The Kontsevich Space of Rational Curves on Cyclic Covers of $\mathbb{P}^n$

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Abstract of the Dissertation

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In this thesis we consider the space of rational curves on a smooth cyclic cover of $\mathbb{P}^n$. These varieties are the simplest examples of Fano varieties beyond the classical examples of complete intersections in homogeneous spaces. We show that for a general cyclic cover, the Kontsevich moduli stack of stable curves in X is irreducible and has the expected dimension.

Specifically, let X be a general smooth cyclic cover of $\mathbb{P}^n$ of degree r branched over a divisor of degree rd, and let $\overline{\mathcal{M}}_{0,k}(X, e)$ be the Kontsevich moduli stack of stable rational curves of degree e on X. We show that if $2d(r-1) < n$ then this space is irreducible with dimension

$$e(n - (r-1)d + 1) + n - 3 + k$$

Contents

1 Introduction

In this thesis, all schemes are over $\mathbb{C}$.

1.1 Background and Statement of Results

Fano Varieties We will be interested in curves on Fano varieties.

Definition 1.1.1. A smooth projective variety X is *Fano* if the anticanonical divisor class $-K_X$ is ample.

For a detailed introduction to Fano varieties, see [Kol2] Chapter V. Some examples:

- The only Fano curve is $\mathbb{P}^1$.
- The Fano surfaces are the Del Pezzo surfaces: either $\mathbb{P}^1 \times \mathbb{P}^1$, or the blowup of $\mathbb{P}^2$ at up to 8 points in sufficiently general position.
- A smooth hypersurface of degree d in $\mathbb{P}^n$, with $d \leq n$. More generally, a smooth complete intersection of hypersurfaces of degrees $d_1, ..., d_r$ in $\mathbb{P}^n$ with $\sum d_i \leq n$.
- A variety that is homogeneous under the action of a linear algebraic group. For example, the Grassmannian varieties are Fano.

In any dimension there are only finitely many deformation families of Fano varieties, but the number grows fast with the dimension. In dimension 3, there is a complete classification of Fano varieties into a total of 105 families (Iskovskih [Is] and Mori and Mukai [MM]), while in dimension 4 or greater there is currently no complete classification.

We will consider finite cyclic covers of projective space; that is, smooth varieties admitting an action of $\mathbb{Z}/r\mathbb{Z}$ such that the quotient is $\mathbb{P}^n$. Their construction is considered more carefully in 1.4. Such a cyclic cover is uniquely determined by the branch divisor inside of $\mathbb{P}^n$, and is Fano if the degree of the branch divisor is sufficiently low.

These Fano varieties are among the simplest beyond the basic examples given above. In particular, we note that a cyclic double cover of $\mathbb{P}^n$ branched over a divisor of degree 2 is just a quadric hypersurface in $\mathbb{P}^{n+1}$, in which case our resuts are covered by those of [HRS].

Notation 1.1.2. Let $X \to \mathbb{P}^n$ be a finite cyclic cover. We will write r for the degree of the cover. The degree of the branch divisor must be a multiple of r, so we let d be the integer such that the degree of D is rd.

Rational Curves on Fano Varieties A result of Mori states that there are always many rational curves on a Fano variety:

Theorem 1.1.3 (Mori)**.** *Let X be a Fano variety and p a point on X. Then there is a non-constant rational curve $f : \mathbb{P}^1 \to X$ passing through p.*

As we define in 1.2.4, the *Kontsevich moduli space*

$$\overline{\mathcal{M}}_{0,m}(X, \beta)$$

is a compactification of the space of irreducible rational curves on X that parametrises stable genus zero maps from a curve of genus 0 to X, with m marked points, such that the image in X has homology class β. The *expected dimension* of this space is defined by

$$\langle c_1(T_X), \beta\rangle + \dim(X) + m - 3$$

In many cases when X is a Fano variety, it has been shown that $\overline{\mathcal{M}}_{0,m}(X, \beta)$ has the expected dimension, especially when X is a complete intersection in a homogeneous variety. For example, in [KP] we have the following result:

Theorem 1.1.4. *Let X be a projective homogeneous variety. Then $\overline{\mathcal{M}}_{0,m}(X, \beta)$ is irreducible and smooth of the expected dimension.*

In [HRS] a new induction argument was developed, which gave the following:

Theorem 1.1.5 (Harris, Roth, Starr)**.** *Let $X \subseteq \mathbb{P}^n$ be a general hypersurface of degree d, with $n \geq 3$ and $d < \frac{n+1}{2}$. Then $\overline{\mathcal{M}}_{0,m}(X, e)$ is irreducible of the expected dimension.*

A smooth hypersurface in $\mathbb{P}^n$ is Fano when $d \leq n$, and it is conjectured that $\overline{\mathcal{M}}_{0,m}(X, e)$ has the expected dimension when X is a general hypersurface of degree $d \leq n - 1$. Therefore the result of [HRS] applies to half of all possible degrees. Recent work of Beheshti ([B]) has improved this result to cover all degrees $d < n - 2\sqrt{n}$.

In [F] the same result is proved for certain low-degree hypersurfaces in a Grassmannian:

Theorem 1.1.6 (Findley)**.** *Let X be a degree-d hypersurface in the Grassmannian $G(n, k)$ with*

$$d \leq n - \frac{k(n-k)}{2}$$

and $(d, k, n) \neq (2, 2, 4)$. Then $\overline{\mathcal{M}}_{0,m}(X, \beta)$ is irreducible of the expected dimension.

For an outline of the technique and the results obtained, see [BK].

In this thesis, we consider Fano cyclic covers of projective space, and prove the following result using the same general approach:

Theorem 1.1.7. *Let $X \to \mathbb{P}^n$ be a general smooth r-sheeted cyclic cover of projective space branched over a divisor D of degree rd, and let $\overline{\mathcal{M}}_{0,0}(X,e)$ be the Kontsevich moduli space of rational curves of degree e on X. Suppose that*

$$d \leq \frac{n}{2(r-1)}$$

Then $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible of the expected dimension

$$e(n-(r-1)d+1)-n-3$$

We show in 1.4 that a smooth cyclic cover of $\mathbb{P}^n$ of degree r branched over a divisor of degree rd is Fano exactly when

$$d < \frac{n}{r-1}$$

so our results cover half of the possible values of d for a given n and r.

The author would like to thank his advisor Jason Starr for much patient help and advice.

1.2 Parameter Spaces of Lines

We begin with some background on the moduli spaces that we will use. The simplest example of a moduli space of rational curves is the *Grassmannian.*

Definition 1.2.1. A *flat family of lines* in $\mathbb{P}^n$ over B is a closed subscheme

$$Z \subset B \times \mathbb{P}^n$$

such that the projection $Z \to B$ is flat and projective, and such that for each closed point $p \in B$ the fibre $Z_p \subseteq \mathbb{P}^n$ is a line.

The functor sending a variety B to the set of all flat families over B is represented by a smooth projective variety called the *Grassmannian*, denoted $G(n,1)$. For an explicit contruction showing that $G(n,1)$ is a projective variety, see [GH] Chapter 1.5.

Examples The Grassmannian $G(2,1)$ parametrising lines in $\mathbb{P}^2$ is again $\mathbb{P}^2$. The Grassmannian $G(3,1)$ of lines in $\mathbb{P}^3$ is a quadric hypersurface in $\mathbb{P}^5$ (the Klein quadric).

The Fano Scheme of Lines on a Hypersurface Let X be a smooth hypersurface of degree d in $\mathbb{P}^n$. The lines in X are parametrised by a projective variety, called the *Fano scheme* of lines on X. The Fano scheme may be described as a closed subscheme of the Grassmannian $G(n,1)$, as follows (cf. [EH] Chapter 8).

The space $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))$ parametrises hypersurfaces of degree d in $\mathbb{P}^n$, and $G(n,1)$ parametrises lines in $\mathbb{P}^n$. Define the *universal hypersurface* $\Phi \subseteq \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d)) \times G(n,1)$ by

$$\Phi = (X, L : X \text{ a hypersurface of degree } d, \text{ and } L \text{ a line in } X)$$

This is a closed subscheme of $\Phi \subseteq \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d)) \times G(n,1)$, as can be checked locally. Let pr_1, pr_2 be the projections of Φ onto $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))$ and $G(n,1)$ respectively.

Definition 1.2.2. Let X be a hypersurface of degree d in $\mathbb{P}^n$, represented by a point p_X in $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))$. We define the *Fano scheme* $F(X)$ of lines in X to be the fibre

$$F(X) = pr_1^{-1}(p_X) \subseteq G(n,1)$$

The Fano scheme represents the functor that sends a variety B to the set of flat families $Z \subseteq B \times X$ such that the fibre over each closed point of B is a line in X.

Example When $X = \mathbb{P}^n$, the Fano scheme of lines in X is the Grassmannian $G(n,1)$.

Example Let X be a smooth quadric hypersurface in $\mathbb{P}^n$, which we consider as the projectivisation of some $n+1$ dimensional vector space V. We consider the equation of the quadric hypersurface as describing a bilinear form on V. Then X itself is the set of all isotropic vectors through the origin in V, and the Fano scheme of lines in X is the subscheme of the Grassmannian $G(n+1,2)$ parametrising totally isotropic planes - the *orthogonal Grassmannian.*

When $n = 3$, the Fano scheme of lines on the surface X is $\mathbb{P}^1 \times \mathbb{P}^1$; when $n \geq 4$, the Fano scheme is a connected homogeneous projective variety. See [FH] Chapter 18.

We will see that a smooth cyclic double cover of$\mathbb{P}^n$ branched over a quadric hypersurface is, in fact, isomorphic to a quadric hypersurface in $\mathbb{P}^{n+1}$. Therefore our results on cyclic covers reduce in the simplest case to this example.

The Kontsevich Moduli Space To study rational curves on a variety X, it is natural to consider the corrresponding moduli space. If X is a smooth variety then there is a quasiprojective variety whose points correspond to rational curves $f : \mathbb{P}^1 \to X$ such that f is a birational isomorphism (that is, not a constant map or a multiple cover of its image). There are several ways to compactify this moduli space; in all cases, the points in the compactification parametrise curves that are degenerate in some way.

Definition 1.2.3. Let X be a smooth projective variety. An *m-pointed stable map to X* is a map $f : C \to X$ from a connected reduced curve C together with a set of marked points $p_1, ..., p_m$ on C, satisfying the following conditions:

- C has at worst nodal singularities.
- Every irreducible component of C of genus 0 contracted by f contains at least three nodes or marked points.
- Every irreducible component of genus 1 contracted by f contains at least one node or marked point.

Note that if C is irreducible then an m-pointed stable curve is just a curve with a free choice of m points.

Definition 1.2.4. For all positive integers m, g and all curve classes $\beta \in H^2(X, \mathbb{Z})$ on X, the *Kontsevich moduli stack*

$$\overline{\mathcal{M}}_{g,m}(X, \beta)$$

is the stack parametrising families of stable m-pointed curves $f : C \to X$ on X, of genus g, whose image $f(C)$ has class β.

In this thesis we will only be concerned with the case of rational curves, so we will always take $g = 0$. Moreover, we will be considering the case when X is a cyclic cover of projective space. We show in 1.4.2 that in this case $\mathrm{Pic}(X) = \mathbb{Z}$, and so $H^2(X, \mathbb{Z})$ is generated by the the curve class of a line L. In cases such as this when $\mathrm{Pic}\, X = \mathbb{Z}$, we define for any positive integer e

$$\overline{\mathcal{M}}_{0,m}(X, e) := \overline{\mathcal{M}}_{0,m}(X, e \cdot L)$$

For many choices of X, the dimension of the Kontsevich space depends only on the curve class and the first Chern class of the tangent bundle of X.

Definition 1.2.5. The *expected dimension* of $\overline{\mathcal{M}}_{0,m}(X, \beta)$ is

$$\langle c_1(T_X), \beta \rangle + \dim(X) + m - 3$$

The Kontsevich moduli stack was first defined in [Kon]. For a detailed proof of the existence and properties of this stack, see [FP]. We summarise the results here.

Theorem 1.2.6. *The space $\overline{\mathcal{M}}_{0,m}(X, \beta)$ is a Deligne-Mumford stack. If $\overline{\mathcal{M}}_{0,m}(X, \beta)$ has the expected dimension is a local complete intersection, and so is Cohen-Macaulay. In particular, a map from $\overline{\mathcal{M}}_{0,m}(X, \beta)$ to a smooth* *variety is flat if it has constant fibre dimension equal to the expected fibre dimension.*

Furthermore, let V be the variety parametrising maps $f : \mathbb{P}^1 \to X$, up to automorphism of $\mathbb{P}^1$, that are birational to their image. Then V is an open substack of $\overline{\mathcal{M}}_{0,0}(X, \beta)$.

Example The Grassmannian $G(n, 1)$ is the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, 1)$.

Example The simplest non-trivial modui space of stable maps is the space of *completed conics.* As motivation, consider the problem of determining how many conics in $\mathbb{P}^2$ are tangent to five lines $L_1, ..., L_5$ in general position. The moduli space of conics in $\mathbb{P}^2$ is $\mathbb{P}^5$, and tangency to a line L_i is a quadratic condition on the coefficients of a conic. Therefore the conics tangent to five lines correspond to points in the intersection of five quadric hypersurfaces $\lambda_1, ..., \lambda_5$ in $\mathbb{P}^5$, and we might assume that for general lines the intersection $\lambda_1 \cap ... \cap \lambda_5$ consists of $2^5 = 32$ points, by Bezout's theorem. However, the correct answer is that there is only one conic tangent to five given lines in $\mathbb{P}^2$.

To explain this discrepancy, notice that the space of conics includes the space of double lines, parametrised by some closed subscheme $D \subseteq \mathbb{P}^5$ isomorphic to $\mathbb{P}^2$, and that every double line intersects each of our five lines with multiplicity two; that is, any double line is "tangent" to any other line L_i. So even if the five lines are in general position, the five quadric hypersurfaces $\lambda_1, ..., \lambda_5$ all contain D. Since the hypersurfaces do not intersect transversely, our calculation is invalid.

We can remedy this by bowing up the subscheme of double lines. Indeed, the parameter space of conics $\mathbb{P}^5$ contains a *boundary divisor* B parametrising singular conics - those that are the union of two lines. Inside of B, the space D of doube lines has codimension 2, rather than codimension 1 as we might expect. This corresponds to the fact that as two transversely-intersecting lines are rotated to a double line, the information of the intersection point is lost.

By blowing up, we restore this information, and the intersection $\lambda_1 \cap ... \cap \lambda_5$ becomes a single point. The blowup of $\mathbb{P}^5$ along D is the space of completed conics, and it is easy to see that it is exactly the moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$.

1.3 Outline of Proof

Throughout, we will use the following notation. The map $\pi : X \to \mathbb{P}^n$ will be a smooth cyclic cover of degree r. That is, the space X admits an action of the cyclic group $G = \mathbb{Z}/r\mathbb{Z}$, and π exhibits $\mathbb{P}^n$ as the quotient variety of X by the action of G. The map π will be branched over a smooth divisor

$D \subseteq \mathbb{P}^n$, and the degree of D will be a multiple of r; let d be the integer such that the degree of D is rd.

Our proof of the irreducibility of $\overline{\mathcal{M}}_{0,0}(X,e)$ follows an induction argument first put forward in [HRS], where a similar result was proved for hypersurfaces in $\mathbb{P}^n$. The new results in this thesis are mostly concerned with establishing the dimension of $\overline{\mathcal{M}}_{0,0}(X,1)$, the space of lines ($e = 1$). Once this is established, the argument of [HRS] is adapted to deduce the result for curves of all degree.

The space $\overline{\mathcal{M}}_{0,1}(X,e)$, parametrizing curves on X with a marked point, comes with an evaluation map to X. The key step in our proof is that the evaluation map is flat almost everywhere, and fails to be flat in a very limited manner:

Theorem 1.3.1. *Let X be a general smooth cyclic cover of $\mathbb{P}^n$ of degree r, branched over a hypersurface of degree rd. Suppose that $d \leq \frac{n}{2(r-1)}$. Then the evaluation map*

$$ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$$

is flat away from a finite set of points, none of which lie on the branch divisor D. The fibre dimension at every flat point is the expected dimension $e(n-(r-1)d+1)-2$, while the fibre dimension over every non-flat point is $e(n-(r-1)d+1)-1$

In section 2 we prove this theorem for lines, when $e = 1$. We first relate lines on X to lines on $\mathbb{P}^n$ satisfying a certain tangency condition with the branch divisor, and we describe the parameter space of such lines as a closed subscheme of a Grassmannian. Inside this subscheme we have the sublocus of lines that lie entirely within the branch divisor. The dimension of the space of lines in a general branch divisor can be computed with an incidence variety argument; by considering how this sublocus sits within the space of lines on X, we compute the dimension of the fiber $ev^{-1}(p)$ for any p in the branch divisor.

Next, we use an indirect argument to limit the fibre dimension of the evaluation map at non-flat points, for a general cyclic cover. If a family of cyclic covers contained points at which the fibre dimension increased by more than one, then some member of the family would exhibit the same behavior on a point of the branch divisor. We consider the total space of the universal family of hypersurfaces in $\mathbb{P}^n$, and compute that this bad behavior occurs in sufficiently high codimension; therefore a general family of cyclic covers contains no such member.

In section 3 we extend this to the case of degree e curves on X, computing the dimension of $\overline{\mathcal{M}}_{0,0}(X,e)$. Again, we prove that the evaluation map

$$ev_e : \overline{\mathcal{M}}_{0,1}(X,e) \to X$$

is flat away from finitely many points. For $e \geq 2$, the space $\overline{\mathcal{M}}_{0,1}(X, e)$ contains a boundary divisor of reducible curves, whose dimension we know by induction on e. The key result in this step is Mori's bend-and-break theorem, which states that a positive-dimensional family of curves through two fixed points must contain reducible curves in codimension one. This allows us to relate the dimension of a fibre $ev_e^{-1}(p)$ to the dimension of its intersection with the boundary divisor.

Bend-and-break results only apply to positive-dimensional families, so applying this induction argument requires a sufficiently high lower bound on the dimension of $\overline{\mathcal{M}}_{0,0}(X, e)$ for all $e > 1$. This is the source of our restriction $2d(r-1) \leq n$.

Finally, in sections 4 and 5 we show that these spaces are irreducible. The boundary divisor of $\overline{\mathcal{M}}_{0,0}(X, e)$ parametrises reducible curves, and admits a decomposition - the *Behrend-Manin decomposition* - whose components correspond to combinatorial arrangements of lower-degree curves. In particular we have the sublocus parametrising *trees* - reducible curves that are a union of lines.

We show that every irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$ contains a reducible curve that is a union of lines (a tree). The locus of such trees is not irreducible: its irreducible components correspond to the isomorphism classes of the dual graph of the tree. However, we argue that a tree of one isomorphism class deforms into a tree of another while remaining within a single irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$. From this we deduce the irreducibility of $\overline{\mathcal{M}}_{0,0}(X, e)$.

1.4 Cyclic Covers

In this section we say a little about the structure of cyclic covers, and perform two computations that will be necessary later. Firstly, we describe the relationship between lines on X and lines on $\mathbb{P}^n$, characterising those lines in $\mathbb{P}^n$ that lift to reducible curves in X. Secondly, we compute a lower bound for the dimension of the space of degree e curves on X.

Good descriptions of the structure of cyclic covers can be found in [P] and [AV]. We recall their construction here. Let $\pi : X \to \mathbb{P}^n$ be a finite morphism that exhibits $\mathbb{P}^n$ as the quotient of X by the action of a finite cyclic group $\mathbb{Z}/r\mathbb{Z}$. That is, $\mathbb{P}^n$ is covered by open affine neighbourhoods $\operatorname{Spec} A$ whose preimage in X is an affine neighbourhood $\operatorname{Spec} A[z]/(z^r - g)$, for some non-zero divisor g. The group $\mathbb{Z}/r\mathbb{Z}$ acts on z as multiplication by ζ, an rth root of unity.

Suppose that $n \geq 2$ and that X is smooth, so that the branch divisor D is irreducible. The action of $\mathbb{Z}/r\mathbb{Z}$ induces a splitting of $\pi_* \mathcal{O}_X$ into the direct sum of eigensheaves

$$\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \cdots \oplus \mathcal{L}^{r-1}$$

where $\mathcal{L}$ is a positive line bundle. Choose coordinates so that $\mathbb{P}^n = \operatorname{Proj} k[x_0, ..., x_n]$; then this coherent sheaf corresponds to a free graded $k[x_0, ..., x_n]$-module of rank r. If we write e_i for the module generator of $\mathcal{L}^i$, then the algebra structure is detemined by the relations

$$e_1^i = e_i \qquad e_1^r = g$$

for some $g \in \mathcal{O}_X$. If d is the degree of $\mathcal{L}$, then g is a homogeneous polynomial of degree rd that cuts out the branch divisor of π.

We will write $D_{\mathbb{P}^n}$ for the branch divisor in $\mathbb{P}^n$ and D_X for its preimage in X. As schemes, $D_{\mathbb{P}^n}$ and D_X are isomorphic; the restriction $\pi|_{D_X} : D_X \to D_{\mathbb{P}^n}$ is an isomorphism.

Picard Group and Canonical Divisor In order to compute the Picard group of the cyclic cover X, we will use the following theorem.

Theorem 1.4.1 (Grothendieck-Lefschetz Theorem)**.** *Let X be a smooth projective variety and D an ample divisor on X. If the dimension of X is at least 4 then the induced map*

$$\operatorname{Pic}(X) \to \operatorname{Pic}(D)$$

is an isomorphism.

(See [G2] Exposé XII, corollary 3.6). Now we can relate the Picard group of a cyclic cover to that of the branch divisor.

Theorem 1.4.2. *Let $X \to \mathbb{P}^n$ be a smooth cyclic cover of degree r branched over a hypersurface $D_{\mathbb{P}^n} \subseteq \mathbb{P}^n$ of degree rd. Then the Picard group of X is $\mathbb{Z}$, generated by the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$. The pullback $\pi^* D_{\mathbb{P}^n}$ is rD_X, so*

$$\mathcal{O}_X(D_X) = \pi^* \mathcal{O}_{\mathbb{P}^n}(d)$$

Proof. Locally the map π is induced by a finite map of rings

$$k[x_1, ..., x_n] \to \frac{k[x_1, ..., x_n, z]}{z^r - g}$$

The function g, defining $D_{\mathbb{P}^n}$, pulls back to z^r, where z defines D_X. Thus $\pi^* D_{\mathbb{P}^n}$ is rD_X. It follows that D_X is ample, since the pullback of an ample divisor by a finite morphism is ample.

Now by Grothendieck-Lefschetz, $\operatorname{Pic}(D_X) = \operatorname{Pic}(X)$ and $\operatorname{Pic}(D_{\mathbb{P}^n}) = \operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}$. But D_X and $D_{\mathbb{P}^n}$ are isomorphic as schemes, so $\operatorname{Pic}(X) = \mathbb{Z}$. □

Corollary 1.4.3. *Let X be as above. Then canonical divisor on X is*

$$\pi^* \mathcal{O}_{\mathbb{P}^n}(-(n+1) + (r-1)d)$$

and X is Fano whenever $(r-1)d \leq n$.

Proof. Consider an affine patch of $\mathbb{P}^n$ over which $\pi : X \to \mathbb{P}^n$ is induced by a map of rings

$$k[x_1, ..., x_n] \to \frac{k[x_1, ..., x_n, z]}{z^r - g}$$

for some polynomial $g(x_1, ..., x_n)$. Since g is smooth, we may (after shrinking the open set) take $dg, dx_2, ..., dx_n$ as a local basis for the cotangent bundle of $\mathbb{P}^n$, and $dz, dx_2, ..., dx_n$ as a basis for the cotangent bundle of X. Now consider the non-vanishing differential n-form on $\mathbb{P}^n$

$$dg \wedge dx_2 \wedge ... \wedge dx_n$$

When pulled back to X, this form vanishes to order $r - 1$ on D_X, since $dg = rz^{r-1}dz$. Therefore

$$K_X = \pi^* K_{\mathbb{P}^n} + (r - 1)D_X$$

From the previous calculation, $\mathcal{O}_X(D_X) = \pi^*\mathcal{O}_{\mathbb{P}^n}(d)$, so

$$\omega_X = \pi^*\mathcal{O}_{\mathbb{P}^n}(-(n + 1) + (r - 1)d)$$

Since the Picard group of X is $\mathbb{Z}$, the canonical divisor is antiample exactly when $-(n + 1) + (r - 1)d < 0$, as required.

□

Curves on Cyclic Covers By a *line* in X, we mean a curve that has degree one with respect to the invertible sheaf $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. A line in X is mapped under π to a line in $\mathbb{P}^n$; its translations under the action of the cyclic group map to the same line. In general, a line in $\mathbb{P}^n$ lifts to an irreducible degree r curve in X, but certain lines lift to reducible curves in X.

Definition 1.4.4. Let L be a line in $\mathbb{P}^n$ and D a hypersurface. We say that L is *r-multiply tangent to D* either if L is contained in D, or if every point of the intersection $L \cap D$ has multiplicity divisible by r.

With this definition we can characterise the lines in $\mathbb{P}^n$ that lift to lines in X.

Theorem 1.4.5. *Let L be a line in $\mathbb{P}^n$, so that $\pi^{-1}L$ is a degree r curve in X. Then $\pi^{-1}L$ is reducible as the union of r lines if and only if L is r-multiply tangent to $D_{\mathbb{P}^n}$. Specifically, if L is contained in $D_{\mathbb{P}^n}$ then $\pi^{-1}L$ is a single line with multiplicity r; otherwise, $\pi^{-1}(L)$ is the union of r distinct curves of degree one.*

Proof. Let p be a point in $L \cap D$, and choose affine coordinates $x_1, ..., x_n$ on an open set of $\mathbb{P}^n$ so that p is the origin and L is the line $x_2 = ... = x_n = 0$. Let g be the (dehomogenised) equation of degree rd that defines D. Then the preimage of L in X is the scheme

$$\operatorname{Spec} \frac{k[x_1, z]}{z^r - g(x_1, 0, ..., 0)}$$

In the formal local ring at p, $g(x_1, 0, ..., 0)$ is of the form $x_1^k u$ for some integer k and some unit u. This k, the lowest power of x_1 occurring in g, is also the multiplicity of the intersection point $p \in L \cap D$.

If the preimage of L comprises r distinct lines, then $z^r - g(x_1, 0, ..., 0)$ must split into linear factors in the formal local ring. Therefore $g(x_1, 0, ..., 0)$ must be an rth power. In the formal local ring, a unit u is an rth power, so the lowest power of x_1 is a multiple of r as required. Conversely, if every point in the intersection $L \cap D$ has intersection multiplicity divisible by r then $z^r - g(x_1, 0, ..., 0)$ splits into linear factors, at least formally locally. But this implies that the normalisation of $\pi^* L$ is an unbranched cover of L. Since L is $\mathbb{P}^1$, the only irreducible unbranched cover is the identity map, and therefore $\pi^* L$ is the union of r irreducible curves. □

Note also that if L intersects D with multiplicity r at a point and pulls back to a reducible curve in X, then each of the r irreducible rational components of $\pi^* L$ will intersect D with multiplicity one.

The Derivative Map Consider the map of cotangent sheaves on X

$$d\pi : \pi^* \Omega_{\mathbb{P}^n} \to \Omega_X$$

Away from the branch divisor, this is an isomorphism, so we consider a point $p \in D$. If D is non-singular, cut out on some neighbourhood of p by a function g, then we can choose coordinates so that $dg, dx_2, ..., dx_n$ is a basis for the cotangent space of $\mathbb{P}^n$. Recall that locally X looks like

$$\operatorname{Spec} k[x_1, ..., x_n, z]/(z^r - g)$$

With this notation, we can take $dz, dx_2, ..., dx_n$ as a basis for the cotangent space of X at p. Then we have

$$dg \mapsto r z^{r-1} dz \qquad dx_i \mapsto dx_i$$

We take the corresponding dual basis for the tangent space, and we compute that

$$\frac{\partial}{\partial z} \mapsto r z^{r-1} \frac{\partial}{\partial g}$$

Therefore we have

$$0 \to T_X \to \pi^* T_{\mathbb{P}^n} \to F \to 0$$

where F is a torsion sheaf supported on the branch divisor. We will describe the cokernel F more carefully. Let I be the ideal sheaf of D in X - in the notation above, I is the subsheaf of $\mathcal{O}_X$ generated locally by z. Then there is a filtration

$$0 = I^{r-1}F \subseteq I^{r-2}F \subseteq ... \subseteq IF \subseteq F$$

The quotient F/IF is the normal bundle of D in $\mathbb{P}^n$, being the subbundle of $T_{\mathbb{P}^n}$ generated locally by $\frac{\partial}{\partial g}$. If $r = 2$ (a cyclic double cover) this is the entire filtration. Otherwise, the subquotient $I^k F/I^{k+1}F$ is generated locally by $z^k \frac{\partial}{\partial g}$ and annihilated by z. Therefore the multiplication map

$$\left(\frac{I}{I^2}\right)^{\otimes k} \otimes \left(\frac{F}{IF}\right) \to \frac{I^k F}{I^{k+1}F}$$

is an isomorphism, and we have

$$\frac{I^k F}{I^{k+1}F} = N_{D/\mathbb{P}^n} \otimes \left(N_{D/X}^\vee\right)^{\otimes k}$$

1.5 Dimension of Moduli Spaces

Our main result concerns the dimension of the moduli space of genus 0 stable curves on a cyclic cover of $\mathbb{P}^n$. As an illustration of the ideas involved, let us consider the simpler case of rational curves in hypersurfaces.

Let D be a hypersurface in $\mathbb{P}^n$ defined by some homogeneous equation $F(x_0, ..., x_n)$ of degree d. By straightforward calculation we can try to determine the dimension of the space $\mathrm{Mor}(\mathbb{P}^1, D)$ that parametrises maps $f : \mathbb{P}^1 \to D$, as follows.

A map from $\mathbb{P}^1$ to D whose image has degree e is given by a set of homogeneous equations

$$g_0(s,t), g_1(s,t), ..., g_n(s,t)$$

each of degree e, such that $F(g_0, ..., g_n) = 0$. This polynomial $F(g_0, ..., g_n)$ is a function of the two variables s and t, and has degree de. Each of the $de+1$ monomials $s^i t^{de-i}$ has a coefficient that is itself a polynomial in the coefficients of $g_0, ..., g_n$. We therefore expect that, *in general*, the demand $F(g_0, ..., g_n) = 0$ imposes $de + 1$ conditions on the coefficients of the g_i.

Each g_i is a polynomial of degree e in two variables; such polynomials are parametrised by a space of dimension $e + 1$. We require $n + 1$ of these polynomials, with two $(n + 1)$-tuples giving the same map if they differ by a scalar; therefore we expect that in general the dimension of the space $\mathrm{Mor}(\mathbb{P}^1, D)$ should be

$$(e+1)(n+1) - (de+1) - 1$$

For specific hypersurfaces D this is incorrect (for example, when D is the Fermat hypersurface) but it turns out that this is the correct dimension when D is general. However, we are far from a proof: it is not at all clear that the $de+1$ conditions on the coefficients of the g_i must be independent.

In [HRS] the argument is made precise in the case $e = 1$. In section 2.2 we outline this argument, and later extend the result to compute the dimension of the space of lines on a cyclic cover of $\mathbb{P}^n$, but for higher degrees we need more sophisticated machinery.

The following theorem gives a lower bound for the dimension of the spaces $\overline{\mathcal{M}}_{0,0}(X, e)$ in terms of the tangent bundle:

Theorem 1.5.1. *Let C be a curve and X a smooth projective variety. Then there is a space* $\mathrm{Mor}(C, X)$ *parametrising morphisms $f : C \to X$, and the dimension of this space at the point corresponding to f is at least $\chi(f^*T_X)$. Furthermore, if the dimension of* $\mathrm{Mor}(C, X)$ *is exactly $\chi(f^*T_X)$ at a point p, then locally* $\mathrm{Mor}(C, X)$ *is a complete intersection.*

For a proof of this theorem, see [FP] section 5 lemma 9, or [D1] theorem 2.6.

Theorem 1.5.2. *Let $f : C \to X$ be a rational curve of degree e on X such that the domain C is irreducible and such that the image of f in X not contained in the branch divisor D.The Euler characteristic of f^*T_X is*

$$\chi(f^*T_X) = e(n - (r-1)d + 1) + n$$

Proof. In this case, we can compute the Euler characteristic directly. Recall from the previous section that we had an exact sequence

$$0 \to T_X \to \pi^*T_{\mathbb{P}^n} \to F \to 0$$

From the proof of 1.4.5, C intersects the branch divisor at de points, each with multiplicity one. Therefore $\mathrm{Tor}_1(F, \mathcal{O}_C) = 0$ and exactness is preserved when we pull back to C. Then we have

$$\chi(f^*T_X) = \chi(f^*\pi^*T_{\mathbb{P}^n}) - \chi(f^*F)$$

The sheaf F has length $r-1$ at each point and C intersects its support transversely, so the Euler characteristic of f^*F is $(r-1)de$. We also compute the Euler characteristic of $f^*\pi^*T_{\mathbb{P}^n}$ by pulling the Euler sequence back to C

$$0 \to \mathcal{O}_C \to O(e)^{\oplus n+1} \to f^*\pi^*T_{\mathbb{P}^n} \to 0$$

and find that it is equal to $(n+1)(e+1)-1$. Therefore we have

$$\chi(f^*T_X) = e(n-(r-1)d+1)+n$$

as required. $\square$

However, the space $\overline{\mathcal{M}}_{0,0}(X,e)$ also parametrizes stable curves with reducible domain. In particular, one or more components may be mapped entirely into the branch divisor. Therefore we repeat the calculation via Chern classes.

Lemma 1.5.3. *Let D be a divisor on a smooth projective variety X, and let F be a coherent sheaf on X that is the pushforward of a line bundle on D. Then $c_1(F) = D$.*

Proof. First, let X be any smooth variety and let $Y \subseteq X$ be a closed subset of codimension 2. Let

$$i : X - Y \to X$$

be the inclusion. Then i^* is an isomorphism between $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X-Y)$ - that is, between the first Chow groups - and for any line bundle L on X we have $c_1(i^*L) = i^*c_1(L)$. The same is true when L is any locally free sheaf, by the splitting principle. Finally the same holds when L is an arbitrary coherent sheaf, by taking a free resolution. Therefore we can compute the first Chern class on the complement of a codimension 2 subset.

Now let D be a divisor on X and F the pushforward of a line bundle on D. Take Y to be a subset of D of codimension 1 such that $F|_{D-Y}$ is trivial. Then Y has codimension 2 on X, so we can compute the first Chern class of F on $X-Y$. But on $X-Y$, the sheaf F is equal to the structure sheaf of D, and we can compute its first Chern class with the exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

$\square$

Theorem 1.5.4. *Let X be a cyclic cover of $\mathbb{P}^n$ and let $f : C \to X$ be a rational curve in X of degree e. Then the Euler characteristic of f^*T_X is*

$$\chi(f^*T_X) = e(n-(r-1)d+1)+n$$

Therefore every irreducible component of $M_{0,0}(X,e)$ has dimension at least $e(n-d+1)+n-3$.

Proof. By Riemann-Roch, the Euler characteristic of the vector bundle V on a curve of genus g is

$$\chi(V) = c_1(V) + \mathrm{rank}(V)(1-g)$$

So we can compute the first chern class of T_X on X, and pull it back to C. Recall that the Picard group of X is $\mathbb{Z} \cdot H$, generated by the pullback of a hyperplane class from $\mathbb{P}^n$. We have

$$c_1(T_X) = c_1(\pi^* T_{\mathbb{P}^n}) - c_1(F)$$

and from the Euler sequence we compute that

$$c_1(\pi^* T_{\mathbb{P}^n}) = (n+1)H$$

The sheaf F has rank $r-1$ and the branch divisor is linearly equivalent to dH, so $c_1(F) = (r-1)dH$ and

$$c_1(T_X) = (n - (r-1)d + 1)H$$

Finally we pull back to the degree e curve C, and note that the space of automorphisms of $\mathbb{P}^1$ has dimension 3.

□

1.6 Gromov-Witten Invariants

We mention here an important corollary of the irreducibility of $\overline{\mathcal{M}}_{0,0}(X, e)$. First, let F be the Fano scheme of lines in $\mathbb{P}^n$. Let $Z_1, ..., Z_r$ be a collection of linear subvarieties of codimensions $c_1, ..., c_r$ respectively.

For each i, we ask which lines in $\mathbb{P}^n$ intersect Z_i. We expect that the locus inside of F parametrising such lines has codimension $c_i - 1$. Therefore if we demand that

$$(c_1 - 1) + (c_2 - 1) + ... + (c_r - 1) = \dim F$$

then we expect that ony finitely many lines in $\mathbb{P}^n$ intersect every linear subspace Z_i, and it is natural to ask how many such lines there are.

In order to compute this number, let σ_i be the Poincare dual of the subvariety Z_i, and let F_r be the scheme parametrising lines in $\mathbb{P}^n$ together with r fixed points, which comes with r evauation maps

$$ev_i : F_r \to \mathbb{P}^n$$

We can express the required number as the integral

$$\int ev_1^* \sigma_1 \cup ... \cup ev_r^* \sigma_r$$

where the integral is taken over the fundamental class of F_r.

Now let X be an arbitrary variety, and replace F_r by the Kontsevich moduli space $\overline{\mathcal{M}}_{0,r}(X, \beta)$. Since this moduli space is a stack, we must replace the fundamental class with the *virtual fundamental class*. With this generalisation we can compute the value of the above integral; the result is the *Gromov-Witten invariant* corresponding to $X, \beta, Z_1, ..., Z_r$.

If the Gromov-Witten invariant is still equal to the number of lines intersecting $Z_1, ..., Z_r$, then the invariant is said to be *enumerative*. For a general X and $\overline{\mathcal{M}}_{0,r}(X, e)$, the invariants may not be enumerative (and they may not even be integers). When $\overline{\mathcal{M}}_{0,0}(X, e)$ has pure expected dimension, however, this is the case (see, for example, [FP] Lemma 14)

Theorem 1.6.1. *Let S be a smooth variety such that $\overline{\mathcal{M}}_{0,0}(X, \beta)$ is irreducible of the expected dimension. Then the Gromov-Witten invariants are enumerative.*

Therefore a corollary of our results is that the Gromov-Witten invariants on a general smooth cyclic cover of $\mathbb{P}^n$ are enumerative.

2 Dimension of the Space of Lines

In this chapter we prove that the space of lines on X has dimension equal to the lower bound that we computed in 1.5.4.

Theorem 2.0.2. *Every irreducible component of* $\overline{\mathcal{M}}_{0,0}(X,1)$ *has dimension* $2n-(r-1)d-2$

The space of lines is more easily parametrised than the space of curves of higher degree. In the next chapter we use this result as the basis for an induction argument that computes the dimension of $\overline{\mathcal{M}}_{0,0}(X,e)$ for all e.

2.1 The Spaces M and M^*

Recall that in theorem 1.4.5 we showed that a line in $\mathbb{P}^n$ lifts to one or more lines in X exactly when it is r-multiply tangent to the branch divisor. Instead of studying the space of lines in X, we will study the lines in $\mathbb{P}^n$ satisfying this tangency condition, since the latter is much easier to parametrize. We will also consider the space of lines with a choice of base point, and the evaluation map to X that comes with it.

Our key technical results relate to the flatness of the evaluation map. Since flatness implies constant fibre dimension, this will allow us to compute the dimension from the dimension of a general fibre. However, in order to prove that the evaluation map is flat away from a finite set of points, we must consider a blowup of the space of lines.

Therefore we make the following definition:

Definition 2.1.1. Let X be a cyclic r-sheeted cover of $\mathbb{P}^n$ branched over a divisor D of degree rd. We define the space M to be the parameter space of lines L in $\mathbb{P}^n$ together with a divisor Q of degree d on the line L, satisfying the following two conditions:

- Firstly, either that $D\cap L$ is all of L, or that every point of the divisor $D\cap L$ on L has multiplicity divisible by r; that is, L is r-multiply tangent to D
- Secondly, if L is not contained in D then we require that $rQ = D\cap L$. If

The first condition, as we have seen, picks out those lines in $\mathbb{P}^n$ that lift to a line or lines in X. The additional information of the d intersection points in Q gives a blow-up of the space of lines on X. If L does not lie in D then $L\cap D$ has degree rd, and is r-divisible, so the divisor Q is already determined. However, if L is contained in D then the space parametrising degree d divisors on L is $\mathbb{P}^d$. Below we will construct M explicitly as a subvariety of a Grassmannian.

We also define the pointed version of M.

Definition 2.1.2. Let M^* be the space parametrising lines $L \subseteq \mathbb{P}^n$ and a degree d divisor Q on L that satisfy the same conditions as in the definition of M, together with a choice of base point p on the line L.

Note that the choice of base point is independent of the divisor Q in the definition of M. The space M^* is a $\mathbb{P}^1$ bundle over M, and comes with an evaluation map $ev : M^* \to \mathbb{P}^n$ that sends each line to the chosen base point.

Definition 2.1.3. Let $M(D)$ (respectively, $M^*(D)$) be the closed subscheme of M (respectively, M^*) parametrising lines that lie entirely within the branch divisor D. Let $ev_D : M^*(D) \to \mathbb{P}^n$ be the evaluation map.

We study the space M in place of $\overline{\mathcal{M}}_{0,0}(X, 1)$. Now we can state the main theorem of this chapter:

Theorem 2.1.4. *For a general cyclic cover X, the evaluation map ev : $M^* \to X$ is flat over the complement of a finite set of points $S \subset X$, and no point of S is contained in the branch divisor D. The fibre dimension over the flat locus is $n - (r-1)d - 1$. Over the points of S, the fibre dimension is at most $n - (r-1)d$ (that is, it increases by at most one)*

We prove this result in the next section in theorems 2.3.3 and 2.3.4.

Before we begin the proof, we show that the results about the space M will imply the results we need concerning $\overline{\mathcal{M}}_{0,0}(X, 1)$. To relate M and $\overline{\mathcal{M}}_{0,0}(X, 1)$, we make the following temporary definition:

Definition 2.1.5. Let N be the parameter space of lines L in $\mathbb{P}^n$ such that either $D \cap L$ is all of L, or every point of the divisor $D \cap L$ on L has multiplicity divisible by r. Let N^* be the parameter space of such lines together with a chosen base point.

There is a map $M \to N$ that just forgets the divisor Q; this map is an isomorphism away from $M(D)$. On the other hand, there is also a finite map $\overline{\mathcal{M}}_{0,0}(X, 1) \to N$; in fact, N is the quotient of $\overline{\mathcal{M}}_{0,0}(X, 1)$ by the action of the cyclic group $\mathbb{Z}/r\mathbb{Z}$.

The corresponding maps from $\overline{\mathcal{M}}_{0,1}(X, 1)$ and M^* to N commute with the evaluation maps.

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,1}(X,1) \arrow[r] \arrow[d, "ev"'] & N^* \arrow[d, "ev"'] & M^* \arrow[l] \arrow[d, "ev"'] \\
X \arrow[r, "\pi"'] & \mathbb{P}^n & \mathbb{P}^n \arrow[l, "="']
\end{tikzcd}$$

Via N, we show that the lower bound of 1.5.4 for the dimension of $\overline{\mathcal{M}}_{0,0}(X, 1)$ also applies to M.

Lemma 2.1.6. *Every irreducible component of N has dimension at least $2n-(r-1)d-2$. If every component of N has dimension $2n-(r-1)d-2$, then so does every component of $\overline{\mathcal{M}}_{0,1}(X,1)$.*

Furthermore, the equivalent statement holds in fibres of the evaluation map. Let p be a point in $\mathbb{P}^n$ and q a point in $\pi^{-1}(p)$. Then every irreducible component of $ev^{-1}(p) \subseteq N^*$ has dimension at least $n-(r-1)d-2$.

Proof. This follows from the fact that the map $\overline{\mathcal{M}}_{0,1}(X,1) \to N^*$ is finite everywhere. We computed a lower bound for the dimension of $\overline{\mathcal{M}}_{0,1}(X,1)$, and fibers of the evaluation map, in 1.5.4, and this lower bound transfers to N. □

Lemma 2.1.7. *Every irreducible component of M not contained in $M(D)$ has dimension at least $2n-(r-1)d-2$.*

Proof. This follows from the previous lemma and the fact that $M \to N$ is finite away from $M(D)$. □

2.2 Lines on Hypersurfaces

In order to prove theorem 2.1.4 above, we construct M^* explicitly as a subvariety of a Grassmannian. Then we will examine the sublocus of lines that lie entirely within the branch divisor $D_{\mathbb{P}^n} \subseteq \mathbb{P}^n$. Therefore we begin by considering lines in hypersurfaces.

Let $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(k))$ be the parameter space of degree-k hypersurfaces in $\mathbb{P}^n$. For any such hypersurface D, let $\overline{\mathcal{M}}_{0,1}(D,1)$ be the parameter space of pointed lines in D. This space comes with an evaluation map

$$ev_D : \overline{\mathcal{M}}_{0,1}(D,1) \to D$$

and for a point $p \in D$ we consider the fibre $ev_D^{-1}(p)$.

Definition 2.2.1. The *expected dimension* of a fibre $ev_D^{-1}(p)$ is $n-k-1$

We consider the pairs of a hypersurface D and a point $p \in D$ such that the evaluation map ev_D has dimension greater than expected. Consider the total space of the universal family of hypersurfaces

$$Z \subseteq \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(k)) \times \mathbb{P}^n$$

Definition 2.2.2. For $i \geq 0$, let $Z_i \subseteq Z$ be the locus of pairs $(D, p \in D) \in Z$ such that $ev_D^{-1}(p)$ has dimension greater than or equal to $(n-k-1)+i$ (that is, i greater than the expected dimension).

Therefore Z_0 contains the points at which the fibre dimension is at least the expected dimension; Z_1 contains the points at which it jumps up by at least one; and so on. The following results results are proved in [HRS] Section 2 (specifically, Corollary 2.2.4 below is [HRS] Theorem 2.1) so we just give an outline of the computation here.

Theorem 2.2.3. *The space Z_0 is the entire total space Z. The subspace $Z_1 \subseteq Z$ has codimension at least n, and $Z_2 \subseteq Z$ has codimension strictly greater than n.*

Proof. Let p be a fixed point in $\mathbb{P}^n$. Choose affine coordinates $x_1, ..., x_n$ on a neighbourhood of $p = (0, ..., 0)$. Then any hypersurface D of degree k is cut out by an equation

$$\Phi = \Phi_0 + \Phi_1(x_1, ..., x_n) + ... + \Phi_k(x_1, ..., x_n)$$

where each Φ_i is a homogeneous polynomial of degree i, and the hypersurfaces through p are those with $\Phi_0 = 0$.

We are interested in lines though p that lie in D, so suppose $\Phi_0 = 0$ and let $[a_1, ..., a_n]$ be a point in the boundary $\mathbb{P}^{n-1}$ of this affine chart, specifiying a line through p. Then the specified line lies on D exactly when

$$\Phi_i(a_1, ..., a_n) = 0$$

for all i. In other words, the lines through p are parametrised by $\mathbb{P}^{n-1}$, and for a fixed hypersurface D there is a set of homogeneous polynomials Φ_i whose mutual vanishing set in $\mathbb{P}^{n-1}$ parametrises the lines that lie in D.

If for each i the polynomial Φ_i is not in the radical of the ideal generated by $\Phi_1, ..., \Phi_{i-1}$ - that is, if the vanishing set of Φ_i is not contained in the mutual vanishing set of the lower-degree polynomials - then by the Hauptidealsatz the codimension of $Z(\Phi_1) \cap ... \cap Z(\Phi_k)$ is exactly k inside the boundary $\mathbb{P}^{n-1}$. In this case the space of lines on D through p has the expected dimension $n - k - 1$.

So we need to show that a relation between the polynomials Φ_i occurs in codimension n. This implies that Z_1 has codimension n in each of the fibres of the forgetful map $Z \to \mathbb{P}^n$, and therefore that Z_1 has codimension n in Z. For this computation, we refer to [HRS] Theorem 2.4.

□

Corollary 2.2.4. *For a general hypersurface D of degree k, the evaluation map $ev_D : \overline{\mathcal{M}}_{0,1}(D, 1) \to D$ has the expected fibre dimension at every point. Furthermore, if a hypersurface contains a point at which the fibre dimension increases by two or more over the expected dimension, then that hypersurface is in a subset of codimension two inside the parameter space of all hypersurfaces.*

Proof. This follows immediately from the theorem. The map from the total space to the parameter space of degree k hypersurfaces is projective, and has fibers of dimension $n-1$. Therefore the image of Z_1 is closed of codimension at least 1, while the image of Z_2 has codimension at least 2. □

Now let X be a cyclic cover, branched over a hypersurface of degree rd. We have seen that the branch locus is an ample divisor. The non-flat locus of the evaluation map $M^* \to X$ is a closed subset of X, so if it has any components of dimension greater than zero then it must intersect the branch locus. Therefore we will show that the evaluation map is flat everywhere on the branch divisor; this implies that the non-flat locus consists of a finite set of points disjoint from D.

Our proof will consider the relationship between lines in X and lines in D. We first consider some simple cases to illustrate the idea.

A Toy Example As a simple analogy, let $X = \mathbb{P}^2$ and let D be a line in X. The space of all lines in X is again $\mathbb{P}^2$, with D represented by a single point. Let M be the space of pairs (L, q) with L a line in X and q a point in $L \cap D$. Then M is the blowup of $\mathbb{P}^2$ at the point corresponding to D. The subspace of M corresponding to lines contained in D is a divisor in M: it is exactly the exceptional divisor of the blowup.

Furthermore, let p be a point in D and consider only those lines in M that pass through p. This sublocus is reducible, consiting of two copies of $\mathbb{P}^1$ intersecting at a point: one component parametrises lines in $\mathbb{P}^2$ through p, on which the marked point q is p itself, while the other component is the exceptional divisor representing the line D with a varying choice of q. Therefore after restricting to the fibre of the evaluation map over p, we find that the lines contained in D have codimension zero.

The Case d=1 Now let X be a cyclic cover of degree r branched over a divisor $D \subseteq \mathbb{P}^n$ of degree r; that is, the case $d = 1$. Then M parametrises lines L in $\mathbb{P}^n$ together with a single marked point $q \in L \cap D$ such that L is r-multiply tangent to D. In this case, this means either that $L \subseteq D$ or that L is tangent to D to order r at the unique intersection point q.

Let p be a point in D. The space of lines L through p is $\mathbb{P}^{n-1}$, and to demand that $L \cap D = r \cdot p$ is to impose $r - 1$ additional conditions; therefore the dimension of such lines is $n - r$.

Inside of this space, we have the subspace of lines through p contained completely in D, which for a general branch divisor has dimension $n - r - 1$, by 2.2.4. If we add an additional free choice of marked point q, the dimension of the subspace is $n - r$.

Recall that $M^*(D)$ was the subspace of M^* parametrising lines in M that lie completely in D. If X is a cyclic cover of degree r branched over a divisor D of degree r, then for any point $p \in D$ we have the fiber $ev^{-1}(p) \subseteq M^*$. We can then summarise this example by saying that the subset

$$M^*(D) \cap ev^{-1}(p)$$

has codimension zero in $ev^{-1}(p)$.

2.3 Flatness of the Evaluation Map

The following two theorems describe how $M(D)$ sits inside M.

Theorem 2.3.1. *Let $X \to \mathbb{P}^n$ be a cyclic cover ramified over a divisor D of degree rd, and let M and $M(D)$ be as above. Then $M(D)$ is a locally principal closed subscheme of M. In particular, its codimension is at most one. Similarly, $M^*(D)$ is a locally principal subscheme of M^*.*

Proof. Our proof relies on the simple observation that, in general, $D \cap L$ is a degree rd divisor on L; the lines L contained completely in D can be identified as the lines on which D vanishes at $rd + 1$ points.

To begin, let G be the Grassmannian of lines in $\mathbb{P}^n$, with tautological vector bundle S. The total space of the projectivisation of S, which we will denote by $\mathbb{P}S$, comes with a map $\sigma : \mathbb{P}^S \to G$. There is also a map $\iota : \mathbb{P}S \to \mathbb{P}^n$ that embeds every fibre of σ_1 as a line. The divisor D is cut out by some global section $F \in H^0(\mathcal{O}(rd), \mathbb{P}^n)$. Pulled back to $\mathbb{P}S$, F defines a divisor of degree rd on a general fibre. On those lines completely contained in D, the section F vanishes.

The space of lines in $\mathbb{P}^n$ together with a choice of degree d divisor is the projective bundle $\sigma_2 : \mathbb{P}\operatorname{Sym}^d S \to G$. We can construct the product $\mathbb{P}S \times_G \mathbb{P}\operatorname{Sym}^d S$.

$$\begin{tikzcd}
\mathbb{P}S \times_G \mathbb{P}\operatorname{Sym}^d S \arrow[r, "\sigma_2'"'] \arrow[d, "\sigma_1'"'] & \mathbb{P}S \arrow[r, "\iota"'] \arrow[d, "\sigma_1"'] & \mathbb{P}^n \\
\mathbb{P}\operatorname{Sym}^d S \arrow[r, "\sigma_2"'] & G &
\end{tikzcd}$$

We have two divisors on this product space. The first, which we denote Δ, has a d-to-1 map to $\mathbb{P}\operatorname{Sym}^d S$, cutting out a degree d divisor on each fibre. The second divisor is that cut out by the pullback of F in the sheaf ${\sigma_2'}^* \iota^* \mathcal{O}(rd)$.

Consider $r\Delta$, the thickened tautological divisor. We restrict ${\sigma_2'}^* \iota^* \mathcal{O}(rd)$ to $r\Delta$, and push forward to $\mathbb{P}\operatorname{Sym}^d S$. The zero set of F in this pushforward is a subset of $\mathbb{P}\operatorname{Sym}^d S$ parametrising lines L and divisors Q of degree d on L such that the divisor cut out by F on L is rQ. That is, the vanishing locus of F is exactly the space M.

Now let H be some hyperplane in $\mathbb{P}^n$. Over a general point of G, the divisor H pulls back to a single point. Over a general point of $\mathbb{P}\operatorname{Sym}^d S$, H is a single point disjoint from the d points selected. We restrict ${\sigma_2'}^* \iota^* \mathcal{O}(rd)$ to $r\Delta + H$, and push forward to $\mathbb{P}\operatorname{Sym}^d S$. This time, F cuts out the locus of lines L and divisors Q; but F must vanish completely on such a line.

We have an exact sequence on $\mathbb{P}S \times_G \mathbb{P}\operatorname{Sym}^d S$

$$0 \to K \to {\sigma_2'}^* \iota^* \mathcal{O}(rd)|_{r\Delta+H} \xrightarrow{\rho} {\sigma_2'}^* \iota^* \mathcal{O}(rd)|_{r\Delta} \to 0$$

Write $F|_{r\Delta+H}$ and $F|_{r\Delta}$ for the section of the last two line bundles induced by F. If we push forward to $\mathbb{P}\operatorname{Sym}^d S$ and restrict to the subset of $\mathbb{P}\operatorname{Sym}^d S$ on which $F|_{r\Delta}$ vanishes (that is, to M), then the subset of lines in D (that is, the space $M(D)$) is the vanishing set of $F|_{r\Delta+H}$.

The pushforward of $\mathcal{O}(rd)|_{r\Delta+H}$ is not rank one. However, it is easy to compute that the kernel K is $\mathcal{O}(rd)(-H)|_H$. The pushforward of this sheaf is a line bundle, since H is degree one. If we restrict to the vanishing set of $F|_{r\Delta}$ then $F|_{r\Delta+H}$ is in the kernel of ρ, and so comes from a section of K. Therefore $M(D)$ is cut out inside of M by a section of a line bundle, as required.

□

We can also describe how $ev_D^{-1}(p)$ sits inside $ev^{-1}(p)$.

Theorem 2.3.2. *Let M^* and $M^*(D)$ be as above, with evaluation map $ev : M^* \to \mathbb{P}^n$. Let p be a point on the branch divisor D. Then $ev_D^{-1}(p) = M^*(D) \cap ev^{-1}(p)$ is codimension zero in $ev^{-1}(p) \subseteq M^*$.*

Proof. In the above proof, let H be a hyperplane containing p. If L is a line through p then $H \cap L = p$, and F vanishes at p. Now we consider the locus in $\mathbb{P}\operatorname{Sym}^d S$ on which $F|_{r\Delta}$ vanishes (the lines in X), and the locus of lines in D is exactly the sublocus on which $F|_{r\Delta+H}$ vanishes.

Now restrict attention to those lines in X passing through p. It is easy to see that at every corresponding point of $\mathbb{P}\operatorname{Sym}^d S$, the sublocus of lines in D is cut out by a zero divisor, as follows. At a point where p is not in the support of the degree d divisor, then locally the sublocus is the whole space. If p is in the degree d divisor then F may be a non-zero section of $\mathcal{O}(rd)|_{r\Delta+H}$ that vanishes on $\mathcal{O}(rd)|_{r\Delta}$; in this case, F is a zero divisor.

□

We can use these results to relate the dimension of the space of lines in D - which we computed in 2.2.4 - to the dimension of the space of lines in all of X.

Theorem 2.3.3. *For a general cyclic cover X, the evaluation map $ev : M^* \to \mathbb{P}^n$ is flat over the complement of a finite set of points $S \subseteq \mathbb{P}^n$, none of which lie in D. The fibre dimension over the flat locus is $n-(r-1)d-1$.*

Proof. A non-flat locus of positive dimension would intersect the ample branch divisor D; therefore it is sufficient to show that the evaluation map is flat over every point of D.

By 2.1.7, the fibre dimension over a point not in D is at least $n-(r-1)d-1$. We show that the fibre dimension over every $p \in D$ is exactly

$n-(r-1)d-1$. By upper-semicontinuity, this implies that the general fibre has dimension exactly $n-(r-1)d-1$, and also shows that the evaluation map is flat over the branch divisor.

Let p be a point in D and let $ev^{-1}(p)$ be the fibre over p in M^*. Let $ev_D^{-1}(p)$ be the fibre over p in $M^*(D)$. By 2.1.7, the components of $ev^{-1}(p)$ have dimension at least $n-(r-1)d-1$. We aim to show that all components have exactly this dimension.

By 2.2.4, the space of pointed lines in a general branch divisor D (of degree rd) will have an evaluation map with fibre dimension $n-rd-1$ over all points (cf [HRS] for curves in hypersurfaces more generally). Since the lines in $M^*(D)$ also come with a choice of d points, the dimension of $ev_D^{-1}(p)$ will be $n-(r-1)d-1$.

Let Z be an irreducible component of $ev^{-1}(p)$. If Z lies entirely in $M^*(D)$ (that is, if every line parametrised by a point of Z lies entirely in the branch divisor) then Z is an irreducible component of $ev_D^{-1}(p)$, and has dimension $n-(r-1)d-1$ by the previous paragraph. Otherwise, by 2.3.1 the intersection $Z \cap ev_D^{-1}(p)$ is codimension one in Z.

We know that the dimension of $ev_D^{-1}(p)$ is $n-(r-1)d-1$, so the lower bound for the dimension of Z implies that $\dim Z$ is either $n-(r-1)d-1$ or $n-(r-1)d$. In the latter case, we would have $Z \cap ev_D^{-1}(p) = ev_D^{-1}(p)$. But there are certainly points in $ev_D^{-1}(p)$ that are not the limit of points in $Z - ev_D^{-1}(p)$. To see this, note that any line through p that is not contained in the branch divisor must contain p in the degree d divisor Q. By continuity, the same is true of any line parametrised by a point of $Z \cap ev_D^{-1}(p)$. But there are clearly points in $ev_D^{-1}(p)$ parametrising lines in which p is not in the support of Q. Therefore $Z \cap ev_D^{-1}(p)$ is codimension one in $ev_D^{-1}(p)$, and the dimension of Z is $n-(r-1)d-1$ as required.

□

Next we prove that the fibers of the evaluation map over the non-flat locus have dimension at most one greater than over a general point.

Theorem 2.3.4. *For a general branch divisor X, the fibre dimension of $ev : M^* \to \mathbb{P}^n$ over the non-flat locus is $n-(r-1)d$ (that is, one greater than expected).*

Proof. First, consider $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(rd))$, the moduli space of degree d hypersurfaces in $\mathbb{P}^n$. The trivial $\mathbb{P}^n$-bundle over this space contains the tautological divisor (where the intersection of each fibre with the divisor is the corresponding hypersurface in $\mathbb{P}^n$).

$$\begin{array}{ccc} \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(rd)) \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ \downarrow & & \\ \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(rd)) & & \end{array}$$

This divisor is ample in the total space $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(rd)) \times \mathbb{P}^n$: the push-forward of the corresponding line bundle is $\mathcal{O}(1)$ on one factor and $\mathcal{O}(rd)$ on the other.

Now let B be a one-dimensional subscheme of $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(rd))$, parametrising a one-dimensional family of branch divisors. For each branch divisor in the family, the corresponding cyclic cover has a zero-dimensional non-flat locus. In the total space of the family, the non-flat locus is one-dimensional, and therefore it intersects the ample branch divisor. It follows that if there is a one-dimensional family on which the fibre dimension of the evaluation map jumps up by some amount k, then the branch divisor of a member of that family must be a hypersurface on which the evaluation fibre dimension also jumps up by k.

Now suppose that for a general cyclic cover (that is, for a general choice of branch divisor) the fibre dimension jumps by two or more at a non-flat point. Then a general one-dimensional family of cyclic covers would contain a branch divisor on which the fibre dimension jumps by two or more. Above we showed that in the fibre over a point of the branch divisor, the sublocus of lines in the branch divisor was codimension zero in the locus of all lines in the double cover. Therefore every one-dimensional family of hypersurfaces would contain a hypersurface on which the fibre dimension jumps by two or more. That is, the locus of hypersurfaces of degree rd on which the fibre can jump by two would be codimension one in the space of all hypersurfaces. But this contradicts 2.2.4.

□

3 Dimension of the Space of Degree e Curves

3.1 Bend and Break

In 1979, Mori introduced the bend-and-break theorem in [Mo], and used it to prove Hartshorne's conjecture that in characteristic zero, the only smooth projective varieties with ample tangent bundle are the projective spaces. He also proves in [Mo]:

Theorem 3.1.1 (Mori)**.** *Every Fano variety contains a rational curve.*

The bend-and-break theorem states that reducible curves occur in certain families in codimension one. For us, this is the key ingredient is calculating the dimension of the space of curves on X. Since reducible curves of degree e are constructed from curves of strictly lower degree, we can proceed by induction: we compute the dimension of the space of reducible curves, and then use bend-and-break to deduce the dimension of the entire space.

In this section we state the bend and break theorem, and give an outline of the proof. The first step is the following, which states that a projective subvariety does not deform to a point.

Theorem 3.1.2 (Rigidity Lemma)**.** *Let X, B, Y be varieties with X projective and B connected. Suppose we have a morphism*

$$f : X \times B \to Y$$

If there is some point $b \in B$ such that $f(X \times \{b\})$ is constant, then F factors through B (that is, it contracts every fibre)

Proof. Consider an affine open neighborhood $U \subseteq Y$ containing $f(X \times \{b\})$. Its preimage $f^{-1}(U)$ is open in $X \times B$. The complement of $f^{-1}(U)$ is closed; since X is projective, the image of this complement under the projection $X \times B \to B$ is also closed. Furthermore, this image does not contain b.

Therefore the points of $X \times B$ that do not map into U all lie over a proper closed subset of B. We can throw out this proper subset to replace B by B', so that $X \times B'$ maps into U. But U is affine and each fibre $X \times \{b\}$ is projective, so every fibre is mapped to a point.

Finally, to show that this holds over all of B, we choose some point $x \in X$ and define a map $X \times B \to Y$ by composing projection to $\{x\} \times B$ with f. This composition agrees with f on $X \times B'$; since all of our spaces are separated, it is true everywhere. □

Theorem 3.1.3. *Let X be a variety, C a curve (possibly reducible), and $C \times B$ a family over a curve B. If $f : C \times B \to X$ is a rational map that does not extend to an everywhere-defined morphism, then X contains a rational curve.*

Proof. First we assume that the curves C and B are smooth, by replacing them with their normalisations if necessary. Rational maps of smooth varieties can be extended over codimension one; since $C \times B$ is two-dimensional, we can extend f to a morphism if we blow up repeatedly at a finite set of points (see, for example, [Kol] Theorem 2.13). Each blowup generates an exceptional divisor isomorphic to $\mathbb{P}^1$. Once f has been extended to a morphism, the exceptional divisor of the final blowup cannot be mapped to a point, or else it can be blown down and would not have been necessary. Therefore the exceptional divisor of the final blowup is a non-constant map from $\mathbb{P}^1$ to X. □

Now we can state the bend-and-break theorem.

Theorem 3.1.4. *Let X be a smooth projective variety and let C be a curve on X. If C deforms with a fixed point p, then there is a rational curve on X through p.*

Proof. Let B be a curve, not necessarily projective, in the base of the deformation. Let B' be its closure, so we have a rational map $f : C \times B' \to X$. If f extends to a morphism, then f maps $\{p\} \times B'$ to a point. By the rigidity lemma and the projectivity of B', this would mean that f is a constant map $C \to X$.

Therefore f does not extend to a morphism on $C \times B'$; it is not defined on some points of $C \times (B' - B)$. Furthermore, we did not use the fact that C is projective and can remove any point in C other than p; therefore the points at which f is not defined must include a point in $\{p\} \times (B' - B)$. Then when we blow up $C \times B$ to extend f, we obtain an exceptional divisor that is a non-constant rational curve through p. □

We will use the theorem in the following form, which can be deduced from theorem 3.1.4. For a proof, see See [D1] Proposition 3.2

Theorem 3.1.5. *Let C be a curve on X, and let M be the moduli space of deformations of C that fix two points. Then M contains reducible curves in codimension one.*

3.2 Dimension of $\overline{\mathcal{M}}_{0,0}(X, e)$

Now we can compute the dimension of the space of lines of any degree. Our argument mimics the one applied in [HRS] and in [CS] to the case of Fano hypersurfaces.

Let X be a cyclic cover of $\mathbb{P}^n$ branched over a divisor D of degree rd. Then for every positive integer e we can make the following claim:

Claim 3.2.1. *The evaluation map $\overline{\mathcal{M}}_{0,1}(X, e) \to X$ is flat away from a finite set of points S, and the fibres of this map have dimension $e(n-(r-1)d+1)-$*

2. *Over the points of* S*, the fibre dimension is at most* $e(n-(r-1)d+1)-1$*; that is, it increases by at most one.*

We have already considered the case $e = 1$ above. Below we show that the result follows for all e by induction, under certain conditions.

Theorem 3.2.2. *Let* X *be as above. Suppose that* $2d(r-1) \leq n$ *and that the claim of 3.2.1 holds for* $e = 1$*. Then the claim holds for all* $e \geq 1$*.*

To begin, consider the map $\overline{\mathcal{M}}_{0,2}(X, e) \to X \times X$ induced by the two evaluation maps. From our lower bound for the dimension of $\overline{\mathcal{M}}_{0,2}(X, e)$, we see that the fibres of this map have dimension at least

$$e(n-(r-1)d+1)-n-1$$

We are going to apply bend-and-break to these fibres, and for this we need this dimension to be at least one. This will be the case when e is sufficiently large; specifically, when

$$e \geq \frac{n+2}{n-(r-1)d+1}$$

(cf the *threshold degree* defined in [HRS] Definition 5.4). We have proved the case $e = 1$ in the previous chapter, but we need this induction argument to apply whenever $e \geq 2$. Therefore we require

$$2d(r-1) \leq n$$

For the remainder of this thesis, we assume that n, d, r satisfy this inequality.

Given the conditions $2d(r-1) \leq n$ and $e \geq 2$, the dimension of the fibres is at least one, and bend-and-break applies: the fiber in $\overline{\mathcal{M}}_{0,2}(X, e)$ over every pair of points $(p, q) \in X \times X$ contains reducible curves in codimension one.

Lemma 3.2.3. *The fibre in* $\overline{\mathcal{M}}_{0,1}(X, e)$ *over every point* p *contains reducible curves in codimension one.*

Proof. Let

$$ev : \overline{\mathcal{M}}_{0,2}(X, e) \to X$$

be evaluation onto the first point. It suffices to show that $ev^{-1}(p)$ contains reducible curves in codimension 1. This fibre has a second map to X, evaluating at the second marked point, and by bend-and-break each fiber of this map contains reducible curves in codimension one. Therefore the entire fibre $ev^{-1}(p)$ contains reducible curves in codimension one. $\square$

Now we can prove theorem 3.2.2 by induction on e, We assume that the claim of 3.2.1 holds for curves of degree less than or equal to $e-1$. Consider the following commutative diagram, for some fixed a with $1 \leq a \leq e-1$:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,2}(X,a) \times_X \overline{\mathcal{M}}_{0,1}(X,e-a) \arrow[rr, hook] \arrow[d, "\phi"'] \arrow[dr, "\psi"] & & \overline{\mathcal{M}}_{0,1}(X,e) \arrow[d, "ev_e"] \\
\overline{\mathcal{M}}_{0,2}(X,a) \arrow[r, "\rho"'] & \overline{\mathcal{M}}_{0,1}(X,a) \arrow[r, "ev_a"'] & X
\end{tikzcd}$$

The space $\overline{\mathcal{M}}_{0,2}(X,e)$ consists of degree-e curves with one marked point. The product space is the boundary divisor containing those degree-e curves that are reducible as a degree-a curve and a degree-$(e-a)$ curve (which may themselves be reducible) in such a way that one marked point lies on each. The product is taken over the intersection point of the two. We will compute the fiber dimension of the boundary divisor over a point of X.

Theorem 3.2.4. *In the above diagram, the map ψ has constant fibre dimension $(e-a)(n-(r-1)d+1)-1$ over all points.*

Proof. By our induction assumption, we know that the dimension of the generic fibre of ψ is $(e-1)(n-(r-1)d+1)-1$. Therefore every irreducible component of a fibre of ψ has dimension at least this large.

The product space consists of degree e curves that are the union of a degree a curve and a degree $e-a$ curve, with the product taken over the intersection point. The map ϕ, which forgets the second irreducible component, is the base change of a map $\overline{\mathcal{M}}_{0,1}(X,e-a) \to X$. By our induction assumption, it is flat away from S with fibre dimension $(e-a)(n-d+1)-2$, and its fibre dimension increases by one over S. The map $\rho : M_{0,2}(X,a) \to M_{0,1}(X,a)$ forgets the interection point.

Let p be a point in $\overline{\mathcal{M}}_{0,1}(X,a)$, corresponding to a curve C of degree a. Let Z be an irreducible component of $\psi^{-1}(p)$. We aim to show that Z must have dimension

$$(e-a)(n-(r-1)d)-1$$

Consider the image $\phi(Z)$, which by construction lies in $\rho^{-1}(p)$. This image contains points of $M_{0,2}(X,a)$ corresponding to the curve C, with an additional choice of intersection point. The possible intersection points in the irreducible component Z are either all of C, or a single point.

If the possible intersection points are all of C, then $\psi^{-1}(p) = (\rho \circ \phi)^{-1}(p)$ has fibre dimension one greater than the generic fiber of ϕ. In this case, the dimension of Z is $(e-a)(n-(r-1)d+1)-1$ as required.

If all of the intersection points in Z are a single point in S, then $\rho^{-1}(p)$ is a single point and the dimension of Z is the dimension of a fibre of ϕ over a point of S. Again, this is $(e-a)(n-(r-1)d+1)-1$.

Finally, the intersection points in Z cannot all be a single point outside of S. If this were the case then Z would have dimension $(e-a)(n-(r-1)d+1)-2$. But this contradicts our lower bound for the dimension of Z. □

Theorem 3.2.5. *The evaluation map $ev_1 : \overline{\mathcal{M}}_{0,1}(X,e) \to X$ has constant fibre dimension*

$$e(n-(r-1)d+1)-2$$

away from a finite set of points S. Over the points of S, the fibre dimension increases by at most one.

Proof. Considering the commutative diagram above, we will show that

$$ev_e : \overline{\mathcal{M}}_{0,2}(X,e) \to X$$

from the space of curves with two marked points to X, evaluating on just the first marked point, has fibre dimension $e(n-(r-1)d+1)-1$ away from S. We know from bend-and-break that every fibre of ev_e intersects the image of $\overline{\mathcal{M}}_{0,2}(X,a) \times_X \overline{\mathcal{M}}_{0,1}(X,e-a)$ in codimension 1, for some a; therefore it is enough to show that

$$ev_a \circ \psi : \overline{\mathcal{M}}_{0,2}(X,a) \times_X \overline{\mathcal{M}}_{0,1}(X,e-a) \to X$$

has constant fibre dimension $e(n-(r-1)d+1)-3$ over X away from S.

But the map $ev_a : \overline{\mathcal{M}}_{0,1}(X,a) \to X$ has constant fibre dimension

$$a(n-(r-1)d+1)-2$$

away from S, by our induction assumption. We have also shown that ψ has constant fibre dimension $(e-a)(n-(r-1)d+1)-1$, so we are done.

Furthermore, the fibre dimension of ev_a increases by one over the non-flat points S. Therefore the same is true of $ev_a \circ \psi$, which will then have fibre dimension at most $e(n-(r-1)d+1)-1$ over a point of S. □

4 Irreducibility of the Space of Lines

In this section, we prove that $\overline{\mathcal{M}}_{0,1}(X,1)$ is irreducible. As in the previous section, the map $\pi : X \to \mathbb{P}^n$ is a cyclic cover of degree r branched over a hypersurface D of degree rd, satisfying the inequality $2d(r-1) \leq n$.

We start with the following simple observation.

Lemma 4.0.6. *Let $f : Y \to Z$ be a map of projective varieties, with Z irreducible. If the general fibre of f is irreducible and if every irreducible component of Y is dominant over Z, then Y is irreducible.*

Proof. Suppose $Y_1, ..., Y_n$ are distinct irreducible components of Y. For each i, let U_i be the dense open subset of Y_i that is not contained in any other Y_j for $j \neq i$.

By assumption each U_i is dominant over Z; its image $f(U_i)$ is a constructible set containing a dense open subset V_i. Then the intersection $V_1 \cap ... \cap V_n$ is a dense open subset of Z, and for each $p \in V_1 \cap ... \cap V_n$ the fibre $f^{-1}(p)$ contains a point from each of the disjoint sets U_i. In particular, $f^{-1}(p)$ is reducible if $n > 1$. □

Now consider the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$. The space X is irreducible, and the evaluation map is flat away from a finite set of points S. If $\overline{\mathcal{M}}_{0,1}(X,e)$ has any irreducible components that are not dominant over X, then they must lie entirely over some point of S. But this is impossible: the dimension of the fibres over S is less than than the lower bound for the dimension of $\overline{\mathcal{M}}_{0,1}(X,e)$ that we computed in theorem 1.5.4. So the evaluation map is dominant on every irreducible component.

Therefore to prove irreducibility of $\overline{\mathcal{M}}_{0,1}(X,e)$, it suffices to prove that a general fibre of the evaluation map is irreducible. We will prove this directly in the case $e = 1$ in this chapter, in theorem 4.2.6, by considering lines in $\mathbb{P}^n$. In the next chapter we will consider the *boundary divisor* of $\overline{\mathcal{M}}_{0,1}(X,e)$ on which the curve is the union of two curves of smaller degree, and prove the general case by induction on e.

4.1 The Principal Parts Bundle

In this section we prove the following fact, which we use in the next section to prove irreducibility of the space of lines.

Theorem 4.1.1. *Let D be a hypersurface of degree d in $\mathbb{P}^n$. Let p be a point not contained in D, and consider the space of lines through p. The subspace of lines intersecting D at a single point (with multiplicity d) is connected.*

To prove this, we will need to consider the *principal parts bundle.* If F is a locally free sheaf on a smooth variety X, the principal parts bundle $P^k(F)$

is another locally free sheaf on X that records the behaviour of sections of F to order k. We describe its construction below.

First, let C be a smooth curve and L a line bundle on C. Let p be a point on C and m the maximal ideal of $\mathcal{O}_{C,p}$. A section α of L, defined on a neighborhood of p, can be considered as an element of

$$L_p/m^k L_p$$

for any $k \geq 0$. To say that α "vanishes to order k" at p is to say that it is zero in this ring.

For any $k \geq 0$, the principal parts bundle of order k will be a locally free sheaf $P^k(L)$ on C whose fiber at every point p is exactly the quotient $L_p/m^k L_p$ (as a complex vector space). For any germ α in the stalk of L at a point p, there is a corresponding section in the stalk of $P^k(L)$; this section is zero exactly when α vanishes to order k at p. We think of $P^k(L)$ as recording the behavior of sections of L to order k.

Given a point $p \in \mathbb{P}^n$, consider the family of lines through p. Pull back the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$ to the total space of this family. Then the (relative) bundle of principal parts $P^k(\mathcal{O}(d))$ will be a locally free sheaf on the total space. It records the vanishing of sections of $\mathcal{O}(d)$ along each line in the family, up to order k. Our hypersurface D is given by a particular global section of $\mathcal{O}(d)$, and the points at which the line is tangent to degree d are exactly those points at which the induced global section of $P^d(\mathcal{O}(d))$ vanishes.

Then to prove that this vanishing set is connected, we will define the set D_k to be the set of points in D at which a line through p is tangent to D at order k. By considering a filtration of the principal parts bundle, we will describe each D_k as a closed subscheme of D_{k-1}, and show that each D_k is connected.

Construction of Principal Parts Bundle The construction is given in [G1] Section 16.7. To begin with, let $f : C \to S$ be a smooth family of curves over some base, and let π_1, π_2 be the projections from $C \times_S C$

$$\begin{tikzcd} C \times_S C \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] & C \arrow[d, "f"'] \\ C \arrow[r, "f"'] & S \end{tikzcd}$$

Let $\Delta : C \to C \times_S C$ be the diagonal, and I the associated sheaf of ideals.

Definition 4.1.2. If F is a locally free sheaf on C, then the *principal parts bundle* of F is the coherent sheaf on C defined by

$$P^k(F) := \pi_{1*}\left(\frac{\pi_2^*(F)}{I^k\pi_2^*(F)}\right)$$

The principal parts bundle admits an important filtration. For all $k > 0$ we have a short exact sequence

$$0 \to \frac{I^{k-1}}{I^2} \to \frac{\mathcal{O}_{C\times C}}{I^k} \to \frac{\mathcal{O}_{C\times C}}{I^{k-1}} \to 0$$

We tensor this exact sequence with the locally free sheaf π_2^*F and push forward by π_1, both of which preserve exactness. The resulting short exact sequence contains the *restriction map*

$$\rho_k : P^k(F) \to P^{k-1}(F)$$

and the kernel of the restriction map is

$$\pi_{1*}\left(\pi_2^*F \otimes \left(\frac{I}{I^2}\right)^{\otimes(k-1)}\right) = F \otimes \pi_{1*}\left(\frac{I}{I^2}\right)^{\otimes(k-1)}$$

But $\pi_{1*}\left(I/I^2\right)$ is equal to $\Delta^*(I/I^2)$, the sheaf of relative differentials $\Omega_{C/S}$. Therefore for each k we have the short exact sequence

$$0 \to F \otimes \Omega_{C/S}^{\otimes(k-1)} \to P^k(F) \xrightarrow{\rho_k} P^{k-1}(F) \to 0$$

It is worth considering the affine case more carefully. Let $B \to A$ be a morphism of rings, and let $\pi_1^\#$ and $\pi_2^\#$ be the maps from A to the tensor product $A \otimes_B A$ that send a to $a \otimes 1$ and $1 \otimes a$ respectively.

$$\begin{tikzcd}
A \otimes_B A & A \arrow[l, "\pi_2^\#"'] \\
A \arrow[u, "\pi_1^\#"] & B \arrow[l] \arrow[u]
\end{tikzcd}$$

Then $A \otimes_B A$ has an A-bimodule structure, via the two different A-module structures induced by π_1 and π_2.

We write $I \subseteq A \otimes_B A$ for the kernel of the multiplication map $A \otimes A \to A$. Now let M be an A-module. The principal parts bundle of the sheaf associated to M is a coherent sheaf corresponing to the module

$$\frac{A \otimes_B A}{I^k} \otimes_{\pi_2} M$$

where by $\otimes_{\pi_2}$ we mean the tensor product of A-modules with respect to the second projection $\pi_2 : A \to A \otimes_B A$. We then consider this tensor product as an A-module via the action of A induced by the *first* projection π_1. Note that if $k = 1$, the two A-module structures on $A \otimes_B A$ are identical, and therefore $P^1(F) = F$.

We can try to define a map

$$M \to \frac{A \otimes_B A}{I^k} \otimes_{\pi_2} M$$

sending m to $(1 \otimes 1) \otimes m$. But this map is *not* A-linear, because we are considering the second module with the action of A via π_1.

Returning to the global case, there is in general no globally-defined $\mathcal{O}_C$-linear map from a sheaf F on C to its principal parts bundle. However, such a map exists when F is the pullback of a sheaf from the base S.

Theorem 4.1.3. *Let $f : C \to S$ be a map of varieties and let G be a locally free sheaf on S. Then for each k there is a canonical $\mathcal{O}_C$-linear map $d_k : f^*G \to P^k(f^*G)$ that commutes with the restriction maps*

$$\begin{tikzcd} f^*G \arrow[r, "d_k"'] \arrow[dr, "d_{k-1}"'] & P^k(f^*G) \arrow[d, "\rho_k"'] \\ & P^{k-1}(f^*G) \end{tikzcd}$$

Proof. First consider the affine case $f : \operatorname{Spec} A \to \operatorname{Spec} B$, as above. Let G be a coherent sheaf corresponding to a B-module N, f^*G corresponds to the A-module $N \otimes_B A$. Then we can define a map

$$N \otimes_B A \to \frac{A \otimes_B A}{I^k} \otimes_{\pi_2} (N \otimes A)$$

defined by

$$n \otimes a \mapsto (a \otimes 1) \otimes n$$

This map is well-defined, and is A-linear if we consider $A \otimes A$ as an A-module via the *left* action. Therefore this extends to a global morphism of A-modules from f^*G to $P^k(f^*G)$. □

Corollary 4.1.4. *Let $f : C \to S$ be a family of curves and let F be a locally free sheaf on C. Then there is a natural $\mathbb{C}$-linear map of vector spaces*

$$H^0(C, F) \to H^0(C, P^k(F))$$

which commute with the restriction maps $\rho_k : P^k(F) \to P^{k-1}(F)$ and which at every point $p \in C$ sends an element of the stalk F_p to its kth order expansion at p on the fibre over $f(p)$.

Proof. Let G be the trivial vector bundle on S given by

$$H^0(C, F) \otimes_S \mathcal{O}_S$$

Then f^*G is a trivial vector bundle on C. By 4.1.3 there is a map of sheaves on C

$$d_k : f^*G \to P^k(f^*G)$$

There is also another map of sheaves on C

$$\delta : f^*G \to F$$

Since the formation of the principal parts bundle is functorial, this gives a map

$$P^k(\delta) : P^k(f^*G) \to P^k(F)$$

Composing these two, we have a map of sheaves

$$P^k(\delta) \circ d_k : f^*G \to P^k(F)$$

In particular, there is a map between the spaces of global sections

$$H^0(C, f^*G) \to H^0(C, P^k(F))$$

Since the space of global sections of f^*G is exactly $H^0(C, F)$, this gives the required map. $\square$

We use the principal parts bundle to examine the space of lines through a point p that are tangent to a hypersurface D to high degree. Let D be a hypersurface in $\mathbb{P}^n$ of degree d, and let $p \in \mathbb{P}^n$ be a point not in D.

Definition 4.1.5. For all $k \leq d$, let $D_k \subseteq D$ be the subset of points $q \in D$ such that the line through p and q intersects D at q with multiplicity k or greater.

Clearly D_k is empty if $k > d$, and $D_1 = D$. We show in 4.1.9 that each D_k is cut out by a global section of an ample line bundle inside of D_{k-1}, and so has a natural subcheme structure. For this, we will need some preliminary lemmas.

Lemma 4.1.6 (Enriques-Severi-Zariski)**.** *Let X be an integral projective variety of dimension $n \geq 2$, and let D be an ample divisor. Then D is connected.*

Proof. ([H] III Corollary 7.9) We may assume that X is normal, by passing to the normalisation if necessary. For any $m > 0$ there is an exact sequence

$$0 \to \mathcal{O}_X(-mD) \to \mathcal{O}_X \to \mathcal{O}_{mD} \to 0$$

Therefore, in particular, we have an exact sequence

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{mD}) \to H^1(X, \mathcal{O}_X(-mD))$$

By Serre duality and the ampleness of D, the higher cohomology of $\mathcal{O}_X(-mD)$ vanishes for $m >> 0$. Since $H^0(X, \mathcal{O}_X)$ is one-dimensional, so is $H^0(X, \mathcal{O}_{mD})$. In particular, this implies that D is connected. $\square$

Definitions 4.1.7. Let p be a point in $\mathbb{P}^n$ and let S be the parameter space of lines through p, so S is isomorphic to $\mathbb{P}^{n-1}$. Let C be the total space of the universal family of lines through p. Let

$$f : C \to S$$

be the map from the total space to the parameter space.

The total space C is isomorphic to the blowup of $\mathbb{P}^n$ at p, with a birational map $\beta : C \to \mathbb{P}^n$. The Picard group $\operatorname{Pic} C$ is generated by the exceptional divisor E and the pullback of a hyperplane section, which we denote H.

Lemma 4.1.8. *The relative canonical bundle $\Omega_{C/S}$ on C is $\mathcal{O}_C(-H - E)$*

Proof. The Picard group of C is $\mathbb{Z}H \oplus \mathbb{Z}E$. Write

$$\Omega_{C/S} = \mathcal{O}_C(aH + bE)$$

for integers a and b.

A fiber of $f : C \to S$ over a point $s \in S$ is a single line through p, of divisor class $H - E$. The relative canonical bundle restricts to the canonical sheaf on this fibre, so $(aH + bE) \cdot (H - E) = -2$.

On the other hand, let ι be the inclusion of the exceptional divisor E. We have a short exact sequence

$$0 \to N^\vee_{E/C} \to \iota^*\Omega_{C/S} \to \Omega_{E/S} \to 0$$

The sheaf $\Omega_{E/S}$ is zero, since E is a section of the map $C \to S$. Therefore the pullback of the relative canonical sheaf $\Omega_{C/S}$ to E is the conormal bundle of E in C, which is $\mathcal{O}_E(1)$. So we have $(aH + bE) \cdot E = 1$.

Combining these, we have $a = -1$ and $b = -1$, and therefore $\Omega_{C/S}$ is equal to $\mathcal{O}_C(-H - E)$ as claimed. $\square$

Now we have enough to prove 4.1.1.

Theorem 4.1.9. *The subvariety $D_k \subseteq D$ is connected for all $k \geq 1$, and is non-empty for $k \leq d$.*

Proof. The hypersurface $D \subseteq \mathbb{P}^n$ is given by a global section g of $\mathcal{O}_{\mathbb{P}^n}(d)$. Since it does not contain p, this hypersurface is isomorphic to its preimage $\beta^* D$ in C, which is given by the corresponding global section $\beta^* g$ of $\beta^* \mathcal{O}_{\mathbb{P}^n}(d)$. Consider the relative principal parts bundle

$$P^k(\beta^* \mathcal{O}_{\mathbb{P}^n}(d))$$

which records the vanishing of sections of $\beta^* \mathcal{O}_{\mathbb{P}^n}(d)$ on the fibers of the family $C \to S$; that is, on lines through p. Since $\beta^* \mathcal{O}_{\mathbb{P}^n}(d)$ is not the pullback of a sheaf from S, it does not admit a morphism to its principal parts bundle. However, by theorem 4.1.4 there is a map

$$H^0(C, \beta^* \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, P^k(\beta^* \mathcal{O}_{\mathbb{P}^n}(d)))$$

for each integer k. We let g_k be the image of g under this map.

Therefore for each k we have a global section g_k of $P^k(\beta^* \mathcal{O}_{\mathbb{P}^n}(d))$ on C, and by construction of the degree k principal parts bundle, the vanishing locus of g_k inside D_{k-1} is exactly D_k. We prove by induction that each D_k is connected. The base case $k = 1$ follows since D_1 is the hypersurface D, so assume that D_{k-1} is connected and that $k \geq 2$. Recall that for each k we have an exact sequence of locally free sheaves

$$0 \to \beta^* \mathcal{O}_{\mathbb{P}^n}(d) \otimes \Omega_{C/S}^{\otimes(k-1)} \to P^k(\beta^* \mathcal{O}_{\mathbb{P}^n}(d)) \xrightarrow{\rho_k} P^{k-1}(\beta^* \mathcal{O}_{\mathbb{P}^n}(d)) \to 0$$

The section g_k is a global section of the middle term, and its image in the last term is g_{k-1}. By 4.1.8 the divisor class of $\Omega_{C/S}$ is $-H - E$ on C, so this sequence becomes

$$0 \to \mathcal{O}_C\left((d-k+1)H - (k-1)E\right) \to P^k(\beta^* \mathcal{O}_{\mathbb{P}^n}(d)) \xrightarrow{\rho_k} P^{k-1}(\beta^* \mathcal{O}_{\mathbb{P}^n}(d)) \to 0$$

Now if we restrict this exact sequence to D_{k-1}, then the restriction of g_k is a global section of the middle term that is (by construction) sent to zero in the last term. Therefore it is the image of a global section of $\mathcal{O}_C\left((d-k+1)H - (k-1)E\right)|_{D_{k-1}}$. Since D_{k-1} does not intersect E, we have

$$\mathcal{O}_C\left((d-k+1)H - (k-1)E\right)|_{D_{k-1}} = \mathcal{O}_C\left((d-k+1)H\right)|_{D_{k-1}}$$

which is ample. Therefore D_k is the zero locus of a global section of an ample line bundle, and is connected by 4.1.6.

□

4.2 The case $e = 1$

Now we show that the general fibre of $ev : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is irreducible. Recall that a line $L \subseteq \mathbb{P}^n$ is called r-multiply tangent to a hypersurface D if either $L \subseteq D$, or all points of $L \cap D$ have multiplicity divisible by r.

Definition 4.2.1. Let Y be the space of tuples (D, L, Q, p) where

- D is a hypersurface in $\mathbb{P}^n$ of degree rd,
- L is a line in $\mathbb{P}^n$ and Q is an effective divisor of degree d on L with $rQ \subseteq L \cap D$, and
- p is a closed point on L.

Lemma 4.2.2. *This space Y is smooth and irreducible.*

Proof. Note that there is a forgetful map from Y to the space of lines L with chosen points $q_1, ..., q_d, p$. The fibres of this forgetful map consist of those hypersurfaces of degree rd that vanish to order r at each of the points $q_1, ..., q_d$.

If q is a point in the divisor Q of mutiplicity k, we can consider those hypersurfaces D such that $L \cap D$ contains $rk \cdot q$. This imposes rk independent linear conditions on $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))$ (with each condition determining a linear factor of the polynomial Φ that cuts out the hypersurface). Since the degree of Q is rd, the space of hypersurfaces that are r-multiply tangent to L at Q will be a linear subspace of $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(rd))$ of codimension rd. Therefore the forgetful map exhibitss Y as a projective bundle over a smooth irreducible space. $\square$

The space Y also has a forgetful map $e : Y \to Z$ to the space Z that parametrises pairs (D, p). We are interested in the fibres of this map. Specifically, we want to show that a general fibre of this map is irreducible. By generic smoothness, the general fibre is smooth, so it suffices to show that a general fibre is connected.

We will show that the general fibre cannot consist of two or more disconnected components by picking out a specific component in each fibre in a well-defined and continuously varying way. Once we have done so, the union of all of these components, as D and p vary, will form an irreducible component of Y. Since Y is irreducible, this will show that there is only one connected component in the general fibre. A formal statement of this stategy is as follows:

Theorem 4.2.3. *Let $e : Y \to Z$ be a morphism of smooth irreducible schemes over an algebraically closed field. Let Y' be irreducible and $i : Y' \to Y$ a morphism such that $e \circ i : Y' \to Z$ is dominant with irreducible general fibre. Then the general fibre of e is also irreducible.*

See [dJS] Lemma 3.2 for details (see also [G1] Proposition 4.5.9)

Corollary 4.2.4. *Let D be a general hypersurface of degree rd in $\mathbb{P}^n$ and let p be a general point not in D. Then the space of lines through p that are r-multiply tangent to D is irreducible.*

Proof. Let Y be as in 4.2.1, and Y' the sublocus of Y in which Q consists of a single point q with mutliplicity d. Then Y' parametrizes tuples (D, L, p, q) such that the line L passes through p and is tangent to D at exactly one point q, with multiplicity rd. Let W be the space of pairs (p, q). Then the forgetful map $Y' \to W$ exhibits Y' as a projective bundle over a smooth irreducible space, so Y' itself is smooth.

Now let Z be the space of pairs (D, p). The space Y' also has a forgetful map to Z. By theorem 4.1.9, this forgetful map is surjective and every fibre is connected. However, since this map is between smooth varieties over $\mathbb{C}$, a general fibre is also smooth, and therefore a general fibre is irreducible.

Therefore the inclusion $Y' \to Y$ and the map $Y \to Z$ satisfy the conditions of theorem 4.2.3. We conclude that the general fibre of the forgetful map $Y \to Z$ is irreducible. □

Corollary 4.2.5. *Let M^* be as defined in section 2.1, with evaluation map $ev : M^* \to \mathbb{P}^n$, and let p be a general point of $\mathbb{P}^n$ not in the branch divisor. Then the fibre $ev^{-1}(p)$ is irreducible.*

Proof. The points in M^* parametrise lines in $\mathbb{P}^n$ that are r-multiply tangent to the branch divisor D. Therefore $ev^{-1}(p)$ is irreducible by 4.2.4. □

Now we can show that for a general X, the fiber of the evaluation map $\overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is irreducible over a general point.

Theorem 4.2.6. *Let $\pi : X \to \mathbb{P}^n$ be a general cyclic cover and let p be a general point in $\mathbb{P}^n$ not in the branch divisor. Let $q_1, ..., q_r \in X$ be the r points in the preimage $\pi^{-1}(p)$. Then for each i the fibre $ev^{-1}(q_i) \subseteq \overline{\mathcal{M}}_{0,1}(X, 1)$ is irreducible.*

Proof. The space $\overline{\mathcal{M}}_{0,1}(X, 1)$ contains a boundary divisor of reducible curves. For a general point p, the boundary divisor intersects $ev^{-1}(p)$ in codimension one. We assume that this is the case.

Then it suffices to prove that the space of irreducible curves in $v^{-1}(q_i)$ is irreducible. By 1.4.5, no irreducible curve passes through both q_i and q_j for $i \neq j$. Therefore $ev^{-1}\pi^{-1}(p)$ in $\overline{\mathcal{M}}_{0,1}(X, 1)$ is the *disjoint* union of the fibers $ev^{-1}(q_i)$.

The fibre $ev^{-1}(p)$ in M^* is the quotient of $ev^{-1}\pi^{-1}(p)$ in $\overline{\mathcal{M}}_{0,1}(X, 1)$ by the action of $\mathbb{Z}/r\mathbb{Z}$, and is therefore isomorphic to any one of these r disjoint components. Therefore these components are irreducible, as required. □

5 Irreducibility of $\overline{\mathcal{M}}_{0,0}(X, e)$

5.1 Trees

Recall from 1.2.3 that the space $\overline{\mathcal{M}}_{0,k}(X, e)$ parametrises stable curves on X of genus zero and degree e. In order to stratify stable curves, we encode their combinatorial data in its *dual graph*.

Notation 5.1.1. For our purposes, all graphs will be trees - that is, they will contain no loops - and will be finite. All graphs also come with the following data:

- For each vertex v, a positive integer d_v which we call the *degree* of the vertex.
- A list $p_1, ..., p_k$ of vertices which we call the *marked points* of the graph. The set of marked points may be empty, and the elements of the set may not be distinct.

We use *dual graph* to mean a collection of data of this type.

Now for each stable curve C on X we construct the corresponding *dual graph of* C by considering the reducible components of the domain curve.

Definition 5.1.2. Let $C \to X$ be a stable map of a genus-zero curve to X. The dual graph of C is a graph (in the above sense) constructed from the following data:

- a vertex for every irreducible component of C;
- for each such vertex v, an integer d_v equal to the degree of the corresponding irreducible component;
- for every intersection point between two components of C, an edge between the corresponding vertices;
- and for every marked point on C, a marking on the corresponding vertex.

The curves we consider have genus zero, so the dual graph does indeed contain no loops, as well as no self-intersections. The *total degree* is of the dual graph is defined to be the sum of the integers d_v over all vertices, equal to the degree of the curve itself.

Notation 5.1.3. We use the word *tree* to mean a stable map from a curve C (possible reducible) to X. A *linear tree* is a tree that is the union of degree 1 curves, so its dual graph has degree 1 at every vertex.

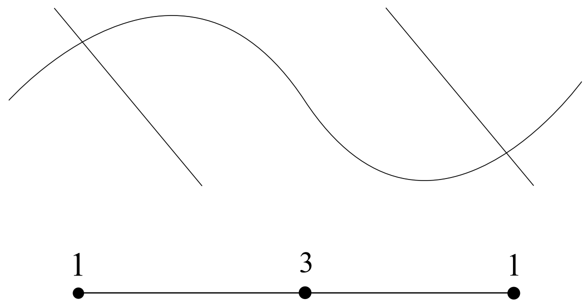

Figure 1: A reducible curve and corresponding dual graph

We will be interested in how trees deform in families, and what happens to the corresponding dual graphs. An irreducible degree e curve deforming in a family might degenerate, at a specific point in the family, into the union of two or more irreducible components with total degree e. At that point, the dual graph changes from a single vertex to a graph with several vertices. Conversely, a reducible curve might deform into a curve with fewer components of higher degree, and the dual graph would contract.

In order to formalise these ideas, we begin by defining *subgraphs* and *contractions* of dual graphs in the obvious way.

Definition 5.1.4. Let G be a graph as defined above. A *subgraph* H of G is a graph such that: the vertices of H are a subset of the vertices of G; the edges of H are exactly those edges of G connecting vertices in H; the degree of each vertex in H is the same as its degree in G; and the marked vertices of H are exactly those marked vertices of G that are also vertices in H, including duplicates.

Definition 5.1.5. Let G be a dual graph and H a non-empty subgraph. We say that G' is a *contraction* of G along H if: the vertices of G' are those vertices in G that are not in H, together with one additional vertex v_H; the edges of G' are those edges of G connecting two vertices that are not in H, together with one edge from v to v_H for every edge connecting a vertex v to a vertex in H; the degree of v_H is the total degree of the subgraph H, and the degrees of the other vertices are their degrees in G;

Definition 5.1.6. Let G be a dual graph and let G' be the dual graph obtained by contracting G along a finite series of subgraphs. Then we say that G' is a *deformation* of G, and that G is a *specialisation* of G'.

Note that a specialisation *increases* the number of vertices in the graph, while a deformation decreases this number. In a family of curves, the number of irreducible components is upper semicontinuous, so specialisation to a particular reducible member of the family will increase the number of components. The dual graph of the reducible curve will be a specialisation of

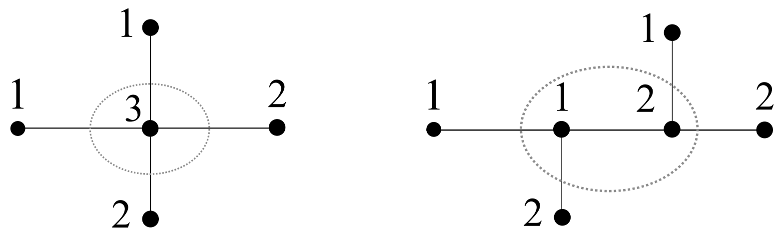

Figure 2: Specialisation and deformation

the dual graph of a general curve. Similarly, if a reducible curve is deformed in a family, then we often find that a general deformation will result in a single irreducible curve.

Definition 5.1.7. For a dual graph G, a tree C is said to be of *type* G if its dual graph is either G itself or any specialisation of G.

Let G be an isomorphism class of trees, of total degree e and with m marked points. The space of trees of type G is a subspace of $\overline{\mathcal{M}}_{0,m}(X,e)$, and we will write $\overline{\mathcal{M}}_{0,m}(X,G)$ for this subspace. If G' is a specialization of G then

$$\overline{\mathcal{M}}_{0,m}(X,G') \subseteq \overline{\mathcal{M}}_{0,m}(X,G)$$

If G consists of a single vertex (with degree e) then $\overline{\mathcal{M}}_{0,m}(X,G)$ is the entire space of degree e curves, since all dual graphs of total degree e deform to G.

5.2 The Case $e > 1$

In this section we prove by induction that the space $\overline{\mathcal{M}}_{0,1}(X,e)$ is irreducible for all $e > 1$, having proved the case $e = 1$ above. We adapt the argument described in [HRS].

The space X is irreducible. We have already shown that the evaluation map $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat away from a finite set of points S, so every non-dominant irreducible component must lie entirely over a point of S. We have shown in the previous chapter that the fibers of the evaluation map over the points of S have dimension $e(n-(r-1)d+1)-1$ (that is, one greater than a general fibre). But we also have a lower bound on the dimension of the irreducible components of the moduli space:

$$\dim \overline{\mathcal{M}}_{0,1}(X,e) \geq e(n-(r-1)d+1)+n-2$$

Therefore no irreducible component lies entirely over a point of S, and the evaluation map is dominant on every component. As we observed for the case $e = 1$, the irreducibility of $\overline{\mathcal{M}}_{0,1}(X,e)$ will then follow if we can show that a general fibre of the evaluation map is irreducible.

In order to show that a general fibre of $ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$ is irreducible, we prove the following facts.

1. If G is a dual graph, then the space $\overline{\mathcal{M}}_{0,0}(X,G)$ of trees of type G is irreducible. In particular, the space of linear trees of a given isomorphism type is irreducible.

2. Every irreducible component of a general fibre of the evaluation map contains a linear tree. By considering the codimension of linear trees inside $\overline{\mathcal{M}}_{0,1}(X,e)$, we will show that in a general fibre, each irreducible component contains an entire irreducible component of the locus of linear trees.

3. A general linear tree is a smooth point of the space $M_{0,1}(X,e)$.

We give proofs of these facts in the next section. Together, they are enough to prove the case $e = 2$.

Theorem 5.2.1. *The space $\overline{\mathcal{M}}_{0,1}(X,2)$ is irreducible.*

Proof. (Assuming 1, 2 and 3 above) As with the case $e = 1$, it is enough to show that a general fibre of the evaluation map is irreducible. Let Z be an irreducible component of a general fibre $ev^{-1}(p)$. We know that Z contains a linear tree, by (2). This tree comprises two lines intersecting at a single point, and its dual graph contain two vertices joined by a single edge. Since the locus of such trees is irreducible by (1), Z contains all such trees.

But this is true for any other irreducible component Z', so all irreducible components of the fibre must contain all trees. Since a general tree in this intersection is a smooth point of $\overline{\mathcal{M}}_{0,1}(X,2)$ by (3), it cannot lie on the intersection of two components. Therefore there is only one irreducible component. □

When the degree is greater than two, there are many different isomorphism types of linear trees. In order to conclude that the irreducible components of the fibre intersect, we need to show the following additional fact:

4. A linear tree in an irreducible component of a general fibre $ev^{-1}(p)$, of a particular isomorphism type, can be deformed into a linear tree of any other isomorphism type of the same degree, in such a way that the deformation does not cross into a different component of the fibre.

As an example, consider the case $e = 3$ with one marked point. A linear tree of degree 3 has a dual graph that can take one of two isomorphism types: in both cases the underlying graph is the unique tree on three vertices, but they differ depending on whether or not the marked point is on the central component or one of the others.

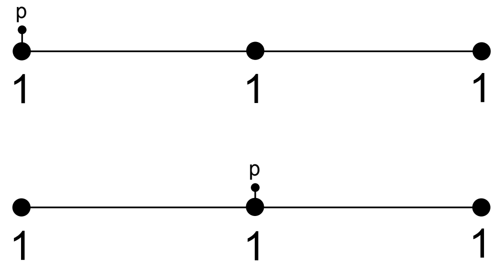

Figure 3: Isomorphism types of linear trees in $\overline{\mathcal{M}}_{0,1}(X, 3)$

We show that every irreducible component of a fibre over a point contains an entire irreducible component of the space of linear trees. But this space of trees has two components which do not intersect. To show that both types of linear tree lie in the same component of the fibre, we show that one can be deformed into the other.

Specifically, we perform three steps. First we deform the tree into the union of a line and an irreducible conic. Next, we slide the marked point along the conic to a new position. Finally, we specialise back to a linear tree, this time with the marked point on the other component.

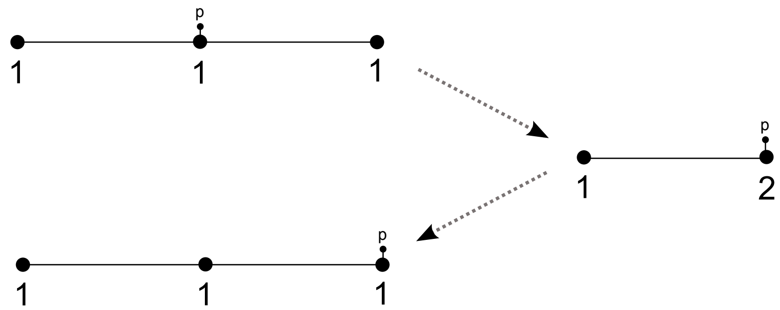

Figure 4: A deformation followed by a specialisation

By choosing the conic carefully, we can ensure that this process does not cross into a new irreducible component of $ev^{-1}(p)$. Therefore all irreducible components of the fibre contain all linear trees. Since a general such tree is a smooth point of the moduli space, there must be only one irreducible component.

5.3 Proofs

Lemma 5.3.1. *If $\overline{\mathcal{M}}_{0,1}(X, e)$ is irreducible, then so is $\overline{\mathcal{M}}_{0,m}(X, e)$ for any $m \geq 0$.*

Proof. The forgetful map

$$\overline{\mathcal{M}}_{0,m}(X,e) \to \overline{\mathcal{M}}_{0,m-1}(X,e)$$

has one-dimensional fibres over all points, and is a $\mathbb{P}^1$ bundle over the dense open subset of points corresponding to irreducible curves. Therefore one is irreducible if and only if the other is. $\square$

In this section, we prove the facts 1 through 4 above, and conclude that $\overline{\mathcal{M}}_{0,m}(X,e)$ is irreducible for all e and m. The argument is by induction on e, and we may assume that we have proved the irreducibility of $\overline{\mathcal{M}}_{0,m}(X,a)$ for all m and for all $a < e$. We have already proved the case $e = 1$ in section 4.

Theorem 5.3.2. *Let G be an isomorphism class of trees of total degree e and with one marked point . Suppose that G has k vertices with $k \geq 2$. Let $\overline{\mathcal{M}}_{0,1}(X,G)$ be the space of pointed curves of type G. Then $\overline{\mathcal{M}}_{0,1}(X,G)$ is irreducible.*

Furthermore, let

$$ev_G : \overline{\mathcal{M}}_{0,1}(X,G) \to X$$

be the evaluation map. Then ev_G is flat away from a finite set of points S, and the general fibre is irreducible. Away from S the fibre dimension is

$$e(n-(r-1)d+1)-2-(k-1)$$

That is, the curves of type G through a point p have codimension $k-1$ in the space of all degree e curves through p. Over the non-flat points S, the fiber dimension increases by one.

Proof. Note that we have $k \geq 2$ in this theorem; the case $k = 1$ is the claim that $\overline{\mathcal{M}}_{0,1}(X,e)$ itself is irreducible, which is the goal of this section. By our induction hypothesis, we may assume this theorem for all G of total degree less than e, for all k including $k = 1$.

The proof is very similar to the proof of theorem 3.2.4. Since G has at least two vertices, a curve of type G is reducible and is the union of two curves of types G_1 and G_2. Let G_i have k_i vertices and total degree e_i, with $k_1 + k_2 = k$ and $e_1 + e_2 = e$. Assume without loss of generality that the marked point lies in the curve of type G_1. Consider the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,2}(X,G_1) \times_X \overline{\mathcal{M}}_{0,1}(X,G_2) \arrow[d, "\phi"'] \arrow[dr, "\psi"] \arrow[rr, hook] & & \overline{\mathcal{M}}_{0,1}(X,e) \arrow[d] \\
\overline{\mathcal{M}}_{0,2}(X,G_1) \arrow[r, "\rho"'] & \overline{\mathcal{M}}_{0,1}(X,G_1) \arrow[r, "ev_{G_1}"'] & X
\end{tikzcd}$$

First, the map ϕ is the base change of the evaluation map $M_{0,1}(X, G_2) \to X$, so by induction its generic fibre dimension is

$$e_2(n - (r - 1)d + 1) - 3 - (k_2 - 1)$$

Therefore the generic fibre of ψ has dimension one greater than this. The dimension of the generic fibre is a lower bound for the fibre over any point, so we have

$$\dim \psi^{-1}(p) \geq e_2(n - (r - 1)d + 1) - 2 - (k_2 - 1)$$

We argue that ψ has this constant fibre dimension over all points.

Consider a point p in $\overline{\mathcal{M}}_{0,1}(X, G_1)$, representing a curve C of type G_1, and take an irreducible component Z of $\psi^{-1}(p)$ inside of the product $\overline{\mathcal{M}}_{0,2}(X, G_1) \times_X \overline{\mathcal{M}}_{0,2}(X, G_2)$. The curves in this component are the union of C with another curve of type G_2.

We can consider the evaluation map to X that takes a reducible curve of this form to the intersection point. The image of the irreducible component Z - the possible intersection points of curves in Z - consists of either a single point in C, or all of C.

If the possible intersection points cover all of C, then the dimension of the irreducible component Z is one greater than the generic fibre of ϕ.

If the image is a single point in S, then the dimension of Z is equal to the dimension of ϕ over this point; by assumption, the fibre dimension over a non-flat point in S is also one greater than the generic fibre of ϕ. Either way, the dimension of Z is

$$e_2(n - (r - 1)d + 1) - 2 - (k_2 - 1)$$

as required. Finally, the possible intersection points cannot just be a single point of C away from S, since then the dimension of Z would be below the lower bound. We conclude that ψ has constant fibre dimension.

Now, by our induction assumption, ev_{G_1} has constant fibre dimension equal to

$$e_1(n - (r - 1)d + 1) - 2 - (k_1 - 1)$$

away from S, and this fibre dimension increases by exactly one over S. Adding this to the computed constant fibre dimension of ψ, we see that $\overline{\mathcal{M}}_{0,1}(X, G) \to X$ has the claimed fibre dimension.

Next we note that a general fibre of the evaluation map

$$\overline{\mathcal{M}}_{0,2}(X, G_1) \times_X \overline{\mathcal{M}}_{0,1}(X, G_2) \to X$$

is irreducible, because by assumption a general fibre of both ev_{G_1} and ψ is irreducible.

Finally, we observe that no irreducible component of the product $\overline{\mathcal{M}}_{0,2}(X, G_1) \times_X \overline{\mathcal{M}}_{0,2}(X, G_2)$ lies over a point of the non-flat locus S, because the fibre dimension over these points is only one greater than over the flat locus, and not large enough to contain an entire component. Therefore every component is dominant over X. Since a general fibre is irreducible, we conclude that $\overline{\mathcal{M}}_{0,1}(X, G)$ is irreducible.

□

It remains to prove theorem 5.3.2 when $k = 1$ and G has only one vertex; that is, for the whole space $\overline{\mathcal{M}}_{0,1}(X, e)$. To show that a general fibre of the evaluation map is irreducible, we consider the locus of linear trees.

Theorem 5.3.3. *Let $ev : \overline{\mathcal{M}}_{0,1}(X, e) \to X$ be the evaluation map, and let p be a general point in X. Let Z be an irreducible component of the fiber $ev^{-1}(p)$. If Z contains a curve of type G, then it contains all curves of type G through p. Furthermore, Z contains a linear tree, so in particular Z contains an entire irreducible component of the locus of linear trees in $ev^{-1}(p)$.*

Proof. A fiber of $ev : \overline{\mathcal{M}}_{0,1}(X, e) \to X$ over a point not in S has dimension $e(n - (r - 1)d + 1) - 2$. By theorem 5.3.2, the fibre contains curves with k components in codimension $k - 1$. Since the dimension of Z is greater than $k - 1$, it contains a linear tree of some isomorphism type. Theorem 5.3.2 also states that the locus of such trees in the fibre is irreducible, and so lies entirely in Z. □

Theorem 5.3.4. *For any dual graph G, a general linear tree of type G is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, e)$.*

This follows from the following theorem:

Theorem 5.3.5 (Kollar-Miyaoka-Mori)**.** *Let X be a smooth projective variety. Then there exists a dense open subset $U \subseteq X$ such that any rational curve of degree e on X that intersects U is free, and in particular is a smooth point of the moduli space $M_{0,0}(X, e)$.*

(see [D1] section 4.2, prop. 4.14). That is, any non-free curves on X are restricted to a proper closed subset. Of course, this theorem is vacuous if X contains no rational curves at all. In the case we are considering, the flatness of the evaluation map and our dimension calculations show that a general linear tree is free.

To finish, we show that every irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$ contains linear trees of all isomorphism types. We will need the some lemmas relating to the deformation of linear trees.

Definition 5.3.6. Let G be a dual graph with degree 1 at every vertex. A *conic deformation* of G is a dual graph K that is obtained by contracting a subgraph of total degree 2 inside of G. Two such dual graphs G_1, G_2 are said to be connected by a conic deformation if there is some dual graph K that is a conic deformation of both of them.

Theorem 5.3.7. *Let G_1, G_2 be two dual graphs of linear trees, both of total degree e (so each has e vertices of degree 1) and both with k marked points. Then G_1 and G_2 differ by a finite series of conic deformations.*

Proof. Let G' be the dual graph on e vertices each of degree 1, with a central vertex v_0 to which every other vertex is connected. Let G' have k marked points all equal to v_0.

It suffices to prove that all linear trees differ from G' by a series of conic deformations. Let G be a linear tree and let v_0 be a vertex of G with the most edges. If v_0 is not connected to every vertex then there is some vertex connected to both v_0 and some other vertex. Then there is conic deformation that increases the number of edges connected to v_0. We conclude that after a finite series of conic deformations, v_0 will be connected to every other vertex. Then if any of the marked points are not v_0, there is a conic deformation that increases the number of marked points equal to v_0.

□

This is a purely combinatorial fact about graphs. Next we show that these transformations of graphs can be realised as the deformation of trees on X. We start with a lemma.

Lemma 5.3.8. *Let $f : Y \to Z$ be a flat dominant morphism of projective varieties. Let B be a curve in Z, and let p be a point on B. Let q be a point in the fibre $f^{-1}(p)$. Then there is an irreducible curve B' in Y passing through q such that B' is a finite cover of B. That is, we can lift a finite cover of B to Y.*

Proof. Since f is flat and projective, we can embed Y into some projective space $\mathbb{P}^n_Z$. Let $\mathbb{P}^n_p$ be the projective space fibre over p. Then Noether normalization tells us that there is some linear projection from $\mathbb{P}^n_p$ to $\mathbb{P}^m$ that is both well-defined and finite when restricted to $f^{-1}(p) \subseteq Y$.

This linear projection is well-defined over some open subset of Z (not just over p). Furthermore, since fibre dimension is upper semicontinuous, this projection is finite on some open subset of Y containing $f^{-1}(p)$. By properness, the closed locus on which the projection is not finite lies over some closed subset of Z not containing p. We conclude that by removing a closed subset of Z, we may factor f through a finite morphism to a projective space over Z.

Then the claim is immediate. We can lift B to $\mathbb{P}^m_Z$, and then take its preimage in Y. If this preimage is reducible, take an irreducible component.

□

Now we can show that the conic deformations can be realised as deformations. For any integer e and any dual graph G, we write ev_e and ev_G for the evaluation maps

$$\begin{aligned} ev_e : \quad & \overline{\mathcal{M}}_{0,1}(X, e) \to X \\ ev_G : \quad & \overline{\mathcal{M}}_{0,1}(X, G) \to X \end{aligned}$$

Theorem 5.3.9. *Let p be a general point in X and let $ev_e^{-1}(p)$ be the fibre of the evaluation map over p. Let C be a linear tree in X contained in the fibre. Let G be the dual graph of C, and let the graph K be a conic deformation of G. Then there is some family of stable curves in X including C and contained in the fibre, such that the dual graph of a general member of the family is K.*

Proof. The curve C is represented by a point in $ev_G^{-1}(p)$. We have seen in 5.3.2 that this is a divisor in $ev_K^{-1}(p)$. Therefore we can take a curve in $ev_K^{-1}(p)$ that passes through the point corresponding to C that does not lie entirely in $ev_G^{-1}(p)$; this gives us the required family.

□

Theorem 5.3.10. *Let p be a general point in X. Let K be a conic deformation of a dual graph G, and let C be a curve passing through p whose dual graph is K. Then there is some family of stable curves in X including C such that every member of the family passes through p, and such that one member of the family has dual graph G.*

Proof. Again, by 5.3.2 the fibre $ev_G^{-1}(p)$, containing curves of dual graph G, is a divisor in the fibre $ev_K^{-1}(p)$. So we can take a curve in $ev_K^{-1}(p)$ passing through the required points. □

Theorem 5.3.11. *A general fibre of $ev_e : \overline{\mathcal{M}}_{0,1}(X, e) \to X$ is irreducible.*

Proof. The base of this deformation consists of three irreducible curves. By construction, their intersection points are smooth points of $\overline{\mathcal{M}}_{0,0}(X, e)$. Therefore this entire deformation takes place in a single irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$. But it is also clear that manoeuvres of this sort can deform a linear tree into a tree of any other isomorphism type. Therefore all linear trees are in the same irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$. Since a general such tree is a smooth point by 5.3.4, we conclude that $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible. □

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