# On geometric and motivic realizations of variations of Hodge structure over Hermitian symmetric domains

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Abstract of the Dissertation

# On geometric and motivic realizations of variations of Hodge structure over Hermitian symmetric domains

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Based on the work of Gross [1] and Sheng and Zuo [2], Friedman and Laza [3] show that over every irreducible Hermitian symmetric domain there exists a canonical variation of real Hodge structure of Calabi-Yau type. The first part of the thesis concerns motivic realizations of the canonical Calabi-Yau variations over irreducible Hermitian symmetric domains of tube type. In particular, we show that certain rational descents of the canonical variations of Calabi-Yau type over irreducible tube domains of type $A$ can be realized as sub-variations of Hodge structure of certain variations which are naturally associated to families of abelian varieties of Weil type. The situations for tube domains of type $D^{\mathbb{H}}$ are also discussed.

The second part of the thesis aims to understand the exceptional isomorphism between the Hermitian symmetric domains of type $\mathrm{II}_4$ and of type $\mathrm{IV}_6$ geometrically. We shall give some geometric constructions relating both of the domains to quaternionic covers of genus three curves.

*To my parents, Fuliang Zhang and Yueyun Liu.*

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# Chapter 1

# Introduction

Hodge theory is a very powerful tool for studying moduli spaces. Specifically, by associating to an algebraic variety the Hodge structure on its cohomology we obtain a period map from a moduli space to a certain period domain; we ask whether this period map is injective, and what is the image. For principally polarized abelian varieties and K3 surfaces, the period map is birational. In general, there is a highly non-trivial obstruction for period maps to be dominant, namely Griffiths transversality. For period domains, Griffiths transversality is trivial only in the case of principally polarized abelian varieties or K3 surfaces. More generally, one can consider Mumford-Tate subdomains of period domains. Following [4], a Mumford-Tate domain is defined as the orbit of a fixed Hodge structure by the group of real points of the corresponding Mumford-Tate group; Mumford-Tate domains are natural objects for the study of global variations of Hodge structure. Deligne has showed that if a Mumford-Tate domain is *unconstrained* (i.e. Griffiths transversality is trivial) then it must be a Hermitian symmetric domain. Furthermore, Friedman and Laza [3] have noticed that if the periods satisfy certain algebraic relations then the corresponding Mumford-Tate subdomain is a Hermitian symmetric domain (c.f. [3] Theorem 1.4).

In the situation that the image of a period map is a Hermitian symmetric domain $\mathcal{D}$, the moduli space is typically birational to an arithmetic quotient $\Gamma\backslash\mathcal{D}$ of $\mathcal{D}$, and one can obtain a lot of geometric information on the moduli space using the rich structures of $\Gamma\backslash\mathcal{D}$. For example, we can compare various compactifications of the moduli spaces with the natural compactifications of $\Gamma\backslash\mathcal{D}$ (e.g. Baily-Borel compactification, Toroidal compactification). Also, using the automorphic forms associated to $\Gamma\backslash\mathcal{D}$ it can be shown that the moduli spaces $\mathcal{A}_g$ of $g$-dimensional principally polarized abelian varieties and the moduli spaces $\mathcal{F}_d$ of degree $d$ polarized K3 surfaces are of general type for $g, d \gg 0$.

It is thus very interesting to study unconstrained Mumford-Tate subdomains (which must be Hermitian symmetric domains) of period domains, or equivalently, the induced variations of Hodge structures which will be called *Hermitian variations of Hodge structure* (see also Definition 2.1 of [3]). Satake [5] and Deligne [6] (especially Table 1.3.9) classify Hermitian $\mathbb{Q}$-variations of Hodge structure which give families of abelian varieties. Based on the earlier work of Gross [1] and Sheng and Zuo [2], Friedman and Laza [3] complete the classification of Hermitian $\mathbb{Q}$-variations of Hodge structure of Calabi-Yau (CY) type (c.f. [3] Definition 2.3) which remain irreducible over $\mathbb{R}$.

The question we are interested in this thesis is how to attach moduli meanings to certain arithmetic quotients of unconstrained Mumford-Tate subdomains $\Gamma\backslash\mathcal{D}$. Beyond the classical cases such as abelian varieties and lattice-polarized K3 surfaces, there are only a small number of such examples (i.e. the corresponding moduli spaces are birational to $\Gamma\backslash\mathcal{D}$) to our knowledge: $n$ points on $\mathbb{P}^1$ with $n \leq 12$ ([7]), algebraic curves of genus 3 or 4 ([8]), cubic surfaces ([9]), cubic threefolds ([10]) and some examples of Calabi-Yau varieties ([11], [12], [13], [14], [15]).

Between the study of abstract Hermitian variations and the question on their geometric realizations, there is an immediate problem: does the abstract Hermitian variations of Hodge structure (specify $\mathbb{Q}$-descents for them first if necessary) occur in algebraic geometry as sub-variations of rational Hodge structure of those coming from families of algebraic varieties. We shall refer this as motivic realizations of the abstract Hermitian variations of Hodge structure.

The first part of this thesis concerns motivic realizations of certain Hermitian $\mathbb{R}$-variations of Hodge structure of CY type. More specifically, over every irreducible Hermitian symmetric domain there is a canonical Hermitian $\mathbb{R}$-variation of Hodge structure of CY type (see [3] Theorem 2.22). Following Corollary 2.29 of op. cit., we say a canonical variation is *primitive* if it is of CY threefold type and the associated domain is of tube type. There are four primitive Hermitian variations of Hodge structure of CY type, which are over the $(A_3, \alpha_3)$ domain, the $(C_3, \alpha_3)$ domain, the $(D_6^{\mathbb{H}}, \alpha_6)$ domain and the $(E_7, \alpha_7)$ domain respectively (recall that irreducible Hermitian symmetric domains are classified by pairs $(\Delta,\ \nu)$ consisting of a connected Dynkin diagram $\Delta$ and a special root $\nu$ (see for example [6] 1.2.6), we use such pairs to denote the isomorphic classes of irreducible Hermitian symmetric domains). The guiding question is then whether these primitive CY variations are realizable geometrically, or from the immediate point of view, motivically. (N.B. the $(C_3, \alpha_3)$ case is classical and well-known, see Section 9 of [1].)

We shall give a motivic realization for the canonical CY variations of real

Hodge structure over Hermitian symmetric domains of type $(A_{2n-1}, \alpha_n)$ using abelian varieties of Weil type ($n \in \mathbb{Z}^+$), which can also be viewed as a generalization of the previous work done by Lombardo [16] and Cacciatori and Filippini [17]. Also, we will discuss the situations for the $(D^{\mathbb{H}}_{2n}, \alpha_{2n})$ cases ($n \geq 2$). For the $(E_7, \alpha_7)$ case, the question is still wildly open; one big issue is that there is no family of abelian varieties over the $(E_7, \alpha_7)$ domain.

In a different but related direction, we explore the geometric realizations for the exceptional isomorphisms between Hermitian symmetric domains of type $(D^{\mathbb{H}}_4, \alpha_4)$ and of type $(D^{\mathbb{R}}_4, \alpha_1)$ (in the standard Siegel's notation, the first domain is of type $\mathrm{II}_4$ and the second is of type $\mathrm{IV}_6$). There are in total five such exceptional isomorphisms (see for example [18] Page 20), and among them the one we consider here involves domains of the highest dimension. This will be the second part of the thesis.

The idea is to interpret both domains as moduli spaces of some common geometric objects. On one hand, the domain of type $\mathrm{II}_4$ parameterizes abelian 8-folds with totally definite quaternionic multiplication (c.f. [19] or Section 9.5 of [20]); on the other hand, it is well-known that type $\mathrm{IV}_6$ domains parameterizes certain lattice-polarized K3 surfaces. Now the question is to find a moduli space which can be associated to both domains. We consider $\mathcal{M}_{3,Q}$, the moduli space of quaternionic covers of genus three smooth projective curves. According to the earlier work by van Geemen and Verra [21], a generic abelian 8-fold with totally definite quaternionic multiplication is isogenous to a certain Prym variety determined by a quaternionic cover of a genus three curve. When the genus three curve $C$ is non-hyperelliptic (i.e. a smooth plane quartic), we prove that a quaternionic cover corresponds (up to finite choices) to a syzygetic tetrad of bitangent lines of $C$ which can be obtained (in 10 ways) from any pair of bitangents. We then construct a family of K3 surfaces by taking double covers of $\mathbb{P}^2$ branched over a plane quartic together with two bitangent lines, and show that they are parameterized by a certain arithmetic quotient of the type $\mathrm{IV}_6$ domain.

# Chapter 2

# Variations of Hodge structure over Hermitian symmetric domains

## 2.1 Hermitian symmetric domains

### 2.1.1 Hermitian symmetric spaces and their automorphisms

#### Hermitian symmetric spaces

Let us start with the definition of Hermitian symmetric spaces. Recall that a Hermitian manifold is a pair $(M, g)$ consisting of a complex manifold $M$ together with a Hermitian metric $g$ on $M$.

**Definition 2.1.1.** A Hermitian manifold $(M, g)$ is *symmetric* if

(1) $(M, g)$ is homogeneous. That is, the holomorphic isometry group $\mathrm{Is}(M, g)$ acts transitively on $M$.

(2) For any point $p \in M$, there exists an involution $s_p$ (i.e. $s_p$ is a holomorphic isometry and $s_p^2 = \mathrm{id}$) such that $p$ is an isolated fixed point of $s_p$. (Such an involution $s_p$ is called a *symmetry* at $p$.)

A connected symmetric Hermitian manifold is called a *Hermitian symmetric space*.

If $(M, g)$ is homogeneous, then to check Condition (2) it suffices to construct a symmetry $s_p$ at some point $p$ of $M$.

Also, the automorphism group $\mathrm{Is}(M, g)$ consists of holomorphic isometries of $M$; if we denote the underlying smooth manifold of $M$ by $M^\infty$, then

$$\mathrm{Is}(M, g) = \mathrm{Is}(M^\infty, g) \cap \mathrm{Hol}(M)$$

(intersection inside $\mathrm{Aut}(M^\infty)$; $\mathrm{Is}(M^\infty, g)$ is the group of isometries of $(M^\infty, g)$ as a Rimannian manifold, and $\mathrm{Hol}(M)$ is the group of automorphisms of $M$ as a complex manifold). We shall take a closer look at $\mathrm{Is}(M, g)$ later because it will play an important role in classifying Hermitian symmetric spaces.

If there is no ambiguity, we shall simply use $M$ to denote the Hermitian manifold $(M, g)$.

**Example 2.1.2.** Here are three basic examples of Hermitian symmetric spaces.

(a) The complex upper half plane $\mathcal{H}_1$.

(b) The projective line $\mathbb{P}^1$ (or the Riemann sphere endowed with the restriction of the standard metric on $\mathbb{R}^3$).

(c) Any quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a discrete additive subgroup $\Lambda \subset \mathbb{C}$ (endowed with the natural complex structure and Hermitian metric inherited from $\mathbb{C}$).

Let us work out the example of upper half plane $\mathcal{H}_1$ and leave the others as exercises. It is easy to see that $\mathcal{H}_1$, endowed with the metric $\frac{dxdy}{y^2}$, is a Hermitian manifold. The natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}_1$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d},$$

identifies $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ with the group $\mathrm{Is}(\mathcal{H}_1)$ of automorphisms of $\mathcal{H}_1$. Since for any $x + iy \in \mathcal{H}_1$,

$$x + iy = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} i,$$

$\mathcal{H}_1$ is homogeneous. The isomorphism $z \mapsto -\frac{1}{z}$ is an involution at the point $i \in \mathcal{H}_1$. Because $\mathcal{H}_1$ is also connected, it is a Hermitian symmetric space.

These examples are different from each other in several aspects (compactness, curvature, whether simply connected, etc.). In fact, each of them represents a different type of Hermitian symmetric spaces.

**Definition 2.1.3.** Let $M$ be a Hermitian symmetric space.

(1) $M$ is said to be of *Euclidean type* if it is isomorphic to $\mathbb{C}^n/\Lambda$ for some discrete additive subgroup $\Lambda \subset \mathbb{C}^n$.

(2) $M$ is said to be *irreducible* if it is not Euclidean and can not be written as a product of two Hermitian symmetric spaces of lower dimensions.

(3) $M$ is said to be of *compact type* (resp. *noncompact type*) if it is the product of compact (resp. noncompact) irreducible Hermitian symmetric spaces. Moreover, Hermitian symmetric spaces of noncompact type are also called *Hermitian symmetric domains*.

Every Hermitian symmetric space can be decomposed uniquely into a product of Hermitian symmetric spaces of these three types (c.f. Chapter VIII of [22], especially Proposition 4.4, Theorem 4.6 and Proposition 5.5).

**Theorem 2.1.4** (Decomposition Theorem)**.** *Every Hermitian symmetric space $M$ decomposes uniquely as*

$$M = M_0 \times M_- \times M_+,$$

*where $M_0$ is a Euclidean Hermitian symmetric space and $M_-$ (resp. $M_+$) is a Hermitian symmetric space of compact type (resp. of noncompact type). Moreover, $M_-$ (resp. $M_+$) is simply connected and decomposes uniquely as a product of compact (resp. noncompact) irreducible Hermitian symmetric spaces.*

We shall be especially interested in Hermitian symmetric domains. As a general property, every Hermitian symmetric domain can be embedded into some $\mathbb{C}^n$ as a bounded domain (via the Harish-Chandra embeddings). Conversely, every bounded domain $\mathcal{D} \subset \mathbb{C}^n$ has a canonical Hermitian metric (called the Bergman metric) which makes $\mathcal{D}$ a Hermitian symmetric domain. For instance, the complex upper half plane $\mathcal{H}_1$ can also be realized as the unit ball $\mathcal{D}_1 \subset \mathbb{C}$.

### Automorphism groups of Hermitian symmetric domains

Let $(\mathcal{D}, g)$ be a Hermitian symmetric domain. Endowed with the compact-open topology, the group $\mathrm{Is}(\mathcal{D}^\infty, g)$ of isometries has a natural structure of (real) Lie group. Being a closed subgroup of $\mathrm{Is}(\mathcal{D}^\infty, g)$, the group $\mathrm{Is}(\mathcal{D}, g)$ is also a Lie group. Let us denote by $\mathrm{Is}(\mathcal{D}, g)^+$ (resp. $\mathrm{Is}(\mathcal{D}^\infty, g)^+$, $\mathrm{Hol}(\mathcal{D})^+$) the connected component of $\mathrm{Is}(\mathcal{D}, g)$ (resp. $\mathrm{Is}(\mathcal{D}^\infty, g)$, $\mathrm{Hol}(\mathcal{D})$) containing the identity.

**Proposition 2.1.5.** *Let $(\mathcal{D}, g)$ be a Hermitian symmetric domain. The inclusions*

$$\mathrm{Is}(\mathcal{D}^\infty, g) \supset \mathrm{Is}(\mathcal{D}, g) \subset \mathrm{Hol}(\mathcal{D})$$

*gives equalities*

$$\mathrm{Is}(\mathcal{D}^\infty, g)^+ = \mathrm{Is}(\mathcal{D}, g)^+ = \mathrm{Hol}(\mathcal{D})^+.$$

*Proof.* See Lemma 4.3 of [22] and Proposition 1.6 of [23]. □

Following Chapter IV Theorem 2.5 and Theorem 3.3 of [22], we can recover the smooth structure of $\mathcal{D}$ as a quotient Lie group of $\mathrm{Is}(\mathcal{D}, g)^+$ by the stabilizer of a certain point.

**Theorem 2.1.6.** *Notations as above.*

*(1)* $\mathrm{Is}(\mathcal{D}, g)^+$ *is a adjoint (i.e. semisimple with trivial center) Lie group.*

*(2) For any point* $p \in \mathcal{D}$*, the subgroup* $K_p$ *of* $\mathrm{Is}(\mathcal{D}, g)^+$ *fixing* $p$ *is compact.*

*(3) The map*

$$\mathrm{Is}(\mathcal{D}, g)^+/K_p \to \mathcal{D}, \quad gK_p \mapsto g \cdot p$$

*is an* $\mathrm{Is}(\mathcal{D}, g)^+$*-equivariant diffeomorphism. In particular,* $\mathrm{Is}(\mathcal{D}, g)^+$ *(and hence* $\mathrm{Hol}(\mathcal{D})^+$ *or* $\mathrm{Is}(\mathcal{D}^\infty, g)^+$*) acts transitively on* $\mathcal{D}$*.*

In particular, every irreducible Hermitian symmetric domain is diffeomorphic to $H/K$ for a unique pair $(H, K)$ (obtained as above) with $H$ a connected noncompact simple adjoint Lie group and $K$ a maximal connected compact Lie group (c.f. Chapter VIII Section 6 of [22]). Conversely, given such a pair $(H, K)$ we can easily get a homogenous smooth manifold $H/K$. The natural question is how to endow $H/K$ with a complex structure and a compatible Hermitian metric so that it is a Hermitian symmetric domain.

We should point out that one can figure out the answer in terms of such pairs $(H, K)$ and accordingly classify the corresponding irreducible Hermitian symmetric domains (see for example Section 2.1 of [18] and references therein). However, here we shall answer the question from the viewpoint of Shimura data. Specifically, we shall replace the Lie group $H$ by an algebraic group $G$, replace cosets of $K$ by certain homomorphisms $u : U_1 \to G$ from the circle group $U_1$ to $G$, and then answer the question in terms of the new pairs $(G, u)$.

To conclude this subsection (and as an initial step to produce a Shimura datum), we discuss how to associate a $\mathbb{R}$-algebraic group $G$ to the real Lie group $\mathrm{Hol}(\mathcal{D})^+$ in such a way that $G(\mathbb{R})^+ = \mathrm{Hol}(\mathcal{D})^+$. The superscript $^+$ in $G(\mathbb{R})^+$denotes the neutral connected component relative to the real topology (v.s. the Zariski topology). We shall follow [24] for the terminologies on algebraic groups, and also refer the readers to it for the related background materials. For example, we say an algebraic group is *simple* if it is non-commutative and has no proper normal algebraic subgroups, while *almost simple* if it is non-commutative and has no proper normal connected algebraic subgroup (N.B. an almost simple algebraic group can have finite center).

**Proposition 2.1.7.** *Let $(\mathcal{D}, g)$ be a Hermitian symmetric domain, and let $\mathfrak{h} = \mathrm{Lie}(\mathrm{Hol}(\mathcal{D})^+)$. There is a unique connected adjoint real algebraic subgroup $G$ of $\mathrm{GL}(\mathfrak{h})$ such that (inside $\mathrm{GL}(\mathfrak{h})$)*

$$G(\mathbb{R})^+ = \mathrm{Hol}(\mathcal{D})^+.$$

*Moreover, $G(\mathbb{R})^+ = G(\mathbb{R}) \cap \mathrm{Hol}(\mathcal{D})$ (inside $\mathrm{GL}(\mathfrak{h})$); therefore $G(\mathbb{R})^+$ is the stabilizer in $G(\mathbb{R})$ of $\mathcal{D}$.*

*Proof.* See [23] Proposition 1.7. □

### 2.1.2 Classification of Hermitian symmetric domains

Consider the circle group $U_1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Motivated by the following fact, one can think of a point of $\mathcal{D}$ as a homomorphism $U_1 \to G$.

**Theorem 2.1.8.** *Let $\mathcal{D}$ be Hermitian symmetric domain. For each $p \in \mathcal{D}$, there exists a unique homomorphism $u_p : U_1 \to \mathrm{Hol}(\mathcal{D})^+$ such that $u_p(z)$ fixes $p$ and acts on $T_p\mathcal{D}$ as multiplication by $z$.*

*Proof.* See Theorem 1.9 of [23]. □

*Remark* 2.1.9. Using the uniqueness of $u_p$ one can easily see that $\mathrm{Hol}(\mathcal{D})^+$ acts on the set of $u_p$'s via conjugation. Specifically, given two different points $p \neq p'$ we choose $f \in \mathrm{Hol}(\mathcal{D})^+$ with $f(p) = p'$, then $f \circ u_p(z) \circ f^{-1}$ $(z \in U_1)$ satisfies the conditions in Theorem 2.1.8 for $p'$, and so $u_{p'} = f \circ u_p \circ f^{-1}$.

**Example 2.1.10.** Let $p = i \in \mathcal{H}_1$. In Example 2.1.2 we have seen that $\mathrm{Hol}(\mathcal{H}_1) = \mathrm{PSL}_2(\mathbb{R})$. The associated real algebraic group (c.f. Proposition 2.1.7) is $(\mathrm{PGL}_2)_{\mathbb{R}}$: $\mathrm{PGL}_2(\mathbb{R})^+ = \mathrm{PSL}_2(\mathbb{R})$. (N.B. the group $\mathrm{PSL}_2$ is not an algebraic group. )

To define $u_i : U_1 \to \mathrm{PSL}_2(\mathbb{R})$ we first consider the following homomorphism

$$h_i : U_1 \to \mathrm{SL}_2(\mathbb{R}), z = a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It is easy to verify that $h_i(z)$ fixes $i$. Since

$$\frac{d}{dw}\left(\frac{aw+b}{-bw+a}\right)\bigg|_{w=i} = \frac{a^2+b^2}{(a-ib)^2} = \frac{z}{\bar{z}} = z^2,$$

$h_i(z)$ acts on the tangent space $T_i\mathcal{H}_1$ as multiplication by $z^2$. So for $z \in U_1$, we choose a square root $\sqrt{z} \in U_1$ and set $u_i(z) = h_i(\sqrt{z})$. The homomorphism $u_i : U_1 \to \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm I$ is independent of the choice of $\sqrt{z}$ because

$h_i(-1) = -I$. By construction $u_i(z)$ fixes $i$ and acts on the tangent space $T_i\mathcal{H}_1$ as multiplication by $z$.

Since $G(\mathbb{R})^+$ ($= \mathrm{Hol}(\mathcal{D})^+$) acting transitively on $\mathcal{D}$, set-theoretically we can view $\mathcal{D}$ as the $G(\mathbb{R})^+$-conjugacy class of $u_p : U_1 \to G(\mathbb{R})$. (We will see that $u_p$ is an algebraic homomorphism later). Conversely, given an abstract pair $(G, u : U_1 \to G)$ with $G$ a real adjoint algebraic group and $u$ an algebraic homomorphism we want to ask the following questions.

**Question 2.1.11.** For a pair $(G, u)$ as above, we let $\mathcal{D}$ be the $G(\mathbb{R})^+$-conjugacy class of $u$. Denote by $K_u$ the subgroup of $G(\mathbb{R})^+$ fixing $u$. There is a bijection $G(\mathbb{R})^+/K_u \to \mathcal{D}$ and so the space $\mathcal{D}$ has a natural smooth structure.

(1) Under what conditions can $\mathcal{D}$ be given a nice complex structure (or a Hermitian structure)? Under what additional conditions is $\mathcal{D}$ a Hermitian symmetric space?

(2) Under what conditions is $K_u$ compact?

(3) Under what conditions is $\mathcal{D}$ a Hermitian symmetric domain (i.e. of noncompact type)?

To answer these questions, let us first study those pairs $(G, u)$ coming from Hermitian symmetric domains.

### Representation of $U_1$

Let $T$ be an algebraic torus defined over a field $k$, and let $K$ be a Galois extension of $k$ splitting $T$. The character group $X^*(T)$ is defined by $X^*(T) = \mathrm{Hom}(T_K, \mathbb{G}_m)$. If $r$ is the rank of $T$, then $X^*(T)$ is a free abelian group of rank $r$ which comes equipped with an action of $\mathrm{Gal}(K/k)$. In general, to give a representation $\rho$ of $T$ on a $k$-vector space $V$ amounts to giving an $X^*(T)$-grading $V_K = \bigoplus_{\chi \in X^*(T)} V_\chi$ on $V_K := V \otimes_k K$ with the property that

$$\sigma(V_\chi) = V_{\sigma\chi}, \quad \text{all } \sigma \in \mathrm{Gal}(K/k), \quad \chi \in X^*(T).$$

Here $V_\chi$ is the $K$-subspace of $V_K$ on which $T(K)$ acts through $\chi$:

$$V_\chi = \{v \in V_K \mid \rho(t)(v) = \chi(t) \cdot v, \ \ \forall\, t \in T(K)\}.$$

For instance, we can regard $U_1$ as a real algebraic torus. As a $\mathbb{R}$-algebraic group, the $K$-valued point (with $K$ a $\mathbb{R}$-algebra) of $U_1$ is

$$U_1(K) = \left\{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \in M_{2\times 2}(K) \;\middle|\; a^2 + b^2 = 1 \right\}.$$

In particular, $U_1(\mathbb{R})$ is the circle group and $U_1(\mathbb{C})$ can be identified with $\mathbb{C}^*$ through

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a+ib, \quad \text{conversely } z \mapsto \begin{pmatrix} \frac{1}{2}(z+\frac{1}{z}) & \frac{1}{2i}(z-\frac{1}{z}) \\ -\frac{1}{2i}(z-\frac{1}{z}) & \frac{1}{2}(z+\frac{1}{z}) \end{pmatrix}.$$

Noting that $X^*(U_1) \cong \mathbb{Z}$ and complex conjugation acts on $X^*(U_1)$ as multiplication by $-1$, we obtain the following proposition.

**Proposition 2.1.12.** *Consider a representation $\rho$ of $U_1$ on a $\mathbb{R}$-vector space $V$. Then $V_{\mathbb{C}} = \bigoplus_{n\in\mathbb{Z}} V_{\mathbb{C}}^n$ with the property that $\overline{V_{\mathbb{C}}^n} = V_{\mathbb{C}}^{-n}$, where $V_{\mathbb{C}}^n = \{v \in V_{\mathbb{C}} \mid \rho(z)(v) = z^n \cdot v, \quad \forall z \in \mathbb{C}^*\}$. Moreover, if $V$ is irreducible, then it must be isomorphic to one of the following types.*

*(a) $V \cong \mathbb{R}$ with $U_1$ acting trivially (so $V_{\mathbb{C}} = V_{\mathbb{C}}^0$).*

*(b) $V \cong \mathbb{R}^2$ with $z = x+iy$ acting as $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^n$ for some $n > 0$ (so $V_{\mathbb{C}} = V_{\mathbb{C}}^n \oplus V_{\mathbb{C}}^{-n}$).*

*In particular, every real representation of $U_1$ is a direct sum of representations of these types.*

*Remark* 2.1.13. Let $V$ be a $\mathbb{R}$-representation of $U_1$ and write $V_{\mathbb{C}} = \bigoplus_{n\in\mathbb{Z}} V_{\mathbb{C}}^n$ as above. Because $\overline{V_{\mathbb{C}}^0} = V_{\mathbb{C}}^0$, the weight space $V_{\mathbb{C}}^0$ is defined over $\mathbb{R}$; in other words, the complexification the real subspace $V^0$ of $V$ defined by $V \cap V_{\mathbb{C}}^0$ is $V_{\mathbb{C}}^0$: $V^0 \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}^0$. The natural homomorphism $V/V^0 \to V_{\mathbb{C}}/\bigoplus_{n\leq 0} V_{\mathbb{C}}^n \cong \bigoplus_{n>0} V_{\mathbb{C}}^n$ is a $\mathbb{R}$-linear isomorphism.

The representations of $U_1$ have the same description no matter we regard it as a Lie group or an algebraic group, and so every homomorphism $U_1 \to \mathrm{GL}(V)$ of Lie groups is algebraic. In particular, the homomorphism $u_p$ is algebraic for any $p \in \mathcal{D}$. Let $K_p$ be the subgroup of $G(\mathbb{R})^+$ fixing $p$. By Theorem 2.1.8, $u_p(z)$ acts on the $\mathbb{R}$-vector space

$$\mathrm{Lie}(G)/\,\mathrm{Lie}(K_p) \cong T_p\mathcal{D}$$

as multiplication by $z$, and it acts on $\mathrm{Lie}(K_p)$ trivially. Suppose $T_p\mathcal{D} \cong \mathbb{C}^k$ and identify it with $\mathbb{R}^{2k}$ by $(a_1+ib_1, \cdots, a_k+ib_k) \mapsto (a_1, b_1, \cdots, a_k, b_k)$, then it is easy to write down the matrix of multiplication by $z = x+iy$ and conclude that $T_p\mathcal{D}$ (as a real representation of $U_1$) splits into a direct sum of $\mathbb{R}^2$'s as in the Part $(b)$ of the previous proposition with $n = 1$.

Accordingly, we can determine the representation $\mathrm{Ad} \circ u_p : U_1 \to G \to \mathrm{GL}(\mathrm{Lie}(G))$ (which is the induced action of $u_p(z)$ on $T_p\mathcal{D}$). It splits into a direct sum of 1-dimensional real vector spaces (as in Part $(a)$ of Proposition 2.1.12) and 2-dimensional spaces (as in Part $(b)$ with $n = 1$). Taking the complexification of the representation $\mathrm{Lie}(G)$, we obtain the following proposition.

**Proposition 2.1.14.** *Notations as above. Only the characters $z$, $1$ and $z^{-1}$ occur in the representation of $U_1$ on $\mathrm{Lie}(G)_{\mathbb{C}}$ defined through $u_p$.*

### Cartan involutions

Let $G$ be a connected algebraic group defined over $\mathbb{R}$, and let $g \mapsto \bar{g}$ denote complex conjugation on $G(\mathbb{C})$.

**Definition 2.1.15.** An involution of $G$ is said to be *Cartan* if the group

$$G^{(\theta)}(\mathbb{R}) := \{g \in G(\mathbb{C}) \mid g = \theta(\bar{g})\}$$

is compact.

**Example 2.1.16.** Let $G = \mathrm{SL}_2$, and let $\theta$ be the conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since

$$\theta\left(\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix},$$

we have

$$\mathrm{SL}_2^{(\theta)}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \middle| d = \bar{a},\ c = -\bar{b} \right\} = \mathrm{SU}_2(\mathbb{R}).$$

Clearly, the group $\mathrm{SU}_2(\mathbb{R})$ is compact, and hence $\theta$ is a Cartan involution for $\mathrm{SL}_2$.

**Theorem 2.1.17.** *A connected $\mathbb{R}$-algebraic group $G$ has a Cartan involution if and only if it is reductive, in which case any two Cartan involutions are conjugate by an element of $G(\mathbb{R})$.*

*Proof.* See Chapter I Theorem 4.2 and Corollary 4.3 of [25]. □

**Example 2.1.18.** Let $G$ be a connected $\mathbb{R}$-algebraic group.

(a) The identity map is a Cartan involution if and only if $G(\mathbb{R})$ is compact. Moreover, it is the only Cartan involution of $G$.

(b) Let $G = \mathrm{GL}(V)$ with $V$ a real vector space of dimension $n$. Fix a basis of $V$, then $G$ has an involution given by $\theta : M \mapsto (M^t)^{-1}$. On $G(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$, $M = \theta(\bar{M})$ if and only if $M\bar{M}^t = I$ (i.e. $M \in \mathrm{U}(n)$). Thus $\theta$ is a Cartan involution. Note that different choices of bases give different Cartan involutions, and the previous theorem says that all Cartan involutions of $G$ arise in this way.

(c) (Chapter I Corollary 4.4 of [25]) Let $G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation. Then $G$ is reductive if and only if it is stable under $g \mapsto g^t$ for a suitable choice of a basis for $V$, in which case the restriction of $g \mapsto (g^t)^{-1}$ to $G$ is a Cartan involution. Furthermore, all Cartan involutions of $G$ arise in this way from the choices of a basis of $V$.

(d) Let $\theta$ be an involution of $G$. Then there is a unique real form $G^{(\theta)}$ of $G_{\mathbb{C}}$ such that complex conjugation on $G^{(\theta)}(\mathbb{C})$ is $g \mapsto \theta(\bar{g})$. So the Cartan involutions of $G$ correspond to the compact forms of $G_{\mathbb{C}}$.

Now let us go back to Hermitian symmetric domains. Let $\mathcal{D}$ be a Hermitian symmetric domain. As before, $G$ is the associated real adjoint algebraic group as in Proposition 2.1.7, and $u_p : U_1 \to G$ is an algebraic homomorphism attached to a point $p \in \mathcal{D}$.

**Proposition 2.1.19.** *The conjugation by $u_p(-1)$ is a Cartan involution of $G$.*

*Proof.* Let $s_p$ be a symmetry at $p$. Denote by $\mathrm{Inn}(s_p)$ the conjugation of $G$ by $s_p$. $\mathrm{Inn}(s_p)$ is an involution because $s_p^2 = \mathrm{id}$. According to Section V.2 of [22], the real form of $G_{\mathbb{C}}$ defined by the involution $\mathrm{Inn}(s_p)$ (c.f. Example 2.1.18 (*d*)) is that associated to the compact dual of the symmetric space. As a result, a symmetry at a point of a symmetric space gives a Cartan involution of $G$ if and only if the space is of noncompact type. In particular, $\mathrm{Inn}(s_p)$ is Cartan. On the other hand, both $u_p(-1)$ and $s_p$ fix $p$ and acts as multiplication by $(-1)$ on $T_p\mathcal{D}$, and hence $u_p(-1) = s_p$ (c.f. Proposition 1.14 of [23], which also implies the uniqueness of symmetries at a point of a Hermitian symmetric domain.) $\square$

Note that Example 2.1.16 is cooked up in this way.

### Classification of Hermitian symmetric domains in terms of real groups

We will classify (pointed) Hermitian symmetric domains in this section. Let $\mathcal{D}$ be a Hermitian symmetric domain. We have already proven part of the following theorem.

**Theorem 2.1.20.** *Let $G$ be the associated adjoint real algebraic group of $\mathcal{D}$. The homomorphism $u_p : U_1 \to G$ attached to a point $p \in \mathcal{D}$ satisfies the following properties:*

- *(a) only the character $z$, $1$ and $z^{-1}$ occur in the representation of $U_1$ on $\mathrm{Lie}(G)_{\mathbb{C}}$ defined by $u_p$;*
- *(b) The conjugation of $G$ by $u_p(-1)$ is a Cartan involution;*
- *(c) $u_p(-1)$ does not projects to $1$ in any simple factor of $G$.*

*Proof.* See Proposition 2.1.14 and 2.1.19 for Part $(a)$ and $(b)$. Suppose $u_p(-1)$ projects to 1 for some simple factor $G_1$ (which corresponds to a noncompact irreducible factor of $\mathcal{D}$, see Theorem 2.1.4), then the Cartan involution $\mathrm{Inn}(u_p(-1))$ is the identity map on $G_1$. But by Example 2.1.18 $(a)$, this implies that $G_1(\mathbb{R})$ is compact, which is a contradiction. □

The properties $(a)$, $(b)$, $(c)$ in Theorem 2.1.20 turns out to be an answer to Question 2.1.11.

**Theorem 2.1.21.** *Let $G$ be a real adjoint algebraic group, and let $u : U_1 \to G$ be a homomorphism satisfying $(a)$, $(b)$ and $(c)$ of Theorem 2.1.20. Then the set $\mathcal{D}$ of conjugates of $u$ by elements of $G(\mathbb{R})^+$ has a natural structure of a Hermitian symmetric domain such that $G(\mathbb{R})^+ = \mathrm{Hol}(\mathcal{D})^+$ and $u(-1)$ is the symmetry at $u$ (regarded as a point of $\mathcal{D}$).*

*Proof.* (Sketch) Let $K_u$ be the subgroup of $G(\mathbb{R})^+$ fixing $u$ (i.e. the centralizer of $u$). By $(b)$, $\theta := \mathrm{Ad}(u(-1))$ is a Cartan involution for $G$. So $G^{(\theta)}(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid g = u(-1) \cdot \bar{g} \cdot u(-1)^{-1}\}$ is compact. Since $K_u \subset G(\mathbb{R})^+$, $\bar{g} = g$ for any $g \in K_u$, and so $K_u \subset G^{(\theta)}(\mathbb{R})$. As $K_u$ is closed, it is also compact. The natural bijection $\mathcal{D} \cong (G(\mathbb{R})^+/K_u) \cdot u$ endows $\mathcal{D}$ with the structure of a smooth (homogeneous) manifold.

With this structure, the (real) tangent space at $u$ is $T_u\mathcal{D} = \mathrm{Lie}(G)/\mathrm{Lie}(K_u)$. Note that $\mathrm{Lie}(G)$ a real representation of $U_1$ via $\mathrm{Ad} \circ u$. Using the notations in Proposition 2.1.12, $(a)$ gives that $\mathrm{Lie}(G)_{\mathbb{C}} = \mathrm{Lie}(G)_{\mathbb{C}}^{-1} \oplus \mathrm{Lie}(G)_{\mathbb{C}}^{0} \oplus \mathrm{Lie}(G)_{\mathbb{C}}^{1}$. Clearly, $K_u = \mathrm{Lie}(G)_{\mathbb{C}}^{0} \cap \mathrm{Lie}(G)$. Using the natural isomorphism

$$\mathrm{Lie}(G)/\mathrm{Lie}(K_u) \to \mathrm{Lie}(G)_{\mathbb{C}}/\mathrm{Lie}(G)_{\mathbb{C}}^{0} \oplus \mathrm{Lie}(G)_{\mathbb{C}}^{-1} \cong \mathrm{Lie}(G)_{\mathbb{C}}^{1},$$

the tangent space $T_u\mathcal{D}$ can be identified with $\mathrm{Lie}(G)_{\mathbb{C}}^{1}$, the complex vector space of $\mathrm{Lie}(G)_{\mathbb{C}}$ on which $u(z)$ acts as multiplication by $z$. This endows $T_p\mathcal{D}$ with a $\mathbb{C}$-vector space structure. (In particular, the corresponding almost complex structure $J$ is $u(i)$.) Since $\mathcal{D}$ is homogenous, this induces a structure of an almost complex manifold on $\mathcal{D}$, which is integrable (c.f. [26] 8.7.9).

The action of $K_u$ on $\mathcal{D}$ induces an action of it on $T_u\mathcal{D}$. As $K_u$ is compact, there is a $K_u$-invariant positive definite form on $T_u\mathcal{D}$ (c.f. Proposition 1.18 of [23]), which is compatible with the complex structure $J$ of $T_u\mathcal{D}$ because $J = u(i) \in K_u$. Now use the homogeneity of $\mathcal{D}$ to move the bilinear form to each tangent space, which will make $\mathcal{D}$ into a Hermitian manifold. It is not difficult to see that $u(-1)$ is the symmetric at $u$ and $\mathcal{D}$ is a Hermitian symmetric space.

Finally, $\mathcal{D}$ is a Hermitian symmetric domain because of $(b)$ and $(c)$. The proof is quite similar with that of Theorem 2.1.20. □

*Remark* 2.1.22. As we saw in the proof of Theorem 2.1.21, the condition $(b)$ guarantees that $K_u$ is compact. If further assuming $(a)$ holds, then one can endow $\mathcal{D}$ with a structure of Hermitian symmetric space. The space $\mathcal{D}$ is a Hermitian symmetric domain because of $(b)$ and $(c)$.

As a corollary, we can classify Hermitian symmetric domains in terms of such pairs.

**Corollary 2.1.23.** *There is a natural one-to-one correspondence between isomorphism classes of pointed Hermitian symmetric domains and pairs $(G, u)$ consisting of a real adjoint algebraic group $G$ and a non-trivial homomorphism $u : U_1 \to G$ satisfying $(a)$, $(b)$, $(c)$ in Theorem 2.1.20.*

### Classification of Hermitian symmetric domains in terms of Dynkin diagrams

Let us now focus on irreducible Hermitian symmetric domains. The irreducibility of the domain implies that the associated adjoint algebraic group is a simple algebraic group. Let $G$ be a simple adjoint group over $\mathbb{R}$, and let $u$ be a homomorphism $U_1 \to G$ satisfying $(a)$ and $(b)$ of Theorem 2.1.20 (N.B. the condition $(c)$ then holds trivially).

**Lemma 2.1.24.** *Let $G_{\mathbb{C}}$ be the scalar extension of $G$ from $\mathbb{R}$ to $\mathbb{C}$, and $\mu = u_{\mathbb{C}} : \mathbb{G}_m \to G_{\mathbb{C}}$. Then*

*(1) $G_{\mathbb{C}}$ is also simple;*

*(2) Only the characters $z$, $1$, $z^{-1}$ occur in the action of $\mathrm{Ad} \circ \mu : \mathbb{G}_m \to \mathrm{Lie}(G_{\mathbb{C}})$.*

*Proof.* See Page 21 of [23] or Page 478 of [27] for Part (1). Part (2) follows from $(a)$ of Theorem 2.1.20. □

**Proposition 2.1.25.** *The map $(G, u) \mapsto (G_{\mathbb{C}}, u_{\mathbb{C}})$ defines a bijection between the sets of isomorphism classes of pairs consisting of*

*(1) a simple adjoint algebraic group over $\mathbb{R}$ and a conjugacy class of $u : U_1 \to G$ satisfying (a) and (b) in Theorem 2.1.20, and*

*(2) a simple adjoint algebraic group over $\mathbb{C}$ and a conjugacy class of cocharacters satisfying (2) of Lemma 2.1.24.*

*Proof.* See Proposition 1.24 of [23]. □

**Example 2.1.26.** Let $\mu$ be a cocharacter of $(\mathrm{PGL}_2)_{\mathbb{C}}$. Let $\theta$ be the conjugation of $\mathrm{PGL}_2(\mathbb{C})$ by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The same computation as in Example 2.1.16 shows that the involution $\bar{\theta}$, defined by $g \mapsto \theta(\bar{g})$, is complex conjugation on the compact form $\mathrm{PGU}_2$ of $(\mathrm{PGL}_2)_{\mathbb{C}}$. Consider another involution on $\mathrm{PGL}_2(\mathbb{C})$ given by $g \mapsto \mu(-1) \circ \bar{\theta}(g) \circ \mu(-1)^{-1}$. By Example 2.1.18 $(d)$, there is a real form $H$ of $(\mathrm{PGL}_2)_{\mathbb{C}}$ such that complex conjugation on $H(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$ is the involution as above. Also define $u := \mu|_{U_1}$ which takes value in $H(\mathbb{R})$. As $\mu(-1)^2 = \mathrm{id}$, the conjugation by $u(-1)$ is an involution of $H$. By construction, it is a Cartan involution. In this way, we obtain a pair $(H, u)$ as in (1) of Proposition 2.1.25.

In particular, if $\mu$ is the scalar extension of $u_i$ which is defined in Example 2.1.10, then

$$\mu(-1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As $\mu(-1) \circ \bar{\theta}(g) \circ \mu(-1)^{-1} = \bar{g}$, the corresponding real form $H$ of $(\mathrm{PGL}_2)_{\mathbb{C}}$ is $(\mathrm{PGL}_2)_{\mathbb{R}}$. Also, it is clear that $u = u_i$.

Let $G_{\mathbb{C}}$ be a simple algebraic group. We choose a maximal torus $T$, and let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ (resp. $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$) be the character (resp. cocharacter) group. Note that there is a natural pairing $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$ between $X^*(T)$ and $X_*(T)$ (see Page 335 of [24]). Choose a set of simple roots $(\alpha_i)_{i \in I}$. The nodes of the Dynkin diagram of $(G_{\mathbb{C}}, T)$ are also indexed by $I$. Recall that highest root is the unique root $\tilde{\alpha} = \sum_{i \in I} n_i \alpha_i$ such that, for any other root $\sum_{i \in I} m_i \alpha_i$, $n_i \geq m_i$. We say that an root $\alpha_i$ (or the corresponding node) is *special* if $n_i = 1$ in the expression of $\tilde{\alpha}$.

**Theorem 2.1.27.** *The isomorphism classes of irreducible Hermitian symmetric domains are classified by the special nodes on connected Dynkin diagrams.*

*Proof.* Notations as above. By Theorem 2.1.21 and Proposition 2.1.25, it suffices to construct a bijection between the conjugacy classes of $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$

satisfying (2) of Lemma 2.1.24, and special nodes of the Dynkin diagram of $G_{\mathbb{C}}(\mathbb{C})$. Since all maximal tori are conjugate, we can assume that $\mu$ is in the cocharacter group $X_*(T) \subset X_*(G_{\mathbb{C}})$ of $T$. Moreover, there is a unique representative $\mu$ such that $\langle \alpha_i, \mu \rangle \geq 0$ for all $i \in I$ because the Weyl group acts transitively and freely on the Weyl chambers. Now (2) of Lemma 2.1.24 is equivalent to $\langle \alpha, \mu \rangle \in \{-1, 0, 1\}$ for all roots $\alpha$. Since $\mu$ is non-trivial, not all values can be 0, so there must be a (unique) simple root $\alpha_i$ such that $\langle \alpha_i, \mu \rangle = 1$, which is in fact a special root (otherwise $\langle \tilde{\alpha}, \mu \rangle > 1$). The other direction easily follows from the fact that $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ is a perfect pairing. $\square$

The special roots of connected Dynkin diagrams are listed in the following table.

| Type | $\widetilde{\alpha}$ | Special root |
|---|---|---|
| $A_n$ | $\alpha_1 + \cdots + \alpha_n$ | $\alpha_1, \cdots, \alpha_n$ |
| $B_n$ | $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$ | $\alpha_1$ |
| $C_n$ | $2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$ | $\alpha_n$ |
| $D_n$ | $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ | $\alpha_1,\ \alpha_{n-1},\ \alpha_n$ |
| $E_6$ | $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ | $\alpha_1$,$\alpha_6$ |
| $E_7$ | $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ | $\alpha_7$ |
| $E_8, F_4, G_2$ | | none |

Table 2.1: Special roots of connected Dynkin diagrams.

## 2.2 Hermitian symmetric domains and Hodge structures

Let $\mathcal{D}$ be a Hermitian symmetric domain. In this section, we describe how to use the associated pair $(G, u)$ (c.f. Proposition 2.1.7 and Theorem 2.1.8) to construct variations of Hodge structure over $\mathcal{D}$. See [4] Chapters I, II, III and [23] Chapter 2 for the background of Hodge structures and variations of Hodge structures.

Given a pair $(G, u)$ as in Theorem 2.1.20, we have seen in Theorem 2.1.21 that the $G(\mathbb{R})^+$-conjugacy class of $u$ has a natural structure of a Hermitian symmetric domain. Here we would like to consider a slightly more general situation. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be the Deligne torus. We consider the following pairs $(\mathbf{G}, h)$ where $\mathbf{G}$ is a reductive (e.g. Page 16 of [24]) algebraic group over $\mathbb{R}$ and $h : \mathbb{S} \to \mathbf{G}$ is an algebraic homomorphism. We denote by $X$ the conjugacy class of $h$ by elements of $\mathbf{G}(\mathbb{R})$ (not $\mathbf{G}(\mathbb{R})^+$). Note that one can produce such a pair from a Hermitian symmetric domain $\mathcal{D}$. Specifically, we set $\mathbf{G}$ to be the adjoint algebraic group $G$ in Proposition 2.1.7 and define $h$ by $h(z) = u(z/\bar{z})$ with $u = u_p$ (c.f. Theorem 2.1.8) for some $p \in \mathcal{D}$ (See also Example 2.1.10). In this case, $\mathcal{D}$ will be a connected component of $X$, the $G(\mathbb{R})$-conjugacy class of $h$. (c.f. Proposition 4.9 of [23].)

Let $Z_h$ be the centralizer of $h$ in $\mathbf{G}(\mathbb{R})$. Then the orbit map identifies $X$ with $\mathbf{G}(\mathbb{R})/Z_h$. We view $X$ as a homogenous manifold via this identification.

For any real representation $\rho : \mathbf{G} \to \mathrm{GL}(V)$ and any $h' \in X$, the composition $\rho \circ h' : \mathbb{S} \to \mathbf{G} \to \mathrm{GL}(V)$ defines a real Hodge structure on $V$ (e.g. Page 26 of [23]). In other words,

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} V_{h'}^{p,q},$$

where $V_{h'}^{p,q} = \{v \in V_{\mathbb{C}} \mid (\rho(h'(z)))(v) = z^{-p}\bar{z}^{-q} \cdot v,\ \forall z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*\}$. In particular, over $\mathbb{R}$ we have the weight space decomposition:

$$V = \bigoplus_{n \in \mathbb{Z}} V_{n,h'}, \quad V_{n,h'} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} V_{h'}^{p,q}$$

(i.e. $v \in V_{n,h'}$ if and only if $(\rho(h'(r)))(v) = r^n \cdot v$ for all $r \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^*$, here $\mathbb{G}_m$ is mapped into $\mathbb{S}$ via $\mathbb{G}_m \xrightarrow{w} \mathbb{S}$, $r \mapsto r^{-1}$).

*Remark* 2.2.1. For a Hodge structure $\varphi : \mathbb{S} \to \mathrm{GL}(V)$, the standard convention in the theory of Shimura variety is $\varphi_{\mathbb{C}}(z_1, z_2)(v^{p,q}) = z_1^{-p} z_2^{-q} \cdot v^{p,q}$ (c.f. (1.1.1.1) of [6]). Meanwhile, a different convention $\varphi_{\mathbb{C}}(z_1, z_2)(v^{p,q}) = z_1^{p} z_2^{q} \cdot v^{p,q}$ is largely used in Hodge theory (e.g. Page 31 of [4]). We shall use different conventions in different contexts.

**Lemma 2.2.2.** *The following statements are equivalent.*

*(1) For all representations $(V, \rho)$ of $\mathbf{G}$, the weight space decomposition of $V$ induced by $h' \in X$ is independent of the $h'$.*

*(2) For any $h' \in X$, the real Hodge structure on $\mathrm{Lie}(\mathbf{G})$ defined by $\mathrm{Ad} \circ h'$ is pure of weight $0$.*

*Proof.* See 1.1.13($\alpha$) of [6]. See also Lemma 5.1 of [28]. □

Assume $X$ satisfies one of the properties in the previous lemma, then for any representation $(V, \rho)$ the weight spaces $V_{n,h'}$ are independent of $h' \in X$, and so we have a trivial vector bundle $X \times (V_n)_{\mathbb{C}} \to X$ for every weight $n$. Furthermore, the Hodge filtration $F^{\bullet}_{n,h'}$ on $(V_n)_{\mathbb{C}}$ induced by $\rho \circ h'$ defines (as $h'$ varies) a filtration on $X \times (V_n)_{\mathbb{C}}$ by subbundles $\mathcal{F}^{\bullet}_n$.

We want to put a complex structure on $X$ such that (1) $\mathcal{F}_n^p$ will be a holomorphic subbundle ($0 \le p \le n$); (2) $\mathcal{F}_n^p$'s satisfy Griffiths transversality for the natural connection on $X \times (V_n)_{\mathbb{C}}$. To do this, we need the following axiom. Recall that a type of a Hodge structure $V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ is the set of $(p,q)$ such that $V^{p,q}$ is non-empty.

(**Axiom I**) The Hodge structure on $\mathrm{Lie}(\mathbf{G})$ given by $\mathrm{Ad} \circ h'$ for any $h' \in X$ is of type $\{(-1,1),(0,0),(1,-1)\}$.

Note that pairs $(\mathbf{G}, h)$ coming from Hermitian symmetric domains clearly satisfy Axiom I (c.f. (a) of Theorem 2.1.20). Also, if Axiom I is satisfied, then Lemma 2.2.2 (2) automatically holds.

Assuming Axiom I, we can endow $X$ with a complex structure as follows (compare to Theorem 2.1.21). Let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. Also fix $h' \in X$ and denote the Hodge structure on $\mathfrak{g}$ induced by $h' \in X$ by $\{\mathfrak{g}_{\mathbb{C}}^{p,q}\}$. Then there is a natural isomorphism $T_{h'}X \cong \mathfrak{g}/\mathfrak{g}^{0,0}$, where $\mathfrak{g}^{0,0}$ is the real descent of $\mathfrak{g}_{\mathbb{C}}^{0,0}$. Because $T_{h'}X = \mathfrak{g}/\mathfrak{g}^{0,0} \subset \mathfrak{g}_{\mathbb{C}}^{1,-1} \oplus \mathfrak{g}_{\mathbb{C}}^{-1,1}$, $\mathrm{Ad}(h'(i))$ acts on $T_{h'}X$ as multiplication by $-1$. Define $J_{h'} = \mathrm{Ad}(h'(\mathrm{e}^{\frac{\pi \mathrm{i}}{4}}))$. Since $J_{h'}^2 = -\mathrm{id}$, this defines a complex structure on $T_{h'}X$. Moving $J_{h'}$ around using the homogeneity of $X$, we obtain an almost complex structure on $X$.

**Theorem 2.2.3.** *Let $\mathbf{G}$ be a reductive group over $\mathbb{R}$ and let $X$ be the $\mathbf{G}(\mathbb{R})$-conjugacy class of an algebraic homomorphism $h : \mathbb{S} \to \mathbf{G}$. If $(\mathbf{G}, X)$ satisfies Axiom I, then the almost complex structure defined by $\{J_{h'}\}$ is integrable.*

*For any representation $V$ of $\mathbf{G}$ and any integers $n$ and $p$, $\mathcal{F}_n^p$ is a holomorphic vector bundle on $X$ with respect to this complex structure. Moreover, $\mathcal{F}_n^\bullet$ satisfies Griffiths transversality for the connection $\nabla = 1 \otimes \mathrm{d} : (V_n)_{\mathbb{C}} \otimes \mathcal{O}_X \to (V_n)_{\mathbb{C}} \otimes \Omega_X^1$. (In other words, $(X \times (V_n)_{\mathbb{R}}, \nabla, \mathcal{F}_n^\bullet)$ forms a real variation of Hodge structure of weight $n$ over $X$. )*

*Proof.* See Proposition 1.1.14 of [6] and Proposition 5.9 of [23]. See also Proposition 5.3 of [28] and Theorem 3.7 of [29]. □

*Remark* 2.2.4. Axiom I is one of Deligne's axioms in the definition of a Shimura datum, for which we refer the readers to Page 53 of [23]. Also, see Page 44 and 50 of op. cit. for the definition of a connected Shimura datum. For the connections between (connected) Shimura data and Hermitian symmetric domains, see Proposition 4.8, Proposition 5.7 and Corollary 5.8 of op. cit..

## 2.3 Locally symmetric domains

### 2.3.1 Quotients of Hermitian symmetric domains

We start by defining some special (discrete) subgroups for an algebraic group or a Lie group. Let $G$ be an algebraic group over $\mathbb{Q}$. For an injective homomorphism $r : G \to \mathrm{GL}_n$, we let

$$G(\mathbb{Z})_r = \{g \in G(\mathbb{Q}) \mid r(g) \in \mathrm{GL}_n(\mathbb{Z})\}.$$

Note that $G(\mathbb{Z})_r$ is independent of $r$ up to commensurability (c.f. Corollary 7.13 of [30]), so $r$ can sometimes be omitted from the notation. A subgroup $\Gamma$ of $G(\mathbb{Q})$ is *arithmetic* if it is commensurable with $G(\mathbb{Z})_r$ for some $r$. (In other words, $\Gamma \cap G(\mathbb{Z})_r$ has finite index in both $\Gamma$ and $G(\mathbb{Z})_r$.) Note that every arithmetic subgroup $\Gamma$ contains a torsion free subgroup of $G(\mathbb{Q})$ of finite index (c.f. Proposition 17.4 of [30]).

As an example, let us consider

$$\Gamma(N) := r(G(\mathbb{Q})) \cap \{A \in \mathrm{GL}_n(\mathbb{Z}) \mid A \equiv I \bmod N\},$$

and define a congruence subgroup of $G(\mathbb{Q})$ to be any subgroup containing $\Gamma(N)$ as a subgroup of finite index. Although $\Gamma(N)$ depends on the choice of the embedding $r$, congruence subgroups do not. Every congruence subgroup is an arithmetic subgroup.

Recall that a *lattice* of a Lie group is a discrete subgroup of finite covolume with respect to an equivariant measure. Consider a connected adjoint Lie group $H$ with no compact factors (e.g. $\mathrm{Hol}(\mathcal{D})^+$ for a Hermitian symmetric domain $\mathcal{D}$), and Let $\Gamma$ be a subgroup of $H$. If there exists a simply connected (c.f. Page 199 of [24]) algebraic group $G$ over $\mathbb{Q}$ and a surjective homomorphism $\varphi : G(\mathbb{R}) \to H$ with compact kernel such that $\Gamma$ is commensurable with $\varphi(G(\mathbb{Z}))$, then we also say that $\Gamma$ is an arithmetic subgroup of $H$. In fact, such a subgroup is a lattice of $H$ (c.f. Page 484 of [27]), and so $\Gamma$ is an *arithmetic lattice*.

We now discuss the quotient of a Hermitian symmetric domain $\mathcal{D}$ by a certain discrete subgroup $\Gamma$ of $\mathrm{Hol}(\mathcal{D})^+$ (e.g. a lattice or an arithmetic lattice). If $\Gamma$ is torsion free, then it acts freely on $\mathcal{D}$, and there is a unique complex structure on $\Gamma\backslash\mathcal{D}$ such that the natural quotient map $\mathcal{D} \to \Gamma\backslash\mathcal{D}$ is holomorphic. In this case, $\mathcal{D}$ is also the universal covering space of $\Gamma\backslash\mathcal{D}$ with $\Gamma$ the group of Deck transformations; the choice of a point of $\mathcal{D}$ determines an isomorphism of $\Gamma$ with the fundamental group of $\Gamma\backslash\mathcal{D}$. Moreover, it is easy to see that for each $p \in \Gamma\backslash\mathcal{D}$, there is an involution $s_p$ defined in a neighborhood of $p$ having $p$ as an isolated point. (In other words, $\Gamma\backslash\mathcal{D}$ is "locally symmetric".)

Note that a discrete group $\Gamma$ of $\mathrm{Hol}(\mathcal{D})^+$ is a lattice (i.e. $\Gamma\backslash\mathrm{Hol}(\mathcal{D})^+$ has finite volume) if and only if $\Gamma\backslash\mathcal{D}$ has finite volume.

Let $H$ be a connected semisimple Lie group with finite center. We say that a lattice $\Gamma$ in $H$ is irreducible if $\Gamma\cdot N$ is dense in $H$ for every noncompact closed normal subgroup $N$ of $H$. If we further assume that $H$ has trivial center and no compact factor, then any lattice in $H$ can be decomposes into irreducible lattices as in Theorem 3.1 of [27]. In particular, one can decompose locally symmetric domains as follows.

**Theorem 2.3.1.** *Let $\mathcal{D}$ be a Hermitian symmetric domain with $H = \mathrm{Hol}(\mathcal{D})^+$. Let $\Gamma$ be a lattice in $H$. Then $\mathcal{D}$ can be written uniquely as a product $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_r$ of Hermitian symmetric domains such that $\Gamma_i := \Gamma \cap \mathrm{Hol}(\mathcal{D}_i)^+$ is an irreducible lattice in $\mathrm{Hol}(\mathcal{D}_i)^+$ and $\Gamma_1\backslash\mathcal{D}_1 \times \cdots \times \Gamma_r\backslash\mathcal{D}_r$ is a finite covering of $\Gamma\backslash\mathcal{D}$.*

*Proof.* See Theorem 3.2 of [27]. □

Recall that a connected semisimple algebraic group can be written as an almost direct product of its almost simple subgroups (called almost simple factors) (c.f. Theorem 17.16 of [24]). We say a simply connected or adjoint algebraic group $G$ over $\mathbb{Q}$ is of compact (reps. noncompact) type if $G_i(\mathbb{R})$ is compact (resp. noncompact) for every almost simple factor $G_i$ of $G$ (see also Definition 3.7 of [27]).

**Theorem 2.3.2.** *Let $\mathcal{D}$ be a Hermitian symmetric domain with $H = \mathrm{Hol}(\mathcal{D})^+$. Let $\Gamma$ be a lattice in $H$. If $\mathrm{rank}(\mathrm{Hol}(\mathcal{D}_i)^+) \geq 2$ in Theorem 2.3.1, then there exists a simply connected algebraic group $G$ of noncompact type over $\mathbb{Q}$ and a surjective homomorphism $\varphi : G(\mathbb{R}) \to H$ with compact kernel such that $\Gamma$ is commensurable with $\varphi(G(\mathbb{Z}))$. (In particular, $\Gamma$ is an arithmetic lattice of $H$.) Moreover, such a pair $(G, \varphi)$ is unique up to a unique isomorphism.*

*Proof.* See Theorem 3.13 of [27]. □

A few remarks on the algebraic structure of locally symmetric domains. Recall that there is a functor $X \mapsto X^{\mathrm{an}}$ associating to a smooth complex algebraic variety $X$ a complex manifold $X^{\mathrm{an}}$. This functor is faithful, but far from surjective on objects or on arrows. However, if we restrict the functor to closed subvarieties of the projective spaces $\mathbb{P}^n_{\mathbb{C}}$, then it produces an equivalence of categories between smooth projective complex varieties and closed submanifolds of $(\mathbb{P}^n_{\mathbb{C}})^{\mathrm{an}}$ (Chow's theorem). By the Baily-Borel theorem, every quotient $\Gamma\backslash\mathcal{D}$ of a Hermitian symmetric domain $\mathcal{D}$ by a torsion free arithmetic subgroup $\Gamma$ of $\mathrm{Hol}(\mathcal{D})^+$ can be realized (canonically) as a Zariski open subvariety of a projective variety, and hence has a canonical structure of an algebraic variety. Now by a *locally symmetric variety* we mean a smooth complex algebraic variety $X$ such that $X^{\mathrm{an}}$ is isomorphic to $\Gamma\backslash\mathcal{D}$ for a Hermitian symmetric domain $\mathcal{D}$ and a torsion free subgroup $\Gamma \subset \mathrm{Hol}(\mathcal{D})^+$ (see also Footnote 15 on Page 488 of [27]).

To obtain an interesting arithmetic theory, one needs to put further restrictions on a locally symmetric variety $X$. When $X^{\mathrm{an}} \cong \Gamma\backslash\mathcal{D}$ for an arithmetic subgroup $\Gamma$ of $\mathrm{Hol}(\mathcal{D})^+$, we call $X$ an *arithmetic locally symmetric variety*. The group $\Gamma$ is usually a lattice, so by Margulis arithmeticity theorem nonarithmetic locally symmetric varieties can only occur in very few cases. For an arithmetic locally symmetric variety $X$ with $X^{\mathrm{an}} \cong \Gamma\backslash\mathcal{D}$, we let $(G, \varphi)$ be the pair associated to $\Gamma\backslash\mathcal{D}$ as in Theorem 2.3.2. If there exists a congruence subgroup $\Gamma_0$ of $G(\mathbb{Z})$ such that $\Gamma$ contains $\varphi(\Gamma_0)$ as a subgroup of finite index, then $X$ will have very rich arithmetic structures; such arithmetic locally symmetric varieties are called *connected Shimura varieties*. We refer the readers to Chapters 4 and 5 of [23] for the more formal definitions of connected Shimura varieties and Shimura varieties.

### 2.3.2 Variations of Hodge structure on locally symmetric domains

We shall describe variations of Hodge structure over locally symmetric domains following [6] and [27] (especially [27] Chapter 8). In what follows, we shall always let $\mathcal{D}$ be a Hermitian symmetric domain and let $\Gamma$ be an torsion free arithmetic lattice of $\mathrm{Hol}(\mathcal{D})^+$, and use $\mathcal{D}(\Gamma)$ to denote the arithmetic locally symmetric variety.

According to Theorem 2.3.1, $\mathcal{D}$ decomposes uniquely into a product $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_r$ such that $\Gamma_i = \Gamma \cap \mathrm{Hol}(\mathcal{D}_i)^+$ is an irreducible lattice of $\mathrm{Hol}(\mathcal{D}_i)^+$ and the map $\mathcal{D}(\Gamma_1) \times \cdots \times \mathcal{D}(\Gamma_r) \to \mathcal{D}(\Gamma)$ is a finite covering. We further assume that

$$\mathrm{rank}(\mathrm{Hol}(\mathcal{D}_i)^+) \geq 2 \tag{2.3.3}$$

for each $1 \leq i \leq r$. According to Margulis arithmeticity theorem, there exists a pair $(G, \varphi)$ where $G$ is a simply connected $\mathbb{Q}$-algebraic group and $\varphi : G(\mathbb{R}) \to \mathrm{Hol}(\mathcal{D})^+$ is a surjective homomorphism with compact kernel such that $\varphi(G(\mathbb{Z}))$ is commensurable with $\Gamma$; moreover, such a pair is unique up to a unique homomorphism. (c.f. Theorem 2.3.2.)

We also fix a point $o \in \mathcal{D}$. By Theorem 2.1.8, there exists a unique homomorphism $u : U_1 \to \mathrm{Hol}(\mathcal{D})^+$ such that $u(z)$ fixes $o$ and acts on $T_o\mathcal{D}$ as multiplication by $z$.

Let

$$G^{\mathrm{ad}}_{\mathbb{R}} = G_{\mathrm{c}} \times G_{\mathrm{nc}},$$

where $G^{\mathrm{ad}}_{\mathbb{R}}$ is the quotient of $G_{\mathbb{R}}$ by its center and $G_{\mathrm{c}}$ (resp. $G_{\mathrm{nc}}$) is the product of the compact (resp. noncompact) simple factors of $G^{\mathrm{ad}}_{\mathbb{R}}$. The homomorphism $\varphi : G(\mathbb{R}) \to \mathrm{Hol}(\mathcal{D})^+$ factors through $G_{\mathrm{nc}}$ and defines an isomorphism of Lie groups $G_{\mathrm{nc}}(\mathbb{R})^+ \to \mathrm{Hol}(\mathcal{D})^+$. Now we define $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ by

$$\bar{h}(z) = (h_{\mathrm{c}}(z), h_{\mathrm{nc}}(z)) \in G_{\mathrm{c}}(\mathbb{R}) \times G_{\mathrm{nc}}(\mathbb{R}), \tag{2.3.4}$$

where $h_{\mathrm{c}}(z) = 1$ and $h_{\mathrm{nc}}(z) = u(z/\bar{z})$ in $G_{\mathrm{nc}}(\mathbb{R})^+ \cong \mathrm{Hol}(\mathcal{D})^+$. Note that $\mathbb{G}_m$ can be embedded into $\mathbb{S}$ via the exact sequence

$$0 \to \mathbb{G}_m \xrightarrow{w} \mathbb{S} \to U_1 \to 0,$$

which is defined on the real valued points by $r \overset{w}{\mapsto} r^{-1}$and $z \mapsto z/\bar{z}$ respectively. It is clear that $\bar{h}$ factors through $\mathbb{S}/\mathbb{G}_m$. Moreover, the $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugates of $\bar{h}$ can be identified with $\mathcal{D}$ through $g\bar{h}g^{-1} \mapsto g \cdot o$.

**Proposition 2.3.5.** *Notations as above. The pair $(G, \bar{h})$ associated to the arithmetic locally symmetric domain $\mathcal{D}(\Gamma)$ and a point $o \in \mathcal{D}$ satisfies the following properties.*

*(1) The Hodge structure on $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})$ defined by $\mathbb{S} \xrightarrow{\bar{h}} G^{\mathrm{ad}}_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}}))$ is of type $\{(1,-1), (0,0), (-1,1)\}$;*

*(2) The conjugation by $\bar{h}(i)$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$.*

*Proof.* By definition, $h_{\mathrm{nc}}(z) = u(z/\bar{z})$ under the identification $G_{\mathrm{nc}}(\mathbb{R})^+ \cong \mathrm{Hol}(\mathcal{D})^+$. Because $G^{\mathrm{ad}}_{\mathbb{R}}$ has trivial center, $\bar{h}$ satisfies (1) and (2) if and only if $u$ satisfies $(a)$ and $(b)$ of Theorem 2.1.20. ☐

Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ and let $h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ be a homomorphism. To state the main results of this subsection, we need to define the weight homomorphism

$$w_h := h \circ w$$

where $w : \mathbb{G}_m \to \mathbb{S}$ is given as above by $r \mapsto r^{-1}$ (note that to give a Hodge structure on a $\mathbb{Q}$-vector space $V$ amounts to giving a homomorphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that $w_h$ is defined over $\mathbb{Q}$), and consider the following condition on $h$.

(**Axiom II***) The conjugation by $h(i)$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}/w_h(\mathbb{G}_m)$.

Axiom II* can be motivated from the following fact. Let $V$ be a faithful representation of $\mathbf{G}$, if $w_h$ is defined over $\mathbb{Q}$, then the homomorphism $h : \mathbb{S} \to$

$\mathbf{G}_{\mathbb{R}}$ defines a rational Hodge structure on $V$; assume that $\mathbf{G}$ is the Mumford-Tate group of $V$, then $V$ is polarizable if and only if $(\mathbf{G}, h)$ satisfies Axiom II*. (c.f. 1.1.18($a$) of [6] and Proposition 6.4 of [27]. Roughly speaking, a Cartan involution produces a bilinear form invariant under the group action, but the Mumford-Tate group $\mathbf{G}$ only preserves a polarization up to scalar, so we consider a Cartan involution on the quotient of the Mumford-Tate group by $w_h(\mathbb{G}_m)$.)

A Hodge structure is said of CM type if it is polarizable and its Mumford-Tate group is a torus. Also, by a variation of integral Hodge structure we mean a variation of rational Hodge structure that admits an integral structure (i.e. the local system of $\mathbb{Q}$-vector spaces comes from a local system of $\mathbb{Z}$-modules). Finally, we denote by $\mathbf{G}^{\mathrm{der}}$ the derived subgroup (c.f. Page 187 of [24]) of $\mathbf{G}$.

**Theorem 2.3.6** (Summary 8.6 of [27])**.** *Let $\mathcal{D}(\Gamma)$ be an arithmetic locally symmetric domain satisfying (2.3.3). Let $G$ be the simply connected $\mathbb{Q}$-algebraic group associated to $\mathcal{D}(\Gamma)$ as in Theorem 2.3.2. Choose a point $o \in \mathcal{D}$ and define $\bar{h}$ as in (2.3.4). To give*

*a polarizable variation of integral Hodge structure on $\mathcal{D}(\Gamma)$ such that some fiber is of CM type and the monodromy representation has finite kernel*

*is the same as giving*

*a triple $(\mathbf{G}, h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}, \rho : G \to \mathrm{GL}(V))$, where $V$ is a $\mathbb{Q}$-representation of $G$ and $\mathbf{G} \subset \mathrm{GL}(V)$ is a reductive algebraic group defined over $\mathbb{Q}$, such that*

*(1) The homomorphism $h$ satisfies Axiom II* and $w_h$ is defined over $\mathbb{Q}$;*

*(2) The representation $\rho$ factors through $\mathbf{G}$ and $\rho(G) = \mathbf{G}^{\mathrm{der}}$;*

*(3) The composition $\mathrm{Ad} \circ h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \to \mathbf{G}^{\mathrm{ad}}_{\mathbb{R}} \cong G^{\mathrm{ad}}_{\mathbb{R}}$ is equal to $\bar{h}$.*

*Proof.* See Pages $510 - 511$ of [27]. $\square$

*Remark* 2.3.7. The reductive group $\mathbf{G}$ should be thought of as the generic Mumford-Tate group of a polarizable variation of Hodge structure on $\mathcal{D}(\Gamma)$. Also, we need to assume the variation of Hodge structure is polarizable and integral so that $\rho(G) = \mathbf{G}^{\mathrm{der}}$ (c.f. Theorem 6.22 of [27]).

For every arithmetic locally symmetric variety, there exists a triple $(\mathbf{G}, h, \rho)$ satisfying the conditions in Theorem 2.3.6, and hence there is a polarizable variation of integral Hodge structure on the variety. See Pages $512 - 514$ of [27] for details.

## 2.4 Hermitian variations of Hodge structure of abelian variety type

In this section, we show how to construct a family of abelian varieties (equivalently, polarizable variations of integral Hodge structure of Hodge level 1) on an arithmetic locally symmetric variety $\mathcal{D}(\Gamma)$ following Chapter 10 and 11 of [27]. For simplicity, we assume that $\mathcal{D}$ is irreducible. Also, we assume that $\mathrm{rank}(\mathrm{Hol}(\mathcal{D})^+) \geq 2$ as in (2.3.3).

According to Theorem 2.3.2, there is a unique simply connected $\mathbb{Q}$-algebraic group $G$ of non-compact type and a surjective homomorphism $\varphi : G(\mathbb{R}) \to \mathrm{Hol}(\mathcal{D})^+$ with compact kernel such that $\varphi(G(\mathbb{Z}))$ is commensurable with $\Gamma$. Note that $\varphi$ factors through $G^{\mathrm{ad}}_{\mathbb{R}}(\mathbb{R})$ and induces an isomorphism of Lie groups $G^{\mathrm{ad}}_{\mathbb{R}}(\mathbb{R})^+ \to \mathrm{Hol}(\mathcal{D})^+$. Fix a point $o \in \mathcal{D}$ and let $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ be as defined in (2.3.4), then $\varphi(\bar{h}(z))$ fixes $o \in \mathcal{D}$ and acts on $T_o\mathcal{D}$ as multiplication by $z/\bar{z}$.

By Theorem 2.3.6, variations of Hodge structure on $\mathcal{D}(\Gamma)$ corresponds to certain representations of $G$. We now define "symplectic representations" and show that they corresponds to families of abelian varieties on $\mathcal{D}(\Gamma)$.

Let $V$ be a rational vector space and $\psi$ be a nondegenerate alternating form on $V$. Denote by $\mathrm{GSp}(V, \psi)$ the group of symplectic similitudes (the algebraic subgroup of $\mathrm{GL}(V)$ whose elements preserves $\psi$ up to scalar). The derived subgroup of $\mathrm{GSp}(V, \psi)$ is the symplectic group $\mathrm{Sp}(V, \psi)$. Also, let $\mathcal{D}(\psi)$ be the set of Hodge structures which are of type $\{(-1, 0), (0, -1)\}$ and are polarized by $2\pi i\psi$.

**Definition 2.4.1.** A homomorphism $G \to \mathrm{GL}(V)$ with finite kernel is a *symplectic representation* of $(G, \bar{h} : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}})$ if there exists a pair $(\mathbf{G}, h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}})$ consisting of a reductive $\mathbb{Q}$-algebraic group $\mathbf{G}$ and a homomorphism $h$, a nondegenerate alternating form $\psi$ on $V$, and a factorization of $G \to \mathrm{GL}(V)$ through $\mathbf{G}$:

$$G \xrightarrow{\phi} \mathbf{G} \xrightarrow{\xi} \mathrm{GL}(V)$$

such that

(1) $\xi \circ h \in \mathcal{D}(\psi)$;

(2) $\phi(G) = \mathbf{G}^{\mathrm{der}}$ and $\xi(\mathbf{G}) \subset \mathrm{GSp}(V, \psi)$;

(3) The composition $\mathrm{Ad} \circ h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \to \mathbf{G}^{\mathrm{ad}}_{\mathbb{R}} \cong G^{\mathrm{ad}}_{\mathbb{R}}$ is equal to $\bar{h}$.

Recall that a family on a connected complex manifold is said to be faithful if the monodromy representation is injective.

**Theorem 2.4.2.** *Let $\mathcal{D}(\Gamma)$ be an arithmetic locally symmetric variety with $\mathcal{D}$ irreducible (for simplicity only) and $\mathrm{rank}(\mathrm{Hol}(\mathcal{D})^+) \geq 2$, and let $(G, \bar{h})$ be the pair associated to $\mathcal{D}(\Gamma)$ and a point $o \in \mathcal{D}$ as above. There exists a faithful family of abelian varieties on $\mathcal{D}(\Gamma)$ having a fiber of CM type if and only if $(G, \bar{h})$ admits a symplectic representation.*

*Proof.* See Theorem 11.8 of [27]. □

In the rest of this subsection we study symplectic representations. Because $\mathcal{D}$ is irreducible, there is no harm to study symplectic representations over $\mathbb{R}$. Let $\rho : G \to \mathrm{GL}(V)$ be a symplectic representation of $(G, \bar{h})$. After scalar extension to $\mathbb{R}$, we assume that $G$ is an almost simple and simply connected $\mathbb{R}$-algebraic group without compact factors and view $V$ as a real representation of $G$. If $V$ is irreducible, then $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{R}$ (Shur's Lemma), and so there are three possibilities:

$$\mathrm{End}_G(V) = \begin{cases} \mathbb{R} & \text{(real type)}, \\ \mathbb{C} & \text{(complex type)}, \\ \mathbb{H} & \text{(quaternionic type)}. \end{cases}$$

Accordingly, we have (here $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$)

$$V_{\mathbb{C}} = \begin{cases} V_+ & \text{(real type)}, \\ V_+ \oplus V_-, V_+ \ncong V_- & \text{(complex type)}, \\ V_+ \oplus V_-, V_+ \cong V_- & \text{(quaternionic type)}, \end{cases}$$

where $V_\pm$ are irreducible complex $G(\mathbb{C})$-representations and $V_+^\vee \cong V_-$. In practice, one can use Theorem (IV.E.4) of [4] to distinguish these cases.

We now classify the irreducible real symplectic representations of the pairs $(G, \bar{h})$. Define $\bar{\mu} : \mathbb{G}_m \to G^{\mathrm{ad}}_{\mathbb{C}}$ by $\bar{\mu}(z) = \bar{h}_{\mathbb{C}}(z, 1)$, where $\bar{h}_{\mathbb{C}} : \mathbb{G}_m \times \mathbb{G}_m \to G^{\mathrm{ad}}_{\mathbb{C}}$ is the complexification of $\bar{h}$. Let $u : U_1 \to \mathrm{Hol}(\mathcal{D})^+ \cong G^{\mathrm{ad}}(\mathbb{R})^+$ be the homomorphism associated to the point $o \in \mathcal{D}$ as in Theorem 2.1.8. Because $\bar{h}_{\mathbb{C}}(z_1, z_2) = u_{\mathbb{C}}(z_1/z_2)$ as in (2.3.4), the homomorphism $\bar{\mu}$ is the scalar extension of $u$: $\bar{\mu}(z) = \bar{h}_{\mathbb{C}}(z, 1) = u_{\mathbb{C}}(z)$.

Fix a maximal torus $T$ of $G^{\mathrm{ad}}_{\mathbb{C}}$, and let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ (resp. $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$) be the character (resp. cocharacter) group. There is a natural pairing $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$ between $X^*(T)$ and $X_*(T)$. Let $R \subset X^*(T)$ (resp. $R^\vee \subset X_*(T)$) be the corresponding root system (resp. coroot system). We also denote by $Q(R)$ the lattice generated by $R$. (In this case $Q(R) = X^*(T)$, but we will not use this.)

Recall that the lattice of weights is $P(R) = \{\varpi \in X^*(T)_{\mathbb{Q}} \mid \langle \varpi, \alpha^\vee \rangle \in \mathbb{Z} \text{ all } \alpha^\vee \in R^\vee\}$. Choose a set $B = \{\alpha_1, \cdots, \alpha_n\}$ of simple roots such that $\langle \alpha, \bar{\mu} \rangle \geq 0$ for all $\alpha \in B$, then the fundamental weights are the dual basis $\{\varpi_1 \cdots, \varpi_n\}$ of $\{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$, and the dominant weights are the elements $\sum n_i \varpi_i$ with $n_i \in \mathbb{N}$.

Also, there is a unique permutation $\tau$ of simple roots (or the corresponding Dynkin diagram or the fundamental weights) such that $\tau^2 = \mathrm{id}$ and the map $\alpha \mapsto -\tau(\alpha)$ extends to the action of the Weyl group. Usually $\tau$ is called the opposition involution. Explicitly, $\tau$ acts nontrivially on the root systems of type $A_n$ ($\alpha_i \leftrightarrow \alpha_{n+1-i}$), $D_n$ with $n$ odd ($\alpha_{n-1} \leftrightarrow \alpha_n$) and $E_6$ ($\alpha_1 \leftrightarrow \alpha_6$), and trivially on the other root systems.

**Theorem 2.4.3.** *Notations as above. Let $V$ be an irreducible real representation of $G$, and $\varpi$ be the highest weight of an irreducible summand $W$ (e.g. $V_+$ or $V_-$) of $V_{\mathbb{C}}$. The representation $V$ is a symplectic representation of $(G, \bar{h})$ if and only if*

$$\langle \varpi + \tau(\varpi), \bar{\mu} \rangle = 1. \tag{2.4.4}$$

*Proof.* (Sketch) (Step 1) By Lemma 1.3.3 of [6] or Proposition 10.4 of [27], a representation $\rho : G \to \mathrm{GL}(V)$ is a symplectic representation if there exist a pair $(\mathbf{G}, h : \mathbb{S} \to \mathbf{G})$ and a factorization $\rho = \xi \circ \phi$ of $\rho$ as in definitionrefsymplectic rep, such that (1) $\xi \circ h$ is of type $\{(-1, 0), (0, -1)\}$; (2) $\phi(G) = \mathbf{G}^{\mathrm{der}}$; and (3) $\mathrm{Ad} \circ h = \bar{h}$. In other words, the nondegenerate alternating form $\psi$ is not needed in the first place.

(Step 2) Consider the projective system $(T_n, T_{nd} \to T_n)$, where the index set is $\mathbb{N} - \{0\}$ (ordered by divisibility), $T_n = \mathbb{G}_m$, and $T_{nd} \to T_n$ is given by $z \mapsto z^d$. Denote by $\tilde{\mathbb{G}}_m$ its inverse limit.

By 1.3.4 of [6], $\tilde{\mathbb{G}}_m$ is the algebraic universal covering of $\mathbb{G}_m$, so we can lift $\bar{\mu} : \mathbb{G}_m \to G_{\mathbb{C}}^{\mathrm{ad}}$ to $\tilde{\mu} : \tilde{\mathbb{G}}_m \to G_{\mathbb{C}}$. Then $W \subset V_{\mathbb{C}}$ is a representation of $\tilde{\mathbb{G}}_m$. According to (10.2) of [27], such a representation $\tilde{\mathbb{G}}_m \to \mathrm{GL}(W)$ can be represented by a homomorphism $f : T_n \to \mathrm{GL}(W)$ and defines a gradation $W = \oplus W_r$ ($r \in (1/n)\mathbb{Z}$) with $f(z)$ acting on $W_r$ by multiplication by $z^{nr}$. We call the $r$ for which $W_r \neq 0$ the weights of the representation of $\tilde{\mathbb{G}}_m$ on $W$. One can check that the weights do not depend on the representative $f$.

The most important observation here is as follows: the nontrivial irreducible representation $W$ occurs in a symplectic representation if and only if $\tilde{\mu}$ has exactly two weights $a$ and $a + 1$ on W. (c.f. Lemma 1.3.5 of [6])

We show the "only if" direction here. For $h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$, we define $\mu_h : \mathbb{G}_m \to \mathbf{G}_{\mathbb{C}}$ by $\mu_h(z) = h_{\mathbb{C}}(z, 1)$ as in 1.1.1 and 1.1.11 of [6]. Because $\mathrm{Ad} \circ h = \bar{h}$ as in definitionrefsymplectic rep (3), we have $\phi_{\mathbb{C}} \circ \tilde{\mu} = \mu_h \cdot \nu$ with $\nu$ in the center of $\mathbf{G}_{\mathbb{C}}$. On $W$, $\mu_h$ has weights 0 and 1 (see (1) of definitionrefsymplectic rep). If

$a$ is the unique weight of $\nu$ on $W$, then the only weights of $\tilde{\mu}$ on $W$ is $a$ and $a+1$. We need the observation in Step 1 for the other direction, see Lemma 10.6 of [27].

(Step 3) Note that the differential of $\tilde{\mu}$ equals the differential of $\bar{\mu}$. The conclusion in Step 2 can be rephrased as follows: if $\varpi$ is the highest weight of $W$, then the representation $W$ occurs in a symplectic representation if and only if $\langle \varpi + \tau(\varpi), \bar{\mu} \rangle = 1$. This is (1.3.6.1) of [6]. In fact, the lowest weight of $W$ is $-\tau(\varpi)$, and the weights $\beta$ of $W$ are of the form $\varpi$ + (a $\mathbb{Z}$-linear combination of roots $\alpha \in R$). Because $\langle \alpha, \bar{\mu} \rangle \in \mathbb{Z}$ for all roots $\alpha$, $\langle \beta, \bar{\mu} \rangle$ takes values $a$ and $a+1$ if and only if $\langle -\tau(\varpi), \bar{\mu} \rangle = \langle \varpi, \bar{\mu} \rangle - 1$, which is clearly equivalent to (2.4.4). □

To apply (2.4.4), we make the following two observations. Because $\varpi + \tau(\varpi) \in Q(R)$, $\langle \varpi + \tau(\varpi), \bar{\mu} \rangle \in \mathbb{Z}$ for every dominant weight $\varpi$. Moreover, $\langle \varpi + \tau(\varpi), \bar{\mu} \rangle > 0$. So only the fundamental weights $\{\varpi_1, \cdots, \varpi_n\}$ can satisfy (2.4.4) (Lemma 1.3.7 of [6]).

Also, by the proof of Theorem 2.1.27 there exists a special node $\alpha_s$ (determined by the irreducible Hermitian symmetric domain $\mathcal{D}$) such that, for simple roots $\alpha \in B = \{\alpha_1, \cdots, \alpha_n\}$,

$$\langle \alpha, \bar{\mu} \rangle = \begin{cases} 1 & \text{if } \alpha = \alpha_s, \\ 0 & \text{if } \alpha \neq \alpha_s. \end{cases}$$

Express a weight $\varpi$ as a $\mathbb{Q}$-linear combination of the simple roots $\{\alpha_i\}$ (c.f. [31]), then $\langle \varpi + \tau(\varpi), \bar{\mu} \rangle = 1$ if and only if the coefficient of $\alpha_s$ in $\varpi + \tau(\varpi)$ equals 1.

**Example 2.4.5.** (Type $A_{n-1}$) In this case,

$$\varpi_i = \frac{n-i}{n}\alpha_1 + \frac{2(n-i)}{n}\alpha_2 + \cdots + \frac{i(n-i)}{n}\alpha_i + \frac{i(n-i-1)}{n}\alpha_{i+1} + \cdots + \frac{i}{n}\alpha_{n-1},$$

for $1 \leq i \leq n-1$. The opposition involution $\tau$ switches the nodes $i$ and $n-i$: $\tau(\varpi_i) = \varpi_{n-i}$, and so

$$\tau(\varpi_i) = \frac{i}{n}\alpha_1 + \frac{2i}{n}\alpha_2 + \cdots + \frac{(n-i)i}{n}\alpha_{n-i} + \frac{(n-i)(i-1)}{n}\alpha_{n+1-i} + \cdots + \frac{n-i}{n}\alpha_{n-1}.$$

If $i \leq n-i$, one can easily compute the coefficient of a simple root $\alpha_j$ in

$\varpi_i + \tau(\varpi_i)$:

$$\text{the coefficient of } \alpha_j \text{ in } \varpi_i + \tau(\varpi_i) = \begin{cases} j & \text{if } 1 \leq j \leq i, \\ i & \text{if } i \leq j \leq n-i, \\ n-j & \text{if } n-i \leq j \leq n-1. \end{cases}$$

The special root $\alpha_s$ can be any $\alpha_j$ with $1 \leq j \leq n-1$. We drop the cases that $\alpha_s = \alpha_1$ and $\alpha_s = \alpha_{n-1}$ so that the assumption (2.3.3) holds. Choose a special root $\alpha_s = \alpha_j$ for $2 \leq j \leq n-2$. It is easy to see that for the coefficient to be 1, $\varpi_i$ must be $\varpi_1$. Similarly, if $i > n-i$, only the fundamental weight $\varpi_{n-1}$ satisfies (2.4.4).

**Example 2.4.6.** (Type $E_6$ and $E_7$) In the $E_6$ case, the special root $\alpha_s = \alpha_1$ or $\alpha_6$, and the opposite involution switches $\alpha_1$ and $\alpha_6$. We seek a fundamental weight $\varpi$ such that $\varpi = a\alpha_1 + \cdots + b\alpha_6$ with $a + b = 1$. But there is no such a fundamental weight for the root system $E_6$, and hence there is no corresponding symplectic representation.

Similarly, there is no symplectic representation associated to the Hermitian symmetric domains of type $E_7$. In fact, $\alpha_s = \alpha_7$ and the opposite involution is trivial in the $E_7$ case. Therefore, a fundamental weight $\varpi$ satisfies (2.4.4) if and only if $\omega = \cdots + \frac{1}{2}\alpha_7$, but no fundamental of $E_7$ is of this form.

If a fundamental weight satisfies (2.4.4), then we call the corresponding node a *symplectic node*. They are listed as follows.

| Type | Symplectic node |
|---|---|
| $(A_n, \alpha_1)$ | $\varpi_1, \cdots, \varpi_n$ |
| $(A_n, \alpha_i)$, $1 < i < n$ | $\varpi_1$, $\varpi_n$ |
| $(B_n, \alpha_1)$, $n \geq 2$ | $\varpi_n$ |
| $(C_n, \alpha_n)$ | $\varpi_1$ |
| $(D_n, \alpha_1)$, $n \geq 4$ | $\varpi_{n-1}$, $\varpi_n$ |
| $(D_4, \alpha_4)$ | $\varpi_1$, $\varpi_3$ |
| $(D_n, \alpha_n)$, $n \geq 5$ | $\varpi_1$ |
| $(E_6, \alpha_1)$ | none |
| $(E_7, \alpha_7)$ | none |

Table 2.2: List of symplectic nodes.

## 2.5 Hermitian variations of Hodge structure of Calabi-Yau (CY) type

In this subsection, we consider Hodge structures of Calabi-Yau type.

**Definition 2.5.1.** A rational (or real) Hodge structure $V$ of Calabi-Yau (CY) type is an effective rational (or real) Hodge structure [1] of weight $n$ such that $V^{n,0}$ is 1-dimensional. If $n = 2$, we also say that $V$ is *of K3 type*.

Based on earlier work of Gross ([1]) and Sheng-Zuo ([2]), Friedman and Laza classified $\mathbb{R}$-variations of Hodge structures (or the $\mathbb{Q}$-variations of Hodge structure which remain irreducible over $\mathbb{R}$) of CY type over irreducible Hermitian symmetric domains in [3] and [32]. In this subsection, let us discuss Friedman-Laza's classification for irreducible Hermitian symmetric domains of tube type. All the irreducible tube domains are tabulated as follows (the first column is standard Siegel's notation; the second column lists the corresponding Dynkin diagrams and special roots (c.f. Theorem 2.1.27); the third column gives the real simply connected algebraic groups for the unique simple adjoint algebraic groups (c.f. Proposition 2.1.7) associated to Hermitian symmetric domains; the fourth column lists the corresponding maximal compact subgroup; the last column gives the real ranks of tube domains).

| Label | $(R, \alpha_s)$ | $G(\mathbb{R})$ | $K$ | $\mathbb{R}$-rank |
|---|---|---|---|---|
| $\mathrm{I}_{n,n}$ | $(A_{2n-1}, \alpha_n)$ | $\mathrm{SU}(n,n)$ | $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))$ | $n$ |
| $\mathrm{II}_{2n}$ | $(D_{2n}, \alpha_{2n})$ | $\mathrm{Spin}^*(2n)$ | $U_1 \times_{\mu_n} \mathrm{SU}(2n)$ | $n$ |
| $\mathrm{III}_n$ | $(C_n, \alpha_n)$ | $\mathrm{Sp}(2n, \mathbb{R})$ | $\mathrm{U}(n)$ | $n$ |
| $\mathrm{IV}_{2n-1}$ | $(B_n, \alpha_1)$ | $\mathrm{Spin}(2, 2n-1)$ | $\mathrm{Spin}(2) \times_{\mu_2} \mathrm{Spin}(2n-1)$ | 2 |
| $\mathrm{IV}_{2n-2}$ | $(D_n, \alpha_1)$ | $\mathrm{Spin}(2, 2n-2)$ | $\mathrm{Spin}(2) \times_{\mu_2} \mathrm{Spin}(2n-2)$ | 2 |
| EVII | $(E_7, \alpha_7)$ | $\mathrm{E}_{7,3}$ | $\mathrm{U}(1) \times_{\mu_3} \mathrm{E}_6$ | 3 |

Table 2.3: Hermitian symmetric domains of tube type

*Remark* 2.5.2. Following [6], we shall sometimes use $D_n^{\mathbb{R}}$ to denote the domain $(D_n, \alpha_1)$, and use $D_{2n}^{\mathbb{H}}$ to denote the domain $(D_{2n}, \alpha_{2n})$.

Let $\mathcal{D}$ be an irreducible Hermitian symmetric domain, and let $\mathbf{D}$ be a classifying space of polarized rational Hodge structures with fixed Hodge numbers. Following Definition 2.1 of [3], we call the variations of Hodge structure induced by an equivariant, holomorphic and horizontal embedding of $\mathcal{D} \hookrightarrow \mathbf{D}$ a *Hermitian variation of Hodge structure*. They are the variations of Hodge structure

[1] In this subsection, we switch to the other convention: if $\varphi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ defines a Hodge structure of weight $n$, then $h(z)$ $(z \in \mathbb{S}(\mathbb{R}))$ acts on $V^{p,q}$ as multiplication by $z^p \bar{z}^q$.

parameterized by Hermitian symmetric domains considered by Deligne [6]. In the terminology of [4], $\mathcal{D} \subset \mathbf{D}$ is an unconstrained Mumford-Tate domain (and hence also a Hermitian symmetric domain).

By Proposition 2.1.7, there is a unique simple adjoint real algebraic group associated to $\mathcal{D}$; we denote by $G$ its algebraic universal covering (N.B. these simply connected algebraic groups are listed in Table 2.3, see also Section 1 of [1]). Choose a reference point $o \in \mathcal{D}$. According to Theorem 2.1.8, there is a homomorphism $u : U_1 \to G^{\mathrm{ad}}$. We define $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}$ by $\bar{h}(z) = u(z/\bar{z})$ as in (2.3.4).

Choosing a suitable arithmetic subgroup of $\mathrm{Hol}(\mathcal{D})^+$, we assume that there is an algebraic group $G_{\mathbb{Q}}$ is defined over $\mathbb{Q}$ with $G_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong G$. To give a Hermitian rational variation of Hodge structure over $\mathcal{D}$, one must give a $\mathbb{Q}$-representation $\rho_{\mathbb{Q}} : G_{\mathbb{Q}} \to \mathrm{GL}(V)$ satisfying the conditions in Theorem 2.3.6. Following Section 2.1 of [3], we assume that the induced real representation $\rho : G \to \mathrm{GL}(V_{\mathbb{R}})$ is irreducible. As variations of real Hodge structure are mainly concerned in this section, we shall focus on the representation $\rho$.

The question is which irreducible representations of $G$ correspond to Hermitian variations of Hodge structure of CY type over $\mathcal{D}$. Suppose $\mathcal{D}$ corresponds to the root system $R$ and the special root $\alpha_i$. We call the corresponding fundamental weight $\varpi_i$ (i.e. $\varpi_i(\alpha_j^\vee) = \delta_{ij}$) a *cominuscule weight*, and call the irreducible representation $V_{\varpi_i}$ of $G(\mathbb{C})$ with highest weight $\varpi_i$ a *cominuscule representation*.

Let $V_{\mathbb{R}}$ be an irreducible $G$-representation. Recall that $V_{\mathbb{R}}$ may be of real type, complex type or quaternionic type. Specifically, $V_{\mathbb{C}}$ may be irreducible (real type) or reducible (complex type or quaternionic type); if $V_{\mathbb{C}}$ is reducible, then we can write $V_{\mathbb{C}} = V_+ \oplus V_-$, where $V_+$ and $V_-$ are irreducible representations of $G(\mathbb{C})$ and $V_+^\vee \cong V_-$. We distinguish the complex case from the quaternionic case depending on whether $V_+ \cong V_-$ (quaternionic type) or not (complex type). We now show that if $V_{\mathbb{R}}$ induces a CY Hermitian variation of Hodge structure over $\mathcal{D}$, then the highest weight of $V_+$ or $V_-$ ($V_+ = V_{\mathbb{C}}$ in the real case) must be a multiple of the corresponding cominuscule weight.

In what follows, let us focus on tube domains $\mathcal{D}$.

**Lemma 2.5.3.** *Let $\mathcal{D}$ be an irreducible Hermitian symmetric domain of tube type, and let $G$ be defined as above. Suppose $\mathcal{D}$ corresponds to $(R, \alpha_i)$ (so $\varpi_i$ is the corresponding cominuscule weight), and let $V_{n\varpi_i}$ be the irreducible representation of $G(\mathbb{C})$ with highest weight $n\varpi_i$ ($n \in \mathbb{N}^+$). Then there exists a real $G$-representation $V_{\mathbb{R}}$ such that $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V_{n\varpi_i}$.*

*Proof.* The condition that $\mathcal{D}$ is of tube type is equivalent to that $\tau(\alpha_i) = \alpha_i$ where $\tau$ is the opposition involution. Let $V_{\varpi_i}$ be a cominuscule representation.

Because the dual representation has highest weight $\tau(\varpi_i)$, we have $V_{\varpi_i} \cong V_{\varpi_i}^\vee$. Now one can verify the reality of the representation using Theorem IV.E.4 of [4]. The same argument works for $V_{n\varpi_i}$. $\square$

**Example 2.5.4.** Let $\mathcal{D}$ be an tube domain corresponding to $(A_5, \alpha_3)$. Then $G = \mathrm{SU}(3,3)$ and the cominuscule weight is $\varpi_3$. Because $\tau(\varpi_3) = \varpi_3$, the cominuscule representation can not be of complex type. We now determine whether it is of real type or quaternionic type using Theorem IV.E.4 of [4].

In the root system $A_5$, we have

$$2\varpi_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5.$$

(Denote the coefficients of $\alpha_i$ by $m_i$.) The only noncompact root in this case is $\alpha_3$ (c.f. Page 335 of [33]). Because

$$\sum_{\alpha_i \text{ compact}} m_i = 1 + 2 + 2 + 1 = 6$$

is even, the cominuscule representation is of real type.

In the proof of 2.1.27 we see that there is a $\mu \in X_*(G^{\mathrm{ad}}_{\mathbb{C}})$ ($\mu = u_{\mathbb{C}}$) such that

$$\langle \alpha, \mu \rangle = \begin{cases} 1 & \text{if } \alpha = \alpha_i, \\ 0 & \text{if } \alpha \neq \alpha_i, \end{cases}$$

where $\alpha_i$ is the special root associated to the domain $\mathcal{D}$. Following [3], we shall use $H_0$ to denote $\mu$, and use $\varpi(H_0)$ to denote the pairing $\langle \varpi, H_0 \rangle$.

**Proposition 2.5.5.** *Notations as above. Let $\rho : G \to \mathrm{GL}(V_{\mathbb{R}})$ be an irreducible representation and $\lambda$ be the highest weight of an irreducible factor $V_+$ of $V_{\mathbb{C}}$. Possibly replace $V_+$ with $V_-$, we can assume that $\tau\lambda(H_0) \leq \lambda(H_0)$. Then a necessary condition for $\rho$ to arise from a CY Hermitian variation of Hodge structure over $\mathcal{D}$ is*

$$\varpi(H_0) < \lambda(H_0) \text{ for all weights } \varpi \neq \lambda \text{ of } V_+.$$

*Furthermore, this condition implies that $\lambda$ is a multiple of the fundamental cominuscule weight $\varpi_i$ associated to the domain $\mathcal{D}$. In particular, if $\mathcal{D}$ is a tube domain, then such representations $V_{\mathbb{R}}$ are of real type.*

*Proof.* Consider the following commutative diagram:

$$\begin{tikzcd} U_1 \arrow[r, "i"] \arrow[d, "2:1"'] & \mathbb{S} \arrow[r, "h"] \arrow[d, "p"'] & \mathbf{G}_{\mathbb{R}} \arrow[r] \arrow[d, "\mathrm{Ad}"'] & \mathrm{GL}(V_{\mathbb{R}}) \\ U_1 \arrow[r, equal] & U_1 \arrow[r, "u"] & G^{\mathrm{ad}} & \end{tikzcd}$$

where $i : U_1 \hookrightarrow \mathbb{S}$ is the kernel of the norm map $\mathrm{Nm} : \mathbb{S} \to \mathbb{G}_m$ ($\mathrm{Nm}(z) = z\bar{z}$), and $p : \mathbb{S} \to U_1$ is defined by $z \mapsto z/\bar{z}$. (Note also that $\mathbf{G}_{\mathbb{R}}^{\mathrm{ad}} \cong G^{\mathrm{ad}}$.) In the situation considered here, the Hodge decomposition on $V_{\mathbb{C}}$ is the weight decomposition of $V_{\mathbb{C}}$ with respective to $U_1(\mathbb{C}) \cong \mathbb{C}^*$ via $h_{\mathbb{C}} \circ i_{\mathbb{C}}$: $V^{p,q}$ corresponds to the eigenspace for the character $z^{p-q}$. If $V_{\mathbb{R}}$ is of real type, then by the above diagram the weights of $\mathbb{G}_m$ on $V_{\mathbb{C}}$ via $h_{\mathbb{C}} \circ i_{\mathbb{C}}$ are $\{2\varpi(H_0) \mid \varpi \in \chi(V_+)\}$, where $\chi(V_+)$ denotes the weights of the irreducible $G(\mathbb{C})$-representation $V_+$ (note that $V_+ = V_{\mathbb{C}}$ in this case). If $V_{\mathbb{R}}$ is of complex or quaternionic case, then $V_{\mathbb{C}} = V_+ \oplus V_-$, and the weights of $h \circ i$ on $V_{\mathbb{C}}$ are $\{\pm 2(\varpi(H_0) - c) \mid \varpi \in \chi(V_+)\}$, where the constant $c$ comes from the action of the center of $\mathbf{G}_{\mathbb{R}}$ on $V_+$ (c.f. 2.1.2 of [3]).

Since all the other weights of $V_+$ are obtained from $\lambda$ by subtracting positive roots, it follows that

$$\max_{\varpi \in \chi(V_+)} \varpi(H_0) = \lambda(H_0).$$

Using the description of the weights of $h_{\mathbb{C}} \circ i_{\mathbb{C}}$ on $V_{\mathbb{C}}$, we see that the CY condition ($\dim_{\mathbb{C}} V^{n,0} = 1$) implies that the above maximal is attained only for the highest weight $\lambda$. In other words, for other weights $\varpi \neq \lambda$ of $V_+$, $\varpi(H_0) < \lambda(H_0)$.

Let $\alpha_i$ be the special root associated to $\mathcal{D}$. By applying the reflection in another simple root $\alpha_j \neq \alpha_i$, we get

$$s_{\alpha_j}(\lambda)(H_0) = (\lambda - \lambda(\alpha_j^\vee) \cdot \alpha_j)(H_0) = \lambda(H_0) - \lambda(\alpha_j^\vee) \cdot \alpha_j(H_0) = \lambda(H_0).$$

Because $s_{\alpha_j}(\lambda) \in \chi(V_+)$, $s_{\alpha_j}(\lambda) = \lambda$, which is equivalent to $\lambda(\alpha_j^\vee) = 0$. Now we can conclude that $\lambda = n\varpi_i$. The last assertion follows from the previous lemma. □

Let $\mathcal{D}$ be an irreducible Hermitian symmetric domain of tube type, and let $\alpha_i$ (resp. $\varpi_i$) be the corresponding special root (resp. cominuscule weight). As shown in [1] and [3], if $V_{\mathbb{R}}$ is an irreducible representation of $G$ such that $V_{\mathbb{C}}$ has highest weight $n\varpi_i$ ($n \in \mathbb{N}^+$) as a $G(\mathbb{C})$-representation, then $V_{\mathbb{R}}$ induces a CY $\mathbb{R}$-variation of Hodge structures over $\mathcal{D}$. If $V_{\mathbb{C}}$ is the cominuscule representation (i.e. has highest weight $\varpi_i$), then we call the induced $\mathbb{R}$-variation of Hodge structure of CY type the canonical CY $\mathbb{R}$-variation of Hodge structure over $\mathcal{D}$.

**Theorem 2.5.6.** *For every irreducible Hermitian symmetric domain $\mathcal{D}$ of tube type, there exists a canonical $\mathbb{R}$-variation of Hodge structure $\mathcal{V}$ of CY type parameterized by $\mathcal{D}$. Any other irreducible CY $\mathbb{R}$-variation of Hodge structure can be obtained from the canonical one by taking the unique irreducible factor of* $\mathrm{Sym}^{\bullet}\mathcal{V}$ *of CY type.*

*Proof.* See Section 3 of [1] or Theorem 2.22 of [3]. □

*Remark* 2.5.7. For an irreducible tube domain $\mathcal{D}$, the weight of the canonical $\mathbb{R}$-variation of Hodge structure is also equal to the real rank of $\mathcal{D}$ which can be found in Table 2.3.

*Remark* 2.5.8. One may wonder what happens if the irreducible domain $\mathcal{D}$ is not of tube type. Let $V_+$ be a cominuscule representation of $G(\mathbb{C})$. Sheng and Zuo [2] have noted $V_+$ carries a $\mathbb{C}$-Hodge structure of CY type, and so $V_+ \oplus V_+^\vee$ will carry a $\mathbb{R}$-Hodge structure. However, this Hodge structure is typically not of CY type. To fix this, one needs to apply the operation "half twist" defined by van Geemen [34]. See Section 2.1.3 of [3].

# Chapter 3

# On motivic realizations of the canonical variations of Hodge structure of Calabi-Yau type over Hermitian symmetric domains of tube type

Over each irreducible Hermitian symmetric domain, there exists a canonical $\mathbb{R}$-variation of Hodge structure of CY type which can descend to a $\mathbb{Q}$-variation of Hodge structure up to some choices. This chapter concerns motivic realizations of the canonical $\mathbb{R}$-variations of Hodge structure of Calabi-Yau type $\mathcal{V}_{\mathbb{R}}$ over irreducible Hermitian symmetric domains of tube type. Namely, we specify $\mathbb{Q}$-descents $\mathcal{V}_{\mathbb{Q}}$ for $\mathcal{V}_{\mathbb{R}}$ and investigate the possibility of realizing $\mathcal{V}_{\mathbb{Q}}$ as sub-variations of rational Hodge structure of those which are naturally associated with families of abelian varieties.

Up to isomorphism, there are six irreducible Hermitian symmetric domains of tube type: $(A_{2n-1}, \alpha_n)$ $(n \geq 1)$, $(B_n, \alpha_1)$ $(n \geq 2)$, $(C_n, \alpha_n)$ $(n \geq 1)$, $(D_n^{\mathbb{R}}, \alpha_1)$ $(n \geq 3)$, $(D_{2n}^{\mathbb{H}}, \alpha_{2n})$ $(n \geq 2)$ and $(E_7, \alpha_7)$. The canonical CY $\mathbb{R}$-variation of Hodge structure over type $B_n$ and $D_n^{\mathbb{R}}$ tube domains have weight 2, hence are of classical K3 surface type and less interesting to us. The $C_n$ case is also well-known, see [1] Section 9. At the other extreme, Satake and Deligne showed that there is no variation of Hodge structure of abelian variety type over the $(E_7, \alpha_7)$ domain, and hence the canonical CY variation of Hodge structure can not come from variations of Hodge structure of abelian variety type.

We shall focus on the remaining $A_{2n-1}$ case and the $D_{2n}^{\mathbb{H}}$ case. In particular, we will give motivic realizations for the canonical real variations of Hodge structure of CY type over the $(A_{2n-1}, \alpha_n)$ domains and the $(D_4^{\mathbb{H}}, \alpha_4)$ domain. We will also give motivic realizations for the irreducible CY real variations of Hodge structure contained in $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$ ($\mathcal{V}_{\mathbb{R}}$ is the canonical real CY variation) over the $(D_{2n}^{\mathbb{H}}, \alpha_{2n})$ domains ($n \geq 3$).

## 3.1 The $A_{2n-1}$ case

We shall give motivic realizations of the canonical variations of Hodge structure of Calabi-Yau type over the $(A_{2n-1}, \alpha_n)$ domains ($n \geq 1$) in this section. Our starting point is the following theorem in [16] Section 3: let $(X, F, E)$, where $F$ is an imaginary quadratic field and $E$ is a polarization of $X$, be an abelian fourfold of Weil type with discriminant 1, then $H^2(X, \mathbb{Q})$ contains a sub-Hodge structure of K3 type. Since any such $(X, F, E)$ is a member of a family of abelian fourfolds $\pi : \mathcal{X} \to \mathcal{D}$ of Weil type with discriminant 1 parameterized by a Hermitian symmetric domain $\mathcal{D}$ of type $A_3$ (see 5.3-5.11 of [35]), we obtain, over every point $s$ of $\mathcal{D}$, a rational Hodge structure of K3 type which is a sub-Hodge structure of $H^2(\mathcal{X}_s, \mathbb{Q})$. We ask whether this forms a Hermitian variation of rational Hodge structure, and, if so, how to compare it (after a scalar extension to $\mathbb{R}$) with the canonical K3 variation of real Hodge structure on $\mathcal{D}$.

After reviewing the definition of abelian varieties of Weil type, we generalize Lombardo's theorem to abelian $2n$-folds of Weil type with discriminant $(-1)^n$ for any $n \in \mathbb{Z}^+$ (c.f. Theorem 3.1.2). For geometric reasons, we are especially interested in the $n = 3$ case. Using the same methods and some standard techniques, we then show how to construct the $\mathbb{Q}$-descents of the canonical variations of CY type over Hermitian symmetric domains of type $A_{2n-1}$ from certain families of abelian $2n$-folds of Weil type (c.f. Theorem 3.1.13). A generalization to certain arithmetic locally symmetric domains is also discussed.

### 3.1.1 Abelian varieties of Weil type

**Definition 3.1.1.** Let $(X, E)$ be a complex polarized abelian variety of dimension $2n$ and let $F \hookrightarrow \mathrm{End}(X) \otimes \mathbb{Q}$ be an imaginary quadratic field. For convenience, we also use $E$ to denote the alternating form from $H_1(X, \mathbb{Q}) \times H_1(X, \mathbb{Q})$ to $\mathbb{Q}$ induced by the polarization. The abelian variety $X$ is said to be *of Weil type* if for all $k \in F$ the action of $k$ on the tangent space $T_0X$ has $n$ eigenvalues $\sigma(k)$ and $n$ eigenvalues $\overline{\sigma(k)}$ (here we fix an embedding $\sigma : F \hookrightarrow \mathbb{C}$), and $E(k_*x, k_*y) = \sigma(k)\overline{\sigma(k)}E(x, y)$ for $x, y \in H_1(X, \mathbb{Q})$.

**Discriminant**

Let $(X, F, E)$ be an abelian variety of Weil type of dimension $2n$ and $F = \mathbb{Q}(\sqrt{-d})$ $(d \in \mathbb{Z}^+)$. The imaginary quadratic field $F$ acts on $H_1(X, \mathbb{Q})$ and $H^1(X, \mathbb{Q})$ via the maps $k_*$ and $k^*$ $(k \in F)$ respectively. According to Lemma 5.2.2 and 5.2.4 of [35], the map

$$H : H_1(X, \mathbb{Q}) \times H_1(X, \mathbb{Q}) \to F, \quad (x, y) \mapsto E((\sqrt{-d})_* x, y) + \sqrt{-d}\ E(x, y)$$

is a nondegenerate Hermitian form on the $F$-vector space $H_1(X, \mathbb{Q})$ of signature $(n, n)$. (One can also define an $F$-Hermitian form on $H^1(X, \mathbb{Q})$ in the same way using the dual alternating form of $E$, which has the same discriminant and signature as $H$; we will also $H$ to denote this Hermitian form.) Let $\Psi$ be the Hermitian matrix corresponding to $H$ with respect to some $F$-basis of $H_1(X, \mathbb{Q})$. By Lemma 5.2.3 of op. cit.,

$$\det(\Psi) \in \mathbb{Q}^*/\mathrm{Nm}_{F/\mathbb{Q}}(F^*)$$

does not depend on the choice of the $F$-basis, nor the lattice defining $X$. So $\mathrm{discr}(X, F, E) := \det(\Psi)$ (viewed as an element in $\mathbb{Q}^*/\mathrm{Nm}_{F/\mathbb{Q}}(F^*)$) defines an isogeny invariant called the *discriminant* of $(X, F, E)$.

### 3.1.2 Sub-Hodge structures of CY type

Let $F = \mathbb{Q}(\varphi)$ with $\varphi^2 = -d$ $(d \in \mathbb{Z}^+)$. We now prove the following theorem. Note that the $n = 2$ case has been proved in Section 3 of [16], and the case when $n = 3$ and $F = \mathbb{Q}(\sqrt{-3})$ has also been discussed in [17].

**Theorem 3.1.2.** *Let $(X, F, E)$ be a polarized abelian $2n$-fold of Weil type. If*

$$\mathrm{discr}(X, F, E) = (-1)^n,$$

*then $H^n(X, \mathbb{Q})$ contains a rational sub-Hodge structure of CY $n$-fold type.*

To prove the theorem, we need the following two lemmas from linear algebra.

**Wedge product over $F$**

Let $F = \mathbb{Q}(\varphi)$ with $\varphi^2 = -d$ $(d \in \mathbb{Z}^+)$ and $V$ be an $F$-vector space. We shall construct a $\mathbb{Q}$-linear map from $\bigwedge_F^l V$ to $\bigwedge_{\mathbb{Q}}^l V$ $(l \in \mathbb{Z}^+)$.

Let $W = V^* := \mathrm{Hom}_F(V, F)$ be the dual space of $V$, note that $W^* \cong V$. Because $\bigwedge_F^l W^* \cong (\bigwedge_F^l W)^*$, we have the following map

$$\bigwedge_F^l V \xrightarrow{\cong} \bigwedge_F^l \mathrm{Hom}_F(W, F) \xrightarrow{\cong} \mathrm{Hom}_F(\bigwedge_F^l W, F).$$

Let $\mathrm{Tr} : F \to \mathbb{Q}$, $z \mapsto z + \bar{z}$ be the trace map. Then the map

$$\omega : W^* \to W^{*\mathbb{Q}} := \mathrm{Hom}_{\mathbb{Q}}(W, \mathbb{Q}), \quad f \mapsto \mathrm{Tr} \circ f$$

is an isomorphism of $\mathbb{Q}$-vector spaces. Similarly, the map

$$\mathrm{Hom}_F(\bigwedge_F^l W, F) \xrightarrow{\mathrm{Tr}\circ} \mathrm{Hom}_{\mathbb{Q}}(\bigwedge_F^l W, \mathbb{Q})$$

is also a $\mathbb{Q}$-linear isomorphism. As the $\mathbb{Q}$-multilinear map $W \times \cdots \times W \to \bigwedge_F^l W$, $(w_1, \cdots, w_l) \mapsto w_1 \wedge_F \cdots \wedge_F w_l$ is alternating, it must factor through $\bigwedge_{\mathbb{Q}}^l W$, and so we obtain a $\mathbb{Q}$-linear map $\bigwedge_{\mathbb{Q}}^l W \to \bigwedge_F^l W$ which is clearly surjective. Taking the duals (as $\mathbb{Q}$-vector spaces), we get an injection

$$\mathrm{Hom}_{\mathbb{Q}}(\bigwedge_F^l W, \mathbb{Q}) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}(\bigwedge_{\mathbb{Q}}^l W, \mathbb{Q}).$$

The space on the right is $(\bigwedge_{\mathbb{Q}}^l W)^{*\mathbb{Q}} \cong \bigwedge_{\mathbb{Q}}^l (W^{*\mathbb{Q}})$, and is thus isomorphic to $\bigwedge_{\mathbb{Q}}^l W^*$ (via $\omega$). Since $\bigwedge_{\mathbb{Q}}^l W^* \cong \bigwedge_{\mathbb{Q}}^l V$, we have constructed a $\mathbb{Q}$-linear map

$$\bigwedge_F^l V \xrightarrow{\cong} \mathrm{Hom}_F(\bigwedge_F^l W, F) \xrightarrow{\mathrm{Tr}\circ} \mathrm{Hom}_{\mathbb{Q}}(\bigwedge_F^l W, \mathbb{Q}) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}(\bigwedge_{\mathbb{Q}}^l W, \mathbb{Q}) \xrightarrow{\cong} \bigwedge_{\mathbb{Q}}^l V.$$

Because the map is a composition of injective maps, we also have the following lemma.

**Lemma 3.1.3.** *There exists a natural injective $\mathbb{Q}$-linear map $\bigwedge_F^l V \to \bigwedge_{\mathbb{Q}}^l V$.*

### The star operator

Let $F = \mathbb{Q}(\sqrt{-d})$ as before, and let $V$ be an $F$-vector space of dimension $2n$ equipped with a nondegenerate $F$-Hermitian from $H$. In this subsection, we will construct an $F$-conjugate linear endomorphism $\star$ on $\bigwedge_F^n V$.

Fix a generator of $\bigwedge_F^{2n} V$, which gives an isomorphism $\gamma : \bigwedge_F^{2n} V \xrightarrow{\cong} F$.

We define an $F$-linear isomorphism $\rho$ by

$$\rho : \bigwedge_F^n V \to (\bigwedge_F^n V)^*, \quad x \mapsto [y \mapsto \gamma(x \wedge_F y)].$$

On $\bigwedge_F^n V$, we have an induced Hermitian form $\tilde{H}$ given by

$$\tilde{H}(v_1 \wedge_F \cdots \wedge_F v_n, w_1 \wedge_F \cdots \wedge_F w_n) = \det(H(v_i, w_j)).$$

Using this, we define an $F$-conjugate linear bijection

$$\tau : \bigwedge_F^n V \to (\bigwedge_F^n V)^*, \quad x \mapsto [y \mapsto \tilde{H}(y, x)].$$

The $F$-conjugate linear (hence $\mathbb{Q}$-linear) endomorphism $\star$ is defined by

$$\star = \rho^{-1} \circ \tau. \tag{3.1.4}$$

Equivalently, for all $x, y \in \bigwedge_F^n V$, $\tilde{H}(y, x) = \gamma(\star(x) \wedge_F y)$.

Let us now compute $\star\star$. To do that, we choose a basis $\{e_1, \cdots, e_{2n}\}$ of $V$ that diagonalize the Hermitian form $H$, fix $e_1 \wedge_F \cdots \wedge_F e_{2n}$ as a generator of $\bigwedge_F^{2n} V$, and let $\Psi$ be the corresponding Hermitian matrix.

**Lemma 3.1.5.** *Notations as above. We have* $\star\star = (-1)^n \det(\Psi) \cdot \mathrm{id}$.

*Proof.* See Lemma 3.21 of [3]. □

*Remark* 3.1.6. If we multiply the generator of $\bigwedge_F^{2n} V$ by some $c \in E$, we multiply $\star$ by $c$, and hence we replace $\star\star$ by $\mathrm{Nm}_{F/\mathbb{Q}}(c) \star\star$ (because $\star c = \bar{c}\star$).

### The CY 3-folds case

Now let us prove Theorem 3.1.2 for $n = 3$, which will be completed in two steps. We have constructed an injective $\mathbb{Q}$-linear map $\bigwedge_F^3 H^1(X, \mathbb{Q}) \hookrightarrow \bigwedge_{\mathbb{Q}}^3 H^1(X, \mathbb{Q})$ in Lemma 3.1.3 (with $l = 3$ here). Let us denote it by $i$. In the first step, we show that the subspace $i(\bigwedge_F^3 H^1(X, \mathbb{Q}))$ is a sub-Hodge structure of $\bigwedge_{\mathbb{Q}}^3 H^1(X, \mathbb{Q})$.

For that, let us consider the following map

$$\varphi_3^* : \bigwedge_{\mathbb{Q}}^3 H^1(X, \mathbb{Q}) \to \bigwedge_{\mathbb{Q}}^3 H^1(X, \mathbb{Q}),$$

which is defined by

$$v_1 \wedge v_2 \wedge v_3 \mapsto \varphi^* v_1 \wedge \varphi^* v_2 \wedge v_3 + \varphi^* v_1 \wedge v_2 \wedge \varphi^* v_3 + v_1 \wedge \varphi^* v_2 \wedge \varphi^* v_3.$$

**Proposition 3.1.7.** *Notations as above. Then* $Im(i) = \mathrm{Ker}(\varphi_3^* + 3d \cdot \mathrm{id})$.

*Proof.* Recall that the injective map $i$ is a composition of several $\mathbb{Q}$ or $F$-linear maps. Write these maps in terms of $\mathbb{Q}$ and $F$-basis, we have that

$$i(v_1 \wedge_F v_2 \wedge_F v_3) = \frac{1}{4} v_1 \wedge v_2 \wedge v_3 - \frac{1}{4d}(\varphi^* v_1 \wedge \varphi^* v_2 \wedge v_3 + \varphi^* v_1 \wedge v_2 \wedge \varphi^* v_3 + v_1 \wedge \varphi^* v_2 \wedge \varphi^* v_3).$$

It is then easy to verify that $i(\bigwedge_F^3 H^1(X,\mathbb{Q})) \subset \mathrm{Ker}(\varphi_3^* + 3d \cdot \mathrm{id})$. For the equality, it suffices to show that these spaces have the same dimension. As $\dim X = 6$, $\dim_{\mathbb{Q}} H^1(X,\mathbb{Q}) = 12$, and hence $\dim_{\mathbb{Q}}(\bigwedge_F^3 H^1(X,\mathbb{Q})) = 40$.

Let $H^1(X,F) = V_+ \oplus V_-$ be the decomposition of $H^1(X,F) = H^1(X,\mathbb{Q}) \otimes_{\mathbb{Q}} F$ into the subspaces on which $\varphi^*$ acts as multiplication by $\varphi$ and $\bar{\varphi}$ respectively. We have that $\dim_F H^1(X,F) = 12$ and $\dim_F V_+ = \dim_F V_- = 6$ (because $\overline{V_+} = V_-$).

By direct computation, the minimal polynomial of $\varphi_3^*$ is $(T+3d)(T-d) = 0$. So $\bigwedge_{\mathbb{Q}}^3 H^1(X,\mathbb{Q}) = H_+^3 \oplus H_-^3$, where $H_+^3$ (resp. $H_-^3$) is the $+d$ eigenspace (resp. the $-3d$ eigenspace) of $\varphi_3^*$. Note that we have showed that $Im(i) \subset H_-^3$. Now tensor the decomposition by $F$, we have on one hand that

$$(\bigwedge_{\mathbb{Q}}^3 H^1(X,\mathbb{Q})) \otimes_{\mathbb{Q}} F = (H_+^3 \otimes_{\mathbb{Q}} F) \oplus (H_-^3 \otimes_{\mathbb{Q}} F) := H_{+,F}^3 \oplus H_{-,F}^3,$$

and on the other hand that

$$(\bigwedge_{\mathbb{Q}}^3 H^1(X,\mathbb{Q})) \otimes_{\mathbb{Q}} F = \bigwedge_F^3 H^1(X,F) = \bigwedge_F^3 V_+ \oplus (\bigwedge_F^2 V_+ \otimes_F V_-) \oplus (V_+ \otimes \bigwedge_F^2 V_-) \oplus \bigwedge_F^3 V_-.$$

It is easy to show that $\bigwedge_F^3 V_+ \subset H_{-,F}^3$, $(\bigwedge_F^2 V_+ \otimes_F V_-) \subset H_{+,F}^3$, $(V_+ \otimes \bigwedge_F^2 V_-) \subset H_{+,F}^3$ and $\bigwedge_F^3 V_- \subset H_{-,F}^3$, and so

$$H_{+,F}^3 = (\bigwedge_F^2 V_+ \otimes_F V_-) \oplus (V_+ \otimes \bigwedge_F^2 V_-), \quad H_{-,F}^3 = \bigwedge_F^3 V_+ \oplus \bigwedge_F^3 V_-.$$

The proposition then follows from the following dimension counting:

$$\dim_{\mathbb{Q}} \mathrm{Ker}(\varphi_3^* + 3d \cdot \mathrm{id}) = \dim_{\mathbb{Q}} H_-^3 = \dim_F H_{-,F}^3 = 20 + 20 = 40.$$

$\square$

Let $S = i(\bigwedge_F^3 H^1(X,\mathbb{Q})) \subset \bigwedge_{\mathbb{Q}}^3 H^1(X,\mathbb{Q}) = H^3(X,\mathbb{Q})$. Using Proposition 3.1.7, we have $S = H_-^3$, the $-3d$ eigenspace of $\varphi_3^*$. Because $\varphi \in \mathrm{End}(X) \otimes \mathbb{Q}$, the induced map $\varphi^*$ preserves the Hodge structure on $H^1(X,\mathbb{Q})$. It is then not difficult to see that the endomorphism $\varphi_3^*$ of $\bigwedge_{\mathbb{Q}}^3 H^1(X,\mathbb{Q}) = H^3(X,\mathbb{Q})$ is a morphism of Hodge structures, and hence $S$ is a sub-Hodge structure of $H^3(X,\mathbb{Q})$. Let us now compute the Hodge numbers of $S$.

**Lemma 3.1.8.** *The subspace $S$ is a sub-Hodge structure of $H^3(X,\mathbb{Q})$ with* $\dim S^{3,0} = \dim S^{0,3} = 2$.

*Proof.* We have shown that $S \subset H^3(X,\mathbb{Q})$ is a sub-Hodge structure. Let $H^1(X,\mathbb{C}) = V_+ \oplus V_-$ be the decomposition of $H^1(X,\mathbb{C})$ into the $\varphi = i\sqrt{d}$ and the $\bar{\varphi} = -i\sqrt{d}$ eigenspaces of $\varphi^*$. Also let $V_\pm^{1,0} = V_\pm \cap H^{1,0}(X)$, and $V_\pm^{0,1} = V_\pm \cap H^{0,1}(X)$. Because $X$ is of Weil type $(3,3)$, we have that $V_\pm = V_\pm^{1,0} \oplus V_\pm^{0,1}$ and $\dim_{\mathbb{C}} V_\pm^{1,0} = \dim_{\mathbb{C}} V_\pm^{0,1} = 3$ (see for instance the proof of Lemma 5.2.6 of [35]).

By Proposition 3.1.7 we know that $S = H_-^3$. The same argument as there shows that $H_-^3 \otimes_{\mathbb{Q}} \mathbb{C} = \bigwedge_{\mathbb{C}}^3 V_+ \oplus \bigwedge_{\mathbb{C}}^3 V_-$. It follows that

$$\dim_{\mathbb{C}} S^{3,0} = \dim_{\mathbb{C}} \bigwedge_{\mathbb{C}}^3 V_+^{1,0} + \dim_{\mathbb{C}} \bigwedge_{\mathbb{C}}^3 V_-^{1,0} = 2,$$

$$\dim_{\mathbb{C}} S^{0,3} = \dim_{\mathbb{C}} \bigwedge_{\mathbb{C}}^3 V_+^{0,1} + \dim_{\mathbb{C}} \bigwedge_{\mathbb{C}}^3 V_-^{0,1} = 2.$$

$\square$

It is often convenient to identify $\bigwedge_F^3 H^1(X,\mathbb{Q})$ with $S$. In the second step, we will further decompose $S$ or $\bigwedge_F^3 H^1(X,\mathbb{Q})$ using the star operator. To be specific, we first choose an $F$-basis $\{e_1, \cdots, e_6\}$ of $H^1(X,\mathbb{Q})$ in which the matrix of the Hermitian form $H$ (defined in Section 3.1.1) is $\mathrm{diag}(a,1,1,-1,-1,-1)$ with $a \in \mathbb{Q}^+$ (see for example [35] 5.4). The element $e_1 \wedge_F \cdots \wedge_F e_6$ is a generator of $\bigwedge_F^6 H^1(X,\mathbb{Q})$, so it defines an isomorphism

$$\gamma : \bigwedge_F^6 H^1(X,\mathbb{Q}) \xrightarrow{\cong} F.$$

Then we define the endomorphism $\star$ on $\bigwedge_F^3 H^1(X,\mathbb{Q})$ in the same way as in

(3.1.4). Using the notations there (with $V = H^1(X,\mathbb{Q})$ and $n = 3$), we have

$$\tilde{H}(v,w) = \gamma(\star(w) \wedge_F v)$$

for all $v, w \in \bigwedge_F^3 H^1(X,\mathbb{Q})$. Recall also that $\star$ is $F$-conjugate linear and satisfies

$$\star\star = a \cdot \mathrm{id}.$$

**Lemma 3.1.9.** *The $\mathbb{Q}$-linear operator $\star$ is a morphism of rational Hodge structures. In other words, $\star \in \mathrm{End}_{\mathrm{Hod}}(S)$.*

*Proof.* The space $H_1(X,\mathbb{R})$ has the structure of a complex vector space via the identification $H_1(X,\mathbb{R}) \cong T_0X$. Taking the dual, we obtain a complex structure on $H^1(X,\mathbb{R})$ which will be denoted by $J$. Then the representation

$$h : \mathbb{C}^* \to \mathrm{GL}(H^1(X,\mathbb{R})), \quad a + bi \mapsto a\mathrm{id} + bJ$$

defines the Hodge structure on $H^1(X,\mathbb{Q})$. It is not difficult to see that the Hodge structure on $S$ corresponds to the representation

$$h_3 : \mathbb{C}^* \to \mathrm{GL}(S \otimes_{\mathbb{Q}} \mathbb{R}), \quad h_3(z)(v_1 \wedge_{\mathbb{C}} v_2 \wedge_{\mathbb{C}} v_3) = h(z)v_1 \wedge_{\mathbb{C}} h(z)v_2 \wedge_{\mathbb{C}} h(z)v_3,$$

where $v_1, v_2, v_3 \in H^1(X,\mathbb{R})$ (note that $S \otimes_{\mathbb{Q}} \mathbb{R} = \bigwedge_{\mathbb{C}}^3 H^1(X,\mathbb{R})$). Let us denote by $h_6$ the representation of $\mathbb{C}^*$ on $\bigwedge_{\mathbb{C}}^6 H^1(X,\mathbb{R})$ defined in a similar manner. According to [35] Lemma 5.2.6, we have $h_6(z)(v \wedge_{\mathbb{C}} w) = z^3\bar{z}^3(v \wedge_{\mathbb{C}} w) = |z|^6(v \wedge_{\mathbb{C}} w)$ for all $v, w \in S \otimes_{\mathbb{Q}} \mathbb{R}$. Using the properties of the Hermitian form $H$, one can easily verify that $\tilde{H}(h_3(z)v, h_3(z)w) = (z\bar{z})^3\tilde{H}(v,w) = |z|^6\tilde{H}(v,w)$. Putting these observations together, we have for all $z \in \mathbb{C}^*$ and $v, w \in S \otimes_{\mathbb{Q}} \mathbb{R}$ that (we use the same letters to denote the $\mathbb{R}$-extensions of $\gamma$, $\star$, $H$ and $\tilde{H}$ here)

$$\gamma(\star(h_3(z)w) \wedge_{\mathbb{C}} h_3(z)v) = \tilde{H}((h_3(z)v, h_3(z)w) = |z|^6\tilde{H}(v,w),$$

and that

$$\gamma((h_3(z)(\star(w)))\wedge_{\mathbb{C}} h_3(z)v) = \gamma(h_6(z)(\star(w)\wedge_{\mathbb{C}} v)) = |z|^6\gamma(\star(w)\wedge_{\mathbb{C}} v) = |z|^6\tilde{H}(v,w).$$

It follows that $\star \circ h_3(z) = h_3(z) \circ \star$, and hence $\star \in \mathrm{End}_{\mathrm{Hod}}(S)$ as claimed. □

We are now ready to prove the $n = 3$ case for Theorem 3.1.2. Let $(X,F,E)$ be a 6-dimensional polarized abelian variety of Weil type with $\mathrm{discr}(X,F,E) = -1$. We have shown that $H^3(X,\mathbb{Q})$ has a sub-Hodge structure $S$ with $\dim_{\mathbb{C}} S^{3,0} = \dim_{\mathbb{C}} S^{0,3} = 2$ in Lemma 3.1.8. As $\det(H) = -a$, we have $\mathrm{discr}(X,F,E) = -a$, and so $a \equiv 1$ in $\mathbb{Q}^*/\mathrm{Nm}_{F/\mathbb{Q}}(F^*)$. In other words, there exists a nonzero element

$c \in F$ such that $a = \mathrm{Nm}_{F/\mathbb{Q}}(c)$. As explained before, if we multiply the generator $e_1 \wedge_F \cdots \wedge_F e_6$ of $\bigwedge_F^6 H^1(X,\mathbb{Q})$ by $c$, we replace $\star\star$ by $\mathrm{Nm}_{F/\mathbb{Q}}(c)\star\star$, and hence we can assume that $\star\star = \mathrm{id}$ (note that originally we have $\star\star = a \cdot \mathrm{id}$).

By Lemma 3.1.9, we have $S = T_+ \oplus T_-$ with $T_\pm = \mathrm{Ker}(\star \mp \mathrm{id})$ be sub-Hodge structures. Multiplication of $\bigwedge_F^3 H^1(X,\mathbb{Q})$ by $\varphi \in F$ induces an isomorphism of Hodge structures $T_+ \to T_-$ (it is well-defined because $\star\varphi = -\varphi\star$ and the inverse map is multiplication by $\varphi/(-d)$). It is now clear that $T_+ \cong T_-$ is a Hodge structure of CY type.

### The general cases

When $n = 1$, the abelian variety of Weil type $X$ has dimension 2, and so the weight 1 Hodge structure $H^1(X,\mathbb{Q})$ has Hodge numbers $[2,2]$. As in the weight 3 case, we can define a morphism $\star \in \mathrm{End}_{\mathrm{HS}}(H^1(X,\mathbb{Q}))$ of rational Hodge structures such that $\star\star = \mathrm{id}$. Then it is easy to see that the sub-Hodge structure $T_+ = \mathrm{Ker}(\star - \mathrm{id})$ of $H^1(X,\mathbb{Q})$ has Hodge numbers $[1,1]$.

For $n \geq 2$, the strategy of the proof is the same as the CY 3-fold case. One more thing we need is a proper generalization of the map $\varphi_3^*$. For a fixed $n$ and $v_1, \cdots, v_n \in H^1(X,\mathbb{Q})$, we pick up $2l$ $(0 \leq 2l \leq n)$ elements from $\{v_1, \cdots, v_n\}$ and apply the map $\varphi^*$ to them in the expression $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. For instance, if the elements $v_1, v_2, \cdots, v_{2l}$ are chosen, then we get $\varphi^* v_1 \wedge \varphi^* v_2 \wedge \cdots \wedge \varphi^* v_{2l} \wedge v_{2l+1} \wedge \cdots \wedge v_n$. The sum of such $\binom{n}{2l}$ elements of $\bigwedge_{\mathbb{Q}}^n H^1(X,\mathbb{Q})$ will be denoted by $\mathcal{E}_{2l}(v_1 \wedge \cdots \wedge v_n)$. We now define the map $\varphi_n^*$ by

$$\varphi_n^* : \bigwedge_{\mathbb{Q}}^n H^1(X,\mathbb{Q}) \to \bigwedge_{\mathbb{Q}}^n H^1(X,\mathbb{Q}), \quad v_1 \wedge \cdots \wedge v_n \mapsto \mathcal{E}_2(v_1 \wedge \cdots \wedge v_n).$$

Since $\varphi^*$ is a morphism of Hodge structures, the map $\varphi_n^*$ respects the Hodge structure on $\bigwedge_{\mathbb{Q}}^n H^1(X,\mathbb{Q}) = H^n(X,\mathbb{Q})$.

Write down the map $i : \bigwedge_F^n H^1(X,\mathbb{Q}) \hookrightarrow \bigwedge_{\mathbb{Q}}^n H^1(X,\mathbb{Q})$ constructed in Lemma 3.1.3 in terms of $\mathbb{Q}$ and $F$-basis, we obtain

$$i(v_1 \wedge_F \cdots \wedge_F v_n) = \frac{1}{2^{n-1}} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(-d)^l} \, \mathcal{E}_{2l}(v_1 \wedge \cdots \wedge v_n).$$

(One can see from this that the composition of the map $i$ with the natural surjective map $\bigwedge_{\mathbb{Q}}^n H^1(X,\mathbb{Q}) \twoheadrightarrow \bigwedge_F^n H^1(X,\mathbb{Q})$ is the identity map on $\bigwedge_F^n H^1(X,\mathbb{Q})$, which confirms that the map $i$ is injective.) Direct calculation shows that $Im(i) \subset \mathrm{Ker}(\varphi_n^* + \binom{n}{2} d \cdot \mathrm{id})$.

Let $H^1(X,F) = V_+ \oplus V_-$ be the decomposition of $H^1(X,F)$ into the $\varphi$ and

$\bar{\varphi}$ eigenspaces of $\varphi^* : H^1(X, F) \to H^1(X, F)$. As in the proof of Lemma 3.1.7, we have that $\dim_F V_+ = \dim_F V_- = 2n$. Also, we have

$$(\bigwedge_{\mathbb{Q}}^{n} H^1(X, \mathbb{Q})) \otimes_{\mathbb{Q}} F = \bigwedge_{F}^{n} H^1(X, F) = \bigoplus_{r+s=n} (\bigwedge_{F}^{r} V_+ \otimes_F \bigwedge_{F}^{s} V_-).$$

Using this, it is easy to see that $\varphi_n^* \otimes_{\mathbb{Q}} F : \bigwedge_F^n H^1(X, F) \to \bigwedge_F^n H^1(X, F)$ acts on the space $(\bigwedge_F^r V_+ \otimes_F \bigwedge_F^{n-r} V_-) \oplus (\bigwedge_F^{n-r} V_+ \otimes_F \bigwedge_F^r V_-)$ as multiplication by $2r(n-r)d - \binom{n}{2}d$. Therefore, the corresponding minimal polynomial is a product of distinct linear factors which all have rational coefficients. It follows that the linear operator $\varphi_n^*$ has the same minimal polynomial, and thus $\bigwedge_{\mathbb{Q}}^n H^1(X, \mathbb{Q})$ splits into a direct sum of eigenspaces of $\varphi_n^*$. It is then not difficult to show that

$$\dim_{\mathbb{Q}}(\mathrm{Ker}(\varphi_n^* + \binom{n}{2} d \cdot \mathrm{id})) = \dim_F(\bigwedge_{F}^{n} V_+ \oplus \bigwedge_{F}^{n} V_-) = 2\binom{2n}{n}.$$

Since $\dim_{\mathbb{Q}} Im(i) = 2\binom{2n}{n}$, we have proven that $Im(i) = \mathrm{Ker}(\varphi_n^* + \binom{n}{2} d \cdot \mathrm{id})$.

An immediate corollary is that $S := \bigwedge_F^n H^1(X, \mathbb{Q})$ is a sub-Hodge structure of $\bigwedge_{\mathbb{Q}}^n H^1(X, \mathbb{Q})$. The condition that $X$ is of Weil type guarantees that $\dim_{\mathbb{C}} S^{n,0} = \dim_{\mathbb{C}} S^{0,n} = 2$ (See Lemma 3.1.8). The rest of the proof is the same as that for the CY 3-fold case. Specifically, we choose an $F$-basis of $H^1(X, \mathbb{Q})$ which diagonalizes the Hermitian form $H$, which allows us to define an $F$-conjugate linear operator $\star$ as in (3.1.4). Thanks to the assumption that $\mathrm{discr}(X, F, E) = (-1)^n$, we have that $\star\star = \mathrm{id}$. The same argument as in Lemma 3.1.9 shows that $\star \in \mathrm{End}_{\mathrm{HS}}(S)$. Let $T_\pm := \mathrm{Ker}(\star \mp \mathrm{id})$. The sub-Hodge structure $T_+$ and $T_-$ are isomorphic to each other, and so each of them is of CY type.

### The abelian varieties of Weil type coming form algebraic curves

Some abelian varieties of Weil type can be constructed from algebraic curves (more precisely, from unramified Galois coverings of algebraic curves with certain Galois groups). One example is given in [36]. Let $C_{n+1}$ be a smooth projective algebraic curve of genus $n+1$. An element of $\mathrm{Pic}^0(C_{n+1})$ of order four generates a subgroup $G = \mathbb{Z}/4\mathbb{Z}$ and defines a $4:1$ covering of $C_{n+1}$ and an intermediate $2:1$ cover

$$C_{n+1} \leftarrow C_{2n+1} \leftarrow C_{4n+1}.$$

The Prym variety $P_n$ of the double cover $C_{4n+1}/C_{2n+1}$ is a principally polarized abelian variety of dimension $2n$. The covering automorphisms of $C_{4n+1}/C_{2n+1}$ induces an action of $\mathbb{Z}[i]$ on $P_n$. By Theorem 5.3 of op. cit., the abelian variety $(P_n, \mathbb{Q}(\sqrt{-1}))$ (with the natural polarization) is of Weil type $(n,n)$. Another example is discussed in [37] Section 11, where the (generalized) Prym variety $Q$ of an unramified Galois cover of a genus 4 curve with Galois group $\mathbb{Z}/3\mathbb{Z}$ is shown to be of Weil type $(3,3)$ with field $\mathbb{Q}(\sqrt{-3})$ and discriminant $-1$.

Combining these observations with Theorem 3.1.2, we obtain the following corollary.

**Corollary 3.1.10.** *Let $P_n$, $Q$ be as above. Then $H^n(P_n, \mathbb{Q})$ (resp. $H^3(Q, \mathbb{Q})$) contains a rational sub-Hodge structure of CY $n$-fold type (resp. of CY 3-fold type).*

Let us also mention that sometimes the converse statements hold (but we will not need them). All examples we know along this line are listed as follows.

1. A generic abelian 4-fold of Weil type $(X, \mathbb{Q}(\sqrt{-1}))$ with discriminant 1 is isogenous to the Prym variety associated to an unramified Galois cover of a genus 3 curve with Galois group $\mathbb{Z}/4\mathbb{Z}$ (see [36]).

2. A generic abelian 6-fold of Weil type $(X, \mathbb{Q}(\sqrt{-3}))$ with discriminant $-1$ is isogenous to the (generalized) Prym variety of an unramified Galois cover of a genus 4 curve with Galois group $\mathbb{Z}/3\mathbb{Z}$ (see [37]).

3. A generic abelian 6-fold of Weil type $(X, \mathbb{Q}(\sqrt{-1}))$ with discriminant $-1$ is isogenous to the Prym variety associated to an unramified Galois cover of a genus 4 curve with Galois group $\mathbb{Z}/4\mathbb{Z}$ (see [38]).

### 3.1.3 Hermitian CY variations of Hodge structures and abelian varieties of Weil type

#### Hermitian symmetric domains of $A_{2n-1}$

Let $\mathcal{D}$ be an irreducible Hermitian symmetric domain, and denote its automorphism group by $\mathrm{Hol}(\mathcal{D})$. There exists a unique simple real algebraic group $\overline{G}$ such that $\overline{G}(\mathbb{R})^+ = \mathrm{Hol}(\mathcal{D})^+$ (see [27] Page 478), where the superscript $+$ denotes the identity component. Taking the algebraic universal cover of $\overline{G}$, we obtain a simply connected, simple real algebraic group $G$ whose real points acts transitively on $\mathcal{D}$. Let $K$ be a maximal compact subgroup of $G(\mathbb{R})$, then $K$ fixes a unique point of $\mathcal{D}$ and $\mathcal{D} \cong G(\mathbb{R})/K$.

According to 1.2.6 of [6] (see also [27] Page 479), the irreducible Hermitian symmetric domains are classified by pairs $(\Delta, \nu)$, where $\Delta$ is a connected Dynkin diagram and $\nu$ is a special node of $\Delta$.

For the Hermitian symmetric domains $\mathcal{D}$ of type $A_{2n-1}$, the corresponding special node is the $n$-th node $\alpha_n$. In this case, we have that $G(\mathbb{R}) = \mathrm{SU}(n,n)$ and $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))$. Note also that the real rank of $\mathcal{D}$ is $n$ and the complex dimension is $n^2$.

### The canonical variations of real Hodge structures of CY type and cominuscule representations

We now recall the construction of the canonical CY variations of real Hodge structure over the irreducible tube domains $\mathcal{D}$ following Section 3 of [1]. Let $U_1$ be the circle group. Let $\epsilon$ be the unique element of order 2 in the center of $G(\mathbb{R})$ which is contained in the connected component of the center of $K$. Now we define the real reductive group $G_1$ to be the quotient of $\mathbb{G}_m \times G$ by the central subgroup generated by the involution $-1 \times \epsilon$.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be the Deligne torus; then we have $\mathbb{S} = \mathbb{G}_m \times U_1/\langle -1 \times -1\rangle$. A point of $\mathcal{D}$ determines a homomorphism $U_1 \hookrightarrow K \hookrightarrow G(\mathbb{R})$ (c.f. also [23] Proposition 5.7). Since this maps the element $-1$ of $U_1$ to the element $\epsilon$ of $G(\mathbb{R})$, it determines a homomorphism $h : \mathbb{S} \to G_1$ which is the identity on $\mathbb{G}_m$. The $G_1(\mathbb{R})$-conjugacy class $Y$ of $h$ has two connected components, each of which is isomorphic to $\mathcal{D}$.

Suppose that the tube domain $\mathcal{D}$ corresponds to the pair $(\Delta, \nu)$. Let $U_\mathbb{R}$ be a real irreducible representation of $G$ such that $U_\mathbb{C} := U_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ is the irreducible representation of $G(\mathbb{C})$ determined by $\nu$ (i.e. the highest weight of $U_\mathbb{C}$ is the fundamental weight corresponding to the node $\nu$). Such a representation $U_\mathbb{C}$ is called a cominuscule representation. Because $\epsilon$ acts as $(-1)^n$ on $U_\mathbb{R}$, with $n = \mathrm{rank}(\mathcal{D})$, $U_\mathbb{R}$ extends uniquely to a representation of $G_1$ such that $\lambda \in \mathbb{G}_m$ acts as multiplication by $\lambda^n$. According to Proposition 1.1.14 of [6] (see also Theorem 2.2.3), the representation $U_\mathbb{R}$ of $G_1$, when combined with the morphisms $h' \in Y$, give rise to a polarizable variation of real Hodge structure on $Y$ (and hence on $\mathcal{D}$).

By Section 4 of [1] or Theorem 2.22 of [3], the variation of real Hodge structure on $\mathcal{D}$ given by $U_\mathbb{R}$ is pure of weight $n = \mathrm{rank}(\mathcal{D})$ and is of CY type. We call it the *canonical CY variations of real Hodge structures* over $\mathcal{D}$. ( We only need the cases when the domain $\mathcal{D}$ is of tube type, which is equivalent to saying that $U_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ is still irreducible. See [3] for the non-tube domain cases.)

**A family of abelian varieties of Weil type**

Let $F = \mathbb{Q}(\varphi)$ with $\varphi^2 = -d$ ($d \in \mathbb{Z}^+$). Also let $V$ be the vector space $F^{2n}$ and $H$ the $F$-Hermitian form of signature $(n, n)$ on $V$ defined by

$$H(v, w) = v_1\bar{w}_1 + \cdots + v_n\bar{w}_n - v_{n+1}\bar{w}_{n+1} - \cdots - v_{2n}\bar{w}_{2n} \tag{3.1.11}$$

for $v = (v_1, \cdots, v_{2n})$ and $w = (w_1, \cdots, w_{2n})$. Then $V_\mathbb{R} := V \otimes_\mathbb{Q} \mathbb{R}$ is the standard representation of the real algebraic group $G = \mathrm{SU}(V_\mathbb{R}, H_\mathbb{R})$, where $H_\mathbb{R}$ denotes the $\mathbb{R}$-bilinear extension of $H$ (note that $G(\mathbb{R}) = \mathrm{SU}(n, n)$ and $G$ has a natural $\mathbb{Q}$-descent). Let $\mathcal{D}$ be an irreducible Hermitian symmetric domain of type $A_{2n-1}$. Up to isomorphism, $\mathcal{D}$ can be described as

$$\mathcal{D} = \{W \subset V_\mathbb{R} \mid \dim_\mathbb{C} W = n, H_\mathbb{R}|_W \text{ is positive definite}\}.$$

Using the previous notations, the element $\epsilon = -I$ in $G(\mathbb{R})$, and so $V_\mathbb{R}$ extends to a representation of $G_1$ such that $\lambda \in \mathbb{G}_m$ acts by $\lambda$. By Proposition 1.1.14 of [6] (see also [27] Section 2.2), the compositions

$$\mathbb{S} \xrightarrow{h'} G_1 \to \mathrm{GL}(V_\mathbb{R}),$$

when $h'$ runs over all elements of $Y$, define a polarizable rational Hodge structure $\mathcal{V}$ on the local system associated to the rational vector space $V$ over $\mathcal{D}$. Moreover, the variation of Hodge structure $\mathcal{V}$ is pure of weight 1. Applying Riemann's Theorem in families (see for example Theorem 2.2 of [39] and its variants) to $\mathcal{V}(1)$ and taking the dual abelian varieties, we obtain a family of abelian $2n$-folds $\pi : \mathcal{X} \to \mathcal{D}$ such that $\mathcal{V}$ is the natural variation of rational Hodge structure of weight 1 associated to $\pi$ (by taking $R^1\pi_*\mathbb{Q}$, see for example Chapter 10 of [40]).

**Lemma 3.1.12.** *Each member of the family $\pi : \mathcal{X} \to \mathcal{D}$ is an abelian variety of Weil type for the field $F$.*

*Proof.* A point $o \in \mathcal{D}$ determines a homomorphism $u : U_1 \hookrightarrow K \hookrightarrow G(\mathbb{R})$ which extends to $h' : \mathbb{S} \to G_1$. The action of $h'(i)$ defines a complex structure $J_o$, or equivalently a Hodge structure of weight one, on $V_\mathbb{R}$. By the construction, the action of $h'(i)$ on $V_\mathbb{R}$ is the same as the action of $u(i)$, which can be described as follows: if the point $o \in \mathcal{D}$ corresponds to a complex subspace $W \subset V_\mathbb{R}$ on which the Hermitian form $H_\mathbb{R}$ is positive definite, then $u(i)$ acts on $W$ as multiplication by $i$ and acts on $W^\perp$ as multiplication by $-i$. With respect to $J_o$, one can see easily that any element $k \in F$ acts on the $n$-dimensional subspace $W$ (resp. $W^\perp$) as scalar multiplication by $k$ (resp. $\bar{k}$). As in [35] Section 3.4, the complex structure $J_o$ also determines the complex structure on $T_0X$ for the corresponding abelian variety $X$ (note that $X$ is unique up to isogeny). Now that $J_o$ commutes with the action of $F$, we have $F \subset \mathrm{End}(X) \otimes \mathbb{Q}$. Moreover, the $\mathbb{R}$-bilinear extension of $E := \mathrm{Im}(H)$ satisfies the Riemann conditions for $J_o$, and induces the required polarization on $X$. It is now not hard to conclude that $X$ is of Weil type. $\square$

### A sub-variation of Hodge structure of CY type

Notations as in the previous sections. Our goal here is to prove the following theorem.

**Theorem 3.1.13.** *Let $\pi : \mathcal{X} \to \mathcal{D}$ be the family of abelian variety constructed as above. Then the natural variation of rational Hodge structure $\bigwedge_{\mathbb{Q}}^n \mathcal{V}$ of weight $n$ associated to $\pi$ contains a variation of rational Hodge structure $\mathcal{U}$ of CY type. Moreover, the scalar extension of $\mathcal{U}$ from $\mathbb{Q}$ to $\mathbb{R}$ is isomorphic to the canonical variation of real Hodge structure of CY type over $\mathcal{D}$.*

*Proof.* We have shown that $\bigwedge_F^n V$ is a $\mathbb{Q}$-subspace of $\bigwedge_{\mathbb{Q}}^n V$ in Lemma 3.1.3. Let $\{e_1, \cdots, e_{2n}\}$ be a standard basis of $V = F^{2n}$. With respect to the generator $e_1 \wedge_F \cdots \wedge_F e_{2n}$ of $\bigwedge_F^n V$, we define the $F$-conjugate linear operator $\star : \bigwedge_F^n V \to \bigwedge_F^n V$ as in (3.1.4). By Lemma 3.1.5, we have $\star\star = \mathrm{id}$. Let $U := \mathrm{Ker}(\star - \mathrm{id})$ be the $+1$ eigenspace of $\star$. We have $U \subset \bigwedge_F^n V \subset \bigwedge_{\mathbb{Q}}^n V$.

The natural action of $G$ on $V_{\mathbb{R}}$ induces an action on $(\bigwedge_F^n V) \otimes_{\mathbb{Q}} \mathbb{R} = \bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$. Denote also by $\star$ its scalar extension to $\mathbb{R}$. The action of $G$ preserves the corresponding volume form of $\bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$, the pairing $\bigwedge_{\mathbb{C}}^n V_{\mathbb{R}} \times \bigwedge_{\mathbb{C}}^n V_{\mathbb{R}} \to \bigwedge_{\mathbb{C}}^{2n} V_{\mathbb{R}}$, and the Hermitian form $H_{\mathbb{R}}$, and hence commutes with $\star$. Being an eigenspace of $\star$, the subspace $U_{\mathbb{R}} := U \otimes_{\mathbb{Q}} \mathbb{R}$ is still a representation of $G$.

As discussed, we can extend this action of $G$ on $U_{\mathbb{R}}$ to $G_1$ such that $\lambda \in \mathbb{G}_m$ acts by $\lambda^n$. Let $h : \mathbb{S} \to G_1$ be the homomorphism defined as before. The compositions of $G_1 \to \mathrm{GL}(U_{\mathbb{R}})$ with the $G_1(\mathbb{R})$-conjugates of $h$ then induces a variation of rational Hodge structure $\mathcal{U}$ on $\mathcal{D}$. It is well-known that the representation $\bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$ has highest weight $\varpi_n$, and hence it is a cominuscule representation for $\mathcal{D}$. Since there exists a $\mathbb{C}$-conjugate linear operator $\star$ with $\star\star = \mathrm{id}$, the representation $\bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$ is defined over $\mathbb{R}$. In other words, we have $U_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$. This shows that $U_{\mathbb{R}}$ is the minimal cominuscule representation; therefore, the induced variation of Hodge structure $\mathcal{U}$ is of CY type and $\mathcal{U} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the canonical one on $\mathcal{D}$.

We claim that with the natural $G$-actions $\bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$ is a subrepresentation of $\bigwedge_{\mathbb{R}}^n V_{\mathbb{R}}$. Recall that the injective map $\bigwedge_{\mathbb{C}}^n V_{\mathbb{R}} \hookrightarrow \bigwedge_{\mathbb{R}}^n V_{\mathbb{R}}$ is a composition of several injective maps. It is not difficult to verify that each of these injective maps commutes with the action of $G$. Since the $\mathbb{R}$-vector space $U_{\mathbb{R}}$ is a $G$-subrepresentation of $\bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$, it is a $G$-subrepresentation of $\bigwedge_{\mathbb{R}}^n V_{\mathbb{R}}$ as well. It also follows from the construction that the inclusion $U_{\mathbb{R}} \hookrightarrow \bigwedge_{\mathbb{R}}^n V_{\mathbb{R}}$ commutes with the action of $\mathbb{G}_m$, and so $U_{\mathbb{R}} \hookrightarrow \bigwedge_{\mathbb{R}}^n V_{\mathbb{R}}$ is a morphism of $G_1$-representations. Composing both representations with all elements $h' : \mathbb{S} \to G_1$ of $Y$, we obtain an inclusion of variations of Hodge structure. $\square$

### 3.1.4 On a generalization to locally symmetric domains

In this section, we discuss an analogue of Theorem 3.1.13 for certain locally symmetric domains.

Let the field $F$, the vector space $V = F^{2n}$ and the Hermitian form $H$ be as in (3.1.11), also let $\mathcal{D}$ be an irreducible Hermitian symmetric domain of type $A_{2n-1}$ with $n \geq 2$. Consider the $\mathbb{Q}$-algebraic group $G = \mathrm{SU}(V, H)$ whose $R$-valued points ($R$ is a $\mathbb{Q}$-algebra) are given by

$$G(R) = \left\{ A \in \mathrm{GL}_{2n}(F \otimes_{\mathbb{Q}} R) \;\middle|\; \overline{A}^t \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} A = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \right\}$$

(The matrix $\overline{A}$ is obtained by taking the conjugate of every entry of $A$ by $\overline{k \otimes r} := \bar{k} \otimes r$). The algebraic group $G$ is simply connected, and $G(\mathbb{R}) = \mathrm{SU}(n, n)$. Since $\mathrm{Hol}(\mathcal{D})^+ \cong \mathrm{PSU}(n, n)$, there exists a surjective homomorphism $\xi : G(\mathbb{R}) \twoheadrightarrow \mathrm{Hol}(\mathcal{D})^+$. Choose a subgroup $\Gamma$ of $\mathrm{Hol}(\mathcal{D})^+$ which is commensurable with $\xi(G(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathbb{Z}))$. By Proposition 3.6 of [23], we can assume that $\Gamma$ is discrete of finite covolume and torsion free. Now the quotient $\Gamma \backslash \mathcal{D}$ is an arithmetic locally symmetric domain in the sense of [27] (see Page 488).

A choice of a reference point of $o \in \mathcal{D}$ gives a homomorphism $\bar{h} : \mathbb{S} \to G_{\mathbb{R}}^{\mathrm{ad}}$ (see (8.4) of [27], Page 510). By Theorem 2.3.6, to give a polarizable variation of rational Hodge structure (which admits an integral structure) $\mathcal{W}$ on $\Gamma \backslash \mathcal{D}$ such that some fiber is of CM-type and the monodromy representation has finite kernel is the same as giving a representation $\rho : G \to \mathrm{GL}(W)$ of $G$, a reductive algebraic group $M \subset \mathrm{GL}(W)$ defined over $\mathbb{Q}$ (thought of as the generic Mumford-Tate group of $\mathcal{W}$), and a morphism of algebraic groups $h : \mathbb{S} \to M_{\mathbb{R}} \subset \mathrm{GL}(W_{\mathbb{R}})$, such that

(1) The morphism $h$ defines a polarizable rational Hodge structure on $W$;

(2) The representation $\rho$ factors through $M$ and $\rho(G) = M^{\mathrm{der}}$;

(3) The composition $\mathrm{Ad} \circ h : \mathbb{S} \to M_{\mathbb{R}} \to M_{\mathbb{R}}^{\mathrm{ad}} \cong G_{\mathbb{R}}^{\mathrm{ad}}$ is equal to $\bar{h}$.

If only pure Hodge structures are concerned, then it suffices to use the subgroup $\mathrm{Hg} = M \cap \mathrm{SL}(W)$ (thought of as the generic Hodge group of $\mathcal{W}$).

The variations of rational Hodge structures on locally symmetric varieties that give families of abelian varieties are classified in Chapter 10 of [27]. In our case, let us consider the natural representation $\rho : G \to \mathrm{GL}(V)$. Let $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ and $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$. It is easy to see that $V_{\mathbb{R}}$ is the standard representation of $G(\mathbb{R}) = \mathrm{SU}(n,n)$, and so we have $V_{\mathbb{C}} \cong V_+ \oplus V_-$ as representations of $G(\mathbb{C})$, where $V_+$ has highest weight $\varpi_1$ and $V_- \cong V_+^*$ has highest weight $\varpi_{2n-1}$. Because both $\varpi_1$ and $\varpi_{2n-1}$ correspond to symplectic nodes of $\mathcal{D}$ (see Page 528 of op. cit.), the representation $\rho$ is a symplectic representation in the sense of Definition 10.12 of op. cit., and hence gives a family of abelian varieties $\pi : \mathcal{X} \to \Gamma\backslash\mathcal{D}$ (see Theorem 11.8 of op. cit.). It is not difficult to see that $\pi$ is a family of abelian varieties of Weil type for the field $F$.

Let $\mathcal{V}$ be the weight 1 variation of rational Hodge structure on $\Gamma\backslash\mathcal{D}$ given by the representation $\rho$, which can also be obtained by taking $R^1\pi_*\mathbb{Q}$ for the family of abelian varieties $\pi : \mathcal{X} \to \Gamma\backslash\mathcal{D}$. Because each fiber of $\pi$ is of Weil type, the generic Hodge group $\mathrm{Hg}(\mathcal{V})$ is semisimple (see B.63 of [41]). We claim that the generic Hodge group $\mathrm{Hg} := \mathrm{Hg}(\bigwedge_{\mathbb{Q}}^n \mathcal{V})$ of $\bigwedge_{\mathbb{Q}}^n \mathcal{V}$ is also semisimple. As is well known, all quotients of a semisimple algebraic group are semisimple, and so the generic Hodge group of $\mathcal{V}^{\otimes n}$, being a quotient of $\mathrm{Hg}(\mathcal{V})$ (see [39] Remark 1.8), is semisimple. Since the generic fiber of $\bigwedge_{\mathbb{Q}}^n \mathcal{V}$ can be viewed as a sub-Hodge structure of that of $\mathcal{V}^{\otimes n}$, Hg is a quotient of $\mathrm{Hg}(\mathcal{V}^{\otimes n})$ (see for example (I.B.7) of [4]), and hence is semisimple.

As in Lemma 3.1.3, we have an inclusion $i : \bigwedge_F^n V \hookrightarrow \bigwedge_{\mathbb{Q}}^n V$. The action of $G$ on $V$ induces the natural actions on $\bigwedge_{\mathbb{Q}}^n V$ and $\bigwedge_F^n V$, and it can be verified that the map $i$ commutes with these $G$-actions. We also define an $F$-conjugate operator $\star$ on $\bigwedge_F^n V$ such that $\star\star = \mathrm{id}$ (see (3.1.4)). It is easy to see that $\star$ commutes with the $G$-action. Let $U = \mathrm{Ker}(\star - \mathrm{id})$, we then have an inclusion $U \subset \bigwedge_F^n V \subset \bigwedge_{\mathbb{Q}}^n V$ of $G$-representations.

Let $U_{\mathbb{R}} = U \otimes_{\mathbb{Q}} \mathbb{R} \subset \bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$. We then have $U_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \bigwedge_{\mathbb{C}}^n V_{\mathbb{R}}$, which has highest weight $\varpi_n$ and hence is a cominuscule representation. By Theorem 2.22 of [3], the $G$-representation $U$ gives rise to a variation of rational Hodge structure $\mathcal{U}$ of CY type.

Consider the representation $G \to \mathrm{GL}(\bigwedge_{\mathbb{Q}}^n V)$ which corresponds to the variation of Hodge structure $\bigwedge_{\mathbb{Q}}^n \mathcal{V}$. Because $\mathrm{Hg} = \mathrm{Hg}(\bigwedge_{\mathbb{Q}}^n \mathcal{V})$ is semisimple, we have $\mathrm{Hg} = \mathrm{Hg}^{\mathrm{der}}$, and hence Hg equals the image of $G$. It follows that we also have $U \subset \bigwedge_F^n V \subset \bigwedge_{\mathbb{Q}}^n V$ as Hg-representations. Composing this with the $\mathrm{Hg}(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to \mathrm{Hg}_{\mathbb{R}}$, we see that the CY type variation of Hodge structure $\mathcal{U}$ is a sub-variation of Hodge structure of $\bigwedge_{\mathbb{Q}}^n \mathcal{V}$.

## 3.2 The $D^{\mathbb{H}}_{2n}$ case

The goal of this section is to prove the following theorem.

**Theorem 3.2.1.** *Let $\mathcal{D}$ be the Hermitian symmetric domain $(D^{\mathbb{H}}_{2n}, \alpha_{2n})$ which has real rank $n$ ($n \geq 2$).*

*(1) When $n = 2$, there exist two families of abelian varieties $\pi_1 : \mathcal{A}_1 \to \mathcal{D}$ and $\pi_2 : \mathcal{A}_2 \to \mathcal{D}$ such that $R^1\pi_{1*}\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi_{2*}\mathbb{Q}$ contains a Hermitian $\mathbb{Q}$-variation of Hodge structure of $K3$ type $\mathcal{V}$. Moreover, $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the canonical $\mathbb{R}$-variation of Hodge structure of CY type (which has weight 2) over $\mathcal{D}$.*

*(2) When $n \geq 2$, there exists a family of abelian varieties $\pi : \mathcal{A} \to \mathcal{D}$ over $\mathcal{D}$ such that $R^{2n}\pi_*\mathbb{Q}$ contains an irreducible Hermitian $\mathbb{Q}$-variation of Hodge structure $\mathcal{V}'$ of CY type. Moreover, $\mathcal{V}' \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the unique irreducible factor of CY type in $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$, where $\mathcal{V}_{\mathbb{R}}$ is the the canonical $\mathbb{R}$-variation of Hodge structure of CY type over $\mathcal{D}$.*

*Remark* 3.2.2. Over Hermitian symmetric domains of type $D^{\mathbb{H}}_{2n}$, one important reason why the rank 2 case (i.e. $n = 2$) is distinguished from the higher rank cases (i.e. $n \geq 3$) is that there are two different symplectic nodes for the rank 2 case while there is only one for the higher rank cases (c.f. Pages $529 - 530$ of [27]). This fact was also noted and used by Abdulali to solve a quite different problem (c.f. [42]).

Also, in the higher rank cases one has to module out the kernel of $\omega_1$ (viewed as a character) from the simply connected groups of $D^{\mathbb{H}}$ type to obtain faithful symplectic representations (c.f. Page 530 and Theorem 10.21 of [27]). More specifically, we should view these faithful representations as representations of the groups $\mathrm{SO}^*$ (c.f. Remark 1.22 of [43]).

*Remark.* In Part (2) of Theorem 3.2.1, we only realize $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$ (not the canonical CY variation $\mathcal{V}_{\mathbb{R}}$). When the rank of the domain is bigger or equal to 3, this is the best our constructions can do. We shall explain the representation theoretic reasons in Remark 3.2.13.

After reviewing some background material on Hermitian symmetric domains of type $D^{\mathbb{H}}_{2n}$ and the groups $\mathrm{Spin}^*$ and $\mathrm{SO}^*$ in Sections 3.2.1 and 3.2.2, we prove the Theorem 3.2.1 for the rank 2 case and higher rank cases in Section 3.2.3 and Section 3.2.4 respectively. The constructions for these two cases are different, but the ideas of the proof are quite similar (and are also similar with the $A_{2n-1}$ case). Specifically, to give a variation of Hodge structure over a Hermitian symmetric domain it suffices to give a Hodge representation, and one can reduce the construction of a sub-variation of Hodge structure to the construction of a subrepresentation. Another key step is to prove the rationality of certain representations (e.g. half-spin representations) using representation theory of spin groups and the ideas from [32].

### 3.2.1 Hermitian symmetric domains of type $D_{2n}^{\mathbb{H}}$

Let $\mathcal{D} = G(\mathbb{R})/K$ be a Hermitian symmetric domain, where $G$ is the $\mathbb{R}$-algebraic group (almost simple and simply connected) associated to $\mathcal{D}$, and $K$ is a maximal compact subgroup of $G(\mathbb{R})$. Recall that irreducible Hermitian symmetric domains are classified by the root system of $G(\mathbb{C})$ together with one of its special roots. In particular, an irreducible Hermitian symmetric domain of type $D_{2n}^{\mathbb{H}}$ (N.B. it has real rank $n$) corresponds to the pair $(D_{2n}, \alpha_{2n}))$ and the associated simply connected algebraic group is $\mathrm{Spin}^*(4n)$ (c.f. Section 1 of [1]). By choosing a suitable arithmetic subgroup of $\mathrm{Hol}(\mathcal{D})$, we can also assume that $G$ is defined over $\mathbb{Q}$.

Following Deligne (see also Theorem 2.3.6), to give a Hermitian $\mathbb{Q}$-variation of Hodge structure over $\mathcal{D}$ one must give a representation $\rho : G \to GL(V)$ defined over $\mathbb{Q}$ and a compatible polarization $Q$ on $V$, so that $\rho(V) \subset \mathrm{Aut}(V, Q)$. As explained in Step 4 of (IV.A) in [4], a compatible polarization typically exists and is unique. Without loss of generality, one can also assume that $\rho$ is irreducible over $\mathbb{Q}$.

The necessary and sufficient conditions for $\rho : G \to GL(V)$ together with a reference point $\varphi : U_1 \to \bar{G}$ ($\bar{G} = G/Z(G)$ is the adjoint group) to give a Hermitian variation of Hodge structure are as follows: there exists a reductive algebraic group $M \subset GL(V)$ defined over $\mathbb{Q}$ (the generic Mumford-Tate group of the variation of Hodge structure) and a homomorphism of algebraic groups $h : \mathbb{S} \to M_{\mathbb{R}} \subset GL(V_{\mathbb{R}})$ ($\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$) such that

(1) the homomorphism $h$ defines a Hodge structure on $V$;

(2) the representation $\rho$ factors through $M$ and $\rho(G) = M^{\mathrm{der}}$;

(3) the induced map $\bar{h} : \mathbb{S}/\mathbb{G}_m \to M_{ad,\mathbb{R}} = \bar{G}$ is equal to $\varphi : U_1 \to \bar{G}$.

*Remark.*

(1) Following [4], we call $\rho$ a Hodge representation.

(2) Subrepresentations of $V$ correspond to sub-variation of Hodge structure and operations on representations correspond to the same operations on Hermitian variations of Hodge structure.

Let us also recall that the canonical $\mathbb{R}$-variation of Hodge structure of CY type $\mathcal{V}_{\mathbb{R}}$ over the $(D_{2n}, \alpha_{2n})$ domain $(n \geq 2)$ is given by a $\mathbb{R}$-representation $S^+_{0,\mathbb{R}}$ of $G(\mathbb{R}) = \mathrm{Spin}^*(4n)$ with the property that $S^+_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is the half-spin representation with highest weight $\varpi_{2n}$ (which is the fundamental cominuscule weight associated to the domain $(D_{2n}, \alpha_{2n})$). The weight of $\mathcal{V}_{\mathbb{R}}$ equals $n$, the real rank of $(D_{2n}, \alpha_{2n})$. Furthermore, any other irreducible $\mathbb{R}$-variation of Hodge structure of CY type can be obtained from $\mathcal{V}_{\mathbb{R}}$ by taking the unique irreducible factor of $\mathrm{Sym}^\bullet \mathcal{V}_{\mathbb{R}}$ of CY type.

### 3.2.2 The groups $\mathrm{Spin}^*$ and $\mathrm{SO}^*$

We now construct a $\mathbb{Q}$-form $H$ of the real algebraic group $\mathrm{SO}^*(2m)$ following [32]. Then the spin double cover $G$ of $H$, which is simply connected gives a $\mathbb{Q}$-form of $\mathrm{Spin}^*(2m)$, and can be associated with type $D^{\mathbb{H}}_m$ domains.

Let $E = \mathbb{Q}(\sqrt{-d})$ $(d \in \mathbb{Z}^+)$ be an imaginary quadratic extension of $\mathbb{Q}$, and let $W$ be an $E$-vector space of dimension $2m$ with an E-basis $e_1, \cdots, e_{2m}$. We write $z = \sum_{i=1}^{2m} z_i e_i$, and similarly for $w \in W$. Suppose that $Q(-,-)$ is a nondegenerate $E$-bilinear form on $W$, written in the standard form

$$Q(z,w) = \sum_{i=1}^{2m} (z_i w_{m+i} + z_{m+i} w_i).$$

Also, let $h$ be the standard $(E, \mathbb{Q})$-Hermitian form of signature $(m, m)$ on $W$ given by

$$h(z,w) = \sum_{i=1}^{m} z_i \bar{w}_i - \sum_{i=1}^{m} z_{m+i} \bar{w}_{m+i}.$$

Now we define $H$ to be the group of $E$-linear isomorphisms of $W$ which have determinant 1 and preserve $Q$ and $h$. The group $H$ is defined over $\mathbb{Q}$ since it is the intersection of $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{SO}(W, Q)$ with $\mathrm{SU}(W, h)$. Moreover, recall that the real group $\mathrm{SO}^*(2m)$ is defined to be the isometry group of a skew-Hermitian form on $\mathbb{H}^m$. By Exercise 1.1.5 (12) of [44], we have $H \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{SO}^*(2m)$.

Define $G$ to be the neutral component of the preimage in $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{Spin}(W, Q)$ of $H$ under the spin double covering map. Then $G$ is a $\mathbb{Q}$-form of $\mathrm{Spin}^*(2m)$.

To conclude this subsection, let us note that there are some natural representations of $G$. The first one is the standard representation $G \to H \to \mathrm{GL}(W)$. Moreover, $G$ also admits two half-spin representations. To construct them, let $W_1$ (resp. $W_2$) to be the $Q$-isotropic $E$-vector subspace of $W$ spanned by $e_1, \cdots, e_m$ (resp. $e_{m+1}, \cdots, e_{2m}$). The half-spin representations are then given by

$$S^+ = \bigwedge_E^{\text{even}} W_1, S^- = \bigwedge_E^{\text{odd}} W_1.$$

### 3.2.3 The rank 2 case

We shall prove Part (1) of Theorem 3.2.1 in this section, using notations from Section 3.2.2 (with $m = 4$). In particular, $\mathcal{D}$ is an irreducible Hermitian symmetric domain of type $(D_4, \alpha_4)$, and $G$ is the simply connected $\mathbb{Q}$-algebraic group associated to $\mathcal{D}$ as constructed in Section 3.2.2. First we show that there are two families of abelian varieties over $\mathcal{D}$.

**Proposition 3.2.3.** *The standard representation $G \to \mathrm{GL}(W)$ and the half-spin representation $G \to \mathrm{GL}(S^-)$ are both Hodge representations giving $\mathbb{Q}$-variations of Hodge structure of abelian variety type over $\mathcal{D}$.*

*Moreover, there exists two families of abelian varieties $\pi_1 : \mathcal{A}_1 \to \mathcal{D}$ and $\pi_2 : \mathcal{A}_2 \to \mathcal{D}$ such that the associated variation of Hodge structure $R^1\pi_{1*}\mathbb{Q}$ (resp. $R^1\pi_{2*}\mathbb{Q}$) corresponds to the Hodge representation $\mathrm{Res}_{E/\mathbb{Q}}W$ (resp. $\mathrm{Res}_{E/\mathbb{Q}}S^-$).*

*Proof.* The representations $G \to \mathrm{GL}(W)$ and $G \to \mathrm{GL}(S^-)$ are both defined over $\mathbb{Q}$. By Summary 10.11 of [27] (see also Table 2.2), there are two symplectic nodes associated to the domain $D_4^{\mathbb{H}}$, namely $\varpi_1$ and $\varpi_3$. By the standard representation theory (e.g. Chapter 19 and 20 of [45]), the representations $(\mathrm{Res}_{E/\mathbb{Q}}W)\otimes_{\mathbb{Q}}\mathbb{R}$ and $(\mathrm{Res}_{E/\mathbb{Q}}S^-)\otimes_{\mathbb{Q}}\mathbb{R}$ satisfy the conditions in Theorem 2.4.3. So they give two Hermitian $\mathbb{Q}$-variation of Hodge structure of abelian variety type, which further give two families of abelian varieties up to choices of the underlying integral structures (c.f. Theorem 11.8 of [27]). □

*Remark* 3.2.4. Recall that there are four types of irreducible polarizable $\mathbb{Q}$-Hodge structures (c.f. Albert's classification and (1.19) – (1.21) of [39]), and correspondingly four types of simple abelian varieties by looking at the associated $\mathbb{Q}$-Hodge structure of weight 1. In our cases, Theorem IV.E.4 of [4] implies that the generic fiber of $\pi_1$ and $\pi_2$ are both of type III; therefore, the generic special Mumford-Tate group (a.k.a. Hodge group) of the Hermitian variations of Hodge structure $R^1\pi_{1*}\mathbb{Q}$ and $R^1\pi_{2*}\mathbb{Q}$ are both semisimple (c.f. Proposition (1.24) of [39]). We also note a general fiber of the family of abelian varieties $\mathcal{A}_1$ is isogenous to a certain Prym variety associated to a quaternionic cover of a genus three curve (c.f. Section 3 of [21]).

Next we show that $\mathrm{Res}_{E/\mathbb{Q}}S^+$ is a $G$-subrepsentation of $\mathrm{Res}_{E/\mathbb{Q}}W \otimes_{\mathbb{Q}} \mathrm{Res}_{E/\mathbb{Q}}S^-$.

**Lemma 3.2.5.**

*(1) $S^+$ is a subrepresentation of $W \otimes_E S^-$.*

*(2) There is a natural inclusion $\mathrm{Res}_{E/\mathbb{Q}}(W \otimes_E S^-) \subset (\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-)$ which also commutes with the $G$-action.*

*Proof.* (1) Let $\mathfrak{g} = \mathrm{Lie}(G)$. Every representation will be viewed as representation of $\mathfrak{g}$ in this proof. Thanks to the complete reducibility, it suffices to construct a surjection $p : W \otimes_E S^- \twoheadrightarrow S^+$ compatible with the action of $\mathfrak{g}$. To define $p$, we use the inclusion $W \subset C(W, Q) \cong \mathrm{End}(S^+ \oplus S^-)$ (where $C(W, Q)$ is the Clifford algebra for $Q$). In other words, there is an action of $W$ on $S^+ \oplus S^-$. By Lemma 20.9 of [45], the action of $W$ exchanges $S^-$ and $S^+$. In other words, we have $W \times S^- \to S^+$, $(w, \xi) \mapsto w(\xi)$ which is clearly $E$-bilinear and hence can be used to define $p$. It is not difficult to check that $p$ is surjective.

Next we check that $p$ is compatible with the action of $\mathfrak{g}$. That is $p(g \cdot (v \otimes \xi)) = g \cdot p(v \otimes \xi)$ for every $g \in \mathfrak{g}$, $v \in W$ and $\xi \in S^-$. To do this, recall that we have $(\mathfrak{g} \subset)\mathfrak{so}(W, Q) \cong \bigwedge_E^2 W \hookrightarrow C(W, Q) \cong \mathrm{End}(S^+ \oplus S^-)$, where the first two maps are morphisms of Lie algebras and the last one is an algebra isomorphism (c.f. Lemma 20.7 of [45]). Without loss of generality we assume that $g = a \wedge b$ for $a, b \in W$. Let us also recall that the multiplication in the Clifford algebra $C(W, Q)$ is defined by $ab + ba = 2Q(a, b)$. Now we have

$$\begin{aligned}
p(g \cdot (v \otimes \xi)) &= p((g \cdot v) \otimes \xi + v \otimes (g \cdot \xi)) \\
&= (g \cdot v)(\xi) + v(g \cdot \xi) \\
&= 2Q(b, v)a(\xi) - 2Q(a, v)b(\xi) + v(ab(\xi)) - Q(a, b)v(\xi) \\
&\quad \text{(By (20.4) and (20.6) of [45])} \\
&= 2Q(b, v)a(\xi) - 2Q(a, v)b(\xi) + (vab)(\xi) - Q(a, b)v(\xi) \\
&= (abv)(\xi) - Q(a, b)v(\xi) \\
&\quad \text{(By the definition of Clifford algebra)} \\
&= (ab)v(\xi) - Q(a, b)v(\xi) \\
&= g \cdot p(v \otimes \xi).
\end{aligned}$$

(2) The proof is essentially the same as Lemma 3.1.3. Replacing wedge product by tensor product causes no essential changes. Clearly, let us denote $\mathrm{Res}_{E/\mathbb{Q}}$ by Res and the $E$-dual vector space using $*$. First observe that there is a natural surjection $\mathrm{Res}(W^*) \otimes_{\mathbb{Q}} \mathrm{Res}(S^{-*}) \twoheadrightarrow \mathrm{Res}(W^* \otimes_E$

$S^{-*}$) which gives by duality an injection

$$\mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}(W^* \otimes_E S^{-*}), \mathbb{Q}) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}(W^*) \otimes_{\mathbb{Q}} \mathrm{Res}(S^{-*}), \mathbb{Q}).$$

Also, for any $E$-vector space $M$ there is a natural isomorphism

$$\mathrm{Res}\,\mathrm{Hom}_E(M, E) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res} M, \mathbb{Q}), \quad f \mapsto \mathrm{Tr} \circ f.$$

So the natural inclusion can be defined as follows.

$$\begin{aligned}
\mathrm{Res}(W \otimes_E S^-) &\cong \mathrm{Res}(\mathrm{Hom}_E(W^*, E) \otimes_E \mathrm{Hom}_E(S^{-*}, E)) \\
&\cong \mathrm{Res}\,\mathrm{Hom}_E(W^* \otimes_E S^{-*}, E) \\
&\cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}(W^* \otimes_E S^{-*}), \mathbb{Q}) \\
&\subset \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}(W^*) \otimes_{\mathbb{Q}} \mathrm{Res}(S^{-*}), \mathbb{Q}) \\
&\cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res} W^*, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res} S^{-*}, \mathbb{Q}) \\
&\cong \mathrm{Res}(W^{**}) \otimes_{\mathbb{Q}} \mathrm{Res}(S^-)^{**} \\
&\cong \mathrm{Res} W \otimes_{\mathbb{Q}} \mathrm{Res} S^-.
\end{aligned}$$

Finally, it is straightforward to check that this map is $G$-equivariant (also after scalar extensions by arbitrary $\mathbb{Q}$-algebras).

$\square$

Now we show that the half-spin representation $S^+$ is defined over $\mathbb{Q}$.

**Lemma 3.2.6.** *There exists a $G$-subrepresentation on a $\mathbb{Q}$-vector space $S_0^+ \subset \mathrm{Res}_{E/\mathbb{Q}} S^+$ such that $S_0^+ \otimes_{\mathbb{Q}} E \cong S^+$.*

*Proof.* As is well known, it suffices to construct an $E$-conjugate linear operator $\star : \mathrm{Res}_{E/\mathbb{Q}} S^+ \to \mathrm{Res}_{E/\mathbb{Q}} S^+$ which is compatible with $G$-action and satisfy $\star \circ \star = \mathrm{id}$. In fact we can use the Hodge star operator associated to the hermitian form $h|W_1$ and the volume form $e_1 \wedge \cdots \wedge e_4$ as defined in Section 3.5 of [3] (see also (3.1.4) and Lemma 3.1.5). One can easily show that $\star$ is $E$-conjugate linear and maps $\bigwedge_E^{2+2k} W_1$ to $\bigwedge_E^{2-2k} W_1$ (here $k = -1, 0, 1$). The more difficult part, which has been done in Section 3 of [32], is to verify that $\star$ is a morphism of $G$-representations (c.f. Page 96 of op. cit.). $\square$

*Remark.* For an arbitrary CM field, the operator $\star$ may not commute with the corresponding group action. To fix this, one should use the "twisted Hodge star operator" defined in Definition 3.10 of [32].

We need a few more lemmas to prove Theorem 3.2.1. In what follows, we shall denote the special Mumford-Tate group (a.k.a. Hodge group) of a $\mathbb{Q}$-Hodge structure $V$ by $\mathrm{Hg}(V)$.

**Lemma 3.2.7.** *Let $V$ be a $\mathbb{Q}$-Hodge structure and $W \subset V$ be a sub-Hodge structure. Then*

*(1) There exists a surjective homomorphism $\mathrm{Hg}(V) \twoheadrightarrow \mathrm{Hg}(W)$.*

*(2) If $\mathrm{Hg}(V)$ is semisimple, then $\mathrm{Hg}(W)$ is also semisimple.*

*Proof.* Part (1) follows from (I.B.7) of [4]. Since any quotient of a semisimple algebraic group is semisimple, Part (2) is clear from Part (1). □

**Lemma 3.2.8.** *Let $V$ be a $\mathbb{Q}$-Hodge structure and $k \in \mathbb{Z}^+$. If $\mathrm{Hg}(V)$ is semisimple, then $\mathrm{Hg}(\bigwedge_{\mathbb{Q}}^k V)$ is semisimple.*

*Proof.* Since $\bigwedge_{\mathbb{Q}}^k V$ is a sub-Hodge structure of $\bigotimes_{\mathbb{Q}}^k V$, by Lemma 3.2.7 it suffices to show that $\mathrm{Hg}(\bigotimes_{\mathbb{Q}}^k V)$ is semisimple. According to (1.8) of [39], we have $\mathrm{Hg}(\bigotimes_{\mathbb{Q}}^k V) = r(\mathrm{Hg}(V))$ where $r : \mathrm{GL}(V) \to \mathrm{GL}(\bigotimes_{\mathbb{Q}}^k V)$ is the natural homomorphism. In other words, there exists a surjective homomorphism $\mathrm{Hg}(V) \twoheadrightarrow \mathrm{Hg}(\bigotimes_{\mathbb{Q}}^k V)$, and hence $\mathrm{Hg}(\bigotimes_{\mathbb{Q}}^k V)$ is semisimple as argued in the previous lemma. □

Finally let us prove Part (1) of Theorem 3.2.1.

*Proof.* By Lemma 3.2.5 and Lemma 3.2.6, we have

$$S_0^+ \subset \mathrm{Res}_{E/\mathbb{Q}} S^+ \subset \mathrm{Res}_{E/\mathbb{Q}}(W \otimes_E S^-) \subset (\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-)$$

as representations of $G$, and $\mathrm{Res}_{E/\mathbb{Q}} W$ (resp. $\mathrm{Res}_{E/\mathbb{Q}} S^+$) corresponds to a family of abelian varieties $\pi_1 : \mathcal{A}_1 \to \mathcal{D}$ (resp. $\pi_2 : \mathcal{A}_2 \to \mathcal{D}$) over $\mathcal{D}$ (c.f. Proposition 3.2.3). Let $A_i$ be the generic fiber of $\pi_i$ ($i = 1, 2$). According to Theorem IV.E.4 of [4], $A_1$ and $A_2$ are both of type III. By Proposition (1.24) of [39], the special Mumford-Tate group of $H^1(A_1 \times A_2, \mathbb{Q})$ is semisimple. The special Mumford-Tate group of $H^2(A_1 \times A_2, \mathbb{Q})$ is also semisimple because $H^2(A_1 \times A_2, \mathbb{Q}) \cong \bigwedge_{\mathbb{Q}}^2 H^1(A_1 \times A_2, \mathbb{Q})$ (c.f. Lemma 3.2.8). Since $H^1(A_1, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(A_2, \mathbb{Q})$ is a sub-Hodge structure of $H^2(A_1 \times A_2, \mathbb{Q})$, $H^1(A_1, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(A_2, \mathbb{Q})$ has a semisimple special Mumford-Tate group as well (Lemma 3.2.7).

As a result, the special Mumford-Tate group of the variation of Hodge structure $R^1\pi_{1*}\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi_{2*}\mathbb{Q}$ (which corresponds to the Hodge representation $(\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-)$) is semisimple. Let us denote it by Hg. By what we recalled in Section 3.2.1, Hg is the image of $G$ in $\mathrm{SL}((\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-))$. Being a $G$-subrepresentation, $S_0^+$ is invariant under the action of the special Mumford-Tate group Hg, and hence corresponds to a sub-Hodge structure (see for example (1.12) of [39]).

Now it suffices to show that $S_0^+ \otimes_{\mathbb{Q}} \mathbb{R}$ gives the canonical $\mathbb{R}$-variation of Hodge structure of CY type. Let us first note that $S_0^+ \otimes_{\mathbb{Q}} \mathbb{C} \cong S_0^+ \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} \mathbb{R} \cong S^+ \otimes_{\mathbb{Q}} \mathbb{R}$. Since $S^+ \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\bigwedge_{\mathbb{C}}^{\mathrm{even}}(W_1 \otimes_{\mathbb{Q}} \mathbb{R})$ (by construction), $S_0^+ \otimes_{\mathbb{Q}} \mathbb{R}$ is the half-spin representation of $G(\mathbb{C}) \cong \mathrm{Spin}(8, \mathbb{C})$ with highest weight $\varpi_4$. Because $\varpi_4$ is the fundamental cominuscule weight associated to the domain $\mathcal{D}$, the theorem follows from Section 3 of [1] or Theorem 2.22 of [3]. □

### 3.2.4 The higher rank cases

Let $\mathcal{D}$ be the irreducible Hermitian symmetric domain of type $(D_{2n}, \alpha_{2n})$ with $n \geq 2$. We will prove Part (2) of Theorem 3.2.1 for $\mathcal{D}$. Still, the notations are the same as in Section 3.2.2 (with $m = 2n$). For instance, $H$ is a $\mathbb{Q}$-form of $\mathrm{SO}^*(2m)$ which admits a spin double cover by $G$ (a simply connected $\mathbb{Q}$-algebraic group associated to $\mathcal{D}$).

We start by constructing a family of abelian varieties over $\mathcal{D}$.

**Proposition 3.2.9.** *The standard representation $H \to \mathrm{GL}(W)$ is a faithful Hodge representation corresponding to a Hermitian $\mathbb{Q}$-variation of Hodge structure of abelian variety type over $\mathcal{D}$. Furthermore, there exists a family of abelian varieties $\pi : \mathcal{A} \to \mathcal{D}$ such that the associated variation of Hodge structure $R^1\pi_*\mathbb{Q}$ is given by the Hodge representation $\mathrm{Res}_{E/\mathbb{Q}}W$.*

*Proof.* By Summary 10.11 of [27], the only symplectic node of $D_{2n}^{\mathbb{H}}$ $(n \geq 3)$ is $\varpi_1$. The rest is the same as the proof of Proposition 3.2.3. □

*Remark.* As in Remark 3.2.4, the generic fiber of $\pi$ is of type III and has a semisimple special Mumford-Tate group.

Next we note the following lemma.

**Lemma 3.2.10.** $\mathrm{Res}_{E/\mathbb{Q}}(\bigwedge_E^m W)$ *is an $H$-subrepresentation of* $\bigwedge_{\mathbb{Q}}^m(\mathrm{Res}_{E/\mathbb{Q}}W)$.

*Proof.* This follows from Lemma 3.1.3 and a similar argument as in Lemma 3.2.5. □

Now we decompose $\bigwedge_E^m W$ by constructing endomorphisms $L$ and $\star$ in $\mathrm{End}_{\mathbb{Q}[H]}(\bigwedge_E^m W)$ and taking the corresponding eigenspaces.

The operator $\star$ is defined in the same way as in (3.1.4) (see also Section 3.5 of [3]). Specifically, note that there are two natural pairings $\wedge : \bigwedge_E^m W \times \bigwedge_E^m W \to \bigwedge_E^{2m} W \cong E$ and $\wedge^m h : \bigwedge_E^m W \times \bigwedge_E^m W \to E$, where $(\wedge^m h)(w_1 \wedge \cdots \wedge w_m, u_1 \wedge \cdots \wedge u_m) = \det(h(w_i, u_j))$. They give an $E$-linear isomorphism

$\varphi : \bigwedge_E^m W \to \bigwedge_E^m W^*$ and an $E$-conjugate-linear isomorphism $\rho : \bigwedge_E^m W \to \bigwedge_E^m W^*$ respectively. The operator $\star$ is then defined by

$$\star = \varphi^{-1} \circ \rho.$$

Furthermore, we can obtain one more $E$-linear isomorphism $\tau : \bigwedge_E^m W \to \bigwedge_E^m W^*$ by considering the pairing $\wedge^m Q : \bigwedge_E^m W \times \bigwedge_E^m W \to E$ defined by $(\wedge^m Q)(w_1 \wedge \cdots \wedge w_m, u_1 \wedge \cdots \wedge u_m) = \det(Q(w_i, u_j))$. Now we define $L$ by

$$L = \varphi^{-1} \circ \tau.$$

Concerning the properties of $\star$ and $L$, we have the following two lemmas.

**Lemma 3.2.11.**

*(1) The $E$-linear operator $L$ commutes with the $H$-action and $L \circ L = \mathrm{id}$.*

*(2) The $E$-conjugate-linear operator $\star$ commutes with the action of $H$ and $\star \circ \star = \mathrm{id}$.*

*Proof.* For Part (1), the action of $H$ preserves the pairing $\wedge : \bigwedge_E^m W \times \bigwedge_E^m W \to \bigwedge_E^{2m} W \cong E$ and the symmetric bilinear form $\wedge^m Q$, and hence commutes with $L$. By Remark (iii) of Theorem 19.2 of [45], $L \circ L = \mathrm{id}$. Part (2) follows from Lemma 3.21 of [3]. □

**Lemma 3.2.12.** *The operators $L$ and $\star$ commutes (i.e. $L \circ \star = \star \circ L$) in $\mathrm{End}_{\mathbb{Q}[H]}(\bigwedge_E^m W)$.*

*Proof.* We start by setting up some notations. Let $\{e_1, \cdots, e_{2m}\}$ be a basis of $W$ such that the symmetric bilinear form $Q$ and the hermitian form $h$ can be expressed in the same form as in Section 3.2.2. Also denote the corresponding dual basis by $\{e_1^*, \cdots, e_{2m}^*\}$. Now define $B : W \to W^*$ by $B(v)(w) = Q(v, w)$, and $F : W \to W^*$ by $F(v)(w) = h(w, v)$. It is clear that $B(e_i) = e_{m+i}^*$, $B(e_{m+i}) = e_i^*$ and that $F(e_i) = e_i^*$, $F(e_{m+i}) = -e_{m+i}^*$ for $1 \le i \le m$.

Using these we can make the operators $\tau$ and $\rho$ more explicit. Specifically, $\tau(e_{l_1} \wedge e_{l_2} \wedge \cdots \wedge e_{l_m}) = B(e_{l_1}) \wedge B(e_{l_2}) \wedge \cdots \wedge B(e_{l_m})$ and $\rho(e_{l_1} \wedge e_{l_2} \wedge \cdots \wedge e_{l_m}) = F(e_{l_1}) \wedge F(e_{l_2}) \wedge \cdots \wedge F(e_{l_m})$.

As for $\varphi$, let $I = \{i_1, i_2, \cdots, i_m\}$ with $1 \le i_1 < i_2 < \cdots < i_m \le 2m$, and $J = \{1, 2, \cdots, 2m\} - I = \{j_1, j_2, \cdots, j_m\}$ with $j_1 < j_2 < \cdots < j_m$. Then it is not difficult to see that $\varphi(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) = \epsilon_{I,J} \cdot e_{j_1}^* \wedge e_{j_2}^* \wedge \cdots \wedge e_{j_m}^* = (-1)^{n+i_1+i_2+\cdots+i_m} e_{j_1}^* \wedge e_{j_2}^* \wedge \cdots \wedge e_{j_m}^*$ (recall that $n = \frac{m}{2}$). So $\varphi^{-1}(e_{j_1}^* \wedge e_{j_2}^* \wedge \cdots \wedge e_{j_m}^*) = (-1)^{n+j_1+j_2+\cdots+j_m} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$.

Now we prove the lemma. Let $I$ and $J$ be the ordered set as above. Clearly, it suffices to verify that $\tau \circ \varphi^{-1} \circ \rho = \rho \circ \varphi^{-1} \circ \tau$ for $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$

$(i_1 < i_2 < \cdots < i_m)$. We first determine which $e_j^*$'s appear for the left-hand-side (i.e. $(\tau \circ \varphi^{-1} \circ \rho)(e_I)$) and the right-hand-side (i.e. $(\rho \circ \varphi^{-1} \circ \tau)(e_I)$). To do this, we define $s(i) \in \{1, 2, \cdots, 2m\}$ for every $1 \leq i \leq 2m$, by $s(i) = m + i$ if $i \leq m$ and $s(i) = i - m$ if $i > m$. If $i \in I$ while $s(i) \notin I$, then $e_i^*$ will appear for both the left-hand-side and the right-hand-side; if $i \in I$ and $s(i) \in I$, then neither $e_i^*$ nor $e_{s(i)}^*$ will be contained in $(\tau \circ \varphi^{-1} \circ \rho)(e_I)$ or $(\rho \circ \varphi^{-1} \circ \tau)(e_I)$; if $i \notin I$ and $s(i) \notin I$, then $e_i^*$ and $e_{s(i)}^*$ will be contained in both the left-hand-side and the right-hand-side. So $(\tau \circ \varphi^{-1} \circ \rho)(e_I)$ and $(\rho \circ \varphi^{-1} \circ \tau)(e_I)$ consist of the same $e_j^*$'s.

Having seen that both $(\tau \circ \varphi^{-1} \circ \rho)(e_I)$ and $(\rho \circ \varphi^{-1} \circ \tau)(e_I)$ can be expressed uniquely, up to a sign, as $e_{l_1}^* \wedge e_{l_2}^* \wedge \cdots \wedge e_{l_m}^*$ with the same sub-indices $l_1 < l_2 < \cdots < l_m$, we verify that the signs are the same, which will verify the lemma. Let $k = \mathrm{Card}(I \cap \{m+1, \cdots, 2m\})$. Then it is straightforward to check that the sign of $(\tau \circ \varphi^{-1} \circ \rho)(e_I)$ is $(-1)^{k+n+i_1+\cdots+i_m+k}$, and the sign for $(\rho \circ \varphi^{-1} \circ \tau)(e_I)$ is $(-1)^{k+n+s(i_1)+\cdots+s(i_m)+(m-k)}$. Since $m$ is an even number, $i \equiv s(i) \pmod 2$ for every $i$, which implies that the two signs are the same. $\square$

We now prove Part (2) of Theorem 3.2.1. Note that $m = 2n$.

*Proof.* Let $S = \ker(L - \mathrm{id})$. Then $S$ is an $H$-subrepresentation of $\bigwedge_E^m W$. By Lemma 3.2.12, $L \circ \star = \star \circ L$, which implies that $L(\star(s)) = \star(L(s)) = \star(s)$ for any $s \in S$. So the restriction of $\star$ to $S$ is well-defined. Let $S_0 = \ker(\star|_S - \mathrm{id}) \subset S$. Since $\star$ is $E$-conjugate-linear, $S_0$ is a $\mathbb{Q}$-subrepresentation of $\mathrm{Res}_{E/\mathbb{Q}} S$.

So we have $S_0 \subset \mathrm{Res}_{E/\mathbb{Q}} S \subset \mathrm{Res}_{E/\mathbb{Q}}(\bigwedge_E^m W) \subset \bigwedge_{\mathbb{Q}}^m(\mathrm{Res}_{E/\mathbb{Q}} W)$ as representations of $H$ (c.f. also Lemma 3.2.10). Recall that the Hodge representation $\mathrm{Res}_{E/\mathbb{Q}} W$ corresponds to a family of abelian varieties $\pi : \mathcal{A} \to \mathcal{D}$ as in Proposition 3.2.9. By Theorem IV.E.4 of [4] and Proposition (1.24) of [39], the special Mumford-Tate group of the Hermitian variation of Hodge structure $R^1\pi_*\mathbb{Q}$ (which corresponds to the Hodge representation $\mathrm{Res}_{E/\mathbb{Q}} W$) is semisimple. So the special Mumford-Tate group of the variation of Hodge structure $R^m\pi_*\mathbb{Q}$ given by the Hodge representation $\bigwedge_{\mathbb{Q}}^m(\mathrm{Res}_{E/\mathbb{Q}} W)$ is semisimple (Lemma 3.2.8), and we shall denote it by Hg. The next part of proof is quite similar as in the rank 2 case. Specifically, by the result of Deligne in Section 3.2.1, Hg is the image of $H$ (or $G$) in $\mathrm{SL}(\bigwedge_{\mathbb{Q}}^m(\mathrm{Res}_{E/\mathbb{Q}} W))$. Being an $H$-subrepresentation, $S_0$ is invariant under the action of the special Mumford-Tate group Hg, and hence corresponds to a sub-Hodge structure.

So it suffices to prove that $S_0$ is the Hodge representation of CY type corresponding to $\mathcal{V}'$. To do this, let us consider $S_{0,\mathbb{R}} := S_0 \otimes_{\mathbb{Q}} \mathbb{R}$. Now that $S_0 \otimes_{\mathbb{Q}} E \cong S$, we have $S_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong S_0 \otimes_{\mathbb{Q}} \mathbb{C} \cong S_0 \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} \mathbb{R} \cong S \otimes_{\mathbb{Q}} \mathbb{R}$. According to Remark (iii) of Theorem 19.2 of [45], $S \otimes_{\mathbb{Q}} \mathbb{R} \subset \bigwedge_{\mathbb{C}}^m W_{\mathbb{R}}$ is the irreducible representation of $H(\mathbb{R}) \cong \mathrm{SO}^*(2m)$ with highest weight $2\varpi_m$, and is isomorphic to an irreducible summand of $\mathrm{Sym}^2(S^+ \otimes_{\mathbb{Q}} \mathbb{R})$. Since $\varpi_m$ is the fundamental cominuscule weight associated to the domain $\mathcal{D}$, the theorem follows from Theorem 2.22 of [3]. $\square$

*Remark* 3.2.13. Note that in Part (2) of the Main theorem we only realize $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$ (not the canonical $\mathcal{V}_{\mathbb{R}}$). This is the best our constructions can do when the rank of the domain is bigger or equal to 3. One important reason is that the half-spin representation with highest weight $\varpi_m$ is not a representation of the orthogonal group $H(\mathbb{C}) \cong \mathrm{SO}(2m, \mathbb{C})$ (c.f. Proposition 23.13 of [45]). Specifically, let $W_{\mathbb{R}}$ be the standard representation corresponding to a Hermitian variation of Hodge structure of abelian variety type, and set $S^+_{0,\mathbb{R}}$ to be the Hodge representation corresponding to the canonical $\mathbb{R}$-variation of Hodge structure of CY type. By Theorem (IV.E.4) of [4], $W_{\mathbb{R}}$ is of quaternion type, i.e. $W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = U \oplus U^*$ with $U \cong U^*$ and $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} U = W_{\mathbb{R}}$. Now suppose we have $S^+_{0,\mathbb{R}} \subset \bigotimes^l_{\mathbb{R}} W_{\mathbb{R}}$ as representations of $G(\mathbb{R}) \cong \mathrm{Spin}^*(2m)$, then $S^+_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a subrepresentation (as representations of $G(\mathbb{C}) \cong \mathrm{Spin}(2m, \mathbb{C})$) of $(\bigotimes^l_{\mathbb{R}} W_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigotimes^l_{\mathbb{C}}(U \oplus U^*)$ which factors through $H(\mathbb{C}) \cong \mathrm{SO}(2m, \mathbb{C})$. Since $\mathrm{SO}(2m, \mathbb{C}) = \mathrm{Spin}(2m, \mathbb{C})/\{\pm 1\}$, $(-1)$ also acts trivially on the half-spin representation $S^+_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ which is a contradiction. Note that this argument also works for other tensor constructions $(\bigotimes^{l_1}_{\mathbb{R}} W_{\mathbb{R}}) \otimes (\bigotimes^{l_2}_{\mathbb{R}} W^*_{\mathbb{R}})$ of $W_{\mathbb{R}}$.

# Chapter 4

# Towards a geometric interpretation of the exceptional isomorphisms between $(D_4, \alpha_4)$ ($\mathrm{II}_4$) and $(D_4, \alpha_1)$ ($\mathrm{IV}_6$)

Aiming to interpret the exceptional isomorphism between the Hermitian symmetric domains of type $\mathrm{II}_4$ and of type $\mathrm{IV}_6$ geometrically, we shall give some geometric constructions relating both of the domains to quaternionic covers of genus three curves in this chapter. We first construct quarternionic covers, quaternionic Pryms and quaternionic Prym maps (in particular, this gives a dominant map from the moduli of quaternionic covers of genus three curves to the type $\mathrm{II}_4$ domain) in Section 4.1. Also, we show that a quaternionic cover of a genus three $C$ determines (up to finite choices) a totally isotropic plane in $J_2(C)$, the group of line bundles $L$ such that $L^{\otimes 2} = \mathcal{O}_C$ (see Lemma 4.1.3 and Lemma 4.1.4). Next we recall the classical theory of theta characteristics in Section 4.2, and use it to show that a totally isotropic plane of $J_2(C)$ correspond to a syzygetic tetrad of bitangent lines of $C$ (i.e. there is a conic passing through all the tangency points). Finally, we construct in Section 4.3 a family of lattice-polarized K3 surfaces, parameterized by a certain arithmetic quotient of the type $\mathrm{IV}_6$ domain, starting from a smooth plane quartic and two bitangent lines (N.B. any pair of bitangents can be extended to a syzygetic tetrad).

## 4.1 Quaternionic covers and quaternionic Pryms

We define quarternionic covers, quaternionic Pryms and quaternionic Prym maps following [21] (see also [46]) in this section.

### 4.1.1 Quaternionic covers

A definite quaternionic $\mathbb{Q}$-algebra $F$ is a skew field with center $\mathbb{Q}$, of dimension 4 over $\mathbb{Q}$, and such that $F \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the Hamilton's quaternions. One can find elements $\mathbf{i}, \mathbf{j}, \mathbf{k} \in F$ such that

$$F = \mathbb{Q} + \mathbb{Q}\mathbf{i} + \mathbb{Q}\mathbf{j} + \mathbb{Q}\mathbf{k}, \quad \mathbf{i}^2 = r, \quad \mathbf{j}^2 = s, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \text{with } r, s \in \mathbb{Q}^-.$$

Such a quaternionic algebra has a canonical involution, it acts as

$$x = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto \bar{x} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

We will use $\mathbb{H}_{\mathbb{Q}}$ to denote the $\mathbb{Q}$-quaternion algebra with $r = s = -1$, and denote by $Q$ the following subgroup of $\mathbb{H}_{\mathbb{Q}}^*$ of order 8:

$$Q = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}, \quad (\pm\mathbf{i})^2 = (\pm\mathbf{j})^2 = (\pm\mathbf{k})^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

**Definition 4.1.1.** Let $C$ be an irreducible smooth projective curve defined over $\mathbb{C}$. A quaternionic cover $\pi : \tilde{C} \to C$ is an unramified Galois cover with Galois group $Q$.

A quaternionic cover $\pi : \tilde{C} \to C$ also determines a 2-dimensional $\mathbb{F}_2$-subspace of $H^1(C, \mathbb{Z}/2)$. In fact, after choosing a reference point $\tilde{p}$ of $\tilde{C}$ (and let $p = \pi(\tilde{p})$), we obtain a surjective group homomorphism $\varphi : \pi_1(C, p) \twoheadrightarrow Q$ whose kernel is $N = \pi_*\pi_1(\tilde{C}, \tilde{p})$. Specifically, let $[\gamma]$ be an element of $\pi_1(C, p)$ and $\tilde{\gamma}$ be the unique lift of $\gamma$ with the starting point $\tilde{p}$, then $\varphi([\gamma])$ is the unique deck transformation sending $\tilde{p}$ to the ending point of $\tilde{\gamma}$ (see Proposition 1.39 of [47]). Next we compose this homomorphism with the quotient map $Q \twoheadrightarrow Q/\{\pm 1\}$. Because $Q/\{\pm 1\}$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, the map $\pi_1(C, p) \to Q \to Q/\{\pm 1\}$ factors through $H_1(C, \mathbb{Z})$:

$$\begin{tikzcd}
\pi_1(C,p) \arrow[r, "\varphi"] \arrow[d] & Q \arrow[r] & Q/\{\pm 1\} \,. \\
H_1(C,\mathbb{Z}) \arrow[urr, dashed, "\bar{\varphi}"] & &
\end{tikzcd}$$

**Lemma 4.1.2.** *The map $\bar{\varphi}$ does not depend on the choice of the reference point $\tilde{p} \in \tilde{C}$.*

*Proof.* Let us denote the quotient map $Q \to Q/\{\pm 1\}$ by $f$. We first claim that the composition $f \circ \varphi : \pi_1(C, p) \to Q/\{\pm 1\}$ stays the same if another point $\tilde{q} \in \tilde{C}$ with $\pi(\tilde{p}) = \pi(\tilde{q})$ is chosen to be the reference point. Let us use $\varphi'$ to denote the map $\pi_1(C, p) \to Q$ for the reference point $\tilde{q}$. Also, we choose an element $[\gamma]$ of $\pi_1(C, p)$ and a path $\tilde{\alpha}$ from $\tilde{p}$ to $\tilde{q}$. Then we have that $\varphi([\gamma])(\tilde{q}) = (\varphi([\gamma]) \circ \varphi([\pi(\tilde{\alpha})]))(\tilde{p})$, and that $\varphi'([\gamma])(\tilde{q}) = \varphi([\pi(\tilde{\alpha}])([\gamma])(1) = (\varphi([\pi(\tilde{\alpha})]) \circ \varphi([\gamma]))(\tilde{p})$. It follows that either $\varphi([\gamma])(\tilde{q}) = \varphi'([\gamma])(\tilde{q})$ or $\varphi([\gamma])(\tilde{q}) = ((-1) \circ \varphi'([\gamma]))(\tilde{q})$. So we have either $\varphi([\gamma]) = \varphi'([\gamma])$ or $\varphi([\gamma]) = (-1) \circ \varphi'([\gamma])$, which verifies the claim.

Next we choose a reference point $\tilde{o}$ with $o := \pi(\tilde{o}) \neq p$. We claim that the map $\bar{\varphi} : H_1(C, \mathbb{Z}) \to Q/\{\pm 1\}$ stays the same for $\tilde{o}$. Let $\varphi'' : \pi_1(C, o) \to Q$ be the map corresponding to the reference point $\tilde{o}$. Also let $\beta$ be an arbitrary path from $p$ to $o$, and set $\tilde{\beta}$ to be the unique lift to $\tilde{p}$ with an ending point $\tilde{o}'$. By what we have proven we can also choose $\tilde{o}'$ as the reference point for $\varphi''$, and then it is easy to check that $\varphi([\gamma]) = \varphi''([\beta^{-1} \cdot \gamma \cdot \beta])$ up to a possible action of $(-1) \in Q$ (by looking at where they send the point $\tilde{o}'$). Because $[\gamma]$ and $[\beta^{-1} \cdot \gamma \cdot \beta]$ are sent to the same cycle in $H_1(C, \mathbb{Z})$, the lemma holds. □

We will omit the reference point in what follows. Now using the natural identification $H_1(C, \mathbb{Z}/2) \cong H_1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2$, it is easy to see that the homomorphism $\bar{\varphi}$ induces a surjective homomorphism $H_1(C, \mathbb{Z}/2) \to Q/\{\pm 1\}$ which is $\mathbb{Z}/2$-linear. Then we obtain by duality an $\mathbb{F}_2$-subspace of $H^1(C, \mathbb{Z}/2) \cong \mathrm{Hom}(H_1(C, \mathbb{Z}/2), \mathbb{Z}/2)$ of dimension two.

Next we consider $J_2(C)$, which is the group of line bundles $L$ such that $L^{\otimes 2} = \mathcal{O}_C$. Since $\mathrm{Jac}(C) = H^0(C, \omega_C)^*/H_1(X, \mathbb{Z})$, we have

$$J_2(C) \cong H_1(C, (1/2)\mathbb{Z})/H_1(C, \mathbb{Z}) \cong H_1(C, \mathbb{Z}/2).$$

There is also a non-degenerate alternating bilinear form on $J_2(C)$ called the Weil pairing (see Page 284 of [48]):

$$\lambda : J_2(C) \times J_2(C) \to \mathbb{Z}/2.$$

Note that via the above identification the Weil pairing $\lambda$ on $J_2(C)$ corresponds to intersection of cycles on $H_1(C, \mathbb{Z}/2)$ (see Page 287 of [48]).

Using Poincaré duality, one can identify $H^1(C, \mathbb{Z}/2)$ with $H_1(C, \mathbb{Z}/2)$ canonically, and so a quaternionic cover $\pi : \tilde{C} \to C$ determines an $\mathbb{F}_2$-subspace of $J_2(C)$ of dimension two. We shall denote this subspace by $V_\pi$.

**Lemma 4.1.3.** *The subspace $V_\pi$ is totally isotropic for the Weil pairing $\lambda$.*

*Proof.* Let us consider the alternating form $\lambda|_{V_\pi \times V_\pi}$ on the $\mathbb{F}_2$-subspace $V_\pi$ of dimension two. By Theorem XV.8.1 of [49], either $\lambda|_{V_\pi \times V_\pi}$ is non-degenerate or $\lambda|_{V_\pi \times V_\pi} = 0$. We want to show that the first case is impossible.

Now let us assume that $\lambda|_{V_\pi \times V_\pi}$ is non-degenerate. Let $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ be a set of generators of $\pi_1(C)$ satisfying $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$. Note that, if we denote by $\bar{\alpha}_i$ (resp. $\bar{\beta}_i$) the cycle in $H_1(C, \mathbb{Z}/2)$ corresponding to $\alpha_l$ (resp. $\beta_l$) where $1 \leq l \leq g$, then among $\bar{\alpha}$'s and $\bar{\beta}$'s only $\bar{\alpha}_l \cap \bar{\beta}_l \neq 0$ for the intersection form on $H_1(C, \mathbb{Z}/2)$. Because the mapping class group maps onto the the symplectic group (w.r.t the intersection pairing) on $H_1(C, \mathbb{Z}/2)$, we can always choose the generators so that $V_\pi = \{0, \bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_1 + \bar{\beta}_1\}$. By construction, this implies that the map $\pi_1(C) \xrightarrow{\varphi} Q \longrightarrow Q/\{\pm 1\}$ is given by

$$\alpha_1 \mapsto \bar{i}, \quad \beta_1 \mapsto \bar{j}, \quad \alpha_l, \beta_l \mapsto \bar{1} \ (l \geq 2).$$

But we then have $\varphi([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]) = -1$, which is a contradiction. $\square$

Conversely, we have the following lemma.

**Lemma 4.1.4.** *Given an $\mathbb{F}_2$-subspace $W$ of dimension 2 in $J_2(C)$ which is totally isotropic with respect to the Weil pairing $\lambda$, there exists a quaternionic cover $\pi : \tilde{C} \to C$ corresponding to $W$ (i.e. $V_\pi = W$).*

*Proof.* Using the same notations and argument in the proof of Lemma 4.1.3, we can choose a set of generators $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ of $\pi_1(C)$ such that $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$ and such that $W = \{0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_1 + \bar{\alpha}_2\}$. Now it is easy to check that the map $\pi_1(C) \to Q/\{\pm 1\}$ defined by

$$\alpha_1 \mapsto \bar{i}, \quad \alpha_2 \mapsto \bar{j}, \quad \alpha_l, \beta_s \mapsto \bar{1} \ (l \geq 3, 1 \leq s \leq g)$$

corresponds to $W$. This map can be lifted to a surjective map $\varphi : \pi_1(C) \to Q$; specifically, we define $\varphi$ by assigning minus signs to the image of $\alpha_l$, $\beta_s$ arbitrarily, and one can easily check that $\varphi([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]) = 1$. Taking the quotient of the universal cover of $C$ by the kernel $N$ of $\varphi$, we obtain a quaternionic cover $N \backslash C^{\text{univ}} \to C$ of C. $\square$

We conclude this subsection by noting that there exists a moduli space $\mathcal{M}_{g,Q}$ of quaternionic covers of genus $g$ curves (see [21] Section 1). The space $\mathcal{M}_{g,Q}$ is irreducible (Corollary 1.5 of op. cit.) and admits a natural morphism $\mathcal{M}_{g,Q} \to \mathcal{M}_g$ which is finite. In particular, we have $\dim \mathcal{M}_{3,Q} = 3g - 3$.

### 4.1.2 Quaternionic Pryms

In this subsection, we review the construction and properties of quaternionic Pryms following Section 2 of [21]. Let $\pi : \tilde{C} \to C$ be a quaternionic cover of a genus $g$ curve $C$. It is easy to see that the genus of $\tilde{C}$ is $8g-7$. The quotient of $\tilde{C}$ by $\{\pm 1\} \subset Q$ is a curve of genus $4g-3$, which will be denoted by $\hat{C}$:

$$\tilde{C} \xrightarrow{2:1} \hat{C} = \tilde{C}/\{\pm 1\} \xrightarrow{4:1} C.$$

The Prym variety $P := \mathrm{Prym}(\tilde{C}/\hat{C})$ is then called the *quaternionic Prym* associated to the quaternionic cover $\pi : \tilde{C} \to C$.

Recall that a polarized abelian variety of quaternionic type is a tripe $(A, E, F)$ where $A$ is an abelian variety, $E$ is a polarization on $A$ and $F$ is a definite quaternionic algebra over $\mathbb{Q}$ endowed with an embedding $F \subset \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, such that $\imath_E(x) = \bar{x}$ for all $x \in F$. Here $\imath_E$ is the Rosati involution and $^-$ denotes the canonical involution on $F$.

Concerning quaternionic Pryms (equipped with the natural principal polarization, see for instance [48] Page 297), the algebra is $\mathbb{H}_{\mathbb{Q}}$. To describe the action of $\mathbb{H}_{\mathbb{Q}}$ on $P$, we introduce the ring

$$\mathbb{H}_{\mathbb{Z}} := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}_{\mathbb{Q}} \mid a, b, c, d \in \mathbb{Z}\}$$

and also the $\mathbb{H}_{\mathbb{Z}}$-module

$$M := \{a\zeta + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}_{\mathbb{Q}} \mid a, b, c, d \in \mathbb{Z}\} \quad \text{with} \quad \zeta = (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})/2.$$

**Proposition 4.1.5.** *The quaternionic Prym $P$ is a $4(g-1)$-dimensional principally polarized abelian variety of quaternionic type for the algebra $\mathbb{H}_{\mathbb{Q}}$. Moreover, we have $H_1(P, \mathbb{Z}) \cong (M \oplus \mathbb{H}_{\mathbb{Z}}^{g-2})^2$ as $\mathbb{H}_{\mathbb{Z}}$-modules.*

*Proof.* See Proposition 2.4 and 2.6 of [21]. □

Let us denote by $\Lambda$ the $\mathbb{H}_{\mathbb{Z}}$-module $(M \oplus \mathbb{H}_{\mathbb{Z}}^{g-2})^2$. Fix an isomorphism

$$\mathfrak{M} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \mathbb{H}_{\mathbb{Q}}^{2g-2}$$

and let $\mathfrak{M}$ be the image of $\Lambda$ in $\mathbb{H}_{\mathbb{Q}}^{2g-2}$. (If one takes the obvious isomorphism induced by $\mathbb{H}_{\mathbb{Z}} \subset \mathbb{H}_{\mathbb{Q}}$ and $M \subset \mathbb{H}_{\mathbb{Q}}$, then $\mathfrak{M} = \Lambda$.) Also define a $(2g-2) \times (2g-2)$ matrix $T$ with entries in $\mathbb{H}_{\mathbb{Q}}$ as in [19] Section 2.2. For such a pair $(\mathfrak{M}, T)$, we put

$$G(\mathfrak{M}, T) := \{U \in \mathrm{Mat}_{2g-2}(\mathbb{H}_{\mathbb{Q}}) \mid UT{}^t\overline{U} = T, \mathfrak{M}U = \mathfrak{M}\}$$

as in Section 2.7 of [19] or Section 9.8 of [20]. Note that $G(\mathfrak{M}, T)$ acts (described in [19] Section 2.7) properly and discontinuously on the Hermitian symmetric domain

$$\mathcal{H}_{2g-2} := \{Z \in \mathrm{Mat}_{2g-2}(\mathbb{C}) \mid {}^tZ = -Z, I_{2g-2} - {}^t\overline{Z}Z > 0\}$$

which is of type $\mathrm{II}_{2g-2}$. The quotient $G(\mathfrak{M}, T)\backslash\mathcal{H}_{2g-2}$ is a normal complex analytic space of dimension $(g-1)(2g-3)$ and parameterizes $4(g-1)$-dimensional polarized abelian varieties of quaternionic type for the algebra $\mathbb{H}_{\mathbb{Q}}$ which are associated to the pair $(\mathfrak{M}, T)$ (see [19] Theorems 1, 2 or [20] Propositions 9.5.3, 9.5.4, 9.8.2). In particular, quaternionic Pryms corresponds to certain points of $G(\mathfrak{M}, T)\backslash\mathcal{H}_{2g-2}$.

### 4.1.3 Quaternionic Prym maps

Let $\mathcal{R}_g$ be the moduli space of étale double covers of genus $g$ curves. There is a natural morphism

$$r : \mathcal{M}_{g,Q} \to \mathcal{R}_{4g-3}, \quad [\tilde{C} \to C] \mapsto [\tilde{C} \to \hat{C} = \tilde{C}/\{\pm 1\}]$$

which associates to the quaternionic cover $\tilde{C} \to C$ the étale double cover $\tilde{C} \to \hat{C}$. Also, we have the Prym map

$$p : \mathcal{R}_{4g-3} \to G(\mathfrak{M}, T)\backslash\mathcal{H}_{2g-2}, \quad [\tilde{C} \to \hat{C}] \mapsto \mathrm{Prym}(\tilde{C}/\hat{C}).$$

Composing them, we obtain the quaternionic Prym map $q = p \circ r : \mathcal{M}_{g,Q} \to G(\mathfrak{M}, T)\backslash\mathcal{H}_{2g-2}$.

From now on, we focus on the case when $g = 3$. Following [21] Section 3, we now study the locus of quaternionic Pryms via the map $q = p \circ r : \mathcal{M}_{3,Q} \to G(\mathfrak{M}, T)\backslash\mathcal{H}_4$. For convenience, we shall simply use $\mathcal{H}$ to denote the domain $\mathcal{H}_4$ in what follows.

**Proposition 4.1.6.** *A general member of the polarized abelian 8-folds of quaternionic type parametrized by $G(\mathfrak{M}, T)\backslash\mathcal{H}$ is a quaternionic Prym.*

*Proof.* We will show that the image $Im(q)$ of $q$ has dimension 6, which is the dimension of $G(\mathfrak{M}, T)\backslash\mathcal{H}$. By Section 1.6 of [21], the map $r$ is surjective. Because the automorphism group of a curve of genus at least 2 is finite, a given $\hat{C}$ is in the image of at most finitely many points in $\mathcal{M}_{3,Q}$, and so the map $r$ is also quasi-finite. Note also that $\dim \mathcal{M}_{3,Q} = \dim(G(\mathfrak{M}, T)\backslash\mathcal{H}) = 6$. Therefore it suffices to show that the differential or the codifferential of the Prym map $p$ has maximal rank at a point $\sigma : \tilde{C} \to \hat{C}$ in the image of the map $r$. Since the codifferential of $p$ at $\sigma$ is the multiplication map $S^2H^0(\hat{C}, \omega_{\hat{C}} \otimes \eta) \to H^0(\hat{C}, \omega_{\hat{C}}^{\otimes 2})$ with $\eta$ the order two line bundle defining $\sigma$, the proposition follows from Lemma 3.3 and Theorem 3.6 of op.cit.. $\square$

## 4.2 Theta characteristics and bitangents

In this section, we will give a geometric interpretation of the totally isotropic planes of $(J_2(C), \lambda)$ (which are associated with quaternionic covers of $C$ as in Lemma 4.1.3 and Lemma 4.1.4) via theta characteristics.

Let $C$ be an irreducible smooth projective curve of genus $g$ defined over $\mathbb{C}$. A theta characteristic on $C$ is a line bundle $L$ such that $L \otimes L \cong \omega_C$ (see for example [48] Page 287). As two theta characteristics differ by an element of $J_2(C)$ and the group $J_2(C)$ is isomorphic to $(\mathbb{Z}/2)^{2g}$, there are $2^{2g}$ theta characteristics. It also follows that the set $\mathrm{TCh}(C)$ of theta characteristics on $C$ is an affine space over the $\mathbb{F}_2$-vector space $J_2(C)$.

Furthermore, using the Weil pairing $\lambda$ on $J_2(C)$ one can associate to every theta characteristic $L$ a quadratic form $q_L$ on $J_2(C)$:

$$q_L : J_2(C) \to \mathbb{F}_2, \quad \epsilon \mapsto h^0(C, L \otimes \epsilon) - h^0(C, L) \ (\mathrm{mod} 2).$$

Note that $q_L$ is a quadratic form whose associated bilinear form is the Weil pairing (see Page 290 of [48]); in other words, we have the Riemann-Mumford relation: for $\eta, \epsilon \in J_2(C)$,

$$q_L(\eta + \epsilon) + q_L(\eta) + q_L(\epsilon) = \lambda(\eta, \epsilon).$$

Using this, one can also show that $L \mapsto Q_L$ gives a natural bijection between the set of theta characteristics on $C$ and the set of quadratic forms associated to $(J_2(C), \lambda)$. Now we review the theory of quadratic forms over $\mathbb{F}_2$ briefly. Every concept defined for quadratic forms will have a meaning for theta characteristics.

### 4.2.1 Quadratic forms over $\mathbb{F}_2$

Let $V$ be an $\mathbb{F}_2$-vector space of dimension $2g$, equipped with a non-degenerate alternating bilinear form

$$\langle\ ,\ \rangle : V \times V \to \mathbb{F}_2.$$

Note that the pairing induces an isomorphism $V \cong V^*$ which associates a vector $v$ the linear form $v^* := \langle v, - \rangle$.

A quadratic form associated to $(V, \langle\ ,\ \rangle)$ is a map $Q : V \to \mathbb{F}_2$ such that

$$Q(u + v) = Q(u) + Q(v) + \langle u, v \rangle,$$

for all $u, v \in V$. Let us denote by $Q(V)$ the set of all quadratic forms associated to $(V, \langle\ ,\ \rangle)$. This set $Q(V)$ is a principal homogenous space over $V^*$. In other words, for any fixed $Q_0 \in Q(V)$ we have a bijection

$$V^* \to Q(V), \quad v^* \mapsto Q_0 + v^*, \text{conversely } Q \mapsto Q + Q_0.$$

Choose a standard symplectic basis $\{e_1, \cdots, e_{2g}\}$ of $V$, that is, a basis with respect to which the intersection matrix of $\langle\ ,\ \rangle$ is

$$\begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}.$$

The *Arf invariant* of $Q \in Q(V)$ is then defined by

$$\mathrm{Arf}(Q) = \sum_{i=1}^{g} Q(e_i)Q(e_{g+i}).$$

A quadratic form $Q$ is called *odd* (resp. *even*) if $\mathrm{Arf}(Q) = 1$ (resp. $\mathrm{Arf}(Q) = 0$). The set of even and odd quadratic forms will be denoted by $Q(V)_+$ and $Q(V)_-$ respectively. There are $2^{g-1}(2^g+1)$ even quadratic forms and $2^{g-1}(2^g-1)$ odd quadratic forms. By [50] Section 5.1.2 we also have the following lemma

**Lemma 4.2.1.** *For any $v \in V$ and quadratic forms $Q$, $Q_0$, $Q_1$, $Q_2$ we have*

*(1)* $\mathrm{Arf}(Q + v^*) = \mathrm{Arf}(Q) + Q(v)$;

*(2)* $\mathrm{Arf}(Q_0 + Q_1 + Q_2) = \mathrm{Arf}(Q_0) + \mathrm{Arf}(Q_1) + \mathrm{Arf}(Q_2) + \langle v_1, v_2\rangle$ *where* $v_1^* = Q_0 + Q_1$ *and* $v_2^* = Q_0 + Q_2$.

Next we define syzygetic triads and tetrads of odd quadratic forms.

**Definition 4.2.2.** A triad $Q_0$, $Q_1$, $Q_2$ of odd quadratic forms is called a *syzygetic triad* if $Q_0 + Q_1 + Q_2$ is odd. A tetrad $Q_0$, $Q_1$, $Q_2$, $Q_3$ of odd quadratic forms is called a *syzygetic tetrad* if $Q_0 + Q_1 + Q_2 + Q_3 = 0$.

For any $v \in V$, $v \neq 0$, we also consider the *Steiner set* $S_v$ defined by

$$S_v := \{Q \in Q(V)_- \mid Q(v) = 0\} = \{Q \in Q(V)_- \mid Q + v^* \in Q(V)_-\}.$$

The structure of the Steiner sets are described as follows.

**Proposition 4.2.3.** *There are $2^{2g} - 1$ Steiner sets. Each Steiner set $S_v$ consists of $2^{g-1}(2^{g-1} - 1)$ elements paired by $Q \mapsto Q + v^*$.*

*If $S_v$ and $S_w$ are two different Steiner sets, then we have*

$$\#(S_v \cap S_w) = \begin{cases} 2^{g-1}(2^{g-2} - 1) & \text{if } \langle v, w\rangle = 0; \\ 2^{g-2}(2^{g-1} - 1) & \text{if } \langle v, w\rangle = 1. \end{cases}$$

*Proof.* See [50] Proposition 5.4.7 and Lemma 5.4.8. □

We now make the following observation.

**Lemma 4.2.4.** *Let $Q_1, Q_2 \in Q(V)_-$. Then there exists a syzygetic tetrad containing $Q_1$ and $Q_2$. Moreover, the number of ways in which the pair $Q_1$, $Q_2$ can be extended to a syzygetic tetrad of odd quadratic forms is equal to $2^{2g-2} - 2^{g-1} - 2$.*

*Proof.* Write $Q_1 + Q_2 = v^*$, then we have $Q_2 = Q_1 + v^*$. Because $Q_1, Q_2 \in Q(V)_-$, we have $\mathrm{Arf}(Q_1) = \mathrm{Arf}(Q_2)$, which by Lemma 4.2.1 implies that $Q_1(v) = 0$. Similarly, $Q_2(v) = 0$. Let $Q_0$ be an odd quadratic form, then $Q_0$, $Q_1$, $Q_2$ is a syzygetic triad if and only if $\mathrm{Arf}(Q_0+Q_1+Q_2) = \mathrm{Arf}(Q_0+v^*) = 1$. By Lemma 4.2.1, we have $\mathrm{Arf}(\mathrm{Q}_0 + \mathrm{v}^*) = \mathrm{Arf}(Q_0) + Q_0(v)$. So $Q_0$, $Q_1$, $Q_2$ is syzygetic if and only if $Q_0(v) = 0$, or equivalently, $Q_0 \in S_v - \{Q_1, Q_2\}$. By Proposition 4.2.3, there are $2^{g-1}(2^{g-1} - 1) - 2 = 2^{2g-2} - 2^{g-1} - 2$ possibilities for $Q_0$. For a chosen $Q_0$, there is a unique way to extend $Q_0$, $Q_1$, $Q_2$ to a syzygetic tetrad which is $Q_0$, $Q_1$, $Q_2$, $Q_0 + Q_1 + Q_2$. □

To conclude this section, we show that all the isotropic planes can be obtained from the syzygetic tetrads of odd quadratic forms. Specifically, for a syzygetic tetrad $Q_0$, $Q_1$, $Q_2$, $Q_3$ of odd quadratic forms we consider the set

$$\{0, Q_0 + Q_1, Q_0 + Q_2, Q_0 + Q_3\}.$$

Using the identification $V \cong V^*$ defined via $\langle\ ,\ \rangle$, we view the set $\{0, Q_0 + Q_1, Q_0 + Q_2, Q_0 + Q_3\}$ as a subset of $V$. Using the condition that $Q_0 + Q_1 + Q_2 + Q_3 = 0$, one can easily check that this map is well-defined and that $\{0, Q_0 + Q_1, Q_0 + Q_2, Q_0 + Q_3\}$ is an $\mathbb{F}_2$-subspace of $V$. Write $Q_0 + Q_1 = v_1^*$ and $Q_0 + Q_2 = v_2^*$, then Lemma 4.2.1 implies that $\langle v_1, v_2\rangle = 0$, and so $\{0, Q_0 + Q_1, Q_0 + Q_2, Q_0 + Q_3\}$ is a isotropic plane of $V$.

**Proposition 4.2.5.** *Every isotropic plane of $V$ can be obtained from $2^{g-3}(2^{g-2} - 1)$ syzygetic tetrads in the above way.*

*Proof.* Let $\{0, v_1, v_2, v_1+v_2\}$ be an isotropic plane of $V$, then we have $\langle v_1, v_2\rangle = 0$. Let us choose an odd quadratic form $Q \in Q(V)_-$, and consider the tetrad $Q$, $Q+v_1^*$, $Q+v_2^*$, $Q+v_1^*+v_2^*$. This is a syzygetic tetrad of odd quadratic forms if and only if $Q(v_1) = Q(v_2) = 0$ (i.e. $Q \in S_{v_1} \cap S_{v_2}$). By the construction, $\{0, v_1, v_2, v_1 + v_2\}$ must be obtained from syzygetic tetrads of this form.

By Proposition 4.2.3, there are $2^{g-1}(2^{g-2} - 1)$ possibilities for $Q$. However, if we replace $Q$ by $Q+v_1^*$ or $Q+v_2^*$ or $Q+v_1^*+v_2^*$, and add 0, $v_1^*$, $v_2^*$ and $v_1^*+v_2^*$ as above, then we still obtain the same syzygetic tetrad $Q$, $Q + v_1^*$, $Q + v_2^*$,

$Q+v_1^*+v_2^*$. The other elements in $S_{v_1} \cap S_{v_2}$ will give different syzygetic tetrads. So there are $2^{g-1}(2^{g-2}-1)/4 = 2^{g-3}(2^{g-2}-1)$ syzygetic tetrads corresponding to the isotropic plane $\{0, v_1, v_2, v_1+v_2\}$ as claimed. $\square$

When $g=3$, $2^{g-3}(2^{g-2}-1) = 1$ and hence we have the following corollary.

**Corollary 4.2.6.** *If $g = 3$, then there is a one-to-one correspondence between the set of isotropic planes of $V$ and the set of syzygetic tetrads of odd quadratic forms on $V$.*

### 4.2.2 Theta characteristics and bitangents

Let us return to the situation when $(V, \langle\ ,\ \rangle) = (J_2(C), \lambda)$. Using the bijection $L \mapsto Q_L$ between the set of theta characteristics on $C$ and the set of quadratic forms associated to $(J_2(C), \lambda)$, we can now talk about odd theta characteristics, syzygetic tetrads of odd theta characteristics, etc.. Note also that the theta character $L$ is odd (resp. even) if and only if and only if $h^0(C, L)$ is odd (resp. even).

We shall further assume that $C$ is a smooth non-hyperelliptic curve of genus $g = 3$. Via the canonical embedding, we view $C$ as a quartic curve in $\mathbb{P}^2$ and we have $\omega_C \cong \mathcal{O}_C(1)$. A bitangent line $l$ of $C$ is a line such that $l \cap C = 2p+2q$ for some points $p, q \in C$. The correspondence

$$l \mapsto \mathcal{O}_C\left(\frac{1}{2}(l \cap C)\right)$$

establishes a bijection between the set of bitangents and the set of odd theta characteristics of $C$.

Now the concepts defined for quadratic forms on $(J_2(C), \lambda)$ has a meaning for bitangents of $C$. In particular, we can rephrase Corollary 4.2.6 as follows: there is a bijection between the set of isotropic planes of $J_2(C)$ and the set of syzygetic tetrads of bitangents of $C$. Also, let us note that four bitangents to $C$ form a syzygetic tetrad exactly when the 8 points of tangency are the complete intersection of $C$ with a conic.

## 4.3 K3 surfaces and plane quartics together with two bitangents

We have seen that a quaternionic cover corresponds (up to finite choices) to a syzygetic tetrad of bitangent lines, which can be obtained from an arbitrary pair of bitangents. Motivated by this, we construct a family of K3 surfaces starting from a plane quartic (i.e. a non-hyperelliptic genus three curve) together with two bitangent lines. We shall show that these K3 surfaces are polarized by the lattice $U(4) \perp D_{12}$ of rank 14, and hence are parameterized by a certain arithmetic quotient of a Hermitian symmetric domain of type $\mathrm{IV}_6$.

### 4.3.1 Standard notations and facts of lattices and K3 surfaces

By a *lattice* we mean a free $\mathbb{Z}$-module $L$ together with a symmetric bilinear form $(-,-)$. The basic invariant of a lattice is its signature. A lattice is *even* if $(x,x) \in 2\mathbb{Z}$ for every $x \in L$. The direct sum of lattices is always assumed orthogonal, and will be denoted using $\perp$. For a lattice $M \subset L$, $M_L^{\perp}$ denotes the orthogonal complement of $M$ in $L$. Given two lattices $L$ and $L'$ and a lattice embedding $L \hookrightarrow L'$, we call it a *primitive embedding* if and only if $L'/L$ is torsion free.

We shall the following lattices frequently: the root lattices $A_n (n \geq 1)$, $D_m (m \geq 4)$, $E_r (r = 6,7,8)$ and the hyperbolic plane $U$. Given a lattice $L$, $L(n)$ denotes the lattice with the same underlying $\mathbb{Z}$-module as $L$ but with the bilinear form multiplied by $n$.

**Notation 4.3.1.** Let L be an even lattice. We define:

- $L^* := \{y \in L \otimes \mathbb{Q} \mid (x,y) \in \mathbb{Z} \text{ for all } x \in L\}$, the dual lattice;
- $A_L := L^*/L$, the discriminant group endowed with the induced quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$;
- $\mathrm{disc}(L)$: the determinant of the intersection matrix with respect to an arbitrary $\mathbb{Z}$-basis of $L$.
- $O(L)$: the group of isometries of $L$;
- $O(q_L)$: the automorphisms of $A_L$ that preserve the quadratic form $q_L$;
- $O_-(L)$: the group of isometries of $L$ of spinor norm 1;
- $\widetilde{O}(L)$: the group of isometries of $L$ that induce the identity on $A_L$;
- $O^*(L) = O_-(L) \cap \widetilde{O}(L)$.
- $\Delta(L)$: the set of roots of $L$ ($\delta \in L$ is a *root* if $(\delta,\delta) = -2$).
- $W(L)$: the *Weyl group*, i.e. the group of isometries generated by reflections $s_\delta$ in root $\delta$, where $s_\delta(x) = x - 2\frac{(x,\delta)}{(\delta,\delta)}\delta$.

For a surface $S$, the intersection form gives a natural lattice structure on the torsion-free part of $H^2(S,\mathbb{Z})$ and on the Néron-Severi group $\mathrm{NS}(S)$. For a K3 surface $S$, we have $H^1(S,\mathcal{O}_S)=0$, and we identify $\mathrm{Pic}(S)$ and $\mathrm{NS}(S)$. Both $H^2(S,\mathbb{Z})$ and $\mathrm{Pic}(S)$ are torsion-free. The natural map $c_1:\mathrm{Pic}(S)\to H^2(S,\mathbb{Z})$ is a primitive embedding. We shall use $\Lambda_{K3}$ to denote the unique even unimodular lattice $E_8^2\perp U^3$ of signature $(3,19)$, which is isomorphic to $H^2(S,\mathbb{Z})$ for any K3 surface $S$.

### 4.3.2 The K3 surface associated to a smooth plane quartic together with two bitangent lines

Let $C\subset\mathbb{P}^2$ be a smooth plane quartic curve (we also assume that $C$ has no hyperflex, which is a codimension 1 condition in $\mathcal{M}_3$), and let $L_1=V(l_1)$ and $L_2=V(l_2)$ be two bitangent lines (non-hyperflex) of $C$. By taking the double cover $\pi_1:X\to\mathbb{P}^2$ of $\mathbb{P}^2$ branched along the sextic curve $C+L_1+L_2$ and resolving the singularities $\pi_2:\widetilde{X}\to X$, we obtain a a smooth K3 surface $\widetilde{X}$ together with a morphism $\pi:=\pi_2\circ\pi_1$ from $\widetilde{X}$ to $\mathbb{P}^2$. (Note that $\widetilde{X}$ can also be obtained by first resolving the singularities and then take the double cover. More specifically, start with $X_0=\mathbb{P}^2$ and $B_0=C+L_1+L_2$. Blow up a singular point of $B_0$. Let $\epsilon_1:X_1\to X_0$ be the resulting surface, and let the strict transform of $B_0$ together with the exceptional divisor of $\epsilon_1$ reduced mod 2 be the new branch divisor $B_1$. Repeat the process until $B_N$ is smooth. Set $X'=X_N$ and $B'=B_N$ and then take the double cover of $X'$ branched along $B'$. See [51] Chapter III Theorem 7.2.)

#### The Picard lattice

We shall show that $\mathrm{Pic}(\widetilde{X})=U(4)\perp D_{12}$. First let us set up some notations and find a $\mathbb{Z}$-basis for $\mathrm{Pic}(\widetilde{X})$.

Suppose $L_1\cap L_2=\{p_0\}$, $C\cap L_1=\{p_1,p_2\}$ and $C\cap L_2=\{p_3,p_4\}$. It is easy to see that $p_0$ is an $A_1$-singularity and $p_1,\cdots,p_4$ are all $A_3$-singularities. We choose the notations as follows.

- $\gamma$ is the exceptional curve obtained by resolving $p_0$;
- $\alpha_4$ (resp. $\beta_4$) is the strict transformation of $L_1$ (resp. $L_2$);
- $\alpha_1$, $\alpha_2$, $\alpha_3$ are the exceptional curves obtained by resolving $p_1$. Moreover, $\alpha_3$ corresponds to the middle node of $A_3$; in other words, $\alpha_3$ is the exceptional curve that meets $\alpha_4$;
- $\alpha_5$, $\alpha_6$, $\alpha_7$ are the exceptional curves obtained by resolving $p_2$, and $\alpha_5$ meets $\alpha_4$;

- $\beta_1$, $\beta_2$, $\beta_3$ are the exceptional curves obtained by resolving $p_3$, and $\beta_3$ meets $\beta_4$;
- $\beta_5$, $\beta_6$, $\beta_7$ are the exceptional curves obtained by resolving $p_4$, and $\beta_5$ meets $\beta_4$.

Consider the sub-lattice of $\mathrm{Pic}(\widetilde{X})$ generated by $\{\alpha_1, \cdots, \alpha_7, \beta_1, \cdots, \beta_7, \gamma\}$. Note that the only relation among them (the total transformation of $L_1$ is linearly equivalent to that of $L_2$) is

$$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 = \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7.$$

Let

$$x := \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7.$$

We also have $x = \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7$. It is not difficult to see that the set

$$\{\alpha_1, \cdots, \alpha_6, \beta_1, \cdots, \beta_6, \gamma, x\}$$

is a $\mathbb{Z}$-basis of this sub-lattice. The intersection matrix with respect to this (ordered) basis is as follows.

$$\begin{pmatrix}
-2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}$$

Note additionally that the morphism $\widetilde{X} \to \mathbb{P}^2$ is given by the class $h := \gamma + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$.

In what follows, we shall use $M$ to denote the abstract rank 14 lattice spanned by $\{\alpha_1, \cdots, \alpha_6, \beta_1, \cdots, \beta_6, \gamma, x\}$ with the intersection form given by the above matrix. There is a natural lattice embedding $j : M \hookrightarrow \mathrm{Pic}(\widetilde{X})$ as described above.

**Lemma 4.3.2.** *The Picard lattice* $\mathrm{Pic}(\widetilde{X})$ *coincides with* $M$ *via the embedding* $j$*. In particular,* $j$ *is a primitive embedding.*

*Proof.* Let us consider the elliptic fibration $\widetilde{X} \to \mathbb{P}^1$ defined by the function $\pi^*(l_1/l_2)$. It is not difficult to see that the elliptic fibration contains two fibers of type $\mathrm{I}_2^*$ (which corresponds to the two bitangents), and admits a 2-section $\gamma$. Using the associated Jacobian fibration, we have that $\mathrm{disc}(\mathrm{Pic}(\widetilde{X})) = 2^2 \cdot \mathrm{disc}(\mathrm{Pic}(J(\widetilde{X}))) = \pm 64$. Meanwhile, one can easily compute that $\mathrm{disc}(M) = -64$. The lemma then follows from the following standard fact on lattices: $\mathrm{disc}(M) = \mathrm{disc}(\mathrm{Pic}(\widetilde{X})) \cdot (\mathrm{Pic}\,\widetilde{X} : M)^2$. □

Conversely, we make the following observation.

**Proposition 4.3.3.** *Assume that* $S$ *is a K3 surface such that* $\mathrm{Pic}(S)$ *is isomorphic to the lattice* $M$*. Then* $S$ *is the double cover of* $\mathbb{P}^2$ *branched over a reducible curve* $C+L_1+L_2$*, where* $C$ *is a smooth plane quartic and* $L_i$ $(i = 1, 2)$ *are bitangent lines (non-hyperflex) of* $C$*.*

*Proof.* By assumption, there exist $\alpha_1, \cdots, \alpha_6, \beta_1, \cdots, \beta_6, \gamma, x \in \mathrm{Pic}(S)$ satisfying the numerical conditions given by the above $14 \times 14$ intersection matrix. By changing $\alpha_i$ (resp. $\beta_j$ or $\gamma$) to $-\alpha_i$ (resp. $-\beta_j$ or $-\gamma$) (i.e. apply $s_{\alpha_i}$, $s_{\beta_j}$ or $s_\gamma$), we can assume that the classes $\alpha_i$, $\beta_j$ and $\gamma$ are all effective.

Let $h = \gamma + x$. Without loss of generality, we assume that $h$ is nef (which can be achieved by acting by $\pm W(S)$). By replacing $h$ with $-h$, let us further assume that $h$ is effective. Noting that $(h, h) = 2$, the class $h$ defines a degree 2 polarization for $S$. We claim that $h$ is base point free. Otherwise, Mayer's theorem (c.f. [52] Chapter 5 Theorem 27) implies that $h = 2E + R$ with $(E, E) = 0$, $(R, R) = -2$ and $(E, R) = 1$ for some $E, R \in \mathrm{Pic}(S)$. This is a contradiction. Namely, let us write $E = \sum_{i=1}^6 a_i\alpha_i + \sum_{j=1}^6 b_j\beta_j + c\gamma + dx$ for some integers $a_i$, $b_j$, $c$, $d$. Now we have on one hand that $(E, h) = (E, 2E + R) = 1$, on the other hand that $(E, h) = 2d$, which is impossible. So the linear system defined by $h$ gives a degree two map $\pi : S \to \mathbb{P}^2$ branched along a sextic curve $B$.

Let $\alpha_7 = x - (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)$ and $\beta_7 = x - (\beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6)$. It is not difficult to see that $(h)^\perp_M \cap \Delta(M) = \pm\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \beta_1, \beta_2, \beta_3, \beta_5, \beta_6, \beta_7, \gamma\}$. It follows that these are classes of irreducible rational curves, which are contracted by $\pi$ to four $A_3$-singularities and one $A_1$-singularity for the sextic $B$. Since $(\alpha_4, h) = (\beta_4, h) = 1$, the classes $\pi_*\alpha_4$ and $\pi_*\beta_4$ are represented by two lines $L_1$ and $L_2$ respectively. Moreover, since $(\alpha_4, \gamma) = (\beta_4, \gamma) = 1$, $L_1$ and $L_2$ intersects at the $A_1$-singularity of $B$. Similarly, the conditions that $(\alpha_4, \alpha_3) = (\alpha_4, \alpha_5) = 1$ and $(\beta_4, \beta_3) = (\beta_4, \beta_5) = 1$ imply that each line passes through two $A_3$-singularities of $B$. The only possibility is that $B$ contains $L_1$ and $L_2$ as components. □

Next we use the results in [53] (especially Corollary 1.10.2 and Corollary 1.13.3) to determine $M$. For that, we need to compute various invariants of $M$.

**Lemma 4.3.4.** *The lattice $M$ has rank* 14*, determinant* $-64$ *and signature* $(1, 13)$*.*

*Proof.* These can be easily computed from the intersection matrix of $M$. $\square$

**Lemma 4.3.5.** *The discriminant group $A_M$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/4\mathbb{Z})^{\oplus 2}$.*

*Proof.* Let us denote by $\alpha_i^*$ (resp. $\beta_i^*$, $\gamma^*$, $x^*$) be the dual element of $\alpha_i$ (resp. $\beta_i$, $\gamma$, $x$). This means $\alpha_i^*$ is the element of $M \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the pairing between $\alpha_i^*$ and $\alpha_i$ equals 1, and the other pairings of $\alpha_i^*$ with the elements of the basis $\{\alpha_1, \cdots, \alpha_6, \beta_1, \cdots, \beta_6, \gamma, x\}$ is 0. The coefficients (for the above basis) of the dual elements can be read from the rows or columns of the inverse intersection matrix (with respect to the ordered basis $\{\alpha_1, \cdots, \alpha_6, \beta_1, \cdots, \beta_6, \gamma, x\}$):

$$\begin{pmatrix}
-\frac{3}{2} & -1 & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/4 \\
-1 & -\frac{3}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/4 \\
-2 & -2 & -4 & -3 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 \\
-\frac{3}{2} & -\frac{3}{2} & -3 & -3 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 \\
-1 & -1 & -2 & -2 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-\frac{1}{2} & -\frac{1}{2} & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & -1 & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 3/4 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{3}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 3/4 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & -4 & -3 & -2 & -1 & 0 & 3/2 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & -3 & -3 & -2 & -1 & 0 & 3/2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & -2 & -2 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & -1 & -1 & -1 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
\frac{3}{4} & \frac{3}{4} & \frac{3}{2} & \frac{3}{2} & 1 & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{2} & \frac{3}{2} & 1 & \frac{1}{2} & \frac{1}{2} & -1
\end{pmatrix}.$$

We now construct an group isomorphism $A_M = M^\vee/M \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/4\mathbb{Z})^{\oplus 2}$ by $\alpha_1^* - \beta_1^* \mapsto (1, 0, 0, 0)$, $\alpha_4^* \mapsto (0, 1, 0, 0)$, $\alpha_1^* \mapsto (0, 0, 1, 0)$ and $x^* \mapsto (0, 0, 0, 1)$. To verify this, we first check that there is no non-trivial relations among $\alpha_1^* - \beta_1^*$, $\alpha_4^*$, $\alpha_1^*$ and $x^*$ in $M^\vee/M$. Then we observe that all the other dual basis (module out $M$) can be generated by these four elements: $\alpha_2^* \equiv \alpha_4^* - \alpha_1^*$, $\beta_1^* = \alpha_1^* - (\alpha_1^* - \beta_1^*)$, $\beta_2^* \equiv 2x^* + \alpha_4^* - \alpha_1^* + (\alpha_1^* - \beta_1^*)$, $\alpha_3^* \equiv \beta_3^* \equiv \gamma^* \equiv 2\alpha_1^*$, $\alpha_6^* \equiv \alpha_4^*$, $\beta_4^* \equiv \beta_6^* \equiv 2x^* + \alpha_4^*$. $\square$

Using the above explicit isomorphism (i.e. the discriminant group $A_M$ is identified with $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/4\mathbb{Z})^{\oplus 2}$ by $\alpha_1^* - \beta_1^* \mapsto (1, 0, 0, 0)$, $\alpha_4^* \mapsto (0, 1, 0, 0)$,

$\alpha_1^* \mapsto (0,0,1,0)$ and $x^* \mapsto (0,0,0,1)$.), we derive a formula for the quadratic form $q_M$.

**Lemma 4.3.6.** *For $a, b \in \{0,1\}$ and $c, d \in \{0,1,2,3\}$, we have*

$$q_M(a,b,c,d) \equiv a^2 + b^2 + \frac{c^2}{2} + d^2 + ab + ac + bc + bd + \frac{3}{2}cd \in \mathbb{Q}/2\mathbb{Z}.$$

**Proposition 4.3.7.** *The lattice $M$ is isomorphic to $U(4) \perp D_{12}$.*

*Proof.* We shall use Corollary 1.13.3 of [53]. The two lattices have the same signature and isomorphic discriminant group. It remains to show that the discriminant forms are isomorphic to each other as well.

There are three classes of quadratic forms $w_{p,k}^{\epsilon}$, $u_k$ and $v_k$ on a finite abelian group. We refer the readers to Section 1.5.3 of [54] for the definitions. We also need Table A.2 and Table A.3 of op. cit. later. The significance of these forms is that every nondegenerate quadratic form on a finite abelian group is isomorphic an orthogonal direct sum of them (c.f. [53] Propositions 1.8.1 and 1.8.2). For the lattice $M$, we have obtained a formula of the discriminant quadratic form $q_M$. In particular, we find out that the values of $q_M$ have at worst denominator 2, which rules out a couple possibilities (e.g. there can be no direct summand $w_{2,2}^{\epsilon}$, $\epsilon \in \{\pm 1, \pm 5\}$ or $u_4$ or $v_4$ for $q_M$). The possible expressions for $q_M$ are as follows: $v_1 \perp u_2$, $u_1 \perp u_2$, $w_{2,1}^{1} \perp w_{2,1}^{-1} \perp u_2$, $w_{2,1}^{1} \perp w_{2,1}^{1} \perp u_2$, $w_{2,1}^{-1} \perp w_{2,1}^{-1} \perp u_2$, $v_1 \perp v_2$, $u_1 \perp v_2$, $w_{2,1}^{1} \perp w_{2,1}^{-1} \perp v_2$, $w_{2,1}^{1} \perp w_{2,1}^{1} \perp v_2$ and $w_{2,1}^{-1} \perp w_{2,1}^{-1} \perp v_2$. By counting the numbers of 0, 1, $\frac{1}{2}$, $-\frac{1}{2}$ for each expression and compare with those for $q_M$, the only possibility left is $v_1 \perp u_2$, and hence $q_M \cong v_1 \perp u_2$. Now that the discriminant form of $U(4) \perp D_{12}$ is also isomorphic to $v_1 \perp u_2$ (see [54] Table A.2 and Table A.3), we complete the proof. ☐

**Proposition 4.3.8.** *The lattice $M$ admits a unique primitive embedding $M \hookrightarrow \Lambda_{K3}$ into the K3 lattice $\Lambda_{K3}$. The orthogonal complement $T := M^{\perp}_{\Lambda_{K3}}$ is isomorphic to $U \perp U(4) \perp D_4$.*

*Proof.* The first statement follows from Theorem 1.14.4 of [53]. The second statement can be proven in a similar way as above (note that $q_T = -q_M$ and the lattice $D_4$ has discriminant form $v_1$ by [54] Table A.2). ☐

### $M$-polarized K3 surfaces and the period map

**Definition 4.3.9.** The lattice $M$ as above. An *$M$-polarized K3 surface* is a pair $(S, j)$ consisting of a K3 surface $S$ and a primitive lattice embedding $j : M \hookrightarrow \mathrm{Pic}(S)$. The embedding $j$ is called the *$M$-polarization* of $S$. If the polarization is understand, we simply say that $S$ is an $M$-polarized K3 surface.

It is a standard fact (see for example [55]) that the moduli space of $M$-polarized K3 surfaces is a quotient $\Gamma\backslash\mathcal{D}$ for a certain Hermitian symmetric domain of type IV and a certain arithmetic subgroup $\Gamma$. Specifically, the condition of $M$-polarization determines a tower of primitive embeddings

$$M \hookrightarrow \mathrm{Pic}(S) \hookrightarrow H^2(S,\mathbb{Z}) \cong \Lambda_{K3}.$$

It follows that the periods of $M$-polarized K3 surfaces belong to the following subdomain of the period domain of K3 surfaces:

$$\mathcal{D} = \{\omega \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid (\omega,\omega) = 0, (\omega,\bar{\omega}) > 0, \omega \perp M\}^0.$$

Conversely, since $\mathrm{Pic}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$, every point of $\mathcal{D}$ corresponds to an $M$-polarized K3 surface. Let $T$ be the orthogonal complement of $M$ (i.e. the transcendental lattice). It is convenient to identify $\mathcal{D}$ with the domain

$$\{\omega \in \mathbb{P}(T \otimes \mathbb{C}) \mid (\omega,\omega) = 0, (\omega,\bar{\omega}) > 0\}^0.$$

Note that $\mathcal{D}$ is a bounded symmetric domain of type $\mathrm{IV}_6$. The group $O^*(T)$ acts on $\mathcal{D}$ naturally.

In our situation, we consider the space $\widetilde{\mathcal{M}}$ parametrizing $(C, L_1, L_2, \sigma, \sigma_1, \sigma_2)$, where $C$ is a smooth plane quartic curve, $L_1$ and $L_2$ are two bitangents (non-hyperflex), $\sigma : \{1,2\} \to \{L_1, L_2\}$ labels $L_1$ and $L_2$ and $\sigma_i : \{1,2\} \to C \cap L_i$ $(i = 1,2)$ labels the tangency points. The K3 surface $\widetilde{X}$ (together with the labelings) carries a natural $M$-polarization $j : M \hookrightarrow \mathrm{Pic}(\widetilde{X})$. Therefore, there is a well-defined map $\widetilde{\mathcal{P}} : \widetilde{M} \to O^*(T)\backslash\mathcal{D}$ sending $(C, L_1, L_2, \sigma, \sigma_1, \sigma_2)$ to the period of $\widetilde{X}$.By Torelli theorem for lattice-polarized K3 surfaces, $\widetilde{\mathcal{P}}$ is injective. Moreover, Proposition 4.3.3 and subjectivity of the period map implies that $\widetilde{\mathcal{P}}$ is birational.

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