Analysis and Design of Configuration Memory Linkage Systems Using Homotopy Continuation Methods

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Abstract of the Thesis

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Shape memory materials have different shapes, volume, stiffness, and electrical resistances in response to the change of temperature or electromagnetic fields. In this research, we developed methods for designing planar linkages that can be actuated with springs made from shape memory materials.The resulting linkages are called the configuration memory linkage systems, which can memorize linkage configurations at different temperatures. The change in ambient temperatures would effect the change in the stiffness of the springs and thereby moves the linkage from one configuration to another. Homotopy continuation methods are used to solve the system of polynomial equations for linkage design.

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Chapter 1 Introduction

Homotopy continuation methods have been proven very reliable and efficient to approximate all isolated solutions of polynomial systems in engineering field during the past decades(see[2]). In general, small changes in the coefficients of a polynomial system will result in small changes to the roots of that system. That is the basic theory of homotopy continuation methods.

In chapter 2, we will talk about the theory of homotopy continuation methods. They can trace the solutions from one polynomial system to anther polynomial system. With the new package Bertini, actually the system does not have to be polynomial. It can include since and cosine functions as long as we can find the start solutions to the start system. In addition, general linear product decompositions are used when we use the total degree homotopy and the total degree is too large.

In chapter 3, we will talk about the applications of multistable mechanisms and the characteristics of shape memory alloys in the engineering field.

In chapter 4, four bar linkages with torsional and linear shape memory alloy springs are discussed. By using homotopy continuation method, we can find all the isolated solutions, which are also the equilibrium positions. In addition, for given several equilibrium or stable positions, synthesis problems are formulated by using homotopy continuation methods. Several examples of analysis and synthesis are given.

In chapter 5, we will do analysis of multi-linkage system. First, a four by four linkage without shape memory alloy springs and with shape memory alloy springs are studied and be compared. Second, the formulations of a general case of multi-linkage are discussed in the last part.

In chapter 6, future work about the applications of shape memory alloy materials are presented.

Chapter 2 Basic Theory of Homotopy Continuation Methods

2.1 Polynomials systems

Systems of polynomial equations often arise in science and engineering. There are three types of polynomials systems: polynomials in one variable, multivariate polynomial systems and trigonometric equations[1]. In general, polynomials in one variable have the form

$$f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d, \tag{1}$$

where $a_0, \cdots, a_d$ are the coefficients and the integer powers of x, namely 1, x, x^2, $\cdots$, x^d, are monomials. Usually in science and engineering the coefficients are real numbers although sometimes they may be complex. If we consider $f(x)$ as a function that maps complex numbers to complex numbers, the number of solutions will equal to exactly d points, counting multiplicities.

Multivariate polynomial systems are more popular in science and engineering compared to polynomials in one variable. There are several ways to define this type of polynomial systems[2]. One is the standard list of polynomials, given in any kinds of formats, such as standard form, various factored forms. Another one is to consider a polynomial systems as a particular basis for an ideal in a polynomial ring, $k[x_x, \cdots, x_n]$. Given a basis, the ideal generated from the basis can be defined to be the basis itself along with all sums of elements in the ideal and products of ideal elements with ring elements[2]. More details about the ideal in general and ideals in particular are discussed in [3].

Trigonometric functions are often used to formulate problems in geometry and kinematics. In most cases, one can converted them to polynomials using the polynomial relations, such as $\sin_\theta^2 + \cos_\theta^2 = 1$. However, not all trigonometric expressions can be converted to polynomials. One example is $x + \sin x$. That problem could be solved by using approximating technique in some softwares[4].

2.2 Basic theory of homotopy continuation methods

In 1996, A new area called Numerical Algebraic Geometry was first developed by Andrew Sommese and Charles Wampler[6]. The numerical path following techniques are very reliable to approximate all isolated solutions of polynomial systems. Also, polynomial homotopy methods are globally convergent methods for finding all the isolated solutions to polynomial equations[4]. In[6], compared to the isolated solutions, new techniques was developed to approximate all positive dimensional irreducible components of the solution set of a polynomial system,

As discussed in [6], there are two stages in homotopy continuation methods. Frist, we need one target system and one start system. The target system are the polynomial equations which we want to solve. The start system is constructed to help us to solve the target system during the homotopy path tracking process. The target system and start system should have exactly as many regular solutions as the root count, counting multiplicities. Then, this start system can be embedded in the following homotopy[6]

$$h(x,t) = \gamma(1-t)g(x) + tf(x) = 0, \quad t \in [0,1], \tag{2}$$

As t moves from 0 to 1, we can see that $h(x)$ will move from the start system to our target system. As a result, the second stage is that the homotopy continuation methods will trace the paths of the solutions of the start system towards the solutions of the target system. This method uses the fact that small changes in the coefficients of a polynomial systems can lead to small changes of the roots of systems[4]. The random complex number γ can overcome the weakness of the homotopy method.

After defining the homotopy, we need to solve it. For polynomials in one variable, we can simply solve them by finding the eigenvalues of the companion matrix. For multivariate polynomial systems, one can use predictor-corrector methods to trace the solution paths defined by the homotopy. More details about the Euler prediction and Newton correction are discussed in [1]. The adaptive step size control method can determines the step length to avoid path crossing in Newton's method while doing quadratic convergence[6]. In[2], three new numerical technique are discussed to apply homotopy continuation. The first one is to increase precision as needed in the path tracking process in order to decrease the computational time of using high precision. The second method is compute the scheme structure of an ideal supported at a single point. The last one is to approximate all solutions of a certain class of two-point boundary value problems from homotopy continuation.

Since the homotopy continuation methods are numerical, we need some softwares to do the algorithms of homotopy continuation. There are a number of packages discussed in [1]. Homlab can run in the Matlab and use mainly general linear product homotopy and parameter homotopy. Hompack, Polsys_plp are sequence of sophisticated continuation algorithms, written in Fortran. This code can only deal with isolated solutions of square systems. PHoM is a C++ code which use polyhedral homotopies. Again, this package can only find isolated solutions of square systems. PHCpack uses a variety of homotopies including all kinds of structures, except polynomial products. The algorithms of PHCpack can handle positive dimensional solutions in addition to isolated roots. Bertini is a C code and most recent package for computation in numerical algebraic geometry. Bertini can apply algorithms for computing and manipulating structures of an algebro-geometric nature. Bertini can be found on the official Bertini website. In this thesis, we will use Bertini as our tool to do homotopy continuation methods.

2.3 Polynomial structures

A very important part of homotopy continuation methods is to choose an appropriate start systems. In this paper, we will discuss only zero dimensional systems in which the number of unknowns equal to the number of equations. In general, there are two types of homotopies: homotopy without providing start systems and user defined homotopy. In the first case, Bertini will automatically create a start system, and solve the start system to find the start points at the beginning of each path[7]. This kind of homotopy is called total degree homotopy. The total degree homotopy uses the least information of the structure of the target systems[1]. Given a polynomial by the number of variables n and a list of degrees d_i, $i = 1, \cdots, n$, a simple system for total degree homotopy can be created as

$$g(z) = \begin{Bmatrix} z_1^{d_1} - 1 \\ z_2^{d_2} - 1 \\ \cdots \\ z_n^{d_n} - 1 \end{Bmatrix} = 0. \tag{3}$$

From the classical Bezout Theorem, we know that the number of finite, non singular solutions to a generic member of the total degree family is $N = d_1 \cdots d_n$ [1]. This start system can be solved easily by getting d_i roots for z_i, we can see that all of these roots are non singular. This start system will have the same number of solutions as the most general member of the total degree family. However, in some case the total degree N could be very large. Thus the computational time could be very large. We need to construct a start system from which fewer path will be traced.

In[1], several start systems are discussed from easier to specificity. Of course, the total degree start system is the easiest one. On the top of this one is multihomogeneous start system and linear products start system. The start system with the fewest paths is monomial products and coefficient parameter. Those are the most difficult ones to solve. In this paper, we will use only general linear products start system. For example, given the following target system

$$f(x_1, x_2) = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} x_1^2 + 4x_2 + 5 \\ x_1^2 + 2x_1x_2^2 + 8 \end{Bmatrix} = 0 \tag{4}$$

If we let $\langle x, y \rangle$ be the linear combination of monomials $ax + by + c$ for arbitrary complex constants[4],[5], after examining the structure of polynomials of the target systems, we can have

$$f_1 \in \langle x_1 \rangle \langle x_1, x_2 \rangle \tag{5}$$

$$f_2 \in \langle x_1 \rangle \langle x_2 \rangle \langle x_1, x_2 \rangle \quad . \tag{6}$$

Then, we will have the following start system

$$g(x_1, x_2) = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} = \begin{Bmatrix} (a_1 x_1 + a_2)(a_3 x_1 + a_4 x_2 + a_5) \\ (b_1 x_1 + b_2)(b_3 x_2 + b_4)(b_5 x_1 + b_6 x_2 + b_7) \end{Bmatrix} = 0 \tag{7}$$

Coefficient $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, b_6, b_7$ are random complex numbers. By solving all the combinations of the linear equations, we can learn that the number of solutions is 5, which is lower than the total degree $2 \times 3 = 6$. We can also express x_2 in term of x_1 from the first equation, and substitute it in the second equation. Again, we can get the number of solutions to be 5. However, In some large polynomial systems, the number of the solutions could be largely reduced by the general linear product method. Note that we should use complex numbers for all the coefficient in order to make the start system generic.

Chapter 3 Multistable Mechanisms and Shape Memory Materials

3.1 Application of multistable mechanisms

Multistable mechanisms are useful in many devices such as relays, threshold switches, valves and memory cells, etc. [8]. One advantage of such mechanisms is that they do not need extra force or energy to maintain their stabilities in every stable state. A Magnetic Shape Memory Alloys based actuator is proposed to keep a stable position when no current is applied to reduce the heat losses in the coils[9].

In [13], by capturing the essential parameters of bistability of our mechanisms, one can determine multistable behavior. A desired number of stable positions can be satisfied by using the above methodology. In [4], the stable and equilibrium positions of mechanisms are found by polynomial homotopy continuation method. In [5], the author formulate and solve the synthesis equations for a compliant four-bar linkage with certain specified equilibrium positions.

3.2 Shape memory materials

A shape memory alloy is an alloy that can remember its original shape. It is widely used in the area of aircraft, piping, automotive, robotics, medicine, optometry, orthopedic surgery, dentistry and engines. It is useful because its shape, stiffness, position, even the natural frequency will change in response to the temperature or the electromagnetic fields.

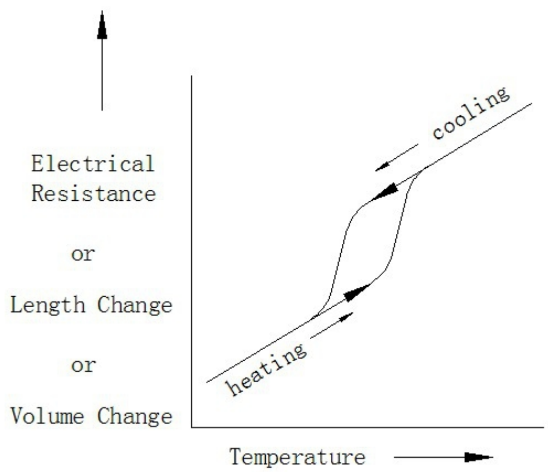

Fig 3.1 Martensite

Fig 3.1 is an example of martensite. As we can see, the changes of characteristics are different between heating and cooling process[23]. Nickel-titanium alloys have been found to be the best choice of all SMAs[14].

By using a self-tuning fuzzy PID controller a new progress of a SMA position control system is proposed in[16]. In[17], the applications of the SMA materials for passive, active and semi-active controls of civil structures is presented. A low-profile bidirectional folding actuator based on SMA sheets is presented for meso- and microscale systems[18]. In addition, the Shape Memory Alloy can be used as artificial muscles as actuators to design artificial limbs that are lightweight, compact and dexterous, which can mimic human body[19]. In [20], Shape Memory Alloy for automotive applications is discussed in details. In [21], the author present the design and dynamics of a new Shape Memory Alloy actuator which possesses impressive payload lifting capabilities.

Chapter 4 Configuration Memory Four Bar Linkage Systems

4.1 Background

4.1.1 Traditional methods for synthesis of four bar linkage systems

In [22], there are two methods for synthesis of four bar linkage in general. One is graphical synthesis, and another one is analytical synthesis technique.

For two prescribed positions and three prescribed positions, we can simply use graphical method by picking up the moving pivots and we will have infinite number of solutions. Another analytical method for two prescribed positions and three prescribed positions is to use dyad[22].

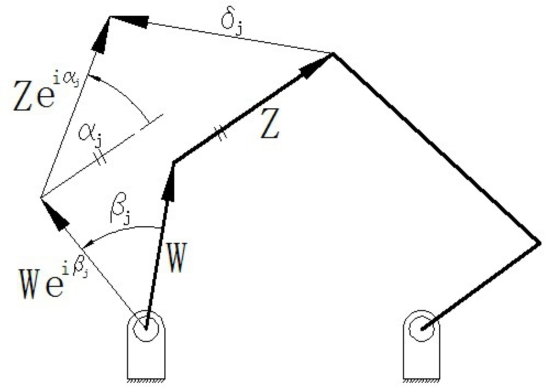

Fig 4.1 Synthesis using dyads

From Fig 4.1, we can formulate the synthesis equations by the closed loop:

$$W(e^{i\beta_j}-1)+Z(e^{i\alpha_j}-1)=\delta_j,\quad j=2,3,4,5 \tag{1}$$

For four and five prescribed positions, we need advanced analytical methods. In [22], By handling four vector equations of (1), synthesis for four prescribed positions can be solved. Based on the synthesis of four prescribed positions, synthesis of five prescribed positions is formulated by using circle and center point curves. More detailed are discussed in [22].

4.1.2 Seudo-rigid body made of compliant four bar linkage systems

Compliant mechanisms are linkage systems whose positions are determined by the elastic deformation of joint and link elements[10]. Pigoski and Duffy [11] studied the planar two-spring system consisting of point connect to ground by two compliant limbs. Their analysis concluded as many as six equilibrium positions for a given external load. Sun et al. [12] studied a planar three-spring system with a moving platform and found a maximum 54 equilibrium positions.

In [4] and [5], Su and McCarthy studied a four bar linkage system with only two torsional springs and formulated analysis and synthesis problems(see Fig 4.2).

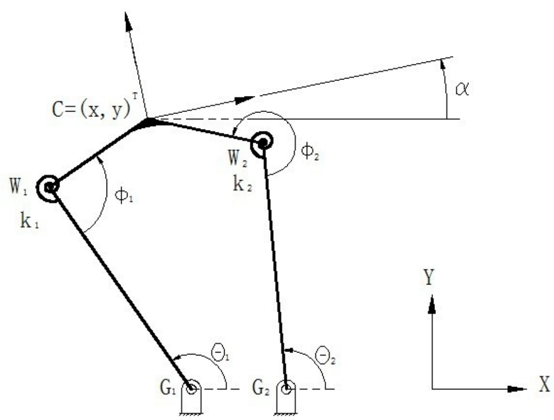

Fig 4.2 A compliant four-bar linkage

Next, we will make some declarations[4]. $[R(\alpha)]$ is the 2 by 2 rotation matrix of angle α and $e(\theta_i)=[\cos\theta_i,\sin\theta_i]^T$. In addition, w_1 and w_2 are coordinates of two moving pivots in the moving plane, while W_1 and W_2 are coordinates of two moving pivots in the fixed plane. Also, r_1 and r_2 are the lengths of left and right link from point G to point W.

From [4], we can get the analysis equations:

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} : [R(\alpha)](w_2 - w_1) - r_2 e(\theta_2) + (G_1 - G_2) + r_1 e(\theta_1) = 0 \tag{2}$$

$$\begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} : [R(\alpha)](w_2 - w_1)v_1 - r_2 e(\theta_2)v_2 + r_1 e(\theta_1) = 0 \tag{3}$$

$$P_5 : T_{in} - k_1 \Delta\phi_1 (v_1 - 1) - k_2 \Delta\phi_2 (v_1 - v_2) = 0 \tag{4}$$

The first and second equations come from the closed loop by using vectors. The third and fouth equations are derived by taking derivatives of the first and second equations. The last one is an equilibrium equation and comes from the principle of virtual work. With five equations and five unknowns in [4], six solutions are found by using homotopy continuation methods.

From [5], we will have the following synthesis equations:

$$K_{1j} :\ W_1^j - G_1 - [R(\Delta\alpha^j - \Delta\phi_1^j)](W_1^0 - G_1) = 0 \quad j = 1, \cdots n-1 \tag{5}$$

$$K_{2j} :\ W_2^j - G_2 - [R(\Delta\alpha^j - \Delta\phi_2^j)](W_2^0 - G_2) = 0 \quad j = 1, \cdots n-1 \tag{6}$$

$$E_j :\ k_1 \Delta\phi_1^j (v_1^j - 1) + k_2 \Delta\phi_2^j (v_1^j - v_2^j) = 0 \quad j = 1, \cdots n-1 \tag{7}$$

$$V_j :\ (W_1^j - W_2^j)v_1^j - (W_1^j - G_1) + (W_2^j - G_2)v_2^j = 0 \quad j = 1, \cdots, n-1 \tag{8}$$

The first part and second part are kinematic equations. The third part are equilibrium equations. The last part are velocity equations. In total, there are 7(n-1) scalar equations with 9+4(n-1) independent unknowns. As, a result, the maximal number of equilibrium positions for synthesis is four.

4.2 Analysis of four bar linkages with SMA springs

4.2.1 Analysis of four bar linkage with one spring

First, let us consider torsional springs. There are many types of four bar linkage. In this thesis, we will take crank-crank linkage as an examples, since it has a large range of motion. In Figure 4.2, we can see that we have four choices to put the springs. The position with input angle $\theta_1 = 125^\circ$ will be defined as the undeformed states for all four cases. The length from point G_1 to point G_2 is 10 and the length from point W_1 to point W_2 is 23. The lengths of left and right bars are both 26.

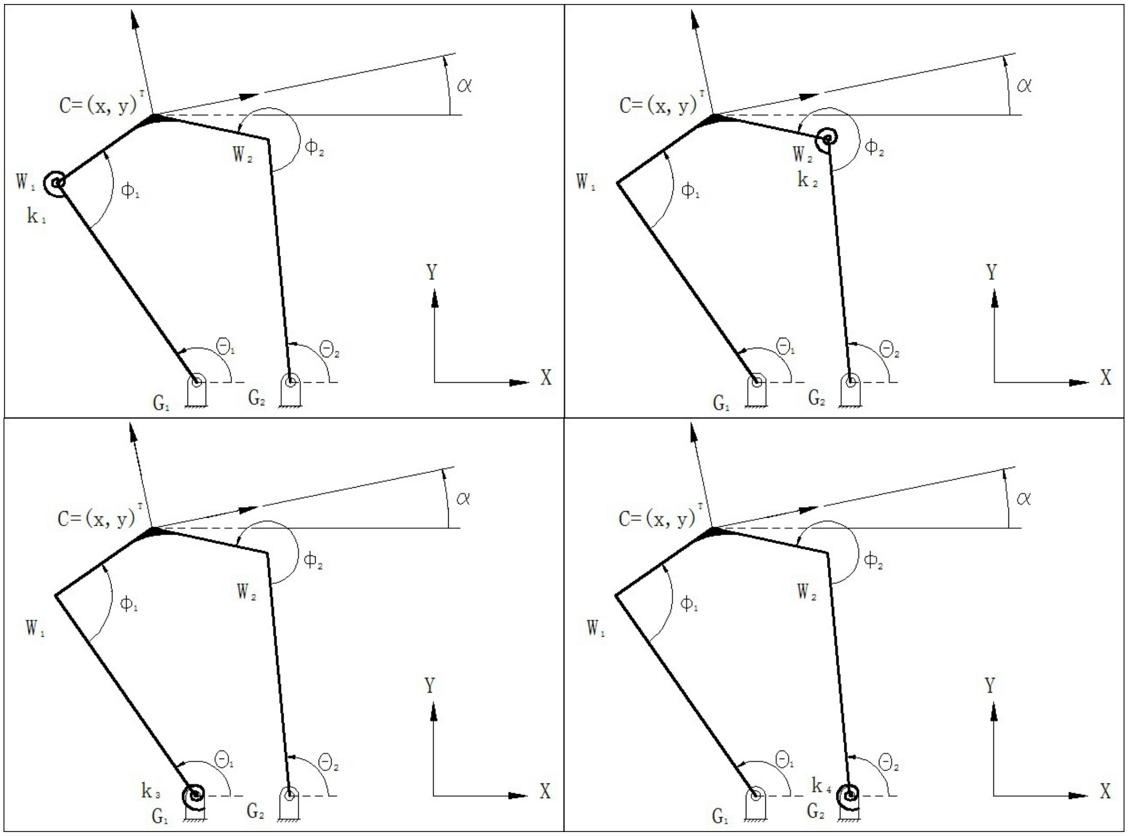

Fig 4.3 Crank-crank Linkage with One Torsional Spring

We know that the energy of torsional spring can be calculated as

$$V = \frac{1}{2} k \Delta\alpha^2 \tag{9}$$

k is the torsional stiffness and $\Delta\alpha$ is the change of the angle. If we set θ_1 as the input, we will have the function V with respect to θ_1. The range of θ_1 could be from $-\infty$ to $+\infty$, and so does angle θ_2. As a result, the energy of the two springs on the bottom can go to infinity and that is not what we want. The input θ_1 should have an limited range. In this example, the angle ϕ_1 and ϕ_2 have an limited range. However, that is not always true, which will increase the complexity of the analysis. Now we set the range of input angle θ_1 to be $(-50^{\circ}, 310^{\circ})$ as an example. Use (9), we can draw the energy curves for all the four cases in Fig 4.3. Here, all the values of stiffness are set to be unit one as an example.

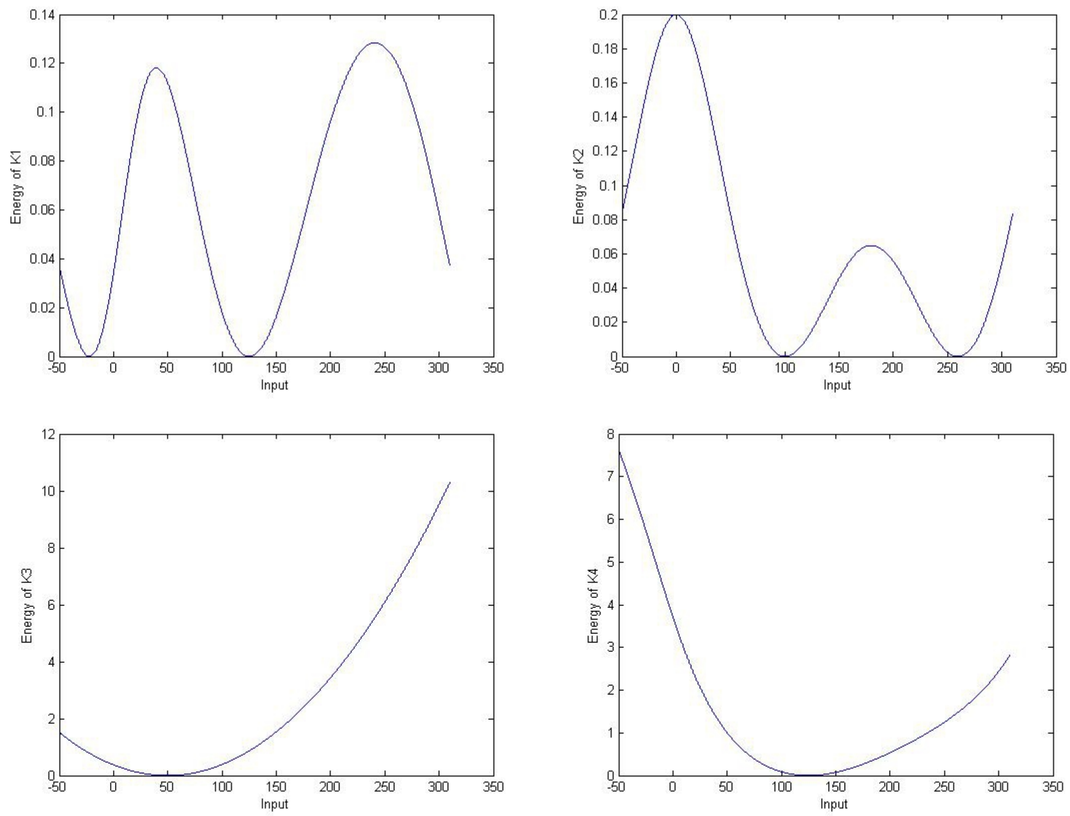

Fig 4.4 Energy curves of four cases for torsional springs

From the Fig 4.4, we can see that there are two stable positions on the two top joints, while there are only one stable positions on the two bottom joints. As a result, to make a four bar linkage have more stable positions, top joints play an important role. For other types of four bar linkage, we can still use this method to find out the joints which can make more stable positions.

Next, we will talk about the linear springs used in the four bar linkage. To compare with the case of torsional springs, we use the same linkage in Fig 4.3 as an example. To put the linear springs on the linkage, two end points are needed. There are infinite ways to choose the end points on the four bar linkage. To better compare with the torsional springs, the end points are chosen a limited length of a away along the limb from the four joints(see Fig 4.5 below). The energy of each linear springs are

$$V_i = \frac{1}{2} k_i \Delta l_i^2 \quad i = 1,2,3,4 \tag{10}$$

The Δl_i can be calculated using law of cosines. Again, we use the same parameters as the example in(Fig 4.3).

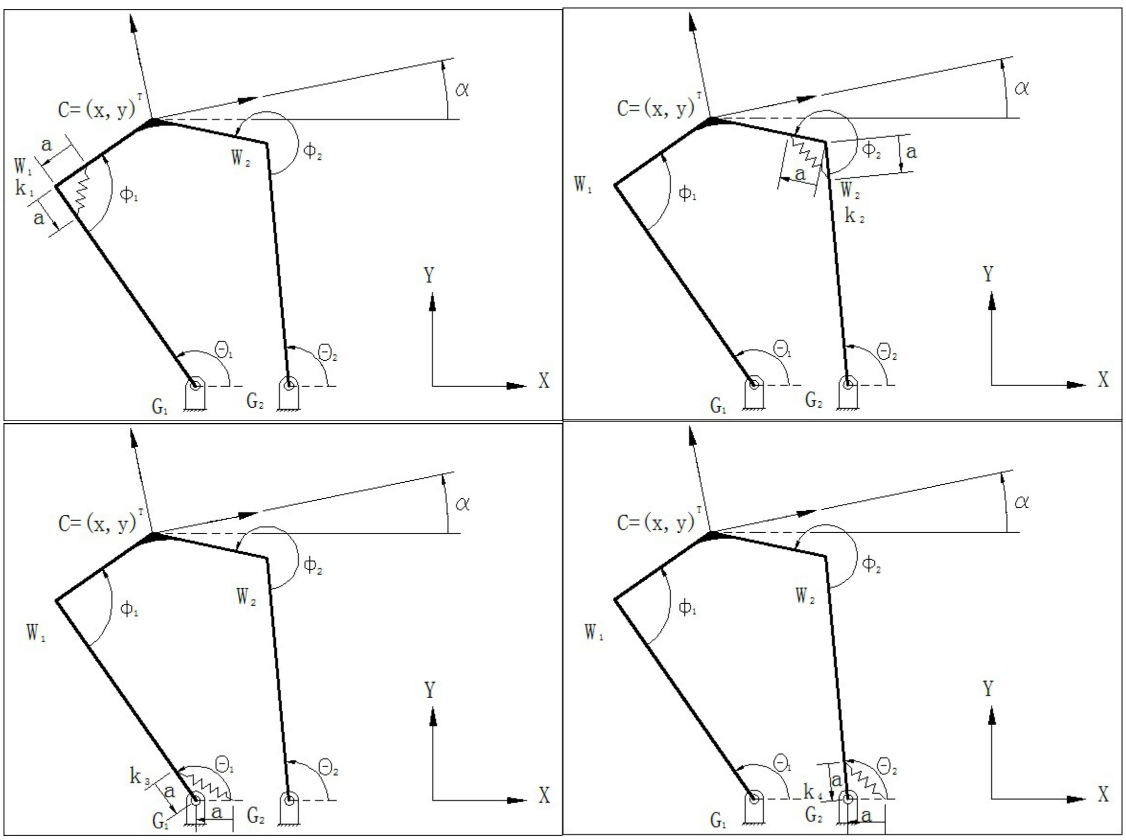

Fig 4.5 Crank-crank Linkage with One Linear Spring

When we use linear spring, even the range of input θ_1 is from $-\infty$ to $+\infty$, the lengths of all the springs are limited, which are different from torsional springs. The value of a is set to be 2 and the undeformed length of all linear springs correspond to the position where the input θ_1 equal to 125°. Use (10), we can calculate the energy curves fo all four cases as

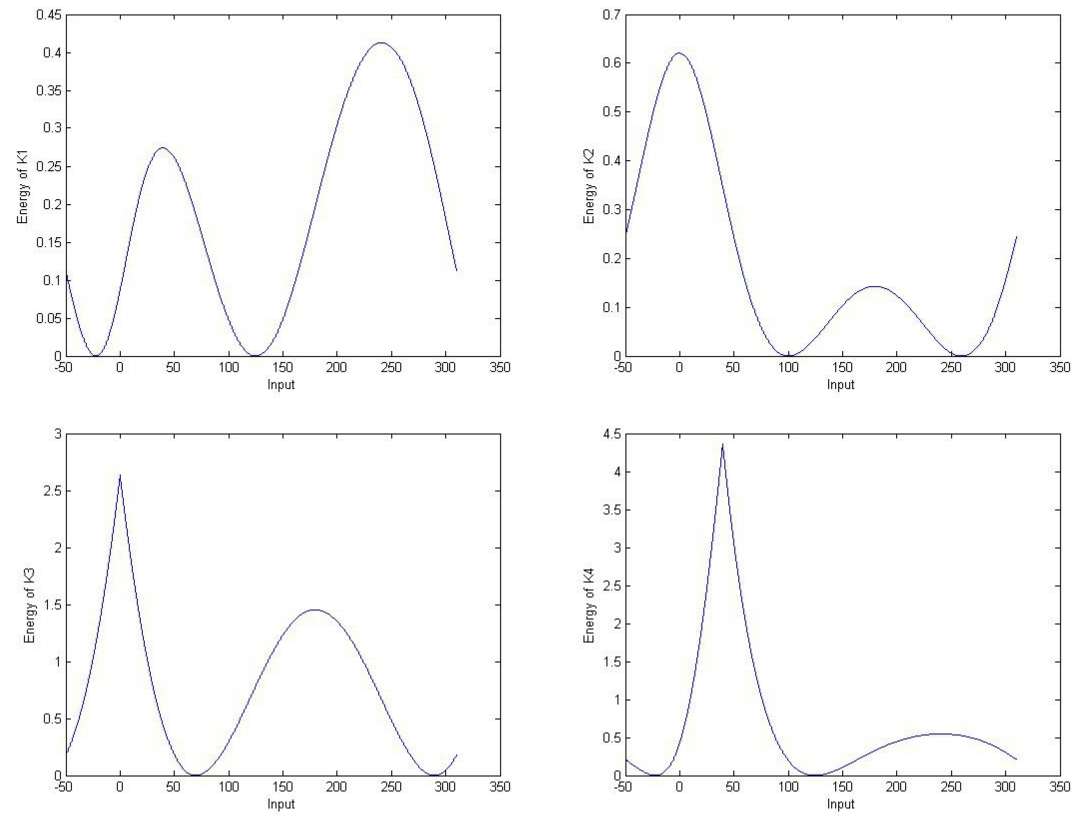

Fig 4.6 Energy curves of four cases for linear springs

From Fig 4.6, we can see that when using linear springs, all four energy curves have two stable positions. The top two curves are almost the same as the cases of torsional springs, because the ranges of the angels on the top joints are very small. For the bottom joints, the lengths of linear springs will change periodically as the input angle change from $-\infty$ to $+\infty$. As a result, we can see that linear springs can get better results for generating multistable positions.

4.2.2 Analysis of four bar linkage with multiple SMA springs

Using only one SMA spring can give us more than one stable positions. However, when the stiffness changes the stable positions will not change, since they should stay still to maintain their lowest energy. So, using more than one SMA springs, we will have more stable positions.

Based on the analysis of [4], we will do more analysis of four bar linkage with Shape Memory Alloy springs. Let us consider the torsional springs first. Actually, we do not need to use four springs for all the joints. Here, we study the general case.

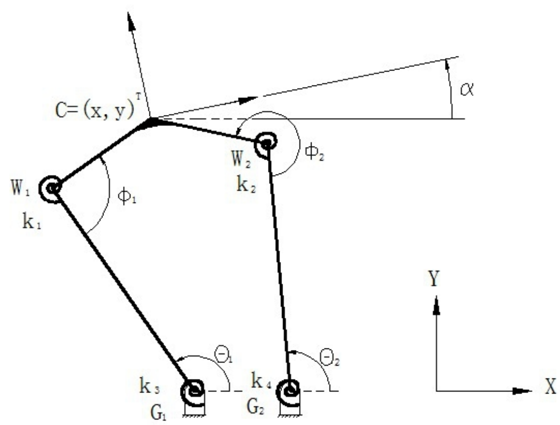

Fig 4.7 Four bar linkage with four torsional springs

Fig 4.7 is a four bar linkage with four torsional springs on the four joints. The potential energy of torsional springs is

$$V = \frac{1}{2}k_1(\Delta\phi_1)^2 + \frac{1}{2}k_2(\Delta\phi_2)^2 + \frac{1}{2}k_3(\Delta\theta_1)^2 + \frac{1}{2}k_4(\Delta\theta_2)^2 . \tag{11}$$

Without providing external force and torque, in equilibrium positions the derivative of the potential energy with respect to input angle θ_1 should equal to zero. That is

$$\frac{dV}{d\theta_1} = k_1\Delta\phi_1 \frac{d\phi_1}{d\theta_1} + k_2\Delta\phi_2 \frac{d\phi_2}{d\theta_1} + k_3\Delta\theta_1 + k_4\Delta\theta_2 \frac{d\theta_2}{d\theta_1} = 0 \tag{12}$$

With (2), (3) and (12), we can solve these equations and get the equilibrium positions by using homotopy continuation methods[4].

Since we use Shape Memory Alloy springs, each springs will have two levels of stiffness. When the stiffness changes, the corresponding equilibrium position will change. In total, we will have $2\times2\times2\times2$ =16 sets of equilibrium positions.

If we define $v_1 = \frac{d\alpha}{d\theta_1}$ and $v_2 = \frac{d\theta_2}{d\theta_1}$, we can rewrite (12) as

$$\frac{dV}{d\theta_1} = k_1\Delta\phi_1(v_1 - 1) + k_2\Delta\phi_2(v_1 - v_2) + k_3\Delta\theta_1 + k_4\Delta\theta_2 v_2 = 0 \tag{13}$$

Since we have the mixed terms of sine, cosine and $\Delta\phi_i^j$, we need to approximate $\Delta\phi_i^j$ in term of since and cosine functions using the following formula[4]:

$$\Delta\phi_i^j \approx \sin\Delta\phi_i^j (c_1 + c_2\cos\Delta\phi_i^j + c_3\cos^2\Delta\phi_i^j) \quad i = 1,2 \quad j = 1,\cdots n-1 \tag{14}$$

The stability of an equilibrium point can be checked by the second derivative of the potential energy. From Eq. (13), we compute

$$\frac{d^2V}{d\theta_1^2} = k_1 \frac{d[\Delta\phi_1(v_1 - 1)]}{d\theta_1} + k_2 \frac{d[\Delta\phi_2(v_1 - v_2)]}{d\theta_1} + k_3 \frac{d[\Delta\theta_1]}{d\theta_1} + k_4 \frac{d[\Delta\theta_2 v_2]}{d\theta_1} \tag{15}$$

The position is stable only if $\frac{d^2V}{d\theta_1^2} > 0$.

Next, we will consider the linear springs. If we change all the torsional springs in Fig 4.7 into linear springs, we will get a new four bar linkage(Fig 4.8). Again, we do not actually need to use all of them in general.

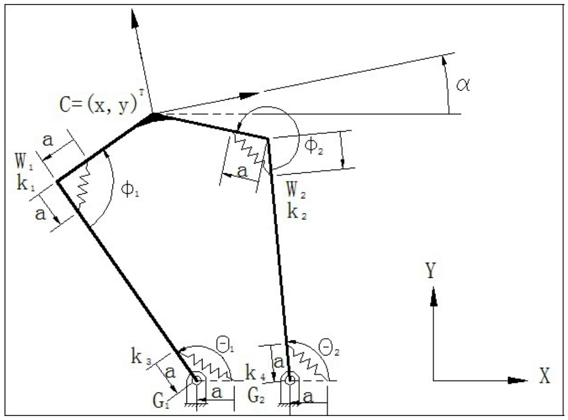

Fig 4.8 Four bar linkage with four linear springs

The lengths of linear springs are l_1, l_2, l_3, l_4, The potential energy of linear springs will be:

$$V = \frac{1}{2}k_1(\Delta l_1)^2 + \frac{1}{2}k_2(\Delta l_2)^2 + \frac{1}{2}k_3(\Delta l_3)^2 + \frac{1}{2}k_4(\Delta l_4)^2 \tag{16}$$

Again, we can take first derivative of energy and set to equal zero. That is

$$\frac{dV}{d\theta_1} = k_1 \Delta l_1 \frac{dl_1}{d\theta_1} + k_2 \Delta l_2 \frac{dl_2}{d\theta_1} + k_3 \Delta l_3 \frac{dl_3}{d\theta_1} k_4 + \Delta l_4 \frac{dl_4}{d\theta_1} = 0 \tag{17}$$

By combining this equation (17) with (2) and (3), we can solve this system by using homotopy continuation method. Before using homotopy continuation method, we need to transform (17) to polynomials by solving

$$l_1, l_2, l_3, l_4, \frac{dl_1}{d\theta_1}, \frac{dl_2}{d\theta_1}, \frac{dl_3}{d\theta_1}, \frac{dl_4}{d\theta_1}.$$

By the law of cosine, we will have

$$\begin{cases} l_1 = \sqrt{a^2 + a^2 - 2aa\cos\phi_1} \\ l_2 = \sqrt{a^2 + a^2 - 2aa\cos\phi_2} \\ l_3 = \sqrt{a^2 + a^2 - 2aa\cos\theta_1} \\ l_4 = \sqrt{a^2 + a^2 - 2aa\cos\theta_2} \end{cases} \tag{18}$$

By taking derivative of (18), we will have

$$\begin{cases} \dfrac{dl_1}{d\theta_1} = \dfrac{\sqrt{2}a\sin\phi_1}{2\sqrt{1-\cos\phi_1}}\dfrac{d\phi_1}{d\theta_1} \\ \dfrac{dl_2}{d\theta_1} = \dfrac{\sqrt{2}a\sin\phi_2}{2\sqrt{1-\cos\phi_2}}\dfrac{d\phi_2}{d\theta_1} \\ \dfrac{dl_3}{d\theta_1} = \dfrac{\sqrt{2}a\sin\theta_1}{2\sqrt{1-\cos\theta_1}}\dfrac{d\theta_1}{d\theta_1} \\ \dfrac{dl_4}{d\theta_1} = \dfrac{\sqrt{2}a\sin\theta_2}{2\sqrt{1-\cos\theta_2}}\dfrac{d\theta_2}{d\theta_1} \end{cases} \tag{19}$$

To eliminate the non-polynomial terms of $\sqrt{1-\cos\phi_i}$ and $\sqrt{1-\cos\theta_i}$ $i=1,2$, we introduce new terms $m_i, i=1,2,3,4$ by adding new equations

$$\begin{cases} N_1 = m_1^2 + \cos\phi_1 - 1 = 0 \\ N_2 = m_2^2 + \cos\phi_2 - 1 = 0 \\ N_3 = m_3^2 + \cos\theta_1 - 1 = 0 \\ N_4 = m_4^2 + \cos\theta_2 - 1 = 0 \end{cases} \tag{20}$$

By substituting $m_i, i=1,2,3,4$ into (17), and clearing the denominators, we will have a purely polynomial equation.

To check stability, we can compute the second derivative of energy. At last, if we use user-defined homotopy from Bertini, actually we do not need to transform the non polynomial terms to polynomials. However, in that case, we need to find a good start system, and solve it. That is a challenging process since there is no general way for solving non polynomial systems.

4.2.3 Examples of analysis with SMA springs

Let us use the four bar linkage in Fig 4.7 and Fig 4.8 as an example. The system parameters are

$$G_{1x}=0, G_{1y}=0, G_{2x}=10, G_{2y}=0, r_1=r_2=26, \theta_1^0=125.00^\circ, \theta_2^0=95.24^\circ.$$

$$w_{1x}=-11.5, w_{1y}=-5, w_{2x}=11.5, w_{2y}=-5\,.$$

First, let us use only two torsional springs on k_1 and k_3 here as an example. The first set of stiffness is $k_1 = 100, k_3 = 3$, and the second set of stiffness is $k_1 = 100, k_3 = 18$.

The four bar linkage in [4] has a stable position where the energy is zero. Here, we will make a little change to increase the range of stable positions with changed stiffness. We will set undeformed angles to be:

$$\phi_1^0 = 66.52^\circ, \phi_2^0 = 286.28, \theta_1^0 = 50^\circ, \theta_2^0 = 95.24^\circ$$

So the angle relations should be

$$\theta_i = \alpha - \Delta\phi_i + \theta_{si}^0 - \alpha_s^0 + \phi_{si}^0 - \phi_1^0,\ i = 1,2 \tag{21}$$

where $\theta_{si}^0, \alpha_s^0, \phi_{si}^0, i = 1,2$ are the angles in the situation of $\theta_1 = 125.00^\circ$.

In (14), we use approximations to express $\Delta\phi_i$ in term of $\sin\phi_i$ and $\cos\phi_i$. In this example, we will have $\Delta\alpha, \sin\alpha, \cos\alpha$ at the same time. We need to do the approximation by using (14) too. The error is about 1.5% when the angle is between -90° and $+90^\circ$. So, we choose the range of θ_1 to be $[-50^\circ, 310^\circ]$, and the range of 0.5α will be $[-90^\circ, 90^\circ]$. Then we can use half angle functions in (14). In this example, the change of $\phi_i, i = 1,2$ will be in the range of $[-90^\circ, 90^\circ]$. So, we will have two more constrains

$$-180^\circ < \alpha < 180^\circ \tag{22}$$

$$-90^\circ < \Delta\phi_i < 90^\circ,\ i = 1,2 \tag{23}$$

By formulating the start system(see[4]), we can solve the polynomial systems. By running the homotopy code in Bertini, we will get 10 solutions for the case of $k_1 = 100, k_3 = 3$ and 8 solutions for the case of $k_1 = 100, k_3 = 18$. By checking the constraints of (22) and (23), we can only have three solutions for each set of springs.

By checking the second derivative of energy, we found that in each case, there are two stable positions and one unstable position. Using the parameters in Table 4.1 and Table 4.2, these positions are shown in Fig 4.9 and Fig 4.10.

Table 4.1 Solutions of torsional springs for $k_1 = 100, k_3 = 3$

case 1	$\cos(0.5\alpha)$	$\sin(0.5\alpha)$	v_1	v_2	$\sin(\Delta\phi_1)$	$\sin(\Delta\phi_2)$	$\cos(\Delta\phi_1)$	$\cos(\Delta\phi_2)$
a	0.9987	0.0517	0.5596	0.9597	0.0762	-0.108	0.9971	0.9941
b	0.4352	-0.900	1.6846	1.4747	0.0511	0.5636	0.9987	0.8261
c	0.9173	-0.398	1.0125	1.3923	0.4669	0.4692	0.8843	0.8831

Table 4.2 Solutions of torsional springs for $k_1 = 100, k_3 = 18$

case 2	$\cos(0.5\alpha)$	$\sin(0.5\alpha)$	v_1	v_2	$\sin(\Delta\phi_1)$	$\sin(\Delta\phi_2)$	$\cos(\Delta\phi_1)$	$\cos(\Delta\phi_2)$
a	0.9959	-0.089	0.6045	1.0745	0.2827	0.1091	0.9592	0.9941
b	0.6405	-0.767	1.6356	1.6216	0.2462	0.5909	0.9692	0.8067
c	0.8899	-0.456	1.1212	1.4556	0.4599	0.5069	0.8879	0.8621

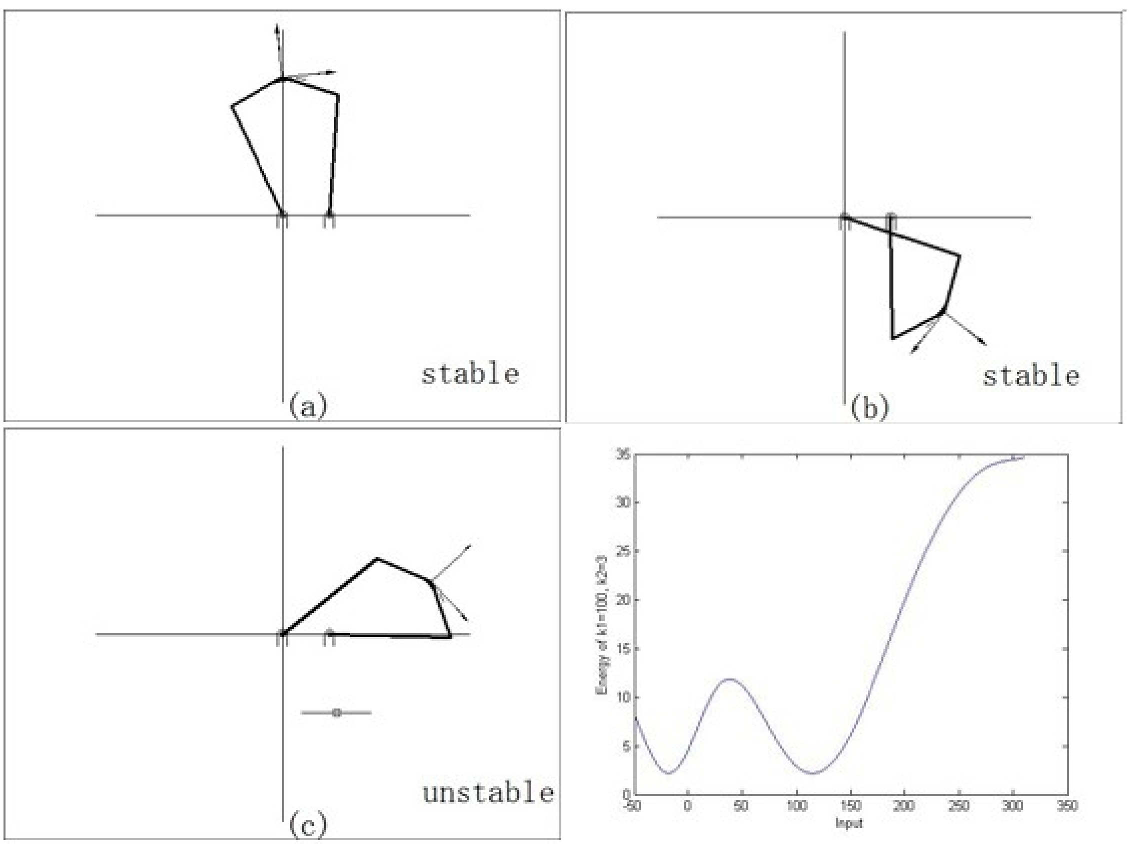

Fig 4.9 Three equilibrium configurations of a compliant four bar linkage of case 1.

Positions (a) and (b) are stable and (c) is unstable. The last one is the energy curve.

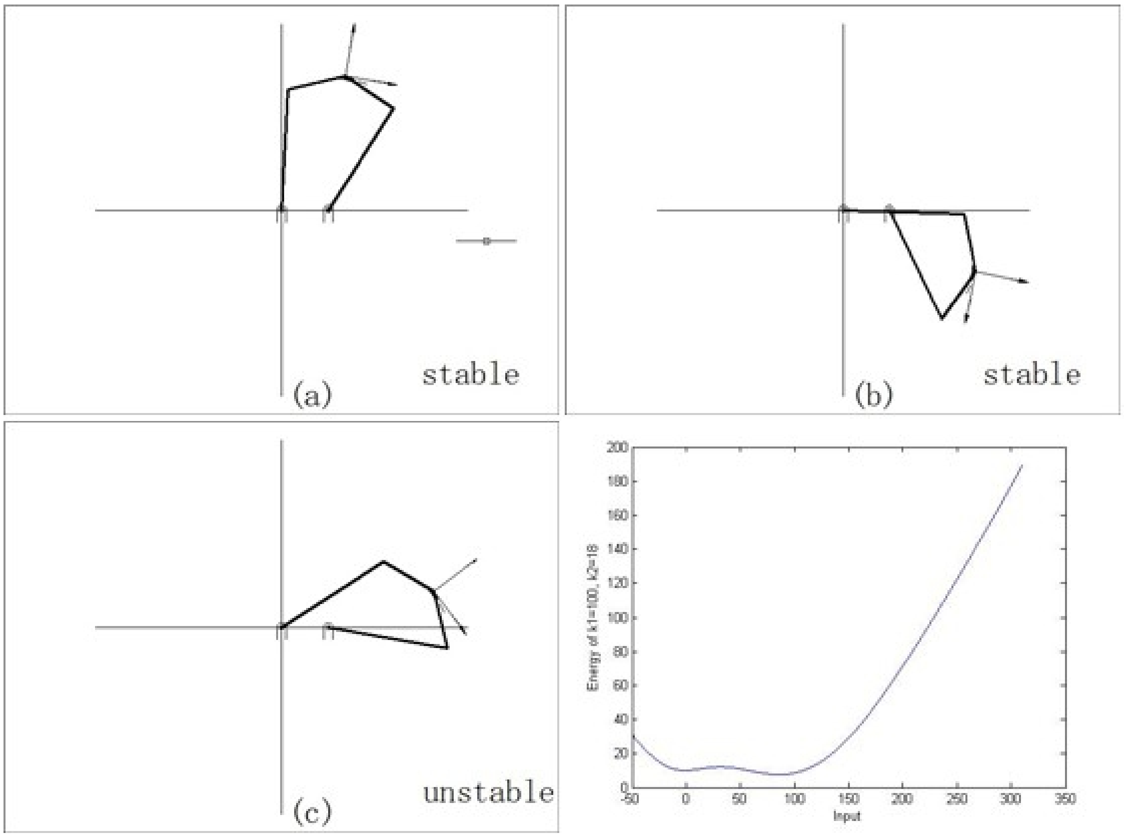

Fig 4.10 Three equilibrium configurations of a compliant four bar linkage of case 2.

Positions (a) and (b) are stable and (c) is unstable. The last one is the energy curve.

Let us compare the two energy curves in Fig 4.9 and Fig 4.10 . By changing stiffness from case 1 to case 2, one stable position will move to another stable position and vice versa.

Next, we will use only two linear springs on k_1 and k_3. All other parameters are same as above. The first set of stiffness is $k_1 = 100, k_3 = 1$, and the second set of stiffness is $k_1 = 20, k_3 = 20$. Since the total degree is only 4608, we can use Bertini to apply total degree homotopy. It takes less than 5 minutes to finish the whole process. The undeformed length of each spring will be chosen as 2.19 and 2.29. In addition, a=2(see Fig 4.8). From (20) and (23), we can see that there are two constraints

$$m_i > 0, i = 1, 3 \tag{24}$$

$$-90^\circ < \Delta\phi_i < 90^\circ,\ \ i = 1,2 \tag{25}$$

There are 32 solutions for case 1 and 20 solutions for case 2. Using the constraints of (24) and (25), we will have 4 solutions for case 1 and 3 solutions for case 2.

Table 4.3 Solutions of linear springs for $k_1 = 100, k_3 = 1$

case 1	$\cos(\alpha)$	$\sin(\alpha)$	v_1	v_2	$\sin(\phi_1)$	$\sin(\phi_2)$	$\cos(\phi_1)$	$\cos(\phi_2)$
a	0.9823	0.1873	0.5633	0.9275	0.9211	-0.993	0.3894	0.1174
b	-0.742	-0.671	1.6608	1.3952	0.9108	-0.653	0.4124	0.7574
c	0.6778	-0.735	1.0184	1.3958	0.9971	-0.714	-0.076	0.6998
d	-0.034	0.9994	0.9688	0.6135	0.6101	-0.994	0.7924	0.1054

Table 4.4 Solutions of linear springs for $k_1 = 20, k_3 = 20$

case 2	$\cos(\alpha)$	$\sin(\alpha)$	v_1	v_2	$\sin(\phi_1)$	$\sin(\phi_2)$	$\cos(\phi_1)$	$\cos(\phi_2)$
a	-0.961	0.278	1.3357	0.8708	0.7209	-0.828	0.6931	0.5609
b	0.9576	-0.288	0.6487	1.1249	0.998	-0.888	0.0557	0.4606
c	0.5895	0.8077	0.7679	0.6886	0.6869	-0.998	0.7267	-0.069

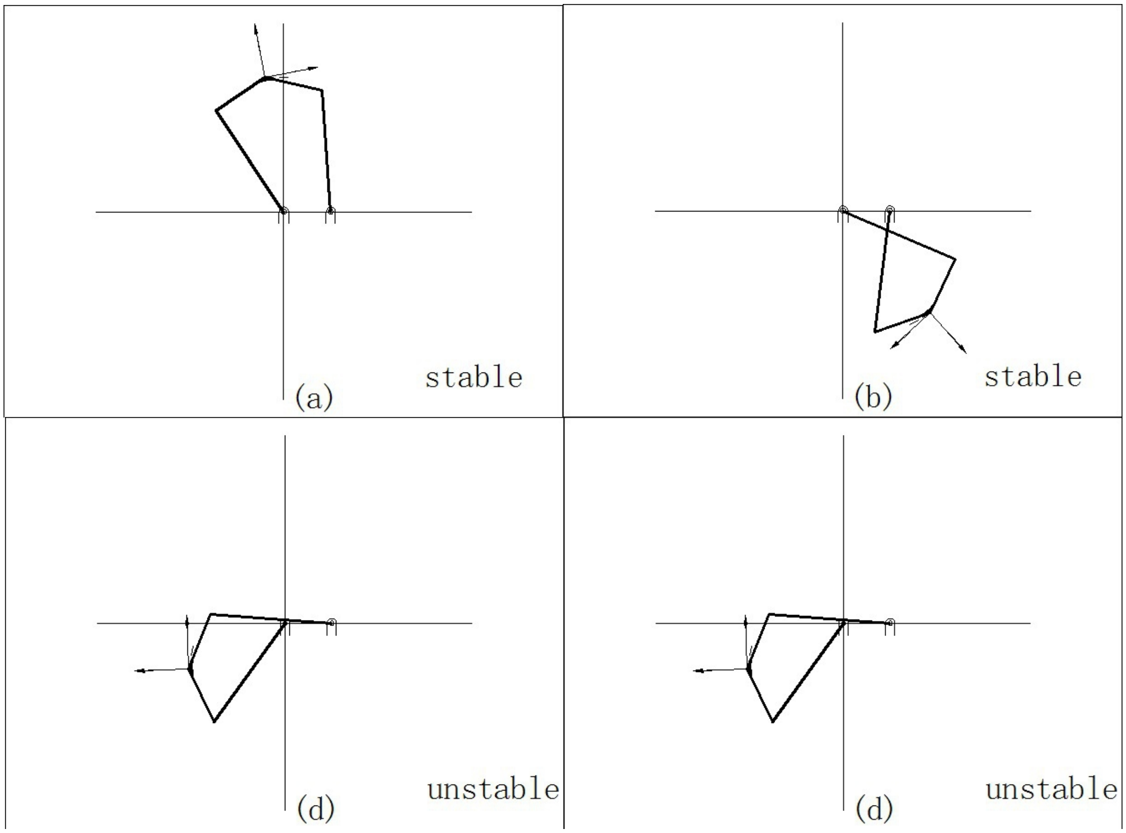

Fig 4.11 Four equilibrium configurations of $k_1 = 100, k_3 = 1$

In Fig 4.11, Positions (a) and (b) are stable, while (c) and (d) are unstable. Fig 4.12 below is the energy curve.

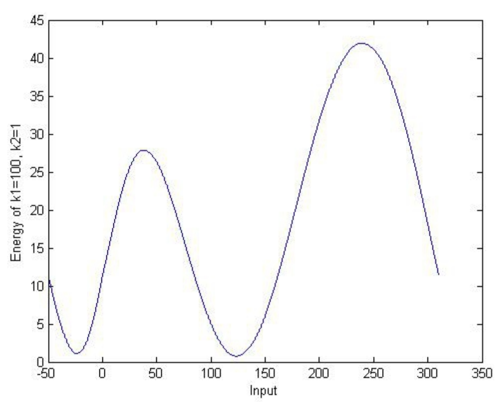

Fig 4.12 Energy curve of $k_1 = 100, k_3 = 1$

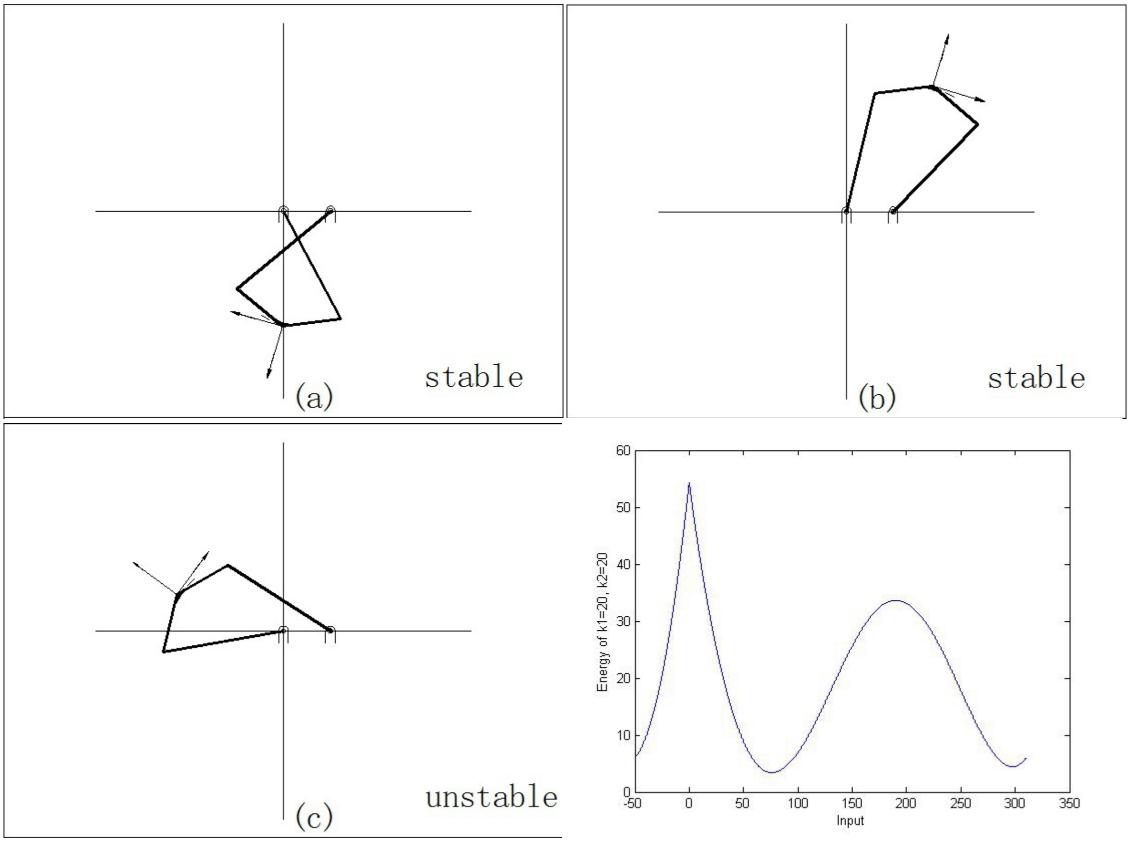

Fig 4.13 Four equilibrium configurations of $k_1 = 20, k_3 = 20$. Positions (a) and (b) are stable and (c) is unstable. The last one is the energy curve.

By comparing the two energy curves above, we can see that, if we change stiffness from $k_1 = 100, k_3 = 1$ to $k_1 = 20, k_3 = 20$, the stable position of (a) and (b) in Fig 4.11 will move to the stable positions of (b) and (a) in Fig 4.13 respectively and and vice versa. By comparing the torsional springs and linear springs, we can see that linear springs can give us larger displacement of stable positions with changed stiffness.

4.3 Synthesis of four bar linkage with SMA springs

4.3.1 Synthesis of four bar linkage with SMA springs for general case

In this section, we will try to design a four bar linkage to make it pass through n

prescribed stable positions, and determine the maximal value of n. In total, there are three constraints: geometric equations, equilibrium equations and stability inequalities.

There are three methods for solving those equations and inequalities: direct search methods, homotopy continuation method and analytical methods. In the first part, we will introduce direct search methods.

First, let us consider the geometric equations. For one prescribed position, we need eight values to determine the corresponding four bar linkage, which are the coordinates of two moving pivots and two fixed pivots.

$$G_{1x}, G_{1y}, G_{2x}, G_{2y}, w_{1x}, w_{1y}, w_{2x}, w_{2y}$$

As long as these eight independent values are determined, we can get the four bar linkage. So, we can say that the solutions of four bar synthesis of one prescribed position are in an eight dimensional space.

Table 4.5 show the relationship between independent unknowns and the number of prescribed configurations.

Table 4.5 Relationship

No. of Prescribed configurations	1	2	3	4	5
No. of Independent unknowns	8	6	4	2	0

Since these unknowns are independent, the coefficients of stiffness in the equilibrium equations can be solved in terms of those independent unknowns.

Next, we will consider the equilibrium equations. Now, we will use only two SMA springs and both have two levels of stiffnesses. One set of stiffnesses corresponds to one equilibrium configuration. The first stable position is the undeformed natural state.

Let us rewrite the equilibrium equations and inequalities as

$$c_{ai}k_{ai} + c_{bj}k_{bj} = 0, i = 1,2 \ j = 1,2 \tag{26}$$

$$q_{ai}k_{ai} + q_{bj}k_{bj} > 0, i = 1,2 \ j = 1,2 \tag{27}$$

The coefficient $c_{ai}, c_{bj}, q_{ai}, q_{bj}, i=1,2\ j=1,2$ can be determined by the four bar linkage and be expressed in term of all the independent unknowns. All the values of stiffness should be greater than zero, and as a result, (26) and (27) can be expressed as:

$$\frac{k_{ai}}{k_{bj}} + \frac{c_{bj}}{c_{ai}} = 0, i=1,2\ j=1,2 \tag{28}$$

$$-q_{ai}\frac{c_{bj}}{c_{ai}} + q_{bj} > 0, i=1,2\ j=1,2 \tag{29}$$

Now, if we want to design a four bar linkage to pass through 5 stable positions, there will be only one solution for the linkage. In general, this solution will not satisfy (28) and (29). Next, we can try 4 stable positions. Since the first one is undeformed state, we will have three equations of (28) and three inequalities of (29). We will consider the ratio of stiffness in (28) as one unknowns. In fact, as long as (28) satisfy the following inequalities (30), there will be real solutions for the values of stiffness.

$$\frac{c_{bj}}{c_{ai}} < 0, i=1,2\ j=1,2 \tag{30}$$

From Table 4.5, we can see that there are two independent unknowns, which should satisfy (29) and (30). We can use direct search method to find the feasible region in a two-dimensional plane for the two independent unknowns with the constraints ((29) and (30). So the maximal number of stable configurations for synthesis using two springs is four.

For three stable configurations and two stable configurations synthesis, there will be more independent unknowns, which means that the solutions are in a four and six dimensional space for the constraints of (29) and (30).

Next, if all the stiffnesses are predetermined, there will be $n-1$ equations of (28) for n specified stable configurations. The number of independent unknowns will be $2\times(5-n)$. To have solutions, n should satisfy $2\times(5-n)\geq(n-1)$, which is $n\leq 3.67$. So for predetermined values of stiffness, the maximal number of stable configurations for synthesis using two springs is three.

In the second part, we will try to use homotopy continuation method to solve the problem above. Since this method can only handle polynomials equations, we need to transform all the inequalities to equations. First, since all the values of stiffness should be greater than zero, to eliminate those constraints we need to predetermine all the values of stiffness in (28). For inequalities (29), we can choose a positive number N_k to make the left side of (29) equal to N_i.

$$-q_{ai}\frac{c_{bj}}{c_{ai}}+q_{bj}=N_k, i=1,2\ j=1,2, k=1,2,3,4 \tag{31}$$

From the direct search method, we can figure out that, if all the stiffnesses are predetermined, there will be $n-1$ equations of (28) and $n-1$ equations of (31) for n specified stable configurations. The number of independent unknowns will be $2\times(5-n)$. To have solutions, n should satisfy $2\times(5-n)\geq(2n-2)$, which is $n\leq 3$. So for predetermined values of stiffness, by using the homotopy continuation methods, the maximal number of stable configurations for synthesis using two springs is three. Actually, when we use the homotopy continuation method more equations with the same number of unknowns are needed because of some non-linear parts(see [4] and [5] for more details).

In the third part, we will talk about the analytical method. It takes two steps, First, we need to do synthesis of four bar linkage without caring about stability. After that, we can get all the coefficients of (28). However, all the coefficient should satisfy (29) and (30). If they do, then we can get all the ratios of stiffness. Otherwise, this method fails. We need to go to the first step to do the synthesis again.

If we compare the above three methods, we could find out that the direct search methods will have solutions for probability one, but need lots of work. The analytical methods need little work,but have large chance of failure. So, next section, we will give an example for the homotopy continuation methods.

4.3.2 Example of synthesis of four bar linkage with SMA springs

In this section, we will give an example for synthesis of four bar linkage with SMA springs using homotopy continuation methods.

First given two prescribed positions in Table 4.6:

Table 4.6 Two prescribed stable configurations

Position	Orientation(degree)	Stability
$C^0 = (-30,30)$	45	Stable
$C^1 = (20,40)$	0	Stable

Next, we need to give all the equations for homotopy continuation methods. In this example, we use two torsional SMA springs for k_1 and k_3 in Fig 4.8. Since there are two stable configurations,, we will use only one set of stiffness(see Table 4.7):

Table 4.7 Stiffness for each spring

Springs	k_1	k_3
Stiffness	40	1

The first stable position is undeformed state, so we do not need to care about it. We have one equilibrium equation and stable equation. In order to handle the derivative parts in the above two equations, we need two more equations by taking the first and second derivative of equation (2). In addition, one equation from (5) and one equation from (6) are needed. Because of the cosine and since function, we need to introduce one more equation to treat the cosine and since part as two unknowns.

Now we can put all the equations all together: From last section, we know that for two position synthesis, there are six independent unknowns. However, there are only one equilibrium equation and one stable equation. We need to specify four independent unknowns. First, we choose the first moving pivot to be (-10,-5). The value of N is chosen as 1.5. The length of the input link is $r_1 = 39.8$ and the length of the next grounded links is also $r_2 = 39.8$.

Now, from the geometric relationship, we can get the following values:

$$G_{1x} = -9, G_{1y} = 0, \Delta\phi_1 = 34.62^\circ, \Delta\alpha_1 = -45^\circ, \theta_1 = 61.50^\circ, \theta_1^0 = 141.12^\circ$$

$$P_1: W_2^1 - G_2 - [R(\Delta\alpha - \Delta\phi_2)](W_2^0 - G_2) = 0$$
$$P_2: k_1\Delta\phi_1(v_1 - 1) + k_3(\Delta\alpha - \Delta\phi_1) = 0$$
$$P_3: k_1\Delta\phi_1\omega_1 + k_1(v_1 - 1)^2 + k_3 - N = 0$$
$$P_4: (W_1^1 - W_2^1)v_1 - (W_1^1 - G_1) + (W_2^1 - G_2)v_2 = 0$$
$$P_5: r_2 e(\theta_2)v_2^2 - r_2 e^\perp(\theta_2)\omega_2 - r_1 e(\theta_1) - [R(\alpha)](w_2 - w_1)v_1^2 + [J][R(\alpha)](w_2 - w_1)\omega_1 = 0$$
$$P_6: \sin^2\Delta\phi_2 + \cos^2\Delta\phi_2 - 1 = 0$$
$$P_7: (W_{2x}^0 - G_{2x})^2 + (W_{2y}^0 - G_{2y})^2 - r_2{}^2 = 0$$

The term of $e(\theta_2), e^\perp(\theta_2)$ can be solved by the following equations[5]:

$$W_i - G_i - r_i e(\theta_i) = 0,\ i = 1,2$$

The first vector equation is geometric constraint equation. The second and third are equilibrium and stable equation. The fourth and fifth are equations used for the new derivative parts. The last two scaler equations are used to eliminate non-linear parts. Since P_1, P_4 and P_5 are vector equations, in total we have ten equations with ten unknowns: $G_{2x}, G_{2y}, w_{2x}, w_{2y}, v_1, v_2, \omega_1, \omega_2, \sin\Delta\phi_2, \cos\Delta\phi_2$.

By using homotopy continuation method, we can find all the isolated solutions. In this example, four solutions are found in Table 4.8.

Table 4.8 Solutions of example

solutions	G_{2x}	G_{2y}	w_{2x}	w_{2y}
1	13.85	-30.76	-28.26	-37.66
2	-7.626	1.117	-11.80	-75.40
3	-44.08	16.80	-27.28	-38.38
4	-8.226	0.074	-22.65	-51.76

Next, we will draw the four solutions in Fig 4.14.

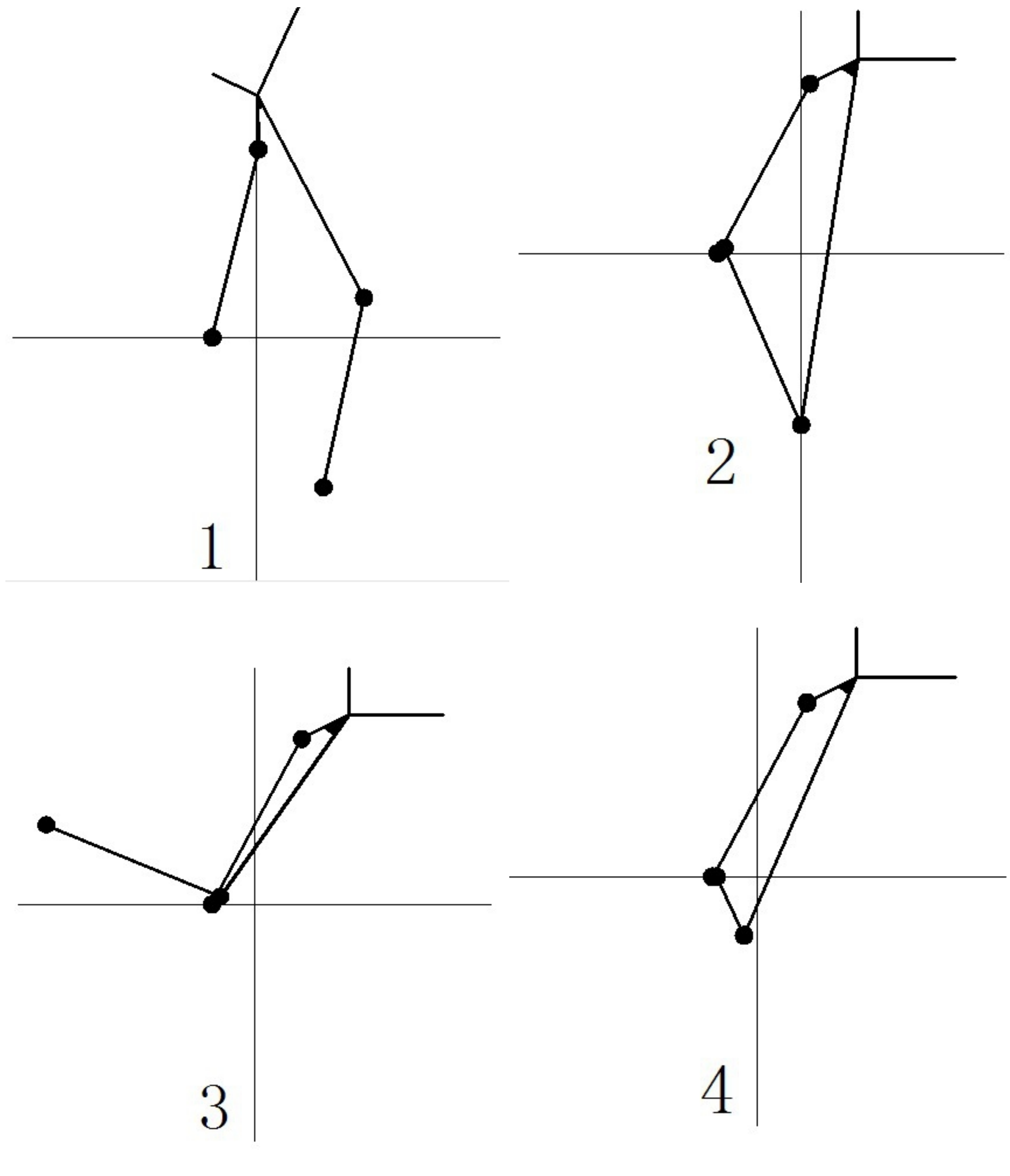

Fig 4.14 Four solutions

If we do the synthesis for three stable configurations using homotopy continuation methods, we may get few solutions since we have more constraints.

Chapter 5 Configuration Memory Sculpture Systems

In this chapter, shape memory alloy springs will be added on sculpture systems to make them memorize some configurations. When temperature of the shape memory alloys springs changes, one configuration can move to another configuration.

5.1 Configuration memory four by four linkage system

5.1.1 Analysis of four by four linkage system without springs

In this section, we will discuss a four by four linkage system (see Fig 5.1).

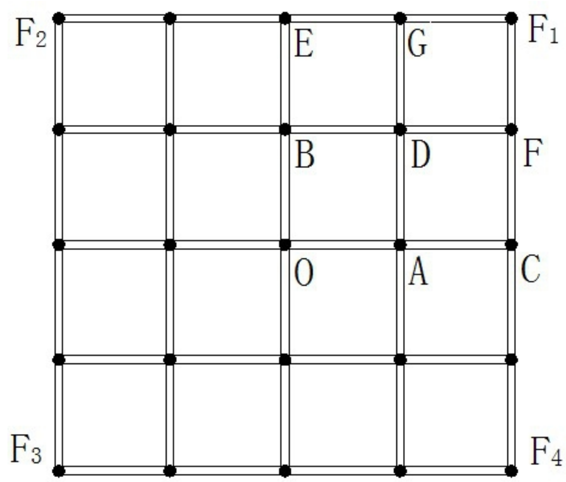

Fig 5.1 Four by four linkage

Four points F_1, F_2, F_3, F_4 are fixed on the horizontal $X-Y$ plane and have the same Z coordinates. In addition, the whole linkage system is symmetric along the $X-Z$ and $Y-Z$ planes. All the lengths of limb are L. Now, our task is to determine different stable status of the linkage by pulling the center O up to different levels.

Next, we will define all the points:

$$O=(0,\ \ 0,\ z_1\ ), A=(d,\ 0,\ z_4\ ), B=(0,\ d,\ z_4\ )$$

$$C=(2c,\ 0,\ z_3\ ), D=(b,\ b,\ z_2\ ), E=(0,\ 2c,\ z_3\ )$$

$$F=(2e,\ f,\ z_5\ ), G=(f,\ 2e,\ z_5\ ), F_1=(2a,\ 2a,\ 0\ )$$

We have nine unknowns: $z_2, z_3, z_4, z_5, b, c, d, e, f$. So nine equations are need to determine the linkage system.

Now, we define $|AB|=\sqrt{(A_x-B_x)^2+(A_y-B_y)^2+(A_z-B_z)^2}$. From the length of limbs we will have the following equations:

$$P_i: \quad \begin{cases} |OA|^2-L^2=0 \\ |AC|^2-L^2=0 \\ |AD|^2-L^2=0 \\ |CF|^2-L^2=0 \\ |DF|^2-L^2=0 \\ |FF_1|^2-L^2=0 \end{cases} \quad i=1,2,3,4,5,6 \tag{1}$$

Suppose the weight of each limb is mg, and the total potential energy will be

$$V=2mg(z_1+4z_2+3z_3+4z_4+6z_5) \tag{2}$$

Next, we will determine the degree of freedom. Since this linkage is symmetric, we can only consider the rod OA, AC, AD, CF, DF, FF_1 . Each rod has six degree of freedom, the maximum of degree of freedom will be $6\times 6=36$. There are eight knots including multiplicities and $8\times 3=24$ freedoms will be removed. In addition, because of the symmetry of the linkage, point A, point C and point D can only move in a fixed plane, and this will remove 3 freedoms. Also, we do not need to care about the rotations of each rod along the centric line, and this will remove 6 freedoms. So, the remaining degree of freedom is $36-24-3-6=3$. We can choose three independent unknowns as our degree of freedom. Here, we choose z_2, z_3, z_4.

According to the principle of virtual work, a system will be in equilibrium if for all virtual displacements the work done on the system by the external forces equals the change in the potential energy of the system, that is

$$\frac{\partial V}{\partial z_2}\delta z_2 + \frac{\partial V}{\partial z_3}\delta z_3 + \frac{\partial V}{\partial z_4}\delta z_4 = F_2\delta z_2 + F_3\delta z_3 + F_4\delta z_4 \tag{3}$$

Given that all the external forces equal zero, we will have the following equations from equations(3):

$$V_i: \quad \begin{cases} \dfrac{\partial V}{\partial z_2} = 2mg(4 + 6\dfrac{\partial z_5}{\partial z_2}) = 0 \\ \dfrac{\partial V}{\partial z_3} = 2mg(3 + 6\dfrac{\partial z_5}{\partial z_3}) = 0 \\ \dfrac{\partial V}{\partial z_4} = 2mg(4 + 6\dfrac{\partial z_5}{\partial z_4}) = 0 \end{cases} \quad i = 1,2,3 \tag{4}$$

From (4), we have new unknowns of the derivative parts. These unknowns can be solved by taking derivative of six equations of (1) to z_2, z_3, z_4 respectively. As a result, we have new 18 equations with new 18 unknowns. After solving those equations analytically, we can express the derivative parts in term of unknowns.

To check stability, we need to consider the Hessian matrix of potential energy.

$$H = \begin{bmatrix} \dfrac{\partial^2 V}{\partial z_2^2} & \dfrac{\partial^2 V}{\partial z_2 \partial z_3} & \dfrac{\partial^2 V}{\partial z_2 \partial z_4} \\ \dfrac{\partial^2 V}{\partial z_3 \partial z_2} & \dfrac{\partial^2 V}{\partial z_3^2} & \dfrac{\partial^2 V}{\partial z_3 \partial z_4} \\ \dfrac{\partial^2 V}{\partial z_4 \partial z_2} & \dfrac{\partial^2 V}{\partial z_4 \partial z_3} & \dfrac{\partial^2 V}{\partial z_4^2} \end{bmatrix} \tag{5}$$

A equilibrium configuration is stable only if the Hessian matrix H is positive definite, that is all of its eigenvalues are positive.

5.1.2 Examples of four by four linkage system without springs

Now, an example of analysis of four by four linkage without springs is given here. The parameters are $a = 1$, $L = 1.1$. First, we pull z_1 to a height of $z_1 = -1.0$. By using homotopy method, we get two solutions. Now we pick the solution with the lowest energy, and trace that solution by changing $z_1 = -0.7,\ -0.4,\ -0.1,\ 0.2,\ 0.5$ using user defined homotopy to find out how the linkage will move when we pull the point z_1 straight up. Next, we will draw the linkage for six cases from Table 5.1.

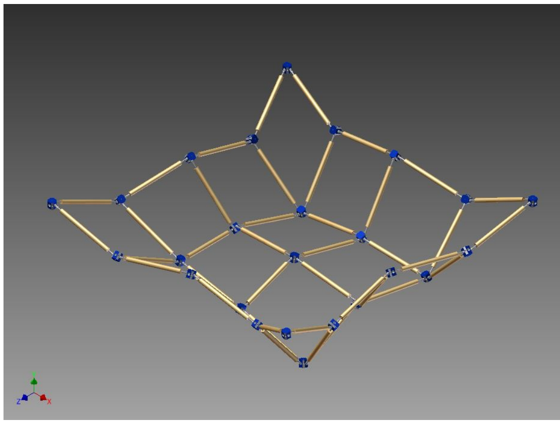

Fig 5.2 case 1

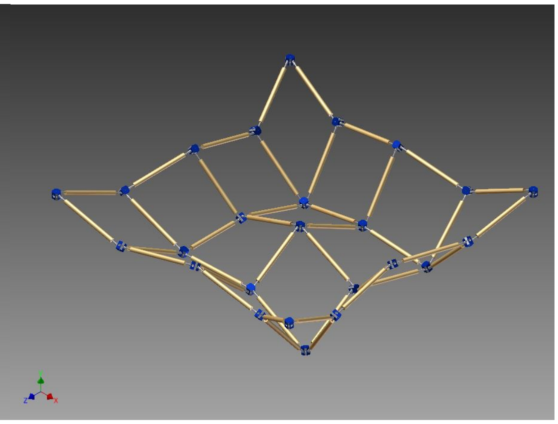

Fig 5.3 case 2

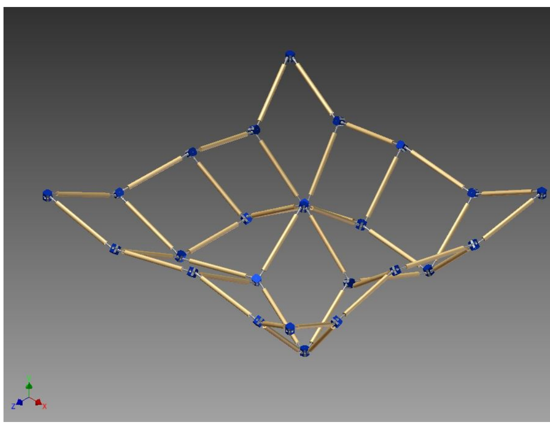

Fig 5.4 case 3

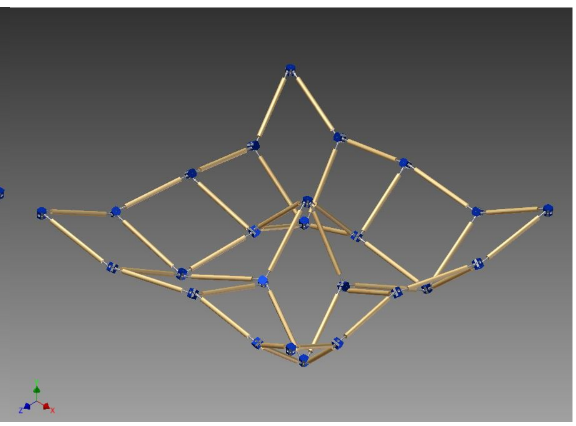

Fig 5.5 case 4

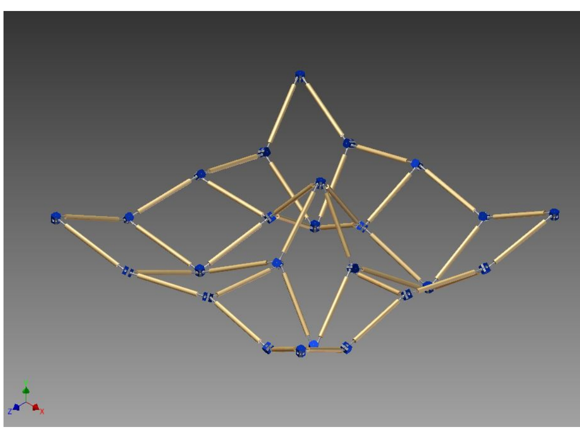

Fig 5.6 case 5

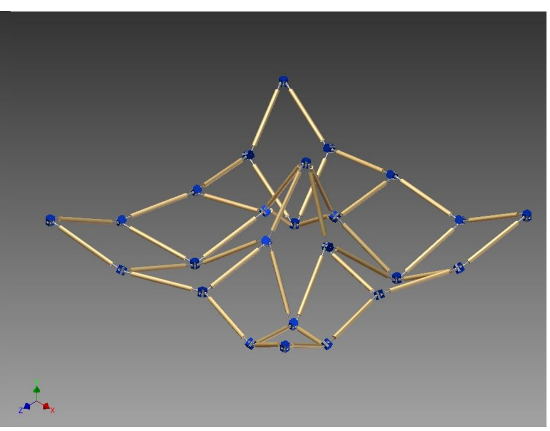

Fig 5.7 case 6

From Fig 5.2 to Fig 5.7 we can see the changes of linkage for different level of the center. Table 5.1 is the solutions for six cases above.

Table 5.1 Solutions of six cases

No.	z_2	z_3	z_4	z_5	b	c	d	e	f
1	-1.4	-0.77	-1.56	-0.56	1.08	0.85	0.94	0.89	1.08
2	-1.39	-0.76	-1.43	-0.56	1.07	0.85	0.82	0.89	1.08
3	-1.37	-0.75	-1.24	-0.56	1.04	0.84	0.7	0.89	1.08
4	-1.32	-0.74	-1.02	-0.56	0.99	0.84	0.61	0.89	1.08
5	-1.24	-0.73	-0.76	-0.55	0.92	0.82	0.54	0.88	1.08
6	-1.12	-0.7	-0.48	-0.54	0.83	0.79	0.5	0.86	1.08

5.1.3 Analysis of four by four linkage system with springs

In the first case, we add four springs in our four by four linkage system (see Fig 5.8).

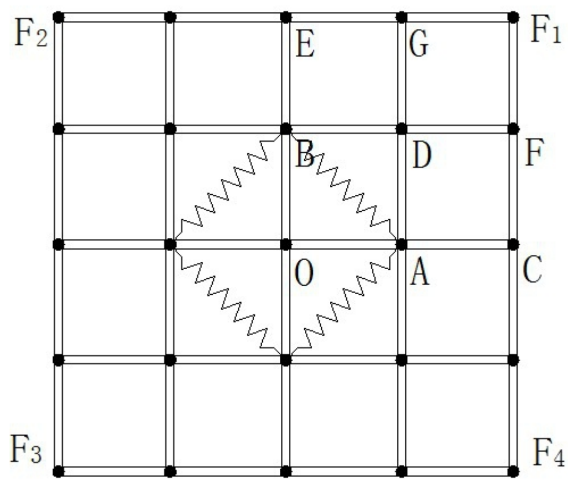

Fig 5.8 Four by four linkage with four springs

The stiffness is k_1, and the new potential energy will be

$$V = 2mg(z_1 + 4z_2 + 3z_3 + 4z_4 + 6z_5) + 2k_1(\sqrt{2}d - l_0)^2 \tag{6}$$

and l_0 is the undeformed length of all springs. By taking partial derivative of (6) to z_2, z_3, z_4, we can have the following equations:

$$V_i: \quad \begin{cases} \dfrac{\partial V}{\partial z_2} = 2mg(4 + 6\dfrac{\partial z_5}{\partial z_2}) + 4\sqrt{2}k_1(\sqrt{2}d - l_0)\dfrac{\partial d}{\partial z_2} = 0 \\ \dfrac{\partial V}{\partial z_3} = 2mg(3 + 6\dfrac{\partial z_5}{\partial z_3}) + 4\sqrt{2}k_1(\sqrt{2}d - l_0)\dfrac{\partial d}{\partial z_3} = 0 \\ \dfrac{\partial V}{\partial z_4} = 2mg(4 + 6\dfrac{\partial z_5}{\partial z_4}) + 4\sqrt{2}k_1(\sqrt{2}d - l_0)\dfrac{\partial d}{\partial z_4} = 0 \end{cases} \quad i = 1,2,3 \tag{7}$$

All other equations are the same as the linkage without springs.

For the second case, we add eight springs in our four by four linkage system (see Fig 5.9).

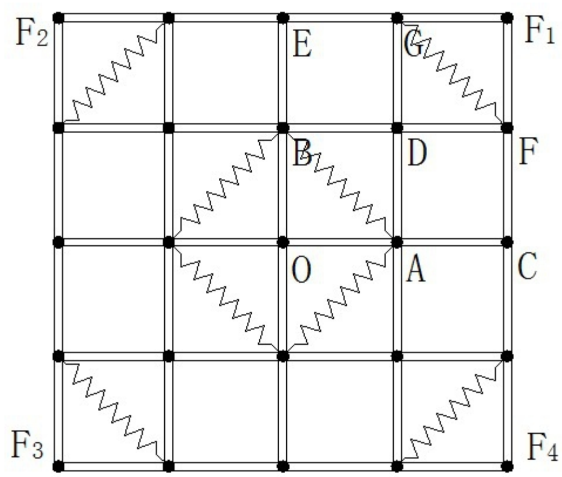

Fig 5.9 Four by four linkage with eight springs

The stiffness of the new springs is k_2, and the new potential energy will be

$$V = 2mg(z_1 + 4z_2 + 3z_3 + 4z_4 + 6z_5) + 2k_1(\sqrt{2}d - l_0)^2 + 2k_2(\sqrt{2}(2e - f) - u_0)^2 \tag{8}$$

and u_0 is the undeformed length of all new springs. By taking partial derivative of (8) to z_2, z_3, z_4, we can have the following equations:

$$\begin{cases} \frac{\partial V}{\partial z_2} = 2mg(4+6\frac{\partial z_5}{\partial z_2}) + 4\sqrt{2}k_1(\sqrt{2}d - l_0)\frac{\partial d}{\partial z_2} + 4\sqrt{2}k_2(\sqrt{2}(2e-f)-u_0)(2\frac{\partial e}{\partial z_2} - \frac{\partial f}{\partial z_2}) = 0 \\ \frac{\partial V}{\partial z_3} = 2mg(3+6\frac{\partial z_5}{\partial z_3}) + 4\sqrt{2}k_1(\sqrt{2}d - l_0)\frac{\partial d}{\partial z_3} + 4\sqrt{2}k_2(\sqrt{2}(2e-f)-u_0)(2\frac{\partial e}{\partial z_3} - \frac{\partial f}{\partial z_3}) = 0 \\ \frac{\partial V}{\partial z_4} = 2mg(4+6\frac{\partial z_5}{\partial z_4}) + 4\sqrt{2}k_1(\sqrt{2}d - l_0)\frac{\partial d}{\partial z_4} + 4\sqrt{2}k_2(\sqrt{2}(2e-f)-u_0)(2\frac{\partial e}{\partial z_4} - \frac{\partial f}{\partial z_4}) = 0 \end{cases}$$

Still, all other equations are the same as the four by four linkage without springs. By using homotopy continuation method, we can solve all those equations.

5.1.4 Example of analysis with four SMA springs

Now, we will give some examples of four by four linkage with four springs. The undeformed length of all springs is 1.6, and $z_1 = 0.5$. we will try different values of stiffness to see how the number of solutions change.

Table 5.2 Number of solutions with different stiffness

Stiffness	1	2	4	8	16	32
No. of solutions	2	2	8	8	8	8

From Table 5.2, we can see that as the stiffness changes from 2 to 4, the number of solutions will jump from 2 to 8, which means that the existence of springs can increase the number of solutions. When the values of stiffness passed 4, the number of solutions will stay on 8, which means that existence of springs can generate only limited number of solutions.

Next, we will pick the stiffness of 1 and 16 to design configuration memory linkage. For stiffness of 1, we have two solutions and we will pick the solution with the lower energy, which is also a stable configuration, as our first configuration. Now, we will use homotopy continuation method to trace that solution to stiffness 16. After doing that, we will have a new corresponding solution of stiffness 16. Table 5.3 are the two solutions and case 1 is for stiffness 1, while case 2 is for stiffness 16.

Table 5.3 Solutions for two case above

Case	z_2	z_3	z_4	z_5	b	c	d	e	f
1	-1.10	-0.73	-0.42	-0.56	0.84	0.83	0.60	0.88	1.08
2	-0.83	-0.79	-0.08	-0.58	0.79	0.89	0.93	0.91	1.08

Next, we can do the simulations for the two solutions.

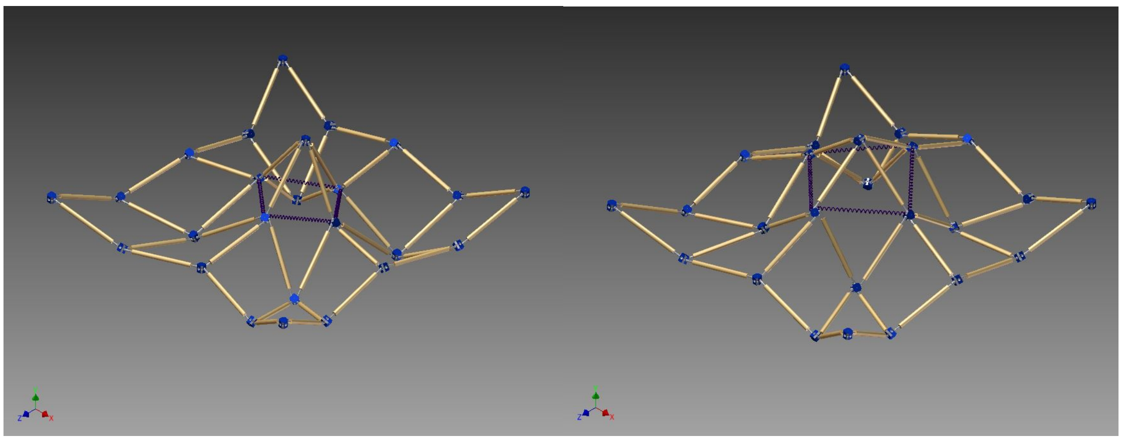

Fig 5.10 Solutions for the two cases above

In Fig 5.10, the first configuration is for the case of stiffness 1. The second is for the case of stiffness 16. When the stiffness change from 1 to 16, the first configuration will move to second configuration. When the stiffness goes back to 1, the configuration will also go back its initial state.

5.2 General case of square linkage systems

Now, we will consider the general case of square linkage systems. To make only one center point O, the linkage systems should have even number of rods along each side. So, suppose we have $2n \times 2n$ linkage systems and what will try to do is to determine the state in different height of the center point O. See Fig 5.11.

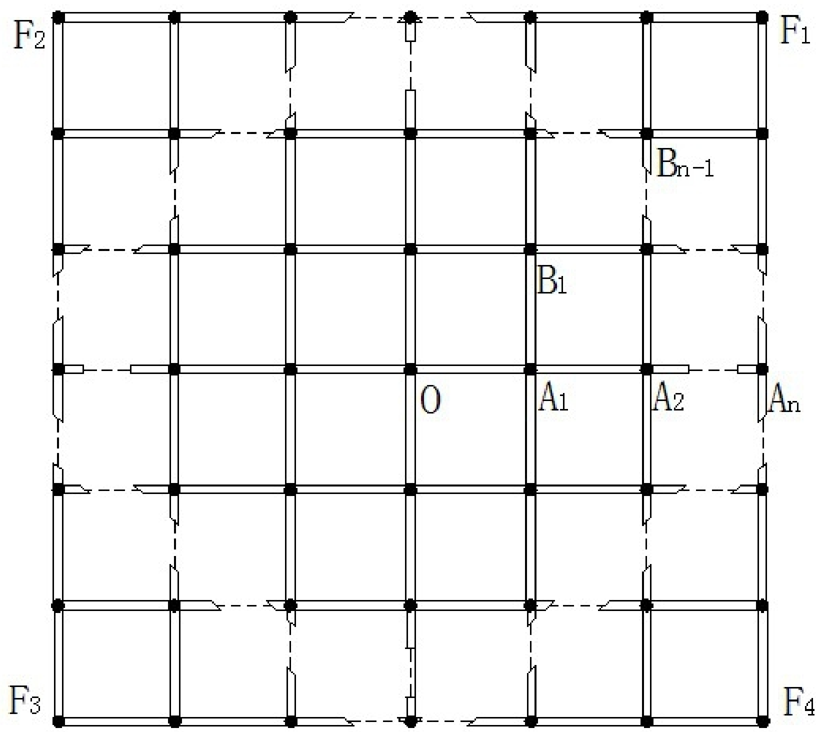

Fig 5.11 General Square Linkage System

Since this is a symmetric system, we only need to consider one triangular part (see Fig 5.12).

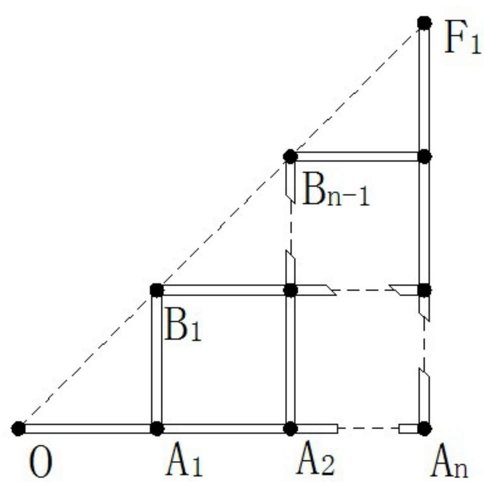

Fig 5.12 Triangular part

We have $0.5n^2+1.5n-1$ undefined points except point O and point F_1, which will give us $1.5n^2+4.5n-3$ unknowns. However, because of symmetric relationship, $A_1\cdots A_n, B_1\cdots B_{n-1}$ will remove $2n-1$ unknowns. As a result, the number of unknowns will actually be $1.5n^2+2.5n-2$. From the length of each rod, we will have n^2+n equations:

$$P_i: \quad \begin{cases} |OA_1|^2-L^2=0 \\ |A_1A_2|^2-L^2=0 \\ |A_2A_3|^2-L^2=0 \\ |A_3A_4|^2-L^2=0 \\ \vdots \\ |B_{n-1}F_1|^2-L^2=0 \end{cases} \quad i=1,2,3,\cdots,n^2+n \tag{9}$$

So, we need $0.5n^2+1.5n-2$ more equations. Next, we will discuss the degree of freedoms. Each rod has six degree of freedom, the maximum of degree of freedom will be $6n^2+6n$. There are $1.5n^2+0.5n+1$ knots including multiplicities and so $4.5n^2+1.5n+3$ freedoms will be removed. In addition, because of the symmetry of the linkage, point $A_1\cdots A_n, B_1\cdots B_{n-1}$ can only move in a fixed plane, and this will remove $2n-1$ freedoms. Also, we do not need to care about the rotations of each rod along the centric line, and this will remove n^2+n freedoms. So, the remaining degree of freedom is $0.5n^2+1.5n-2$, which is equal to the number of equations we need in the beginning. . We can choose $0.5n^2+1.5n-2$ independent unknowns as our degree of freedom. Here, we choose z coordinates of points which are counted from the bottom to the top and from left to the right in Fig 4.6, which is $z_2\cdots z_{m+1}$. Let $m=0.5n^2+1.5n-2$.

The potential energy of the linkage systems will be:

$$V=mg(\lambda_1z_1+\lambda_2z_2+\lambda_3z_3+\cdots+\lambda_{m+2}z_{m+2}) \tag{10}$$

where $\lambda_1 \cdots \lambda_{m+1}$ are coefficients by counting all the rods in the linkage system and $z_1 \cdots z_{m+1}$ are z coordinates of points which are counted from the bottom to the top and from left to the right except O and F_1 in Fig 4.6.

According to the principle of virtual work, we will have the following equations:

$$V_i: \quad \begin{cases} \dfrac{\partial V}{\partial z_2} = mg(\lambda_2 + \lambda_{m+2}\dfrac{\partial z_{m+2}}{\partial z_2}) = 0 \\ \dfrac{\partial V}{\partial z_3} = mg(\lambda_3 + \lambda_{m+2}\dfrac{\partial z_{m+2}}{\partial z_3}) = 0 \\ \qquad \vdots \\ \dfrac{\partial V}{\partial z_{m+1}} = mg(\lambda_{m+1} + \lambda_{m+2}\dfrac{\partial z_{m+2}}{\partial z_{m+1}}) = 0 \end{cases} \qquad i = 1,2,3 \cdots m \tag{11}$$

Because of the new unknowns of $\dfrac{\partial z_{m+2}}{\partial z_2} \cdots \dfrac{\partial z_{m+2}}{\partial z_{m+1}}$, we need more equations. By taking derivative of (9) to $z_2 \cdots z_{m+1}$, we will have $m(n^2+n)$ new equations and $m(n^2+n)$ new unknowns. In total, we have $(1.5n^2 + 2.5n - 2) + m(n^2+n)$ equations and the same number of unknowns. Each equation except in (11) will have the degree of 2. So the total degree will be $2^{(n^2+n)(m+1)}$. Even given a small value 3 to n, the total degree will be 2^{84}, which is very large. So, we can use user defined homotopy to decrease the number of paths to be traced(see [4] and [5].

For the case with springs, we can simply change the equations of potential energy parts. Sometimes we can get nonlinear parts such as root square of unknowns. In that case, we need to introduce a new value to substitute them into the equations.

Chapter 6 Future Work

Numerical homotopy continuation methods have been developed for over twenty years, but are not widely used in the engineering field. In the future, multi-linkage systems with shape memory springs can be developed by using homotopy methods. These can be used in the switches and valves. We can also use shape memory materials in the toy to make it have more movement without being provided with energy. The main advantage of shape memory alloy is that it does not need extra power to change it the state. The homotopy continuation methods are best for find all the isolated solutions of complicated systems. By combining this two merits, we can devise more economic and efficient mechanisms to satisfy our need.

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