Uniqueness of Ricci Flow Solution on Non-compact Manifolds and Integral Scalar Curvature Bound

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Abstract of the Dissertation

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In this dissertation, we prove two results. The first is about the uniqueness of Ricci flow solution. B.-L. Chen and X.-P. Zhu first proved the uniqueness of Ricci flow solution to the initial value problem by assuming bilaterally bounded curvature over the space-time. Here we show that, when the initial data has bounded curvature and is non-collapsing, the complex sectional curvature bounded from below over the space-time guarantees the short-time uniqueness of solution.

The second is about the integral scalar curvature bound. A. Petrunin proved that for any complete boundary free Riemannian manifold, if the sectional curvature is bounded from below by negative one, then the integral of the scalar curvature over any unit ball is bounded from above by a constant depending only on the dimension. We ask whether this is true when replacing the sectional curvature with Ricci curvature in the condition. We show that, essentially, there is no counter-example with warped product metric. The application to the uniqueness of Ricci flow is also discussed.

To my parents.

Contents

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Chapter 1

Riemannian Geometry

1.1 Introduction

In this Chapter we provide the necessary background at the same time conventions we adopted in Riemannian Geometry. Mostly, our conventions agree with those in Petersen [1]. We also assuming readers have the minimal understanding of the basics of the differential manifold and tensor, which can be found in e.g. Spivak [2]. And throughout this dissertation, all manifolds, compact or complete noncompact, are boundary free unless otherwise indicated.

1.1.1 Curvatures of the Riemannian Metric

Let (M, g) be a n-dimensional Riemannian manifold with a Riemannian metric g. Let ∇ be the Levi-Civita connection with respect to g, the dvol be the correspondent to g. And in some cases we append g to ∇ and dvol, i.e. ∇^g and dvol_g to emphasize the relations therein.

The basic formulas of Riemannian curvature tensor, curvature operator, ricci curvature, and scalar curvature, are listed below. The Riemannian curvature is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.1}$$

and $\mathrm{rm}(X, Y, Z, W) = g(R(X, Y)Z, W)$. The sectional curvature at the plane spanned by X, Y is defined by

$$\sec(X, Y) := \frac{\mathrm{rm}(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

In our convention, sectional curvature and scalar curvature of standard sphere is positive.

And we define the curvature operator

$$\Re(X \wedge Y) : \wedge^2 TM \to \wedge^2 TM$$

such that

$$g(\Re(X \wedge Y), Z \wedge W) = \mathrm{rm}(X, Y, W, Z), \tag{1.2}$$

where

$$g(X \wedge Y, Z \wedge W) := g(X, Z)g(Y, W) - g(X, W)g(Y, Z)$$

The Riemannian curvature tensor has a lot symmetries including the first and second Bianchi identities, which are listed as follows.

$$\mathrm{rm}(X, Y, Z, W) = -\,\mathrm{rm}(Y, X, Z, W) = \mathrm{rm}(Z, W, X, Y). \tag{1.3}$$

The first Bianchi identity

$$\mathrm{rm}(X, Y, Z, W) + \mathrm{rm}(Y, Z, X, W) + \mathrm{rm}(Z, X, Y, W) = 0. \tag{1.4}$$

The second Bianchi identity

$$\nabla_X \mathrm{rm}(Y, Z, U, V) + \nabla_Y \mathrm{rm}(Z, X, U, V)\nabla_Z(\mathrm{rm}(X, Y, U, V) = 0. \tag{1.5}$$

The Ricci curvature is the trace of the Riemannian curvature tensor, which is defined as

$$\mathrm{rc}(X, Y) = \sum_{i=1}^{n} \mathrm{rm}(e_i, X, Y, e_i), \tag{1.6}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of the tangent space T_pM. The scalar curvature is in turn the trace of the Ricci curvature.

$$\mathrm{sc} = \sum_{i=1}^{n} \mathrm{rc}(e_i, e_i). \tag{1.7}$$

A very important class of Riemannian manifold is the Einstein manifold, which is the manifold with constant Ricci curvature, i.e.,

$$\mathrm{rc}(X, Y) = \lambda g(X, Y) \tag{1.8}$$

for any $X, Y \in \Gamma(TM)$. The following Schur's lemma says that the isotropic Ricci curvature must be constant.

Lemma 1.1.1 (Schur's Lemma)**.** *If* $\mathrm{rc} = f(p)g$ *for some smooth function* $f : M^n \to \mathbb{R}$ *and for any* $p \in M$*, then* f *must be a constant function.*

The proof of the above lemma is a simply application of the following corollary of the Bianchi identities.

$$\sum_i^n \nabla_{e_i} \mathrm{rc}(e_i, X) = \frac{1}{2}\nabla_X \mathrm{sc}\,. \tag{1.9}$$

Here we introduce a less well known curvature which is called *complex sectional curvature*. The complex sectional curvature is important from the view point of the Ricci flow since its positivity is preserved along the Ricci flow in compact manifold (see e.g. Ni and Wolfson [3] or Andrews and Hopper [4]) and complete noncompact manifold with curvature bounded from below (see Corollary 3.3.3). We will use it in the following chapters.

Let $T^cM = TM \otimes \mathbb{C}$ be the complexified tangent bundle. We extend both metric g and curvature tensor rm to be $\mathbb{C}$-multilinear maps

$$g : S^2(\Gamma(T^cM)) \to \mathbb{C} \quad \mathrm{rm} : S^2(\wedge^2\Gamma(T^cM)) \to \mathbb{C}.$$

Definition 1.1.2. For any 2-dimensional complex subspace $\sigma^c \in T^c_pM$, the complex sectional curvature is defined by

$$\sec^c(\sigma^c) := \mathrm{rm}(u, v, \bar{v}, \bar{u}) = g(\Re(u \wedge v), \overline{u \wedge v}),$$

for any unitary orthogonal vectors $u, v \in \sigma^c$ i.e. $g(u, \bar{u}) = g(v, \bar{v}) = 1$ and $g(u, \bar{v}) = 0$, where, e.g. $\bar{u}$ is the complex conjugate of u.

Let $u = x + iy$ and $v = z + iw$, where $x, y, z, w \in T_pM$. Then the unitary condition becomes

$$g(x, x) + g(y, y) = 1, \quad g(z, z) + g(y, y) = 1. \tag{1.10}$$

And the orthogonal condition becomes

$$g(x, z) + g(y, w) = 0, \quad g(x, w) + g(y, z) = 0. \tag{1.11}$$

Remark 1.1.3. Although its name contains the work "complex", it is a "real" curvature well defined on any Riemannian metric. By a straightforward calculation below, we can see that for any σ^c, say spanned by $x + iy$ and $z + iw$ — a pair of unitary orthogonal complex vectors, $\sec^c(\sigma^c)$ is always a real number:

$$\begin{aligned}\sec^c(\sigma^c) = \operatorname{rm}(x,z,z,x) + \operatorname{rm}(x,w,w,x)\\ + \operatorname{rm}(y,z,z,y) + \operatorname{rm}(y,w,w,y) + 2\operatorname{rm}(x,y,z,w)\end{aligned} \tag{1.12}$$

When σ^c is a complex plane spanned by a pair of orthonormal real vector x, z over the complex number field and σ is a real plane spanned by the pair, the $\sec^c(\sigma^c)$ is exactly the sectional curvature $\sec(\sigma)$. So, for any constant c,

$$\sec^c \geq c \Rightarrow \sec \geq c. \tag{1.13}$$

Though the reverse is not true in general, when the sectional curvature is pointwise 1/4-pinched, then the complex sectional curvature is nonnegative (Ni and Wolfson [3, Proposition 3.2]).

We have the following relation between the curvature operator and complex sectional curvature, by an argument similar to the one in Andrews and Hopper [4, Chap. 14 Proposition 14.9].

$$\Re \geq c \Rightarrow \sec^c \geq 2c. \tag{1.14}$$

In particular, the positive(nonnegative) curvature operator implies the positive(nonnegative) complex sectional curvature. The reverse is not true neither.

In dimension two, all curvatures are actually the one — the Gaussian curvature. In dimension three, we have the following relations. Suppose $\mu \leq \nu \leq \lambda$ are the eigenvalues of the curvature operator, then the $\mu + \nu \leq \mu + \lambda \leq \nu + \lambda$ are the eigenvalues of the Ricci curvature. And, in contrast to those in higher dimension, the positive sectional curvature does imply the positive curvature operator. Actually, we have the following property.

Proposition 1.1.4. *On any Riemannian three manifold, the following inequality is true pointwisely:*

$$\sec \geq (>)c \Rightarrow \Re \geq (>)c.$$

Proof. The key to the proof is the fact that

$$\operatorname{rm}(e_1,e_2,e_3,e_1) = \operatorname{rm}(e_1,e_2+e_3,e_2+e_3,e_1) - \operatorname{rm}(e_1,e_2,e_2,e_1) - \operatorname{rm}(e_1,e_3,e_3,e_1). \tag{1.15}$$

We may first choose a plane $\sigma \subset T_pM$ spanned by orthonormal basis $\{e_1, e_2\}$, such that $\sec(\sigma)$ realizes the minimum of the sectional curvature at p. Then we pick up another unit vector e_3 in T_pM, which is orthogonal to σ. Then $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ is an orthonormal basis of $\wedge^2 T_pM$. Using the fact mentioned above, we a direct computation shows that $g(\mathfrak{R}(\varphi), \varphi) \geq c$ for any $\varphi = a\, e_1 \wedge e_2 + b\, e_1 \wedge e_3 + c\, e_2 \wedge e_3 \in \wedge^2 T_p M$, $a^2 + b^2 + c^2 = 1$. ◻

We conclude this section by the scaling properties of those curvatures.

Proposition 1.1.5. *Let $\tilde{g} = k^2 g$ for some nonzero constant k. Then, we have*

$$\widetilde{\mathfrak{R}} = \frac{1}{k^2}\,\mathfrak{R} \tag{1.16}$$

$$\widetilde{\sec}(\sigma) = \frac{1}{k^2}\,\sec(\sigma) \tag{1.17}$$

$$\widetilde{\mathrm{rm}}(X, Y, Z, W) = k^2\,\mathrm{rm}(X, Y, Z, W) \tag{1.18}$$

$$\widetilde{\mathcal{R}}(X, Y)Z = \mathcal{R}(X, Y)Z \tag{1.19}$$

$$\widetilde{\mathrm{rc}}(X, Y) = \mathrm{rc}(X, Y) \tag{1.20}$$

$$\widetilde{\mathrm{sc}} = \frac{1}{k^2}\,\mathrm{sc} \tag{1.21}$$

$$\widetilde{\sec}^c(\sigma) = \frac{1}{k^2}\,\sec^c(\sigma). \tag{1.22}$$

Proof. It is easy to see that $\widetilde{\mathfrak{R}} = c\,\mathfrak{R}$ for some constant c, which we can use definition to determine. The only subtle calculation is that, $\tilde{g}(X \wedge Y, Z \wedge W) = k^4 g(X \wedge Y, Z \wedge W)$. Others are by definition. ◻

1.1.2 Comparison Geometry

In this section, we mention only a few results from the comparison geometry, of which we will make use in later chapters. For a complete introduction to this field, the interested readers are directed to e.g. Cheeger and Ebin [5] do Carmo [6] [7] Petersen [1] among many others.

We begin by defining the geodesics and the exponential map.

Definition 1.1.6. A smooth path $\gamma : (a, b) \to M$ is called a *geodesic* if

$$\nabla_{\gamma'(t)} \gamma'(t) = 0.$$

Definition 1.1.7. For any $p \in M$ we define the *exponential map* $\exp_p : T_p M \to M$ such that for any $\xi \in T_p M$

$$\exp_p(\xi) := \gamma_\xi(1)$$

where $\gamma_\xi(t)$ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = \xi$.

Remark 1.1.8. Of course the exponential map is ill defined when the Riemannian metric is not complete — the geodesic might not be defined on $t = 1$. However, it perfectly makes sense when the Riemannian manifold is a complete metric space, of which the case in what we really concern in this paper. By Hopf-Rinow theorem, we know the geodesic completeness (every geodesic is well defined on $\mathbb{R}$) is equivalent to the metric completeness.

A fundamental fact about the exponential map is that its differential at T_pM (after identifying T_0T_pM with T_pM) is identity hence is a local diffeomorphism around $0 \in T_pM$. However in general, it is not a diffeomorphism. To describe in what extend it is a diffeomorphism, we introduce the notion of the injectivity radius.

Definition 1.1.9. Let (M, g) be any Riemannian manifold. For any $p \in M$, the *injectivity radius* of p is defined by

$$\mathrm{inj}_g(p) := \sup\{r \in \mathbb{R} : \exp_p|_{B_0(r) \subset T_pM} \text{ is injective.}\}$$

Now we can state various comparison theorems which compares the diameter of the Riemannian manifold or the volume of balls in a manifold with those in the space form.

Theorem 1.1.10 (Günther-Bishop, see Chavel [7] Theorem 3.7)**.** *Let (M, g) be any complete Riemannian manifold with sectional curvature less than or equal to K. Then for any $x \in M$, all $r \leq \min\{\mathrm{inj}_g(x), \pi/\sqrt{K}\}$ we have*

$$\mathrm{Vol}_g(B_r(x)) \geq \mathrm{Vol}_r^K \tag{1.23}$$

where Vol_r^K is the volume of the ball of radius r in the space form of constant curvature K, and $\pi/\sqrt{K} := +\infty$ when $K \leq 0$. Furthermore, the equality is achieved when $B_r(x)$ is isometric to the ball of radius r in the space form.

Now, let us turn to the theorems with the assumption of that the Ricci curvature is bounded from below. This is an extreme powerful assumption which leads to many important results.

The first one of those is the Bonnet-Myers theorem below (see e.g. Petersen [1] for proof.), which tells us that, if the Ricci curvature is bounded from below by a positive number, then the Riemannian manifold has bounded diameter.

Theorem 1.1.11 (Bonnet-Myers)**.** *Let (M^n, g) be a complete Riemannian n-manifold with Ricci curvature $\mathrm{rc} \geq (n-1) > 0$. Then its diameter is no greater than π, hence it is compact and has finite fundamental group.*

The Ricci curvature is closely related to the volume of balls in Riemannian manifold. One of the most powerful theorem in the Riemannian geometry is the Bishop-Gromov relative volume comparison theorem as follows.

Theorem 1.1.12 (Bishop-Gromov)**.** *Let (M^n, g) is a complete Riemannian manifold with* $\mathrm{rc} \geq (n-1)k$ *for any real number k. Then the volume ratio*

$$\frac{\mathrm{Vol}(B_r(p))}{v(n,k,r)}$$

is nonincreasing along r and has limit 1 *as $r \to 0^+$, where* $\mathrm{Vol}(B_r(p))$ *is the volume of the ball $B_r(p) \subset M$ centered at any $p \in M$ with radius r and $v(k,n,r)$ is the volume of the ball with radius r in the space form with constant sectional curvature k.*

There are numerous consequences of the above theorem which we have neither intention nor space to mention in this paper. But, we will use this theorem again and again.

Another important theorem in understanding the structure of the complete noncompact Riemannian manifolds with nonnegative Ricci curvature is the famous Cheeger-Gromoll splitting theorem below. Before stating it, we need the following definition.

Definition 1.1.13. A geodesic $\gamma : \mathbb{R} \to M$ is called a *line* if for any $s, t \in \mathbb{R}$, $\mathrm{dist}(\gamma(s), \gamma(t)) = |s - t|$.

Theorem 1.1.14 (Cheeger-Gromoll)**.** *Let (M^n, g) be a complete Riemannian manifold with nonnegative Ricci curvature. If it contains a line, then (M, g) splits isometrically to a direct product Riemannian manifold $N \times \mathbb{R}$.*

1.1.3 Compactness Theorem of Cheeger-Gromov Type

Because the injectivity radius lower bound is crucial to apply the Cheeger-Gromov compactness theorem, we first introduce an estimate of it.

The injectivity radius positive lower bound is controlled by the curvature in some complicated way, in particular when the curvature is positive. For compact manifold, the injectivity radius lower bound is studied extensively, e.g. Klingenberg's famous injectivity radius estimate for the compact Riemannian manifold with positive sectional curvature. However, for our purpose, we only discuss the following estimate of injectivity radius for complete non-compact Riemannian manifold with positive (and nonnegative) bounded sectional curvature.

Theorem 1.1.15 (Toponogov[8] also Maeda[9]). *If M is a complete noncompact Riemannian manifold, and for all point $p \in M$ and all tangent two-plane σ_p its sectional curvature $K(\sigma_p)$ satisfies the inequality $0 \le K(\sigma_p) \le \lambda$, then there exists a constant $i_0 > 0$ such that for all $q \in M$ the injectivity radius $i(q)$ satisfies*

$$i(q) \ge i_0.$$

Furthermore, if $0 < K(\sigma_p) \le \lambda$ for all σ_p, then for all $q \in M$,

$$i(q) \ge \frac{\pi}{\sqrt{\lambda}}.$$

Remark 1.1.16. In the case of strictly positive curvature, the non-compactness assumption is crucial to the theorem due to the example of Berger's spheres. Berger's sphere are a one parameter family of complete metrics on topological three sphere, whose sectional curvatures are between $[\epsilon^2, 4 - \epsilon^2]$, where ϵ is the parameterPetersen [1]. And further, when $\epsilon \to 0$, the three-spheres reduce to a two-sphere with constant curvature, in particular, the injectivity radius becomes zero while the curvature stays positive and bounded. Hence above theorem does not apply to the compact Riemannian manifold!

Remark 1.1.17. For the nonnegative curvature case, the bound of the injectivity radius in the above theorem still depends on the metric in addition to the curvature bound. This can be seen from the example of the direct product metrics of the Berger spheres and $\mathbb{R}$. Or, even simpler, from a flat cylinder $S^1 \times \mathbb{R}$.

Now let us turn to the central topic of this subsection — the (pointed) Cheeger-Gromov convergence and Cheeger-Gromov compactness theorem.

Definition 1.1.18. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of open sets in manifold M. We call it is an *exhaustion* if

- $\cup U_i = M$, and
- the closure $\overline{U}_i$ is compact and $\overline{U}_i \subset U_{i+1}$.

Definition 1.1.19. Let (M_i, g_i, p_i), $p_i \in M$ be a sequence of pointed complete Riemannian n-manifolds. We say they are convergent in the sense of pointed Cheeger-Gromov type, if there exists a pointed complete Riemannian n-manifold (M, g, p), $p \in M$, an exhaustion $\{U_i\}$ of M, and a sequence of diffeomorphism $\varphi_i : U_i \to \varphi_i(U_i) \subset M_i$ satisfy the following conditions.

- $p \in U_0$, $p_i \in \varphi_i(U_i)$, and $\varphi_i(p) = p_i$.

- Restricted to every compact set $K \subset M$, $\varphi_i^*(g_i) \to g$ in C^∞ topology.

Similarly, we can define the $C^{1,\alpha}$ convergence etc.

The following are sufficient conditions for the Cheeger-Gromov convergence.

Theorem 1.1.20 (Cheeger-Gromov Compactness Theorem). *Let (M_i, g_i, p_i) be a sequence of complete Riemannian n-manifolds. And there exist constants C_o, c, and $C_k > 0$, $i \in \mathbb{N}$, such that, for all i, k the following conditions are satisfied.*

- $|\mathrm{rm}_{g_i}|_{g_i} \leq C_0$.
- $|\nabla^k \mathrm{rm}_{g_i}|_{g_i} \leq C_k$.
- $\mathrm{inj}_{g_i}(p_i) \geq c > 0$.

Then there exists a subsequence of above pointed Riemannian manifolds, which converges in C^∞-topology in the sense of pointed Cheeger-Gromov to a limit pointed complete Riemannian n-manifold, say (M, g, p), on which the above restrictions are preserved. I.e.,

- $|\mathrm{rm}_g|_g \leq C_0$.
- $|\nabla^k \mathrm{rm}_g|_g \leq C_k$.
- $\mathrm{inj}_g(p) \geq c > 0$.

At the end of this section, let us state a lemma of Cheeger-Gromov-Taylor([Cheeger et al. [10] see also Wang [11]) which allows us to replace the condition of injectivity radius lower bound with volume ratio lower bound in above theorem.

Lemma 1.1.21 (Cheeger-Gromov-Taylor). *Let (M, g) be any complete Riemannian n-manifold whose sectional curvature is between -1 and 1. And $p \in M$ be any point. If there exists $c > 0$ and $r_o > 0$ such that, for any $0 < r < r_0$ the volume of ball $B(p, r) \subset M$ satisfying,*

$$\frac{\mathrm{Vol}(B(p, r))}{r^n} \geq c,$$

then $\mathrm{inj}(p) \geq i_0 > 0$ *for some constant i_0 depends only on n, r_0, and c.*

1.2 Curvature of Warped Product Metrics

1.2.1 Second Fundamental Form

We follow the definitions of Wu et al. [12] for the second fundamental form. Let $M \hookrightarrow \widetilde{M}$ be an immersion. $\mathcal{N}(M)$ be normal vector fields to M, and $\mathcal{T}(M)$ be tangent vector field on M. Without further indication, the $\widetilde{\ }$ed quantities are of $\widetilde{M}$ and untilded of M. E.g. g and $\tilde{g}$ are metrics of M and $\widetilde{M}$ respectively, ∇ and $\widetilde{\nabla}$ are the correspondent Levi-Civita connections. So are the curvatures.

Definition 1.2.1. For any $\nu \in \mathcal{N}(M)$, the second fundamental form of ν , denoted by II_ν, is defined by

$$\mathrm{II}_\nu(X, Y) = \tilde{g}(\widetilde{\nabla}_X \nu, Y) \quad \forall X, Y \in \mathcal{T}(M)$$

Remark 1.2.2. The covariant derivative $\widetilde{\nabla}_X \nu$ is well defined though ν is only defined along M. The reason is that the covariant derivative of a vector fields depending only on its value along a curve.

Remark 1.2.3. II_ν is a symmetric $(0, 2)$-tensor fields on M and

$$\mathrm{II}_\nu(X, Y) = -\tilde{g}(\nu, \widetilde{\nabla}_X Y).$$

And II_ν depends only on $\nu(x)$ where $x \in M$.

The second fundamental forms have other descriptions. One can define it as shape operator:

$$\mathcal{S} : T_xM \otimes T_xM \to T_x^\perp M$$

s.t. $\tilde{g}(S(v, w), \nu) = \mathrm{II}_\nu(v, w)$. Or for any $\nu \in \mathcal{N}(M)$, define

$$\mathcal{A}_\nu : T_xM \to T_xM$$

by $g(A_\nu(v), w) = \mathrm{II}_\nu(v, w)$.

Here we introduce two orthogonal projections:

$$\top : T_x\widetilde{M} \to T_xM \qquad \perp : T_x\widetilde{M} \to T_x^\perp M$$

Another perspective to the second fundamental form is that it is the difference of $\widetilde{\nabla}$ and $\perp \widetilde{\nabla}$. Since it is not hard to prove that $\nabla_X Y = \perp (\widetilde{\nabla}_X Y)$. We have

$$\mathcal{S}(X, Y) = \perp \widetilde{\nabla}_X Y - \widetilde{\nabla}_X Y = \nabla_X Y - \widetilde{\nabla}_X Y.$$

1.2.2 Curvatures of Submanifolds

We first state the relations between the second fundamental form and the Riemannian curvatures of M and $\widetilde{M}$.

Theorem 1.2.4 (Gauss Equations)**.** *Let $X, Y, Z, W \in \Gamma(\mathcal{T}M)$, then*

$$\begin{aligned} \tilde{g}(\widetilde{\mathcal{R}}(X,Y)Z,W) = g(\mathcal{R}(X,Y)Z,W) &+ \tilde{g}(\mathcal{S}(X,Z),\mathcal{S}(Y,W)) \\ &- \tilde{g}(\mathcal{S}(X,W),\mathcal{S}(Y,Z)) \end{aligned} \tag{1.24}$$

Theorem 1.2.5 (Codazzi Equations)**.**

$$\begin{aligned} (\widetilde{\mathcal{R}}(X,Y)Z)^{\perp} = \widetilde{\nabla}_X\mathcal{S}(Y,Z) - \widetilde{\nabla}_Y\mathcal{S}(X,Z) &- \mathcal{S}(\nabla_X Y,Z) \\ + \mathcal{S}(\nabla_Y X,Z) - \mathcal{S}(Y,\nabla_X Z) &+ \mathcal{S}(X,\nabla_Y Z) \end{aligned} \tag{1.25}$$

Remark 1.2.6*.* The Gauss equation are about the tangential components of the Riemannian curvature, the Codazzi equations are the perpendicular components.

When M is a hypersurface, II is the same as $\mathcal{S}$. When we assume further that $\widetilde{M}$ has constant sectional curvature, while the Gauss equation remains the same, the the Codazzi equation becomes

$$(\widetilde{\mathcal{R}}(X,Y)Z)^{\perp} = \Big((\nabla_X \mathrm{II})(Y,Z)) - (\nabla_Y \mathrm{II})(X,Z)\Big)\nu \tag{1.26}$$

where ν is unit normal vector field on M. (Yes! ∇_X NOT $\widetilde{\nabla}_X$ here!) The proof of the above equation can be found in Section 1.2.3 — the part of evaluating Codazzi equation for $r = c$ in metric $\mathrm{d}\, r^2 + g_r$ — where we only need to change $\widetilde{\nabla} r$ as ν.

The geometric meaning of Gauss equations and Codazzi equations is that they are integrable conditions of isometric embedding of submanifold with prescribing second fundamental form with codmension one. Details can be found in Wu etc [12] along with many interesting results of the submanifold.

For those submanifolds with codimension greater then one, we have ricci equation, which describes the curvature like $(\widetilde{\mathcal{R}}(X,Y)\nu)^{\perp}$. Interested reader can find the formula from Spivak [13], though we do not need in the thesis.

Remark 1.2.7*.* $(\widetilde{\mathcal{R}}(X,Y)\nu)^{\top}$ is in fact Codazzi equations by Bianchi identity. And the Riemannian curvature involving three normal vectors has nothing to do with the submanifold.

1.2.3 Geometry of Warped Product Metric

Definition 1.2.8. We call g a warped product metric on a manifold M if it satisfies the following conditions:

1. M is diffeomorphic to $I \times N$ for some interval I and manifold N.
2. There exist a smooth function $f : I \mapsto \mathbb{R}^+$ and a Riemannian metric h on N s.t. $g = \mathrm{d}\,r^2 + f^2(r)h$ on coordinate system $(r, p) \in I \times N$.

We call (N, h) the warped component, the direction tangential to N the radial direction, and ∇r the axial direction. We will also denote the metric $f^2(r)h$ by g_r.

From the definition, (N, g_r) is a hypersurface in (M, g) for any fixed r. And r is a smooth distance function, hence $\nabla r = \partial_r$. Then we have the Hessian of r and the curvature equations in the direction of ∇_r, [see 1, p. 47].

$$\mathfrak{L}_{\partial_r} g = 2\,\mathrm{Hess}\,r \tag{1.27}$$

$$(\mathfrak{L}_{\partial_r}\,\mathrm{Hess}\,r)(X, Y) - \mathrm{Hess}^2\,r(X, Y) = -\,\mathfrak{R}(X, \partial_r, \partial_r, Y) \tag{1.28}$$

For the convenience of the calculation, we introduce $\phi = \log f$. Then the metric becomes

$$g = \mathrm{d}\,r^2 + \mathrm{e}^{2\phi(r)}\,h. \tag{1.29}$$

The volume form of g is

$$\mathrm{dvol} = \mathrm{e}^{n\phi}\;\mathrm{d}\,r \wedge \mathrm{dvol}_h. \tag{1.30}$$

By simple calculation, we have $f''/f = \phi'' + (\phi')^2$ and $f'/f = \phi'$. Because $f > 0$ is a smooth function, ϕ is smooth wherever f is defined. Using the curvature equation (1.27), the Gauss equation (1.24), and the Codazzi equation (1.25), curvatures can be written as below. The computation is straightforward and not too different from those in Petersen [1, pp.], where it is about the rotationally symmetric metric.

$$\mathfrak{R}(\nabla r \wedge X) = -(\phi'' + (\phi')^2)\nabla r \wedge X. \tag{1.31}$$

Let $\{E_a\}$, $1 \leq a \leq n(n-1)/2$ be a o.n. frame of $\wedge^2 T_x N$, which diagonalize the curvature operator $\mathfrak{R}^h$, with correspondent eigenvalues λ_a's. It is easy to see

$$\mathfrak{R}(E_a) = (\lambda_a \mathrm{e}^{-2\phi} - (\phi')^2)E_a. \tag{1.32}$$

Let $\{e_\alpha\}$, $0 \leq \alpha \leq n+1$ be an o.n. frame for g and $e_0 = \nabla r$. Hence $\{e_i\}$

$i = 1, 2, \cdots, n$ are orthogonal frame on (N, h). And the sectional curvature

$$\sec(e_0, e_i) = -(\phi'' + (\phi')^2), \quad \sec(e_i, e_j) = \mathrm{e}^{-2\phi} \sec^h(e_i, e_j) - (\phi')^2.$$

We can choose $\{e_i\}$ further s.t. they diagonalize rc^h, then for any $i \neq j$,

$$\mathrm{rc}(e_0, e_0) = -n(\phi'' + (\phi')^2) \quad \mathrm{rc}(e_0, e_i) = 0 = \mathrm{rc}(e_i, e_j). \tag{1.33}$$

and

$$\mathrm{rc}(e_i, e_i) = \mathrm{rc}^h(e_i, e_i) - (n-1)(\phi')^2 - (\phi'' + (\phi')^2) = \mathrm{rc}^h(e_i, e_i) - n(\phi')^2 - \phi''. \tag{1.34}$$

Thus, the scalar curvature is

$$\mathrm{sc} = \mathrm{e}^{-2\phi} \mathrm{sc}^h - 2n(\phi'' + (\phi')^2) - n(n-1)(\phi')^2 = \mathrm{e}^{-2\phi} \mathrm{sc}^h - 2n\phi'' - n(n+1)(\phi')^2. \tag{1.35}$$

We will need the following property of the geodesics of on the warped metric in later chapters.

Lemma 1.2.9. *Assume that $|\phi'(r)| \leq \rho$ for some constant $\rho \geq 0$ and all $r \in \mathbb{R}$. Then any unit ball $B_1(r_0, p) \subset M$ centered in (r_0, p) with respect to metric g is contained in $[r_0 - 1, r_0 + 1] \times B_C^{r_0}(p)$, where $B_C^{r_0}(p)$ is ball centered at $p \in N$ with radius C in N with respect to g_{r_0} and $C = \mathrm{e}^{\rho}$.*

Proof. Let $\gamma(t) = (r(t), y(t))$, where $0 \leq t \leq \tau$ and $\gamma(0) = (r_0, p)$, be a length minimizing geodesic w.r.t. g whose length is less than 1. We will estimate the $\max_{0 \leq t \leq \tau} |r(t) - r(0)|$ and $\max_{0 \leq t \leq \tau} \mathrm{dist}_{r_0}(y(t), y(0))$ where dist_{r_0} is the distance function induced by g_{r_0} on N. Let $L^{\gamma}(0, \tau)$ be the length of the geodesic segment $\gamma(t)$ with $0 \leq t \leq \tau$, and $|\cdot|_{r(t)}$ be the norm induced by the metric $g_{r(t)}$. Obviously,

$$1 \geq L^{\gamma}(0, \tau) = \int_0^{\tau} |\gamma'(t)|_g^2 \,\mathrm{d}\, t = \int_0^{\tau} \sqrt{r'^2 + |y'|_{r(t)}^2} \,\mathrm{d}\, t. \tag{1.36}$$

On the one hand, for any $t \in [0, \tau]$, by equation (1.36),

$$1 \geq \int_0^t |r'(s)| \,\mathrm{d}\, s \geq \left| \int_0^t r'(s) \,\mathrm{d}\, s \right| = |r(t) - r(0)|.$$

So we have $\max_{0 \leq t \leq \tau}(|r(t) - r(0)|) \leq 1$

On the other hand, by equation (1.36) again,

$$1 \geq \int_0^t |y'|_{r(s)} \,\mathrm{d}\, s = \int_0^t \mathrm{e}^{\phi(r(s))} |y'(s)|_h \,\mathrm{d}\, s.$$

where $|\cdot|_h$ means the norm induced by metric h. Next, we estimate $\phi(r(s))$. Note

$$\begin{aligned}\phi(r(t)) - \phi(r(0)) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\phi(r(s))\,\mathrm{d}\,s = \int_0^t \frac{\mathrm{d}}{\mathrm{d}r}\phi(r)\frac{\mathrm{d}}{\mathrm{d}s}r(s)\,\mathrm{d}\,s \\ &= \int_{r(0)}^{r(t)} \frac{\mathrm{d}}{\mathrm{d}r}\phi(r)\,\mathrm{d}\,r \geq -c|r(t) - r(0)| \geq -c\end{aligned}$$

Combine above two inequalities, we get

$$1 \geq \int_0^t \mathrm{e}^{\phi(r_0)-\rho}\,|y'(s)|_h\,\mathrm{d}\,s \geq \mathrm{e}^{-\rho}\int_0^t \mathrm{e}^{\phi(r_0)}\,|y'(s)|_h\,\mathrm{d}\,s = \mathrm{e}^{-\rho}\int_0^t |y'(s)|_{r_0}\,\mathrm{d}\,s.$$

Note that the last integral is the length of the curve $y(s) \subset N$, $0 \leq s \leq t$ with respect to metric g_{r_0}, consequently the distance between $y(0)$ and $y(t)$ is less than e^{ρ} .

When (r_1, q) is in $B_1(r_0, p)$, let $(r(s), y(s))$, $0 \leq s \leq t$ be a length minimizing geodesic segment s.t. $(r(0), y(0)) = (r_0, p)$ and $(r(t), y(t)) = (r_1, q)$. By definition, the length of the geodesic segment is less than 1. By arguments above, we know $|r_1 - r_0| \leq 1$ and $\mathrm{dist}_{r_0}(p, q) \leq \mathrm{e}^{\rho}$, where dist_{r_0} is the distance function induced by metric g_{r_0}. In other words, $(r_1, q) \in [r_0 - 1, r_0 + 1] \times B^{r_0}_{e^{\rho}}(p)$ which completes the proof.

□

Chapter 2

Basics of Ricci Flow

Let M be a differentiable manifold, I be an interval, and $g(t) \equiv g_t \equiv g(x,t)$ where $(x,t) \in M \times I$, be a family of complete Riemannian metrics on M, which is parameterized by $t \in I$. Let rc_t be ricci curvature of g_t. The ricci flow is a weakly parabolic system of partial differential equations defined by

$$\frac{\partial}{\partial t} g_t = -2\,\mathrm{rc}_t\,. \tag{2.1}$$

The one of the purposes of the this dissertation is to study a uniqueness of the solution to the above equation in the case that the underlying manifold is complete non-compact.

The Ricci flow was first introduced by Richard Hamilton in his celebrated 1982 paper[14] and was used to deform Riemannian metrics. Thereafter, as a powerful tool, ricci flow had greatly changed the landscape of differential geometry. For example, it is a crucial tool to prove Poincaré conjecture and differentiable sphere theorem. See, for example, [15] and [16]. Since there are so many progresses in the topic of the Ricci flow after it was introduced, it is impossible to discuss all of them here. The interested readers are directed to e.g. Andrews and Hopper [4], Chow et al. [17, 18, 19, 20] etc. In this chapter, we only provide the minimal background of Ricci flow for our results in the subsequent chapters.

The Ricci flow as a system of partial differential equations, the short time existence and uniqueness of its solution to the initial value problem are among the most fundamentals. When the underlying manifold is compact without boundary, they were first proved by Hamilton in his first paper on this subject mentioned above.

Theorem 2.0.10. *Let (M, g) be a compact Riemannian manifold. Then there exists a unique solution to the equation (2.1) which satisfies the initial condition $g(0) = g$*

on some short time interval $[0, \epsilon)$.

The obstacle prevents us using standard parabolic equation theory to prove the short time existence here is that Ricci flow is a weakly parabolic system i.e. the Ricci curvature as an operator acting on the metric, is degenerate. Shortly after Hamilton's proof was published, DeTurck[21] first realized the reason behind it is that the Ricci curvature is diffeomorphism invariant. He, by introducing a time dependent diffeomorphism evolving along with the Ricci flow, transformed the equation to a strict parabolic one, where the classical theory of existence is available. This technique now is commonly named as DeTurk's trick. Together with the help from the harmonic heat flow, Hamilton gave a new proof to the uniqueness in [22] using DeTurck's trick. DeTurk's trick is also applied with modification to the same question in the case of that the underlying manifold is either compact with boundary (see Gianniotis [23]) or complete noncompact (see Shi [24], Chen and Zhu [25]).

At the birth of the Ricci flow, Hamilton had already use it to study the topology of Riemannian manifolds. As a result, he proved the following sphere theorem.

Theorem 2.0.11 (R. Hamilton)**.** *Let* (M^3, g) *be a compact Riemannian three manifold with strictly positive Ricci curvature. Then* M^3 *admits a positive constant sectional curvature.*

The application of the Ricci flow in this respect culminate at the proof of Poincaré conjecture and Thurston's geometrization conjecture by Perelman (Perelman [15, 26, 27] see also Kleiner and Lott [28], Cao and Zhu [29], Morgan and Tian [30]) and the differentiable sphere theorem by Brendle and Schoen (Brendle and Schoen [16], see also Ni and Wolfson [3], Andrews and Hopper [4]). Along the course of proving above celebrated theorems, many features of the Ricci flow are discovered and a lot exciting new methods are invented, however, we have no intention to enumerate all has been achieved but only those related to the uniqueness of solution in the complete noncompact manifolds.

2.1 Evolution Equations and Derivative Estimate

Though Ricci flow itself is only weakly parabolic, the evolution equations for various curvatures are strictly parabolic, which we list as below. The derivation can be easily found in Hamilton [31] or Chow et al. [17] etc.

The involution of the inverse of the metric is

$$\partial_t g^{ij} = 2g^{ik}R_{kl}g^{lj}. \tag{2.2}$$

The evolution equation for the volume form is

$$\partial_t \,\mathrm{dvol} = -\,\mathrm{sc}\ \mathrm{dvol}. \tag{2.3}$$

The evolution of the Riemannian curvature tensor is

$$\begin{aligned}\partial_t R_{ijkl} = \Delta R_{ijkl} &+ g^{pq}(R^r_{ijp}R_{rqkl} - 2R^r_{pik}R_{jqrl} + 2R_{pirl}R^r_{jqk}) \\ &- (R^p_i R_{pjkl} + R^p_j R_{ipkl} + R^p_k R_{ijpl} + R^p_l R_{ijkp}).\end{aligned} \tag{2.4}$$

Remark 2.1.1. Above equation can be further simplified using so called Ulenbeck's trick, which reader can find from Chow and Knopf [32].

The evolution equation of Ricci curvature is

$$\partial_t R_{jk} = \Delta R_{jk} + 2g^{pq}g^{rs}R_{pjkr}R_{qs} - 2g^{pq}R_{jp}R_{qk}. \tag{2.5}$$

The evolution equation of scalar curvature is

$$\partial_t \,\mathrm{sc} = \Delta\,\mathrm{sc} + 2|\mathrm{rc}|^2. \tag{2.6}$$

Just like the classic heat equation, maximum principle is one of the most powerful tool to extract information out of the various evolution equations here. As a simple consequence of the evolution equation of scalar curvature, its lower bound is preserved along the Ricci flow on any compact manifold.

Proposition 2.1.2. *Let $(M, g(t))$ be a Ricci flow solution on any compact manifold M. If the scalar curvature is bounded from below for $g(0)$, then it is bounded by the same constant from below for any $t > 0$.*

Proof. Apply maximum principle for equation (2.6). □

Using maximum principle, in particular Böhm and Wilking's refinement (Böhm and Wilking [33]), on compact manifolds, the positive curvature operator (Böhm and Wilking [33]), the positive isotropic curvature (Brendle and Schoen [16], Nguyen [34]), and the positive complex sectional curvature (Ni and Wolfson [3]) are preserved along the Ricci flow. These facts are key to prove the following long standing conjectures.

Theorem 2.1.3 (Böhm and Wilking[33])**.** *Compact manifolds with positive curvature operators are diffeomorphic to spaces forms.*

Theorem 2.1.4 (Brendle and Schoen[16], see also [3])**.** *Any compact Riemannian manifold with positive pointwise* $1/4$*-pinched sectional curvature is diffeomorphic to a spherical space form.*

Another important feature of Ricci flow is that the uniform bound of the Riemannian curvature implies the uniform bound of their derivatives. This estimate is first done by Shi [24], which is named after him now.

Theorem 2.1.5 (W.-X. Shi[24] see also Chow and Knopf [32])**.** *Let* $(M^n, g(t))$ *be a solution of the Ricci flow for which the maximum principle holds (this is true when M is compact!). Then for each* $\alpha > 0$ *and every* $m \in \mathbb{N}$*, there exists a constant* C_m *depending only on m and* $\max\{\alpha, 1\}$ *such that, if*

$$|\mathrm{rm}|_{g(t)} \leq K \quad \textit{for all } x \in M \textit{ and } t \in [0, \frac{\alpha}{K}],$$

then

$$|\nabla^m \mathrm{rm}|_{g(t)} \leq \frac{C_m K}{t^{m/2}} \quad \textit{for all } x \in M \textit{ and } t \in (0, \frac{\alpha}{K}].$$

2.2 Hamilton's Compactness Theorem

The Hamilton's compactness theorem is especially useful in analyzing singularities. The definition of the Cheeger-Gromov type convergence for the Ricci flow is very similar to that for the Riemannian manifold. However, with the help of Shi's Derivative estimate, the Hamilton's compactness theorem has relaxed assumption comparing with the correspondent one in Riemannian geometry while maintains the similar conclusion . We first state the definition of the convergence for the Ricci flow solutions.

Definition 2.2.1. Let $(M_i^n, g_i(t), p_i)$, $p_i \in M$ and $t \in (-T, 0]$ be a sequence of pointed complete Ricci flow solutions. We say they are *convergent in the sense of pointed Cheeger-Gromov type*, if there exist a pointed complete Ricci flow solution $(M^n, g(t), p)$, $p \in M$ and $t \in (-T, 0]$, an exhaustion $\{U_i\}$ of M, and a sequence of diffeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subset M_i$ satisfying the following conditions.

- $p \in U_0$, $p_i \in \varphi_i(U_i)$, and $\varphi_i(p) = p_i$.
- Restricted to every compact set $K \subset M$, $\varphi_i^*(g_i(t)) \to g(t)$ in $C^\infty(K \times (-T, 0])$ topology, for any $t \in (-T, 0]$.

As an application of the compactness theorem of Cheeger-Gromov type (Theorem 1.1.20), we have the following compactness theorem for the Ricci flow (Hamilton [31] also Wang [11]).

Theorem 2.2.2. *Let $(M_i^n, g_i(t), p_i)$, $t \in (-T, 0]$ be a sequence of pointed complete Ricci flow solutions. And there exist constants C_o, c, and $C_k > 0$, $i \in \mathbb{N}$, such that, for all i, k the following conditions are satisfied.*

- $|\mathrm{rm}|_{g_i(t)} \le C_0$.
- $\mathrm{inj}_{g_i(0)}(p_i) \ge c > 0$.

Then there exists a subsequence of pointed Ricci flow solutions, which converges in C^∞-topology in the sense of pointed Cheeger-Gromov to a limit pointed complete Ricci flow solution, say $(M, g(t), p)$, for any $t \in (-T, 0]$.

Sketch of the proof. At the first glance, to apply Theorem 1.1.20 for $(M_i, g_i(0), p_i)$, it seems the higher derivative bounds are missing. However, since we are working on the Ricci flow, the Shi's derivative estimate (Theorem 2.1.5), comes to help us to get them. To extend the convergence to $t < 0$, we need the positive lower bound for the injectivity radius, which can be obtain through Ricci flow again. Equation (2.3) and (2.1) plus the assumption of bounded curvature lend us a control of the growth of the volume as well as the distance. Then applying the lemma of Cheeger-Gromov-Taylor (Lemma 1.1.21), we have what we desire. □

2.3 Blowup Technique

Blowup technique is the major tool to deal with the singularities occurring in the Ricci flow solution. Here we briefly summarize this idea in the context of compact manifolds. And in the Chapter 3, we will use a version of this technique to serve our own purpose.

On compacts manifolds, Hamilton proved the following long time existence theorem (Chow and Knopf [32, Theorem 7.1]).

Theorem 2.3.1. *Let $(M, g(t))$, $t \in [0, T)$ be a smooth Ricci flow solution on a compact manifold. And T is the maximum time the solution can exist. Then*

$$\lim_{t \to T^-} \sup_M |\mathrm{rm}|_{g(t)} = +\infty. \tag{2.7}$$

Remark 2.3.2. There are even better results in this direction, for instance, Šešum [35], Wang [36] etc. Actually, X.-X. Chen (see Wang [11]) conjectures that above theorem can be true when conclusion changed to that the scalar curvature becomes infinite as $t \to T^-$.

Hence, to study the singularities, it is important to study the high curvature region. The strategy here are summarized as follows.

1. Find a sequence of points (p_i, t_i) in the spacetime on which the curvature goes to infinity while $t_i \to T$.
2. Dilate the Ricci flow solutions at our choices of the time from the sequence according to the curvature.
3. Make sure above sequence of pointed Ricci flow solutions subconverge to a limit Ricci flow solution.
4. Study the properties of the limit, derive whatever conclusion about the Ricci flow as we want.

The choice of the sequence of the spacetime points is obviously highly non-unique. However, to apply Theorem 2.2 in the hope of obtaining subconvergent Ricci flow solutions in our scheme, the choice must satisfy the following conditions.

1. $t_i \to T$ as $i \to \infty$.
2. $|\mathrm{rm}_{g(t_i)}(p_i)| \to \infty$ as $i \to \infty$.
3. $|\mathrm{rm}_{g(t)}(p)| \le |\mathrm{rm}_{g(t_i)}(p_i)|$ for all $(p, t) \in M \times [0, t_i]$.

The first two requirements are self-evident, while the last one is related to the dilation defined below. With this choice, we get a sequence of Ricci flow solutions with bounded curvature. In the case that M is compact, by a standard compactness argument, all the above conditions can be satisfied while picking points.

Now we describe the parabolic dilation as follows. Let $(M, g(t))$, $t \in [0, T]$ be a solution of the Ricci flow and Q be a positive real number. The new metric

$$\tilde{g}(t) := Qg(Q^{-1}t + T) \tag{2.8}$$

is called a parabolic dilation of $g(t)$. By straightforward computation, $\tilde{g}(t)$ is a Ricci flow solution defined on $[-QT, 0]$. And the Riemannian curvature $\widetilde{\mathrm{rm}} = Q^{-1}\,\mathrm{rm}$.

Take above $Q = |\mathrm{rm}_{g_i(t_i)}(p_i)| \equiv Q_i$, we get a sequence of dilated Ricci flow $\tilde{g}_i(t)$, $t \in [-Q_i t_i, 0]$ whose Riemannian curvature is uniformly bounded by -1. We claim that, when $T < \infty$, the injectivity radii of $\tilde{g}_i(0)$'s

$$\mathrm{inj}_{\tilde{g}_i(0)}(p_i) \ge c > 0.$$

To prove above claim, we need Perelman's celebrated no local collapsing theorem (see Perelman [26], for detailed exposition see e.g. Chow et al. [18]), stated below.

Definition 2.3.3. Let (M, g) be any complete Riemannian manifold. Given $\rho \in (0, \infty]$ and $\kappa > 0$, we say that the metric g is *κ-noncollapsed below the scale ρ* if for any metric ball $B(p, r)$, $r < \rho$, satisfying $|\mathrm{rm}| \leq \frac{1}{r^2}$ everywhere in the ball, we have

$$\frac{\mathrm{Vol}(B(p,r))}{r^n} \geq \kappa. \tag{2.9}$$

In particular, when $\rho = \infty$, we say that g is κ-noncollapsed at all scales.

By definition, we have following scaling property.

Proposition 2.3.4. *If g is κ-noncollapsed below the scale ρ, then for any $k > 0$, $k^2 g$ is κ-noncollapsed below the scale $k\rho$.*

Theorem 2.3.5 (No local collapsing)**.** *Let $(M^n, g(t))$, $t \in [0, T)$ be a Ricci flow solution on a compact manifold with $T < \infty$. Then for any $\rho > 0$, there exists a constant $\kappa = \kappa(n, g(0), T, \rho) > 0$ such that $g(t)$ is κ-noncollapsed below the scale ρ for all $t \in [0, T)$.*

The proof of the above theorem can be found in the references mentioned earlier.

Applying Theorem 2.3.5 to $g(t_i)$ then Proposition 2.3.4 to $\tilde{g}_i(0)$ at p_i, we have, for any $r < \sqrt{Q_i}$, hence for $r < 1$ when i is sufficiently large,

$$\frac{\mathrm{Vol}(B_{\tilde{g}_i(0)}(p_i, r))}{r^n} \geq \kappa > 0, \tag{2.10}$$

We also notice that κ depends only on $g(0)$, dimension, and T, which, in particular is independent of i. Then apply Lemma 1.1.21 of Cheeger-Gromov-Taylor, the uniform positive lower bound of injectivity radius for $\tilde{g}_i(0)$ at p_i is obtained. Therefore, applying Theorem 2.2.2, above dilated solutions subconverge to a limit pointed Ricci flow solution, say $(M, g_\infty(t))$ on $(-\infty, 0]$.

2.4 Ancient Solution

As shown in the last section, the limit of the dilation of a finite time singular Ricci flow solutions is so called ancient solution defined as follows.

Definition 2.4.1. We call a Ricci flow solution $(M, g(t))$ defined on $t \in (-\infty, T]$ for some finite T an *ancient solution*.

Remark 2.4.2. We can also define accordingly the eternal or the immortal solutions for those solutions defined on $(-\infty, \infty)$ or (T, ∞) respectively. Those definition is useful for e.g. solutions with singularity at the time infinity, the limit of dilations of which are eternal solutions.

Understanding the properties of the ancient solutions are important for understanding the singularities of the Ricci flow solutions. Here we state several properties of the ancient solutions for our later use.

Lemma 2.4.3 (Cabezas-Rivas and Wilking [37])**.** *Let $(M, g(t))$ be a non-flat ancient solution which is complete noncompact for any $t \in (-\infty, 0]$. Assume further that $g(t)$ has bounded curvature operator and nonnegative complex sectional curvature. Then, for all t,*

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B_{g(t)}(\cdot, r))}{r^n} = 0$$

Remark 2.4.4. A version of the above theorem was first proved in Perelman [38] with additional assumption of nonnegative curvature operator and κ-noncollapsing, of which the later is removed in Ni [39].

In dimension three, even the assumption of positive curvature of the above theorem is redundant. Using his local pinching estimate on complete Ricci flow solution, B-L Chen is able to prove the following result in Chen [40].

Lemma 2.4.5. *Any ancient smooth complete Ricci flow solution (even without bounded curvature) on three manifold must have nonnegative sectional curvature.*

Note that the nonnegative sectional curvature actually is equivalent to the nonnegative curvature operator in dimension three (1.1.4). Therefore in Lemma 2.4.3, the nonnegativity assumption on complex sectional curvature can be safely removed.

Chapter 3

Uniqueness of Ricci Flow Solution on Noncompact Manifolds

In this chapter we discuss the uniqueness of Ricci flow solution on noncompact manifolds.

3.1 Introduction

3.1.1 Fundamentals and Ideas from Classic Heat Equation

As the Ricci flow is a (weakly) parabolic equation, it is helpful to have its prototype — the classic heat equation — in comparison. Specifically, we consider the following Cauchy problem.

$$\begin{cases} \frac{\partial}{\partial t}u - \triangle u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

One cannot take the uniqueness of solution to the above problem for granted. It is Tychonoff who had not only discovered a non-zero solution with vanishing initial data, but also proved the uniqueness by assuming solutions with at most exponential growth (see e.g. John[41] page 211 and 217)

Following the spirit of Tychonoff, Chen and Zhu established a uniqueness theorem for Ricci flow in the class of metrics with bounded curvature.

Theorem 3.1.1 (Chen-Zhu [25])**.** *Let (M^n, g) be a complete noncompact Riemannian manifold of dimension n with bounded curvature. Let $g_1(t)$ and $g_2(t)$ be two solutions to the Ricci flow on $M \times [0,T]$ with g as their common initial data and with bounded curvatures. Then $g_1 = g_2$ for all $(x,t) \in M^n \times [0,T]$.*

This theorem is the basis of our research on this problem.

Also along this spirit, Kotschwar [42] proved the following theorem.

Theorem 3.1.2. *Let* (M, g_0) *is a complete noncompact Riemannian metric with volume growth*

$$\mathrm{Vol}_{g_0}(B_{g_0}(x_0, r)) \leq V_0 e^{V_0 r}$$

for some constant V_0 *and any* $r > 0$. *If* $g_1(t)$ *and* $g_2(t)$ *are two solutions of Ricci flow with* $g_1(0) = g_2(t) = 0$ *on* $M \times [0, T]$, *for which*

$$\gamma^{-1} g_0(x) \leq g_1(x, t), \ g_2(x, t) \leq \gamma g_0(x)$$

and

$$|\mathrm{rm}(x, t)|_{g_1(t)} + |\mathrm{rm}(x, t)|_{g_2(t)} \leq \frac{K_0}{t^\delta}(r_0^2(x) + 1),$$

for some constants γ, K_0, *and* $\delta \in (0, 1/2)$, *then* $g_1(t) = g_2(t)$ *for all* $t \in [0, T]$.

His method is completely different from Chen-Zhu's. Instead, he deals the ricci flow directly by forcing an energy to zero, which is defined by weighted L^2- norm of the difference of metrics, connections, and curvatures of two solutions. Hence these two solutions are identical.

Let us go back to the classic heat equation once more. A different condition to ensure the uniqueness of solution to equation (3.1) is assuming that the solution is bounded from below.

Theorem 3.1.3 (Widder see [41])**.** *With further assumption* $u(x, t) \geq 0$, *the solution to Equation* (3.1) *is unique.*

Remark 3.1.4. Both the conditions imposed in Tychonoff's and Widder's theorem are physically natural — remember that the heat equation was first derive for describing the propagation of heat, and the solution is simply the temperature which is the density of thermal energy! It is not only very reasonable but also a physical law to require the total energy is finite or no temperature can be lower than absolute zero!

Considering the heat equation on Riemannian manifolds, whether Widder's type of uniqueness theorem is true actually depends on the curvature. A typical sufficient condition is Ricci curvature being bounded from below, which is state below(see Li and Yau [43, Theorem 5.1] or [44]).

Theorem 3.1.5 (Li-Yau and Donnelly)**.** *Let* (M, g) *be a complete Riemannian manifold with Ricci curvature at any* $x \in M$ *satisfying*

$$\mathrm{rc}_x \geq -C \operatorname{dist}^2(p, x)$$

for some constant C, where $p \in M$ is some fixed point. The problem (3.1) has unique solution within the category of functions bounded from below.

Furthermore, A condition which ensures non-uniqueness is that the sectional curvature of a geodesic ball with radius R is bounded from above by $-CR^{2+\epsilon}$ for some positive constant C and ϵ (Azencott [45]). Actually the following theorem provide a sufficient and necessary condition for the Widder's type of uniqueness theorem (Murata [46]).

Theorem 3.1.6. *Let (M, g) be a simply connected smooth complete Riemannian manifold, in which there exists a point p, positive constants a, b, and a positive continuous function $k : [0, +\infty) \to \mathbb{R}^+$, such that for any $R > 0$, $q \in \partial B(p, R)$, and orthonormal vectors $X, Y \in T_pM$, the sectional curvature* $\sec(X, Y)$ *satisfies*

$$-bk(R) \leq sec(X, Y) \leq -k(R) \leq -aR^2$$

Then the Widder's theorem is true for (M, g) if and only if

$$\int_1^\infty \frac{\mathrm{d}\, r}{\sqrt{k(r)}} = \infty$$

3.1.2 Curvature Bounded from Below

Back to the uniqueness of the Ricci flow solution, one may be curious whether we can weaken the condition of bounded curvature. In particular, can we replace the bilateral bound of the curvature by unilateral bound? I.e., can we have the Widder's type of uniqueness theorem?

By the above discussion, on complete Riemannian manifold, we notice that to prove the uniqueness theorem for classic heat equation, in addition to the assumption of solution bounded from below, we need also the curvature bounded from below. And the evolution equations of the curvature in the case of Ricci flow is a parabolic equation, hence one might speculate that assuming curvature bounded from below might be a good assumption ensuring the uniqueness.

Indeed, we prove the following theorem, which is the main theorem of this chapter.

Theorem 3.1.7 (Sheng and Wang [47])**.** *Let (M^n, g) be a complete noncompact n-dimensional Riemannian manifold, with the complex sectional curvature* $\sec^c \geq -1$ *and the scalar curvature* $\mathrm{sc} \leq s_0$ *for some $s_0 > 0$. In addition, the volume of every unit ball is bounded from below uniformly by some constant $v_0 > 0$. Let $g_i(t)$ $i = 1, 2$ be two solutions to the Ricci flow on $M \times [0, T]$ with same initial data*

$g_i(0) = g$. If both $g_i(t)$ are complete Riemannian metrics with $\sec^c(g_i(t)) \geq -1$ for all $t \in [0, T]$, then there exists a constant $C_n > 0$ depending only on dimension n, s.t. $g_1(t) = g_2(t)$ on $M \times [0, \min\{\epsilon, T\}]$, where $\epsilon = \min\{v_0/2C_n, 1/(n-1)\}$.

Remark 3.1.8. We need the volume lower bound for unit ball of initial metric to obtain injectivity radius lower bound of $g(t)$ when $t > 0$. This is crucial to establish Cheeger-Gromov convergence of ricci flows in our proof. Actually, the curvature condition alone cannot guarantee the volume lower bound, which is shown by the following example. Consider a rotationally symmetric metric $g = dr^2 + e^{-2r} ds^2$ on a half-cylinder. Its curvature is $-\frac{(e^{-r})''}{e^{-r}} = -1$. Closing up one end at $r = 0$ by putting on a cap, smoothing if necessary, we get a complete metric with bounded curvature. But the volume of unit ball goes to zero when r increases to infinity. By crossing with $\mathbb{R}^n$ we get examples for higher dimensions.

With Theorem 3.1.1, above theorem is an immediate corollary of the following one.

Theorem 3.1.9. *Let (M^n, g) be as in above theorem. Let $g(t)$ be a ricci flow solution on $M^n \times [0, T]$ with $g(0) = g$. If $\sec_t^c \geq -1$ and $g(t)$ is complete for any $t \in [0, T]$, then rm_t is uniformly bounded on $M \times [0, \min\{\epsilon, T\}]$ for the same ϵ in above theorem.*

Remark 3.1.10. Readers may wonder if the uniqueness is true on $[0, T]$. Unfortunately, our method is not strong enough to prove it. Actually, Cabezas-Rivas and Wilking had constructed an immortal complete ricci flow solution with positive curvature operator (hence with positive complex sectional curvature) which is bounded if and only if $t \in [0, 1)$ (See [37] Theorem 4 b). And by Theorem 1.1.15, it has a uniform lower bound for the volume of any unit ball in the initial metric. So it does satisfy the assumptions of Theorem 3.1.7. However, our method to prove the uniqueness is to show that the curvature is bounded, which is not true in general for the whole interval of $[0, T]$ as shown by the above example. Nonetheless, the question is still open.

Remark 3.1.11. Authors are in debt to Cabezas-Rivas and Wilking [37] for following their use of the complex sectional curvature and adapting many of their arguments to serve our own purpose.

3.1.3 Curvature Bounded from Above

We are also curious whether we can get uniqueness for Ricci flow solution, assuming curvature bounded from above,

Indeed, we have the following positive answer.

Proposition 3.1.12. *Let (M, g) be a complete noncompact Riemannian manifold with bounded sectional curvature. Let $g_1(t)$ and $g_2(t)$ $t \in [0, T]$ with $g_1(0) = g = g_2(0)$ be two solutions of Ricci flow of which both have sectional curvatures bounded from above. Then $g_1 = g_2$ for any $t \in [0, T]$.*

The above result is a corollary of the following theorem (Chen [40, Corollary 2.3]).

Theorem 3.1.13. *Let $g(t)$ $t \in [0, T]$ be a complete smooth solution to Ricci flow on M^n. If $\mathrm{sc}_{g_0} \geq -K$, then, for any $t \in [0, T]$, we have*

$$\mathrm{sc}(t) \geq -\frac{n}{2t + n/K}.$$

By the above theorem, both g_1 and g_2 have bounded sectional curvature. The conclusion is immediate in light of Theorem 3.1.1.

3.1.4 Uniqueness in Lower Dimension — Strong Uniqueness Theorem

Though the Ricci flow is only weakly parabolic and non-linear, it is actually "better" than the classic heat equation, at least in the lower dimensions. Actually in dimension two or three, there are uniqueness theorems whose assumptions are curvature bounded on only the initial metric and the completeness.

In dimension two, Giesen and Topping [48], Topping [49] proved the following theorem.

Theorem 3.1.14 (Topping)**.** *For any initial metric on a two dimensional manifold, which could be noncomplete and/or with unbounded curvature, the Ricci flow has one and only one instantaneously complete solution.*

In dimension three, Chen [40] proved the following theorem.

Theorem 3.1.15. *For any complete noncompact initial metric, if it has nonnegative and bounded sectional curvature, then on some time interval depending only on the initial curvature bound, there exists a unique complete Ricci flow solution.*

These two theorems raise a question on our metaphor of on the classic heat equation and the Ricci flow — is the completeness of the metric is the true "physical" condition we should impose to guarantee the uniqueness?

Question 3.1.16. *Can we prove any strong uniqueness theorem for Ricci flow on manifold with dimension ≥ 4? In particular, is the Euclidean solution the only solution to Ricci flow with Euclidean initial data on $\mathbb{R}^n$ when $n \geq 4$?*

Actually this question is important when we try to use Ricci flow as a tool to study the classification of manifold with curvature conditions. We do wish Einstein metrics would be fixed points of Ricci flow!

By the discussion of uniqueness with unilateral curvature bound above, if the answer to the question is *no*, there must be a solution e.g. with the Euclidean initial data and with unbounded sectional curvature from both sides! This fact might shed a light on either constructing a second solution or proving the uniqueness?

3.2 Proof of Theorem 3.1.9

The idea of the proof is as follows. We first show that the volume lower bound for balls with fixed radius is preserved along the solution of Ricci flow for $t \in [0, \epsilon]$, where $\epsilon > 0$ is exact the one in the statement of the theorem. Then with this volume lower bound we prove that scalar curvature is bounded by C/t with some constant $C > 0$. If not, by a clever choice of base points due to Perelman, rescaling the metrics both in space and time around the base points, we get a sequence of Ricci flow solution which converges to an ancient solution of Ricci flow where the volume ratio $\mathrm{Vol}(B_r)/r^n > v_0 > 0$ on $t = 0$. However, while the limit satisfies the premises of Lemma 2.4.3, it contradicts the conclusion of the lemma, which is $\lim_{r\to\infty} \mathrm{Vol}(B_r)/r^n = 0$. With the curvature lower bound assumption, we obtain that Riemannian curvature is bounded by C/t, which allows us to apply Theorem 3.2.7 to conclude that the solution has bounded curvature. Hence, applying Theorem 3.1.1, the uniqueness of solution is obtained.

3.2.1 Estimate Volume Lower Bound

We first establish uniform volume lower bound for unit balls along the Ricci flow.

Lemma 3.2.1. *Let (M, g) be a complete Riemannian n-manifold whose unit balls have volume lower bound $v_0 > 0$. Let $(M, g(t))$ be a solution of ricci flow on $t \in [0, T]$ with $g(0) = g$, and its sectional curvature is bounded from below by -1 for any $t \in [0, T]$, then there exists a constant $C_n > 0$ depending only on dimension n, s.t. The volume of any ball $B_t(p, \mathrm{e}) \subset (M, g(t))$ bounded from below by $v_0/2$ for $t \in [0, \epsilon]$, where $\epsilon = \min\{v_0/2C_n, 1/(n-1)\}$.*

Proof of Lemma 3.2.1. Because sectional curvature is bounded uniformly from below by -1, $\mathrm{rc}_t \geq -(n-1)g(t)$, we have the following estimate for distance function.

Lemma 3.2.2. *For any fixed points $p, q \in M$,*

$$\mathrm{dist}_t(p,q) \leq \mathrm{dist}_0(p,q)\, \mathrm{e}^{(n-1)t}\,. \tag{3.2}$$

Proof. Let $\gamma : [0,1] \to M$ be a length minimizing geodesic with respect to $g(0)$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Let $|\gamma|_t$ be the length of the curve γ with respect to $g(t)$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\gamma|_t &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{g_t(\dot\gamma(s),\dot\gamma(s))}\,\mathrm{d}\,s = \int_0^1 \frac{-\,\mathrm{rc}(\dot\gamma(s),\dot\gamma(s))}{\sqrt{g_t(\dot\gamma(s),\dot\gamma(s))}}\,\mathrm{d}\,s\\ &\leq \int_0^1 (n-1)\|\dot\gamma(s)\|_t\,\mathrm{d}\,s = (n-1)|\gamma|_t.\end{aligned}$$

By integrating above differential inequality, we get

$$\mathrm{dist}_t(p,q) \leq |\gamma|_t \leq |\gamma|_0\mathrm{e}^{(n-1)t} = \mathrm{dist}_0(p,q)\mathrm{e}^{(n-1)t}\,.$$

□

Therefore $B_0(p,1) \subset B_t(p,\mathrm{e})$ for any $0 \leq t \leq 1/(n-1)$ and any $p \in M$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}_t(B_0(p,1)) &= -\int_{B_0(p,1)} \mathrm{sc}_t\ \mathrm{dvol}_t\\ &= -\int_{B_t(p,\mathrm{e})} \mathrm{sc}_t\ \mathrm{dvol}_t + \int_{B_t(p,\mathrm{e})-B_0(p,1)} \mathrm{sc}_t\ \mathrm{dvol}_t\\ &\geq -C(n) + \int_{B_t(p,\mathrm{e})} -n(n-1)\,\mathrm{dvol}_t.\end{aligned}$$

In last step we use $\mathrm{sc}_t \geq -n(n-1)$ and Petrunin's estimate as below.

Theorem 3.2.3 (Petrunin[50])**.** *Let M be a complete Riemannian manifold whose sectional curvature is at least -1. Then*

$$\int_{B(p,1)} \mathrm{sc} \leq C(n)$$

for any $p \in M$, where $B(p,1)$ is the unit ball centered in p and $C(n)$ is a constant depending only on dimension n.

Then, applying Bishop-Gromov volume comparison,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}_t(B_0(p,1)) \geq -C(n) - n(n-1)\,\mathrm{Vol}_{\mathrm{H}}(\mathrm{e}) =: -C'(n),$$

where $\mathrm{Vol}_{\mathrm{H}}(r)$ is the volume of a ball with radius r in simply connected hyperbolic space of constant curvature -1. Integrating above,

$$\mathrm{Vol}_t(B_t(p,\mathrm{e})) \geq \mathrm{Vol}_t(B_0(p,1)) \geq \mathrm{Vol}_0(B_0(p,1)) - C'(n)t \geq v_0 - C(n)t.$$

Hence, let $\epsilon = \min\{v_0/2C'(n), 1/(n-1)\}$,

$$\mathrm{Vol}_t(B_t(p,\mathrm{e}) \geq v_0/2 \text{ for any } t \in [0,\epsilon].$$

□

3.2.2 Curvature Uniform Bound

We have the following lemma which is similar to one in Cabezas-Rivas and Wilking [37].

Lemma 3.2.4. *Let $(M^n, g(t))$ be a complete solution of ricci flow for $M \times [0,\epsilon]$ with* $\sec^c \geq -1$. *If there is a constant $v' > 0$ such that volume of balls* $\mathrm{Vol}_t(B_t(p,\mathrm{e})) > v'$, *for any $(t,p) \in [0,\epsilon] \times M$, then there exists a constant $C > 0$, so that*

$$\mathrm{sc}_t(p) \leq \frac{C}{t}$$

for any $(p,t) \in M \times (0,\epsilon]$.

Proof. Prove by contradiction. Suppose there is a sequence $(p_k, t_k) \subset M \times (0,\epsilon]$ such that,

$$\mathrm{sc}_k(p_k) > \frac{4^k}{t_k}.$$

where and hereafter, $B_k = B_{g(t_k)}$, $B_t = B_{g(t)}$, $\mathrm{sc}_t = \mathrm{sc}_{g(t)}$, $\mathrm{sc}_k = \mathrm{sc}_{g(t_k)}$, $\mathrm{dist}_t = \mathrm{dist}_{g(t)}$, and $\mathrm{dist}_k = \mathrm{dist}_{g(t_k)}$. We can pick points in space-time to blow up by following trick first introduced by Perelman in [26].

Claim 3.2.5. *For any sufficiently large integer k, there exists $\bar{p}_k \in M$ and $\bar{t}_k \in (0,t_k]$ satisfy the following equations*

$$\mathrm{sc}_t(x) \leq 2\,\mathrm{sc}_{\bar{t}_k}(\bar{p}_k) \quad \text{for all} \begin{cases} x \in B_{\bar{t}_k}(\bar{p}_k, \frac{k}{\sqrt{\mathrm{sc}_{\bar{t}_k}(\bar{p}_k)}}) \text{ and,} \\ t \in (\bar{t}_k - \frac{k}{\mathrm{sc}_{\bar{t}_k}(\bar{p}_k)}, \bar{t}_k] \end{cases} \tag{3.3}$$

$$\mathrm{sc}_{\bar{t}_k}(\bar{p}_k) \geq \frac{4^k}{t_k}. \tag{3.4}$$

Proof of Claim 3.2.5. We start searching for $(\bar{p}_k, \bar{t}_k)$ from (p_k, t_k) — if it satisfies equation (3.3), we are done. Otherwise we can find x_1 and τ_1 s.t.

$$\mathrm{dist}_k(x_1, p_k) \leq \frac{k}{\sqrt{\mathrm{sc}_k(p_k)}}, \ \tau_1 \in (t_k - \frac{k}{\mathrm{sc}_k(p_k)}, t_k], \text{ and } \mathrm{sc}_{\tau_1}(x_1) \geq 2\,\mathrm{sc}_k(p_k).$$

If (x_1, τ_1) is not what we are looking for, we can find x_2 and τ_2, s.t.

$$\mathrm{dist}_{\tau_1}(x_1, x_2) \leq \frac{k}{\sqrt{\mathrm{sc}_{\tau_1}(x_1)}}, \ \tau_2 \in (\tau_1 - \frac{k}{\mathrm{sc}_{\tau_1}(x_1)}, \tau_1], \text{ and } \mathrm{sc}_{\tau_2}(x_2) \geq 2\,\mathrm{sc}_{\tau_1}(x_1).$$

We claim that, along this way only for finite steps, we can find desired $(\bar{t}_k, \bar{p}_k)$. Otherwise, we get a sequence of (x_k, τ_k) and $\mathrm{sc}_{\tau_k}(x_k) \to \infty$ as $k \to \infty$. Here readers may worry that τ_i would go below 0 thus ill-defined. However, by our construction, not only this cannot happen, but also, the above sequence is well bounded in both space and time, as shown in the following discussion. For the convenience, let $\tau_0 := t_k$ and $x_0 := p_k$.

Firstly,

$$\begin{aligned} t_k \geq \tau_{i+1} &\geq t_k - \sum_{l=0}^{i} \frac{k}{\mathrm{sc}_{\tau_l}(x_l)} \\ &\geq t_k - t_k \sum_{l=0}^{i} \frac{k}{2^l\,\mathrm{sc}_k(p_k)} \geq t_k\left(1 - \frac{2k}{\mathrm{sc}_k(p_k)}\right) =: \epsilon_k > 0. \end{aligned}$$

for sufficiently large k, where ϵ_k is a constant independent of i.

Secondly, x_i is also bounded with respect to a background metric g_k. Based on assumption, ricci curvature is bounded from below by $-(n-1)$, and by definition $\tau_i < t_k$, then by equation (3.2), for any integer $l \geq 1$ $\mathrm{dist}_k(x_{l-1}, x_l) \leq \mathrm{e}^{(n-1)t_k}\,\mathrm{dist}_{\tau_{l-1}}(x_{l-1}, x_l)$. Thus

$$\begin{aligned} \mathrm{dist}_k(p_k, x_i) &\leq \mathrm{e}^{(n-1)t_k} \sum_{l=1}^{i} \mathrm{dist}_{\tau_{l-1}}(x_{l-1}, x_l) \\ &\leq \mathrm{e}^{(n-1)t_k} \sum_{l=1}^{i} \frac{k}{\sqrt{\mathrm{sc}_{\tau_{l-1}}(x_{l-1})}} \\ &\leq \mathrm{e}^{(n-1)t_k} \sum_{l=1}^{\infty} \frac{k}{(\sqrt{2})^{l-1}\sqrt{\mathrm{sc}_k(p_k)}} \\ &=: C_k \leq \infty, \end{aligned}$$

where C_k is a constant independent of i.

Therefore, (x_i, τ_i) subconverges to a limit, say (x_∞, τ_∞). Then, by continuity of scalar curvature, $\mathrm{sc}_{\tau_\infty}(x_\infty) = \infty$, which is absurd. $\square$

Now we consider the volume ratios. By Bishop-Gromov relative volume comparison theorem, for any $0 < r \le \mathrm{e}$,

$$\frac{\mathrm{Vol}_{\bar{t}_k}(\bar{p}_k, r)}{r^n} = \frac{\mathrm{Vol}_{\bar{t}_k}(\bar{p}_k, r)}{\mathrm{Vol}_{\mathrm{H}}(r)} \frac{\mathrm{Vol}_{\mathrm{H}}(r)}{r^n} \ge \frac{\mathrm{Vol}_{\bar{t}_k}(\bar{p}_k, \mathrm{e})}{\mathrm{Vol}_{\mathrm{H}}(\mathrm{e})} \frac{\mathrm{Vol}_{\mathrm{H}}(r)}{r^n} \ge \frac{v'}{\mathrm{Vol}_{\mathrm{H}}(\mathrm{e})} \frac{\mathrm{Vol}_{\mathrm{H}}(r)}{r^n}.$$

Using relative volume comparison theorem again for euclidean and hyperbolic space, for any $0 < r$,

$$c(n) \ge \frac{r^n}{\mathrm{Vol}_{\mathrm{H}}(r)}$$

where $c(n)$ is a constant depending only on dimension n. So

$$\frac{\mathrm{Vol}_{\bar{t}_k}(\bar{p}_k, r)}{r^n} \ge \frac{v'}{\mathrm{Vol}_{\mathrm{H}}(e)} \frac{1}{c(n)} =: v'' > 0$$

where $v'' = v''(v_0, n)$ is a constant depending only on v_0 and n.

Let $Q_k = \mathrm{sc}_{\bar{t}_k}(\bar{p}_k)$. We rescale metric $g_{\bar{t}_k}$ on $B_{\bar{t}_k}(\bar{p}_k, k/\sqrt{Q_k})$ for k large. In this case, $k/\sqrt{Q_k} < e$, so above lower bound for volume ratio is true for $B_{\bar{t}_k}(\bar{p}_k, r)$ with any $r < k/\sqrt{Q_k}$. We define a new metric by parabolic rescaling

$$\tilde{g}_k(x, s) := Q_k g(x, \bar{t}_k + Q_k^{-1} s).$$

Hereafter $\widetilde{\mathrm{sc}}_k$ is the scalar curvature of $\tilde{g}_k$, so are the other $\tilde{}$ed quantities. Then by definition, $\tilde{g}_k$ is a solution to ricci flow on $(-k, 0] \times B_{\tilde{g}_k(0)}(\bar{p}_k, k)$. Note by equation (3.3), scalar curvature of $\tilde{g}_k$ is bounded from above by 2. And the complex sectional curvature

$$\widetilde{\sec}^c \ge -\frac{1}{Q_k} > -1. \tag{3.5}$$

Consequently, for some constant $c(n) > 0$ depending only on dimension,

$$|\widetilde{\mathrm{rm}}| \le \widetilde{\mathrm{sc}} + c(n) \le c(n). \tag{3.6}$$

Because volume ratio is scaling-invariant,

$$\frac{\mathrm{Vol}_{\tilde{g}_k(0)}(B_{\tilde{g}_k(0)}(\bar{p}_k, r))}{r^n} \ge v'' > 0 \tag{3.7}$$

for any $0 < r \leq k$. Therefore, combining equation (4.12) and (3.7) with injectivity radius estimate of Cheeger-Gromov-Taylor[10] (see also Cabezas-Rivas and Wilking [37] Theorem C.3), we have

$$\mathrm{inj}_{\tilde{g}(0)}(\bar{p}_k) \geq c(v'', n) > 0.$$

Together with equation (3.6), applying Hamilton's compactness theorem (see [31] also [18]), the sequence of pointed ricci flow solution $(B_{\tilde{g}_k}(\bar{p}_k, k), \tilde{g}_k(s), \bar{p}_k)$ subconverge to a complete ancient solution $(M_\infty, g_\infty(s), p_\infty)$ for $s \in (-\infty, 0]$ in the pointed Cheeger-Gromov sense. From the convergence and $\widetilde{\mathrm{sc}}_k(\bar{p}_k) = 1$, $\mathrm{sc}_{g_\infty(0)}(p_\infty) = 1$, hence the limit metric is non-flat. For the same reason, its Riemannian curvature is bounded from equation (3.6) and its complex sectional curvature is nonnegative from equation (3.5). It is also noncompact for every $s \in (-\infty, 0]$. We first observe the diameter of $g_\infty(0)$ is infinity. And we pick a sequence of points $y_i \in M_\infty$ for $i = 1, 2, ...$, s.t. $\mathrm{dist}_{g_\infty(0)}(p_\infty, y_i) = i$. Because Ricci curvature is nonnegative, using Lemma 3.2.2 again, we show $\mathrm{dist}_{g_\infty(-s)}(p_\infty, y_i) \geq i \to \infty$. With above observations, we can apply Lemma 4.5 in Cabezas-Rivas and Wilking [37] to this ancient solution, and get, for any $s \in (-\infty, 0]$,

$$\lim_{r \to \infty} \frac{\mathrm{Vol}_{g_\infty(s)}(B_{g_\infty}(\cdot, r))}{r^n} = 0.$$

However, this contradicts to the fact that, for any $r > 0$,

$$\frac{\mathrm{Vol}_{g_\infty(0)}(B_{g_\infty(0)}(p_\infty, r))}{r^n} > 0$$

from equation (3.7). □

Now we are ready to show the curvature is bounded.

Proposition 3.2.6. *Let (M^n, g) be a complete noncompact n-dimensional Riemannian manifold, with complex sectional curvature $\sec^c \geq -1$ and $\mathrm{sc} \leq s_0$ for some $s_0 > 0$. Let $g(t)$ be a solution to the ricci flow on $M \times [0, \epsilon]$ with $g(0) = g$. If for any $t \in [0, \epsilon]$ the volume of every unit ball in $g(t)$ is bounded from below uniformly by some constant $v_0 > 0$, and $g(t)$ is complete Riemannian metrics with $\sec^c(g_i(t)) \geq -1$, then the curvature operator of $g(t)$ is bounded from above uniformly.*

Proof. The proof follows from the following theorem.

Theorem 3.2.7 (B.L.Chen[40] or M.Simon[51])**.** *There is a constant $C = C(n)$ with the following property. Suppose we have a smooth solution to the ricci flow on $M^n \times [0, T]$ such that $B_t(x_0, r_0)$, $0 \leq t \leq T$, is compactly contained in M and*

1. $|\operatorname{rm}| \leq r_0^{-2}$ *on* $B_0(x_0, r_0)$ *at* $t = 0$;

2.
$$|\operatorname{rm}|_t(x) \leq \frac{K}{t}$$
where $K \geq 1$, $\operatorname{dist}_t(x_0, x) < r_0$, *whenever* $0 \leq t \leq T$.

Then we have
$$|\operatorname{rm}|_t(x) \leq \mathrm{e}^{CK} (r_0 - \operatorname{dist}_t(x_0, x))^{-2}$$
whenever $0 \leq t \leq T$, $\operatorname{dist}_t(x_0, x) < r_0$.

Let x_0 be any point in M. By our assumption, the ricci flow is complete for any t hence any ball in M is compactly contained. Again by assumption, $|\operatorname{rm}_0|_0$ is bounded. Thus we may choose r_0 small enough so that condition 1 is satisfied. Finally, lemma 3.2.4 gives condition 2, i.e.

$$|\operatorname{rm}|_t(x) \leq \operatorname{sc}_t + c(n) \leq \frac{C}{t}.$$

Consequently, there exists a C depending on only dimension n, such that

$$|\operatorname{rm}_t(x)| \leq C$$

whenever $0 \leq t \leq \epsilon$ and $\operatorname{dist}_t(x_0, x) \leq \frac{C'}{r_0^2}$. For $x_0 \in M$ is arbitrary, we have rm is uniformly bounded on $M \times [0, \epsilon]$. □

Proof of theorem 4.4.1. The proof follows from Lemma 3.2.4 and Proposition 3.2.6. □

3.3 Corollaries and Open Questions

3.3.1 Preservation of Nonnegative Complex Sectional Curvature

If we assume further, that (M^n, g) has nonnegative complex sectional curvature, together with upper bound for scalar curvature, it has nonnegative sectional curvature by equation (1.13). Then g has a positive lower bound for injectivity radius by using Toponogov's injectivity radius estimate (Theorem 1.1.15). Because the sectional curvature is bounded from above, say by a positive constant C_0, by the volume comparison (Theorem 1.1.10), the volume of any ball with fixed radius i_0 has a uniform lower bound. Rescaling the metric if necessary, by Theorem 4.4.1, we get the following corollary.

Corollary 3.3.1. *Let (M^n, g) be a complete noncompact n-dimensional Riemannian manifold, with complex sectional curvature* $\sec^c \geq 0$ *and* $\mathrm{sc} \leq s_0$ *for some $s_0 > 0$. Let $g_i(t)$ $i = 1, 2$ be two ricci flow solutions on $M \times [0, T]$, with same initial data $g_i(0) = g$. If both of them are complete Riemannian metrics and* $\sec^c(g_i(t)) \geq -C$ *for every $t \in [0, T]$ and some constant C, then there exists a constant $\epsilon > 0$, s.t. $g_1(t) = g_2(t)$ on $M \times [0, \min\{\epsilon, T\}]$.*

Note that Cabezas-Rivas and Wilking [37, Theorem 1] proved the following theorem for the existence of Ricci flow solution with nonnegative complex sectional curvature.

Theorem 3.3.2. *Let (M, g) be a complete non-compact Riemannian manifold with nonnegative complex sectional curvature. Then there exists a constant T depending only on n and g such that the Ricci flow has a has a solution on the time interval $[0, T]$, with $g(0) = g$ and with $g(t)$ having nonnegative complex sectional curvature.*

Together with Corollary 3.3.1, we know the nonnegativity of complex sectional curvature is preserved along the Ricci flow as long as the complex sectional curvature is bounded from below. The precise statement is as follows.

Corollary 3.3.3. *Let $(M, g(t))$ be a complete non-compact Ricci flow solution whose initial metric g has nonnegative complex sectional curvature and bounded Riemannian curvature. If the complex sectional curvature of $g(t)$ is bounded from below, then there exists a constant T depending only on n and g such that, for any $t \in [0, T]$, $g(t)$ has nonnegative and bounded complex sectional curvature.*

3.3.2 Solutions on Three Manifolds

Next let us restrict ourselves in dimension three. Here, the sectional curvature, the complex sectional curvature, and the curvature operator bounded from below are all equivalent. Then we have the following corollary.

Corollary 3.3.4. *Let (M, g) be a smooth complete Riemannian three manifold with bounded sectional curvature. And the volume of unit balls in (M, g) is uniformly bounded by $v_0 > 0$. Let $g_i(t)$, $i = 1, 2$ $t \in [0, T]$ be two complete solutions of ricci flow with $g_i(0) = g$. If their sectional curvatures are both bounded from below for any $t \in [0, T]$, then there exists a constant C_n depending on dimension n, s.t. $g_1(t) = g_2(t)$ on $M \times [0, \min\{\epsilon, T\}]$, where $\epsilon = \min\{v_0/2C_n, 1/(n-1)\}$.*

Another about the Ricci flow solution of dimension three is that the nonnegativity of sectional curvature is preserved (Chen [40]). Then by Corollary 3.3.1, we obtain the following strong uniqueness which is first proved by Chen [40].

Theorem 3.3.5 (Chen [40])**.** *In dimension three, when the initial metric has nonnegative and bounded sectional curvature, the ricci flow solution on some time interval* $[0, \epsilon]$ *is unique.*

When we consider the uniqueness of solution with the warped product metric, we are able to prove with the assumption of Ricci curvature bounded from below. Details please see Theorem 4.4.1.

3.3.3 Open Questions

Compare to the great achievements have been established in compact manifolds, the realm of the Ricci flow on complete noncompact manifolds is largely a virgin land. Indeed, finding the weakest sufficient conditions for the uniqueness of the solution is still a fundamental question to yet to be answered. Here we point out some questions for further investigation along this direction.

There are several directions to extend the Theorem 3.1.7.

Question 3.3.6. *Can we replace the* ϵ *by* T *in the conclusion of Theorem 3.1.7?*

Question 3.3.7. *Can we replace the complex sectional curvature lower bound with the sectional curvature lower bound in the condition of the same theorem?*

In our proof of Theorem 3.1.7, we only use the complex sectional curvature in one place — a result of ancient solution with nonnegative complex sectional curvature. If only we can replace the nonnegative complex sectional curvature by nonnegative sectional curvature there, we are done.

Question 3.3.8. *Can we replace the complex sectional curvature lower bound with the Ricci curvature lower bound in dimension three again in the same theorem?*

When dimension reduces to three, in our proof to Theorem 3.1.7, sectional curvature lower bound is only needed in proving that the non-collapsing condition is preserved, and the Ricci lower bound is sufficient elsewhere. This is actually related to question 4.1.2. And we have got a result in the special case, see Theorem 4.4.1.

One might be curious whether we can drop the lower curvature bound on the initial metric in Proposition 3.1.12. Indeed, the above lower bound for the scalar curvature is also true when $K = \infty$, which becomes $-n/2t$. This speculation leads to the following question.

Question 3.3.9. *Let* $g_1(t)$ *and* $g_2(t)$ *be two smooth solutions to* (3.2) *on* $t \in [0, T]$ *with* $g_1(0) = g_2(0) = g$*. Assume further sectional curvatures* $-C/t < \sec_i(t) < C'$ *for* $i = 1, 2$ *and* $t \in (0, T]$*. Is* $g_1(t) = g_2(t)$ *on* $t \in (0, T]$*?*

Note that the above question is confirmed in dimension 2 by Chen and Yan [52] and Giesen and Topping [48], Topping [49].

Chapter 4

Integral Scalar Curvature Bound

In this chapter, we study the integral scalar curvature bound of on Riemannian manifolds with warped product metrics and Ricci curvature bounded from below. After we get the main theorem — Theorem 4.3.13, we apply it to the uniqueness of Ricci flow solution.

4.1 Introduction

4.1.1 Sectional Curvature Bounded from Below

Our project is inspired by the following theorem proved by Petrunin.

Theorem 4.1.1 (Petrunin [50]). *Let M be any complete n-dimensional Riemannian manifold whose sectional curvature is no less than -1. Then, there exists a constant $C(n)$ depending only on dimension, such that*

$$\int_{B_1(p)} \mathrm{sc} \le C(n)$$

for any $p \in M$, where sc *is the scalar curvature, $B_1(p)$ is the unit ball centered in p.*

When the sectional curvature is bounded from below, as an obvious algebraic consequence, the scalar curvature is bounded from below too. However, it is a surprise that the integral of the scalar curvature is bounded from the opposite direction!

Though Petrunin proved the theorem in a completely different way, we have an intuitive viewpoint towards it, which is stated as follows. On the one hand, by volume comparison theorem, the curvature lower bound assures the volume of any unit ball is bounded from above. On the other hand, the integral of scalar curvature can be large only if the scalar curvature itself is large. However, the scalar curvature controls the volume on infinitesimal level in a way that the larger is the scalar curvature the smaller is the volume of the region on which it lives. Precisely we have (c.f. e.g. Gray [53])

$$\mathrm{Vol}(B_p(r)) = c_n r^n \Big(1 - \frac{\mathrm{sc}(p)}{6n+12} r^2 + O(r^4)\Big).$$

By looking at the integral as the sum of products of the integrand and the volume of subdivided domains, the above theorem seems very reasonable.

4.1.2 Ricci Curvature Bounded from below

Indeed, when assuming instead the Ricci curvature bounded from below, the above intuitive discussion goes through too. So we make the following conjecture.

Conjecture 4.1.2. *Let M be any complete n-dimensional Riemannian manifold whose Ricci curvature is at least $-(n-1)$. Then, there exists a constant $C(n)$ depending only on dimension, such that*

$$\int_{B_1(p)} \mathrm{sc} \le C(n)$$

for any $p \in M$, where sc *is the scalar curvature, $B_1(p)$ is the unit ball centered in p.*

In fact, the above question is, when the assumption is strengthened to nonnegative Ricci curvature, in essential a special case (the case of $k = 1$) of one of Yau's open problems stated as below .

Question 4.1.3. (Yau [54, p.278 Problem 9]). *Given an n-dimensional complete manifold with nonnegative Ricci curvature. Let $B(r)$ be the geodesic ball around some point p and σ_k be the k^{th} elementary symmetric function of the Ricci tensor, then is it true that $r^{-n+2k} \int_{B_r} \sigma_k$ has an upper bound when r tends to infinity?*

As pointed out by Yau, the above is a generalization of the Cohn-Vossen inequality shown below, which itself is an extension of the Gauss-Bonnet formula on complete non-compact surface. The following inequality

$$\iint_{M^2} K \,\mathrm{d}\,A \le 2\pi\chi(M^2),$$

where K is the Gauss curvature and $\chi(M^2)$ is the Euler characteristic number, is true when the integration on the left is finite.

It is easy to see that Yau's open problem has an affirmative answer for $k = 1$ if our Conjecture is true. Actually, when $k = 1$,

$$r^{-n+2k} \int_{B_r} \sigma_k = r^{2-n} \int_{B_r(p)} \mathrm{sc}\ \mathrm{dvol} = \int_{B_1^{r^2}(p)} \mathrm{sc}_{r^2 g}\ \mathrm{dvol}_{r^2 g} \le C(n)$$

where the Conjecture is used in the last step.

Remark 4.1.4. However, the answer to Yau's question is "NO" for all $k \geq 2$ due to the counter-examples discovered in Yang [55]. Nonetheless, the special case when $k = 1$, which is exactly our question, is still open.

4.2 Examples

We will discuss several examples including those with direct product metrics and Einstein metrics in this section.

We first see the case of direct product metrics. Suppose that the conjecture is FALSE in some dimension n — in other words, we have a sequence of Riemannian n-manifolds (N_i, h_i) and a sequence of points $p_i \in N_i$ such that the Ricci curvature $\mathrm{rc}_{h_i} \geq -1$ and

$$\int_{B_1^i(p_i)} \mathrm{sc}_i\ \mathrm{dvol}_i > i, \tag{4.1}$$

Where the $B^i(p_i)$'s are unit balls in (N_i, h_i). Then we consider product metrics $g_i = \mathrm{d}\,r^2 + h_i$. It is easy to see that the integral of scalar curvature over the region $[-1, 1] \times B^i(p_i)$ is unbounded. So we get another counter example in product metric in dimension $n + 1$. This example shows that in Theorem 4.3.13, the assumption that Conjecture 4.1.2 is true for lower dimension manifold is necessary.

The following example shows that Ricci curvature bounded from below is necessary in the conjecture, and further, even when it is replaced with scalar curvature bounded from below, it is not sufficient.

Let $(N, h) = (\mathbb{R}^2 \times \mathbb{S}^2, h = h_1 \oplus h_2)$ where h_1 is the standard hyperbolic metric with sectional curvature -1 and h_2 is the metric of round sphere with curvature 2. Then the scalar curvature of h is 1. And the volume of the ball B_r^h with radius r is

$$\mathrm{Vol}_h(B_r^h) \geq c\mathrm{e}^r\,.$$

Let us consider the following sequence of rescaled metrics $h_i = \frac{1}{i^2}h$. Then

$$\int_{B_1^i} \mathrm{sc}_i \, \mathrm{dvol}_i = 2i^2 \operatorname{Vol}_{h_i}(B_1^i) = 2i^{-2} \operatorname{Vol}_h(B_i^h) \geq ci^{-2}\mathrm{e}^i \to +\infty, \tag{4.2}$$

as $i \to \infty$, which contradicts the conclusion in the conjecture. And note that the $\mathrm{rc}_i \to -\infty$ in the direction of the hyperbolic component, which violates the assumption of the conjecture. We also see that the scalar curvature is positive for all i.

Base on the metric h same as the above, we examine a new sequence of the product metrics $\mathrm{d}\,r^2 + \frac{1}{i^2}h$, $i \in \mathbb{N}$. Then, for large i,

$$\int_{B_1^i} \mathrm{sc}_i \, \mathrm{dvol}_i \geq \int_0^1 \int_{B_{i/2}^h} 2i^{-2} \, \mathrm{dvol}_h \, \mathrm{d}\,r \geq ci^{-2}\mathrm{e}^{i/2} \to +\infty,$$

as $i \to \infty$. And the Ricci curvature in the axial direction is zero, but in the radial direction goes to $-\infty$ as $i \to \infty$. Therefore to prove the conjecture even for direct product metrics, which is a special case of warped product metrics, rc bounded from below is needed in all directions. This is used to justify our assumption for the Theorem 4.3.13 (see Remark 4.3.14.)

Now we turn to the Einstein metrics.

Proposition 4.2.1. *Let (M, g) be any complete Einstein manifold. Then there exists a constant $C = C(n)$ depending on the dimension n such that, for any unit ball $B_1 \subset M$,*

$$\int_{B_1} \mathrm{sc} \, \mathrm{dvol} \leq C.$$

Proof. There is nothing to prove if the Einstein constant is non-positive. For the positive case, the point is to show the integral upper bound depending only on n. Without loss of generality, assume the Ricci curvature $\mathrm{rc} = (n-1)c^2 0$ for some constant c. Thanks to the Bonnet-Myers theorem, the diameter of (M, g) is bounded from above by π/c. Consequently, with the help of the Bishop-Gromov volume comparison,

$$\int_{B_1} \mathrm{sc} \, \mathrm{dvol} \leq \int_M \mathrm{sc} \, \mathrm{dvol} = \int_{B_{\pi/c}} \mathrm{sc} \, \mathrm{dvol} \leq \mathrm{sc} \operatorname{Vol}(S^n_{\pi/c}) \leq n(n-1)c^2\omega(n)\Big(\frac{\pi}{c}\Big)^n,$$

where $\operatorname{Vol}(S^n_{\pi/c})$ is the volume of the standard n-sphere with radius π/c, $\omega(n)$

is the volume of the standard n-sphere with radius 1. So, for all $c \geq 1$,

$$\int_{B_1} \mathrm{sc}\,\mathrm{dvol} \leq C(n).$$

When $0 < c < 1$, compare with the volume of unit Euclidean ball using Bishop-Gromov theorem,

$$\int_{B_1} \mathrm{sc}\,\mathrm{dvol} = n(n-1)c^2 \int_{B_1} \mathrm{dvol} \leq n(n-1)\mathrm{V}(1) = C(n).$$

where V(1) is the volume of unit Euclidean ball. □

We conclude this section by an example that the premise of Petrunin's theorem fails however our conjecture's applies.

Let $N = \mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$, which admits an Einstein metric, say h, with positive Ricci curvature (Tian [56]). And the Euler characteristic number of N is 11. By the following lemma, the sectional curvature of h is negative somewhere.

Lemma 4.2.2 (Gursky and Lebrun [57]). *Let (N, h) be a compact 4-dimensional Einstein manifold of nonnegative sectional curvature. Then $\chi(M) \leq 9$.*

Consider the warped metric $g = \mathrm{d}\,r^2 + \mathrm{e}^{-2r}\,h$ on $\mathbb{R}\times N$.By Equations (1.32) , (1.33), and (1.34),

$$\mathfrak{R}(E_a) = (\lambda_a \mathrm{e}^{2r} - 1)E_a, \quad \mathrm{rc}(e_i, e_i) = \mathrm{rc}_h(e_i, e_i) - n, \quad \text{and} \quad \mathrm{rc}(e_0, e_0) = -n.$$

So the curvature operator approach to $+\infty$ or $-\infty$ as $r \to +\infty$ when λ_a is positive or negative, while the Ricci curvature is bounded from below. Therefore this metric has the properties we claimed above. By equation (1.35), we also notice that the scalar curvature

$$\mathrm{sc} = \mathrm{e}^{2r}\,\mathrm{sc}_h - n(n+1) \to +\infty \quad \text{as} \quad r \to \infty.$$

To summarize, this example has unbounded sectional curvature from below, hence the Petrunin's theorem does not apply. In the meantime, its Ricci curvature is bounded from below, which makes it stay in the scope of our conjecture. And, though its scalar curvature is unbounded from above, according to Corollary 4.3.17, the integral of the scalar curvature on any unit ball is bounded from above! This is indeed a supportive evidence to our conjecture.

4.3 The Case of Warped Product Metrics

4.3.1 Ricci Curvature Lower Bound in Axial Direction

We first consider the case that $I = \mathbb{R}$ and N is compact. As we have done in the section 1.2, we introduce a function $\phi(r) = \log f(r)$, so $g_r = e^{2\phi(r)} h$.

We first prove a technical lemma. We will see later that this is where the completeness assumption on the metric is applied.

Lemma 4.3.1. *Let $a : [0, +\infty) \to \mathbb{R}$ be a smooth function s.t. $a' + a^2 \leq C^2$ for some constant $C \geq 0$. Then, when $C > 0$, $-C \leq a \leq \max\{a(0), C\}$, when $C = 0$, $0 \leq a \leq a(0)$.*

Proof. The case $C > 0$. Consider the correspondent ODE

$$A' + A^2 = C^2.$$

When $A(0) = \pm C$, $A \equiv \pm C$. When $A(0) \neq \pm C$, its solution is

$$A = C + \frac{2C}{Ke^{2Cr} - 1} \quad \text{where } K \text{ is a constant satisfying } A(0) = C(1 + \frac{2}{K-1}).$$

Then let us has a closer look of A with different initial values other than $\pm C$.

1. When $A(0) > C$. Then $K > 1$. Hence $A(r)$ is decreasing and $A \to C$ as $r \to \infty$. So $A(0) \geq A \geq C$ as long as $r \geq 0$.

2. When $-C < A(0) < C$. Then $K < 0$. So $A(r)$ is increasing and $A \to C$ as $r \to \infty$. So $C \geq A \geq -C$ as long as $r \geq 0$.

3. When $A(0) < -C$. Then $0 < K < 1$. So $A(r)$ is decreasing and $A \to -\infty$ as $r \to r_0$ for some $0 < r_0 < \infty$.

Let $A(0) = a(0)$, by comparison principle of ordinary differential equations, $a(r) \leq A(r)$ as long as $A(r)$ exists. So, when $a(0) < -C$, $a(r)$ becomes discontinuous at some finite r, which contradicts to our assumption on a. Consequently, $a(0) \geq -C$, thus $a(r) \leq A(r) \leq \max\{a(0), C\}$. Not only so, we can further conclude that $a(r) \geq -C$. Otherwise, there exists some $\tilde{r} > 0$ s.t. $a(\tilde{r}) < -C$. Due to the fact that the differential inequality satisfied by a is translation invariant, it becomes discontinuous for the same reason mention as above.

Conclusion for $C = 0$ follows a similar argument. $\square$

Next, we state a property of the warped product metric with curvature bounded from below.

Lemma 4.3.2. *Let* (M, g) *be a complete smooth Riemannian* $n+1$*-manifold, where* $M = \mathbb{R} \times N$, $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)}\, h$ *is warped product metric, and* (N, h) *is a complete smooth Riemannian n-manifold. If* $\mathrm{rc}(\nabla_r, \nabla_r) \geq -n\rho^2$ *for some constant* ρ*, then, for any* $r \in \mathbb{R}$, $|\phi'| \leq \rho$.

Proof. From the assumption $\mathrm{rc}(\nabla r, \nabla r) \geq -n\rho^2$ and equation (1.33), we have

$$-\phi'' - (\phi')^2 \geq -\rho^2. \tag{4.3}$$

Using Lemma 4.3.1, given any r_1, we have

$$-\rho \leq \phi'(r) \leq \max\{\phi'(r_1), \rho\}$$

for any $r \geq r_1$. Let $\psi(r) := \phi(-r)$ and $s = -r$. The metric becomes $g = \mathrm{d}\, s^2 + \mathrm{e}^{2\psi(s)}\, h$. This is the same metric in different coordinate system, thus the curvature condition again gives

$$-\psi'' - (\psi')^2 \geq -\rho^2.$$

Apply Lemma 4.3.1 again, for any given s_0, we have

$$-\rho \leq \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \leq \max\{\frac{\mathrm{d}\psi(s_0)}{\mathrm{d}s}, \rho\} \tag{4.4}$$

for any $s \geq s_0$. In the meantime, by definition, we have

$$\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{\mathrm{d}\phi(-s)}{\mathrm{d}s} = -\phi'(-s) = -\phi'(r),$$

Hence, when taking $s_0 = -r_2$ for some arbitrary $r_2 > r_1$, we are able to rewrite equation (4.4) as

$$\min\{\phi'(r_2), \rho\} \leq \phi'(r) \leq \rho, \tag{4.5}$$

for any $r \leq r_2$. Therefore, $|\phi'(r)| \leq \rho$ for any $r \in (r_1, r_2)$. Note that r_1 and r_2 are arbitrary, the desired conclusion is obtained. □

Now we are ready to state the main result of the current subsection.

Proposition 4.3.3. *Let* (N, h) *be a compact smooth Riemannian n-manifold with bounded Ricci curvature* $-(n-1) \leq \mathrm{rc}^h \leq (n-1)$ *and diameter D. Let* $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)}\, h$ *be a complete smooth metric on* $M = \mathbb{R} \times N$*. If the ricci curvature of* g $\mathrm{rc}(\nabla r, \nabla r) \geq -n\rho^2$ *for some constant* $\rho \geq 0$*, then the integral of the scalar curvature of* g *over any unit ball is bounded from above by a constant depending on only* ρ *and D.*

Remark 4.3.4. For an arbitrary compact Riemannian manifold (N, h) the Ricci curvature is bounded by compactness. Hence we can always by rescaling to make the curvature satisfies the condition in Proposition 4.3.3 and hide the scaling factor as a additive constant in $\phi(r)$. However, we have to pay the price of increasing the diameter D.

Proof. Let $(r_0, p) \in M$ be any point and B_1 be the unit ball in M centered at (r_0, p). Using equation (1.35), we have

$$\int_{B_1} \mathrm{sc}\ \mathrm{dvol} = \int_{B_1} \mathrm{e}^{-2\phi}\ \mathrm{sc}^h\ \mathrm{dvol} + \int_{B_1} -2n\phi'' - n(n+1)(\phi')^2\,\mathrm{dvol}, \tag{4.6}$$

of which we will estimate the last two integrals respectively.

The first integral

$$\int_{B_1} \mathrm{e}^{-2\phi}\ \mathrm{sc}^h\ \mathrm{dvol} \le \int_{B_1} \mathrm{e}^{-2\phi}\,(\mathrm{sc}^h + n(n-1))\,\mathrm{dvol}.$$

We first notice the fact, which follows from the assumption on rc^h, that the scalar curvature of h is $-n(n-1) \le \mathrm{sc}^h \le n(n-1)$, So the integrand of the right hand side in above equation is nonnegative. Let $A = \mathrm{e}^{\rho}$, $I = [r_0 - 1, r_0 + 1]$, and $B_A^{r_0}$ be the ball with radius A centered at $p \in N$ with respect to g_{r_0}. Then applying Lemma 1.2.9, we have

$$\begin{aligned}
\int_{B_1} \mathrm{e}^{-2\phi}\ \mathrm{sc}^h\ \mathrm{dvol} &\le \int_{I\times B_A^{r_0}} \mathrm{e}^{-2\phi}\,(\mathrm{sc}^h + n(n-1))\,\mathrm{dvol} \\
&\le 2n(n-1)\int_{I\times B_A^{r_0}} \mathrm{e}^{-2\phi(r)}\ \mathrm{dvol} \\
&= 2n(n-1)\int_{r_0-1}^{r_0+1}\int_{B_A^{r_0}} \mathrm{e}^{(n-2)\phi(r)}\ \mathrm{dvol}^h\ \mathrm{d}\,r \\
&\le 2n(n-1)\int_{r_0-1}^{r_0+1} \mathrm{e}^{(n-2)(\phi(r_0)+\rho)}\int_{B_A^{r_0}} \mathrm{dvol}^h\ \mathrm{d}\,r.
\end{aligned} \tag{4.7}$$

In the last step, we use the fact that

$$\phi(r) \le \phi(r_0) + \rho, \tag{4.8}$$

which follows from the fact $|\phi'| \le \rho$ (Lemma 4.3.2). When $\mathrm{e}^{\phi(r_0)} D \le A$, in particular $N \in B_A^{r_0}$, so

$$\begin{aligned}\int_{r_0-1}^{r_0+1} \mathrm{e}^{(n-2)(\phi(r_0)+\rho)} \int_{B_A^{r_0}} \mathrm{dvol}^h \,\mathrm{d}\, r &= \int_{r_0-1}^{r_0+1} \mathrm{e}^{(n-2)(\phi(r_0)+\rho)} \int_N \mathrm{dvol}^h \,\mathrm{d}\, r \\ &\leq 2\Big(\frac{A}{D}\Big)^{(n-2)} \mathrm{e}^{(n-2)\rho} \,\mathrm{Vol}^h(N) \leq 2\Big(\frac{A}{D}\Big)^{(n-2)} \mathrm{e}^{(n-2)\rho}\, \mathrm{V}_{-1}(D)\end{aligned} \tag{4.9}$$

where $\mathrm{V}_{-1}(D)$ is the volume of unit ball in hyperbolic space with constant sectional curvature -1. And in the last inequality we apply the volume comparison theorem since $\mathrm{rc}^h \geq -(n-1)$.

Now let us consider the case $\mathrm{e}^{\phi(r_0)} D \geq A$. Note that the eigenvalue of Ricci curvature of $g_{r_0} = \mathrm{e}^{2\phi(r_0)} h$ and the eigenvalue of Ricci curvature of h are differed by a multiplicative constant $\mathrm{e}^{2\phi(r_0)}$. Thus, by assumption that $\mathrm{rc}_h \geq -(n-1)$, $\mathrm{rc}_{r_0} \geq -(n-1)\mathrm{e}^{-2\phi(r_0)} \geq -(n-1)\big(\frac{D}{A}\big)^2$. By volume comparison theorem, it follows from equation (4.7) that

$$\begin{aligned}&\int_{r_0-1}^{r_0+1} \mathrm{e}^{(n-2)(\phi(r_0)+\rho)} \int_{B_A^{r_0}} dv^h \,\mathrm{d}\, r \\ =&\int_{r_0-1}^{r_0+1} \mathrm{e}^{-2(\phi(r_0)+(n-2)\rho)} \,\mathrm{d}\, r \int_{B_A^{r_0}} \mathrm{e}^{n\phi(r_0)} \,\mathrm{dvol}^h \\ \leq& 2\Big(\frac{D}{A}\Big)^2 \mathrm{e}^{(n-2)\rho} \int_{B_A^{r_0}} \mathrm{dvol}^{r_0} \\ \leq& 2\Big(\frac{D}{A}\Big)^2 \mathrm{e}^{(n-2)\rho} \,\mathrm{Vol}_H(A).\end{aligned} \tag{4.10}$$

In the last step, we apply the volume comparison theorem and $\mathrm{Vol}_H(A)$ is the volume of the ball with radius A in the hyperbolic space with constant curvature $-(D/A)^2$. Therefore, by equation (4.7), (4.9), and (4.10),

$$\int_{B_1} \mathrm{e}^{-2\phi} \,\mathrm{sc}^h \,\mathrm{dvol} \leq C_0, \tag{4.11}$$

where

$$C_0 := 4n(n-1) \cdot \max\{\Big(\frac{A}{D}\Big)^{(n-2)} \mathrm{e}^{(n-2)\rho}\, \mathrm{V}_{-1}(D), \Big(\frac{D}{A}\Big)^2 \mathrm{e}^{(n-2)\rho} \,\mathrm{Vol}_H(A)\}.$$

We turn to the second integral in the right hand side of equation (4.6). It is obvious that

$$\int_{B_1} -2n\phi'' - n(n+1)(\phi')^2 \,\mathrm{dvol} \le \int_{B_1} -2n\phi'' \,\mathrm{dvol} \le \int_{B_1} -2n\phi'' + 4n\rho^2 \,\mathrm{dvol}. \tag{4.12}$$

Note that, by equation (4.3) and the fact $|\phi'| \le \rho$,

$$-\phi'' \ge -\rho^2 - (\phi')^2 \ge -2\rho^2.$$

Thus, the integrand in the last integral in equation (4.12) is positive. Then, apply Lemma 1.2.9 again, we have

$$\begin{aligned} \int_{B_1} -2n\phi'' + 4n\rho^2 \,\mathrm{dvol} &\le \int_{I\times B_A^{r_0}} -2n\phi'' + 4n\rho^2 \,\mathrm{dvol} \\ &= \int_{r_0-1}^{r_0+1} \int_{B_A^{r_0}} (-2n\phi'' + 4n\rho^2)\mathrm{e}^{n\phi(r)} \,\mathrm{dvol}^h \,\mathrm{d}\,r \end{aligned} \tag{4.13}$$

Let r be any number in $[r_0-1, r_0+1]$, by equation (4.8)

$$\int_{B_A^{r_0}} \mathrm{e}^{n\phi(r)} \,\mathrm{dvol}^h \le \int_{B_A^{r_0}} \mathrm{e}^{n(\phi(r_0)+\rho)} \,\mathrm{dvol}^h \tag{4.14}$$

When $\mathrm{e}^{\phi(r_0)} D \le A$, the $N \in B_A^{r_0}$, So

$$\int_{B_A^{r_0}} \mathrm{e}^{n(\phi(r_0)+\rho)} \,\mathrm{dvol}^h \le \int_N \Big(\frac{A}{D}\Big)^n \mathrm{e}^{n\rho} \,\mathrm{dvol}^h = \Big(\frac{A}{D}\Big)^n \mathrm{e}^{n\rho} \,\mathrm{Vol}^h(N) \tag{4.15}$$

When $\mathrm{e}^{\phi(r_0)} D \ge A$, again we have $\mathrm{rc}_{r_0} \le -(n-1)(D/A)^2$. By volume comparison theorem, we have

$$\int_{B_A^{r_0}} \mathrm{e}^{n(\phi(r_0)+\rho)} \,\mathrm{dvol}^h = \mathrm{e}^{n\rho} \int_{B_A^{r_0}} \mathrm{dvol}^{r_0} \le \mathrm{e}^{n\rho} \,\mathrm{Vol}_H(A). \tag{4.16}$$

Combining equation (4.14),(4.15), and (4.16), we have

$$\int_{B_A^{r_0}} \mathrm{e}^{n\phi(r)} \,\mathrm{dvol}^h \le \max\{\Big(\frac{A}{D}\Big)^n \mathrm{e}^{n\rho} \,\mathrm{Vol}^h(N), \mathrm{e}^{n\rho} \,\mathrm{Vol}_H(A)\} =: C_1 \tag{4.17}$$

Then, together with equation (4.13) , we obtain

$$\int_{B_1} - 2n\phi'' + 4n\rho^2 \,\mathrm{dvol} \le C_1 \int_{r_0-1}^{r_0+1} -2n\phi'' + 4n\rho^2 \,\mathrm{d}\,r$$
$$\le 2nC_1(|\phi'(r_0-1)| + |\phi'(r_0+1))| + 4\rho^2) \le 4n\rho C_1(1+2\rho)$$

Together with (4.12), we get

$$\int_{B_1} -2n\phi'' - n(n+1)(\phi')^2 \,\mathrm{dvol} \le 4n\rho C_1(1+2\rho) \tag{4.18}$$

Therefore, combining equation (4.16) and (4.11), we obtain

$$\int_{B_1} \mathrm{sc}\ \mathrm{dvol} \le C := 2n(n-1)C_0 + 4n\rho C_1(1+2\rho)$$

By checking the prove carefully, the constant C depends on only ρ and D. □

Remark 4.3.5. Though the special case $\rho = 0$ is covered by our proof, it is easier to proof directly. It follows from $|\phi'| \le \rho$ that $\phi' \equiv 0$. Thus the metric is $g = \mathrm{d}\,r^2 + e^{2c}h$ a direct product metric. Then the conclusion of our proposition follows immediately.

4.3.2 Ricci Curvature Lower Bound in All Directions

In this section, we show, in essential, that no counter-example to the Conjecture 4.1.2 can be found among the complete warped product metrics. For precise statement see Theorem 4.3.13.

We first show that the Ricci curvature bounded from below for a warped product metric implies that the Ricci curvature of the warped component is also bounded from below via two different ways. However, we will not use the first but the second one.

Proposition 4.3.6. *Let $g = \mathrm{d}\,r^2 + \mathrm{e}^{2\phi(r)}h$ be a complete smooth warped product metric on $\mathbb{R} \times N$ with Ricci curvature bounded from below by $-n\rho^2$, where ρ is nonnegative constant. Then the Ricci curvature of h is also bounded from below.*

Proof. By equation (1.34),

$$\mathrm{rc}(e_i, e_i) = \mathrm{rc}^h(e_i, e_i) - n(\phi')^2 - \phi'' \ge -n\rho^2.$$

Hence, for any $r \in \mathbb{R}$,

$$\mathrm{rc}^h(\mathrm{e}^{\phi(r)}\, e_i, \mathrm{e}^{\phi(r)}\, e_i) \geq \mathrm{e}^{-2\phi(r)}\, (n(\phi'(r))^2 + \phi''(r) - n\rho^2).$$

Remember that $\{e_i\}$ are orthonormal vectors with respect to g, hence $\{\mathrm{e}^{\phi}\, e_i\}$ are orthonormal vectors with respect to h. Without loss of generality, we may pick $\{e_i\}$ in a way that rc^h is diagonalized by them. Therefore,

$$\mathrm{rc}^h \geq \sup_{r \in \mathbb{R}} \{\mathrm{e}^{-2\phi(r)}\, (n(\phi'(r))^2 + \phi''(r) - n\rho^2)\}. \tag{4.19}$$

□

Proposition 4.3.7. *Let $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)}\, h$ be a complete smooth warped product metric on $\mathbb{R} \times N$ with Ricci curvature bounded from below by $-n\rho^2$, where ρ is a nonnegative constant. If there exist $p \in N$, a unit vector (w.r.t. h) $v \in T_pN$ and $\lambda > 0$ such that $\mathrm{rc}^h(v, v)|_p = -(n-1)\lambda$, then, for any $r \in \mathbb{R}$*

$$\lambda \mathrm{e}^{-2\phi(r)} \leq \frac{\rho(1 + n\rho)\mathrm{e}^{2\rho}}{n-1}. \tag{4.20}$$

Remark 4.3.8. The above proposition says that a complete warped product metric can only collapse when it has nonnegative curvature.

Proof. Without loss of generality, we may assume that, among $\{e_i\}$, $e_1 = \mathrm{e}^{-\phi(r)}\, v$. Therefore,

$$\mathrm{rc}(e_1, e_1) = -(n-1)\lambda \mathrm{e}^{-2\phi} - n(\phi')^2 - \phi'' \geq -n\rho^2 \tag{4.21}$$

Drop the square term, we get

$$-(n-1)\lambda \mathrm{e}^{-2\phi(r)} \geq \phi''(r) - n\rho^2.$$

Fix any $s \in \mathbb{R}$, then for any $r \in [s-1, s+1]$, it follows from $|\phi'| \leq \rho$ (Lemma 4.3.2) that $\phi(r) \leq \phi(s) + \rho$. Thus, by integrating above inequality, we have

$$\begin{aligned} -2(n-1)\lambda \mathrm{e}^{-2(\phi(s)+\rho)} &\geq \int_{s-1}^{s+1} \phi''(r) - n\rho^2 \,\mathrm{d}\, r \\ &= -\phi'(s+1) + \phi'(s-1) - 2n\rho^2 \geq -2\rho - 2n\rho^2. \end{aligned}$$

Therefore,

$$\mathrm{e}^{-2\phi(s)} \leq \frac{\rho(1 + n\rho)\mathrm{e}^{2\rho}}{(n-1)\lambda}. \tag{4.22}$$

□

The corollaries below follow immediately from Proposition 4.3.7.

Corollary 4.3.9. *Let $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)} h$ be a complete smooth warped product metric on $\mathbb{R} \times N$ with Ricci curvature bounded from below by $-n\rho^2$, where ρ is nonnegative constant. Then the Ricci curvature of Riemannian metric $\mathrm{e}^{2\phi(r)} h$ is bounded from below by a constant $-C = -C(n, \rho)$ depending only on dimension and ρ.*

Proof. When $\mathrm{rc}_h \geq 0$ the proof is trivial. Otherwise, we can apply Proposition 4.3.7. □

Corollary 4.3.10. *Let $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)} h$ be a complete smooth warped product metric on $\mathbb{R} \times N$ with Ricci curvature bounded from below by $-n\rho^2$, where ρ is nonnegative constant. Then the Ricci curvature of h is bounded from below by*

$$- C(n, \rho) \inf_{r \in \mathbb{R}} \mathrm{e}^{2\phi(r)} \tag{4.23}$$

for a constant $C(n, \rho) \geq 0$ depending only on dimension n and ρ.

Proof. We only need to show Equation (4.23) is true when rc_h is not nonnegative, which follows immediately from Proposition 4.3.7. □

We have the following simple corollaries if Conjecture 4.1.2 is true.

Proposition 4.3.11. *Assume that for some fixed dimension n, the Conjecture 4.1.2 is true. Let (N, h) be a complete Riemannian n-manifold whose Ricci curvature is at least $-(n-1)$. Let $B_\epsilon \subset N$, $\epsilon < 1$, be any ball with radius ϵ, then*

$$\int_{B_\epsilon} \mathrm{sc}\ \mathrm{dvol} \leq C(n)\epsilon^{n-2}. \tag{4.24}$$

Proof. Prove by the argument of rescaling. Let $h' = \epsilon^{-2} h$ and all primed quantities below are all with respect to h'. So the ball B_ϵ is exactly the B'_1, and the eigenvalue of the Ricci curvature rc' $\lambda' = \epsilon^2 \lambda$, where the latter is the correspondent one of rc, thus $\mathrm{rc}' \geq -\epsilon^2(n-1) \geq -(n-1)$. So, applying conjecture on (N, h'),

$$\int_{B_\epsilon} \mathrm{sc}\ \mathrm{dvol} = \int_{B'_1} \epsilon^{-2}\, \mathrm{sc}\, \epsilon^n\ \mathrm{dvol}' \leq C(n)\epsilon^{n-2}.$$

□

Proposition 4.3.12. *Assume that for some fixed n, the Conjecture 4.1.2 is true. Let (N,h) be a complete smooth Riemannian n-manifold whose Ricci curvature is at least $-(n-1)r^2$. Let $B_{1/r}\subset N$ be any ball with radius $1/r$, then*

$$\int_{B_{Kr^{-1}}} \mathrm{sc}\ \mathrm{dvol} \leq C(n,K)r^{2-n}. \tag{4.25}$$

Proof. Similar to the proof of the Proposition 4.3.11, we consider the rescaled metric $h' = r^2 h$. And all primed quantities are of h'. Thus,

$$\int_{B_{Kr^{-1}}} \mathrm{sc}\ \mathrm{dvol} = \int_{B'_K} r^{2-n}\, \mathrm{sc}'\ \mathrm{dvol}' \tag{4.26}$$

And $\mathrm{rc}' \geq -(n-1)$, hence $\mathrm{sc}' \geq -n(n-1)$.

When $K \leq 1$, apply Proposition 4.3.11, theorem to Equation (4.26). We get

$$\int_{B_{Kr^{-1}}} \mathrm{sc}\ \mathrm{dvol} \leq r^{2-n}K^{n-2}C(n).$$

When $K > 1$, from Equation (4.26), we have

$$\int_{B_{Kr^{-1}}} \mathrm{sc}\ \mathrm{dvol} \leq r^{2-n}\int_{B'_K} \mathrm{sc}' + n(n-1)\,\mathrm{dvol}'. \tag{4.27}$$

To estimate the right hand side, we need the following fact. We can always choose a finite number of points $p'_i \in B'_K$, such that the balls $B'_{1/2}(p_i)$ and $B'_1(p_i)$, whose centers are p_i's and radii measured by h' are $1/2$ and 1 respectively, satisfy the following property.

$$\cup_i B'_1(p_i) \supset B'_K \quad \text{and} \quad B'_{1/2}(p_i)\cap B'_{1/2}(p_j) = \emptyset \text{ for any } i \neq j.$$

The reason is as follows. Firstly, consider any m disjoint balls with radius $1/2$ in $B'_{K+1/2}$. Applying the Bishop-Gromov relative volume comparison theorem, we get a upper bound of m as below. Let $B'_{1/2}(p_i)$ be the ball with the smallest volume, we have

$$m \leq \frac{\mathrm{Vol}(B'_{K+1/2}))}{\mathrm{Vol}(B'_{1/2}(p_i))} \leq \frac{v(n,-1,K+1/2)}{v(n,-1,1/2)} =: C(n,K)$$

where $v(n,k,r)$ is the volume of unit ball with radius r in the n-dimensional space form of constant sectional curvature k. In particular such disjoint balls with radius $1/2$ are of finite number. Secondly, we enlarge the radii of above

m balls to get m unit balls. If these unit balls do not cover B'_K, we can add at least one more ball $B'_{1/2}(p) \subset B_{K+1/2}$. Since the number of such balls is finite, we eventually get a covering consists of a number of unit balls, and the number of them is bounded by $C(n, k)$.

Note the integrand of the right hand side of Equation (4.27) is nonnegative. So, by Conjecture 4.1.2 and volume comparison theorem,

$$\int_{B_{Kr^{-1}}} \mathrm{sc}\ \mathrm{dvol} \leq r^{2-n} \sum_i \int_{B'_1(p_i)} \mathrm{sc}' + n(n-1)\,\mathrm{dvol}' \leq C(n, K) r^{2-n}.$$

Combining both cases, the desired bound is obtained. □

Now we can state and prove the main theorem of this chapter.

Theorem 4.3.13. *Assume that for some fixed n, the Conjecture 4.1.2 is true where the upper bound of the integral of scalar curvature is $\mathcal{S}$. Let (N, h) be any complete n-dimensional smooth Riemannian manifold. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function then $g = \mathrm{d}\,r^2 + e^{2\phi(r)}\,h$ be a complete warped metric on $M = \mathbb{R} \times N$. If the Ricci curvature of g is at least $-n\rho^2$, then, on (M, g), the integral of scalar curvature over any unit ball is bounded from above by a constant depending only on the dimension, S, and ρ.*

Remark 4.3.14. Though we see from Proposition 4.3.3 the lower bound of the Ricci curvature in axial direction can give us the upper integral scalar curvature bound, in light of the example in Section 4.2, the bound itself must depend on the diameter and curvature of (N, h). So to prove the conjecture even for the warped product metrics, we have to assume the Ricci curvature lower bound in all directions.

Proof. Throughout this proof, $C(n, \rho)$ is a generic constant depending on only the dimension n and ρ.

Let $(r_0, p) \in M$ be any point and B_1 be the unit ball in M centered at (r_0, p). Using equation (1.35), we have

$$\int_{B_1} \mathrm{sc}\ \mathrm{dvol} = \int_{B_1} e^{-2\phi}\ \mathrm{sc}^h\ \mathrm{dvol} + \int_{B_1} -2n\phi'' - n(n+1)(\phi')^2\,\mathrm{dvol}, \tag{4.28}$$

of which we will estimate the last two terms respectively.

The first term

$$\int_{B_1} e^{-2\phi}\ \mathrm{sc}^h\ \mathrm{dvol} \leq \int_{B_1} e^{-2\phi}\ \mathrm{sc}^h + C(n, \rho)\,\mathrm{dvol}.$$

By Corollary 4.3.9, the scalar curvature of $\exp(2\phi)h$ i.e. $\exp(-2\phi)\,\mathrm{sc}_h$ is bounded from below, so by adding a constant, the integrand of the right hand side is nonnegative. Let $A = \mathrm{e}^{\rho}$, $I = [r_0 - 1, r_0 + 1]$, and $B_A^{r_0}$ be the ball with radius A centered at $p \in N$ with respect to g_{r_0}. Since $|\phi'| \leq \rho$ (Lemma 4.3.2), we may apply Lemma 1.2.9 to above inequality and have

$$\int_{B_1} \mathrm{e}^{-2\phi}\ \mathrm{sc}^h\ \mathrm{dvol} \leq \int_{I\times B_A^{r_0}} \mathrm{e}^{-2\phi}\ \mathrm{sc}^h + C(n,\rho)\,\mathrm{dvol}. \tag{4.29}$$

Because $|\phi'| \leq \rho$, $\phi(r) \leq \phi(r_0) + \rho$ for any $r \in [r_0 - 1, r_0 + 1]$. Thus,

$$\begin{aligned}\int_{I\times B_A^{r_0}} C(n,\rho)\,\mathrm{dvol} &\leq C(n,\rho) \int_{r_0-1}^{r_0+1} \int_{B_A^{r_0}} \mathrm{e}^{n\phi(r)}\ \mathrm{dvol}_h \\ &\leq C(n,\rho) \int_{r_0-1}^{r_0+1} \int_{B_A^{r_0}} \mathrm{e}^{n(\phi(r_0)+\rho)}\ \mathrm{dvol}_h \\ &= 2C(n,\rho)\mathrm{e}^{n\rho} \int_{B_A^{r_0}} \mathrm{dvol}_{r_0} \\ &= 2C(n,\rho)\,\mathrm{Vol}_{r_0}(B_A^{r_0}).\end{aligned}$$

By Corollary 4.3.9, the Ricci curvature of g^{r_0} is bounded from below by a constant depending only on n and ρ. Consequently, applying volume comparison theorem, the volume of $B_A^{r_0}$ is bounded from above by a constant depending on n and ρ. Therefore,

$$\int_{I\times B_A^{r_0}} C(n,\rho)\,\mathrm{dvol} \leq C(n,\rho). \tag{4.30}$$

We turn to the other term of equation (4.29). Note

$$\begin{aligned}\int_{I\times B_A^{r_0}} \mathrm{e}^{-2\phi}\ \mathrm{sc}_h\ \mathrm{dvol} &= \int_{r_0-1}^{r_0+1} \int_{B_A^{r_0}} \mathrm{e}^{(n-2)\phi(r)}\ \mathrm{sc}_h\ \mathrm{dvol}_h\ \mathrm{d}\,r \\ &= \int_{r_0-1}^{r_0+1} \mathrm{e}^{(n-2)\phi(r)} \int_{B_A^{r_0}} \mathrm{sc}_h\ \mathrm{dvol}_h\ \mathrm{d}\,r.\end{aligned} \tag{4.31}$$

Because of $B_A^{r_0} = B^h_{A\exp(-\phi(r_0))}$

$$\int_{B_A^{r_0}} \mathrm{sc}_h\ \mathrm{dvol}_h = \int_{B^h_{A\exp(-\phi(r_0))}} \mathrm{sc}_h\ \mathrm{dvol}_h. \tag{4.32}$$

By Corollary 4.3.10, the Ricci curvature of h

$$\mathrm{rc}_h \geq -C(n,\rho) \inf_{r\in\mathbb{R}} \mathrm{e}^{2\phi(r)} \geq -(n-1)C(n,\rho)\mathrm{e}^{2\phi(r_0)}\,.$$

Note

$$A\mathrm{e}^{-\phi(r_0)} = A\sqrt{C(n,\rho)} \cdot \frac{1}{\sqrt{C(n,\rho)}\,\mathrm{e}^{\phi(r_0)}}$$

It follows from Proposition 4.3.12 that, with $r = \exp(\phi(r_0))\sqrt{C(n,\rho)}$ and $K = A\sqrt{C(n,\rho)}$,

$$\int_{B^h_{A\exp(-\phi(r_0))}} \mathrm{sc}_h \;\mathrm{dvol}_h \leq C(n,\rho)\mathrm{e}^{(2-n)\phi(r_0)}\,.$$

Thus, together with Equations (4.31) and (4.32), and using the fact $|\phi'| \leq \rho$ again, we have

$$\begin{aligned}\int_{I\times B^{r_0}_A} &\mathrm{e}^{-2\phi}\;\mathrm{sc}_h\;\mathrm{dvol}\\ &\leq C(n,\rho)\mathrm{e}^{(2-n)\phi(r_0)}\int_{r_0-1}^{r_0+1}\mathrm{e}^{(n-2)\phi(r)}\;\mathrm{d}\,r\\ &\leq C(n,\rho)\mathrm{e}^{(2-n)\phi(r_0)}\int_{r_0-1}^{r_0+1}\mathrm{e}^{(n-2)\phi(r_0)+\rho}\;\mathrm{d}\,r\\ &\leq C(n,\rho).\end{aligned} \tag{4.33}$$

Therefore, it follows from Equations (4.29), (4.30), and (4.33) that

$$\int_{B_1}\mathrm{e}^{-2\phi}\;\mathrm{sc}^h\;\mathrm{dvol} \leq C(n,\rho). \tag{4.34}$$

We turn to the second term of the right hand side of the Equation (4.28). It is obvious that

$$\int_{B_1} -2n\phi'' - n(n+1)(\phi')^2\,\mathrm{dvol} \leq \int_{B_1} -2n\phi''\,\mathrm{dvol} \leq \int_{B_1} -2n\phi'' + 2n\rho^2\,\mathrm{dvol}. \tag{4.35}$$

By equation (4.3), the integrand in the last integral in equation (4.35) is nonnegative. Then, apply Lemma 1.2.9 again, we have

$$\begin{aligned}\int_{B_1} -2n\phi'' + 2n\rho^2 \,\mathrm{dvol} &\le \int_{I\times B_A^{r_0}} -2n\phi'' + 2n\rho^2 \,\mathrm{dvol} \\ &= \int_{r_0-1}^{r_0+1} \int_{B_A^{r_0}} (-2n\phi'' + 2n\rho^2)\mathrm{e}^{n\phi(r)} \,\mathrm{dvol}^h \,\mathrm{d}\,r\end{aligned} \tag{4.36}$$

Use the fact $\phi(r) \le \phi(r_0) + \rho$ for any $r \in [r_0 - 1, r_0 + 1]$ again,

$$\begin{aligned}&\int_{B_1} - 2n\phi'' + 2n\rho^2 \,\mathrm{dvol} \\ &\quad\le \int_{r_0-1}^{r_0+1} \int_{B_A^{r_0}} (-2n\phi'' + 2n\rho^2)\mathrm{e}^{n(\phi(r0)+\rho)} \,\mathrm{dvol}_h \,\mathrm{d}\,r \\ &\quad= \int_{r_0-1}^{r_0+1} \mathrm{e}^{n\rho} \int_{B_A^{r_0}} -2n\phi''(r) \,\mathrm{dvol}_{r_0} \,\mathrm{d}\,r + \int_{r_0-1}^{r_0+1} \mathrm{e}^{n\rho} \int_{B_A^{r_0}} 2n\rho^2 \,\mathrm{dvol}_{r_0} \,\mathrm{d}\,r\end{aligned} \tag{4.37}$$

The second term at the end of above equation is bounded from above by $C(n, \rho)$ by the same argument to obtain the Equation (4.30). For the first, use the fact $|\phi'| \le \rho$, we have

$$\begin{aligned}&\int_{r_0-1}^{r_0+1} \mathrm{e}^{n\rho} \int_{B_A^{r_0}} -2n\phi''(r) \,\mathrm{dvol}_{r_0} \,\mathrm{d}\,r \\ &\quad= -2n\Big(\phi'(r_0 + 1) - \phi'(r_0 - 1)\Big)\mathrm{e}^{n\rho} \,\mathrm{Vol}_{r_0}(B_A^{r_0}) \\ &\quad\le 2n\rho\mathrm{e}^{n\rho} \,\mathrm{Vol}_{r_0}(B_A^{r_0}) \le C(n, \rho).\end{aligned} \tag{4.38}$$

In the last step, we apply again the argument used for obtaining the equation (4.30). Therefore,

$$\int_{B_1} -2n\phi'' - n(n + 1)(\phi')^2 \,\mathrm{dvol} \le C(n, \rho). \tag{4.39}$$

Then the conclusion follows from the Equations (4.28), (4.34), and (4.39). □

Using above theorem repeatedly, we have the same conclusion for the doubly warped metric.

Corollary 4.3.15. *Assume that for some fixed n,m, the Conjecture 4.1.2 is true where the upper bound of the integral of scalar curvature is $\mathcal{S}_n$ and $\mathcal{S}_m$ respectively. Let (N_1, h_1) and (N_2, h_2) be any complete smooth Riemannian manifolds* *with dimension n and m. Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be smooth functions then $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)}\, (h_1 + \mathrm{e}^{2\psi(r)}\, h_2)$ be a complete doubly warped metric on $M = \mathbb{R} \times N_1 \times N_2$. If the Ricci curvature of g is at least $-n\rho^2$, then, on (M, g), the integral of scalar curvature over any unit ball is bounded from above by a constant depending only on the dimension.*

Proof. We only need to show that the of conjecture is true for $h_1 + \mathrm{e}^{2\psi(r)}\, h_2$, which is obvious since this is a direct product metric. Then Theorem 4.3.13 applies. □

And we are able to verify the conjecture for three dimensional warped metrics.

Corollary 4.3.16. *Let (N^2, h) be any complete Riemann surface. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function then $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)}\, h$ be a complete warped metric on $M \cong \mathbb{R} \times N$. If the Ricci curvature of g is at least $-n\rho^2$, then, on (M, g), the integral of scalar curvature over any unit ball is bounded from above by a constant $C(n, \rho)$.*

Proof. Since the sectional curvature and the Ricci curvature are the same on Riemann surface, the Conjecture is true by Petrunin's Theorem 4.1.1. The conclusion follows from the Theorem 4.3.13. □

The conjecture is also true for those warped metrics whose warped components (N, h) are Einstein.

Corollary 4.3.17. *Let (N, h) be any complete smooth Einstein manifold. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function then $g = \mathrm{d}\, r^2 + \mathrm{e}^{2\phi(r)}\, h$ be a complete warped metric on $M \cong \mathbb{R} \times N$. If the Ricci curvature of g is at least $-n\rho^2$, then, on (M, g), the integral of scalar curvature over any unit ball is bounded from above by a constant $C(n, \rho)$.*

Proof. This follows immediately from Proposition 4.2.1. □

4.4 Application to Uniqueness of Ricci Flow Solution

In this section, we apply our estimate to the integral curvature of the Ricci flow, and obtain the uniqueness for the Ricci flow solution in the class of warped product metrics with Ricci curvature bounded from below.

Theorem 4.4.1. *Let (M^3, g) be a complete Riemannian three-manifold with bounded sectional curvature, in which there is positive uniform volume lower bound for any* *unit ball. Suppose* $(M^3, g_i(t))$ $i = 1, 2$ *be two Ricci flow solutions with same initial data* $g_i(0) = g$. *If at any time* t, *both* g_i *are warped product Riemannian metrics and both have Ricci curvatures bounded from below, then there exist a short time interval on which* $g_1(t) = g_2(t)$.

Remark 4.4.2. Here the $g(t)$ is warped metric means that there exists a diffeomorphism $\varphi_t : M \to \mathbb{R} \times N^2$ and a smooth function $f_t : \mathbb{R} \to \mathbb{R}^+$ such that $g(t) = \varphi_t^*(\mathrm{d}\, r^2 + f_t^2(r)h_t)$, where $\mathrm{d}\, r^2 + f_t^2(r)h_t$ is a complete warped product metric on $\mathbb{R} \times N$.

Proof. The proof follows from a variance of the proof of Theorem 3.1.7 when replacing the complex sectional curvature lower bound with Ricci curvature lower bound.

In the proof of the Theorem 3.1.7, we use the complex sectional curvature bound in the following places. Firstly the proof of Lemma 3.2.1, where we need it to apply Petrunin's theorem to get time independent lower bound for the volume of unit balls. Here, the Petrunin's theorem can be replaced by the Corollary 4.3.16 where our new assumption is sufficient.

Secondly, the place we applies the Hamilton's compactness for the Ricci flow to get the convergence of Ricci flow solutions. Precisely we use the complex sectional curvature lower bound and the scalar curvature upper bound to get the two-side-bound for the sectional curvature. In fact, this is can be achieved in dimension three with Ricci curvature lower bound plus scalar curvature upper bound.

The third place we need the complex sectional curvature is to use the ancient solution lemma. Indeed, this lemma is automatically true due to Lemma 2.4.5. So we can apply Lemma 2.4.3 then get a contradiction.

The last place is where we use the Theorem 3.2.7 to prove the Proposition 3.2.6. And in dimension three, the scalar curvature bounds the sectional curvature with the help of Ricci curvature lower bound. $\square$

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