorphism between $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$ and $(\mathcal{C}(E'), \mathcal{J}^{(\nabla',I)})$.*

Proof. The isomorphism $A : (E_{\mathbb{C}}, \nabla^{0,1}) \longrightarrow (E'_{\mathbb{C}}, \nabla'^{0,1})$ induces the biholomorphism $\tilde{A} : Gr_n(E_{\mathbb{C}}) \longrightarrow Gr_n(E'_{\mathbb{C}})$ that is defined by $\tilde{A}(< e_1, ..., e_n >_{\mathbb{C}}) =< Ae_1, ..., Ae_n >_{\mathbb{C}}$. Since A is a real map, $\tilde{A}$ restricts to a biholomorphism between the set $\{P \in Gr_n(E_{\mathbb{C}}) | P \oplus \overline{P} = E_{\mathbb{C}}|_{\pi_{Gr(P)}}\}$ in $Gr_n(E_{\mathbb{C}})$ and the corresponding one in $Gr_n(E'_{\mathbb{C}})$. The proposition then follows from Theorem 4.2.1 and Proposition 4.1.1, which show that these sets are respectively biholomorphic to $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$ and $(\mathcal{C}(E'), \mathcal{J}^{(\nabla',I)})$. $\square$

Now suppose that E and E' are also equipped with respective metrics g and g' and that the above connections preserve the appropriate metrics. If we $\mathbb{C}$-bilinearly extend g and g' to $E_{\mathbb{C}}$ and $E'_{\mathbb{C}}$, we then have

Proposition 4.4.2. *Let $A : (E_{\mathbb{C}}, \nabla^{0,1}) \longrightarrow (E'_{\mathbb{C}}, \nabla'^{0,1})$ be an isomorphism of holomorphic vector bundles that is orthogonal with respect to g and g'. Then A induces a fiber preserving biholomorphism between $(\mathcal{T}(E, g), \mathcal{J}^{(\nabla,I)})$ and $(\mathcal{T}(E', g'), \mathcal{J}^{(\nabla',I)})$.*

Proof. Similar to the proof of Proposition 4.4.1, the isomorphism $A : (E_{\mathbb{C}}, \nabla^{0,1}) \longrightarrow (E'_{\mathbb{C}}, \nabla'^{0,1})$ induces a biholomorphism $\tilde{A} : Gr_n(E_{\mathbb{C}}) \longrightarrow Gr_n(E'_{\mathbb{C}})$. Since A is an orthogonal map, $\tilde{A}$ maps the space of maximal isotropics, $MI(E_{\mathbb{C}})$, in $Gr_n(E_{\mathbb{C}})$ to the one in $Gr_n(E'_{\mathbb{C}})$. The proposition then follows from Proposition 4.2.2, which shows that $(\mathcal{T}(E, g), \mathcal{J}^{(\nabla,I)})$ and $(\mathcal{T}(E', g'), \mathcal{J}^{(\nabla',I)})$ are respectively biholomorphic to $MI(E_{\mathbb{C}})$ and $MI(E'_{\mathbb{C}})$. $\square$

In the following two sections we consider some applications of the above propositions.

4.4.1 Cohomology Independence

Let $(E, g, J) \longrightarrow (M, I)$ be a holomorphic Hermitian bundle fibered over a complex manifold and let $\overline{\partial}$ be the standard $\overline{\partial}$-operator on $E^{1,0} \oplus E^{*1,0}$, where $E^{1,0}$ is the $+i$ eigenbundle of J. If we choose $D \in \Gamma(T^{*0,1} \otimes \wedge^2 E^{*1,0})$ to satisfy $\overline{\partial} D = 0$ then, as described in Section 3.2, $\nabla^{Ch(0,1)} + g^{-1}D$ is a $\overline{\partial}$-operator on $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$ and, for $\nabla = \nabla^{Ch} + g^{-1}D + \overline{g^{-1}D}$, the twistor space $(\mathcal{T}(E), \mathcal{J}^{(\nabla,I)})$ is a complex manifold. If we now let $B \in \Gamma(\wedge^2 E^{*1,0})$ then $\nabla^{Ch(0,1)} + g^{-1}(D + \overline{\partial} B)$ is another $\overline{\partial}$-operator on $E_{\mathbb{C}}$ and it is natural to wonder, as in Section 3.2, whether the associated twistor space is biholomorphic to the previous one. In other words, does the above give a well defined mapping from the Dolbeault cohomology group $H^{0,1}(\wedge^2 E^{*1,0})$ to the isomorphism classes of complex structures on $\mathcal{T}$?

By using Proposition 4.4.2, we will show here that such a mapping does indeed exist. As a first step, let us consider the section of $O(E_{\mathbb{C}}, g)\ exp(g^{-1}B)$, which equals $(1 + g^{-1}B)$ since $(g^{-1}B)^2 = 0$. We then have

Proposition 4.4.3. *The map $exp(-g^{-1}B) : (E_{\mathbb{C}}, \nabla^{Ch(0,1)} + g^{-1}D) \longrightarrow (E_{\mathbb{C}}, \nabla^{Ch(0,1)} + g^{-1}(D + \overline{\partial} B))$ is an isomorphism of holomorphic vector bundles.*

Proof. Let $(\nabla^{Ch(0,1)} + g^{-1}D)v = 0$ and consider

$$\begin{aligned}
&(\nabla^{Ch(0,1)} + g^{-1}(D + \overline{\partial} B))(1 - g^{-1}B)v \\
&= -\nabla^{Ch(0,1)}(g^{-1}Bv) + (g^{-1}\overline{\partial} B)v \\
&= -(\nabla^{Ch(0,1)} g^{-1}B)v - g^{-1}B\nabla^{Ch(0,1)}v + (g^{-1}\overline{\partial} B)v.
\end{aligned}$$

Since the first and last terms cancel, we are left with

$$-g^{-1}B\nabla^{Ch(0,1)}v = -g^{-1}B(-g^{-1}Dv) = 0.$$

This then proves the proposition. $\square$

By Proposition 4.4.2, we can now conclude that the twistor spaces mentioned above are biholomorphic:

Proposition 4.4.4. *$exp(-g^{-1}B)$ induces a fiber preserving biholomorphism between $(\mathcal{T}, \mathcal{J}^{(\nabla,I)})$ and $(\mathcal{T}, \mathcal{J}^{(\nabla',I)})$, where $\nabla^{0,1} = \nabla^{Ch(0,1)} + g^{-1}D$ and $\nabla'^{0,1} = \nabla^{Ch(0,1)} + g^{-1}(D + \overline{\partial}B)$.*

As a corollary, we have

Proposition 4.4.5. *The map $[D] \longrightarrow [\mathcal{J}^{(\nabla,I)}]$, where $\nabla^{0,1} = \nabla^{Ch(0,1)} + g^{-1}D$, from the Dolbeault cohomology group $H^{0,1}(\wedge^2 E^{*1,0})$ to the isomorphism classes of complex structures on $\mathcal{T}(E,g)$ is well defined.*

4.4.2 Changing the Metric

In the previous example we worked with a fixed metric g; but what if we were to choose another metric g' on E that is compatible with J–then is it true that $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch},I)})$ and $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'},I)})$ are biholomorphic? This is part of a more general question that was posed in Section 3.2: in that section we used a fixed metric, g, to define $\overline{\partial}$-operators on $E_{\mathbb{C}}$ and thus complex structures on $\mathcal{T}(g)$ –but if we were to choose another metric g' then do we obtain new complex manifolds by considering $\mathcal{T}(g')$?

To address these questions, let us first recall some of the details of that section. Let $(E,J) \longrightarrow (M,I)$ be a holomorphic vector bundle, considered as a real bundle with fiberwise complex structure J, that is fibered over a complex manifold. Defining $<,>$ and $\overline{\partial}$ to be the standard inner product and $\overline{\partial}$-operator on $E^{1,0} \oplus E^{*1,0}$, let us consider the $\overline{\partial}$-operator $\bar{\partial} + \mathcal{D}'^{0,1}$, where $\mathcal{D}'^{0,1} \in \Gamma(T^{*0,1} \otimes \mathfrak{so}(E^{1,0} \oplus E^{*1,0}))$. If g is a metric on E that is compatible with J then, as in Section 3.2, we can use the orthogonal isomorphism

$$1 \oplus g : (E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}, \frac{g}{2}) \longrightarrow (E^{1,0} \oplus E^{*1,0}, <,>)$$

to obtain the $\overline{\partial}$-operator $\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$ on $E_{\mathbb{C}}$ as well as the complex structure $\mathcal{J}^{(\nabla^{Ch}+\mathcal{D}_g,I)}$ on $\mathcal{T}(g)$. (Here, $\mathcal{D}_g = \mathcal{D}_g^{0,1} + \overline{\mathcal{D}_g^{0,1}}$.)

Similarly, if g' is another metric that is compatible with J then we have the complex structure $\mathcal{J}^{(\nabla^{Ch'}+\mathcal{D}_{g'},I)}$ on $\mathcal{T}(g')$. The goal then is to use Proposition 4.4.2 to show that the complex manifolds $\mathcal{T}(g)$ and $\mathcal{T}(g')$ are equivalent under a fiberwise biholomorphism.

First note, that if we compose the map $(1 \oplus g)$ with $(1 \oplus g')^{-1}$ then we obtain the following isomorphism of holomorphic vector bundles:

$$\begin{aligned} (E_{\mathbb{C}}, \nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}) &\longrightarrow (E_{\mathbb{C}}, \nabla^{Ch'(0,1)} + \mathcal{D}_{g'}^{0,1}) \\ v^{1,0} + v^{0,1} &\longrightarrow (v^{1,0} + g'^{-1}gv^{0,1}), \end{aligned}$$

where we have used the decomposition, $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$. As this is an orthogonal map from $(E_{\mathbb{C}}, g)$ to $(E_{\mathbb{C}}, g')$, by Proposition 4.4.2 we have

Proposition 4.4.6. *There exists a fiber preserving biholomorphism between* $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch}+\mathcal{D}_g, I)})$ *and* $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'}+\mathcal{D}_{g'}, I)})$.

In particular, if we set $\mathcal{D}'^{(0,1)}$ to zero, we have:

Proposition 4.4.7. *Let* $(E, J) \longrightarrow (M, I)$ *be a holomorphic vector bundle that is equipped with two Hermitian metrics* g *and* g'. *Then* $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch}, I)})$ *and* $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'}, I)})$ *are biholomorphic.*

Part II

Applications

Chapter 5

Holomorphic Subvarieties

Having given examples and explored various properties of holomorphic twistor spaces, we will now present our first application. We will use holomorphic sections of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$ to decompose (M, I) into different types of holomorphic subvarieties. We will then develop tools from twistor spaces in order to study different properties of these subvarieties. The importance of our results will especially be demonstrated when we consider bihermitian manifolds in Chapter 8.

5.1 The $M_{\leq s}$ and $M_{(\leq r,\pm)}$

To begin, let $E \longrightarrow (M, I)$ be a real rank $2n$ vector bundle fibered over a complex manifold and equipped with a connection ∇ that has (1,1) curvature. Further, suppose that J and K are respectively parallel and holomorphic sections of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$—so that $\nabla J = 0$ and $K\nabla K = \nabla_I K$ (see Section 3.4). Our main theorem of this section is that the degeneracy loci of the real bundle maps $[J, K]$, $J+K$ and $J-K$ are holomorphic subvarieties of M:

Theorem 5.1.1. *Let J and K respectively be parallel and holomorphic sections of $(\mathcal{C}, \mathcal{J}^{(\nabla,I)}) \longrightarrow (M, I)$. The following are holomorphic subvarieties of M:*

$$1)\ M_{\leq s} = \{x \in M |\ Rank[J, K]|_x \leq 2s\}$$
$$2)\ M_{(\leq r,\pm)} = \{x \in M |\ Rank(J \pm K)|_x \leq 2r\}.$$

Notation 5.1.2. *We will similarly define M_s and $M_{(r,\pm)}$ as above but with the appropriate $\leq$ signs replaced with $=$.*

The reason that the above real bundle maps yield holomorphic subvarieties is that they are in fact holomorphic when restricted to the appropriate holomorphic bundles, which we now describe.

First consider the holomorphic bundle $E_{\mathbb{C}}$ that is equipped with the $\overline{\partial}$-operator $\nabla^{0,1}$ (see Lemma 2.4.2). Since J is parallel, $E_J^{1,0}$ and $E_J^{0,1}$ are two holomorphic subbundles of $E_{\mathbb{C}}$, and since K is holomorphic, by Proposition 3.4.2, $E_K^{0,1}$ is a third. Now the holomorphicity of K, as explained in Section 3.4.1, can also be used to show that the connection $\nabla' = \nabla + \frac{1}{2}(\nabla K)K$

has (1,1) curvature, so that $\nabla'^{0,1}$ is another $\overline{\partial}$-operator on $E_{\mathbb{C}}$. As $\nabla' K = 0$, $E_K^{1,0}$ and $E_K^{0,1}$ are holomorphic subbundles. (Note that since $(\nabla^{0,1}K)E_K^{0,1} = 0$, $\nabla^{0,1} = \nabla'^{0,1}$ when acting on $E_K^{0,1}$.)

Given these bundles, we have

Proposition 5.1.3. *The following are holomorphic*

$$\begin{array}{ll} 1)\ J+K : E_K^{0,1} \longrightarrow E_J^{0,1} & 2)\ J-K : E_K^{0,1} \longrightarrow E_J^{1,0} \\ 3)\ J+K : E_J^{1,0} \longrightarrow E_K^{1,0} & 4)\ J-K : E_J^{0,1} \longrightarrow E_K^{1,0} \end{array}$$
$$5)\ [J,K] : E_K^{0,1} \longrightarrow E_K^{1,0}.$$

Proof. The proofs of 1) and 2) are straightforward. To prove 3), let $e \in \Gamma(E_J^{1,0})$ satisfy $\nabla^{0,1}e = 0$ and consider

$$\nabla'^{0,1}(J+K)e = (i+K)\nabla'^{0,1}e = \frac{1}{2}(i+K)(\nabla^{0,1}K)Ke.$$

Since $(i+K)\nabla^{0,1}K = 0$, the map given in 3) is holomorphic.

The proof of 4) is similar, and that of 5) follows by composing the maps in 1) and 4) or the maps in 2) and 3). □

Note that Theorem 5.1.1 immediately follows from the above proposition.

Although we proved the holomorphicity of $[J,K]$ by composing, say, the maps given in 1) and 4), we should stress that it does not depend on the (1,1) condition on the curvature of ∇, R^{∇}—but only depends on the (1,1) condition on $R^{\nabla'}$:

Proposition 5.1.4. *Let ∇ be a connection on $E \longrightarrow (M,I)$ and let J and K be sections of $\mathcal{C}$ such that $\nabla J = 0$ and $K\nabla K = \nabla_I K$. Moreover, assume that $R^{\nabla'}$, where $\nabla' = \nabla + \frac{1}{2}(\nabla K)K$, is (1,1). Then*

$$[J,K] : E_K^{0,1} \longrightarrow E_K^{1,0}$$

is holomorphic, where each of the bundles is equipped with the $\overline{\partial}$-operator $\nabla'^{0,1}$.

Proof. Let $e \in \Gamma(E_K^{0,1})$ satisfy $\nabla'^{0,1}e = 0$ and consider

$$\nabla'^{0,1}[J,K]e = -(i+K)(\nabla'^{0,1}J)e.$$

Since $\nabla J = 0$, this equals

$$-\frac{1}{2}(i+K)[(\nabla^{0,1}K)K, J]e,$$

which is zero since

$$(i+K)\nabla^{0,1}K = \nabla^{0,1}K(i-K) = 0.$$

Hence $[J,K]$ is holomorphic. □

As an example, if (M,g,I) is not an SKT manifold then the curvature of ∇^- (Section 3.3.1) is not (1,1) but the curvature of $(\nabla^-)' = \nabla^- + \frac{1}{2}(\nabla^- I)I = \nabla^{Ch}$ is always so. (Note that $I\nabla^- I$ still equals $\nabla_I^- I$.)

5.1.1 Metric Case

Let us now further suppose that E is equipped with a fiberwise metric g, ∇ is a metric connection and J and K are sections of $\mathcal{T}(E,g)$. Then note:

- The holomorphicity of map 3) in Proposition 5.1.3 can be derived from that of 1). The reason is that map 3) equals $-g^{-1}(J+K)^t g$, where $J+K : E_K^{0,1} \longrightarrow E_J^{0,1}$. Similarly, the holomorphicity of map 4) can be derived from that of 2).

- The maps

  · $[J,K]g^{-1} : E_K^{*1,0} \longrightarrow E_K^{1,0}$
  · $g[J,K] : E_K^{0,1} \longrightarrow E_K^{*0,1}$

  respectively define holomorphic sections of $\wedge^2 E_K^{1,0}$ and $\wedge^2 E_K^{*0,1}$.

5.2 The $M^{\#}$ and M^{δ}

Consider the setup of Theorem 5.1.1 of a real rank $2n$ bundle $E \longrightarrow (M,I)$ and the respective parallel and holomorphic sections J and K of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)}) \longrightarrow (M,I)$. We will now introduce a slightly finer decomposition of the holomorphic subvarieties $M_{\leq s}$ and $M_{(\leq r,\pm)}$.

Definition 5.2.1. Let $M^{\#}$ stand for any of the following:

$$
\begin{aligned}
&1)\ M^{(m_1,*)} = \{x \in M|\ dimKer(J+K)|_x = 2m_1\}\\
&2)\ M^{(*,m_{-1})} = \{x \in M|\ dimKer(J-K)|_x = 2m_{-1}\}\\
&3)\ M^{(m_1,m_{-1})} = M^{(m_1,*)} \cap M^{(*,m_{-1})}.
\end{aligned}
$$

Notation 5.2.2. *So far we have introduced the $M_{\leq s}$, $M_{(\leq r,\pm)}$ and $M^{\#}$. We will let M^{δ} stand for any of these holomorphic subvarieties.*

To relate the $M^{\#}$ to the other subvarieties, first note that we can decompose $M_s = \{x \in M|\ Rank[J,K]|_x = 2s\}$ as follows:

$$
M_s = \bigcup_{m_1+m_{-1}=n-s} M^{(m_1,m_{-1})}. \tag{5.2.1}
$$

This is immediate from

Lemma 5.2.3. *Let V be an even dimensional real vector space and let J and K be elements of $\mathcal{C}(V)$. The $ker[J,K] = ker(J+K) \oplus ker(J-K)$.*

Proof. If we restrict JK to $ker[J,K]$ then it squares to 1. Hence

$$
ker[J,K] = W_1 \oplus W_{-1},
$$

where $JK|_{W_1} = 1$ and $JK|_{W_{-1}} = -1$. The lemma then follows from the fact that $W_1 = ker(J+K)$ and $W_{-1} = ker(J-K)$. $\square$

Also note:

1) Locally, $M_s = M^{(m_1,m_{-1})}$, for some (m_1, m_{-1}) that satisfy $m_1 + m_{-1} = n - s$. The reason is that the dimensions of $ker(J + K)$ and $ker(J - K)$ cannot locally increase.

2) $M_{(r,+)} = M^{(n-r,*)}$ and $M_{(r,-)} = M^{(*,n-r)}$.

5.2.1 Metric Case

Let us now further suppose that E is equipped with a fiberwise metric g, ∇ is a metric connection and J and K are sections of $\mathcal{T}(E, g)$. The following gives some additional properties of the $M^{\#}$.

Proposition 5.2.4.

1) Given $x \in M$, the following is an orthogonal and $J, K-$invariant splitting of E_x:

$$Im[J, K] \oplus ker(J + K) \oplus ker(J - K).$$

Moreover the $rank[J, K] = 4k$.

2) M is a disjoint union of the following open subsets:

$$\bigcup_{m_1=even} M^{(m_1,*)} \text{ and } \bigcup_{m_1=odd} M^{(m_1,*)}.$$

3) $\bigcup_{m_1=even} M^{(m_1,*)} = \{x \in M |\ J_x$ and K_x induce the same orientations$\}$.

4) $\bigcup_{m_1=odd} M^{(m_1,*)} = \{x \in M |\ J_x$ and K_x induce opposite orientations$\}$.

Parts $2) - 4)$ are also true if we were to replace $M^{(m_1,*)}$ with $M^{(*,m_{-1})}$ and J_x with $-J_x$.

Example 5.2.5. Suppose that $rankE = 4$ and J and K induce the same orientations on E. Then by the above proposition, $M = M^{(0,0)} \cup M^{(0,2)} \cup M^{(2,0)}$. □

Proposition 5.2.4 follows immediately from the following brief background on the algebraic interaction of two complex structures.

Some Background: Let (V, g) be an even dimensional real vector space equipped with a positive definite metric and let J and $K \in \mathcal{T}(V, g)$. Consider the orthogonal and $J, K-$invariant splitting:

$$V = Im[J, K] \oplus ker(J + K) \oplus ker(J - K). \tag{5.2.2}$$

We then have:

Proposition 5.2.6.

1) *$Im[J, K]$ is an $\mathbb{H}$-module and is 4k real dimensional.*

2) *J and K induce the same orientation on V if and only if $\dfrac{dimKer(J + K)}{2}$ is even.*

Remark 5.2.7. *The above proposition would not necessarily be true if we were to assume that J and K are two general elements of $\mathcal{C}(V)$. For instance, take $V =< v_1, v_2 >_{\mathbb{R}}$ and define J and K via the equations*

- $Jv_1 = v_2$, $Jv_2 = -v_1$
- $Kv_1 = -rv_2$, $Kv_2 = r^{-1}v_1$, *where* $r \in \mathbb{R}_{>0} - \{1\}$.

Then J and K are elements of $\mathcal{C}(V)$ that induce opposite orientations on V and yet the $ker(J + K) = 0$. Moreover, the $rank[J, K] = 2$ and not a multiple of four. Note that $J, K \notin \mathcal{T}(V, g)$ for any metric g because the eigenvalues of JK do not have norm 1.

Proof of Proposition 5.2.6. Let us begin by diagonalizing JK:

$$V \otimes \mathbb{C} = (V_{c_1} \oplus \overline{V_{c_1}}) \oplus ... \oplus (V_{c_l} \oplus \overline{V_{c_l}}) \oplus (V_1 \otimes \mathbb{C}) \oplus (V_{-1} \otimes \mathbb{C}),$$

where $c \in \mathbb{C} - \{\pm 1\}$ satisfies $c\bar{c} = 1$, $V_{\pm 1} \subset V$ and $JKv_\lambda = \lambda v_\lambda$ for $v_\lambda \in V_\lambda$.

Now set $2e = c + \bar{c}$ and $V_c \oplus \overline{V_c} = V_e \otimes \mathbb{C}$ for $V_e \subset V$. We then obtain the following orthogonal and $J, K-$invariant decomposition:

$$V = V_{e_1} \oplus ... \oplus V_{e_l} \oplus V_1 \oplus V_{-1}.$$

Note:

- $\{J, K\}v = 2\epsilon v$, for $v \in V_\epsilon$
- $Im[J, K] = V_{e_1} \oplus ... \oplus V_{e_l}$, $ker(J + K) = V_1$ and $ker(J - K) = V_{-1}$.

The claim then is that V_e—and thus $Im[J, K]$—is an $\mathbb{H}$-module. The reason is that if we let $J' = \frac{JK - e}{f}|_{V_e}$, where $e^2 + f^2 = 1$, then $(J')^2 = -1$ and $\{J', K\} = 0$ when acting on V_e. ($\{J', J\} = 0$ as well.)

Although the rest of the proof of the proposition follows from this claim, we will now give a more direct proof of the fact that $rank[J, K] = 4k$. To see this just note that $g[J, K] : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}^*$ (where $V_{\mathbb{C}} := V \otimes \mathbb{C}$) is skew and sends $V_J^{1,0}$ to $V_J^{*1,0}$ and $V_J^{0,1}$ to $V_J^{*0,1}$. □

Remark 5.2.8. *If we consider the algebra iD_∞, generated by two complex structures J_0 and K_0 over $\mathbb{R}$, then one may use the above proof to derive its orthogonal representations. (These representations are by definition the ones where J_0 and K_0 act by orthogonal transformations with respect to some metric.)*

Corollary 5.2.9. *The irreducible, orthogonal representations of iD_∞ are:*

$$\mathbb{R}[t]/(p) \oplus K_0\mathbb{R}[t]/(p),$$

where J_0K_0 acts by t and

1) $p = t \pm 1$
2) $p = (t - c)(t - \bar{c})$, *for* $c \in \mathbb{C} - \{\pm 1\}, c\bar{c} = 1$.

In [11], we not only derive the orthogonal representations of iD_∞ but the indecomposable ones as well.

Having given some basic properties of the M^δ in the above propositions, we will present our major results about them in Chapter 7. There, we determine lower bounds on the dimensions of the M^δ as well as necessary conditions for there to exist curves in M that lie in certain $M^\#$. We will give the applications of these results, for the case when M is a bihermitian manifold, in Chapter 8. Our present focus is to derive them by first considering in the next chapter an example of the general setup of Theorem 5.1.1 where the base manifold is itself the total space of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)}) \longrightarrow (M, I)$. We will then use this example in Section 7.1 to establish a twistor point of view of the general M^δ—by realizing them as the intersection of $K(M)$ with the corresponding $\mathcal{C}^\delta$ in $\mathcal{C}$. Some of these $\mathcal{C}^\delta$ will be shown to be complex submanifolds of $\mathcal{C}$ and by determining their dimensions and describing their tangent bundles, we will derive the results mentioned above. (We will also describe a metric version of this setup where $\mathcal{C}$ is replaced with $\mathcal{T}$.)

In the next chapter, we will first consider the twistor spaces $\mathcal{C}(V)$ and $\mathcal{T}(V, g)$ which are associated to vector spaces and then those associated to vector bundles and will be focusing on studying the above properties of the $\mathcal{C}^\delta$ and $\mathcal{T}^\delta$.

Chapter 6

Stratifications of Twistor Spaces

6.1 Twistor Spaces of Vector Spaces

6.1.1 $\mathcal{C}(V)$-case

Let V be a $2n$ dimensional real vector space and let $\mathcal{C} := \mathcal{C}(V)$ be its twistor space with complex structure $I_{\mathcal{C}}$. To obtain an example of the setup of Theorem 5.1.1, consider the trivial bundle $E = \mathcal{C} \times V \longrightarrow \mathcal{C}$ along with its trivial connection d. By Proposition 3.4.13, ϕ, defined by $\phi|_K = K$, is a natural holomorphic section of $(\mathcal{C}(E), \mathcal{J}^{(d,I_{\mathcal{C}})})$. As for a parallel section, we will choose the constant section J, a fixed element of $\mathcal{C}$, so that $dJ = 0$. By Theorem 5.1.1, we then have

Proposition 6.1.1. *Given $J \in \mathcal{C}$, the following are holomorphic subvarieties of $\mathcal{C}$:*

$$\begin{aligned} &1)\ \mathcal{C}_{\leq s}(J) = \{K \in \mathcal{C}|\ Rank[J,K] \leq 2s\} \\ &2)\ \mathcal{C}_{(\leq r,\pm)}(J) = \{K \in \mathcal{C}|\ Rank(J \pm K) \leq 2r\}. \end{aligned}$$

We also have the subvarieties $\mathcal{C}^{\#}(J)$ that correspond to the $M^{\#}$ of Definition 5.2.1. More explicitly, $\mathcal{C}^{\#}(J)$ will stand for any of the following:

$$\begin{aligned} &1)\ \mathcal{C}^{(m_1,*)}(J) = \{K \in \mathcal{C}|\ dimKer(J + K) = 2m_1\} \\ &2)\ \mathcal{C}^{(*,m_{-1})}(J) = \{K \in \mathcal{C}|\ dimKer(J - K) = 2m_{-1}\} \\ &3)\ \mathcal{C}^{(m_1,m_{-1})}(J) = \mathcal{C}^{(m_1,*)} \cap \mathcal{C}^{(*,m_{-1})}. \end{aligned}$$

Notation 6.1.2. *When referring to the above subvarieties, we will usually drop the "(J)" factors and will denote all of them by $\mathcal{C}^{\delta}$.*

We will now be studying different properties of the $\mathcal{C}^{\delta}$. In particular, we will show that the $\mathcal{C}^{\#}$ are complex submanifolds that form several stratifications of $\mathcal{C}$ and will determine their dimensions and describe their tangent bundles.

To accomplish this, we will be using the following holomorphic embedding of $\mathcal{C}$ into the Grassmannians of n-planes in $V_{\mathbb{C}} = V \otimes \mathbb{C}$:

Lemma 6.1.3. *The map*

$$\psi : \mathcal{C} \longrightarrow Gr_n(V_{\mathbb{C}})$$
$$K \longrightarrow V_K^{0,1},$$

where $V_K^{0,1}$ is the $-i$ eigenspace of K, is a holomorphic embedding whose image is open in $Gr_n(V_{\mathbb{C}})$.

Proof. See Proposition 4.1.1. □

We then have

Proposition 6.1.4.

$$1)\ \psi(\mathcal{C}_{(\le r,+)}) = \{W \in Im\psi |\ dim_{\mathbb{C}}(V_J^{1,0} \cap W) \ge n-r\}$$
$$2)\ \psi(\mathcal{C}^{(m_1,*)}) = \{W \in Im\psi |\ dim_{\mathbb{C}}(V_J^{1,0} \cap W) = m_1\}$$
$$3)\ \psi(\mathcal{C}^{(*,m_{-1})}) = \{W \in Im\psi |\ dim_{\mathbb{C}}(V_J^{0,1} \cap W) = m_{-1}\}.$$

Analogous formulas hold for $\mathcal{C}_{(\le r,-)}$ and $\mathcal{C}^{(m_1,m_{-1})}$.

We thus find that ψ maps $\mathcal{C}^{(m_1,*)}$, $\mathcal{C}^{(*,m_{-1})}$ and $\mathcal{C}_{(\le r,\pm)}$ to open subsets of either $Gr^{(s)}$ or $Gr^{(\ge s)}$ in $Gr_n(V_{\mathbb{C}})$, where $Gr^{(s)} = \{W \in Gr_n(V_{\mathbb{C}})|\ dim_{\mathbb{C}}(V^0 \cap W) = s\}$ for some $V^0 \in Gr_n(V_{\mathbb{C}})$. $Gr^{(s)}$ is a type of Schubert cell in $Gr_n(V_{\mathbb{C}})$ and we will now review some of its properties.

$\mathbf{Gr^{(s)}}$: Let V be a real vector space of dimension $2n$ and let $V^0 \in Gr_n(V_{\mathbb{C}})$. As above, define $Gr^{(s)} = \{W \in Gr_n(V_{\mathbb{C}})|\ dim_{\mathbb{C}}(V^0 \cap W) = s\}$. We will now introduce certain holomorphic charts for $Gr_n(V_{\mathbb{C}})$ that will, in particular, be used to show that $Gr^{(s)}$ is a complex submanifold.

To begin, let $W \in Gr^{(s)}$ and split

$$V_{\mathbb{C}} = W \oplus W' = (W_1 \oplus W_2) \oplus (W_1' \oplus W_2'),$$

where $W_1 = V^0 \cap W$, $W_1 \oplus W_2' = V^0$ and W_2 and W_1' are appropriate complements.

Now consider the corresponding holomorphic chart for $Gr_n(V_{\mathbb{C}})$ about W:

$$\rho : End(W, W') \longrightarrow Gr_n(V_{\mathbb{C}})$$
$$A \longrightarrow Graph(A) = \{w + Aw \in V_{\mathbb{C}}|\ w \in W\}.$$

We then have

Proposition 6.1.5. *Let* $A = \begin{array}{cc} & \begin{array}{cc} W_1 & W_2 \end{array} \\ \begin{array}{c} W_1' \\ W_2' \end{array} & \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \end{array} \in End(W, W')$. *Then* $Graph(A) \cap V^0 = \{w + Aw \in V_{\mathbb{C}}|\ w \in ker a_1\}$ *and its dimension equals that of* $ker a_1$.

Proof. Let $w + Aw \in Graph(A)$ and set $w = w_1 + w_2 \in W_1 \oplus W_2$. Then $w + Aw \in V^0$ if and only if $w_2 + a_1 w_1 + a_2 w_2 = 0$, which in turn is equivalent to $w = w_1 \in ker a_1$. □

If we define

$$End_t(W,W') = \{ \begin{array}{cc} & \begin{array}{cc} W_1 & W_2 \end{array} \\ \begin{array}{c} W_1' \\ W_2' \end{array} & \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \end{array} \in End(W,W') |\ dimKera_1 = t\}$$

we then have

Corollary 6.1.6. *The map*

$$End_t(W,W') \longrightarrow Gr^{(t)} \cap Im\rho$$
$$A \longrightarrow Graph(A)$$

is well defined and bijective (here, $t \in \{0,1,...,s\}$). Moreover, when $t = s$ this map gives a holomorphic chart for $Gr^{(s)}$ about W.

Using the above corollary, it is straightforward to show:

Corollary 6.1.7.

1) *$Gr^{(s)}$ is a complex submanifold of $Gr_n(V_{\mathbb{C}})$ of dimension $n^2 - s^2$.*
2) *$\overline{Gr^{(s)}}$ equals $Gr^{(\geq s)}$ and is a holomorphic subvariety of dimension $n^2 - s^2$.*

Properties of the $\mathcal{C}^\delta$: By then combining Proposition 6.1.4 and Corollary 6.1.7, we obtain

Proposition 6.1.8.

1) *$\mathcal{C}^{(m_1,*)}$ is a complex submanifold of $\mathcal{C}$ of dimension $n^2 - m_1^2$.*
2) *$\mathcal{C}_{(\leq r,\pm)}$ is a holomorphic subvariety of dimension $n^2 - (n-r)^2$.*

An analogous result holds for $\mathcal{C}^{(*,m_{-1})}$.

Let us now consider some properties of $\mathcal{C}^{(m_1,m_{-1})} = \mathcal{C}^{(m_1,*)} \cap \mathcal{C}^{(*,m_{-1})}$.

Proposition 6.1.9. *$\mathcal{C}^{(m_1,m_{-1})}$ is nonempty if and only if $m_1 + m_{-1} \leq n$.*

Proof. If $K \in \mathcal{C}^{(m_1,m_{-1})}$ then by Lemma 5.2.3, $dimKer[J,K] = 2(m_1 + m_{-1}) \leq 2n$.

Conversely, given $m_1, m_{-1} \in \mathbb{Z}_{\geq 0}$ such that $m_1 + m_{-1} \leq n$, we will define a $K \in \mathcal{C}^{(m_1,m_{-1})}$ as follows. First consider the J-invariant splitting:

$$V = \bigoplus_{i\in\{1,2,...,l\}} < v_i, Jv_i > \oplus V_1 \oplus V_{-1},$$

where $dimV_1 = 2m_1$ and $dimV_{-1} = 2m_{-1}$.

Now define $K \in \mathcal{C}$ by setting

- $Kv_i = -rJv_i$ and $KJv_i = r^{-1}v_i$, where $r \in \mathbb{R} - \{0, \pm 1\}$

- $Kw_1 = -Jw_1$ and $Kw_{-1} = Jw_{-1}$, $\forall\ w_1 \in V_1$ and $w_{-1} \in V_{-1}$.

So defined, one may check that K is indeed an element of $\mathcal{C}^{(m_1,m_{-1})}$. $\square$

Supposing that $m_1 + m_{-1} \leq n$, we will now show that $\mathcal{C}^{(m_1,*)}$ and $\mathcal{C}^{(*,m_{-1})}$ intersect transversally, thus proving, in particular, that $\mathcal{C}^{(m_1,m_{-1})}$ is a complex manifold. To show this, we will first describe $T_K\mathcal{C}^{(m_1,*)}$ and $T_K\mathcal{C}^{(*,m_{-1})}$ in $T_K\mathcal{C} = \mathfrak{gl}_{\{K\}} := \{A \in \mathfrak{gl}(V) |\ \{A, K\} = 0\}$.

Proposition 6.1.10.

1) $T_K\mathcal{C}^{(m_1,*)} = \{A \in \mathfrak{gl}_{\{K\}} |\ A : Ker(J + K) \to Im(J + K)\}$

2) $T_K\mathcal{C}^{(*,m_{-1})} = \{A \in \mathfrak{gl}_{\{K\}} |\ A : Ker(J - K) \to Im(J - K)\}$

3) *If* $m_1 + m_{-1} \leq n$ *then* $\mathcal{C}^{(m_1,*)}$ *and* $\mathcal{C}^{(*,m_{-1})}$ *intersect transversally.*

Proof. To prove Part 1) of the proposition, let $K \in \mathcal{C}^{(m_1,*)}$ and $v \in ker(J + K)$ and suppose $K(t)$ is a curve in $\mathcal{C}^{(m_1,*)}$ that satisfies $K(0) = K$.

As the rank of $J + K(t)$ is independent of t, one may extend v to a curve $v(t)$ in V so that

$$(J + K(t))v(t) = 0.$$

Taking $\frac{d}{dt}|_{t=0}$ of the above expression gives

$$K'v = -(J + K)v'(0),$$

which shows that

$$T_K\mathcal{C}^{(m_1,*)} \subset \{A \in \mathfrak{gl}_{\{K\}} |\ A : Ker(J + K) \to Im(J + K)\}.$$

That these subspaces are indeed equal then follows from the fact that they have the same dimensions.

The proof of Part 2) of the proposition is similar and that of Part 3) is straightforward. $\square$

Corollary 6.1.11. *For* $m_1 + m_{-1} \leq n$, $\mathcal{C}^{(m_1,m_{-1})}$ *is a complex submanifold of dimension* $n^2 - m_1^2 - m_{-1}^2$.

Since by Equation 5.2.1

$$\mathcal{C}_s = \bigcup_{m_1+m_{-1}=n-s} \mathcal{C}^{(m_1,m_{-1})},$$

it follows that $\mathcal{C}_s$ is a disjoint union of complex submanifolds of varied dimensions.

Lastly note that by then using the different $\mathcal{C}^{\#}$ we can stratify $\mathcal{C}$ in several ways, i.e., they can be used to decompose $\mathcal{C}$ into disjoint unions of complex submanifolds.

6.1.2 $\mathcal{T}$-case

Let (V, g) be a $2n$ dimensional real vector space with a positive definite metric and let $\mathcal{T} := \mathcal{T}(V, g)$ be the associated twistor space. Similar to the $\mathcal{C}-$case of the previous section, we have

Proposition 6.1.12. *Given $J \in \mathcal{T}$, the following are holomorphic subvarieties of $\mathcal{T}$:*

$$1)\ \mathcal{T}_{\leq s}(J) = \{K \in \mathcal{T} |\ Rank[J,K] \leq 2s\}$$
$$2)\ \mathcal{T}_{(\leq r,\pm)}(J) = \{K \in \mathcal{T} |\ Rank(J \pm K) \leq 2r\}.$$

We also have the subvarieties $\mathcal{T}^{\#}(J)$ that correspond to the $M^{\#}$ of Definition 5.2.1 and which will stand for $\mathcal{T}^{(m_1,*)}(J)$, $\mathcal{T}^{(*,m_{-1})}(J)$ and $\mathcal{T}^{(m_1,m_{-1})}(J)$.

Notation 6.1.13. *When referring to the above subvarieties, we will usually drop the "(J)" factors and will denote all of them by $\mathcal{T}^{\delta}$.*

We will now study the $\mathcal{T}^{\delta}$ in two ways. The first will be to embed them into a certain space of maximal isotropics associated to $V_{\mathbb{C}} := V \otimes \mathbb{C}$ and the second will be to study them directly inside of $\mathcal{T}$ by using special charts.

Maximal Isotropics:
To begin, let

$$MI(V_{\mathbb{C}}) = \{W \in Gr_n(V_{\mathbb{C}}) |\ g(w_1, w_2) = 0,\ \forall\ w_1, w_2 \in W\}$$

be the space of maximal isotropics in $V_{\mathbb{C}}$. Considering it as a complex submanifold of $Gr_n(V_{\mathbb{C}})$, we have

Lemma 6.1.14. *The map*

$$\psi : \mathcal{T} \longrightarrow MI(V_{\mathbb{C}})$$
$$K \longrightarrow V_K^{0,1},$$

where $V_K^{0,1}$ is the $-i$ eigenspace of K, is a biholomorphism.

Proof. See Proposition 4.1.2. □

Proposition 6.1.15.

$$1)\ \psi(\mathcal{T}_{(\leq r,+)}) = \{W \in MI(V_{\mathbb{C}}) |\ dim_{\mathbb{C}}(V_J^{1,0} \cap W) \geq n - r\}$$
$$2)\ \psi(\mathcal{T}^{(m_1,*)}) = \{W \in MI(V_{\mathbb{C}}) |\ dim_{\mathbb{C}}(V_J^{1,0} \cap W) = m_1\}$$
$$3)\ \psi(\mathcal{T}^{(*,m_{-1})}) = \{W \in MI(V_{\mathbb{C}}) |\ dim_{\mathbb{C}}(V_J^{0,1} \cap W) = m_{-1}\}.$$

Analogous formulas hold for $\mathcal{T}_{(\leq r,-)}$ and $\mathcal{T}^{(m_1,m_{-1})}$.

It follows that ψ maps $\mathcal{T}^{(m_1,*)}$, $\mathcal{T}^{(*,m_{-1})}$ and $\mathcal{T}_{(\leq r,\pm)}$ to either $MI^{(s)}$ or $MI^{(\geq s)}$ in $MI(V_{\mathbb{C}})$, where $MI^{(s)} = \{W \in MI(V_{\mathbb{C}}) |\ dim_{\mathbb{C}}(V^0 \cap W) = s\}$ for some $V^0 \in MI(V_{\mathbb{C}})$. $MI^{(s)}$ is a type of Schubert cell in $MI(V_{\mathbb{C}})$ and we will now review some of its properties.

$\mathbf{MI^{(s)}}$: As above let (V, g) be a real vector space of dimension $2n$ with a positive definite metric and for $V^0 \in MI(V_{\mathbb{C}})$, define $MI^{(s)} = \{W \in MI(V_{\mathbb{C}}) |\ dim_{\mathbb{C}}(V^0 \cap W) = s\}$.

To study the $MI^{(s)}$, we will begin by defining certain holomorphic charts for $MI(V_{\mathbb{C}})$ that will be used, in particular, to show that the $MI^{(s)}$ are complex submanifolds.

If we let $W \in MI^{(s)}$ we then have:

Proposition 6.1.16. *One may split*

$$V_{\mathbb{C}} = W \oplus W' = (W_1 \oplus W_2) \oplus (\overline{W_1} \oplus W_2'),$$

where

- W' *is a maximal isotropic*
- $W_1 = V^0 \cap W$ *and* $W_1 \oplus W_2' = V^0$
- $g(\overline{w_1}, w_2) = 0$, $\forall\ \overline{w_1} \in \overline{W_1}$ *and* $w_2 \in W_2$.

Proof. Using Lemma 6.1.14, let $V^0 = V_J^{0,1}$ and $W = V_K^{0,1}$, where $J, K \in \mathcal{T}$ satisfy $dimKer(J-K) = 2s$.

Consider then the following orthogonal and J, K-invariant splitting:

$$V = \tilde{V} \oplus ker(J-K),$$

where $\tilde{V} = Im(J-K)$.

If we complexify, we may further split

$$\tilde{V} \otimes \mathbb{C} = (\tilde{V})_J^{0,1} \oplus (\tilde{V})_K^{0,1}$$

and

$$ker(J-K) \otimes \mathbb{C} = V_J^{0,1} \cap V_K^{0,1} \oplus V_J^{1,0} \cap V_K^{1,0}.$$

Thus

$$V_{\mathbb{C}} = (V_J^{0,1} \cap V_K^{0,1} \oplus (\tilde{V})_K^{0,1}) \oplus (V_J^{1,0} \cap V_K^{1,0} \oplus (\tilde{V})_J^{0,1}),$$

which satisfies the conditions listed in the proposition. □

Given the above splitting for $V_{\mathbb{C}}$, consider the corresponding holomorphic chart for $MI(V_{\mathbb{C}})$ about W:

$$\begin{aligned} \rho : End^g(W, W') &\longrightarrow MI(V_{\mathbb{C}}) \\ A &\longrightarrow Graph(A) = \{w + Aw \in V_{\mathbb{C}} |\ w \in W\}, \end{aligned}$$

where $End^g(W, W') = \{A \in End(W, W') |\ g(Aw, \tilde{w}) = -g(w, A\tilde{w})\ \forall w, \tilde{w} \in W\}$.

We will now describe how each $Graph(A)$ intersects V^0:

Proposition 6.1.17. *Given* $A = \begin{array}{c c} & \begin{array}{cc} W_1 & W_2 \end{array} \\ \begin{array}{c} \overline{W_1} \\ W_2' \end{array} & \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \end{array} \in End^g(W, W')$, *the* $Graph(A) \cap V^0 = \{w + Aw \in V_{\mathbb{C}} |\ w \in kera_1\}$ *and its dimension equals that of* $kera_1$.

Proof. This follows from Proposition 6.1.5. □

If we define

$$End^g_t(W,W') = \{ \begin{array}{cc} & \begin{array}{cc} W_1 & W_2 \end{array} \\ \begin{array}{c} \overline{W_1} \\ W_2' \end{array} & \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \end{array} \in End^g(W,W') |\ dimKera_1 = t\}$$

then by the above proposition we have:

Corollary 6.1.18. *The map*

$$\begin{aligned} End^g_t(W,W') &\longrightarrow MI^{(t)} \cap Im\rho \\ A &\longrightarrow Graph(A) \end{aligned}$$

is well defined and bijective. Moreover, when $t = s$ this map gives a holomorphic chart for $MI^{(s)}$ about W.

Note that in appropriately chosen bases, $A = \begin{array}{cc} & \begin{array}{cc} W_1 & W_2 \end{array} \\ \begin{array}{c} \overline{W_1} \\ W_2' \end{array} & \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \end{array} \in End^g(W,W')$ is a skew matrix. Hence

$$End^g(W,W') = \begin{cases} \bigcup_{t\in\{0,2,4,\ldots,s\}} End^g_t(W,W') & \text{if } s \text{ is even} \\ \bigcup_{t\in\{1,3,5,\ldots,s\}} End^g_t(W,W') & \text{if } s \text{ is odd.} \end{cases}$$

It is then straightforward to prove the following proposition:

Proposition 6.1.19.

1) *$MI(V_{\mathbb{C}})$ is a disjoint union of the following two open subsets:* $\displaystyle\bigcup_{t=even} MI^{(t)}$ *and* $\displaystyle\bigcup_{t=odd} MI^{(t)}$.

2) *$MI^{(s)}$ is a complex submanifold of dimension* $\dfrac{n(n-1)-s(s-1)}{2}$.

3) *$\overline{MI^{(s)}}$ equals* $\displaystyle\bigcup_{k\in\mathbb{Z}_{\geq 0}} MI^{(s+2k)}$ *and is a holomorphic subvariety of dimension* $\dfrac{n(n-1)-s(s-1)}{2}$.

4) *$MI^{(\geq s)} = \overline{MI^{(s)}} \cup \overline{MI^{(s+1)}}$ and is a holomorphic subvariety of $MI(V_{\mathbb{C}})$.*

Properties of the $\mathcal{T}^\delta$: If we return to the setup of (V,g) with a fixed element $J \in \mathcal{T}$ then by Proposition 6.1.15 we have a result analogous to Proposition 6.1.19 but for $\mathcal{T}$ instead of $MI(V_{\mathbb{C}})$. For future reference we provide the details:

Proposition 6.1.20.

1) $\mathcal{T}$ *is a disjoint union of the following two open subsets:*
$$\bigcup_{m_1=even} \mathcal{T}^{(m_1,*)} \text{ and } \bigcup_{m_1=odd} \mathcal{T}^{(m_1,*)}.$$

2) $\mathcal{T}^{(m_1,*)}$ *is a complex submanifold of dimension* $\dfrac{n(n-1)-m_1(m_1-1)}{2}$.

3) $\overline{\mathcal{T}^{(m_1,*)}}$ *equals* $\bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathcal{T}^{(m_1+2k,*)}$ *and is a holomorphic subvariety of dimension* $\dfrac{n(n-1)-m_1(m_1-1)}{2}$.

4) $\mathcal{T}_{(\leq r,+)} = \overline{\mathcal{T}^{(n-r,*)}} \cup \overline{\mathcal{T}^{(n-r+1,*)}}$ *and is a holomorphic subvariety of* $\mathcal{T}$.

Analogous results hold for $\mathcal{T}_{(\leq r,-)}$ *and* $\mathcal{T}^{(*,m_{-1})}$.

Another way to prove Part 1) of the above proposition is to use the following:

Proposition 6.1.21.

1) $\bigcup_{m_1=even} \mathcal{T}^{(m_1,*)} = \{K \in \mathcal{T} | \ J$ *and* K *induce the same orientations*$\}$.

2) $\bigcup_{m_1=odd} \mathcal{T}^{(m_1,*)} = \{K \in \mathcal{T} | \ J$ *and* K *induce opposite orientations*$\}$.

The above holds true if we were to replace $\mathcal{T}^{(m_1,*)}$ *with* $\mathcal{T}^{(*,m_{-1})}$ *and* J *with* $-J$.

Proof. The proof follows from Proposition 5.2.4. □

Remark 6.1.22. *Below, we will present an alternative derivation of (the entire) Proposition 6.1.20 without using maximal isotropics.*

Let us now consider some properties of $\mathcal{T}^{(m_1,m_{-1})} = \mathcal{T}^{(m_1,*)} \cap \mathcal{T}^{(*,m_{-1})}$.

Proposition 6.1.23. $\mathcal{T}^{(m_1,m_{-1})}$ *is nonempty if and only if* $n - m_1 - m_{-1} = 2k$ $(k \in \mathbb{Z}_{\geq 0})$.

Proof. If $K \in \mathcal{T}^{(m_1,m_{-1})}$ then by Lemma 5.2.3, $dimKer[J,K] = 2(m_1 + m_{-1})$ and by Proposition 5.2.6, $rank[J,K] = 4k$. Hence $2n = 4k + 2(m_1 + m_{-1})$. The proof of the rest of the proposition is straightforward. □

Supposing that $n - m_1 - m_{-1} = 2k$, we will now show that $\mathcal{T}^{(m_1,*)}$ and $\mathcal{T}^{(*,m_{-1})}$ intersect transversally. To do so, we will first describe $T_K\mathcal{T}^{(m_1,*)}$ and $T_K\mathcal{T}^{(*,m_{-1})}$ in $T_K\mathcal{T} = \mathfrak{o}_{\{K\}} := \{A \in \mathfrak{o}(V,g) | \ \{A, K\} = 0\}$. If we split $V = Im[J,K] \oplus ker(J+K) \oplus ker(J-K)$ and let P_0, P_1 and P_{-1} be the corresponding projection operators, we then have

Proposition 6.1.24.

1) $T_K\mathcal{T}^{(m_1,*)} = \{A \in \mathfrak{o}_{\{K\}} | \ P_1AP_1 = 0\}$

2) $T_K\mathcal{T}^{(*,m_{-1})} = \{A \in \mathfrak{o}_{\{K\}} | \ P_{-1}AP_{-1} = 0\}$

3) *If* $n - m_1 - m_{-1} = 2k$*, for* $k \in \mathbb{Z}_{\geq 0}$*, then* $\mathcal{T}^{(m_1,*)}$ *and* $\mathcal{T}^{(*,m_{-1})}$ *intersect transversally.*

Proof. The proof is similar to that of Proposition 6.1.10. $\square$

Corollary 6.1.25. *For $n - m_1 - m_{-1} = 2k$ ($k \in \mathbb{Z}_{\geq 0}$), $\mathcal{T}^{(m_1,m_{-1})}$ is a complex submanifold of dimension $\frac{1}{2}(n(n-1) - m_1(m_1-1) - m_{-1}(m_{-1}-1))$.*

Since

$$\mathcal{T}_s = \bigcup_{m_1+m_{-1}=n-s} \mathcal{T}^{(m_1,m_{-1})},$$

we find that $\mathcal{T}_s$ is a disjoint union of complex submanifolds of varied dimensions.

Other Charts for $\mathcal{T}(V,g)$: We will now describe a C^∞ chart for $\mathcal{T}$ about $K \in \mathcal{T}^{(m_1,m_{-1})}(J)$ that can be used to derive Proposition 6.1.20 without using maximal isotropics.

There is of course the standard chart that is induced from the map

$$\begin{aligned} \mathfrak{o}_{\{K\}} &\longrightarrow \mathcal{T} \\ A &\longrightarrow exp(A)Kexp(-A), \end{aligned}$$

where $\mathfrak{o}_{\{K\}} = \{A \in \mathfrak{o}(V,g) |\ \{A,K\} = 0\}$. However, this chart has the disadvantage that it does not allow us to immediately determine in which $\mathcal{T}^{(m'_1,m'_{-1})}$ $exp(A)Kexp(-A)$ lies. Instead, let us use J and K to decompose $\mathfrak{o}(V,g)$ into

$$\mathfrak{u}_J + \mathfrak{u}_K \oplus \mathfrak{o}_{\{J\}} \cap \mathfrak{o}_{\{K\}},$$

where $\mathfrak{u}_K = \{A \in \mathfrak{o}(V,g) |\ [A,K] = 0\}$. (Note that this is an orthogonal splitting of $\mathfrak{o}(V,g)$, where the metric used is $-tr$.) Letting $\mathcal{D}_J$ be a complement to $\mathfrak{u}_J \cap \mathfrak{u}_K$ in $\mathfrak{u}_J$, consider then the map

$$\begin{aligned} \mu : \mathcal{D}_J \oplus \mathfrak{o}_{\{J\}} \cap \mathfrak{o}_{\{K\}} &\longrightarrow \mathcal{T} \\ (A,B) &\longrightarrow exp(A) \cdot exp(B) \cdot K, \end{aligned}$$

where $exp(B) \cdot K := exp(B)Kexp(-B)$. This map is a local diffeomorphism from a neighborhood about the origin in the domain to one about the point K in the range. We will use it to define C^∞ charts for the $\mathcal{T}^{\#}(J)$ about K as follows: First consider the orthogonal and J,K-invariant splitting of $V = V_0 \oplus V_1 \oplus V_{-1}$, where $V_{\pm 1} = ker(J \pm K)$, and the corresponding splitting of $\mathfrak{o}_{\{J\}} \cap \mathfrak{o}_{\{K\}} = \mathfrak{o}_{\{J\}} \cap \mathfrak{o}_{\{K\}}|_{V_0} \oplus \mathfrak{o}_{\{J\}}|_{V_1} \oplus \mathfrak{o}_{\{J\}}|_{V_{-1}}$. We then have:

Proposition 6.1.26. *Suppose $K \in \mathcal{T}^{(m_1,m_{-1})}(J)$ and let $A \in \mathcal{D}_J$ and $B = B_0 + B_1 + B_{-1} \in \mathfrak{o}_{\{J\}} \cap \mathfrak{o}_{\{K\}}|_{V_0} \oplus \mathfrak{o}_{\{J\}}|_{V_1} \oplus \mathfrak{o}_{\{J\}}|_{V_{-1}}$. There exists a neighborhood N in $\mathfrak{o}_{\{J\}} \cap \mathfrak{o}_{\{K\}}$ about the origin such that if $B \in N$ then*

1) *$\mu(A,B) \in \mathcal{T}^{(m_1,*)}(J)$ if and only if $B_1 = 0$*
2) *$\mu(A,B) \in \mathcal{T}^{(*,m_{-1})}(J)$ if and only if $B_{-1} = 0$*
3) *$\mu(A,B) \in \mathcal{T}^{(m_1,m_{-1})}(J)$ if and only if $B_1 = 0$ and $B_{-1} = 0$.*

Proof. First note that since $A \in \mathfrak{u}_J$, the $dimKer(J \pm exp(A) \cdot exp(B) \cdot K) = dimKer(J \pm exp(B) \cdot K)$. Focusing on the proof of Part 1) of the proposition, let us split $J = J_0 \oplus J_1 \oplus J_{-1}$ according to the decomposition of $V = V_0 \oplus V_1 \oplus V_{-1}$. It then follows that for small enough

$B \in \mathfrak{o}_{\{J\}} \cap \mathfrak{o}_{\{K\}}$, the $ker(J + exp(B) \cdot K) = ker(J_1 + exp(B_1) \cdot K_1)$. Hence the $dimKer(J + exp(B) \cdot K) = 2m_1$ if and only if $J_1 = -exp(B_1) \cdot K_1$. As $K_1 = -J_1$ and $B_1 \in \mathfrak{o}_{\{J\}}|_{V_1}$, this is equivalent to $exp(2B_1) = 1$ (as endomorphisms of V_1). For small enough B, this in turn holds if and only if $B_1 = 0$. The other parts of the proposition are proved similarly. □

This then defines charts for the $\mathcal{T}^{\#}(J)$ that together with Proposition 6.1.12 can be used to derive Proposition 6.1.20 without using maximal isotropics.

6.2 Twistors of Bundles

We will now introduce the holomorphic subvarieties $\mathcal{C}^\delta$ and $\mathcal{T}^\delta$ for the case when the twistor spaces are associated to bundles. Their properties will be used in Chapter 7 to derive our main results about the general M^δ.

6.2.1 $\mathcal{C}$-case

Let $E \longrightarrow (M, I)$ be a real rank $2n$ bundle fibered over a complex manifold and equipped with

- a connection ∇ that has (1,1) curvature
- a section J of $\pi : \mathcal{C}(E) \longrightarrow M$ satisfying $\nabla J = 0$.

We will use J to decompose $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)})$ into complex submanifolds and holomorphic subvarieties as follows. First consider the bundle π^*E with the connection $\pi^*\nabla$ that has (1,1) curvature with respect to $\mathcal{J}^{(\nabla,I)}$. By Proposition 3.4.13, ϕ, defined by $\phi|_K = K$, is a natural holomorphic section of $(\mathcal{C}(\pi^*E), \mathcal{J}^{(\pi^*\nabla,\mathcal{I})}) \longrightarrow \mathcal{C}(E)$, where $\mathcal{I} = \mathcal{J}^{(\nabla,I)}$. Since π^*J is a parallel section, by Theorem 5.1.1 we obtain the following holomorphic subvarieties in $\mathcal{C} := \mathcal{C}(E)$.

Proposition 6.2.1. *Let $J \in \Gamma(\mathcal{C})$ satisfy $\nabla J = 0$. The following are holomorphic subvarieties of $(\mathcal{C}, \mathcal{J}^{(\nabla,I)})$:*

$$1)\ \mathcal{C}_{\leq s}(J) = \{K \in \mathcal{C} |\ Rank[J, K] \leq 2s\}$$
$$2)\ \mathcal{C}_{(\leq r,\pm)}(J) = \{K \in \mathcal{C} |\ Rank(J \pm K) \leq 2r\}.$$

We also have the subvarieties $\mathcal{C}^{\#}(J)$ that correspond to the $M^{\#}$ of Definition 5.2.1 and which will stand for $\mathcal{C}^{(m_1,*)}(J)$, $\mathcal{C}^{(*,m_{-1})}(J)$ and $\mathcal{C}^{(m_1,m_{-1})}(J)$.

Notation 6.2.2. *When referring to the above subvarieties, we will usually drop the "(J)" factors and will denote all of them by $\mathcal{C}^\delta$.*

Using the fact that the $\mathcal{C}^{\#}$ are C^∞ fiber bundles together with the results of Propositions 6.1.8 and 6.2.1, we arrive at the following:

Proposition 6.2.3.

1) The $\mathcal{C}^{\#}$ are complex submanifolds of $\mathcal{C}$ and have the following codimensions:

a) $codim_{\mathbb{C}}\mathcal{C}^{(m_1,*)} = m_1^2$

b) $codim_{\mathbb{C}}\mathcal{C}^{(*,m_{-1})} = m_{-1}^2$

c) $codim_{\mathbb{C}}\mathcal{C}^{(m_1,m_{-1})} = m_1^2 + m_{-1}^2$.

2) $\mathcal{C}_{(\leq r,\pm)}$ is a holomorphic subvariety of $\mathcal{C}$ of codimension $(n-r)^2$.

We can also describe $T_K\mathcal{C}^{\#}(J)$ as follows. First recall that ∇ induces a splitting of $T\mathcal{C}$ into $V\mathcal{C} \oplus H^{\nabla}\mathcal{C}$, where $V_K\mathcal{C} = T_K\mathcal{C}(E_{\pi(K)})$ and $H^{\nabla}\mathcal{C}$ is a certain horizontal distribution (see Lemma 2.2.4). Since $\nabla J = 0$, we have

Proposition 6.2.4. $T_K\mathcal{C}^{\#}(J) = V_K\mathcal{C}^{\#}(J) \oplus H_K^{\nabla}\mathcal{C}$.

Note that $V_K\mathcal{C}^{\#}(J) = T_K\mathcal{C}(E_{\pi(K)})^{\#}(J)$ was already described in Proposition 6.1.10.

Proof of Proposition 6.2.4. By Section 2.2, a general element of $H_K^{\nabla}\mathcal{C}$ is given by $\frac{dK(t)}{dt}|_{t=0}$, where $K(t)$ is the parallel translate of $K = K(0)$ (using ∇) along some curve in M. The proof of the proposition then follows from the fact that since $\nabla J = 0$, $K(t) \in \mathcal{C}^{(m_1,m_{-1})}(J)$ for some (m_1, m_{-1}). □

6.2.2 $\mathcal{T}$-case

Let $(E,g) \longrightarrow (M,I)$ be a real rank $2n$ bundle fibered over a complex manifold and equipped with a fiberwise metric. Also let

- ∇ be a metric connection on E that has (1,1) curvature and
- J be a section of $\pi : \mathcal{T}(E,g) \longrightarrow M$ that satisfies $\nabla J = 0$.

If we consider $\mathcal{T} := \mathcal{T}(E,g)$ with its complex structure $\mathcal{J}^{(\nabla,I)}$ then, similar to the $\mathcal{C}-$case of the previous section, we have

Proposition 6.2.5. *Let $J \in \Gamma(\mathcal{T})$ satisfy $\nabla J = 0$. The following are holomorphic subvarieties of $(\mathcal{T}, \mathcal{J}^{(\nabla,I)})$:*

$$1)\ \mathcal{T}_{\leq s}(J) = \{K \in \mathcal{T} |\ Rank[J,K] \leq 2s\}$$
$$2)\ \mathcal{T}_{(\leq r,\pm)}(J) = \{K \in \mathcal{T} |\ Rank(J \pm K) \leq 2r\}.$$

We also have the subvarieties $\mathcal{T}^{\#}(J)$ that correspond to the $M^{\#}$ of Definition 5.2.1 and which will stand for $\mathcal{T}^{(m_1,*)}(J)$, $\mathcal{T}^{(*,m_{-1})}(J)$ and $\mathcal{T}^{(m_1,m_{-1})}(J)$.

Notation 6.2.6. *When referring to the above subvarieties, we will usually drop the "(J)" factors and will denote all of them by $\mathcal{T}^{\delta}$.*

Note it is straightforward to show that the $\mathcal{T}^{\#}$ are in fact C^{∞} fiber bundles. Using this together with the results of Propositions 6.1.20 and 6.2.5, we arrive at

Proposition 6.2.7.

1) The $\mathcal{T}^{\#}$ are complex submanifolds of $\mathcal{T}$ and have the following codimensions:

a) $codim_{\mathbb{C}}\mathcal{T}^{(m_1,*)} = \frac{m_1(m_1-1)}{2}$

b) $codim_{\mathbb{C}}\mathcal{T}^{(*,m_{-1})} = \frac{m_{-1}(m_{-1}-1)}{2}$

c) $codim_{\mathbb{C}}\mathcal{T}^{(m_1,m_{-1})} = \frac{m_1(m_1-1)+m_{-1}(m_{-1}-1)}{2}$.

2) $\overline{\mathcal{T}^{(m_1,*)}} = \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathcal{T}^{(m_1+2k,*)}$ *and is a holomorphic subvariety of codimension* $\frac{m_1(m_1-1)}{2}$.

3) $\mathcal{T}_{(\leq r,+)} = \mathcal{T}^{(\geq n-r,*)} = \overline{\mathcal{T}^{(n-r,*)}} \cup \overline{\mathcal{T}^{(n-r+1,*)}}$.

Parts 2) and 3) are also true if we were to replace $\mathcal{T}_{(\leq r,+)}$ *with* $\mathcal{T}_{(\leq r,-)}$ *and* $\mathcal{T}^{(m_1,*)}$ *with* $\mathcal{T}^{(*,m_{-1})}$.

Let us now describe $T_K\mathcal{T}^{\#}(J)$. First recall that ∇ induces a splitting of $T\mathcal{T}$ into $V\mathcal{T} \oplus H^{\nabla}\mathcal{T}$, where $V_K\mathcal{T} = T_K\mathcal{T}(E_{\pi(K)})$ and $H^{\nabla}\mathcal{T}$ is a certain horizontal distribution (see Lemma 2.2.5). We then have

Proposition 6.2.8. $T_K\mathcal{T}^{\#}(J) = V_K\mathcal{T}^{\#}(J) \oplus H^{\nabla}_K\mathcal{T}$.

Note that $V_K\mathcal{T}^{\#}(J) = T_K\mathcal{T}(E_{\pi(K)})^{\#}(J)$ was already described in Proposition 6.1.24.

Proof of Proposition 6.2.8. The proof is similar to that of Proposition 6.2.4. $\square$

Chapter 7

General Theorems about the M^δ

7.1 A Twistor Point of View

Let us return to the general setup of Chapter 5 and consider the respective parallel and holomorphic sections J and K of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)}) \longrightarrow (M, I)$. We are now prepared to carry out the set of ideas that were laid out at the end of that chapter—to realize the $M^\delta := \{M_{\leq s}, M_{(\leq r,\pm)}, M^\#\}$ as the intersection of certain complex submanifolds and subvarieties in $\mathcal{C}$ and to use this point of view to derive a number of corollaries about the M^δ.

To begin, we will realize the $M^\#$ as the intersection of complex submanifolds in $\mathcal{C}$: First note that since $\nabla J = 0$, by the previous section the $\mathcal{C}^\#(J)$ are complex submanifolds of $\mathcal{C}$. Secondly, since $K : M \longrightarrow \mathcal{C}$ is holomorphic, we can use it to holomorphically embed M into $\mathcal{C}$. We then have:

$$K(M^\#) = K(M) \cap \mathcal{C}^\#(J)$$

or alternatively

$$M^\# = K^{-1}(\mathcal{C}^\#(J)). \tag{7.1.1}$$

To realize the other M^δ inside of $\mathcal{C}$, note that in the above equations we can respectively replace $M^\#$ with $M_{\leq s}$ or $M_{(\leq r,\pm)}$ and $\mathcal{C}^\#(J)$ with $\mathcal{C}_{\leq s}(J)$ or $\mathcal{C}_{(\leq r,\pm)}(J)$.

If we now equip E with a fiberwise metric g and choose ∇ to be a metric connection and $J, K \in \Gamma(\mathcal{T})$ then it is clear that Equation 7.1.1 and its surrounding discussion would still be true if we were to replace $\mathcal{C}$ with $\mathcal{T}$. This then allows us to view the M^δ as the intersection of complex submanifolds and subvarieties inside the twistor space $\mathcal{T}$.

By using the properties of the subvarieties $\mathcal{C}^\delta$ and $\mathcal{T}^\delta$ which were derived in the previous section, we will now demonstrate two of our main corollaries of this twistorial point of view of the M^δ.

7.2 Bounds on the M^δ

As a first corollary, we will bound the dimensions of the M^δ. We will find that the bounds depend on whether J and K are sections of $\mathcal{C}$ or of $\mathcal{T}$.

7.2.1 $\mathcal{C}-$case

Let us consider the setup of Theorem 5.1.1 where J and K are respectively parallel and holomorphic sections of $(\mathcal{C}(E), \mathcal{J}^{(\nabla,I)}) \longrightarrow (M, I)$. Since $K : M \longrightarrow \mathcal{C}$ is a holomorphic map and, by the above discussion, $M_{(\leq r,\pm)} = K^{-1}(\mathcal{C}_{(\leq r,\pm)}(J))$, it follows from general theory that if $M_{(\leq r,\pm)}$ is nonempty in M then the $codim M_{(\leq r,\pm)} \leq codim \mathcal{C}_{(\leq r,\pm)}(J)$. As we have already determined the codimensions of the $\mathcal{C}_{(\leq r,\pm)}(J)$ in Proposition 6.2.3, we have

Proposition 7.2.1. *Let $dim_{\mathbb{C}} M = m$. If $M_{(\leq r,\pm)}$ is nonempty then the complex dimension of each of its components is $\geq m - (n-r)^2$.*

Remark 7.2.2. *The above proposition can also be proved in another way by applying the following lemma (see [8]) to the first two holomorphic bundle maps given in Proposition 5.1.3.*

Lemma 7.2.3. *Let E and F be two rank n holomorphic vector bundles fibered over a complex manifold N and let $A : E \longrightarrow F$ be a holomorphic bundle map. The complex codimension of*

$$N_{\leq k} = \{x \in N |\ Rank A|_x \leq k\}$$

is $\leq (n-k)^2$.

Either by using an analysis that is similar to the one first used to derive Proposition 7.2.1 or by deriving it directly from that proposition, we have:

Proposition 7.2.4. *Let $dim_{\mathbb{C}} M = m$. If $M^{\#}$ is nonempty then the complex dimension of each of its components is bounded as follows:*

$$1)\ dim M^{(m_1,*)} \geqslant m - m_1^2$$
$$2)\ dim M^{(*,m_{-1})} \geqslant m - m_{-1}^2$$
$$3)\ dim M^{(m_1,m_{-1})} \geqslant m - m_1^2 - m_{-1}^2.$$

7.2.2 $\mathcal{T}$-case

Let us now consider the setup of Section 5.2.1 so that J and K are respectively parallel and holomorphic sections of $(\mathcal{T}(E,g), \mathcal{J}^{(\nabla,I)}) \longrightarrow (M, I)$. We will first focus on the $M_{(\leq r,+)}$. By using Proposition 5.2.4, we may split it into two disjoint open subsets:

$$M_{(\leq r,+)} = M^{(\geq n-r,*)}_{even} \cup M^{(\geq n-r+1,*)}_{even},$$

where

$$M^{(\geq m_1,*)}_{even} := \bigcup_{k \in \mathbb{Z}_{\geq 0}} M^{(m_1+2k,*)}.$$

To bound the dimensions of $M^{(\geq m_1,*)}_{even}$, we will express this subvariety as $K^{-1}(\mathcal{T}^{(\geq m_1,*)}_{even}(J))$, where $\mathcal{T}^{(\geq m_1,*)}_{even}(J)$ is analogously defined and is, by Proposition 6.2.7, a holomorphic subvariety of $\mathcal{T}$. As in the previous section, it then follows that the $codim M^{(\geq m_1,*)}_{even} \leq codim \mathcal{T}^{(\geq m_1,*)}_{even}(J)$ and since we have already determined the codimensions of the $\mathcal{T}^{(\geq m_1,*)}_{even}(J)$ in Proposition 6.2.7, we obtain:

Theorem 7.2.5. *Let* $dim_{\mathbb{C}}M = m$. *If* $M_{even}^{(\geq m_1,*)}$ *is nonempty then the complex dimension of each of its components is* $\geq m - \frac{m_1(m_1-1)}{2}$.

An analogous statement is true for $M_{even}^{(*,\geq m_{-1})}$.

Either by using a similar analysis or by deriving it directly from the above theorem, we obtain:

Theorem 7.2.6. *If* $M^{\#}$ *is nonempty then the complex dimension of each of its components is bounded as follows:*

$$1)\ dimM^{(m_1,*)} \geqslant m - \frac{m_1(m_1-1)}{2}$$
$$2)\ dimM^{(*,m_{-1})} \geqslant m - \frac{m_{-1}(m_{-1}-1)}{2}$$
$$3)\ dimM^{(m_1,m_{-1})} \geqslant m - \frac{m_1(m_1-1)+m_{-1}(m_{-1}-1)}{2}.$$

We will now list some cases that we will focus on in Chapter 8.

Corollary 7.2.7. *The following are bounds on the complex dimensions of some of the* $M^{\#}$*:*

$$1)\ dimM^{(1,2)}, dimM^{(2,1)} \geqslant m-1$$
$$2)\ dimM^{(2,2)} \geqslant m-2$$
$$3)\ dimM^{(2,3)}, dimM^{(3,2)} \geqslant m-4$$
$$4)\ dimM^{(*,2)}, dimM^{(2,*)} \geqslant m-1.$$

Remark 7.2.8. *Note that the bounds given in 1) follow from those in 4) since if* $M^{(1,2)}$ *is nonempty then it is open in* $M^{(*,2)}$.

7.3 The $M^{\#}$ along Curves

We will now apply the twistor point of view of Section 7.1 to derive another corollary about the $M^{\#}$. Unlike the previous one, this corollary will not use the holomorphicity of the twistor space $\mathcal{C}$, rather it will just use the descriptions of $T_K\mathcal{C}^{\#}(J)$ and $T_K\mathcal{T}^{\#}(J)$ that were given in Propositions 6.2.4 and 6.2.8. To describe it, consider the setup of a rank $2n$ real vector bundle $E \longrightarrow M$ equipped with a connection ∇. Also let J and K be sections of $\pi : \mathcal{C}(E) \longrightarrow M$ such that $\nabla J = 0$. Note that we do not impose any conditions on K or on the curvature of ∇; nor do we require M to be even dimensional.

Given x in $M^{\#}$ and $v \in T_xM$, the goal that we are currently working on is to derive necessary and sufficient conditions for there to exist a curve γ in M such that $\gamma'(0) = v$ and $\gamma(t) \in M^{\#}$ for at least small $t \in \mathbb{R}$. We are interested in these conditions because they can be used to derive both upper and lower bounds on the dimensions of the $M^{\#}$ (see the discussion below and [12]).

As a first step, we will now show how to use twistor spaces to give a natural geometrical derivation of certain necessary conditions on $\nabla_v K$.

Proposition 7.3.1. *Let $x \in M^{\#}$ and suppose that there is a curve γ in M such that $\gamma(0) = x$, $\gamma(t)$ lies in $M^{\#}$ for small $t \in \mathbb{R}$ and $\gamma'(0) = v$. Then*

$$\begin{aligned}
&1)\ \textit{for } M^{\#} = M^{(m_1,*)}, \nabla_v K : Ker(J+K) \longrightarrow Im(J+K)\\
&2)\ \textit{for } M^{\#} = M^{(*,m_{-1})}, \nabla_v K : Ker(J-K) \longrightarrow Im(J-K)\\
&3)\ \textit{for } M^{\#} = M^{(m_1,m_{-1})}, \nabla_v K : Ker(J+K) \longrightarrow Im(J+K) \textit{ and}\\
&\qquad\qquad\qquad\qquad\qquad\quad \nabla_v K : Ker(J-K) \longrightarrow Im(J-K).
\end{aligned}$$

Proof. As $\gamma(t) \in M^{\#}$, $K(\gamma(t)) \in \mathcal{C}^{\#}(J)$, so that $K_* v \in T_K \mathcal{C}^{\#}(J)$. Since by Proposition 6.2.4

$$T_K \mathcal{C}^{\#}(J) = V_K \mathcal{C}^{\#}(J) \oplus H_K^{\nabla} \mathcal{C},$$

the vertical projection P^{∇} of $K_* v$ lies in $V_K \mathcal{C}^{\#}(J) = T_K \mathcal{C}(E_x)^{\#}(J)$. Now by Proposition 2.2.6, $P^{\nabla} = \pi^* \nabla \phi$, where $\phi \in \Gamma(\pi^* End E)$ is defined by $\phi|_j = j$, and since

$$\pi^* \nabla_{K_* v} \phi = \pi^* \nabla_{K_* v} (\pi^* K) = \nabla_v K,$$

we find that $\nabla_v K \in T_K \mathcal{C}(E_x)^{\#}(J)$. The proof of the proposition then follows from Proposition 6.1.10. $\square$

Remark 7.3.2. *Although Proposition 7.3.1 can certainly be proved by more direct methods that do not involve twistor spaces, we are currently using the twistor point of view of the $M^{\#}$ to derive stronger results (at least in the case when there are certain differential conditions imposed on K).*

Let us now further suppose that E is equipped with a fiberwise metric g and J and K are sections of $\pi : \mathcal{T}(E, g) \longrightarrow M$. In this case we have derived a result analogous to Proposition 7.3.1. To state it, let us first orthogonally split $E = Im[J, K] \oplus ker(J + K) \oplus ker(J - K)$ at the point $x \in M$ and define P_0, P_1 and P_{-1} to be the corresponding projection operators. We then have:

Proposition 7.3.3. *Let $x \in M^{\#}$ and suppose that there is a curve γ in M such that $\gamma(0) = x$, $\gamma(t)$ lies in $M^{\#}$ for small $t \in \mathbb{R}$ and $\gamma'(0) = v$. Then*

$$\begin{aligned}
&1)\ \textit{for } M^{\#} = M^{(m_1,*)}, P_1(\nabla_v K)P_1 = 0\\
&2)\ \textit{for } M^{\#} = M^{(*,m_{-1})}, P_{-1}(\nabla_v K)P_{-1} = 0\\
&3)\ \textit{for } M^{\#} = M^{(m_1,m_{-1})}, P_1(\nabla_v K)P_1 = 0 \textit{ and } P_{-1}(\nabla_v K)P_{-1} = 0.
\end{aligned}$$

Proof. The proof follows directly from Proposition 7.3.1 and the fact that $Im(J + K) = Im[J, K] \oplus ker(J - K)$. Note that if we also assume that $\nabla g = 0$ then one can alternatively derive the above proposition by replacing $\mathcal{C}$ in the proof of Proposition 7.3.1 with $\mathcal{T}$ and by using the description of $T_K \mathcal{T}(E_x)^{\#}(J)$ given in Proposition 6.1.24. $\square$

It follows that if there exists a $v \in T_x M$ such that $P_1(\nabla_v K)P_1 \neq 0$ then M cannot equal $M^{(m_1,*)}$ along any curve γ that satisfies $\gamma'(0) = v$, i.e., the dimension of $ker(J + K)$ along any such γ must always change. Hence Propositions 7.3.1 and 7.3.3 can be used to derive upper bounds on the dimensions of the $M^{\#}$; see [12] for explicit examples.

We will now consider the holomorphic twistor spaces of bihermitian manifolds and will apply Theorem 7.2.6 and Proposition 7.3.3 to study certain Poisson structures on the manifold.

Chapter 8

Bihermitian Manifolds

8.1 Subvarieties and Bundle Maps

Let (M, g, J_+, J_-) be a bihermitian manifold, as described in Section 3.3.2, so that

$$\nabla^+ J_+ = 0 \text{ and } \nabla^- J_- = 0,$$

where $\nabla^\pm = \nabla^{Levi} \pm \frac{1}{2} g^{-1} H$ for a closed three form H. To build holomorphic subvarieties inside of (M, J_+), first note that by Corollary 3.3.4, the total space of $(\mathcal{T}(TM), \mathcal{J}^{(\nabla^-, J_+)}) \longrightarrow (M, J_+)$ is a complex manifold. Also, since $\nabla^- J_- = 0$, J_- is a parallel section of $\mathcal{T}$ and, by Proposition 3.4.11, J_+ is a holomorphic section. (Note that the holomorphicity condition on J_+ is equivalent to $J_+ \nabla^- J_+ = \nabla^-_{J_+} J_+$, which in turn can be shown to be equivalent to the integrability condition on J_+.) As these results are also true if we were to interchange $+$ with $-$, by Theorem 5.1.1 we have

Proposition 8.1.1. *Let (M, g, J_+, J_-) be a bihermitian manifold. The following are holomorphic subvarieties of M with respect to both J_+ and J_-:*

$$1)\ M_{\leq s} = \{x \in M |\ Rank[J_+, J_-]|_x \leq 2s\}$$
$$2)\ M_{(\leq r, \pm)} = \{x \in M |\ Rank(J_+ \pm J_-)|_x \leq 2r\}.$$

Let us now fix the complex structure J_+ on M. As in the general case of Section 5.1, to derive the holomorphicity of the above subvarieties in (M, J_+), we will first consider the following holomorphic bundles: $T^{1,0}_-$ and $T^{0,1}_-$, equipped with the $\overline{\partial}$-operator $\nabla^{-(0,1)}$, as well as $T^{1,0}_+$ and $T^{0,1}_+$, equipped with the $\overline{\partial}$-operator $(\nabla^-)'^{(0,1)} := \nabla^{-(0,1)} + \frac{1}{2}(\nabla^{-(0,1)} J_+) J_+$. By Proposition 3.4.9, this latter (0,1) connection equals $\nabla^{Ch(0,1)}$, where ∇^{Ch} is the Chern connection on TM that is associated to (g, J_+). (Note that $\nabla^{Ch(0,1)} = \nabla^{-(0,1)}$ on $T^{0,1}_+$.)

By Proposition 5.1.3, we then have:

Proposition 8.1.2. *Let (M, g, J_+, J_-) be a bihermitian manifold. The following are holo-*

morphic maps between the specified bundles that are fibered over (M, J_+):

$$1)\ J_+ + J_- : T_+^{0,1} \longrightarrow T_-^{0,1} \qquad 2)\ J_+ - J_- : T_+^{0,1} \longrightarrow T_-^{1,0}$$
$$3)\ J_+ + J_- : T_-^{1,0} \longrightarrow T_+^{1,0} \qquad 4)\ J_+ - J_- : T_-^{0,1} \longrightarrow T_+^{1,0}$$
$$5)\ [J_+, J_-] : T_+^{0,1} \longrightarrow T_+^{1,0}.$$

With the appropriate holomorphic structures on the bundles, the above statement is also true if we were to interchange $+$ with $-$.

Corollary 8.1.3. *$[J_+, J_-]g^{-1} \in \Gamma(\wedge^2 TM)$ induces a holomorphic section of $\wedge^2 T_+^{1,0} \longrightarrow (M, J_+)$ and of $\wedge^2 T_-^{1,0} \longrightarrow (M, J_-)$.*

Remark 8.1.4. *Note that Corollary 8.1.3 was first derived in [19] by using other methods and thus the holomorphicity of $M_{\leq s}$ is already known in the literature. However, the holomorphicity of the other maps in Proposition 8.1.2 as well as the holomorphicity of $M_{(\leq r, \pm)}$ are new to the literature.* [1]

Remark 8.1.5. *By Proposition 5.1.4, the holomorphicity of $[J_+, J_-]$, as given in Part 5) of Proposition 8.1.2, is independent of the bihermitian condition $dH = 0$.*

8.2 Bounds and Poisson Geometry

Let us consider the $M^{\#}$ in a bihermitian manifold that correspond to those of Definition 5.2.1. It immediately follows that their dimensions are bounded by the ones given in Theorem 7.2.6. The present goal is to use known Poisson structures on M to derive additional bounds on the dimensions of the $M^{\#}$ that are stronger than some of the bounds given in that theorem. At the same time, we will use these latter bounds in Section 8.3 to derive an existence result about certain $M^{\#}$ in $\mathbb{CP}^3$ that cannot be derived by using the bounds from Poisson geometry.

8.2.1 Motivation for Bounds

Before we specify the additional bounds for the $M^{\#}$, we would like to use Proposition 7.3.1 to motivate the reason that one should search for other bounds in the first place. Indeed, one way to obtain lower bounds on the dimensions of an $M^{\#}$ about a smooth point x is to find curves such as γ in M that satisfy $\gamma(0) = x$ and $\gamma(t) \in M^{\#}$ (for at least small $t \in \mathbb{R}$). Although Proposition 7.3.1 does not give sufficient conditions for the existence of such curves, it does give ones that are necessary. To specify them, first consider the following orthogonal and J_+, J_- invariant splitting:

$$T_xM = T_0 \oplus T_1 \oplus T_{-1} := Im[J_+, J_-] \oplus ker(J_+ + J_-) \oplus ker(J_+ - J_-).$$

Letting P_0, P_1 and P_{-1} be the corresponding projection operators, we then have

[1] We note here that Marco Gualtieri has derived the holomorphicity of the $M_{(\leq r, \pm)}$ by using the generalized geometry description of bihermitian geometry.

Proposition 8.2.1. *If there exists a curve γ of the above type and if $\gamma'(0) = v$ then*

1) *for* $M^{\#} = M^{(m_1,*)}$, $P_1[g^{-1}H_v, J_+]P_1 = 0$
2) *for* $M^{\#} = M^{(*,m_{-1})}$, $P_{-1}[g^{-1}H_v, J_+]P_{-1} = 0$
3) *for* $M^{\#} = M^{(m_1,m_{-1})}$, $P_1[g^{-1}H_v, J_+]P_1 = 0$ *and* $P_{-1}[g^{-1}H_v, J_+]P_{-1} = 0$.

Proof. The proof follows from Proposition 7.3.3 and the fact that $\nabla^- J_+ = -[g^{-1}H, J_+]$. □

Remark 8.2.2. *Note that if we found a $w \in T_xM$ such that $P_1[g^{-1}H_w, J_+]P_1 \neq 0$ then there are no curves in M that satisfy $\gamma'(0) = w$ and that lie in $M^{(m_1,*)}$ for at least small $t \in \mathbb{R}$. We will apply this result in [12] to derive upper bounds on the dimensions of the $M^{\#}$ in specific examples.*

Now our main point is that we have proved

Proposition 8.2.3. *The following holds true:*

1) $P_1[g^{-1}H_v, J_+]P_1 = 0 \ \ \forall v \in T_1^{\perp}$
2) $P_{-1}[g^{-1}H_v, J_+]P_{-1} = 0 \ \ \forall v \in T_{-1}^{\perp}$
3) $P_1[g^{-1}H_v, J_+]P_1 = 0$ *and* $P_{-1}[g^{-1}H_v, J_+]P_{-1} = 0 \ \ \forall v \in T_0$.

Proof. First note that by Section 3.3.2, the three form H in the connections $\nabla^{\pm} = \nabla^{Levi} \pm \frac{1}{2}g^{-1}H$ is (1,2) + (2,1) with respect to both J_+ and J_-. This is equivalent to the conditions: $J_+[g^{-1}H, J_+] = [g^{-1}H_{J_+}, J_+]$ and $J_-[g^{-1}H, J_-] = [g^{-1}H_{J_-}, J_-]$.

Now consider

$$\begin{aligned} P_1[g^{-1}H_v, J_+]P_1 &= J_-J_+P_1[g^{-1}H_v, J_+]P_1 \\ &= J_-P_1[g^{-1}H_{J_+v}, J_+]P_1. \end{aligned}$$

As $P_1J_+ = -P_1J_-$, this equals

$$-J_-P_1[g^{-1}H_{J_+v}, J_-]P_1 = P_1[g^{-1}H_{J_-J_+v}, J_+]P_1.$$

Hence

$$P_1[g^{-1}H_{(J_-J_+-1)v}, J_+]P_1 = 0,$$

which proves Part 1) of the proposition; the other parts are proved similarly.

□

Given $v \in T_0$, for instance, it is then natural to ask whether there exists a curve γ in M such that $\gamma'(0) = v$ and $\gamma(t) \in M^{(m_1,m_{-1})}$ for at least small $t \in \mathbb{R}$. The present goal is to show that such curves do indeed exist by using real Poisson geometry.

8.2.2 Background

Before we describe its connection to bihermitian geometry, let us first give some brief background on Poisson geometry. Let (N, λ) be a real manifold that is equipped with a section of $\wedge^2 T$. λ is then a real *Poisson structure* if $\{f,g\} = \lambda(df, dg)$ defines a Lie bracket on $C^\infty(N)$. Given a complex structure on N, one can similarly define a holomorphic Poisson structure.

The fact about the real Poisson manifold (N, λ) that we will need in order to obtain the desired curves in a bihermitian manifold is the following. If we let $v \in Im\lambda_x$, where $\lambda_x : T_x^*N \longrightarrow T_xN$, and extend it to a vector field $\lambda(df)$ then, as λ is Poisson, the flow of this vector field will preserve λ. Hence there exists a curve γ in N such that $\gamma'(0) = v$ and $\gamma(t)$ lies in a constant rank locus of λ for at least small $t \in \mathbb{R}$. If we define $N_k = \{x \in N | \, Rank\lambda_x = k\}$ to be such a locus, we then have

Proposition 8.2.4. *Let (N, λ) be a real Poisson manifold. If N_k is nonempty then $dimN_k \geq k$.*

Remark 8.2.5. *Another, though closely related, way to derive the proposition is to first note that the $Im\lambda$ is an integrable generalized distribution for N. It can then be shown that the leaves of the corresponding foliation lie in the constant rank loci of λ, yielding the bounds given above.*

8.2.3 Poisson Bounds for the $M^\#$

With this background on Poisson geometry, let us return to the bihermitian setup (M, g, J_+, J_-). In this case, the following Poisson structures were found by Lyakhovich and Zabzine [20] as well as Nigel Hitchin [19] (see also [1]).

Proposition 8.2.6. *The following are real Poisson structures on M:*

$$\sigma = [J_+, J_-]g^{-1} \quad \textit{and} \quad \lambda_\pm = (J_+ \pm J_-)g^{-1}.$$

Moreover, if we let σ_+ be the (2,0) component of σ with respect to J_+ then it is a holomorphic Poisson structure on (M, J_+). The same statement is true if we interchange $+$ with $-$.

Consequently, the $M_{\leq s}$ and $M_{(\leq r, \pm)}$ of Proposition 8.1.1 are the degeneracy loci of the above Poisson structures and the $M^\#$ refine the structure of the constant rank loci.

If we reconsider the vector $v \in T_0$ from the end of Section 8.2.1 then we can now understand the reason that there exists a curve γ in M such that $\gamma'(0) = v$ and $\gamma(t)$ lies in $M^{(m_1, m_{-1})}$, for at least small $t \in \mathbb{R}$. The reason is that since σ is Poisson, there exists such a curve that lies in M_s, a constant rank locus of $[J_+, J_-]$, and by Section 5.2, M_s is locally $M^{(m_1, m_{-1})}$ for some (m_1, m_{-1}).

By also considering the other real Poisson structures given in the above proposition together with Proposition 8.2.4, we have derived the following bounds on the $M^\#$.

Proposition 8.2.7. *Let $dim_{\mathbb{C}}M = m$. If $M^\#$ is nonempty then the complex dimension of each of its components is bounded as follows:*

$$1) \dim M^{(m_1,*)} \geqslant m - m_1$$
$$2) \dim M^{(*,m_{-1})} \geqslant m - m_{-1}$$
$$3) \dim M^{(m_1,m_{-1})} \geqslant m - (m_1 + m_{-1}).$$

Remark 8.2.8. *Note that the bounds in 1) and 2) modify some of those in 3). For example, if $M^{(m_1,1)}$ is nonempty then its complex dimension is really $\geqslant m - m_1$ and not just $m - m_1 - 1$. The reason is that by Proposition 5.2.4, $M^{(m_1,1)}$ is open in $M^{(m_1,*)}$. Also note that if $M^{(1,*)}$ is nonempty then its complex dimension is m.*

To compare the above bounds to the ones derived from twistor space, note that only a few of the twistor bounds of Theorem 7.2.6, which we list in Corollary 7.2.7, are stronger than the Poisson bounds given in Proposition 8.2.7. Though, in the next section we will give a corollary of the twistor bounds that cannot be derived by using the Poisson bounds alone.

Remark 8.2.9. *It can be shown that the fact that σ and $\lambda_\pm$ are Poisson structures, as stated in Proposition 8.2.6, does not depend on the bihermitian condition $dH = 0$. Hence the bounds of Proposition 8.2.7 are true regardless of this condition. This is to be compared to the bounds derived from holomorphic twistor spaces (Theorem 7.2.6), where the condition $dH = 0$ was certainly used.*

8.3 Existence of the $M^{\#}$ and the Bounds from $\mathcal{T}$

In the following, we will use the twistor bounds of Theorem 7.2.6 to derive information about the constant rank loci of holomorphic Poisson structures on $\mathbb{CP}^3$ and about the existence of certain $M^{\#}$ in the manifold. If we let $M = \mathbb{CP}^3$ and let I be its standard complex structure, then first note that there exists a holomorphic Poisson structure, $\tilde{\sigma}$, on M that vanishes only on points and complex curves [23]. Now it is possible to show, as described below, that there are bihermitian structures (g, J_+, J_-) on $\mathbb{CP}^3$ such that $J_+ = I$ and the constant rank loci of $[J_+, J_-]$ are the same as those for $Re\tilde{\sigma}$. Given such a structure, our claim is that the associated subvarieties $M^{(2,1)}$ and $M^{(1,2)}$, as defined in Definition 5.2.1, must be empty in M. In particular, M_0, the zero rank locus of $[J_+, J_-]$ and of $\tilde{\sigma}$, must be either $M^{(0,3)}$ or $M^{(3,0)}$ but can never (even locally) be $M^{(2,1)}$ or $M^{(1,2)}$. The reason is that if $M^{(2,1)}$ were to exist then its complex dimension, by Theorem 7.2.6, must be greater than or equal to two. However, the dimension of M_0 is less than or equal to one, and as $M^{(2,1)}$ is contained inside of M_0, it follows that $M^{(2,1)}$ cannot exist. (The same argument is true if we replace $M^{(2,1)}$ with $M^{(1,2)}$.)

Note that the bound for the complex dimension of $M^{(2,1)}$ given in Proposition 8.2.7 is only greater than or equal to one (see Remark 8.2.8); thus we really needed the twistor bounds from Theorem 7.2.6 to come to our conclusion. Lastly, there are methods given, for instance, in [15] that can be used to produce bihermitian structures on $\mathbb{CP}^3$ with the above properties; it can be checked that in these examples, M_0 is indeed equal to $M^{(0,3)}$.

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