Almost-Kähler Anti-Self-Dual Metrics

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Abstract of the Dissertation

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We show the existence of strictly almost-Kähler anti-self-dual metrics on certain 4-manifolds by deforming a scalar-flat Kähler metric. On the other hand, we prove the non-existence of such metrics on certain other 4-manifolds by means of Seiberg-Witten theory. In the process, we provide a simple new proof of the fact that any almost-Kähler anti-self-dual 4-manifold must have a non-trivial Seiberg-Witten invariant.

To my parents

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Chapter 1

Introduction

In this thesis, we study almost-Kähler anti-self-dual metrics on a 4-manifold M. These metrics can be obtained by deforming a scalar-flat Kähler metric. Many examples of scalar-flat Kähler metrics have been constructed by Kim, LeBrun, Pontecorvo [12], [15] and Rollin, Singer [30]. And the deformation theory of such metrics has been studied [20].

In the opposite direction, we show that an almost-Kähler anti-self-dual 4-manifold has a unique solution of the Seiberg-Witten equation for an explicit perturbation form. Combining with Liu's theorem [22], we get useful topological information when we also assume M admits a positive scalar curvature metric. In particular, we show that if a simply connected 4-manifold admits an almost-Kähler anti-self-dual metric and also a metric with positive scalar curvature, then it is diffeomorphic to one of $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ for $n \geq 10$.

Chapter 2

Geometry of almost-Kähler anti-self-dual Metrics

Let M be a compact, oriented smooth 4-dimensional manifold. A smooth fiber-wise linear map $J : TM \to TM$ on M is called an almost-complex structure when J satiesfies $J^2 = -1$. If a smooth manifold M admits J, its structure group reduces to $GL(2, \mathbb{C})$ and we can think of TM as a complex vector bundle. Therefore, Chern classes of (M, J) are well-defined. It is well-known that we have the following relation between the Pontryagin and Chern classes [25]

$$p_1(TM) = c_1^2(TM) - 2c_2(TM).$$

On the other hand, by the Hirzebruch signature theorem, we have

$$\tau(M) = \frac{1}{3}\langle p_1(TM), M\rangle,$$

where $\tau(M)$ is the signature of the manifold. From this, we get

$$c_1^2 = 2\chi + 3\tau,$$

where χ is the Euler characteristic of M.

Let (M, ω) be a symplectic 4-manifold. The symplectic form ω is a closed and nondegenerate 2-form. By this, we mean $d\omega = 0$, and for each $x \in M$ and nonzero

$v \in T_xM$, there exists nonzero $w \in T_xM$ such that $\omega(v, w) \neq 0$. We say ω are compatible with J if

$$\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$$

for each $x \in M$ and $v_1, v_2 \in T_xM$ and $\omega(v, Jv) > 0$ for all nonzero $v \in T_xM$. It is known that the space of almost-complex structures which is compatible with ω is nonempty and contractible [24]. In this case, we can define an associated riemannian metric by

$$g(v, w) = \omega(v, Jw).$$

Then g is a positive symmetric bilinear form and J is compatible with g, that is, $g(v, w) = g(Jv, Jw)$.

Definition 1. *An almost-Kähler structure on a symplectic 4-manifold (M, ω) is a pair (g, J) such that J is compatible with ω and g is defined by $g(v, w) = \omega(v, Jw)$.*

Suppose e_i, $1 \leq i \leq 4$, are orthonormal basis of 1-forms on an oriented riemannian 4-manifold. Then we define $e_i \wedge e_j$, $i < j$, as an orthonormal basis of 2-forms and define the Hodge-star operator by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle vol.$$

It can be checked directly $*^2 = 1$ on 2-forms. According to this, the bundle of 2-forms decomposes as self-dual and anti-self-dual 2-forms

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \tag{1}$$

where Λ^+ is the $+1$ eigenspace of the Hodge-star operator $*$ and Λ^- is the -1 eigenspace of $*$. The Hodge-star operator $*$ is conformally invariant, and therefore this decomposition depends only on the conformal structure of M. Note that a 2-form correspond to a skew-adjoint transformation of T_xM using the metric. In this respect, the decomposition (1) corresponds to the following decomposition of the Lie algebra of $SO(4)$

$$Lie(SO(4)) = Lie(SO(3) \oplus SO(3)).$$

According to this, we have the decomposition of the curvature operator $R : \Lambda^2 \to \Lambda^2$,

$$R = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right)$$

where the first block represents a map from Λ^+ to Λ^+ and $\mathring{r} = r - \frac{s}{4}g$ is the trace-free part of the Ricci curvature r and s is the scalar curvature. We say g is anti-self-dual (ASD) if $W_+ = 0$. This condition depends only on the conformal structure since W_+ invariant under the conformal change of the metric.

Here is one useful way to express an almost-Kähler structure.

Lemma 1. *Let (M, g, ω, J) be an almost-Kähler structure. Then ω is a self-dual harmonic 2-form ω with $|\omega| = \sqrt{2}$.*

Proof. The easiest way to see this is to use an orthonormal basis. Since J is orthogonal with respect to g, there exists an orthonormal basis of the form $\{e_1, e_2 = J(e_1), e_3, e_4 = J(e_3)\}$. Then the corresponding 2-form ω is

$$e_1 \wedge e_2 + e_3 \wedge e_4.$$

This is self-dual and $|\omega| = \sqrt{2}$. Since ω is closed and self-dual, we get

$$d^*\omega = - * d * \omega = - * d\omega = 0,$$

and therefore,

$$\Delta\omega = (dd^* + d^*d)\omega = 0$$

Thus, we can conclude ω is a self-dual harmonic 2-form with $|\omega| = \sqrt{2}$. $\square$

On an oriented riemannian 4-manifold, there is well-known Weitzenböck formula [3] for self-dual 2-forms

$$\Delta\omega = \nabla^*\nabla\omega - 2W_+(\omega, \cdot) + \frac{s}{3}\omega. \tag{2}$$

If we take an inner product with ω in (2), we get

$$< \Delta\omega, \omega >=< \nabla^*\nabla\omega, \omega > -2W_+(\omega, \omega) + \frac{s}{3}|\omega|^2.$$

Using the fact $|\omega| = \sqrt{2}$ and ω is harmonic, we get

$$0 = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - 2W_+(\omega, \omega) + \frac{s}{3}|\omega|^2$$

$$0 = |\nabla\omega|^2 - 2W_+(\omega, \omega) + \frac{2s}{3}$$

Then, if g is an almost-Kähler ASD metric, then this simplifies to the following,

$$0 = |\nabla\omega|^2 + \frac{2s}{3}. \tag{3}$$

This formula tells us that the scalar curvature of an almost-Kähler ASD metric is always nonpositive. Moreover, we can identify the zero set of s with the zero set of $\nabla\omega = 0$. Also, it tells us that the scalar curvature vanishes identically if and only if (M, g, ω, J) is a Kähler manifold. Note that anti-self-dual metric is real analytic in suitable coordinates. Thus, $\nabla\omega$ is also real analytic and therefore, it does not vanish on an open set. Moreover, it can only vanishes on the union of real analytic subvarieties. Thus, we can conclude $s \neq 0$ on an open dense subset and therefore, it's not Kähler on this set.

When we have an almost-complex structure J on M, the complexified tangent vector bundle decomposes as

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1},$$

where

$$T^{1,0} = \{Z = X - iJ(X) \in T^{\mathbb{C}} | X \in TM\}$$

$$T^{0,1} = \{Z = X + iJ(X) \in T^{\mathbb{C}} | X \in TM\}.$$

We define $v^{1,0}$ and $v^{0,1}$ by

$$v^{1,0} = \frac{1}{2}(v - iJv), \;\; v^{0,1} = \frac{1}{2}(v + iJv).$$

There is a natural almost-complex structure J^* on TM^* defined by

$$J^* f^*(v) = f(J(v))$$

Using this, the complexified cotangent bundle decomposes as

$$T^*_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1},$$

where

$$\Lambda^{0,1} = \{f \in Hom(TM, \mathbb{C}) = f(J(X)) = -if(X)\}$$

$$\Lambda^{1,0} = \{f \in Hom(TM, \mathbb{C}) = f(J(X)) = if(X)\}.$$

When we have a metric g such that $g(X,Y) = g(JX, JY)$, it induces a map

$$(TM, J) \to (TM^*, J^*),$$

where $v^* = g(v, \cdot)$. Then we can check this map is complex anti-linear. Then, it can be shown there is a complex-isomorphism

$$T^{1,0} \cong \Lambda^{0,1}.$$

One useful way to characterize $\Lambda^{0,1}$ is that $\Lambda^{0,1}$ is the annihilator of $T^{1,0}$. Suppose $f \in \Lambda^{0,1}$ and $v - iJv \in T^{1,0}$. Then if we evaluate complex linearly, we have

$$f(v - iJv) = f(v) - if(J(v)) = f(v) + i^2 f(v) = 0$$

When we evaluate 2-forms on complexified vectors, we extend it complex linearly on both factors. Accordingly, we define extended metric on $T^{\mathbb{C}}$ by

$$g_{\mathbb{C}}(iX, Y) = ig_{\mathbb{C}}(X, Y)$$

$$g_{\mathbb{C}}(X, iY) = ig_{\mathbb{C}}(X, Y),$$

where $X, Y \in TM$. In this convention, we have

$$g_{\mathbb{C}}(T^{1,0}, T^{1,0}) = 0.$$

Recall ω is related to g by $\omega(X, Y) = g(JX, Y)$. This can be extended complex bilinearly on both factors and therefore, we can think of ω as an element of $\Lambda^2_{\mathbb{C}}$. We claim ω associated with metric $g_{\mathbb{C}}$ is an element of $\Lambda^{1,1}$.

Lemma 2. *Suppose (M, g, ω, J) is an almost-Kähler 4-manifold. Then ω is a (1,1)-form.*

Proof. We use $g = g_{\mathbb{C}}$. It is enough to show that $\omega(X, Y) = 0$ for $X, Y \in T^{1,0}$. Since $JX \in T^{1,0}$ when $X \in T^{1,0}$, the Lemma follows from the fact $g(T^{1,0}, T^{1,0}) = 0$. Or since $\omega(JX, JY) = \omega(X, Y)$, we can conclude $\omega \in \Lambda^{1,1}$. $\square$

Following the outline given in [32], we show that when $d\omega = 0$, the zero set of Nijenhuis tensor N is equal to the zero set of $\nabla\omega$. Then by Armstrong's theorem [1], we can conclude that $\nabla\omega$ should vanish somewhere, or equivalently, scalar curvature s should vanish somewhere unless it satisfies $5\chi + 6\tau = 0$.

Lemma 3. *Suppose (M, g, ω, J) is an almost-Kähler 4-manifold. Then, we have $\nabla\omega \in \Lambda^{2,0} \oplus \Lambda^{0,2} \otimes T^*_{\mathbb{C}}$.*

Proof. It is enough to show that

$$\nabla_X\omega(Y, Z) = 0$$

for $Y \in T^{1,0}$ and $Z \in T^{0,1}$. Using this fact and torsion-free property of the metric, we have

$$\nabla_X\omega(Y, Z) = X\omega(Y, Z) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z)$$

$$= Xg(JY, Z) + g(JZ, \nabla_X Y) - g(JY, \nabla_X Z)$$

$$= g(\nabla_X JY, Z) + g(JY, \nabla_X Z) + g(JZ, \nabla_X Y) - g(JY, \nabla_X Z)$$

$$= ig(\nabla_X Y, Z) - ig(Z, \nabla_X Y) = 0.$$

□

Lemma 4. *Suppose (M, g, ω, J) is an almost-Kähler structure. For $Y, Z \in T^{1,0}$ and $X \in T^{\mathbb{C}}$, we have*

$$(\nabla_X\omega)(Y, Z) = 2ig(\nabla_X Y, Z).$$

And for $Y, Z \in T^{0,1}$, we have

$$(\nabla_X\omega)(Y, Z) = -2ig(\nabla_X Y, Z).$$

Proof. According to the definition, we have

$$(\nabla_X \omega)(Y, Z) = X\omega(Y, Z) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z)$$

$$= Xg(JY, Z) + g(JZ, \nabla_X Y) - g(JY, \nabla_X Z)$$

$$= g(\nabla_X JY, Z) + g(JZ, \nabla_X Y).$$

For, $Y, Z \in T^{1,0}$, we have $JY = iY$ and $JZ = iZ$. Thus, we have

$$(\nabla_X \omega)(Y, Z) = 2ig(\nabla_X Y, Z).$$

For, $Y, Z \in T^{0,1}$, we have $JY = -iY$ and $JZ = -iZ$. Thus, we get

$$(\nabla_X \omega)(Y, Z) = -2ig(\nabla_X Y, Z).$$

$\square$

Recall that the Nijenhuis tensor is a map

$$N : \alpha \to (d\alpha)^{0,2},$$

where $\alpha \in \Lambda^{1,0}$. Thus, we can think of N as a section of $\Lambda^{2,0} \otimes \Lambda^{1,0}$.

By the torsion-free property of the Levi-Civita connection and Lemma 4, for $X, Y, Z \in T^{0,1}$, we have

$$g([X, Y], Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$$

$$= -\frac{1}{2i}(\nabla_X \omega)(Y, Z) + \frac{1}{2i}(\nabla_Y \omega)(X, Z).$$

If the Nijenhuis tensor vanishes, then $[X, Y] \in T^{0,1}$ and therefore, we have $g([X, Y], Z) = 0$. Thus, we get

$$(\nabla_X \omega)(Y, Z) - (\nabla_Y \omega)(X, Z) = 0$$

On the other hand, by definition we have

$$d\omega(X, Y, Z) = (\nabla_X \omega)(Y, Z) - (\nabla_Y \omega)(X, Z) + (\nabla_Z \omega)(X, Y).$$

Thus, if $d\omega = 0$ and N vanishes, we can conclude

$$(\nabla_Z \omega)(X, Y) = 0$$

for $X, Y, Z \in T^{0,1}$. Also, from Lemma 3, we have $(\nabla_X \omega)(Y, Z) = 0$ and $(\nabla_Y \omega)(X, Z) = 0$ for $X, Y \in T^{0,1}$ and $Z \in T^{1,0}$. Then from $d\omega = 0$, we get

$$(\nabla_Z \omega)(X, Y) = 0.$$

Combining these two, we can conclude $(\nabla_Z \omega)(X, Y) = 0$ for $Z \in T^{\mathbb{C}}$ and $X, Y \in T^{0,1}$.

When $N = 0$, we also have $[T^{1,0}, T^{1,0}] \in T^{1,0}$ and therefore $g([X, Y], Z) = 0$ holds for $X, Y, Z \in T^{1,0}$. In the same way, we can show $(\nabla_X \omega)(Y, Z) = 0$ for for $Z \in T^{\mathbb{C}}$ and $X, Y \in T^{1,0}$. Combining with Lemma 3, we can conclude when $d\omega = 0$ holds, the zero set of N is equal to the zero set of $\nabla \omega$.

Armstrong showed a certain topological condition hold when the Nijenhuis tensor is nowhere vanishing. We'll see almost all of our examples do not satisfy this topological condition, and therefore, we can conclude $\nabla_X \omega$ should vanish somewhere. Equivalently, the scalar curvature should vanish somewhere.

Theorem 1. *(Armstrong) Let (M, g, J) be a compact smooth 4-manifold with an almost-complex structure J. If the Nijenhuis tensor N is nowhere vanishing, then we have $5\chi + 6\tau = 0$.*

Proof. We saw $N \in \Lambda^{1,0} \otimes \Lambda^{2,0}$. If there is a nowhere vanishing section of $\Lambda^{1,0} \otimes \Lambda^{2,0}$, then there is also such one of $\Lambda^{0,1} \otimes \Lambda^{0,2}$. And the Euler characteristic of this bundle is zero. Note that on a 4-dimensional manifold, Euler class = second Chern class c_2. Since $\Lambda^{0,1}$ is a rank-2 vector bundle and $\Lambda^{0,2}$ is a line bundle, the second Chern class of the bundle $\Lambda^{0,1} \otimes \Lambda^{0,2}$ can be expressed by

$$c_2(\Lambda^{0,1} \otimes \Lambda^{0,2}) = c_1(\Lambda^{0,2})^0 c_2(\Lambda^{0,1}) + c_1(\Lambda^{0,2}) c_1(\Lambda^{0,1}) + c_1(\Lambda^{0,2})^2 c_0(\Lambda^{0,1}) = 2c_1^2 + c_2.$$

Here we used the fact that $c_1(E) = c_1(\wedge^2 E)$ for a rank 2 vector bundle and $c_1(M) := c_1(T^{1,0}) = c_1(\Lambda^{0,2})$ and $c_2(M) = c_2(\Lambda^{0,1})$. Since we have $c_1^2 = 2\chi + 3\tau$ and $c_2(M) =$ the Euler class, we can conclude that

$$c_2(\Lambda^{0,1} \otimes \Lambda^{0,2})(M) = 2c_1^2 + c_2 = 2(2\chi + 3\tau) + \chi = 5\chi + 6\tau.$$

Thus, when there is a nowhere vanishing section of $\Lambda^{0,1} \otimes \Lambda^{0,2}$, we have

$$c_2(\Lambda^{0,1} \otimes \Lambda^{0,2})(M) = 5\chi + 6\tau = 0.$$

$\square$

Example 1. *In the next chapter, we show that the following examples admit almost-Kähler anti-self-dual metrics for certain values* n*.*

1) $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$*.*

In this case, $\chi = 3 + n$ *and* $\tau = 1 - n$*. Using this, we have*

$$5\chi + 6\tau = 5(3 + n) + 6(1 - n) = 21 - n$$

Thus, except $n = 21$*, we can conclude the scalar curvature of an almost-Kähler ASD metric should vanish somewhere.*

2) $S^2 \times T^2 \# n\overline{\mathbb{C}P^2}$*.*

In this case, we have $\chi = n$ *and* $\tau = -n$*. Thus, we have*

$$5\chi + 6\tau = 5n - 6n = -n$$

Thus, for $n \geq 1$*, the scalar curvature should vanish somewhere.*

3) $S^2 \times \Sigma_{\mathbf{g}} \# n\overline{\mathbb{C}P^2}$*.*

In this case, we have $\chi = -4(\mathbf{g} - 1) + n$ *and* $\tau = -n$*. Thus, we have*

$$5\chi + 6\tau = -20\mathbf{g} + 5n + 20 - 6n = -20\mathbf{g} - n + 20$$

Thus $5\chi + 6\tau$ *is always negative for* $n \geq 0$ *and* $\mathbf{g} \geq 2$*.*

Suppose a smooth 4-manifold M admits an almost-complex structure J and a compatible metric g. Then the complex-valued 2-forms decompose as

$$\Lambda^2 T^*_{\mathbb{C}} = (\Lambda^{2.0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1}.$$

On the other hand, for self-dual 2-forms on an oriented riemannian manifold, we have the following decomposition

$$\Lambda^2 T^*_{\mathbb{C}} = \Lambda^+_{\mathbb{C}} \oplus \Lambda^-_{\mathbb{C}}.$$

As before, we define an associated 2-form ω by $\omega(v, w) = g(Jv, w)$. When it is not required $d\omega = 0$, (M, g, ω, J) is called an almost-Hermitian manifold. In this case, two decompositions are compatible in the following way.

Lemma 5. *Let (M, g, ω, J) be an almost-Hermitian manifold. Then we have*

$$\Lambda^+_{\mathbb{C}} = \mathbb{C}\omega \oplus \Lambda^{2.0} \oplus \Lambda^{0,2},$$

$$\Lambda^-_{\mathbb{C}} = \Lambda^{1,1}_0,$$

where $\Lambda^{1,1}_0$ is the orthogonal complement of ω in the space of $\Lambda^{1,1}$.

Chapter 3

Deformation of scalar-flat Kähler metrics

Many examples of scalar-flat Kähler metrics are known [12], [15], [30] and their deformation theory has been studied [20]. Deformation theory of ASD metrics has been known in a purely differential geometric point of a view on a 4-manifold M using Atiyah-Singer Index theorem.

In this chapter, we study them together at the twistor level using short exact sequences of sheaves. This chapter mainly rely on [20].

In some sense scalar-flat Kähler metrics are special because they are one of extremal Kähler metrics, which is introduced by Calabi and moreover, this metric is especially anti-self-dual [20]. Since this is a central fact in our discussion, we include the proof.

Proposition 1. *Let g be a Kähler metric on a complex surface. Then g is anti-self-dual if and only if its scalar curvature $s = 0$.*

Proof. Let us begin by considering the curvature tensor of a Kähler manifold. On a Kähler manifold, we have $\nabla J = 0$, where ∇ is the Levi-Civita Connection. We claim that $R(X, Y)$ is a $(1, 1)$-form. Since $\nabla J = 0$, we have $R(X, Y)JZ = JR(X, Y)Z$. Using this, it follows that

$$< R(X,Y)JZ, JW >=< JR(X,Y)Z, JW >=< R(X,Y)Z, W >$$

On the other hand, by the first Bianchi identity of the curvature operator, we have

$$< R(JX, JY)Z, W >=< R(Z, W)JX, JY >=< R(Z, W)X, Y >=< R(X, Y)Z, W > .$$

Thus, we can conclude that

$$R \in \Lambda^{1,1} \otimes \Lambda^{1,1}.$$

On the other hand, $\Lambda^{1,1}$ decomposes as

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}.$$

Thus the curvature operator $R|_{\Lambda^+} : \Lambda^+ \to \Lambda^+$ is $\mathbb{R}\omega \to \mathbb{R}\omega$. So the corresponding matrix has only one component, which is the trace of this map. Note that W_+ is the trace-free part of $R|_{\Lambda^+}$ and therefore the trace of $W_+ + \frac{s}{12}I$ is $\frac{s}{4}$. Thus, $R|_{\Lambda^+} : \Lambda^+ \to \Lambda^+$ have the following form.

$$W_+ + \frac{s}{12}I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{s}{4} \end{pmatrix}$$

Therefore, we have

$$W_+ = \begin{pmatrix} -\frac{s}{12} & 0 & 0 \\ 0 & -\frac{s}{12} & 0 \\ 0 & 0 & \frac{s}{6} \end{pmatrix}$$

Thus, $s = 0$ if and only if $W_+ = 0$. $\square$

Let us consider the geometry of scalar-flat Kähler metrics briefly. This will be useful later when they are compared with almost-Kähler ASD metrics. We define the Ricci form ρ by

$$\rho(X,Y) = Ric(JX,Y).$$

Then ρ are real closed (1,1) form. The important theorem on a Kähler manifold is that the first Chern class of the manifold is represented by $\frac{1}{2\pi}\rho$

$$c_1(M) := c_1(K^{-1}) = [\frac{\rho}{2\pi}].$$

We can decompose a $(1,1)$ form φ, by $\varphi = \frac{1}{2}(\Lambda\varphi)\omega + \varphi_0$, where $\Lambda\varphi = \langle\varphi,\omega\rangle$ and $\varphi_0 \in \wedge_0^{1,1}$. From this, we can write the Ricci from ρ as $\rho = \frac{1}{4}s\omega + \rho_0$. Using the fact $[\rho] = 2\pi c_1$ and $d\mu = \frac{[\omega]^2}{2}$, we get

$$2\pi c_1 \cdot [\omega] = [\rho] \cdot [\omega] = \frac{1}{4}[s\omega] \cdot [\omega] = \int_M \frac{1}{2}s \; d\mu$$

$$4\pi c_1 \cdot [\omega] = \int_M s d\mu.$$

This gives us the following proposition.

Proposition 2. *Suppose (M,ω) is a Kähler manifold with the scalar curvature s. Then we have the following identity*

$$\int_M s d\mu = 4\pi c_1 \cdot [\omega]. \tag{4}$$

Yau proved that if $c_1 \cdot [\omega] \geq 0$, or equivalently if the total scalar curvature satisfies

$\int_M s d\mu \geq 0$, then either $\Gamma(M, O(K^m)) = 0, \forall m > 0$ or $c_1 = 0$, where $c_1 \in H^2(M, \mathbb{R})$ [35]. Suppose $u \in \Gamma(M, O(K^m))$. Then Yau showed that

$$vol(D) = -m \int s d\mu,$$

where D is the zero locus of the section u. If we assume $c_1 \cdot [\omega] \geq 0$, then we can conclude $vol(D) = 0$. Thus, u is a nowhere-zero section. This implies K^m is trivial and therefore, $c_1 = 0$. By the same argument, we can conclude if $c_1 \cdot [\omega] = 0$, then $\Gamma(M, O(K^m)) = 0, \forall m \neq 0$ or $c_1 = 0$ [20].

Theorem 2. *(Yau) Let (M, J) be a complex surface which admits a Kahler metric ω such that $c_1 \cdot [\omega] = 0$. If $c_1 \neq 0$, that is, if M is not covered by a complex torus or K3 surface, then M is a ruled surface or its blown up.*

For a scalar-flat Kähler metric, we have $c_1 \cdot [\omega] = 0$ from Proposition 2. Therefore, by Yau's theorem, we can conclude that a scalar-flat Kähler surface is either covered by $K3$ or T^4 or it is a ruled surface or its blown up.

There are well-known obstructions for the existence of constant scalar curvature Kähler metrics. Namely, these are Matsushima-Lichnerowicz obstruction [21], [23] and Futaki obstruction [8], [9]. Briefly, the M-L obstruction tells us that if the scalar curvature of a Kähler manifold is constant, the Lie algebra of its holomorphic vector fields is the same with the complexification of Killing fields up to parallel vector fields.

Theorem 3. *(Matsushima-Lichnerowicz) Suppose (M, ω) is a compact Kähler manifold with constant scalar curvature. Then the space of holomorphic vector fields is* *direct sum of the space of parallel* $(1,0)$ *vector fields and the space of vector fields of the form* $(\bar{\partial}_g f)^\sharp$ *where* f *is the solution of the following equation*

$$\Delta^2 f + 2\langle dd^c f, \rho\rangle = 0.$$

And the imaginary part of $(\bar{\partial}_g f)^\sharp$ *correspond to a Killing field with a zero.*

Let us explain this theorem briefly following [19]. Suppose $g_{\mathbb{C}}$ be a complex-bilinear extension of g. Then we define $(\bar{\partial}_g f)^\sharp$ by

$$g_{\mathbb{C}}(v, (\bar{\partial}_g f)^\sharp) = (\bar{\partial} f)(v).$$

Note that $(\bar{\partial}_g f)^\sharp$ is a $(1,0)$-vector field and it is a holomorphic vector field when it satisfies $\bar{\partial}(\bar{\partial}_g f)^\sharp = 0$. When the scalar curvature s is constant, we have

$$\bar{\partial}(\bar{\partial}_g f)^\sharp = 0 \iff \Delta^2 f + 2\langle dd^c f, \rho\rangle = 0$$

and the latter equation is a real equation. Thus, when a complex-valued function f is a solution of this equation, it's real and imaginary parts are also solutions. And it's proved that when f is a real solution, the imaginary part of $(\bar{\partial}_g f)^\sharp$ is a Killing field with zero and every such Killing filed arises in this way.

Another obstruction for constant scalar curvature Kähler metrics is the Futaki obstruction [8], [9]. Suppose (M, J) is a complex manifold and $H^{1,1}(M, \mathbb{R})^+$ be the cohomology classes of Käher forms on (M, J). By the Hodge theorem, every de-Rham class of 2-forms has a unique harmonic representative. The difference between the ricci form ρ and it's harmonic part ρ_H is expressed by the Ricci potential

$$\rho = \rho_H + dd^c\phi_\omega.$$

Then the Futaki character is defined by

$$\mathcal{F}(\Xi, [\omega]) = \int_M \Xi(\phi_\omega) d\mu,$$

where $\mathcal{F} : \mathfrak{b}(M) \times H^{1,1}(M, \mathbb{R})^+ \longrightarrow \mathbb{C}$ and $\mathfrak{b}(M)$ is the space of holomorphic vector fields on M. It is known that $\mathcal{F}$ depends on $[\omega]$ rather than ω.

By taking an inner product with ω in $\rho = \rho_H + dd^c\phi_\omega$, we can conclude that if s is constant, then $\Delta\phi_\omega = 0$. Then we can conclude ϕ_ω is constant since we have

$$\int_M \langle \delta d\phi_\omega, \phi_\omega \rangle d\mu = \int_M \langle d\phi_\omega, d\phi_\omega \rangle d\mu = 0.$$

If we normalize ϕ_ω further such that $\int \phi_\omega = 0$, then we can conclude the ricci potential $\phi_\omega = 0$ and thus the Futaki character $\mathcal{F}$ is zero.

When our holomorphic vector field is given by $2(\bar{\partial}_g f)^\sharp$ with $\int_M f d\mu = 0$, the Futaki character can be expressed by [20]

$$\mathcal{F}(\Xi, [\omega]) = -\frac{1}{2} \int_M f s d\mu.$$

Proposition 3. *[20] The differentiation of the Futaki character with respect to* ω

has the following formula,

$$\frac{d}{dt}\mathcal{F}(\Xi,[\omega+t\alpha])|_{t=0}=2(f\rho,\alpha_H),$$

where $\Xi = 2(\bar{\partial}_g f)^\sharp$ *with* $\int_M f d\mu = 0$ *and* α *is a closed* $(1,1)$*-form and* α_H *is its harmonic part. Here we denote global* L^2 *inner product as* $(,)$.

Remark 1. *If a compact smooth 4-manifold* M *admits a scalar-flat Kähler metric, then we have* $c_1 \cdot [\omega] = 0$. *Thus, by Yau's theorem, either* $b_+ = 1$ *or* M *is covered* T^4 *or* $K3$. *Suppose* $c_1 \neq 0$. *Since* ω *is a self-dual harmonic 2-form, we can conclude* c_1 *belongs to* H^-. *Thus, we have* $c_1^2 < 0$. *Let us consider* $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$. *Then we have* $\chi = 3+n$ *and* $\tau = 1-n$. *Thus, we have*

$$c_1^2 = 2\chi + 3\tau = 2(3+n)+3(1-n) = 9-n.$$

Thus, we can conclude that $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ *with* $n \leq 9$ *cannot admit scalar-flat Kähler metrics.*

For the existence part, LeBrun constructed explicit scalar-flat Kähler metrics on a ruled surface and its blown up, $S^2 \times \Sigma_{\mathbf{g}}\#n\overline{\mathbb{C}P^2}$ for $n \geq 2$ and $\mathbf{g} \geq 2$ [15] and Kim, Pontercorvo extended this result for $n \geq 1$ [13]. Kim, LeBrun and Pontecorvo constructed scalar-flat Kähler metrics on $\mathbb{C}P^2\#14\overline{\mathbb{C}P^2}$ and $S^2 \times T^2\#n\overline{\mathbb{C}P^2}$ for $n \geq 6$ [12] using the result of Donaldson-Friedman [5] . And Rollin and Singer improved this result on $\mathbb{C}P^2\#10\overline{\mathbb{C}P^2}$ [30]. Note that from the above remark, 10 is the minimum number for which scalar-flat Kähler metrics can exist on $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$. Also Kim, Pontercorvo proved any one-point blow up of a non-minimal scalar-flat Kähler surface also admits a scalar-flat Kähler metric [13]. Therefore, we get the following.

Theorem 4. *[12] [13] [30] Suppose M is diffeomorphic to one of $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ for $n \geq 10$. Then M admits a scalar-flat Kähler metric.*

We study deformation theory of a scalar-flat Kähler metric. We deform this metric in two different categories, namely in scalar-flat Kähler metrics and ASD metrics. The obstruction as ASD metrics lies in

$$Coker DW_+ \cong H^2(Z, \Theta_Z),$$

where Z is the twistor space of M. For a scalar-flat Kähler metric, Pontecorvo's result [29] gives us an additional structure on the twistor space. Before stating this result, let us explain about the twistor space briefly.

A 4-dimensional riemannian manifold M with an ASD metric has its companion complex 3-dimensional manifold Z. This Z is the total space of the sphere bundle of self-dual 2-forms on M and so we have a bundle map $\pi : Z \to M$. The Levi-Civita connection on M induces the connection on TZ. Using this connection, we can split $TZ \cong \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal part which is isomorphic to $\pi^*(TM)$ and $\mathcal{V}$ is the vertical part. Since fibers are 2-spheres, $\mathcal{V}$ has a natural almost-complex structure. Fiber over $x \in M$ is a space of self-dual 2-forms with unit length at x. Given a metric g on M, self-dual 2-forms and almost-complex structure correspond via the map $\omega(v, w) = g(Jv, w)$. Thus, we can think of the fiber over x as the set of all linear maps $J_x : TM_x \to TM_x$ such that $J_x^2 = -1$. Suppose $z \in Z$ and $\pi(z) = x$.

Then $\mathcal{H}_z = \pi^*(TM_x)$ and since z itself represent a J_x in TM_x, we can assign this J_x on $\mathcal{H}_z$. Thus, TZ admits an almost-complex structure. The remarkable fact is that when g is ASD, this almost-complex structure on Z is integrable and therefore, the twistor space Z becomes a complex manifold [2]. In addition to this, we have a fiberwise antipodal map

$$\omega \to -\omega$$

and this gives us a fixed-point free anti-holomorphic involution σ on the total space Z.

Suppose M admits a scalar-flat Kähler metric g. In this case, we have the following Pontecorvo's result [29]. An almost-complex structure J and its conjugate $-J$ give us embeddings of M into Z as complex hypersurfaces and we denote them by Σ and $\overline{\Sigma}$ and their sum by $D = \Sigma + \overline{\Sigma}$. Then the anti-holomorphic involution σ interchanges Σ and $\overline{\Sigma}$ and so we have

$$\sigma(\Sigma + \overline{\Sigma}) = \overline{\Sigma} + \Sigma.$$

Thus, we get a real bundle $D = [\Sigma + \overline{\Sigma}]$. The main result is the following,

$$[D] \cong K_Z^{-\frac{1}{2}}.$$

Conversely, let $Z \to M$ be a twistor fibration and suppose we have a complex hypersurface $\Sigma \subset Z$ which meets every fiber at one point. Then Σ is diffeomorphic to M and we can think of M as a complex surface induced from Σ. Suppose that we have $[D] \cong K_Z^{-\frac{1}{2}}$, where $D = \Sigma + \overline{\Sigma}$. Then there is a metric g in the conformal class $[g]$ such that (M, g, J) is scalar-flat Kähler.

Using this result, we have the following, which is originally discovered by Boyer [4] in a different way.

Theorem 5. *[4], [29] Let M be a compact, smooth manifold with an ASD metric g and assume the first Betti number $b_1(M)$ is even. Suppose there is a complex structure J such that $g(v, w) = g(Jv, Jw)$. Then the conformal class of g has a unique scalar-flat Kähler metric.*

From this theorem, we can conclude that deforming scalar-flat Kähler metrics is equivalent to deforming ASD hermitian conformal structures. And the latter correspond to the deformation of the pair (Z, D) preserving the real structure. Therefore, when obstruction vanishes, the moduli space of the ASD hermitian conformal structures is the real slice of $H^1(Z, \Theta_{Z,D})$.

Lemma 6. *When obstruction vanishes, the deformation of scalar-flat Kähler metrics corresponds to the real slice of $H^1(Z, \Theta_{Z,D})$, where $\Theta_{Z,D}$ is the sheaf of holomorphic vector fields on Z which are tangent to D. And its obstruction lies in $H^2(Z, \Theta_{Z,D})$.*

Remark 2. *Note that deformation of scalar-flat Kähler metrics with a fixed complex structure corresponds to the sheaf $\Theta_Z \otimes \mathcal{I}_D$.*

Here we denote the ideal sheaf of D by $\mathcal{I}_D$. LeBrun and Singer identifies $H^i(Z, \Theta \otimes \mathcal{I}_D)$ using the Penrose transform [20].

Theorem 6. *(LeBrun and Singer) Let M be a compact, smooth 4-manifold with a* *scalar-flat Kähler metric. Let us consider the following elliptic operator.*

$$0 \longrightarrow \Lambda^-(M) \xrightarrow{S} \Lambda^+(M) \longrightarrow 0,$$

where

$$S(\alpha) = d^+\delta\alpha - \frac{1}{2}\langle\rho, \alpha\rangle\omega$$

for $\alpha \in \Lambda^-$. Then we have

$$H^1(Z, \Theta \otimes \mathcal{I}_D) \cong KerS$$

and

$$H^2(Z, \Theta \otimes \mathcal{I}_D) \cong CokerS.$$

The operator S is elliptic and the index of S is $-\tau(M)$.

Remarkably, this $cokerS$ turns out to be closely related with the M-L obstruction and the Futaki character.

Proposition 4. *[20] Suppose (M, ω) is a compact Kähler manifold with a scalar-flat Kähler metric. Assume also M is not Ricci flat. Then $cokerS$ is identified with C^∞ functions such that*

$$\Delta^2 f = -2\langle dd^c f, \rho\rangle$$

and

$$(f\rho, \alpha)_{L^2} = 0,$$

where α is any ASD harmonic 2-form and ρ is the Ricci form.

In particular, if there is no holomorphic vector field, then $cokerS = 0$ and therefore, the obstruction vanishes.

Lemma 7. *(Pontecorvo) Let M be a compact scalar-flat Kähler surface. Then, the normal bundle of $D \subset Z$ is given by*

$$N_D = K_D^{-1},$$

where $D \cong \Sigma + \overline{\Sigma}$ and Z is the twistor space of M.

Proof. Recall by Pontecorvo's theorem, we have

$$[D] \cong K_Z^{-\frac{1}{2}}.$$

On the other hand, by the adjunction formula, the normal bundle and K_D are given by

$$N_D = [D]|_D,$$

$$K_D = (K_Z \otimes [D])|_D.$$

Since $[D] \cong K_Z^{-\frac{1}{2}}$, we have the following,

$$N_D = K_Z^{-\frac{1}{2}}|_D$$

$$K_D = (K_Z \otimes K_Z^{-\frac{1}{2}})|_D = K_Z^{\frac{1}{2}}|_D.$$

Thus, we get

$$N_D \cong K_D^{-1}$$

$\square$

Proposition 5. *[20] Let (M, ω) be a compact scalar-flat Kähler surface and let Z be its twistor space. Suppose $c_1 \neq 0$. If $H^2(Z, \Theta \otimes \mathcal{I}_D) = 0$, then $H^2(Z, \Theta_{Z,D}) = H^2(Z, \Theta_Z) = 0$.*

Proof. By the Serre Duality, we have

$$H^2(M, \Theta_M) \cong H^0(M, \Omega(K)).$$

Note that the normal bundle of a rational curve on a ruled surface is trivial. Using $K_{\mathbb{P}^1} = \mathcal{O}(-2)$ and the adjunction formula, we have

$$K_M|_{\mathbb{P}^1} = \mathcal{O}(-2)$$

Similarly, we have

$$\Omega_M|_{\mathbb{P}^1} = \Omega|_{\mathbb{P}^1} \oplus Id = \mathcal{O}(-2) \oplus Id$$

Thus, we get

$$\Omega(K)|_{\mathbb{P}^1} = (\mathcal{O}(-2) \oplus Id) \otimes \mathcal{O}(-2) = \mathcal{O}(-2) \oplus \mathcal{O}(-4)$$

Since M is a ruled surface and $u \in H^0(M, \Omega(K))$ is zero when restricted on the rational curve, we can conclude that $H^0(M, \Omega(K)) = 0$. Thus, we get $H^2(D, \Theta_D) = 0$, where $D = \Sigma + \overline{\Sigma}$ and Σ is a complex hypersurface in Z induced by a complex structure on M.

By Lemma 7, the normal bundle of D in Z is K_D^{-1}. Also, by Yau's theorem, we have $\Gamma(M, O(K^l)) = 0, \forall l \neq 0$. Then, by the Serre duality we get

$$H^2(D, N_D) = H^2(D, K_D^{-1}) \cong H^0(D, \mathcal{O}(K_D^2)) = 2H^0(M, \mathcal{O}(K_M^2)) = 0.$$

Let us consider the following short exact sequences.

$$0 \longrightarrow \Theta_{Z,D} \longrightarrow \Theta_Z \longrightarrow N_D \longrightarrow 0 \tag{5}$$

$$0 \longrightarrow \Theta_Z \otimes \mathcal{I}_D \longrightarrow \Theta_{Z,D} \longrightarrow \Theta_D \longrightarrow 0. \tag{6}$$

From (6), we get the following long exact sequence,

$$\cdots \longrightarrow H^2(Z, \Theta_Z \otimes \mathcal{I}_D) \longrightarrow H^2(Z, \Theta_{Z,D}) \longrightarrow H^2(D, \Theta_D) \longrightarrow \cdots$$

Thus, if we have $H^2(Z, \Theta_Z \otimes \mathcal{I}_D) = 0$, then we can conclude that

$$H^2(Z, \Theta_{Z,D}) = 0.$$

Similarly, from (5), we get

$$\cdots \longrightarrow H^2(Z, \Theta_{Z,D}) \longrightarrow H^2(Z, \Theta_Z) \longrightarrow H^2(D, N_D) \longrightarrow \cdots$$

Thus, when we have $H^2(Z, \Theta_{Z,D}) = 0$, we get $H^2(Z, \Theta_Z) = 0$. $\square$

Thus, if we can show $Coker S$ vanishes, then we can easily conclude that obstructions of deformation of scalar-flat Kähler metrics and ASD metrics vanish. From the explicit formula of S, it's possible to calculate $Coker S$ in some cases and it's been shown that this actually vanishes for non-minimal scalar-flat Kähler surfaces [12], [20]. Assuming obstruction vanishes, let us then count dimensions. First let us consider the short exact sequence (6). Since $D \cong \Sigma + \overline{\Sigma}$, we have

$$\chi(Z, \Theta_{Z,D}) = \chi(Z, \Theta_Z \otimes \mathcal{I}_D) + 2\chi(M, \Theta_M).$$

When obstruction vanishes, we can think of $-\chi$ as the dimension of the moduli space. We saw index S $=-\tau$, which implies that

$$\chi(Z, \Theta_Z \otimes I_D) = \tau.$$

Thus, the dimension of scalar-flat Kähler metrics is given by

$$-\chi(Z, \Theta_{Z,D}) = -\tau - 2\chi(M, \Theta_M).$$

By the index theorem, $\chi(M, \Theta_M)$ is given by

$$\chi(M, \Theta_M) = \int_M (1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12})(2 + c_1 + \frac{c_1^2 - 2c_2}{2})$$

$$= (\frac{c_1^2 + c_2}{6} + \frac{c_1^2}{2} + \frac{c_1^2}{2} - c_2)(M) = (\frac{7}{6}c_1^2 - \frac{5}{6}c_2)(M)$$

$$= \frac{7(2\chi + 3\tau) - 5\chi}{6} = \frac{9\chi + 21\tau}{6} = \frac{3\chi}{2} + \frac{7\tau}{2}.$$

Therefore, the expected dimension of the moduli space of scalar-flat Kähler metrics is

$$-\chi(Z, \Theta_{Z,D}) = -\tau - 2\chi(M, \Theta_M) = -\tau - (3\chi + 7\tau) = -3\chi - 8\tau$$

Let us consider another short exact sequence (5). By Lemma 7, we have $N_D = K_D^{-1}$. From this, we get

$$\chi(Z, \Theta_Z) = \chi(Z, \Theta_{Z,D}) + 2\chi(M, K_M^{-1})$$

Let us calculate $\chi(M, K_M^{-1})$. This is given by the Riemann-Roch formula,

$$\chi(L) = \chi(\mathcal{O}_M) + \int_M \frac{c_1(L)\,(c_1(L) + c_1(K^{-1}))}{2},$$

where L is any holomorphic line bundle on M. If $L = K^{-1}$, we get

$$\chi(K^{-1}) = \chi(\mathcal{O}_M) + c_1(K^{-1}) \cdot c_1(K^{-1}) = \chi(\mathcal{O}_M) + c_1^2(M).$$

Since $\chi(\mathcal{O}_M) = \frac{c_1^2 + c_2}{12}$, we have

$$\chi(K^{-1}) = \frac{c_1^2 + c_2}{12} + c_1^2.$$

Using $c_1^2(M) = 2\chi + 3\tau$ and $c_2(M) = \chi$, we get

$$\chi(M, K_M^{-1}) = \frac{2\chi + 3\tau + \chi}{12} + 2\chi + 3\tau$$

$$= \frac{1}{4}\chi + \frac{1}{4}\tau + 2\chi + 3\tau.$$

Then we have

$$\chi(Z, \Theta_Z) = \chi(Z, \Theta_{Z,D}) + 2\chi(M, K_M^{-1})$$

$$= 3\chi + 8\tau + \frac{1}{2}\chi + \frac{1}{2}\tau + 4\chi + 6\tau$$

$$= \frac{1}{2}(15\chi + 29\tau).$$

Thus, when obstruction vanishes, we can conclude the dimension of the moduli space of ASD metrics is given by

$$-\frac{1}{2}(15\chi + 29\tau).$$

Theorem 7. *Suppose (M, g, J, ω) is a compact scalar-flat Kähler manifold. Assume it is not Ricci-flat and obstruction of deformations vanish. Then the dimension of the moduli of scalar-flat Kähler metrics is $-3\chi - 8\tau$ and the dimension of the moduli of ASD metrics is $-\frac{1}{2}(15\chi + 29\tau)$.*

Remark 3. *The dimension of the moduli of ASD metrics, $-\frac{1}{2}(15\chi + 29\tau)$, is given first by I. M. Singer from Atiyah-Singer Index theorem [7]. Also the dimension of the moduli of scalar-flat Kähler metrics, $(-3\chi - 8\tau)$, has been known to experts, for example in another version of [13], but it seems it has not been written down in detail.*

In the following lemma, we show that there is a unique almost-Kähler ASD metric in each conformal class which is close to one containing a scalar-flat Kähler metric.

Lemma 8. *Let M be a compact, smooth 4-dimensional manifold with which admits* *a scalar-flat Kähler metric. Assume obstruction of deformation vanishes. If g is a metric which is obtained from deformation of a scalar-flat Kähler metric, then there is a unique almost-Kähler ASD metric in the conformal class of g.*

Proof. We can find a unique metric in the conformal class of g such that the corresponding self-dual 2-form ω' has $|\omega'| = \sqrt{2}$. This ω' is non-degenerate since it is close to the Kähler form. $\square$

Example 2. *Let us consider $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$. It is non-minimal and therefore obstruction vanishes as proved in [20]. The dimension the moduli space of scalar-flat Kähler metrics is given by*

$$-3\chi - 8\tau = -3(3+n) + 8(n-1)$$

$$= -9 - 3n + 8n - 8 = -17 + 5n$$

On the other hand, the dimension of the moduli space of ASD metrics is given by

$$-\frac{(15\chi + 29\tau)}{2} = -\frac{(15(3+n) + 29(1-n))}{2}$$

$$= -\frac{(45 + 15n + 29 - 29n)}{2} = 7n - 37$$

Therefore, we have

$$7n - 37 - (-17 + 5n) \geq 0 \iff n \geq 10$$

and equality holds if and only if $n = 10$. This tells us that for $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, the dimension of the moduli space of almost-Kähler ASD metrics is greater than or equal *to the dimension of the moduli of scalar-flat Kähler metrics if and only if* $n \geq 10$ *and they are equal when* $n = 10$. *Thus, when* $n > 10$, *there exists a strictly almost-Kähler ASD metric.*

Example 3. *Let us consider* $S^2 \times T^2 \# n\overline{\mathbb{C}P^2}$. *Using the fact that* $\chi(S^2 \times T^2) = \chi(S^2)\chi(T^2) = 0$ *and* $\chi(\mathbb{C}P^2) = 3$, *we get*

$$\chi(S^2 \times T^2 \# n\overline{\mathbb{C}P^2}) = \chi(S^2 \times T^2) + n(\chi(\overline{\mathbb{C}P^2}) - 2) = n$$

Since $\tau = -n$, *the dimension of the moduli space of scalar-flat Kähler metrics is given by*

$$-3\chi - 8\tau = -3n + 8n = 5n$$

and the dimension the moduli space of ASD metrics is equal to

$$-\frac{(15\chi + 29\tau)}{2} = -\frac{(15(n) + 29(-n))}{2} = 7n$$

Thus, when $n \geq 1$, *we have a strictly almost-Kähler ASD metric from a deformation if there is a scalar-flat Kähler metric. Note that LeBrun, Kim and Pontecorvo showed the existence of a scalar-flat Kähler metric when* $n \geq 6$ *[12].*

By a similar calculation, we can show that the dimension of the moduli space of almost-Kähler ASD metrics is greater than the one of scalar-flat Kähler metrics in case of $S^2 \times \Sigma_{\mathbf{g}} \# n\overline{\mathbb{C}P^2}$ *for* $\mathbf{g} \geq 2$. *And note that LeBrun showed the existence of scalar-flat Kähler metric explicitly when* $n \geq 2$ *[15] and Kim, Pontecorvo showed the existence of such a metric for* $n \geq 1$ *[13].*

Example 4. *Let us consider $M = S^2 \times \Sigma_{\mathbf{g}}$, where $\mathbf{g} \geq 2$.*

When $S^2 \times \Sigma_{\mathbf{g}}$ admits a standard product Kähler metric, there is a holomorphic vector field. Let us consider proejctivization of a rank 2 holomorphic vector bundle $\mathbb{V}$ over $\Sigma_{\mathbf{g}}$. When $\mathbb{V}$ is a stable vector bundle over $\Sigma_{\mathbf{g}}$, then it can be shown that there is no holomorphic vector field on the ruled surface $P(\mathbb{V}) \to \Sigma_{\mathbf{g}}$ [6]. By Narasimhan-Seshadri theorem [27], there is a flat connection on $P(\mathbb{V})$ and therefore, universal cover of $P(\mathbb{V})$ is $S^2 \times \mathcal{H}^2$. Then locally, $P(\mathbb{V})$ is $S^2 \times \Sigma_{\mathbf{g}}$, where the standard Kähler metric is given. Therefore, $P(\mathbb{V})$ admits a scalar-flat Kähler metric.

On the other hand, suppose $P(\mathbb{V})$ admits a scalar-flat Kähler metric. It is shown in [14] that $P(\mathbb{V})$ is locally riemannian product $S^2 \times \Sigma_{\mathbf{g}}$ with the standard metric. We briefly discuss the proof. Note that in this case, $\tau = 0$, and therefore, we have $b_+ = b_-$ and $W_+ = W_-$. Thus, there is a self-dual harmonic 2-form ω and also an anti-self-dual harmonic 2-form φ. From the equation (2), we can conclude $\nabla\omega = 0$. For an anti-self-dual harmonic form ρ, we have

$$0 =< \nabla^*\nabla\varphi, \varphi > -2W_-(\varphi, \varphi) + \frac{s}{3}|\varphi|^2.$$

Thus, we can conclude $\nabla\varphi = 0$. Since there are two parallel 2-forms, the holonomy is a subgroup of $SO(2) \times SO(2)$ and thus we get the conclusion.

In sum, we can conclude scalar-flat Kähler metrics on $P(\mathbb{V})$ correspond to the following representation up to conjugation.

$$\rho : \pi_1(M) \to SO(3) \times SO(2,1).$$

Note that $\pi_1(M)$ has $2\mathbf{g}$ generators and has 1 relation. Fundamental group $\pi_1(M)$ is expressed by

$$\pi_1(M) =< a_1.b_1, ...a_{\mathbf{g}}, b_{\mathbf{g}} | a_1 b_1 a_1^{-1} b_1^{-1} ... a_{\mathbf{g}} b_{\mathbf{g}} a_{\mathbf{g}}^{-1} b_{\mathbf{g}}^{-1} = 1 > .$$

Thus, the dimension of this representation is

$$6(2\mathbf{g} - 2) = 12\mathbf{g} - 12.$$

We can also count the dimension of the moduli space of scalar-flat Kähler metrics from $3\chi - 8\tau$. When the vector bundle is stable, there is no holomorphic vector field, and so we can deform the scalar-flat Kähler metric on it. In this case, $\tau = 0$, and therefore, the dimension of the moduli space is -3χ. Since $\chi = -4(\mathbf{g} - 1)$, we have

$$-3\chi = 3 \times 4(\mathbf{g} - 1) = 12\mathbf{g} - 12.$$

Also, we saw the dimension of the moduli space of ASD metrics is given by

$$-\frac{1}{2}(15\chi + 29\tau).$$

Again, since $\tau = 0$, it's equal to

$$-\frac{1}{2}15\chi = \frac{1}{2}15 \times 4(\mathbf{g} - 1) = 30(\mathbf{g} - 1).$$

This also can be calculated by considering the corresponding representation. ASD metrics on $S^2 \times \Sigma_{\mathbf{g}}$ or $S^2 \tilde{\times} \Sigma_{\mathbf{g}}$, where $S^2 \tilde{\times} \Sigma_{\mathbf{g}}$ is a twisted product, is conformally flat since $\tau = 0$. The conformal group acts on the universal covering space, $S^4 - S^1$. Then the dimension of ASD metric is the same with the dimension of the following representation space up to conjugation.

$$\rho : \pi_1(M) \to SO(5,1).$$

Since $dimSO(5,1) = 15$, the dimension which comes from this representation is given by $15(2\mathbf{g} - 2)$, which is the same as before.

Chapter 4

Seiberg-Witten invariants

In the previous section, we saw the existence of strictly almost Kähler metrics. In this section, changing the direction, we show the Seiberg-Witten invariant can give us useful topological information for manifolds which admit a strictly almost-Kähler ASD metric and also a metric with positive scalar curvature.

We begin by explaining Seiberg-Witten theory briefly. Suppose a smooth, compact 4-manifold admits an almost-complex structure J and a compatible metric g. Then, the complexified tangent bundle $TM \otimes \mathbb{C}$ decomposes as $T^{1,0} \oplus T^{0,1}$. From the natural embedding $U(2) \hookrightarrow Spin^c(4)$, we get the canonical $Spin^c$-structure and its positive and negative spinor bundles are given by

$$\mathbb{V}_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$$

$$\mathbb{V}_- = \Lambda^{0,1}.$$

These spinor bundles inherits a hermitian inner product from g on M. Also this bundles have Clifford action of $\Lambda^{p,q}$ given by

$$v \cdot (w^1 \wedge \cdots \wedge w^k) = \sqrt{2} v^{0,1} \wedge w^1 \wedge \cdots \wedge w^k - \sqrt{2} \sum_{i=1}^{k} \langle w^i, \overline{v^{1,0}} \rangle w^1 \wedge \cdot \cdot \hat{w^i} \cdot \cdot \wedge w^k,$$

where $\langle , \rangle$ is the hermitian inner product which is complex linear on the first variable and anti linear on the second variable.

For this $Spin^c$-structure, the determinant line bundle is the anti-canonical line bundle $\Lambda^{0,2}$. Suppose a $U(1)$-connection A on K^{-1} be given. Then A on K^{-1} and Spin connection on $\mathbb{S}_\pm$ which is induced by Levi-Civita connection determines a connection on $\mathbb{V}_\pm$ since on the contractible open set, we have [11]

$$\mathbb{V}_\pm \cong \mathbb{S}_\pm \otimes L^{\frac{1}{2}}.$$

Then using this connection on $\mathbb{V}_\pm$, we define the Dirac operator

$$D_A : C^\infty(\mathbb{V}_+) \to C^\infty(\mathbb{V}_-),$$

where D_A is given by

$$D_A : C^\infty(\mathbb{V}_+) \xrightarrow{\nabla A} C^\infty(T^*M \otimes \mathbb{V}_+) \xrightarrow{cl} C^\infty(\mathbb{V}_-).$$

Using orthonormal basis, it is given by $D_A(\Phi) = \Sigma e_i \cdot (\nabla_A \Phi)(e_i)$.

The perturbed Seiberg-Witten equation is defined by

$$\begin{cases} F_A^+ = \sigma(\Phi) + i\epsilon \\ D_A(\Phi) = 0 \end{cases}$$

Here Φ is a section of $C^\infty(\mathbb{V}_+)$ and F_A^+ is the self-dual part of the curvature form of the connection A on K^{-1} and ϵ is a perturbation self-dual 2-form. Since the Lie algebra of $U(1)$ is $i\mathbb{R}$, $F_A^+ \in \Lambda^2 \otimes i\mathbb{R}$ is a purely imaginary self-dual 2-form.

The Seiberg-Witten equation is invariant under the action of the gauge group, $Map(M, S^1)$. In order to get a well-defined invariant, we need to consider the gauge group action. We call (A, Φ) a reducible solution if $\Phi \equiv 0$. Otherwise, we call (A, Φ) an irreducible solution. The gauge group does not act freely on reducible solutions, and therefore, we only consider irreducible solutions.

Let us fix a unitary connection A_0 on K^{-1}. Then for any given unitary connection A on K^{-1}, there is a gauge transformation so that after the gauge transformation on A, we have $A = A_0 + \theta$ and $d^*\theta = 0$. Thus, the moduli space

$$\mathcal{M}_g^* = \{(A, \Phi) \in L_k^p(\Lambda^1) \times L_k^p(\mathbb{V}^+) | D_A\Phi = 0, F_A^+ = \sigma(\Phi) + i\epsilon, \Phi \neq 0\}/Map(M, S^1) \tag{7}$$

can be rewritten as follows,

$$\mathcal{M}_g^* = \{(A, \Phi) | D_A\Phi = 0, F_A^+ = \sigma(\Phi) + i\epsilon, d^*(A - A_0) = 0, \Phi \neq 0\}/U^1 \rtimes H^1(M, \mathbb{Z}),$$

where $U^1 \rtimes H^1(M, \mathbb{Z})$ is a 1-dimensional group. In order to define a well-defined map, we choose the space L_k^p, where $p > 4$ and $k \geq 1$. Here L_k^p denote the Sobolev space

$$L_k^p(\Omega) = \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k\}.$$

Then for generic ϵ, $\mathcal{M}_g^*$ is a smooth manifold of dimension 0. We refer to [31] for proofs in detail.

Lemma 9. *The space of irreducible solutions of the equations*

$$\begin{cases} D_A\Phi = 0 \\ d^*(A - A_0) = 0 \end{cases}$$

in L^p_k-topology is a Banach manifold $\mathcal{B}^p_k$.

Lemma 10. *Let us consider the map*

$$\wp : (A, \Phi) \to F^+ - \sigma(\Phi), \tag{8}$$

where $(A, \Phi) \in \mathcal{B}^p_k$ and $F^+_A - \sigma(\Phi) \in L^p_{k-1}(i\Lambda^+)$. Then for generic $\epsilon \in \Lambda^+$, $\wp^{-1}(i\epsilon)/(U^1 \rtimes H^1(M, \mathbb{Z}))$ is a smooth manifold of dimension 0.

Proof. For index, we need to consider only highest order terms. The index of $d\wp$ is equal to the index of $D_A \oplus d^+ \oplus d^*$. It is known that

$$index_{\mathbb{C}} D_A = \frac{c_1^2(M) - \tau}{8}$$

Thus, over the real, we have

$$index D_A = \frac{c_1^2(M) - \tau}{4}$$

The index of $d^+ \oplus d^*$ is $b^1 - b_+ - 1 = -(\frac{\chi+\tau}{2})$. Thus, the index of $D_A + d^+ + d^*$ is given by

$$\frac{c_1^2(M) - \tau}{4} - \frac{\chi + \tau}{2} = \frac{c_1^2(M) - (2\chi + 3\tau)}{4},$$

which is zero since $c_1^2 = 2\chi + 3\tau$. Note that we always have $\int < d^*\theta, c >= 0$, where c is constant. Thus, the image of $d^*(A - A_0)$ is perpendicular to constants. Thus, it is codimension 1-subspace of L^p_{k-1}. Therefore, we add 1 to this index. Then following [31], it can be shown for generic ϵ, $\wp^{-1}(i\epsilon)$ is a 1-dimensional smooth manifold and therefore $\wp^{-1}(i\epsilon)/(U^1 \rtimes H^1(M, \mathbb{Z}))$ is a smooth manifold of dimension 0.

□

In sum, we have shown that the dimension of moduli space (7) is zero. Using the gauge-fixing Lemma, and the bound of a spinor field Φ which comes from the Seiberg-Witten equation intrinsically, we get compactness of $\mathcal{M}^*$ and we recover regularity from elliptic bootstrapping. Since the dimension of the moduli space is zero and the moduli space is compact, it consists of finite points.

If (A, Φ) is reducible, then $F_A^+ = i\epsilon$. If the orthogonal projection of ϵ onto the self dual harmonic 2-forms is not equal to $-2\pi c_1^+$, then there is no such a solution. But this is a closed condition, and therefore, we can conclude that for a generic 2-form, there is no reducible solution.

Definition 2. *[16] Let g be a smooth Riemannian metric on a 4-dimensional manifold M which admits an almost-complex structure J and let $\epsilon \in L^p_{k-1}(\Lambda^+)$. If $[\epsilon_H] \neq -2\pi c_1^+$, then we say (g, ϵ) is a good pair. Here, ϵ_H is a harmonic part of ϵ and c_1^+ means its projection to self-dual part.*

Definition 3. *[16] We call (g, ϵ) an excellent pair if $i\epsilon$ is a regular value for the map $\wp$.*

When $b_+ > 1$, using standard cobordism argument, it can be shown that the number of solutions of the Seiberg-Witten equation up to gauge equivalence modulo 2 is independent of the choice of the excellent pair. We define this as a Seiberg-Witten invariant. It depends only on the smooth structure of M.

When $b_+ = 1$, it cannot be guaranteed that the SW-invariant is independent of the metric and the perturbation form. The important fact is that there are exactly two path components of the good pairs. By the same argument, the SW-invariant is constant for the same path component. The following remark will be important in our argument.

Lemma 11. *[16] Suppose M is a smooth riemannian 4-manifold which admits an almost-complex structure J and $b_+ = 1$. Assume $c_1(K^{-1}) \neq 0$, and $c_1^2 \geq 0$, where K^{-1} is the anti-canonical bundle. Then $(g, 0)$ is a good pair for any metric g and therefore, all the pairs $(g, 0)$ belongs to the same path component.*

We show that there is a non-trivial solution of the Seiberg-Wittten equation for the pair (g, ϵ), where g is an almost-Kähler anti-self-dual metric and ϵ is an explicit perturbation form, which will be given shortly. Using an almost-Kähler metric, Taubes proved there is only one solution for a large perturbation form [34]. And this unique solution consists of special connection A_0 on the anti canonical bundle K^{-1}, which was discovered independently by Blair and Taubes, and a simple positive spinor $\Phi_0 = (r, 0) \in \mathbb{V}^+$.

LeBurn calculated the curvature form of this connection.

Proposition 6. *[18] Let M be a symplectic 4-manifold with almost-Kähler metric* *g. Then the curvature form of the Blair-Taubes connection A_0 is given by*

$$iF_{A_0} = \eta + \frac{s+s^*}{8}\omega + \frac{s-s^*}{8}\hat{\omega},$$

where s is the scalar curvature and s^* is the star-scalar curvature and $\eta = W^+(\omega)^\perp \in \Lambda^{2,0} \oplus \Lambda^{0,2}$ is orthogonal to ω and $\hat{\omega} \in \Lambda^-$.

From this formula, we get simple expression of the self-dual part of iF_{A_0} when $W^+ = 0$,

$$iF^+_{A_0} = \frac{s+s^*}{8}\omega.$$

By definition, we have

$$s^* = 2R(\omega, \omega)$$

Since $\omega \in \Lambda^+$, from the decomposition of the curvature operator, we get

$$R(\omega, \omega) = W^+(\omega, \omega) + \frac{s}{12}|\omega|^2.$$

Since we have ASD metric, we get

$$s^* = 2R(\omega, \omega) = \frac{s}{3}$$

Thus, $iF^+_{A_0}$ is given in the following way when the metric is almost-Kähler anti-self-dual.

$$iF^+_{A_0} = \frac{s+s^*}{8}\omega = \frac{s+\frac{s}{3}}{8}\omega = \frac{s}{6}\omega. \tag{9}$$

In the Kähler case, we saw that the following identity (4) holds

$$\int_M s d\mu = 4\pi c_1 \cdot [\omega]$$

In the symplectic case, there is a corresponding formula which is discovered by Blair. In this paper, we can prove this identity using the curvature formula of the Blair-Taubes connection.

Lemma 12. *[18] On symplectic 4-manifolds, we have the following identity*

$$\int_M \frac{s+s^*}{2} d\mu = 4\pi c_1 \cdot [\omega].$$

Proof. By Proposition 6, we have

$$iF_{A_0} = \eta + \frac{s+s^*}{8}\omega + \frac{s-s^*}{8}\hat{\omega}$$

Since

$$iF_{A_0} = 2\pi c_1(K^{-1}),$$

we get

$$4\pi c_1 \cdot \omega = 2iF_{A_0} \wedge \omega = \int_M \frac{s+s^*}{4}\omega \wedge \omega = \int_M \frac{s+s^*}{2} d\mu.$$

□

If the metric is almost-Kähler anti-self-dual, then s^* is equal to $\frac{s}{3}$. And therefore, we get

$$4\pi c_1 \cdot [\omega] = \int_M \frac{s+s^*}{2} d\mu = \int_M \frac{s+\frac{s}{3}}{2} d\mu = \int_M \frac{2}{3} s d\mu. \tag{10}$$

On the other hand, by the Weitzenböck formula (3) for a self dual 2-form, we have

$$0 = |\nabla \omega|^2 + \frac{2}{3}s.$$

Thus, the scalar curvature s is non-positive and moreover, $s = 0$ if and only if $\nabla \omega = 0$. Therefore, for a strictly almost-Kähler metric, we have

$$4\pi c_1 \cdot [\omega] = \int_M \frac{2}{3} s d\mu < 0.$$

Lemma 13. *Suppose we have an almost-Kähler ASD metric on a 4-manifold M. Then we have $c_1 \cdot [\omega] \leq 0$. Moreover, $c_1 \cdot [\omega] = 0$ if and only if g is a Kähler metric.*

For an almost-Kähler metric, Taubes showed that the constant section u_0 of $\Lambda^{0,0}$ with unit length satisfies the Dirac equation. The connection A_0 on K^{-1} induces a covariant derivative ∇_{A_0} on $\mathbb{V}_+$ and $\nabla_{A_0} u_0 \in \mathbb{V}_+ \otimes T^*_{\mathbb{C}}$. As it is shown by Taubes [34], A_0 can be chosen so that the following holds

$$\nabla_{A_0} u_0|_{\Lambda^{0,0}} = 0.$$

Lemma 14. *[34] Let A_0 be a connection on K^{-1} for which we have $\nabla_{A_0} u_0 \in T^*_{\mathbb{C}} \otimes \Lambda^{0,2}$. Then $d\omega = 0$ if and only if $D_{A_0} u_0 = 0$, where D_{A_0} is the Dirac operator on $\mathbb{V}_+$.*

Proof. Since $u_0 \in \Lambda^{0,0}$, we have

$$\omega \cdot u_0 = -2iu_{0,}$$

where $\cdot$ is Clifford multiplication. Taking the Dirac operator D_{A_0}, we get

$$D_{A_0}(\omega \cdot u_0) = -2iD_{A_0}u_0$$

By the Leibniz rule, we have

$$D_{A_0}(\omega \cdot u_0) = (D_{A_0}\omega) \cdot u_0 + \sum_{i=1}^{4} e_i \cdot (\omega \cdot (\nabla_{A_0}u_0)(e_i)).$$

Since ω is a self-dual harmonic 2-form and $D_{A_0} = d + d^*$ when restricted on the forms, we have

$$D_{A_0}\omega = d\omega + d^*\omega = 0.$$

Since $(\nabla_{A_0}u_0)(e_i) \in \Lambda^{0,2}$ and $\Lambda^{0,2}$ is a $2i$-eigenspace of ω, we have

$$2iD_{A_0}u_0 = -2iD_{A_0}u_0$$

$$4iD_{A_0}u_0 = 0 \Longrightarrow D_{A_0}u_0 = 0.$$

$\square$

Given $\Phi \in \mathbb{V}_+$, we define $\sigma(\Phi) \in End(\mathbb{V}_+)$ [28] by

$$\sigma(\Phi) : \rho \to \langle \rho, \Phi \rangle \Phi - \frac{1}{2}|\Phi|^2\rho$$

On the other hand, $\Lambda^2_+ \otimes \mathbb{C}$ induces an endomorphism of $\mathbb{V}_+$ by Clifford multiplication. Let us write $\Phi = (\alpha, \beta)$, where $\alpha \in \Lambda^{0,0}$ and $\beta \in \Lambda^{0,2}$. Then we claim the following self-dual 2-form induces $\sigma(\Phi)$,

$$\frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega + \frac{1}{2}(\bar{\alpha}\beta - \alpha\bar{\beta}).$$

This can be checked directly. Here we use $e_i \wedge e_j$, $i < j$ are orthonormal basis and $|dz_i| = |\overline{dz_i}| = \sqrt{2}$. Also note that above 2-form is a purely imaginary self-dual 2-form. Thus, the Seiberg-Witten equation can be written as

$$\begin{cases} F_A^+ = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega + \frac{1}{2}(\bar{\alpha}\beta - \alpha\bar{\beta}) + i\epsilon \\ D_A\Phi = 0 \end{cases}$$

Recall we have the following curvature formula (9) for A_0

$$iF_{A_0}^+ = \frac{s + s^*}{8}\omega = \frac{s + \frac{s}{3}}{8}\omega = \frac{s}{6}\omega.$$

For the spinor solution $\Phi = (r, 0)$, the corresponding self-dual 2-form is given by

$$\sigma(\Phi) = \frac{ir^2\omega}{4}.$$

Therefore, the ϵ corresponding to $(A_0, (r, 0))$ in the Seiberg-Witten equation becomes

$$\epsilon = -(\frac{s}{6} + \frac{r^2}{4})\omega.$$

In the following, we show the Blair-Taubes connection A_0 and positive spinor $\Phi_0 = (r, 0)$ is a unique solution with respect to the almost-Kähler ASD metric g and this particular ϵ.

Theorem 8. *Let M be a symplectic 4-manifold with an almost-Kähler ASD metric g. Then there is a unique non-trivial solution of the Seiberg-Witten equation for the pair (g, ϵ) and for $r \geq \sqrt{\frac{4|s|}{3}}$, where*

$$\epsilon = -(\frac{s}{6} + \frac{r^2}{4})\omega.$$

Proof. Let us consider following perturbed Seiberg-Witten equation

$$\begin{cases} D_A\Phi = 0 \\ F_A^+ = \sigma(\Phi) + i\epsilon. \end{cases}$$

A section of $\Phi \in \mathbb{V}_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ can be written as

$$\Phi = (\alpha, \beta).$$

The Weitzenböck formula for $D_A^* D_A$ is given by

$$D_A^* D_A \Phi = \nabla_A^* \nabla_A \Phi + \frac{s}{4}\Phi + \frac{F_A}{2} \cdot \Phi,$$

where $\cdot$ is Clifford multiplication. Since $\Phi \in \mathbb{V}^+$, we have $\frac{F_A}{2} \cdot \Phi = \frac{F_A^+}{2} \cdot \Phi$. From the Seiberg Witten equation, we get

$$0 = \nabla_A^* \nabla_A \Phi + \frac{s}{4}\Phi + \frac{\sigma(\Phi)}{2} \cdot \Phi + \frac{i\epsilon}{2} \cdot \Phi. \tag{11}$$

The the self-dual 2-form ω acts on $\Phi \in \mathbb{V}_+$ by

$$\omega \cdot \Phi = \omega \cdot (\alpha, \beta) = (-2i\alpha, 2i\beta) = -2i(\alpha, -\beta).$$

We take an inner product with Φ in (11), and using the fact that

$$\langle(\alpha, -\beta), (\alpha, \beta)\rangle = |\alpha|^2 - |\beta|^2,$$

we have

$$0 = \langle \nabla_A^* \nabla_A \Phi, \Phi\rangle + \frac{s}{4}|\Phi|^2 + \frac{|\Phi|^4}{4} - (\frac{s}{6} + \frac{r^2}{4})(|\alpha|^2 - |\beta|^2)$$

Let us write

$$\phi = \sigma(\Phi).$$

Then ϕ is a self-dual 2-form, and ϕ and Φ are related in the following way [17].

$$|\phi|^2 = \frac{1}{8}|\Phi|^4,$$

$$|\nabla \phi|^2 \leq \frac{1}{2}|\Phi|^2 |\nabla \Phi|^2.$$

Since ϕ is a self-dual 2-form and we have an almost-Kähler ASD metric, using the Weitzenböck formula for self-dual 2-form (2), we get

$$\langle \triangle \phi, \phi\rangle = \langle \nabla^* \nabla \phi, \phi\rangle + \frac{s}{3}|\phi|^2 = \frac{1}{2} \triangle |\phi|^2 + |\nabla \phi|^2 + \frac{s}{3}|\phi|^2$$

Since $\int_M \langle \triangle \phi, \phi\rangle d\mu = \int_M \langle (d\delta + \delta d)\phi, \phi\rangle d\mu = \int_M \langle d\phi, d\phi\rangle d\mu + \int_M \langle \delta\phi, \delta\phi\rangle d\mu$, which is non-negative and $\int \triangle|\phi|^2 = 0$, we get

$$\int_M \left(|\nabla\phi|^2 + \frac{s}{3}|\phi|^2\right) d\mu \geq 0$$

Expressing this in terms of Φ, we have

$$\int_M \left(\frac{1}{2}|\Phi|^2|\nabla\Phi|^2 + \frac{s}{24}|\Phi|^4\right) d\mu \geq \int_M \left(|\nabla\phi|^2 + \frac{s}{3}|\phi|^2\right) d\mu \geq 0$$

This means

$$\int_M \left(|\Phi|^2|\nabla\Phi|^2 + \frac{s}{12}|\Phi|^4\right) d\mu \geq 0$$

Note that we can rewrite the Weitzenböck formula in the following way.

$$0 = \int_M \left(\langle \nabla_A^*\nabla_A\Phi, \Phi\rangle + \frac{s}{4}|\Phi|^2 + \frac{|\Phi|^4}{4} + \frac{i\epsilon}{2}\cdot\Phi\right) d\mu$$

$$= \int_M \left(\frac{1}{2}\triangle|\Phi|^2 + |\nabla\Phi|^2 + \frac{s}{4}|\Phi|^2 + \frac{|\Phi|^4}{4} - (\frac{s}{6} + \frac{r^2}{4})(|\alpha|^2 - |\beta|^2)\right) d\mu$$

$$= \int_M \left(|\nabla\Phi|^2 + \frac{s}{12}|\Phi|^2 + \frac{s}{6}|\Phi|^2 + \frac{|\Phi|^4}{4} - (\frac{s}{6} + \frac{r^2}{4})(|\alpha|^2 - |\beta|^2)\right) d\mu$$

By multiplying $|\Phi|^2$, we can conclude

$$\int_M |\Phi|^2\left(\frac{s}{6}|\Phi|^2 + \frac{|\Phi|^4}{4} - (\frac{s}{6} + \frac{r^2}{4})(|\alpha|^2 - |\beta|^2)\right) d\mu \leq 0$$

By expanding terms, we get

$$\int_M |\Phi|^2\left(\frac{s}{6}(|\alpha|^2 + |\beta|^2) + \frac{1}{4}(|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2) - \frac{s}{6}|\alpha|^2 + \frac{s}{6}|\beta|^2 - \frac{r^2}{4}|\alpha|^2 + \frac{r^2}{4}|\beta|^2\right) d\mu \leq 0$$

$$\int_M |\Phi|^2 \left(\frac{s}{3}|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r^2)^2 + \frac{r^2(|\alpha|^2 - r^2)}{4} + \frac{1}{4}|\beta|^4 + \frac{1}{2}|\alpha|^2|\beta|^2 + \frac{r^2}{4}|\beta|^2\right) d\mu \leq 0$$

By subtracting some positive terms, we have

$$\int_M |\Phi|^2 \left(\frac{s}{3}|\beta|^2 + \frac{r^2(|\alpha|^2 - r^2)}{4} + \frac{r^2}{4}|\beta|^2\right) d\mu \leq 0$$

If we choose r so that

$$\frac{s}{3} + \frac{r^2}{4} \geq 0,$$

namely, if

$$r \geq \sqrt{\frac{4|s|}{3}},$$

then we can conclude that

$$\int_M \frac{|\Phi|^2 r^2(|\alpha|^2 - r^2)}{4} d\mu \leq 0$$

Since r is constant, we have

$$\int_M \frac{|\Phi|^2(|\alpha|^2 - r^2)}{4} d\mu \leq 0 \Longrightarrow \int_M |\Phi|^2|\alpha|^2 d\mu \leq r^2 \int_M |\Phi|^2 d\mu$$

Using that fact $|\Phi|^2$=$|\alpha|^2 + |\beta|^2$, we have

$$\int_M |\alpha|^2|\alpha|^2 d\mu \leq \int_M |\Phi|^2|\alpha|^2 d\mu \leq r^2 \int_M \left(|\alpha|^2 + |\beta|^2\right) d\mu.$$

Thus, we get

$$\int_M \left(|\alpha|^2|\alpha|^2 - r^2|\alpha|^2\right) d\mu \leq r^2 \int_M |\beta|^2 d\mu.$$

On the other hand, $|\alpha|^2 - |\beta|^2$ is given by

$$|\alpha|^2 - |\beta|^2 = -2i\langle F_A, \omega\rangle + 4(\frac{s}{6} + \frac{r^2}{4}).$$

Here we used the fact that $F_A^+ = \sigma(\Phi) + i\epsilon$ and $\langle \sigma(\Phi), \omega\rangle = i(\frac{|\alpha|^2 - |\beta|^2}{2})$.

Since c_1 is represented by $[\frac{iF_A}{2\pi}]$ for any connection A, we have $iF_A = iF_{A_0} + d\gamma$ for the Blair-Taubes connection A_0 and for some 1-form γ. Since we know the curvature of the Blair-Taubes connection, we have

$$|\alpha|^2 - |\beta|^2 = -\frac{4s}{6} - 2\langle d\gamma, \omega\rangle + 4(\frac{s}{6} + \frac{r^2}{4}).$$

Thus, we get

$$|\alpha|^2 - |\beta|^2 - r^2 = -2\langle d\gamma, \omega\rangle$$

Since ω is a self-dual harmonic 2-form, by integrating, we have

$$\int_M \left(|\alpha|^2 - r^2\right) d\mu = \int_M \left(|\beta|^2 - 2\langle \gamma, d^*\omega\rangle\right) d\mu = \int_M |\beta|^2 d\mu$$

Using this equality and $\int_M (|\alpha|^2|\alpha|^2 - r^2|\alpha|^2)\, d\mu \leq r^2 \int_M |\beta|^2 d\mu$, we have

$$\int_M \left(|\alpha|^4 - r^2|\alpha|^2\right) d\mu \leq r^2 \int_M \left(|\alpha|^2 - r^2\right) d\mu$$

This implies

$$\int_M (|\alpha|^2 - r^2)^2 d\mu = \int_M \left(|\alpha|^4 - r^2|\alpha|^2 - r^2|\alpha|^2 + r^4\right) d\mu \leq 0.$$

This means $|\alpha|^2 = r^2$ and from the equality $\int_M (|\alpha|^2 - r^2)\, d\mu = \int |_M \beta|^2 d\mu$, we can conclude that $\beta = 0$. Then up to gauge equivalence we have that $\alpha = r$. Since the Dirac equation is invariant under the gauge transformation, we have

$$D_A \Phi_0 = 0,$$

where $\Phi_0 = (r, 0)$. Since $D_A \Phi_0 = D_{A_0} \Phi_0 + \frac{1}{2}\theta \cdot \Phi_0$, where $\cdot$ is Clifford multiplication, we get

$$\theta \cdot \Phi_0 = 0.$$

This implies that $\theta^{0,1} = 0$. On the other hand, since θ is a purely-imaginary 1-form, we can conclude $\theta = 0$. Thus, up to gauge transformation , we get the standard solution $(A_0, (r, 0))$. $\square$

Lemma 15. *Let g be an almost-Kähler ASD metric and assume*

$$\epsilon = -(\frac{s}{6} + \frac{r^2}{4})\omega.$$

Then, (g, ϵ) is an excellent pair.

Proof. We already showed that for this pair, there is a unique solution $(A_0, (r, 0))$ up to gauge equivalence. We show that this solution is nondegenerate. This is equivalent to showing the map $\wp$ in (8) is surjective. We saw the index of this map is 1. Let us consider the linearization of the following equations $\wp$ at $\Big((A_0, (r, 0), -(\frac{s}{6} + \frac{r^2}{4})\omega\Big)$

$$d^*(A - A_0) = 0, D_A \Phi = 0, F_A^+ = \sigma(\Phi) + i\epsilon.$$

Linearization $D\wp$ is onto if and only if dimension of the kernel of $D\wp$ is 1. Suppose $(\theta, (u, \psi)) \in kerD\wp$. We saw that any solution $\Phi = (\alpha, \beta)$ of the equations $D_A\Phi = 0$ and $F_A^+ = \sigma(\Phi) + i\epsilon$ must be $|\alpha|^2 = r^2, \beta = 0$. Thus (u, ψ) satisfies the linearization of the pair of the equations $|\alpha|^2 = r^2$ and $\beta = 0$. The linearization of these equations evaluated at $(r, 0)$ is the following

$$r(u + \bar{u}) = 0, \quad \psi = 0.$$

Note that $u + \bar{u} = 0$ implies that the real part of u is zero. Thus, u is purely imaginary. Also note that from the linearization of the Dirac equation, we get

$$D_{A_0}(u) = -\frac{1}{2}\theta \cdot (r, 0).$$

Since $D_{A_0}(u) = \sqrt{2}\bar{\partial}u$, we get

$$\bar{\partial}u = C\theta^{0,1},$$

where C is an explicit constant. This implies that $du = \theta$. Since $d^*\theta = 0$, we get $d^*du = 0$. By taking an L^2 inner product with u, we can conclude $du = 0$, and therefore, u is constant and $\theta = 0$. Since u is purely imaginary and constant, we get the dimension of ker $D\wp$=1. $\square$

Lemma 16. *Let M be a smooth, compact 4-manifold with a strictly almost-Kähler anti-self-dual metric g. Then $(g, 0)$ and (g, ϵ) are good pairs respectively and they can be path connected through good pairs, and therefore, they belong to the same path component.*

Proof. We can use the Blair-Taubes connection A_0 in order to get $2\pi c_1^+$. Since $b_+ = 1$, we can think of $\frac{\omega}{\sqrt{2}}$ as a basis for $\mathcal{H}^+$. Thus, the harmonic part of iF_A^+ is

$$\left(\int_M \langle \frac{s\omega}{6}, \frac{\omega}{\sqrt{2}} \rangle d\mu\right) \frac{\omega}{\sqrt{2}} = \frac{s_0}{6}\omega,$$

where $s_0 = \int s d\mu$. Since g is a strictly almost-Kähler ASD metric, we have $s_0 < 0$. In particular, this means $2\pi c_1^+ \neq 0$. Thus, $(g, 0)$ is a good pair. We claim, for t such that $0 \leq t \leq 1$, $(g, t\epsilon)$ is a good pair. Suppose not. Then there is $t_0 \in [0, 1]$ such that the following holds

$$\frac{s_0}{6}\omega = t_0(\frac{s_0}{6} + \frac{r^2}{4})\omega.$$

Then we can rewrite this as

$$(1 - t_0)\frac{s_0}{6} = \frac{t_0 r^2}{4}\omega.$$

Since $s_0 < 0$, we can easily check this does not hold. □

From this, we can conclude that $(g, 0)$ and (g, ϵ) belong to the same path component. And therefore, for the chamber which contains $(g, 0)$, where g is a strictly almost-Kähler ASD metric, the SW invariant is non-zero. On the other hand, using the Weitzecböck formula of the Seiberg-Witten equation, it's well-known if M admits a Riemannian metric $\tilde{g}$ with positive scalar curvature, then the SW invariant is zero for the chamber which contains $(\tilde{g}, 0)$.

Lemma 17. *[16] Suppose g is a Riemannian metric on M with positive scalar curvature. Then, for the chamber which contains $(g, 0)$, the SW invariant is zero.*

Therefore, $(g, 0)$ and $(\tilde{g}, 0)$ belong to the different chambers. On the other hand, by Lemma 11, if $c_1 \neq 0$ and $c_1^2 \geq 0$, all the pairs $(g, 0)$ belong to the same chamber. Thus, we can conclude if M admits a strictly almost-Kähler anti-self-dual metric and also a metric with positive scalar curvature, then $c_1^2 < 0$. Liu's theorem [22] tells us about symplectic manifolds which admits a positive scalar curvature metric.

Theorem 9. *(Liu) Let M be a symplectic four manifold. If M admits a positive scalar curvature metric, then M is diffeomorphic to either a rational, ruled surface or its blown up.*

Theorem 10. *Suppose a symplectic 4-manifold (M, ω) admits an almost Kähler ASD metric. If M also admits a metric of positive scalar curvature, then it is diffeomorphic to one of the following,*

$$\begin{cases} \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2} \text{ for } n \geq 10 \\ S^2 \times \Sigma_{\mathbf{g}}, S^2 \tilde{\times} \Sigma_{\mathbf{g}}, \text{ where } \Sigma_{\mathbf{g}} \text{ is a Riemann surface with genus } \mathbf{g} \geq 2 \\ (S^2 \times \Sigma_{\mathbf{g}}) \# n\overline{\mathbb{C}P^2} \text{ for } n \geq 1 \\ (S^2 \times T^2) \# n\overline{\mathbb{C}P^2} \text{ for } n \geq 1 \end{cases}$$

Proof. If a symplectic manifold (M, ω) admits a metric with positive scalar curvature, then $b_+ = 1$. By assumption, (M, ω) admits an almost-Kähler ASD metric g. Then g is either strictly almost-Kähler ASD or scalar-flat Kähler one.

Suppose g is strictly almost-Kähler ASD metric. Then this implies scalar curvature s should be negative somewhere. Then from (10), we can conclude $c_1 \neq 0$. From the argument above, we can conclude $c_1^2 < 0$. Applying this on the list of manifolds of Liu's theorem, we get the conclusion in this case. Note that by Gromov-Lawson [10], all of these examples admit a metric with positive scalar curvature.

Suppose g is a scalar-flat Kähler metric. Then by Yau's theorem, M has either $c_1 = 0$, or it is a ruled surface or its blown up. Suppose $c_1 \neq 0$. Then, M is differeomorphic to one of the following. $S^2 \times S^2$, $S^2 \times T^2$, $S^2 \times \Sigma_{\mathbf{g}}$, where $\mathbf{g} \geq 2$, and their twisted form, and their blown ups. Note that all of these has $b_+ = 1$. Then, from (4), we have $c_1 \cdot [\omega] = 0$. Since $c_1 \neq 0$, and $b_+ = 1$, we have $c_1^2 < 0$. Thus, we also get the conclusion in this case.

Suppose M admits a scalar-flat Kähler metric and $c_1 = 0$. Then, universal cover of M is to either T^4 or $K3$. If M admits a positive scalar curvature metric, then by lifting this metric to T^4 or $K3$, we have a metric of positive scalar curvature on T^4 or $K3$. But it is known that T^4 does not admit such a metric by Schoen and Yau [33], and $K3$ also does not admit such a metric by the Lichnerowicz formula of Dirac operator. Thus, we can conclude $c_1 \neq 0$. $\square$

Proposition 7. *Suppose $\mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}$ admits an almost-Kähler ASD metric. Then $n \geq 10$.*

Proof. In this case, $c_1 \neq 0$. If $n \leq 9$, then $c_1^2 \geq 0$ and therefore, all the pairs $(g, 0)$ belong to the same chamber. Since these manifolds admit a positive scalar curvature metric, we can conclude any almost-Kähler anti-self-dual metric is scalar-flat Kähler. Then, we have $c_1^2 < 0$, which is a contradiction. Thus, we can conclude $\mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}$ for $n \leq 9$ does not admit an almost-Kähler anti-self-dual metric. $\square$

Proposition 8. *Let M be a $K3$ or T^4 as a smooth manifold. If M admits an almost-Kähler anti-self-dual metric g, then g is hyperKähler.*

Proof. For an almost-Kähler anti-self-dual metric, we have $s \leq 0$. Since $c_1 = 0$ on T^4 or $K3$, we can conclude $s = 0$ from the equation (10). Then this implies that g is Kähler. On the other hand, we have

$$c_1^2 = 2\chi + 3\tau = \frac{1}{4\pi^2} \int_M \left(\frac{|s|^2}{24} + 2|W^+|^2 - |ric_0|^2 \right) d\mu.$$

Since g is scalar-flat anti-self-dual, we have $W^+ = s = 0$, and therefore, $ric_0 = 0$. Thus, g is Ricci-flat Kähler. $\square$

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