Tritangents to Spherical Curves

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Abstract of the Dissertation

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A generic smooth immersed closed curve in the 2-sphere has a finite number of circles tangent to it in three points. We present a classification of these tritangent circles and study how each class changes during deformations of the curve. We relate these classes to finite type invariants of the curve and derive a relation among the numbers of tritangent circles in the classes and finite type invariants of the curve.

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1 Introduction

Fabricius-Bjerre [6] obtained a formula for the difference between the number of external double tangents to a planar curve, where the tangencies occur on the same side of the tangent line, and internal double tangents, where the tangencies occur on both sides of the line. Ferrand [5] improved upon this by using finite order invariants introduced by Arnold [3] to split the formula. In addition to the side of contact, Ferrand's formulas involved the orientation of the line and the curve at the point of tangency. Ferrand also derived similar results for great circles tangent in two points to spherical curves. A similar problem is to count the number of circles, not necessarily great, that are tangent to a spherical curve in three points.

For a generic immersed oriented curve in the sphere, there is a finite number of circles tangent to it in three points. Tangencies are counted with multiplicity, so osculating circles tangent in an additional point should also be counted. Each circle is oriented by the cyclic order its tangencies make with the curve. This splits the set of all circles tangent in three points into four classes. Let τ^i be the signed count of the circles tangent in three points plus half the number of circles osculating in one point and tangent at another, where the orientation at exactly i tangencies agrees with that of the curve. These numbers turn out to be invariants of the curve of order two.

The image of a spherical curve is a graph on the sphere. The double points are the vertices, the arcs between the double points are the edges, and the connected components of the complement are the faces. Each face has an index, which measures how many times the curve winds around it. The index may be extended to a function on the edges and vertices by averaging over adjacent faces. Let F, E, and V be the sums of the index cubed over each face, edge, and vertex, respectively. An appropriate linear combination of these numbers recovers the τ^i.

$$\tau^0 = -\tau^3 = -\frac{1}{3}F + \frac{2}{3}E - V$$
$$\tau^1 = -\tau^2 = F - \frac{2}{3}E + \frac{1}{3}V$$

In Section 2, we classify circles tangent to a spherical curves in three points. We define a sign which determines each circle's contribution to its class. Finally, we introduce a surface with singularities which is the moduli space of circles tangent to a curve. In Section 3, we enumerate the deformations of a curve which affect tritangent circles. We also define the index of a point in the sphere with respect to a curve. Finally, in Section 4, we derive the combinatorial formulas presented above.

2 Definitions

2.1 Tritangent circles

Let $\gamma : S^1 \to S^2$ be an immersed oriented spherical curve. The tangency between a circle and a curve is characterized by the order of contact, k. If the order is 1, the tangency is simple. If $k > 1$, the tangency is osculating.

If γ is generic, there is a finite number of circles that are simply tangent to it in exactly three points. We will call such a circle a **simple tritangent**. Tangencies should be counted with multiplicity. A simple tangency counts as a single tangency. However, a second order tangency should be counted as two tangencies. The curve γ also has a finite number of circles tangent in two points where one of the tangencies is second order and the other one is simple. We call such a circle an **osculating tritangent**. Finally, at each vertex (extremum of curvature) of a curve, the osculating circle makes third order contact with the curve. This type of tangency has multiplicity three and we call the circle a **vertex tritangent**.

A simple tritangent circle has a natural orientation such that starting at any tangency point, the next mutual tangency point on the circle following its orientation is the same as the next mutual tangency point on γ following the orientation of γ. An osculating or vertex tritangent circle inherits its orientation from the curve at the point of osculation.

2.2 Sign

An oriented circle C separates the sphere into two sides. The orientation of the circle and the orientation of the sphere allow us to call one of the sides left and the other right. Let t be a velocity vector for C at x. Then, a vector v at x points to the left if the frame (t, v) gives the orientation of the sphere.

At each simple tangency point in a tritangent, the curve is locally on one side of the tritangent. If the curve is on the left, we call the tangency **left**. Otherwise, the tangency is **right**.

We introduce a sign to distinguish different tritangents. The **sign of a simple tangency** is positive if the tangency is left and negative if it is right. The **sign of a simple tritangent** is the product of the signs of its three tangencies. The **sign of an osculating tritangent** is the opposite of the sign of the non-osculating simple tangency. At an osculating tangency, the curve crosses from one side of the osculating cirlce to the other, so we can view it as the limiting behavior of two tangencies, one form the left and one from the right, coming together and merging. We will use $\sigma(C)$ to denote the sign of a tritangent.

2.3 Splitting the set of tritangent circles

Given an oriented circle C and a curve γ tangent to it, the orientations of C and γ may agree or disagree at the point of tangency. If they agree, we say the tangency is **direct**. Otherwise, it is **indirect**.

The set of simple tritangent circles splits into four subsets according to how many tangencies are direct. Let T^i be the set of simple tritangent circles with exactly i direct tangencies. We define the following quantities

$$t^i = \sum_{C \in T^i} \sigma(C)$$

The set of osculating tangencies splits in a similar manner. Let S^-, S^+ be sets of osculating tritangents with the non-osculating tangency indirect and direct, respectively. Again, we define the signed sum

$$s^\pm = \sum_{C \in S^\pm} \sigma(C)$$

For the purposes of counting tritangents, we need to consider a linear combination of the numbers of simple and osculating tritangents. We define

$$\begin{aligned} \tau^0 &= t^0 \\ \tau^1 &= t^1 \\ \tau^2 &= t^2 + \frac{s^-}{2} \\ \tau^3 &= t^3 + \frac{s^+}{2} \end{aligned}$$

An osculating tangency may be considered as the merging of two tangencies, one from the left and one from the right, both of which are direct, so they only contribute to τ^2 and τ^3. The reason for the factor of $\frac{1}{2}$ will become clear when we examine deformations in Section 3.

2.4 Spherical geometry

The full Möbius group is the group of orientation-preserving and orientation-reversing conformal transformations of the sphere. These are Möbius transformations as well as reflections about circles. Under each transformation, circles are sent to circles. Each transformation is a diffeomorphism, and therefore preserves order of contact between curves. Furthermore, the side of contact is preserved. While a reflection with respect to a tangent circle reverses the side of contact, it also reverses the orientation of the sphere. The directness of a tangency is clearly preserved. It follows that the τ^i are invariant under the action of this group.

Each oriented circle in the sphere has a signed curvature. Let S^2 be the unit sphere in $\mathbb{R}^3$ and C a circle in the sphere. Such a circle has a radius, r, as measured in $\mathbb{R}^3$. The unsigned geodesic curvature is defined by

$$k_g(C) = \frac{\sqrt{1-r^2}}{r}$$

This measures how far a circle is from being a great circle. If C is oriented, we can improve this definition by introducing a **signed curvature**, κ. The circle C bounds two disks in S^2. These disks inherit an orientation from S^2. Each disk also induces an orientation on C. Let D be the disk such that the orientation it induces on the boundary agrees with the given orientation on C. If C is a great circle, define its signed curvature to be 0. Otherwise, define

$$\kappa(C) = \frac{2\pi - \operatorname{area} D}{|2\pi - \operatorname{area} D|} k_g(C)$$

Finally, we define the signed curvature of a curve at a point as the signed curvature of the osculating circle to the curve at that point.

2.5 Surface of tangent circles

2.5.1 Big front and its singularities

Denote by M, the space of unoriented circles in the sphere. This is the open unit ball in $\mathbb{R}^3$ blown up at the origin. Consider the unit sphere S^2 embedded in $\mathbb{R}^3$. Any plane that intersects this sphere does so in a circle or a point. Conversely, any circle in the sphere determines the plane that cuts it out. Let B be the unit ball in $\mathbb{R}^3$. Then any point $x \in B \setminus 0$ can be seen as a vector v from the origin and uniquely defines the plane passing through x and normal to v. Furthermore, every plane not passing through the origin arises in this way. So, the circles in the sphere which are not great circles are in a one-to-one correspondence with the points of $B \setminus 0$. Great circles are in a one-to-one correspondence with planes through the origin, and these are defined by their normal directions. But, we only need the normal direction without regard for sign since the plane through the origin and normal to a vector w is the same as the plane normal to the vector $-w$.

Another description of M is the tautological line bundle over $\mathbb{R}P^2$. The zero section is the space of great circles and the fiber over each point is the set of circles with the same center on the sphere as the center of the base point great circle.

There is a natural compatification, $\overline{M}$, obtained by adding the circles of radius 0. This is the closed unit ball blown up at the origin with the circles of radius 0 making up the points of the boundary sphere.

Let γ be a spherical curve parametrized by arclength. Fix a point x on γ. Let $\mathcal{C}_x$ be the family of oriented circles tangent to γ at x. The orientation of each circle is given by the velocity vector of γ at x. Each circle in this family is determined by its signed curvature λ.

The disjoint union $\coprod_{x\in\gamma} \mathcal{C}_x$ is a cylinder. We can give it coordinates (s, λ), where s is the arclength (modulo the total length of the curve) and λ is the signed curvature. Then, let $\mu : S^1 \times \mathbb{R} \to M$ be the map that sends circles tangent to γ to their corresponding points in M. For a fixed λ_0, the image of $\mu(s, \lambda_0)$ is a front of the original curve. The full image $S = \mu(S^1 \times \mathbb{R})$ is called the big front. It follows that μ is a Legendrian map and its singularities are well classified [2].

Theorem 1. *The differential $d\mu$ has full rank at C if C is not an osculating circle. It is rank 1 if C is an osculating circle.*

We will prove this theorem below. The map μ is never an immersion as it fails to have full rank at osculating circles, where it falls by 1. In fact, away from the vertices of the curve (points where the curvature has an extremum) the image of the set of osculating circles is a cusp edge. Since every point on γ has a single osculating circle, the cusp edge is the image of a circle. However, it is not diffeomorphic to one. At vertices of γ, the image of μ has a swallowtail.

A neighborhood of a vertex of a curve looks like a parabola. The vertex of the parabola is the vertex of the curve (hence the name). The centers of

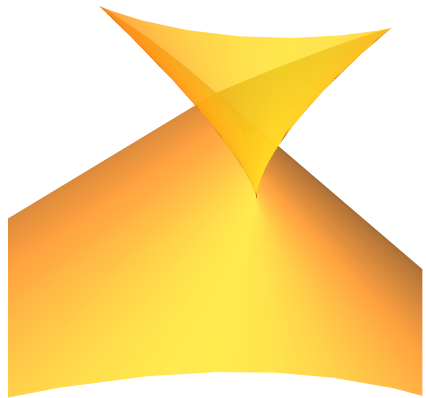


Figure 1: The swallowtail singularity has two cusp edges meeting in a cusp and a self-intersection edge.

osculating circles to the parabola trace out a cusp in the sphere. The cusp point is the circle that osculates at the vertex. This lifts to the swallowtail in the big front. It is a cusp of the cusp edge. Additionally, there is a family of circles tangent to the parabola in two points, one on each side of the vertex. This corresponds to a self-intersection edge on the big front. This self-intersection edge terminates in the swallowtail point (see Figure 1).

Each point on the big front S corresponds to a circle tangent to γ in one or more points. The tangencies are classified by their order (simple is 1, osculating is 2, and osculating at a vertex is 3). A generic curve will never be tangent to a circle to higher order than 3. So, the only singularities on the big front of a generic curve are cusp edges, swallowtails, and self-intersections of the surface. The following list summarizes the possibilities as well as their codimensions on the surface.

Codimension 0

- S_1: Smooth sheet

Codimension 1

- $S_{1,1}$: Self-intersection edge
- S_2: Cusp edge

Codimension 2

- $S_{1,1,1}$: Triple point (simple tritangent)
- $S_{2,1}$: Cusp edge / smooth sheet intersection (osculating tritangent)
- S_3: Swallowtail point (vertex tritangent)

Finally, the S_1 part of the surface has a coorientation. Let C be a circle tangent to γ that is not osculating. If we perturb C in a direction transverse to the curve, it will locally intersect γ in two points or in no points. Perturb it so that it intersects it in two points. This produces a new circle C' that lies on one side of the big front. We will call this side positive.

2.5.2 Local model for big front

Let p be any point in the complement of γ. We may treat it as the south pole and stereographically project the rest of the sphere to the plane tangent at the north pole. Circles tangent to γ in the sphere will map to circles or lines tangent to the image of γ in the stereographic projection.

The space of circles (but not lines!) in the plane can be identified with the upper half space $M' = (\mathbb{R}^3)^+$. Given a circle C with center (x, y) and radius r, we send it to the point (x, y, r).

Much like in the sphere, there is a signed planar curvature. It agrees with the usual notion of curvature, except it is negative if the tangent vector to the curve is turning clockwise as we move along the curve. The circles tangent to γ in the stereographic projection are thus parametrized by (s, λ) where s is the arclength (modulo the total length) and $\lambda \in \mathbb{R}^*$ is the signed planar curvature. So, the set of circles tangent to γ in the plane is the disjoint union of two cylinders. Let ν be the map that sends them to M'.

Let S' be the image of ν inside M'. This pair (M', S') is a local model for the pair (M, S) away from the projection point p. In fact, we have the following obvious lemma.

Lemma 1. *Let $E \subset M$ be the set of circles tangent to p. There exists a diffeomorphism of pairs, $(M \setminus E, S \setminus E) \to (M', S')$.*

□

Proof of Theorem 1. It suffices to prove the theorem locally. Let U be a sufficiently small neighborhood of C in M. We may choose a point p in the sphere such that there are no circles through p that lie in U. We stereographically project from p and pass to the local model.

Let γ be the stereographic projection of the curve in the sphere. The space of all circles tangent to γ is parametrized by (s, λ), where s is the arclength and λ is the signed planar curvature. Denote by $\kappa(s)$ the signed planar curvature of $\gamma(s)$. Let $T(s)$ be the unit tangent vector to γ and $N(s)$ the unit normal vector. The circle C is mapped to some circle in the plane tangent to γ. Suppose it is tangent at $\gamma(s_0)$ and has planar curvature λ_0.

We restrict $d\nu$ to the level sets of constant s. Let G be the set of circle tangent to γ at s_0. The center of such a circle is a function of its curvature λ and is given by

$$\gamma(s_0) + \frac{1}{\lambda} N(s_0)$$

Differentiating this with respect to λ we get $\frac{-1}{\lambda^2}$. Since our curvature is never 0 or ∞, this derivative is never 0 or ∞ either. This implies that the rank of $d\nu$ is always at least 1.

On the other hand, if we restrict $d\nu$ to the level sets of constant λ, the rank may drop. Let H be the set of circles tangent to γ with curvature λ_0. The center of each circle in H is given by

$$\gamma(s) + \frac{1}{\lambda_0} N(s)$$

We differentiate this with respect to s and apply the Frenet-Serret formula $N' = -\kappa T$. The derivative is

$$T(s) - \frac{\kappa}{\lambda_0} T(s)$$

Since our curve is smooth, T is never 0. It follows that this quantity vanishes if and only if $\kappa = \lambda_0$. In other words, the rank of the differential drops by 1 only at osculating circles. $\square$

2.5.3 Structure of fronts

A horizontal section of the big front at height r is two curves parallel to γ and offset by a distance r to the right and to the left. These are fronts of γ. One of these has smaller curvature in absolute value than γ at corresponding points. It always looks like γ. The other curve has a greater curvature in absolute value at corresponding points. For small r, it also looks like γ. As

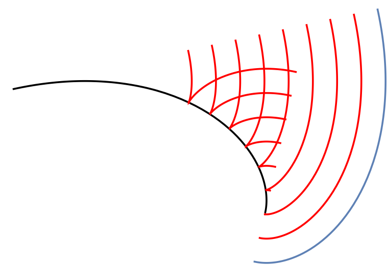


Figure 2: The original curve is in blue, the evolute is in black, and the fronts of the curve are in red.

r increases past the radius of curvature at a point, the front inverts in a neighborhood of that point. This can be seen most easily with an arc of a circle. Parallels of an arc of a circle are nested arcs until the offset is equal to the radius of the circle. After that, the parallels are curved in the opposite direction. For a curve with monotonically varying curvature, the front will have a cusp at the point where it meets the evolute (the curve consisting of the centers of curvature of γ). At this point the front has infinite curvature. This is most easily understood as follows. A point on γ and on its front have the same center of curvature. However, as r increases, the front gets closer to the center, so its curvature must increase in absolute value. This is illustrated in Figure 2.

When this picture is assembled into the big front, there are two components. The fronts with smaller curvature in absolute value form a smooth wall emerging from the curve. The fronts with larger curvature in absolute value form a wall with singularities. The singularities are made up of the cusps of each front. This is the cusp edge and it projects down to the evolute.

The two components of the big front are ruled by rays corresponding to circles tangent at a given point. The neighborhood of one such ray is picture in Figure 3.

2.5.4 Interpretation of tritangent sign

The surface S' is cooriented away from the osculating circles. This defines a normal vector to the surface at each smooth point. At a triple point of S', the three normal vectors form a frame.

Theorem 2. *The frame formed by the coorientation vectors at a triple point*

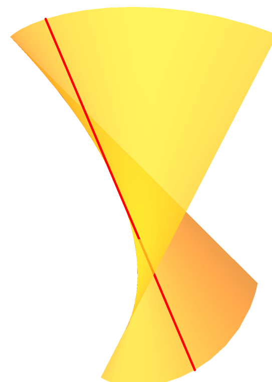


Figure 3: Neighborhood of a ray on the big front.

of S' is positively oriented if and only if the sign of the corresponding tritangent is positive.

Flowing along a vertical vector in M' corresponds to increasing or decreasing the radius of the circle. This produces concentric circles which is not integrable for any curve. A vertical vector will never be tangent to S'. So, if we want to indicate the coorientation of S' at a smooth point, it suffices to specify a vertical direction.

Given a point x on γ, the set of circles tangent to γ at x form two rays, each making an angle of $\frac{\pi}{4}$ with the xy-plane. One of the rays contains the osculating circle to γ at x. We will say this ray is marked by the osculating point. The coorientation of the big front points upward on the unmarked ray and on the marked ray until the mark. After the mark it points downward.

Lemma 2. *Let C be a circle tangent to γ at x such that the disk in the plane bounded by C has the same orientation as the plane. The tangency at x is right if and only if C corresponds to a point on the unmarked ray or on the marked ray before the mark.*

□

Proof of Theorem 2. Let C be a tritangent circle tangent at x, y, and z. The points x, y, and z are vertices of a tetrahedron. The top vertex of the tetrahedron is the triple point corresponding to C. The edges of the tetrahedron make angles of $\frac{\pi}{4}$ with the plane.

Suppose C bounds a disk that has the same orientation as the plane. By Lemma 2, the tangency at any of the points x, y, or z is right if and only if the vertical component of the coorienting normal points up. Since the normal vector is determined by its vertical component, the orientation of the frame is determined by the sides of tangency.

On the other hand, if C has the opposite orientation, we see that a tangency is right if and only if the vertical component points down. This means the coorienting normals point in the opposite direction to the case above. However, since C has the opposite oriention, the order of the vectors in the frame changes. It follows that the orientation of the frame depends only on the sides of tangency, and not on the orientation induced on the tritangent circle.

The frame in $(\mathbb{R}^3)^+$ given by $(\partial_x, \partial_y, -\partial_z)$ is the positive one. It is easy to check that a small counter-clockwise oriented circle with three right tangencies corresponds to a negatively oriented frame. The theorem follows. $\square$

3 Moves

Let Ω be the space of all smooth spherical curves with the Whitney topology. It has two connected components corresponding to the regular homotopy classes of curves in the sphere.

A smooth immersed curve may have singularities in the form of self-tangencies and triple points. These are singularities of the immersion. A generic curve has no self-tangencies and no triple points. Generic curves form an open dense subset in the space of all curves. Curves which violate one of the above conditions make up a statified hypersurface with singularities called the **discriminant**.

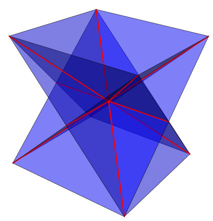
The big front for a non-generic curve may have additional singularities. These are classified in [2]. We will examine each one in the next section.

A one-parameter family of curves is said to be generic if at most a finite number of its members have one of the degeneracies listed above and no member has more than one degeneracy. When a family of curves passes through a degeneracy, the curve undergoes a **move**. In particular, there are two perturbations of a degenerate curve. One of these will be called the negative deformation, and the other the positive deformation. The direction of the move is the transition from the negative deformation to the positive one. The directions we choose are based on the ones introduced by Arnold [3].

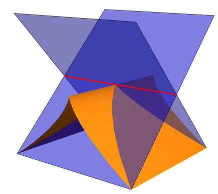
When a family of curves passes through a self-tangency or a triple point, the numbers τ^i may change. However, the τ^i are invariant under the remaining moves.

3.1 τ-preserving moves

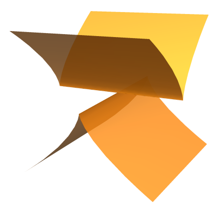
For a generic curve, a circle is said to be tritangent to it if it is tangent in three points. As we have seen, these three points may be allocated between simple tangencies, osculating tangencies (which count as two tangencies), and vertex tangencies at vertices (which count as three). For a one parameter family, there will be members with circles which are tangent in four points. In a sense, these form a stratum of codimension -1 in S. The different ways this can occur is enumerated below. Figure 4 shows the big front at the moment of each move. In each case, the numbers τ^i do not change.



(a) $S_{1,1,1,1}$

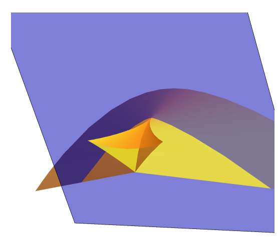


(b) $S_{2,1,1}$

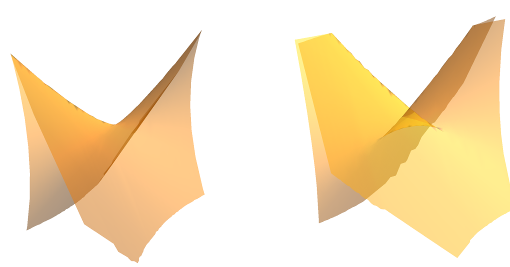


(c) $S_{2,2}$

(d) $S_{2,2}$



(e) $S_{3,1}$



(f) S_4 and moment after

Figure 4: Big front during moves.

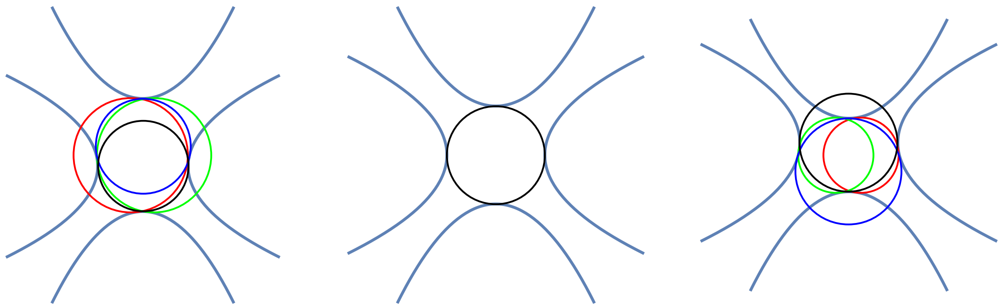


Figure 5: As the top branch is lowered, the red, blue, green, and black circles merge into one and move apart.

3.1.1 $S_{1,1,1,1}$: Quadruple sheet intersection

During this move, a single circle is tangent to the curve in four distinct points. Between any three branches of the curve in the quadruple contact, there is a simple tritangent circle. So, this move involves four simple tritangent circles merging. They immediately move apart again, and nothing else happens (see Figure 5). On the big front, a smooth sheet passes through a triple point and keeps going.

3.1.2 $S_{2,1,1}$: Cusp edge and two sheet intersection

A circle tangent to the curve in two points and osculating at a third appears as part of a move which creates two new simple tritangents (see Figure 6). On the big front this looks like two smooth sheets intersecting in a line. This line then moves through a cusp edge intersecting it transversally and continues on. There are two new triple points. Because the triple points form on opposite sides of the cusp, they will differ by a single tangency and come in with opposite signs. Two osculating tritangents move past each other, but are otherwise unaffected.

3.1.3 $S_{2,2}$: Two cusp edge intersection

A circle may be osculating for two different points of the curve. On the big front this looks like two cusp edges meeting. This can happen in two ways. Either four new osculating tritangents are created (see Figure 7), or

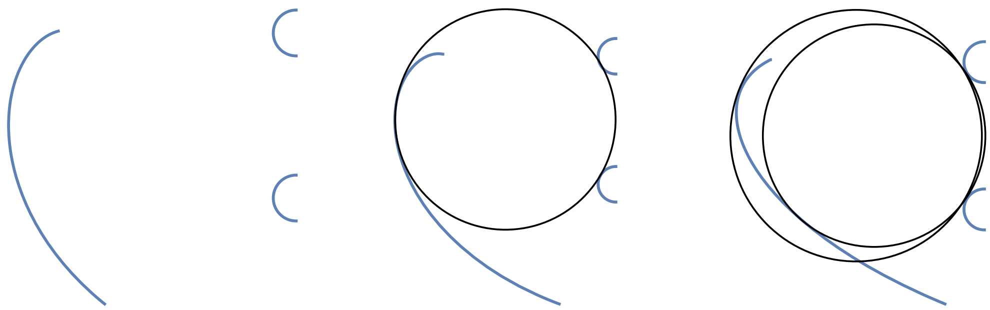


Figure 6: Two tritangents are created during the move.

two disappear and two appear.

In either case, the move acts only on osculating tritangents. These either appear or disappear in pairs. For each pair, the individual osculating tritangents lie on opposite sides of a cusp edge and come in with opposite signs.

3.1.4 $S_{3,1}$: Swallowtail and sheet intersection

A circle may be an osculating circle at a vertex and simply tangent in another point. On the big front this looks like a smooth sheet crossing a swallowtail. This creates a new simple tritangent and two osculating tritangents (see Figure 8). The new simple tritangent is tangent to the same branch in two places on either side of the vertex, so it can only belong to T^2 or T^3. The osculating tritangents are identical in sign and belong to the same $S^{\pm}$. The factor of $\frac{1}{2}$ in the definition of τ was chosen earlier so that the newly formed simple tritangent and osculating tritangents would cancel.

3.1.5 S_4: Swallowtail pair creation/annihilation

For a generic curve, the points of minimum and maximum curvature alternate. During a deformation of the curve, a minimum and maximum can merge and annihilate. On the big front, this looks like two swallowtails which share a cusp edge merging and annihilating each other. At the same time, two osculating tritangents annihilate as well. The inverse of this is two vertices forming along with two osculating tritangents. These osculating tritangents are always oriented the same way. Furthermore, the non-osculating

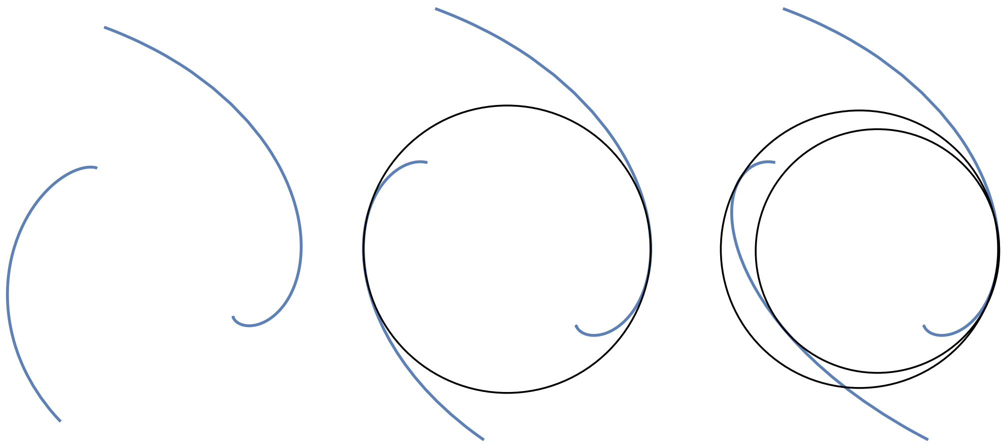


Figure 7: Four osculating tritangents are created (two not pictured).

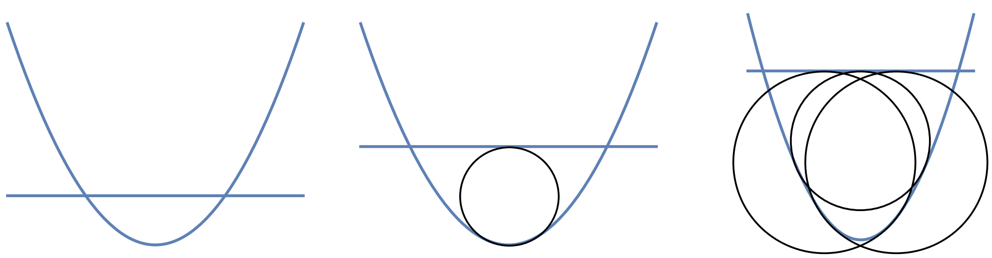


Figure 8: A simple tritangent and two osculating tritangents are created.

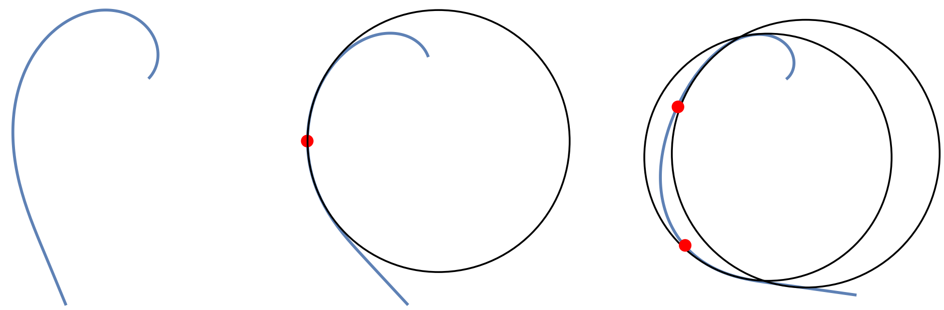


Figure 9: The osculating circle at the degenerate vertex splits into two osculating tritangents. The vertices are shown in red.

tangencies are always on opposite sides and both are direct. So, the newly formed osculating tritangents both belong to S^+ and come in with opposite signs (see Figure 9).

3.2 Self-tangencies

If the orientations of the two branches of a curve in a self-tangency agree, we say the self-tangency is **direct**. Otherwise, it is **indirect**. In both cases, the positive deformation is the one with two more double points.

During a self-tangency, the change in the τ^i depends on a function defined on the complement of the curve called the index.

3.2.1 Index of a curve

Let α be a planar curve. There is a function, called the winding number or the index, $\mathrm{ind} : \mathbb{R}^2 \setminus \alpha \to \mathbb{Z}$, which counts the number of times the curve winds around the given point. This function depends only on the connected component of the complement, and jumps by 1 between adjacent components.

We will need an analagous function on the sphere. Let γ be an oriented curve in the sphere and y be a point in $S^2 \setminus \gamma$. We take the stereographic projection from y and obtain a curve in the plane, γ_y. We orient the plane so that a positively oriented frame in the sphere is sent to a positively oriented frame in the plane. We define $\mathrm{ind}_y(x)$ to be the index of x with respect to the curve γ_y. This naive definition is not invariant under different choices of y.

A curve in the plane has an integer regular homotopy class. This is called the rotation number or turning number of the curve, and we will denote it by ω. It is defined as the degree of the tangential Gauss map from the curve to the unit circle which sends each point to the unit velocity vector at the point as the curve is traversed in the direction of its orientation. We have the following theorem due to Arnold [1].

Theorem 3. *The following quantity is independent of choice of y.*

$$\mathrm{ind}_y(x) - \frac{\omega(\gamma_y)}{2}$$

If we choose y and x to lie in the same connected component of the complement, then $\mathrm{ind}_y(x) = 0$. We are motivated to make the following definition.

$$\mathrm{ind}(x) = -\frac{\omega(\gamma_x)}{2}$$

We will no longer need the planar index, so ind will always refer to the spherical version. This function is $\frac{1}{2}\mathbb{Z}$ valued.

Observe that $\mathrm{ind}(x)$ jumps by 1 when x is moved from one connected component to an adjacent one. Let A and B be adjacent components of the complement and β a path from a point in A to a point in B that crosses a mutual boundary component only once. Since the boundary component is an arc of γ, it is oriented. If it crosses β from left to right, then $\mathrm{ind}(x)$ jumps by $+1$. If it crosses from right to left, then $\mathrm{ind}(x)$ jumps by -1. This tells us that $\mathrm{ind}(x)$ is completely determined by its value on a single point in the complement of the γ.

3.2.2 Counting circles in a self-tangency

When a curve undergoes a self-tangency move, tritangent circles both appear and disappear. Denote the two branches in the move by L and R and the point of contact by x. Using a Möbius transformation followed by an appropriate stereographic projection, we may arrange so the branches have opposite planar curvature sign and the L branch is on the left.

Theorem 4. *Let C be a circle through x and tangent to both branches. Suppose C is simply tangent to the curve in exactly one other point on branch Y.*

If the curvature of C is such that C is locally between L and R in a neighborhood of x, there exist two tritangent circles close to C in the negative deformation, but not the positive one. Otherwise, if C is locally to one side of both L and R, there exist two tritangent circles close to C in the positive deformation, but not the negative one.

Proof. In the planar model for the big front, a self-tangency of the curve corresponds to two rays of self-tangency of the surface. The two rays emerge from the self-tangency point at an angle of $\frac{\pi}{4}$ with the horizontal. A circle C that passes through x, is tangent to both branches, and is tangent at another point of the curve at y corresponds to the transverse intersection between one of these rays and another sheet of the surface. We focus on one of the rays since the picture is symmetric.

Let $\mathcal{F}_L(h)$ and $\mathcal{F}_R(h)$ be fronts of L and R a distance h to the right of their respective branches. These two fronts are tangent for all h and the points of tangency form a ray, $\rho(h)$. Let $\rho(o)$ be the osculating circle for R at x. The neighborhood of ρ is pictured in Figure 3. Below o, $\mathcal{F}_R(h)$ has the same curvature sign in a neighborhood of $\rho(h)$ as the branch R. Above o, the curvature sign is opposite. Furthermore, for $h > o$, since the center of curvature of L at x is to the left of the center of curvature of R at x, locally, the curvature of $\mathcal{F}_L(h)$ is always less in absolute value than the curvature of $\mathcal{F}_R(h)$ (see Figure 10).

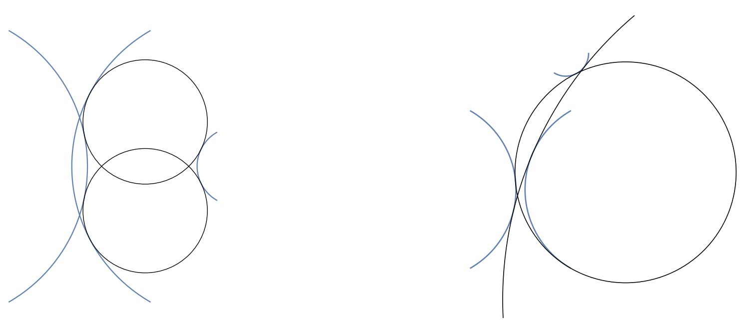
As the move resolves either to the negative or positive deformation, $\mathcal{F}_L(h)$ intersects $\mathcal{F}_R(h)$ in two points. This means ρ bifuracates into two pieces of self-intersection of the surface. So, a third sheet which transversally intersects ρ will acquire two triple points. If the curvature of C is such that it is locally to the right of R, the sheet transverse to ρ intersects it below o. Otherwise, it intersects it above. □

The appearing and disappearing circles in the theorem above are illustrated in Figure 11. We need a method of counting these. Let C be a circle simply tangent to γ at some point. The tangency is either left or right and either direct or indirect. We use l to denote left tangencies and r to denote right tangencies. We also use a superscript $+$ to denote direct tangencies and a superscript $-$ to denote indirect tangencies. For example, an indirect left tangency is denoted by l^-.

For any point $x \in S^2 \setminus \gamma$ and any tangent vector v at x, we define L_v^- as the set of circles through x, tangent to and oriented by v, and simply

(a) $h < o$ (b) $h > o$

Figure 10: Fronts $\mathcal{F}_L(h)$ and $\mathcal{F}_R(h)$ near a self-tangency.



(a) Appearing tritangents (b) Disappearing tritangents

Figure 11: Circles appearing and disappearing during self-tangency move.

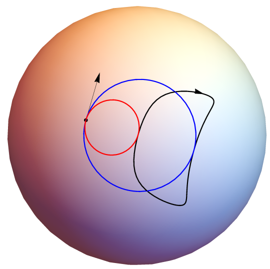


Figure 12: The larger blue circle is a tangent of type l^+. The smaller red circle is a tangent of type l^-. Two r^+ circles are not pictured.

tangent at exactly one other point with l^- tangency. We define L_v^+, R_v^-, and R_v^+ analagously (see Figure 12).

Lemma 3.

$$\left|L_v^-\right| - \left|R_v^-\right| = \left|R_v^+\right| - \left|L_v^+\right| = 2\,\mathrm{ind}(x)$$

Proof. Stereographically project from x. Circles through x and tangent to v are sent to lines with a fixed direction. The lemma now follows from the definition of the turning number by counting the preimages of the tangential Gauss map. □

Let x be as in Theorem 4. Pick any v tangent to both branches. Keeping x fixed, perturb γ to obtain the negative deformation γ^- such that x is between the branches (see Figure 13). We may now apply the lemma to determine the number of circles that are affected by the move. This will count four extra circles, namely the ones through x and tangent to the branches. We can see these as follows. We focus on one branch in the self-tangency. Let B be a branch of γ passing through x and B' the same branch in γ^-. Stereographically projecting from x sends the osculating circle to B at x to a vertical line. This line is an asymptote for the image of B under the projection (see Figure 14). The image of B approaches the asymptote from opposite sides. The projection sends B' to an arc that curves around instead of going off to infinity. Circles tangent to γ and passing through x are sent

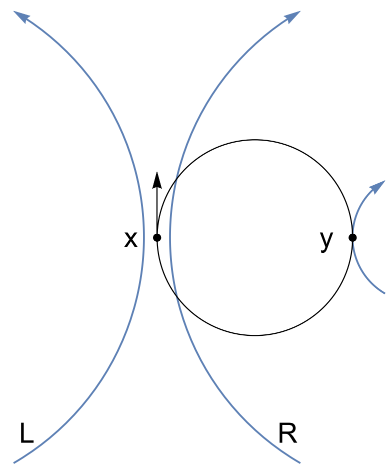


Figure 13: The circle passes through x with a vertical tangency, and is tangent to the curve at y.

to vertical tangents in the image. So, to determine how many extra circles are produced by perturbing γ to γ^-, it suffices to determine how many new vertical tangents are formed. There are two such tangent lines. Repeating this for the second branch, we count a total of four extra circles.

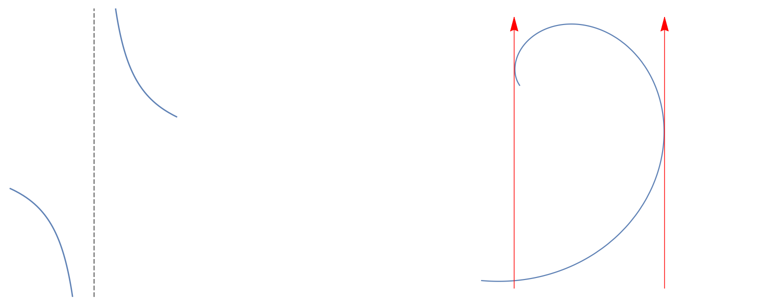
3.2.3 Direct self-tangency

Theorem 5. *Let γ^- be the negative deformation of a direct self-tangency, and x a point between the branches. The move effects the following change in the τ^i.*

$$\Delta\tau^0 = \Delta\tau^2 = -\Delta\tau^1 = -\Delta\tau^3 = 2\,\mathrm{ind}(x)$$

Let γ be a curve with a self-tangency and γ^- a negative deformation. We may choose γ^- to agree with γ outside a small neighborhood, U, around the point of self-tangency. Let $x \in U$ be the point between the branches in γ^- and v a vector at x pointing in the same direction as both of the branches.

Let C be a circle through x, tangent to v, and tangent to γ^- at some point y. We orient C so that it agrees with v. Suppose C makes an l^- tangency at y and $\kappa(C) > \kappa(R)$ (see Figure 13). When we carry through the move, C will split into two tritangent circles (see Figure 11a). These new tritangents are oriented based on the order they pass the branches of the curve. Only



(a) Projection of branch and its asymptote

(b) Projection of perturbed branch and its extra tangent circles

Figure 14: Counting extra circles tangent to a branch perturbation.

one of them will have the same orientation as C. For concreteness, suppose the curve passes through y, then branch L, and then branch R. The newly formed circles are then of type $l^+l^+l^-$ and $r^-r^-r^+$. So, τ^2 increases by 1 and τ^1 decreases by 1.

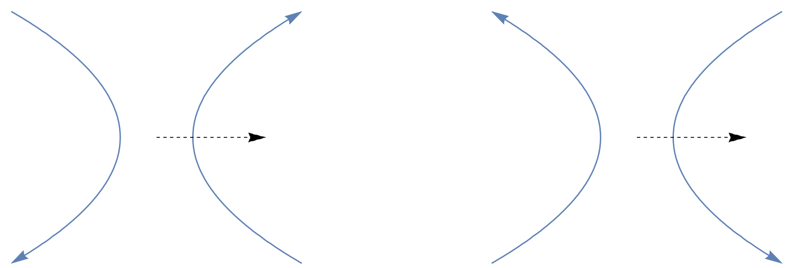
We assumed that branch L followed y on γ, however it could happen that it is the other way around. If the curve passes through y, then branch R, and then branch L, the newly formed circles have the same type as before, but their positions are swapped and the cyclic order of their tangencies is changed. Either way, τ^2 increases by 1 and τ^1 decreases by 1.

If $\kappa(C) < \kappa(L)$, then y is necessarily on the other side of the picture, and the newly formed circles are of types $r^+r^+l^-$ and $l^-l^-r^+$. The change in the τ^i is the same as above.

Finally, if $\kappa(L) < \kappa(C) < \kappa(R)$, then two circles disappear (see Figure 11b). The disappearing circles are of type $l^-l^+r^+$ and $r^+r^-l^-$. Since the circles are disappearing, the τ^i change in the opposite direction. We see that τ^2 increases by 1 and τ^1 decreases by 1 as before.

In the example above, we chose the tangency at y, $\kappa(C)$, and which branch, L or R, followed y on γ. We saw that, in fact, $\kappa(C)$ and the order of the branches were irrelevant. We can run the above argument for any starting circle C. The net effect depends only on its tangency at y.

We group all of the circles together by tangency type at y. The following table summarizes how the tangency of C at y translates into changes to the τ^i.



(a) Left-handed indirect self-tangency

(b) Right-handed indirect self-tangency

Figure 15: Indirect self-tangencies.

Tangency at y	$\Delta\tau^0$	$\Delta\tau^1$	$\Delta\tau^2$	$\Delta\tau^3$
l^-		-1	1	
r^-		1	-1	
l^+	-1			1
r^+	1			-1

Each l^- tangency increases τ^2 and each r^- tangency decreases it. The difference $l^- - r^-$ is the net change in τ^2. If we apply Lemma 3 to (x, v), we see that the number of circles with l^- tangencies minus the number of circles with r^- tangencies is twice the index of the point x. However, this also counts four extra circles.

There are two circles passing through x and tangent to the L branch. These make r^- and l^+ contact with γ^-. There are two circles tangent to the R branch. These make l^- and r^+ contact. Since these factor in with opposite signs in Lemma 3, we may ignore them.

3.2.4 Indirect self-tangency

This move can happen in two inequivalent ways. Let x be a point between the two branches in the negative deformation. If we move from x to the other side of one of the branches, the branch crosses our path. If the branch crosses from left to right, we say the self-tangency is **right-handed**. Otherwise, it is **left-handed** (see Figure 15).

Theorem 6. *Let γ^- be the negative deformation of an indirect self-tangency, and x a point between the branches. If the self-tangency is left-handed, the* *move effects the following change in the τ^i.*

$$\Delta\tau^1 = -\Delta\tau^2 = 4\,\mathrm{ind}(x) - 4$$

If the self-tangency is right-handed, the move effects the following change in the τ^i.

$$\Delta\tau^1 = -\Delta\tau^2 = 4\,\mathrm{ind}(x) + 4$$

We follow the same procedure as for a direct self-tangency. When we choose v, we can choose it to be either vertical direction. The count of circles is not affected. The following table summarizes the results.

Tangency at y	$\Delta\tau^1$	$\Delta\tau^2$
l^-	1	-1
r^-	-1	1
l^+	-1	1
r^+	1	-1

We need to account for the extra circles. For a left-handed tangency, there are two circles passing through x and tangent to the L branch. These make l^- and r^+ contact with γ^-. There are two circles tangent to the R branch. These also make l^- and r^+ contact. So, we are overcounting by 4 circles. For a right-handed tangency, there are two circles passing through x and tangent to the L branch. These make l^+ and r^- contact with γ^-. There are also two circles tangent to the R branch. These also make l^+ and r^- contact. So, we are undercounting by 4 circles.

3.3 Triple points

Consider a degenerate curve where three branches pass through the same point. This can be perturbed in two ways. In one deformation, the inscribed circle between the three branches has an odd number of direct tangencies. In the other deformation, it has an even number. We choose the one with the even number to be positive.

This move can happen in four distinct ways. Consider the triangle formed by the branches of the curve which vanishes during the move. This triangle has an inscribed circle which is oriented by the order the curve passes the three branches. The circle bounds a disk contained in the vanishing triangle. If the orientation induced on this disk from the boundary circle is the same as the orientation of the sphere, then we say the move is **proper**. Otherwise,

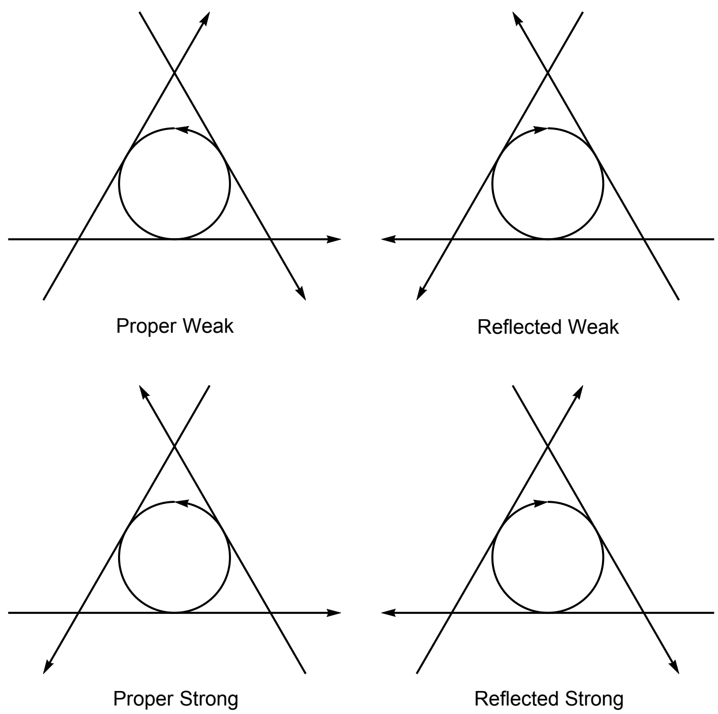


Figure 16: Negative deformation of triple point move. The inscribed circle is a tritangent and its orientation is determined by the order the curve passes the branches.

we say it is **reflected**. Let v_i be velocity vectors to the three branches at the moment of the move. If there exist positive non-zero coefficients c_i such that $\sum c_i v_i = 0$, we say the move is **strong**. Otherwise, it is **weak** (see Figure 16).

The triple point move always involves exactly four tritangent circles. There is one circle inscribed in the vanishing triangle and three circles outside the triangle and tangent to it on each side. The move acts on all tangencies in the same way. The directness changes, but the side does not. In the negative deformation of a proper strong move, the inscribed circle is of type $r^+r^+r^+$ and the three outer circles are all of type $l^+l^-l^-$. After the move, the inscribed circle is of type $r^-r^-r^-$ and the three outer circles are all of type $l^-l^+l^+$. The sign of the inscribed circle is negative in both cases. So, this move increases τ^3 by 1, decreases τ^1 by 3, increases τ^2 by 3, and decreases τ^0 by 1. The effect of each move is given in the table below.

Triple point	$\Delta\tau^0$	$\Delta\tau^1$	$\Delta\tau^2$	$\Delta\tau^3$
Proper Strong	-1	-3	3	1
Reflected Strong	1	3	-3	-1
Proper Weak	1	-1	1	-1
Reflected Weak	-1	1	-1	1

4 Formulas

We have analyzed how the numbers τ^i change under various moves. We will now derive an explicit formula for them. We do this by exhibiting a function that has the same jumps during curve moves.

4.1 Extending the index

The double points and arcs of a spherical curve γ form a graph on the sphere. The complement of the curve is separated into faces. The index was defined in Section 3.2.1 as a function on these faces. This definition extends to the graph as follows. Assign to each edge the average of the two adjacent faces and to each vertex the average of the four adjacent faces. Viro [4] constructs finite type invariants out of sums of powers of the index over the faces, edges, and vertices of such a graph. We will use a similar strategy to obtain formulas for the τ^i.

We define the following sums.

$$F = \sum_{f \in \text{faces}} \text{ind}(f)^3$$
$$E = \sum_{e \in \text{edges}} \text{ind}(e)^3$$
$$V = \sum_{v \in \text{vertices}} \text{ind}(v)^3$$

Theorem 7. *For any move,*

$$\Delta\tau^0 = -\Delta\tau^3 = \Delta\left(-\frac{1}{3}F + \frac{2}{3}E - V\right)$$
$$\Delta\tau^1 = -\Delta\tau^2 = \Delta\left(F - \frac{2}{3}E + \frac{1}{3}V\right)$$

The numbers F, E, and V are invariant under any homotopy of the curve which does not pass through self-tangencies or triple points. To determine how these numbers change during a self-tangency or triple point move, we pick a point in the complement of the curve and fix an index of i for it. As long as a branch of the curve does not cross over the point during the move, its index will remain unchanged. So, it suffices to check how the numbers change locally in a small neighborhood of the move.

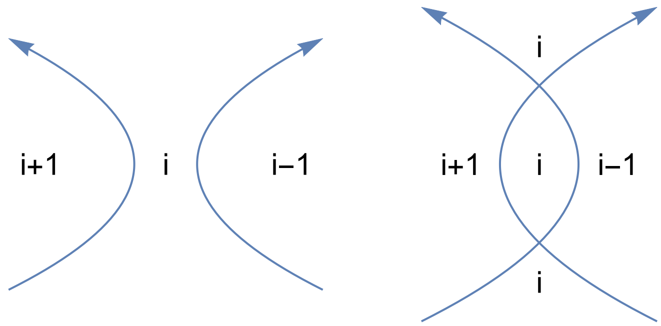


Figure 17: Change in face indices during direct self-tangency.

Fix an index of i for the face between the two branches in the negative deformation of a direct self-tangency. Since the index jumps by 1 when crossing over an arc of the curve, the remaining faces must have index $i+1$ and $i-1$. The edges have indices $i+\frac{1}{2}$ and $i-\frac{1}{2}$. There are locally no vertices. During the move, the central face is broken into two faces and a bigon forms between them. The faces on the left and right are shrunk slightly as a result of the move, but their indices are unaffected. So, we can determine the remaining indices. All of the new faces have index i. The change in F is $2i^3$. The two edges are broken as well. There are now a total of six edges. Two of the new edges have index $i+\frac{1}{2}$ and the other two have index $i-\frac{1}{2}$. The change in E is $4i^3+3i$. Two new vertices form. They both have index i, so the change in V is $2i^3$. This is illustrated in Figure 17. Putting it all together, $\Delta\left(-\frac{1}{3}F+\frac{2}{3}E-V\right)=2i$ and $\Delta\left(F-\frac{2}{3}E+\frac{1}{3}V\right)=-2i$.

The other moves are analyzed similarly. The following table summarizes the results.

Move	ΔF	ΔE	ΔV
Proper Strong	$-9(i^2-3i+3)$	$-18i^2+54i-\frac{93}{2}$	$-3(3i^2-9i+7)$
Reflected Strong	$9(i^2+3i+3)$	$18i^2+54i+\frac{93}{2}$	$3(3i^2+9i+7)$
Proper Weak	$3i^2+3i+1$	$6i^2+6i+\frac{7}{2}$	$3i^2+3i+1$
Reflected Weak	$-3i^2+3i-1$	$-6i^2+6i-\frac{7}{2}$	$-3i^2+3i-1$
Direct	$2i^3$	$i(4i^2+3)$	$2i^3$
Indirect Left	$i^3+(i-2)^3$	$4i^3-12i^2+15i-7$	$2(i-1)^3$
Indirect Right	$i^3+(i+2)^3$	$4i^3+12i^2+15i+7$	$2(i+1)^3$

4.2 Normalization

The previous theorem shows that the jumps in the τ^i and in the given linear combinations of F, E, and V agree. This means they differ by at most a constant. We strengthen the result by showing that the constant is 0.

Theorem 8.

$$\tau^0 = -\tau^3 = -\frac{1}{3}F + \frac{2}{3}E - V$$
$$\tau^1 = -\tau^2 = F - \frac{2}{3}E + \frac{1}{3}V$$

There are two regular homotopy classes of curves in the sphere. These are represented by the circle and the figure eight. To prove the theorem, it suffices to verify it for one example in each class.

Lemma 4. *The ellipse has no simple or osculating tritangents.*

Proof. Since stereographic projection preserves tangencies, we may work in the plane. Both the ellipse and circle are cut out by degree two polynomials. By Bezout's Theorem, they intersect in at most 4 points, counted with multiplicty. A tangency counts as two points and an osculating tangency counts as three. A simple tritangent would need to have 6 point tangency and an osculating tritangent would need to have 5 point tangency. □

Lemma 5. *There exists a figure eight curve with no simple or osculating tritangents.*

Proof. Consider a unit sphere in $\mathbb{R}^3$ resting with its south pole on the origin of the xy-plane. Viviani's curve is obtained by intersecting this sphere with a cylinder of radius $\frac{1}{2}$, standing on the xy-plane, and passing through the center of the sphere (see Figure 18). This is a figure eight curve in the sphere. When projected down along the z direction, it forms a circle C of radius $\frac{1}{2}$. Any circle in the sphere will project to an ellipse or a chord of the unit circle. Suppose there was a tritangent, either simple or osculating. Projection preserves tangencies, and never decreases the order of contact. So, the image of the tritangent would be either an ellipse or a chord that is tangent to C. The image cannot be a chord because the preimage of the chord is a circle which is simply tangent to the figure eight in at most two points. The image cannot be an ellipse because it would have to be tangent to C in at least 5 points, counted with multiplicity. This would violate Bezout's Theorem. □

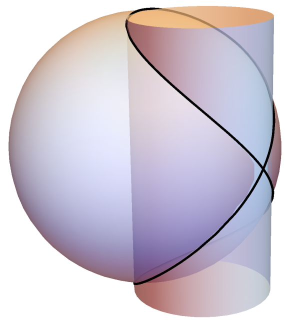


Figure 18: Viviani's curve.

We can easy check that the right hand side of the equations in Theorem 8 is 0 for both the ellipse and the figure eight. The result follows.

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