A Data Driven Design Methodology for SS Dyads and its Application to Unified Synthesis of Planar, Spherical and Spatial Linkages

A Dissertation presented

by

Xin Ge

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mechanical Engineering

Stony Brook University

December 2016

Stony Brook University

The Graduate School

Xin Ge

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Qiaode Jeffrey Ge - Dissertation Advisor
Professor and Interim Chair, Mechanical Engineering Department

Anurag Purwar - Co-Advisor
Research Associate Professor, Mechanical Engineering Department

Nilanjan Chakraborty - Chairperson of Defense
Assistant Professor, Mechanical Engineering Department

Dimitris Samaras - Committee Member
Associate Professor, Computer Science Department

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

A Data Driven Design Methodology for SS Dyads and its Application to Unified Synthesis of Planar, Spherical and Spatial Linkages

by

Xin Ge

Doctor of Philosophy

in

Mechanical Engineering

Stony Brook University

2016

This research deals with a general kinematic synthesis problem of how to generate the motion of a spatial platform subject to one or more spherical constraints. Each spherical constraint can be modeled by a SS dyad, which is a spatial two-link kinematic chain connected by two spherical (S) joints. It defines a five-degree-of-freedom motion of the end link such that a point stays on the surface of a sphere. A spherical constraint degenerates into a plane constraint when the radius of the sphere becomes infinite. In this way, a plane constraint is considered as a special case of a spherical constraint and a one-degree-of-freedom motion of a moving platform defined by a combination of five spherical and/or plane constraints can be modeled by a 5-SS platform linkage.

In addition to a SS dyad, a spherical constraint can also be realized alternatively with a TS dyad, where T denotes a universal joint, or a three-link Revolute-

Revolute-Spherical chain, or RRS chain, where R denotes a revolute joint. The plane constraint can be realized by RRS, RPS and PRS kinematic chains, where P denotes a prismatic joint. In this way, a SS dyad based formulation could lead to a unified synthesis of platform linkages formed by all these kinematic chains.

Traditionally, the problem of mechanism synthesis separates into type synthesis and dimensional synthesis. Type synthesis deals with the selection of a mechanism type based on given task; dimensional synthesis deals with the determination of link dimensions. While dimensional synthesis is highly amendable to mathematical treatment, type synthesis remains elusive and highly dependent on designer's prior experience, despite of recent theoretical advancement in graph theory or topology. Furthermore, recent research indicates that there are mechanism synthesis problems that one can not separate type and dimensional synthesis, i.e., slight variation of input data might lead to different mechanism types. This calls for a data driven simultaneous type and dimensional synthesis approach. The central idea to this approach is to synthesize geometric constraints for a given task. This includes the location of the moving point as well as the location and radius associated with the spherical constraint. Once the geometric constraints are determined from the given task, the next step is to figure out kinematic chains such as TS and RRS that can be used to generate the geometric constraints.

In this dissertation, we first consider a simpler version of the problem, which is to design a planar mechanism such that its moving link is constrained by one or more circular constraints. Instead of separating type and dimensional synthesis by selecting joint types, either revolute (R) or prismatic (P) joint before dimensional synthesis, we present a unified design equation such that the selection of joint types and determining link lengths can be carried out simultaneously. In the process, we developed a linear representation of the design equations that can be extended to the synthesis of spherical four-bar linkages.

The spherical constraints in 5-SS linkage synthesis problem are extensions of circular constraints from two to three dimensions. This leads to linear form of design equations for SS dyads that are similar to that of planar and spherical RR dyads but with additional variables and bilinear constraints. In contrast to the existing approach that leads to a system of polynomial design equations, which can be solved using homotopy method, this linear formulation leads to a generalized eigenvalue problem that can be more readily solved than the homotopy method.

It turns out the SS dyad formulation can be applied to the synthesis of spatial RR dyads whose moving and fixed axes in general neither intersect nor are in parallel. Planar and spherical dyads can be viewed as special cases of a spatial RR dyad when the two axes either intersect or are in parallel. In all RR dyads, a point on the moving axis traces out a circle. Furthermore, as a circle can be obtained as intersection of two spheres or a plane and a sphere, the circular constraints of spatial RR dyads can be obtained as intersection of spherical and plane constraints. Thus, planar, spherical and Bennett 4R linkages, which are composed of different types of RR dyads, may be treated as special cases of spatial 5-SS linkages. This leads to a new and unified methodology for synthesizing planar, spherical, and spatial RR dyads without a priori knowledge whether the given input positions are planar, spherical or spatial in nature.

Table of Contents

List of Figures

List of Tables

Acknowledgements

My deepest gratitude goes first and foremost to Professor Qiaode Jeffrey Ge, my advisor, for his mentorship, patient support and continuous encouragement when I am confused in research, also for his broad and immense knowledge so that I can always find the best tracks in my research. It was an honor working with him, and I will never forget these days.

I want to thank my co-advisor, Professor Anurag Purwar. He supported and guided me during my years of study, for his sincere advices and kindness, so that I have confidence in continuing my study.

I would like to express my thanks to my committee members, Professor Nilanjan Chakraborty and Professor Dimitris Samaras, for taking their precious time in attending my presentation, their insight feedback and comment on my proposal.

There is also thanks to Professor Ping Zhao and Professor Xiangyun Li, for their initial and friendly help, providing suggestions when I meet obstacles.

The research is financially founded by NSF grant under Award No. 1563413 and I want to express my thanks to their support.

Last but not the least, my thanks goes to my family, for raising me and supporting me for all these years.

Previously Published Work

Ge, X., Purwar, A., Ge, Q. J., 2016. "From 5-SS Platform Linkage to Four-Revolute Jointed Planar, Spherical and Bennett Mechanisms". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, Paper No. DETC2016-60574.

Zhao, P., Ge, X., Zi, B., Ge, Q. J., 2016. "Planar Linkage Synthesis for Mixed Exact and Approximated Motion Realization Via Kinematic Mapping". ASME Journal of Mechanisms and Robotics, **8**(5):051004-051004-8, DOI:10.1115/1.4032212.

Ge, Q. J., Ge, X., Purwar, A., Li, X., 2015. "A Null-Space Analysis Method for Solving Bilinear Equations in Kinematic Synthesis of Planar and Spherical Dyads". The 14th IFToMM World Congress, DOI: 10.6567/IFToMM. 14TH.WC.OS2.026.

Ge, X., Ge, Q. J., Gao, F., 2015. "A Novel Algorithm for Solving Design Equations for Synthesizing Platform Linkages". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2015-47981

Zhao, P., Ge, X., Zi, B., Ge, Q. J., 2015. "Planar Linkage Synthesis for Mixed Exact and Approximated Motion Realization Via Kinematic Mapping". In ASME

International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2015-47231.

Li, X., Ge, X., Purwar, A., Ge, Q. J., 2014. "A Unified Algorithm for Analysis and Simulation of Planar Four-Bar Motions Defined With R- and P-Joints". ASME Journal of Mechanisms and Robotics, **7**(1):011014-011014-7, DOI:10.1115/1.4029295.

Chapter 1

Introduction and Background

This paper focuses on the rigid body guidance problem which is to determine the geometric parameters of mechanisms, parallel manipulators and platform robotic systems to guide the end point through a number of specified positions. This problem has been studied extensively for planar, spherical, as well as spatial mechanisms known as the dimensional synthesis or geometric design [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The process to solve the problem of kinematic synthesis for rigid body guidance is to formulate the design equations for a given type of mechanism and find a set of design parameters associated with the dimensions of links and joints such that the end-effector is guided through a set of prescribed positions. Such equations is formulated by the constraints information of movement of the end-effector. The synthesis problems can be transferred to that of the study of the end-effector motion under a single or multiple geometric constraints, such as the motion for which a point on the moving body stays on a circle or a line in plane kinematics or for which a moving point stays on a sphere or a plane in spatial kinematics, or the moving point is constrained by combination of sphere and plane in some other spatial configurations.

For a simple case as example, the Burmester problem, is to find the geometric dimensions of a planar 4R linkages (where R denotes a revolute joint) for a set of five positions of the coupler link [12]. Innocenti [5] presented a polynomial solution to the spatial counterpart of this classical Burmester problem, which is to find the center and radius of each of the five spheres that constrain the legs of a spatial platform while it guides a spatial body through seven task positions.

In this paper, we restrict our attention to such kinds of geometric constraints that the design parameters may be divided into two groups, one is associated with the coordinates of the moving points and the other is associated with the coordinates of the geometric constraint. The inner relationship of these design parameters are bilinear. We found the bilinear relationship representing the relation of design parameters of the constraint equations of each linkage discussed and developed a general and simple as while as reliable method to solve for these design parameters.

Generally, in the field of kinematic synthesis, solving a system of bilinear equations is typically using traditional methods or procedures developed for systems of polynomial equations. These methods could be symbolic, numeric or geometric. Dialytic elimination [13] and Grobner bases [14] are symbolic methods with roots in algebraic geometry. They can be used for eliminating variables from a polynomial system and thereby transform the problem into that of finding roots of univariate polynomials [15]. They are most effective for systems with small number of polynomials and with low degree. In addition, the resulting high-degree univariate polynomial could have its numerical problems such as ill-conditioning that reduces its practical effectiveness [16, 17]. The numeric methods include local search and iterative techniques such as the Newton's method as well as homotopy continuation method. Local methods require good initial guess and are often difficulty to obtain all solutions. Homotopy based methods have shown to be most effective in finding all the roots of a polynomial system and have made major progress in the past decade [18, 19, 20], however, it still requires an initial guess. The advantage is that with enough iterations and consumption of time, one can always find full set of solutions. The backside is clear that the whole process could be time or memory consuming.

We formulate our design equations by concept of dyad synthesis. The term “dyad synthesis” is often used to describe the synthesis process resulting from a bilinear systems of equations of constraint, although for the spatial case a single constraint may be realized by a series of more than two links and joints [11, 21, 1, 22] which can be referred to the term “triad”.

There are many research on the algorithm for solving such bilinear systems. Angeles and Bai [23] have developed a bilinear formulation on the study of the spherical four-bar linkage synthesis. They provided a two-step synthesis method, which sequentially deals with equation-solving by a semigraphical approach and branching detection. M.Plecnik and J. M. McCarthy introduced a bilinear structure of certain six-bar linkages [24]. The result is then solved using the polynomial numerical homotopy software $BERTINI$. For bennett 4R mechanism synthesis, Alba Perez and J. M. McCarthy [25] use the cylindroid[26] associated with the Bennett's linkage to simplify the design equation.

We present a unified methodology both on formulation and the way to solve for such problems with circular, linear, planar and spherical constraints. Our formulation of constraint equation is not used for a single, but a category of linkages. This means that our algorithm is data driven based. In other words, it determines types and dimensions simultaneously based on the given data instead of intuition. Traditionally, people separate the two problems by picking a type of linkage based on experience first and then finding the dimensions of that specific linkage. If the chosen linkage is not correct, one has to pick another one and repeat the process. We obtained our design equation by substituting the original design parameters by a set of new intermediate parameters and formulate new matrix equations based on the inner relations of new parameters. This not only can unify the algorithm itself but also indicates the type of input data. Thus it makes this algorithm capable of dealing with more special cases, such as planar and spherical, in a unified way. It has potential to be applied to other linkages as well.

The construction of the paper is as following. We first present the different types of planar, spherical and spatial dyads as well as develop the geometric constraint design equations for each type. Then we start to talk about the linkages constructed by these dyads separately, the unified method for five positions synthesis of planar and spherical four bar linkages, seven position synthesis of spatial five SS platform linkage, application of seven position synthesis of 5-SS linkage to planar, spherical and bennett four bar linkage. Next we develop the six and five position synthesis with one or more additional constrains of spatial 5-SS platform linkage. Finally a unified algorithm is developed for five position synthesis of general planar, spherical and spatial positions. This algorithm combines the linkage synthesis problems mentioned in the first part and the 5-SS linkage together. The corresponding algorithm is described in each part and numeric examples are given at the end of each chapter.

Chapter 2

Types of Planar, Spherical and Spatial Dyads

2.1 Introduction

In this chapter, we extend the idea of dyad synthesis and formulate the geometric constraints of planar RR, PR, RP dyad, spherical RR dyad and spatial SS dyad. Furthermore, we combine the geometric constraints of planar and spherical dyads into one general homogeneous equation in order to unify the synthesis method of both categories of dyads.

2.2 Planar RR, PR and RP Dyads

Planar dyads are links with both revolute (R) or prismatic (P) end joints as well as the combination of both. The constraints can be found from the illustrations. Fig.2.1 to 2.3 illustrate the planar RR, PR and RP dyads separately. Solid small circles represent revolute joints (R joints) while blocks represent prismatic joints (P joints). Specifically, the end link of a RR dyad moves along a circle of which the center point is its fixed joint, because the distance of end link and fixed link is constant. The prismatic block of PR dyad moves along a straight line, thus geometric constraint for PR dyad is a line. For RP dyad, the end link, the prismatic joint, although goes along a moving line, the line is always tangent to a fixed circle with fixed joint its center.

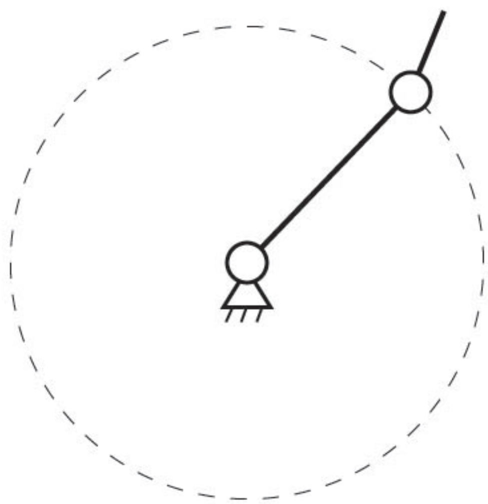

Figure 2.1: Planar RR Dyad: one joint moves along a circle; another joint stays on the center of the circle.

The design equation related to each of the dyads is discussed as following.

Let $\mathbf{X} = (X_1, X_2, X_3)$ (where $X_3 \neq 0$) be a vector of homogeneous coordinates of the moving pivot in the fixed plane F, and let $\mathbf{a} = (a_1, a_2, a_3, a_0)$ be another set of homogeneous coordinates associated with a circle that constrains the movement of the point $\mathbf{X}$, where the center of the circle is given by $(a_1/a_0, a_2/a_0)$. Then a planar RR dyad is defined by the following equation of a circle:

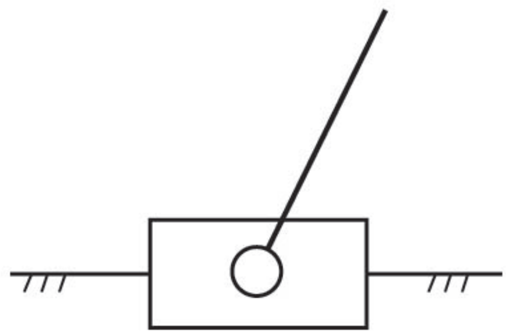

Figure 2.2: Planar PR Dyad: prismatic joint moves on a line.

$$a_1X_1 + a_2X_2 + a_3X_3 = a_0(\frac{X_1^2 + X_2^2}{2X_3}), \tag{2.1}$$

with the radius r being given by

$$r^2 = (2a_0a_3 - a_1^2 - a_2^2)/a_0^2. \tag{2.2}$$

As for a PR dyad, the constraint equation is a line equation. This is the case when $a_0 = 0$ in Equation 2.1.

$$a_1X_1 + a_2X_2 + a_3X_3 = 0. \tag{2.3}$$

A RP dyad can be represented by an infinity of lines tangent to concentric

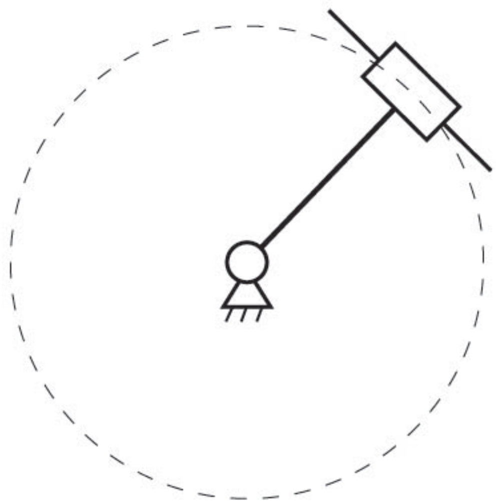

Figure 2.3: Planar RP Dyad: prismatic joint moves on a line tangent to a circle.

circles of different radii, thus one could use a line passing through the center of those circles as the line constraint without losing generality. In this case, due to the duality of point and line geometry in the plane [27], a linear constraint of the same form as 2.3 can be used if we replace the point coordinates $\mathbf{X} = (X_1, X_2, X_3)$ with homogeneous line coordinates $\mathbf{L} = (L_1, L_2, L_3)$.

2.3 Spherical RR Dyad

A spherical RR dyad is shown in Figure 2.4. The constraint requires the moving joint b to be staying on a circle with axis AO. Since the angle between fixed pivot axis AO and the moving pivot axis BO is constant, we can utilize this feature to constrain a spherical RR dyad.

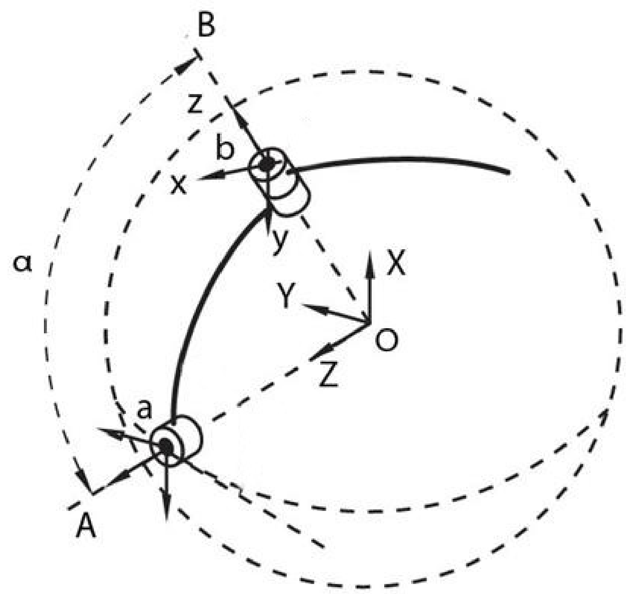

Figure 2.4: Spherical RR Dyad: origin O of fixed frame X-Y-Z is located at the center of sphere; rotational axis AO of fixed joint a; moving frame x-y-z; rotational axis BO of moving joint b; angle α between fixed and moving axes.

Define $\mathbf{a} = (a_1, a_2, a_3)$ as a unit vector representing the location of the fixed pivot A and $\mathbf{X} = (X_1, X_2, X_3)$ is another unit vector defining the location of the moving pivot B of the end link. The origin of the global coordinate is located at center point O. α is the angle between AO and BO. Thus we have the relationship,

$$\mathbf{a} \cdot \mathbf{X} = \cos\alpha \tag{2.4}$$

This leads to a linear equation.

$$a_1X_1 + a_2X_2 + a_3X_3 + d = 0 \tag{2.5}$$

where $d = -\cos\alpha$ with α being the constant angle between $\mathbf{a}$ and $\mathbf{X}$.

2.4 Spatial RR Dyad

A spatial RR dyad is a dyad composed of two revolute joints. There are three cases based on the relationships of the two rotational axes of revolute joints. They define three kinds of ruled surfaces, hyperboloid, cylinder and conical surface. A ruled surface is defined if through every point on the surface there is a straight line that lies on. In the case where the two axes of the revolute joints are neither parallel or intersecting, it can be categorized into a general spatial RR dyad, as shown in Figure 2.5.

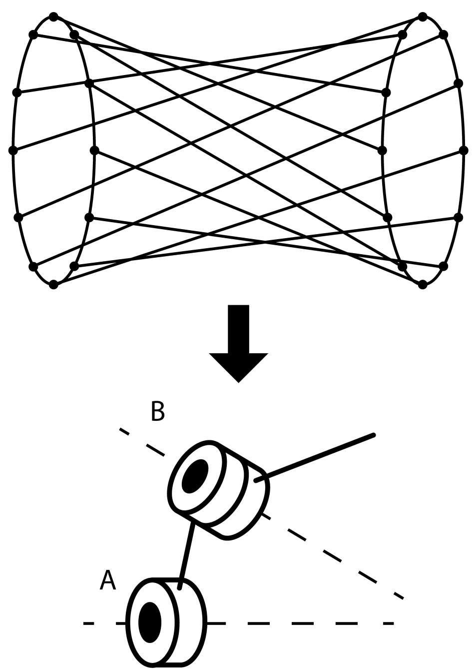

Figure 2.5: A General Spatial RR Dyad from a Circular Hyperboloid Surface of One Sheet: Two points move along two parallel circles with same rotational axis at the same speed. The line defined by these two points at different positions. The equivalent RR dyad of hyperboloid, where A and B represent the axes of fixed and moving joints.

The figure shows two points move along two parallel circles with same rotational axis at the same speed. The line defined by these two points moves on the surface of a circular hyperboloid of one sheet. The figure shows different positions of this line. The lower figure is the equivalent RR dyad of this hyperboloid, where A and B represent the axes of fixed joint and moving joint defined by axis of circles and direction of moving line separately.

When axes of fixed and moving joints are parallel, it becomes a planar RR dyad shown in Figure 2.6. Two points move along two parallel circles with same rotational axis at the same speed. The line defined by these two points are parallel to the axis of the two circles and moves on the surface of a circular cylinder. The figure shows different positions of this line. The lower figure is the equivalent RR dyad of this cylinder, where A and B represent the axes of fixed joint and moving joint defined by axis of hyperboloid and direction of moving line separately.

Another special case is when the two axes are intersecting into one point. This becomes a spherical RR dyad in Figure 2.7. Spherical RR dyad can be generated by a circular conical surface. Conical surface is formed by the union of all the straight lines that pass through a fixed point, the apex or vertex. The lower figure shows the equivalent spherical RR dyad, where A is the rotational axis of fixed joint defined by axis of two circles and B is the axis of moving joint defined by any one of straight line on the conical surface.

One more to be mentioned is that our 5-SS linkage synthesis algorithm is derived from spherical and planar constraints. Thus we can represent a circle by the intersection of a plane and a sphere or two spheres, Figure 2.8.

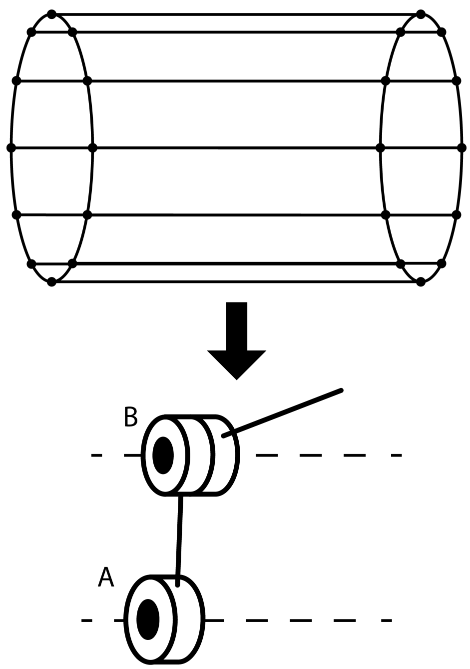

Figure 2.6: A Planar Spatial RR Dyad from a Circular Cylinder Surface: Two points move along two parallel circles with same rotational axis at the same speed. The line defined by these two points are parallel to the axis at different positions. The equivalent planar RR dyad, where A and B represent the axes of fixed and moving joints.

2.5 Spherical and Planar Constraints of Spatial Dyads

In this section, we will review the leg types (leg in this paper refers to a spatial dyad or triad) by spherical or planar constraints as cited in [28]. Both constraints can be represented by the same homogeneous equation, leading to a unified representation of all six types of legs for a platform linkage.

Figure 2.9 shows that a spatial SS leg is associated with a spherical constraint.

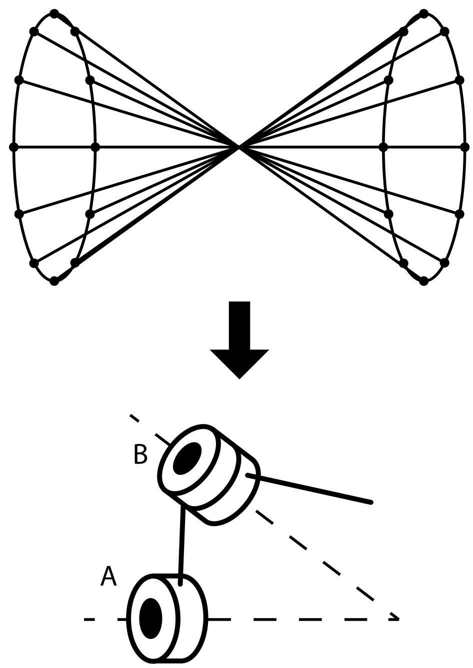

Figure 2.7: A Spherical Spatial RR Dyad from a Circular Conical Surface: Two points move along two parallel circles with same rotational axis at the same speed. Different positions of the line defined by two moving points intersect into one point. The equivalent spherical RR dyad, where A and B represent the axes of fixed and moving joints.

One of the spherical joint in SS leg is fixed and defines the center of the sphere while another moves on the surface of the sphere. Furthermore, a spherical joint can be substituted or decomposed by one or more other joints, e.g. a T (universal) joint or two R (revolute) joints. The former becomes a TS leg, Figure 2.10, and the latter is a RRS chain, Figure 2.11. They all constrain a point on the moving platform to the surface of a sphere. For a RRS leg, the axes of two revolute joints intersect and form a spherical RR dyad. The point on the platform, coincident with S joint here, always stays on the surface of a sphere defined by the spherical

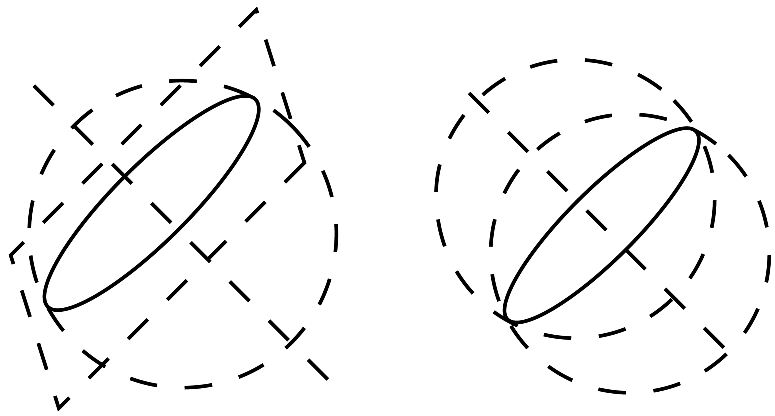

Figure 2.8: Circle Defined by the Intersection of a Plane and a Sphere or the Intersection of Two Spheres.

RR dyad.

While the length of the SS leg which is the radius of the constraint sphere becomes infinite, the sphere can be treated as a plane. This relationship can be seen algebraically from the design equation.

There are many leg types associated with a plane constraint, e.g. RRS, RPS and PRS legs. A RRS leg is shown in Figure 2.12. The axes of two revolute joints are parallel to each other (intersect at infinity). A point on the end effector, coincident with S joint, stays on a plane in the fixed frame. RPS and PRS legs are shown in Figure 2.13 and Figure 2.14 separately. The axes of R and P joints are perpendicular to each other.

When a point of a moving platform is found to be constrained on a sphere, we can apply SS, TS or RRS (with two intersecting R-axes) leg, to trace the task motion. For RRS leg, the design parameters can vary as long as it satisfies the same spherical constraint. For SS and TS legs, their design parameters are determined once the sphere constraints are found.

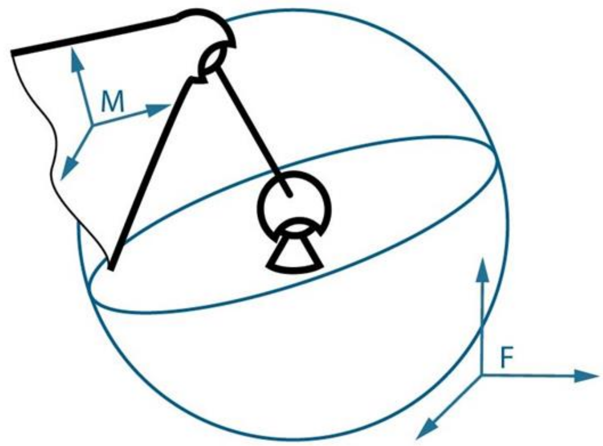

Figure 2.9: SS Leg: fixed spherical joint defines center of sphere; moving spherical joint moves on the surface of the sphere; fixed frame F; moving frame M.

Likewise, when the resulting constraint becomes a plane constraint, we can use RRS (with two parallel R-axes), RPS and PRS legs to generate such constraint. This establishes the basis for formulating the platform linkage synthesis problem that unifies the choice of the six legs.

The design equation is similar to that of the planar and spherical dyads. Let $\mathbf{X} = (X_1, X_2, X_3, X_4)$ (where $X_4 \neq 0$) denote the homogeneous coordinates of a point in the fixed frame. Then the sphere equation in homogeneous form can be written as,

$$2a_1X_1 + 2a_2X_2 + 2a_3X_3 + a_4X_4 = a_0(\frac{X_1^2 + X_2^2 + X_3^2}{X_4}). \tag{2.6}$$

When $a_0 \neq 0$, the center of the sphere is given by,

$$\mathbf{a} = (a_1/a_0, a_2/a_0, a_3/a_0) \tag{2.7}$$

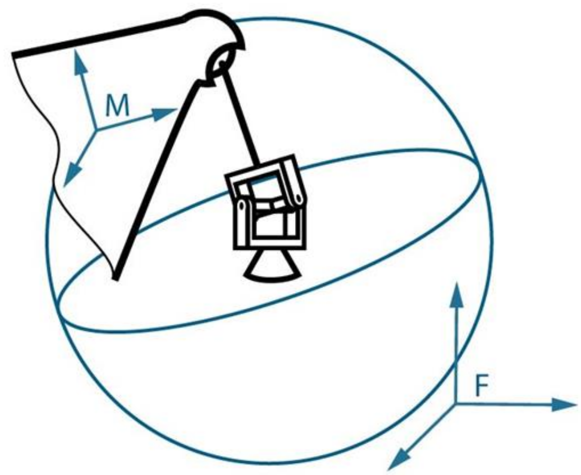

Figure 2.10: TS Leg: fixed universal joint defines center of sphere; moving spherical joint moves on the surface of the sphere; fixed frame F; moving frame M.

and the square of the radius r of the sphere is

$$r^2 = (a_1^2 + a_2^2 + a_3^2 + a_0 a_4)/a_0^2. \tag{2.8}$$

When $a_0 = 0$, the sphere equation reduces to the equation of a plane:

$$2a_1X_1 + 2a_2X_2 + 2a_3X_3 + a_4X_4 = 0 \tag{2.9}$$

In this case, (a_1, a_2, a_3) becomes the directional vector of the plane.

Hence, Equation 2.6 is a unified representation for both a sphere and a plane in homogeneous form.

Planar and spherical dyads are special cases of spatial legs, in other words, one can always use the spatial legs to substitute either planar or spherical dyads and obtain an equivalent planar or spherical linkages. Thus Equation 2.6 can

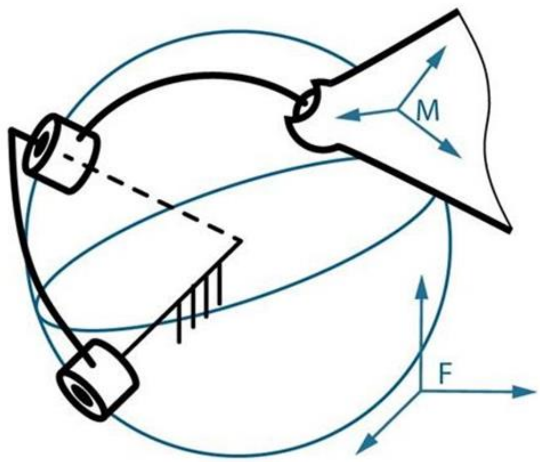

Figure 2.11: Spherical RRS Chain: axes of two revolute joints intersect into center of sphere; the spherical joint moves on the surface of the sphere; fixed frame F; moving frame M.

be treated as the unified design equation planar, spherical and spatial legs. The numerical verification and examples will be given in later chapters.

2.6 Summary

This chapter discusses the summary of the geometric constraints imposed by different types of dyads as well as the resulting design equations drawn from these constraints. We present circle, line, sphere and plane constraints while they can further be simplified to the intersections of spheres and planes. These dyads are the fundamentals in formulating any mechanical linkages in later chapters. One may note that in spatial situations, a sphere or a plane may extend to represent a triad instead of dyad, thus this method is potential to apply to more mechanisms.

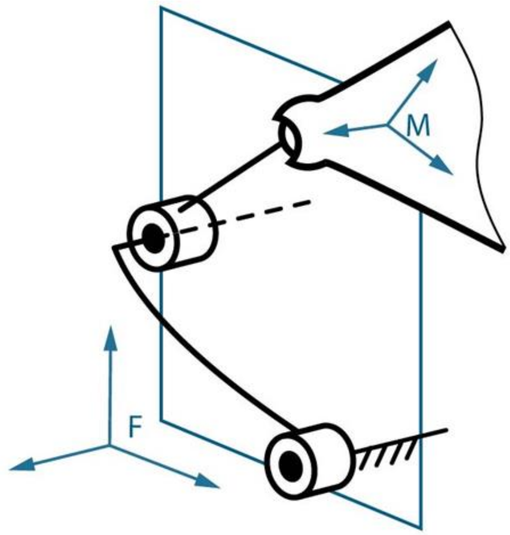

Figure 2.12: Planar RRS Chain: axes of two revolute joints are perpendicular to a plane; the spherical joint moves on the surface of the plane; fixed frame F; moving frame M.

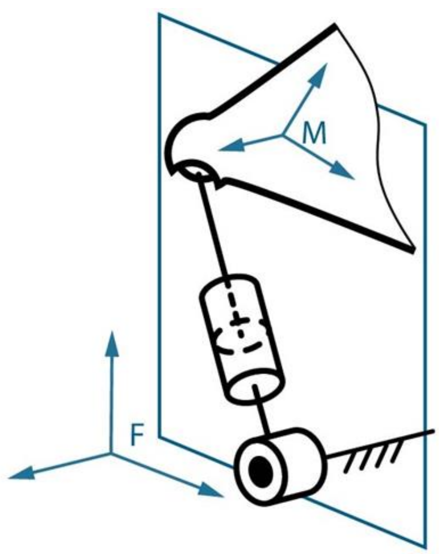

Figure 2.13: Planar RPS Chain: axis of revolute joint is perpendicular to a plane; moving direction of cylindrical joint is parallel to the plane; spherical joint moves on the plane fixed frame F; moving frame M.

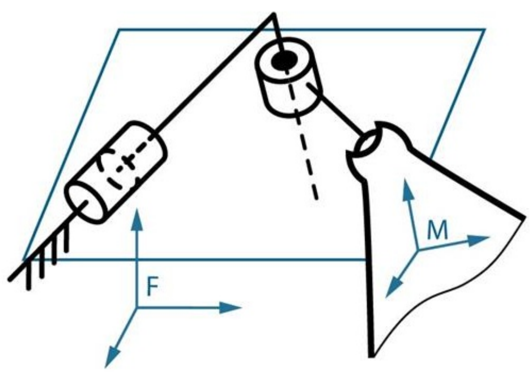

Figure 2.14: Planar PRS Chain: moving direction of cylindrical joint is parallel to a plane; axis of revolute joint is perpendicular to the plane; spherical joint moves on the plane; fixed frame F; moving frame M.

Chapter 3

Task Driven Type and Dimensional Synthesis of Planar and Spherical Linkages

3.1 Introduction

This section introduces and develops the task driven planar and spherical linkage synthesis method. This method acts as the fundamental for synthesis of the other complex linkages in future chapters. There are some initial development and previous work in this problem ([29], [30], [31], [32], [33]), however, it cannot be extended onto other linkages such as spatial 4R or 5-SS linkages. Thus we utilize the basic formulation of the equations and develop an extendable bilinear form for the algorithm throughout this paper.

3.2 Planar Quaternions

In mathematics and mechanics, planar quaternions are well used to represent planar rigid body displacement ([34], [22], [35]). We use planar quaternions to represent the displacement because it allows us to visualize the results as different ruled surfaces, thus provides a more intuitive method in synthesizing this problem.

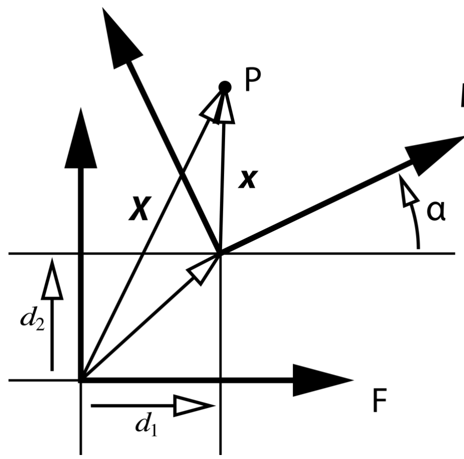

Figure 3.1: Transformation of Coordinate: fixed coordinate F; moving coordinate M; $\mathbf{x}$ is the coordinate of point P in moving frame; $\mathbf{X}$ is the coordinate of point P in fixed frame; d_1, d_2 are the translations of moving frame; α is the rotation of moving frame.

Consider a planar displacement with $\mathbf{d} = (d_1, d_2)$ being the vector of translation and α the angle of rotation. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{X} = (X_1, X_2, X_3)$ denote the homogeneous coordinates of a point in the moving frame M and fixed frame F, respectively. See Figure 3.2. The relationship between $\mathbf{x}$ and $\mathbf{X}$ for a planar displacement parameterized by (d_1, d_2, α) is given as follows:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & d_1 \\ \sin\alpha & \cos\alpha & d_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{3.1}$$

Introducing a planar quaternion, $\mathbf{z} = (z_1, z_2, z_3, z_4)$, formulated as such that (see, for example, [22, 36, 37]):

$$
\begin{aligned}
z_1 &= (d_1/2)\cos(\alpha/2) - (d_2/2)\sin(\alpha/2), \\
z_2 &= (d_1/2)\sin(\alpha/2) + (d_2/2)\cos(\alpha/2), \\
z_3 &= \sin(\alpha/2), \\
z_4 &= \cos(\alpha/2),
\end{aligned}
\tag{3.2}
$$

Then one can rewrite Equation 3.1 in quaternion form as

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} z_4^2 - z_3^2 & -2z_3z_4 & 2(z_1z_3 + z_2z_4) \\ 2z_3z_4 & z_4^2 - z_3^2 & 2(z_2z_3 - z_1z_4) \\ 0 & 0 & z_3^2 + z_4^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{3.3}
$$

3.3 A Unified Representation for Planar Dyads

Substituting Equation 3.3 into Equation 2.1, we obtain, after some algebra:

$$
A_0P_0 + A_1P_1 + A_2P_2 + A_3P_3 + \cdots + A_9P_9 = 0, \tag{3.4}
$$

where

$$
\begin{aligned}
&A_0 = z_3^2 + z_4^2, && A_1 = z_1z_3 - z_2z_4, \\
&A_2 = A_6 = (z_3^2 - z_4^2)/2, && A_3 = -A_5 = z_3z_4, \\
&A_4 = z_2z_3 + z_1z_4, && A_7 = -(z_1^2 + z_2^2), \\
&A_8 = z_1z_3 + z_2z_4, && A_9 = z_2z_3 - z_1z_4,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&P_0 = 2a_3x_3 - a_0(x_1^2 + x_2^2)/x_3, \\
&P_1 = a_0x_1, \qquad P_2 = a_1x_1, \qquad P_3 = a_2x_1, \\
&P_4 = a_0x_2, \qquad P_5 = a_1x_2, \qquad P_6 = a_2x_2, \\
&P_7 = a_0x_3, \qquad P_8 = a_1x_3, \qquad P_9 = a_2x_3.
\end{aligned}
\tag{3.6}
$$

It has been well established that Equations 3.4, 3.5, 3.6 define a circular hyperboloid of one sheet in the projective three-space (z_1, z_2, z_3, z_4) when $a_0 \neq 0$ and that they define a hyperbolic paraboloid when $a_0 = 0$ (see [22]).

Note that these coefficients are homogeneous and that the nine coefficients P_i $(i = 1, 2, \ldots, 9)$ are bilinear in the original design parameters $(a_0, a_1, a_2, x_1, x_2, x_3)$. Furthermore, they satisfy the relations:

$$
\begin{aligned}
\frac{P_1}{P_7} = \frac{P_2}{P_8} = \frac{P_3}{P_9} = \frac{x_1}{x_3} = \lambda_1, \\
\frac{P_4}{P_7} = \frac{P_5}{P_8} = \frac{P_6}{P_9} = \frac{x_2}{x_3} = \lambda_2.
\end{aligned}
\tag{3.7}
$$

and (when $a_0 \neq 0$)

$$
\begin{aligned}
\frac{P_2}{P_1} = \frac{P_5}{P_4} = \frac{P_8}{P_7} = \frac{a_1}{a_0} = \mu_1, \\
\frac{P_3}{P_1} = \frac{P_6}{P_4} = \frac{P_9}{P_7} = \frac{a_2}{a_0} = \mu_2.
\end{aligned}
\tag{3.8}
$$

These two relationships indicate we can choose whether to solve moving coordinate x_i or fixed coordinate a_i first depending on the additional design requirements. They also indicate that $P_0, P_7, P_8, P_9, \lambda_1, \lambda_2$ can be treated as intermediate design parameters.

It has been shown in [29, 38] that for a RP dyad, one can obtain a linear equation of the same form as 3.4 where the coefficients P_i are bilinear in terms of (a_0, a_1, a_2), which defines a fixed point, and (l_1, l_2, l_3), which defines a line in the end link:

$$\begin{aligned}
&P_0 = a_0 l_3 \\
&P_1 = 0, \qquad P_2 = 0, \qquad P_3 = 0, \\
&P_4 = a_0 l_1, \quad P_5 = a_1 l_1, \quad P_6 = a_2 l_1, \\
&P_7 = a_0 l_2, \quad P_8 = a_1 l_2, \quad P_9 = a_2 l_2.
\end{aligned} \tag{3.9}$$

In summary, any planar dyad, RR, PR, and RP, can be represented by a special bilinear equation of the form 3.4 with two relations 3.7 or 3.8. Furthermore, when $P_1 = P_4 = P_7 = 0$, we obtain a PR dyad; when $P_1 = P_2 = P_3 = 0$, we obtain a RP dyad.

It is also worth noting that there have been other attempts to solve the design equations for planar dyads [39, 40, 41, 42, 43]. In [29], we have presented a different formulation using only 8 homogeneous coordinates, i.e., $(P_0, P_1, P_2 + P_6, P_3 - P_5, P_7, P_8, P_9)$. This can be clearly seen from 3.5 that the terms associated with P_2 and P_6 can be combined, so are those for P_3 and P_5. This formulation reduces the solution of the design equation into that of finding the roots of two quadratic equations. Unfortunately, this formulation can not be extended to spherical dyads. Therefore we present this bilinear formulation which covers synthesis method of planar, spherical and spatial linkages in this paper.

3.4 Design Equation Extended to Spherical RR Dyad

Our goal is to formulate the design equation of spherical RR dyad and find out whether the same algorithm of planar dyad synthesis can be applied in this case. Fortunately, the design equation of spherical RR dyad is very similar to that of planar dyads in its formulation.

Referring to Figure 2.4, let $\mathbf{s} = (s_x, s_y, s_z)$ denotes a unit vector in the direction of the rotation axis of fixed joint and θ denotes its rotation angle. One can represent the rotation with *Euler−Rodrigues parameters* ([1, 22]) by quaternion formulation:

$$\begin{aligned} q_1 &= s_x \sin(\theta/2), \quad q_2 = s_y \sin(\theta/2), \\ q_3 &= s_z \sin(\theta/2), \quad q_4 = \cos(\theta/2). \end{aligned} \tag{3.10}$$

Let $\mathbf{x} = (x_1, x_2, x_3)$ denotes a unit vector representing the moving joint B in the moving coordinate frame attached to the end effector. Then we have

$$\begin{aligned} X_1 &= [(q_4^2 + q_1^2 - q_2^2 - q_3^2)x_1 + 2(q_1q_2 - q_4q_3)x_2 \\ &\quad +2(q_1q_3 + q_4q_2)x_3]/S^2, \\ X_2 &= [2(q_1q_2 + q_4q_3)x_1 + (q_4^2 - q_1^2 + q_2^2 - q_3^2)x_2 \\ &\quad +2(q_2q_3 - q_4q_1)x_3]/S^2, \\ X_3 &= [2(q_1q_3 - q_4q_2)x_1 + 2(q_2q_3 + q_4q_1)x_2 \\ &\quad +(q_4^2 - q_1^2 - q_2^2 + q_3^2)x_3]/S^2 \end{aligned} \tag{3.11}$$

where $S^2 = q_1^2+q_2^2+q_3^2+q_4^2$. To simplify the equation, we use unit quaternions to make $S^2 = 1$.

Substituting 3.11 into 2.5, we can obtain a linear equation in the same form as 3.4, where,

$$\begin{aligned}
A_0 &= q_1^2 + q_2^2 + q_3^2 + q_4^2, & A_1 &= q_4^2 + q_1^2 - q_2^2 - q_3^2,\\
A_2 &= 2(q_1q_2 + q_4q_3), & A_3 &= 2(q_1q_3 - q_4q_2),\\
A_4 &= 2(q_1q_2 - q_4q_3), & A_5 &= q_4^2 - q_1^2 + q_2^2 - q_3^2,\\
A_6 &= 2(q_2q_3 + q_4q_1), & A_7 &= 2(q_1q_3 + q_4q_2),\\
A_8 &= 2(q_2q_3 - q_4q_1), & A_9 &= q_4^2 - q_1^2 - q_2^2 + q_3^2),
\end{aligned} \tag{3.12}$$

and the coefficients P_i $(i = 1, 2, \ldots, 9)$ are bilinear in (a_1, a_2, a_3) and (x_1, x_2, x_3):

$$\begin{aligned}
&P_0 = d,\\
&P_1 = a_1x_1, \quad P_2 = a_2x_1, \quad P_3 = a_3x_1,\\
&P_4 = a_1x_2, \quad P_5 = a_2x_2, \quad P_6 = a_3x_2,\\
&P_7 = a_1x_3, \quad P_8 = a_2x_3, \quad P_9 = a_3x_3.
\end{aligned} \tag{3.13}$$

It can be easily verified that the coefficients P_i $(i = 1, 2, \ldots, 9)$ satisfy the same relations as 3.7 and 3.8. Thus Equation 3.4 together with the relations 3.7 or 3.8 provide a unified representation of the design equation for both planar and spherical dyads [44].

3.5 Solving the Unified Design Equation

There are five design parameters in the design equations of planar and spherical linkages, therefore the maximum number of positions for exact synthesis problem is five. We are going to develop the algorithm for five position synthesis to find the type and dimensions of dyads of which the end effectors pass all the positions.

Consider a set of five displacements of the end-effector. If they are planar displacements, we use 3.5 to obtain the coefficients A_{ij} $(i = 1, 2, \ldots, N,\ j = 0, 1, 2, \ldots, 9)$ and if they are spherical displacements, we use 3.12 instead.

The system of five linear equations of the form 3.4 can be written in matrix form as

$$[A]\mathbf{P} = 0, \tag{3.14}$$

where $[A]$ is a 5×10 matrix and $\mathbf{P}^T = [P_0 \quad P_1 \quad P_2 \quad \ldots \quad P_9]$. In view of 3.7 and 3.8, the goal is to solve the intermediate design parameters p_i first and then use 3.6 and 3.13 to obtain the original design parameters for planar and spherical dyads, respectively. In the case of planar dyads, one may use 3.9 instead of 3.6 if $P_1 = P_2 = P_3 = 0$.

In order to find the intermediate design parameters $(P_0, P_7, P_8, P_9, \lambda_1, \lambda_2)$, we propose a Gaussian elimination based method that is different from the one presented in [31]. The formulation of design equations in [31] works only for planar dyads. In the previous section, we have shown that both planar dyads and spherical dyads can be represented by the same design equations using intermediate parameters. Thus the advantage of our new method is unifying the synthesis problem for both planr and spherical dyads. This unified method is presented in [44] using null-space analysis to find the eigenvectors corresponding to zero eigenvalues which can then construct the basis vectors of intermediate design parameters. Based on the bilinear formulation of the equation and combined with 3.7 or 3.8, the design parameters are solved. The null-space analysis has been used in our previous papers, but this method is not capable to differentiate input positions, for example, whether the position are on a plane or from one sphere, where the general algorithm is not as efficient. Therefore we use Gaussian elimination instead of null-space analysis and prove it to be effective to tell the types of input positions without preprocessing in the following chapters.

In linear algebra, Gaussian elimination (also known as row reduction) is an algorithm for solving systems of linear equations by elementary row operations [45]. For each row in a matrix, if the row does not consist of only zeros, then the left-most non-zero entry is called the leading coefficient (or pivot) of that row. One can always order the rows so that for every non-zero row, the leading coefficient is to the right of the leading coefficient of the row above. If this is the case, then matrix is said to be in row echelon form [46]. Furthermore, a row echelon form can become reduced row echelon form (also called row canonical form) if every leading coefficient is 1 and is the only nonzero entry in its column.

For five general arbitrary positions, to simplify the calculation, first we subtract each equation from position 2 of 3.4 by position 1. This will result 4 new linear equations in the form

$$\sum_{i=1}^{9}(A_{k,i} - A_{1,i})p_i = 0 \tag{3.15}$$

where $A_{k,i}$ denote the coefficients associated with the $k^{th}(k = 2, 3, 4, 5)$ position in 3.4. This subtraction also delete the first term p_0 because the coefficient of p_0 in each equation is the same. After substituting $h_{k,i}$ for $A_{k,i} - A_{1,i}$ and rewrite 3.15,

$$[H]\mathbf{P} = 0 \tag{3.16}$$

where $[H]$ is the matrix with only $h_{k,i}$ as its elements.

Next step is to obtain the reduced row echelon form (RREF) of Matrix $[H]$ by Gaussian elimination.

$$RREF([H]) = \begin{bmatrix} 1 & 0 & 0 & 0 & k_{1,5} & k_{1,6} & \cdots & k_{1,9} \\ 0 & 1 & 0 & 0 & k_{2,5} & k_{2,6} & \cdots & k_{2,9} \\ 0 & 0 & 1 & 0 & k_{3,5} & k_{3,6} & \cdots & k_{3,9} \\ 0 & 0 & 0 & 1 & k_{4,5} & k_{4,6} & \cdots & r_{4,9} \end{bmatrix} \tag{3.17}$$

Thus p_1, p_2, p_3, p_4 can be represented by the rest of p_5, p_6, p_7, p_8, p_9,

$$\begin{aligned} p_1 &= -k_{1,5}p_5 - k_{1,6}p_6 - \ldots - k_{1,9}p_9, \\ p_2 &= -k_{2,5}p_5 - k_{2,6}p_6 - \ldots - k_{2,9}p_9, \\ p_3 &= -k_{3,5}p_5 - k_{3,6}p_6 - \ldots - k_{3,9}p_9, \\ p_4 &= -k_{4,5}p_5 - k_{4,6}p_6 - \ldots - k_{4,9}p_9. \end{aligned} \tag{3.18}$$

Consider 3.7 and 3.8 after some algebra manipulation, we can have the new form of the relationships,

$$\begin{aligned} p_1 - \lambda_1 p_7 = 0, p_2 - \lambda_1 p_8 = 0, p_3 - \lambda_1 p_9 = 0, \\ p_4 - \lambda_2 p_7 = 0, p_5 - \lambda_2 p_8 = 0, p_6 - \lambda_2 p_9 = 0 \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} p_2 - \mu_1 p_1 = 0, p_5 - \mu_1 p_4 = 0, p_8 - \mu_1 p_7 = 0, \\ p_3 - \mu_2 p_1 = 0, p_6 - \mu_2 p_4 = 0, p_9 - \mu_2 p_7 = 0 \end{aligned} \tag{3.20}$$

Then substitute 3.18 into 3.19 and 3.20,

$$[B_1]\mathbf{F} = 0 \tag{3.21}$$

and

$$[B_2]\mathbf{F} = 0 \tag{3.22}$$

$[B_1], [B_2]$ and F are of the following form,

$$[B_1] = \begin{bmatrix} -k_{1,5} & -k_{1,6} & \cdots & -k_{1,8} & -k_{1,9} \\ -k_{2,5} & -k_{2,6} & \cdots & -k_{2,8} - \lambda_1 & -k_{2,9} \\ -k_{3,5} & -k_{3,6} & \cdots & -k_{3,8} & -k_{2,9} - \lambda_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & -\lambda_2 & 0 \\ 0 & 0 & \cdots & 0 & -\lambda_2 \end{bmatrix}$$

$$[B_2] = \begin{bmatrix} \mu_1 k_{1,5} - k_{2,5} & \mu_1 k_{1,6} - k_{2,6} & \cdots & \mu_1 k_{1,8} - k_{2,8} & \mu_1 k_{1,9} - k_{2,9} \\ \mu_1 k_{4,5} + 1 & \mu_1 k_{4,6} & \cdots & \mu_1 k_{4,8} & \mu_1 k_{4,9} \\ 0 & 0 & \cdots & 1 & 0 \\ \mu_2 k_{1,5} - k_{3,5} & \mu_2 k_{1,6} - k_{3,6} & \cdots & \mu_2 k_{1,8} - k_{3,8} & \mu_2 k_{1,9} - k_{3,9} \\ \mu_2 k_{4,6} & \mu_2 k_{4,6} + 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$F = [p_5, p_6, p_7, p_8, p_9]^T \tag{3.23}$$

Both B_1 and B_2 are 6×5 matrices, thus 3.21and 3.22 are over-determined systems. For $\mathbf{F}$ to have finite, non-trivial solutions, there should be only four independent rows in either matrix, because there are five variables in homogeneous form in $\mathbf{F}$, and thus the left matrix must be rank-deficient ([47], [48], [49]). In order for the matrix to be rank-deficient, one can find two of the biggest square sub-matrices (by dropping off one row at each time) to obtain two 5×5 matrices and calculate their determinants. When the determinants equal to zero simultaneously, λ_1 and λ_2 are the solutions of moving joint or μ_1 and μ_2 the are solutions of fixed joint. There are also other research on solving such a over-determined general eigenvalue problem ([50], [51], [52]).

Once λ_i or μ_i are found, we can determine $\mathbf{F}$ by null-space of the B_i using various methods such as Singular Value Decomposition ([53], [54]). The eigenvectors corresponding to the zero eigenvalues are the solutions of $\mathbf{F}$. The resulting $\mathbf{F}$ can be substituted back into 3.18 to obtain all the intermediate parameters p_i that satisfy 3.15, while P_0 is to be obtained by arithmetic average after substituting $P_1 ... P_9$ to 3.4. One can then obtain the original design parameters by inverting 3.6 or 3.13.

3.6 Numerical Examples of Planar and Spherical Linkages

3.6.1 Example 1: Planar Dyads Synthesis

Now consider five planar task positions given in Table 3.1. The substitution of the data in the table into relationship 3.2 yields five planar quaternions $\mathbf{z}_i$ $(i = 1, 2, 3, 4, 5)$ in Table 3.2, which are then substituted into 3.5 to obtain the

Table 3.1: Example 1: Five Planar Task Positions

Position	d_1	d_2	α
1	3.6700	0.6457	$-77.5362°$
2	2.7965	1.5640	$-56.6879°$
3	2.5562	1.7066	$-35.2713°$
4	3.7451	1.1415	$-38.6715°$
5	4.5797	0.5694	$-63.1693°$

Table 3.2: Example 1: Planar Quaternions Representation

Position	z_1	z_2	z_3	z_4
1	-1.4007	1.2285	-0.6261	0.7796
2	-1.3520	0.8593	-0.4747	0.8801
3	-1.2004	0.9595	-0.3029	0.9530
4	-1.1585	1.5779	-0.3311	0.9435
5	-1.4418	1.8015	-0.5237	0.8518

matrix $[A]$.

They define three planar $4R$ linkages as well as three slider-crank mechanisms. The solutions of moving joints and intermediate parameters are listed in Table 3.4 and 3.5. Three constraint circles and one constraint line as well as their respective circle points are computed using 3.6 and are shown in Table 3.6. Figure 3.2 shows one resulting four bar linkage in this example.

3.6.2 Example 2: Spherical Dyads Synthesis

There are five spherical displacements given in Table 3.3. After constructing the matrix $[A]$ using 3.12 and 3.14, the first step is to use Gaussian elimination t find the linear relationship of p_i. We then construct the matrices $[B_1], [B_2], [B_3]$ and solve the resulting generalized eigenvalue problem, the moving joint coordinates

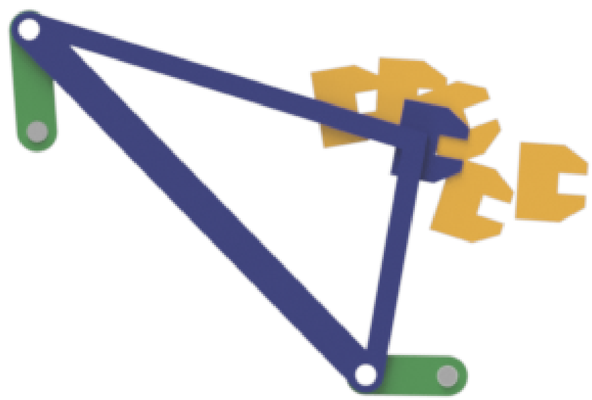

Figure 3.2: A resulting four bar linkage and five planar positions.

and corresponding intermediate design parameters are shown in Table 3.7 and 3.8. We can then invert 3.13 to obtain the original design parameters shown in Table 3.9. Figure 3.3 shows one of spherical 4R linkages generated from this example.

3.7 Summary

In this chapter, we derived the design equations of planar and spherical four bar linkage synthesis problem from the geometric constraints of planar and spherical dyads. Later we have shown that the design equations of both cases have the same bilinear formulation with different coefficients. Based on the design equations and the bilinear relationship of design parameters, we have developed an unified method to solve the synthesis problems of both categories of linkages.

Figure 3.3: A resulting spherical four bar linkage and five spherical positions.

Table 3.3: Example 2: Five spherical displacements in quaternion form.

Position	q_1	q_2	q_3	q_4
1	0.2456	0.4356	0.7485	0.4356
2	0.1027	0.3194	0.7237	0.6030
3	0.1167	0.2550	0.5669	0.7746
4	0.1759	0.3038	0.4195	0.8371
5	0.2456	0.5060	0.2456	0.7895

Table 3.4: Example 1: Solutions of Moving Joints

	(λ_1, λ_2)
PR	$(0.9997, -2.9994)$
RR	$(0.3812, -1.8718)$
RR	$(2.2086, -1.0049)$
RR	$(-1.9998, -2.9999)$

Table 3.5: Example 1: Solutions of Intermediate Parameters

	$p_1 \dots p_9$	p_0
PR	$(0.0000, 0.1346, 0.2690, -0.0001, -0.4038, -0.8071, 0.0000, 0.1346, 0.2691)$	0.0671
RR	$(-0.0314, -0.1277, -0.1052, 0.1544, 0.6273, 0.5168, -0.0825, -0.3351, -0.2761)$	0.3179
RR	$(-0.1757, -0.6968, 0.2257, 0.0799, 0.3170, -0.1027, -0.0796, -0.3155, 0.1022)$	0.4983
RR	$(0.3221, 0.0001, 0.3221, 0.4831, 0.0001, 0.4831, -0.1610, -0.0000, -0.1610)$	0.6142

Table 3.6: Example 1: Homogeneous Coordinates of the Constraint Circle (or Line) and the Circle Point

	a_0, a_1, a_2, a_3	x_1, x_2, x_3
PR	$0.0000, 0.1346, 0.2690, 0.1343$	$0.9997, -2.9994, 1.0000$
RR	$1.0000, 4.0668, 3.3503, -5.5287$	$0.3812, -1.8718, 1.0000$
RR	$1.0000, 3.9659, -1.2846, -8.2698$	$2.2086, -1.0049, 1.0000$
RR	$1.0000, 0.0000, 1.0000, 0.0000$	$-1.9998, -2.9999, 1.0000$

Table 3.7: Example 2: Solutions of Moving Joints

(λ_1, λ_2)
$(-0.0030, 0.5771)$
$(-0.3877, 0.4882)$
$(0.8812, -0.6568)$
$(-0.0028, -0.5639)$

Table 3.8: Example 2: Solutions of Intermediate Design Parameters

$p_1 \ldots p_9$	p_0
$(0.0000, -0.0017, 0.0000, -0.0003, 0.4842, -0.0000, -0.0005, 0.8388, -0.0001)$	-0.2562
$(-0.0611, 0.2977, -0.0751, 0.0769, -0.3751, 0.0947, 0.1576, -0.7683, 0.1939)$	-0.3224
$(-0.3420, -0.2499, 0.1826, 0.2531, 0.1850, -0.1351, -0.3876, -0.2833, 0.2069)$	-0.8120
$(-0.0007, -0.0000, 0.0000, -0.3761, -0.0001, -0.0015, 0.6555, 0.0001, 0.0026)$	0.8679

Table 3.9: Example 2: The Dimensions of the Four Resulting Spherical RR Dyads

a_1, a_2, a_3, d	x_1, x_2, x_3
$0.0009, -1.000, 0.0001, -0.2562$	$-0.0030, 0.5771, 1.0000$
$0.1953, -0.9507, 0.2408, -0.3224$	$-0.3877, 0.4882, 1.0000$
$-0.7423, -0.5398, 0.3970, -0.8121$	$0.8812, -0.6568, 1.0000$
$0.9999, 0.0013, 0.0142, 0.8679$	$-0.0028, -0.5639, 1.0000$

Chapter 4

Task Driven Type and Dimensional Synthesis of 5-SS Platform Linkage

4.1 Introduction

A 5-SS platform linkage is a mechanism with five legs of which the moving joints are linked onto one rigid platform and fixed joints are located on the ground as shown in Figure 4.1. It is a single degree of freedom mechanism. We use geometric constraints to formulate the design equation of legs which can be constructed to 5-SS platform linkage. We will show the similarity in synthesis algorithm of this spatial mechanism with that of planar and spherical linkages. Therefore it is an extension of our method from two to three dimensions.

4.2 Dual Quaternion Formulation

Dual quaternions are used to represent spatial rigid body displacements ([55], [56]). A spatial displacement of a rigid body is represented by the transformation of a moving frame M attached to the moving body with respect to a fixed frame

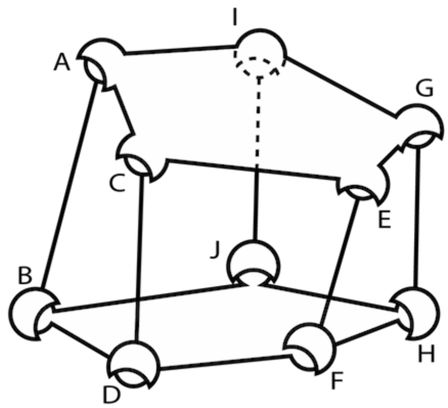

Figure 4.1: A 5-SS Platform Linkage: A, C, E, G, I are moving joints and B, D, F, H, J are fixed joints.

F:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_x & \mathbf{r}_y & \mathbf{r}_z & \mathbf{d} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{4.1}$$

where the vectors $\mathbf{X} = (X_1, X_2, X_3, X_4)$ and $\mathbf{x} = (x_1, x_2, x_3, x_4)$ represent homogeneous coordinates of a point in F and M, respectively and unit vectors $\mathbf{r}_x$, $\mathbf{r}_y$, $\mathbf{r}_z$ represent the axes of M with respect to F and thus define the rotation matrix $[R]$. The vector $\mathbf{d}$ represents the translation of the origin of M relative to F. This transformation and notations are shown in Figure 4.2.

In this chapter, we follow [57] and use a dual quaternion formulation for the homogeneous transform 4.1. A dual quaternion $\hat{\mathbf{Q}} = (\mathbf{q}, \mathbf{g})$ involves a real part

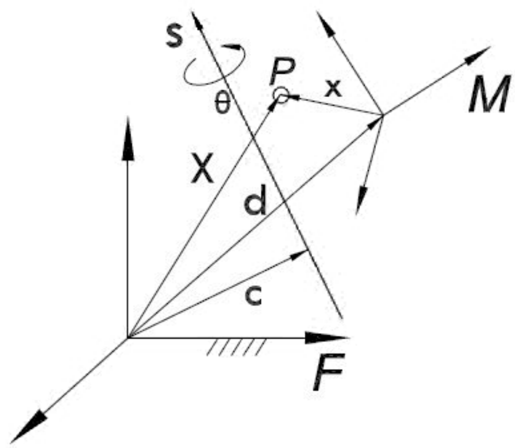

Figure 4.2: Spatial Displacement Definition: $\mathbf{x}$ and $\mathbf{X}$ are the coordinates of a point P relative to moving and fixed frame; $\mathbf{d}$ is the distance between origin of moving and fixed frame; s defines a rotational axis and θ defines the rotational angle.

$\mathbf{q} = (q_1, q_2, q_3, q_4)$ and a dual part $\mathbf{g} = (g_1, g_2, g_3, g_4)$ (see also [1] and [22]). The real part $\mathbf{Q}$ can be constructed with rotation axis $\mathbf{s} = (s_x, s_y, s_z)$ and rotation angle θ, from the ration matrix $[R]$ using Cayley's formula [22]:

$$\mathbf{Q} = (s_x \sin\frac{\theta}{2}, s_y \sin\frac{\theta}{2}, s_z \sin\frac{\theta}{2}, \cos\frac{\theta}{2}) \tag{4.2}$$

The dual part $\tilde{\mathbf{Q}}$ is given by the formula:

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -d_3 & d_2 & d_1 \\ d_3 & 0 & -d_1 & d_2 \\ -d_2 & d_1 & 0 & d_3 \\ -d_1 & -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}. \tag{4.3}$$

Thus the rotation matrix $[R]$ in 4.1 can be parameterized with the unit quaternion $\mathbf{q} = (q_1, q_2, q_3, q_4)$ as:

$$[R] = \begin{bmatrix} q_4^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_4) & 2(q_1q_3 + q_2q_4) \\ 2(q_1q_2 + q_3q_4) & q_4^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_1q_4) \\ 2(q_1q_3 - q_2q_4) & 2(q_2q_3 + q_1q_4) & q_4^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \tag{4.4}$$

where $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$.

The translation vector $\mathbf{d} = (d_1, d_2, d_3)$ can be expressed as

$$\mathbf{d} = -2 \begin{bmatrix} g_4q_1 - g_1q_4 + g_2q_3 - g_3q_2 \\ g_4q_2 - g_2q_4 + g_3q_1 - g_1q_3 \\ g_4q_3 - g_3q_4 + g_1q_2 - g_2q_1 \end{bmatrix}. \tag{4.5}$$

where the real and dual components satisfy the relation $q_1g_1 + q_2g_2 + q_3g_3 + q_4g_4 = 0$.

4.3 The Unified Design Equation for Spherical and Planar Constraints

It has been shown in chapter 2 that both sphere and plane can be formulated by a unified homogeneous equation. In the following steps, we substitute given data in dual quaternion representation into this unified equation and obtain the design equation of such constraints. Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ denotes the homogeneous coordinate of a point on the moving platform that stays on a sphere (or a plane) defined by 2.6 and let the movement of the platform be defined by dual quaternion components (q_1, q_2, q_3, q_4) and (g_1, g_2, g_3, g_4). It follows from [57] that the following linear equation in p_i $(i = 0, 1, 2, \ldots, 16)$ is the design equation for the spherical (or plane) constraint:

$$p_0 + \sum_{i=1}^{16} A_i p_i = 0, \tag{4.6}$$

where

$$\begin{aligned}
A_1 &= 2(-g_2q_1 - g_3q_2 + g_2q_3 + g_1q_4),\\
A_2 &= -q_1^2 + q_2^2 + q_3^2 - q_4^2,\\
A_3 &= 2(-q_1q_2 - q_3q_4),\\
A_4 &= 2(-q_1q_3 + q_2q_4),\\
A_5 &= 2(g_3q_1 - g_4q_2 - g_1q_3 + g_2q_4),\\
A_6 &= 2(-q_1q_2 + q_3q_4),\\
A_7 &= q_1^2 - q_2^2 + q_3^2 - q_4^2,\\
A_8 &= 2(-q_2q_3 - q_1q_4),\\
A_9 &= 2(-g_2q_1 + g_1q_2 - g_4q_3 + g_3q_4),\\
A_{10} &= 2(-q_1q_3 - q_2q_4),\\
A_{11} &= 2(-q_2q_3 + q_1q_4),\\
A_{12} &= q_1^2 + q_2^2 - q_3^2 - q_4^2,\\
A_{13} &= 4g_1^2 + 4g_2^2 + 4g_3^2 + 4g_4^2,\\
A_{14} &= 4g_4q_1 - 4g_3q_2 + 4g_2q_3 - 4g_1q_4,\\
A_{15} &= 4g_3q_1 + 4g_4q_2 - 4g_1q_3 - 4g_2q_4,\\
A_{16} &= -4g_2q_1 + 4g_1q_2 + 4g_4q_3 - 4g_3q_4,
\end{aligned} \tag{4.7}$$

and each of the p_i $(i = 0, 1, 2, \ldots, 16)$ are defined by the choice of the moving joint $\mathbf{x} = (x_1, x_2, x_3, x_4)$ as well as the spherical constraint $(a_0, a_1, a_2, a_3, a_4)$ as follows:

$$
\begin{aligned}
p_1 = a_0x_1x_4,\ \ p_2 = a_1x_1x_4,\ \ p_3 = a_2x_1x_4,\ \ p_4 = a_3x_1x_4 \\
p_5 = a_0x_2x_4,\ \ p_6 = a_1x_2x_4,\ \ p_7 = a_2x_2x_4,\ \ p_8 = a_3x_2x_4 \\
p_9 = a_0x_3x_4,\ p_{10} = a_1x_3x_4,\ p_{11} = a_2x_3x_4,\ p_{12} = a_3x_3x_4 \\
p_{13} = a_0x_4^2,\ \ p_{14} = a_1x_4^2,\ \ p_{15} = a_2x_4^2,\ \ p_{16} = a_3x_4^2 \\
p_0 = a_0x_1^2 + a_0x_2^2 + a_0x_3^2 - a_4x_4^2.
\end{aligned} \tag{4.8}
$$

As both x_i, a_i are homogeneous, there are only 7 independent design parameters. This allows a maximum of seven positions to be used for exact type and dimensional synthesis. Typically, one tries to solve x_i and a_i directly from the design equation by using algebraic methods and homotopy for solving a set of polynomial equations, see for example, McCarthy and Soh [11]. Homotopy is basically an iteration method. The advantage is that if choosing a suitable initial value and after enough iteration, one can always find the full solution set. Thus it is easy to conclude that homotopy is a time-consuming method. Although there are many software helping users to decide and narrow down the initial values, it is still not the best choice of solving polynomial system directly. Our method is to first reformulate the polynomial system and then allows us to solve it with any general package solvers for higher order equations. The advantage is fast, simple, extendable and unified.

In [57], an initial development of our approach is taken that it treats the design equation as a combination of the linear equation in 17 unknowns as given by 4.6 and the following 9 bilinear constraints:

$$p_2p_5 - p_1p_6 = 0,\ p_1p_7 - p_3p_5 = 0,\ p_1p_8 - p_4p_5 = 0,$$
$$p_2p_9 - p_1p_{10} = 0,\ p_1p_{11} - p_3p_9 = 0,\ p_1p_{12} - p_4p_9 = 0,$$
$$p_2p_{13} - p_1p_{14} = 0.\ p_1p_{15} - p_3p_{13} = 0,\ p_1p_{16} - p_4p_{13} = 0. \tag{4.9}$$

For a set of seven positions, we obtain a set of seven linear equations in the form 4.6. One can eliminate p_0 by subtracting equation 2 to 7 by equation 1 and obtain six linear equations in the form:

$$\sum_{i=1}^{16}(A_{k,i} - A_{1,i})p_i = 0 \tag{4.10}$$

where $A_{k,i}$ denote the coefficients associated with the k^{th} position of i^{th} equation in 4.6.

Let $\mathbf{P} = [p_1, \ldots, p_{16}]^T$ and $h_{ki} = A_{k,i} - A_{1,i}$, then we can assemble 6 linear equations in matrix form as

$$[H]\mathbf{P} = 0, \tag{4.11}$$

Thus the problem reduces to the solution of the system of linear equation 4.11 subject to the same group of bilinear constraints 4.9. In [57], a homotopy based algorithm was then developed for the problem.

We can rewrite 4.6 as a bilinear matrix equation,

$$[a_0 \dots a_3] \begin{bmatrix} h_1 & \dots & h_{13} \\ \vdots & \ddots & \vdots \\ h_4 & \dots & h_{16} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = 0 \tag{4.12}$$

This indicates that, from the formulation, we can either solve for a_i or x_i first depending on whether there are more constraints on the moving or fixed joints. This will be presented in the chapter of six or five position synthesis that one can choose to formulate based on either x or a first for different problems.

The nine constraint equations in 4.9 are equivalent to the following relations among the sixteen parameters p_i:

$$\begin{aligned} \frac{p_1}{p_{13}} = \frac{p_2}{p_{14}} = \frac{p_3}{p_{15}} = \frac{p_4}{p_{16}} = \frac{x_1}{x_4} = \lambda_1, \\ \frac{p_5}{p_{13}} = \frac{p_6}{p_{14}} = \frac{p_7}{p_{15}} = \frac{p_8}{p_{16}} = \frac{x_2}{x_4} = \lambda_2, \\ \frac{p_9}{p_{13}} = \frac{p_{10}}{p_{14}} = \frac{p_{11}}{p_{15}} = \frac{p_{12}}{p_{16}} = \frac{x_3}{x_4} = \lambda_3. \end{aligned} \tag{4.13}$$

We can also obtain another relationship in terms of a_0, a_1, a_2, a_3 by assuming $a_0 \neq 0$.

$$\begin{aligned} \frac{p_2}{p_1} = \frac{p_6}{p_5} = \frac{p_{10}}{p_9} = \frac{p_{14}}{p_{13}} = \frac{a_1}{a_0}, \\ \frac{p_5}{p_1} = \frac{p_7}{p_5} = \frac{p_{11}}{p_9} = \frac{p_{15}}{p_{13}} = \frac{a_2}{a_0}, \\ \frac{p_4}{p_1} = \frac{p_8}{p_5} = \frac{p_{12}}{p_9} = \frac{p_{16}}{p_{13}} = \frac{a_3}{a_0}. \end{aligned} \tag{4.14}$$

4.4 Solving the Design Equation of Spherical and Planar Constraints

This section is to solve the problem of determining the type and dimensions of a spatial platform linkage such that an fixed end effector located on the platform guides through a set of seven given spatial positions. This kind of linkage is composed of legs whose moving joints are constrained onto the surface of either spheres or planes.

We obtain the reduced row echelon form (RREF) using Gaussian elimination to solve spherical and planar design equations which is similar to that in planar and spherical algorithm. For seven position synthesis, matrix $[H]$ of 4.11 can be factored into:

$$RREF([H]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & k_{1,7} & \cdots & k_{1,15} & k_{1,16} \\ 0 & 1 & 0 & 0 & 0 & 0 & k_{2,7} & \cdots & k_{2,15} & k_{2,16} \\ 0 & 0 & 1 & 0 & 0 & 0 & k_{3,7} & \cdots & k_{3,15} & k_{3,16} \\ 0 & 0 & 0 & 1 & 0 & 0 & k_{4,7} & \cdots & k_{4,15} & k_{4,16} \\ 0 & 0 & 0 & 0 & 1 & 0 & k_{5,7} & \cdots & k_{5,15} & k_{5,16} \\ 0 & 0 & 0 & 0 & 0 & 1 & k_{6,7} & \cdots & k_{6,15} & k_{6,16} \end{bmatrix} \tag{4.15}$$

Therefore the first six intermediate design parameters $p_i(i = 1, 2, \ldots, 6)$ can be represented by the rest of $p_i(i = 7, 8, \ldots, 16)$ as the following,

$$p_1 = -k_{1,7}p_7 - k_{1,8}p_8 - \ldots - k_{1,16}p_{16},$$

$$
\begin{aligned}
p_2 &= -k_{2,7}p_7 - k_{2,8}p_8 - \ldots - k_{2,16}p_{16}, \\
&\vdots \\
p_6 &= -k_{6,7}p_7 - k_{6,8}p_8 - \ldots - k_{6,16}p_{16}.
\end{aligned}
\tag{4.16}
$$

Reformulating 4.13 and 4.14,

$$
\begin{gathered}
p_1 - \lambda_1 p_{13} = 0, p_2 - \lambda_1 p_{14} = 0, p_3 - \lambda_1 p_{15} = 0, p_4 - \lambda_1 p_{16} = 0, \\
p_5 - \lambda_2 p_{13} = 0, p_6 - \lambda_2 p_{14} = 0, p_7 - \lambda_2 p_{15} = 0, p_8 - \lambda_2 p_{16} = 0, \\
p_9 - \lambda_3 p_{13} = 0, p_{10} - \lambda_3 p_{14} = 0, p_{11} - \lambda_3 p_{15} = 0, p_{12} - \lambda_3 p_{16} = 0
\end{gathered}
\tag{4.17}
$$

and

$$
\begin{gathered}
p_2 - \mu_1 p_1 = 0, p_6 - \mu_1 p_5 = 0, p_{10} - \mu_1 p_9 = 0, p_{14} - \mu_1 p_{13} = 0, \\
p_5 - \mu_2 p_1 = 0, p_7 - \mu_2 p_5 = 0, p_{11} - \mu_2 p_9 = 0, p_{15} - \mu_2 p_{13} = 0, \\
p_4 - \mu_3 p_1 = 0, p_8 - \mu_3 p_5 = 0, p_{12} - \mu_3 p_9 = 0, p_{16} - \mu_3 p_{13} = 0
\end{gathered}
\tag{4.18}
$$

Substituting 4.16 into 4.17 or 4.18 to obtain new homogeneous matrix equations,

$$[B_1]\mathbf{F} = 0 \tag{4.19}$$

and

$$[B_2]\mathbf{F} = 0 \tag{4.20}$$

where $[B_1]$, $[B_2]$ and $\mathbf{F}$ are of the following form,

$$[B_1] = \begin{bmatrix} -k_{1,7} & -k_{1,8} & \cdots & -k_{1,14} & -k_{1,15} & -k_{1,16} \\ -k_{2,7} & -k_{2,8} & \cdots & -k_{2,14}-\lambda_1 & -k_{2,15} & -k_{2,16} \\ -k_{3,7} & -k_{3,8} & \cdots & -k_{3,14} & -k_{3,15}+\lambda_1 & -k_{3,16} \\ -k_{4,7} & -k_{4,8} & \cdots & -k_{4,14} & -k_{4,15} & -k_{4,16}+\lambda_1 \\ -k_{5,7} & -k_{5,8} & \cdots & -k_{5,14} & -k_{5,15} & -k_{5,16} \\ -k_{6,7} & -k_{6,8} & \cdots & -k_{6,14}-\lambda_2 & -k_{6,15} & -k_{6,16} \\ 1 & 0 & \cdots & 0 & \lambda_2 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \lambda_2 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & \lambda_3 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \lambda_3 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \lambda_3 \end{bmatrix}$$

$$[B_2] = \begin{bmatrix} -k_{2,5} + \mu_1 k_{1,5} & \cdots & -k_{2,15} + \mu_1 k_{1,15} & -k_{2,16} + \mu_1 k_{1,16} \\ -k_{6,5} + \mu_1 k_{5,5} & \cdots & -k_{6,15} + \mu_1 k_{5,15} & -k_{6,16} + \mu_1 k_{5,16} \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ -k_{5,5} + \mu_2 k_{1,5} & \cdots & -k_{5,15} + \mu_2 k_{1,15} & -k_{5,16} + \mu_2 k_{1,16} \\ 1 + \mu_2 k_{5,5} & \cdots & \mu_2 k_{5,15} & \mu_2 k_{5,16} \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & 0 \\ -k_{4,5} + \mu_3 k_{1,5} & \cdots & -k_{4,15} + \mu_3 k_{1,15} & -k_{4,16} + \mu_3 k_{1,16} \\ \mu_3 k_{5,5} & \cdots & \mu_3 k_{5,15} & \mu_3 k_{5,16} \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\mathbf{F} = [p_7, p_8, \cdots, p_{16}]^T \tag{4.21}$$

λ_i and μ_i are the unknowns to be found and they are the coordinates of moving joints $x_1/x_4, x_2/x_4, x_3/x_4$ and fixed joints $a_1/a_0, a_2/a_0, a_3/a_0$.

Both B_1 and B_2 are 10×12 matrices, thus 4.19 and 4.20 are over-determined systems, in other words the number of rows is more than the number of columns. For $\mathbf{F}$ to have finite, non-trivial solutions, there should be only nine independent rows in either matrix, because there are ten variables in homogeneous form in $\mathbf{F}$, requiring the left matrix to be rank-deficient. In order for the matrix to be rank-deficient, one can find three of the biggest square sub-matrices (by dropping off two of any three rows at each time) to obtain two 10×10 matrices and calculate their determinants. When the determinants equal to zero simultaneously, $\lambda_1, \lambda_2, \lambda_3$ are the solutions of moving joints or μ_1, μ_2, μ_3 the are solutions of fixed joints. The solution to three equations can be obtained using $NSolve$ in Mathematica in seconds for a seven-position synthesis problem.

Once λ_i or μ_i are found, we can determine $\mathbf{F}$ by null-space of the B_i using various methods such as Singular Value Decomposition. The eigenvectors corresponding to the zero eigenvalues are the solutions of $\mathbf{F}$. The resulting $\mathbf{F}$ can be substituted back into 4.16 to obtain all the intermediate parameters p_i that satisfy 4.10, while P_0 is to be obtained by arithmetic average after substituting $P_1...P_{16}$ to 4.6. One can then obtain the original design parameters by inverting 4.8.

4.5 Numerical Example for Seven General Position Synthesis

Take the example in [58] as shown in table 4.1 for seven general given task positions synthesis, which are in the form of rotation axis vector $\mathbf{s}$=(s_x, s_y, s_z), rotation angle θ and translation vector $\mathbf{d}$=(d_1, d_2, d_3).

We obtain 20 real solutions of p_i. a_i and x_i can be computed by inverting 4.8 and are listed in Table 7.3. The final types of mechanism can be determined by constructing any five of these legs together. Figure 4.3 shows a resulting 5-SS linkage passing each of the seven positions.

Table 4.1: Seven General Task Positions: given as rotaional axis (s_x, s_y, s_z), rotational angle θ and translation vector (d_1, d_2, d_3).

	s_x	s_y	s_z	θ	d_1	d_2	d_3
1	0.0000,	0.0000,	0.0000,	0.0000	0.0000,	0.0000,	0.0000
2	-0.0863,	0.0096,	-0.9962,	1.7088	1.0000,	-0.7423,	-0.1337
3	-0.3509,	-0.7019,	0.6197,	2.0294	0.3182,	-0.5085,	-0.7922
4	-0.3775,	0.8612,	0.3402,	2.9993	-0.1788,	-1.7842,	-1.0429
5	-0.5828,	-0.3469,	0.7347,	3.1035	-1.2580,	0.8362,	-1.4992
6	0.0075,	-0.1487,	-0.9888,	1.2080	-3.5939,	2.7283,	-2.0334
7	0.8987,	0.1616,	0.4075,	2.3064	-0.0497,	0.5700,	-1.4858

Notes: The units of s_x, s_y, s_z, d_x, d_y, d_z are meters; the unit of θ is radian.

4.6 Summary

In this chapter, we extended the synthesis methodology of planar and spherical linkages to spatial 5-SS linkage. We presented a design equation with bilinear relationship of its design parameters. We formulated the bilinear relationship into matrix equations and solved the design parameters. This algorithm can further be found that it combines the planar and spherical linkages with spatial 5-SS linkages together and thus can be derived into an unified algorithm for designing all the linkages.

Table 4.2: General Example: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000, -7.9666, 2.5182, -4.8173, 7306.7500	51.3313, 26.9291, -62.1552
2	1.0000, 0.0730, -0.5605, 0.2412, 660.5880	-5.3925, 3.2024, 25.0794
3	1.0000, 0.8993, -0.9070, 0.1314, 118.9910	3.4589,3.3524,-9.6636
4	1.0000, -3.2436, -34.9680, -7.2182, 164.5690	5.4835,-5.0920,14.7184
5	1.0000, -4.0713, -2.5601, -3.6966, 69.1985	-1.3251,-7.2066,5.0705
6	1.0000, -48.9526, -37.5513, -43.9814, 29.3375	-0.0679,5.1452,-4.5168
7	1.0000, 75.5422, 37.6131, -87.4322, 22719.7000	-43.3100,-113.5570,-109.9560
8	1.0000, -0.4049, -0.8840, -1.2398, 9.3185	1.6293,1.8374,-1.7462
9	1.0000, -0.1483, 2.6789, -0.4008, 12.0059	0.1609,-0.6353,2.4775
10	1.0000, -7.7352, -9.6332, -10.4381, -11.4725	-0.4713,1.5841,-1.9811
11	1.0000, -1.4532, -0.4130, -1.1780, 4.5181	1.3606,0.0850,-1.0281
12	1.0000, 0.8104, -0.9742, -2.7162, 4.8647	2.3574,0.6639,0.2455
13	1.0000, 1.2795, 0.7159, -1.2141, 16.3161	-0.6459,4.1420,0.9058
14	1.0000, -0.3764, -0.2693, -2.2550, 6.1162	2.1435,-0.9265,-0.1024
15	1.0000, -3.4210, -0.2940, 1.4624, 7.5653	0.7026,-0.3222,-0.6562
16	1.0000, -2.5613, -4.1576, -8.7596, -15.0158	-0.3029,0.2047,-0.9218
17	1.0000, -7.8391, -0.0888, 9.6491, 25.0373	0.3670,-0.5877,-0.9344
18	1.0000, 0.2611, 2.4585, -3.4241, 0.5437	-1.5558,1.1520,0.2327
19	1.0000, -3.8199, -3.7258, 4.3851, 15.4123	0.8757,2.4774,2.7771
20	1.0000, 0.9735, 2.9069, -3.0423, 0.6578	-0.9779,1.0618,0.4360

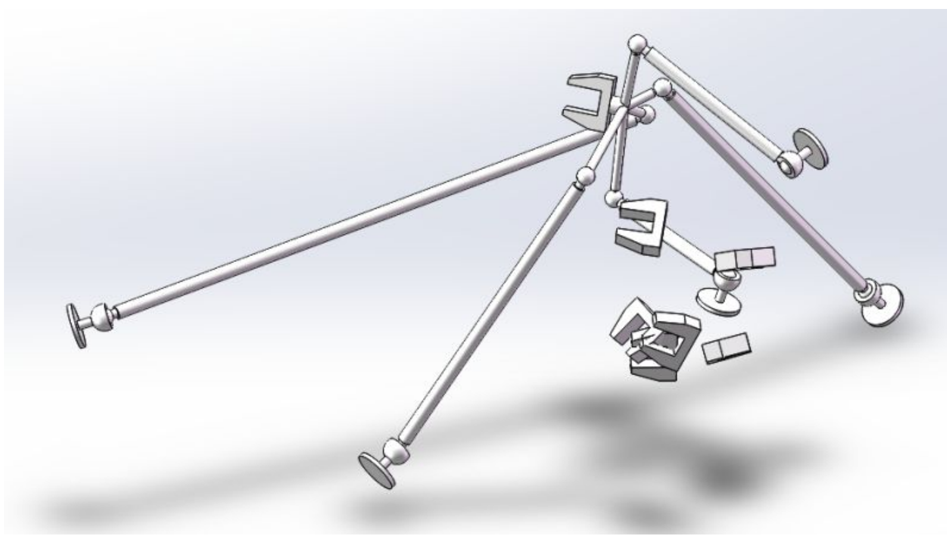

Figure 4.3: An Example of 5-SS Linkage and Seven Positions

Chapter 5

From Spatial 5-SS Platform Linkage to Planar and Spherical 4R Linkage

5.1 Introduction

In this chapter, we are going to develop and present the relationship among spatial legs and planar and spherical dyads [28]. The basis indicating a relationship existing is that planar and spherical dyds are special cases of spatial dyads. There are multiple constructions of the spatial leg types equivalent to planar links, spherical RR link or spatial RR link. The combinations of these links will form various types of linkages. There are several benefits to build this relation from the spatial 5-SS platform linkage to planar and spherical linkages synthesis, in addition to simplifying the algorithm, it also extends the original circle and unit sphere constraints into the more general cases, spatial plane and sphere. This allows the input positions to be given without considering preproessing to fit different design equations which greatly simplifies the design process. There will be one unified design equation and algorithm for all the linkages mentioned in our previous papers.

5.2 Degree of Freedom and Over-constraint Linkages

The degrees of freedom (DOF) of a rigid body is defined as the number of independent movements it has. A spatial rigid body has six DOF, the movement of x, y, z directions and the rotations around them. Grubler gives a formula to calculate DOF of a system of rigid bodies and links connected by different joints ([59] - [60]), known as Grubler-Kutzbach criterion (or mobility formula). For the planar and spherical cases, the formula is given by,

$$DOF = 3(n - j - 1) + \sum_{i=1}^{j} w_i. \tag{5.1}$$

and the spatial case is,

$$DOF = 6(n - j - 1) + \sum_{i=1}^{j} w_i. \tag{5.2}$$

where n is the number of the total members of the bodies and links, j is the number of the total joints, w_i is the DOF of the i^{th} joint.

Take 5-SS linkage for example, $n = 7, j = 10, w_i = 3$, thus $DOF = 6(7 - 10 - 1) + 3 \times 10 = 6$. Note that the rotation of the SS leg around its axial direction does not influence the input and output motion of the linkage, called passive or redundant degree of freedom, thus they are not considered. The $DOF = 6 - 5 = 1$.

The degree of freedom of a planar four bar or spherical 4R linkage by 5.1 is $DOF = 3(4 - 4 - 1) + 4 = 1$. It means they are actually one DOF linkages. But if they are considered as a spatial linkage, using 5.2, one will have $DOF = 6(4 - 4 - 1) + 1 \times 4 = -2$. It indicates that they are actually over-constraint linkages in spatial case. This means they cannot move which is clearly incorrect. If a over-constraint linkage is supposed to move, the links and joints should meet certain criteria until the degree of freedom becomes one or more.

The summary is that a general over-constraint linkage is nothing but a rigid body, only when its links and joints meet certain criteria, it becomes movable. A planar 4R linkage has one DOF because it's a over-constraint spatial 4R linkage of which all four of the rotational axes are parallel. A spherical 4R linkage has all rotational axes intersect into one point. A Bennett's linkage has its own special criteria for the dimensions of the links so that it can move.

5.3 Planar and Spherical Linkages in terms of Spatial 5-SS Platform Linkage

We present the planar and spherical RR link by combinations of the spatial leg types using planar and spherical constraints. Specifically, for instance, when a linkage is fully constructed, the joint formed by a couple of SS dyads is constrained to move on a circle which behave as a revolute joint. Thus, planar four bar and spherical 4R linkages are able to be constructed by spatial legs and the same algorithm can be applied for linkage synthesis. Furthermore, this method has the potential to extend to more linkages.

5.3.1 Representation of Planar RR Dyad by Spherical and Planar Constraints

A planar RR dyad can be represented by the combination of a plane and a sphere constraints as illustrated in Figure 5.1, where the S joint can be replaced by a T joint. In the figure, OB is the spatial SS dyad, A'B is the equivalent planar RR dyad while A' is the fixed joint, B is the moving joint and line AO is the rotational axis of its fixed pivot. The joint B is moving on a circle defined by sphere O and a plane. SS dyad can also be substituted by other spatial legs whose moving joints move on the surface of a sphere.

For example, in the case of planar position synthesis, one can find multiple spheres and parallel planes. Since the constraint manifold of planar linkage is a spatial plane, we can find infinite planes as long as they have the same normal direction, in other words, they are parallel planes. They define linkages with different translation on the direction of rotational axes, but they all pass through the same given positions. The intersection of spheres and planes are circles defining the planar RR dyads. Although there are multiple solutions, the intersection circles all define a finite set of RR dyads.

When any two of the RR links construct a 4R linkage, if all of the rotational axes are parallel to each other(both fixed and moving pivots), it becomes a planar 4R linkage, otherwise, it may be a spherical 4R linkage or a bennett 4R linkage which we will show later. The design parameters of RR links can further be calculated from their geometry properties.

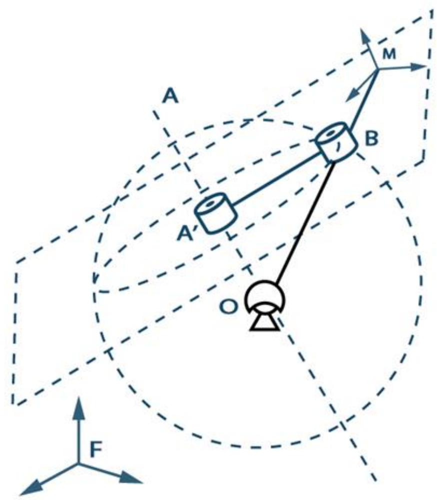

Figure 5.1: Planar RR Leg Constrained by a Sphere and a Plane: spatial SS dyad OB; equivalent planar RR dyad A'B; rotational axis AO of fixed joints; moving frame M; fixed frame F.

5.3.2 Representation of Spherical RR Dyad by Spherical and Planar Constraints

The spherical RR dyad shares some similarity with the planar RR dyad. The difference is the fixed and moving rotational axes are not parallel, they intersect into one point O as shown in Figure 5.2. The illustration is the case where the spherical dyad is constrained by a plane and a sphere. AO is the fixed axis defined by the center of the sphere and the directional vector of the plane, OB is the SS leg (S can be replaced by a T joint) as well as the moving pivot axis of the spherical dyad. Joint B is located on the intersection circle of the plane and sphere.

In spherical position synthesis, the solutions can be multiple groups of spheres of which the centers in each group lie on a same line passing a common center or

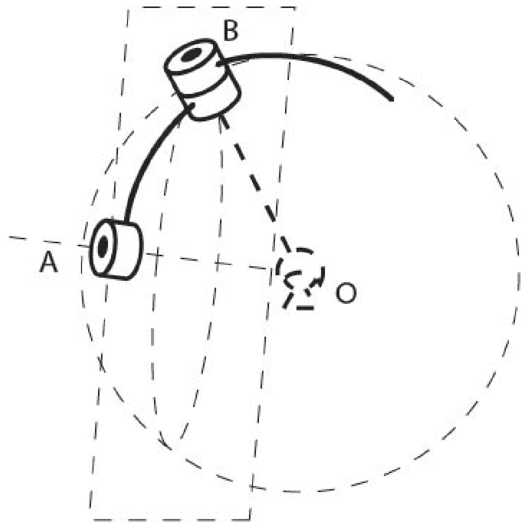

Figure 5.2: Spherical RR Leg Constrained by a Sphere and a Plane: Spatial SS dyad OB; equivalent spherical RR dyad AB; rotational center O.

finite set of parallel planes. It is straightforward to obtain the fixed pivot axis, one can find a circle by intersection of sphere and plane or two sphereshowever to obtain the moving pivot axis requires more information. One method is to treat coupler link and end effector as a rigid body and translate all seven possible positions for end effector onto one position, thus the origin of moving frame will be found on a plane of which the normal direction is OB. This method is going to be discussed in detail in the next chapter for spatial RR dyad synthesis in Bennett's linkage. The simplest method is to verify the locations for the given positions being on a surface of sphere. Once we know the given positions are spherical movements, we can use OB as the axis for the moving pivot because O is the rotational center of the linkage. The verification of input positions of different cases are developed in the last chapter.

To obtain a 4R spherical linkage, one can pick two spherical RR dyads while the fixed axes and moving axis of two spherical RR dyads intersect into one point, i.e. rotational center.

5.4 Seven Planar and Spherical Position Synthesis Examples

5.4.1 Seven Planar Position Synthesis

Table 5.1: Seven Planar Task Positions: given as rotaional axis (s_x, s_y, s_z), rotational angle θ and translation vector (d_1, d_2, d_3).

	s_x s_y s_z θ	d_1 d_2 d_3
1	-0.6515,-0.7232,0.2286,2.5453	3.1016,0.9856,-0.5777
2	-0.7617,-0.5857,0.2768,2.6375	2.6772,-0.1732,-0.2852
3	-0.8465,-0.4285,0.3157,2.7486	2.5387,-0.3868,-0.1754
4	-0.8350,-0.4543,0.3103,2.7300	3.2552,0.5430,-0.7586
5	-0.7303,-0.6304,0.2628,2.6070	3.7249,1.3641,-1.1224
6	-0.7005,-0.6684,0.2497,2.5814	2.8605,0.3533,-0.4092
7	-0.7884,-0.5430,0.2888,2.6672	3.6032,1.0066,-1.0408

Notes: The units of s_x, s_y, s_z, d_x, d_y, d_z are meters; the unit of θ is radian.

We call it "general" to make the difference from the traditional planar synthesis because our method will find planar linkage lying on spatial planes. This algorithm not only gives the original design parameters for a planar linkage, but also provides the orientation of the plane where the linkage is located. Thus this is a general linkage synthesis algorithm for spatial planar four bar linkage.

Table 5.1 gives seven planar task positions. There are 11 solutions as shown in Table 7.2, two of them represent planar dyads while others can construct

Table 5.2: Planar Position Example: design parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	0.0000, -0.2002, 0.0188, -0.2160, -0.9554	-1.1528, -510.5240, 0.0000
2	0.0000, 0.2002, -0.0188, 0.2160, 0.9554	-23.6118, 54.6819, 0.0000
3	0.0000, -0.2002,0.0188,-0.2160,-0.9554	8.9950, -41.5867, 0.0000
4	0.0000, -0.2002, 0.0188, -0.2160, -0.9554	2.35318, -22.8586, 0.0000
5	0.0000, 0.2002, -0.0188, 0.2160, 0.9554	-6.8741, 3.0252, 0.0000
6	0.0000, -0.2002,0.0188,-0.2160,-0.9554	1.0357, -3.0201, 0.0000
7	0.0000, 0.2002,-0.0188,0.2160,0.9554	0.6000, -2.5640, 0.0000
8	0.0000, 0.2002,-0.0188,0.2160,0.9554	2.75513, -0.6102, 0.0000
9	0.0000, 0.2002,-0.0188,0.2160,0.9554	-1.1509, -3.0936, 0.0000
10	0.0000, -0.2002,0.0188,-0.2160,-0.9554	0.3867, -1.8745, 0.0000
11	0.0000, -0.2002, 0.0188, -0.2160, -0.9554	-1.3097, -0.5883, 0.0000

multi DOF mechanism or one DOF mechanism by adding other constraints. We pick the set so that all the task positions are lying on the solution planes. One may observe that the 11 solutions are on a same plane with the different moving pivots. Noted that there actually exist infinite solutions, the result of 11 solutions is only a sub-set of the infinite solutions. It is caused by the numerical issues and the solving algorithm of the polynomial equations which can be improved in the future.

Figure 5.3 illustrates the resulting four bar linkage generated from this example. This synthesis method verified the effectiveness of our algorithm. A unified and more efficient process is developed in five position synthesis.

5.4.2 Seven Spherical Position Synthesis

Seven spherical task positions are given in Table 5.3 of the same format. Four solutions are obtained as shown in Table 5.4.

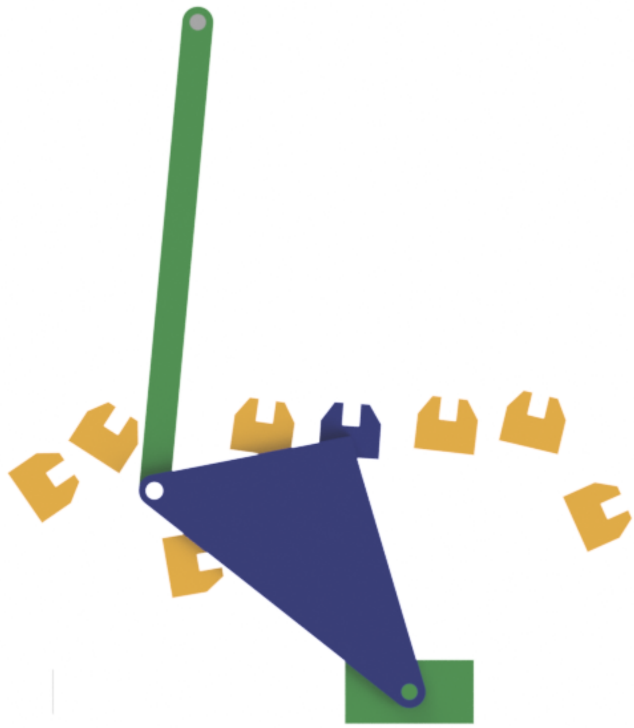

Figure 5.3: The resulting planar four bar linkage.

The solutions define two spherical RR dyads, specifically two rotational axes for the fixed joint. The moving joint axes are obtained by linking moving joint to the center of spheres. Link length can be found further. The numerical results of two equivalent spherical dyads are shown in Table 5.5 (**a** is the direction vector of fixed pivot axis, **x** is the direction vector of moving pivot axis). Figure 5.4 shows the resulting spherical 4R linkage.

5.5 Summary

In this chapter, we applied seven position from planar and spherical linkages into our algorithm of 5-SS linkage synthesis. From the solution point of view, we were able to obtain the planar or spherical linkages that meet the design

Table 5.3: Seven Spherical Task Positions: given as rotaional axis (s_x, s_y, s_z), rotational angle θ and translation vector (d_1, d_2, d_3).

	s_x	s_y	s_z	θ	d_1	d_2	d_3
1	-0.2728,	0.4839,	0.8315,	4.0429	-0.0069,	0.6097,	2.7820
2	-0.1287,	0.4004,	0.9072,	4.4361	0.4390,	0.5358,	3.0623
3	-0.1845,	0.4032,	0.8963,	4.9137	0.5778,	0.4099,	3.1069
4	-0.3216,	0.5554,	0.7669,	5.1255	0.4107,	0.3629,	3.0574
5	-0.4002,	0.8245,	0.4002,	4.9615	-0.0844,	0.3541,	2.7224
6	-0.2512,	0.4701,	0.8461,	5.0522	0.5144,	0.3867,	3.0912
7	-0.4008,	0.7642,	0.5052,	5.0846	0.0823,	0.3515,	2.8761

Notes: The units of s_x, s_y, s_z, d_x, d_y, d_z are meters; the unit of θ is radian.

Table 5.4: Spherical Example: design parameters a_i and x_i.

	a_0	a_1	a_2	a_3	a_4	$\frac{x_1}{x_4}$	$\frac{x_2}{x_4}$	$\frac{x_3}{x_4}$
1	1.0000,	0.0000,	0.4149,	1.8909,	387.5581	-0.2479,	-10.1364,	15.8819
2	1.0000,	0.0000,	1.0289,	1.4390,	13.6207	1.5863,	1.0138,	-3.2643
3	1.0000,	0.0000,	-1.0730,	0.0122,	5.3682	-0.2652,	-0.3564,	-1.1654
4	1.0000,	0.0000,	-1.9914,	0.1773,	6.8826	-0.2698,	-0.3598,	-1.1602

requirements. This indicated that the planar and spherical are the special cases of spatial linkages and thus helped us to develop a more general method for synthesizing these linkages.

Table 5.5: The Dimensions of the Two Resulting Spherical RR Dyads: fixed joints axes **a** and one pair of moving joints axes **x**.

a	**x**
$(0.8400, 0.4140, 1.8870)^T + \mu(0.9999, -0.0002, -0.0040)$	$(0.8400, 0.4140, 1.8870)^T + \lambda(-0.8660, 0.4990, 0.0020)$
$(0.8400, 0.4140, 1.8870)^T + \mu(-0.7414, 0.5418, -0.3957)$	$(0.8400, 0.4140, 1.8870)^T + \lambda(0.3988, -0.9156, 0.0511)$

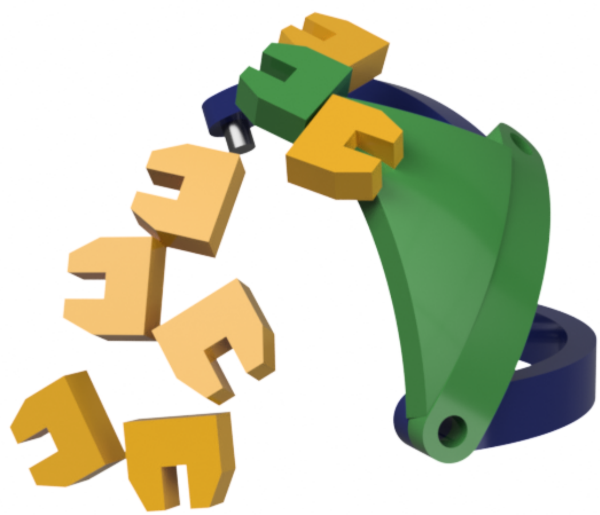

Figure 5.4: The Resulting Spherical 4R Linkage.

Chapter 6

From Spatial 5-SS Platform Linkage to Bennett 4R Linkage Synthesis

6.1 Introduction

In this chapter, we focus on a special spatial linkage, Bennett 4R linkage. It is composed of two spatial RR dyads. The core in this chapter is to demonstrate how to obtain a spatial RR dyad with our method to verify its effectiveness.

6.2 Condition for Bennett 4R Linkage

According to [61], bennett 4R linkage is the only movable spatial 4R-chain. It is constructed by two spatial RR dyads, while the resulting linkage is a over-constraint system, however it is movable when it meets certain criteria following the notation in Figure 6.1:

$$\begin{aligned} l_1 = l_3, l_2 = l_4, \\ \alpha_1 = \alpha_3, \alpha_2 = \alpha_4, \end{aligned} \tag{6.1}$$

$$\frac{l_1}{sin(\alpha_1)} = \frac{l_2}{sin(\alpha_2)}.$$

while for $i = 1, 2, 3, 4$, l_i are the link length, α_i are the angles between joint axis $A_i A_{i+1} (A_4 = A_1)$.

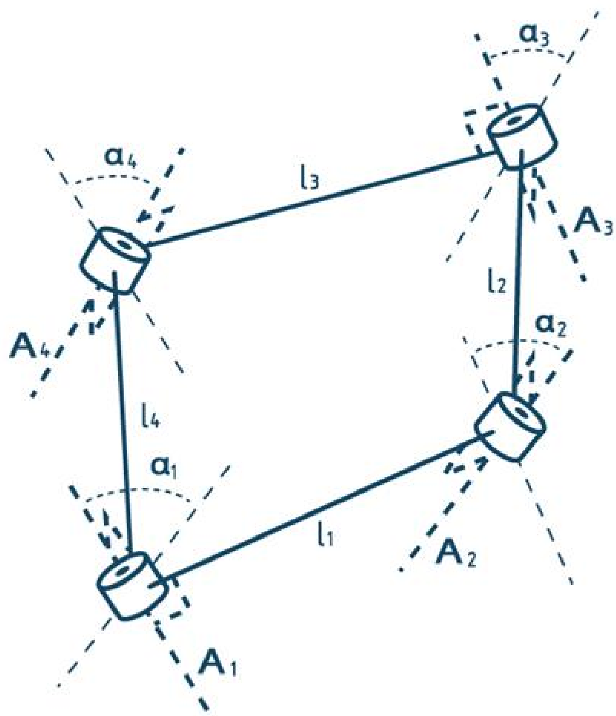

Figure 6.1: Bennett 4R Linkage Notation: link length l_1, l_2, l_3, l_4; rotational axes of joints A_1, A_2, A_3, A_4; angle between rotational axes $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

6.3 Rotation About Arbitrary Axis in 3D

Our method to obtain the moving axis of spatial RR link involves basic knowledge of spatial transformation. Let's review the rotation matrix of an arbitrary axis in 3D space [62].

Define a point $D = (d_1, d_2, d_3)$ in space, then the transformation matrix to move the point to the origin is,

$$[T_d] = \begin{bmatrix} 1 & 0 & 0 & -d_1 \\ 0 & 1 & 0 & -d_2 \\ 0 & 0 & 1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then define T_{xz} the matrix to rotate a unit vector $v = (v_1, v_2, v_3)$ about the z axis to the xz plane. T_z is the matrix to rotate the unit vector in the xz plane to the z axis and $R_z(\theta)$ is rotational matrix by θ around z axis,

$$[T_{xz}] = \begin{bmatrix} \frac{v_1}{\sqrt{v_1^2+v_2^2}} & \frac{v_2}{\sqrt{v_1^2+v_2^2}} & 0 & 0 \\ -\frac{v_2}{\sqrt{v_1^2+v_2^2}} & \frac{v_1}{\sqrt{v_1^2+v_2^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$[T_z] = \begin{bmatrix} v_3 & 0 & -\sqrt{v_1^2+v_2^2} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{v_1^2+v_2^2} & 0 & v_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$[R_z(\theta)] = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus we can define the line by a point $P = (p_1, p_2, p_3)$ and a unit vector $S = (s_1, s_2, s_3)$, then the matrix for rotation about an arbitrary line in 3D is given by $T_l = T_d^{-1} T_{xz}^{-1} T_z^{-1} R_z(\theta) T_z T_{xz} T_d$,

$$[T_l] = \begin{bmatrix} T_{1,1} & T_{1,2} & T_{1,3} & T_{1,4} \\ T_{2,1} & T_{2,2} & T_{2,3} & T_{2,4} \\ T_{3,1} & T_{3,2} & T_{3,3} & T_{3,4} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.2}$$

where

$$\begin{aligned}
T_{1,1} &= s_1^2 + (s_2^2 + s_3^2)cos\theta \\
T_{1,2} &= s_1 s_2 (1 - cos\theta) - s_3 sin\theta \\
T_{1,3} &= s_1 s_3 (1 - cos\theta) + s_2 sin\theta \\
T_{1,4} &= (p_1(s_2^2 + s_3^2) - s_1(p_2 s_2 + p_3 s_3))(1 - cos\theta) + (p_2 s_3 - p_3 s_2) sin\theta \\
T_{2,1} &= s_1 s_2 (1 - cos\theta) + s_3 sin\theta \\
T_{2,2} &= s_2^2 + (s_1^2 + s_3^2)cos\theta \\
T_{2,3} &= s_2 s_3 (1 - cos\theta) - s_1 sin\theta
\end{aligned}$$

$$
\begin{aligned}
T_{2,4} &= (p_2(s_1^2+s_3^2)-s_2(p_1s_1+p_3s_3))(1-cos\theta)+(p_3s_1-p_1s_3)sin\theta \\
T_{3,1} &= s_1s_3(1-cos\theta)-s_2sin\theta \\
T_{3,2} &= s_2s_3(1-cos\theta)+s_1sin\theta \\
T_{3,3} &= s_3^2+(s_1^2+s_2^2)cos\theta \\
T_{3,4} &= (p_3(s_1^2+s_2^2)-s_3(p_1s_1+p_2s_2))(1-cos\theta)+(p_1s_2-p_2s_1)sin\theta
\end{aligned}
$$

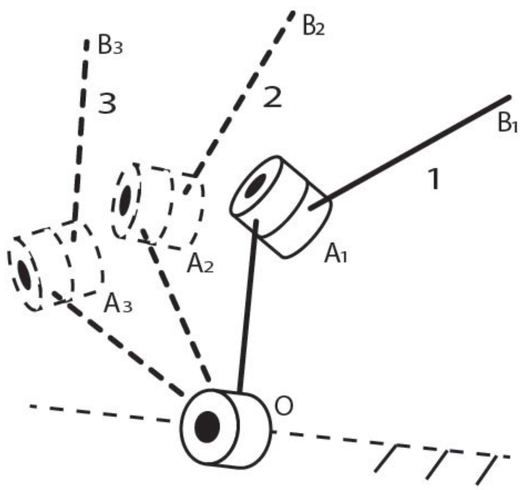

Figure 6.2: Equivalent Spatial RR Links for First Three Positions: fixed joint O; moving joints A_1, A_2, A_3; point B_1, B_2, B_3 on end link.

6.4 Synthesis of Spatial RR Dyad

In this section, we focus on how to find the rotational axis of moving pivot. The fixed pivot axis is obtained by the same method in previous chapters.

Figure 6.2 presents the solutions for first three positions after the spatial dyads synthesis. Note that the fixed joint O has been determined during the solution solving. A_1, A_2, A_3 are moving pivots for each position. B_1, B_2, B_3 are origin of moving frame for which we will show they rotate around a moving axis. There are four more positions not shown in the figure but the methods are the same.

First step is to rotate A_3, A_2 to A_1 about axis of fixed pivot O and find the matrix T_l, formulated in 6.2. Define $A_1 = (a_{1,1}, a_{1,2}, a_{1,3})$ and $A_2 = (a_{2,1}, a_{2,2}, a_{2,3})$, the rotation of joint A_2 around axis O to joint A_1 is,

$$A_1 = T_l A_2 \tag{6.3}$$

There are three possible equations for 6.3 while there is only one unknown parameter, θ. One can pick any one of the equations to obtain θ and it should be verified to meet the other two.

Next step is to rotate all the moving joints onto one joint, i.e. A_1, using 6.3. Note that we treat links OA_i and A_iB_i as a single rigid body as rotating. In other words, the rotational axis of B_i is also the axis of joint O.

Fig.6.3 shows the poses of dyads of first three positions after rotation around fixed axis. A_2' and A_3' are new positions of A_2 and A_3, B_2' and B_3' are new positions of B_2 and B_3. Again we actually have the points relative to seven dyads, $A_1, A_2', A_3', ..., A_7'$ and $B_1, B_2', B_3', ..., B_7'$.

We will find a plane and a sphere of which the points $B_1, B_2', B_3', ..., B_7'$ are on their surface. Thus the normal direction of the plane and the center of the sphere will define the axis of moving joint of the dyad relative to the first position.

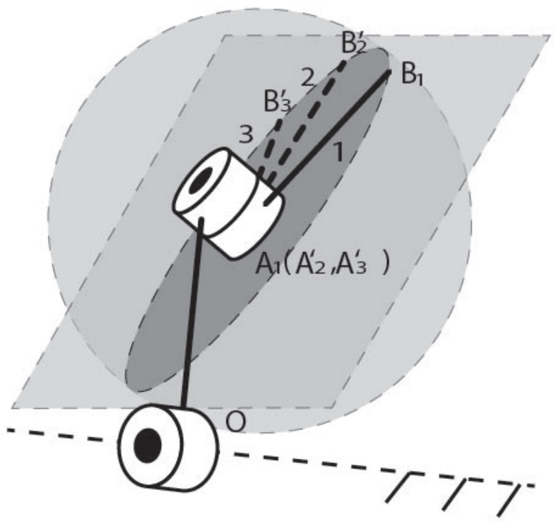

Figure 6.3: Spatial RR Dyad after Rotation about Axis of Fixed Joint O.

6.5 Bennett 4R Linkage Synthesis

A 4R Bennett linkage is very similar to a 4R planar linkage, the only difference is the revolute axes of each joint are not parallel to each other, the length and the angles of the legs should meet certain criteria. We explain our method by numerical example.

Table 6.1 shows 7 positions of a Bennett mechanism.

There are two solution sets of $\lambda_1, \lambda_2, \lambda_3$, as shown in Table 6.2, each leads to one solution. Therefore there are only two solutions which had been verified in [25].

Again these solutions behave very interesting. The points on each sphere lie on a same plane. Thus there exist two separate circles, each defines a revolute joint for the fixed pivot of a spatial RR dyad. The moving pivot can then be

Table 6.1: Seven Bennett Linkage Task Positions: given as rotaional axis (s_x, s_y, s_z), rotational angle θ and translation vector (d_1, d_2, d_3).

	s_x	s_y	s_z	θ	d_1	d_2	d_3
1	0.0000,	0.4579,	-0.8889,	2.7464	-0.0243,	0.1258,	-0.1170
2	0.0000,	0.9701,	-0.2425,	2.8485	-0.0084,	0.2084,	-0.0233
3	0.0000,	0.2609,	0.9653,	2.7135	0.0119,	0.0661,	0.1442
4	0.0000,	0.9954,	0.0962,	2.8525	0.0025,	0.2104,	0.0116
5	0.0000,	0.9996,	-0.0282,	2.8533	-0.0014,	0.2111,	-0.0011
6	0.0000,	0.6672,	-0.7448,	2.7910	-0.0240,	0.1675,	-0.0883
7	0.0000,	0.9137,	0.4062,	2.8380	0.0124,	0.2004,	0.0455

Notes: The units of s_x, s_y, s_z, d_x, d_y, d_z are meters; the unit of θ is radian.

obtained by translating the moving frames onto the first position. One can find that the moving frames have a common rotational axis.

The two result RR dyads are shown in Table 6.4 (**a** is the direction vector of fixed pivot axis, **x** is the direction vector of moving pivot axis). The correctness can be verified by the definition of Bennett 4R mechanism criteria. The final linkage and seven positions are shown in Figure 6.4.

Table 6.2: Bennett 4R Linkage Synthesis Example: real solutions of $\lambda_1, \lambda_2, \lambda_3$

	λ_1	λ_2	λ_3
1	3.0668	0.1000	-3.1654
2	3.0703	-0.1180	3.4346

Table 6.3: Bennett 4R Linkage Synthesis Example: original design parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000,5.4984,0.0583,-6.1470,-39.8236	3.0668,0.1000,-3.1654
2	1.0000,-1.6892,-0.2938,-1.7364,39.4918	3.0703,-0.1180,3.4346

Table 6.4: The dimensions of the revolute axes of bennett mechanism

a	**x**
$(5.498, 0.058, -6.147)^T + \mu(-0.666, 0.016, 0.744)$	$(-0.162, 0.300, -0.149)^T + \lambda(0.741, 0.504, -0.442)$
$(-1.689, -0.293, -1.736)^T + \mu(0.696, 0.028, 0.717)$	$(0.096, -0.023, -0.089)^T + \lambda(0.216, 0.343, 0.913)$

6.6 Summary

The geometric constraint of spatial RR dyads can be found by the intersection of two spheres or a sphere and a plane. Therefore we developed the synthesis algorithm for linkages constructed by spatial RR dyads, Bennett 4R linkage, from the method of 5-SS linkage. From solution point of view, we were able to obtain the RR dyads and find the Bennett linkage directly using our 5-SS linkage synthesis algorithm.

Figure 6.4: The Result Bennett Linkage and Seven Positions

Chapter 7

Finite Position Synthesis of Spatial 5-SS Platform Linkage

7.1 Introduction

This chapter focuses on six and five finite position synthesis. The design equation is started as 4.10. For six position synthesis, we add one additional equation where all moving or fixed pivots of the 5-SS linkage legs are located on and a plane on which all moving pivots of legs are tended to be. In the development of five position synthesis, we use two additional equations to constrain the positions of moving or fixed pivots. In our formulation, the additional equations are given at the final step, thus the algorithm itself remains intact. This means the form of the additional constraints are very flexible. Numerical example for each case is given at the end of this chapter.

7.2 The Reordered Design Equation for a SS Chain

In this problem, we reordered the terms in 4.6 and obtained the new order of coefficients A_i,

$$p_0 + \sum_{i=1}^{16} A_i p_i = 0, \tag{7.1}$$

where

$$\begin{aligned}
A_1 &= 4g_4q_1 - 4g_3q_2 + 4g_2q_3 - 4g_1q_4,\\
A_2 &= 4g_3q_1 + 4g_4q_2 - 4g_1q_3 - 4g_2q_4,\\
A_3 &= -4g_2q_1 + 4g_1q_2 + 4g_4q_3 - 4g_3q_4,\\
A_4 &= 4g_1^2 + 4g_2^2 + 4g_3^2 + 4g_4^2,\\
A_5 &= -q_1^2 + q_2^2 + q_3^2 - q_4^2,\\
A_6 &= 2(-q_1q_2 - q_3q_4),\\
A_7 &= 2(-q_1q_3 + q_2q_4),\\
A_8 &= 2(-g_2q_1 - g_3q_2 + g_2q_3 + g_1q_4),\\
A_9 &= 2(-q_1q_2 + q_3q_4),\\
A_{10} &= q_1^2 - q_2^2 + q_3^2 - q_4^2,\\
A_{11} &= 2(-q_2q_3 - q_1q_4),\\
A_{12} &= 2(g_3q_1 - g_4q_2 - g_1q_3 + g_2q_4),\\
A_{13} &= 2(-q_1q_3 - q_2q_4),\\
A_{14} &= 2(-q_2q_3 + q_1q_4),\\
A_{15} &= q_1^2 + q_2^2 - q_3^2 - q_4^2,\\
A_{16} &= 2(-g_2q_1 + g_1q_2 - g_4q_3 + g_3q_4),
\end{aligned} \tag{7.2}$$

and each of the p_i $(i = 0, 1, 2, \ldots, 16)$ are defined by the choice of the moving point $\mathbf{x} = (x_1, x_2, x_3, x_4)$ as well as the spherical constraint $(a_0, a_1, a_2, a_3, a_4)$ as follows:

$$p_1 = a_1x_4^2, \quad p_2 = a_2x_4^2, \quad p_3 = a_3x_4^2, \quad p_4 = a_0x_4^2 \tag{7.3}$$
$$p_5 = a_1x_1x_4, \quad p_6 = a_2x_1x_4, \quad p_7 = a_3x_1x_4, \quad p_8 = a_0x_1x_4$$
$$p_9 = a_1x_2x_4, \quad p_{10} = a_2x_2x_4, \quad p_{11} = a_3x_2x_4, \quad p_{12} = a_0x_2x_4$$
$$p_{13} = a_1x_3x_4, \quad p_{14} = a_2x_3x_4, \quad p_{15} = a_3x_3x_4, \quad p_{16} = a_0x_3x_4$$
$$p_0 = a_0x_1^2 + a_0x_2^2 + a_0x_3^2 - a_4x_4^2.$$

As both x_i, a_i are homogeneous, there are only 7 independent design parameters. The reason to reorder the coefficients A_i is to prepare for the unified algorithm for planar, spherical and spatial five position synthesis in the next chapter.

For N position synthesis, we have N linear equations each in the form 7.1. One can subtract the jth equation ($j = 2, 3, \ldots, N$) from the first equation to obtain four equations in the form:

$$(A_{j1} - A_{11})p_1 + (A_{j2} - A_{12})p_2 + (A_{j3} - A_{13})p_3 + \\ \cdots + (A_{j16} - A_{116})p_{16} = 0. \tag{7.4}$$

The reordered 7.1 is in bilinear form of x_i and a_i, by observing 7.3 one can obtain two new similar relations among the sixteen parameters p_i. One of them is based on x_i, another one is based on a_i, as following:

$$\frac{p_5}{p_1} = \frac{p_6}{p_2} = \frac{p_7}{p_3} = \frac{p_8}{p_4} = \frac{x_1}{x_4} = \lambda_1,$$
$$\frac{p_9}{p_1} = \frac{p_{10}}{p_2} = \frac{p_{11}}{p_3} = \frac{p_{12}}{p_4} = \frac{x_2}{x_4} = \lambda_2, \tag{7.5}$$

$$\frac{p_{13}}{p_1} = \frac{p_{14}}{p_2} = \frac{p_{15}}{p_3} = \frac{p_{16}}{p_4} = \frac{x_3}{x_4} = \lambda_3.$$

and (assuming $a_0 \neq 0$)

$$\begin{aligned}
\frac{p_1}{p_4} = \frac{p_5}{p_8} = \frac{p_9}{p_{12}} = \frac{p_{13}}{p_{16}} = \frac{a_1}{a_0} = \mu_1, \\
\frac{p_2}{p_4} = \frac{p_6}{p_8} = \frac{p_{10}}{p_{12}} = \frac{p_{14}}{p_{16}} = \frac{a_2}{a_0} = \mu_2, \\
\frac{p_3}{p_4} = \frac{p_7}{p_8} = \frac{p_{11}}{p_{12}} = \frac{p_{15}}{p_{16}} = \frac{a_3}{a_0} = \mu_3.
\end{aligned} \tag{7.6}$$

Let h_i denote 16 components in 7.4 represented in terms of $A_{ji} - A_{1i}$ ($j = 2, 3, \ldots, N, i = 1, 2, \ldots, 16$) and we can rewrite 7.4 into

$$\mathbf{h}\mathbf{P}^T = 0, \tag{7.7}$$

where $\mathbf{h} = (h_1, h_2, \ldots, h_{16})$ and $\mathbf{P} = (p_1, \ldots, p_{16})$.

For a set of N given positions, we obtain a set of $N-1$ vectors $\mathbf{h}_j$ ($j = 2, 3, \ldots, N$). These $N-1$ linear equations can be assembled into a matrix equation as:

$$[H]\mathbf{P}^T = 0, \tag{7.8}$$

where $[H]$ is the $(N-1) \times 16$ matrix consisting $\mathbf{h}_j$. Thus the problem reduces to the solution of the system of linear equation 7.8 subject to the same group of bilinear constraints 7.5 or 7.6.

In order to solve 7.8, we first use Gaussian elimination to obtain the reduced row echelon form of $[H]$. This final form is unique [63]. A matrix is in reduced row echelon form when it satisfies: (1) The matrix is in row echelon form (2) The leading coefficient in each row is 1 and is the only non-zero entry in its column.

7.3 Six Position Synthesis for Spatial 5-SS Linkages with One Additional Constraint

This section will talk about six position synthesis of 5-SS platform linkage. Because there are seven original independent design variables, two additional constraint equations are needed in order to obtain finite solutions. In our formulation, the additional equations are given at the final step, thus the algorithm itself remains intact. This means the form of the additional constraints are very flexible.

In general six position synthesis, the reduced row echelon form of $[H]$ is a 5 by 16 matrix in the form,

$$RREF([H]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & k_{1,6} & \cdots & k_{1,16} \\ 0 & 1 & 0 & 0 & 0 & k_{2,6} & \cdots & k_{2,16} \\ 0 & 0 & 1 & 0 & 0 & k_{3,6} & \cdots & k_{3,16} \\ 0 & 0 & 0 & 1 & 0 & k_{4,6} & \cdots & r_{4,16} \\ 0 & 0 & 0 & 0 & 1 & k_{5,6} & \cdots & r_{5,16} \end{bmatrix} \tag{7.9}$$

Thus $p_1, p_2, \cdots, p_5$ can be represented by the linear combination of other 11 variables $p_6, p_7, \cdots, p_{16}$.

$$p_1 = -k_{1,6}p_6 - k_{1,7}p_7 - \ldots - k_{1,16}p_{16},$$

$$p_2 = -k_{2,6}p_6 - k_{2,7}p_7 - \ldots - k_{2,16}p_{16},$$

$$p_3 = -k_{3,6}p_6 - k_{3,7}p_7 - \ldots - k_{3,16}p_{16},$$
$$p_4 = -k_{4,6}p_6 - k_{4,7}p_7 - \ldots - k_{4,16}p_{16},$$
$$p_5 = -k_{5,6}p_6 - k_{5,7}p_7 - \ldots - k_{5,16}p_{16}. \tag{7.10}$$

Considering 7.5 and 7.6 after algebraic manipulation, we have

$$\begin{gathered} p_5 - \lambda_1 p_1 = 0, p_6 - \lambda_1 p_2 = 0, p_7 - \lambda_1 p_3 = 0, p_8 - \lambda_1 p_4 = 0, \\ p_9 - \lambda_2 p_1 = 0, p_{10} - \lambda_2 p_2 = 0, p_{11} - \lambda_2 p_3 = 0, p_{12} - \lambda_2 p_4 = 0, \\ p_{13} - \lambda_3 p_1 = 0, p_{14} - \lambda_3 p_2 = 0, p_{15} - \lambda_3 p_3 = 0, p_{16} - \lambda_3 p_4 = 0 \end{gathered} \tag{7.11}$$

and

$$\begin{gathered} p_1 - \mu_1 p_4 = 0, p_5 - \mu_1 p_8 = 0, p_9 - \mu_1 p_{12} = 0, p_{13} - \mu_1 p_{16} = 0, \\ p_2 - \mu_2 p_4 = 0, p_6 - \mu_2 p_8 = 0, p_{10} - \mu_2 p_{12} = 0, p_{14} - \mu_2 p_{16} = 0, \\ p_3 - \mu_3 p_4 = 0, p_7 - \mu_3 p_8 = 0, p_{11} - \mu_3 p_{12} = 0, p_{15} - \mu_3 p_{16} = 0 \end{gathered} \tag{7.12}$$

We can substitute 7.10 into 7.11 and 7.12 to formulate the homogeneous matrix equations,

$$[B_1]\mathbf{F} = 0 \tag{7.13}$$

and

$$[B_2]\mathbf{F} = 0 \tag{7.14}$$

where $[B_1]$, $[B_2]$ and $\mathbf{F}$ are of the following form,

$$[B_1] = \begin{bmatrix} -k_{5,6} + \lambda_1 k_{1,6} & \cdots & -k_{5,15} + \lambda_1 k_{1,15} & -k_{5,16} + \lambda_1 k_{1,16} \\ 1 - \lambda_1 k_{2,6} & \cdots & \lambda_1 k_{2,15} & \lambda_1 k_{2,16} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_3 k_{3,6} & \cdots & 1 - \lambda_3 k_{3,15} & \lambda_3 k_{4,16} \\ \lambda_3 k_{4,6} & \cdots & \lambda_3 k_{4,15} & 1 - \lambda_3 k_{4,16} \end{bmatrix}$$

$$[B_2] = \begin{bmatrix} -k_{1,6} + \mu_1 k_{4,6} & \cdots & -k_{1,15} + \mu_1 k_{4,15} & -k_{1,16} + \mu_1 k_{4,16} \\ -k_{5,6} & \cdots & -k_{5,15} & -k_{5,16} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & -\mu_3 \end{bmatrix}$$

$$\mathbf{F} = [p_6, p_7, \cdots, p_{16}]^T \tag{7.15}$$

$[B_1]$ and $[B_2]$ are 12×11 rectangular matrices. One can drop any one row of the 12 by 11 matrix to obtain two 11 by 11 square matrix. Again if $\mathbf{F}$ has non-trivial solutions, the determinants of the square matrices should be zero. Then we can add an additional equation of either x_i (for formulation of $[B_1]$) or a_i (for formulation of $[B_2]$) to obtain finite solutions.

The non-trivial solution of $\mathbf{F}$ can then be found using null space analysis method. p_1, p_2, p_3, p_4, p_5 are obtained by substituting $\mathbf{F}$ back into 7.10. Since p_0 is independent, it can be found by the arithmetic average. The original design parameters are obtained by inner relationship of p_i.

7.4 Five Position Synthesis for Spatial 5-SS linkages, Planar and Spherical 4R Linkages with Two Additional Constraints

This section develops the general spatial, planar and spherical five position synthesis based on our seven position algorithm. Any equation of either fixed joints x_i or a_i or the combination of two can be given to obtain a finite set of solutions.

In this problem, because we want to obtain the exact types and dimensions of linkages corresponding to the different categories of input displacements, the additional constraints are not totally arbitrary. Although it is true that any two arbitrary additional constraint equations are allowed in our algorithm to narrow the solution number, one may not obtain a planar or spherical linkage. It is necessary to give the correct constraint equations to find the planar or spherical linkages. The rule for choosing the correct constraints relies on determining the type of input displacements.

Gaussian elimination shows its advantage because the result of reduced row echelon form of $[H]$ can indicate the types of input positions from its formulation. If the input is general five positions, $[H]$ is reduced to a 4 by 16 matrix in the form,

$$RREF([H]) = \begin{bmatrix} 1 & 0 & 0 & 0 & k_{1,5} & k_{1,6} & \cdots & k_{1,16} \\ 0 & 1 & 0 & 0 & k_{2,5} & k_{2,6} & \cdots & k_{2,16} \\ 0 & 0 & 1 & 0 & k_{3,5} & k_{3,6} & \cdots & k_{3,16} \\ 0 & 0 & 0 & 1 & k_{4,5} & k_{4,6} & \cdots & k_{4,16} \end{bmatrix} \tag{7.16}$$

When the input positions are pure planar displacements, then $[H]$ is reduced to the following,

$$RREF([H]) = \begin{bmatrix} 1 & 0 & k_{1,3} & 0 & 0 & k_{1,6} & \cdots & k_{1,16} \\ 0 & 1 & k_{2,3} & 0 & 0 & k_{2,6} & \cdots & k_{2,16} \\ 0 & 0 & 0 & 1 & 0 & k_{3,6} & \cdots & k_{3,16} \\ 0 & 0 & 0 & 0 & 1 & k_{4,6} & \cdots & k_{4,16} \end{bmatrix} \tag{7.17}$$

For pure spherical displacements, reduced row echelon form of $[H]$ is,

$$RREF([H]) = \begin{bmatrix} 1 & 0 & 0 & k_{1,4} & 0 & k_{1,6} & \cdots & k_{1,16} \\ 0 & 1 & 0 & k_{2,4} & 0 & k_{2,6} & \cdots & k_{2,16} \\ 0 & 0 & 1 & k_{3,4} & 0 & k_{3,6} & \cdots & k_{3,16} \\ 0 & 0 & 0 & 0 & 1 & k_{4,6} & \cdots & k_{4,16} \end{bmatrix} \tag{7.18}$$

Therefore reduced row echelon form will tell by its form the type of input displacements. Furthermore, $(-k_{1,3}, -k_{2,3}, 1)$ in 7.17 is the directional vector of the plane where five planar displacements are located, $(-k_{1,4}, -k_{2,4}, -k_{3,4})$ in 7.18 is the rotational center of five spherical displacements. The algebraic proof can be referred to Appendix A. The advantages are that it has the same effect as doing preprocessing and being a part of our unified algorithm as well.

Next we are going to add the constraints and find the corresponding linkages based on different types of displacements.

7.4.1 Five General Displacements

In this case, p_1, p_2, p_3, p_4 are represented by the linear combination of other 12 variables $p_5, p_6, \cdots, p_{16}$,

$$\begin{aligned} p_1 &= -k_{1,5}p_5 - k_{1,6}p_6 - \ldots - k_{1,16}p_{16}, \\ p_2 &= -k_{2,5}p_5 - k_{2,6}p_6 - \ldots - k_{2,16}p_{16}, \\ p_3 &= -k_{3,5}p_5 - k_{3,6}p_6 - \ldots - k_{3,16}p_{16}, \\ p_4 &= -k_{4,5}p_5 - k_{4,6}p_6 - \ldots - k_{4,16}p_{16}. \end{aligned} \tag{7.19}$$

The same relations of 7.11 or 7.12 formulate the homogeneous matrix equations in the form of 7.13 or 7.14, where $[B_1]$, $[B_2]$ and $\mathbf{F}$ are of the new form,

$$[B_1] = \begin{bmatrix} 1-\lambda_1 k_{1,5} & \lambda_1 k_{1,6} & \lambda_1 k_{1,7} & \cdots & \lambda_1 k_{1,15} & \lambda_1 k_{1,16} \\ \lambda_1 k_{2,5} & 1-\lambda_1 k_{2,6} & \lambda_1 k_{2,7} & \cdots & \lambda_1 k_{2,15} & \lambda_1 k_{2,16} \\ \lambda_1 k_{3,5} & \lambda_1 k_{3,6} & 1-\lambda_1 k_{3,7} & \cdots & \lambda_1 k_{3,15} & \lambda_1 k_{3,16} \\ \lambda_1 k_{4,5} & \lambda_1 k_{4,6} & \lambda_1 k_{4,7} & \cdots & \lambda_1 k_{3,15} & \lambda_1 k_{3,16} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \lambda_3 k_{2,5} & \lambda_3 k_{2,6} & \cdots & 1-\lambda_3 k_{2,14} & \lambda_3 k_{2,15} & \lambda_3 k_{2,16} \\ \lambda_3 k_{3,5} & \lambda_3 k_{3,6} & \cdots & \lambda_3 k_{3,14} & 1-\lambda_3 k_{3,15} & \lambda_3 k_{4,16} \\ \lambda_3 k_{4,5} & \lambda_3 k_{4,6} & \cdots & \lambda_3 k_{4,14} & \lambda_3 k_{4,15} & 1-\lambda_3 k_{4,16} \end{bmatrix}$$

$$
[B_2] = \begin{bmatrix} -k_{1,5} + \mu_1 k_{4,5} & \cdots & -k_{1,15} + \mu_1 k_{4,15} & -k_{1,16} + \mu_1 k_{4,16} \\ 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & -\mu_3 \end{bmatrix}
$$

$$
\mathbf{F} = [p_5, p_6, \cdots, p_{16}]^T \tag{7.20}
$$

Both $[B_1]$ and $[B_2]$ are 12×12 square matrices. If $\mathbf{F}$ has non-trivial solutions, the square matrices should be rank-deficient, i.e. determinants to be zero. One can then add any two constraint equations of either moving joints x_i combined with determinant of $[B_1]$ equals zero or fixed joints a_i with determinant of $[B_2]$ equals zero in order to obtain finite solutions. The original parameters are obtained by the same method in six position synthesis. One can also add one constraint of moving joints and another of fixed joints combined with $[B_1]$ or $[B_2]$ as in the case of planar displacements.

One can prove algebraically that the polynomial equation, obtained by calculating determinant of either $[B_1]$ or $[B_2]$, has an order of four. Therefore if we add two additional linear equations, such as two planes, there can be no more than four solutions. If one wants to obtain more solutions, we can add additional constraint equations of higher order such as spheres or more complex surfaces. This is shown in the examples that two solutions are obtained by adding two plane constraints while six solutions are obtained when adding a plane and a sphere constraints.

Previously, the two additional equations we added are either constraining moving joints x_i or fixed joints a_i. Now we have developed the case where one linear equation of x_i and one linear equation of a_i are added as following,

$$\begin{aligned} t_1x_1 + t_2x_2 + t_3x_3 + t_4 &= 0, \\ t'_1a_1 + t'_2a_2 + t'_3a_3 + t'_4 &= 0. \end{aligned} \tag{7.21}$$

In this case, we have to convert one of the two equations. Let's develop the process separately in the following.

First let's convert the equation of x_i by substituting x_3 using (when $t_3 \neq 0$),

$$x_3 = -\frac{t_1}{t_3}x_1 - \frac{t_2}{t_3}x_2 - \frac{t_4}{t_3} \tag{7.22}$$

Noted that the vector $\mathbf{P}$ in 7.8 is,

$$\begin{aligned} \mathbf{P}^T = [&a_1, a_2, a_3, a_0, a_1x_1, a_2x_1, a_3x_1, a_0x_1, a_1x_2, a_2x_2, a_3x_2, a_0x_2, \\ &a_1x_3, a_2x_3, a_3x_3, a_0x_3]^T \end{aligned} \tag{7.23}$$

After substituting 7.22 into 7.23, we can rewrite $\mathbf{P}$ as,

$$\mathbf{P}^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_0 \\ a_1 x_1 \\ a_2 x_1 \\ a_3 x_1 \\ a_0 x_1 \\ a_1 x_2 \\ a_2 x_2 \\ a_3 x_2 \\ a_0 x_2 \\ -\frac{t_1}{t_3} a_1 x_1 - \frac{t_2}{t_3} a_1 x_2 - \frac{t_4}{t_3} a_1 \\ -\frac{t_1}{t_3} a_2 x_1 - \frac{t_2}{t_3} a_2 x_2 - \frac{t_4}{t_3} a_2 \\ -\frac{t_1}{t_3} a_2 x_1 - \frac{t_2}{t_3} a_2 x_2 - \frac{t_4}{t_3} a_2 \\ -\frac{t_1}{t_3} a_2 x_1 - \frac{t_2}{t_3} a_2 x_2 - \frac{t_4}{t_3} a_2 \end{bmatrix} \tag{7.24}$$

It can be further rewritten as a matrix form,

$$\mathbf{P}^T = [W']\mathbf{P}'^T \tag{7.25}$$

where $[W']$ and $\mathbf{P}'$ are,

$$[W'] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{t_4}{t_3} & 0 & 0 & 0 & -\frac{t_1}{t_3} & 0 & 0 & 0 & -\frac{t_2}{t_3} & 0 & 0 & 0 \\ 0 & -\frac{t_4}{t_3} & 0 & 0 & 0 & -\frac{t_1}{t_3} & 0 & 0 & 0 & -\frac{t_2}{t_3} & 0 & 0 \\ 0 & 0 & -\frac{t_4}{t_3} & 0 & 0 & 0 & -\frac{t_1}{t_3} & 0 & 0 & 0 & -\frac{t_2}{t_3} & 0 \\ 0 & 0 & 0 & -\frac{t_4}{t_3} & 0 & 0 & 0 & -\frac{t_1}{t_3} & 0 & 0 & 0 & -\frac{t_2}{t_3} \end{bmatrix} \tag{7.26}$$

$$\begin{aligned} \mathbf{P}'^T &= [a_1, a_2, a_3, a_0, a_1x_1, a_2x_1, a_3x_1, a_0x_1, a_1x_2, a_2x_2, a_3x_2, a_0x_2]^T \\ &= [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}]^T \end{aligned} \tag{7.27}$$

We can substitute $\mathbf{P}'$ into 7.8 to obtain the new matrix equation,

$$[H]\mathbf{P}^T = [H][W']\mathbf{P}'^T = 0 \tag{7.28}$$

Now we can use our method to solve the intermediate parameters p_i by Gaussian Elimination and the bilinear relationship. Noted that because there are only 12 elements in $\mathbf{P}'$, we only need 9 of the bilinear relationships excluding the 3 relationships relating to x_3. One can construct a similar matrix equation as in 7.14 while $[B_2]$ is a 9×8 matrix. We can find two 8×8 submatrix and calculate the determinant polynomials. Combined with the additional constraint equation of a_i in 7.21, we can obtain the fixed joint coordinates a_i. The rest design parameters can be solved using our previous method.

In some situations, one may want to sole for moving joints x_i first. Therefore, we will convert the additional equation of a_i in 7.21. a_0 can be written as the combination of a_1, a_2, a_3(when $t'_4 \neq 0$),

$$a_0 = -\frac{t'_1}{t'_4}a_1 - \frac{t'_2}{t'_4}a_2 - \frac{t'_3}{t'_4}a_3 \tag{7.29}$$

After substituting a_0 into 7.23, $\mathbf{P}$ is,

$$\mathbf{P}^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ -\frac{t_1'}{t_4'}a_1 - \frac{t_2'}{t_4'}a_2 - \frac{t_3'}{t_4'}a_3 \\ a_1x_1 \\ a_2x_1 \\ a_3x_1 \\ -\frac{t_1'}{t_4'}a_1x_1 - \frac{t_2'}{t_4'}a_2x_1 - \frac{t_3'}{t_4'}a_3x_1 \\ a_1x_2 \\ a_2x_2 \\ a_3x_2 \\ -\frac{t_1'}{t_4'}a_1x_2 - \frac{t_2'}{t_4'}a_2x_2 - \frac{t_3'}{t_4'}a_3x_2 \\ a_1x_3 \\ a_2x_3 \\ a_3x_3 \\ -\frac{t_1'}{t_4'}a_1x_3 - \frac{t_2'}{t_4'}a_2x_3 - \frac{t_3'}{t_4'}a_3x_3 \end{bmatrix} \tag{7.30}$$

Rewrite $\mathbf{P}$ in matrix form,

$$\mathbf{P}^T = [W'']\mathbf{P}''^T \tag{7.31}$$

where $[W'']$ and $\mathbf{P}''$ are,

$$
[W''] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{t'_1}{t'_4} & -\frac{t'_2}{t'_4} & -\frac{t'_3}{t'_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{t'_1}{t'_4} & -\frac{t'_2}{t'_4} & -\frac{t'_3}{t'_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{t'_1}{t'_4} & -\frac{t'_2}{t'_4} & -\frac{t'_3}{t'_4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{t'_1}{t'_4} & -\frac{t'_2}{t'_4} & -\frac{t'_3}{t'_4}
\end{bmatrix} \tag{7.32}
$$

$$
\begin{aligned}
\mathbf{P}''^T &= [a_1, a_2, a_3, a_1x_1, a_2x_1, a_3x_1, a_1x_2, a_2x_2, a_3x_2, a_1x_3, a_2x_3, a_3x_3]^T \\
&= [p_1, p_2, p_3, p_5, p_6, p_7, p_9, p_{10}, p_{11}, p_{13}, p_{14}, p_{15}]^T
\end{aligned} \tag{7.33}
$$

We can substitute $\mathbf{P}''$ into 7.8 to obtain the new matrix equation,

$$[H]\mathbf{P}^T = [H][W'']\mathbf{P}''^T = 0 \tag{7.34}$$

Then we use our method to solve the intermediate parameters p_i by Gaussian Elimination and the bilinear relationship. Noted that because there are only 12 elements in $\mathbf{P}''$, we only need 9 of the bilinear relationships excluding the 3 relationships relating to a_0. One can construct a similar matrix equation as in 7.13 while $[B_1]$ is a 9×8 matrix. We can find two 8×8 submatrix and calculate the determinant polynomials. Combined with the additional constraint equation of x_i in 7.21, we can obtain the moving joint coordinates x_i. The rest design parameters can be solved using our previous method.

7.4.2 Five Planar Displacements

After Gaussian Elimination, we can obtain the directional vector $(-k_{1,3}, -k_{2,3}, 1)$ of the plane where the planar linkage should be located on. Therefore the additional constraints should be two parallel planes with this normal direction to constrain the moving and fixed joints. There are no other directions of constraint planes if three or more planar displacements are given (see Appendix B).

In order to obtain a planar four bar linkage, the additional constraints should be given such that the moving joints and fixed joints are located on parallel planes, as the following (k_4 and k_5 can be any value),

$$-k_{1,3}X_1 - k_{2,3}X_2 + X_3 + k_4 = 0$$

$$-k_{1,3}a_1 - k_{2,3}a_2 + a_3 + k_5 = 0 \tag{7.35}$$

Note that the first equation in 7.35 is using (X_1, X_2, X_3) as moving joints measured in the fixed frame. Before next step, we will need to convert it into (x_1, x_2, x_3) by substituting any one of 4.1 into the equation and then it becomes,

$$K_1x_1 - K_2x_2 + K_3x_3 + K_4 = 0 \tag{7.36}$$

where K_1, K_2, K_3, K_4 are new coefficients consist of $k_{1,3}, k_{2,3}, k_4$ and the quaternion elements in 4.1.

Now this problem turns into the first situation in general five position problem where we need to convert one of the additional equations. In this problem, moving joints coordinate x_i can be any value. If we want to solve x_i first, there will be infinite solutions. Therefore we have to convert the equation of x_i in order to solve a_i first.

We apply the same method and steps from 7.22 to 7.28 and use our method to solve the intermediate parameters p_i by Gaussian Elimination and the bilinear relationship. One can construct a similar matrix equation as in 7.14 while $[B_2]$ is a 9×8 matrix. We can find two 8×8 submatrix and calculate the determinant polynomials. Combined with the additional constraint equation of a_i in 7.35, we can obtain the fixed joint coordinates a_i. The rest design parameters can be solved using our previous method.

7.4.3 Five Spherical Displacements

One of the characteristics of spherical 4R linkage is that its links can be moving on the surfaces of concentric spheres whose center is the rotational center of the given spherical displacements. There can be no other rotational center of four or more spherical displacements (see Appendix B). We have known that the reduced row echelon form will find the rotational center of the five spherical displacements. Thus one can always find a set of concentric spheres with the center of $(-k_{1,4}, -k_{2,4}, -k_{3,4})$ while the radius can be any size. Any one of these spheres defines a surface where the spherical 4R linkages can be found and these spherical linkages are equivalent in each surface. The additional spherical constraint added for moving joints is,

$$(X_1 + k_{1,4})^2 + (X_2 + k_{2,4})^2 + (X_3 + k_{3,4})^2 + t_3 = 0 \qquad (7.37)$$

where t_3 can be any value related to the radius of the sphere. Note that the constraint is using (X_1, X_2, X_3) as moving joints measured in the fixed frame. Same as the planar displacements synthesis, we can convert it by substituting any one of 4.1 into the equation to obtain the additional constraint of x_i.

Next step is to find the planes by imposing the constraint $a_0 = 0$. These planes intersect with the sphere into circles defining the rotational links. In this case, the additional constraint is used to constrain the moving joints x_i, so that they are on a surface of a sphere. We apply 7.29 to 7.34 and use our method to solve the intermediate parameters p_i by Gaussian Elimination and the bilinear relationship. One can construct a similar matrix equation as in 7.13 while $[B_1]$ is a 9×8 matrix. We can find two 8×8 submatrix and calculate the determinant polynomials.

One can then combine the constraint equation of 7.37 with the two determinants. Similarly, the non-trivial solution of $\mathbf{F}'$ can be found using null space analysis method.

7.5 Numerical Examples

In this section, we offer the examples to demonstrate the effectiveness of this bilinear equation synthesis algorithm for one general case and three special cases to prove the effectiveness of our algorithm.

7.5.1 Example 1: Six Position with One Additional Plane where Moving Joints are Located

Table 7.1 gives six arbitrary spatial task positions. We pick an additional plane in moving frame coordinate $0.4326x + 0.6587y - 0.5746z + 2.9487 = 0.0000$ so that all moving joints are located on this plane.

We obtain six solutions shown in Table 7.2. Solution 1-5 represent spatial leg types of spherical constraints while solution 6 represents a spatial leg type of planar constraint. Figure 7.1 illustrates one resulting 5-SS linkage and six input positions.

Table 7.1: Six Arbitrary Spatial Task Positions

	s_x s_y s_z θ	d_1 d_2 d_3
1	0.0000, 0.0000, 0.0000, 0.0000	0.0000, 0.0000, 0.0000
2	-0.0863, 0.0096, -0.9962, 1.7088	1.6535, -0.7423, -0.1337
3	-0.3656, -0.7019, 0.6197, 2.0276	0.3182, -0.5085, -0.7922
4	-0.3705, 0.8766, 0.3402, 2.9993	-0.1768, -1.8879, -1.8768
5	-0.5828, -0.3469, 0.7347, 3.1035	-1.2585, 0.8366, -1.4992
6	0.0075, -0.1487, -0.9888, 1.2080	-3.5963, 2.7283, -2.0334

Notes: The units of s_x, s_y, s_z, d_x, d_y, d_z are meters; the unit of θ is radian.

Table 7.2: Example 1: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000, 0.6564, -0.7823, -0.0740, 85.2721	-0.9032, 3.2331, 8.1578
2	1.0000, -0.6445, 1.8969, -0.3072, 20.8373	-0.5463, -0.6268, 4.0016
3	1.0000, 2.4787, 7.1532, -0.9771, 50.0279	1.4110, -1.9391, 3.9710
4	1.0000, -3.6114, -2.6387, -2.6076, 5.3054	0.0235, -2.9864, 1.7255
5	1.0000, 0.3800, 0.6772, -2.4766, 91.4318	5.7093, -5.9361, 2.6254
6	0.0000, 0.0344, 0.0160, -0.0394, 0.2628	-4.2652, -12.1239, -11.9787

7.5.2 Example 2: Six Position with One Additional Plane where Fixed Joints are Located

We use the same six arbitrary task positions as given in Table 7.1and add a plane $-0.6475x - 0.6491y + 0.3276z + 1.0504 = 0.0000$ in fixed frame coordinate where the fixed joints are located on. There are six solutions as shown in Table 7.3. All of them are spherical constraints. Figure **??** illustrates one of the resulting 5-SS linkage and six positions.

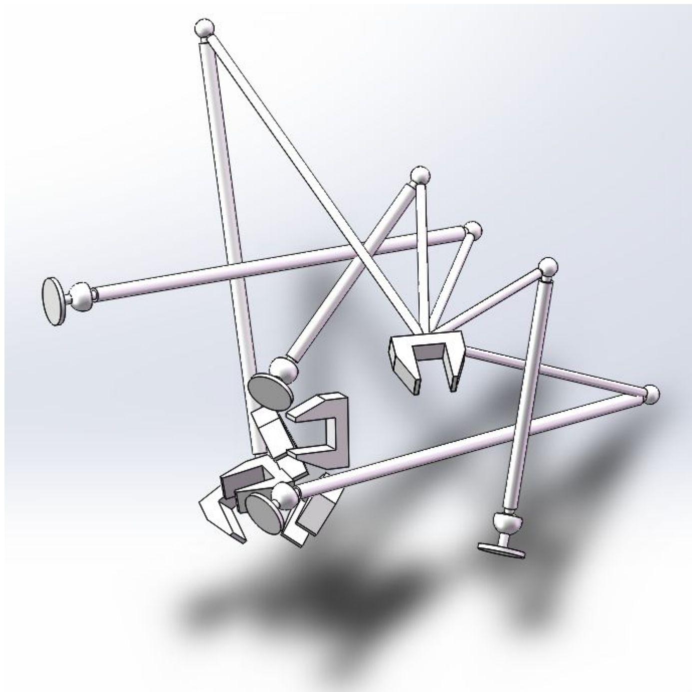

Figure 7.1: Example 1: 5-SS linkage and six positions.

7.5.3 Example 3: Five Position with Two Additional Planes where Moving Joints are Located

We use the first five positions of example 1 for this example and add two planes to constrain the moving coordinates $0.2809x - 0.8677y + 0.6489z - 2.6758 = 0.0000$ and $0.3509x - 0.1654y - 0.2584z + 1.0849 = 0.0000$. The intersection of these two planes defines a line where the moving joints are located.

There are four solutions as in Table7.4. All of them are spherical constraints.

Table 7.3: Example 2: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000,-0.9091,-1.9522,-8.8714,-12.2000	-0.4507,-0.2628,-0.6206
2	1.0000,1.2033,-1.0005,-2.8103,6.7844	2.4451,0.8821,0.6602
3	1.0000,-0.6135,2.1429,-0.1728,11.7035	-0.0634,-0.5663,2.8368
4	1.0000,1.0849,0.1172,-0.8296,15.8050	0.1255,3.7098,1.1342
5	1.0000,2.7137,-0.0968,1.9653,55.3359	2.0813,4.6345,-4.6517
6	1.0000,0.0163,0.4739,-2.2351,43.7082	4.1068,-4.2104,0.9668

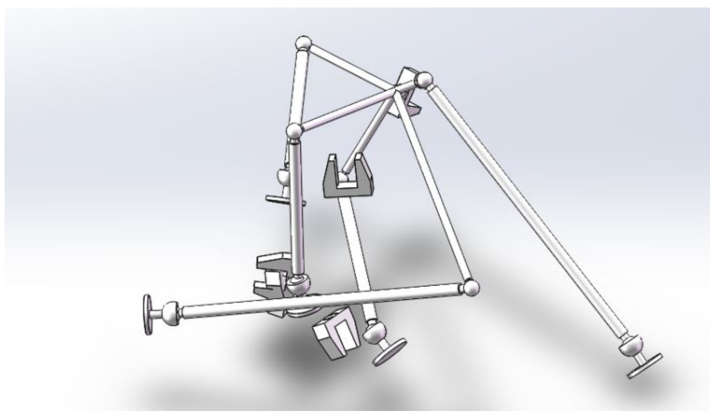

Figure 7.2: Example 2: 5-SS linkage and six positions.

7.5.4 Example 4: Five Position with Two Additional Planes where Fixed Joints are Located

We take the same five positions of example 3 adding two planes in fixed frame coordinate $0.0239x+0.8953y+0.1673z-3.7979 = 0.0000$ and $-0.3385x+0.6210y-0.0238z-3.2052 = 0.0000$ where the fixed joints are located. The intersection of these two planes defines a line where the moving joints are located.

There are two solutions shown in Table7.5. All of them are spherical constraints.

Table 7.4: Example 3: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000,1.5327,-0.9765,-1.7139,168.6620	6.0527,5.5186,8.8825
2	1.0000,-0.5064,1.4661,-0.3797,17.8212	-0.6946,-0.5918,3.63282
3	1.0000,-0.8105,-0.4466,-1.5723,17.4204	-2.7275,-2.43289,2.05111
4	1.0000,0.3147,-1.0656,-0.4104,236.1020	-11.3330,-10.2260,-4.6441

Table 7.5: Example 4: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000,-1.7402,4.2255,0.3352,39.6111	-0.1987,-0.7651,6.1079
2	1.0000,0.8717,5.3952,-6.2968,-4.4397	-1.2593,0.9287,0.0740

7.5.5 Example 5: Five Position with One Plane and One Sphere Constraining Fixed Joints

In this example, we use the same five positions and add a plane and a sphere equation to show the flexibility of our algorithm. Thus the fixed joints are located on the intersecting circle of two surfaces. The equations of the plane and the sphere are,

$$a1^2 + a2^2 + a3^2 - 8.2446a1 - 7.9071a2 - 1.0815a3 - 15.1953 = 0,$$

$$0.5516a1 + 0.6781a2 + 0.4378a3 - 4.8977 = 0.0000$$

There are six solutions shown in Table 7.6. Figure 7.3 shows the 5-SS linkage composed of the first five solutions. The fixed joints are located on a circle as expected.

Table 7.6: Example 5: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000,7.1434,5.0327,-5.6090,6.6718	0.5350,-0.8933,0.3660
2	1.0000,-1.4949,8.0213,0.6449,42.8956	0.6251,-0.9559,5.6250
3	1.0000,6.0007,6.0785,-5.7891,5.4698	0.3944,0.0255,0.8345
4	1.0000,4.4929,7.1764,-5.5900,2.6903	-0.0466,0.5445,0.8159
5	1.0000,1.9297,1.3896,6.6014,18.9326	-0.4390,0.3989,-1.2454
6	1.0000,6.3240,-1.0376,4.8247,898.9550	13.9298,16.9887,-19.2690

7.5.6 Example 6: Five Planar Position Synthesis

Figure 7.7 shows five planar displacements. After Gaussian elimination, we can obtain the directional vector $(0.6633, -0.0871, 1.0000)$. Therefore we give the following two additional constraints,

$$0.6633X_1 - 0.0871X_2 + X_3 = 0.0000,$$

$$0.6633a_1 - 0.0871a_2 + a_3 = 0.0000$$

We can obtain two solutions as shown in 7.8. The revolute axes of equivalent 4R linkage are shown in 7.9. Figure 7.4 shows the resulting equivalent four bar linkage and five spatial planar positions in this example.

7.5.7 Example 7: Five Spherical Position Synthesis

Figure 7.10 shows five spherical displacements. After Gaussian elimination, we can obtain the rotaional center of the five displacements $(0.8402, 0.4147, 1.8875)$.

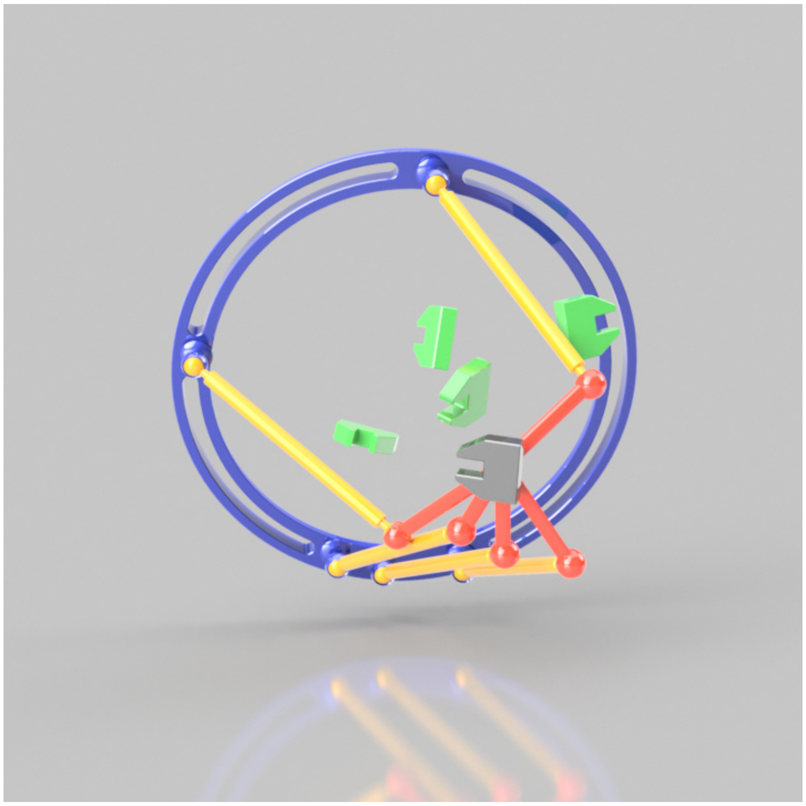

Figure 7.3: Example 5: 5-SS linkage composed of first five solutions.

Therefore we give the following additional constraint,

$$(X_1 - 0.8402)^2 + (X_2 - 0.4147)^2 + (X_3 - 1.8875)^2 - 4 = 0.0000 \tag{7.38}$$

We can obtain four solutions as shown in 7.11 while they define two spherical dyads as shown in 7.12. Figure 7.5 illustrates the resulting equivalent spherical 4R linkage and five spherical positions.

Table 7.7: Five Planar Task Positions

	s_x s_y s_z θ	d_1 d_2 d_3
1	-0.4979,-0.5562,-0.6653,4.0458	2.3145,0.2803,2.0023
2	-0.5393,-0.5746,-0.6154,4.1930	3.4349,-0.2631,1.2117
3	-0.6155,-0.6039,-0.5062,4.4450	3.3433,-0.4117,1.2595
4	-0.6496,-0.6145,-0.4475,4.5503	4.2406,0.4564,0.7399
5	-0.7096,-0.6269,-0.3214,4.7266	4.8847,1.2343,0.3805

Notes: The units of s_x, s_y, s_z, d_x, d_y, d_z are meters; the unit of θ is radian.

Table 7.8: Five Planar Displacement Example: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000,2.9664,-1.4592,-2.0948,-14.3887	-1.2800,-3.8532,-0.0204
2	1.0000,5.5082,-3.1621,-3.9292,-49.4149	-3.1281,-6.8070,3.5207

7.6 Summary

In this chapter, we started from the synthesis algorithm of 5-SS linkage and developed an unified method for solving five planar, spherical or spatial position synthesis problems. The reduced row echlon form after Gaussian elimination was able to identify the sypes of input data. This indication would decide how to choose the additional constraints in order to obtain the corresponding planar, spherial or spatial linkages.

Table 7.9: The dimensions of the revolute axes of planar 4R linkage

a	**x**
$(2.9664, -1.4592, -2.0948)^T + \mu(0.6633, -0.0871, 1.0000)$	$(2.8602, -0.5136, -1.9420)^T + \lambda(0.6633, -0.0871, 1.0000$
$(5.5082, -3.1621, -3.9292)^T + \mu(0.6633, -0.0871, 1.0000)$	$(6.9286, -1.2260, -4.7027)^T + \lambda(0.6633, -0.0871, 1.0000$

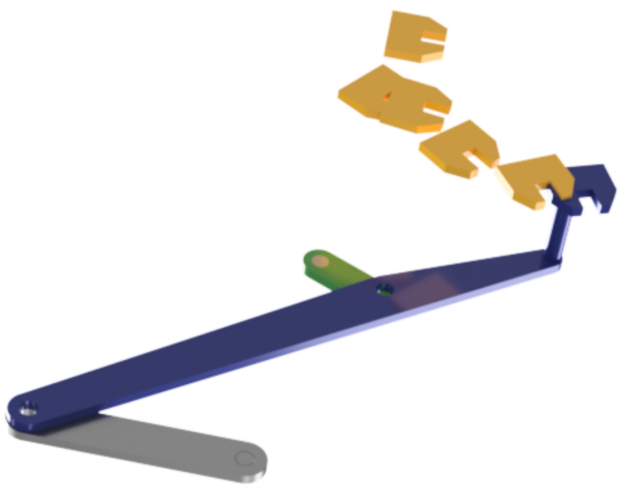

Figure 7.4: 5-SS linkage and five spatial planar positions.

Table 7.10: Five Spherical Task Positions

	s_x	s_y	s_z	θ	d_1	d_2	d_3
1	0.5260	-0.8401	-0.1320	1.5788	-0.2603	0.0296	2.3303
2	0.7861	-0.2435	-0.5679	0.3275	1.0426	0.4159	3.1183
3	-0.6938	0.6956	0.1861	2.1183	1.0414	1.2588	0.9916
4	-0.1678	0.9704	0.1730	4.4286	-0.3686	0.7080	1.9773
5	-0.5734	0.1397	0.8072	4.7938	0.5445	-0.3281	2.8448

Notes: The units of s_x, s_y, s_z, d_x, d_y, d_z are meters; the unit of θ is radian.

Table 7.11: Five Spherical Displacement Example: Design Parameters a_i and x_i

	a_0 a_1 a_2 a_3 a_4	$\frac{x_1}{x_4}$ $\frac{x_2}{x_4}$ $\frac{x_3}{x_4}$
1	1.0000,-33.7529,-6.8529,53.8351	1.2275,-1.6328,-1.5425
2	1.0000,-1.0498,-0.4042,2.9065	0.6704,-0.7008,-2.8991
3	1.0000,-33.7529,-6.8529,-158.2160	-1.7580,0.9199,-0.7882
4	1.0000,-1.0498,-0.4042,-4.3395	-1.2009,-0.0120,0.5682

Table 7.12: The dimensions of the revolute axes of spherical 4R linkage

a	**x**
$(0.8402, 0.4147, 1.8875)^T + \mu(1.0000, -33.7529, -6.8529)$	$(0.8402, 0.4147, 1.8875)^T + \lambda(1.0000, -1.3974, 0.2891)$
$(0.8402, 0.4147, 1.8875)^T + \mu(1.0000, -1.0498, -0.4042)$	$(0.8402, 0.4147, 1.8875)^T + \lambda(1.0000, -0.0791, -1.6528)$

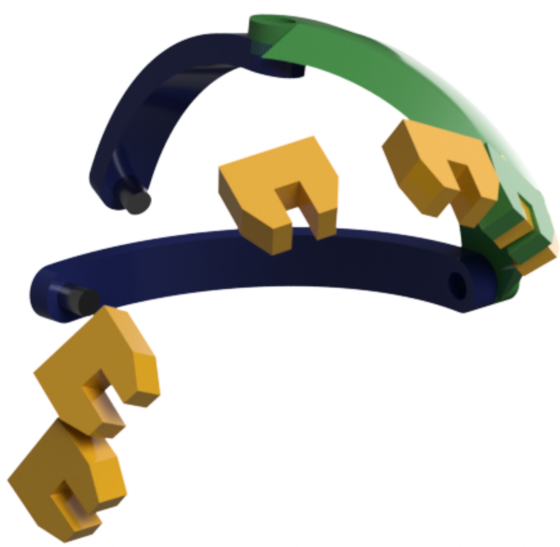

Figure 7.5: Equivalent spherical 4R linkage and five spherical positions.

Chapter 8

Conclusion and Future Work

The purpose of this dissertation is to unify the dimensional and type synthesis in planar, spherical and spatial linkages of which the design parameters of constraint equations have bilinear relationship and thus to provide a simple and efficient method to solve the problem. Such bilinear relationship proves the possibility of our task driven design methodology to be extended from planar to spatial linkages. By developing a constraint based kinematic geometry that leads to our task driven design of linkages and robotic systems, we are able to simultaneously solve the dimensional and type synthesis for a set or finite numbers of given poses in space.

We first presented a unified synthesis method of both planar four bar and spherical 4R linkages for five given task positions. It built on the previous works of Ge and Ping ([31], [64], [29]) and modified the design equation formulation into biliner form so as to prepare the extension to spatial linkages. We obtained the same results in five arbitrary positions for exact synthesis which proves the reliability and effectiveness of our new approach. This method can also be applied to finite positions, e.g. four or three, with additional constraints.

Based on the new approach, we were able to extend it into spatial linkages.

We took spatial 5-SS platform linkage as our start. It met the requirement of our task driven design method with seven arbitrary poses and proved its correctness and efficiency in the numerical example. One of the benefit of this new approach is that the bilinear formulation allows us to find the connection and unify planar, spherical and spatial linkages.

Therefore, in the third part of this dissertation, we looked for a synthesis method of spatial 5-SS linkage to obtain the same results of planar and spherical linkages and that of Bennett's linkage. We sought for the direct relationship from spatial to other linkages, thus we started by seven position synthesis.

Finally we developed the finite position synthesis of 5-SS linkage, e.g. six or five positions, with one or two additional constraints. This shows the full extension of our approach from planar and spherical linkages to spatial linkages. Then a unified methodology was proposed for five planar, spherical and spatial position synthesis. Our data driven method is able to tell the types of input positions and apply the algorithm based on different cases to obtain corresponding linkages without any preprocessing steps.

This methodology proves by itself the effectiveness and efficiency in synthesizing planar four bar, spherical four bar and spatial 5-SS linkages. It is meaningful to extend this method to other linkages in the future. We have developed circular, linear, spherical and planar constraints in this dissertation, but there are more constraints that can be formulated. Therefore, developing a unified algorithm for a larger category of linkages is possible.

Another work is to research on the robustness of this algorithm. It is valuable to find the relationship of input data to the linkage dimensions. One example is to find out how a small change of input data will interfere the solution. Another valuable research can be finding if there are any patterns in changing the input data that can cause less changes to the solutions.

It is hoped that the results of these research will lead to more possibilities in the linkage design as well as other fields connected to kinematics.

Appendix A

Proof for Directional Vector and Rotational Center in Reduced Row Echelon Form

Figure A.1 illustrates one planar displacement M and fixed frame F. $d = (d_{i,1}, d_{i,2}, d_{i,3})(i = 1, 2, 3, \cdots, N)$ is the distance vector from fixed frame origin to moving frame origin. $v = (v_1, v_2, v_3)$ is the directional vector of plane of planar displacement. For five position synthesis, the relationship of each d and v are,

$$\begin{aligned}
v_1 d_{1,1} + v_2 d_{1,2} + v_3 d_{1,3} + v_4 = 0 \\
v_1 d_{2,1} + v_2 d_{2,2} + v_3 d_{2,3} + v_4 = 0 \\
v_1 d_{3,1} + v_2 d_{3,2} + v_3 d_{3,3} + v_4 = 0 \\
v_1 d_{4,1} + v_2 d_{4,2} + v_3 d_{4,3} + v_4 = 0 \\
v_1 d_{5,1} + v_2 d_{5,2} + v_3 d_{5,3} + v_4 = 0
\end{aligned} \tag{A.1}$$

where v_4 relates to the translation of the plane.

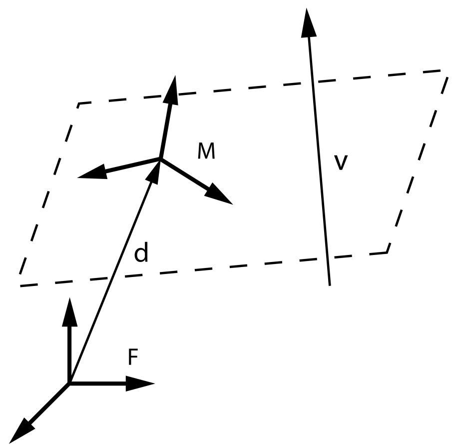

Figure A.1: Planar Displacement: fixed frame F; moving frame M; distance vector d from fixed frame origin to moving frame origin; directional vector of plane v.

The first four coefficients in 7.2 by Euler angles and distances are,

$$A_{i,1} = -2d_{i,1}, \quad A_{i,2} = -2d_{i,2}, A_{i,3} = -2d_{i,3},$$
$$A_{i,4} = d_{i,1}^2 + d_{i,2}^2 + d_{i,3}^2 \qquad (i = 1, 2, 3, 4, 5) \tag{A.2}$$

Subtracting A.1 and A.2 into 7.4 to eliminate $d_{i,3}$ by assuming $v_3 \neq 0$ (v_1 or v_2 can also be assumed to be zero),

$$A_{i,1} - A_{1,1} = -2(d_{i,1} - d_{1,1}),$$
$$A_{i,2} - A_{1,2} = -2(d_{i,2} - d_{1,2}),$$
$$A_{i,3} - A_{1,3} = 2(\frac{v_1}{v_2}(d_{2,1} - d_{1,1}) + \frac{v_2}{v_3}(d_{2,2} - d_{1,2})), \tag{A.3}$$
$$A_{i,4} - A_{1,4} = (d_{i,1}^2 + d_{i,2}^2 + (-\frac{v_4}{v_3} - \frac{v_1}{v_3}d_{i,1} - \frac{v_2}{v_3}d_{i,2})^2)$$

$$-(d_{1,1}^2 + d_{1,2}^2 + (-\frac{v_4}{v_3} - \frac{v_1}{v_3}d_{1,1} - \frac{v_2}{v_3}d_{1,2})^2)$$
$$(i = 2, 3, 4, 5)$$

Finally we use 7.8 to formulate new $[H]$ and do the Gaussian elimination to obtain the reduced row echelon form,

$$RREF([H]) = \begin{bmatrix} 1 & 0 & -\frac{v_1}{v_3} & 0 & \cdots \\ 0 & 1 & -\frac{v_2}{v_3} & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{A.4}$$

It proves the first two row elements in the third column is the directional vector of the plane defining the translation of given planar displacements.

Another case is that the inputs are spherical displacements as in figure A.2. F and M represent fixed frame and moving frame. d is the distance vector from fixed frame origin pointing to moving frame origin. O is the rotational center of spherical displacement. s is the distance vector from origin of fixed frame to rotational center. Vector r is pointing from rotational center to origin of moving frame and its length is the radius of the sphere where displacement is moving on.

For five displacements, the relations of three vectors are,

$$d = s + r \quad (i = 1, 2, 3, 4, 5) \tag{A.5}$$

Subtracting A.5 and A.2 into 7.4 to eliminate d and simplify,

$$A_{i,1} - A_{1,1} = -2(r_{i,1} - r_{1,1}),$$

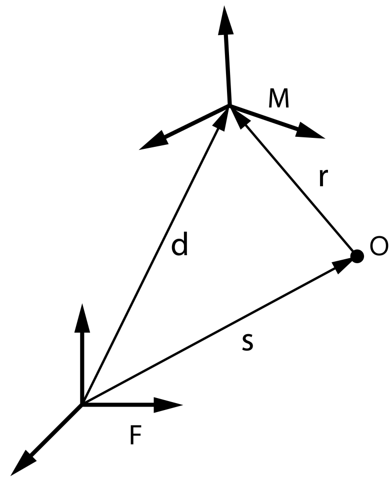

Figure A.2: Spherical Displacement: fixed frame F; moving frame M; distance vector $d = (d_{i,1}, d_{i,2}, d_{i,3})$ from fixed frame origin to moving frame origin; distance vector $s = (s_1, s_2, s_3)$ from origin of fixed frame to rotational center O of spherical displacement; vector $r = (r_{i,1}, r_{i,2}, r_{i,3})$ from rotational center to origin of moving frame.

$$A_{i,2} - A_{1,2} = -2(r_{i,2} - r_{1,2}),$$

$$A_{i,3} - A_{1,3} = -2(r_{i,3} - r_{1,3}), \tag{A.6}$$

$$\begin{aligned} A_{i,4} - A_{1,4} = & -2(r_{i,1} - r_{1,1})s_1 - 2(r_{i,2} - r_{1,2})s_2 \\ & -2(r_{i,3} - r_{1,3})s_3 \qquad (i = 2, 3, 4, 5) \end{aligned} \tag{A.7}$$

Finally we use 7.8 to formulate new $[H]$ and do the Gaussian elimination to obtain the reduced row echelon form,

$$RREF([H]) = \begin{bmatrix} 1 & 0 & 0 & -s_1 & \cdots \\ 0 & 1 & 0 & -s_2 & \cdots \\ 0 & 0 & 1 & -s_3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{A.8}$$

It proves that the first three row elements in the forth column can represent the rotational center of given spherical displacements.

Appendix B

Proof for Uniqueness of Directional Vector and Rotational Center

Figure B.1 illustrates five planar displacements $M_i(i = 1, 2, 3, 4, 5)$ and points $A_i(i = 1, 2, 3, 4, 5)$ attached to each of the moving frame. The coordinate of each point relative to attached moving frame is the same.

It is known that three spatial points can define a plane. If there is a plane constraint, all five points should be on that plane at the same time. Let's assume A_1, A_2, A_3 are on a same plane and A_3, A_4, A_5 are on another plane. If the direction of two planes are not the same, it means that five displacements can move on different directions of planes. It conflicts with the fact that all five displacements are planar displacements which means they translate on a single plane. Thus there should be only one direction of plane in five planar displacements synthesis problem. That's the reason, in our algorithm, we can find infinite parallel plan constraints.

Furthermore, as long as there are three or more planar displacements, the direction of plane constraints to be found is always unique.

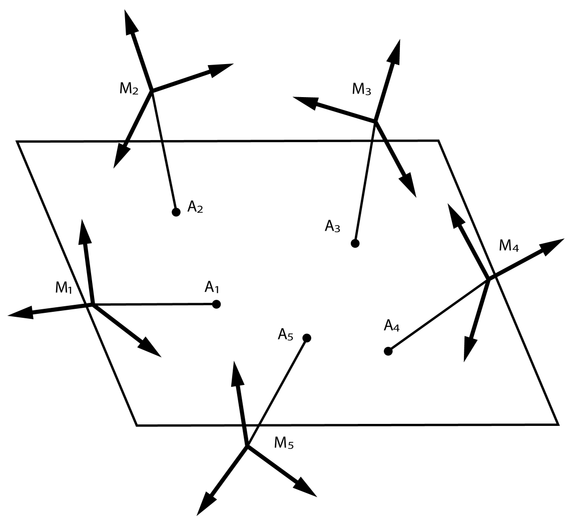

Figure B.1: Five Planar Displacements: moving frame $M_i(i = 1, 2, 3, 4, 5)$; point $A_i(i = 1, 2, 3, 4, 5)$ attached to the moving frame.

There are five spherical displacements $M_i(i = 1, 2, 3, 4, 5)$ in Figure B.2. $A_i(i = 1, 2, 3, 4, 5)$ are the points attached to each of the moving frame. The coordinate of each point relative to attached moving frame is the same.

The proof of this problem is similar to the planar displacements. It is known that four spatial points can define a sphere. If there is a sphere constraint to be found, all five points should be on that sphere at the same time. Let's assume A_1, A_2, A_3, A_4 are on a sphere of one center and A_2, A_3, A_4, A_5 are on another sphere of another center. If the centers of two spheres are not the same, it means that five displacements can rotate around different rotational centers. It conflicts with the fact that all five displacements are spherical displacements which means they rotate around the same rotational center. Thus there should be only one rotational center in five spherical displacements synthesis problem. This explains the reason that one can choose any radius of the sphere constraint of the same center.

Furthermore, as long as there are four or more planar displacements, the rotational center to be found is always unique.

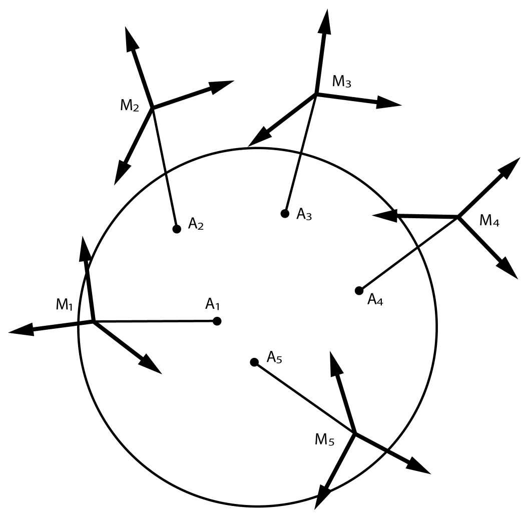

Figure B.2: Five Spherical Displacements: moving frame $M_i(i = 1, 2, 3, 4, 5)$; point $A_i(i = 1, 2, 3, 4, 5)$ attached to the moving frame.

Bibliography

[1] Bottema, O., and Roth, B., 1979. *Theoretical Kinematics*. North-Holland Publishing.

[2] Chen, P., and Roth, B., 1969. "Design equations for the finitely and infinitesimally separated position synthesis of binary links and combined link chains". *ASME. Journal of Engineering for Industry,* **91**(1), pp. 209–219.

[3] Suh, C. H., and Radcliffe, C. W., 1978. *Kinematics and Mechanisms Design*. John Wiley Sons Inc.

[4] Tsai, L.-W., and Morgan, A. P., 1985. "Solving the kinematics of the most general six- and five-degree of freedom manipulators by continuation methods". *ASME. Journal of Mechanisms Transmissions and Automation in Design,* **107**(2), pp. 189–200.

[5] Innocenti, C., 1995. "Polynomial solution of the spatial burmester problem". *Journal of Mechanical Design,* **117**(1), pp. 64–68.

[6] Liao, Q., and McCarthy, J. M., 1997. "On the seven position synthesis of a 5-ss platform linkage". *Journal of Mechanical Design,* **123**(1), pp. 74–79.

[7] Kong, X., and Gosselin, C. M., 2004. "Type synthesis of 3-dof translational parallel manipulators based on screw theory". *Journal of Mechanical Design,* **126**(1), pp. 83–92.

[8] Kong, X., and Gosselin, C. M., 2006. "Type synthesis of 4-dof sp-equivalent parallel manipulators: A virtual chain approach". *Mechanisms and Machine Theory,* **41**(11), pp. 1306–1319.

[9] Lee, E., and Mavroidis, C., 2004. "Geometric design of spatial prr manipulators". *Mechanisms and Machine Theory,* **39**(4), pp. 395–408.

[10] Li, Q., Huang, Z., and Herve, J. M., 2004. "Type synthesis of 3r2t 5-dof parallel mechanisms using the lie group of displacements". *IEEE Robotics and Automation,* **20**(2), pp. 173–180.

[11] McCarthy, J. M., and Soh, G. S., 2012. *Geometric Design of Linkages*. Springer.

[12] Burmester, L., 1886. *Lehrbuch der Kinematik*. Verlag Von Arthur Felix.

[13] Huang, C., and Chang, Y.-J., 2000. "Polynomial solution to the five-position synthesis of spatial cc dyads via dialytic elimination". In Proceedings of the ASME Design Technical Conferences, pp. 10–13.

[14] Buchberger, B., and Winkler, F., 1998. *Gröbner bases and applications*, Vol. 251. Cambridge University Press.

[15] Liao, Q., and McCarthy, J., 1995. "Solving polynomial systems for the kinematic analysis and synthesis of mechanisms and robot manipulators". *ASME Journal of Mechanical Design,* **117.B**, pp. 71–79.

[16] Wilkinson, J., 1959. "The evaluation of the zeros of ill-conditioned polynomials. part i". *Journal Numerische Mathematik*, pp. 150–166.

[17] Wilkinson, J., 1959. "The evaluation of the zeros of ill-conditioned polynomials. part ii". *Journal Numerische Mathematik*, pp. 167–180.

[18] Brunnthaler, K., Schrocker, H.-P., and Husty, M., 2006. "Synthesis of spherical four-bar mechanisms using spherical kinematic mapping". *Advances in Robot Kinematics*, pp. 377–385.

[19] Sommese, A. J., and II, C. W. W., 2005. *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*. World Scientific.

[20] Bates, D., Hauenstein, J., Sommese, A., and Wampler, C., 2013. *Numerically solving polynomial systems with Bertini*. SIAM.

[21] Sandor, and Erdman, A. G., 1984. *Advanced Mechanism Design: Analysis and Synthesis Vol. 2*. Pearson.

[22] Mccarthy, J. M., 1990. *Introduction to Theoretical Kinematics*. The MIT Press.

[23] Bai, S., and Angeles, J., 2012. "A robust solution of the spatial burmester problem". *ASME Journal of Mechanisms and Robotics,* **4**(3).

[24] Plecnik, M. M., and Mccarthy, J. M., 2014. "A numerical synthesis of six-bar linkages for mechanical computation". *ASME Journal of Mechanisms and Robotics,* **6**(3).

[25] Perez, A., and Mccarthy, J. M., 2003. "Dimensional synthesis of bennett linkages". *Journal of Mechanical Design,* **125**(1), pp. 98–104.

[26] Huang, C., 1997. "The cylindroid associated with finite motions of the bennett mechanism". *Journal of Mechanical Design,* **119**(4), pp. 521–524.

[27] McCrea, W. H., 2006. *Analytical geometry of three dimensions.* Dover Publications.

[28] Ge, X., Purwar, A., and Ge, Q. J., 2016. "From 5-ss platform linkage to four-revolute jointed planar, spherical and bennett's mechanisms". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, Paper No. DETC2016-60574.

[29] Ge, Q. J., Zhao, P., and Purwar, A., 2013. "A task driven approach to unified synthesis of planar four-bar linkages using algebraic fitting of a pencil of g-manifolds". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2013-12977.

[30] Ge, Q. J., Zhao, P., and Purwar, A., 2013. "Decomposition of planar burmester problems using kinematic mapping". In *Advances in Mechanisms, Robotics and Design Education and Research.* Springer, pp. 145–157.

[31] Ge, Q. J., Zhao, P., Purwar, A., and Li, X., 2012. "A novel approach to algebraic fitting of a pencil of quadrics for planar 4r motion synthesis". *Journal of Computing and Information Science in Engineering,* **12**(4).

[32] Purwar, A., Anantwar, S., and Zhao, P., 2012. "An interactive approach to designing planar parallel manipulators using image space representation". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2012-70880, American Society of Mechanical Engineers, pp. 589–604.

[33] Zhao, P., Li, X., Purwar, A., Thakkar, K., and Ge, Q. J., 2013. "A task driven approach to the synthesis of spherical 4r linkages using algebraic fitting method". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2013-13179, American Society of Mechanical Engineers.

[34] Blaschke, W., 1960. *Kinematics and Quaternions*, Vol. 4. VEB Deutscher Verlag der Wissenschaften.

[35] Collins, C., and McCarthy, J., 1998. "The quartic singularity surfaces of planar platforms in the clifford algebra of the projective plane". *Mechanism and Machine Theory,* **33**(7), pp. 931–944.

[36] Ravani, B., and Roth, B., 1983. "Motion synthesis using kinematic mappings". *Journal of mechanisms, Transmissions, and Automation in Design,* **105**(3), pp. 460–467.

[37] Ravani, B., and Roth, B., 1984. "Mappings of spatial kinematics". *Journal of Mechanisms, Transmissions, and Automation in Design,* **106**(3), pp. 341–347.

[38] Li, X., Zhao, P., Ge, Q. J., and Purwar, A., 2013. "A task driven approach to simultaneous type synthesis and dimensional optimization of planar parallel manipulator using algebraic fitting of a family of quadrics". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2013-13197, American Society of Mechanical Engineers.

[39] Larochelle, P., 1996. "Synthesis of planar rr dyads by constraint manifold projection". In Proceedings of the ASME Design Engineering Technical Conferences, DOI: 96-DETC/MECH-1187.

[40] Larochelle, P., 2000. "Approximate motion synthesis via parametric constraint manifold fitting". In *Advances in robot kinematics.* Springer, pp. 103–110.

[41] Larochelle, P. M., 2003. "Approximate motion synthesis of open and closed chains via parametric constraint manifold fitting: Preliminary results". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI: 10.1115/DETC2003/DAC-48814, pp. 1049–1057.

[42] Hayes, M. J. D., Luu, T., and Chang, X., 2004. "Kinematic mapping application to approximate type and dimension synthesis of planar mechanisms". In *On Advances in Robot Kinematics.* Springer, pp. 41–48.

[43] Hayes, M. J. D., and Rusu, S. R., 2011. "Quadric surface fitting applications to approximate dimensional synthesis". In 13th World Congress in Mechanism and Machine Science, pp. 19–23.

[44] Ge, Q. J., Ge, X., Purwar, A., and Li, X., 2015. "A null-space analysis method for solving bilinear equations in kinematic synthesis of planar and

spherical dyads". In Proceedings of the 14th IFToMM World Congress,DOI: 10.6567/IFToMM.14TH.WC.OS2.026.

[45] Lazard, D., 1983. "Gröbner bases, gaussian elimination and resolution of systems of algebraic equations". In European Conference on Computer Algebra, Springer, pp. 146–156.

[46] Noble, B., and Daniel, J. W., 1988. *Applied linear algebra*, Vol. 3. Prentice-Hall New Jersey.

[47] Strang, G., 2011. "Introduction to linear algebra".

[48] Brannigan, M., 1982. "The strict chebyshev solution of overdetermined systems of linear equations with rank deficient matrix". *Numerische Mathematik,* **40**(3), pp. 307–318.

[49] Lawson, C. L., and Hanson, R. J., 1995. *Solving least squares problems*, Vol. 15. SIAM.

[50] Das, S., and Neumaier, A., 2013. "Solving overdetermined eigenvalue problems". *SIAM Journal on Scientific Computing,* **35**(2), pp. A541–A560.

[51] Wilkinson, J. H., 1965. *The algebraic eigenvalue problem*, Vol. 87. Clarendon Press Oxford.

[52] Parlett, B. N., 1980. *The symmetric eigenvalue problem*, Vol. 7. SIAM.

[53] Lathauwer, L. D., Moor, B. D., and Vandewalle, J., 2000. "A multilinear singular value decomposition". *SIAM journal on Matrix Analysis and Applications,* **21**(4), pp. 1253–1278.

[54] Golub, G. H., and Reinsch, C., 1970. "Singular value decomposition and least squares solutions". *Numerische mathematik,* **14**(5), pp. 403–420.

[55] Perez, A., and McCarthy, J. M., 2004. "Dual quaternion synthesis of constrained robotic systems". *Journal of Mechanical Design,* **126**(3), pp. 425–435.

[56] Kenwright, B., 2012. "A beginners guide to dual-quaternions". *WSCG.*

[57] Li, X., Ge, Q. J., and Gao, F., 2014. "A unified algorithm for geometric design of platform linkages with spherical and plane constraints". In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2014-35218.

[58] Innocenti, C., 1995. “Polynomial solution of the spatial burmester problem”. *Journal of Mechanical Design,* **117**(1), pp. 64–68.

[59] Grubler, M., 1883. “Allgemeine eigenschaften der zwanglaufigen ebenen kinematische kette: I”. *Civilingenieur,* **29**(1), pp. 167–200.

[60] Huang, Z., Kong, L. F., and Fang, Y. F., 1997. Mechanism theory and control of parallel manipulators.

[61] Brunnthaler, K., Schröcker, H.-P., and Husty, M., 2005. “A new method for the synthesis of bennett mechanisms”. In International Workshop on Computational Kinematics, pp. 4–6.

[62] Murray, G., 2013. “Rotation about an arbitrary axis in 3 dimensions”. *Online] http://inside. mines. edu*.

[63] Noble, B., and Daniel, J. W., 1988. *Applied Linear Algebra*. Pearson.

[64] Zhao, P., Ge, Q. J., and Purwar, A., 2014. “On the complete synthesis of finite positions with constraint decomposition via kinematic mapping”. In ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DOI:10.1115/DETC2014-34216.