Asset Pricing in Intraday Trading

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Abstract of the Dissertation

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The dissertation consists of three parts. In first part, a general equilibrium for asset pricing is proposed which incorporates asymmetric information as the key element determining security prices. The concepts of completeness, arbitrage, state price deflator and equivalent martingale measure are extended. It is shown in the model that in a so-called quasi-complete market, agents with differential information can reach an agreement on a universal equilibrium price. And as a consequence, information asymmetry can lead to mispricing as well. Second part investigates the market intrinsic time, such as volume clock and transaction clock. The normality of asset returns are shown to be recovered when time is measured by volume or transactions. A multivariate subordinated Brownian motion model is used to estimate dependency structure with market intrinsic time as subordinator. Third part considers a multivariate mean-reverting Lévy model on asset price and volatility together. The model assumes common jump factor among prices and volatilities. Empirical analysis on estimation and option pricing are conducted.

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$\mathcal{T}$	Time Horizon
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$	Probability Space Equipped with Filtration $\mathbb{F}$
I	The Set of Agents
$\mathbf{P}^{(i)}$	Individual Prior Belief of Agent i
$U^{(i)}$	Utility of Agent i
$\mathbb{F}^{(i)} = \{\mathcal{F}_t^{(i)}\}$	Endowed Information Filtration of Agent i
$e^{(i)}$	Endowment of Agent i
$\overline{\mathcal{F}}_t$	The Smallest Sigma-algebra containing all $\mathcal{F}_t^{(i)}$
$\underline{\mathcal{F}}_t$	The Intersection of all $\mathcal{F}_t^{(i)}$
e	Aggregated Endowment
J	The Set of Securities
δ	Dividend Process of Securities
$\mathcal{E}$	The Economy
S	Security Price
δ^θ	The Dividend Process Generated by Trading Strategy θ
c	Consumption Process
$\{\mathcal{G}_t\}$	The Information Filtration Generated by S
L	The Space of all $\mathbb{F}$-adapted Processes
$L^{(i),S}$	The Space of Processes Adapted to the Filtration $\{\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\}$
EMM	Equivalent Martingale Measure
Quasi-EMM	Quasi Equivalent Martingale Measure

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Chapter 1

Multi-period Equilibrium Model with Asymmetric Information

1.1 Introduction and Motivation

General equilibrium theory deals with an economy consisting of multiple agents in a market endowed with initial resources and willing to exchange commodities with others. It considers the behavior of the economy as a closed and inter-related system. In a general equilibrium perspective, each agent in the market optimizes his/her behavior to achieve maximum consumption utility. Agents' optimal behavior represents the behavior of the economy. Equilibrium prices are determined endogenously. The existence of such an equilibrium is based on the assumption of perfect competition among agents. In other words, the model assumes all individuals are price-takers, i.e. that they have zero price impact. This type of model is often called Walrasian equilibrium, from the Walras (1898) theory of markets. The modern version of general equilibrium theory was formalized by Arrow and Debreu (1954, 1956) and McKenzie (1954). In the Arrow-Debreu model, the market is static and deterministic. Radner (1968, 1972, 1979) and Jordan and Rander (1982) explores the competitive equilibrium in the case of uncertainty. In a Radner-type economy, different market agents are allowed to have different information. Agents in such an economy maximize their expected utility with respect to the their own information. The work opens the possibility of applying general equilibrium to financial markets to explain the prices of financial assets. Breeden (1979) develops the Consumption-Based Capital Asset Pricing Model (CCAPM) that connects continuous-time general equilibrium to characteristics of returns on securities. Cox, Ingersoll and Ross (1985) also examine the price behavior in a general equilibrium. Duffie and Zame (1989) further study and extend the approach.

Demarzo and Skiadas (1998) further investigate asset pricing in economies with asymmetric information, although not in a dynamic setting. The concept of quasi-completeness is introduced to describe the feasibility of consumption conditioned on the agents individual information. Also, in Yannelis (1991) and Glycopantis and Yannelis (2006), the core of an economy and its related concepts are introduced and discussed to study information asymmetry. In Lengwiler (2009), Heer and Maussner (2009), Black and Glaser (2010), Starr (2011), and Ludvigson (2011), the modern general equilibrium pricing theory are elaborated and reviewed. Recently, Biais, Bossaerts and Spatt (2004, 2010) develop a two-date equilibrium with differential information and demonstrates that information is partially revealed by the price. Fama and French (2007) investigate the effects of disagreement and preference differences on asset prices in a CAPM setting. Ostrovsky (2012) uses an extended sequential auction model based on market scoring rules to study information aggregation with partially informed trades. Iyer, Johari and Moallemi (2014) develop a market with heterogeneous traders and a market maker to give a condition for information aggregation. Equilibrium under information asymmetry has also been studied in many other papers, including Bernardo, Antonio and Judd (2000), Cao and Ou-yang (2009), Breon-Drish (2010), Gao, and Song and Wang (2013), among others. Meanwhile there also exist many literature investigate in the extensions of CAPM theory, including Campbell and Cochrane (1995), Dionne (2011), Breeden and Litzenberger (2015), Barberis and Greenwood (2015), among others.

Our work is in line with the formalization of the Consumption-based Capital Pricing Models in Duffie (2010) and the Demarzo and Skiadas (1998) model with differential information among agents. We model asset price formation in a multi-period general equilibrium framework. The concept of equivalent martingale measure under quasi-completeness as in Demarzo and Skiadas (1998) and related concepts will be rephrased in a CCAPM context. In our framework, the state-price can be extended to a broader sense, so that it stays universal while difference agent views it asymmetrically. It provides a convenient tool for asset valuation. Also, the similar result that agents will come to asymmetric betas in their beta form asset pricing, as the CAPM in Demarzo and Skiadas (1998). The difference is that our beta is based on consumption, thus not requiring the strict mean-variance utility.

1.1.1 Efficiency

It is nature to ask the question that what properties the equilibria has if it exists. Efficiency is one of the important aspect. How efficient is the market if the market is on a equilibria? To answer this question, I define efficiency in two different concepts: (i) Informational efficiency, and (ii) allocative efficiency.

Informational Efficiency addresses the information revealed by price. The more information price is revealing, the more informationally efficient market is. Efficient Market Hypothesis asserts that price reflects public/all information. In our equilibrium, the information revealed by price is represented by the filtration generated by equilibrium price. The informational efficiency is about how large the filtration is.

Allocative Efficiency addresses how optimal the equilibrium allocation is. An allocation is (Pareto) efficient if there does not exist an redistribution which can benefit at least one without making any other worse off.

1.1.2 Factoring Private Information into Price

From Capital Asset Pricing Model (CAPM) to Fama-French three factor model, then to Arbitrage Price Theory (APT), the bridge is built by applying cross-sectional test to show the factors are actually risk factors, in supporting the Market Efficient Hypothesis. Although CAPM has already been explained in general equilibrium with symmetric information, fator model is stil lacking an explanation on how the factors are incorporated into price.

We assert that the factors is coming from private information. Intuitively, consider that if a equilibrium price is more informative than dividend, it is also revealing some private information. Suppose the additional private information is generated by some signal process, then the equilibrium price must incorporate the signal as risk factor.

Our goal is to develop a general equilibrium pricing model in which market efficiency (both informational and allocative) can be investigated, and risk factors can be priced into equilibrium price. We will start with basic concept and model structure. And then I will provide the relation between the key concepts: arbitrage, state-price deflator, martingale measure. From there, I will represent security prices. Finally I will discuss the unfinished work and potential extension.

1.2 The Model

In this section, We formulate a general equilibrium model with information asymmetry in discrete time. In our setting, there are finite number of agents in economy. As in other general equilibrium models, the agents are assumed to be price takers. Agents make their consumption plans based on not only their own private information but also the information generated by the price process. An agent prices a consumption process by her own pricing function, which is obtained from maximizing her utility. That means, for some consumption process, the prices as seen by different agents are allowed to be different. However since agents are price takers and the market must clear, it is shown that there exists a so-called quasi pricing kernel such that the pricing function of each agent is the optimal projection of the quasi pricing kernel onto her own information filtration. Equivalently, there exists a probability measure, a so-called quasi equivalent martingale measure, under which a security price is viewed as a martingale for each agent (i.e., conditioned on individual information filtration).

1.2.1 Uncertainty

There are $T+1$ dates: $0, 1, ..., T$. Denote by $\mathcal{T} = \{0, 1, ..., T\}$ the time horizon. Then uncertainty is modeled by the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T\}$. $\mathcal{F}_t$ denotes the information up to time t and $\mathcal{F}_T = \mathcal{F}$. Ω denotes the state space. $\mathbf{P}$ is the physical probability measure. Let L be the space of all $\mathbb{F}$-adapted processes.

1.2.2 Economy

In our economy, there are n agents. Denote by $I = \{1, ..., n\}$ the set of agents. Each agent $i \in I$ is characterized by individual belief (represented by probability measure $\mathbf{P}^{(i)}$), utility function $U^{(i)}$, endowment $e^{(i)}$, and information flow, represented by the filtration $\mathbb{F}^{(i)} = \{\mathcal{F}_t^{(i)}\}_{t \in \mathcal{T}}$.

Each individual filtration satisfies $\mathcal{F}_t^{(i)} \subseteq \mathcal{F}_t$. It means each agent can access some subset of the total information, which allows that information available to agents is not symmetric. Denote the common information by $\underline{\mathcal{F}}_t = \bigcap_{i \in I} \mathcal{F}_t^{(i)}$, and the largest information that can be accessed by all agents as whole by $\overline{\mathcal{F}}_t = \bigvee_{i \in I} \mathcal{F}_t^{(i)}$ which is the smallest σ-algebra contains all $\mathcal{F}_t^{(i)}$. Each agent knows her endowment process very well. In another word, $e^{(i)}$ is $\mathbb{F}^{(i)}$-adapted for each $i \in I$. Denote the aggregated endowment by $e(t) = \sum_{i \in I} e^{(i)}(t)$.

There exist m assets; denote by $J = \{1, ..., m\}$ the set of assets. Each asset $j \in J$ is associated with its dividend process $\delta_j = (\delta_j(t))$. $\delta = (\delta_1(t), \delta_2(t), ..., \delta_m(t))$ denotes dividend process, which is adapted to $\mathbb{F}$. Assume that all agents can observe dividends, that is, $\sigma(\delta_u : u \leq t) \subseteq \underline{\mathcal{F}}_t$ for all $t \in \mathcal{T}$.

Definition 1 (Economy)**.** The economy is defined as a collection

$$\mathcal{E} = \{(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P}), (\mathbf{P}^{(i)}, U^{(i)}, \mathbb{F}^{(i)}, e^{(i)})_{i \in I}, \delta\}.$$

For simplicity, we also assume all agents share the same (physical) probability measure, that is, $\mathbf{P}^{(i)} = \mathbf{P}$ for all $i \in I$.

1.2.3 Security Prices

The security price at time t is denoted by $S(t) = (S_1(t), ..., S_m(t))$. $(S(t))$ is adapted to $\mathbb{F}$. Security price is public information, thus given price process S, each agent i observes the information generated by S as well as private information $\mathcal{F}^{(i)}$. Denote by $\mathcal{G}_t = \sigma(S_u : u \leq t)$ for $t \in \mathcal{T}$ the filtration generated by prices. Assume that all agents are price-takers, thus the securities' prices S is the exact price at which agents trade.

1.2.4 Trading Strategies and Consumption

Agents in the economy will choose their trading strategies to optimize their satisfaction (which will be represented by utility later). However, the trading strategies each agent can access are restricted by the information she can access. More information will give more choice of trading strategies. This means that an agent's trading strategies are adapted to the information available to her.

Definition 2 (Feasible trading strategies)**.** A trading strategy $\theta^{(i)} = (\theta^{(i)}(t))_{t \in \mathcal{T}}$ is feasible for agent i given price S if $\theta^{(i)}(t)$ is $\mathcal{G}_t \vee \mathcal{F}_t^{(i)}$-measurable, and the consumption process $c(t;\theta,S) \in \mathbb{R}_{++}$ which is given by the budget constraint:

$$\theta^{(i)}(t) \cdot S(t) = \sum_{\tau=0}^{t-1} \theta^{(i)}(\tau) \cdot \big(S(\tau+1) - S(\tau) + \delta(\tau+1)\big) \\ + \sum_{\tau=0}^{t} e^{(i)}(t) - \sum_{\tau=0}^{t} c(t;\theta,S) \ a.s.$$

or equivalently,

$$c(t;\theta,S) = e(t) + \theta(t-1) \cdot \big(S(t) + \delta(t)\big) - \theta(t) \cdot S(t), \text{ for } t = 1, ..., T, \text{ and}$$
$$c(0;\theta,S) = e(0) - \theta(0) \cdot S(0).$$

The budget constraint makes sure that the consumption cannot exceed the sum of endowment, dividends and trading profit. Additionally, for convenience, we can define a strategy-generated dividend process as follows.

Definition 3. Let $\theta(-1) = 0$, then define the dividend process generated by trading strategy θ as

$$\delta^{\theta}(t) = \theta(t-1) \cdot \big(S(t) + \delta(t)\big) - \theta(t) \cdot S(t). \ t \in \mathcal{T}$$

The dividend generated by trading strategy θ can be interpreted as the trading profit generated from price changes and dividends. Then the feasible consumption set for agent i and a given price process S can be written as

$$X^{(i),S} = \left\{ e^{(i)} + \delta^{\theta} \in L_+ : \theta \in L^{(i),S} \right\}.$$

1.3 Arbitrage, State Price Deflator, Martingale

Definition 4 (Arbitrage)**.** We call a trading strategy an *arbitrage* for given (δ, S) if $\delta^{\theta} > 0$.

Recall L denotes the space of all $\mathbb{F}$-adapted processes and similarly denote by $L^{(i),S}$ the space of process adapted to the filtration generated by $\{\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\}_{t\in\mathcal{T}}$. Then $M^{(i),S} = \{\delta^{\theta} : \theta \in L^{(i),S}\}$ and $M = \{\delta^{\theta} : \theta \in L\}$ are linear subspaces of the space of L.

Proposition 1. *There is no arbitrage strategy in L' if and only if there is a strictly increasing linear function $F : L \to \mathbb{R}$ such that $F(\delta^\theta) = 0$ for any $\theta \in L'$. Here L' can be L or $L^{(i),S}$ for some $i \in I$.*

Corollary 1. *If there is no arbitrage in $\overline{L}$ the space of all processes adapted to the filtration generated by $\{\overline{\mathcal{F}}_t \vee \mathcal{G}_t\}_{t\in\mathcal{T}}$ for a given S, then there is no arbitrage in $L^{(i),S}$ for all $i \in I$.*

Remark. When (δ, S) admits arbitrage in the trading strategy space L, it is not necessary that there exists an arbitrage in a smaller space $L' \subset L$. That means an agent could not be able to find arbitrage due to the lack of information. See the following example.

Example 1 (Arbitrage in case of information asymmetry)**.** Let time $\mathcal{T} = 0, 1, 2, 3$, and let there be one risky asset which pays dividend $\delta(t), t = 0, 1, 2,$ where $\delta(t)$ satisfies

$$\delta(t) = f(t) + \epsilon_D(t),$$

where $f(t)$ is the fundamental value satisfying

$$f(t) = f(t-1) + \epsilon_f(t), t = 1, 2, 3 \text{ and } f(0) = 1,$$

and $\epsilon_D(t)$ is pure noise with $\epsilon_D(0) = 0$, and $\epsilon_D(t), t = 1, 2, 3$ are i.i.d Bernoulli distributed

$$\epsilon_D(t) = \begin{cases} 0, & \text{with probability } p \\ 1, & \text{with probability } q \end{cases}$$

and $\epsilon_f(t)$ is the pure noise with $\epsilon_f(0) = 0$ and $\epsilon_f(t), t = 1, 2, 3$ are i.i.d Bernoulli distributed

$$\epsilon_f(t) = \begin{cases} 0, & \text{with probability } \mu \\ 1, & \text{with probability } \nu \end{cases}$$

Then this example can be illustrated by the tree with $(\delta(t), f(t), \epsilon_D(t))$ as nodes, as shown in Figure 1.1.

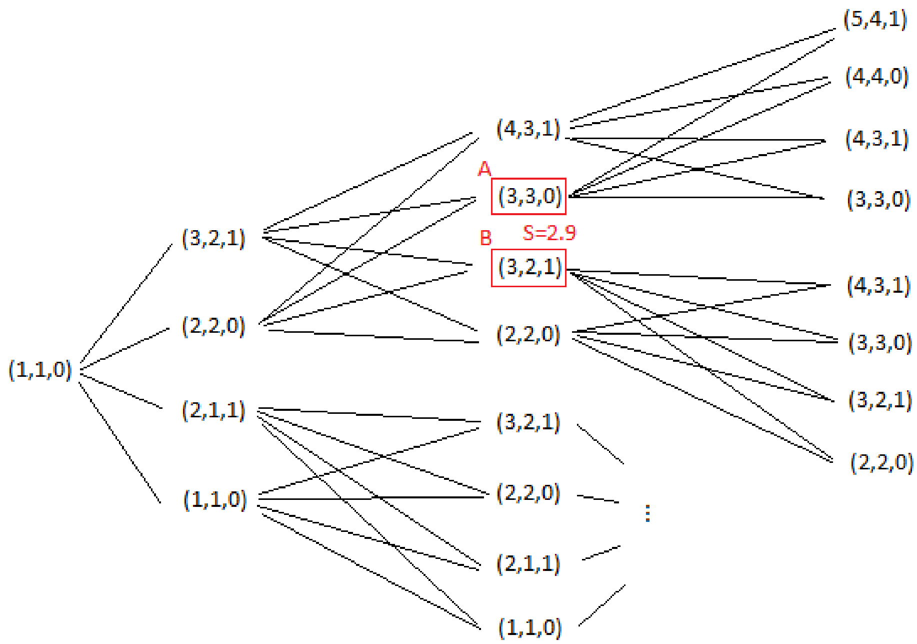


Figure 1.1: Example. Arbitrage opportunity in case of information asymmetry.

Now suppose there two agents. One is uninformed and can only observe dividends, and the other is an insider who knows the true fundamental. Thus the uninformed agent cannot distinguish the two nodes A,B as shown in the red boxes, and the insider can distinguish between the two nodes. Let price given by probability measure $p = 0.2, q = 0.8, \mu = 0.5, \nu = 0.5$, then the prices for the non-informed agent for state A and B are the same, $S = 2.9$. However, the insider can easily obtain an arbitrage opportunity by making a strategy as (i) Buy if state A happens, (ii) Sell if state B happens, (iii) No trade if none of them happens.

Classic Case

Lemma 1. *For each linear function $F : L \to \mathbb{R}$, there exists a unique π in L such that for any $x \in L$*

$$F(x) = \mathbb{E}^{\mathbf{P}} \left[\sum_{t=0}^{T} \pi(t) x(t) \right].$$

π is called the Riesz representation of F. If F is strictly increasing, then π is strictly positive.

Proof. Apply Riesz representation theorem to the Hilbert space L, where inner product is defined as $\langle x, y \rangle = \mathbb{E}^{\mathbf{P}} \left[\sum_{t=0}^{T} x(t) y(t) \right], \ x, y \in L.$ $\square$

We call a strictly positive process π, which is adapted to $\mathbb{F}$, a state-price deflator if, for all $t \in \mathcal{T}$,

$$S(t) = \frac{1}{\pi(t)} \mathbb{E}^{\mathbf{P}} \left[\sum_{j=t+1}^{T} \pi(j) \delta(j) \ \middle| \mathcal{F}_t \right].$$

Proposition 2. *A strict positive process $\pi \in L$ is a state-price deflator if and only if, for any trading strategy $\theta \in L$,*

$$\theta(t) \cdot S(t) = \frac{1}{\pi(t)} \mathbb{E}^{\mathbf{P}} \left[\sum_{j=t+1}^{T} \pi(j) \delta^{\theta}(j) \ \middle| \mathcal{F}_t \right], \quad t < T.$$

Suppose there exists a risk-free short-rate process $r^f = (r^f(t))_{t \in \mathcal{T}}$, define the discount factor

$$R(s, t) = \prod_{j=s}^{t-1} (1 + r^f(j)).$$

Definition 5 (Equivalent Martingale Measure)**.** We call a probability measure $\mathbf{Q}$ an equivalent martingale measure (EMM) if, $\mathbf{Q}$ is equivalent to $\mathbf{P}$ and

$$S(t) = \mathbb{E}^{\mathbf{Q}} \left[\sum_{j=t+1}^{T} \frac{1}{R(t, j)} \delta(j) \ \middle| \mathcal{F}_t \right], \quad t < T.$$

Theorem 1. *π is a state-price deflator if and only if there exists an equivalent martingale measure $\mathbf{Q}$ with the density process ξ such that*

$$\xi(t) = \frac{R(0,t)\pi(t)}{\pi(0)},$$

where the density process ξ is defined by

$$\xi(t) = \mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} \;|\mathcal{F}_t\right].$$

Proof. See Duffie (2001) page 30. $\square$

Case of Asymmetric Information

In the classic case of symmetric information, a security's price equals the expectation of aggregated future discounted dividend flow under the equivalent martingale measure, and also equals the expectation of aggregated discounted future dividend flows under the physical probability measure. To extend the analysis to asymmetric information, we allow each agent to have her own state-price deflator, which is the optimal projection of the universal state-price deflator to her own information.

Let us first investigate some process $X \in L$ for a smaller filtration $\mathbb{H} = \{\mathcal{H}_t\}_{t\in\mathcal{T}}$, where $\mathcal{H}_t \subseteq \mathcal{F}_t$ for all t. Denote by $X^{\mathbb{H}}$ the optimal projection of X onto $\mathbb{H}$, that is,

$$X^{\mathbb{H}}(t) = \mathbb{E}^{\mathbf{P}}\left[X(t) \;|\mathcal{H}_t\right].$$

Some simple facts:

Fact 1. $X^{\mathbb{H}} = X \iff X$ is adapted to $\mathbb{H}$.

Fact 2. $X^{\mathbb{H}}$ is $\mathbb{H}$-martingale if X is $\mathbb{F}$-martingale under same probability measure.

Fact 3. If X is a $\mathbb{F}$-martingale, and adapted to $\mathbb{H}$, then X is a $\mathbb{H}$-martingale.

For an equivalent probability measure $\mathbf{Q}$, we define its density process $\xi^{\mathbb{H}}$ on some smaller filtration $\mathbb{H}$ as the optimal projection of ξ onto $\mathbb{H}$, that is

$$\xi^{\mathbb{H}}(t) = \mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} \,\middle|\mathcal{H}_t\right] = \mathbb{E}^{\mathbf{P}}\left[\xi(t) \,\middle|\mathcal{H}_t\right].$$

Claim. If $\mathbf{Q}$ is an equivalent martingale measure, and π is the corresponding state-price deflator as in Theorem 1, and $R(s,t)$ is $\mathcal{H}_t$-measurable for all $0 \le s < t \le T$, then

$$\xi^{\mathbb{H}}(t) = \frac{R(0,t)\pi^{\mathbb{H}}(t)}{\pi^{\mathbb{H}}(0)}.$$

If there exists no arbitrage in L, and let π be the state-price deflator, then the price viewed by the partially informed agent at time t can be represented by

$$S(t) = \mathbb{E}[S(t) \,|\mathcal{H}_t] = \mathbb{E}^{\mathbf{P}}\left[\frac{1}{\pi(t)}\sum_{j=t+1}^{T}\pi(j)\delta(j) \,\middle|\mathcal{H}_t\right],$$

where $\mathcal{H}_t$ is the information set can be accessed at time t, and $S(t)$ is $\mathcal{H}_t$-measurable, which means security prices are public information. Moreover, for any strategy in $L^{\mathbb{H}}$, the space of $\mathbb{H}$-adapted processes,

$$\theta(t) \cdot S(t) = \mathbb{E}^{\mathbf{P}}\left[\frac{1}{\pi(t)}\sum_{j=t+1}^{T}\pi(j)\delta^{\theta}(j) \,\middle|\mathcal{H}_t\right].$$

Now, bearing in mind that all agents are price takers, we can start to construct the martingale measure and state-price deflator for asymmetric information. Assume that the risk-free rate is known to all agents.

Assumption. $R(s,t), s \le t \le T$, is $\mathcal{F}_t^{(i)}$-measurable for all $i \in I$.

Definition 6 (Quasi Equivalent Martingale Measure)**.** We call an equivalent probability measure $\mathbf{Q}$ a quasi equivalent martingale measure (Quasi-EMM) if, for all $i \in I$,

$$S(t) = \mathbb{E}^{\mathbf{Q}}\left[\sum_{j=t+1}^{T}\frac{1}{R(t,j)}\delta(j) \,\middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right], \quad t < T.$$

Denote the density processes of $\mathbf{Q}$ respect to $\mathbf{P}$ for the filtrations $\mathcal{F}^{(i)} \vee \mathcal{G}_t,\ i \in I$ by

$$\xi^{(i)}(t) = \mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}\ \big|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right].$$

Under a Quasi-EMM, a security's price equals the expectation of discounted future dividends conditioned on individual information for every agent. All agents in our economy agree on the same (fair) prices of securities, which are exactly consistent with the assumption that all agents are price takers. Correspondingly, we can define a quasi state-price deflator as follows.

Definition 7. (Quasi State-price Deflator) A strictly positive process π is called a quasi state-price deflator if, for all $i \in I$,

$$S(t) = \frac{1}{\pi^{(i)}(t)}\mathbb{E}^{\mathbf{P}}\left[\sum_{j=t+1}^{T} \pi^{(i)}(j)\delta(j)\ \big|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right],$$

where $\pi^{(i)}$ is the optimal projection of π onto $\{\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\}_{t\in\mathcal{T}}$.

Proposition 3. *If π is a quasi state-price deflator, then for any trading strategy $\theta^{(i)} \in L^{(i),S},\ i \in I$,*

$$\theta^{(i)}(t) \cdot S(t) = \frac{1}{\pi^{(i)}(t)}\mathbb{E}^{\mathbf{P}}\left[\sum_{j=t+1}^{T} \pi^{(i)}(j)\delta^{\theta^{(i)}}(j)\ \big|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right], \quad t < T.$$

The following theorem shows the relationship between a quasi-EMM and a quasi state-price deflator, which is similar to the classic case of symmetric information.

Theorem 2. *π is a quasi state-price deflator if and only if there exists an quasi equivalent martingale measure $\mathbf{Q}$ with the density process ξ such that*

$$\xi^{(i)}(t) = \frac{R(0,t)\pi^{(i)}(t)}{\pi^{(i)}(0)}, \quad \textit{for all } i \in I,$$

where the density process ξ is defined by

$$\xi^{(i)}(t) = \mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} \;\middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right], i \in I.$$

The following proposition gives a strong condition for the existence of a quasi state-price deflator, which is actually for the case of symmetric information.

Proposition 4. *If there exists a quasi state-price deflator, then there is no arbitrage in $L^{(i),S}$ for all $i \in I$. If there is no arbitrage in $\overline{L}$, then there exists a quasi state-price deflator.*

Pricing Consumption Process

The use of quasi state-price deflator π is to price any consumption process, that is, the price of a consumption process c in L_+ is given by $\mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi(t)\right]$. Thus π will be also called a pricing kernel.

If $c \in X^{(i),S}$ for some i, then the price for this particular agent i is given by

$$\mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi(t)\right] = \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi^{(i)}(t)\right].$$

But for some other agent j in whose information c is not accessible, then the price will be

$$\mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi^{(j)}(t)\right],$$

which does not necessarily coincide with $\Pi(c)$. The difference between them comes from asymmetry of information, thus for agent j to access this consumption c, she will have to pay a fee to obtain more information. This, fee, which we call an information fee η, should be

$$\eta = \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi(t)\right] - \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi^{(i)}(t)\right] > 0.$$

This argument can be extended to any consumption $c \in L_+$. The price of any consumption $c \in L_+$ for an agent i is

$$\mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi^{(i)}(t)\right].$$

And the information fee η is

$$\eta = \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi(t)\right] - \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} c(t)\pi^{(i)}(t)\right] \geq 0.$$

Thus, $\eta = 0$ when $c \in X^{(i),S}$.

1.4 Individual Agent Optimality

The objective of each agent is to maximize her individual utility by choosing the optimal feasible trading strategy. Then the optimization problem for agent i can be written as

$$\max_{c \in X^{(i),S}} U^{(i)}(c). \tag{1.1}$$

The following claims show the relation between arbitrage and individual optimization problem.

Proposition 5. *The optimization problem (1.1) for agent i and given S has a solution $\Rightarrow$ there exists no arbitrage in $L^{(i),S}$.*

Corollary 2. *There exists no arbitrage in $L^{(i),S}$ and $U^{(i)}$ is continuous $\Rightarrow$ Optimization problem for agent i given S has a solution.*

First-Order Condition. If the individual optimization (1.1) has a strictly positive solution c^*, and $U^{(i)}$ is continuously differentiable at c^*, then

$$\nabla U^{(i)}(c^*; \delta^\theta) = 0, \quad \forall \theta \in L^{(i),S},$$

where $\nabla U(x; y)$ denotes the $\nabla U(x)$ at y, i.e.

$$\nabla U(x; y) \equiv \lim_{\alpha \to 0} \frac{U^{(i)}(x + \alpha y) - U^{(i)}(x)}{\alpha}.$$

Note that $\nabla U^{(i)}(c^*;\cdot) : L \to \mathbb{R}$ is a linear function. The following proposition shows that this linear function gives the pricing kernel for the individual agent.

Proposition 6. *Suppose the individual optimization problem (1.1) has a strict positive solution c^* and U has a strictly positive continuous derivatives at c^*. Then the Riesz representation π of $\nabla U^{(i)}(c^*;\cdot)$ satisfies that*

$$S(t) = \frac{1}{\pi^{(i)}(t)} \mathbb{E}^{\mathbf{P}} \left[\sum_{j=t+1}^{T} \pi^{(i)}(j)\delta(j) \;\middle|\; \mathcal{F}_t^{(i)} \vee \mathcal{G}_t \right].$$

If we restrict utility satisfying additive form, then we can have the following. Suppose $U^{(i)}$, for each $i \in I$, has the additive form:

$$U^{(i)}(c) = \mathbb{E}^{\mathbf{P}} \left[\sum_{t=0}^{T} u_t^{(i)}(c(t)) \right].$$

Then for any $t \leq \tau$,

$$S(t) = \frac{1}{u_t'(c^*(t))} \mathbb{E}^{\mathbf{P}} \left[S(\tau)u_\tau'(c^*(\tau)) + \sum_{j=t+1}^{\tau} \delta(j)u_j'(c^*(j)) \;\middle|\; \mathcal{F}^{(i)} \vee \mathcal{G}_t \right].$$

1.5 Equilibrium Asset Pricing

Definition 8. A security-spot market equilibrium (SSE) is a collection $\left\{(\theta^{(i)})_{i\in I}, S\right\}$, such that, for each i, $\theta^{(i)}$ solves individual optimization

$$\max_{c \in X^{(i),S}} U^{(i)}(c), \tag{1.2}$$

under market clearing condition

$$\sum_{i \in I} \theta^{(i)} = 0. \tag{1.3}$$

If a SSE equilibrium exists, that means the equilibrium consumption

$$c^* = (c^{(1)*}, c^{(2)*}, ..., c^{(n)*})$$

solves all individual optimization problems. From last section, we have the result that there is no arbitrage opportunity for each individual agent. Also, each $\nabla U^{(i)}(c^{(i)*}; \cdot)$ gives this agent i a pricing kernel $\pi^{(i)}$, which is adapted to individual information filtration, up to a positive multiplier. In order to construct a universal pricing kernel π, such that $\pi^{(i)}$ equals to the optimal project of π on individual information. If this universal pricing kernel π exists, then π is a quasi state-price deflator, and $\Pi(c) \leq \Pi(e)$ implies $\Pi^{(i)}(c) \leq \Pi^{(i)}(e^{(i)})$ for all $c \in L_+^{(i),S}$, where $\Pi(\cdot) = \langle \pi, \cdot \rangle$, $\Pi^{(i)}(\cdot) = \langle \pi^{(i)}, \cdot \rangle$.

Proposition 7. *Suppose that there exist an equilibrium satisfying (1.2)(1.3), then* $(\pi^{(1)}, ..., \pi^{(n)})$ *obtained from Proposition 6 admits a quasi state-price deflator.*

Definition 9. We say the market is quasi-complete if for each $i \in I$,

$$L^{(i),S} = \left\{ (\theta(t) \cdot \delta(t)) : \theta \in L^{(i),S} \right\}.$$

When the market is quasi-complete, the existence of the universal π will reduce the individual optimization to the following form:

$$\max_{c \in L_+^{(i),S}} U^{(i)}(c) \quad \text{subject to } \Pi(c) \leq \Pi(e^{(i)}). \tag{1.4}$$

Since $U^{(i)}$ is strictly increasing, there is a Lagrange multiplier $\lambda^{(i)}$ such that the optimization above is equivalent to

$$\max_{c \in L_+^{(i),S}} \lambda^{(i)} U^{(i)}(c) - \left(\Pi(c) - \Pi(e^{(i)}) \right).$$

Define the utility function $U_\lambda : L_+ \to \mathbb{R}$ by

$$U_\lambda(x) = \max_{c^{(i)} \in L_+^{(i),S}, i \in I} \sum_{i \in I} \lambda^{(i)} U^{(i)}(c^{(i)}) \quad \text{subject to } \sum_{i \in I} c^{(i)} \leq x.$$

Proposition 8. *Suppose that there exists an equilibrium satisfying (1.2)(1.3) for a quasi-complete market, then the equilibrium consumption solves*

$$\max_{c^{(i)} \in L_+^{(i),S}, i \in I} \sum_{i \in I} \lambda^{(i)} U^{(i)}(c^{(i)}) \quad \textit{subject to } \sum_{i \in I} c^{(i)} \leq \sum_{i \in I} e^{(i)}.$$

Corollary 3. *Moreover, if for each i that $U^{(i)}$ is of additive form*

$$U^{(i)}(c) = \mathbb{E}^{\mathbf{P}} \left[\sum_{t=0}^{T} u_t^{(i)}(c(t)) \right],$$

then U_λ is also of additive form

$$U_\lambda(c) = \mathbb{E}^{\mathbf{P}} \left[\sum_{t=0}^{T} u_{\lambda t}(c(t)) \right].$$

where

$$u_{\lambda t}(x) = \max_{c^{(i)} \in L_+^{(i),S}, i \in I} \sum_{i \in I} \lambda^{(i)} u^{(i)}(c^{(i)}) \quad \textit{subject to } \sum_{i \in I} c^{(i)} \leq x.$$

for any $t \leq \tau$,

$$S(t) = \frac{1}{u_t'(c^*(t))} \mathbb{E}^{\mathbf{P}} \left[S(\tau) u_\tau'(c^*(\tau)) + \sum_{j=t+1}^{\tau} \delta(j) u_j'(c^*(j)) \, \middle| \mathcal{F}^{(i)} \vee \mathcal{G}_t \right].$$

1.6 State-Price Beta Model

Denote by $\mathbb{E}_{i,t}$ the expectation conditioned on $\mathcal{F}_t^{(i)} \vee \mathcal{G}_t$, by $\mathrm{Var}_{i,t}$ the variance conditioned on $\mathcal{F}_t^{(i)} \vee \mathcal{G}_t$, and by $\mathrm{Cov}_{i,t}$ the covariance conditioned on $\mathcal{F}_t^{(i)} \vee \mathcal{G}_t$.

Now, let us define the capital returns generated by a trading strategy θ by

$$r^\theta(t) = \frac{\theta(t-1) \cdot (S(t) + \delta(t))}{\theta(t-1) \cdot S(t-1)}$$

Denote by $r^{(i),0}(t)$ the risk-free return for agent i. Thus $r^{(i),0}$ is the return of a strategy θ_0, such that,

$$\theta_0 \in L^{(i),S}, \text{ and } \operatorname{corr}^{\mathbf{P}}_{i,t-1}(r^{(i),0}, \pi(t)) = 0,$$

where $\operatorname{corr}_{i,t}$ is the correlation conditioned on $\mathcal{F}^{(i)}_t \vee \mathcal{G}_t$.

Each agent also believes in her own market portfolio. That is, for each agent i, she constructs the optimal market portfolio from $L^{(i),S}$ by maximizing the the correlation with π conditioned on their own information. That is , at time t, the market portfolio is given by

$$\max_{\theta \in L^{(i),S}} \operatorname{corr}^{\mathbf{P}}_{i,t-1}\left(r^{\theta}(t), \pi(t)\right).$$

Let $r^{(i),M}(t)$ denote the market return. Then the maximization implies

$$\pi(t) = r^{(i),M}(t) + \epsilon(t),$$

where $\operatorname{Cov}^{\mathbf{P}}_{i,t-1}\left(\epsilon(t), r^{\theta}(t)\right) = 0$ for all $\theta \in L^{(i),S}$. Note that

$$\operatorname{Cov}^{\mathbf{P}}_{i,t-1}\left(r^{\theta}(t), \pi(t)\right) = \pi^{(i)}(t-1)\left(1 - \frac{\mathbb{E}^{\mathbf{P}}_{i,t-1}\left[r^{\theta}(t)\right]}{\mathbb{E}^{\mathbf{P}}_{i,t-1}\left[r^{(i),0}(t)\right]}\right).$$

Then we can obtain the beta form of the CAPM:

$$\mathbb{E}^{\mathbf{P}}_{i,t-1}\left[r^{\theta}(t) - r^{(i),0}(t)\right] = \beta^{\theta}_{i,t-1}\mathbb{E}^{\mathbf{P}}_{i,t-1}\left[r^{(i),M}(t) - r^{(i),0}(t)\right]$$

where

$$\beta^{\theta}_{i,t-1} = \frac{\operatorname{Cov}^{\mathbf{P}}_{i,t-1}\left(r^{\theta}(t), r^{(i),M}(t)\right)}{\operatorname{Var}^{\mathbf{P}}_{i,t-1}\left(r^{(i),M}(t)\right)}.$$

1.7 Examples and Discussion

1.7.1 Examples

Example 2 (Quasi-EMM and Price in Case of No Private Information)**.** Let time $\mathcal{T} = 0, 1, 2$, and there is one risky asset which pays dividend $\delta(t)$ satisfying

$$\delta(t) = f(t) + \epsilon_D(t), t = 0, 1, 2,$$

where $f(t)$ is the fundamental value satisfying

$$f(t) = f(t-1) + \epsilon_f(t), t = 1, 2, \text{ and } f(0) = 1,$$

and $\epsilon_D(t)$ is noise with $\epsilon_D(0) = 0$, and $\epsilon_f(t)$ is noise with $\epsilon(0) = 0$. Assume that $\epsilon_D(1), \epsilon_D(2), \epsilon_f(1), \epsilon_f(2)$ are jointly normally distributed.

Now suppose no agents has private information, i.e., the only information agents can access are generated by dividend and price.

Let (Quasi-)EMM be the probability measure under which $\epsilon_D(1)$, $\epsilon_D(2)$, $\epsilon_f(1)$, $\epsilon_f(2)$ are independent standard normally distributed $N(0,1)$. Then the price given by the following is valid,

$$\begin{aligned} S(1) &= \mathbb{E}^{\mathbf{Q}}\left[\delta_2 \middle|\ \delta(1), S(1)\right] = \frac{1}{3}\delta(1) + \frac{2}{3} \\ S(0) &= \mathbb{E}^{\mathbf{Q}}[\delta(1) + \delta(2)] = 2 \end{aligned}$$

In this case, price reveals no additional information beyond dividends.

Example 3 (Quasi-EMM and Price in Case of Symmetric Private Information)**.** Within the framework as in Example 2, we additionally introduce one signal process $y(t)$ satisfying

$$y(t) = f(t) + \epsilon_y(t), t = 1, 2, \text{ and } y(0) = 1,$$

where $\epsilon_y(t)$ is noise. All agents can access signal y as their private information.

Case 1. Let Quasi-EMM be the probability measure under which $\epsilon_D(1)$, $\epsilon_D(2)$, $\epsilon_f(1)$, $\epsilon_f(2)$, $\epsilon_y(1)$ are independent standard normally distributed $N(0,1)$. Then the price given by the following is valid,

$$S(1) = \mathbb{E}^{\mathbf{Q}}\left[\delta_2 \middle|\ \delta(1), S(1), y(1)\right] = \frac{1}{3}\delta(1) + \frac{1}{3}y(1) + \frac{1}{3}$$

$$S(0) = \mathbb{E}^{\mathbf{Q}}[\delta(1) + \delta(2)] = 2$$

Case 2. Let Quasi-EMM be the probability measure under which $\epsilon_f(1)$, $\epsilon_D(1)$, $\epsilon_y(1)$, $\epsilon_f(2)$, $\epsilon_D(2)$ are jointly normally distributed with mean $\mathbf{0}$ and covariance matrix

$$\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & 0 & 0 \\ \frac{3}{4} & 1 & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the price given by the following is valid,

$$S(1) = \mathbb{E}^{\mathbf{Q}}\left[\delta_2 | \ \delta(1), S(1), y(1)\right] = \frac{1}{2}\delta(1) + \frac{1}{2}$$

$$S(0) = \mathbb{E}^{\mathbf{Q}}[\delta(1) + \delta(2)] = 2$$

In this example, we can see when there exists private information, either case can happen: (i) the signal is incorporated into price as factor, and (ii) price is very inefficient and reveals no private information.

Example 4 (Quasi-EMM and Price in Case of Asymmetric Private Information)**.** As in Example 3, we introduce not one but two signal processes $y_1(t)$ and $y_2(t)$ satisfying

$$y_1(t) = f(t) + \epsilon_1(t), t = 1, 2, \text{ and } y_1(0) = 1,$$

$$y_2(t) = f(t) + \epsilon_2(t), t = 1, 2, \text{ and } y_2(0) = 1,$$

where $\epsilon_y(t)$ is noise. There are two agents in economy, agent 1 can access signal y_1, and agent 2 can access signal y_2.

Let Quasi-EMM be the probability measure under which $\epsilon_D(1)$, $\epsilon_D(2)$, $\epsilon_f(1)$, $\epsilon_f(2)$, $\epsilon_1(1)$, $\epsilon_2(1)$ are independent standard normally distributed $N(0,1)$. Then the price given by the following is valid,

$$
\begin{aligned}
S(1) &= \mathbb{E}^{\mathbf{Q}}\left[\delta_2 \middle|\ \delta(1), S(1), y_1(1)\right] \\
&= \mathbb{E}^{\mathbf{Q}}\left[\delta_2 \middle|\ \delta(1), S(1), y_2(1)\right] \\
&= \frac{1}{4}\delta(1) + \frac{1}{4}y_1(1) + \frac{1}{4}y_2(1) + \frac{1}{4} \\
S(0) &= \mathbb{E}^{\mathbf{Q}}[\delta(1) + \delta(2)] = 2
\end{aligned}
$$

In this example, price reflects aggregated private information.

1.7.2 Mis-pricing

Suppose we have a pricing kernel π and that its optimal projections on individual information filtration are $\pi^{(i)}$'s. Consider a derivative written by agent w and it is traded at time 0. This derivative pays dividend δ^{θ}, where θ is a trading strategy in $L^{(w),S}$. That means the writer, agent w, can replicate this derivative and that the value of this derivative is 0. Agent w want to sell this derivative to other agents. The question is what would be the fair price of this derivative for some agent i.

If $\theta \in L^{(i),S}$, agent i can replicate this derivative easily, and also the consumption process $c = e^{(i)} + \delta^{\theta}$ cannot increase the utility of this agent i. Thus agent i will not pay any positive price for this derivative.

If $\theta \notin L^{(i),S}$, agent i cannot reach this dividend with a trading strategy based on her own information. Thus agent i will buy this derivative if the consumption process $c = e^{(i)} + \delta^{\theta}$ can increase her maximum utility, that is, $U^{(i)}(c) > U^{(i)}(c^{(i)*})$. The maximum price at which agent i will to pay will be $\Pi^{(i)}(c) - \Pi^{(i)}(c^{(i)*})$. In this case, agent w is just taking advantage of better information than agent i. The true value of this derivative is in fact 0, since some agent can get this dividend without any cost. This mis-pricing can be regarded as a bubble, since agents still want to buy this derivative even though they know the price of this derivative is higher than its true value.

Example 5 (Mis-pricing due to Information Asymmetry)**.** Let us consider a imprecise but illustrative example. Suppose there is one stock and its price follows a geometric Brownian motion with the drive $B(t)$. There is a bank account with risk-free rate 1. Say there is a European option with maturity time T and payoff $(S(T)-K)^+$.

One agent w in market can observe $\{B(s) : s \leq t\}$ at any time t, then agent A can perfect hedge this option, and the value is given by $p^w = \mathbb{E}^{\mathbf{Q}}[(S(T)-K)^+]$. Equivalently, the value of the derivative V paying dividend $p^w - (S(T)-K)^+$ at time T is 0 for agent A.

However, another agent b can only observe $\{\tilde{B}(s) : s \leq t\}$ at time t, where $\tilde{B}(t) = B(t)\mathbf{1}_{\{a<B(t)<b\}}$. Then at any time t, the volatility of $S(T)$ conditioned on $\sigma(\tilde{B}(u) : u \leq t)$ is larger that the volatility of $S(T)$ conditioned on $\sigma(B(u) : u \leq t)$. Thus the value of the option for agent b, p^b, is greater than p^w.

A trade may occur if agent w writes an option sell to be agent b at price p^w. In this case, although the price of the option is higher than its real value p^w for the writer, a trade can still be made.

Chapter 2

Market Time Changes: Lévy Subornadition and Empirical Facts

2.1 Introduction

The asset price returns in financial market generally do not admit the normality assumption as in many classic theory, such as Black-Scholes model. The heavy-tailness of financial returns have been documented in many empirical literature. As pioneered by Clark (1973), the relation between the trading volume and the normality are studied in the subordinated process framework. The normality as claimed by Clark (1973) can be recovered by view the price process in the stochastic volume time. The stochastic time-changed model is closely link to the concept of stochastic volatility model. The asset price volatility exhibits time-varying and clustering effects. In discrete-time model, the volatility clustering is modeled by Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) process. In continuous-time setting, the stochastic volatility models are used. Here, one is prone to ask that whether the non-Gaussian tail and clustering volatility is the consequence of stochastic market trading activity.

These days the availability of high-frequency data makes it possible to analyze the relation between the market events and the asset returns. It is the fact that the asset price is driven by the trades made by every market participants. The decision of the trades is triggered not by the chronological timing but by the market events, i.e., the incremental of price reaching a threshold or the bid-ask spread being small at some level. The asset price is not chronological driven but event-driven. It implies that a better way to measure time is to use to some market intrinsic measurement.

In this chapter, we first examine the one-dimensional asset return by using a subordinated Brownian motion process with the market intrinsic time as the subordinator. Note that these processes are pure jump processes. Then we move to model the multivariate asset returns by assuming the dependency among the marginal subordinators. In a continuous time model, roughly speaking, the two pure jump process can only be dependent if they can possibly jump together. Therefore, we adopt the factor-based multivariate subordinator model as introduced in Luciano and Semeraro (2010). The common jump is factorized in to contribute the dependency among marginal asset returns. Unlike Luciano and Semeraro (2010), the explicit sample subordinator processes are used to conduct empirical analysis.

2.2 Univariate stochastic time change

2.2.1 Preliminaries

Let $(X(t))_{t\geq 0}$ be a $\mathbb{R}^d$-valued Lévy process. The characteristic function of a Lévy process can be expressed by its so-called Lévy triplet (A, ν, γ).

Theorem 3 (Lévy -Khinchin representation). *Let $(X(t))$ be a Lévy process on $\mathbb{R}^d$ with Lévy triplet (A, ν, γ). Then the characteristic function of X can be written as*

$$\mathbb{E}\left[e^{\langle u, X(t)\rangle}\right] = e^{t\Psi_X(u)}, \quad u \in \mathbb{R}^d$$

with the characteristic exponent

$$\Psi(u) = -\frac{1}{2}\langle u, Au\rangle + i\langle \gamma, u\rangle + \int_{\mathbb{R}^d} (e^i\langle u, x\rangle - 1 - i\langle z, x\rangle \mathbf{1}_{|x|\leq 1})\nu(dx). \quad (2.1)$$

A Lévy process is called a *subordinator* if it is a almost surely non-decreasing.

Theorem 4 (Subordination of a Lévy Process). *Given a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X(t))_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic exponent $\Psi(u)$ and Lévy triplet (A, ν, γ). And let $(G(t))_{t\geq 0}$ be a subordinator with Laplace exponent $l(u)$ and triplet $(0, \rho, b)$. Then the process $(Y(t))_{t\geq 0}$ defined for each $w \in \Omega$ by $Y(t, \omega) = X(G(t, \omega), \omega)$ is a Lévy process. And its characteristic function is*

$$\mathbb{E}\left[e^{iuY_t}\right] = e^{tl(\Psi(u))}. \quad (2.2)$$

The triplet (A^Y, ν^Y, γ^Y) *of* Y *is given by*

$$A^Y = bA, \tag{2.3}$$

$$\nu^Y(B) = b\nu(B) + \int_0^\infty p_s^X(B)\rho(ds), \quad \forall B \in \mathcal{B}(\mathbb{R}^d) \tag{2.4}$$

$$\gamma^Y = b\gamma + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x p_s^X(dx), \tag{2.5}$$

where p_t^X *is the probability distribution of* $X(t)$*.*

The following thoerem shows that under some very general technical conditions, a Lévy process can be expressed as a subordinated Brownian motion process.

Theorem 5. *let* ν *be a Lévy measure on* $\mathbb{R}$ *and* $\mu \in \mathbb{R}$*. There exist a Lévy process* $(Y(t))_{t\geq 0}$ *with Lévy measure* ν *such that* $Y(t) = B(G(t)) + \mu G(t)$ *for some subordinator* $(G(t))_{t\geq 0}$ *and some Brownian motion* $(B(t))_{t\geq 0}$ *independent from* G *if and only if the following conditions are satisfied:*

1. ν *is absolutely continuous with density* $\nu(x)$*.*

2. $\nu(x)e^{-\mu x} = \nu(-x)e^{\mu x}$ *for all* x*.*

3. $\mu(\sqrt{u})e^{-\mu\sqrt{u}}$ *is a complete monotonic function on* $(0, \infty)$*.*

2.2.2 Examples

Here are several commonly-used examples of subordinators.

Tempered stable subordinator

Let us consider a class of subordinator called *tempered stable subordinator*. The Lévy measure of tempered stable subordinator, ρ, can be written by

$$\rho(x) = \frac{c^{-\lambda x}}{x^{\alpha+1}} \mathbf{1}_{x>0}, \tag{2.6}$$

where c and λ are positive constants and $0 \leq \alpha < 1$. The Laplace exponent of tempered stable subordinator in general case ($0 < \alpha < 1$) is

$$l(u) = c\Gamma(-\alpha)\left[(\lambda - u)^{\alpha} - \lambda^{\alpha}\right], \tag{2.7}$$

and for $\alpha = 0$

$$l(u) = -c \log\left(1 - \frac{u}{\lambda}\right). \tag{2.8}$$

Two special cases commonly used are

1. Gamma subordinator when $\alpha = 0$.
2. Inverse Gaussian subordinator when $\alpha = 1/2$.

Gamma subordinator

A Gamma subordinator G with parameter (λ, c) can characterized by its Laplace transform

$$\mathbb{E}\left[e^{uG(t)}\right] = \left(1 - \frac{u}{\lambda}\right)^{-ct}. \tag{2.9}$$

The Lévy density is

$$\rho(x) = \frac{ce^{-\lambda x}}{x}\mathbf{1}_{x>0}. \tag{2.10}$$

The value of G at time 1, $G(1)$, follows a Gamma distribution. The probability density function of $G(1)$ is given by

$$f_{G(1)}(x) = \frac{\lambda^{c}}{\Gamma(c)}x^{c-1}e^{-\lambda x}. \tag{2.11}$$

Let us denote W a Brownian motion independent of G. Then the subordinated Brownian motion $Y = (W(G(t)))$ by the Gamma subordinator is called Variance Gamma (VG) process. By letting the mean rate the Gamma subordinator to be unit, the Variance Gamma process has three parameters:

- σ, the volatility of Brownian motion,
- θ, the drift of Brownian motion,

- κ, the variance of the Gamma subordinator.

The characteristic exponent Ψ of a VG process Y with parameters (σ, θ, κ) is given by

$$\Psi(z) = -\frac{1}{\kappa} \log \left(1 + \frac{1}{2} z^2 \sigma^2 \kappa - i\theta\kappa z \right). \tag{2.12}$$

And the probability density function of $Y(t)$ is given by

$$f_{Y(t)}(x) = C|x|^{\frac{t}{\kappa}-\frac{1}{2}} e^{\theta x/\sigma^2} K_{\frac{t}{\kappa}-\frac{1}{2}}(B|x|), \tag{2.13}$$

where K is the modified Bessel function of the second kind, and

$$B = \frac{\theta^2 + \sigma^2/\kappa}{\sigma^2}, \quad C = \sqrt{\frac{\sigma^2\kappa}{2\pi}} \frac{(\theta^2\kappa + 2\sigma^2)^{\frac{1}{4}-\frac{\theta}{2\kappa}}}{\Gamma(t/\kappa)} \tag{2.14}$$

Inverse Gaussian subordinator

A Inverse Gaussian subordinator G with parameter (λ, c) can characterized by its Laplace transform

$$\mathbb{E}\left[e^{uG(t)}\right] = \exp\left(-2ct\sqrt{\pi}(\sqrt{\lambda - u} - \sqrt{\lambda})\right). \tag{2.15}$$

The Lévy density is

$$\rho(x) = \frac{ce^{-\lambda x}}{x^{3/2}} \mathbf{1}_{x>0}. \tag{2.16}$$

The value of G at time 1, $G(1)$, follows a Gamma distribution. The probability density function of $G(1)$ is given by

$$f_{G(1)}(x) = \frac{ct}{x^{3/2}} e^{2ct\sqrt{\pi\lambda}} e^{-\lambda x - \pi c^2 t^2/x}. \tag{2.17}$$

Let us denote W a Brownian motion independent of G. Then the subordinated Brownian motion $Y = (W(G(t)))$ by the Inverse Gaussian subordinator is called Normal Inverse Gaussian (NIG) process. By letting the mean rate the Inverse Gaussian subordinator to be unit, the NIG process has three parameters as in VG case: (i) σ, the volatility of Brownian motion, (ii) θ, the drift of Brownian motion, and (iii) κ, the variance of the Inverse Gaussian subordinator.

The characteristic exponent Ψ of a NIG process Y with parameters (σ, θ, κ) is given by

$$\Psi(z) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 + z^2\sigma^2\kappa - 2i\theta\kappa z}. \tag{2.18}$$

And the probability density function of $Y(t)$ is given by

$$f_{Y(t)}(x) = Ce^{\theta x/\sigma^2}\frac{K_1(B\sqrt{x^2 + t^2\sigma^2/\kappa})}{\sqrt{x^2 + t^2\sigma^2/\kappa}}, \tag{2.19}$$

where K is the modified Bessel function of the second kind, and

$$B = \frac{\theta^2 + \sigma^2/\kappa}{\sigma^2}, \quad C = \frac{t}{\pi}e^{t/\kappa}\sqrt{\frac{\theta^2}{\kappa\sigma^2} + \frac{1}{\kappa^2}} \tag{2.20}$$

Generalized Inverse Gaussian subordinator

A Generalized Inverse Gaussian (GIG) distribution is obtained by introducing an additional parameter to inverse Gaussian distribution. The probability density function of a GIG distribution with three parameters (λ, a, b) is given by

$$f(x) = \frac{(a/b)^{\lambda/2}}{2K_\lambda(\sqrt{ab})}x^{\lambda-1}e^{-\frac{1}{2}(ax+\frac{b}{x})}\mathbf{1}_{x>0}, \tag{2.21}$$

where K_λ is modified Bessel function of the second kind with index λ.

The characteristic function of GIG distribution is

$$\Psi(u) = \frac{1}{K_\lambda(ab)}\left(1 - \frac{2iu}{b^2}\right)^{-\frac{\lambda}{2}} K_\lambda\left(ab\sqrt{1 - 21ub^{-2}}\right), \tag{2.22}$$

where K_λ denotes the modified Bessel function of the third kind with index λ.

Although GIG distribution is infinitely divisible, it is not closed under convolution. As a static analog of Brownian subordination, generalized hyperbolic (GH) model can be obtained by mixing the mean and variance of a normal distribution. That is, GH distribution is a normal mean-variance mixture of GIG laws,

$$\mu G + \sqrt{G}W, \tag{2.23}$$

where W is a standard normal distribution. GH law is also infinite divisible.

2.2.3 Market Intrinsic Time

Although physical time is the most nature measure for us to understand the scheduling of dynamics or evolving, the market itself, as the complex system which aggregates all the activities of human tradings and algorithmic tradings, has its own internal clock. Market is driven by tradings. And tradings, thinking as the activities in the exchanges and other platforms, is scheduled not in a chronological manner, but event-based. Many trade activities react not only to the physical time schedules but also events. Therefore it is nature to consider to analyze the market behavior in a event-based time.

The market intrinsic time has been discussed for a long time. Mandelbrot and Taylor (1967) and Clark (1970; 1973) started to investigate financial series using a different time clock. Jones, Kaul and Lipson (1994) studied the relation between asset price volatility and number of trades. And Ané and Geman (2000) also provided a empirical analysis of the normality of the asset returns under transaction time. Easley et al. (2012) discussed the use of volume time, and further argued that the information contained in volume can be used to identify informed trades. Transaction time and volume time will be the two time measures for viewing financial series here.

The financial returns generally exhibit several stylish characteristics which coincide with the properties of stochastic time changes (by transaction or volume). The asset returns exhibit heavy-tailness and volatility clustering when it is sampled by physical time. Also, the volatility are relatively higher at the beginning and ending of the market hours in each day. This can be caused by the event-based driver of the asset price. The price is driven by trades, which is crowded at some time period. Also transactions and volumes exhibits a seasonality pattern daily. They are more concentrated at the beginning and ending hours every day.

2.2.4 Empirical Analysis

Data description

In this section we conduct the empirical tests against Lobster high frequency data[1]. We select 5 stocks tick-by-tick data starting from 10/01/2013 to 09/20/2014, that is 252 trading days. The stocks include PFE, MSFT, GE,

[1] https://lobsterdata.com/

GIS and C. The data include both visible trade and invisible trades. Therefore time-changing by both visible volumes and invisible volumes are considered.

Basic characteristics and normality

The main statements are: when returns are sampled in market intrinsic time,

1. Do returns recover normal?
2. Can the volatility clustering effect be removed?
3. Do empirical variances of returns scale linearly?

Intraday Pattern

The five stocks generally exhibits the similar intraday seasonality pattern of the market intrinsic time. As shown in Figure 2.1, during the beginning and the ending of the market open hours, the speed of the market intrinsic time is faster than physical one. And during the mid-day, market intrinsic times eclipse slower. Figure 2.2 shows the intraday pattern of volatility. The volatility in the first two market hours is significantly higher than the rest of the day.

Normality

The asset returns are much more close to normal distribution when it is sampled in market intrinsic time. Figure 2.3 and Figure 2.4 shows the empirical distribution of the asset returns sampled in different time measures. The qq-plots are shown in Figure 2.5. We can see that asset returns in physical time exhibits more heavy-tailness than the ones in volume and transaction time. However market intrinsic times do not fully recover the normality.

The discrepancy among the distribution of returns in volume time, visible-volume time and transaction time are quite small. Including invisible traded volume into time measurement does not recover extra normality in asset returns.

The distance to the closest normal distribution of the asset returns are shown in Table 2.1 and Table 2.2. From the Hellinger distance and total

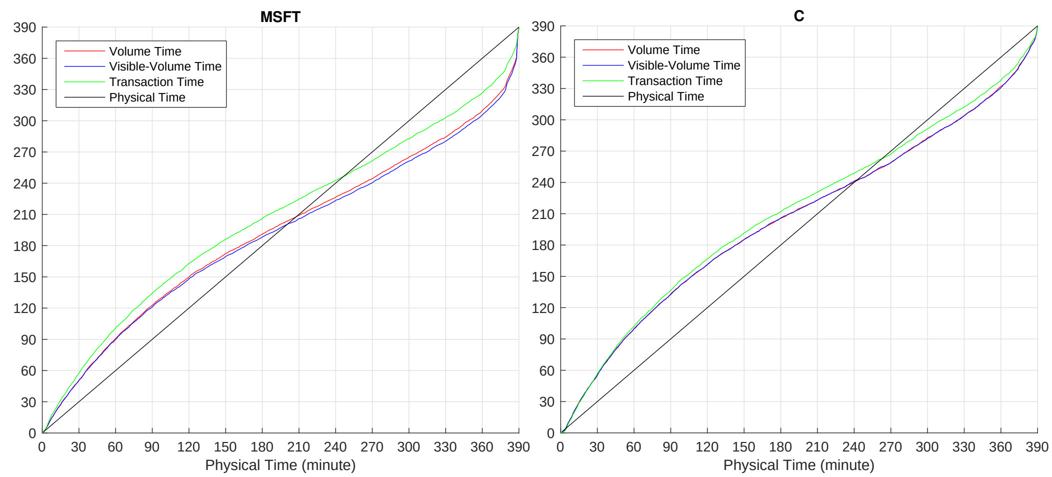


Figure 2.1: Intraday market intrinsic time seasonality pattern.

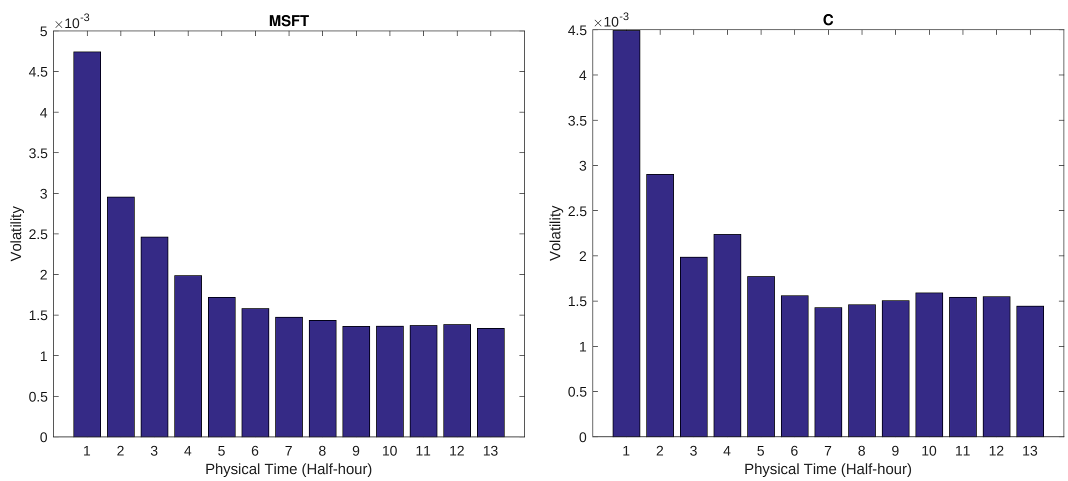


Figure 2.2: Intraday volatility seasonality pattern.

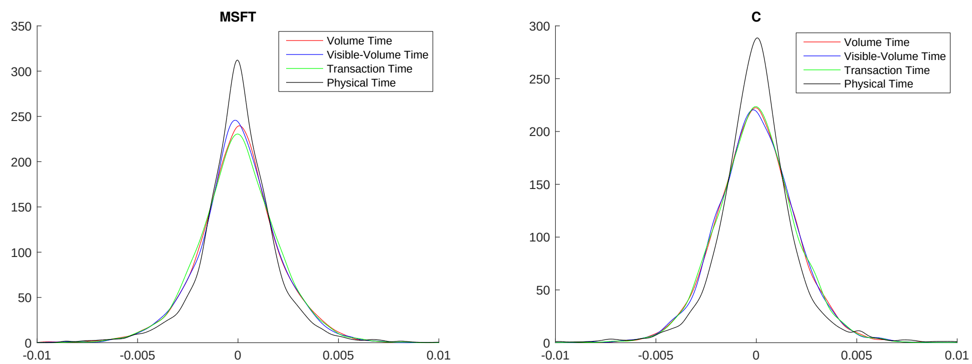


Figure 2.3: Empirical probability density of asset returns in different time measures.

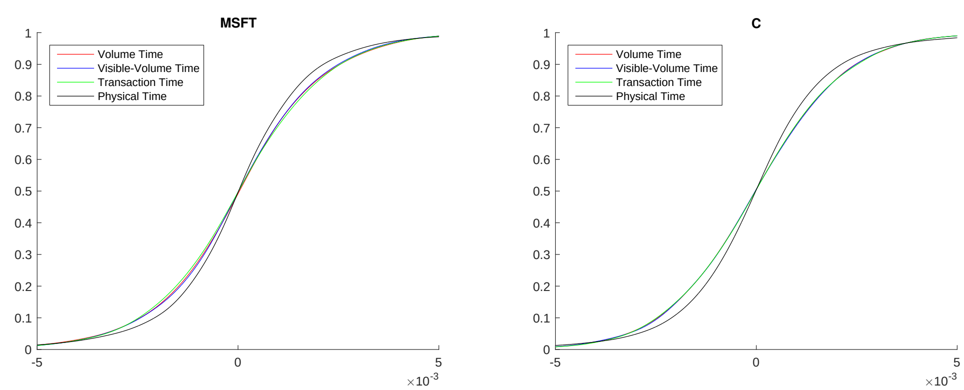
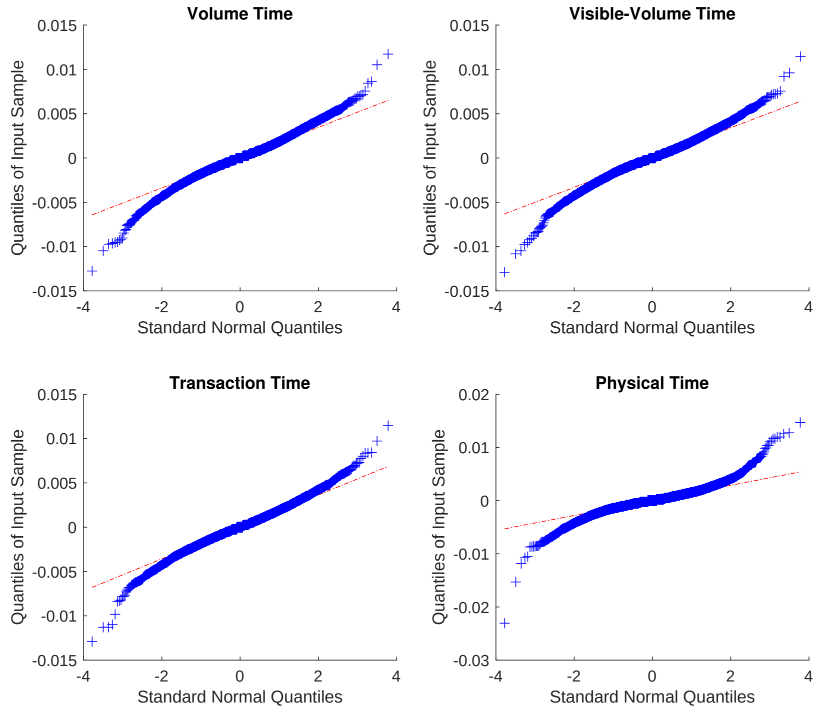
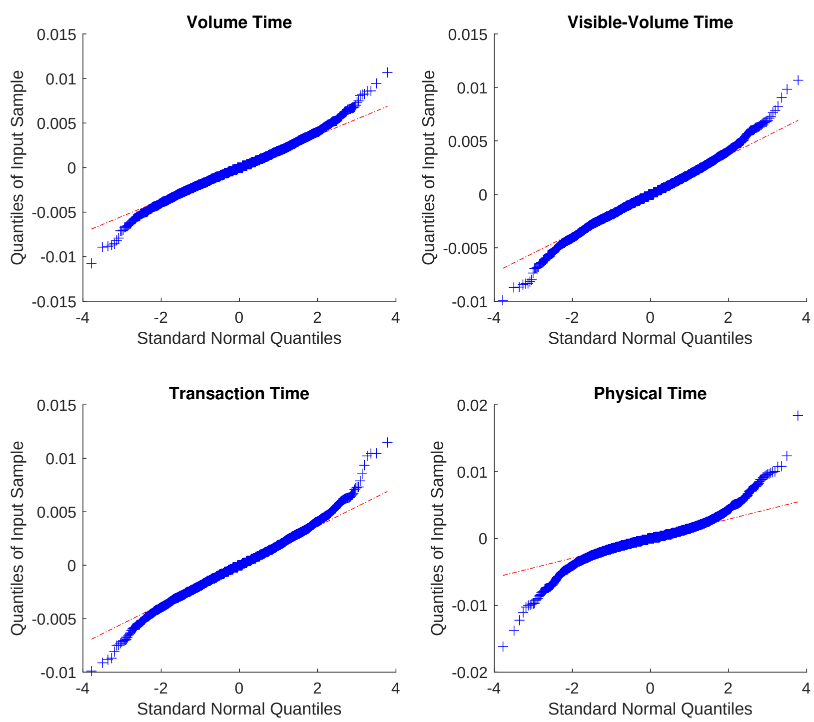


Figure 2.4: Empirical CDF of asset returns in different time measures.



(a) MSFT



(b) C

Figure 2.5: QQ Plots of asset returns in different time measures.

variance distance, we can clearly see that returns in the market intrinsic time are much more closer to normality. The visible-volume time and all-volume time have the similar distance to normal distribution. The transaction time recovers the most normality.

Table 2.1: Distances to normal distribution, MSFT

	Hellinger	Total Variance	K-S Statistic
Visible-Volume Time	0.068819	0.074262	0.496369
All Volume Time	0.069555	0.074938	0.496526
Transaction Time	0.055198	0.054044	0.496328
Physical Time	0.131942	0.144132	0.495722

Table 2.2: Distances to normal distribution, C

	Hellinger	Total Variance	K-S Statistic
Visible-Volume Time	0.042037	0.033756	0.496283
All-Volume Time	0.045214	0.033718	0.496369
Transaction Time	0.044115	0.035834	0.496225
Physical Time	0.128710	0.139106	0.495373

GARCH Effect

Here we focus on the question how much the volatility clustering effects add to the heavy-tailness of the asset returns. First likelihood ratio test are conducted to check whether the returns in different time clock admit GARCH model. The results are summerized in Table 2.3 and Table 2.4. For MSFT, the asset returns in every time clock all admit GARCH model. For C, asset returns admit GARCH model except in all-volume time clock. More details on the results from other stocks can be found in appendix.

Table 2.3: Likelihood-Ratio Test for GARCH(1,1) model, MSFT

	H	p-value	Statistic
Visible-Volume	1	0.000	194.03
All-Volume	1	0.000	171.44
Transaction	1	0.000	123.68
Physical	1	0.000	415.33

Table 2.4: Likelihood-Ratio Test for GARCH(1,1) model, C

	H	p-value	Statistic
Visible-Volume	1	0.000	46.67
All-Volume	0	0.161	3.65
Transaction	1	0.000	16.93
Physical	1	0.000	278.64

To obtain more idea on the contribution of GARCH effect to non-normality, we also calculate the distance to normal distribution from the innovations filtered by GARCH model. The results are shown in Table 2.5 and Table 2.6. For the physical time case, the asset returns can achieve a better normality by filtering from GARCH model. But for the market intrinsic time clock, the normality is hardly improved by filtering returns from GARCH model. Also, in term of normality recovery, filtering with GARCH model cannot recover as same as change-of-time. It is not confirmed that time-changing by volume or transaction clock removes volatility clustering effect. However it is clear that the volatility clustering does not contribute no-Gaussian tails as much as market time variation.

Table 2.5: Distances to closest distribution after GARCH(1,1) filtering, MSFT. The number is parentheses are the improvement comparing to the distances before GARCH filtering as in Table 2.1.

	Hellinger	Total Variance
Visible Volume	0.063059 (.005760)	0.064093 (.010169)
All Volume	0.065452 (.004103)	0.066555 (.008383)
Transaction	0.054010 (.001188)	0.048464 (.005580)
Physical Time	0.122227 (.009715)	0.129129 (.015003)

Table 2.6: Distances to normal distribution after GARCH(1,1) filtering, C. The number is parentheses are the improvement comparing to the distances before GARCH filtering as in Table 2.2.

	Hellinger	Total Variance
Visible Volume	0.044119 (-.002082)	0.032082 (-.008281)
All Volume	0.049152 (-.003938)	0.035807 (-.011496)
Transaction	0.044404 (-.000289)	0.034796 (-.008281)
Physical Time	0.123426 (.005284)	0.129516 (.010396)

Table 2.7: Estimated parameters of Gamma subordinator.

	MSFT		C	
	λ	c	λ	c
Volume	2.1099	0.4740	1.8009	0.5553
Visible-Volume	2.1225	0.4711	1.8258	0.5477
Transaction	2.3801	0.4202	2.0804	0.4807

Distribution of market time clock

The market intrinsic time subordinators are fitted into the parametric distributions described in Section 2.2. We adopt maximum likelihood estimation (MLE) method. All subordinators are normalized to unit expectation. That is, for each subordinator G, $\mathbb{E}(G(1)) = 1$. Table 2.7, 2.8 and 2.9 respectively show the estimated parameters for Gamma, inverse Gaussian and generalized inverse Gaussian subordinators. For the cases in fitting into GIG subordinator that the parameter b is very close to 0 ($< 10^{-6}$), GIG is practically degenerated to the Gamma distribution.

The survival functions are shown in Figure 2.6, of which the zoomed-in plots are shown in Fgireu 2.7 . The empirical distributions of market intrinsic time start from Gamma law when t is near 0. And for very large t, it move towards inverse Gaussian (IG) law.

2.3 Factor-based Multivariate Subordinator

2.3.1 Preliminaries

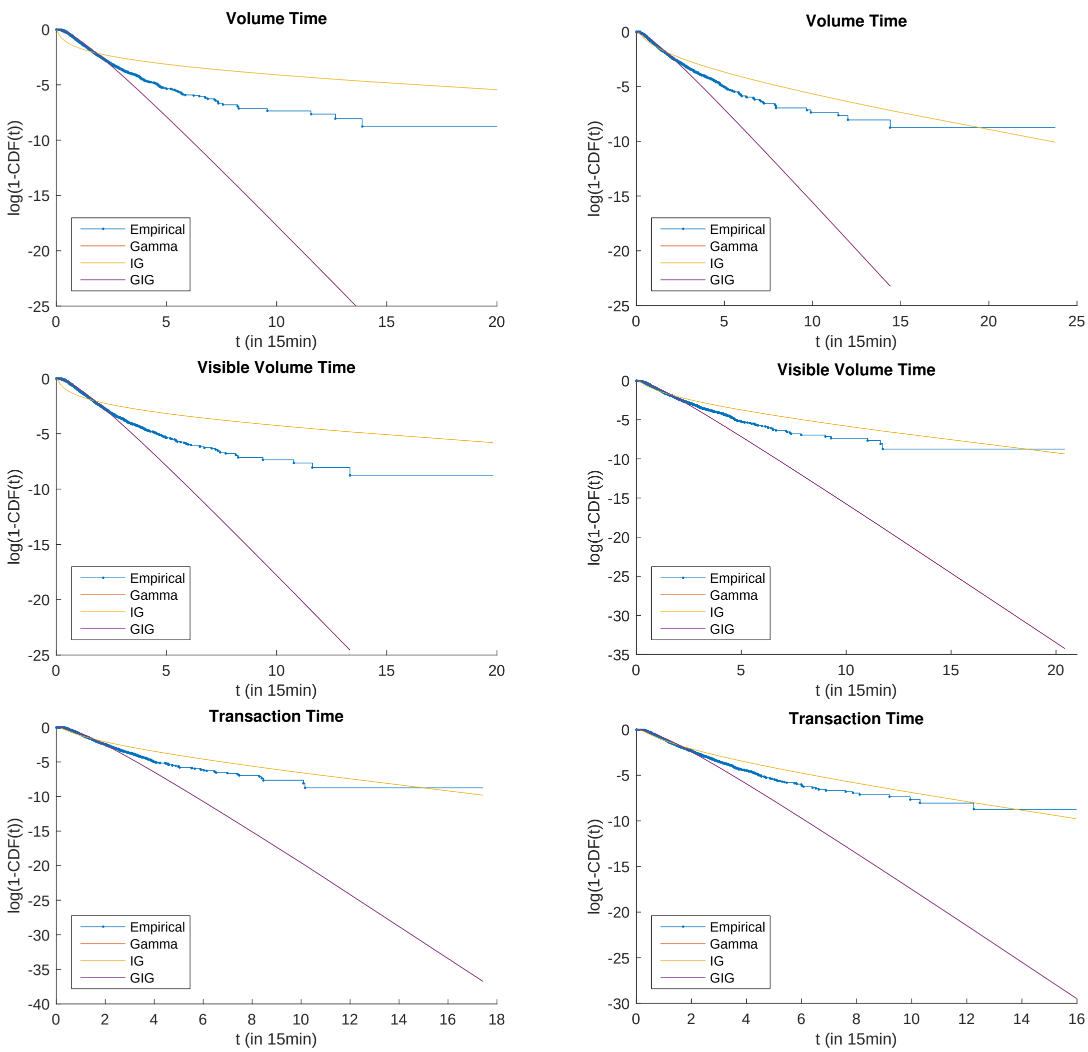
Theorem 6. *Let $G = (G_1(t), G_2(t), ..., G_n(t))_{t\geq 0}$ be a multivariate subordinator with Lévy triplet $(0, \nu_G, \gamma_G)$. Let $(X(t))_{t\geq 0} = (X_1, ..., X_n)$ be a Lévy process on $\mathbb{R}^n$ with independent marginal components and triplet $(\gamma_X, \Sigma_X, \nu_X)$,*

Table 2.8: Estimated parameters of Inverse Gaussian subordinator. The mean parameter μ are normalized to be unit.

	MSFT	C
	λ	λ
Volume	0.1255	0.48270
Visible-Volume	0.1645	0.51181
Transaction	0.6824	0.75822

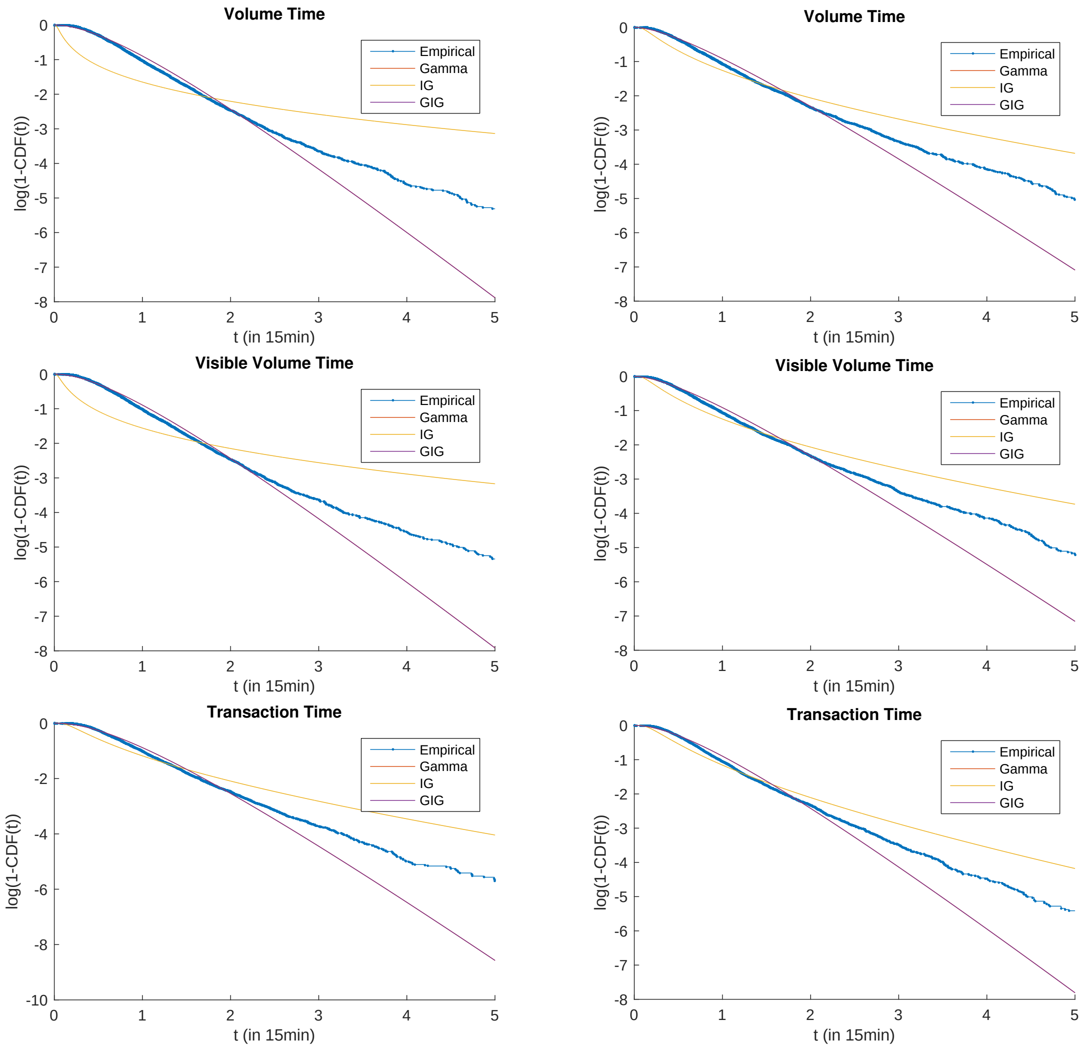
Table 2.9: Estimated parameters of Generalized Inverse Gaussian subordinator.

	MSFT			C		
	λ	a	b	λ	a	b
Volume	2.1103	4.2210	$< 10^{-6}$	1.8009	3.6018	$< 10^{-6}$
Visible-Volume	2.1229	4.2466	$< 10^{-6}$	1.8262	3.6527	$< 10^{-6}$
Transaction	2.3800	4.7601	$< 10^{-6}$	2.0804	4.1609	$< 10^{-6}$



(a) MSFT (b) C

Figure 2.6: Survival functions of the distributions of the subordinators. The left column is for MSFT. The right column is for C. The curves for Gamma subordinator are overlapped with GIG ones.



(a) MSFT (b) C

Figure 2.7: Survival functions of the distributions of the subordinators (Zoomed-in). The left column is for MSFT. The right column is for C. The curves for Gamma subordinator are overlapped with GIG ones.

where $\Sigma_X = \mathrm{diag}(\sigma_1, ..., \sigma_n)$ *and denote by* Ψ_{X_i} *the characteristic function of* X_i*. Let* ρ_s *be the distribution of* $X(s)$*. Define the subordinated process* $Y = (Y(t))_{t\geq 0}$ *by*

$$Y(t) = (X_1(G_t(t)), X_2(G_2(t)), ..., X_n(G_n(t))), \quad t \geq 0. \tag{2.24}$$

Then the process Y *is a Lévy process and the characteristic function of* Y *is*

$$\mathbb{E}\left[e^{i\langle u, Y(t)\rangle}\right] = \exp\left(t\Psi_G(\mathrm{Log}\,\widehat{\Psi_X}(u))\right), \tag{2.25}$$

where Ψ_G *is the characteristic exponent of* G*, and*

$$\mathrm{Log}\,\widehat{\Psi_X}(u) = \left(\log \Psi_{X_1}(u), ..., \log \Psi_{X_n}(u)\right).$$

Moreover, the Lévy triplet $(\Sigma_Y, \nu_G, \gamma_G)$ *is given by*

$$\Sigma_Y = \mathrm{diag}(\gamma_1^G \sigma_1, ..., \gamma_n^G \sigma_n), \tag{2.26}$$

$$\nu_Y(B) = v_{(1)}(B) + \nu_{(2)}(B), \quad \forall B \in \mathcal{B}(\mathbb{R}^n \backslash 0), \tag{2.27}$$

$$\gamma_Y = \int_{\mathbb{R}_+^n} \nu_G(ds) \int_{|x|\leq 1} x\rho_s(dx) + (\gamma_1^G \gamma_1^X, ..., \gamma_n^G \gamma_n^X)^\top, \tag{2.28}$$

where $\gamma_G = (\gamma_1^G, ..., \gamma_n^G)$*,* $\gamma_G = (\gamma_1^X, ..., \gamma_n^X)$*, and* $\nu_{(1)}, \nu_{(2)}$ *are defined by* $\nu_{(1)}(0) = 0, \nu_{(2)}(0) = 0$ *and for* $B \in \mathcal{B}(\mathbb{R}^n \backslash 0)$ *by*

$$\nu_{(1)}(B) = \int_{\mathbb{R}_+^n} \rho_s(B)\nu_G(ds), \tag{2.29}$$

$$\nu_{(1)}(B) = \int_B \gamma_i^G \mathbf{1}_{A_1}(x)\nu_{X_1}(dx) + ... + \gamma_i^G \mathbf{1}_{A_n}(x)\nu_{X_n}(dx), \tag{2.30}$$

where $x \in \mathbb{R}$*,* $\nu_{X_j}, j = 1, ..., n$ *are the Lévy measures of the independent marginal process of* X *and for* $j = 1, ..., n$*,*

$$A_j = \left\{x = (x_1, ..., x_n)^\top \in \mathbb{R}^n : x_k = 0 \textit{ for } k \neq j, k = 1, ..., n\right\}.$$

2.3.2 Subordination of Brownian Motions

One-factor multivariate subordinator

Let X_j, $j = 1, ..., n$ and Z be independent subordinators. X_j, $j = 1, ..., n$ are individual subordinators, and Z is the common subordinator factor. Then multivariate subordinator $G = (G(t))_{t \geq 0}$ is defined by

$$G(t) = (X_1(t) + \alpha_1 Z(t), ..., X_n(t) + \alpha_n Z(t)), \tag{2.31}$$

where $\alpha_j > 0$, $j = 1, ..., n$. Let Ψ_j and Ψ_Z respectively denote the characteristic exponents of the processes X_j and Z, namely, for $u \in \mathbb{C}$ and $\Re u \leq 0$,

$$\Psi_j(u) = ib_j x + \int_{\mathbb{R}_+} (e^{iux} - 1)\nu_j(dx), \tag{2.32}$$

$$\Psi_Z(u) = ib_Z x + \int_{\mathbb{R}_+} (e^{iux} - 1)\nu_Z(dx). \tag{2.33}$$

Then the characteristic exponent Ψ_G of G is given by

$$\Psi_G(u) = \sum_{j=1}^{n} \Psi_j(u_j) + \Psi_Z\left(\sum_{j=1}^{n} \alpha_j u_j\right), \quad u \in \mathbb{C}^n \text{ with } \Re u \leq 0. \tag{2.34}$$

From now on, we will only consider the subordinator processes X_j and Z of zero drift term.

Marginal Specification

Gamma subordinator

Let X_j, for $j = 1, ..., n$, and Z all have Gamma subordinator, as follows,

$$X_j(1) \sim Gamma\left(\frac{1}{\alpha_j - a}, \alpha_j\right), \tag{2.35}$$

$$Z(1) \sim Gamma\,(a, 1), \tag{2.36}$$

where the parameters should satisify

$$\frac{1}{a} > \alpha_j > 0, \quad j = 1, ..., n. \tag{2.37}$$

Then the marginal subordinators $G_j = X_j + \alpha_j Z$ are also Gamma, with distribution

$$G_j(1) \sim Gamma(\frac{1}{\alpha_j}, \alpha_j), \quad , j = 1, ..., n. \tag{2.38}$$

Inverse Gaussian subordinator

Let X_j, for $j = 1, ..., n$, and Z all have IG subordinator, as follows,

$$X_j(1) \sim IG\left(1 - a\sqrt{\alpha_j}, \alpha_j\right), \tag{2.39}$$

$$Z(1) \sim IG\left(a, 1\right), \tag{2.40}$$

where the parameters should satisfy

$$\frac{1}{\sqrt{\alpha_j}} > a > 0, \quad j = 1, ..., n. \tag{2.41}$$

Then the marginal subordinators $G_j = X_j + \alpha_j Z$ are also IG, with distribution

$$G_j(1) \sim IG\left(1, \frac{1}{\sqrt{\alpha_j}}\right), \quad , j = 1, ..., n. \tag{2.42}$$

Generalized Inverse Gaussian subordinator

In GIG case, it is impossible to have both X_j and Z distributed according to GIG laws since GIG distribution family is note cosed under convolution. Therefor we will assume Z is Gamma subordinator as

$$Z(1) \sim Gamma\left(a, \frac{1}{2}\right), \tag{2.43}$$

and X_j follows the form $X_j(1) = R_j + V_j$ with

$$R_j \sim GIG\left(-\lambda, a, \frac{1}{\sqrt{\alpha_j}}\right), \text{ and,} \tag{2.44}$$

$$V_j \sim Gamma\left(\lambda - a, \frac{1}{2\alpha_j}\right), \tag{2.45}$$

where R_j and V_j are independent. The parameters should satisfy

$$\lambda > a > 0, \text{ and } \alpha_j > 0, \ j = 1, ..., n. \tag{2.46}$$

Then the marginal subordinators $G_j = X_j + \alpha_j Z$ are also IG, with distribution

$$G_j(1) \sim GIG\left(\lambda, \delta_j, \frac{1}{\sqrt{\alpha_j}}\right), \quad j = 1, ..., n. \tag{2.47}$$

Independent Brownian motion

Let $W_j = (W_j(t))_{t\geq 0}$, $j = 1, ..., n$ be n independent standard Brownian motions, and let $(W(t))_{t\geq 0}$ be

$$W(t) = (\mu_1 t + \sigma_1 W_1(t), ..., \mu_n t + \sigma_n W_n(t), \quad \mu_j \in \mathbb{R}, \sigma_j > 0. \tag{2.48}$$

Then the Lévy triplet of W is $(\mu, \Sigma, 0)$, where $\mu = (\mu_1, ..., \mu_n)$ and $\Sigma = \text{diag}(\sigma_1, ..., \sigma_n)$.

Now we consider the multivariate process Y generated by subordinating W with G,

$$Y(t) = \big(\mu_1 G_1(t) + \sigma_1 W_1(G_1(t)), ..., \mu_n G_n(t) + \sigma_n W_n(G_n(t))\big)^\top, \tag{2.49}$$

with the subordinator G is independent of the Brownian motion W.

Then we can have that the characteristic function of the Lévy process Y is

$$\mathbb{E}\left[e^{i\langle z, Y(t)\rangle}\right] = \exp\left(t\Psi_G(\log \psi_W(z))\right), \tag{2.50}$$

where ψ_W is the characteristic function of W, and Ψ_G is the characteristic exponent of G. The Lévy triplet $(\gamma_Y, \Sigma_Y, \nu_Y)$ of Y is given by

$$\gamma_Y = \int_{\mathbb{R}_+^n} \nu_G(ds) \int_{|x|\leq 1} x\rho_s(ds), \tag{2.51}$$

$$\Sigma_Y = 0, \tag{2.52}$$

$$\nu_Y = \int_{\mathbb{R}_+^n} \rho_s(B)\nu_G(ds), \quad B \in \mathcal{B}(\mathbb{R}^n \backslash 0). \tag{2.53}$$

Dependent Brownian motion

In this section, we generalize the on-factor model from Section 3.2.1 by allowing the Brownian motion W having dependent marginals. Denote by W^ρ the Brownian motion with correlation matrix $\rho = (\rho_{jk})_{j,k=1,...,n}$. Then the Lévy triplet $(\mu^\rho, \Sigma^\rho, 0)$ is

$$\mu = (\mu_1, ..., \mu_n) \tag{2.54}$$

$$\Sigma_\rho = (\sigma_j \sigma_k \rho_{jk})_{j,k=1,...,n} \,. \tag{2.55}$$

Now we assume that the process Y^ρ is combined with two components: the individual movement and the common jumps with correlated jump size. That is,

$$Y^\rho(t) = \begin{pmatrix} W_1(X_1(t)) + W_1^\rho(\alpha_1 Z(t)) \\ \vdots \\ W_n(X_n(t)) + W_n^\rho(\alpha_n Z(t)), \end{pmatrix} \tag{2.56}$$

Denote by $Y^\perp(t) = (W_1(X_1(t)), ..., W_n(X_n(t)))$ the individual part, and $Y^\parallel(t) = (W_1^\rho(\mu_1 Z(t)), ..., W_n^\rho(\mu_n Z(t))$ the common jump part.

Then, we can have the characteristic function of Y^ρ is given by

$$\psi_{Y^\rho(t)}(z) = \psi_{Y^\perp(t)}(z)\psi_{Y^\parallel(t)}(z). \tag{2.57}$$

2.3.3 Linear Correlation

The correlation between subordinator margins can be given by the following relation,

$$\operatorname{Cov}(G_j(t), G_k(t)) = \alpha_j \alpha_k \operatorname{Var}(Z(t)), \tag{2.58}$$

$$\operatorname{Var}(G_j(t)) = \operatorname{Var}(X_j(t)) + \alpha_j^2 \operatorname{Var}(Z(t)). \tag{2.59}$$

The correlation is simply

$$\rho_{G(t)}(j,k) = \frac{\alpha_j \alpha_k \operatorname{Var}(Z(t))}{\sqrt{\left(\operatorname{Var}(X_j(t)) + \alpha_j^2 \operatorname{Var}(Z)\right)\left(\operatorname{Var}(X_k(t)) + \alpha_k^2 \operatorname{Var}(Z)\right)}}. \tag{2.60}$$

Then the variance and covariance between margins of the subordinated process Y can be written by

$$\operatorname{Cov}(Y_j(t), Y_k(t)) = \mu_j \mu_k \operatorname{Cov}(G_j(t), G_k(t)) = \mu_j \mu_k \alpha_j \alpha_k \operatorname{Var}(Z(t)), \tag{2.61}$$

and

$$\operatorname{Var}(Y_j(t)) = \mathbb{E}\left[\operatorname{Var}(Y_j(t)|G_j(t))\right] + \operatorname{Var}(\mathbb{E}\left[Y_j(t)|G_j(t)\right]) \tag{2.62}$$

$$= \sigma_j^2 \mathbb{E}[G_j(t)] + \mu_j^2 \operatorname{Var}(G_j(t)). \tag{2.63}$$

Therefore, the correlations of Y are given by

$$\rho_{Y(t)}(k,j) = \frac{\mu_j \mu_k \alpha_j \alpha_k \operatorname{Var}(Z(t))}{\sqrt{\operatorname{Var}(Y_j(t)) \operatorname{Var}(Y_k(t))}}. \tag{2.64}$$

Similarly, for the general case Y^ρ, the linear correlation coefficients of Y^ρ can be written by

$$\rho_{Y^\rho(t)}(k,j) = \frac{\mu_j \mu_k \alpha_j \alpha_k \operatorname{Var}(Z(t)) + \rho_{jk}\sigma_j\sigma_j\sqrt{\alpha_j\alpha_k}\mathbb{E}[Z(t)]}{\sqrt{\operatorname{Var}(Y_j^\rho(t)) \operatorname{Var}(Y_k^\rho(t))}}. \tag{2.65}$$

Since it is sufficient to characterize any Lévy process L by its law at time 1, hereafter we shall also use L denote the random variable $L(1)$ when there is no confusion. For example, Y^ρ can denote either the process Y^ρ or the process at time 1, $Y^\rho \equiv Y^\rho(1)$, depending on the context.

In this one-factor model, Y^ρ, the linear correlation is provided from two sources: (i) the common jumps Z in in subordinator G, and (ii) the correlation of the jump size given by W^ρ. With known distribution of marginal subordinators, the parameter a serves as the level of common shock, indicating the the common variance contributed to the variance of each marginal. Therefore the range of a is limited by the minimal variance among all marginal subordinator. For the case of independent Brownian motion, $Y(t)$, the linear dependency is only introduced by the common jumps. Thus $\rho_{Y(t)}$ cannot reach 1 when the marginal subordinators have different variance.

Comparing to model the asset return by multivariate distributions, such as multivariate general hyperbolic distribution, the advantages here are: First, each marginal are allowed to have its own parameters. Second, it is able to model independence.

The linear correlations for the specific marginal are given as follows.

Gamma marginal

For the case of Gamma marginal subordinator, the correlation is given by

$$\rho_{Y^\rho}(j,k) = a \cdot \frac{\mu_j \mu_k \alpha_j \alpha_k + \rho_{jk}\sigma_j\sigma_k\sqrt{\alpha_j\alpha_k}}{\sqrt{(\sigma_j^2 + \mu_j^2\alpha_j)(\sigma_k^2 + \mu_k^2\alpha_k)}}, \tag{2.66}$$

with the constraint

$$0 < a < \alpha_j, \ j = 1, ..., n. \tag{2.67}$$

Inverse Gaussian marginal

For the case of Inverse Gaussian marginal subordinator, the correlation is given by

$$\rho_{Y^\rho}(j,k) = a \cdot \frac{\mu_j \mu_k \alpha_j \alpha_k + \rho_{jk}\sigma_j\sigma_k\sqrt{\alpha_j\alpha_k}}{\sqrt{\left(\sigma_j^2 + \frac{\mu_j^2}{\sqrt{\alpha_j}}\right)\left(\sigma_k^2 + \frac{\mu_k^2}{\sqrt{\alpha_k}}\right)}}, \tag{2.68}$$

with the constraint

$$\frac{1}{\sqrt{\alpha_j}} > a > 0, \quad j = 1, ..., n. \tag{2.69}$$

2.3.4 Empirical Analysis

In this section, we mainly exhibit the empirical results on transaction market clock since it recovers the most normality of asset returns. The more results for other market times (volume and visible-volume) can be found in appendix.

The data used here are the same as in Section 2.4.1.

Summary statistics

The basic statistics and sample correlation coefficients of asset returns are reported in Table 2.10 and Table 2.11. Similarly , for the time-changed returns the sample moments are correlations are reported in Table 2.12 and Table 2.13.

Table 2.10: Sample moments of original asset returns.

	Mean (10^{-4})	Std. Dev.	Skewness	Kurtosis
MSFT	0.2069	0.0020	-0.0861	11.1120
C	-0.0172	0.0020	0.0882	9.1638
PFE	-0.1640	0.0018	-0.0675	9.2125
GE	-0.2062	0.0016	-0.4057	9.7757
GIS	0.1792	0.0018	0.0329	8.8859

Table 2.11: Sample correlation coefficients of original asset returns.

	MSFT	C	PFE	GE	GIS
MSFT	1.0000				
C	0.1876	1.0000			
PFE	0.0362	0.0507	1.0000		
GE	0.0469	0.0935	0.0439	1.0000	
GIS	0.0452	0.0870	0.0691	0.0566	1.0000

Table 2.12: Sample moments of time-changed asset returns.

	Mean (10^{-4})	Std. Dev.	Skewness	Kurtosis
MSFT	0.2057	0.0020	-0.0944	4.5846
C	-0.0318	0.0020	0.1061	4.2108
PFE	-0.2715	0.0018	-0.1529	4.7207
GE	-0.2197	0.0016	-0.0539	4.1403
GIS	0.0564	0.0018	-0.0702	6.6587

Table 2.13: Sample correlation coefficients of time-changed asset returns.

	MSFT	C	PFE	GE	GIS
MSFT	1.0000				
C	-0.0011	1.0000			
PFE	0.0110	0.0041	1.0000		
GE	-0.0102	0.0182	-0.0121	1.0000	
GIS	0.0139	-0.0120	0.0147	0.0152	1.0000

Estimation of dependency

Since the common jump is one contributor for the dependency, the marginal subordinator processes are shown in Figure 2.8 in order to have a visual impression of how the jumps cluster cross-sectionally.

In Table 2.14, we report the estimated parameters, α_j's, for the cases of Gamma and IG marginal subordinators. The maximal possible value of the common jump parameter a is dominated by the parameters of subordinators, as shown in Table 2.15.

Table 2.14: The estimated parameters, α_j's, for specific distribution of marginal subordinators.

	MSFT	C	PFE	GE	GIS
Gamma	0.4202	0.4807	0.4704	0.5571	0.3072
IG	2.1474	1.7394	1.2923	6.9194	0.4870

To get the optimal value of a, we minimize the following error function between sample correlation and the model correlation

$$J(a,\rho) = \left\| \widehat{\rho}_{Y^\rho} - \rho_{Y^\rho}(a,\rho;\widehat{\mu},\widehat{\sigma},\widehat{\alpha}) \right\|_F , \tag{2.70}$$

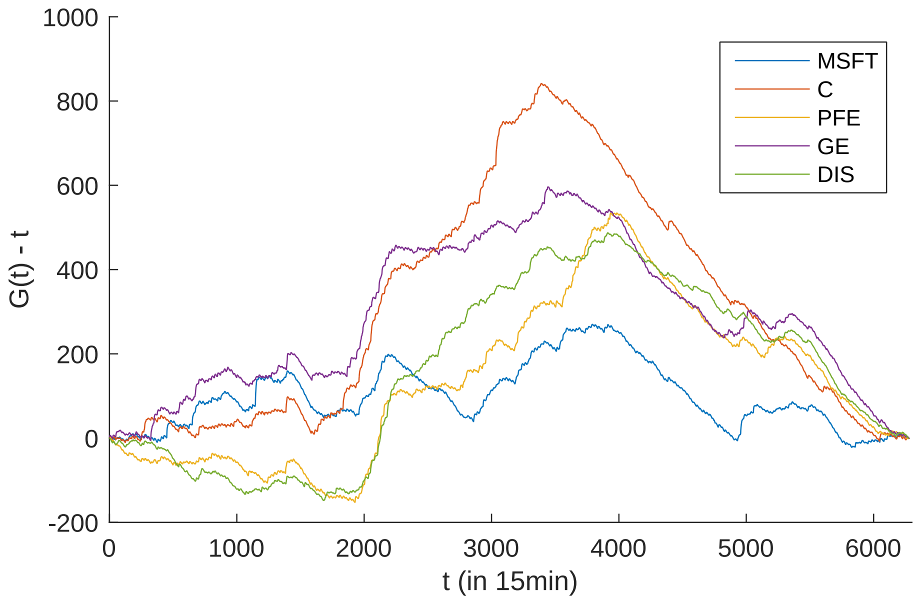


Figure 2.8: Sample marginal subordinator processes.

Table 2.15: The estimated parameters, α_j's, for specific distribution of marginal subordinators.

	maximal a
Gamma	0.3072
IG	0.4870

where $\| \cdot \|_F$ is the Frobenius norm, $\widehat{\rho}_{Y^\rho}$ is the sample correlation matrix of the original asset returns, ρ_{Y^ρ} is the model correlation matrix, $\widehat{\mu}$ and $\widehat{\sigma}$ respectively are the estimated mean and standard deviation of time-changed asset returns, and $\widehat{\alpha}$ are the estimated parameters of marginal subordinators.

The estimated model correlations are provided in Table 2.16 and Table 2.17. The errors are reported in Table 2.19 for each specifications. The estimated correlations of W^ρ is reported in Table 2.20 and Table 2.21.

The differences between sample and model correlation for the Gamma case are reported in Table 2.18. And the differences for the IG case in every entry are all very small ($< 10^{-4}$).

Both Gamma marginal and IG marginal model fit the sample correlation matrix very well. In the Gamma case, the common jump parameter a almost reaches the it maximal possible value, which means common jump component contributes the linear dependency almost as much as the maximum. But still there is a relatively large difference between the sample correlation of (MSFT, C) and the model correlation of the pair.

Note that the model correlation matrix has a as a common multiplier at off-diagonal entries. And the maximal value of a is given by the vairances of marginal subordinator. In some sense, a serves as a shrinkage parameter of the correlation matrix.

Table 2.16: Estimated correlation matrix of the asset returns, Gamma case.

	MSFT	C	PFE	GE	GIS
MSFT	1.0000				
C	0.1378	1.0000			
PFE	0.0362	0.0507	1.0000		
GE	0.0469	0.0934	0.0439	1.0000	
GIS	0.0452	0.0868	0.0690	0.0565	1.0000

Table 2.17: Estimated correlation matrix of the asset returns, IG case.

	MSFT	C	PFE	GE	GIS
MSFT	1.0000				
C	0.1876	1.0000			
PFE	0.0362	0.0507	1.0000		
GE	0.0469	0.0935	0.0439	1.0000	
GIS	0.0452	0.0870	0.0691	0.0566	1.0000

Table 2.18: Difference between sample and model correlation matrix of the asset returns. Gamma Case.

	MSFT	C	PFE	GE	GIS
MSFT	0				
C	-0.0498	0			
PFE	-0.0000	-0.0000	0		
GE	-0.0000	-0.0001	-0.0000	0	
GIS	-0.0000	-0.0002	-0.0001	-0.0001	0

Table 2.19: Estimated value of a, and the error.

	a	Error
Gamma	0.3069	0.0705
IG	0.2562	$< 10^{-4}$

Table 2.20: Estimated correlation of the underlying Brownian motion W^{ρ}.

	MSFT	C	PFE	GE	GIS
MSFT	1.0000				
C	0.9990	1.0000			
PFE	0.2651	0.3471	1.0000		
GE	0.3157	0.5883	0.2792	1.0000	
GIS	0.4096	0.7360	0.5917	0.4454	1.0000

Table 2.21: Estimated correlation of the underlying Brownian motion W^{ρ}.

	MSFT	C	PFE	GE	GIS
MSFT	1.0000				
C	0.3789	1.0000			
PFE	0.0851	0.1319	1.0000		
GE	0.0480	0.1051	0.0567	1.0000	
GIS	0.1724	0.3689	0.3399	0.1204	1.0000

Chapter 3

Multivariate Log-Price-Volatility Model

3.1 Preliminary

3.1.1 Stationary Processes with Long-Range Dependence within Semi-martingale Framework

In this section, we will give the construction of univariate semimartingales with long-range dependence following the results in Barndorff-Nielsen and Stelzer (2009) and Barndorff-Nielsen and Basse-O'Connor (2011).

A Quasi Ornstein-Uhlenbeck (QOU) processes $(X(t))$ is a process driven by a measurable process $(N(t))$,

$$X(t) = X(0) - \lambda \int_0^t X(s)ds + N(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where λ is some positive real number. Thus the process X is a stationary solution to the Langevin equation

$$dX(t) = -\lambda X(t)dt + dN(t). \tag{3.2}$$

For a process Z with finite second moments, that is, Z is $\mathcal{L}^2$-process. Denote by $V(Z,t) = \mathrm{Var}(Z(t))$ the variance function of Z. In addition, if Z is stationary, let $R(Z,t) = \mathrm{Cov}(Z(t), Z(0)))$ denote the autocovariance function and let $\bar{R}(Z,t) = R(Z,0) - R(Z,t)$ be the complementary autocovariance function.

If Y is a $\mathcal{L}^2$-process and its autocovariance function $R(Y,t)$ is regularly varying at ∞ of order $-\alpha$, $0 < \alpha < 1$, then Y is said to have long-range dependence of order α. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is regularly varying at ∞ of index $\beta \in \mathbb{R}$ if, for $t \to \infty$,

$$f(t) \sim t^{\beta} l(t),$$

where l is slowly varying, i.e., for all $a > 0$,

$$\lim_{t\to\infty} l(at)/l(t) = 1.$$

Let Z be a mean zero Lévy process with Lévy measure ν and Gaussian component σ^2. Then denote by $\mathcal{I}(\nu, \sigma)$ the space of all $g : \mathbb{R} \to \mathbb{R}$, integrable with respect to the Lebesgue measure on R and satisfying:

$$\int_{-\infty}^{\infty} \left(g^2(s)\sigma^2 + \int_{-\infty}^{\infty} (|ug(s)|^2 \wedge |ug(s)|)\nu(du) \right) ds < \infty. \tag{3.3}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be measurable on R, and $f(x) =$ for $x < 0$. Suppose that $g_{t,f}(x) := f(t - x) - f(-x)$, $t \in \mathbb{R}$, is a function in $\mathcal{I}(\nu, \sigma)$ for all t in $\mathbb{R}$. A QOU process X is constructed by the Langevin equation

$$dX(t) = \lambda X(t)dt + dN_{Z,f}(t), \tag{3.4}$$

where $\lambda > 0$, and $(N_{z,f}(t))$ is given by

$$N_{Z,f}(t) = \int_{-\infty}^{\infty} g_{t,f}(s)dZ(s) = \int_{\infty}^{\infty} (f(t - s) - f(-s))dZ(s). \tag{3.5}$$

That is,

$$X(t) = X(0) - \lambda \int_0^t X(s)ds + N_{Z,f}(t). \tag{3.6}$$

Assume that for some $\eta \in (-1, -\frac{1}{2})$,

1. $f'(t)$ exists for large t, and $f'(t) \sim ct^\eta$ for some $c \neq 0$, and
2. $|f(t)| \leq rt^\eta$ for all $t > 0$ and some $r > 0$.

Then for $t \to \infty$, $R(X, t) = \mathrm{Cov}(X(t), X(0)) \sim D(c, \eta, \lambda)t^{2\eta+1}$ for some positive constant $D(c, \alpha, \lambda)$. By setting $\alpha := -2\eta - 1$, $X(t)$ has a long-range dependence of order α, $0 < \alpha < 1$.

Definition 10 (Semimartingale with long-range dependence (SLRD))**.** Let Z be a Lévy process with zero mean, and

$$f(t) = f_{\delta,H}(t) := (\delta \wedge t)^{H-\frac{1}{2}},$$

for some $\delta > 0$ and $\frac{1}{2} < H < 1$. Then

$$X(t) = X(0) - \lambda \int_0^t X(s)ds + \int_{-\infty}^{\infty} \left[f_{\delta,H}(t - s) - f_{\delta,H}(-s)\right] dZ(s) \tag{3.7}$$

is a semimartingale with long-range dependence of order $\alpha = 2 - 2H$.

3.1.2 Multivariate Variance Gamma Model

In Equation 3.7, the driving process Z is a zero-mean Lévy process. In the multivariate case of N dimension, we assume that

$$Z(t) = (Z_1(t), ..., Z_N(t))^\top \tag{3.8}$$

is an N-dimensional zero-mean Lévy process. In order to capture the tail dependencies in financial prices while keeping a relatively simple structure, here we choose Z to be zero-mean Multivariate Variance Gamma (MVG) process. Recall first Univariate Variance Gamma (VG) Process with parameters (θ, σ, v) is given by the representation

$$Z(t) = \theta G(t) + \sigma B(G(t)), \tag{3.9}$$

where $B = (B(t))_{t\geq 0}$ is standard Brownian motion, $(G(t))_{t\geq 0}$ is Gamma subordinator process independent of B with $\mathbb{E}(G(1)) = 1$ and $\mathrm{Var}(G(1)) = v$. By letting $a = \theta v$, $b = \sigma^2 v$ and $c = v$, we re reparametrize the univariate VG process by (a, b, c), denoting as $Z \coloneqq VG(a, b, c)$. The characteristic funtion of Z is now given by

$$\psi_{VG(a,b,c)(t)}(u) = \left(\frac{1}{1 - iau + \frac{1}{2}bu^2} \right)^{tc}.$$

Under the new parametrization, $Z(t) = a\widetilde{G}(t) + bB(\widetilde{G}(t))$, where B is standard Brownian motion and $\widetilde{G}$ is Gamma process with $\mathbb{E}(G(1)) = \mathrm{Var}(G(1)) = c$. Note that for N independent VG processes $Z_j = VG(a, b, c_j)$, $j = 1, ..., N$, we have

$$Z_1(t) + ... + Z_N(t) \overset{d}{=} VG(a, b, c_1 + ... + c_N)(t).$$

N-dimensional VG process

Given N univariate VG processes $\widetilde{Z}_i = VG(\theta_i, \sigma_i, v_i)$, $i = 1, ..., N$, a multivariate VG (MVG) process $Z = (Z_1, ..., Z_N)$ can be constructed with the same marginals

$$Z_i \overset{d}{=} \widetilde{Z}_i, \quad i = 1, ..., N,$$

and having the dependence structure given by

$$Z_i = A_i + Y_i, i = 1, ..., N. \tag{3.10}$$

where $A = (A_i, ..., A_N)$ is a N-dimensional process with dependent components:

$$A_i(t) = \theta_i \frac{v_i}{v_0} G_{v_0}(t) + \sigma_i \sqrt{\frac{v_i}{v_0}} B_i(G_{v_0}(t)) \tag{3.11}$$

$$\stackrel{d}{=} VG\left(\theta \frac{v_i}{v_0}, \sigma_i \sqrt{\frac{v_i}{v_0}}, v_0\right)(t), \quad i = 1, .., N, \tag{3.12}$$

where $B(t) = (B_1(t), ..., B_N(t)),\ t \geq 0$ is the standard N-dimensional Brownian motion with $dB_i(t)dB_j(t) = \rho_{ij}dt$, and G_{v_0} is Gamma process independent of B with mean-variance parameters $(1, v_0)$, $v_0 \geq \vee_{i=1}^N v_i$; And $Y = (Y_1, ..., Y_N)$ is a N-dimensional process with independent univariate VG marginal components

$$Y_i = VG\left(\theta_i \left(1 - \frac{v_i}{v0}\right), \sigma_i \sqrt{1 - \frac{v_i}{v0}}, \frac{v_i v_0}{v_0 - v_i}\right),\ i = 1, ..., N. \tag{3.13}$$

Y is also independent from A. A is viewed as the common factor, while Y as idiosyncratic terms.

The distributional tail dependence between Z_i and Z_j is relatively limited as $v_0 \geq \vee_{i=1}^N v_i$. That is the maximal tail correlation will obtained when $v_0 = \vee_{i=1}^N v_i$. When $v_o \to \infty$, Z_i and Z_j become independent. The centeral distributional dependence will be governed by the correlation coefficients ρ_{ij}.

Most importantly, $Z = (Z_1, ..., Z_N)$ is a a N-dimensional Lévy process, with joint characteristic function given by, for $u = (u_1, .., u_N) \in \mathbb{R}^N$,

$$\psi_{Z(t)}(u) = \mathbb{E}\left[e^{i\langle u, Z(t)\rangle}\right] \tag{3.14}$$

$$= \left(1 - i\sum_{n=1}^N u_n\theta_n v_n + \frac{1}{2}u\Sigma u^\top\right)^{-\frac{1}{v_0}} \prod_{n=1}^N \left(1 - iu_n\theta_n v_n + \frac{1}{2}\sigma^2 v_n u_n^2\right)^{\frac{1}{v_0} - \frac{1}{v_n}}, \tag{3.15}$$

where $\Sigma = (\Sigma_{ij})_{i,j=1,...,N}$ is $N \times N$ matrix with entries

$$\Sigma_{ii} = \sigma_i^2 v_i, \tag{3.16}$$

$$\Sigma_{ij} = \sigma_i\sigma_j\rho_{ij}\sqrt{v_i v_j} \quad \text{for } i \neq j. \tag{3.17}$$

The correlation for $Z_i(t)$ and $Z_j(t)$ is given by

$$\operatorname{corr}(Z_i(t), Z_j(t)) = \frac{\theta_i\theta_j v_i v_j + \sigma_i\sigma_j\rho_{ij}\sqrt{v_i v_j}}{v_0\sqrt{(\theta^2 v_i + \sigma_i^2)(\theta^2 v_j + \sigma_j^2)}}. \tag{3.18}$$

Now we can use MVG process $\widetilde{Z}$ to generate the multivariate semitmartingale with different order of long-range dependence for the marginal processes. Let

$$f_i(t) = f_{\delta,H_i}(t) = (\delta \wedge t)^{H_i - \frac{1}{2}}, \quad i, ..., N, \tag{3.19}$$

for some $\delta > 0$ and $\frac{1}{2} < H_i < 1$. For each $i = 1, ..., N$, the processes R_i, defined by

$$R_i(t) = R_i(0) - \lambda_i \int_0^t R_i(s)ds + \int_{-\infty}^{\infty} (f_i(t-s) - f_i{-}s)\, dZ_i(s), \tag{3.20}$$

is a semimartingale with long range dependence of order $\alpha_i = 2 - 2H_i$. Then the multivariate process $R = (R_1, ..., R_N)$ is a N-dimensional semimartingale, so-called Multivariate Roger Process. The increments $R(t+h) - R(t)$ could have heavy-tailed and skewed distributions, with heavy-tailed copula dependence.

3.1.3 Multivariate Lévy Processes with Long-Range Dependence

In section, we will further generalize the model following the results in Barndorff-Nielsen and Stelzer (2013). The general model for Multivariate Roger Process will be so-called Barndorff-Nielsen-Stelzer (BNS) process. A BNS process R is generated by the following stochstic differential equation

$$dR(t) = a(t)dt + b(\epsilon(t-))dB(t) + \psi(dL(t)), \quad R(0) = 0, \tag{3.21}$$

where $B = (B_1, ..., B_N)$ is standard N-dimensional Brownian motion, $(L(t))$ is N-dimensional Lévy process, ϵ is multivariate volatility process, and a, b, ψ are some deterministic operators.

The goal here is to describe important cases when $R(t)$ is multivariate semimartingale with possibly different long-range dependence order in each marginal. We start with some notations:

- $\mathbb{R}^{N,M}$, the space of real N, M matrices
- $\mathbb{C}^{N,M}$, the space of complex N, M matrices
- $\mathcal{S}^N$, the space of symmetric matrices in $\mathbb{R}^{N,N}$
- $\mathcal{S}^N_+$, the cone of positive definite matrices in $\mathbb{R}^{N,N}$

- $\bar{\mathcal{S}}_+^N$, the cone of positive semidefinite matrices in $\mathbb{R}^{N,N}$
- $I^{N,N}$, the identity matrix in $\mathbb{R}^{N,N}$
- $\mathcal{I}^N$, the group of invertible matrices in $\mathbb{R}^{N,N}$
- $\mathcal{L}^N$, the space of linear operators from $\mathbb{R}^{N,N}$ to $\mathbb{R}^N$
- $\mathcal{LS}^N$, the space of linear operators from $\mathcal{S}^N$ to $\mathbb{R}^N$
- $\varsigma(A)$, the set of all eigenvalues of $A \in \mathbb{C}^{N,N}$
- $\langle A, B\rangle_{\mathbb{R}^{N,N}} = tr(A^\top B)$, the scalar product on $\mathbb{R}^{N,N}$
- $\langle A, B\rangle_{\mathbb{C}^{N,N}} = tr(A^\top B)$, the scalar product on $\mathbb{C}^{N,N}$
- $vec : \mathbb{R}^{N,N} \to \mathbb{R}^{N^2}$, the bijective linear operator that stacks the columns of a matrix below on other
- $vech(A)$, for $A \in \mathcal{S}^N$, the vector consisting of the upper diagonal part including the diagonal
- If L is a $\mathbb{R}^{M,N}$-valued Lévy process, A is a $\mathbb{R}^{K,M}$-valued Lévy process, and A is a $\mathbb{R}^{N,L}$-valued Lévy process , then the matrix integral $\int_0^t A(s)dL(s)B(s)$ stands for an $K \times L$ matrix with (i,j) entry

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\int_0^t A_{ip}(s)dL_{p,q}(s)B_{qj}(s).$$

We next give the definition of multivariate stochastic volatility (MSV) Ornsterin-Uhlenbeck (OU) process. Let

- $(a(t))$ be a predictable $\mathbb{R}^N$-valued process;
- L be a matrix subordinator, independent of the N-dimensional Brownian motion $(B(t))$. That is, L is a $\mathbb{R}^{N,N}$-valued Lévy process which increments are positive semidefinite and let $\mathbb{E}\left(\max\left(\log\|L(1)\|, 0\right)\right) < \infty$;
- $A \in \mathbb{R}_-^{N,N}$, the subspace of $\mathbb{R}^{N,N}$ with elements $\tilde{A}$ having $\varsigma(\tilde{A}) \subset (-\infty, 0) + i(-\infty, \infty)$;

- $b : \mathcal{S}_+^N \to \mathbb{R}^{N,N}$ be the continuous function , wich that $b(x)b(x)^\top = x$ for all $x \in \mathcal{S}^N$ (A nature choice for B is the positive definite square root);

- $\psi \in \mathcal{L}^N$.

Then $R(t) = (R_1(t), ..., R_N(t))^\top$, $t \geq 0$, is said to be SVOU process with parameters (a, b, ψ, A, L), or an $SVOU(a, b, \psi, A, L)$-process if

$$dR(t) = a(t)dt + b(\epsilon(t-))dB(t) + \psi(dL(t)), \quad R(0) = 0, \tag{3.22}$$

where ϵ is $\mathcal{S}_+^N$-valued OU process, that is, it is the unique stationary solution to

$$d\epsilon(t) = \left(A\epsilon(t-) + \epsilon(t-)A^\top\right) dt + dL(t). \tag{3.23}$$

The solution of Equation 3.23 is

$$\epsilon(t) = \int_{-\infty}^{t} e^{A(t-s)} dL(s) e^{A^\top (t-s)}, \quad t \geq 0. \tag{3.24}$$

The process ϵ is a mean reverting stochastic volatility positive definite matrix process introducing stochastic dependence between marginals. ϵ increases only with the jumps which represents random shocks to the volatilities and the correlations. The leverage term $\psi(dL(t))$ correlated those volatility jumps with the price jumps which constitutes the leverage effect.

Next we will give the definition of infinitely divisible independently scattered random measure, shortly called, Lévy bases.[1]. Let $\mathcal{B}_{\mathcal{B}}(\mathbb{R}_-^{N,N} \times \mathbb{R})$ be the space of bounded Borel sets on the product space $\mathbb{R}_-^{N,N} \times \mathbb{R}$.

Definition 11 (Lévy basis)**.** λ is a Lévy basis if

1. $\lambda : \mathbb{R}_-^{N,N} \times \mathbb{R} \to \mathcal{S}^N$ with $\lambda(B)$ being a Borel set of $\mathcal{S}^N$ for all $B \in \mathcal{B}_{\mathcal{B}} = \mathcal{B}_{\mathcal{B}}(\mathbb{R}_-^{N,N} \times \mathbb{R})$;

2. for every $B \in \mathcal{B}_{\mathcal{B}}$, $\lambda(B)$ is infinitely divisible random matrix;

[1]See Barndorff-Nielsen, Shephard, Neil (2012)

3. for every disjoint sets $B_1, ..., B_k$ in $\mathcal{B}_\mathcal{B}$, the random mappings $\lambda(B_1), ..., \lambda(B_k)$ are independent, $k = 2, .3, ...$;

4. for every sequence of disjoint sets $B_1, B_2, ...$ in $\mathcal{B}_\mathcal{B}$, such that its union is also in $\mathcal{B}_\mathcal{B}$ then $\sum_{k=1}^{\infty} \lambda(B_k)$ exists almost surely and

$$\sum_{k=1}^{\infty} \lambda(B_k) = \lambda\left(\sum_{k=1}^{\infty} B_k\right)$$

almost surely.

The Lévy basis λ is called Homogeneous (in time) and Factorizable (into one underlying infinitely divisible distribution and one probability distribution on $\mathbb{R}_-^{N,N}$), if its characterisitc function has the form

$$\mathbb{E}\left(e^{i\langle u,\lambda(B)\rangle}\right) = \exp\left(\phi(u)\Pi(B)\right), \tag{3.25}$$

for all $u \in \mathbb{R}^N$, $B \in \mathcal{B}_\mathcal{B}$, where

1. $\Pi = \pi \times Leb$, π is a probability measure on $\mathbb{R}_-^{N,N}$, and Leb is the Lebeseque measure on R;

2. $\phi(u)$ is the log-characteristic function of an infinitely divisible distribution on $\mathbb{R}^N$ with Lévy triplet (g, S, ν), $g \in \mathbb{R}^N$, $S \in \mathcal{S}_+^N$, ν is a Lévy measure, that is, a Borel measure on $\mathbb{R}^N$ with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^N}(\|x\|^2 \wedge 1)\nu(dx) < \infty$. Thus $\phi(u)$ has the form:

$$\phi(u) = iu^\top g - \frac{1}{2}u^\top Su + \int_{\mathbb{R}^N} \left(\exp(iu^\top x) - 1 - iu^\top x\mathbf{1}_{\|x\|\leq 1}\right)\nu(dx); \tag{3.26}$$

3. (g, S, ν, π) is the generating quadruple for the Lévy basis λ.

Now we introduce vector-valued Barndorff-Nielsen-Stelzer (BNS) processes, knows as supOU processes as random mixture of OU processes where mixing is over various mean reverting coefficients.

Theorem 7 (BNS)**.** *Let λ be a $\mathbb{R}^N$-valued Lévy basis on $\mathbb{R}_-^{N,N} \times \mathbb{R}$ with generating quadruple (g, S, ν, π) satisfying*

$$\int_{\|x\|>1} \log(\|x\|)\nu(dx) < \infty, \tag{3.27}$$

and assume there exist measurable functions $\rho : \mathbb{R}_-^{N,N} \to (0, \infty)$ and $\kappa : \mathbb{R}_-^{N,N} \to [1, \infty)$ such that

$$\|As\| \leq \kappa(A)e^{-\rho(A)s} \quad \forall s \in \mathbb{R}_+, \pi\text{-almost surely,} \tag{3.28}$$

and

$$\int_{\mathbb{R}_-^{N,N}} \frac{\kappa(A)^2}{\rho(A)}\pi(dA) < \infty. \tag{3.29}$$

Then the process $(X(t))_{t\in\mathbb{R}}$ given by

$$X(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)}\lambda(dA, ds) \tag{3.30}$$

is well defined for all $t \in \mathbb{R}$ and stationary. The distribution of X_t is infinitely divisible with characteristic triplet (g_X, S_X, ν_X) given by

$$g_X = \int_{\mathbb{R}_-^{N,N}} \int_{\mathbb{R}_+} \left(e^{As}g + \int_{\mathbb{R}^N} x(\mathbf{1}_{\|e^{As}x\|\leq 1} - \mathbf{1}_{\|x\|\leq 1})\nu(dx) \right) ds\ \pi(dA), \tag{3.31}$$

$$S_X = \int_{\mathbb{R}_-^{N,N}} \int_{\mathbb{R}_+} e^{As}Se^{A^*s}ds\ \pi(dA), \tag{3.32}$$

$$\nu_X(B) = \int_{\mathbb{R}_-^{N,N}} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \mathbf{1}_{e^{As}x\in B}\nu(dx)ds\ \pi(dA) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^N). \tag{3.33}$$

Specials Cases

Let $\mathbb{R}^{N,N}_{-,Normal}$ be the space of Gaussian distributed matrices $A \in \mathbb{R}^{N,N}$ with $\varsigma(A) \subset (-\infty) + i(-\infty, \infty)$.

Case 1. Suppose $\pi(\mathbb{R}^{N,N}_{-,Normal}) = 1$, then (3.28)(3.29) are satisfied with

$$\kappa(A) = 1,$$

and

$$\rho(A) = -\max(\Re(\varsigma(A))).$$

Case 2. Suppose there are diagonalizable A_k, $k = 1, ..., K$ in $\mathbb{R}^{N,N}_{-}$, such that $\pi(\{\lambda A_k, k = 1, ..., K, \lambda \neq 0\}) = 1$, (3.28)(3.29) are satisfied with

$$\kappa(A) \equiv c \geq 1,$$

and

$$\rho(A) = -\max(\Re(\varsigma(A))).$$

We are ready now to give some examples of BNS process with long-range dependence (LRD). Here by long-range dependence, it is understood as at least for one $i = 1, ..., N$,

$$\sum_{h=1}^{\infty} |\mathrm{Cov}(X_i(h), X_i(0))| = \infty. \tag{3.34}$$

BNS Example 1.

Let λ be a Lévy basis with generating quadruple (g, S, ν, π) with ν satisfying

$$\int_{\mathbb{R}^N} \|x\|^2 \nu(dx) < \infty,$$

and

$$\pi \stackrel{d}{=} G(\alpha, \beta)B,$$

where B is a fixed diagonalizable matrix in $\mathbb{R}_-^{N,N}$, and $G(\alpha,\beta)$ is a $Gamma(\alpha,\beta)$ distributed random variable with $\alpha > 1$ and $\beta > 0$. That is, the density of $G(\alpha,\beta)$ is given by

$$f_{G(\alpha,\beta)}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{\beta x}, \ x \geq 0.$$

Then the process X, given by

$$X(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)} \lambda(dA) ds, \ t \in \mathbb{R} \tag{3.35}$$

is a BNS process (and thus stationary and infinite divisible).

Furthermore,

$$\mathrm{Cov}(X(h), X(0)) = -\frac{\beta^\alpha}{\alpha-1} (\beta I_N - Bh)^{1-\alpha} \mathfrak{B}^{-1}(S) + \int_{\mathbb{R}^N} x x^\top \nu(dx), \tag{3.36}$$

where $\mathfrak{B} : \mathbb{R}^{N,N} \to \mathbb{R}^{N,N}$ given by

$$\mathfrak{B}(A) = BA + AB^\top.$$

Thus for $\alpha \in (1,2)$, X exhibits LRD.

BNS Example 2.

This is an extension of BNS Example 1, when π is concentrated on several rays: let $w_i, i = 1, ..., m$ be positive weights, with total sum 1 and fix diagonalizable $B_i \in \mathbb{R}_-^{N,N}, i = 1, ..., m$. Let $\pi_i \overset{d}{=} G(\alpha_i, \beta_i) B_i$ where $G(\alpha_i, \beta_i)$ is a $Gamma(\alpha_i, \beta_i)$-distributed random variable with $\alpha_i > 1$ and $\beta_i > 0$. Let ν be as Example 1 with

$$\int_{\mathbb{R}^N} \|x\|^2 \nu(dx) < \infty,$$

and let

$$\pi = \sum_{i=1}^{m} w_i \pi_i.$$

Then the process

$$X(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)} \lambda(dA) ds, \ t \in \mathbb{R} \tag{3.37}$$

is a BNS process with

$$\mathrm{Cov}(X(h), X(0)) = -\sum_{i=1}^{m} \frac{w_i \beta_i^{\alpha_i}}{\alpha_i - 1} (\beta_i I_N - B_i h)^{1-\alpha_i} \mathfrak{B}_i^{-1} \left(S + \int_{\mathbb{R}^N} x x^\top \nu(dx) \right), \tag{3.38}$$

where $\mathfrak{B} : \mathbb{R}^{N,N} \to \mathbb{R}^{N,N}$ given by

$$\mathfrak{B}_i(A) = B_i A + A B_i^\top.$$

Thus if for at least one i, $\alpha_i \in (1, 2)$, X exhibits LRD.

BNS Example 3.

Let λ be a Lévy basis with generating quadruple (g, S, ν, π) with ν satisfying $\int_{\mathbb{R}^N} \|x\|^2 \nu(dx) < \infty$. Let π be a measure on the space $\mathcal{D}_-^{N,N}$ of $N \times N$ diagonal matrices with strictly negative entries on the diagonal, thus identified as a measure on $(-\infty, 0)^N$. And let π have a probability density with independent marginals

$$\pi(da_1, ..., da_N) = \prod_{i=1}^{N} \pi_i(da_i), \quad a_i < 0, i = 1, ..., N,$$

with $(-\pi_i) \stackrel{d}{=} G(\alpha_i, \beta_i) B_i$ and $G(\alpha_i, \beta_i)$ is a $Gamma(\alpha_i, \beta_i)$-distributed random variable with $\alpha_i > 1$ and $\beta_i > 0$, $i = 1, ..., N$.

Then the process

$$X(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)} \lambda(dA) ds, \; t \in \mathbb{R} \tag{3.39}$$

is a BNS process. It is stationary and infinitely divisible. The autocovariance function of the i-th component can be written as

$$\mathrm{Cov}(X_i(h), X_i(0)) = \frac{\beta_i^{\alpha_i}}{2(\alpha_i - 1)} (\beta_i + h)^{1-\alpha_i} \left(S_{ii} + \int_{\mathbb{R}} x_i^2 \nu_i(dx_i) \right), \tag{3.40}$$

and thus we have LRD in i-th components if $\alpha_i \in (1, 2)$.

3.1.4 Matrix-valued Lévy Processes

In this section, we give the definition of a matrix-valued Lévy basis and matrix-valued BNS process, which will be stationary and infinitely divisible.

Let $\mathcal{B}_{\mathcal{B}}(\mathbb{R}_{-}^{N,N} \times \mathbb{R})$ be the space of bounded Borel sets on the product space $\mathbb{R}_{-}^{N,N} \times \mathbb{R}$. The the space

$$\mathcal{LB}_{\mathbb{M}} = \mathcal{LB}_{\mathbb{M}} \left(\mathbb{R}_{-}^{N,N} \times \mathbb{R}, \mathcal{S}^N \right)$$

of all random mappings from $\mathbb{R}_{-}^{N,N} \times \mathbb{R}$ to $\mathcal{S}^N$.

Lévy basis will be defined as follows.

Definition 12 (Matrix-valued Lévy basis). $\lambda_{\mathbb{M}} \in \mathcal{LB}_{\mathbb{M}}$ is a matrix-valued Lévy basis if

1. $\lambda_{\mathbb{M}}(B)$ is a Borel set of $\mathcal{S}^N$ for all $B \in \mathcal{B}_{\mathcal{B}}$;
2. $\lambda_{\mathbb{M}}(B)$ is an infinitely divisible random matrix for every $B \in \mathcal{B}_{\mathcal{B}}$;
3. for every set of disjoint sets $B_1, ..., B_k$ in $\mathcal{B}_{\mathcal{B}}$, the random mappings $\lambda_{\mathbb{M}}(B_1), ..., \lambda_{\mathbb{M}}(B_k)$ are independent, $k = 2, 3, 4...$;
4. for every sequence of disjoint sets $B_1, B_2, ...$ in $\mathcal{B}_{\mathcal{B}}$, such that its union is also in $\mathcal{B}_{\mathcal{B}}$, then $\sum_{k=1}^{\infty} \lambda_{\mathbb{M}}(B_k)$ exists almost surely and

$$\sum_{k=1}^{\infty} \lambda_{\mathbb{M}}(B_k) = \lambda_{\mathbb{M}} \left(\sum_{k=1}^{\infty} B_k \right)$$

almost surely.

The Lévy basis $\lambda_{\mathbb{M}} \in \mathcal{LB}$ is called Homogeneous (in time) and Factorizable (into one underlying infinitely divisible distribution and one probability distribution on $\mathbb{R}_{-}^{N,N}$), if its characteristic function has the form

$$\mathbb{E}(\exp(i\ tr(u_{\mathbb{M}} \lambda_{\mathbb{M}}(B)))) = \exp(\phi_{\mathbb{M}}(u_{\mathbb{M}}) \Pi(B)), \quad \text{for all } u_{\mathbb{M}} \in \mathcal{S}_N, B \in \mathcal{B}_{\mathcal{B}}. \tag{3.41}$$

where

1. $\Pi = \pi \times Leb$, π is a probability measure on $\mathbb{R}_-^{N,N}$, and Leb is the Lebesgue measure on $\mathbb{R}$;

2. $\phi_{\mathbb{M}}(u_{\mathbb{M}})$ is the log-characteristic function of an infinitely divisible distribution on $\mathcal{S}^N$ with Lévy triplet $(g_{\mathbb{M}}, 0, \nu_{\mathbb{M}})$, $g_{\mathbb{M}} \in \mathcal{S}^N$, $\nu_{\mathbb{M}}$ is a Lévy measure on $\mathcal{S}^N$, that is a Borel measure on $\mathcal{S}^N$ with $\nu_{\mathbb{M}}(0) = 0$ and

$$\int_{\mathcal{S}^N} (\|x\|^2 \wedge 1)\nu_{\mathbb{M}}(dx) < \infty,$$

   and thus $\psi_{\mathbb{M}}(u_{\mathbb{M}})$ has the form

$$\psi_{\mathbb{M}}(u_{\mathbb{M}}) = i\ tr(u_{\mathbb{M}} g_{\mathbb{M}}) + \int_{\mathcal{S}^N} \left(e^{i\ tr(u_{\mathbb{M}} g_{\mathbb{M}})} - 1\right) \nu_{\mathbb{M}}(dx); \tag{3.42}$$

3. $(g_{\mathbb{M}}, \nu_{\mathbb{M}}, \pi)$ is the generating triplet for the matrix-value Lévy basis $\lambda_{\mathbb{M}}$.

For a given $\lambda_{\mathbb{M}} \in \mathcal{LB}_{\mathbb{M}}$, we can define a matrix-valued Lévy process $L_{\mathbb{M}}(t)$, $t \in \mathbb{R}$, as follows

1. $L_{\mathbb{M}}(0) = 0$;

2. $L_{\mathbb{M}}(t, A_{\mathbb{M}}) = \lambda_{\mathbb{M}}(A_{\mathbb{M}} \times (0, t])$, $t > 0$, for any measure rectangle $A_{\mathbb{M}} \times (0, t) \in \mathcal{B}_{\mathcal{B}}$ with $A_{\mathbb{M}}$ being a Borel set in $\mathbb{R}_-^N$;

3. $L_{\mathbb{M}}(t, A_{\mathbb{M}}) = \lambda_{\mathbb{M}}(A_{\mathbb{M}} \times (t, 0])$, $t < 0$, for any measure rectangle $A_{\mathbb{M}} \times (t, 0] \in \mathcal{B}_{\mathcal{B}}$ with $A_{\mathbb{M}}$ being a Borel set in $\mathbb{R}_-^N$.

Thus $L_{\mathbb{M}}$ has a Lévy triplet $(g_{\mathbb{M}}, 0, \nu_{\mathbb{M}})$ and will be called $\lambda_{\mathbb{M}}$ Lévy process.

We next construct a BNS stochastic volatility model. To this end, we shall reformulate Theorem 7 (construction of BNS process) for matrix-valued BNS process.

A $N \times N$ matrix-valued Lévy basis $\lambda_{\mathbb{M}} \in \mathcal{LB}_{\mathbb{M}}$ will be identified by the $vec(\lambda_{\mathbb{M}})$, which is a $\mathbb{R}^{N^2}$-valued Lévy basis given by $vec(\lambda_{\mathbb{M}})(B) = vec(\lambda_{\mathbb{M}}(B))$ for every Borel set on $\mathbb{R}^{N,N}$. Recall that vec is a Hilbert space isometry between $\mathbb{R}^{N,N}$ (equipped with the scalar product) and $\mathbb{R}^{N^2}$ (equipped with the Euclidean scalar product).

Matrix-valued Barndorff-Nielsen-Stelzer (BNS) processes can be constructed by the following theorem.

Theorem 8 (BNS matrix-value process). *Let $\lambda_{\mathbb{M}} \in \mathcal{LB}_{\mathbb{M}}$ be Lévy basis with generating triplet $(g_{\mathbb{M}}, \nu_{\mathbb{M}}, \pi)$, where $\nu_{\mathbb{M}}$ takes non-zeros values only on the space of positive definite matrices $\mathcal{S}_+^N$ and*

$$\int_{\|x\|>1} \log(\|x\|)\nu_{\mathbb{M}}(dx) < \infty. \tag{3.43}$$

Let $\rho : \mathbb{R}_-^{N,N} \to (0,\infty)$ and $\kappa : \mathbb{R}_-^{N,N} \to [1,\infty)$ be measurable functions such that

$$\|A^s\| \le \kappa(A)\exp(-\rho(A)s), \quad \forall s \in \mathbb{R}_+, \pi - \textit{almost surely}, \tag{3.44}$$

and

$$\int_{\mathbb{R}_-^{N,N}} \frac{\kappa(A)^2}{\rho(A)}\pi(dA) < \infty. \tag{3.45}$$

Then the process $\mathbb{Y} = (\mathbb{Y}(t))_{t\in\mathbb{R}}$, given by

$$\mathbb{Y}(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)}\lambda_{\mathbb{M}}(dA, ds)e^{A^\top(t-s)}, \tag{3.46}$$

is well-defined for all $t \in \mathbb{R}$. Furthermore, Y is stationary infinitely divisible matrix process with log-characteristic function

$$\log\left(\mathbb{E}\left(e^{i\ tr(u_{\mathbb{M}}\mathbb{Y}(1))}\right)\right) = i\ tr(u_{\mathbb{M}}g_{\mathbb{Y}}) + \int_{\mathbb{R}_-^{N,N}} \left(\exp(i\ tr(u_{\mathbb{M}}x) - 1)\nu_{\mathbb{Y}}(dx)\right),\ u_{\mathbb{M}} \in \mathbb{R}_-^{N,N} \tag{3.47}$$

where

$$g_{\mathbb{Y}} = \int_{\mathbb{R}_-^{N,N}} \int_0^\infty e^{As}g_{\mathbb{M}}e^{A^\top s}ds\ \pi(dA), \tag{3.48}$$

$$\nu_{\mathbb{Y}}(B) = \int_{\mathbb{R}_-^{N,N}} \int_0^\infty \int_{\mathcal{S}_+^N} \mathbf{1}_{\exp(As+A^\top s)\in B}\nu_{\mathbb{M}}(dx)ds\ \pi(dA),\ \forall B \in \mathcal{B}(\mathcal{S}^N). \tag{3.49}$$

We call $\mathbb{Y}$ $\lambda_{\mathbb{M}}$-underlying BNS Lévy process.

Specials Cases

Let $\mathbb{R}^{N,N}_{-,Normal}$ be the space of Gaussian distributed matrices $A \in \mathbb{R}^{N,N}$ with $\varsigma(A) \subset (-\infty) + i(-\infty, \infty)$.

Case 1. Suppose $\pi(\mathbb{R}^{N,N}_{-,Normal}) = 1$, then (3.44)(3.45) are satisfied with

$$\kappa(A) = 1,$$

and

$$\rho(A) = -\max(\Re(\varsigma(A))).$$

Case 2. Suppose there are diagonalizable A_k, $k = 1, ..., K$ in $\mathbb{R}^{N,N}_{-}$, such that $\pi(\{\lambda A_k, k = 1, ..., K, \lambda \neq 0\}) = 1$, (3.44)(3.45) are satisfied with

$$\kappa(A) \equiv c \geq 1,$$

and

$$\rho(A) = -\max(\Re(\varsigma(A))).$$

We are ready now to give some examples of BNS process with long-range dependence (LRD). Here by long-range dependence, it is understood as at least for one $i = 1, ..., N$,

$$\sum_{h=1}^{\infty} |\mathrm{Cov}(\mathbb{Y}_{ii}(h), \mathbb{Y}_{ii}(0))| = \infty. \tag{3.50}$$

Matrix-valued BNS Example 1.

Let $\lambda_{\mathbb{M}} \in \mathcal{LB}$ be a Lévy basis with generating triplet $(g_{\mathbb{M}}, \nu_{\mathbb{M}}, \pi)$, where $\nu_{\mathbb{M}}$ takes non-zeros values only on the space of positive definite matrices and

$$\int_{\|x\| \geq 1} \log(\|x\|) \nu_{\mathbb{M}}(dx) < \infty,$$

and $\pi \overset{d}{=} G(\alpha,\beta)B$ for a fixed diagonalizable $B \in \mathbb{R}_-^{N,N}$, and $G(\alpha,\beta)$ is a *Gamma*(α,β)-distributed random variable with $\alpha > 1$, $\beta > 0$. Then the process

$$\mathbb{Y}(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)} \lambda_{\mathbb{M}}(dA, ds) e^{A^\top (t-s)},\ t \in \mathbb{R}, \tag{3.51}$$

is a BNS matrix-valued process. Furthermore, the auto-covariance function is

$$\begin{aligned}\mathrm{Cov}(vec(\mathbb{Y}(h)), vec(\mathbb{Y}(0))) = &- \frac{\beta^\alpha}{\alpha - 1}\big[\beta I_{N^2} - (B \otimes I_N + I_N \otimes B)h\big]^{1-\alpha} \\ &\mathfrak{B}^{-1}\left(\int_{\mathcal{S}^N} vec(x) vec(x)^\top \nu_{\mathbb{M}}(dx)\right),\end{aligned} \tag{3.52}$$

where $\mathfrak{B} : \mathbb{R}^{N^2,N^2} \to \mathbb{R}^{N^2,N^2}$ defined by

$$\mathfrak{B}(A) = (B \otimes I_N + I_N \otimes B)A + A(B^\top \otimes I_N + I_N \otimes B^\top).$$

Thus, for $\alpha \in (1,2)$, $\mathbb{Y}$ exhibits long-range dependence.

Matrix-valued BNS Example 2.

This is an extension of Matrix-valued BNS Example 1, when π is concentrated on several rays: let $w_i, i = 1, ..., m$ be positive weights, with total sum 1 and fix diagonalizable $B_i \in \mathbb{R}_-^{N,N}, i = 1, ..., m$. Let $\pi_i \overset{d}{=} G(\alpha_i, \beta_i)B_i$ where $G(\alpha_i,\beta_i)$ is a *Gamma*(α_i,β_i)-distributed random variable with $\alpha_i > 1$ and $\beta_i > 0$. Let $\nu_{\mathbb{M}}$ be as Example 1 with

$$\int_{\mathbb{R}^N} \|x\|^2 \nu_{\mathbb{M}}(dx) < \infty,$$

and let

$$\pi = \sum_{i=1}^{m} w_i \pi_i.$$

Then the process

$$\mathbb{Y}(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)} \lambda_{\mathbb{M}}(dA, ds) e^{A^\top (t-s)},\ t \in \mathbb{R} \tag{3.53}$$

is a BNS matrix-valued process with

$$\begin{aligned}\mathrm{Cov}(vec(\mathbb{Y}(h)), vec(\mathbb{Y}(0))) = -\sum_{i=1}^{m} \frac{w_i \beta_i^{\alpha_i}}{\alpha_i - 1} \left(\beta_i I_{N^2} - (B_i \otimes I_N + I_N \otimes B_i)h\right)^{1-\alpha_i} \\ \mathfrak{B}_i^{-1} \left(\int_{\mathcal{S}^N} vec(x) vec(x)^\top \nu(dx) \right),\end{aligned} \tag{3.54}$$

where $\mathfrak{B} : \mathbb{R}^{N^2,N^2} \to \mathbb{R}^{N^2,N^2}$ defined by

$$\mathfrak{B}(A) = (B \otimes I_N + I_N \otimes B)A + A(B^\top \otimes I_N + I_N \otimes B^\top).$$

Thus if for at least one i, $\alpha_i \in (1,2)$, X exhibits LRD.

Matrix-valued BNS Example 3.

Let $\lambda_{\mathbb{M}} \in \mathcal{LB}_{\mathbb{M}}$ be a Lévy basis with generating triplet $(g_{\mathbb{M}}, \nu_{\mathbb{M}}, \pi)$ with $\nu_M M$ satisfying $\int_{\|x\|\geq 1} \log(\|x\|)\nu(dx) < \infty$. Let π be a measure on the space $\mathcal{D}_-^{N,N}$ of $N \times N$ diagonal matrices with strictly negative entries on the diagonal, thus identified as a measure on $(-\infty, 0)^N$. And let π have a probability density with independent marginals

$$\pi(da_1, ..., da_N) = \prod_{i=1}^{N} \pi_i(da_i), \quad a_i < 0, i = 1, ..., N,$$

with $(-\pi_i) \stackrel{d}{=} G(\alpha_i, \beta_i) B_i$ and $G(\alpha_i, \beta_i)$ is a $Gamma(\alpha_i, \beta_i)$-distributed random variable with $\alpha_i > 1$ and $\beta_i > 0$, $i = 1, ..., N$.

Then the process

$$\mathbb{Y}(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} e^{A(t-s)} \lambda(dA) ds, \ t \in \mathbb{R} \tag{3.55}$$

is a BNS process. It is stationary and infinitely divisible. The autocovariance function of the i-th component can be written as

$$\mathrm{Cov}(\mathbb{Y}_{ii}(h), \mathbb{Y}_{ii}(0)) = \frac{\left(\frac{\beta_i}{2}\right)^{\alpha_i}}{2(\alpha_i - 1)} \left(\frac{\beta_i}{2} + h\right)^{1-\alpha_i} \int_{\mathbb{R}} x_i^2 \nu_i(dx_i), \tag{3.56}$$

and thus we have LRD in i-th components if $\alpha_i \in (1,2)$.

3.2 Multivariate LPV Model

Denote the price process of n securities $S = (S(t))$ by

$$S(t) = (S_1(t), ..., S(n, t)),\ t \geq 0. \tag{3.57}$$

Let the dynamics of S be determined by a $N = 2n$ dimensional mean reverting BNR process

$$R(t) = (R_1(t), ..., R_N(t)),\ t \geq 0, \tag{3.58}$$

with the first n components denote the log-price process, that is,

$$R_i(t) = \log(S_i(t)),\ t \geq 0, i = 1, ..., n, \tag{3.59}$$

and the second n components describe the dynamics of the price log-volatility,

$$R_{i+n}(t) = V_i(t),\ t \geq 0, i = 1, ..., n. \tag{3.60}$$

Thus $R(t) = (\log S_1(t), ..., \log S_n(t), V_1(t), ..., V_n(t))$ is called log-price log-volatility (LPV) process. The joint dynamics of log-price and price volatility is determined by the N-dimensional mean reverting BNS process on $\mathbb{R}$

$$dR(t) = a(t)dt + b(\epsilon(t-))dB(t) + \psi(dL(t)), \quad R(0) = 0, \tag{3.61}$$

where

1. $B(t) = (B_1(t), ..., B_N(t))$ is standard N-dimensional Brownian motion;
2. $a(t)$ is a predictable $\mathbb{R}^N$-valued process;
3. $\lambda_{\mathbb{M}} \in \mathcal{LB}_{\mathbb{M}}$ is Lévy basis with generating triplet $(g_{\mathbb{M}}, \nu_{\mathbb{M}}, \pi)$, independent of $(B(t))$, satisfying the conditions in Theorem 8;
4. $L(t)$ is the underlying BNS Lévy process defined as in Theorem 8;
5. $\psi : \mathcal{S}^N \to \mathbb{R}^N$ is a linear operator from $\mathcal{S}^N$ to $\mathbb{R}^N$.;
6. $b : \mathcal{S}_+^N \to \mathbb{R}^{N,N}$ is a continuous mapping, such that $x = b(x)b(x)^\top$;
7. ϵ is mean reverting stochastic volatility matrix process, given by,

$$\epsilon(t) = \int_{\mathbb{R}_-^{N,N}} \int_{-\infty}^{t} \exp(A(t-s))\lambda_{\mathbb{M}}(dA, ds) \exp(A^\top(t-s)),\ t \geq 0. \tag{3.62}$$

Then $R(t)$ follows a BNS stochastic volatility model with leverage with parameters $(a, b, \psi, g_{\mathbb{M}}, \nu_{\mathbb{M}}, \pi)$.

3.2.1 Stochastic Volatility Model with Leverage Effect

In this section we give an example of mean reverting joint log-price and price volatility stochastic volatility model with leverage effect.

Let $k \in \mathbb{N}$, $\Sigma \in \mathcal{S}_{+}^{N,N}$ and let $\mathbf{M}$ be a $k \times N$ random matrix with i.i.d. standard normal entries. Then the matrix

$$\mathbf{W}_{\mathcal{N}} = \sqrt{\Sigma}\mathbf{M}\mathbf{M}^{\top}\sqrt{\Sigma}$$

is said to be Wishart distributed with parameters k, N and Σ, denoted by

$$\mathbf{W}_{\mathcal{N}} \stackrel{d}{=} Wishart_N(k, \Sigma).$$

We next define a compound Poisson Matrix Subordinator (PMS) with intensity $\Lambda > 0$ and i.i.d. $Wishart_N(k, \Sigma)$ distributed jumps, denoted by $PMS_{\Lambda,\mathbf{W}_{\mathcal{N}}}(t)$. The resulting multivariate stochastic volatility model is called OU-Wishart (OUW) model, introduced in Karbe, Pfaffel, and Stelzer (2012).

The multivariate LPV model is constructed by

$$dR(t) = a(t)dt + b(\epsilon(t-))dB(t) + \psi(dL(t)), t \geq 0, \quad R(0) = 0, \tag{3.63}$$

with the specifications

1. B is standard N-dimensional Brownian motion;
2. $a(t) = \tilde{a} + \tilde{b}(\epsilon(t))$ with $\tilde{a} \in \mathbb{R}^N$, $\tilde{b} \in \mathcal{LS}^N$;
3. $\psi \in \mathcal{LS}^N$;
4. $b(\epsilon(t)) = \sqrt{\epsilon(t)}$;
5. $(L(t))$ is the independent of B matrix Lévy OUW subordinator, that is, $L(t) \stackrel{d}{=} PMS_{\Lambda,\mathbf{W}_{\mathcal{N}}}(t)$.

Thus

$$dR(t) = (\tilde{a} + \tilde{b}(\epsilon(t-)))dt + \sqrt{\epsilon(t-)}dB(t) + \psi(dL(t)), \tag{3.64}$$

$$d\epsilon(t) = (A\epsilon(t) + \epsilon(t)A^{\top})dt + dL(t), \tag{3.65}$$

where $A \in \mathbb{R}^{N,N}$ is a real matrix.

Note that the covariance matrix process $\epsilon(t)$ provides a stochastic correlations among the log-price and price volatilities in R. $\epsilon(t)$ is mean reverting and increases only with jumps, representing the arrivals of new information, resulting in positive or negative shocks in correlations. The leverage effect is obtained since the shocks $\psi(dL(t)$ is correlated with the jumps in $\epsilon(t)$.

3.2.2 Multivariate LPV with Leverage Effect and Heavy-tailed Distribution

We will further introduce heavy-tailedness in each component of the LPV process $R(t)$ by replacing the Gaussian random matrix with Variance Gamma (VG) matrix with independent but not identically distributed entries.

Recall the definition of univariate Variance Gamma (VG) random variable: Z is a VG random variable with parameters θ, σ, v, denoted by $(VG)(\theta, \sigma, v)$, if Z has the normal mean-variance mixture representation

$$Z = \theta G + \sigma\sqrt{G}W,$$

where W is standard normal random variable, G is independent of W Gamma random variable $\mathbb{E}(G) = 1$ and $\mathrm{Var}(G) = v$. By letting $a = \theta v$, $b = \sigma^2 v$ and $c = v$, we re reparametrize the univariate VG process by (a, b, c), denoting as $Z \coloneqq VG(a, b, c)$. The characteristic funtion of Z is now given by

$$\psi_{VG(a,b,c)(t)}(u) = \left(\frac{1}{1 - iau + \frac{1}{2}bu^2}\right)^{tc}.$$

Under the new parametrization, $Z(t) = a\widetilde{G}(t) + bW(\widetilde{G}(t))$, where W is standard Brownian motion and $\widetilde{G}$ is Gamma process with $\mathbb{E}(G(1)) = \mathrm{Var}(G(1)) = c$. Note that for N independent VG processes $Z_j = VG(a, b, c_j)$, $j = 1, ..., N$, we have

$$Z_1(t) + ... + Z_N(t) \stackrel{d}{=} VG(a, b, c_1 + ... + c_N)(t).$$

Let $k \in \mathbb{N}$, $\Sigma \in \mathcal{S}_+^{N,N}$ and let $\mathbf{M}$ be a $k \times N$ random matrix with independent Variance Gamma (VG) entries

$$\mathbf{M}_{ij} = Z_{ij} \stackrel{d}{=} VG(A_{ij}, B_{ij}, C_{ij}) \tag{3.66}$$

where A, B and C be the $k \times N$ matrices with entries (A_{ij}), (B_{ij}) and (C_{ij}) respectively, $i = 1, ..., k$, $j = 1, ..., N$. Then the matrix

$$\mathbf{W}_{VG} = \sqrt{\Sigma}\mathbf{M}\mathbf{M}^\top\sqrt{\Sigma}$$

is said to be Wishart distributed with parameters k, N, Σ, A, B, C, denoted by

$$\mathbf{W}_{VG} \stackrel{d}{=} Wishart_{VG}(k, N, \Sigma, A, B, C). \tag{3.67}$$

We next define a compound Poisson matrix Subordinator (PMS) with intensity $\Lambda > 0$ and i.i.d. $Wishart_{VG}(k, N, \Sigma, A, B, C)$ distributed jumps, denoted by

$$PMS_{\Lambda,\mathbf{W}_{VG}}(t), \quad t \geq 0. \tag{3.68}$$

The resulting multivariate LPV model is given by

$$dR(t) = (\tilde{a} + \tilde{b}(\epsilon(t)))dt + \sqrt{\epsilon(t)}dB(t) + \psi(dL(t)), \tag{3.69}$$

with $\epsilon(t)$ is the covariance process of OU-type with values in $\mathcal{S}^N$ satisfying

$$d\epsilon(t) = (A\epsilon(t) + \epsilon(t)A^\top)dt + dL(t), \tag{3.70}$$

where $(L(t))$ is the independent of $(B(t))$ Matrix-valued Lévy subordinator, and in this case

$$L \stackrel{d}{=} PMS_{\Lambda,\mathbf{W}_{VG}}. \tag{3.71}$$

The subordinator L is called a OU-type Wishart-VG (OUW-VG) subordinator. And this stochastic volatility model will be called OU-Wishart-VG (OUW-VG) model.

3.2.3 Multivariate LPV with Leverage Effect, Heavy-tailedness and LRD

We start with the multivariate volatility model as in the previous example:

$$dR(t) = (\tilde{a} + \tilde{b}(\epsilon(t)))dt + \sqrt{\epsilon(t)}dB(t) + \psi(dL(t)), \tag{3.72}$$

$$d\epsilon(t) = (A\epsilon(t) + \epsilon(t)A^\top)dt + dL(t), \tag{3.73}$$

$$L \stackrel{d}{=} PMS_{\Lambda,\mathbf{W}_{VG}}. \tag{3.74}$$

Then we borrow the LRD construction in Matrix-valued BNS Example 2. Let $w_i, i = 1, ..., m$ be positive weights with total sum 1, and fix diagonalizable $A_i \in \mathbb{R}_-^{N,N}, i = 1, ..., m$, and let $\pi_i \stackrel{d}{=} G(\alpha_i, \beta_i)A_i$ with $G(\alpha, \beta_i)$ a $Gamma(\alpha_i, \beta_i)$ distributed random variable with $\alpha_i > 1$ and $\beta_i > 0$. Let $\pi = \sum_{i=1}^m \pi_i w_i$. Then $\epsilon(t), t \geq 0$, is a BNS matrix-valued process, and thus stationary and infinitely divisible. Furthermore, the auto-covariance function is

$$\mathrm{Cov}(vec(\epsilon(h)), vec(\epsilon(0))) = -\sum_{i=1}^{m} C(i)\frac{w_i\beta_i^{\alpha_i}}{\alpha_i - 1}\left(\beta_i I_{N^2} - (B_i \otimes I_N + I_N \otimes B_i)h\right)^{1-\alpha_i}, \tag{3.75}$$

with some constants $C(i), i = 1, ..., m$. Thus, long-range dependence is modeled in

1. the log-price covariances,
2. the covariances among log-prices and price volatilities,
3. the covariances between price volatilities.

3.2.4 Multivariate LPV with Extended LRD in Leverage Effect

In this model, we will extend the LRD to the leverage effect from previous model. We start with the multivariate stochastic volatility model given in previous section

$$dR(t) = (\tilde{a} + \tilde{b}(\epsilon(t)))dt + \sqrt{\epsilon(t)}dB(t) + \psi(dL(t)), \tag{3.76}$$
$$d\epsilon(t) = (A\epsilon(t) + \epsilon(t)A^{\top})dt + dL(t), \tag{3.77}$$
$$L \stackrel{d}{=} PMS_{\Lambda,\mathbf{W}_{VG}}. \tag{3.78}$$

Next we will impose LRD structure on the OUW-VG subordinator L.

Let $Z = L - \mathbb{E}(L)$ be the zero-mean centralized subordinator, that is, $Z(t) = (Z_{ij}(t))_{i,j=1,...,N}$ with

$$Z_{ij}(t) = L_{ij}(t) - \mathbb{E}(L_{ij}(t)). \tag{3.79}$$

For each (i, j), let

$$f_{ij}(t) := (\delta\wedge)^{H_{ij}-\frac{1}{2}},$$

for some $\delta > 0$ and $\frac{1}{2} < H_{ij} < 1$. Then

$$X_{ij}(t) = X_{ij}(0) - \int_0^t X_{ij}(s)ds + |int_{-\infty}^{\infty}\left(f_{ij}(t-s) - f_{ij}(-s)\right)dZ_{ij}(s), \tag{3.80}$$

is a semimartingale with long range dependence of order $2 - 2H_{ij}$.

Now by replacing $L = \mathbb{E}(L) + Z$ with

$$\widehat{L} = \mathbb{E}(L) + X, \tag{3.81}$$

we have $\widehat{L}$ is a matrix semimartingale with entries $\widehat{L}_{ij}$ exhibiting long-range dependence of order $2 - 2H_{ij}$. The LPV process now has the LRD dynamics

$$dR(t) = (\tilde{a} + \tilde{b}(\epsilon(t)))dt + \sqrt{\epsilon(t)}dB(t) + \psi(d\widehat{L}(t)), \tag{3.82}$$

with $\epsilon(t)$ is the covariance process of OU-type with values in $\mathcal{S}_+^N$ satisfying

$$d\epsilon(t) = (A\epsilon(t) + \epsilon(t)A^\top)dt + d\widehat{L}(t). \tag{3.83}$$

Due to the leverage term $\psi(d\widehat{L}(t))$, we have introduced in $R(t)$ long-range dependence in leverage effect. If $\psi = 0$, our model will have LRD in covariance process $\epsilon(t)$.

3.3 Estimation of Multivariate LPV process

3.3.1 Specification

We start with parameters estimation in Section 3.2.1, mean reverting joint log-price and price volatility (LPV) model with leverage effect. Recall that in our multivariate LPV model, the LVP process is given by

$$R(t) = \left(\log \frac{S_1(t)}{S_1(0)}, ..., \log \frac{S_n(t)}{S_n(0)}, V_1(t), ..., V_n(t)\right), \tag{3.84}$$

with $S(t) = (S_1(t), ..., S_n(t))$ denotes the price process, and $V(t) = (V_1(t), ..., V_n(t))$ denotes the price-log-volatility, that is,

$$V_i(t) = \log \mathbf{V}_i(t), \tag{3.85}$$

where $\mathbf{V}_i(t)$ is the integrated volatility process. Namely, $d\mathbf{V}_i(t)$ is the standard deviation of the i-th stock return $d(\log S_i(t))$ in $(t, t+dt]$. The joint dynamics of LPV is determined by N-dimensional mean-reverting process R satisfying

$$dR(t) = (a + b(\epsilon(t)))dt + \sqrt{\epsilon(t)}dB(t) + \psi(dL(t)), \tag{3.86}$$

where $\epsilon(t)$ is the covariance process of OU type with valued in $\mathcal{S}_+^N$ with dynamics given by

$$d\epsilon(t) = (A\epsilon(t) + \epsilon(t)A^\top)dt + dL(t). \tag{3.87}$$

In Equation (3.86)(3.87),

1. B is standard N-dimensional Brownian motion;
2. $a \in \mathbb{R}^N$, $b \in \mathcal{LS}^N$;
3. $\psi \in \mathcal{LS}^N$;
4. $(L(t))$ is the independent of B matrix Lévy subordinator, that is a $\mathcal{S}_N$-valued Lévy process such that all the process increments $L(t) - L(s), t > s \geq 0$ have values in the space $\mathcal{S}_+^N$. The log-characteristic function of the subordinator $L(t), t \geq 0$ is given by, for $M \in \mathcal{S}_+^N$,

$$\begin{aligned} \log \Phi_L(M) &= \log \mathbb{E}\left(e^{i\ tr(ML(t))}\right) \\ &= i\ tr(g_L ML(t)) + \int_{\mathcal{S}_+^N} \left(\exp(i\ tr(MA)) - 1\right) \nu(dA) \end{aligned} \tag{3.88}$$

where $g_L \in \mathcal{S}_+^N$, and ν_L is a Lévy measure on $\mathcal{S}_N$ which is non-zero only on $\mathcal{S}_+^N$, and

$$\int_{\|A\| \leq 1, A \in \mathcal{S}^N} (\|A\|)\nu_L(dA) < \infty. \tag{3.89}$$

Given the matrix subordinator L and the matrix $A \in \mathbb{R}^{N,N}$, the covariance process $\epsilon(t), t \geq 0$ is defined as the strong unique solution of the SDE (3.87), given by

$$\epsilon(t) = e^{At}\epsilon(0)e^{A^\top t} + \int_0^t e^{At} dL(s) e^{A^\top (t-s)}. \tag{3.90}$$

We now borrow few definitions and results from [KMS]. By use of (3.90), we can obtain a close form expression for the *realized volatility*

$$\epsilon_{realized}(t) := \int_0^t \epsilon(s) ds \tag{3.91}$$

$$= \mathcal{P}^{-1}\big(\epsilon(t) - \epsilon(0) - L(t)\big), \tag{3.92}$$

where $\mathcal{P} : \mathcal{S}_+^N \to \mathcal{S}_+^N$ is a invertible operator given by

$$\mathcal{P}(M) = AM + M^\top A. \tag{3.93}$$

Consider now that the risk premium parameter b and the leverage operate ψ in (3.86) have a special diagonal form: for $M \in \mathcal{S}^N$, $M = (M_{ij})_{i,j=1,...,N}$,

$$b(M) = (b(M)_1, ..., b(M)_N)^\top = (b_1 M_{1,1}, ..., b_N M_{NN})^\top \tag{3.94}$$

for some $(b_1, ..., b_N)^\top \in \mathbb{R}^N$, and

$$\psi(M) = (\psi(M)_1, ..., \psi(M)_N)^\top = (\psi_1 M_{1,1}, ..., \psi_N M_{NN})^\top \tag{3.95}$$

for some $(\psi_1, ..., \psi_N)^\top \in \mathbb{R}^N$. Then the marginal dynamics of $R(t)$ is given by

$$\begin{aligned}&\big(R_i(t_j), 0 \le t_j, j = 1, ..., J\big) \stackrel{d}{=} \\ &\left(a_i t_j + b_i \epsilon_{realized}(t_j) + \int_0^{t_j} \sqrt{\epsilon_{ii}(s)}dB_i(s) + \psi_i(L(t_j)), 0 \le t_j, j = 1, ..., J\right).\end{aligned} \tag{3.96}$$

Now let us consider the special case when A is diagonal. Then the i-th marginal of $R(t)$ has finite distribution given by

$$\begin{aligned}&\big(R_i(t_j), 0 \le t_j, j = 1, ..., J\big) \stackrel{d}{=} \\ &\left(a_i t_j + b_i \epsilon_{realized}(t_j) + \int_0^{t_j} \sqrt{\epsilon_{ii}(s)}dB_i(s) + \psi_i(L(t_j)), 0 \le t_j, j = 1, ..., J\right),\end{aligned} \tag{3.97}$$

where ϵ_{ii} is given by

$$d\epsilon_{ii}(t) = (2A_{ii}\epsilon_{ii}(t))dt + dL_{ii}(t). \tag{3.98}$$

The parameters in the model (3.97)(3.98) can be estimated by using the multivariate characteristic method by fitting the sample characteristic function to the theoretical one

$$\Phi_{(R(t),\epsilon(t))}(r, \varepsilon) \coloneqq \mathbb{E}\left(\exp(r^\top R(t) + tr\big(\varepsilon^\top \epsilon(t)\big))\right), \ t \ge 0 \tag{3.99}$$

for $r \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}^{N,N}$. Then the parameters can be determined by minimizing the error

$$\Delta(\widehat{\Phi}, \Phi) \coloneqq \sup_{|(r,\varepsilon)| \le 1} \left|\widehat{\Phi}_{(R(t),\epsilon(t))}(r, \varepsilon) - \Phi_{(R(t),\epsilon(t))}(r, \varepsilon)\right| \tag{3.100}$$

Note that the closeness of $\Delta(\widehat{\Phi}, \Phi)$ to zeros will guarantee the closeness of sample mixed moments (of nay order) to the corresponding theoretical mixed moments.

The characteristic function Φ in Equation (3.99) have a explicit form

$$\begin{aligned}\log \Phi_{(R(t),\epsilon(t))}(r,\varepsilon) =& i\left[r^\top(R(0)+at) + tr\big(R(0)\mathcal{U}(t,\mathcal{P},\varepsilon)\big) + tr\big(R(0)\mathcal{M}(t,\mathcal{P},r)\big)\right] \\ &+ \int_0^1 \log \Phi_L\big(\mathcal{U}(s,\mathcal{P},\varepsilon) + \Psi(r) + \mathcal{M}(s,\mathcal{P},r)\big)ds,\end{aligned} \tag{3.101}$$

where

1. $\mathcal{P}$ is the invertible operator from $\mathcal{S}_+^N$ to $\mathcal{S}_+^N$, given by $\mathcal{P}(M) = AM + M^\top A$;

2. $\mathcal{P}^*$ is the invertible adjoint to $\mathcal{P}$ operator, $\mathcal{P}^* : \mathcal{S}_+^N \to \mathcal{S}_+^N$,

$$\mathcal{P}^*(M) = A^\top M + MA;$$

3. $\mathcal{U}(t, A, \varepsilon) \coloneqq \exp(tA^\top)\varepsilon \exp(tA)$;

4.
$$\mathcal{M}(t,A,\mathcal{P},r) \coloneqq e^{tA^\top}\mathcal{P}^{-*}\left(b^*(r) + \frac{i}{2}rr^\top\right)e^{tA} - \mathcal{P}^{-*}\left(b^*(r) + \frac{i}{2}rr^\top\right), \tag{3.102}$$

   where b^* is the adjoint operator of b.

5. $\Psi(r) = (\Psi(r)_{ij})_{i,j=1,\dots,N}$, with entries

$$\Psi(r)_{ij} = 0, \text{ for } i \neq j, \text{ and} \tag{3.103}$$
$$\Psi(r)_{ii} = \psi_i r_{ii}. \tag{3.104}$$

One-Asset Log-Price-Volatility (LPV) Example.

In the case of one asset, that is $n = 1, N = 2$, and

$$R_1(t) = \log S(t) \coloneqq X(t), \quad \text{the log-price process,} \tag{3.105}$$

$$R_2(t) = V(t), \quad \text{the log-cumulative volatility.} \tag{3.106}$$

The LPV process $R = (X, V)$ is assumed to follow the bivariate stochastic volatility model

$$\begin{aligned} dR(t) &= \begin{pmatrix} dX(t) \\ dV(t) \end{pmatrix} \\ &= \left(\begin{pmatrix} a_X \\ a_V \end{pmatrix} - \begin{pmatrix} b_X \epsilon_{XX}(t) \\ b_V \epsilon_{VV}(t) \end{pmatrix} \right) dt \\ &\quad + \sqrt{\begin{pmatrix} \epsilon_{XX}(t) & \epsilon_{XV}(t) \\ \epsilon_{XV}(t) & \epsilon_{VV}(t) \end{pmatrix}} \begin{pmatrix} dB_X(t) \\ dB_V(t) \end{pmatrix} \\ &\quad + \begin{pmatrix} \Psi_{XX} dL_{XX}(t) & \Psi_{XV} dL_{XV}(t) \\ \Psi_{XV} dL_{XV}(t) & \Psi_{VV} dL_{VV}(t) \end{pmatrix}, \end{aligned} \tag{3.107}$$

where

$$\begin{aligned} d \begin{pmatrix} \epsilon_{XX}(t) & \epsilon_{XV}(t) \\ \epsilon_{XV}(t) & \epsilon_{VV}(t) \end{pmatrix} &= \begin{pmatrix} 2A_X \epsilon_{XX}(t) & (A_X + A_V)\epsilon_{XV}(t) \\ (A_X + A_V)\epsilon_{XV}(t) & 2A_V \epsilon_{VV}(t) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} dL_{XX}(t) & dL_{XV}(t) \\ dL_{XV}(t) & dL_{VV}(t) \end{pmatrix}. \end{aligned} \tag{3.108}$$

We will use, however, an equivalent to (3.107)(3.108), slightly re-parameterized model

$$\begin{aligned} dR(t) &= \begin{pmatrix} dX(t) \\ dV(t) \end{pmatrix} \\ &= \left(\begin{pmatrix} a_X \\ a_V \end{pmatrix} - \frac{1}{2} \begin{pmatrix} b_X \epsilon_{XX}(t) \\ b_V \epsilon_{VV}(t) \end{pmatrix} \right) dt \\ &\quad + \sqrt{\begin{pmatrix} \epsilon_{XX}(t) & \epsilon_{XV}(t) \\ \epsilon_{XV}(t) & \epsilon_{VV}(t) \end{pmatrix}} \begin{pmatrix} dB_X(t) \\ dB_V(t) \end{pmatrix} \\ &\quad + \begin{pmatrix} \Psi_{XX} dL_{XX}(t) & \Psi_{XV} dL_{XV}(t) \\ \Psi_{XV} dL_{XV}(t) & \Psi_{VV} dL_{VV}(t) \end{pmatrix}, \end{aligned} \tag{3.109}$$

where

$$\begin{aligned} d \begin{pmatrix} \epsilon_{XX}(t) & \epsilon_{XV}(t) \\ \epsilon_{XV}(t) & \epsilon_{VV}(t) \end{pmatrix} &= \left[\begin{pmatrix} c_X & 0 \\ 0 & c_V \end{pmatrix} + \begin{pmatrix} 2A_X \epsilon_{XX}(t) & (A_X + A_V)\epsilon_{XV}(t) \\ (A_X + A_V)\epsilon_{XV}(t) & 2A_V \epsilon_{VV}(t) \end{pmatrix} \right] dt \\ &\quad + \begin{pmatrix} dL_{XX}(t) & dL_{XV}(t) \\ dL_{XV}(t) & dL_{VV}(t) \end{pmatrix}. \end{aligned} \tag{3.110}$$

The initial values of the above model (3.109)(3.110) are

$$R(0) = \begin{pmatrix} X(0) \\ V(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.111}$$

$$\epsilon(0) = \begin{pmatrix} \epsilon_{XX}(0) & \epsilon_{XV}(0) \\ \epsilon_{XV}(0) & \epsilon_{VV}(0) \end{pmatrix} = \begin{pmatrix} \epsilon_{XX,0} & \epsilon_{XV,0} \\ \epsilon_{XV,0} & \epsilon_{VV,0} \end{pmatrix} := \epsilon_0, \tag{3.112}$$

where ϵ_0 is a 2×2 positive definite matrix. Then the constraints on the parameters of model (3.109)(3.110) are

$$c_X \geq 0,\ c_V \geq 0,\ A_X < 0,\ A_Z < 0,\ \text{and },$$

$$\begin{pmatrix} \Psi_{XX} & \Psi_{XV} \\ \Psi_{XV} & \Psi_{VV} \end{pmatrix} \in \mathbb{R}^{2,2}.$$

We will then follow the model in Section 3.2.1: mean-reverting joint LPV stochastic vlarility model with leverage effect. Let

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XV} \\ \Sigma_{XV} & \Sigma_{VV} \end{pmatrix} \tag{3.113}$$

is a 2×2 positive definite matrix, and let $\mathbf{M}$ be a 2×2 matrix with i.i.d. standard mornal entries. Then the matrix

$$\mathbf{W}_{\mathcal{N}} := \sqrt{\Sigma}\mathbf{M}\mathbf{M}^{\top}\sqrt{\Sigma} \tag{3.114}$$

is Wishart distributed $\mathbf{W}_{\mathcal{N}} \overset{d}{=} \mathit{Wishart}_2(2, \Sigma)$.

The density function of $\mathbf{W}_{\mathcal{N}} = \begin{pmatrix} \mathbf{W}_{XX} & \mathbf{W}_{VX} \\ \mathbf{W}_{VX} & \mathbf{W}_{VV} \end{pmatrix}$ is given by

$$f_{\mathbf{W}_{\mathcal{N}}}(w) = \frac{1}{4|\Sigma|\pi}|w|^{-\frac{1}{2}}2^{-\frac{1}{2}tr(\Sigma^{-1}w)}, \tag{3.115}$$

where $|\Sigma|$ is the deteminant of Σ. The mean and variance of $\mathbf{W}_{\mathcal{N}}$ are given by

$$\mathbb{E}(\mathbf{W}_{\mathcal{N}}) = 2\Sigma, \tag{3.116}$$

$$\mathrm{Var}(\mathbf{W}_{\mathcal{N}}) = 2\begin{pmatrix} 2\Sigma_{XX}^2 & \Sigma_{XV} + \Sigma_{XX}\Sigma_{VV} \\ \Sigma_{XV} + \Sigma_{XX}\Sigma_{VV} & 2\Sigma_{VV}^2 \end{pmatrix}. \tag{3.117}$$

And the characteristic function of $\mathbf{W}_{\mathcal{N}}$ is, for $\forall\Theta \in \bar{\mathcal{S}}_+^2$,

$$\mathbb{E}\left(e^{i\ tr(\Theta^\top \mathbf{W}_{\mathcal{N}})}\right) = |I - 2i\Theta\Sigma|. \tag{3.118}$$

Then we can define the compound Poisson Matrix Subordinator (PMS)

$$L(t) = \begin{pmatrix} L_{XX}(t) & L_{XV}(t) \\ L_{XV}(t) & L_{VV}(t) \end{pmatrix} \tag{3.119}$$

with intensity $\Lambda > 0$ and i.i.d $Wishart_2(2, \Sigma)$-distributed jumps, denoted by $PMS_{\Lambda, \mathbf{W}_{\mathcal{N}}}(t)$, $t \geq 0$.

The closed form of the moment generating function (m.g.f) of the bivariate LPV process $R(t)$, as in (3.107)(3.108) with specific subordinator (3.119), is given as follows. For $t \geq 0$, the m.g.f of $R(t)$ is

$$\mathbb{M}_{R(t)}(z) = \mathbb{E}\left(e^{z^\top R(t)}\right), \text{ for } z = \begin{pmatrix} z_X \\ z_V \end{pmatrix} \in \mathbb{C}^2,$$

when the expectation exists,

$$\begin{aligned} \log \mathbb{M}_{R(t)}(z) =& tz^\top a + tr\Big(\epsilon(0)\mathcal{M}(t, A, \mathcal{P}, z)\Big) + \int_0^t tr\Big(c\mathcal{M}(s, A, \mathcal{P}, z)\Big) ds \\ &+ \Lambda \int_0^t \frac{1}{\left|I - 2\mathcal{M}(t, A, \mathcal{P}, z) + \widetilde{\Psi}(z)\Sigma\right|} ds - \Lambda t, \end{aligned} \tag{3.120}$$

where

$$a = \begin{pmatrix} a_X(t) \\ a_V(t) \end{pmatrix}, c = \begin{pmatrix} c_X(t) & 0 \\ 0 & c_V(t) \end{pmatrix}, \widetilde{\Psi}(z) = \begin{pmatrix} \Psi_{XX} z_X & \Psi_{XV} z_X \\ \Psi_{XV} z_V & \Psi_{VV} z_V \end{pmatrix}.$$

We further simplify the model by letting A be a diagonal matrix

$$A = \begin{pmatrix} A_X & 0 \\ 0 & A_V \end{pmatrix} = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix}. \tag{3.121}$$

Then we can obtain the closed-form m.g.f $\mathbb{M}_{R(t)}(z)$. Set

$$B \coloneqq B(z, \Sigma) = \frac{1}{4\kappa} \begin{pmatrix} z_X^2 - z_X & z_X z_V \\ z_X z_V & z_V^2 - z_V \end{pmatrix} \Sigma, \tag{3.122}$$

$$C \coloneqq C(\Psi, z, \Sigma) = \begin{pmatrix} \Psi_{XX} z_X & \Psi_{XV} z_X \\ \Psi_{XV} z_V & \Psi_{VV} z_V \end{pmatrix} \Sigma, \tag{3.123}$$

$$
\begin{align}
b_0 &:= 1 + 4|B - C| + 2tr(B - C), \tag{3.124}\\
b_1 &:= -8|B| + 4tr(B)tr(C) - 4tr(BC) - 2tr(B), \tag{3.125}\\
b_2 &:= 4tr(B), \tag{3.126}\\
D &:= \sqrt{4b_0b_2 - b_1^2}. \tag{3.127}
\end{align}
$$

Consider two cases $D \neq 0$ and $D = 0$. For $D \neq 0$, we have

$$
\begin{aligned}
\log \mathbb{M}_{R(t)}(z) =& tz^\top a + \left(e^{2\kappa t} - 1\right) tr\left(\epsilon(0)\frac{1}{4\kappa}\begin{pmatrix} z_X^2 - z_X & z_X z_V \\ z_X z_V & z_V^2 - z_V \end{pmatrix}\right) \\
&+ \frac{1}{4\kappa}\left(c_X(z_X^2 - z_X) + c_V(z_V^2 - z_V)\right)\left(\frac{1}{2\kappa}\left(e^{2\kappa t} - 1\right) - t\right) \\
&+ \frac{\Lambda}{2\kappa b_0}\left\{\frac{b_1}{D}\left[\arctan\left(\frac{2b_2 + b_1}{D}\right) - \arctan\left(\frac{2b_2e^{2\kappa t} + b_1}{D}\right)\right]\right. \\
&\qquad\left. + \frac{1}{2}\log\left(\frac{b_0 + b_1 + b_2}{b_0 + b_1e^{2\kappa t} + b_2e^{4\kappa t}}\right)\right\} \\
&+ \Lambda t\left(\frac{1}{2b_0} - 1\right).
\end{aligned} \tag{3.128}
$$

For $D = 0$, we have

$$
\begin{aligned}
\log \mathbb{M}_{R(t)}(z) =& tz^\top a + \left(e^{2\kappa t} - 1\right) tr\left(\epsilon(0)\frac{1}{4\kappa}\begin{pmatrix} z_X^2 - z_X & z_X z_V \\ z_X z_V & z_V^2 - z_V \end{pmatrix}\right) \\
&+ \frac{1}{4\kappa}\left(c_X(z_X^2 - z_X) + c_V(z_V^2 - z_V)\right)\left(\frac{1}{2\kappa}\left(e^{2at} - 1\right) - t\right) \\
&+ \frac{\Lambda}{2\kappa b_0}\left[\frac{b_1}{2b_2e^{2\kappa t} + b_1} - \frac{b_1}{2b_2 + b_1}\right. \\
&\qquad\left. + \frac{1}{2}\log\left(\frac{b_0 + b_1 + b_2}{b_0 + b_1e^{2\kappa t} + b_2e^{4\kappa t}}\right)\right] \\
&+ \Lambda t\left(\frac{1}{2b_0} - 1\right).
\end{aligned} \tag{3.129}
$$

To sum up, this model (3.107)(3.108)(3.119)(3.121) has 12 parameters

$$
\mathscr{P} := \left\{\begin{pmatrix} a_X \\ a_V \end{pmatrix}, \begin{pmatrix} \Psi_{XX} & \Psi_{XV} \\ \Psi_{XV} & \Psi_{VV} \end{pmatrix}, \begin{pmatrix} c_X & 0 \\ 0 & c_V \end{pmatrix}, \kappa, \Lambda, \begin{pmatrix} \Sigma_{XX} & \Sigma_{XV} \\ \Sigma_{XV} & \Sigma_{VV} \end{pmatrix}\right\}, \tag{3.130}
$$

where

1. $a = \begin{pmatrix} a_X \\ a_V \end{pmatrix}$ is the logterm instantaneous mean return for $R(t) = \begin{pmatrix} X(t) \\ V(t) \end{pmatrix}$;

2. $\Psi = \begin{pmatrix} \Psi_{XX} & \Psi_{XV} \\ \Psi_{XV} & \Psi_{VV} \end{pmatrix}$ is the mixing leverage effects;

3. $c = \begin{pmatrix} c_X & 0 \\ 0 & c_V \end{pmatrix}$ is the logterm mean of the instantaneous change $d\epsilon(t)$ in the covariance matrix $\epsilon(t) = \begin{pmatrix} \epsilon_{XX}(t) & \epsilon_{XV}(t) \\ \epsilon_{XV}(t) & \epsilon_{VV}(t) \end{pmatrix}$;

4. κ is the mean-reverting parameter of the covariance matrix;

5. Λ and $\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XV} \\ \Sigma_{XV} & \Sigma_{VV} \end{pmatrix}$ are the parameters of $L(t)$, the $PMS_{\Lambda,\mathbf{w}_{\mathcal{N}}}$ subordinator with intensity $\Lambda > 0$ and i.i.d $\mathit{Wishart}_2(2, \Sigma)$-distributed jumps.

3.3.2 Estimation

Sample Integrated Volatility

Given sample price process $S(t), t \geq 0$, the integrated volatility over the period $[t-1, t]$ is estimated by the sample realized volatility

$$\widehat{\sigma}_h(t) = \sum_{s=1}^{h} r^2 \left(t - 1 + \frac{s-1}{h}, t - 1 + \frac{s}{h} \right), \tag{3.131}$$

where $r(t-\delta, t)$ is the log return of the price $S(t)$ over the period $(t-\delta, t]$,

$$r(t-\delta, t) = \log(S(t)) - \log(S(t-\delta)).$$

For notation simplicity, let $r(t) \coloneqq r(t-1, t)$.

Estimation Procedure

The sample values of the LPV process $R(t) = (X(t), V(t))^\top, t = 0, ..., T$ is given by

$$\widehat{R}_h(t) = \begin{pmatrix} \widehat{X}(t) \\ \widehat{V}_h(t) \end{pmatrix} \coloneqq \begin{pmatrix} \sum_{s=0}^{t} r(s) \\ \log \sum_{s=0}^{t} \widehat{\sigma}_h(s). \end{pmatrix} \tag{3.132}$$

Now we consider a mesh on the upper-left part of unit disk,

$$z_{k,j} = \begin{pmatrix} z_{X,k,j} \\ z_{V,k,j} \end{pmatrix} = \begin{pmatrix} \frac{1}{k}\cos(j\frac{\pi}{2J}) \\ \frac{1}{k}\sin(j\frac{\pi}{2J}) \end{pmatrix}, \quad k = 1, ..., K, j = 0, ..., J \tag{3.133}$$

Then the sample log-m.g.f is given by, for every fixed $k = 1, .., K, j = 1, ..., J$,

$$\widehat{\mathfrak{M}}_{\widehat{R}_\tau,T}(z_{k,j}) = \log\left(\frac{1}{T}\sum_{t=1}^{T}\exp\left(\delta\widehat{X}(t)z_{X,k,j} + \delta\widehat{V}_h(t)z_{V,k,j}\right)\right), \tag{3.134}$$

where δ is the difference operator $\delta X(t) = X(t) - X(t-1)$.

For a parameter set $\mathcal{P} = \left\{a, \Psi, c, \kappa, \Lambda, \Sigma\right\}$, and fixed $k = 1, ..., K, j = 1, ..., J$, the theoretical log-m.g.f. is

$$\mathfrak{M}_R(z_{k,j}; \mathcal{P}) := \log\left(\mathbb{M}_{R(1)}(z_{k,j})\right), \tag{3.135}$$

where $\mathbb{M}_{R(t)}(z)$ is defined as in (3.128)(3.129).

The goal of estimation is to find the optimal $\widehat{\mathcal{P}} = \left\{\widehat{a}, \widehat{\Psi}, \widehat{c}, \widehat{\kappa}, \widehat{\Lambda}, \widehat{\Sigma}\right\}$ such that the error function

$$\sum_{k=1}^{K}\sum_{j=0}^{J}\left(\widehat{\mathfrak{M}}_{\widehat{R}_\tau,T}(z_{k,j}) - \mathfrak{M}_R(z_{k,j}; \mathcal{P})\right)^2 \tag{3.136}$$

is minimized.

3.3.3 Results

We select 5 stocks Lobster high-frequency data starting from 10/01/2013 to 09/20/2014, that is 252 trading days. The stocks include PFE, MSFT, GE, GIS and C. The prices of the stock is given by the mid price instead of the transaction price in order to avoid microstructure noises.

The sampling time interval is chosen to be 1 minute, and $h = 30$. The sample process $\widehat{R}_h(t)$ as in (3.132) is shown in Figure 3.1. We fit each stock price process to the one-asset LPV model as described in previous section. The estimated parameter are shown in Table 3.1.

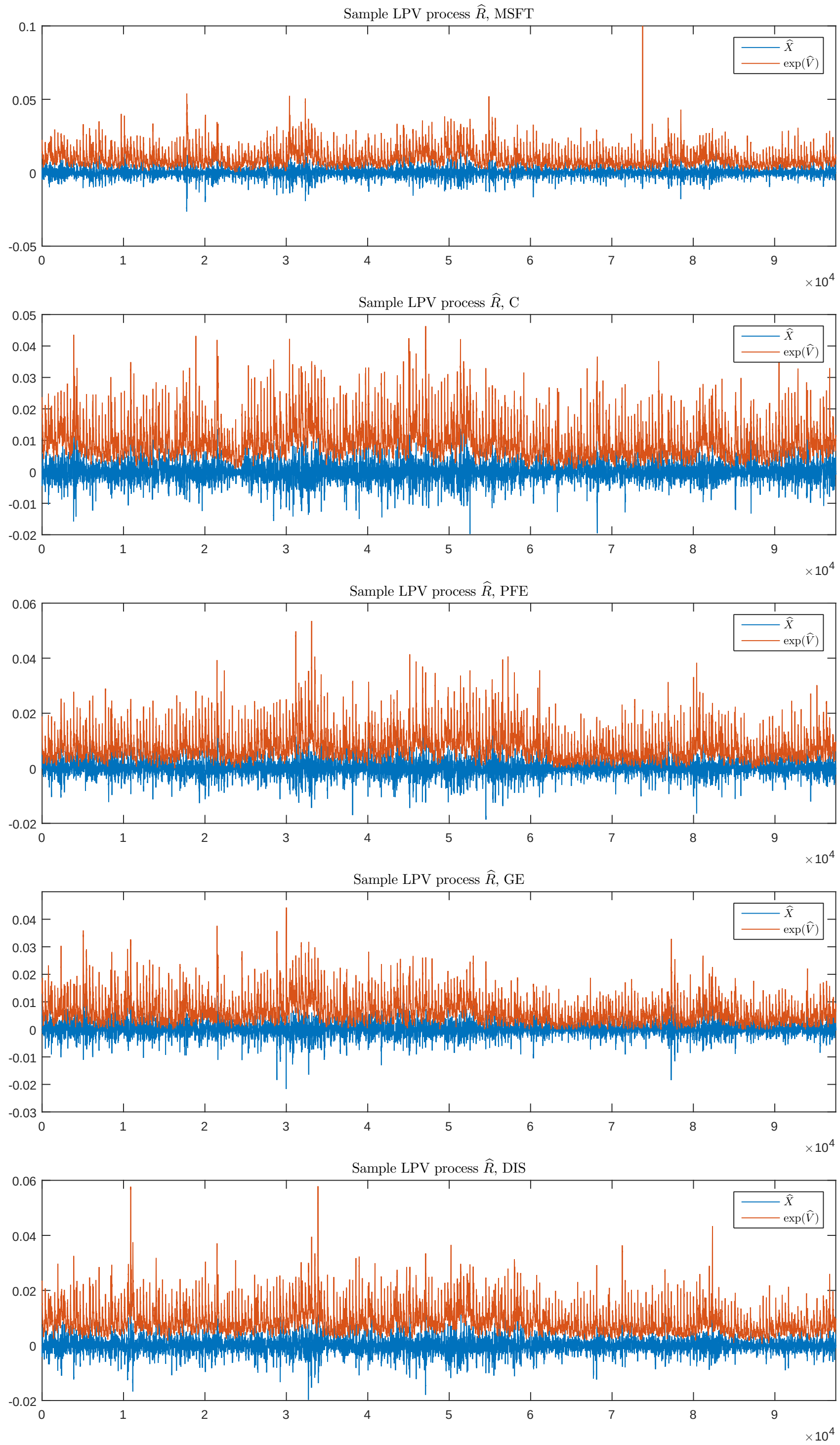


Figure 3.1: Sample LPV processes.

Table 3.1: Estimated Parameters of one-asset LPV process. $\tau = 30$min, and the sampling interval is 1min.

	a	Ψ	c	κ	Λ	Σ
MSFT	$\begin{pmatrix} 0.0021 \\ 0.0029 \end{pmatrix}$	$\begin{pmatrix} 0.1459 & -0.2593 \\ -0.2593 & 0.8731 \end{pmatrix}$	$\begin{pmatrix} 0.0513 & 0 \\ 0 & 0.0594 \end{pmatrix}$	-4.2328	1.5726e-04	$\begin{pmatrix} 0.7770 & 0.0469 \\ 0.0469 & 0.5991 \end{pmatrix}$
C	$\begin{pmatrix} 0.0010 \\ 0.0017 \end{pmatrix}$	$\begin{pmatrix} 0.1412 & -0.2404 \\ -0.2404 & 0.8836 \end{pmatrix}$	$\begin{pmatrix} 0.0234 & 0 \\ 0 & 0.0321 \end{pmatrix}$	-4.2420	7.5252e-05	$\begin{pmatrix} 0.7808 & 0.0436 \\ 0.0436 & 0.6087 \end{pmatrix}$
PFE	$\begin{pmatrix} 0.0021 \\ 0.0035 \end{pmatrix}$	$\begin{pmatrix} 0.1496 & -0.2347 \\ -0.2347 & 0.8784 \end{pmatrix}$	$\begin{pmatrix} 0.0522 & 0 \\ 0 & 0.0680 \end{pmatrix}$	-4.2410	1.6546e-04	$\begin{pmatrix} 0.7662 & 0.0443 \\ 0.0443 & 0.6103 \end{pmatrix}$
GE	$\begin{pmatrix} 0.0010 \\ 0.0034 \end{pmatrix}$	$\begin{pmatrix} 0.0036 & -0.1197 \\ -0.1197 & 0.8600 \end{pmatrix}$	$\begin{pmatrix} 0.0243 & 0 \\ 0 & 0.0592 \end{pmatrix}$	-4.2442	8.8704e-05	$\begin{pmatrix} 0.7797 & 0.0010 \\ 0.0010 & 0.696 \end{pmatrix}$
GIS	$\begin{pmatrix} 0.0002 \\ 0.0010 \end{pmatrix}$	$\begin{pmatrix} 0.6069 & -0.4808 \\ -0.4808 & 0.5137 \end{pmatrix}$	$\begin{pmatrix} 0.0059 & 0 \\ 0 & 0.0148 \end{pmatrix}$	-3.9170	2.2820e-05	$\begin{pmatrix} 0.8092 & 0.0170 \\ 0.0170 & 0.0078 \end{pmatrix}$

3.4 Option Pricing and Calibration

3.4.1 Equivalent Martingale Measure

For a price process $S(t) = (S_1(t), ..., S_n(t)), 0 \leq t \leq T$, and the log integrated volatility process $V(t) = (V_1(t), ..., V_n(t)), 0 \leq t \leq T$, consider the LPV model as in (3.84)(3.86)(3.87)

$$\begin{aligned}
R(t) &= \left(\log \frac{S_1(t)}{S_1(0)}, ..., \log \frac{S_n(t)}{S_n(0)}, V_1(t), ..., V_n(t)\right), \\
dR(t) &= (a + b(\epsilon(t)))dt + \sqrt{\epsilon(t)}dB(t) + \psi(dL(t)), \\
d\epsilon(t) &= (c + A\epsilon(t) + \epsilon(t)A^\top)dt + dL(t).
\end{aligned}$$

Denote the discounted price process by $e^{-rt}S(t), t \geq 0$ for a given constant interest rate r. Then as in Karbe, Pfaffel, and Stelzer (2012) (Theorem 2.10), we have

Theorem 9 (Martingale Conditions)**.** *The discounted price process* $e^{-rt}S(t), 0 \leq t \leq T$ *is a martingale if and only if, for all* $i = 1, ..., n$*,*

$$\int_{\|X\|>1, X\in\mathcal{S}_+^N} e^{\psi_i(X)}\nu_L(X) < \infty, \tag{3.137}$$

$$\tag{3.138}$$

and

$$b_i(X) = \frac{1}{2}X_{ii}, \ \forall X \in \mathcal{S}_+^N, \tag{3.139}$$

$$a_i = r - \int_{\mathcal{S}_+^N} \left(e^{\psi_i(X)} - 1\right)\nu_L(dX). \tag{3.140}$$

The existence of Equivalent Martingale Measure (EMM) is given by the following theorem as in Karbe, Pfaffel, and Stelzer (2012) (Theorem 2.11).

Theorem 10 (Structure Preserving EMM)**.** *Let* $y : \mathcal{S}_+^n \to (0, \infty)$ *such that*

$$\int_{\mathcal{S}_+^n} \left(\sqrt{y(X)} - 1\right)^2 \nu_L(dX) < \infty, \ \textit{and,} \tag{3.141}$$

$$\int_{\|X\|>1, X\in\mathcal{S}_+^N} e^{\psi_i(X)}\nu_L^y(dX) < \infty, \tag{3.142}$$

where

$$\nu_L^y(B) := \int_B y(X)\nu_L(dX), \text{ for } B \in \mathcal{B}(\mathcal{S}_+^n).$$

Define the $\mathbb{R}^N$-valued process $\Theta_t, 0 \leq t \leq T$ by

$$\Theta_t = -\epsilon(t)^{-\frac{1}{2}} \left(a + b(\epsilon(t)) + \frac{1}{2}\begin{pmatrix} \epsilon_{11}(t) \\ \vdots \\ \epsilon_{nn}(t) \end{pmatrix} + \begin{pmatrix} \int_{\mathcal{S}_+^n}\left(e^{\psi_1(X)}-1\right)\nu_L^y(dX) \\ \vdots \\ \int_{\mathcal{S}_+^n}\left(e^{\psi_n(X)}-1\right)\nu_L^y(dX) \end{pmatrix} - \mathbf{1}r \right), \tag{3.143}$$

where $\mathbf{1} = (1, ..., 1)^\top \in \mathbb{R}^n$. Then we have

$$Z = \mathscr{E}\left(\int_0^\cdot \Theta(s)dW(s) + (y-1)*(\mu_L - \rho_L)\right) \tag{3.144}$$

is a density process, and the probability measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ is an equivalent martingale measure. Moreover, $W^{\mathbb{Q}} := W - \int_0^\cdot \Theta(s)ds$ is a $\mathbb{Q}$-standard Brownian motion, and L is an independent driftless $\mathbb{Q}$-matrix subordinator with Lévy measure ν_L^y. The dynamics of R under $\mathbb{Q}$ is given by

$$dR_i(t) = \left(r - \int_{\mathcal{S}_+^n}\left(e^{\psi_1(X)}-1\right)\nu_L^y(dX) - \frac{1}{2}\epsilon_{ii}(t)\right)dt + \left(\sqrt{\epsilon(t)}dB(t)\right)_i + \psi_i(dL(t)), \tag{3.145}$$

$$d\epsilon(t) = (c + A\epsilon(t) + \epsilon(t)A^\top)dt + dL(t). \tag{3.146}$$

3.4.2 Specification and Pricing

One-Asset European Option Pricing

The price of an option with payoff $f(R(T)-s)$ is given by the following theorem.

Theorem 11. *For a fixed $\alpha \in \mathbb{R}^N$, let $g(x) := e^{\langle \alpha, x\rangle} f(x)$ for $x \in \mathbb{R}^N$. Assume that*

$$g \in L^1 \cap L^\infty, \quad \Phi_{R(T)}(\alpha) < \infty,$$

and

$$w \mapsto \Phi_{R(T)}(\alpha + iw),\ w \in L^1,$$

where Φ is the characteristic funtion. Then

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT} f(R(T)-s)\right] = \frac{e^{-\langle \alpha, s\rangle - rT}}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\langle u, s\rangle} \Phi_{R(T)}(\alpha + iu) \widetilde{f}(i\alpha - u) du, \tag{3.147}$$

where $\widetilde{f}$ is the Fourier transform of f.

Now consider the one-asset case of LPV model as in Section 3.3.1, and an European option with payoff $f(S(T)) = (S(T)-K)^+$. The Fourier transform of f is

$$\widetilde{f}(z) = \frac{K^{1+iz}}{iz(1+iz)}, \tag{3.148}$$

for $z \in \mathbb{C}$ with $\Im(z) > 1$. Then we only need the transforms of the marginal model. That is,

$$\log \mathbb{E}[e^{z_X X(t)}] = t z_X a_X + \frac{e^{2\kappa t} - 1}{4\kappa}\left(z_X^2 - z_X\right) \epsilon_{XX}(0) \tag{3.149}$$

$$+ \frac{1}{4\kappa} c_X (z_X^2 - z_X)\left(\frac{1}{2\kappa}\left(e^{2at} - 1\right) - t\right) \tag{3.150}$$

$$+ \frac{\Lambda}{2\kappa b_0} \log\left(\frac{b_0 + b_1}{b_0 + b_1 e^{2\kappa t}}\right) \tag{3.151}$$

$$+ \Lambda t \left(\frac{1}{2b_0} - 1\right), \tag{3.152}$$

where

$$b_0 = 1 + \left(\frac{1}{2a_1}(z_X^2 - z_X) - 2\Psi_{XX}z_X\right)\Sigma_{ZZ} - 2\Psi_{XV}z_V\Sigma_{XV}, \tag{3.153}$$

$$b_1 = -\frac{1}{2\kappa}(z_X^2 - z_X)\Sigma_{XX}. \tag{3.154}$$

3.4.3 Calibration and Results

We will use prices of S&P500 index call options at 2015-09-04, in total 179 options. We calibrate the on-asset LPV model as in Section 3.3.1 by minimizing the root mean of squared error (RMSE) between Black-Scholes implied volatility by market prices and model prices. That is,

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\sigma_{BS}(C_i) - \sigma_{BS}(\widehat{C}_i(\mathcal{P}))\right]^2}, \tag{3.155}$$

where N is the number of options, $C_i, i = 1, ..., N$ are the prices of the options, $\sigma_{BS}(C)$ denotes the Black-Schole implied volatility by price C, and $\widehat{C}_i(\mathcal{P})$ denote the model calculated option price given parameters $\mathcal{P}$.

The calibrated parameters are provided in Table 3.2. The corresponding RMSE is 0.0013. The comparison of market price implied volatilities and model price implied volatilities are shown in Figure 3.2.

Table 3.2: Calibrated Parameters of one-asset LPV process.

Ψ_{XX}	σ_{XV}	c_X	κ	Λ	Σ_{XX}	Σ_{XV}
-4.9749	1.4390	0.7279	-20.0825	0.9444	0.0002	0.0053

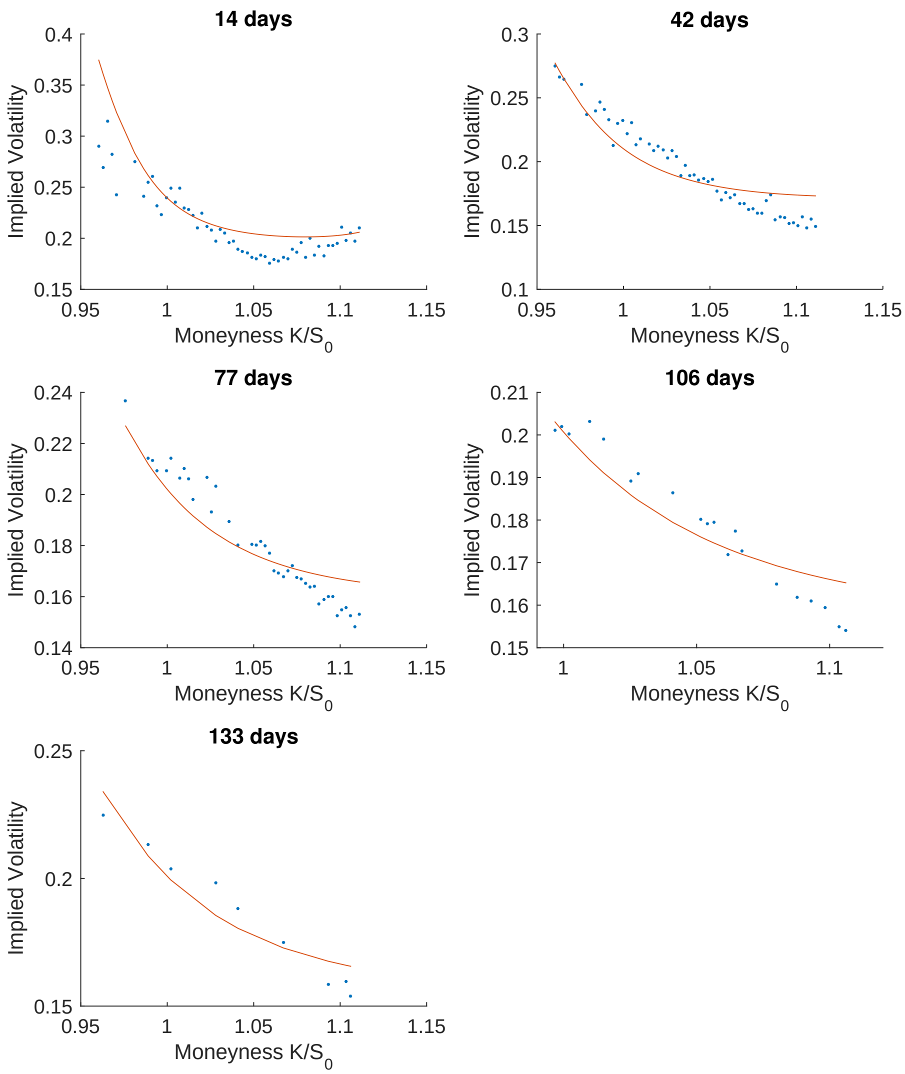


Figure 3.2: Comparison of Black-Schole volatilities implied by market prices (dots) and model prices (solid line).

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Appendix A

Proofs

A.1 Proposition 1

Proof. By use of Hyperplane Separating Theorem. □

A.2 Corollary 1

Proof. $M^{(i),S}$ is a subspace of M for any $i \in I$. □

A.3 Theorem 2

Proof. Let π be a quasi state-price deflator, then we define the equivalent probability measure $\mathbf{Q}$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \xi(T) = \frac{R(0,T)\pi(T)}{\pi(0)},$$

Considering a trading strategy θ that invests one unit at time t in risk-free account and then hold it until time T. That is, $\theta(t) \cdot S(t) = 1$, and $\delta^{\theta}(T) = R(0,T)$. Note that $\theta \in L^{(i),S}$ for all $i \in I$. Thus, we have, for any $i \in I$,

$$\begin{aligned}
\pi^{(i)}(t) &= \mathbb{E}^{\mathbf{P}}\left[\pi^{(i)}(T)R(t,T) \;\middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbf{P}}\left[\pi(T)R(t,T) \;\middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbf{P}}\left[\frac{\pi(0)\xi(T)}{R(0,T)}R(t,T) \;\middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi(0)}{R(0,t)}\mathbb{E}^{\mathbf{P}}\left[\xi(T)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \frac{\pi(0)}{R(0,t)}\mathbb{E}^{\mathbf{P}}\left[\mathbb{E}^{\mathbf{P}}\left[\xi(T)\ |\mathcal{F}_t\right]\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \frac{\pi(0)}{R(0,t)}\mathbb{E}^{\mathbf{P}}\left[\xi(t)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \frac{\pi(0)}{R(0,t)}\xi^{(i)}(t).
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
S(t) &= \frac{1}{\pi^{(i)}(t)}\mathbb{E}^{\mathbf{P}}\left[\sum_{j=t+1}^{T}\pi^{(i)}(j)\delta(j)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \frac{R(0,t)}{\xi^{(i)}(t)}\mathbb{E}^{\mathbf{P}}\left[\sum_{j=t+1}^{T}\frac{\xi^{(i)}(j)}{R(0,j)}\delta(j)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \frac{1}{\xi^{(i)}(t)}\mathbb{E}^{\mathbf{P}}\left[\sum_{j=t+1}^{T}\frac{\xi^{(i)}(j)}{R(t,j)}\delta(j)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \frac{1}{\mathbb{E}^{\mathbf{P}}\left[\xi(T)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right]}\mathbb{E}^{\mathbf{P}}\left[\sum_{j=t+1}^{T}\frac{\xi(T)}{R(t,j)}\delta(j)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbf{Q}}\left[\sum_{j=t+1}^{T}\frac{1}{R(t,j)}\delta(j)\ \middle|\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\right].
\end{aligned}
$$

Therefore, $\mathbf{Q}$ is a quasi-EMM. Note that all of the derivations above are two-way, it completes the proof. □

A.4 Proposition 5

Proof. Suppose there exists an arbitrage strategy $\theta \in L^{(i),S}$, that is $\delta^\theta > 0$. And denote by c^* the the solution of the optimization problem (1.1) for agent i given S. Let θ^* be one of the optimal strategies, that is, $c^* = e + \delta^{\theta^*}$.

Then construct another strategy by $\theta' = \theta^* + \theta$. We have $c' = e + \delta^{\theta'} = e + \delta^{\theta^*} + \delta^{\theta} = c^* + \delta^{\theta} > c^*$. Since U is strictly increasing, $U^{(i)}(c') > U^{(i)}(c^*)$. Contradiction. $\square$

A.5 Corollary 2

Proof. Since there is no arbitrage, $X^{(i),S}$ is bounded, that is, $0 \leq c \leq e$ for all $c \in X^{(i),S}$. Continuous utility function can reach the maximum on $X^{(i),S}$. $\square$

A.6 Proposition 6

Proof. First-Order Condition implies that

$$\begin{aligned} 0 &= \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} \pi(t)\delta^{\theta}(t)\right] \\ &= \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} \mathbb{E}^{\mathbf{P}}\left[\pi(t)\delta^{\theta}(t) \,\middle|\mathcal{F}^{(i)} \vee \mathcal{G}_t\right]\right] \\ &= \mathbb{E}^{\mathbf{P}}\left[\sum_{t=0}^{T} \pi^{(i)}(t)\delta^{\theta}(t)\right]. \end{aligned}$$

Let τ be an arbitrary stopping time such that $\tau \leq T$ and $\{\tau \leq t\}$ is $\mathcal{F}_t^{(i)} \vee \mathcal{G}_t$-measurable. Define the trading strategy θ by, for some asset l, $\theta_k(t) = 0$ for all $k \neq l, t \in \mathcal{T}$, and

$$\theta_l(t) = \begin{cases} 1, & \text{for } t < \tau, \\ 0, & \text{for } t \geq \tau, \end{cases}$$

where θ_k denotes the weight invested in asset k. Note that $\theta \in L^{(i),S}$. Now we can have

$$\mathbb{E}^{\mathbf{P}}\left[-S_l(0)\pi^{(i)}(0) + \sum_{t=1}^{\tau} \pi^{(i)}(t)\delta_l(t) + \pi^{(i)}(\tau)S_l(\tau)\right] = 0.$$

That means the deflated gain process of asset l defined by

$$G_l(t;\pi) = \pi^{(i)}(t)S_l(t) + \sum_{j=1}^{t} \pi^{(i)}(j)\delta_l(j)$$

is a martingale with respect to the filtration $\{\mathcal{F}^{(i)} \vee \mathcal{G}_t\}$. $\square$

A.7 Proposition 7

Proof. Let $\pi^{(i)}$ be the state-price deflator obtained by Proposition 6, $i \in I$, and let

$$A = \big\{\pi : \exists\alpha^{(2)} > 0, ..., \alpha^{(n)} > 0, \text{ such that, for } i = 2, ..., n, \\ \text{the optimal projection of } \pi \text{ onto } \{\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\}_{t\in\mathcal{T}} \text{ is } \alpha^{(i)}\pi^{(i)}\big\}.$$

Suppose $\forall\pi \in A$, $\forall\alpha^{(1)} > 0$, the optimal projection of π onto $\{\mathcal{F}_t^{(i)} \vee \mathcal{G}_t\}_{t\in\mathcal{T}}$ is <u>not</u> $\alpha^{(1)}\pi^{(1)}$.

Now if there exists an equilibrium satisfying (1.2)(1.3), that is, $\exists\theta^{(1)}, ..., \theta^{(n)}$ such that $\theta^{(1)} + ... + \theta^{(n)} = 0$ and $\theta^{(i)}$ maximizes the individual utility. The First-Order Condition implies for all $i \in I$,

$$\langle\alpha^{(i)}\pi^{(i)}, \delta^{\theta^{(i)}}\rangle = \alpha^{(i)}\langle\pi^{(i)}, \delta^{\theta^{(i)}}\rangle = 0.$$

Then we have $\langle\pi, \delta^{\theta^{(i)}}\rangle = \langle\alpha^{(i)}\pi^{(i)}, \delta^{\theta^{(i)}}\rangle = 0$ for $i = 2, ..., n$, however $\langle\pi, \delta^{\theta^{(1)}}\rangle \neq \langle\alpha^{(1)}\pi^{(i)}, \delta^{\theta^{(1)}}\rangle = 0$. By summing up, we have a contradiction

$$0 = \langle\pi, \delta^{\sum_i \theta^{(i)}}\rangle = \langle\pi, \sum_i \delta^{\theta^{(i)}}\rangle = \langle\pi, \delta^{\theta^{(1)}}\rangle \neq 0.$$

The above shows non-existence of quasi state-price deflator implies non-existence of equilibrium. $\square$

A.8 Proposition 8

Proof. For any feasible allocation $(x^{(1)}, ..., x^{(n)})$, we have

$$\begin{aligned}
\sum_{i \in I} \lambda^{(i)} U^{(i)}(c^{(i)*}) &= \sum_{i \in I} \left[\lambda^{(i)} U^{(i)}(c^{(i)*}) - (\Pi(c^{(i)*}) - \Pi(e^{(i)})) \right] \\
&\geq \sum_{i \in I} \left[\lambda^{(i)} U^{(i)}(x^{(i)}) - (\Pi(x^{(i)}) - \Pi(e^{(i)})) \right] \\
&= \sum_{i \in I} \lambda^{(i)} U^{(i)}(x^{(i)}) - \sum_{i \in I} \Pi(x^{(i)} - e^{(i)}) \\
&= \sum_{i \in I} \lambda^{(i)} U^{(i)}(x^{(i)}) - \Pi\left(\sum_{i \in I} x^{(i)} - e \right) \\
&\geq \sum_{i \in I} \lambda^{(i)} U^{(i)}(x^{(i)})
\end{aligned}$$

□