Aspects of T-dually extended Superspaces

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Abstract of the Dissertation

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This dissertation is divided into three main parts where we derive various properties of the T-dually extended superspaces.

In the first part we reformulate the manifestly T-dual description of the massless sector of the closed bosonic string, directly from the geometry associated with the (left and right) affine Lie algebra of the coset space Poincaré/Lorentz. This construction initially doubles not only the (spacetime) coordinates for translations but also those for Lorentz transformations (and their "dual"). As a result, the Lorentz connection couples directly to the string (as does the vielbein), rather than being introduced indirectly through covariant derivatives as previously. This not only reproduces the old definition of T-dual torsion, but automatically gives a general, covariant definition of T-dual curvature (but still with some undetermined connections).

In the second part we give the manifestly T-dual formulation of the massless sector of the classical 3D Type II superstring in off-shell 3D $\mathcal{N} = 2$ superspace, including the action. It has a simple relation to the known superspace of 4D $\mathcal{N} = 1$ supergravity in 4D M-theory via 5D F-theory. The pre-potential appears as part of the vielbein, without derivatives.

In the last and the most involved part we find the pre-potential in the superspace with $AdS_5 \times S^5$ background. The pre-potential appears as part

of the vielbeins, without derivatives. In both subspaces (AdS_5 and S^5) we use Poincaré coordinates. We pick one bulk coordinate in AdS_5 and one bulk coordinate in S^5 to define the space-cone gauge. Such space-cone gauge destroys the bulk Lorentz covariance. However, it still preserves boundary Lorentz covariance (and gives projective superspace) $SO(3,1) \otimes SO(4)$ and so symmetries of boundary CFT are manifest.

To my family and friends.

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Chapter 1

1 General Introduction

T-duality is an important duality that we know exists in string theory. We also know that there should exist a low energy effective theory coming from string theory. This low energy theory is known as supergravity and string theory hints to us that supergravity should exhibit a natural appearance of T-duality. As in usual general relativity also in supergravity the Riemann curvature tensor is of uttermost importance. The T-dual analog of the Riemann tensor was first obtained by Professor Warren Siegel in his paper **Superspace duality in low-energy superstrings** [1]. The derivation of that tensor was not direct. In our first paper **Natural curvature for manifest T-duality** [11] we therefore obtained the T-dual Riemann tensor in a natural way directly using a certain T-dually extended superspace (though this did not require supersymmetry). Our derivation of the Riemann tensor paralleled the original Einstein theory. We also discovered that additional covariance compensators should be added to the original T-dual Riemann tensor.

As we saw in our first paper the T-dually extended superspace has potential to produce objects of interest (like the Riemann tensor) in a very natural way. In the next paper we proceeded to include supersymmetry into the model. One of the easiest models of supergravity is three dimensional supergravity with two kinds of supersymmetry, often written as 3D $\mathcal{N} = 2$ supergravity. It is known that this supergravity naturally comes from an even easier theory of four dimensional supergravity with only one supersymmetry, also known as 4D $\mathcal{N} = 1$ supergravity. The process of going from 4D $\mathcal{N} = 1$ to 3D $\mathcal{N} = 2$ supergravity is called dimensional reduction where one basically forgets about one spatial dimension. It did not take long to realise that this setting was even more interesting with respect to the emergence of the T-dually extended superspace. One can imagine that the four dimensional supergravity with one supersymmetry could be thought of as coming from a curious five dimensional space with one supersymmetry, but with two dimensions of time! It was even more fascinating because this idea parallels some formulations of twelve dimensional two-time theories called F-theory. So we also called our peculiar two-time theory a low dimensional F-theory. The reason why we looked one dimension higher and at two-time theory was the twofold use of the dimensional reduction. Starting from five dimensional F-theory one can dimensionally reduce in the time direction, i.e. forget about one time direction and re-obtain the classical $\mathcal{N} = 1$ supergravity in four dimensions. However, one can alternatively remove one spatial dimension and thus get a very interesting two-time and two-space dimensional theory. Surprisingly, this theory can be naturally re-casted as the theory living at three dimensional T-dually extended superspace. This identification was our starting motivation for the second paper **T-duality off shell in 3D Type II superspace** [12]. The objective was to use the T-dually extended superspace to solve three dimensional supergravity via an object called the pre-potential. One can imagine a pre-potential as the basic vacuum field from which all the physical fields can be obtained by action of covariant derivatives. We succeeded and obtained the right form of pre-potential again in a very natural way.

The long term objective was to look at real superstring theory, which naturally lives in ten dimensions. The real space is however four dimensional. Superstring theory therefore has to be somehow constrained or compactified to four dimensions. In the last paper **Pre-potential in the $AdS_5 \times S^5$ Type IIB superspace** [13] we examined fully fledged ten dimensional $\mathcal{N} = 2$ superstring theory, i.e. IIB supergravity using the T-dually extended superspace. We solved it for flat and also for the $AdS_5 \times S^5$ background. The $AdS_5 \times S^5$ background is a curved, but still maximally symmetric background. This space is of high importance because of the AdS/CFT correspondence. In recent years physicists realised that there exists a dictionary between problems formulated in terms of superstring theory i.e. supergravity living in ten dimensional space and Conformal Field theory living at the boundary of that space, this is the AdS/CFT correspondence. It is of high interest because it shows how to translate hard problems in one theory to hopefully easier problems in another. The natural formulation of objects in the supergravity is therefore very important. That was our motivation to use our T-dually extended superspace to understand solutions in the $AdS_5 \times S^5$ background. The pre-potential solution we obtained for both backgrounds (flat and $AdS_5 \times S^5$) is in a sense united. We started with the T-dualised version of type IIB superstring theory in a flat background. As in usual general relativity this theory could be made curved by using generalised vielbeins. To see the physical content of the theory one needs to solve various constraints, known as the $ABCD$ constraints. In the theory with extended T-dual superspace one also finds a huge gauge symmetry. Therefore we picked the gauge to be a generalised light-cone gauge. This choice fully fixes the gauge freedom. We solved the theory at the linearised level around the flat and the $AdS_5 \times S^5$ background. The particular treatment of the $AdS_5 \times S^5$ background was in its own right interesting. In order to put it on the same footing with the flat space we needed to extend the $AdS_5 \times S^5$ symmetry group (after the Wick rotation) $SO(5) \otimes SO(5)$ to the the double flat space space symmetry group $SO(10) \otimes SO(10)$ (after the Wick rotation). That added some non-trivial contributions into the solutions of the torsion constraints. Finally we found the right pre-potential solution for both flat and the $AdS_5 \times S^5$ background and even more interestingly the $AdS_5 \times S^5$ pre-potential was a certain curvature dependent deformation of the flat solution. We also provided the near horizon limit in the case of $AdS_5 \times S^5$ and a more detailed analysis about the connection with the boundary CFT theory.

Chapter 2

2 Introduction: Natural curvature for manifest T-duality

2.1 Outline

T-duality invariance can be manifested on all the fields of the massless sector of bosonic strings [1]. This was based on the treatment of the compactification scalars, for dimensional reduction of d dimensions, as elements of the coset $SO(d,\, d)\,/SO(d)^2$ [2]. This symmetry was expanded to:

$$SO(D,\, D)\,/\,SO(D-1,\, 1)^2 \tag{2.1}$$

for the full D dimensions to include all fields without compactification, where the symmetry is broken spontaneously to the usual SO(D−1,1), except when partially restored by dimensional reduction. (Generalization to GL groups [4] was also treated, but turned out not to be convenient for supersymmetry, and will not be considered here. For relations to later approaches, and extensions beyond what is needed here, see [3] and references therein.)

We will work on a space with explicit Lorentz coordinates. Dependence of the (background) vielbein on them is completely fixed (up to gauge) by the coset constraints, as applied by fixing the associated parts of the torsion to take their "vacuum" values. Moreover, as in [1], D-dimensional spacetime will be dualized. To do the stringy generalizations (of oscillator algebras together with the Lorentz algebras), we will need to introduce a new current Σ for consistency with the Jacobi identity [5]. (The necessity of this current was first realized in the context of $AdS_5 \times S^5$ [16].) The usual oscillator Lie algebra will become the extended affine Lie algebra (Lorentz and Σ generators included).

The generalized torsion is constructed from this affine Lie algebra in a general background, which acts as the stringy generalization of covariant derivatives. Because of the additional currents, the enlarged vielbein that describes this background includes the Lorentz connection, and the enlarged torsion includes also the curvature. Closure of the algebra implies the orthogonality constraints $E\eta E^{\mathbf{T}} = \eta$ on the vielbein. Solving these together with the coset constraints reduces the vielbein to the usual T-dual generalization of the vielbein and Lorentz connection, as well as a new curvature-like field. There is also an extension of dimensional reduction to the usual D coordinates. At the end we will obtain the same results for the torsion constraints and curvature tensor as previously, but by a much more direct way.

The rest of this paper is organized as follows: In the remainder of the Introduction we summarize the general procedure. In the next section we review the description of fields on general coset spaces, and then apply this to the case of spin for Poincaré/Lorentz to give a "first-quantized" approach to general relativity. The corresponding affine Lie algebra is described in section 3. In section 4 we introduce the vielbein and the coset constraints on the torsion, and orthogonality. The new analysis of Lorentz connections and curvatures is given in section 5, followed by our conclusions.

2.2 Procedure

The general procedure (to be applied in detail below for the present example) is thus:

1. Begin with a coset space G/H. By the usual construction (left and right group multiplication) this comes with two Lie algebras for G, one for "symmetry generators" and one for "covariant derivatives", represented by derivatives on the group space.

2. Generalize to the affine Lie algebras by making the group coordinates functions of the worldsheet coordinate σ. The number of currents is double that of the original Lie algebra, since they are also worldsheet vectors. (I.e., there are τ and σ components, or "left" and "right", depending on the basis. In the present case, the left and right currents are also left-propagating and right-propagating on the worldsheet; this is determined by the definition of the Virasoro operators, which we don't discuss here.) The covariant derivatives and symmetry generators become currents Z and $\tilde{Z}$ that commute with each other, $[Z, \tilde{Z}] = 0$.

3. The zero-modes of this affine Lie algebra define an enlarged ordinary Lie algebra/group, the inhomogeneous version IG of the original group G [16]. For manifest T-duality, double the coordinates to describe this enlarged group space, using the standard construction for the affine Lie algebra of a group [19].

4. Make this group space into a general curved space (describing massless fields) by multiplying the covariant derivative currents Z by a "viel-

bein" E: The group currents Z are thus a basis for general currents $\mathit{\Pi}$ on this space; they define the "vacuum", $\langle \mathit{\Pi} \rangle = Z$. The algebra of these currents $\mathit{\Pi}$ replaces the structure constants of the affine Lie algebra IG with covariant "torsion". Requiring that the inhomogeneous term still gives the group metric imposes orthogonality on the vielbein.

5. The coset constraints are then imposed by requiring that commutators of the currents $\mathit{\Pi}$ of H with arbitrary $\mathit{\Pi}$'s yield the same result as in the coset (before introducing the vielbein). This implies the $\mathit{\Pi}$ for H can be gauged to its coset value, and fixes the H-dependence of the remaining currents. These constraints can be stated as conditions on the torsion.

6. Apply any additional torsion constraints, such as those in ordinary (super)gravity.

7. Finally, to spontaneously break T-duality symmetry and return to the usual coordinates, half of the currents for the symmetry generators $\tilde{Z}$ (forming a subalgebra) are taken as Killing vectors [6]. (This corresponds to removing the coordinates for the inhomogeneous part of IG, reversing step 3 above.) Since they commute with the basis Z for the covariant derivatives, the requirement that they commute with the (curved space) covariant derivatives $\mathit{\Pi}$ implies that the vielbein E is independent of the corresponding coordinates.

Chapter 3

3 Coset spaces and their generalizations

3.1 Group spaces

Coset constructions have proven useful in defining representations of the Poincaré, (anti) de Sitter, and conformal groups, and their supersymmetric generalizations. With these in mind, we now review the general procedure for defining fields on coset spaces.

Cosets are often used to construct nonlinear σ models: There one focuses on the coset space itself, of which the scalar fields are elements. For example, one usually first-quantizes string theory about symmetric backgrounds by treating the spacetime coordinates $X(\tau, \sigma)$ (etc.) as coordinates of a coset space. (Of course, more general backgrounds are also considered, but are less tractable.) The string wave function is then implicitly a scalar functional of these coordinates (at fixed τ).

There is some difficulty with this approach for the superstring, since the ground state, and thus the string field/wave function, is not a scalar. Similar remarks apply to introducing massless backgrounds into the string action, since the coordinates carry "curved" indices, while coupling gravity to fermions requires also "flat" ones.

The generalization that solves this problem is simple: For the coset G/H, keep all the coordinates of G (the "symmetry" or "isometry" group), rather than the usual procedure of immediately going to a unitary gauge where the coordinates of H (the "gauge", "isotropy", or "stabilizer" subgroup) are gauged away. The dependence of the fields on the H coordinates will be fixed, by defining their representations of H, but will be trivial only for scalars.

For this purpose we need to distinguish the differential operators responsible for left and right group multiplication:

$$g' = g_L \, g \, g_R \tag{3.1}$$

Parametrizing any group element g by coordinates α^I in terms of the generators G_I

$$[\, G_I, \, G_J \,] \, = \, -i f_{IJ}{}^K \, G_K \tag{3.2}$$

(e.g., using any exponential parametrization), we can then write the corresponding infinitesimal transformations as

$$\delta g \;=\; i\epsilon_L^I G_I g + g i\epsilon_R^I G_I = (\epsilon_L^I q_I + \epsilon_R^I D_I) g(\alpha) \tag{3.3}$$

where

$$q_I \;=\; L_I{}^M(\alpha)\partial_M, \quad (dg)g^{-1} \;\equiv\; i d\alpha^M L_M{}^I G_I \tag{3.4}$$

$$D_I \;=\; R_I{}^M(\alpha)\partial_M, \quad g^{-1}(dg) \;\equiv\; i d\alpha^M R_M{}^I G_I \tag{3.5}$$

(where $\partial_M \;\equiv\; \partial/\partial\,\alpha^M$) define the symmetry generators q and covariant derivatives D in terms of the vielbein appearing in the differential forms invariant under one or the other type of transformation. Because left and right group multiplication commute, so do the symmetry generators and covariant derivatives:

$$[q_I, D_J] \;=\; 0 \tag{3.6}$$

Thus the "covariant" derivatives are actually *in*variant; they become only covariant in unitary H gauges, due to compensating gauge transformations.

3.2 Fields on coset spaces

We then decompose the basis of generators G_I of the symmetry group G into the generators H_ι of the isotropy group H and the remaining ones T_i of the coset G/H. The representation space for the coset is constructed as follows: Define the linear space with basis elements $|0,\,{}_m\rangle$. Let that space carry the matrix representation $\rho\,(H_\iota)_m{}^k$ of the isotropy subgroup algebra; i.e., we have:

$$H_\iota\,|0,\,{}_m\rangle \;:=\; \rho\,(H_\iota)_m{}^k\,|0,\,{}_k\rangle \tag{3.7}$$

We also have the action of the whole group on this basis:

$$|\alpha,\,{}_m\rangle \;:=\; g(\alpha)\,|0,\,{}_m\rangle \tag{3.8}$$

We can then express the representation of the symmetry generators and covariant derivatives as differential operators on the wave function

$$\psi_m(\alpha) \;:=\; \langle\alpha,\,{}_m|\,\psi\,\rangle \tag{3.9}$$

The wave function $\psi_m(\alpha)$ depends also on the isotropy group coordinates α^ι, but this dependence is fixed: In a convenient exponential parametrization,

$$\begin{aligned} \psi_m(\alpha) &:= \langle 0,\, _m|\; e^{-i\,\alpha^\iota\, H_\iota}\, e^{-i\,\alpha^i\, T_i}\;|\psi\rangle \;=\; \left(e^{-i\,\alpha^\iota\,\rho\,(H_\iota)}\right)_m{}^k\,\langle 0,\, _k|\; e^{-i\,\alpha^i\, T_i}\;|\psi\rangle \\ &= \left(e^{-i\,\alpha^\iota\,\rho\,(H_\iota)}\right)_m{}^k\,\psi_k\,(\alpha^i) \qquad \equiv\; e_m{}^k(\alpha^\iota\,)\,\psi_k\,(\alpha^i) \end{aligned} \tag{3.10}$$

The vielbein $e_m{}^k(\alpha^\iota)$ is dependent only on the coset coordinates α^ι and can be gauged to the identity.

From the above construction we know how the covariant derivatives corresponding to the isotropy subgroup act on $\psi_m(\alpha)$:

$$\begin{aligned} D_\iota\,\psi_m\,(\alpha) &= \langle 0,\, _k|\; \rho\,(H_\iota)_m{}^k\, g^{-1}(\alpha)\;|\psi\rangle \\ &= \rho\,(H_\iota)_m{}^k\psi_k\,(\alpha) \end{aligned} \tag{3.11}$$

We can also calculate the action of the symmetry group generators on the wave function:

$$\begin{aligned} q_I\,\psi_m(\alpha) &= \langle 0,\, _m|\; g^{-1}(\alpha)\,G_I\;|\psi\rangle \\ &= (\,G_I\,\psi\,)_m\,(\alpha) \end{aligned} \tag{3.12}$$

Since we know how the covariant derivatives with respect to the α^ι act, we can therefore solve those constraints and replace partial derivatives (with respect to the α^ι) with matrices in q_I and D_I. The dependence of all objects on the coset coordinates is thus fixed. The remaining covariant derivatives D_i act nontrivially.

3.3 Curved spaces with isotropic coordinates

We can also covariantize the covariant derivatives D_I with respect to (super) Yang-Mills symmetry. (The (super) Yang-Mills gauge group is unrelated to the isotropy gauge group, except for the case of gravity.) We can write the (super) Yang-Mills covariantized covariant derivatives as:

$$\nabla_I \,:=\, D_I \,+\, i\,A_I, \quad [\,\nabla_I,\,\nabla_J\,\} \,=\, f_{IJ}{}^K\,\nabla_K \,+\, i\,F_{IJ} \tag{3.13}$$

In the first-quantized approach to (super)gravity the derivatives are gauge covariantized with respect to the (super-)Poincaré group [5]. The Yang-Mills generators are replaced with partial derivatives with respect to all coordinates:

$$D_I \,\rightarrow\, \nabla_I \,=\, e_I{}^K\,\partial_K \,=\, \hat{e}_I{}^K\,D_K \tag{3.14}$$

The vielbein ${e_I}^K$ or ${\hat{e}_I}^K$ are arbitrary. The local Lorentz transformations are now included with the rest of the coordinate transformations and the covariant derivatives transform under the symmetry transformations as:

$$\nabla' \;=\; e^{\Lambda}\,\nabla\,e^{-\Lambda} \quad \text{where} \quad \Lambda \;:=\; \Lambda^M\,D_M \;\equiv\; \bar{\Lambda}^A\,\nabla_A \tag{3.15}$$

The torsion T is a combination of the structure constants and field strengths of Yang-Mills:

$$[\,\nabla_I,\,\nabla_J\,] \;=\; {T_{IJ}}^K\,\nabla_K \tag{3.16}$$

We divide indices as before, for the isotropy group, which in our case will be the Lorentz groups $SO(D-1,1)^2$, and for the coset space: We can write $\nabla_I \;\equiv\; (\,\nabla_H,\,\nabla_{G/H}\,)$. Using the newly defined indices:

$$\begin{aligned}
[\,\nabla_H,\,\nabla_H\,] &= {f_{H\;H}}^H\,\nabla_H \\
[\nabla_H,\,\nabla_{G/H}] &= {f_{H\;\;G/H}}^{G/H}\,\nabla_{G/H} \\
[\nabla_{G/H},\,\nabla_{G/H}] &= {R_{G/H\;\;G/H}}^H\,\nabla_H \,+\, {T_{G/H\;\;G/H}}^{G/H}\,\nabla_{G/H}
\end{aligned} \tag{3.17}$$

The R in (3.17) is the usual curvature (its stringy analog will be calculated in the Riemann tensor subsection 6.2); ${T_{G/H\;\;G/H}}^{G/H}$ is the usual torsion.

We have required that ∇_H act as in coset space (which in our case will be flat space): The fact that the torsions ${T_{H\,H}}^H$ and ${T_{H\;H}}^{G/H}$ $(=0)$ take their free values implies that ∇_H can be gauged to its free value. The isotropy transformation of the coset part $\nabla_{G/H}$ is fully fixed by the requirement that the torsions ${T_{H\;G/H}}^H$ $(=0)$ and ${T_{H\;G/H}}^{G/H}$ get their free values. (We will see the stringy analog of this in subsection 5.2.) By keeping this dependence on the H coordinates, rather than gauging them away entirely, we have the first-quantized way to define the spin (for arbitrary representations), as a differential operator on that space [5].

Chapter 4

4 Affine Lie algebra and generalized T-duality

4.1 Current algebras

For application to the string, we consider current algebras on the worldsheet, or affine Lie algebras

$$[\, Z_{\mathcal{M}}\,(1),\; Z_{\mathcal{N}}\,(2)\,] \;=\; -i\,\eta_{\mathcal{M}\mathcal{N}}\,\delta'\,(\,2\;-\;1\,)\;-\;i\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\,Z_{\mathcal{P}}\,\delta\,(\,2\;-\;1\,) \tag{4.1}$$

where f is the structure constants of the ordinary Lie algebra. (Note that all the generators are understood as string currents, so they are dependent on the string coordinate $\sigma \;\equiv\; \sigma_1 \;\equiv$ "1". There is an implicit 2π with every $\delta(\sigma)$. Also, for dimensional analysis there is an implicit $1/\alpha'$ with η.) The metric η of the affine (Schwinger) term is invertible as a consequence of our including both components of the current, as should be clear from the Abelian case considered below. Due to our doubling of coordinates for manifest T-duality, the group coordinates $X^{\mathcal{M}}$ carry the same index. Acting on background fields ϕ, these reduce to the group covariant derivatives $D_{\mathcal{M}}$ of the ordinary (non-affine) algebra (with the same structure constants),

$$[Z_{\mathcal{M}}(1), \phi(X(2))] \;=\; -i(D_{\mathcal{M}}\phi)\delta\,(\,2\;-\;1\,) \tag{4.2}$$

(Similar remarks apply to a second Lie algebra $\tilde{Z}$ for which q replaces D and $[Z, \tilde{Z}] = 0$.) We are interested in the affine Poincaré algebra, where the index

$$_{\mathcal{M}} \;:=\; (\;_{MN},\;_{M},\;^{MN}\;) \tag{4.3}$$

has dimension $2D^2$, as we will now describe.

We begin with the current algebra associated with the usual X coordinates. In string theory one naturally gets the interpretation of T-duality as the reflection subgroup of the bigger $O(D,D)$ group. One can rewrite the string oscillator algebra using the explicit $O(D,D)$ vector

$$P_M \;:=\; (\,P_m,\; X'^m\,) \tag{4.4}$$

Using this generalized $O(D,D)$ momentum one gets the algebra

$$[\,P_M\,(1),\; P_N\,(2)\,] \;=\; i\,\eta_{M\,N}\,\delta'\,(\,2\;-\;1\,) \tag{4.5}$$

where η_{MN} is the $O(D,D)$ metric:

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_m{}^n \\ \delta_n{}^m & 0 \end{pmatrix} \tag{4.6}$$

In the future we want to use a different basis for the string oscillator algebra (4.5). Therefore we introduce the left/right vector

$$P_M := (P_{\mathbf{m}}, P_{\tilde{\mathbf{m}}}) \equiv \frac{1}{\sqrt{2}} (P_m + X'_m, P_m - X'_m) \tag{4.7}$$

In this basis the oscillator algebra has the same form as (4.5) except for the form of the metric:

$$\eta_{MN} = \begin{pmatrix} \eta_{\mathbf{m}\,\mathbf{n}} & 0 \\ 0 & -\eta_{\tilde{\mathbf{m}}\,\tilde{\mathbf{n}}} \end{pmatrix} \tag{4.8}$$

4.2 Lorentz

In the next step we want to merge the algebra (4.5) with the Lorentz algebra $so(D-1,1)^2$. The reason is that the metric g and b field are in the coset space $SO(D,D)/SO(D-1,1)^2$. This suggests that the coordinate space should be obtained by modding out by the subgroup $SO(D-1,1)^2$. The left/right basis of (4.7) is then appropriate.

The generators for this Lorentz algebra are denoted as

$$S_{MN} := (S_{\mathbf{mn}}, S_{\tilde{\mathbf{m}}\tilde{\mathbf{n}}}) \tag{4.9}$$

and satisfy the usual commutation relations

$$[S_{\mathbf{mn}} (1), S_{\mathbf{kl}} (2)] = - i\, \eta_{[\mathbf{m}[\mathbf{k}} S_{\mathbf{n}]\mathbf{l}]}\, \delta (2 - 1) \tag{4.10}$$

$$[S_{\mathbf{m}\,\mathbf{n}} (1), S_{\tilde{\mathbf{k}}\,\tilde{\mathbf{l}}} (2)] = 0 \tag{4.11}$$

Same for Left $\rightarrow$ Right

(where $[\ldots]$ is the unweighted anti-symmetrization). Since P and S form the ordinary Poincaré algebra, we have:

$$[S_{\mathbf{m}\,\mathbf{n}} (1), P_{\mathbf{k}} (2)] = i\eta_{\mathbf{k}[\mathbf{m}} P_{\mathbf{n}]}\, \delta (2 - 1) \tag{4.12}$$

$$[S_{\mathbf{m}\,\mathbf{n}} (1), P_{\tilde{\mathbf{k}}} (2)] = 0 \tag{4.13}$$

Same for Left $\rightarrow$ Right

However, the set of generators (S_{MN}, P_M) does not form a closed affine Lie algebra. The Jacobi identity requires a new field Σ such that

$$[\,P,\,P\,] \propto \delta^{'} + \Sigma \quad \text{and} \quad [\,S,\,\Sigma\,] \propto \delta^{'} + \Sigma \tag{4.14}$$

Using the commutators $[\,S,\,[\,P,\,P\,]]$ and the Jacobi identity, we obtain the new set of generators

$$Z_{\mathcal{M}} := (\,S_{MN},\,P_M,\,\Sigma^{MN}\,) \tag{4.15}$$

for which we have the following affine Lie algebra (providing only non-zero commutators):

$$\begin{aligned}
[\,S_{\mathbf{mn}}\,(1),\,S_{\mathbf{kl}}\,(2)\,] &= -i\,\eta_{[\,\mathbf{m}\,[\,\mathbf{k}}\,S_{\mathbf{l}\,]\,\mathbf{n}\,]}\,\delta\,(\,2\,-\,1\,) \qquad (4.16)\\
[\,S_{\mathbf{mn}}\,(1),\,P_{\mathbf{k}}\,(2)\,] &= i\,\eta_{\mathbf{k}\,[\,\mathbf{m}}\,P_{\mathbf{n}\,]}\,\delta\,(\,2\,-\,1\,)\\
[\,S_{\mathbf{mn}}\,(1),\,\Sigma^{\mathbf{kl}}\,(2)\,] &= i\,\delta_{\mathbf{mn}}{}^{\mathbf{kl}}\,\delta'\,(\,2\,-\,1\,) - i\delta_{[\,\mathbf{m}}{}^{[\,\mathbf{k}}\,\eta_{\mathbf{n}\,]\,\mathbf{s}}\Sigma^{\mathbf{l}]\mathbf{s}}\,\delta\,(\,2\,-\,1\,)\\
[\,P_{\mathbf{m}}\,(1),\,P_{\mathbf{n}}\,(2)\,] &= i\,\eta_{\mathbf{mn}}\,\delta'\,(\,2\,-\,1\,) + i\,\eta_{\mathbf{m}\,\mathbf{h}}\,\eta_{\mathbf{n}\,\mathbf{s}}\Sigma^{\mathbf{hs}}\,\delta\,(\,2\,-\,1\,)\\
&\curvearrowright \quad \text{left algebra} \to -\text{ right algebra}\\
[\,\text{left, right}\,\} &= 0.
\end{aligned}$$

Thus we get the general structure of an affine Lie algebra (10.1). (Non-affine stringy Lorentz algebras were considered in [20]. Left and right spin algebras have also been used in [21], but commuting with P. Neither of those had Σ.)

For dealing with antisymmetric pairs of indices we have introduced an implicit metric such that for any two antisymmetric tensors we have

$$A \cdot B \,\equiv\, \tfrac{1}{2} A^{\mathbf{mn}} B_{\mathbf{mn}} \tag{4.17}$$

The identity matrix with respect to this inner product is

$$\delta_{\mathbf{mn}}{}^{\mathbf{pq}} \,\equiv\, \delta_{[\mathbf{m}}{}^{\mathbf{p}} \delta_{\mathbf{n}]}{}^{\mathbf{q}} \tag{4.18}$$

The only nonvanishing terms in the metric and structure constants are (as could be guessed by dimensional analysis)

$$\eta_{PP},\ \eta_{S\Sigma}\,; \quad f_{SPP},\ f_{SS\Sigma} \tag{4.19}$$

where we have lowered the upper index on f with η to take advantage of its total antisymmetry, and used "schematic" notation, replacing explicit indices with their type:

$$\mathcal{M} := (\,{}_{MN},\,{}_{M},\,{}^{MN}\,) := (\,S,\,P,\,\Sigma\,) \tag{4.20}$$

Explicitly these are, for the left-handed algebra,

$$\begin{aligned} (\eta)_{\mathbf{mn}} &= \eta_{\mathbf{mn}}; \quad (\eta)_{\mathbf{mn}}{}^{\mathbf{pq}} = \delta_{\mathbf{mn}}{}^{\mathbf{pq}} \\ f_{\mathbf{mn}}{}^{\mathbf{p}\,\mathbf{q}} &= -\delta_{\mathbf{mn}}{}^{\mathbf{pq}}; \quad f_{\mathbf{mn}\,\mathbf{pq}}{}^{\mathbf{rs}} = \eta_{[\mathbf{m}[\mathbf{p}}\delta_{\mathbf{q}]\mathbf{n}]}{}^{\mathbf{rs}} \end{aligned} \tag{4.21}$$

For the right-handed algebra we change the signs of the corresponding terms in $\eta_{\mathcal{MN}}$ but not in f.

Chapter 5

5 Curved spaces with affine algebras

5.1 Background fields

We now introduce background fields following [1], but using the affine algebra (10.1). Using the vielbein we can write:

$$\mathit{\Pi}_{\mathcal{A}}(1) \;=\; E_{\mathcal{A}}{}^{\mathcal{M}}(X^{\mathcal{M}})Z_{\mathcal{M}} \tag{5.1}$$

Then we get the affine Lie algebra for the $\mathit{\Pi}_{\mathcal{A}}$ operators:

$$[\mathit{\Pi}_{\mathcal{A}}(1), \mathit{\Pi}_{\mathcal{C}}(2)] \;\equiv\; -i\eta_{\mathcal{AC}}\,\delta'\,(\,2\,-\,1\,) - iT_{\mathcal{AC}}{}^{\mathcal{E}}\,\mathit{\Pi}_{\mathcal{E}}\,\delta\,(\,2\,-\,1\,) \tag{5.2}$$

where T is the stringy generalization of the torsion:

$$\begin{aligned} T_{\mathcal{AC}}{}^{\mathcal{E}} \;\;=\;\; & E_{[\mathcal{A}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{\mathcal{C}]}{}^{\mathcal{N}})E^{-1\mathcal{E}}_{\mathcal{N}} + \tfrac{1}{2}\eta^{\mathcal{ED}}E_{\mathcal{D}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{[\mathcal{A}|}{}^{\mathcal{N}})E^{-1\mathcal{F}}_{\mathcal{N}}\eta_{\mathcal{F}|\mathcal{C}]} \\ & + E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{C}}{}^{\mathcal{N}}E^{-1\mathcal{E}}_{\mathcal{P}}f_{\mathcal{MN}}{}^{\mathcal{P}} \end{aligned} \tag{5.3}$$

where $[\,{\scriptstyle\mathcal{A}}\,|\,|\,{\scriptstyle\mathcal{C}}\,]$ indicates antisymmetrization in only those indices. Note that the Jacobi identities imply the total antisymmetry of the torsion, just as for the structure constants.

This torsion can be identified with that of "ordinary" curved-space covariant derivatives (as in subsection 3.3) by use of the strong constraint: We write

$$\nabla_{\mathcal{A}} \;:=\; E_{\mathcal{A}}{}^{\mathcal{M}}D_{\mathcal{M}} \tag{5.4}$$

Using this and the strong constraint

$$(\nabla^{\mathcal{A}}\phi)(\nabla_{\mathcal{A}}\psi) \;=\; 0 \tag{5.5}$$

we get the same torsion in

$$[\,\nabla_{\mathcal{A}},\, \nabla_{\mathcal{C}}\,] = T_{\mathcal{AC}}{}^{\mathcal{D}}\,\nabla_{\mathcal{D}} \tag{5.6}$$

when acting on fields, since the second term in (15.3) can be added for free.

By setting the coefficient of the Schwinger term to be the metric η, the vielbein is forced to obey the orthogonality constraints:

$$E_{\mathcal{A}}{}^{\mathcal{M}}\eta_{\mathcal{MN}}\,E_{\mathcal{C}}{}^{\mathcal{N}} \;\equiv\; \eta_{\mathcal{AC}} \tag{5.7}$$

This choice does not affect the physics, and simplifies many of the expressions. For example, it implies the total antisymmetry of the torsion, when the upper index is implicitly lowered with η:

$$T_{\mathcal{ABC}} \;=\; \tfrac{1}{2}E_{[\mathcal{A}|}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{|\mathcal{B}}{}^{\mathcal{N}})E_{\mathcal{C}]\mathcal{N}} + E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{B}}{}^{\mathcal{N}}E_{\mathcal{C}}{}^{\mathcal{P}}f_{\mathcal{MNP}} \tag{5.8}$$

where we have used $E^{-1}{}_{\mathcal{M}}{}^{\mathcal{A}} \;=\; \eta^{\mathcal{A}\,\mathcal{B}}\eta_{\mathcal{M}\,\mathcal{N}}E_{\mathcal{B}}{}^{\mathcal{N}}$. (Also note that in the first term the antisymmetrization can be written as a cyclic sum without the 1/2, since it is already antisymmetric in the last two indices.) Thus, because of orthogonality, the vielbein is like (the exponential of) a 2-form, while the torsion is a 3-form; similarly, the Bianchi identities are a 4-form.

When solving the orthogonality constraint, note that we are also putting some parts of E to zero or to some particular constant value, which comes from the coset constraints on the torsion, as explained later. We get:

$$E_{\mathcal{A}}{}^{\mathcal{M}} \;=\; \begin{array}{c} \\ {\scriptstyle AB} \\ {\scriptstyle A} \\ {\scriptstyle AB} \end{array}\begin{array}{c} \begin{array}{ccc} {\scriptstyle MN} & \qquad\qquad {\scriptstyle M} & \qquad {\scriptstyle MN} \end{array} \\ \left(\begin{array}{ccc} \delta_{AB}{}^{MN} & 0 & 0 \\ \omega_A{}^{MN} & e_A{}^M & 0 \\ r^{ABMN} - \tfrac{1}{2}\omega^{CAB}\omega_C{}^{MN} & -e_C{}^M\omega^{CAB} & \delta^{AB}{}_{MN} \end{array}\right) \end{array} \tag{5.9}$$

where the new field r has a role to be explained later, and satisfies

$$r^{ABCD} + r^{CDAB} \;=\; 0 \tag{5.10}$$

5.2 Coset constraints

Our aim is to generalize the coset construction described in subsection 3.3 to affine Lie algebras, specifically the affine Poincaré algebra (14.7). Coset space dependence is fixed by the constraint that the covariant derivatives with the Lorentz group indices $S \;\equiv\; {}_{AB}$ act on fields by some particular matrix representation, i.e.,

$$(\nabla_S\,\psi)_{\,S} \;:=\; (M_S)_S{}^S\,\psi_S \tag{5.11}$$

For the covariant derivatives themselves, this implies, as described in section 3.3,

$$[\nabla_S, \nabla_{\mathcal{A}}\,] = f_{S\mathcal{A}}{}^{\mathcal{B}}\nabla_{\mathcal{B}} \qquad (T_{S\mathcal{A}}{}^{\mathcal{B}} = f_{S\mathcal{A}}{}^{\mathcal{B}}) \tag{5.12}$$

I.e., all covariant derivatives are in the same representations of S as in flat space. In particular, this means the subalgebra of ∇_S is unmodified from flat space, so we can choose the gauge

$$\nabla_S = D_S \qquad (E_S{}^{\mathcal{M}} = \delta_S{}^{\mathcal{M}}) \tag{5.13}$$

(However, other gauges, such as lightcone gauges, may also be useful [5].) This gauge was used, in addition to orthogonality, to obtain the expression for the vielbein in (5.9).

The rest of the coset constraint (5.12) gives the action of D_S on the nontrivial components of $E_A{}^{\mathcal{K}}$:

$$D_S\, E_P{}^P \equiv D_{AB}\, e_C{}^K = -\eta_{C[A}\, e_{B]}{}^K + e_C{}^M \eta_{M[A}\, \delta_{B]}{}^K \tag{5.14}$$

$$D_S\, E_P{}^S \equiv D_{AB}\, \omega_C{}^{KL} = -\,\eta_{C[A}\, \omega_{B]}{}^{KL} + \omega_C{}^{MN}\, \eta_{[M[A}\, \delta_{B]}{}^K\, \delta_{N]}{}^L \tag{5.15}$$

Thus in this gauge the dependence on the Lorentz coordinates is fixed for the vielbein, as well as the (residual) gauge parameters. (E.g., the Lorentz gauge parameters still have arbitrary dependence on x.)

Dimensional analysis is useful for further analysis of the torsion. The following table summarizes the torsion engineering dimensions:

Torsion component	Dimension
$T_{SS}{}^{\Sigma}$	-2
$T_{SS}{}^{P}$	-1
$T_{SS}{}^{S}$	0
$T_{SP}{}^{P}$	0
$T_{SP}{}^{S}$	1

Torsion component	Dimension
$T_{PP}{}^{P}$	1
$T_{S\Sigma}{}^{S}$	2
$T_{PP}{}^{S}$	2
$T_{P\Sigma}{}^{S}$	3
$T_{\Sigma\Sigma}{}^{S}$	4

Note that most of the torsions, including all torsions of nonpositive dimension, have already been fixed by the coset constraint.

Chapter 6

6 Relations to previous tensors

6.1 Remaining torsion constraint

The "usual" torsion constraint (generalized to 2D-valued indices)

$$T_{PP}{}^{P} = 0 \tag{6.1}$$

eliminates the last surviving torsion of dimension 1, and gives the constraints that were previously found in [1] by a different method. This can be expanded in schematic notation as

$$0 = T_{\color{red}P\color{blue}P\color{green}P} = \tfrac{1}{2}E_{[\color{red}P\color{black}|}{}^{\mathcal{K}}(D_{\mathcal{K}}E_{|\color{blue}P}{}^{\mathcal{H}})E_{\color{green}P\color{black}]\mathcal{H}} + E_{\color{red}P}{}^{\mathcal{K}}E_{\color{blue}P}{}^{\mathcal{H}}E_{\color{green}P}{}^{\mathcal{L}}f_{\mathcal{KHL}} \tag{6.2}$$

(Colored indices are not summed.)

For comparison, the analog of the torsion that appears in [1] (but taking into account orthogonality):

$$\mathbf{F}_{ABC} := \tfrac{1}{2}e_{[A|}{}^{K}(\partial_K e_{|B}{}^{H})e_{C]H} \tag{6.3}$$

is the same except that the range of indices is over only P, where (in our gauge) $e_A{}^M \equiv E_A{}^M$ and $D_P = \partial_M$ acting on a field. Thus, expanding the indices in (6.2) over (S, P, Σ) will separate it into $\mathbf{F}$ and ω terms.

Using the structure of the vielbein $E_{\mathcal{A}}{}^{\mathcal{M}}$ in (5.9), from the former term of (6.2) we get:

$$\mathbf{F}_{\color{red}P\color{blue}P\color{green}P} + \tfrac{1}{2}E_{[\color{red}P\color{black}|}{}^{S}(D_S E_{|\color{blue}P}{}^{P})E_{\color{green}P\color{black}]P} \tag{6.4}$$

(Repeated schematic indices (S, P, Σ) are summed over the subset indicated.) The latter term in this expression vanishes according to the first condition in (5.14) and structure of the vielbein. The latter term of (6.2) gives:

$$\begin{aligned} E_{\color{red}P}{}^{\mathcal{K}}E_{\color{blue}P}{}^{\mathcal{H}}E_{\color{green}P}{}^{\mathcal{L}}f_{\mathcal{KHL}} &= \tfrac{1}{2}E_{[\color{red}P}{}^{S}E_{\color{blue}P}{}^{P}E_{\color{green}P\color{black}]}{}^{P}f_{SPP} \\ & \quad {\color{red}P \to A} \mid {\color{blue}P \to B} \mid {\color{green}P \to C} \\ &= \tfrac{1}{2}\omega_{[\color{red}A\color{blue}B\color{green}C\color{black}]} \end{aligned} \tag{6.5}$$

We thus get the relation

$$\mathbf{F}_{ABC} + \tfrac{1}{2}\omega_{[ABC]} = 0 \tag{6.6}$$

This agrees with the constraints on $\omega_A{}^{BC}$ in [1],

$$\omega_{[\mathbf{abc}]} = -2\mathbf{F}_{\mathbf{abc}}\,, \quad \omega_{\mathbf{a}\tilde{\mathbf{b}}\tilde{\mathbf{c}}} = -\,\mathbf{F}_{\mathbf{a}\tilde{\mathbf{b}}\tilde{\mathbf{c}}} \tag{6.7}$$

There are also constraints involving the dilaton, which work the same way as previously; these are needed to allow definition of a Ricci tensor and scalar (i.e., field equations and action) independent of those connections that are not fixed by the above constraint.

6.2 Riemann tensor

Previously no full curvature tensor with manifest T-duality was derived, and even those pieces that were found came in an indirect way, not by commutation of covariant derivatives. Here we duplicate the known curvature directly as a torsion, and the missing pieces are identified as corresponding to the new field r^{ABCD}.

From (3.17) the curvature tensor is $T_{P\,P}{}^{S} \equiv R_{G/H\ G/H}{}^{H}$:

$$\begin{aligned} T_{P\,P}{}^{S} \;=\;& E_{[P}{}^{S}(D_S\, E_{P]}{}^{\mathcal{R}})E^{-1}_{\mathcal{R}}{}^{S} + \tfrac{1}{2}\eta^{S\,\Sigma}\, E_{\Sigma}{}^{S}(D_S\, E_{[P}{}^{\mathcal{R}})E^{-1}_{\mathcal{R}}{}^{\mathcal{K}}\,\eta_{\mathcal{K}\,|\,P]} \\ & + E_{[P}{}^{P}(D_P\, E_{P]}{}^{\mathcal{R}})E^{-1}_{\mathcal{R}}{}^{S} + \tfrac{1}{2}\eta^{S\,\Sigma}\, E_{\Sigma}{}^{P}(D_P\, E_{[P}{}^{\mathcal{R}})E^{-1}_{\mathcal{R}}{}^{\mathcal{K}}\,\eta_{\mathcal{K}\,|\,P]} \\ & + \tfrac{1}{2}\eta^{S\,\Sigma}\, E_{\Sigma}{}^{\Sigma}(D_{\Sigma}\, E_{[P}{}^{\mathcal{R}})E^{-1}_{\mathcal{R}}{}^{\mathcal{K}}\,\eta_{\mathcal{K}\,|\,P]} + E_{P}{}^{S} E_{P}{}^{S} E^{-1}_{S}{}^{S}\, f_{S\,S}{}^{S} \\ & + E_{[P}{}^{S}\, E_{P]}{}^{P}\, E^{-1}_{P}{}^{S}\, f_{S\,P}{}^{P} + E_{P}{}^{P} E_{P}{}^{P} E^{-1}_{\Sigma}{}^{S}\, f_{P\,P}{}^{\Sigma} \end{aligned} \tag{6.8}$$

Rewriting using explicit forms of the schematic indices and f, and using (5.14) and (6.3), after some algebra we get the final expression:

$$\begin{aligned} T_{AB}{}^{CD} \;=\;& e_{[A}{}^{M}\,\partial_M\,\omega_{B]}{}^{CD} + \omega_{[A\,|}{}^{C}{}_{H}\,\omega_{B]}{}^{HD} - \tfrac{1}{2}\,\omega_M{}^{CD}\,\omega^M{}_{AB} \\ & - \mathbf{F}_{A\,B}{}^{N}\,\omega_N{}^{CD} + r^{C\,D}{}_{AB} + ((D_{\Sigma})^{CD}\, e_A{}^{K})e_{BK} \end{aligned} \tag{6.9}$$

In the usual representations, $D_{\Sigma} = q_{\Sigma} = \partial_{\Sigma}$; as part of dimensional reduction, we set $q_{\Sigma}\,\phi = 0$. Then the curvature reduces to:

$$\begin{aligned} T_{AB}{}^{C\,D} \;=\;& e_{[A}{}^{M}\,\partial_M\,\omega_{B]}{}^{C\,D} + \omega_{[A|}{}^{C}{}_{H}\,\omega_{B]}{}^{H\,D} - \tfrac{1}{2}\,\omega_M{}^{C\,D}\,\omega^M{}_{A\,B} \\ & - \mathbf{F}_{A\,B}{}^{N}\,\omega_N{}^{C\,D} + r^{C\,D}{}_{A\,B} \end{aligned} \tag{6.10}$$

This form was derived also in [1] up to the antisymmetric $r^{CD}{}_{AB}$ part, required for covariance. Here the curvature tensor was obtained in a more direct way.

r can also be fixed by constraining the corresponding part of the curvature to vanish:

$$T_{\mathbf{abcd}} - T_{\mathbf{cdab}} \ = \ T_{\tilde{\mathbf{a}}\tilde{\mathbf{b}}\tilde{\mathbf{c}}\tilde{\mathbf{d}}} - T_{\tilde{\mathbf{c}}\tilde{\mathbf{d}}\tilde{\mathbf{a}}\tilde{\mathbf{b}}} \ = \ T_{\mathbf{ab}\tilde{\mathbf{c}}\tilde{\mathbf{d}}} - T_{\tilde{\mathbf{c}}\tilde{\mathbf{d}}\mathbf{ab}} \ = \ 0 \tag{6.11}$$

As the final step we reduce the coordinates to the usual half by dimensional reduction, with the conditions

$$q_{\Sigma}\,\phi \ = \ (q_{P_L} - q_{P_R})\,\phi \ = \ 0 \tag{6.12}$$

Here q indicates a Killing vector of the original ("flat") coset space, commuting with all the flat covariant derivatives D. Since q_{Σ} are Abelian, we can always choose coordinates where $q_{\Sigma} = \partial_{\Sigma}$; and since the rest are Abelian mod q_{Σ}, we can also choose coordinates where they are $\partial_{P_L} - \partial_{P_R}$ mod ∂_{Σ} terms. We have also fixed the dependence of the fields on the Lorentz coordinates previously by the coset constraints. In that way the original $2D^2$-dimensional coordinate space is reduced to $\mathbb{R}^D$.

Chapter 7

7 Conclusion: Natural curvature for manifest T-duality

We outline the results we have obtained: We began with the generalized affine algebra $SP\Sigma$ (14.7), enlarging the configuration space to $2D^2$ dimensions. The background fields were introduced via vielbein $E_{\mathcal{A}}{}^{\mathcal{M}}(X^{\mathcal{N}})$. The orthogonality constraints were applied to them. Together with coset constraints on torsions the specific structure of the vielbein was derived (5.9). From dimensional arguments we obtained one particular torsion constraint reproducing that originally obtained in [1]. From the torsion $T_{PP}{}^{S} \equiv R_{G/H\ G/H}{}^{H}$ we got the curvature tensor. The result (6.10) matches the result from [1] except for the antisymmetric part $r^{CD}{}_{AB}$, which can be fixed by an additional constraint. The resulting curvature tensor has explicit $O(D, D)$ index structure, which was our goal.

Various generalizations suggest themselves:

1. supersymmetry (especially AdS),
2. α' corrections, which may clarify the results of [3],
3. the corresponding first-quantization of the string (ghosts, BRST, etc.), and
4. string field theory (with vielbein fields).

Chapter 8

8 Introduction: T-duality off shell in 3D Type II superspace

In previous chapters and in the paper [11] we obtained the curvature tensor (previously discovered in [1]) in a manifestly T-dual way. The aim of the following chapters is to extend the techniques of the (only bosonic) T-dually extended spaces to the supersymmetric case.

The manifest T-duality is in general constructed by doubling the space-time coordinates, as shown in [1]. This doubled space-time is further extended by the coordinates for the Lorentz generators. The dependence of the background vielbein on them is fully fixed (up to gauge) by the coset constraints [1]. This is done by requiring that the associate torsions take their vacuum values. The generalisation of this approach to string theory requires the use of the affine symmetry algebra (oscillator algebra together with the Lorentz algebra). The consistency (closure of the Jacobi identities) of the new affine symmetry algebra requires addition of the new current Σ^{mn} for every Lorentz current S_{mn}. We also add the new coordinates for the current Σ^{mn} (The necessity of this new current was first realized in the context of $\mathrm{AdS}_5 \times \mathrm{S}_5$ [16]). In the supersymmetric case the fermionic coordinates are being doubled as well. For the fermionic current D_μ we need to add a dual current Ω^μ (for the consistency of the affine supersymmetry algebra), see [1]. By this way we obtain the affine supersymmetry algebra with the extra currents Ω^μ, Σ^{mn}.

The generalized torsion is constructed from this affine Lie algebra in a general background, which acts as the stringy generalization of covariant derivatives. Because of the additional currents, the enlarged vielbein that describes this background includes the Lorentz connection, and the enlarged torsion includes also the curvature. Closure of the algebra implies the orthogonality constraints $E\,\eta\,E^T = \eta$ on the vielbein. In the following chapters we solve these just on linear level together with the coset constraints also solved on linear level. There is also an extension of dimensional reduction to the usual D coordinates.

In the following chapters (as a starting point for a bigger program on T-dually extended superspaces) we consider the 3 dimensional T-dually extended superspace. The higher dimensional case is discussed in [17]. We would also see that this (toy) model of 3 dimensional T-dually extended space goes with the idea of lower dimensional F-theory (i.e. lower dimensional analogue of the 12 dimensional F-theory, see [37]). At the end we will show that the physical spectrum (and the structure) of the theory coincides with the $\mathcal{N}=2$ supergravity in 3 dimensions (after the dimensional reduction). That should be expected since as we will show the classical $\mathcal{N}=1$ supergravity in 4 dimensions could be interpreted as to have the same F-theory origin as the T-dual 3D supergravity. So does the 3D $\mathcal{N}=2$ supergravity (after the compactification of 4D $\mathcal{N}=1$ supergravity to 3D). The doubling in this paper is obtained naturally from the compactification of the F-theory along one space direction. We will get: $SO\,(\,3,\,2\,) \rightarrow SO\,(\,2,\,2) \simeq SO\,(\,2,\,1\,) \otimes SO\,(\,2,\,1\,)$ (to be explained in the text) what is the T-dual $\mathcal{N}=2$ string theory and effectively the T-dual 3D $\mathcal{N}=2$ supergravity.

We are following the procedure described in the articles [11], [1] and [17]. The differences are that we are working just to linear order in fields and in the 3D T-dual superspace. On top of that will also find the relation of the T-dually extended theory to the (lower dimensional analog of) F-theory.

Chapter 9

9 F-theory (membrane vs. strings)

9.1 F-theory and its compactification

The F-theory has first been proposed by Cumrun Vafa as 12 dimensional theory, see [37]. The theory is further compactified on the two-torus or more generally on the elliptically fibered Calabi-Yau manifolds. We discuss the 5 dimensional analogue of this theory. We want to motivate the natural identification between the 4D $\mathcal{N} = 1$ supergravity, further compactified to a 3D $\mathcal{N} = 2$ (the 3D $\mathcal{N} = 2$ supergravity is recently discussed in [38]), and the T-dual 3 D $\mathcal{N} = 2$ string theory. Both can be thought to have an origin in higher dimensional F-theory. This theory will be further compactified in two ways. One compactification produces the 4 dimensional M-theory that will effectively become the $\mathcal{N} = 1$ supergravity with the specific chiral compensator that contains a 3-form. This is expected since this $\mathcal{N} = 1$ supergravity is an effective theory of 2-branes (discussion of the lower dimensional supersymmetric membrane theory could be found in [39], (super) membrane theory discussed in [40], [41], [42], [43]). The other compactification gives the 3 dimensional T-dual $\mathcal{N} = 2$ string theory so effectively the T-dual $\mathcal{N} = 2$ supergravity.

9.2 5D vs. 4D vs. 3D - compactifications

The 5 dimensional F-theory is the (supersymmetric) 2-brane theory in the space with the signature $(+, +, +, -, -)$. The Lorentz group is $SO(3, 2)$. We can pick the time direction and compactify the F-theory along one time direction, so we will get the Lorentz group breaking $SO(3, 2) \to SO(3, 1)$. The 4 dimensional $\mathcal{N} = 1$ $SO(3, 1)$ theory is just the 4 dimensional M-theory, which is effectively the 4 dimensional $\mathcal{N} = 1$ supergravity. We can also pick the space direction and compactify the F-theory along this direction, so we will get: $SO(3, 2) \to SO(2, 2) \simeq SO(2, 1) \otimes SO(2, 1)$ what will become the T-dual $\mathcal{N} = 2$ string theory and effectively the T-dual 3D $\mathcal{N} = 2$ supergravity. If we further compactify the 4 dimensional $\mathcal{N} = 1$ supergravity along the space direction we will get the 3 dimensional $\mathcal{N} = 2$ supergravity coupled to a vector multiplet. On the other hand, if we take the T-dual 3D

$\mathcal{N} = 2$ theory and compactify half of the dimensions we would again get the 3D $\mathcal{N} = 2$ supergravity coupled to a vector multiplet. We therefore have the natural identification of the objects from the 4D $\mathcal{N} = 1$ supergravity (further compactified) and the T-dual 3D $\mathcal{N} = 2$ supergravity. We can therefore use the techniques of T-dually extended superspace and derive the 3D $\mathcal{N} = 2$ supergravity coupled to a vector multiplet.

In the 4D ($n = -\frac{1}{3}$ minimal and linearised) supergravity we have the prepotential $H_{\alpha\dot\beta}$ and the scalar prepotential $\mathcal{V}$. The scalar prepotential becomes a particular (chiral) compensator of the form $\phi = \bar{D}^2\mathcal{V}$. That contains a 3-form, see section 4.4.d in [22], or more generally [23]. This is expected since 4D $\mathcal{N} = 1$ supergravity is the effective theory for 2-branes.

The 4D $\mathcal{N} = 1$ gauge transformations are, see section 5.2 in [22] or [7]:

$$\delta H_{\alpha\dot\beta} = D_\alpha \bar{L}_{\dot\beta} - \bar{D}_{\dot\beta} L_\alpha \quad \text{and} \quad \delta\mathcal{V} = D^\alpha L_\alpha + \bar{D}^{\dot\alpha}\bar{L}_{\dot\alpha} \tag{9.1}$$

where D_α and $\bar{D}_{\dot\alpha}$ are usual 4D $\mathcal{N} = 1$ covariant derivatives. We can dimensionally reduce the theory to 3D and obtain the 3D $\mathcal{N} = 2$ theory. Using the dimensional reduction we get:

$$D_\alpha = \tfrac{1}{\sqrt{2}}(D_\alpha + i\,D_{\alpha'}) \quad \text{and} \quad \bar{D}_{\dot\alpha} = \tfrac{1}{\sqrt{2}}(D_\alpha - i\,D_{\alpha'}) \tag{9.2}$$

where D_α and $D_{\alpha'}$ are real 3D $\mathcal{N} = 2$ covariant derivatives. The gauge parameters can be written as:

$$L_\alpha = \tfrac{1}{\sqrt{2}}(\Lambda_\alpha - i\,\Lambda_{\alpha'}) \quad \text{and} \quad \bar{L}_{\dot\alpha} = \tfrac{1}{\sqrt{2}}(\Lambda_\alpha + i\,\Lambda_{\alpha'}). \tag{9.3}$$

The 3D $\mathcal{N} = 2$ gauge transformations thus are:

$$\delta H_{(\alpha\dot\beta)} = \delta H_{\alpha\beta'} = i\,(D_{(\alpha'}\Lambda_{\beta)} + D_{(\alpha}\Lambda_{\beta')}) \tag{9.4}$$

$$\delta H_{[\alpha\dot\beta]} = \delta V = D^\alpha\Lambda_\alpha - D^{\alpha'}\Lambda_{\alpha'} \tag{9.5}$$

$$\delta\mathcal{V} = D^\alpha\Lambda_\alpha + D^{\alpha'}\Lambda_{\alpha'} \tag{9.6}$$

The 4D $\mathcal{N} = 1$ prepotential $H_{\alpha\dot\beta} \equiv (H_{(\alpha\dot\beta)}, H_{[\alpha\dot\beta]})$ is a 4D vector and becomes the 3D vector $H_{(\alpha\beta')}$ and a prepotential V (for a vector multiplet). We also have the 4D prepotential $\mathcal{V}$ (for the chiral compensator $\phi = \bar{D}^2\mathcal{V}$) that becomes the 3D prepotential $\mathcal{V}$. On the other hand the 3D T-dual prepotential (symmetric part) $H_{(\alpha\beta')}$ (after the dimensional reduction to 3D $\mathcal{N} = 2$) is again a vector (describes the conformal supergravity) but the $H_{[\alpha\beta']}$ becomes the prepotential $\mathcal{V}$, see the transformations (10.24), and the prepotential $H_{\alpha\,\beta'}$ is just part of vielbeins, see table (10). Finally the 3D T-dual $\mathcal{N}=2$ prepotential V becomes the prepotential for the vector multiplet in 3D $\mathcal{N}=2$ supergravity.

Therefore we have an identifications between 3D $\mathcal{N}=2$ T-dual supergravity and 3D $\mathcal{N}=2$ supergravity coupled to a vector multiplet: $H_{(\alpha\,\beta')} \to H_{(\alpha\,\beta')}$, $H_{[\alpha\,\beta']} \to \mathcal{V}$ and $V \to V$.

We also have the identification between 4D $\mathcal{N}=1$ supergravity and 3D $\mathcal{N}=2$ supergravity coupled to a vector multiplet: $H_{(\alpha\,\dot{\beta})} \to H_{(\alpha\,\beta')}$, $H_{[\alpha\,\dot{\beta}]} \to V$ and $\mathcal{V} \to \mathcal{V}$.

The situation could be summarised in the following diagram 1:

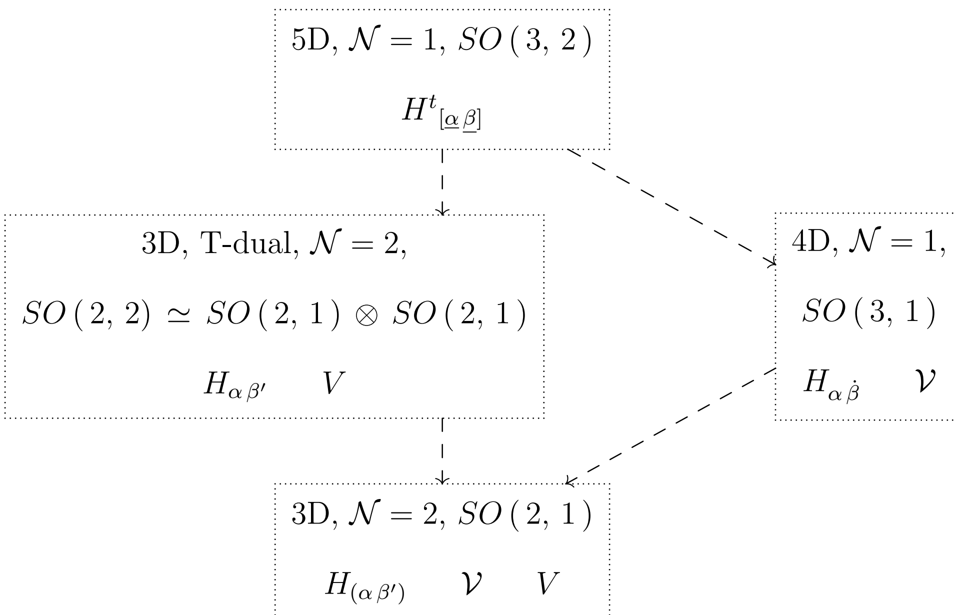

Figure 1: F-theory breaking

where $H^t{}_{[\underline{\alpha}\,\underline{\beta}]}$ is the 5 dimensional prepotential ($\underline{\alpha} \in \{1, ..4\}$,“$t$” means that it is traceless, it has 5 real components).

Chapter 10

10 Algebra

We give very brief outline of the algebraic objects and steps that will lead to the formulation of the linearised T-dual 3D supergravity. The interested reader may see the following references (where the subject is explained in great detail): [11], [1].

10.1 Current algebra of $Z_{\mathcal{M}}$

As in the paper [11], we consider the (super)string generalisation of the string oscillator algebra. Because of the T-duality and the (super)Bianchi identity the current algebra has a structure:

$$[\, Z_{\mathcal{M}}\,(1),\, Z_{\mathcal{N}}\,(2)\,] \;=\; -i\,\eta_{\mathcal{M}\mathcal{N}}\,\delta'\,(\,2\,-\,1\,)\,-\,i\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\,Z_{\mathcal{P}}\,\delta\,(\,2\,-\,1\,) \qquad (10.1)$$

where $Z_{\mathcal{M}} \;:=\; (\,S_{MN},\, D_{\mu},\, P_M,\, \Omega^{\mu},\, \Sigma^{MN}\,)$ is the generalisation of the (super)string oscillators and the metric $\eta_{\mathcal{M}\mathcal{N}}$ (given later). The P_M generators are the $O\,(D,\, D)$ generalisation of string oscillators P_m. In the explicit $O(D,\, D)$ basis are the P_M generators given as: $P_M \;:=\; (\,P_m,\, X'^{\,m}\,)$. For the future purpose we want to use a different left/right basis. In left/right basis the $P_M \;:=\; (\,P_{\mathbf{m}},\, P_{\tilde{\mathbf{m}}}\,) \;=\; \frac{1}{\sqrt{2}}\,(\,P_m\,+\,X'_m,\, P_m\,-\,X'_m\,)$. The Lorentz generators also have the left/right structure: $S_{MN} \;:=\; (\,S_{\mathbf{mn}},\, S_{\tilde{\mathbf{m}}\tilde{\mathbf{n}}})$, where $S_{\mathbf{mn}}$ are generators of left (or equivalently $S_{\tilde{\mathbf{m}}\tilde{\mathbf{n}}}$ right) Lorentz transformations. The $D_{\mu} \;:=\; (\,D_{\mu},\, D_{\tilde{\mu}}\,)$ are the generators of left and right supersymmetry transformations. The generators $\Omega^{\mu} \;:=\; (\,\Omega^{\mu},\, \Omega^{\tilde{\mu}}\,)$ and $\Sigma^{MN} \;:=\; (\,\Sigma^{\mathbf{mn}},\, \Sigma^{\tilde{\mathbf{m}}\tilde{\mathbf{n}}}\,)$ are the new generators, needed to satisfy the Bianchi identity. For further reference see [11], [1].

The full current algebra of $Z_{\mathcal{M}}$ oscillators (10.1) is the affine (super)Lie algebra (14.7) and its explicit form is:

$$\begin{array}{rcl}
[\,S_{\mathbf{mn}}(1),\, S_{\mathbf{kl}}(2)\,] & = & -i\,\eta_{[\,\mathbf{m}\,[\,\mathbf{k}}\,S_{\mathbf{l}\,]\,\mathbf{n}\,]}\;\delta\,(\,2\,-\,1\,) \\
[\,S_{\mathbf{mn}}(1),\, D_{\rho}\,(2)\,] & = & -\,i\,\tfrac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^{\sigma}{}_{\rho}\,D_{\sigma}\,\delta\,(\,2\,-\,1\,) \\
[\,S_{\mathbf{mn}}(1),\, P_{\mathbf{k}}\,(2)\,] & = & i\,\eta_{\mathbf{k}\,[\,\mathbf{m}}\,P_{\mathbf{n}\,]}\,\delta\,(\,2\,-\,1\,) \\
[\,S_{\mathbf{mn}}(1),\, \Omega^{\rho}\,(2)\,] & = & i\,\tfrac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^{\rho}{}_{\sigma}\,\Omega^{\sigma}\,\delta\,(\,2\,-\,1\,) \\
[\,S_{\mathbf{mn}}(1),\, \Sigma^{\mathbf{kl}}\,(2)\,] & = & i\,\delta_{\mathbf{mn}}{}^{\mathbf{kl}}\,\delta'\,(\,2\,-\,1\,)\,-\,i\delta_{[\,\mathbf{m}}{}^{[\,\mathbf{k}}\,\eta_{\mathbf{n}\,]\,\mathbf{s}}\Sigma^{\mathbf{l}]\mathbf{s}}\,\delta\,(\,2\,-\,1\,) \\
\{\,D_{\rho}\,(1),\, D_{\sigma}(2)\,\} & = & 2\,(\,\gamma^{\mathbf{m}}\,)_{\rho\sigma}\,P_{\mathbf{m}}\,\delta\,(\,2\,-\,1\,) \\
[\,D_{\rho}\,(1),\, P_{\mathbf{m}}\,(2)\,] & = & 2\,(\,\gamma_{\mathbf{m}})_{\rho\sigma}\,\Omega^{\sigma}\,\delta\,(\,2\,-\,1\,) \\
\{\,D_{\rho}\,(1),\, \Omega^{\sigma}\,(2)\,\} & = & i\,\delta^{\sigma}_{\rho}\,\delta'\,(\,2\,-\,1\,)\,-\,i\,\tfrac{1}{4}\,(\,\gamma_{\mathbf{mn}}\,)^{\sigma}{}_{\rho}\,\Sigma^{\mathbf{mn}}\,\delta\,(\,2\,-\,1\,) \\
[\,P_{\mathbf{m}}\,(1),\, P_{\mathbf{n}}\,(2)\,] & = & i\,\eta_{\mathbf{mn}}\,\delta'\,(\,2\,-\,1\,)\,+\,i\,\eta_{\mathbf{m\,h}}\,\eta_{\mathbf{n\,s}}\Sigma^{\mathbf{hs}}\,\delta\,(\,2\,-\,1\,) \\
 & \curvearrowright & \text{left algebra} \,\rightarrow\, -\,\text{right algebra} \\
[\,\text{left, right}\,\} & = & 0.
\end{array} \qquad (10.2)$$

The only nonvanishing terms in the metric and structure constants are (as could be guessed by dimensional analysis)

$$\eta_{PP},\ \eta_{S\Sigma},\ \eta_{D\Omega},;\quad f_{SPP}\ ,\ f_{SS\Sigma}\ ,\ f_{DDP}\ ,\ f_{SD\Omega} \qquad (10.3)$$

where we have lowered the upper index on f with η to take advantage of its total (graded)antisymmetry, and used "schematic" notation, replacing explicit indices with their type:

$$\mathcal{M}\ :=\ (\,{}_{MN},\ {}_{\mu},\ {}_{M},\ {}^{\mu},\ {}^{MN}\,)\ :=\ (\,S,\ D,\ P,\ \Omega,\ \Sigma\,) \qquad (10.4)$$

Explicitly these are, for the left-handed algebra,

$$(\eta)_{\mathbf{mn}}\ =\ \eta_{\mathbf{mn}}\,,\ \ (\eta)_{\mathbf{mn}}{}^{\mathbf{pq}}\ =\ \delta_{\mathbf{mn}}{}^{\mathbf{pq}}\,,\ (\eta)_{\sigma}{}^{\rho}\ =\ \delta^{\rho}_{\sigma} \qquad (10.5)$$

$$\begin{array}{rcl}
f_{\mathbf{mn}}{}^{\mathbf{p\,q}}\ =\ -\,\delta_{\mathbf{mn}}{}^{\mathbf{pq}} & || & f_{\mathbf{mn\,pq}}{}^{\mathbf{rs}}\ =\ \eta_{[\mathbf{m}[\mathbf{p}}\delta_{\mathbf{q}]\mathbf{n}]}{}^{\mathbf{rs}} \\
f_{\sigma\rho}{}^{\mathbf{m}}\ =\ 2\,(\,\gamma^{\mathbf{m}}\,)_{\sigma\rho} & || & f_{\mathbf{mn}\,\sigma}{}^{\rho}\ =\ -\,\tfrac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^{\rho}_{\sigma}
\end{array} \qquad (10.6)$$

For the right-handed algebra we change the signs of the corresponding terms in $\eta_{\mathcal{MN}}$ but not in f.

For dealing with antisymmetric pairs of indices we have introduced an implicit metric such that for any two antisymmetric tensors we have

$$A\cdot B\ \equiv\ \tfrac{1}{2}A^{\mathbf{mn}}B_{\mathbf{mn}} \qquad (10.7)$$

The identity matrix with respect to this inner product is

$$\delta_{\mathbf{mn}}{}^{\mathbf{pq}}\ \equiv\ \delta_{[\mathbf{m}}{}^{\mathbf{p}}\delta_{\mathbf{n}]}{}^{\mathbf{q}} \qquad (10.8)$$

10.2 Cartan-Killing metric

Having the Lie algebra $\mathcal{G}$, one can define a symmetric bilinear form:

$$K(X, Y) := \frac{1}{x_\lambda} \operatorname{Tr}(\operatorname{ad}_X \operatorname{ad}_Y) \equiv \frac{1}{x_\lambda} \left\langle E^i \right| \operatorname{ad}_X \operatorname{ad}_Y \left| E_i \right\rangle \tag{10.9}$$

$$\text{where} \quad X, Y \in \mathcal{G} \text{ and } x_\lambda \equiv \text{Dynkin index}$$

$$\text{and} \quad E_i, E^j \in \mathcal{G} \text{ and } \mathcal{G}^*$$

then for $X, Y \in$ basis of $\mathcal{G}$:

$$K(E_i, E_j) \equiv K_{ij} = \frac{1}{x_{\text{ad}}} f_{im}{}^{n} f_{jn}{}^{m} \tag{10.10}$$

$$\text{where } f_{ab}{}^{c} \quad \text{are} \quad \text{struc. cons. of } \mathcal{G}$$

The *Cartan-Killing* metric has many important group theoretical properties. We are interested in it because of the way we will fix the (engineering) dimension 1 torsion constraints. They are fixed using the (generalized) *Cartan-Killing* metric (using its free values). To see that, we need to generalize the *Cartan-Killing* metric (10.10) to the case of the graded algebra (14.7). We use the direct generalisation of the expression (10.10) for the algebra (14.7) in the presence of the background fields (vielbeins). In that case the structure constants are given by (15.3). We get (the Dynkin index $x_{\text{ad}} = 2$):

$$K_{\mathcal{A}\mathcal{B}} = \frac{1}{2} T_{\mathcal{A}\mathcal{C}}{}^{\mathcal{D}} T_{\mathcal{B}\mathcal{D}}{}^{\mathcal{C}} \tag{10.11}$$

We are interested in linearised version of previous equation. Again we expand the vielbeins to the first order and get:

$$K_{\mathcal{A}\mathcal{B}} = \frac{1}{2} f_{\mathcal{A}\mathcal{C}}{}^{\mathcal{D}} f_{\mathcal{B}\mathcal{D}}{}^{\mathcal{C}} + \underbrace{\frac{1}{2} f_{(\mathcal{A}|\mathcal{C}}{}^{\mathcal{D}} T^{(1)}{}_{\mathcal{B}]\mathcal{D}}{}^{\mathcal{C}}}_{K^{(1)}{}_{\mathcal{A}\mathcal{B}}} \tag{10.12}$$

$$+ \mathcal{O}(E^{(2)})$$

$$\text{where} \quad T^{(1)}{}_{\mathcal{A}\mathcal{B}\mathcal{C}} := \frac{1}{2} D_{[\mathcal{A}} E^{(1)}{}_{\mathcal{B}\mathcal{C})} + \frac{1}{2} E^{(1)}{}_{[\mathcal{A}}{}^{\mathcal{M}} f_{\mathcal{M}|\mathcal{B}\mathcal{C})}$$

10.3 Background fields

The aim is to find linearised formulation of the 3D T-dual theory. We are following the approach used in the paper, see [11], section 1.2. We will briefly mention the outline here:

We want to use the T-dual formulation of the stringy generalisation of the oscillatory algebra (14.7). We introduce the background fields via vielbeins. Following [1] but using algebra (10.1) we get:

$$\Pi_{\mathcal{A}}(1) \;=\; E_{\mathcal{A}}{}^{\mathcal{M}}(X^{\mathcal{N}})Z_{\mathcal{M}} \tag{10.13}$$

the affine Lie algebra for the $\Pi_{\mathcal{A}}$ could be compactly written as:

$$[\Pi_{\mathcal{A}}(1), \Pi_{\mathcal{C}}(2)] \;\equiv\; -i\eta_{\mathcal{AC}}\,\delta'\,(\,2\;-\;1\,) - iT_{\mathcal{AC}}{}^{\mathcal{E}}\Pi_{\mathcal{E}}\,\delta\,(\,2\;-\;1\,) \tag{10.14}$$

where $T_{\mathcal{AC}}{}^{\mathcal{E}}$ is a (super)stringy generalisation of torsion, see [11]:

$$\begin{aligned} T_{\mathcal{AC}}{}^{\mathcal{E}} \;&=\; E_{[\mathcal{A}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{\mathcal{C})}{}^{\mathcal{N}})E^{-1\mathcal{E}}_{\mathcal{N}} + \tfrac{1}{2}\eta^{\mathcal{ED}}E_{\mathcal{D}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{[\mathcal{A}|}{}^{\mathcal{N}})E^{-1\mathcal{F}}_{\mathcal{N}}\eta_{\mathcal{F}|\mathcal{C})} \\ &+E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{C}}{}^{\mathcal{N}}E^{-1\mathcal{E}}_{\mathcal{P}}f_{\mathcal{MN}}{}^{\mathcal{P}} \end{aligned} \tag{10.15}$$

where $[\,\mathcal{A}\,|\,|\,\mathcal{C}\,)$ indicates graded antisymmetrization in only those indices. By the $\mathcal{D}_{\mathcal{M}}$ in the (15.3) and in the whole text we mean the group covariant derivatives of the (non-affine) part of algebra (14.7): $[\,\mathcal{D}_{\mathcal{M}},\,\mathcal{D}_{\mathcal{N}}\,\} \;=\; -\,i\,f_{\mathcal{MN}}{}^{\mathcal{E}}\,\mathcal{D}_{\mathcal{E}}$.

Note that the (super)Jacobi identities imply the total graded antisymmetry of the torsion, just as for the structure constants. Torsion (15.3) can be identified with that of "ordinary" curved-space covariant derivatives by use of the strong constraint, as explained in [11], [1].

We can set the coefficient of the Schwinger term to be the metric η, the vielbein is forced to obey the orthogonality constraints:

$$E_{\mathcal{A}}{}^{\mathcal{M}}\eta_{\mathcal{MN}}\,E_{\mathcal{C}}{}^{\mathcal{N}} \;\equiv\; \eta_{\mathcal{AC}} \tag{10.16}$$

This choice does not affect the physics, and simplifies many of the expressions. For example, it implies the total graded antisymmetry of the torsion, when the upper index is implicitly lowered with η:

$$T_{\mathcal{ABC}} \;=\; \tfrac{1}{2}E_{[\,\mathcal{A}\,|}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{|\,\mathcal{B}}{}^{\mathcal{N}})E_{\mathcal{C}\,)\,\mathcal{N}} + E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{B}}{}^{\mathcal{N}}E_{\mathcal{C}}{}^{\mathcal{P}}f_{\mathcal{MNP}} \tag{10.17}$$

where we have used $E^{-1}_{\;\mathcal{M}}{}^{\mathcal{A}} \;=\; \eta^{\mathcal{A}\,\mathcal{B}}\eta_{\mathcal{M}\,\mathcal{N}}E_{\mathcal{B}}{}^{\mathcal{N}}$. (Also note that in the first term the graded antisymmetrization can be written as a cyclic sum without the 1/2, since it is already graded antisymmetric in the last two indices.) Thus, because of orthogonality, the vielbein is like (the exponential of) a super 2-form, while the torsion is a super 3-form; similarly, the Bianchi identities are a super 4-form.

The (super)orthogonality constraint (15.4) could be fully solved for the general structure of the vielbein $E_{\mathcal{A}}{}^{\mathcal{M}}$. However, we are interested just in the linear level. Thus we get the (super)orthogonality constraint for the linearised part of the vielbein $E^{(1)}{}_{\mathcal{A}}{}^{\mathcal{M}}$:

$$E_{\mathcal{A}}{}^{\mathcal{M}} = \delta_{\mathcal{A}}{}^{\mathcal{M}} + E^{(1)}{}_{\mathcal{A}}{}^{\mathcal{M}} + \mathcal{O}\,(\,E^{(2)}\,) \Rightarrow \text{using (15.4)} \quad (10.18)$$

$$E^{(1)}{}_{(\,\mathcal{A}\mathcal{B}]} = 0 \quad (10.19)$$

We would also need the linear level version of the equation (15.5):

$$T_{\mathcal{A}\mathcal{B}\mathcal{C}} = f_{\mathcal{A}\mathcal{B}\mathcal{C}} + T^{(1)}{}_{\mathcal{A}\mathcal{B}\mathcal{C}} + \mathcal{O}\,(\,E^{(2)}\,) \quad (10.20)$$

$$\text{where } T^{(1)}{}_{\mathcal{A}\mathcal{B}\mathcal{C}} := \tfrac{1}{2}\,D_{[\,\mathcal{A}}\,E^{(1)}{}_{\mathcal{B}\mathcal{C}\,)} + \tfrac{1}{2}\,E^{(1)}{}_{[\mathcal{A}}{}^{\mathcal{M}}\,f_{\mathcal{M}\,|\mathcal{B}\mathcal{C}\,)} \quad (10.21)$$

10.4 Further constraints and gauge fixing

Following the discussion in the subsection 4.2 in the paper [11], we get the coset constraint on the torsion piece $T_{\mathcal{S}\,\mathcal{A}}{}^{\mathcal{B}} = f_{\mathcal{S}\,\mathcal{A}}{}^{\mathcal{B}}$ (where we used the $\mathcal{S}$ index as the schematic index (10.4) and $\mathcal{A}, \mathcal{B}$ are general indices). On the linear level the previous condition becomes: $T^{(1)}{}_{\mathcal{S}\,\mathcal{A}\mathcal{B}} = 0$. From this one gets the condition for the linear vielbein: $E^{(1)}{}_{\mathcal{S}}{}^{\mathcal{M}} = 0 + \mathcal{O}\,(\,E^{(2)}\,)$.

We would like to gauge fix some of the remaining gauge freedom. Note that the coset constraints discussed above sets the gauge parameter (defined below) $\lambda_{\mathcal{S}} = 0$. From specific gauge fixing we get the further conditions on the linear vielbein $E^{(1)}$. The gauge transformations are given as (see also [1]):

$$\delta_{\Lambda}\,\Pi_{\mathcal{A}} = [\,-\,i\,\Lambda,\,\Pi_{\mathcal{A}}\,\} \quad (10.22)$$

$$\text{where } \Lambda := \int d1\,\lambda^{\mathcal{M}}\,(\,X\,)\,D_{\mathcal{M}}$$

We are working in the basis where the covariant derivatives satisfy:

$$[\,D_{\mathcal{M}},\,D_{\mathcal{N}}\,\} = i\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\,D_{\mathcal{P}} \quad (10.23)$$

Thus the (linear)gauge transformation of (linear)vielbein are:

$$\delta_{\Lambda}\,E^{(1)}{}_{\mathcal{A}\mathcal{B}} = -\,\tfrac{i}{2}\,D_{[\,\mathcal{A}}\,\lambda_{\mathcal{B}\,)} + f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}\,\lambda_{\mathcal{C}} \quad (10.24)$$

Now, we can pick the following gauge:

$$\gamma_{\mathbf{a}}{}^{\alpha\beta}\, E^{(1)}{}_{\alpha\beta} = 0 \quad \Rightarrow \quad \lambda_{\mathbf{a}} \propto \gamma_{\mathbf{a}}{}^{\alpha\beta}\, D_\alpha\, \lambda_\beta \tag{10.25}$$

$$\gamma^{\mathbf{a}\,\alpha\beta}\, E^{(1)}{}_{\alpha\,\mathbf{a}} = 0 \quad \Rightarrow \quad \lambda^\alpha \propto \gamma^{\mathbf{a}\,\alpha\beta}\, D_{[\mathbf{a}}\, \lambda_{\beta]} \tag{10.26}$$

$$E^{(1)}{}_{\mathbf{a}\,\mathbf{b}} = 0 \quad \Rightarrow \quad \lambda_{\mathbf{a}\,\mathbf{b}} \propto D_{[\mathbf{a}}\, \lambda_{\mathbf{b}]} \tag{10.27}$$

Same for Left $\to$ Right

We can see that by (10.25), (10.26), (10.27) we automatically have expressions for gauge parameters λ_P, λ_Ω, λ_Σ as derivatives of another gauge parameter λ_D. It is unlike the usual $\mathcal{N} = 1$ supergravity where we need first to solve the chirality condition to relate derivatives of Λ with K, see section X.A.1 in [10], also section 5.3 in [22]. Moreover, the (10.25)-(10.27) give the constraints on $E^{(1)}$ and solving those we will get:

$$E^{(1)}{}_{DD} = E^{(1)}{}_{\alpha\beta} = 0 \quad || \quad E^{(1)}{}_{PP} = E^{(1)}{}_{\mathbf{a}\,\mathbf{b}} = 0 \tag{10.28}$$

$$E^{(1)\alpha\beta}{}_{\beta} = 0 \text{ (part of } E^{(1)P}{}_{D})$$

Later (by dimension $-\frac{1}{2}$ constraints) one can see that the $E^{(1)}{}_{PD} = 0$. We thus need to set up the dimensional constraints. The following table 1 summarise the torsion dimensions:

Torsion	Dim.
$T_{SS}{}^{\Sigma}$	-2
$T_{SS}{}^{\Omega}$	$-\frac{3}{2}$
$T_{SS}{}^{P}$	-1
$T_{SD}{}^{\Omega}$	-1
$T_{SS}{}^{D}$	$-\frac{1}{2}$
$T_{SP}{}^{\Omega}$	$-\frac{1}{2}$
$T_{DD}{}^{\Omega}$	$-\frac{1}{2}$
$T_{SS}{}^{S}$	0
$T_{SD}{}^{D}$	0
$T_{SP}{}^{P}$	0
$T_{DD}{}^{P}$	0

Torsion	Dim.
$T_{SD}{}^{S}$	$\frac{1}{2}$
$T_{SP}{}^{D}$	$\frac{1}{2}$
$T_{DD}{}^{D}$	$\frac{1}{2}$
$T_{PP}{}^{\Omega}$	$\frac{1}{2}$
$T_{SP}{}^{S}$	1
$T_{S\Omega}{}^{D}$	1
$T_{DP}{}^{D}$	1
$T_{DD}{}^{S}$	1
$T_{PP}{}^{P}$	1
$T_{S\Omega}{}^{S}$	$\frac{3}{2}$
$T_{DP}{}^{S}$	$\frac{3}{2}$
$T_{D\Omega}{}^{D}$	$\frac{3}{2}$
$T_{PP}{}^{D}$	$\frac{3}{2}$

Torsion	Dim.
$T_{S\Sigma}{}^{S}$	2
$T_{D\Omega}{}^{S}$	2
$T_{PP}{}^{S}$	2
$T_{P\Omega}{}^{D}$	2
$T_{D\Sigma}{}^{S}$	$\frac{5}{2}$
$T_{P\Omega}{}^{S}$	$\frac{5}{2}$
$T_{\Omega\Omega}{}^{D}$	$\frac{5}{2}$
$T_{P\Sigma}{}^{S}$	3
$T_{\Omega\Omega}{}^{S}$	3
$T_{\Omega\Sigma}{}^{S}$	$\frac{7}{2}$
$T_{\Sigma\Sigma}{}^{S}$	4

Table 1: Torsion dimensions

Notice that many of the torsions in the previous table 1 are fixed (to flat structure constants $f_{\mathcal{S}\,\mathcal{A}}{}^{\mathcal{B}}$).

We put the torsions of the negative (engineering) dimensions to 0 (as always in QFT, see the red coloured torsions in the previous table). We also put the (unfixed) torsions of the zero dimension to 0 (see the blue torsions in the previous table), see [11]. We will also put the dimension $\frac{1}{2}$ (unfixed) torsions to 0 (the green torsions in the table). Doing that we produce just algebraic constraints on veilbeins.

The nontrivial dimensional constraints are:

$$T_{D\,D}{}^{\Omega} = 0, \; T_{D\,D}{}^{P} = f_{D\,D}{}^{P}, \; T_{D\,D}{}^{D} = 0, \; T_{P\,P}{}^{\Omega} = 0 \tag{10.29}$$

10.5 Dimensional constraints: solution

The solution of the previous nontrivial dimensional constraints could be given in a full generality, however in the following chapters we are interested just in the linearised case. The tables 9 and 10 summarise the linearised solutions of those four constraints (notice that we have also the possibility of mixed left/right indices):

$T_{D\,D}{}^{\Omega} = 0$ and $\gamma^{\mathbf{a}\,\alpha\,\beta}\,E^{(1)}{}_{\alpha\,\mathbf{a}} = 0$	$\Rightarrow$	$E^{(1)}{}_{P\,D} = 0$
$T_{D\,D}{}^{P} = f_{D\,D}{}^{P}$	$\Rightarrow$	$E^{(1)}{}_{D\,\Omega} = 0$
$T_{P\,P}{}^{\Omega} = 0$ or $T_{D\,D}{}^{D} = 0$	$\Rightarrow$	$E^{(1)}{}_{\Sigma\,D} = E^{(1)\mathbf{a}\mathbf{b}}{}_{\alpha} = -\,2\,\gamma^{[\mathbf{a}}{}_{\alpha\,\rho}\,E^{(1)\rho\,\vert\,\mathbf{b}]}$

Table 2: Unmixed constraints

$T_{D\,D}{}^{\tilde{\Omega}} = 0$	$\Rightarrow$	$E^{(1)}{}_{P\,\tilde{D}} \equiv E^{(1)}{}_{\mathbf{a}\,\tilde{\alpha}} = -\,\frac{1}{2}\,\gamma_{\mathbf{a}}{}^{\beta\,\epsilon}\,D_{\beta}\,E^{(1)}{}_{\epsilon\,\tilde{\alpha}}$
$T_{D\,D}{}^{\tilde{P}} = 0$	$\Rightarrow$	$E^{(1)}{}_{P\,\tilde{P}} \equiv E^{(1)}{}_{\mathbf{a}\,\tilde{\mathbf{b}}} = -\,\frac{1}{2}\,\gamma_{\mathbf{a}}{}^{\beta\,\epsilon}\,D_{\beta}\,E^{(1)}{}_{\epsilon\,\tilde{\mathbf{b}}}$
$T_{D\,\tilde{D}}{}^{P} = 0$	$\Rightarrow$	$E^{(1)}{}_{\Omega\,\tilde{D}} \equiv E^{(1)\alpha}{}_{\tilde{\beta}} = -\,\frac{1}{6}\,\gamma^{\mathbf{a}\,\epsilon\,\alpha}\,D_{[\epsilon}\,E^{(1)}{}_{\mathbf{a}]\,\tilde{\beta}}$
$T_{P\,\tilde{P}}{}^{\Omega} = 0$	$\Rightarrow$	$E^{(1)}{}_{\Omega\,\tilde{P}} \equiv E^{(1)\alpha}{}_{\tilde{\mathbf{a}}} = -\,\frac{1}{6}\,\gamma^{\mathbf{b}\,\epsilon\,\alpha}\,D_{[\epsilon}\,E^{(1)}{}_{\mathbf{b}]\,\tilde{\mathbf{a}}}$
$T_{P\,P}{}^{\tilde{\Omega}} = 0$	$\Rightarrow$	$E^{(1)}{}_{\Sigma\,\tilde{D}} \equiv E^{(1)\mathbf{a}\mathbf{b}}{}_{\tilde{\alpha}} = \eta^{\mathbf{a}\,\mathbf{c}}\,\eta^{\mathbf{b}\,\mathbf{d}}\,D_{[\mathbf{c}}\,E^{(1)}{}_{\mathbf{d}]\,\tilde{\alpha}}$

Table 3: Mixed constraints

From the table 9 we can see that we have one linear relation:

$$E^{(1)}{}_{\Sigma\,D} \propto E^{(1)}{}_{\Omega\,P} \tag{10.30}$$

From the table 10 we have linear relations:

$$\{\, E^{(1)}{}_{P\,\tilde{D}},\, E^{(1)}{}_{P\,\tilde{P}},\, E^{(1)}{}_{\Omega\,\tilde{D}},\, E^{(1)}{}_{\Omega\,\tilde{P}},\, E^{(1)}{}_{\Sigma\,\tilde{D}} \,\} \propto E^{(1)}{}_{D\,\tilde{D}} \tag{10.31}$$

Again, we have automatically obtained the expressions for the vielbeins as derivatives of $E^{(1)}{}_{D\,\tilde{D}}$ vielbein (prepotential). It is unlike the $\mathcal{N} = 1$ supergravity where the prepotential comes as the solution of the bisection condition (or chirality condition in covariant approach), see section X.A.1 in [10] and section 5.2.a and 5.3 in [22].

10.6 Dimension 1 unmixed constraints

To proceed we need to find the constraints for the dimension 1 torsions. We can see that putting those to zero in general introduces the differential constraints, that we do not want (except of the strong constraint and later the equation of motion). However there is a way how to fix dimension 1 torsions without producing differential constraints. We will use the following set of unmixed constraints (again we have two cases for the torsion index structure: mixed and unmixed):

$$\begin{aligned} T^{(1)}{}_{P\,P\,P} &\equiv T^{(1)}{}_{\mathbf{a\,b\,c}} = \vartheta\,\varepsilon_{\mathbf{a\,b\,c}}\,B \\ T^{(1)}{}_{D\,D\,\Sigma} &\equiv T^{(1)}{}_{\alpha\,\beta}{}^{\mathbf{ab}} = \xi\,\gamma^{\mathbf{ab}}{}_{\alpha\,\beta}\,B \\ T^{(1)}{}_{P\,D\,\Omega} &\equiv T^{(1)}{}_{\mathbf{a}\,\alpha}{}^{\beta} = \zeta\,\gamma_{\mathbf{a}}{}^{\beta}_{\alpha}\,B \end{aligned} \tag{10.32}$$

where the new object B is determined from (10.32). Using the linearised Bianchi identity the coefficients in (10.32) are related as:

$$\xi = -\,8\,\zeta - 2\,\vartheta \tag{10.33}$$

Equation (10.21) gives explicit relations for the $T^{(1)}$'s from (10.32):

$$T^{(1)}{}_{\mathbf{a\,b\,c}} = -\,\tfrac{1}{2}\,\eta_{[\mathbf{a}|\,\mathbf{d}}\,\eta_{\mathbf{b}|\,\mathbf{e}}\,E^{(1)\mathbf{de}}{}_{\mathbf{c}]} \tag{10.34}$$

$$T^{(1)}{}_{\alpha\,\beta}{}^{\mathbf{ab}} = D_{(\alpha}\,E^{(1)}{}_{\beta)}{}^{\mathbf{ab}} + 2\,\gamma^{\mathbf{c}}{}_{\alpha\,\beta}\,E^{(1)\mathbf{ab}}{}_{\mathbf{c}} \tag{10.35}$$

$$T^{(1)}{}_{\mathbf{a}\,\alpha}{}^{\beta} = D_{\alpha}\,E^{(1)\,\beta}{}_{\mathbf{a}} + 2\,\gamma_{\mathbf{a}\,\alpha\,\epsilon}\,E^{(1)\,\epsilon\,\beta} + \tfrac{1}{4}\,\varepsilon_{\mathbf{c\,d\,e}}\,\gamma^{\mathbf{e}}{}^{\beta}_{\alpha}\,E^{(1)\,\mathbf{cd}}{}_{\mathbf{a}} \tag{10.36}$$

From (10.34) and first relation of (10.32) we get:

$$2\,\vartheta\,B = -\,\varepsilon_{\mathbf{h\,d\,e}}\,\eta^{\mathbf{h\,c}}\,E^{(1)\mathbf{de}}{}_{\mathbf{c}} \tag{10.37}$$

From (10.36) and requiring that we want just algebraic constraints we get the second equation for (10.32) fixing constants:

$$0 \,=\, -\,3\,\zeta \,-\, \tfrac{1}{8}\,(\,2\,\vartheta \,+\, 3\,\xi\,) \,-\, \tfrac{1}{2}\,\vartheta \tag{10.38}$$

Substituing result (10.33) we have soultion for any ϑ and ζ except when $\vartheta \,=\, -\,6\,\zeta$. That condition would produce the differential constraint on ${E^{(1)}}_{\Omega\,P}$ (see eq. (10.39)). From (10.36) and third of (10.32) we will get fixing of ${E^{(1)}}_{\Omega\,\Omega}$. From (10.35) and second of (10.32) we will get fixing of ${E^{(1)}}_{P\,\Sigma}$. The net result of dimension 1 unmixed algebraic constraints (10.32) is that everything could be expressed in terms of ${E^{(1)}}_{D\,\Sigma}$ and so (see table (9)) by ${E^{(1)}}_{P\,\Omega}$ (and two constants ϑ, ζ s.t. $\vartheta \,\neq\, -\,6\,\zeta$):

$$\begin{aligned}
B \;&=\; -\,\tfrac{1}{\vartheta\,+\,6\,\zeta}\,\gamma^{\mathbf{a}\;\alpha}_{\;\;\beta}\,D_\alpha\,{E^{(1)\beta}}_{\mathbf{a}} && (10.39)\\
{E^{(1)}}_{\Omega\,\Omega} \,=\, E^{(1)\,\alpha\,\beta} \;&=\; \tfrac{1}{12}\,\gamma^{\mathbf{a}\,(\alpha|\,\epsilon}\,D_\epsilon\,{E^{(1)\,\beta)}}_{\mathbf{a}} \,+\, \tfrac{1}{12}\,{\gamma_{\mathbf{a}}}^{\alpha\,\beta}{E^{(1)\mathbf{ab}}}_{\mathbf{b}} && (10.40)\\
{E^{(1)}}_{P\,\Sigma} \,=\, {E^{(1)}}_{\mathbf{c}}{}^{\mathbf{ab}} \;&=\; -\,\tfrac{1}{2}\,{\gamma_{\mathbf{c}}}^{\alpha\,\beta}\,D_\alpha\,{E^{(1)}}_{\beta}{}^{\mathbf{ab}} && (10.41)\\
&\quad +\,(\,\vartheta \,+\, 4\,\zeta\,)\,\eta_{\mathbf{c}\,\mathbf{e}}\,\varepsilon^{\mathbf{e}\,\mathbf{a}\,\mathbf{b}}\,B
\end{aligned}$$

10.7 Dimension 1 mixed constraints

Some of the mixed dimension 1 torsions are determined in terms of ${E^{(1)}}_{\alpha\,\tilde\beta} \equiv {E^{(1)}}_{D\,\tilde D}$ and ${E^{(1)}}_{P\,\Omega}$ already. Using the previous results (tables 9, 10 and results of previous section) we can see that mixed dimension 1 torsions ${T^{(1)}}_{\mathbf{a}\,\tilde\alpha}{}^{\rho} \;\equiv\; {T^{(1)}}_{P\,\tilde D\,\Omega}$ and ${T^{(1)}}_{\tilde\alpha\,\beta}{}^{\mathbf{ab}} \;\equiv\; {T^{(1)}}_{\tilde D\,D\,\Sigma}$ are fully determined, see (10.42). The mixed determined and undetermined torsions are summarised below:

$$\left.\begin{aligned}
{T^{(1)}}_{P\,\tilde D\,\Omega} \equiv {T^{(1)}}_{\mathbf{a}\,\tilde\alpha}{}^{\beta} \;&=\; D_{\mathbf{a}}\,{E^{(1)}}_{\tilde\alpha}{}^{\beta} \,+\, D^{(\beta}\,{E^{(1)}}_{\tilde\alpha)\,\mathbf{a}}\\
{T^{(1)}}_{\tilde D\,D\,\Sigma} \equiv {T^{(1)}}_{\tilde\alpha\,\beta}{}^{\mathbf{ab}} \;&=\; D_{(\tilde\alpha}\,{E^{(1)}}_{\beta)}{}^{\mathbf{ab}} \,+\, D^{\mathbf{ab}}\,{E^{(1)}}_{\tilde\alpha\,\beta}
\end{aligned}\right\} \tag{10.42}$$

$$\left.\begin{aligned}
{T^{(1)}}_{\tilde P\,P\,P} \equiv {T^{(1)}}_{\tilde{\mathbf{a}}\,\mathbf{b}\,\mathbf{c}} \;&=\; D_{[\mathbf{b}}\,{E^{(1)}}_{\mathbf{c}]\,\tilde{\mathbf{a}}} \,-\, \eta_{\mathbf{b}\,\mathbf{d}}\,\eta_{\mathbf{c}\,\mathbf{e}}\,{E^{(1)\mathbf{de}}}_{\tilde{\mathbf{a}}}\\
{T^{(1)}}_{\tilde P\,D\,\Omega} \equiv {T^{(1)}}_{\tilde{\mathbf{a}}\,\alpha}{}^{\beta} \;&=\; D_{(\alpha}\,{E^{(1)\beta)}}_{\tilde{\mathbf{a}}} \,+\, \tfrac{1}{4}\,\gamma_{\mathbf{de}\;\alpha}{}^{\beta}\,{E^{(1)\mathbf{de}}}_{\tilde{\mathbf{a}}}\\
{T^{(1)}}_{P\,D\,\tilde\Omega} \equiv {T^{(1)}}_{\mathbf{a}\,\alpha}{}^{\tilde\beta} \;&=\; D_{[\mathbf{a}}\,{E^{(1)}}_{\alpha]}{}^{\tilde\beta} \,+\, 2\,\gamma_{\mathbf{a}\,\alpha\,\epsilon}\,E^{(1)\,\epsilon\,\tilde\beta}\\
{T^{(1)}}_{\tilde D\,\tilde D\,\Sigma} \equiv {T^{(1)}}_{\tilde\alpha\,\tilde\beta}{}^{\mathbf{ab}} \;&=\; D_{(\tilde\alpha}\,{E^{(1)}}_{\tilde\beta)}{}^{\mathbf{ab}} \,+\, 2\,\gamma^{\tilde{\mathbf{e}}}{}_{\tilde\alpha\,\tilde\beta}\,{E^{(1)}}_{\tilde{\mathbf{e}}}{}^{\mathbf{ab}}
\end{aligned}\right\} \tag{10.43}$$

From the (10.43) is evident that by putting ${T^{(1)}}_{\tilde P\,P\,P} \,=\, 0$ we can determine ${E^{(1)}}_{\Sigma\,\tilde P}$ in terms of ${E^{(1)}}_{P\,\tilde P}$ and so ${E^{(1)}}_{D\,\tilde D}$. Equivalently we can obtain that fixing either of $T^{(1)}{}_{\tilde{P}\,D\,\Omega}$ or $T^{(1)}{}_{D\,D\,\tilde{\Sigma}}$. By putting $T^{(1)}{}_{P\,D\,\tilde{\Omega}} = 0$ we can determine $E^{(1)}{}_{\Omega\,\tilde{\Omega}}$ in terms of $E^{(1)}{}_{D\,\tilde{\Omega}}$ and $E^{(1)}{}_{P\,\tilde{\Omega}}$ and so again in $E^{(1)}{}_{D\,\tilde{D}}$. The dimension 1 mixed constraints give:

$$E^{(1)}{}_{\Sigma\,\tilde{P}} \equiv E^{(1)\mathbf{bc}}{}_{\tilde{\mathbf{a}}} = \eta^{\mathbf{b\,d}}\,\eta^{\mathbf{c\,e}}\,D_{[\mathbf{d}}\,E^{(1)}{}_{\mathbf{e}]\,\tilde{\mathbf{a}}} \equiv \eta\eta\;D_{[P}\,E^{(1)}{}_{P]\,\tilde{P}} \quad (10.44)$$

$$E^{(1)}{}_{\Omega\,\tilde{\Omega}} \equiv E^{(1)\alpha\,\tilde{\beta}} = \tfrac{1}{6}\,\gamma^{\mathbf{a}\,\alpha\,\epsilon}\,D_{[\mathbf{a}}\,E^{(1)}{}_{\epsilon]}{}^{\tilde{\beta}} \equiv \gamma\;D_{[P}\,E^{(1)}{}_{D]\tilde{\Omega}} \quad (10.45)$$

The dimension 1 constraints could be viewed also form another perspective. For that we would need to borrow the expression for the *Cartan-Killing* metric $K_{\mathcal{A}\,\mathcal{B}}$ that is discussed in section 10.2. The expression for the linearised *Cartan-Killing* metric:

$$K^{(1)}{}_{\mathcal{A}\,\mathcal{B}} \equiv \tfrac{1}{2}\,f_{(\,\mathcal{A}\,|\,\mathcal{C}}{}^{\mathcal{D}}\,T^{(1)}{}_{\mathcal{B}\,]\,\mathcal{D}}{}^{\mathcal{C}} \quad (10.46)$$

taking the (10.46) for $\mathcal{A},\,\mathcal{B} \in \{\,\alpha,\,\tilde{\beta}\,\}$ we will get:

$$K^{(1)}{}_{\alpha\,\beta} \propto \varepsilon_{\alpha\,\beta}\,B, \quad K^{(1)}{}_{\tilde{\alpha}\,\tilde{\beta}} \propto \varepsilon_{\tilde{\alpha}\,\tilde{\beta}}\,\tilde{B}, \quad K^{(1)}{}_{\alpha\,\tilde{\beta}} \quad (10.47)$$

then using the exercise XA2.6 in [10] we could write the dimension 1 constraints as:

$$T^{(1)}{}_{\mathbf{a\,b\,c}} \propto \varepsilon_{\mathbf{a\,b\,c}}\,\varepsilon^{\alpha\,\beta}\,K^{(1)}{}_{\beta\,\alpha}\,, \qquad T^{(1)}{}_{\alpha\,\beta}{}^{\mathbf{ab}} \propto \gamma^{\mathbf{ab}}{}_{\alpha\,\beta}\,\varepsilon^{\epsilon\,\sigma}\,K^{(1)}{}_{\sigma\,\epsilon} \quad (10.48)$$

$$T^{(1)}{}_{\mathbf{a}\,\alpha}{}^{\beta} \propto \gamma_{\mathbf{a}}{}^{\beta}_{\alpha}\,\varepsilon^{\epsilon\,\sigma}\,K^{(1)}{}_{\sigma\,\epsilon}\,, \qquad T^{(1)}{}_{\mathbf{a}\,\tilde{\alpha}}{}^{\beta} \propto \gamma_{\mathbf{a}}{}^{\beta\,\epsilon}\,K^{(1)}{}_{\epsilon\,\tilde{\alpha}} \quad (10.49)$$

$$T^{(1)}{}_{\tilde{\alpha}\,\beta}{}^{\mathbf{ab}} \propto \gamma^{\mathbf{ab}}{}_{\beta}{}^{\epsilon}\,K^{(1)}{}_{\epsilon\,\tilde{\alpha}}\,, \qquad T^{(1)}{}_{\mathbf{a}\,\alpha}{}^{\tilde{\beta}} \propto \gamma_{\mathbf{a}\,\alpha\,\epsilon}\,K^{(1)\;\epsilon\,\tilde{\beta}} \quad (10.50)$$

Remaining dimension 1 torsions have to be 0 since we do not have appropriate nonzero *Cartan-Killing* metric. We also put second torsion of (10.50) to 0. Since that does not produce any differential constraints and fixes $E^{(1)}{}_{\Omega\,\tilde{\Omega}}$, see (10.45). Moreover in the spirit of the exercise XA2.6 in [10], we can identify $(\,K^{(1)}{}_{\alpha\,\tilde{\beta}},\,B,\,\tilde{B}\,)$ with a $SO(\,3,\,3\,)$ vector $G^{\alpha\,\beta} = (\,G^{a},\,B,\,\bar{B}\,)$ in $SL(\,4\,)$ notation (form the $\mathcal{N} = 1$ supergravity).

10.8 $\tilde{T} = 0$ constraints

In the previous subsections we discovered that all the vielbeins (mixed and unmixed) (except of $E^{(1)}{}_{\Omega\,\Sigma}$ and $E^{(1)}{}_{\Sigma\,\Sigma}$) could be determined in terms of $E^{(1)}{}_{P\Omega}$ and $E^{(1)}{}_{D\tilde{D}}$. We need further constraint to relate those two undetermined vielbeins. We are following article [1]. There a new torsion was introduced. It came from the requirement of partial integration also in the presence of the new integration measure ϕ^2 (dilaton). Following [1] the new torsion is:

$$\tilde{T}_{\mathcal{A}} := \phi^2 \overleftarrow{\nabla}_{\mathcal{A}} \phi^{-2} \tag{10.51}$$

where $\nabla_{\mathcal{A}} = E_{\mathcal{A}}{}^{\mathcal{M}} D_{\mathcal{M}}$. The torsion (10.51) should vanish, so we get the $\tilde{T}$ torsion constraint: $\tilde{T}_{\mathcal{A}} = 0$. We are interested just in the first order part of $\tilde{T}_{\mathcal{A}}$:

$$\begin{aligned} \tilde{T}_{\mathcal{A}} = 0 + \tilde{T}^{(1)}{}_{\mathcal{A}} + \mathcal{O}(E^{(2)}) \;\Rightarrow\; & \tilde{T}^{(1)}{}_{\mathcal{A}} = D^{\mathcal{B}} E^{(1)}{}_{\mathcal{B}\mathcal{A}} + 2 D_{\mathcal{A}} \phi^{(1)} \\ \text{where} \qquad & \phi = 1 + \phi^{(1)} + \mathcal{O}(\phi^{(2)}) \end{aligned} \tag{10.52}$$

The relation $\tilde{T}^{(1)}{}_S = 0$ gives $D_S \phi^{(1)} = 0$. Using $\tilde{T}^{(1)}{}_D = 0$ we get the relation:

$$\begin{aligned} \tfrac{1}{4} \varepsilon_{\mathbf{a}\mathbf{b}\mathbf{c}} \gamma^{\mathbf{c}\,\beta}{}_{\alpha} E^{(1)\mathbf{ab}}{}_{\beta} = 2 \gamma_{\mathbf{a}\alpha\beta} E^{(1)\beta\mathbf{a}} &= D^{\tilde{\beta}} E^{(1)}{}_{\tilde{\beta}\alpha} + D^{\tilde{\mathbf{a}}} E^{(1)}{}_{\tilde{\mathbf{a}}\alpha} \qquad (10.53) \\ & \quad - D_{\tilde{\beta}} E^{(1)\tilde{\beta}}{}_{\alpha} - 2 D_{\alpha} \phi^{(1)} \\ &= D_{\tilde{\Omega}} E^{(1)}{}_{\tilde{D}D} + D_{\tilde{P}} E^{(1)}{}_{\tilde{P}D} \\ & \quad - D_{\tilde{D}} E^{(1)}{}_{\tilde{\Omega}D} - D_D \phi^{(1)} \qquad (10.54) \end{aligned}$$

Where we used the results of table 9. Using the table 10 for $E^{(1)}{}_{\tilde{\mathbf{a}}\alpha}$ and $E^{(1)\tilde{\beta}}{}_{\alpha}$ we have the relation between $E^{(1)}{}_{P\Omega}$ and $E^{(1)}{}_{D\tilde{D}}$ and linearised dilaton $\phi^{(1)}$:

$$\begin{aligned} \gamma\, E^{(1)}{}_{\Omega P} \equiv 2 \gamma^{\mathbf{a}}{}_{\alpha\beta} E^{(1)\beta}{}_{\mathbf{a}} &= -\tfrac{1}{3} \gamma^{\tilde{\mathbf{a}}\,\tilde{\beta}\tilde{\epsilon}} [D_{\tilde{\mathbf{a}}}, D_{\tilde{\beta}}] E^{(1)}{}_{\tilde{\epsilon}\alpha} \qquad (10.55) \\ & \quad + \tfrac{1}{2} \gamma^{\tilde{\mathbf{a}}\,\tilde{\beta}\tilde{\epsilon}} D_{\tilde{\mathbf{a}}} D_{\tilde{\beta}} E^{(1)}{}_{\tilde{\epsilon}\alpha} - 2 D_{\alpha} \phi^{(1)} \\ &\equiv -\gamma\, [D_{\tilde{P}}, D_{\tilde{D}}] E^{(1)}{}_{\tilde{D}D} \\ & \quad + \gamma\, D_{\tilde{P}} D_{\tilde{D}} E^{(1)}{}_{\tilde{D}D} - D_D \phi^{(1)} \end{aligned}$$

We notice that result (10.55) is exactly the right combination in order to express B from (10.39) in terms of $E^{(1)}{}_{\tilde{D}D}$ and $\phi^{(1)}$. This will be used in next sections:

$$\begin{aligned} B &= -\frac{1}{\vartheta + 6\zeta} \varepsilon^{\nu\alpha} D_{\nu} \Big[\gamma^{\tilde{\mathbf{a}}\,\tilde{\beta}\tilde{\epsilon}} \Big(-\tfrac{1}{6} [D_{\tilde{\mathbf{a}}}, D_{\tilde{\beta}}] + \tfrac{1}{4} D_{\tilde{\mathbf{a}}} D_{\tilde{\beta}} \Big) E^{(1)}{}_{\tilde{\epsilon}\alpha} \qquad (10.56) \\ & \quad - D_{\alpha} \phi^{(1)} \Big] \\ &\equiv \varepsilon D_D \Big[\gamma \Big([D_{\tilde{P}}, D_{\tilde{D}}] - D_{\tilde{P}} D_{\tilde{D}} \Big) E^{(1)}{}_{\tilde{D}D} - D_D \phi^{(1)} \Big] \end{aligned}$$

Using the relation $\tilde{T}^{(1)}{}_P \,=\, 0$ and similar steps we get:

$$
\begin{aligned}
&- 2\,\varepsilon_{\mathbf{a}\,\mathbf{b}\,\mathbf{c}}\,\gamma^{\mathbf{b}\,\alpha}{}_{\epsilon}\,D_\alpha\,E^{(1)\;\epsilon\,\mathbf{c}} \,+\, 2\,D_\alpha\,E^{(1)\alpha}{}_{\mathbf{a}} \,-\, 4\,D_{\mathbf{a}}\,\phi^{(1)} = \\
&- \tfrac{1}{3}\gamma^{\tilde{\mathbf{a}}\;\tilde{\beta}\,\tilde{\epsilon}}\,[\,D_{\tilde{\mathbf{a}}},\,D_{\tilde{\beta}}\,]\,\gamma_{\mathbf{a}}{}^{\sigma\,\alpha}\,D_\sigma\,E^{(1)}{}_{\tilde{\epsilon}\,\alpha} \,+\, \tfrac{1}{2}\,\gamma^{\tilde{\mathbf{a}}\;\tilde{\beta}\,\tilde{\epsilon}}\,D_{\tilde{\mathbf{a}}}\,D_{\tilde{\beta}}\,\gamma_{\mathbf{a}}{}^{\sigma\,\alpha}\,D_\sigma\,E^{(1)}{}_{\tilde{\epsilon}\,\alpha} = \\
&\gamma_{\mathbf{a}}{}^{\sigma\alpha}D_\sigma\Big(-\,2\,\gamma^{\mathbf{b}}{}_{\alpha\,\beta}E^{(1)\,\beta}{}_{\mathbf{b}}\,\Big) \qquad (10.57)\\
&- 2\,\varepsilon_{\mathbf{a}\,\mathbf{b}\,\mathbf{c}}\,\gamma^{\mathbf{b}\,\alpha}{}_{\epsilon}\,D_\alpha\,E^{(1)\;\epsilon\,\mathbf{c}} \,+\, 2\,D_\alpha\,E^{(1)\alpha}{}_{\mathbf{a}} \,-\, 4\,D_{\mathbf{a}}\,\phi^{(1)} = \\
&- 2\,\varepsilon_{\mathbf{a}\,\mathbf{b}\,\mathbf{c}}\,\gamma^{\mathbf{b}\,\alpha}{}_{\epsilon}\,D_\alpha\,E^{(1)\;\epsilon\,\mathbf{c}} \,+\, 2\,D_\alpha\,E^{(1)\alpha}{}_{\mathbf{a}} - \,4\,D_{\mathbf{a}}\,\phi^{(1)} \qquad (10.58)
\end{aligned}
$$

So, from relation $\tilde{T}^{(1)}{}_P \,=\, 0$ we will get no new constraints.

From relations $\tilde{T}^{(1)}{}_{\Omega} \,=\, 0$ and $\tilde{T}^{(1)}{}_{\Sigma} \,=\, 0$ we will get some constraints on unfixed (and unused) vielbeins $E^{(1)}{}_{\Sigma\,\Omega}$ and $E^{(1)}{}_{\Sigma\,\Sigma}$.

10.9 Structure of linearized dilaton

After imposing all the constraints we have found that everything could be expressed in terms of $E^{(1)}{}_{P\,\Omega}$ and $E^{(1)}{}_{D\,\tilde{D}}$. The gamma "trace" part of $E^{(1)}{}_{P\,\Omega}$ is related directly to $E^{(1)}{}_{D\,\tilde{D}}$ by (10.55). Therefore we want equation of motion for the field $E^{(1)}{}_{D\,\tilde{D}}$.

We start with some action S and vary it with respect to vielbein $E^{D\,D}$ and put it to the zero, i.e. ${}^{\delta}/_{\delta\,E^{DD}}\,S \,=\, 0$. The variation produces the dimension 1 antisymmetric tensor. On the other hand in the previous subsection we have seen that $K_{D\,D}$ is the canonical antisymmetric dimension 1 tensor. Therefore we can impose the equations of motion:

$$
\tfrac{\delta}{\delta\,E^{DD}}\,S \;\equiv\; K_{D\,D} \,=\, 0 \qquad (10.59)
$$

To obtain the structure of linearized dilaton we do the variation of the S with respect to $E^{\alpha\,\beta}$ and $E^{\tilde{\alpha}\,\tilde{\beta}}$. We get:

$$
K_{\alpha\,\beta} \,=\, K^{(1)}{}_{\alpha\,\beta} \,=\, 0 \qquad \text{and} \qquad K_{\tilde{\alpha}\,\tilde{\beta}} \,=\, K^{(1)}{}_{\tilde{\alpha}\,\tilde{\beta}} \,=\, 0 \qquad (10.60)
$$

where

$$
K^{(1)}{}_{\alpha\,\beta} \,\propto\, \varepsilon_{\alpha\,\beta}\,B \qquad\qquad K^{(1)}{}_{\tilde{\alpha}\,\tilde{\beta}} \,\propto\, \varepsilon_{\tilde{\alpha}\,\tilde{\beta}}\,\tilde{B} \qquad (10.61)
$$

Equations (10.60) and (10.61) could be rewritten in a different way:

$$
B \,+\, \tilde{B} \,=\, 0 \;\text{ and }\; B \,-\, \tilde{B} \,=\, 0 \qquad (10.62)
$$

where B is given by eq. (10.56). Because the explicit structure of B and $\tilde{B}$ is important for the next considerations we repeat it here:

$$\begin{aligned} B &\propto \varepsilon^{\nu\,\alpha}\, D_\nu\, (\, D^{\tilde{\epsilon}} \,+\, \tfrac{1}{4}\,\gamma^{\tilde{\mathbf{a}}\,\tilde{\beta}\,\tilde{\epsilon}}\, D_{\tilde{\mathbf{a}}}\, D_{\tilde{\beta}}\,)\, E^{(1)}{}_{\tilde{\epsilon}\,\alpha} \,+\, \varepsilon^{\alpha\,\beta}\, D_\beta\, D_\alpha\, \phi^{(1)} && (10.63)\\ &\equiv \varepsilon\; D_D\, (\, D_{\tilde{\Omega}} \,+\, \gamma\; D_{\tilde{P}}\, D_{\tilde{D}}\,)\, E^{(1)}{}_{\tilde{D}\,D} \,+\, D_D\;\, D_D\, \phi^{(1)} \\ \tilde{B} &\propto \varepsilon^{\tilde{\nu}\,\tilde{\epsilon}}\, D_{\tilde{\nu}}\, (\, -\, D^\alpha \,+\, \tfrac{1}{4}\,\gamma^{\mathbf{a}\,\beta\,\alpha}\, D_{\mathbf{a}}\, D_\beta\,)\, E^{(1)}{}_{\tilde{\epsilon}\,\alpha} \,+\, \varepsilon^{\tilde{\alpha}\,\tilde{\beta}}\, D_{\tilde{\beta}}\, D_{\tilde{\alpha}}\, \phi^{(1)} && (10.64)\\ &\equiv \varepsilon\; D_{\tilde{D}}\, (\, -\, D_\Omega \,+\, \gamma\; D_P\, D_D\,)\, E^{(1)}{}_{\tilde{D}\,D} \,+\, D_{\tilde{D}}\;\, D_{\tilde{D}}\, \phi^{(1)} \end{aligned}$$

To analyse the second terms in (10.63) and (10.64) we need the following identities:

$$\gamma^{\mathbf{a}\,\beta\,\alpha}\, D_{\mathbf{a}}\, D_\beta \;=\; 4\, D^\alpha \,-\, \tfrac{1}{2}\, D^2\, \varepsilon^{\alpha\,\epsilon}\, D_\epsilon \qquad (10.65)$$

$$\gamma^{\tilde{\mathbf{a}}\,\tilde{\beta}\,\tilde{\alpha}}\, D_{\tilde{\mathbf{a}}}\, D_{\tilde{\beta}} \;=\; -\,4\, D^{\tilde{\alpha}} \,+\, \tfrac{1}{2}\, \tilde{D}^2\, \varepsilon^{\tilde{\alpha}\,\tilde{\epsilon}}\, D_{\tilde{\epsilon}} \qquad (10.66)$$

where $D^2 \;=\; \varepsilon^{\beta\,\alpha}\, D_\alpha\, D_\beta \;\equiv\; D_D\;\, D_D$ (similarly for $\tilde{D}^2$). Using (10.65) and (10.66) we get:

$$B \;\propto\; -\,\tfrac{1}{8}\, \tilde{D}^2\, (\, \varepsilon^{\alpha\,\nu}\, \varepsilon^{\tilde{\epsilon}\,\tilde{\sigma}}\, D_\nu\, D_{\tilde{\sigma}}\, E^{(1)}{}_{\tilde{\epsilon}\,\alpha}\,) \,+\, D^2\, \phi^{(1)} \qquad (10.67)$$

$$\tilde{B} \;\propto\; -\,\tfrac{1}{8}\, D^2\, (\, \varepsilon^{\alpha\,\nu}\, \varepsilon^{\tilde{\epsilon}\,\tilde{\sigma}}\, D_\nu\, D_{\tilde{\sigma}}\, E^{(1)}{}_{\tilde{\epsilon}\,\alpha}\,) \,+\, \tilde{D}^2\, \phi^{(1)} \qquad (10.68)$$

Then the first of (10.62) becomes the equation:

$$0 \;=\; (\, D^2 \,+\, \tilde{D}^2\,)\, (\, -\, \tfrac{1}{8}\, \varepsilon^{\alpha\,\nu}\, \varepsilon^{\tilde{\epsilon}\,\tilde{\sigma}}\, D_\nu\, D_{\tilde{\sigma}}\, E^{(1)}{}_{\tilde{\epsilon}\,\alpha} \,+\, \phi^{(1)}\,) \qquad (10.69)$$

We can rewrite (10.69) using a new field V:

$$(\, D^2 \,-\, \tilde{D}^2\,)\, V \;=:\; (\, -\, \tfrac{1}{8}\, \varepsilon^{\alpha\,\nu}\, \varepsilon^{\tilde{\epsilon}\,\tilde{\sigma}}\, D_\nu\, D_{\tilde{\sigma}}\, E^{(1)}{}_{\tilde{\epsilon}\,\alpha} \,+\, \phi^{(1)}\,) \qquad (10.70)$$

Using this definition the (10.69) could be written as:

$$0 \;=\; (\, D^2 \,+\, \tilde{D}^2\,)\, (\, D^2 \,-\, \tilde{D}^2\,)\, V \qquad (10.71)$$

The operator $(\, D^2 \,+\, \tilde{D}^2\,)\, (\, D^2 \,-\, \tilde{D}^2\,)$ is acting on the scalar field V. It can be rewritten in a nicer form:

$$\begin{aligned} (\, D^2 \,+\, \tilde{D}^2\,)\, (\, D^2 \,-\, \tilde{D}^2\,)\, V &= 4\, (\Box \,-\, D_\nu\, D^\nu \,+\, D^\nu\, D_\nu && (10.72)\\ &\quad -\, (\, \tilde{\Box} \,-\, D_{\tilde{\nu}}\, D^{\tilde{\nu}} \,+\, D^{\tilde{\nu}}\, D_{\tilde{\nu}}\,)\,)\, V \\ &\equiv 4\, D^{\mathcal{A}}\, D_{\mathcal{A}}\, V && (10.73) \end{aligned}$$

Therefore the first equation of (10.62) could be rewritten as:

$$D^{\mathcal{A}}\, D_{\mathcal{A}} V \;=\; 0 \tag{10.74}$$

and so (10.74) is identically satisfied since it is just the strong constraint.

Using the relation (10.70) we find the structure of the linear dilaton:

$$\phi^{(1)} \;=\; \tfrac{1}{8}\, \varepsilon^{\alpha\,\nu}\, \varepsilon^{\tilde{\epsilon}\,\tilde{\sigma}}\, D_{\nu}\, D_{\tilde{\sigma}}\, E^{(1)}{}_{\tilde{\epsilon}\,\alpha} \,+\, (\, D^2 \,-\, \tilde{D}^2\,)\, V \tag{10.75}$$

We notice that the structure of the linearized dilaton matches the structure of the dilaton field obtained by compactifying the 4D $\mathcal{N} = 1$ supergravity to 3 dimensions, see section 7.2.b in [22].

Chapter 11

11 Conclusion: T-duality off shell in 3D Type II superspace

We outline results we have obtained: we started with the T-dual $\mathcal{N} = 2$ string theory, i.e. effective $\mathcal{N} = 2$ supergravity in 3 dimensions. We knew that this theory should be equivalent to the theory obtained from the classical $\mathcal{N} = 1$ supergravity in 4 dimensions. In the following chapters we first obtained the dimension -1 prepotential as the vielbein component ${E^{(1)}}_{D\tilde{D}} \equiv {E^{(1)}}_{\alpha\tilde{\beta}}$ and the dimension $-\frac{3}{2}$ unconstrained gauge parameter $\Lambda_D \equiv \Lambda_\alpha$ (also $\Lambda_{\tilde{D}}$) without solving any differential constraints. In the usual 4 dimensional $\mathcal{N} = 1$ supergravity they appear only through their derivatives in objects of higher dimension after solving differential constraints, see section X.A.1 in [10]. We have also derived the structure of the $\mathcal{N} = 2$ supergravity in 3 dimensions using the techniques of the T-dually extended superspace. In particular the structure of the linear dilaton ϕ has been derived. It matches the structure obtained from 4D $\mathcal{N} = 1$ and its compactification, see section 7.9 in [22] and [7]. This suggest that T-dualy extended superspace approach could be extended also to higher dimensional cases, see [17].

Chapter 12

12 Introduction: Pre-potential in the $AdS_5 \times S^5$ Type IIB superspace

In the paper [11] we obtained the curvature tensor (previously discovered in [1]) in a manifestly T-dual way. In the paper [12] we extended the techniques for the case of three dimensional $\mathcal{N} = 2$ T-dual extended superspace. There we correctly obtained the pre-potential (as a part of vielbein), structure of linearised dilaton and field equations. The aim of this paper was to look at the full ten dimensional $\mathcal{N} = 2$ T-dually extended superspace in the flat and also in $AdS_5 \times S^5$ background, i.e. IIB string theory expanded around $AdS_5 \times S^5$ background. The AdS was earlier analysed in superspace in papers [24], [25], [14] and [15]. In following chapters discovered the projective (and also the chiral) pre-potential to sit in a certain combination of $H_{S\,\tilde{S}}$ and $H_{D\,\tilde{D}}$. This was first obtained in the flat case and later generalised for the $AdS_5 \times S^5$. We also performed the near horizon limit and derived the equation of motion for the pre-potential in that limit. This limit also picks out the projective pre-potential instead of the chiral pre-potential, even though both pre-potentials are valid bulk solutions. The projective and harmonic superspaces were earlier analysed in [26] and [27].

Chapter 13

13 Motivation

The massive development of the AdS/CFT correspondence (and its generalisations) started with foundational papers [29] and [30]. It is considered to be one of the best achievements in string theory. The correspondence promises the way of computing quantum effects in strongly coupled field theory using the classical (super) gravity theory. The AdS/CFT correspondence (and its generalisations) is currently used in many different areas, like study of confinement, condensed matter systems or investigation of non-equilibrium phenomenas in strongly coupled plasma.

In the AdS/CFT correspondence the strongly coupled quantum field theory (four dimensional $\mathcal{N} = 4$ super Yang-Mills theory) is related to the supergravity theory living on $AdS_5 \times S^5$ space. The strongly coupled conformal field theory is hard to work with. The correspondence provides feasible way to calculate effects in such theory by translating it to weakly coupled supergravity.

It is therefore of high interest to develop suitable framework to handle supergravity living on $AdS_5 \times S^5$ background. This framework could be later used as a base for perturbative calculations. In this paper our aim was to look at type IIB supergravity living on $AdS_5 \times S^5$ background. We are looking for solutions generated via field called pre-potential. The pre-potential is a basic field from which all physical fields (in the massless spectrum) arise. Thus we get the classical solution of supergravity living on the background $AdS_5 \times S^5$. This solution is interesting because of our later aim to use it in perturbative calculations. Our novel approach unites the way how this solution is found. The first interesting observation is universal usage of string based framework of T-dually extended superspaces, that started with foundational paper [1] and later examined in [11] and [12]. Another interesting feature is the way how the $AdS_5 \times S^5$ pre-potential solution was obtained in this paper. The approach is actually an analogy of how we derived the flat ten dimensional pre-potential in this paper. We started with flat supergravity where it was much easier to identify the pre-potential. Inspired by this solution we turned on the $AdS_5 \times S^5$ background, modified the equations and looked for the $AdS_5 \times S^5$ deformed solution. This gave us both technical and also geometrical advantages to look at the pre-potential as coming from the same field solution. The sanity checked worked here, if we flatten back the $AdS_5 \times S^5$ (i.e. the $R \to \infty$ limit) we would get back the flat space pre-potential solution. Moreover by looking at the near horizon limit ($R \to 0$) it was easy to naturally identify the equations of motion and find the correspondence between gravity fields and CFT fields.

We believe that the use of the string based T-dually extended superspaces in various contexts is fruitful and natural way to cast supergravity, as was observed in [11] and [1]. In this particular work we showed how it can be effectively used to naturally find the pre-potential solution in the fully fledged type IIB supergravity on the $AdS_5 \times S^5$ background. That is a novel result that to our best knowledge has never been derived. In the future we would like to look how similar approach might shed some light on the supergravity solutions on different backgrounds, relevant for generalizations of the $AdS_5 \times S^5$. Moreover in this framework the calculations beyond linearized level are also feasible and are left for future work.

Chapter 14

14 Type II superspace, notation and motivation

14.1 10 dimensional type II superspace

The T-duality is important duality we know to exist in (super)string theory. The low energy limit of superstring theory is version of ten dimensional supergravity. The T-duality then forms T-duality symmetry which is the symmetry of such low energy theory. That symmetry can be manifestly realized by doubled spacetime coordinates. This realization was first made in the paper [1].

In the next introduction we are closely following paper [17]. To T-dualize ordinary space we start with space that can be built by the coset construction. In the procedure of T-dualizing the coset construction one starts with ordinary Lie algebra generated by G_I, where by I we mean some particular set of indices. We require this Lie algebra to have non-degenerate metric. As for the usual coset construction, one can exponentialize the Lie algebra and get the Lie group (more precisely the vicinity of the unit element), i.e. one constructs the Lie group element $g(Z^I)$, where Z^I are group coordinates. The covariant derivatives and symmetry generators (Lie derivatives) on that space are obtained by considering left and right actions of the group. Since left and right action on a group element commute so do the covariant derivatives and symmetry generators. More explicitly for ordinary (particle) construction we get:

$$\begin{aligned} \text{Symmetry generators: } \widetilde{\nabla}_I &= L_I{}^M \tfrac{1}{i}\,\partial_M \\ \text{Covariant derivatives: } \overset{\circ}{\nabla}_I &= R_I{}^M \tfrac{1}{i}\,\partial_M \end{aligned} \tag{14.1}$$

where $L_I{}^M$ and $R_M{}^I$ are matrices defining right and left invariant one forms: $(d\,g)\,g^{-1} = i\,d\,Z^M\,L_M{}^I\,G_I$ and $g^{-1}\,(d\,g) = i\,d\,Z^M\,R_M{}^I\,G_I$. The L matrix is used in left action generator and R matrix is used in right action generator in (14.1). It follows from construction that the covariant derivatives $\overset{\circ}{\nabla}_I$ and symmetry generators $\widetilde{\nabla}_J$ satisfy the following Lie algebra:

$$
\begin{aligned}
[\,\widetilde{\nabla}_I,\, \widetilde{\nabla}_J\} &= i\, f_{I\,J}{}^{K}\, \widetilde{\nabla}_K \quad || \quad [\,\overset{\circ}{\nabla}_I,\, \overset{\circ}{\nabla}_J\} = -\,i\, f_{I\,J}{}^{K}\, \overset{\circ}{\nabla}_K \\
[\,\overset{\circ}{\nabla}_I,\, \widetilde{\nabla}_J\} &= 0
\end{aligned}
\tag{14.2}
$$

where we included graded commutators as an immediate superalgebra generalization. The graded bracket in (14.2) is anticommuting for two fermions and commuting for others. Having the covariant derivatives in hands we can curve given space arbitrarily via vielbeins $\nabla_A \,=\, E_A{}^I\, \overset{\circ}{\nabla}_I$.

The next step is to generalize the algebra (14.2) to the (super)string case. It is obtained by string extension of particle one parameter $Z^I\,(\,\tau\,)$ to stringy $Z^I\,(\,\tau,\,\sigma\,)$. By that we get string generalization of (14.2), the string affine Lie algebra:

$$
\begin{aligned}
[\,\widetilde{\triangleright}_I,\, \widetilde{\triangleright}_J\} &= i\, f_{I\,J}{}^{K}\, \widetilde{\triangleright}_K \,+\, i\, \overset{\circ}{\eta}_{I\,J}\, \partial_\sigma\, \delta\,(\,2\,-\,1\,) \\
[\,\overset{\circ}{\triangleright}_I,\, \overset{\circ}{\triangleright}_J\} &= -\,i\, f_{I\,J}{}^{K}\, \overset{\circ}{\triangleright}_K \,-\, i\, \overset{\circ}{\eta}_{I\,J}\, \partial_\sigma\, \delta\,(\,2\,-\,1\,) \\
[\,\widetilde{\triangleright}_I,\, \overset{\circ}{\triangleright}_J\} &= 0
\end{aligned}
\tag{14.3}
$$

where the metric $\overset{\circ}{\eta}_{I\,J}$ is the (graded), constant, non-degenerate (super) Lie group metric and $\partial_\sigma\, \delta\,(\,2\,-\,1\,) \,\equiv\, \partial_{\sigma_2}\, \delta\,(\,2\,-\,1\,) \,\equiv\, \delta'\,(\,2\,-\,1\,)$. In analogy with (14.1) the left and right action generators could be solved explicitly in the string case:

$$
\begin{aligned}
\text{Symmetry generators: } \widetilde{\triangleright}_I \;&:=\; L_I{}^M \left(\tfrac{1}{i}\, \partial_M \,+\, \partial_\sigma\, Z^{\,N}\, B_{N\,M} \right) \\
&\quad -\, \partial_\sigma\, Z^M\, L_M{}^J\, \overset{\circ}{\eta}_{J\,I} \\
\text{Covariant derivatives: } \overset{\circ}{\triangleright}_I \;&:=\; R_I{}^M \left(\tfrac{1}{i}\, \partial_M \,+\, \partial_\sigma\, Z^{\,N}\, B_{N\,M} \right) \\
&\quad +\, \partial_\sigma\, Z^M\, R_M{}^J\, \overset{\circ}{\eta}_{J\,I}
\end{aligned}
\tag{14.4}
$$

where $B_{M\,N}$ is the B-field. We can get the curved version of the string covariant derivatives (in paper [17] also called curved current) via vielbeins:

$$
\triangleright_A := \, E_A{}^I\,(\,Z\,(\,\tau,\,\sigma\,)\,)\, \overset{\circ}{\triangleright}_I \,\equiv\, E_A{}^I\, \overset{\circ}{\triangleright}_I \tag{14.5}
$$

Using (14.5) we can introduce geometric objects like torsions. Looking at stringy affine algebra in the curved background we get:

$$
[\,\triangleright_A,\, \triangleright_B\} \,:=\, -i\, T_{A\,B}{}^{C}\;\, \triangleright_C\;\, \delta\,(\,2\,-\,1\,) \,-\, i\, \eta_{A\,B}\, \partial_\sigma\, \delta\,(\,2\,-\,1\,) \tag{14.6}
$$

where $T_{AB}{}^{C}$ is the string generalization of torsion and η_{AB} is the curved group metric. In general are both functions of vielbeins. We will impose constraint on η_{AB} and require η_{AB} to be a constant metric $\overset{\circ}{\eta}_{AB}$. Such constraint does not impose any restrictions on physical content and makes calculations simpler.

In the next step we will describe particular realization of above construction. For the usual particle we can start with algebra of translations generated by $p_{\mathbf{m}}$, where $\mathbf{m} \in \{1, \ldots, \dim\}$. Next we can include the supersymmetry generator D_μ, where μ is fermionic index and range depends on dimensionality of space. By including D_μ we get the supertranslations. Finally we can add the Lorentz generator $S_{\mathbf{mn}}$ and thus get a particle algebra of super-Poincaré transformations. All previous can be described by nice diagram (see table 4), see also reference paper [17].

translations	$\rightarrow$ supertranslations	$\rightarrow$ super-Poincaré
$p_{\mathbf{m}}$	$D_\mu,\, p_{\mathbf{m}}$	$S_{\mathbf{mn}},\, D_\mu,\, p_{\mathbf{m}}$

Table 4: Particle algebra generators

To build the T-dual analog of table 4 for type II string we first need to double the translation generators and also have in mind that those generators are forming an affine Lie algebra as seen in (14.3). We get the set of string translation generators $P_{\mathbf{m}}$, $P_{\widetilde{\mathbf{m}}}$, they generate left and right translations. Next we will include the supersymmetry generators that are for obvious reasons also doubled to D_μ, $D_{\widetilde{\mu}}$. Including those generates a small issue. Because we consider the non-degenerate group metric η_{IJ} and the affine Lie algebra of supertranslations has to satisfy super-Jacobi identities we need to include another independent fermionic generators (dual currents) Ω^μ, $\Omega^{\widetilde{\mu}}$. The necessity for the dual currents was discussed in great detail in [5]. Finally, if we include the Lorentz generators $S_{\mathbf{mn}}$, $S_{\widetilde{\mathbf{mn}}}$ we also discover yet another set of dual currents $\Sigma^{\mathbf{mn}}$, $\Sigma^{\widetilde{\mathbf{mn}}}$. We summarise the type II generators in diagram 5.

translations	$\rightarrow$ supertranslations	$\rightarrow$ super-Poincaré
$P_{\mathbf{m}}$	$D_\mu,\, P_{\mathbf{m}},\, \Omega^\mu$	$S_{\mathbf{mn}},\, D_\mu,\, P_{\mathbf{m}},\, \Omega^\mu,\, \Sigma^{\mathbf{mn}}$
$P_{\widetilde{\mathbf{m}}}$	$D_{\widetilde{\mu}},\, P_{\widetilde{\mathbf{m}}},\, \Omega^{\widetilde{\mu}}$	$S_{\widetilde{\mathbf{mn}}},\, D_{\widetilde{\mu}},\, P_{\widetilde{\mathbf{m}}},\, \Omega^{\widetilde{\mu}},\, \Sigma^{\widetilde{\mathbf{mn}}}$

Table 5: String affine algebra generators

We can see that the whole set of generators is doubled for type II string affine algebra. That introduces space with high dimensionality. In the end the dimensional reduction, coset constraints and section condition are imposed on fields (living on such high dimensional space) to remove unphysical degrees of freedom. To deal with indices and generators in table 5 we introduce notation for various forms of graded indices, notation can be found in table 6.

$$\begin{array}{ll}
\text{Explicit:} & {}_{M} := \left({}_{\mathbf{mn}},\ {}_{\mu},\ {}_{\mathbf{m}},\ {}^{\mu},\ {}^{\mathbf{mn}} \right) \\
\text{Carried by:} & S_{\mathbf{mn}},\ D_{\mu},\ P_{\mathbf{m}},\ \varOmega^{\mu},\ \varSigma^{\mathbf{mn}} \\
\text{Explicit:} & {}_{\widetilde{M}} := \left({}_{\widetilde{\mathbf{mn}}},\ {}_{\widetilde{\mu}},\ {}_{\widetilde{\mathbf{m}}},\ {}^{\widetilde{\mu}},\ {}^{\widetilde{\mathbf{mn}}} \right) \\
\text{Carried by:} & S_{\widetilde{\mathbf{mn}}},\ D_{\widetilde{\mu}},\ P_{\widetilde{\mathbf{m}}},\ \varOmega^{\widetilde{\mu}},\ \varSigma^{\widetilde{\mathbf{mn}}} \\
 & \\
\text{Symbolic:} & {}_{M} := (\, S,\ D,\ P,\ \varOmega,\ \varSigma \,) \\
 & {}_{\widetilde{M}} := (\, \widetilde{S},\ \widetilde{D},\ \widetilde{P},\ \widetilde{\varOmega},\ \widetilde{\varSigma} \,)
\end{array}$$

$$\text{Multiindices:} \quad {}_{\mathcal{M}} := (\, S,\ \widetilde{S},\ D,\ \widetilde{D},\ P,\ \widetilde{P},\ \varOmega,\ \widetilde{\varOmega},\ \varSigma,\ \widetilde{\varSigma} \,) \quad \equiv (\, {}_{M},\ {}_{\widetilde{M}} \,)$$

Table 6: String affine algebra indices

Using definition from table 6 we can define stringy super-Poincaré covariant derivatives (and string coordinates) in the sense of generic string affine (super) Lie algebra from (14.3), see table 7.

$$\begin{array}{rl}
\text{Covariant derivatives:} & \overset{\circ}{\triangleright}_{M} := (\, S_{\mathbf{mn}},\ D_{\mu},\ P_{\mathbf{m}},\ \varOmega^{\mu},\ \varSigma^{\mathbf{mn}} \,) \\
 & \overset{\circ}{\triangleright}_{\widetilde{M}} := (\, S_{\widetilde{\mathbf{mn}}},\ D_{\widetilde{\mu}},\ P_{\widetilde{\mathbf{m}}},\ \varOmega^{\widetilde{\mu}},\ \varSigma^{\widetilde{\mathbf{mn}}} \,) \\
\text{Coordinates:} & Z^{M} := (\, u^{\mathbf{mn}},\ \theta^{\mu},\ x^{\mathbf{m}},\ \varphi_{\mu},\ \upsilon_{\mathbf{mn}} \,) \\
 & Z^{\widetilde{M}} := (\, u^{\widetilde{\mathbf{mn}}},\ \theta^{\widetilde{\mu}},\ x^{\widetilde{\mathbf{m}}},\ \varphi_{\widetilde{\mu}},\ \upsilon_{\widetilde{\mathbf{mn}}} \,)
\end{array}$$

Table 7: String covariant derivatives and coordinates

The table 7 covariant derivatives satisfy the following explicit string type II affine super-Poincaré algebra, for which the non-zero structure constants and the central charges are:

$$\begin{aligned}
[\, S_{\mathbf{mn}}(1),\, S_{\mathbf{kl}}(2)\,] &= -i\,\eta_{[\,\mathbf{m}\,[\,\mathbf{k}}\, S_{\mathbf{l}\,]\,\mathbf{n}\,]}\;\delta\,(\,2\,-\,1\,) \\
[\, S_{\mathbf{mn}}(1),\, D_{\rho}\,(2)\,] &= -\,i\,\tfrac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^{\sigma}{}_{\rho}\, D_{\sigma}\,\delta\,(\,2\,-\,1\,) \\
[\, S_{\mathbf{mn}}(1),\, P_{\mathbf{k}}\,(2)\,] &= i\,\eta_{\mathbf{k}\,[\,\mathbf{m}}\, P_{\mathbf{n}\,]}\,\delta\,(\,2\,-\,1\,) \\
[\, S_{\mathbf{mn}}(1),\, \Omega^{\rho}\,(2)\,] &= i\,\tfrac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^{\rho}{}_{\sigma}\,\Omega^{\sigma}\,\delta\,(\,2\,-\,1\,) \\
[\, S_{\mathbf{mn}}(1),\, \Sigma^{\mathbf{kl}}\,(2)\,] &= i\,\delta_{\mathbf{mn}}{}^{\mathbf{kl}}\,\delta'\,(\,2\,-\,1\,)\,-\,i\delta_{[\,\mathbf{m}}{}^{[\,\mathbf{k}}\,\eta_{\mathbf{n}\,]\,\mathbf{s}}\Sigma^{\mathbf{l}]\mathbf{s}}\,\delta\,(\,2\,-\,1\,) \\
\{\, D_{\rho}\,(1),\, D_{\sigma}(2)\,\} &= 2\,(\,\gamma^{\mathbf{m}}\,)_{\rho\sigma}\, P_{\mathbf{m}}\,\delta\,(\,2\,-\,1\,) \\
[\, D_{\rho}\,(1),\, P_{\mathbf{m}}\,(2)\,] &= 2\,(\,\gamma_{\mathbf{m}})_{\rho\sigma}\,\Omega^{\sigma}\,\delta\,(\,2\,-\,1\,) \\
\{\, D_{\rho}\,(1),\, \Omega^{\sigma}\,(2)\,\} &= i\,\delta^{\sigma}_{\rho}\,\delta'\,(\,2\,-\,1\,)\,-\,i\,\tfrac{1}{4}\,(\,\gamma_{\mathbf{mn}}\,)^{\sigma}{}_{\rho}\,\Sigma^{\mathbf{mn}}\,\delta\,(\,2\,-\,1\,) \\
[\, P_{\mathbf{m}}\,(1),\, P_{\mathbf{n}}\,(2)\,] &= i\,\eta_{\mathbf{mn}}\,\delta'\,(\,2\,-\,1\,)\,+\,i\,\eta_{\mathbf{m\,h}}\,\eta_{\mathbf{n\,s}}\Sigma^{\mathbf{hs}}\,\delta\,(\,2\,-\,1\,) \\
&\curvearrowright\;\; \text{left algebra} \,\to\, -\ \text{right algebra} \\
[\,\text{left, right}\,\} &= 0.
\end{aligned} \tag{14.7}$$

As indicated above, the algebra for the right generators is the same up to the overall sign. We can assign the canonical dimensions to the generators: $\dim(S,\, D,\, P,\, \Omega, \Sigma) \,=\, (0,\, \frac{1}{2},\, 1,\, \frac{3}{2},\, 2)$. The S generators generate the $SO\,(\,9,1\,)\,\otimes\, SO(\,9,\,1\,)$ algebra, i.e. left (and right) local Lorentz transformations. The D generate left (and right) supersymmetry transformation and P left (and right) translations. The Ω and Σ are the left (and right) dual currents (corresponding to D and S), see also [1]. We can see once again that the only non-vanishing terms (for left handed algebra, similarly for right handed algebra) in the metric and structure constants in (14.7) are (as can be guessed by dimensional analysis):

$$\eta_{PP},\ \eta_{S\Sigma},\ \eta_{D\Omega};\quad f_{SPP}\ ,\ f_{SS\Sigma}\ ,\ f_{DDP}\ ,\ f_{SD\Omega} \tag{14.8}$$

where we have lowered the upper index on f with η to take advantage of its total (graded) antisymmetry. In that notation we explicitly have, for the left-handed algebra:

$$(\eta)_{\mathbf{mn}}\,=\,\eta_{\mathbf{mn}}\,,\ \ (\eta)_{\mathbf{mn}}{}^{\mathbf{pq}}\,=\,\delta_{\mathbf{mn}}{}^{\mathbf{pq}}\,,\ (\eta)_{\sigma}{}^{\rho}\,=\,\delta^{\rho}_{\sigma} \tag{14.9}$$

$$\begin{aligned}
f_{\mathbf{mn}}{}^{\mathbf{p\,q}} = -\,\delta_{\mathbf{mn}}{}^{\mathbf{pq}} \quad &||\quad f_{\mathbf{mn\,pq}}{}^{\mathbf{rs}} = \eta_{[\mathbf{m}[\mathbf{p}}\delta_{\mathbf{q}]\mathbf{n}]}{}^{\mathbf{rs}} \\
f_{\sigma\rho}{}^{\mathbf{m}} = 2\,(\,\gamma^{\mathbf{m}}\,)_{\sigma\rho} \quad &||\quad f_{\mathbf{mn}\,\sigma}{}^{\rho} = -\,\tfrac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^{\rho}_{\sigma}
\end{aligned} \tag{14.10}$$

The type IIA and IIB theories are distinguished by the choice of ten dimensional fermionic coordinates. For IIA theory we pick fermionic coordinates with both ten dimensional chiralities $(Z^{\mu}, Z^{\widetilde{\mu}}) \equiv (\theta^{\mu}, \theta_{\mu})$. Note that now the θ coordinate uses concrete ten dimensional fermionic index (in chiral representation), in contrast with generic fermionic index used in description of θ coordinate in table 7. Moreover those indices are ten dimensional chiral indices with respect to the common (diagonal) local Lorentz group (defined after the dimensional reduction). The IIB theory uses the indices of the same ten dimensional chirality $(Z^{\mu}, Z^{\widetilde{\mu}}) \equiv ({\theta_1}^{\mu}, {\theta_2}^{\mu})$, therefore we denoted them by substript $_1$ and $_2$.

The above construction is very natural from the point of view of the superstring theory, where we know the T-duality is a symmetry of low energy theory. Moreover as we have seen in papers [11] and [12] this description has advantages to naturally introduce objects of interest. For example in the paper [11] we derived the T-dual version of Riemann curvature tensor using T-dually extended space (even though there without supersymmetry). That tensor has been previously discovered in [1] but by indirect methods. In paper [12] we discovered that the pre-potential for $\mathcal{N} = 2$ three dimensional supergravity is naturally part of vielbeins living on the T-dually extended superspace (i.e. the three dimensional version of above construction). It was known how to find the pre-potential for $\mathcal{N} = 2$ three dimensional supergravity before, however some further differential constraints were needed (like the bisection condition) see [22]. In paper [12] all constraints are coming naturally from the torsion constraints (we will discuss them later in this paper as well). We will see that ten dimensional generalization of [11] and [12] is a fruitful way how to treat the pre-potential in ten dimensional flat and even $AdS_5 \times S^5$ space.

In next sections we will use the Wick rotation. We feel free to Wick rotate from Minkowski to Euclidean metric and back because in the following chapters we do not discuss the reality conditions, so the rotation is the matter of convenience.

Our first and most significant use of Wick rotation is in the description of space-cone gauge in the case of $AdS_5 \times S^5$. The procedure is described more precisely in [8] and so we are giving just short overview. In order to introduce the space-cone basis in $AdS_5 \times S^5$ one can first Wick rotate the sphere S^5 to AdS_5. This is done by extension of one S^5 coordinate to complex numbers and then taking it to be purely imaginary. After that we have space $AdS_5 \times AdS_5$. At two corresponding Poincaré patches (at the $AdS_5 \times AdS_5$)

we take two bulk coordinates (two space-like coordinates one from the original AdS_5 and another one from the Wick rotated sphere) and introduce the space-cone coordinates x^+ and x^- as their combinations. The near horizon limit is attained by $x^+ \to 0$. This limit turns the superspace into the projective superspace. In the text we will use those x^+ and x^- coordinates and related space-cone gauge (for vielbeins living on the extended space). The space-cone gauge destroys the explicit local Lorentz covariance but still preserves the boundary (near horizon limit) Lorentz covariance $SO(3,1) \otimes SO(4)$ the symmetries of boundary CFT.

We will also use the Wick rotation whenever we find it easier to explain or define some notion. For example, we define ten dimensional Γ matrices as matrices for $SO(10)$ Lorentz group. To get the gamma matrices for $SO(9,1)$ one can Wick rotate back.

14.2 Gamma matrices

The gamma matrices $(\gamma_{\mathbf{m}})_{\mu\nu}$ used in the algebra (14.7) are the $16 \otimes 16$ block gamma matrices from 10 dimensional $32 \otimes 32$ chiral representation:

$$\Gamma_{\mathbf{m}} = \begin{pmatrix} 0 & (\gamma_{\mathbf{m}})^{\mu\nu} \\ (\gamma_{\mathbf{m}})_{\mu\nu} & 0 \end{pmatrix} \quad \text{where} \quad \{ \Gamma_{\mathbf{m}}, \Gamma_{\mathbf{n}} \} = 2\,\eta_{\mathbf{m}\mathbf{n}}\,\delta. \qquad (14.11)$$

Moreover the block gamma matrices satisfy:

$$\begin{aligned} &(\gamma_{\mathbf{m}})_{\mu\nu} = (\gamma_{\mathbf{m}})_{\nu\mu} \quad || \quad (\gamma_{(\mathbf{m}})^{\mu\nu} (\gamma_{\mathbf{n})})_{\nu\sigma} = 2\,\eta_{\mathbf{m}\mathbf{n}}\,\delta^{\mu}_{\sigma} \\ &(\gamma_{\mathbf{m}})_{(\mu\nu}(\gamma^{\mathbf{m}})_{\sigma)\lambda} = 0 \end{aligned} \qquad (14.12)$$

The IIB fermion generators (in algebra 14.7) are described by $16 \oplus 16$ chiral fermion generators (for left and right generators) with the same 10 dimensional chirality. For the future use we need to look closer at the structure of the matrices $(\gamma_{\mathbf{m}})_{\mu\nu}$ from equation (14.11). The gamma matrices from equation (14.11) could be constructed from $SO(9)$ gamma matrices or equivalently from $SO(8)$ gamma matrices and the chirality matrix. We can go one step down and construct the $SO(8)$ gamma matrices from $SO(6)$ gamma matrices. For the $SO(6)$ gamma matrices we use the Majorana representation of those matrices (they are purely imaginary). Thus we can get the Majorana - Weyl representation of the $SO(8)$ gamma matrices.

For the future reference we will define the following $16 \otimes 16$ matrix $\widetilde{\Gamma}_5$:

$$\widetilde{\Gamma}_5 := \gamma_{\mathbf{10}} \prod_{\mathbf{m}=1}^{4} \gamma_{\mathbf{m}} \tag{14.13}$$

where $\gamma_{\mathbf{m}}$ are block gamma matrices from (14.11).

14.3 Space - cone basis and indices

In the following sections we will use the space-cone basis we introduce it for the gamma matrices we constructed in previous subsection. We first notice that the block gamma matrices $\gamma_{\mathbf{m}}$ in equation (14.11) could have either upper indices $(\gamma_{\mathbf{m}})^{\mu\nu}$ or lower indices $(\gamma_{\mathbf{m}})_{\mu\nu}$. From the construction it follows that those matrices are equal up to the sign.

In the equation (14.11) let us further divide the (either upper or lower) 16 dimensional index μ to $8 \oplus 8$ pieces (they are the $SO(8)$ chiral indices), thus we introduce $\mu := (\mu, \mu')$. In another words we want to look how the block $\gamma_{\mathbf{m}}$ matrices look in the $SO(8)$ (Majorana - Weyl) basis. Furthermore we introduce the following space-cone combinations of the equation (14.11) block gamma matrices:

$$(\gamma_+)_{\mu\nu} = \tfrac{1}{2}(\gamma_{\mathbf{10}} + \gamma_{\mathbf{9}})_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\mu'\nu'} \end{pmatrix} \;||\; (\gamma_-)_{\mu\nu} = \tfrac{1}{2}(\gamma_{\mathbf{10}} - \gamma_{\mathbf{9}})_{\mu\nu} = \begin{pmatrix} \delta_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \tag{14.14}$$

$$(\gamma_+)^{\mu\nu} = \tfrac{1}{2}(\gamma_{\mathbf{10}} + \gamma_{\mathbf{9}})^{\mu\nu} = \begin{pmatrix} \delta^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \;||\; (\gamma_-)^{\mu\nu} = \tfrac{1}{2}(\gamma_{\mathbf{10}} - \gamma_{\mathbf{9}})^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\mu'\nu'} \end{pmatrix} \tag{14.15}$$

For the convenience we also write the remaining gamma matrices using the $SO(8)$ indices:

$$(\gamma_{\mathbf{i}})_{\mu\nu} = \begin{pmatrix} 0 & (\widetilde{\gamma}_{\mathbf{i}})_{\mu\nu'} \\ (\widetilde{\gamma}_{\mathbf{i}})_{\mu'\nu} & 0 \end{pmatrix} \;||\; (\gamma_{\mathbf{i}})^{\mu\nu} = \begin{pmatrix} 0 & (\widetilde{\gamma}_{\mathbf{i}})^{\mu\nu'} \\ (\widetilde{\gamma}_{\mathbf{i}})^{\mu'\nu} & 0 \end{pmatrix} \tag{14.16}$$

where the $\widetilde{\gamma}_{\mathbf{i}}$ are the $SO(8)$ gamma matrices.

In the above introduced $8 \oplus 8$ basis the (14.13) looks like $\sigma_3 \otimes \mathbb{1}$ where $\mathbb{1}$ is the $8 \otimes 8$ unit matrix and σ_3 is the Pauli matrix.

Chapter 15

15 The AdS background in the T-dually extended superspace

15.1 Short review of the theory in curved background

In the treatment of the theory (of T-dually extended superspaces) the curved background is introduced via vielbeins $E_{\mathcal{A}}{}^{\mathcal{M}}(Z^{\mathcal{N}})$, see papers [1], [11], [12]. Note that the index $\mathcal{M}$ was introduced in table 6:

$$\triangleright_{\mathcal{A}} = E_{\mathcal{A}}{}^{\mathcal{M}}(Z^{\mathcal{N}}) \overset{\circ}{\triangleright}_{\mathcal{M}} \tag{15.1}$$

where $\overset{\circ}{\triangleright}_{\mathcal{M}}$ are generators of the flat algebra (14.7). The affine Lie algebra for the curved covariant derivatives $\triangleright_{\mathcal{A}}$ can be written as:

$$[\ \triangleright_{\mathcal{A}}, \triangleright_{\mathcal{C}}\ \} \equiv -i\eta_{\mathcal{AC}}\,\delta'(\,2\,-\,1\,) - iT_{\mathcal{AC}}{}^{\mathcal{E}}\ \triangleright_{\mathcal{E}}\ \delta(\,2\,-\,1\,) \tag{15.2}$$

where $T_{\mathcal{AC}}{}^{\mathcal{E}}$ is the superstring generalisation of the torsion, see [11]:

$$\begin{aligned} T_{\mathcal{AC}}{}^{\mathcal{E}} \ &= \ E_{[\mathcal{A}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{\mathcal{C})}{}^{\mathcal{N}})E^{-1\mathcal{E}}_{\mathcal{N}} + \tfrac{1}{2}\eta^{\mathcal{ED}}E_{\mathcal{D}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{[\mathcal{A}|}{}^{\mathcal{N}})E^{-1\mathcal{F}}_{\mathcal{N}}\eta_{\mathcal{F}|\mathcal{C})} \\ &+ E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{C}}{}^{\mathcal{N}}E^{-1\mathcal{E}}_{\mathcal{P}}f_{\mathcal{MN}}{}^{\mathcal{P}} \end{aligned} \tag{15.3}$$

where $[\,\mathcal{A}\,|\,|\,\mathcal{C}\,)$ indicates graded anti-symmetrization in only those indices. By $D_{\mathcal{M}}$ in (15.3) and in the whole next text we mean the group covariant derivatives of the (non-affine) part of algebra (14.7): $[\,D_{\mathcal{M}},\,D_{\mathcal{N}}\,\} \,=\, i\,f_{\mathcal{MN}}{}^{\mathcal{E}}\,D_{\mathcal{E}}$.

Note that the super-Jacobi identities imply the total graded antisymmetry of the torsion, just as for the structure constants. We can set the coefficient of the Schwinger term (the central charge in algebra (15.2)) to be the flat metric $\overset{\circ}{\eta}$ (now we rename it to η, to simplify notation). After that the vielbein is forced to obey the orthogonality constraints:

$$E_{\mathcal{A}}{}^{\mathcal{M}}\eta_{\mathcal{MN}}\,E_{\mathcal{C}}{}^{\mathcal{N}} \ \equiv\ \eta_{\mathcal{AC}} \tag{15.4}$$

This choice does not affect the physics, and simplifies many of the expressions. For example, it implies the total graded antisymmetry of the torsion, when the upper index is implicitly lowered with η:

$$T_{\mathcal{ABC}} \ =\ \tfrac{1}{2}E_{[\,\mathcal{A}\,|}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{|\,\mathcal{B}}{}^{\mathcal{N}})E_{\mathcal{C}\,)\,\mathcal{N}} \ +\ E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{B}}{}^{\mathcal{N}}E_{\mathcal{C}}{}^{\mathcal{P}}f_{\mathcal{MNP}} \tag{15.5}$$

where we have used $E^{-1}{}_{\mathcal{M}}{}^{\mathcal{A}} = \eta^{\mathcal{A}\,\mathcal{B}}\eta_{\mathcal{M}\,\mathcal{N}}E_{\mathcal{B}}{}^{\mathcal{N}}$. Also note that in the first term the graded anti-symmetrization can be written as a cyclic sum without the 1/2, since it is already graded antisymmetric in the last two indices. Because of orthogonality, the vielbein is like (the exponential of) a super 2-form, while the torsion is a super 3-form. Similarly the Bianchi identities are a super 4-form.

To solve the theory (in terms of pre-potential, to get physical fields and equation of motion) orthogonality condition (15.4) has to be solved explicitly (or at least at linearised level). Moreover there is a huge gauge group invariance that should be fixed:

$$\delta_{\Lambda}\,\triangleright_{\mathcal{A}} = [\,-\,i\,\Lambda,\,\triangleright_{\mathcal{A}}\,\} \qquad \text{where } \Lambda := \int d\,\sigma\,\lambda^{\mathcal{M}}\,(\,Z\,)\,D_{\mathcal{M}} \tag{15.6}$$

At the top of the orthogonality condition and the gauge invariance, we should include the torsion constraints. The torsion constraints are additional constraints on vielbein. They are imposed by putting some of the torsions in (15.5) to zero. Of course, not all torsions in (15.5) are zero. The relevant torsion constraints have been carefully analysed in [17]. All possible constraints on torsions are coming from curved space version of the **A B C D** (first class) constraints: **A** Virasoro, (string world-sheet) diffeomorphism constraints, **B** and **C** and **D** are the first class fermionic κ symmetry constraints, for further details see [9], [32], [33], [34], [35] and [17]. The rule of thumb is that at least the torsions of negative (10 dimensional) engineering dimension should be zero.

We will not try to solve the full nonlinear version of the theory. We linearise the theory around some background. In the papers [11], [12] we linearised around the flat background. In this paper however we linearise the theory around the $AdS_5 \times S^5$ solution of the classical supergravity (reformulated in the language of the doubled algebra).

After the linearisation we rewrite the orthogonality constraints (15.4) and torsions (15.5) using the vielbein expansion $E_{\mathcal{C}}{}^{\mathcal{D}} = \delta_{\mathcal{C}}{}^{\mathcal{D}} + E^{(1)}{}_{\mathcal{C}}{}^{\mathcal{D}} + \mathcal{O}\,(\,E^{(2)}\,)$. Let us for simplicity rename the first fluctuation $E^{(1)}{}_{\mathcal{C}}{}^{\mathcal{D}} \equiv H_{\mathcal{C}}{}^{\mathcal{D}}$. Then the equation (15.4) is just statement that: $H_{(\mathcal{C}}{}^{\mathcal{D}}\,\eta_{D\,|\,E]} \equiv H_{(\,C\,E\,]} = 0$ and the structure of linearised torsion (15.5):

$$\begin{aligned} T_{\mathcal{ABC}} \;&=\; f_{\mathcal{ABC}} \,+\, T^{(1)}{}_{\mathcal{ABC}} \,+\, \mathcal{O}\,(\,T^{(2)}\,) \\ \text{where} \qquad & T^{(1)}{}_{\mathcal{ABC}} \equiv \tfrac{1}{2}\,D_{[\mathcal{A}}\,H_{\mathcal{BC}\,)} \,+\, \tfrac{1}{2}\,H_{[\,\mathcal{A}}{}^{\mathcal{M}}\,f_{\mathcal{BC}\,)\,\mathcal{M}} \end{aligned} \tag{15.7}$$

15.2 $AdS_5 \times S^5$ background

In the expansion (15.7) we need to have the concrete structure constants $f_{\mathcal{A}\mathcal{B}\mathcal{C}}$ (i.e. vacuum values of torsions). We are interested in solving the theory (at least identifying the pre-potential) around this $AdS_5 \times S^5$ background. The relevant structure constants for the T-dually extended superspace in the context of the $AdS_5 \times S^5$ background were analysed in the last section of the paper [17]. We are repeating them here for the convenience. In the next section we will embed this $AdS_5 \times S^5$ version of T-dual algebra (see [17]) to a certain larger algebra that will be actually used in computations. The relevant non-vanishing $AdS_5 \times S^5$ torsions from [17] are:

$$
\begin{aligned}
\text{dim } 0: \quad & T_{S\,S\,\Sigma} = f_{S\,S\,\Sigma} \;||\; T_{S\,D\,\Omega} = f_{S\,D\,\Omega} \;||\; T_{S\,P\,P} = f_{S\,P\,P} \\
& T_{D\,D\,P} = f_{D\,D\,P} \\
\text{dim } 1: \quad & T_{D\,\widetilde{D}\,\Sigma} = R_{D\,\widetilde{D}\,\Sigma} \;||\; T_{P\,\widetilde{D}\,\Omega} = R_{P\,\widetilde{D}\,\Omega} \\
\text{dim } 2: \quad & T_{\Omega\,\Omega\,P} = R_{\Omega\,\Omega\,P} \;||\; T_{\Omega\,\Omega\,P} = R_{\Omega\,\Omega\,\widetilde{P}} \;||\; T_{P\,\widetilde{P}\,\Sigma} = R_{P\,\widetilde{P}\,\Sigma} \\
\text{dim } 3: \quad & T_{\Omega\,\widetilde{\Omega}\,\Sigma} = R_{\Omega\,\widetilde{\Omega}\,\Sigma}
\end{aligned}
\tag{15.8}
$$

note that the left and right index notation was introduced in table 6.

The $f_{\mathcal{A}\mathcal{B}\mathcal{C}}$ in (15.8) are usual flat superspace structure constants for the (flat) T-dually extended superspace with the (common) local Lorentz group $SO(4,1) \otimes SO(5)$. The nontrivial curvatures from table (15.8), for example $R_{D\,\widetilde{D}\,\Sigma}, R_{P\,\widetilde{D}\,\Omega}$, etc. are defined using the dimension 1 torsion $T_{P\,\widetilde{D}\,\Omega}$. The $T_{P\,\widetilde{D}\,\Omega} \equiv T_{\mathbf{a}\,\widetilde{\alpha}}{}^{\beta} = \gamma_{\mathbf{a}\;\alpha\sigma}\,F^{\beta\,\widetilde{\sigma}}$, where the R-R field strength $F_{\Omega\,\widetilde{\Omega}} \equiv F^{\alpha\,\widetilde{\beta}} = \frac{1}{r_{AdS}}\,(\widetilde{\Gamma}_5)^{\alpha\,\beta}$. Note that the $\widetilde{\Gamma}_5$ was defined in (14.13) and the new parameter r_{AdS} is the AdS_5 space radius (note, $r_{AdS} = r_S$, i.e. the radius of S^5 is the same of AdS_5, so that Ricci scalar $R = 0$). More specifically some of the table (15.8) curvatures:

$$
\begin{aligned}
\text{dim } 1: \quad & R_{D\,\widetilde{D}\,\Sigma} \equiv R_{\alpha\,\widetilde{\beta}}{}^{\mathbf{a}\,\mathbf{b}} = T_{\widetilde{\beta}}{}^{\sigma\,[\mathbf{a}}\,\gamma^{\mathbf{b}]}{}_{\alpha\,\sigma} && (15.9)\\
\text{dim } 1: \quad & R_{P\,\widetilde{D}\,\Omega} \equiv R_{\mathbf{a}\,\widetilde{\alpha}}{}^{\beta} = T_{\mathbf{a}\,\widetilde{\alpha}}{}^{\beta} = \gamma_{\mathbf{a}\;\alpha\,\sigma}\,F^{\beta\,\widetilde{\sigma}} \\
\text{dim } 2: \quad & R_{P\,\widetilde{P}\,\Sigma} \equiv R_{\mathbf{a}\,\widetilde{\mathbf{b}}}{}^{\mathbf{cd}} \propto T_{\widetilde{\mathbf{b}}\,\alpha}{}^{\widetilde{\beta}}\,R_{\widetilde{\beta}\,\gamma}{}^{\mathbf{cd}}\,\gamma_{\mathbf{a}}{}^{\alpha\,\gamma} \\
\text{dim } 3: \quad & R_{\Omega\,\widetilde{\Omega}\,\Sigma} \equiv R^{\alpha\,\widetilde{\beta}\;\mathbf{ab}} \propto \left(T^{\widetilde{\mathbf{d}}\,\widetilde{\beta}}{}_{\sigma}\,R_{\widetilde{\mathbf{d}}\,\mathbf{e}}{}^{\mathbf{ab}} + T_{\mathbf{e}}{}^{\widetilde{\beta}\,\nu}\,R_{\widetilde{\sigma}\,\nu}{}^{\mathbf{ab}}\right)\gamma^{\mathbf{e}\;\sigma\,\alpha}
\end{aligned}
$$

where the dim 2 curvature is proportional with the constant $2^{-\frac{D}{2}+1}$ and the dim 3 curvature is proportional with a constant D (where D is 10 in our case).

All the other curvatures in (15.9) are obtained using the appropriate Bianchi identities (one can obtain all curvatures from $T_{P\,\widetilde{D}\,\Omega}$ using Bianchi identities). We note that the torsions (15.8) and curvatures (15.9) are consistent with torsions and curvatures given in the [31].

15.3 Extended $AdS_5 \times S^5$ T-dual algebra

To identify the pre-potential in the generalised vielbeins, i.e. solving the spectrum of the theory (on linearised level) we want to proceed as described in earlier papers [17], [11], [12]. There the vielbeins were introduced as in (15.1) and linearised as above the equation (15.7). This procedure means the expansion of generally curved superspace around some (in those papers just a flat) background. Moreover the gauge invariance was completely fixed (in referenced papers the covariant gauge was considered) and after that the pre-potential was identified as a part of vielbein (acting on by derivatives, the physical spectrum is produced).

Here we want to proceed in similar way. We want to introduce the vielbeins and linearise the theory around the $AdS_5 \times S^5$ background. We have tried to use solely the algebra described in the previous sub-section, i.e. to take the algebra (15.8) and introduce the vielbeins, gauge fix and linearise. Even though we still believe that the pre-potential is sitting in that theory in some combination of vielbeins, it was not easy to identify it. The reason was that to identify the pre-potential we need to find a scalar contraction of some linear combination of vielbeins that is anihilated by the D_v and $D_{\bar{v}}$ operators. The D_v and $D_{\bar{v}}$ are certain combinations of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ (see the index notation above the (14.14), i.e. they are a particular chiral part of the $SO\,(\,8\,)$ chiral decomposition of the 16 supersymmetry translations D_α defined at the beginning, see (14.7) and section above (14.14)).

Because the metric is in $H_{P\,\widetilde{P}}$ vielbein, we expected the pre-potential to be (at least a part of it) in $\mathrm{Tr}\, H_{P\,\widetilde{P}}$. The problem with $\mathrm{Tr}\, H_{P\,\widetilde{P}}$ is that it already has dimension 0. To show that it is annihilated by D_v and $D_{\bar{v}}$ operators we would need to use torsion constraints of dimension $\frac{1}{2}$. Moreover we also know that the pre-potential has to be annihilated by the suitably defined $\mathcal{P}_+$ operator ($\mathcal{P}_+ \propto (\,P_+ + P_{\widetilde{+}}\,)$ where $P_+ \equiv D_{P_+}$ and $P_{\widetilde{+}} \equiv D_{P_{\widetilde{+}}}$), in a light cone basis introduced in (14.14) and in the near horizon limit (defined later)). The high dimensionality of $\mathrm{Tr}\, H_{P\,\widetilde{P}}$ then requires to use at least dimension 1 torsion constraints to prove that $\mathcal{P}_+$ vanishes (in the near horizon limit). That seemed to be problematic to analyse in the theory based just on the algebra (15.8) and (15.9) because of the degauging procedure. The degauging appears since the theory coming from algebras (15.8) and (15.9) is really coming from the full $SO(10) \otimes SO(10)$ so there are some missing Lorentz connections. By simple dimensional analysis it is evident that the missing Lorentz connections are first appearing at dimension 1 (for example the appearance of the full H_{PS} in the dim 1 torsion $T_{PPP} \propto \ldots + H_{[P|S} f_{PP]\Sigma} + \ldots$).

For that reason we extended the algebra (15.8) to include the original (Wick rotated) local Lorentz group $SO(10) \otimes SO(10)$. All the other structure constants and curvatures in (15.9) and (15.8) stay the same. Except now we have separate left local Lorentz $S_{\mathbf{ab}}$ generator together with the right local Lorentz generator $S_{\widetilde{\mathbf{ab}}}$, where $\mathbf{a} \in \{1, \ldots 10\}$. The common (Wick rotated) $SO(5) \otimes SO(5)$ Lorentz group of original $AdS_5 \times S^5$ algebra (15.8) is then the subgroup in the diagonal $SO(10)$ subgroup of $SO(10) \otimes SO(10)$. The extension procedure can be viewed from the different picture. We could start with the full 10 dimensional string superspace as introduced in [1] and [16]. Then introduce the curved version of that space via vielbeins and then linearise around some background as described earlier. The extension of (15.8) and (15.9) to full $SO(10) \otimes SO(10)$ is then just a choice of some particular background that is consistent with the original $AdS_5 \times S^5$. This has an advantage that now we have a natural place where to put the troubling (part) of the pre-potential $\mathrm{Tr}\, H_{P\widetilde{P}}$. Because of the additional $S_{\mathbf{ab}}$ and $S_{\widetilde{\mathbf{ab}}}$ we can have $\mathrm{Tr}\, H_{+\mathbf{a}\widetilde{+\mathbf{b}}} \equiv \mathrm{Tr}\, H_{S\widetilde{S}}$ instead of the $\mathrm{Tr}\, H_{P\widetilde{P}}$. The $\mathrm{Tr}\, H_{P\widetilde{P}} \equiv H_{\mathbf{a}\widetilde{\mathbf{b}}}$ is related to $\mathrm{Tr}\, H_{S\widetilde{S}} \equiv \mathrm{Tr}\, H_{+\mathbf{a}\widetilde{+\mathbf{b}}}$ by an action of $D_{P_-} \equiv P_-$ and $D_{P_{\widetilde{-}}} \equiv P_{\widetilde{-}}$ (they are both invertible operators). The vielbein $H_{S\widetilde{S}}$ is of the dimension -2 and so there is no need to use the higher dimensional torsion constraints. Moreover in the full $SO(10) \otimes SO(10)$ theory we do not have to do the degauging procedure.

This extension comes with the cost. The mixed pieces of the AdS algebra (15.9) are breaking the explicit $SO(10) \otimes SO(10)$ invariance (they are not the $SO(10) \otimes SO(10)$ invariant tensors). We still have present the full $D_S \equiv S$ generators. Those derivatives could hit the (non-invariant) curvatures. The solution of this is to keep the explicit mixed curvature dependence (as generic mixed curvatures) till the S derivatives are not being explicitly evaluated. We will describe this procedure in detail later.

Chapter 16

16 Gauge fixing

We want to fix the space-cone gauge (T-dual super space-cone gauge) for the first fluctuation $H_{\mathcal{AB}}$, i.e. like in the usual light-cone we have $D_{P_-} \equiv P_-$ operator invertible, now we have P_- and $P_{\tilde{-}}$ invertible (where $D_{P_{\tilde{-}}} \equiv P_{\tilde{-}}$).

First we look at the gauge variation (15.6) more closely and at the linearised level:

$$\delta_\Lambda H_{\mathcal{AB}} = D_{[\mathcal{A}}\lambda_{\mathcal{B})} + f_{\mathcal{AB}}{}^{\mathcal{C}}\lambda_{\mathcal{C}} \tag{16.1}$$

In the light-cone gauge we in general pick a vielbein with an P_- or $P_{\tilde{-}}$ index, put that vielbein to zero. In order to maintain that gauge we need to fix the particular gauge parameter. For simplicity we call $P_- \equiv -$ and $P_{\tilde{-}} \equiv \tilde{-}$ then:

$$\begin{aligned} H_{-\mathcal{A}} &= 0 \;\Rightarrow\; \delta_\Lambda H_{-\mathcal{A}} = 0 \\ \Rightarrow &\quad P_-\lambda_{\mathcal{A}} - D_{\mathcal{A}}\lambda_- + f_{-\mathcal{A}}{}^{\mathcal{C}}\lambda_{\mathcal{C}} = 0 \\ \text{then} &\quad \lambda_{\mathcal{A}} = \tfrac{1}{P_-}\left(D_{\mathcal{A}}\lambda_- + f_{-\mathcal{A}}{}^{\mathcal{C}}\lambda_{\mathcal{C}}\right) \end{aligned} \tag{16.2}$$

Note that there are more possibilities to fix the particular gauge parameters $\lambda_{\mathcal{A}}$. To fix $\lambda_{\mathcal{A}}$ we could also put $H_{\tilde{-}\mathcal{A}} = 0$ and use the invertibility of $P_{\tilde{-}}$. Of course we can not fix some gauge parameter twice. We have to decide which vielbeins we are going to fix in this "double" light-cone gauge.

We picked the approach where we used the mixed vielbeins to vanish by the double light-cone gauge fixing, i.e. we put $H_{\tilde{-}\mathcal{A}} = 0$ for $\mathcal{A} \in \{S, D, P, \Omega, \Sigma\} \equiv$ left part of algebra. Together with $H_{-\tilde{\mathcal{A}}} = 0$ for $\tilde{\mathcal{A}} \in \{\tilde{S}, \tilde{D}, \tilde{P}, \tilde{\Omega}, \tilde{\Sigma}\} \equiv$ right part of algebra. By that choice we fully fix the gauge parameters $\lambda_{\mathcal{A}}$ and $\lambda_{\tilde{\mathcal{A}}}$ in terms of λ_-. That parameter can be fixed by the gauge invariance of the gauge invariance.

The motivation for the previously described mixed left right light-cone gauge fixing came from the flat space (just the extended $AdS_5 \times S^5$ space with $r_{AdS} \to \infty$, i.e. the flat $SO(10) \otimes SO(10)$ T-dual superspace). After picking this type of the light-cone gauge the mixed torsion constraints of the type $T_{\tilde{-}\mathcal{AB}} = 0$ are as algebraic as possible:

$$\begin{aligned} T_{\tilde{-}\mathcal{AB}} &= P_{\tilde{-}} H_{\mathcal{AB}} + D_{\mathcal{B}} H_{\tilde{-}\mathcal{A}} + D_{\mathcal{A}} H_{\mathcal{B}\tilde{-}} + H_{\tilde{-}\mathcal{M}}\eta^{\mathcal{MN}} f_{\mathcal{ABN}} \\ &\quad + H_{[A|\mathcal{M}}\eta^{\mathcal{MN}} f_{\mathcal{B})\tilde{-}\mathcal{N}} \end{aligned} \tag{16.3}$$

$$T_{\tilde{-}\mathcal{AB}} = P_{\tilde{-}} H_{\mathcal{AB}} = 0 \;\Rightarrow\; H_{\mathcal{AB}} = 0 \text{ for } \mathcal{A}, \mathcal{B} \,:\, T_{\tilde{-}\mathcal{AB}} = 0 \tag{16.4}$$

where we used our mixed light-cone gauge and $r_{AdS} \to \infty$ of extended algebra in (16.4).

The same as in (16.3) and (16.4) holds if one fully swaps left and right indices. For finite r_{AdS} we can have the mixed structure constants nonzero (i.e. $f_{\mathcal{A}\mathcal{B}}{}^{\tilde{\mathcal{-}}} \neq 0$) and so we would have a right hand side in (16.4). Note also that there could be the contribution from S derivatives hitting the mixed structure constants. Even though the right hand side in (16.4) is not generally vanishing for finite r_{AdS} we found that the mixed left-right light-cone gauge is still useful in the $AdS_5 \times S^5$ case.

Chapter 17

17 Torsion constraints

17.1 $AdS_5 \times S^5$ curvatures and D_S derivatives

As we noted in the introduction section. Because we have enhanced our superspace, we have to take special care when the local Lorentz derivatives $D_S \equiv S$ are hitting the mixed curvatures (15.9). This problem arises because the curvatures in (15.9) are not full $SO(10) \otimes SO(10)$ invariant. The solution is to keep the non-invariant torsions (15.9) generic and explicitly act by the $D_S \equiv S$ derivatives on those torsions. Only after this explicit S action we can evaluate those torsions (or curvatures) and be fixed as in (15.9).

Let's take an example, from the equation (16.3) we can see that in the $AdS_5 \times S^5$ case in the mixed light-cone gauge the $H_{\mathcal{AB}}$ is determined as:

$$H_{\mathcal{AB}} = \frac{1}{P_{\underset{\sim}{-}}} H_{[\mathcal{A}|\mathcal{M}} \eta^{\mathcal{MN}} f_{\mathcal{B})\underset{\sim}{-}\mathcal{N}} \tag{17.1}$$

In many instances in the following chapters we use similar relation as in (17.1) to fix some particular vielbein in terms of another vielbeins. If all structure constants f would be $SO(10) \otimes SO(10)$ invariant tensors then there is not a problem and we can treat the f structure constants as genuine constants also with respect to the S derivatives. In our case however the mixed f structure constants (that we call also the curvatures) in (15.9) explicitly break the $SO(10) \otimes SO(10)$ local Lorentz invariance down to the $SO(5) \otimes SO(5)$ local Lorentz (as it should be in the $AdS_5 \times S^5$ case). One possibility is to restrict our superspace local Lorentz invariance (the S derivatives) to $SO(5) \otimes SO(5)$. Then we would return back to the $PSU(2,2|4)$ that we wanted to avoid in the first place (in order to have $H_{\mathbf{+a}\widetilde{\mathbf{+b}}}$ instead of $H_{\mathbf{a}\tilde{\mathbf{b}}}$). The alternative, that we picked, is to work with the full $SO(10) \otimes SO(10)$ local Lorentz group. But then the structure constants that are breaking that invariance are not invariant tensors and so the action of those S derivatives on the mixed structure constants has to be accounted for. So we should keep the mixed f structure constants and when needed explicitly act by the S derivatives on them. We will evaluate them as the very last step in our calculations. Let's look at the example in (17.1) and look at the action of

$S_{\widetilde{+\mathbf{a}}}$, where $\mathbf{a} \in \{1 \ldots 8\}$:

$$
\begin{aligned}
S_{\widetilde{+\mathbf{a}}}\, H_{\mathcal{A}\mathcal{B}} \;&=\; S_{\widetilde{+\mathbf{a}}}\left(\tfrac{1}{P_{\widetilde{-}}}\, H_{[\mathcal{A}|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\right) f_{\mathcal{B})\,\widetilde{-}\,\mathcal{N}} \\
&\quad + \tfrac{1}{P_{\widetilde{-}}}\, H_{[\mathcal{A}|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\left(S_{\widetilde{+\mathbf{a}}}\, f_{\mathcal{B})\,\widetilde{-}\,\mathcal{N}}\right) \\
&=\; S_{\widetilde{+\mathbf{a}}}\left(\tfrac{1}{P_{\widetilde{-}}}\, H_{[\mathcal{A}|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\right) f_{\mathcal{B})\,\widetilde{-}\,\mathcal{N}} \;+\; \eta_{+-}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{[\mathcal{A}|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, f_{\mathcal{B})\,\widetilde{\mathbf{a}}\,\mathcal{N}}
\end{aligned}
\tag{17.2}
$$

We will evaluate the $f_{\mathcal{B}\,\widetilde{\mathbf{a}}\,\mathcal{N}}$ in the second equation just after all the (possibly future) S derivatives have already acted. We also should bear in mind that whenever we are acting by the S derivative on some vielbein (that is determined by another vielbeins) there might be the above described issue. The second term can (and it will) nontrivially contribute to our calculations.

17.2 Torsion constraints and $H_{S\,S}$ vielbein

The torsion constraints are (mainly) given by the curved version of the **A B C D** (first class) constraints, see [9] and [17]. There are further constraints called $\widetilde{T}_{\mathcal{A}} = 0$ coming from requirement of partial integration in the presence of the dilaton measure, see [1], and [12]. There is also a strong constraint: on every field in the double field theory one has to require $D^{\mathcal{A}}\, D_{\mathcal{A}} = 0$. There are also a dimensional reduction constraints, as we see later.

Our aim is to analyse the necessary constraints consistent with the above constraints by which we can identify the pre-potential. Following the analysis given in [8], we identify the pre-potential as a scalar super-field (given by some super-trace of possibly a combination of vielbeins), that is annihilated by certain combination of the $D_{\nu'}$ and $D_{\widetilde{\nu}'}$. The precise combination of $D_{\nu'}$ and $D_{\widetilde{\nu}'}$ is also going to be determined from the constraints.

As usual, we start to eliminate the lowest dimensional vielbeins. The vielbeins of the lowest dimension are $H_{S\,S}$, $H_{S\,\widetilde{S}}$, $H_{\widetilde{S}\,\widetilde{S}}$. They are of the dimension -2 (we mean the ten dimensional dimension). Using equations (16.3) and (16.4) for indices $\mathcal{A} = S$ and $\mathcal{B} = S$ (also after change left $\leftrightarrow$ right) we immediately get that $H_{S\,S} = 0 = H_{\widetilde{S}\,\widetilde{S}}$. Note, that even in the extended $AdS_5 \times S^5$ superspace the structure constant $f_{S\,\widetilde{\mathcal{A}}\,\mathcal{B}} = 0 \rightarrow f_{S\,\widetilde{-}\,\mathcal{B}} = 0$.

We mention an important observation that will help us simplify future calculations. As we saw in previous sub-chapter, we should keep the mixed structure constants generic and evaluate them at the end. Note however, that the mixed structure constants with the S indices are always zero (like the one we considered here: $f_{\mathcal{S}\widetilde{\mathcal{A}}\mathcal{B}}$). The action of S derivatives on them results in the mixed structure constants again with the S index and such are zero after the evaluation. So specifically, we can evaluate the mixed structure constants with the S indices to zero even before acting by S derivatives on them.

The mixed vielbein $H_{S\,\widetilde{S}}$ is not all zero and the claim is that the part of the pre-potential is in this particular vielbein. To see which parts are possibly nonzero we rewrite the $H_{S\,\widetilde{S}}$ in the double light-cone components:

$$\begin{aligned} H_{S\,\widetilde{S}} \;\equiv\; & \{\, H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}},\, H_{+\mathbf{a}\,\widetilde{-\mathbf{b}}},\, H_{-\mathbf{a}\widetilde{-\mathbf{b}}},\, H_{+\mathbf{a}\,\widetilde{\mathbf{bc}}},\, H_{-\mathbf{a}\,\widetilde{\mathbf{bc}}},\, H_{\mathbf{ab}\,\widetilde{\mathbf{cd}}},\, H_{+-\,\widetilde{+-}},\, H_{+-\,\widetilde{+\mathbf{a}}} \\ & H_{+-\,\widetilde{-\mathbf{a}}},\, H_{+-\,\widetilde{\mathbf{ab}}} \} \oplus \text{ swap} \end{aligned} \tag{17.3}$$

where we might swap left index with the right index in (17.3). Also note that in all previous we have $\mathbf{a} \in \{\,1, \ldots, 8\,\}$. We remind that $P_{+} \equiv + \propto 10 + 9$ and $P_{-} \equiv - \propto 10 - 9$.

We want to use analog of equations (16.3) and (16.4) for the mixed $H_{S\,\widetilde{S}}$ and $r_{AdS} \neq \infty$:

$$\begin{aligned} T_{\tilde{-}\,S\,\widetilde{S}} = 0 \;\; = \;\; & P_{\tilde{-}}\, H_{S\,\widetilde{S}} + D_{\widetilde{S}}\, H_{\tilde{-}\,S} + D_{S}\, H_{\widetilde{S}\,\tilde{-}} \\ & + H_{[\,\tilde{-}|\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{S\,\widetilde{S}]\,\mathcal{N}} \end{aligned} \tag{17.4}$$

In equation (17.5) we have term $H_{\tilde{-}\,S}$ zero by gauge choice. The vielbein $H_{\widetilde{S}\,\tilde{-}}$ is proportional to $f_{S\,\tilde{-}\,\mathcal{N}}$. We see that this term is zero after evaluating $f_{S\,\tilde{-}\,\mathcal{N}} = 0$ by (15.9). Using the (16.3) and (16.4) for $\mathcal{A} = S$ and $\mathcal{B} = -$ and keeping $f_{-\,\tilde{-}\,S}$ nonzero, we get $H_{\mathcal{S}\,-} = \frac{1}{P_{\tilde{-}}}\, f_{-\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, H_{S\,\mathcal{N}}$ and similarly for $H_{\widetilde{\mathcal{S}}\,\tilde{-}}$ and so the third term in (17.5) is also fixed. The (17.5) is then:

$$T_{\tilde{-}\,S\,\widetilde{S}} = 0 = P_{\tilde{-}}\, H_{S\,\widetilde{S}} + D_{S}\Big(\tfrac{1}{P_{-}}\, f_{-\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, H_{\widetilde{S}\,\mathcal{N}} \Big) + H_{S\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\widetilde{S}\,\tilde{-}\,\mathcal{N}} \tag{17.5}$$

Using equations (16.3), (16.4), (17.5) and the mixed light-cone gauge together with keeping the mixed structure constants and evaluating the explicit actions of the S and $\widetilde{S}$ derivatives we get the first important result for the structure of the $H_{\mathcal{S}\widetilde{\mathcal{S}}}$ vielbein (in the AdS case), see table (9) in Appendices. From the table (9) we can see that the possibly nonzero $H_{S\,\widetilde{S}}$ in (17.5) are those for which $f_{\widetilde{S}\,\tilde{-}\,\mathcal{N}}$ are nonzero (after evaluation of mixed structure constants). By simple left $\leftrightarrow$ right swap we get that for $H_{S\,\widetilde{S}}$ to be nonzero also $f_{S\,-\,\mathcal{N}}$ has to be nonzero. From the $[\,S,\, P\,]$ part of $SO\,(\,10\,) \otimes SO\,(\,10\,)$

extended algebra of (14.7) we can see the only possibility: $H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}} \neq 0$. All the other components of $H_{S\,\widetilde{S}} = 0$ by (17.5) and table (9) after evaluation. This is a first hint that we are on the right track. The $H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}} \neq 0$ is the only nonzero part (after evaluation of mixed structure constants) of $H_{S\,\widetilde{S}}$ piece, it has a scalar trace and we can easily relate it to $H_{\mathbf{a}\,\widetilde{\mathbf{b}}} \equiv H_{P\,\widetilde{P}}$, where we expect the part of the pre-potential to be (the symmetric part corresponds to the metric).

To see how $H_{P\,\widetilde{P}}$ is related to $H_{S\,\widetilde{S}}$ consider the third relation from table (9) and after evaluation of mixed structure constants:

$$(1 + \tfrac{1}{2\,(r_{AdS})^2}\,\tfrac{1}{P_-\,P_{\widetilde{-}}})\, H_{+\mathbf{a}\widetilde{+\mathbf{b}}} = \tfrac{1}{P_-}\, H_{\mathbf{a}\,\widetilde{+\mathbf{b}}} \tag{17.6}$$

We want to reduce $H_{\mathbf{a}\,\widetilde{+\mathbf{b}}}$ further to get $H_{\mathbf{a}\,\widetilde{\mathbf{b}}}$. One can naïvely expect to just hit $H_{\mathbf{a}\,\widetilde{+\mathbf{b}}}$ with $P_{\widetilde{-}}$ to get rid of the $\widetilde{S}$ index (or alternatively hit by P_- the $H_{+\mathbf{a}\,\widetilde{\mathbf{b}}}$). It works but one has to be more careful since in the $AdS_5 \times S^5$ space one has the mixed structure constant $f_{-\,\widetilde{\mathbf{b}}\mathcal{N}} \neq 0$. To see what is this structure constant (after the mixed structure constants evaluation) we remind that in (15.9) we saw that dimension 2 structure constant is given as $R_{P\,\widetilde{P}\,\Sigma} \equiv R_{\mathbf{a}\,\widetilde{\mathbf{b}}}{}^{\mathbf{cd}} \propto T_{\widetilde{\mathbf{b}}\,\alpha}{}^{\widetilde{\beta}}\, R_{\widetilde{\beta}\,\gamma}{}^{\mathbf{cd}}\,\gamma_{\mathbf{a}}{}^{\alpha\,\gamma}$. We have to be careful with the indices in the $R_{\mathbf{a}\,\widetilde{\mathbf{b}}}{}^{\mathbf{cd}}$. The Σ indices $^{\mathbf{cd}}$ in (15.9) were indices for $SO(5) \otimes SO(4,1)$ local Lorentz group (or its Wick rotated version $SO(5) \otimes SO(5)$). But the index $\mathcal{N}$ in $f_{-\,\widetilde{\mathbf{b}}\mathcal{N}} \neq 0$ includes the indices for the full $SO(10) \otimes SO(10)$. We have already made the claim that the original local Lorentz group $SO(5) \otimes SO(5)$ is in the *diagonal* subgroup of the $SO(10) \otimes SO(10)$, i.e. we have the following (at the level of algebras) $so(10) \oplus so(10) \equiv \frac{1}{2}(so(10) + so(10)) \oplus \frac{1}{2}(so(10) - so(10)) := so(10)_{\mathbf{D}} \oplus so(10)_{\mathbf{Off}}$. (The meaning of previous is to do the operations on basis. The $so(10) - so(10)$ means for example to combine e.g. Lorentz generators like: $S - \widetilde{S} = S_{\mathbf{Off}}$ and similarly for another generators). Now the $so(5) \oplus so(5) \hookrightarrow so(10)_{\mathbf{D}}$. Let us write the last sequence of algebras more precisely. Using indices, the diagonal subgroup (subalgebra) is $so(10)_{\mathbf{D}} \equiv S^{\mathbf{D}}{}_{\mathbf{ab}} := \frac{1}{2}(S_{\mathbf{ab}} + S_{\widetilde{\mathbf{ab}}}) = (S^{\mathbf{D}}{}_{\mathbf{ij}}, S^{\mathbf{D}}{}_{\mathbf{kl}}, S^{\mathbf{D}}{}_{\mathbf{ik}})$, where $\mathbf{i} \in \{10, 1, 2, 3, 4\}$ and $\mathbf{k} \in \{5, 6, 7, 8, 9\}$ and the $\mathbf{a} = (\mathbf{i}, \mathbf{k}) \equiv \{1, \ldots, 10\}$. The $S^{\mathbf{D}}{}_{\mathbf{ij}}$ and $S^{\mathbf{D}}{}_{\mathbf{kl}}$ are the generators of the $SO(5) \otimes SO(5)$. Moreover, the previous definitions give precise embedding of those operators.

Defining $S^{\mathbf{D}}{}_{\mathbf{ij}}$ and $S^{\mathbf{D}}{}_{\mathbf{kl}}$ we can see that the structure constant $f_{\mathbf{a}\,\widetilde{\mathbf{b}}\mathcal{N}}$ is either 0 or given by the appropriate $R_{P\,\widetilde{P}\,\Sigma_{\mathbf{D}}}$. We included a small subindex to the Σ coordinate just to remind us that the Σ coordinate is now for the $SO(5) \otimes SO(5)$ diagonal subgroup of $SO(10)_{\mathbf{D}}$ only.

Finally, taking definitions of $R_{\mathbf{a}\widetilde{\mathbf{b}}}{}^{\mathbf{cd}}$ and table (15.9) and our definitions we can see that $f_{\mathbf{a}\widetilde{\mathbf{b}}\mathcal{N}}$ is coming from the mixed commutator:

$$[P_{\mathbf{a}}, P_{\widetilde{\mathbf{b}}}] \propto \begin{cases} (\frac{1}{r_{AdS}})^2 S^{\mathbf{D}}{}_{\mathbf{ab}} & \text{if } \mathbf{a} \,\&\, \mathbf{b} \in \{10, 1, 2, 3, 4\} \\ -(\frac{1}{r_{AdS}})^2 S^{\mathbf{D}}{}_{\mathbf{ab}} & \text{if } \mathbf{a} \,\&\, \mathbf{b} \in \{5, 6, 7, 8, 9\} \\ 0 & \text{otherwise} \end{cases} \tag{17.7}$$

The proportionality constant is $\mathbf{c}_1 = -2$. With the previous definition and with $P_- \equiv - = \frac{1}{2}(10 - 9)$ and $\widetilde{\mathbf{b}} \in \{5, 6, 7, 8, 9\}$ we will get the $[P_-, P_{\widetilde{\mathbf{b}}}] \propto S^{\mathbf{D}}{}_{\mathbf{9b}} = \frac{1}{2}(S_{\mathbf{9b}} + S_{\widetilde{\mathbf{9b}}})$ we can also use $P_{\mathbf{9}} = (P_+ - P_-)$ and so $[P_-, P_{\widetilde{\mathbf{b}}}] \propto (S_{+\mathbf{b}} + S_{\widetilde{+\mathbf{b}}} - S_{-\mathbf{b}} - S_{\widetilde{-\mathbf{b}}})$. Knowing the last relation we can proceed and hit the result of (17.6) by $P_{\tilde{-}}$ and relate $H_{+\mathbf{a}\widetilde{+\mathbf{b}}}$ with $H_{\mathbf{a}\widetilde{\mathbf{b}}}$ for the evaluated version. For non-evaluated version we need to do the same for the non-evaluated version of (17.6), that is the third top equation in table (9). For vielbein $H_{\widetilde{+\mathbf{b}}\mathbf{a}}$ needed in this procedure we get:

$$\begin{aligned} 0 = T_{P\widetilde{P}\widetilde{S}} \equiv T_{\mathbf{a}\tilde{-}\widetilde{+\mathbf{b}}} &= P_{[\mathbf{a}} H_{\tilde{-}\widetilde{+\mathbf{b}}]} + H_{[\mathbf{a}|\mathcal{M}} \eta^{\mathcal{MN}} f_{\tilde{-}\widetilde{+\mathbf{b}}]\mathcal{N}} \\ &= -P_{\tilde{-}} H_{\widetilde{+\mathbf{b}}\mathbf{a}} - P_{\mathbf{a}} H_{\tilde{-}\widetilde{+\mathbf{b}}} - H_{\widetilde{+\mathbf{b}}\mathcal{M}} \eta^{\mathcal{MN}} f_{\mathbf{a}\tilde{-}\mathcal{N}} \\ &\quad - H_{\mathbf{a}\mathcal{M}} \eta^{\mathcal{MN}} f_{\tilde{-}\widetilde{+\mathbf{b}}\mathcal{N}} \end{aligned} \tag{17.8}$$

The term $H_{\tilde{-}\mathbf{a}}$ vanishes because of mixed light-cone gauge, the term $H_{\tilde{-}\widetilde{+\mathbf{b}}}$ is fixed by the torsion $T_{-\tilde{-}\widetilde{+\mathbf{b}}} = 0$ and use of mixed light-cone gauge similarly as in equation (A.1). By that we get:

$$H_{\tilde{-}\widetilde{+\mathbf{b}}} = -f_{-\tilde{-}\mathcal{M}} \eta^{\mathcal{MN}} \tfrac{1}{P_-} H_{\widetilde{+\mathbf{b}}\mathcal{N}} \rightsquigarrow 0 \tag{17.9}$$

The last term in (17.8) is just $\eta_{-+}\eta_{\mathbf{a}\mathcal{N}}$, the analog term as in (17.6). The extra mixed term (after evaluation) in (17.8) is:

$$H_{+\mathbf{a}\mathcal{M}} \eta^{\mathcal{MN}} f_{\widetilde{\mathbf{b}}-\mathcal{N}} \propto (\tfrac{1}{r_{AdS}})^2 H_{+\mathbf{a}\widetilde{+\mathbf{b}}} \tag{17.10}$$

Plugging (17.9) into (17.8) and then the result (that is the fixed vielbein $H_{\widetilde{+\mathbf{b}}\mathbf{a}}$) into the third top equation in table (9) we obtain the non-evaluated relation between $H_{+\mathbf{a}\widetilde{+\mathbf{b}}}$:

$$\begin{aligned} P_- H_{+\mathbf{a}\widetilde{+\mathbf{b}}} &= -f_{-\tilde{-}\mathcal{M}} \eta^{\mathcal{MN}} S_{\widetilde{+\mathbf{b}}} (\tfrac{1}{P_{\tilde{-}}} H_{+\mathbf{a}\mathcal{N}}) - f_{-\widetilde{\mathbf{b}}\mathcal{M}} \eta^{\mathcal{MN}} \tfrac{1}{P_{\tilde{-}}} H_{+\mathbf{a}\mathcal{N}} \\ &= -f_{-\tilde{-}\mathcal{M}} \eta^{\mathcal{MN}} \tfrac{1}{P_{\tilde{-}}} P_{\mathbf{a}} \tfrac{1}{P_-} H_{\widetilde{+\mathbf{b}}\mathcal{N}} + f_{\tilde{-}\mathbf{a}\mathcal{M}} \eta^{\mathcal{MN}} \tfrac{1}{P_{\tilde{-}}} H_{\widetilde{+\mathbf{b}}\mathcal{N}} \\ &\quad + \tfrac{1}{P_{\tilde{-}}} H_{\mathbf{a}\widetilde{\mathbf{b}}} \end{aligned} \tag{17.11}$$

after evaluation of the mixed structure constants in (17.11) we get:

$$H_{\mathbf{a}\,\widetilde{\mathbf{b}}} \;=\; \Big(\,P_{-}\,P_{\widetilde{-}} \;+\; \tfrac{1}{(r_{AdS})^2}\,\Big)\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \tag{17.12}$$

17.3 Torsion constraints and $H_{D\,S}$ vielbein

To identify what combination of vielbeins gives the pre-potential, we first repeat the properties we are looking for. We are looking for combination of vielbeins (of the low dimension), that has a scalar contraction and is annihilated by certain combination of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ (see indices defined above (14.14)). Moreover the combination has to be annihilated by the properly defined $\mathcal{P}_{+}$ operator in the $R \to 0$ limit (still to be defined).

To start, we have one nontrivial hint. We showed that the vielbein $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ is nonzero and is related to the $H_{\mathbf{a}\,\widetilde{\mathbf{b}}}$. So we can examine what is the action of the $D_{\alpha'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$, i.e. we look at the torsion constraint:

$$\begin{aligned}
T_{D\,S\,\widetilde{S}} \;\equiv\; T_{\alpha'\,+\mathbf{a}\,\widetilde{+\mathbf{b}}} \;=\; 0 \;&=\; D_{[\alpha'}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}})} \;+\; H_{[\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\,+\mathbf{a}\,\widetilde{+\mathbf{b}})\mathcal{N}} \qquad (17.13)\\
&=\; D_{\alpha'}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \;+\; S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \;+\; S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha'}\\
&\quad +\,H_{\widetilde{+\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,+\mathbf{a}\,\mathcal{N}}\\
&=\; D_{\alpha'}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \;+\; S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \;+\; S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha'}\\
&\quad +\,\tfrac{1}{2}\,(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta}\,H_{\widetilde{+\mathbf{b}}\,\beta} \qquad (17.14)
\end{aligned}$$

In the (17.14) we can see various terms with the S derivatives. If we could evaluate mixed structure constants before an action of S derivatives, those S terms in (17.14) would vanish (because the relevant vielbeins are proportional to vanishing mixed constants as we will see). Now they will nontrivially contribute. We note again that we still have the $f_{\widetilde{S}\,D\,\mathcal{N}} \;\equiv\; f_{\widetilde{+\mathbf{b}}\,\alpha'\,\mathcal{N}} \;=\; 0 \;=\; f_{S\,\widetilde{S}\,\mathcal{N}} \;\equiv\; f_{+\mathbf{a}\,\widetilde{+\mathbf{b}}\,\mathcal{N}}$. The vielbeins $H_{\alpha'\,+\mathbf{a}}$ and $H_{\alpha'\,\widetilde{+\mathbf{b}}}$ are fixed by torsion constraints $T_{\widetilde{P}\,D\,S} \;=\; T_{\widetilde{-}\,\alpha'\,+\mathbf{a}} \;=\; 0 \;=\; T_{P\,D\,\widetilde{S}} \;=\; T_{-\,\alpha'\,\widetilde{+\mathbf{b}}}$ and some few other torsion constraints, as will be shown. We note that as in (16.3) and (16.4) we almost always have the strategy to use invertibility of P_{-} and $P_{\widetilde{-}}$ together with our mixed left-right light cone gauge to eliminate/fix vielbeins. Sometimes its not enough and we need to explore some further torsion constraints. Let's look at the already mentioned set of torsion constraints:

$$\begin{aligned}
T_{\widetilde{P}\,D\,S} \;=\; T_{\widetilde{-}\,\alpha'\,+\mathbf{a}} \;=\; 0 \;&=\; P_{[\widetilde{-}}\,H_{\alpha'\,+\mathbf{a}]} \;+\; H_{[\widetilde{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,+\mathbf{a}]\,\mathcal{N}} \qquad (17.15)\\
&=\; P_{\widetilde{-}}\,H_{\alpha'\,+\mathbf{a}} \;+\; H_{+\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{-}\,\alpha'\,\mathcal{N}} \qquad (17.16)\\
\Rightarrow\; H_{\alpha'\,+\mathbf{a}} \;&=\; f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}} \;\rightsquigarrow\; 0 \qquad (17.17)
\end{aligned}$$

in (17.15) we used just the mixed light cone gauge and $f_{S\,\widetilde{P}\,\mathcal{N}} \,=\, 0$ (in flat case and also in AdS). To evaluate the last term in (17.16) that is present only in AdS case we have to take the AdS curvature $T_{\widetilde{P}\,D\,\widetilde{\Omega}} \,\equiv\, T_{\widetilde{\mathbf{a}}\,\alpha}{}^{\widetilde{\beta}} \,\equiv\, f_{\widetilde{\mathbf{a}}\,\alpha}{}^{\widetilde{\beta}} \,=\, \frac{1}{r_{AdS}}\,(\gamma_{\mathbf{a}})_{\alpha\,\nu}\,\widetilde{\mathit{\Gamma}}_5{}^{\nu\,\beta}$ as discussed above (15.9). For our specific indices we have $\frac{1}{r_{AdS}}\,(\gamma_{-})_{\alpha'\,\nu'}\,\widetilde{\mathit{\Gamma}}_5{}^{\nu'\,\beta}$ but the $(\gamma_{-})_{\alpha'\,\nu'} \,=\, 0$ as we can see in the construction of the light cone basis for the gamma matrices in (14.14). The vielbein $H_{\alpha'\,-\mathbf{a}}$ can be fixed in almost the same set of equations as $H_{\alpha'\,+\mathbf{b}}$. Fixing the vielbein $H_{\alpha'\,\widetilde{+\mathbf{b}}}$ (and similarly $H_{\alpha'\,\widetilde{-\mathbf{b}}}$) is also similar but a bit more profound. For that we first examine torsion analogous to (17.13) but for $H_{\alpha'\,\widetilde{+\mathbf{b}}}$ vielbein:

$$T_{P\,D\,\widetilde{S}} \,\equiv\, T_{-\,\alpha'\,\widetilde{+\mathbf{b}}} \,=\, 0 \;=\; P_{[-}\,H_{\alpha'\,\widetilde{+\mathbf{b}}]} \,+\, H_{[-\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\,\mathcal{N}}\,f_{\alpha'\,\widetilde{+\mathbf{b}}]\,\mathcal{N}} \qquad (17.18)$$

$$=\; P_{-}\,H_{\alpha'\,\widetilde{+\mathbf{b}}} \,+\, S_{\widetilde{+\mathbf{b}}}\,H_{-\,\alpha'} \,+\, H_{\widetilde{+\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\,\mathcal{N}}\,f_{-\,\alpha'\,\mathcal{N}}$$

$$\Rightarrow \qquad H_{\alpha'\,\widetilde{+\mathbf{b}}} \,=\, -\,\tfrac{1}{P_{-}}\,S_{\widetilde{+\mathbf{b}}}\,H_{-\,\alpha'} \qquad (17.19)$$

The (17.18) structure constant $f_{-\,\alpha'\,\mathcal{N}} \,\propto\, (\gamma_{-})_{\alpha'\,\nu'}$ but as before that particular piece of gamma matrix is zero (remember the non-mixed structure constants are not breaking the $SO\,(\,10\,) \otimes SO\,(\,10\,)$ so we can evaluate them without any concern). The other term in (17.18) is $H_{-\,\alpha'}$. That is fixed by the dim $\frac{1}{2}$ torsion constraint $T_{-\,\widetilde{-}\,\alpha'} \,=\, 0$:

$$T_{P\,\widetilde{P}\,D} \,=\, T_{-\,\widetilde{-}\,\alpha'} \,=\, 0 \;=\; P_{[-}\,H_{\widetilde{-}\,\alpha')} \,+\, H_{[-\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{-}\,\alpha')\,\mathcal{N}} \qquad (17.20)$$

$$=\; P_{\widetilde{-}}\,H_{\alpha'\,-} \,+\, H_{\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{-\,\widetilde{-}\,\mathcal{N}} \qquad (17.21)$$

$$\Rightarrow\; H_{\alpha'\,-} \;=\; -\,\tfrac{1}{P_{\widetilde{-}}}\,f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,H_{\alpha'\,\mathcal{N}} \,\rightsquigarrow\, 0 \qquad (17.22)$$

We used in (17.20) the mixed light cone gauge, also the fact that $f_{-\,\alpha'\,\mathcal{M}} \propto (\gamma_{-})_{\alpha'\,\beta'} \,=\, 0$. We note that the (17.20) evaluates to zero because the mixed structure constant $f_{-\,\widetilde{-}\,\mathcal{N}} \,=\, 0$. Moreover by the light-cone gauge the (17.20) term $H_{-\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{-}\,\alpha'\,\mathcal{N}} \,=\, 0$ even in the non-evaluated regime. The reason is that the structure constant $f_{\widetilde{-}\,\alpha'\,\mathcal{N}}$ is zero after evaluation and the action of whatever S on this structure constant produces either zero or the right D index $\equiv\, \widetilde{D}$ (after the summation with the vielbein) i.e. the vielbein $H_{-\,\widetilde{D}}$ that is again zero by the light-cone gauge. Combining (17.19) and (17.22) we can get a fixed version of the $H_{\alpha'\,\widetilde{+\mathbf{b}}}$. By the similar equations as above we can fix $H_{\alpha'\,\widetilde{-\mathbf{b}}}$. That result and more detailed analysis is shown in the Appendix, see table (10).

In the Appendix, we also derived the equations (A.16) and (A.37). Those are the actions of the S and $\widetilde{S}$ derivatives that we need in the equation

(17.14). Putting the results from (A.16) and (A.37) into (17.14) we get fixing of the $D_{\alpha'}\, H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$, we note that this is an important result:

$$\begin{aligned} D_{\alpha'}\, H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \;=\; & -\tfrac{1}{g}\,\tfrac{1}{(r_{AdS})\,P_{\widetilde{-}}}\,(\gamma_{\mathbf{b}})_{\alpha'\,\sigma}\,(\widetilde{\varGamma}_5)^{\sigma\,\beta}\,H_{\widetilde{\beta}\,+\mathbf{a}} \\ & +\tfrac{1}{2\,g}\,(\,1\,-\,\tfrac{1}{f}\,\tfrac{1}{(r_{AdS})^2\,P_{\widetilde{-}}\,P_{-}}\,)(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta}\,H_{\beta\,\widetilde{+\mathbf{b}}} \\ & +\tfrac{1}{f\,g}\,\tfrac{1}{2\,(r_{AdS})^3\,P_{-}\,(P_{\widetilde{-}})^2}\,(\gamma_{\mathbf{a}})_{\alpha'\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\beta}\,H_{\widetilde{\beta}\,+\mathbf{b}} \end{aligned} \tag{17.23}$$

where f and g are defined as follows:

$$\begin{aligned} f \;&:=\; (\,1\,-\,\tfrac{1}{2\,(r_{AdS})^2\,P_{\widetilde{-}}\,P_{-}}\,) \\ g \;&:=\; (\,1\,-\,\tfrac{1}{f}\,\tfrac{1}{2\,(r_{AdS})^2\,P_{\widetilde{-}}\,P_{-}}\,) \end{aligned} \tag{17.24}$$

Changing left $\leftrightarrow$ right in (17.23) we get the equation for $D_{\widetilde{\alpha}'}\, H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$. There is one simplification we can make in equations (17.23). Because only half of the block diagonal γ_+ matrix is nonzero and is proportional to the δ for the nonzero part. The $(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta} \;=\; \delta_{\alpha'\,\nu'}\,(\gamma_{\mathbf{a}})^{\nu'\,\beta} \;\equiv\; (\gamma_{\mathbf{a}})_{\alpha'}{}^{\beta}$.

The observation from (17.23) and its left $\leftrightarrow$ right swap is that the action of the $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ is producing two new vielbeins $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$. This hints that we need some another vielbein, such that the action of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on it will effectively subtract the fields $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$. We found such a vielbein, but before giving it we will look at the flat case superspace first to give a motivation. After that we will generalise it to the $AdS_5\,\times\,S^5$ background.

Chapter 18

18 Flat space solution

18.1 Flat space diagram

To see what could be possibly a missing vielbein that will subtract vielbeins in (17.23) (and its left $\leftrightarrow$ right change) we first solve the same problem in flat space background. That is the extended superspace with $r_{AdS} \to \infty$. Note that in flat superspace ($r_{AdS} \to \infty$) the relation (17.23) simplifies significantly, because there are no r_{AdS} dependent parts. The surviving part after $r_{AdS} \to \infty$ is just the second term on the right hand side of (17.23) with $g = 1$, i.e. $\frac{1}{2}\,(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta}\, H_{\beta\,\widetilde{+\mathbf{b}}}$.

Let us therefore further examine an action of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$ respectively:

$$
\begin{aligned}
T_{D\,D\,\widetilde{S}} \equiv T_{\alpha'\,\beta\,\widetilde{+\mathbf{a}}} = 0 \;&=\; D_{[\alpha'}\,H_{\beta\,\widetilde{+\mathbf{a}})} + H_{[\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}})\,\mathcal{N}} &(18.1)\\
&=\; D_{\alpha'}\,H_{\beta\,\widetilde{+\mathbf{a}}} + S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta} - D_{\beta}\,H_{\widetilde{+\mathbf{a}}\,\alpha'} &(18.2)\\
&\quad + H_{\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}}\,\mathcal{N}}\\
&\quad + H_{\widetilde{+\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,\beta\,\mathcal{N}} - H_{\beta\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{+\mathbf{a}}\,\alpha'\,\mathcal{N}}
\end{aligned}
$$

The mixed terms in the f part of (18.1) are zero (note they are zero also in the AdS background). The structure constant $f_{\alpha'\,\beta\,\mathcal{N}} = 2\,(\gamma_{\mathbf{a}})_{\alpha'\,\beta}\,\delta^{\mathbf{a}}{}_{\mathcal{N}}$ (the same is in the AdS background). The vielbein $H_{\widetilde{+\mathbf{a}}\,\alpha'} \rightsquigarrow 0$ by the table (10). Then the equation (18.1) can be rewritten as:

$$
0 = D_{\alpha'}\,H_{\beta\,\widetilde{+\mathbf{a}}} + S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta} + 2\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta}\,H_{\widetilde{+\mathbf{a}}\,\mathbf{c}} \qquad (18.3)
$$

To evaluate the only S derivative term in (18.3) i.e. $S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta}$ we would need to work a bit, in the AdS superspace. The whole AdS analysis of the actions of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$ is done in the Appendix, see equations (A.41) till (A.65). In this section we would need only $r_{AdS} \to \infty$ limit of that analysis.

In the Appendix we derived the equations (A.59) and (A.65). Those equations are telling us that in the AdS case (and so also in the flat case) the actions of $D_{\widetilde{\alpha}'}$ and $D_{\alpha'}$ result in a combination of $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ and $H_{\alpha\,\widetilde{\beta}}$. This is actually a hint that we should add the trace of $H_{\alpha\,\widetilde{\beta}}$ to the trace of $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ in order to subtract an action of a linear combination of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ (future D^v derivative) on trace $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$. In the rest of this paragraph and next chapter we will look at how the pre-potential is built up in a flat space limit, i.e. we consider equations (A.59) and (A.60) and (A.65) and (A.66) in the limit $r_{AdS} \to \infty$. We find equations that are fixing pre-potential and vanishing D_w derivative.

Thus we repeat the flat space limits of the $D_{\widetilde{\alpha}'}$ and $D_{\alpha'}$ actions on $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$ respectively, i.e. the equations (A.65) and (A.66) in $r_{AdS} \to \infty$ limit:

$$0 = D_{\widetilde{\alpha}'} H_{\beta\,\widetilde{+\mathbf{a}}} + \tfrac{1}{2}\,(\gamma_{\mathbf{a}})_{\alpha'}{}^{\nu} H_{\beta\,\widetilde{\nu}} \tag{18.4}$$

$$0 = D_{\alpha'} H_{\widetilde{\beta}\,+\mathbf{a}} + \tfrac{1}{2}\,(\gamma_{\mathbf{a}})_{\alpha'}{}^{\nu} H_{\nu\,\widetilde{\beta}} \tag{18.5}$$

Similarly, we can also look at the equations (17.23) and (A.59) and (A.60) (in the flat space limit) and together with (18.4) and (18.5) we can observe the following interesting flat space diagram:

$$\begin{array}{ccccc} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{+\mathbf{b}}} & \xrightarrow{D_{\alpha'}} & H_{\mathbf{c}\,\widetilde{+\mathbf{b}}} \\ \big\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \big\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \\ H_{+\mathbf{a}\,\widetilde{\beta}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{\beta}} & & \\ \big\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & & & \\ H_{+\mathbf{a}\,\widetilde{\mathbf{c}}} & & & & \end{array}$$

Figure 2: Flat space diagram

The scheme (2) is nice and actually tells us what we should do next. Recall that the nodes $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ and $H_{\mathbf{c}\,\widetilde{+\mathbf{b}}}$ and $H_{+\mathbf{a}\,\widetilde{\mathbf{c}}}$ could be identified by the use of invertible operators P_{-} and $P_{\widetilde{-}}$ see (17.6). Our original aim was find a field that has a scalar trace and could possibly subtract actions of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$. We will see that the missing field is exactly $H_{\beta\widetilde{\beta}}$ (in the flat space, the only nonzero part of $H_{D\,\widetilde{D}}$). The diagram (2) suggests what to do. We calculate the remaining arrows and fill the square.

To fill the remaining arrows we need to calculate the action of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{\alpha\,\widetilde{\beta}}$ together with some another arrows that will be discussed later. We consider the dimension $\frac{1}{2}$ torsion constraint $T_{D\,D\,\widetilde{D}} \equiv T_{\alpha'\,\beta\,\widetilde{\sigma}} = 0$. We note again that for now on we are working in the flat space. Later we will generalise the procedure for the AdS space:

$$
\begin{aligned}
T_{D\,D\,\widetilde{D}} \equiv T_{\alpha'\,\beta\,\widetilde{\sigma}} = 0 &= D_{(\alpha'}H_{\beta\,\widetilde{\sigma})} + H_{(\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{\sigma}\,)\,\mathcal{N}} && (18.6)\\
&= D_{\alpha'}H_{\beta\,\widetilde{\sigma}} + 2\,(\gamma^{\mathbf{a}})_{\alpha'\,\beta}\,H_{\widetilde{\sigma}\,\mathbf{a}} && (18.7)
\end{aligned}
$$

where we used that $H_{\alpha'\,\beta} = H_{\alpha'\,\widetilde{\sigma}} = 0$ in flat space (see (A.42) and (A.47) and do flat space limit). We also have a left $\leftrightarrow$ right swap of (18.7). The vielbein $H_{\widetilde{\sigma}\,\mathbf{c}}$ in (18.7) is related to $H_{+\mathbf{c}\,\widetilde{\sigma}}$. For that consider torsion constraint $T_{P\,S\,\widetilde{D}} \equiv T_{-\,+\mathbf{c}\,\widetilde{\sigma}} = 0$:

$$
\begin{aligned}
T_{P\,S\,\widetilde{D}} \equiv T_{-\,+\mathbf{c}\,\widetilde{\sigma}} = 0 &= P_{[-}H_{+\mathbf{c}\,\widetilde{\sigma})} + H_{[-\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{+\mathbf{c}\,\widetilde{\sigma}\,)\,\mathcal{N}} && (18.8)\\
&= P_{-}H_{+\mathbf{c}\,\widetilde{\sigma}} + \eta_{+-}\,H_{\widetilde{\sigma}\,\mathbf{c}} && (18.9)
\end{aligned}
$$

where we used left-right light-cone gauge, together with $H_{-\,+\mathbf{c}} = 0$ that is shown in the Appendix and holds even in AdS, see (A.23).

To fill the diagram (2) we need to calculate two more torsion constraints that are providing the actions of $D_{\alpha'}$ on $H_{+\mathbf{a}\,\widetilde{\mathbf{c}}}$ and on $H_{\sigma\,\widetilde{\mathbf{c}}}$. We first consider $T_{D\,S\,\widetilde{P}} \equiv T_{\alpha'\,+\mathbf{a}\,\widetilde{\mathbf{c}}} = 0$:

$$
\begin{aligned}
T_{D\,S\,\widetilde{P}} \equiv T_{\alpha'\,+\mathbf{a}\,\widetilde{\mathbf{c}}} = 0 &= D_{[\alpha'}H_{+\mathbf{a}\,\widetilde{\mathbf{c}})} + H_{[\alpha'\,|\,\mathcal{M}}\eta^{\mathcal{M}\mathcal{N}}\,f_{+\mathbf{a}\,\widetilde{\mathbf{c}})\mathcal{N}} && (18.10)\\
&= D_{\alpha'}H_{+\mathbf{a}\,\widetilde{\mathbf{c}}} + \tfrac{1}{2}\,(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\sigma}\,H_{\widetilde{\mathbf{c}}\,\sigma} && (18.11)\\
&= D_{\alpha'}H_{+\mathbf{a}\,\widetilde{\mathbf{c}}} + \tfrac{1}{2}\,(\gamma_{\mathbf{a}})_{\alpha'}{}^{\sigma}\,H_{\widetilde{\mathbf{c}}\,\sigma} && (18.12)
\end{aligned}
$$

where we used that $H_{\alpha'\,+\mathbf{a}} = 0$ (holds even in the AdS, see table (10)). We also used that $H_{\alpha'\,\widetilde{\mathbf{c}}} = 0$ (that is enough in a flat space to have $S_{+\mathbf{a}}\,H_{\alpha'\,\widetilde{\mathbf{c}}} = 0$). To see that $H_{\alpha'\,\widetilde{\mathbf{c}}} = 0$ we use the torsion $T_{\widetilde{P}\,\widetilde{S}\,D} \equiv T_{\widetilde{-}\,\widetilde{+\mathbf{c}}\,\alpha'} = 0$:

$$
\begin{aligned}
T_{\widetilde{P}\,\widetilde{S}\,D} \equiv T_{\widetilde{-}\,\widetilde{+\mathbf{c}}\,\alpha'} = 0 &= P_{[\widetilde{-}}H_{\widetilde{+\mathbf{c}}\,\alpha')} + H_{[\widetilde{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{+\mathbf{c}}\,\alpha'\,)\,\mathcal{N}} && (18.13)\\
&= P_{\widetilde{-}}H_{\widetilde{+\mathbf{c}}\,\alpha'} + \eta_{+-}\,H_{\alpha'\,\widetilde{\mathbf{c}}} && (18.14)
\end{aligned}
$$

and previously we saw that $H_{\widetilde{+\mathbf{c}}\,\alpha'} = 0$ (even in the AdS case, see table (10)). From (18.14) in the flat case follows that $H_{\alpha'\,\widetilde{\mathbf{c}}} = 0$. Examining the (18.14) in the AdS case one also finds that $H_{\alpha'\,\widetilde{\mathbf{c}}} = 0$ (after evaluation). The (18.12) however could have some additional term in the AdS case. The structure constant $f_{\alpha'\,\widetilde{\mathbf{c}}\,\mathcal{N}} \neq 0$ and so the term proportional to that structure constant in the AdS case is $\frac{1}{r_{AdS}}\,(\gamma_{\mathbf{c}})_{\alpha'\,\sigma}\,(\widetilde{\Gamma}_5)^{\sigma\,\nu}\,H_{+\mathbf{a}\,\widetilde{\nu}}$. That term is nonzero in the AdS case. Moreover, in the (18.10) one finds one more AdS term, coming from evaluated action $S_{+\mathbf{a}}\, H_{\alpha'\,\widetilde{\mathbf{c}}}$. Those terms are not of a big concern right now (doing the flat space first), we will see them later in the section where we generalise to AdS case.

Last torsion constraint to examine in order to fill the (2) is the one that determines the action of $D_{\alpha'}$ on $H_{\beta\,\widetilde{\mathbf{c}}}$. Consider therefore the dimension $\frac{1}{2}$ torsion $T_{D\,D\,\widetilde{P}} \equiv T_{\alpha'\,\beta\,\widetilde{\mathbf{c}}} = 0$:

$$\begin{aligned} T_{D\,D\,\widetilde{P}} \equiv T_{\alpha'\,\beta\,\widetilde{\mathbf{c}}} = 0 \;&=\; D_{[\alpha'} H_{\beta\,\widetilde{\mathbf{c}})} + H_{[\alpha'\,|\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\beta\,\widetilde{\mathbf{c}}\,)\,\mathcal{N}} && (18.15) \\ &=\; D_{\alpha'}\, H_{\beta\,\widetilde{\mathbf{c}}} + 2\,(\gamma^{\mathbf{a}})_{\alpha'\,\beta}\, H_{\widetilde{\mathbf{c}}\,\mathbf{a}} && (18.16) \end{aligned}$$

where we used the $H_{\alpha'\,\beta} = 0$ (holds also in AdS after the evaluation) and $H_{\widetilde{\mathbf{c}}\,\alpha'} = 0$ (also holds in AdS after the evaluation). In the AdS case in the equation (18.16) we have two additional terms. They come from $f_{\beta\,\widetilde{\mathbf{c}}\,\mathcal{N}} \neq 0$ and also $f_{\alpha'\,\widetilde{\mathbf{c}}\,\mathcal{N}} \neq 0$. Those terms will be further analysed in future sections, let just write their structure as $\frac{1}{r_{AdS}}\,(\gamma_{\mathbf{c}})_{\beta\,\nu'}\,(\widetilde{\varGamma}_5)^{\nu'\,\sigma'}\,H_{\widetilde{\sigma}'\,\alpha'}$ and $\frac{1}{r_{AdS}}\,(\gamma_{\mathbf{c}})_{\alpha'\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\sigma}\,H_{\widetilde{\sigma}\,\beta}$. The vielbein $H_{\widetilde{\sigma}'\,\alpha'} = 0$ (in flat case and also in AdS after the evaluation) as can be calculated from torsion constraints $T_{-\,\widetilde{\sigma}'\,\alpha'} = 0$ and $T_{\widetilde{-}\,-\,\widetilde{\sigma}'} = 0$ and the use of the double light-cone gauge. The term $H_{\widetilde{\sigma}\,\beta}$ is nonzero ($H_{\widetilde{\sigma}\,\beta}$ vielbein is a part of a pre-potential).

We can add results of (flat space) equations (18.12) and (18.16) together with (18.7) and their left $\leftrightarrow$ right swaps to the diagram (2) and find the following square diagram:

$$\begin{array}{ccccc}
H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{+\mathbf{b}}} & \xrightarrow{D_{\alpha'}} & H_{\mathbf{c}\,\widetilde{+\mathbf{b}}} \\
\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} \\
H_{+\mathbf{a}\,\widetilde{\beta}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{\beta}} & \xrightarrow{D_{\alpha'}} & H_{\mathbf{c}\,\widetilde{\beta}} \\
\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} \\
H_{+\mathbf{a}\,\widetilde{\mathbf{c}}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{\mathbf{c}}} & \xrightarrow{D_{\alpha'}} & H_{\mathbf{c}\,\widetilde{\mathbf{c}}}
\end{array}$$

Figure 3: Full flat space diagram

As we saw before the (3) nodes $\{\, H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}},\, H_{\mathbf{c}\,\widetilde{+\mathbf{b}}},\, H_{+\mathbf{a}\,\widetilde{\mathbf{c}}},\, H_{\mathbf{c}\,\widetilde{\mathbf{c}}} \,\}$ should be identified (as one node). We proved that using various torsion constraints, mixed light-cone gauge and invertibility of P_- and $P_{\widetilde{-}}$. The same way the nodes $\{ H_{+\mathbf{a}\,\widetilde{\beta}},\, H_{\mathbf{c}\,\widetilde{\beta}} \}$ and independently nodes $\{ H_{\beta\,\widetilde{+\mathbf{b}}},\, H_{\beta\,\widetilde{\mathbf{c}}} \}$ should be identified (as two independent nodes). The vielbein $H_{\beta\,\widetilde{\beta}}$ is then just a single node. After the described identifications the diagram (3) could be rewritten in the simpler and more informative form.

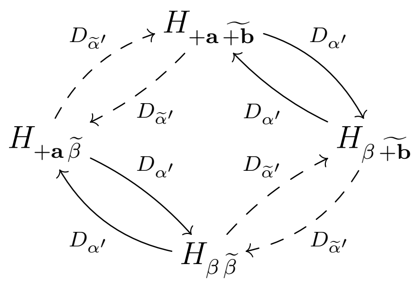

Figure 4: Identified flat space diagram

Note, the dashed arrows stand for action of $D_{\widetilde{\alpha}'}$ and solid arrows stand for action of $D_{\alpha'}$. From the nice flat space diagram (4) it is obvious that in order to have a vanishing derivative we have to combine $D_{\alpha'}$ with $D_{\widetilde{\alpha}'}$ and that combination should act on the combination of traces of $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ with $H_{\beta\,\widetilde{\beta}}$.

18.2 The $\mathbb{H}$ matrix

The diagram (4) could be rewritten in the matrix form. The observation is that each action of the derivatives in the (4) is given by some matrix. The derivatives are mixing fields just as in (4). Let us introduce the $2 \otimes 2$ block matrix $\mathbb{H}$:

$$\mathbb{H} := \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix} \tag{18.17}$$

The action of $D_{\alpha'}$ is then given as the left action of some constant (up to P_- operator) block off diagonal matrix $\mathit{\Gamma}_{\alpha'}$:

$$\begin{aligned} D_{\alpha'} \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix} &\equiv \begin{pmatrix} 0 & \frac{1}{2}(\gamma_{\mathbf{a}})_{\alpha'}{}^{\sigma} \\ 2\,P_-\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta} & 0 \end{pmatrix} \begin{pmatrix} H_{+\mathbf{c}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{c}\,\widetilde{\beta}} \\ H_{\sigma\,\widetilde{+\mathbf{b}}} & H_{\sigma\,\widetilde{\beta}} \end{pmatrix} \\ D_{\alpha'}\,\mathbb{H} &= \mathit{\Gamma}_{\alpha'}\,\mathbb{H} \end{aligned} \tag{18.18}$$

The action of $D_{\widetilde{\alpha}'}$ on $\mathbb{H}$ is given as a right action of similar matrix $\Gamma_{\widetilde{\alpha}'}$:

$$D_{\widetilde{\alpha}'} \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix} \equiv \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}} & H_{+\mathbf{a}\,\widetilde{\sigma}} \\ H_{\beta\,\widetilde{+\mathbf{c}}} & H_{\beta\,\widetilde{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & 2\,P_{\widetilde{-}}\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta} \\ \frac{1}{2}\,(\gamma_{\mathbf{b}})_{\alpha'}{}^{\sigma} & 0 \end{pmatrix}$$

$$D_{\widetilde{\alpha}'}\,\mathbb{H} = \mathbb{H}\,\Gamma_{\widetilde{\alpha}'} \tag{18.19}$$

Now we will proceed to the main step. We arbitrarily linearly combine $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$, i.e. we multiply the $D_{\widetilde{\alpha}'}$ with some unknown nonsingular matrix $\mathcal{M}_{\alpha'}{}^{\beta'}$:

$$\mathcal{D}_v \equiv \mathcal{D}^v{}_{\alpha'} := (\,D_{\alpha'} - \mathcal{M}_{\alpha'}{}^{\beta'}\,D_{\widetilde{\beta}'}\,) \tag{18.20}$$

Moreover we impose that in the matrix version of $D_{\widetilde{\alpha}'}$ action the matrix $\mathcal{M}$ acts as follows:

$$\mathcal{M}_{\alpha'}{}^{\beta'}\,\Gamma_{\widetilde{\beta}'} := \mathbb{A}\,\Gamma_{\widetilde{\alpha}'}\,\mathbb{B} \tag{18.21}$$

for some nonsingular matrices $\mathbb{A}$ and $\mathbb{B}$. Combining (18.18) and (18.19) together with (18.20) and (18.21) we get:

$$\mathcal{D}^v{}_{\alpha'}\,\mathbb{H} = \Gamma_{\alpha'}\,\mathbb{H} - \mathbb{H}\,\mathbb{A}\,\Gamma_{\widetilde{\alpha}'}\,\mathbb{B} \qquad /\,\mathbb{B}^{-1} \tag{18.22}$$

$$\mathcal{D}^v{}_{\alpha'}\,\mathbb{H}\,\mathbb{B}^{-1} = \Gamma_{\alpha'}\,\mathbb{H}\,\mathbb{B}^{-1} - \mathbb{H}\,\mathbb{A}\,\Gamma_{\widetilde{\alpha}'} \qquad /\,\mathrm{Str} \tag{18.23}$$

$$\mathcal{D}^v{}_{\alpha'}\,\mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,) = \mathrm{Str}\Big(\,(\,\mathbb{B}^{-1}\,\Gamma_{\alpha'} - \mathbb{A}\,\Gamma_{\widetilde{\alpha}'}\,)\,\mathbb{H}\Big) \tag{18.24}$$

by the Str we mean the super-trace. We put to zero the $\mathrm{Str}\Big(\,(\,\mathbb{B}^{-1}\,\Gamma_{\alpha'} - \mathbb{A}\,\Gamma_{\widetilde{\alpha}'}\,)\,\mathbb{H}\Big) = 0$ by finding the suitable matrices $\mathbb{B}$ and $\mathbb{A}$ and the matrix $\mathcal{M}$. By that we get the equation:

$$\mathcal{D}^v{}_{\alpha'}\,\mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,) = 0 \tag{18.25}$$

thus the equation (18.25) defines the $\mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,)$ as the scalar field on which particular combination of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ now called $\mathcal{D}^v{}_{\alpha'}$ vanishes. So, we found a pre-potential $V := \mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,)$. We note that even though the equation (18.24) might seem easy to solve just by putting $\mathbb{B}^{-1} = \mathbb{A}$. It is not that simple since $\Gamma_{\alpha'} \neq \Gamma_{\widetilde{\alpha}'}$. Therefore some more involved solution has to be found.

18.3 Solution via the gamma matrix identity

We solve the equation (18.24) using the following identity:

$$\mathcal{A}_{\alpha'}{}^{\sigma'}\,\mathcal{B}_{\mathbf{a}}{}^{\mathbf{c}}\,\mathcal{C}_{\beta}{}^{\nu}\,(\gamma_{\mathbf{c}})_{\sigma'\,\nu} = (\gamma_{\mathbf{a}})_{\alpha'\,\beta} \tag{18.26}$$

has two $SO(4)$ invariant solutions:

$$\mathbb{I}. \ : \ \mathcal{A}_{\alpha'}{}^{\beta'} = \delta_{\alpha'}{}^{\beta'} \ || \ \mathcal{C}_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta} \ || \ \mathcal{B}_{\mathbf{a}}{}^{\mathbf{b}} = \delta_{\mathbf{a}}{}^{\mathbf{b}} \tag{18.27}$$

$$\mathbb{II}. \ : \ \mathcal{A}_{\alpha'}{}^{\beta'} = (\widetilde{\Gamma}_5)_{\alpha'}{}^{\beta'} \ || \ \mathcal{C}_{\alpha}{}^{\beta} = (\widetilde{\Gamma}_5)_{\alpha}{}^{\beta} \ || \ \mathcal{B}_{\mathbf{a}}{}^{\mathbf{b}} = (\widetilde{\Gamma}_5)_{\mathbf{a}}{}^{\mathbf{b}} \tag{18.28}$$

The solution (18.27) is trivial, the solution (18.28) is based on property of the $\widetilde{\Gamma}_5$ matrix: $[\,\widetilde{\Gamma}_5, \gamma_{\mathbf{a}}\,] = 0$ for $\mathbf{a} \in \{10, 1, 2, 3, 4\}$ and $\{\,\widetilde{\Gamma}_5, \gamma_{\mathbf{a}}\,\} = 0$ for $\mathbf{a} \in \{5, 6, 7, 8, 9\}$. The previous follows directly from the definition of $\widetilde{\Gamma}_5$, see (14.13). The new matrix $(\widetilde{\Gamma}_5)_{\mathbf{a}}{}^{\mathbf{b}}$ in (18.28) is defined by the (18.26) to fix the signs. Note that the indices $\mathbf{a}$ in (18.26) have a range: $\mathbf{a} \in \{1, \ldots, 8\}$.

Next, we look explicitly at the equation:

$$\mathrm{Str}\Big((\mathbb{B}^{-1}\,\Gamma_{\alpha'} - \mathbb{A}\,\Gamma_{\widetilde{\alpha}'})\,\mathbb{H} \Big) \equiv \mathrm{Str}\,\mathbb{X}_{\alpha'} = 0 \tag{18.29}$$

let's rename the members of the matrix $\mathbb{H}$:

$$\mathbb{H} \equiv \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix} \equiv \begin{pmatrix} H_{S\,\widetilde{S}} & H_{S\,\widetilde{D}} \\ H_{D\,\widetilde{S}} & H_{D\,\widetilde{D}} \end{pmatrix} \tag{18.30}$$

Let us define the matrices $\mathbb{A}$ and $\mathbb{B}^{-1}$ to be block diagonal matrices. This is a consistent choice with the fact that we want to have a pre-potential build out of $H_{S\,\widetilde{S}}$ and $H_{D\,\widetilde{D}}$. The pre-potential is in (18.25) given as $\mathrm{Str}(\,\mathbb{H}\,\mathbb{B}^{-1}\,)$. We do not want to mix in some off diagonal $\mathbb{H}$ fields by the action of $\mathbb{B}^{-1}$. Thus $\mathbb{A}$ and $\mathbb{B}^{-1}$ are:

$$\mathbb{A} \equiv \begin{pmatrix} A_{S\,\widetilde{S}} & 0 \\ 0 & A_{D\,\widetilde{D}} \end{pmatrix} \ || \ \mathbb{B}^{-1} \equiv \begin{pmatrix} B^{-1}{}_{S\,\widetilde{S}} & 0 \\ 0 & B^{-1}{}_{D\,\widetilde{D}} \end{pmatrix} \tag{18.31}$$

With definitions (18.31) we get the equation (18.29) into the following matrix equation:

$$\begin{aligned} \mathrm{Str}\Bigg(& \begin{pmatrix} \frac{1}{2}\,B^{-1}{}_{S\,\widetilde{S}}\,\gamma\,H_{D\,\widetilde{S}} & \cdots \\ \cdots & 2\,P_{-}\,B^{-1}{}_{D\,\widetilde{D}}\,\gamma\,H_{S\,\widetilde{D}} \end{pmatrix} \\ & - \begin{pmatrix} 2\,P_{\widetilde{-}}\,A_{S\,\widetilde{S}}\,\gamma\,H_{D\,\widetilde{S}} & \cdots \\ \cdots & \frac{1}{2}\,A_{D\,\widetilde{D}}\,\gamma\,H_{S\,\widetilde{D}} \end{pmatrix} \Bigg) = 0 \end{aligned} \tag{18.32}$$

Then from (18.32) we get two equations (since fields $H_{D\,\widetilde{S}}$ and $H_{S\,\widetilde{D}}$ are independent):

$$\tfrac{1}{2}\,B^{-1}{}_{S\,\widetilde{S}} - 2\,P_{\widetilde{-}}\,A_{S\,\widetilde{S}} = 0 \Rightarrow A_{S\,\widetilde{S}} = \tfrac{1}{4\,P_{\widetilde{-}}}\,B^{-1}{}_{S\,\widetilde{S}} \tag{18.33}$$

$$P_{-}\,B^{-1}{}_{D\,\widetilde{D}} - A_{D\,\widetilde{D}} = 0 \Rightarrow A_{D\,\widetilde{D}} = 4\,P_{-}\,B^{-1}{}_{D\,\widetilde{D}} \tag{18.34}$$

Now we are prepared to examine the equation (18.21) using the $\mathbb{A}$ and $\mathbb{B}$ constructed above. Then the matrix equation (18.21) can be (schematically) written:

$$\begin{pmatrix} \frac{1}{4\,P_{\underset{\sim}{-}}}\,B^{-1}{}_{S\,\widetilde{S}} & 0 \\ 0 & 4\,P_{-}\,B^{-1}{}_{D\,\widetilde{D}} \end{pmatrix} \begin{pmatrix} 0 & 2\,P_{\underset{\sim}{-}}\,\gamma \\ \frac{1}{2}\,\gamma & 0 \end{pmatrix} \begin{pmatrix} B_{S\,\widetilde{S}} & 0 \\ 0 & B_{D\,\widetilde{D}} \end{pmatrix} = \\ = \mathcal{M} \begin{pmatrix} 0 & 2\,P_{\underset{\sim}{-}}\,\gamma \\ \frac{1}{2}\,\gamma & 0 \end{pmatrix} \quad (18.35)$$

$$\mathcal{M} \begin{pmatrix} 0 & 2\,P_{\underset{\sim}{-}}\,\gamma \\ \frac{1}{2}\,\gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\,B^{-1}{}_{S\,\widetilde{S}}\,\gamma\,B_{D\,\widetilde{D}} \\ 2\,P_{-}\,B^{-1}{}_{D\,\widetilde{D}}\,\gamma\,B_{S\,\widetilde{S}} & 0 \end{pmatrix} \quad (18.36)$$

We now do the following re-scalings $\mathcal{M} \rightarrow \frac{1}{\lambda}\,\mathcal{M}$ and $B_{S\,\widetilde{S}} \rightarrow \Delta\,B_{S\,\widetilde{S}}$ and $B_{D\,\widetilde{D}} \rightarrow \rho\,B_{D\,\widetilde{D}}$. Rescaled $\mathcal{M}$ and $B_{S\,\widetilde{S}}$ and $B_{D\,\widetilde{D}}$ belong to one of the two solutions of identity (18.26). Then we get the version of (18.36):

$$\mathcal{M} \begin{pmatrix} 0 & 2\,P_{\underset{\sim}{-}}\,\gamma \\ \frac{1}{2}\,\gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\,\frac{\lambda\,\rho}{\Delta}\,B^{-1}{}_{S\,\widetilde{S}}\,\gamma\,B_{D\,\widetilde{D}} \\ 2\,\frac{\lambda\,\Delta}{\rho}\,P_{-}\,B^{-1}{}_{D\,\widetilde{D}}\,\gamma\,B_{S\,\widetilde{S}} & 0 \end{pmatrix} \quad (18.37)$$

Now we want the λ and ρ and Δ to satisfy:

$$\begin{aligned} \frac{\lambda\,\rho}{\Delta} &= 4\,P_{\underset{\sim}{-}} \quad \text{and} \quad \frac{\lambda\,\Delta}{\rho} = \frac{1}{4\,P_{-}} \quad \Rightarrow \quad \lambda = \pm\sqrt{\frac{P_{\underset{\sim}{-}}}{P_{-}}} \\ \frac{\rho}{\Delta} &= \pm\,4\,\sqrt{P_{\underset{\sim}{-}}\,P_{-}} \end{aligned} \quad (18.38)$$

The (18.37) is just a matrix equation:

$$\mathcal{M} \begin{pmatrix} 0 & P_{\underset{\sim}{-}}\,\gamma \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^{-1}{}_{S\,\widetilde{S}}\,P_{\underset{\sim}{-}}\,\gamma\,A_{D\,\widetilde{D}} \\ A^{-1}{}_{D\,\widetilde{D}}\,\gamma\,A_{S\,\widetilde{S}} & 0 \end{pmatrix} \quad (18.39)$$

that can be solved by (18.26). Even though we saw the appearance of the nasty square roots in the (18.38) and so in the definition of $\mathcal{D}^{v}{}_{\alpha'}$ and in the pre-potential via super-trace of $\mathbb{H}\,\mathbb{B}^{-1}$. We will see in the AdS case solution that there is a way how to get rid of it.

Chapter 19

19 $AdS_5 \times S^5$ solution

19.1 $AdS_5 \times S^5$ diagram

In the previous sub-sections we saw how to find the pre-potential in the flat case. We are really interested in the AdS case. Along the way we analysed the flat case in the previous sub-sections we mentioned also changes one has to make in the AdS case. We repeat them here again since they are scattered over the previous flat case sub-sections and in the Appendix. First change has already been worked out in the relation between $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ and $H_{\mathbf{a}\,\widetilde{\mathbf{b}}}$ in (17.12). We also note that there are AdS contributions in equations (17.23) also in (A.59) and (A.60). The nontrivial contributions also appeared in equations (A.65) and (A.66).

We could visualise the relations (17.23) and (A.59) and (A.60) and (A.65) and (A.66) by the similar diagram as used in flat case, see (2). The structure is very similar just with more arrows between nodes. Since the AdS diagram is messier we will not provide it. The idea is however the same as in the flat case. In order to determine the vanishing ${D^v}_{\alpha'}$ derivative and the pre-potential we need to combine $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ for D^v derivative and $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ together with $H_{\alpha\,\widetilde{\beta}}$ for pre-potential.

The only missing derivative in the set of AdS equations: (17.23) and (A.59) and (A.60) and (A.65) and (A.66), is an action of $D_{\alpha'}$ on $H_{\beta\,\widetilde{\sigma}}$. This action can be calculated from $T_{D\,D\,\widetilde{D}} \equiv T_{\alpha'\,\beta\,\widetilde{\sigma}} = 0$ torsion constraint. The AdS contribution in that constraint comes from $f_{\alpha'\,\widetilde{\sigma}\,\mathcal{N}}$ structure constant. We have already analysed this structure constant, see equations (A.48) and (A.49) till (A.52). We can thus directly write the constraint with an extra AdS term:

$$\begin{aligned} T_{D\,D\,\widetilde{D}} \equiv T_{\alpha'\,\beta\,\widetilde{\sigma}} = 0 \;&=\; D_{[\alpha'} H_{\beta\,\widetilde{\sigma})} + H_{[\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, f_{\beta\,\widetilde{\sigma}\,)\,\mathcal{N}} \\ &=\; D_{\alpha'}\, H_{\beta\,\widetilde{\sigma}} + 2\,(\gamma^{\mathbf{a}})_{\alpha'\,\beta}\, H_{\widetilde{\sigma}\,\mathbf{a}} \\ &\quad + \tfrac{1}{r_{AdS}}\,(\gamma^{[\mathbf{c}})_{\sigma\,\rho}\,(\widetilde{\mathit{\Gamma}}_5)^{\rho\,\nu}\,(\gamma^{\mathbf{d}]})_{\nu\,\alpha'}\, H_{\beta\,\mathbf{cd}} \end{aligned} \qquad (19.1)$$

where we again note that the Σ indices in the last expression of the (19.1) second line are from the $SO(5) \otimes SO(5)$ diagonal subgroup. The $H_{\beta\,\mathbf{cd}}$ vielbein has nonzero both $H_{\beta\,+\mathbf{b}}$ and also $H_{\beta\,\widetilde{+\mathbf{b}}}$. The second vielbein is the term already in the matrix $\mathbb{H}$ from the flat section (ultimate goal is to rewrite the AdS case in the terms of matrix $\mathbb{H}$ and use the super-trace trick to get the pre-potential). The field $H_{\beta\,+\mathbf{b}}$ is related to the $H_{\widetilde{\rho}+\mathbf{c}}$ as we saw in table (11).

There is one last piece in the equation (19.1) that we did not relate to the fields in the $\mathbb{H}$ matrix. The field $H_{\widetilde{\sigma}\,\mathbf{a}}$. As we saw in the flat case, that field should be related to $H_{\widetilde{\sigma}\,+\mathbf{a}}$ via P_{-}. We have seen however (for example in (17.12)) that such relations are a bit changed in the AdS case. Consider the following torsion constraint (and use mixed light-cone and $H_{-\,+\mathbf{a}} \rightsquigarrow 0$):

$$
\begin{aligned}
T_{P\,\widetilde{D}\,S} \equiv T_{-\,\widetilde{\beta}+\mathbf{a}} = 0 \;&=\; D_{[-}\,H_{\widetilde{\beta}+\mathbf{a})} \,+\, H_{[-\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{\beta}+\mathbf{a})\mathcal{N}} \qquad (19.2)\\
&=\; P_{-}\,H_{\widetilde{\beta}+\mathbf{a}} \,+\, H_{\widetilde{\beta}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{-\,+\mathbf{a}\mathcal{N}}\\
&\quad +\, H_{+\mathbf{a}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{-\,\widetilde{\beta}\mathcal{N}}\\
&=\; P_{-}\,H_{\widetilde{\beta}+\mathbf{a}} \,+\, \eta_{-\,+}\,H_{\widetilde{\beta}\,\mathbf{a}}\\
&\quad +\, \tfrac{1}{r_{AdS}}\,(\gamma_{-})_{\beta\,\nu}\,(\widetilde{\Gamma}_{5})^{\nu\,\sigma}\,H_{+\mathbf{a}\,\nu}
\end{aligned}
$$

In the table (11) we derived the relation between $H_{+\mathbf{a}\,\nu}$ and $H_{+\mathbf{a}\,\widetilde{\nu}}$. That result together with (19.3) we get:

$$
\left(P_{-} \,-\, \tfrac{1}{P_{\widetilde{-}}}\,\tfrac{1}{(r_{AdS})^2}\right) H_{+\mathbf{a}\,\widetilde{\alpha}} \;=\; H_{\mathbf{a}\,\widetilde{\alpha}} \qquad (19.3)
$$

With the equation (19.3) we succeeded to calculate the last missing derivative $D_{\alpha'}\,H_{\beta\,\widetilde{\sigma}}$ in terms of $\mathbb{H}$ vielbeins:

$$
\begin{aligned}
&D_{\alpha'}\,H_{\beta\,\widetilde{\sigma}} \,-\, 2\,(\,P_{-} \,-\, \tfrac{1}{P_{\widetilde{-}}}\,\tfrac{1}{(r_{AdS})^2}\,)\,(\gamma^{\mathbf{a}})_{\alpha'\,\beta}\,H_{+\mathbf{a}\,\widetilde{\sigma}} \qquad (19.4)\\
&-\,\tfrac{1}{2\,(r_{AdS})^2\,P_{\widetilde{-}}}\,(\widetilde{\Gamma}_{5})_{\sigma}{}^{\nu}\,(\gamma^{\mathbf{d}})_{\nu\,\alpha'}\,(\widetilde{\Gamma}_{5})_{\beta}{}^{\rho}\,H_{\widetilde{\rho}+\mathbf{d}} \,-\, \tfrac{1}{2\,(r_{AdS})}\,(\widetilde{\Gamma}_{5})_{\sigma}{}^{\nu}\,(\gamma^{\mathbf{d}})_{\nu\,\alpha'}\,H_{\beta\,\widetilde{+\mathbf{d}}} \;=\; 0\\
&D_{\widetilde{\alpha}'}\,H_{\widetilde{\beta}\,\sigma} \,-\, 2\,(\,P_{\widetilde{-}} \,-\, \tfrac{1}{P_{-}}\,\tfrac{1}{(r_{AdS})^2}\,)\,(\gamma^{\mathbf{a}})_{\alpha'\,\beta}\,H_{\widetilde{+\mathbf{a}}\,\sigma} \qquad (19.5)\\
&-\,\tfrac{1}{2\,(r_{AdS})^2\,P_{-}}\,(\widetilde{\Gamma}_{5})_{\sigma}{}^{\nu}\,(\gamma^{\mathbf{d}})_{\nu\,\alpha'}\,(\widetilde{\Gamma}_{5})_{\beta}{}^{\rho}\,H_{\rho\,\widetilde{+\mathbf{d}}} \,-\, \tfrac{1}{2\,(r_{AdS})}\,(\widetilde{\Gamma}_{5})_{\sigma}{}^{\nu}\,(\gamma^{\mathbf{d}})_{\nu\,\alpha'}\,H_{\widetilde{\beta}+\mathbf{d}} \;=\; 0
\end{aligned}
$$

Where the $(\widetilde{\Gamma}_{5})_{\sigma}{}^{\nu} := (\gamma^{+})_{\sigma\,\lambda}\,(\widetilde{\Gamma}_{5})^{\lambda\,\nu}$. The AdS equations (17.23) and (A.59) and (A.60) and (A.65) and (A.66) and (19.4) and (19.5) could be summarised in the following diagram:

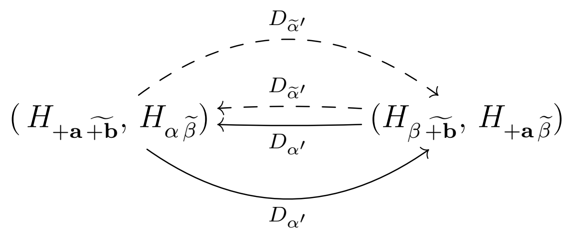

Figure 5: $AdS_5 \times S^5$ space diagram

From the above diagram is obvious that we again have to combine $H_{+\mathbf{a}\widetilde{+\mathbf{b}}}$ and $H_{\alpha\widetilde{\beta}}$ and derivatives $D_{\widetilde{\alpha}'}$ and $D_{\alpha'}$ to get a vanishing derivative on some scalar.

19.2 The $\mathbb{H}$ matrix in $AdS_5 \times S^5$

We want to repeat the chapter on the flat solution via the $\mathbb{H}$ matrix. The $\mathbb{H}$ matrix was defined in (18.30). We want to write the action $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on the $\mathbb{H}$. This was given in components in equations: (17.23) and (A.59) and (A.60) and (A.65) and (A.66) and (19.4) and (19.5) and also graphically in (5). We expect that the resulting matrix equations have pieces given by the flat equations (18.18) and (18.19) plus purely AdS pieces (dependent as powers of $\frac{1}{r_{AdS}}$). We could write those equations in such explicit matrix form, but resulting equations are complicated and unnecessary for our purpose. We instead summarise the right hand side of $D_{\alpha'}\,\mathbb{H}$ and $D_{\widetilde{\alpha}'}\,\mathbb{H}$ using two new matrices $\mathbb{X}_{\alpha'}$ and $\mathbb{Y}_{\widetilde{\alpha}'}$ respectively. We propose matrix from of the AdS equations:

$$D_{\alpha'}\,\mathbb{H} = \mathbb{X}_{\alpha'} \tag{19.6}$$

$$D_{\widetilde{\alpha}'}\,\mathbb{H} = \mathbb{Y}_{\widetilde{\alpha}'} \tag{19.7}$$

The matrices $\mathbb{X}_{\alpha'}$ and $\mathbb{Y}_{\widetilde{\alpha}'}$ are fully fixed by (17.23) and (A.59) and (A.60) and (A.65) and (A.66) and (19.4) and (19.5). In the $r_{AdS} \to \infty$ the $\mathbb{X}_{\alpha'} \to \mathit{\Gamma}_{\alpha'}\,\mathbb{H}$ and $\mathbb{Y}_{\widetilde{\alpha}'} \to \mathbb{H}\,\mathit{\Gamma}_{\widetilde{\alpha}'}$, where the matrices $\mathit{\Gamma}_{\alpha'}$ and $\mathit{\Gamma}_{\widetilde{\alpha}'}$ are given in (18.18) and (18.19).

19.3 Chiral and projective solutions for $AdS_5 \times S^5$

In the next step we repeat the argument we gave in the flat case section but for the AdS equations (19.6) and (19.7). We define:

$$\mathcal{D}_v \equiv \mathcal{D}^v{}_{\alpha'} := (D_{\alpha'} + \mathcal{M}_{\alpha'}{}^{\beta'} D_{\widetilde{\beta}'}) \tag{19.8}$$

now we act by (19.8) on $\mathbb{H}$:

$$\mathcal{D}^v{}_{\alpha'} \, \mathbb{H} = (\mathbb{X}_{\alpha'} + \mathcal{M}_{\alpha'}{}^{\beta'} \, \mathbb{Y}_{\widetilde{\beta}'}) \tag{19.9}$$

we multiply by $\mathbb{B}$ and apply Str:

$$\mathcal{D}^v{}_{\alpha'} \, \mathrm{Str} (\mathbb{H} \mathbb{B}) = \mathrm{Str} \big((\mathbb{X}_{\alpha'} + \mathcal{M}_{\alpha'}{}^{\beta'} \, \mathbb{Y}_{\widetilde{\beta}'}) \, \mathbb{B} \big) \tag{19.10}$$

We will further analyse the structure of (19.10) in next discussion but before we note one change with respect to (18.20). In (18.20) we used $\mathbb{B}^{-1}$ here we are using (yet to be determined) matrix $\mathbb{B}$, the difference is purely conventional. As in the flat case, we want to put the right hand side of (19.10) to zero and by that obtain vanishing $\mathcal{D}^v{}_{\alpha'}$ on some scalar field $\mathrm{Str} (\mathbb{H} \mathbb{B})$, that will be called pre-potential. In the flat space it was crucial that we had the identity (18.26). It was used in the relation (18.21). Similarly in the AdS case the identity (18.26) will also be crucial.

In the solution of the vanishing (19.10) right hand side we still want to maintain the $SO(4) \otimes SO(4)$ invariance. Therefore the $\mathbb{B}$ matrix has a block-diagonal form:

$$\mathbb{B} := \begin{pmatrix} b_{+\mathbf{a}+\mathbf{b}} & 0 \\ 0 & b_{\alpha\beta} \end{pmatrix} \tag{19.11}$$

Let us also simplify the notation for the constants appearing in the equations (17.23) and (19.4) and similarly for their left-right conjugates. In (17.23) we redefine:

$$X_1 := -\tfrac{1}{g} \tfrac{1}{(r_{AdS}) P_{\underset{\sim}{-}}} \quad || \quad X_2 := \tfrac{1}{2g} (1 - \tfrac{1}{f} \tfrac{1}{(r_{AdS})^2 P_- P_{\underset{\sim}{-}}}) \tag{19.12}$$
$$X_3 := + \tfrac{1}{fg} \tfrac{1}{2 (r_{AdS})^3 P_- (P_{\underset{\sim}{-}})^2} \quad ||$$

where the f and g were defined in (17.24). In (19.4) we define:

$$Y_1 := -2 \Big(P_{\underset{\sim}{-}} - \tfrac{1}{P_-} \tfrac{1}{(r_{AdS})^2} \Big) \;||\; Y_2 := -\tfrac{1}{2 (r_{AdS})^2 P_-} \tag{19.13}$$
$$Y_3 := -\tfrac{1}{2 (r_{AdS})}$$

With the definitions (19.11), (19.12) and (19.13) let us rewrite the right hand side of (19.10) explicitly:

$$
\begin{aligned}
0 \;=\;& (\, X_1 \,-\, X_3\,)\, (\widetilde{\Gamma}_5)^{\beta\,\sigma}\, b_{+\mathbf{a}+\mathbf{b}}\, (\gamma^{\mathbf{b}})_{\sigma\,\alpha'} \,-\, \mathcal{M}_{\alpha'}{}^{\sigma'}\, X_2\, b_{+\mathbf{a}+\mathbf{b}}\, (\gamma^{\mathbf{b}})^{\beta}{}_{\sigma'} \\
&+ Y_1\, b^{\beta\,\sigma}\, (\gamma_{\mathbf{a}})_{\sigma\,\alpha'} \,+\, Y_2\, b^{\nu\,\sigma}\, (\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\, (\gamma_{\mathbf{a}})_{\lambda\,\alpha'}\, (\widetilde{\Gamma}_5)_{\nu}{}^{\beta} \qquad (19.14)\\
&+ \mathcal{M}_{\alpha'}{}^{\sigma'}\, Y_3\, b^{\beta\,\sigma}\, (\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\, (\gamma_{\mathbf{a}})_{\lambda\,\sigma'} \\
0 \;=\;& \mathcal{M}_{\alpha'}{}^{\nu'}\, (\, \tilde{X}_1 \,-\, \tilde{X}_3\,)\, (\widetilde{\Gamma}_5)^{\beta\,\sigma}\, b_{+\mathbf{a}+\mathbf{b}}\, (\gamma^{\mathbf{b}})_{\sigma\,\nu'} \,-\, X_2\, b_{+\mathbf{a}+\mathbf{b}}\, (\gamma^{\mathbf{b}})^{\beta}{}_{\alpha'} \\
&+ \mathcal{M}_{\alpha'}{}^{\nu'}\, \tilde{Y}_1\, b^{\beta\,\sigma}\, (\gamma_{\mathbf{a}})_{\sigma\,\nu'} \,+\, \mathcal{M}_{\alpha'}{}^{\nu'}\, \tilde{Y}_2\, b^{\nu\,\sigma}\, (\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\, (\gamma_{\mathbf{a}})_{\lambda\,\nu'}\, (\widetilde{\Gamma}_5)_{\nu}{}^{\beta} \\
&+ Y_3\, b^{\beta\,\sigma}\, (\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\, (\gamma_{\mathbf{a}})_{\lambda\,\alpha'} \qquad (19.15)
\end{aligned}
$$

where $\tilde{X}_1$, $\tilde{X}_3$ and $\tilde{Y}_1$, $\tilde{Y}_2$ are left-right conjugates of the constants defined in (19.12) and (19.13) and X_2 and Y_3 are the same after left-right swap.

The equations (19.14) and (19.15) are the AdS analogies of the flat space equations (18.39). To solve them we first multiply the equation (19.15) by matrix $\mathcal{M}_{\alpha'}{}^{\beta'}$. Thus we get the equation (19.15) into the form:

$$
\begin{aligned}
0 \;=\;& \mathcal{M}^2{}_{\alpha'}{}^{\nu'}\, (\, \tilde{X}_1 \,-\, \tilde{X}_3\,)\, (\widetilde{\Gamma}_5)^{\beta\,\sigma}\, b_{+\mathbf{a}+\mathbf{b}}\, (\gamma^{\mathbf{b}})_{\sigma\,\nu'} \,-\, \mathcal{M}_{\alpha'}{}^{\nu'}\, X_2\, b_{+\mathbf{a}+\mathbf{b}}\, (\gamma^{\mathbf{b}})^{\beta}{}_{\nu'} \\
&+ \mathcal{M}^2{}_{\alpha'}{}^{\nu'}\, \tilde{Y}_1\, b^{\beta\,\sigma}\, (\gamma_{\mathbf{a}})_{\sigma\,\nu'} \,+\, \mathcal{M}^2{}_{\alpha'}{}^{\nu'}\, \tilde{Y}_2\, b^{\nu\,\sigma}\, (\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\, (\gamma_{\mathbf{a}})_{\lambda\,\nu'}\, (\widetilde{\Gamma}_5)_{\nu}{}^{\beta} \qquad (19.16)\\
&+ \mathcal{M}_{\alpha'}{}^{\nu'}\, Y_3\, b^{\beta\,\sigma}\, (\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\, (\gamma_{\mathbf{a}})_{\lambda\,\nu'}
\end{aligned}
$$

The equation (19.16) is almost identical to the (19.14) except of the left-right swapped constants and $\mathcal{M}^2$ matrix. By suitable choice of the $\mathcal{M}$ matrix we can turn (19.16) into (19.14) and thus reduce number of equations by half. By that we get the condition on the matrix $\mathcal{M}$:

$$
\mathcal{M}^2{}_{\alpha'}{}^{\beta'} \;=\; q^2\, \delta_{\alpha'}{}^{\beta'} \qquad (19.17)
$$

where the constant $q^2 \;=\; \frac{P_{-}}{P_{\tilde{-}}}$. By that choice of the matrix $\mathcal{M}^2$ and constant q^2 we turn equation (19.16) into (19.14). Furthermore we should solve relation (19.17) for the matrix $\mathcal{M}$. As in the whole AdS section we ask for the $SO\,(\,4\,)\,\otimes\, SO\,(\,4\,)$ invariance. With that requirement we get two branches for the $\mathcal{M}$ matrix (actually we get four, as we will see, but the $\pm$ is not very important to us):

$$
(\mathcal{M}^2)_{\alpha'}{}^{\beta'} \;=\; \frac{P_{-}}{P_{\tilde{-}}}\, \delta_{\alpha'}{}^{\beta'} \quad\Rightarrow\quad \mathcal{M}_{\alpha'}{}^{\beta'} \;=\; \pm\, \sqrt{\frac{P_{-}}{P_{\tilde{-}}}} \begin{cases} \delta_{\alpha'}{}^{\beta'} \\ (\widetilde{\Gamma}_5)_{\alpha'}{}^{\beta'} \end{cases} \qquad (19.18)
$$

We first notice few nice properties of (19.18). The solution is actually the same as in the flat case, see (18.38). We are in the AdS space but the matrix $\mathcal{M}$ that combines $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ does not depend on the r_{AdS}. Unfortunately we got the same not very nice square root factor in (19.18). We would need to find some way to deal with it.

Having solved one half of equations (19.14) and (19.15). We solve the second half, that is just relation (19.14):

$$\begin{aligned} 0 \;=\;& (X_1 - X_3)\,(\widetilde{\Gamma}_5)^{\beta\sigma}\, b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})_{\sigma\alpha'} - \mathcal{M}_{\alpha'}{}^{\sigma'}\, X_2\, b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})^{\beta}{}_{\sigma'} \\ & + Y_1\, b^{\beta\sigma}\,(\gamma_{\mathbf{a}})_{\sigma\alpha'} + Y_2\, b^{\nu\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\alpha'}\,(\widetilde{\Gamma}_5)_{\nu}{}^{\beta} \\ & + \mathcal{M}_{\alpha'}{}^{\sigma'}\, Y_3\, b^{\beta\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\sigma'} \end{aligned} \qquad (19.19)$$

The claim is that given solution $\mathcal{M}$ the block matrices $b_{+\mathbf{a}+\mathbf{b}}$ and $b_{\alpha\beta}$ are fixed (up to the overall constant). We will again use the same identity (18.26) as in the flat case. We expect the solutions (we have two branches) will be certain r_{AdS} dependent deformation of the original flat space solutions. We also require to maintain the $SO(4) \otimes SO(4)$ invariance of the solution so the most general ansatz for the equation (19.19) is:

$$b_{+\mathbf{a}+\mathbf{b}} := A\,\delta_{\mathbf{a}\mathbf{b}} + B\,(\widetilde{\Gamma}_5)_{\mathbf{a}\mathbf{b}} \;\;||\;\; b_{\alpha\beta} := C\,\delta_{\alpha\beta} + D\,(\widetilde{\Gamma}_5)_{\alpha\beta} \qquad (19.20)$$

Because of later importance we will first solve the $(\widetilde{\Gamma}_5)$ branch of the $\mathcal{M}$ solution (19.18). Later we will also provide solution for the δ branch of the (19.18). We plug $\mathcal{M}$ and (19.20) into (19.19) and solve for A, B, C and D using the identity (18.26), we remind that $q := \pm\sqrt{\frac{P_-}{P_{\widetilde{-}}}}$. We get the following solutions:

the $(\widetilde{\Gamma}_5)_{\alpha'}{}^{\beta'}$ branch: (19.21)

$$\begin{aligned} det &:= \big((X_1 - X_3)^2 - (q\,X_2)^2\big) \\ A &= \tfrac{D}{det}\big((X_1 - X_3)(Y_1 + Y_2) + q^2\, X_2\, Y_3\big) \\ B &= q\,\tfrac{D}{det}\big(X_2\,(Y_1 + Y_2) + (X_1 - X_3)\,Y_3\big) \\ C &= 0 \end{aligned}$$

the $\delta_{\alpha'}{}^{\beta'}$ branch: (19.22)

$$\begin{aligned} det &:= \big((Y_1 + Y_2)^2 - (q\,Y_3)^2\big) \\ B &= 0 \\ C &= q\,\tfrac{A}{det}\big((Y_1 + Y_2)\,X_2 + (X_1 - X_3)\,Y_3\big) \\ D &:= -\tfrac{A}{det}\big((Y_1 + Y_2)(X_1 - X_3) + q^2\, X_2\, Y_3\big) \end{aligned}$$

We can again see that as we do $r_{AdS} \to \infty$ limit in (19.21) we will get the flat solution (18.38), keeping the D (or A in δ branch) r_{AdS} independent in that limit.

19.4 Near horizon limit

In the previous section we found the structure of the linearised pre-potential (19.21) and (19.22) and also the construction of $\mathcal{D}^v{}_{\alpha'}$ that vanishes on the pre-potential (19.8) and (19.18). We will now introduce the complementary derivative $\mathcal{D}^w{}_{\alpha'}$ that is constructed after picking the $\mathcal{D}^v{}_{\alpha'}$ derivative (i.e. picking the matrix $\mathcal{M}$ in (19.18)) and changing the sign in front of the $\mathcal{M}$ (the second linearly independent combination). Thus we have:

$$\mathcal{D}^w{}_{\alpha'} := D_{\alpha'} - \mathcal{M}_{\alpha'}{}^{\beta'} D_{\widetilde{\beta}'} \tag{19.23}$$

The notation for the upper indices v and w in ((19.8) and (19.23)) comes from equivalent notation for d_v and d_w derivatives used in [8], (and also for d_u and $d_{\bar{u}}$, whose analogies are to be defined later). In analogy with the paper [8] we want to define the $\mathcal{P}_+$ operator that has $\mathcal{D}^v{}_{\alpha'}$ and $\mathcal{D}^w{}_{\alpha'}$ as eigenvectors with nonzero eigenvalues. We can solve for $\mathcal{P}_+$ in full generality, i.e. keeping the non-local square root factors in derivatives $\mathcal{D}^v{}_{\alpha'}$ and $\mathcal{D}^w{}_{\alpha'}$. This would introduce the non-local square root factors also into the definition of $\mathcal{P}_+$ and would cause further problems. What we will do instead is to restrict the coordinate dependence of the pre-potential V to be just the $PSU\,(2,\,2\,|\,4)$. This is the same algebra we wanted to use at the beginning of this project, but we were forced to extend it to the full $SO\,(10) \otimes SO\,(10)$ T-dually extended super-algebra. Now, we want to restrict just the coordinate dependence of the pre-potential. Doing so the $P_- = P_{\widetilde{-}}$ **on** pre-potential, not everywhere. That is enough to get rid of the non-local factors in $\mathcal{D}^v{}_{\alpha'}$ and $\mathcal{D}^w{}_{\alpha'}$ as they act on pre-potential. Then we can redefine (19.8) and (19.23) by saying that the new square root free $\mathcal{D}^v$ and $\mathcal{D}^w$ to be our new definitions. With this it is easy to see that the good definition of $\mathcal{P}_+$ is:

$$\mathcal{P}_+ := \tfrac{1}{2}\,(\,P_+ + P_{\widetilde{+}}\,) = P_+ \tag{19.24}$$

where the last equality holds on pre-potential.

Following the definitions in [8] of the AdS boundary limit we propose that any operator $\mathcal{K}$ which is an eigenvector of $\mathcal{P}_+$ operator, i.e. $[\,\mathcal{P}_+,\,\mathcal{K}\,] = \mathbf{c}\,\mathcal{K}$, scales as $R^{\mathbf{c}}$ as we approach the boundary, i.e. $R \to 0$ limit, where R is a radial coordinate on the Poincaré patch. Another way how to state the limit is that by putting the $R \to 0$ we contract the isometry groups $SO\,(\,4,\,1\,)$ and $SO\,(\,4,\,1\,)$ to $ISO\,(\,3,1\,)$ and $ISO\,(\,3,1\,)$ (we Wick rotated the S^5 isometry group for the purpose of this limit). For more details on this limit (that can be stated also through the explicit coordinates on AdS_5 and S^5) see notes [8].

Using the previous definitions of the AdS boundary limit we can analyse the different branches of the $\mathcal{D}^v$ solutions (19.18). Let's first pick the $\delta_{\alpha'}{}^{\beta'}$ branch (let's work with both $\pm$ sub-branches at once). Note that even on pre-potential the $D_{\alpha'} \neq D_{\widetilde{\alpha}'}$ as can be seen from the explicit construction of those derivatives in [17] in the section **5**. Then the commutator is:

$$[\,\mathcal{P}_+,\, D_{\alpha'} \,+\, D_{\widetilde{\alpha}'}] \,=\, \pm\, \tfrac{1}{r_{AdS}}\, (\gamma_+)_{\alpha'\,\beta'}\, (\widetilde{\Gamma}_5)^{\beta'\,\sigma'}\, (D_{\sigma'} \,\pm\, D_{\widetilde{\sigma}'}) \,+\, \ldots \quad (19.25)$$

The $\ldots$ part correspond to the current that vanishes in the supergravity limit (i.e. we do not see string parameter σ) and on pre-potential. We also used the commutators from (14.7) and the mixed AdS commutators from (15.9). We also used the explicit solution for the $PSU\,(\,2,\,2\,|\,4\,)$ (we are on pre-potential) derivatives in terms of τ and σ currents, see section **5** in [17]. More specifically we used that $D_\Omega \,\equiv\, D^{\alpha'} \,=\, \omega^{\alpha'} + \frac{1}{2}\,\frac{1}{r_{AdS}}\,(\widetilde{\Gamma}_5)^{\alpha'\,\beta'}\, D_{\widetilde{\beta}'}$, where the $\omega^{\alpha'}$ is the current proportional to σ derivative and it has to vanish in the supergravity limit. The equation (19.25) is very interesting. It tells us how the $\mathcal{D}^v$ scales for the $\pm\,\delta_{\alpha'}{}^{\beta'}$ branch of (19.18). We also notice that the scaling constant is r_{AdS} dependent and vanishes for $r_{AdS} \to \infty$. More importantly because of the $(\widetilde{\Gamma}_5)$ for fixed r_{AdS} and for fixed sub-branch of $\pm\,\delta_{\alpha'}{}^{\beta'}$ the scaling constant $\mathbf{c}$ is either $+\,(\frac{1}{r_{AdS}})$ for one half of $SO\,(8)$ chiral index α' or $-\,(\frac{1}{r_{AdS}})$ for second half. And this is not good because by [8] the $\mathcal{D}^v$ derivative should scale like $\frac{1}{R}$ and $\mathcal{D}^w$ should scale like R (put $r_{AdS} \,=\, 1$ for simplicity). In (19.25) we can see that just $\frac{1}{2}$ of derivatives scale properly. This boundary limit then distinguishes between two branches of (19.18). In the following we will see that the $(\widetilde{\Gamma}_5)$ branch has exactly right scaling properties so it corresponds to the right solution. Without this boundary limit we did not have a way how to pick a branch in (19.18). In the case of $(\widetilde{\Gamma}_5)$ branch we have one more $(\widetilde{\Gamma}_5)$ matrix in (19.25) thus we get:

$$[\,\mathcal{P}_+,\, D_{\alpha'} \,\pm\, (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'}\, D_{\widetilde{\rho}'}] \,=\, \pm\, \tfrac{1}{r_{AdS}}\, (D_{\alpha'} \,\pm\, (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'}\, D_{\widetilde{\rho}'}) \,+\, \ldots \quad (19.26)$$

The equation (19.26) will give us the correct solution. From (19.26) we can see that for fixed r_{AdS} and for fixed $(\widetilde{\Gamma}_5)$ sub-branch we will have proper scaling for full $SO(8)$ chiral index α'. Because we require $\mathcal{D}^v$ to scale like $\frac{1}{R}$ and $\mathcal{D}^w$ scale like R we have $\mathcal{D}^v{}_{\alpha'} = (D_{\alpha'} - (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'} D_{\widetilde{\rho}'})$ and $\mathcal{D}^w{}_{\alpha'} = (D_{\alpha'} + (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'} D_{\widetilde{\rho}'})$. The positive news is that the blowing-up derivative $\mathcal{D}^v$ is zero on the pre-potential by our construction, so there is no possible singularity arising as we approach the boundary.

Its easy to see how the derivatives D_α and $D_{\widetilde{\alpha}}$ scale. Because the $(\gamma_+)_{\alpha\beta} = 0$ the $[\mathcal{P}_+, D_\alpha] = [\mathcal{P}_+, D_{\widetilde{\alpha}}] = 0$. So they scale like 1. Those derivatives are building up the $\mathcal{D}^u$ and $\mathcal{D}^{\bar{u}}$, analogous derivatives to paper [8] derivatives d_u and $d_{\bar{u}}$. The explicit forms of $\mathcal{D}^u$ and $\mathcal{D}^{\bar{u}}$ won't be needed in this paper so we do not provide them.

19.5 Near horizon limit and field equations

Comparing result with [8] we want to see that the field equations for the pre-potential in the near horizon limit (i.e. in the $R \to 0$) is just of the form $\mathcal{P}_+ V = 0 + \mathcal{O}(R)$. This will be our final confirmation that we discovered the right pre-potential. We first notice that the Lorentz generator scales like $\mathcal{O}(1)$, this can be seen from commutator $[S_{+\mathbf{a}}, \mathcal{P}_+] = [S_{\widetilde{+\mathbf{a}}}, \mathcal{P}_+] = 0$. To see what is $\mathcal{P}_+$ on pre-potential we could directly use some appropriate torsions (remember pre-potential is a linear combination of fields). We found it easier however to use a different approach. Let's look at the torsion constraint (17.13) but for the α index instead of α' (the α index is one of the $SO(8)$ chiral indices):

$$
\begin{aligned}
T_{D S \widetilde{S}} \equiv T_{\alpha +\mathbf{a} \widetilde{+\mathbf{b}}} = 0 &= D_{[\alpha} H_{+\mathbf{a}\widetilde{+\mathbf{b}})} + H_{[\alpha|\mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} f_{+\mathbf{a}\widetilde{+\mathbf{b}})\mathcal{N}} \quad (19.27)\\
&= D_\alpha H_{+\mathbf{a}\widetilde{+\mathbf{b}}} + S_{\widetilde{+b}} H_{\alpha+\mathbf{a}} + S_{+\mathbf{a}} H_{\widetilde{+\mathbf{b}}\alpha} \quad (19.28)
\end{aligned}
$$

First notice that the structure of (19.28) is very different than the structure of (17.13). There is no f term in (19.28) and there is the full derivative term present. Even in the AdS case the f term is missing. This can be seen as follows. The $f_{+\mathbf{a}\widetilde{+\mathbf{b}}\mathcal{N}} = 0$ in AdS and also in flat case and also $f_{\alpha\widetilde{+\mathbf{b}}\mathcal{N}} = 0$. The only possibly nonzero f term is coming from $f_{\alpha+\mathbf{a}\mathcal{N}}$. The $H_{\widetilde{+\mathbf{b}}\mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} f_{\alpha+\mathbf{a}\mathcal{N}} \propto (\gamma_{+\mathbf{a}})_\alpha{}^{\nu'} H_{\nu'\widetilde{+\mathbf{b}}}$. The vielbein $H_{\nu'\widetilde{+\mathbf{b}}}$ is zero (also in the AdS) as was shown in the analysis under (17.13). Next, we can recognise the term $H_{\alpha\widetilde{+\mathbf{a}}}$ as a part of $\mathbb{H}$ matrix (18.17). The vielbein $H_{\alpha+\mathbf{a}}$ has also been analysed in table (11). It is related to $H_{+\mathbf{a}\widetilde{\alpha}}$, see table (11). We need to be more careful with that relation because in (19.28) we again discover the S derivative peculiarity, we saw earlier.

In general all fields in $\mathbb{H}$ (now better viewed as their irreducible pieces) could be obtained from the pre-potential V by an action of appropriate (irreducible) combination of $\mathcal{D}^w$ on the pre-potential. One could analyse in full detail what is the exact structure of those pieces and reproduce famous field content of $AdS_5 \times S^5$ supergravity first discovered in [36] and later used in [28]. This would lead us away from this paper real aim, so we postpone this analysis to next paper. The aim of this section is to show that on pre-potential V the operator $\mathcal{P}_+$ vanishes in the near horizon limit.

For this reason we notice following expansions:

$$
\begin{aligned}
H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} &= \mathbf{c}_0\,V + \mathbf{c}_2\,(\mathcal{D}^w)^2\,V + \mathbf{c}_4\,(\mathcal{D}^w)^4\,V + \mathbf{c}_6\,(\mathcal{D}^w)^6\,V \qquad (19.29)\\
&\quad + \mathbf{c}_8\,(\mathcal{D}^w)^8\,V\\
H_{+\mathbf{a}\,\widetilde{\alpha}} &= \mathbf{d}_1\,\mathcal{D}^w\,V + \mathbf{d}_3\,(\mathcal{D}^w)^3\,V + \mathbf{d}_5\,(\mathcal{D}^w)^5\,V + \mathbf{d}_7\,(\mathcal{D}^w)^7 V \qquad (19.30)\\
H_{\widetilde{+\mathbf{a}}\,\alpha} &= \mathbf{e}_1\,\mathcal{D}^w\,V + \mathbf{e}_3\,(\mathcal{D}^w)^3\,V + \mathbf{e}_5\,(\mathcal{D}^w)^5\,V + \mathbf{e}_7\,(\mathcal{D}^w)^7 V \qquad (19.31)
\end{aligned}
$$

where factors $\mathbf{c_0}$, $\mathbf{c_2}$ ..., $\mathbf{d_1}$, $\mathbf{d_3}$ ... and $\mathbf{e_1}$, $\mathbf{e_3}$ ... are constant factors with appropriate index structure. Note that the $\mathbf{c_0}$ is non-zero. The important observation is that for each term in (19.29), (19.30) and (19.31) we know how it scales in the $R \to 0$ limit, because we know that $\mathcal{D}^w$ scales like R.

Next, we want to combine (19.28) with known scalings of all (19.28) objects to get an information how $D_\alpha V$ scales. On one hand it should scale like $\mathcal{O}(1)$ on the other hand the relation (19.28) relates it to different fields. What we obtain is a nontrivial relation that $D_\alpha V = \mathcal{O}(R)$ as we go to the boundary. It just means that $D_\alpha V = 0$ (and so also $D_{\widetilde{\alpha}} V = 0$) in the near horizon limit. Because of the anti-commutator $\{D_\alpha, D_\beta\} = 2\,(\gamma_-)_{\alpha\beta}\,P_+$. This is enough to see that $P_+ V \equiv \mathcal{P}_+ V = 0$ in the near horizon limit. There are two crucial steps. One is to relate the (19.28) term $S_{\widetilde{+\mathbf{b}}}\,H_{\alpha+\mathbf{a}}$ to $H_{+\mathbf{a}\,\widetilde{\alpha}}$. This is relatively straightforward using table (11) and explicit $S_{\widetilde{+\mathbf{b}}}$ derivative. Second step is to plug expansions (19.29), (19.30) and (19.31) and the scalings of particular pieces into (19.28). Doing that we get the following:

$$
\begin{aligned}
0 &= D_\alpha H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} + S_{\widetilde{+\mathbf{b}}} H_{\alpha\,+\mathbf{a}} + S_{+\mathbf{a}} H_{\widetilde{+\mathbf{b}}\,\alpha} && (19.32)\\
&= D_\alpha H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} + S_{\widetilde{+\mathbf{b}}} \Big(- f_{\widetilde{-}\,\alpha\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}} \tfrac{1}{P_{\widetilde{-}}} H_{+\mathbf{a}\,\mathcal{N}} \Big) + S_{+\mathbf{a}} H_{\widetilde{+\mathbf{b}}\,\alpha} \\
&= D_\alpha H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} - \tfrac{\mathbf{c}}{r_{AdS}} (\gamma_-)_{\alpha\,\nu} \, (\,\widetilde{\mathit{\Gamma}}_5\,)^{\nu\,\sigma} S_{\widetilde{+\mathbf{b}}} \tfrac{1}{P_{\widetilde{-}}} H_{+\mathbf{a}\,\widetilde{\sigma}} + S_{+\mathbf{a}} H_{\widetilde{+\mathbf{b}}\alpha} && (19.33)\\
&= D_\alpha \Big(\mathbf{c}_0\, V + \mathbf{c}_2\, (\mathcal{D}^w)^2\, V + \ldots \Big) && (19.34)\\
&- \tfrac{1}{r_{AdS}} (\gamma_-)_{\alpha\,\nu} \, (\,\widetilde{\mathit{\Gamma}}_5\,)^{\nu\,\sigma} S_{\widetilde{+\mathbf{b}}} \tfrac{1}{P_{\widetilde{-}}} \Big(\mathbf{d}_1\, \mathcal{D}^w\, V + \mathbf{d}_3\, (\mathcal{D}^w)^3\, V + \ldots \Big) \\
&+ S_{+\mathbf{a}} \Big(\mathbf{e}_1\, \mathcal{D}^w\, V + \mathbf{e}_3\, (\mathcal{D}^w)^3\, V + \ldots \Big)
\end{aligned}
$$

The equation (19.33) contains all the right expressions to establish the near horizon limit. By the discussion below (19.26) the ${\mathcal{D}^w}_{\alpha'}$ derivative scales like $\mathcal{O}\,(\,R\,)$ (for the projective branch), we also have the scaling of $S_{+\mathbf{a}}$ and $S_{\widetilde{+\mathbf{b}}}$ that goes like a constant. Applying that knowledge we get the equation (19.33) in the near horizon limit:

$$
0 \;=\; \mathbf{c}_0\, D_\alpha\, V + \mathcal{O}\,(R) \qquad (19.35)
$$

The $\mathbf{c_0}$ is nonzero constant (tensor) so it follows that $D_\alpha V = 0$ at the AdS boundary. From $\{\, D_\alpha,\, D_\beta \,\} = (\gamma_-)_{\alpha\,\beta}\, P_+$ we get the field equation for the pre-potential in the near horizon limit:

$$
\begin{aligned}
0 &= P_+ V + \mathcal{O}\,(R) && (19.36)\\
&\equiv \mathcal{P}_+ V + \mathcal{O}\,(R) && (19.37)
\end{aligned}
$$

Chapter 20

20 Conclusion: Pre-potential in the $AdS_5 \times S^5$ Type IIB superspace

We outline results we have obtained: starting from the 10 dimensional IIB string theory. We embedded the $AdS_5 \times S^5$ background and expanded the theory around this background (we also considered a flat background, i.e. $AdS_5 \times S^5$ with $r_{AdS} \to \infty$). Our aim was to obtain (linearised) pre-potential with desired properties in the case of $AdS_5 \times S^5$ (also in the flat case). We succeeded and obtained pre-potential construction for flat and $AdS_5 \times S^5$ background. We derived only the linearised form, but the vielbein construction makes non-linearisation straightforward perturbation. The pre-potential (in flat and also in $AdS_5 \times S^5$, the projective and chiral) sits in the combination (without further derivatives) of vielbeins $H_{S\,\widetilde{S}}$ and $H_{D\,\widetilde{D}}$. By construction the D^v derivative vanishes in bulk on the pre-potential and the (projective) pre-potential satisfies the near horizon limit field equation $\mathcal{P}_+ V = 0 + \mathcal{O}(R)$ together with vanishing of $\mathcal{D}^u$ and $\mathcal{D}^{\bar{u}}$ on pre-potential in the near horizon limit. This near horizon limit picks out the projective pre-potential instead of chiral pre-potential (both were obtained as valid bulk solutions).

The vanishing of $\mathcal{P}_+$ at the boundary fixes the difference between the conformal weights ($\equiv \Delta$) and $U(1)$ charges ($\equiv \Delta_Y$) of all boundary BPS operators, since $\mathcal{P}_+ \propto \Delta - \Delta_Y$. The $\mathcal{P}_- \propto \Delta + \Delta_Y$ and known expansion of $\mathbb{H}$ in powers of $\mathcal{D}^w$ from V, fixes the conformal weights and the $U(1)$ charges for the boundary BPS operators, the relations important in the AdS/CFT correspondence, see [29], [30].

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Appendices

A $AdS_5 \times S^5$ structure of some vielbeins and their derivatives

A.1 The $H_{S\,\widetilde{S}}$

Using equations (16.3), (16.4), (17.5) and the mixed light-cone gauge together with keeping the mixed structure constants and evaluating the explicit actions of the S and $\widetilde{S}$ derivatives we derived the first important result for the structure of the $H_{\mathcal{S}\widetilde{\mathcal{S}}}$ vielbein (in the AdS case). Note that by the symbol $\rightsquigarrow$ in the in the whole text we denoted the evaluation of the mixed structure constants in the sense described in section (17.2).

$H_{-\mathbf{a}\,\widetilde{-\mathbf{b}}} \,=\, -\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,S_{-a}\,(\,\frac{1}{P_{-}}\,H_{\widetilde{-b}\,\mathcal{N}}\,) \,\rightsquigarrow\, 0$
$H_{-\mathbf{a}\,\widetilde{+\mathbf{b}}} \,=\, -\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}}\,S_{\widetilde{+\mathbf{b}}}\,(\,\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}\,)$ $-\, f_{-\,\widetilde{\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}\,P_{\widetilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}$ $\rightsquigarrow\, -\,\frac{1}{2\,(r_{AdS})^2}\,\frac{1}{P_{-}\,P_{\widetilde{-}}}\,H_{-\mathbf{a}\widetilde{+\mathbf{b}}} \,\Rightarrow\, H_{-\mathbf{a}\,\widetilde{+\mathbf{b}}} \,=\, 0$
$H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \,=\, -\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}}\,S_{\widetilde{+\mathbf{b}}}\,(\,\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\mathcal{N}}\,)$ $-\, f_{-\,\widetilde{\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}\,P_{\widetilde{-}}}\,H_{+\mathbf{a}\mathcal{N}} \,+\, \frac{1}{P_{-}}\,H_{\widetilde{+\mathbf{b}}\,\mathbf{a}}$ $\rightsquigarrow\, -\,\frac{1}{2\,(r_{AdS})^2}\,\frac{1}{P_{-}\,P_{\widetilde{-}}}\,H_{+\mathbf{a}\widetilde{+\mathbf{b}}} \,+\, \frac{1}{P_{-}}\,H_{\widetilde{+\mathbf{b}}\,\mathbf{a}}$ $\Rightarrow\, (\,1 \,+\, \frac{1}{2\,(r_{AdS})^2}\,\frac{1}{P_{-}\,P_{\widetilde{-}}}\,)\,H_{+\mathbf{a}\widetilde{+\mathbf{b}}} \,=\, \frac{1}{P_{-}}\,H_{\mathbf{a}\,\widetilde{+\mathbf{b}}}$
$H_{-\mathbf{a}\,\widetilde{\mathbf{bc}}} \,=\, -\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}}\,S_{\widetilde{\mathbf{bc}}}\,(\,\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}\,) \,\rightsquigarrow\, 0$
$H_{+\mathbf{a}\,\widetilde{\mathbf{bc}}} \,=\, f_{\mathbf{a}\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\,H_{\widetilde{\mathbf{bc}}\,\mathcal{N}}$ $-\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,S_{+\mathbf{a}}\,(\,\frac{1}{P_{-}}\,H_{\widetilde{\mathbf{bd}}\,\mathcal{N}}\,)$ $\rightsquigarrow\, -\,\frac{1}{2\,(r_{AdS})^2}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\,H_{\widetilde{\mathbf{bc}}\,+\mathbf{a}} \Rightarrow\, H_{+\mathbf{a}\,\widetilde{\mathbf{bc}}} \,=\, 0$
$H_{\mathbf{ab}\,\widetilde{\mathbf{cd}}} \,=\, -\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}}\,S_{\widetilde{\mathbf{cd}}}\,(\,\frac{1}{P_{\widetilde{-}}}\,H_{\mathbf{ab}\mathcal{N}}\,) \,\rightsquigarrow\, 0$

Table 8: $H_{S\,\widetilde{S}}$ vielbein

$H_{+-\widetilde{+-}} = f_{-\tilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}\,P_{-}}\,H_{\widetilde{+-}\mathcal{N}}$ $-\,f_{-\tilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,S_{+-}\,(\,\frac{1}{P_{-}}\,H_{\widetilde{+-}\mathcal{N}}\,) \rightsquigarrow 0$
$H_{+-\widetilde{-\mathbf{a}}} = f_{-\tilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}\,P_{-}}\,H_{\widetilde{-\mathbf{a}}\mathcal{N}}$ $-\,f_{-\tilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,S_{+-}\,(\,\frac{1}{P_{-}}\,H_{\widetilde{-\mathbf{a}}\mathcal{N}}\,) \rightsquigarrow 0$
$H_{+-\widetilde{+\mathbf{a}}} = f_{-\tilde{\mathbf{a}}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}\,P_{\tilde{-}}}\,H_{+-\mathcal{N}}$ $-\,f_{-\tilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}}\,S_{\widetilde{+\mathbf{a}}}\,(\,\frac{1}{P_{\tilde{-}}}\,H_{+-\mathcal{N}}\,)$ $\rightsquigarrow \frac{1}{2\,(r_{AdS})^2}\,\frac{1}{P_{-}\,P_{\tilde{-}}}\,H_{+-\widetilde{+\mathbf{a}}} \Rightarrow H_{+-\widetilde{+\mathbf{a}}} = 0$
$H_{+-\widetilde{\mathbf{ab}}} = f_{-\tilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}\,P_{-}}\,H_{\widetilde{\mathbf{ab}}\mathcal{N}}$ $-\,f_{-\tilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,S_{+-}\,(\,\frac{1}{P_{-}}\,H_{\widetilde{\mathbf{ab}}\mathcal{N}}\,) \rightsquigarrow 0$

Table 9: $H_{S\,\widetilde{S}}$ vielbein

In the table (9) (and after the evaluation of mixed structure constants) we have heavily used the structure of the mixed structure constant $f_{\mathbf{a}\,\tilde{\mathbf{b}}\,\mathcal{M}}$ that is analysed in the main text, see analysis before equation (17.8). Moreover we used one more torsion constrain to fix $H_{P\,S}$ and $H_{\widetilde{P}\,\widetilde{S}}$ in the table (9). Let's take an example $H_{P\,S} = H_{\mathbf{a}\,\mathbf{bc}}$. To fix that vielbein we consider $T_{\tilde{-}\,\mathbf{a}\,\mathbf{bc}} = 0$:

$$\begin{aligned} T_{\widetilde{P}\,P\,S} \equiv T_{\tilde{-}\,\mathbf{a}\,\mathbf{bc}} &= 0 \qquad\qquad \text{(A.1)} \\ &= P_{\tilde{-}}\,H_{\mathbf{a}\,\mathbf{bc}} + S_{\mathbf{bc}}\,H_{\tilde{-}\,\mathbf{a}} + P_{\mathbf{a}}\,H_{\mathbf{bc}\,\tilde{-}} + H_{\mathbf{bc}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\tilde{-}\,\mathbf{a}\,\mathcal{N}} \\ &= P_{\tilde{-}}\,H_{\mathbf{a}\,\mathbf{bc}} + f_{\tilde{-}\,\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,H_{\mathbf{bc}\,\mathcal{N}} \\ \Rightarrow\; H_{\mathbf{a}\,\mathbf{bc}} &= -\,f_{\tilde{-}\,\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,H_{\mathbf{bc}\,\mathcal{N}} \end{aligned}$$

A.2 The $H_{S\,\widetilde{D}}$ and $H_{\widetilde{S}\,D}$

In the section (17.3) we analysed vielbein $H_{\alpha'\,+\mathbf{b}}$. By the similar set of equations as in the section (17.3) we can fix $H_{\alpha'\,\widetilde{-\mathbf{b}}}$. We summarise the structure of the fixed vielbeins from the section (17.3) discussion in the table 10.

$H_{\alpha' \, -\mathbf{a}} \, = \, f_{\tilde{-} \, \alpha' \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{\tilde{-}}} \, H_{-\mathbf{a} \mathcal{N}} \rightsquigarrow \, 0$
$H_{\alpha' \, +\mathbf{a}} \, = \, f_{\tilde{-} \, \alpha' \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{\tilde{-}}} \, H_{+\mathbf{a} \mathcal{N}} \rightsquigarrow \, 0$
$H_{\alpha' \, \widetilde{-\mathbf{a}}} \, = \, - \, f_{- \, \tilde{-} \, \mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{-}} \, S_{\widetilde{-\mathbf{a}}} \, \frac{1}{P_{\tilde{-}}} \, H_{\alpha' \, \mathcal{N}} \, \rightsquigarrow \, 0$
$H_{\alpha' \, \widetilde{+\mathbf{a}}} \, = \, - \, S_{\widetilde{+\mathbf{a}}} \, (\, f_{- \, \tilde{-} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{-} \, P_{\tilde{-}}} \, H_{\alpha' \, \mathcal{N}} \,) \, \rightsquigarrow \, 0$

Table 10: $H_{D \, \widetilde{S}}$ vielbein

Similarly we can calculate what is the table (10) with α' swapped with α. We will use the analogous analysis as in section (17.3) except sometimes instead of the equation (17.18) we use $T_{\widetilde{P} \, D \, \widetilde{S}}$ and also we fix the $H_{\widetilde{P} \, \widetilde{S}}$ using $T_{P \, \widetilde{P} \, \widetilde{S}}$ (or some left − right swap of those). Let's look at two such examples and calculate what is $H_{\alpha \, -\mathbf{a}}$ and $H_{\widetilde{\alpha} \, -\mathbf{a}}$ respectively (we also use the mixed light-cone gauge):

$$
\begin{aligned}
T_{\widetilde{P} \, D \, S} \, \equiv \, T_{\tilde{-} \, \alpha \, -\mathbf{a}} \, = \, 0 \quad &= \quad P_{[\tilde{-}} \, H_{\alpha \, -\mathbf{a})} \, + \, H_{[\, \tilde{-} \, | \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, f_{\alpha \, -\mathbf{a}) \mathcal{N}} && \text{(A.2)} \\
&= \quad P_{\tilde{-}} \, H_{\alpha \, -\mathbf{a}} \, + \, H_{-\mathbf{a} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, f_{\alpha \, \tilde{-} \, \mathcal{N}} && \text{(A.3)} \\
\Rightarrow \quad & \quad H_{\alpha \, -\mathbf{a}} \, = \, - \, f_{\alpha \, \tilde{-} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{\tilde{-}}} \, H_{-\mathbf{a} \mathcal{N}} \\
& \quad H_{\alpha \, -\mathbf{a}} \, \rightsquigarrow \, (\gamma_{-})_{\alpha \, \nu} \, (\widetilde{\Gamma}_5)^{\nu \, \sigma} \, \frac{1}{(r_{AdS}) \, P_{\tilde{-}}} \, H_{-\mathbf{a} \, \widetilde{\sigma}}
\end{aligned}
$$

Next, examine:

$$
\begin{aligned}
T_{P \, S \, \widetilde{D}} \, \equiv \, T_{- \, -\mathbf{a} \, \widetilde{\alpha}} \, = \, 0 \quad &= \quad P_{[-} \, H_{-\mathbf{a} \, \widetilde{\alpha})} \, + \, H_{[- \, | \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, f_{-\mathbf{a} \, \widetilde{\alpha}) \mathcal{N}} && \text{(A.4)} \\
&= \quad P_{-} \, H_{-\mathbf{a} \, \widetilde{\alpha}} \, + \, D_{\widetilde{\alpha}} \, H_{- \, -\mathbf{a}} && \text{(A.5)} \\
& \quad + H_{-\mathbf{a} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, f_{\widetilde{\alpha} \, - \, \mathcal{N}} && \text{(A.6)}
\end{aligned}
$$

the $H_{- \, -\mathbf{a}}$ is fixed by $T_{\widetilde{P} \, P \, S} \, \equiv \, T_{\tilde{-} \, - \, -\mathbf{a}} \, = \, 0$:

$$
\begin{aligned}
T_{\widetilde{P} \, P \, S} \, \equiv \, T_{\tilde{-} \, - \, -\mathbf{a}} \, = \, 0 \quad &= \quad P_{[\tilde{-}} \, H_{- \, -\mathbf{a})} \, + \, H_{[\tilde{-} \, | \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, f_{- \, -\mathbf{a}) \mathcal{N}} && \text{(A.7)} \\
&= \quad P_{\tilde{-}} \, H_{- \, -\mathbf{a}} \, + \, H_{-\mathbf{a} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, f_{\tilde{-} \, - \, \mathcal{N}} && \text{(A.8)} \\
\Rightarrow \quad & \quad H_{- \, -\mathbf{a}} \, = \, f_{- \, \tilde{-} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{\tilde{-}}} \, H_{-\mathbf{a} \mathcal{N}} \, \rightsquigarrow \, 0 && \text{(A.9)}
\end{aligned}
$$

plugging (A.9) into the (A.5) we get:

$$
\begin{aligned}
H_{-\mathbf{a} \, \widetilde{\alpha}} \quad &= \quad - \, f_{- \, \tilde{-} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{-}} \, D_{\widetilde{\alpha}} \, \frac{1}{P_{\tilde{-}}} \, H_{-\mathbf{a} \mathcal{N}} && \text{(A.10)} \\
& \quad - \, f_{- \, \widetilde{\alpha} \, \mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, \frac{1}{P_{-}} \, H_{-\mathbf{a} \mathcal{N}} && \text{(A.11)} \\
\Rightarrow \quad & \quad H_{-\mathbf{a} \, \widetilde{\alpha}} \rightsquigarrow \, - \, (\gamma_{-})_{\alpha \, \nu} \, (\widetilde{\Gamma}_5)^{\nu \, \sigma} \, \frac{1}{(r_{AdS}) \, P_{-}} \, H_{-\mathbf{a} \, \sigma} && \text{(A.12)}
\end{aligned}
$$

We notice that combining the result (A.12) with (A.3) we get after the evaluation of the mixed structure constants that $H_{-\mathbf{a}\,\alpha} \rightsquigarrow 0$ and so also $H_{-\mathbf{a}\,\widetilde{\alpha}} \rightsquigarrow 0$. Similar analysis can be made for the rest of the vielbeins (we mean those from table (10), except α' switched with α). Thus we get the table (11).

$H_{\alpha\,-\mathbf{a}} \;=\; -\,f_{\alpha\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}}$ $\Rightarrow\; H_{\alpha\,-\mathbf{a}} \;\rightsquigarrow\; 0$
$H_{\alpha\,+\mathbf{a}} \;=\; f_{\widetilde{-}\,\alpha\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}}$ $\rightsquigarrow\; (\gamma_{-})_{\alpha\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{AdS})\,P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\widetilde{\sigma}}$
$H_{\widetilde{\alpha}\,-\mathbf{a}} \;=\; f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}}\,D_{\widetilde{\alpha}}\,\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}}$ $+\,f_{-\,\widetilde{\alpha}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{-}}\,H_{-\mathbf{a}\,\mathcal{N}}$ $\Rightarrow\; H_{\widetilde{\alpha}\,-\mathbf{a}} \;\rightsquigarrow\; 0$
$H_{\alpha\,\widetilde{+\mathbf{a}}} \;=\; -\,f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,D_{\alpha}\,\frac{1}{P_{-}}\,H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \;+\; f_{\widetilde{-}\,\alpha\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\widetilde{+\mathbf{a}}\,\mathcal{N}}$ $-\,\eta_{\widetilde{-}\,\widetilde{+}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\alpha\,\widetilde{\mathbf{a}}}$ $\Rightarrow\; H_{\alpha\,\widetilde{+\mathbf{a}}} \;\rightsquigarrow\; (\gamma_{-})_{\alpha\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{AdS})\,P_{\widetilde{-}}}\,H_{\widetilde{+\mathbf{a}}\,\widetilde{\sigma}} \;+\; \frac{1}{P_{\widetilde{-}}}\,H_{\alpha\,\widetilde{\mathbf{a}}}$

Table 11: $H_{D\,\widetilde{S}}$ vielbein

Let us repeat our goal. We wanted to determine the actions of $S_{\widetilde{+\mathbf{b}}}$ and $S_{+\mathbf{a}}$ on $H_{\alpha'\,+\mathbf{a}}$ and $H_{\widetilde{+\mathbf{b}}\,\alpha'}$ respectively. We wanted to do that because then the (17.14) gives the action of $D_{\alpha'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ (where at least the part of the pre-potential sits). The action of $S_{\widetilde{+\mathbf{b}}}$ on $H_{\alpha'\,+\mathbf{a}}$ is easily computed using our table (10). Taking the second top relation from the table (10) and by explicitly applying the $S_{\widetilde{+\mathbf{b}}}$ derivative we get:

$$
\begin{aligned}
S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \;&=\; S_{\widetilde{+\mathbf{b}}}\,(\,f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}}) &\text{(A.13)}\\
&=\; \eta_{\widetilde{-}\,\widetilde{+}}\,f_{\widetilde{\mathbf{b}}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}} &\text{(A.14)}\\
&\quad +\,f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{b}}}\,(\,\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}}\,) &\text{(A.15)}\\
\Rightarrow \qquad & S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \;\rightsquigarrow\; -\,(\gamma_{\mathbf{b}})_{\alpha'\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{AdS})\,P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\widetilde{\sigma}} &\text{(A.16)}
\end{aligned}
$$

To evaluate $S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha'}$ we need to work a bit more. One can directly use the last relation in the table (10). We found an easier way however. For that we need an alternative fixing of the vielbein $H_{\widetilde{+\mathbf{b}}\,\alpha'}$. This alternative fixing seems to be more suited for an explicit evaluation of the $S_{+\mathbf{a}}$ action (and

$S_{-\mathbf{a}}$ action). An alternative way how to fix $H_{\alpha'\,\widetilde{+\mathbf{a}}}$ is to use $T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\tilde{-}\,\alpha'\,\widetilde{+\mathbf{a}}}$ instead of one that we used in (17.18) and (17.19). Similarly it will be useful to find an alternative fixing for $H_{\alpha'\,\widetilde{-\mathbf{a}}}$. Again that could be done by considering torsion $T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\tilde{-}\,\alpha'\,\widetilde{-\mathbf{a}}}$. Let's look at this alternative fixing more closely:

$$
\begin{aligned}
T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\tilde{-}\,\alpha'\,\widetilde{-\mathbf{a}}} = 0 \; &= \; P_{[\tilde{-}}\, H_{\alpha'\,\widetilde{-\mathbf{a}})} + H_{[\tilde{-}\,|\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\alpha'\,\widetilde{-\mathbf{a}})\,\mathcal{N}} && \text{(A.17)}\\
&= \; P_{\tilde{-}}\, H_{\alpha'\,\widetilde{-\mathbf{a}}} + D_{\alpha'}\, H_{\widetilde{-\mathbf{a}}\,\tilde{-}} && \text{(A.18)}\\
&\quad + H_{\widetilde{-\mathbf{a}}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\tilde{-}\,\alpha'\,\mathcal{N}}
\end{aligned}
$$

The $H_{\widetilde{-\mathbf{a}}\,\tilde{-}}$ type of vielbein has been fixed in (A.9). Plugging the fixing into (A.18) we get an alternative $H_{\alpha'\,\widetilde{-\mathbf{a}}}$ fixing:

$$
\begin{aligned}
H_{\alpha'\,\widetilde{-\mathbf{a}}} \; &= \; -\, f_{-\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{\tilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} && \text{(A.19)}\\
&\quad -\, f_{\tilde{-}\,\alpha'\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\widetilde{-\mathbf{a}}\,\mathcal{N}}\\
H_{\alpha'\,\widetilde{-\mathbf{a}}} \; &\rightsquigarrow \; 0 && \text{(A.20)}
\end{aligned}
$$

again we can see the behaviour of the $H_{\alpha'\,\widetilde{-\mathbf{a}}}$ in (A.20) as we evaluate the theory, as it should be comparing with its behaviour from the fixing in the table (10). The alternative fixing for the vielbein $H_{\widetilde{+\mathbf{b}}\,\alpha'}$ is calculated similarly:

$$
\begin{aligned}
T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\tilde{-}\,\alpha'\,\widetilde{+\mathbf{a}}} = 0 \; &= \; P_{[\tilde{-}}\, H_{\alpha'\,\widetilde{+\mathbf{a}})} + H_{[\tilde{-}\,|\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\alpha'\,\widetilde{+\mathbf{a}})\,\mathcal{N}} && \text{(A.21)}\\
&= \; P_{\tilde{-}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} + D_{\alpha'}\, H_{\widetilde{+\mathbf{a}}\,\tilde{-}} && \text{(A.22)}\\
&\quad + H_{\widetilde{+\mathbf{a}}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\tilde{-}\,\alpha'\,\mathcal{N}} + H_{\alpha'\,\tilde{\mathbf{a}}}
\end{aligned}
$$

The $H_{\widetilde{+\mathbf{a}}\,\tilde{-}}$ is fixed similarly to (A.9) resulting in:

$$
H_{\widetilde{+\mathbf{a}}\,\tilde{-}} \; = \; f_{-\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \rightsquigarrow 0 \qquad \text{(A.23)}
$$

The $H_{\alpha'\,\tilde{\mathbf{a}}}$ is fixed by the dim $\frac{1}{2}$ torsion constraint $T_{P\,D\,\widetilde{P}} \equiv T_{-\,\alpha'\,\tilde{\mathbf{a}}} = 0$:

$$
\begin{aligned}
T_{P\,D\,\widetilde{P}} \equiv T_{-\,\alpha'\,\tilde{\mathbf{a}}} = 0 \; &= \; P_{[-}\, H_{\alpha'\,\tilde{\mathbf{a}})} + H_{[-\,|\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\alpha'\,\tilde{\mathbf{a}})\,\mathcal{N}} && \text{(A.24)}\\
&= \; P_{-}\, H_{\alpha'\,\tilde{\mathbf{a}}} + P_{\tilde{\mathbf{a}}}\, H_{-\,\alpha'} && \text{(A.25)}\\
&\quad + H_{\alpha'\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\tilde{\mathbf{a}}\,-\,\mathcal{N}}
\end{aligned}
$$

The last vielbein we need to fix is the $H_{-\,\alpha'}$, that is again fixed by the dim $\frac{1}{2}$ torsion constraint $T_{\widetilde{P}\,P\,D} \equiv T_{\tilde{-}\,-\,\alpha'} = 0$:

$$
\begin{aligned}
T_{\widetilde{P}\,P\,D} \equiv T_{\tilde{-}\,-\,\alpha'} = 0 \; &= \; P_{[\tilde{-}}\, H_{-\,\alpha')} + H_{[\tilde{-}\,|\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{-\,\alpha')\,\mathcal{N}} && \text{(A.26)}\\
&= \; P_{\tilde{-}}\, H_{-\,\alpha'} + H_{\alpha'\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, f_{\tilde{-}\,-\,\mathcal{N}} && \text{(A.27)}\\
\Rightarrow \qquad & H_{-\,\alpha'} \; = \; f_{-\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\alpha'\,\mathcal{N}} \rightsquigarrow 0 && \text{(A.28)}
\end{aligned}
$$

Plugging (A.28) into (A.25) and that into (A.22) we finally get an alternative fixing for the $H_{\alpha'\,\widetilde{+a}}$:

$$\begin{aligned} H_{\alpha'\,\widetilde{+\mathbf{a}}} \;=\; & -f_{-\,\tilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\left(\tfrac{1}{P_{\tilde{-}}\,P_{-}}\,P_{\tilde{a}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{\alpha'\,\mathcal{N}} + \tfrac{1}{P_{\tilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{+\mathbf{a}}\,\mathcal{N}}\right) \\ & -f_{\alpha'\,\tilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} - f_{-\,\tilde{\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}\,P_{-}}\,H_{\alpha'\,\mathcal{N}} \end{aligned} \quad \text{(A.29)}$$

Now we are ready to calculate an action of $S_{\widetilde{-\mathbf{a}}}$ and $S_{-\mathbf{a}}$ and $S_{\widetilde{+\mathbf{a}}}$ and $S_{+\mathbf{a}}$ on table (10) vielbeins (with the exception of $S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha'}$ that we want to calculate in the end of this paragraph). We summarise those S (and $\widetilde{S}$) actions in the next tables:

$S_{-\mathbf{b}}\,H_{\alpha'\,-\mathbf{a}} = -(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\,f_{\tilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}$ $-f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{-\mathbf{b}}\,\frac{1}{P_{\tilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}$ $\Rightarrow\; S_{-\mathbf{b}}\,H_{\alpha'\,-\mathbf{a}} \rightsquigarrow 0$
$S_{\widetilde{-\mathbf{b}}}\,H_{\alpha'\,-\mathbf{a}} = -f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\tilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}$ $\Rightarrow\; S_{\widetilde{-\mathbf{b}}}\,H_{\alpha'\,-\mathbf{a}} \rightsquigarrow 0$
$S_{-\mathbf{b}}\,H_{\alpha'\,+\mathbf{a}} = -(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\,f_{\tilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\mathcal{N}}$ $-f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{-\mathbf{b}}\,\frac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\mathcal{N}}$ $\Rightarrow\; S_{-\mathbf{b}}\,H_{\alpha'\,+\mathbf{a}} \rightsquigarrow 0$
$S_{\widetilde{-\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} = -f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\mathcal{N}}$ $\Rightarrow\; S_{\widetilde{-\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \rightsquigarrow 0$
$S_{-\mathbf{b}}\,H_{\alpha'\,\widetilde{-\mathbf{a}}} = -f_{-\,\tilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{-\mathbf{b}}\,\frac{1}{P_{\tilde{-}}}\,D_{\alpha'}\,\frac{1}{P_{-}}\,H_{\widetilde{-\mathbf{a}}\mathcal{N}}$ $-(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\,f_{\tilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,H_{\widetilde{-\mathbf{a}}\mathcal{N}} - f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{-\mathbf{b}}\,\frac{1}{P_{\tilde{-}}}\,H_{\widetilde{-\mathbf{a}}\mathcal{N}}$ $\Rightarrow\; S_{-\mathbf{b}}\,H_{\alpha'\,\widetilde{-\mathbf{a}}} \rightsquigarrow 0$
$S_{\widetilde{-\mathbf{b}}}\,H_{\alpha'\,\widetilde{-\mathbf{a}}} = -f_{-\,\tilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\tilde{-}}}\,D_{\alpha'}\,\frac{1}{P_{-}}\,H_{\widetilde{-\mathbf{a}}\mathcal{N}}$ $-f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\tilde{-}}}\,H_{\widetilde{-\mathbf{a}}\mathcal{N}}$ $\Rightarrow\; S_{\widetilde{-\mathbf{b}}}\,H_{\alpha'\,\widetilde{-\mathbf{a}}} \rightsquigarrow 0$

Table 12: the S action on $H_{D\,\widetilde{S}}$ vielbein

Now we calculate $S_{-\mathbf{b}}\,H_{\alpha'\,\widetilde{+\mathbf{a}}}$. The reasoning will be similar later for the final calculation of the $S_{+\mathbf{b}}\,H_{\alpha'\,\widetilde{+\mathbf{a}}}$ so we first do the former in order to see how it works. Calculation of the $S_{-\mathbf{b}}$ action on $H_{\alpha'\,\widetilde{+\mathbf{a}}}$ is straightforward. It's done using the relation (A.29) and applying $S_{-\mathbf{b}}$, thus we get:

$$
\begin{aligned}
S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \;\; = \;\; & - f_{-\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}} \left(\tfrac{1}{P_{\widetilde{-}}\, P_{-}}\, P_{\widetilde{a}}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{\alpha'\,\mathcal{N}} \,+\, \tfrac{1}{P_{\widetilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \right) \\
& - \tfrac{1}{2}\, (\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\nu\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \qquad\qquad (\text{A.30}) \\
& + f_{\alpha'\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \,-\, f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\, \tfrac{1}{P_{\widetilde{-}}\, P_{-}}\, H_{\alpha'\,\mathcal{N}}
\end{aligned}
$$

Now, we want to evaluate equation (A.30). The terms proportional to $f_{-\,\widetilde{-}\,\mathcal{M}}$ and $f_{\alpha'\,\widetilde{-}\,\mathcal{M}}$ are vanishing by the AdS algebra. We write what's left over after evaluation:

$$
\begin{aligned}
S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \;\; = \;\; & \tfrac{1}{2}\, (\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\, (\gamma_{-})_{\nu\,\sigma}\, (\widetilde{\Gamma}_{5})^{\sigma\,\epsilon}\, \tfrac{1}{(r_{AdS})}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\widetilde{\epsilon}} \qquad\qquad (\text{A.31}) \\
& + \tfrac{1}{2\,(r_{AdS})^{2}}\, \tfrac{1}{P_{\widetilde{-}}\, P_{-}}\, S_{-\mathbf{b}} \left(H_{\alpha'\,+\mathbf{a}} \,+\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \,\pm\, H_{\alpha'\,-\mathbf{a}} \,\pm\, H_{\alpha'\,\widetilde{-\mathbf{a}}} \right)
\end{aligned}
$$

Note, the $\pm$ in last line in (A.31) is explained in the section above (17.8). According to the table (12) all actions of $S_{-\mathbf{b}}$ in the second line of (A.31) are evaluated to zero except of the $S_{-\mathbf{b}}\, H_{\alpha'\,+\mathbf{a}}$ that we want to determine. Then the (A.31) could be rewritten in a way that determines $S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$ (after evaluation):

$$
\begin{aligned}
\left(1 \,-\, \tfrac{1}{2\,(r_{AdS})^{2}}\, \tfrac{1}{P_{\widetilde{-}}\, P_{-}} \right) S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \;\; & = \;\; \ldots \qquad\qquad (\text{A.32}) \\
& = \;\; \tfrac{1}{2}\, (\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\, (\gamma_{-})_{\nu\,\sigma}\, (\widetilde{\Gamma}_{5})^{\sigma\,\epsilon}\, \tfrac{1}{(r_{AdS})}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\widetilde{\epsilon}}
\end{aligned}
$$

We remind that the $H_{\widetilde{+\mathbf{a}}\,\widetilde{\epsilon}}$ vielbein is related to the $H_{\widetilde{+\mathbf{a}}\,\epsilon}$ vielbein by the second top line in the table (11). Similarly to the (A.30) and its evaluated version (A.32) we can calculate an action of $S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$. The result is:

$$
\begin{aligned}
S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \;\; = \;\; & - f_{-\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{\widetilde{-\mathbf{b}}} \left(\tfrac{1}{P_{\widetilde{-}}\, P_{-}}\, P_{\widetilde{a}}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{\alpha'\,\mathcal{N}} \,+\, \tfrac{1}{P_{\widetilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \right) \\
& - f_{\alpha'\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{\widetilde{-\mathbf{b}}}\, \tfrac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \,-\, \eta_{\widetilde{\mathbf{a}}\,\widetilde{\mathbf{b}}}\, f_{-\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{\widetilde{-}}\, P_{-}}\, H_{\alpha'\,\mathcal{N}} \\
& - f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{\widetilde{-\mathbf{b}}}\, \tfrac{1}{P_{\widetilde{-}}\, P_{-}}\, H_{\alpha'\,\mathcal{N}} \qquad\qquad (\text{A.33})
\end{aligned}
$$

and after evaluation, where we again use the results from table (12):

$$
S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \,\rightsquigarrow\, 0 \qquad\qquad (\text{A.34})
$$

Finally the action of $S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$ is calculated as in (A.30). Now we know that by an analogy with the (A.30) and its evaluation we would need to know analogy of the table (12) except now for the $S_{+\mathbf{b}}$. Since calculations are very analogous to those that led to the table (12) we list just the resulting table(s): (13) and (14)

$$\begin{array}{|c|}
\hline
S_{+\mathbf{b}}\, H_{\alpha'\,-\mathbf{a}} \;=\; -\tfrac{1}{2}\,(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\tilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}} \\
+\, f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{+\mathbf{b}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}} \\
\Rightarrow\; S_{+\mathbf{b}}\,H_{\alpha'\,-\mathbf{a}} \;\rightsquigarrow\; 0 \\
\hline
S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,-\mathbf{a}} \;=\; \eta_{\tilde{-}\,\tilde{+}}\, f_{\tilde{\mathbf{b}}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}} \;+\; f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{b}}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}} \\
\Rightarrow\; S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,-\mathbf{a}} \;\rightsquigarrow\; 0 \\
\hline
S_{+\mathbf{b}}\, H_{\alpha'\,+\mathbf{a}} \;=\; -\tfrac{1}{2}\,(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\tilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}} \\
+\, f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{+\mathbf{b}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}} \\
\Rightarrow\; S_{+\mathbf{b}}\,H_{\alpha'\,+\mathbf{a}} \;\rightsquigarrow\; -\,(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\,(\gamma_{-})_{\nu\,\sigma}\,(\widetilde{\Gamma}_{5})^{\sigma\,\lambda}\,\tfrac{1}{(r_{AdS})}\,\tfrac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\,\tilde{\lambda}} \\
\hline
S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \;=\; \eta_{\tilde{-}\,\tilde{+}}\, f_{\tilde{\mathbf{b}}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}} \;+\; f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{b}}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}} \\
\Rightarrow\; S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \;\rightsquigarrow\; (\gamma_{\mathbf{b}})_{\alpha'\,\nu}\,(\widetilde{\Gamma}_{5})^{\nu\,\sigma}\,\tfrac{1}{(r_{AdS})}\,\tfrac{1}{P_{\tilde{-}}}\,H_{+\mathbf{a}\,\tilde{\sigma}} \\
\hline
\end{array}$$

Table 13: the S action on $H_{D\,\widetilde{S}}$ vielbein

$$\begin{array}{|c|}
\hline
S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{-\mathbf{a}}} \;=\; -\tfrac{1}{2}\,(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\tilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \\
+\, f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{+\mathbf{b}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \\
+\,\eta_{-+}\, f_{\mathbf{b}\,\tilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \;+\; f_{-\,\tilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{+\mathbf{b}}\,\tfrac{1}{P_{\tilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \\
\Rightarrow\; S_{+\mathbf{b}}\,H_{\alpha'\,\widetilde{-\mathbf{a}}} \;\rightsquigarrow\; 0 \\
\hline
S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,\widetilde{-\mathbf{a}}} \;=\; \eta_{\tilde{-}\,\tilde{+}}\, f_{-\,\tilde{\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \\
+\, f_{-\,\tilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{b}}}\,\tfrac{1}{P_{\tilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \\
+\,\eta_{\tilde{-}\,\tilde{+}}\, f_{\tilde{\mathbf{b}}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \;+\; f_{\tilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{b}}}\,\tfrac{1}{P_{\tilde{-}}}\,H_{\widetilde{-\mathbf{a}}\,\mathcal{N}} \\
\Rightarrow\; S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,\widetilde{-\mathbf{a}}} \;\rightsquigarrow\; 0 \\
\hline
\end{array}$$

Table 14: the S action on $H_{D\,\widetilde{S}}$ vielbein

Now, we calculate the the missing piece in the equation (17.14), i.e.

$S_{+\mathbf{b}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}}$. In analogy with (A.30) we get:

$$
\begin{aligned}
S_{+\mathbf{b}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}} \;=\;& - f_{-\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{+\mathbf{b}} \left(\tfrac{1}{P_{\tilde{-}}\, P_{-}}\, P_{\tilde{a}}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\alpha'\,\mathcal{N}} \,+\, \tfrac{1}{P_{\tilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \right) \\
& - \eta_{-+}\, f_{\mathbf{b}\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}} \left(\tfrac{1}{P_{\tilde{-}}\, P_{-}}\, P_{\tilde{a}}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\alpha'\,\mathcal{N}} \,+\, \tfrac{1}{P_{\tilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \right) \\
& + \tfrac{1}{2}\, (\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\nu\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \\
& - f_{\alpha'\,\tilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{+\mathbf{b}}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \\
& - f_{-\,\tilde{\mathbf{a}}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{+\mathbf{b}}\, \tfrac{1}{P_{\tilde{-}}\, P_{-}}\, H_{\alpha'\,\mathcal{N}} \\
& + \eta_{-+}\, f_{\tilde{\mathbf{a}}\,\mathbf{b}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \tfrac{1}{P_{\tilde{-}}\, P_{-}}\, H_{\alpha'\,\mathcal{M}}
\end{aligned}
\tag{A.35}
$$

and the (partially) evaluate version of (A.35):

$$
\begin{aligned}
S_{+\mathbf{b}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}} \;\rightsquigarrow\;& - \tfrac{1}{2\,(r_{AdS})^2}\, \tfrac{1}{P_{\tilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,+\mathbf{b}} \\
& - (\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, (\gamma_{-})_{\nu\,\sigma}\, (\widetilde{\mathit{\Gamma}}_5)^{\sigma\,\lambda}\, \tfrac{1}{(r_{AdS})}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\tilde{\lambda}} \\
& + \tfrac{1}{2\,(r_{AdS})^2}\, S_{+\mathbf{b}}\, \tfrac{1}{P_{\tilde{-}}\, P_{-}} \left(H_{\alpha'\,+\mathbf{a}} \,+\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \,\pm\, H_{\alpha'\,-\mathbf{a}} \,\pm\, H_{\alpha'\,\widetilde{-\mathbf{a}}} \right)
\end{aligned}
\tag{A.36}
$$

We can see why we just partially evaluated the equation (A.35). The reason is that last term leads to an action of $S_{+\mathbf{b}}$. Fortunately for us we already computed all those actions in tables (13) and (14) except $S_{+\mathbf{b}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}}$ that we want to calculate. Therefore the (A.36) leads to the evaluated version of the $S_{+\mathbf{b}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}}$:

$$
\begin{aligned}
\left(1 \,-\, \tfrac{1}{2\,(r_{AdS})^2}\, \tfrac{1}{P_{\tilde{-}}\, P_{-}} \right) S_{+\mathbf{b}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}} \rightsquigarrow & - \tfrac{1}{2\,(r_{AdS})^2}\, \tfrac{1}{P_{\tilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,+\mathbf{b}} \\
& - \tfrac{1}{2}\, (\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, (\gamma_{-})_{\nu\,\sigma}\, (\widetilde{\mathit{\Gamma}}_5)^{\sigma\,\lambda}\, \tfrac{1}{(r_{AdS})}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\tilde{\lambda}} \\
& - (\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, (\gamma_{-})_{\nu\,\sigma}\, (\widetilde{\mathit{\Gamma}}_5)^{\sigma\,\lambda}\, \tfrac{1}{2\,(r_{AdS})^3}\, \tfrac{1}{P_{\tilde{-}}\, P_{-}\, P_{\tilde{-}}}\, H_{+\mathbf{a}\,\tilde{\lambda}}
\end{aligned}
\tag{A.37}
$$

where we used results of tables (13) and (14). For completeness we provide (just the evaluated version) the last remaining part of tables (13) and (14), i.e. $S_{\widetilde{+\mathbf{b}}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}}$:

$$
\begin{aligned}
\left(1 \,-\, \tfrac{1}{2\,(r_{AdS})^2}\, \tfrac{1}{P_{\tilde{-}}\, P_{-}} \right) S_{\widetilde{+\mathbf{b}}}\, H_{\alpha'\, \widetilde{+\mathbf{a}}} \;\rightsquigarrow\;& - \tfrac{1}{2\,(r_{AdS})^2}\, \tfrac{1}{P_{\tilde{-}}}\, D_{\alpha'}\, \tfrac{1}{P_{-}}\, H_{\widetilde{+\mathbf{a}}\,+\mathbf{b}} \\
& - \tfrac{1}{2}\, (\gamma_{\mathbf{b}})_{\alpha'\,\nu}\, (\widetilde{\mathit{\Gamma}}_5)^{\nu\,\lambda}\, \tfrac{1}{(r_{AdS})}\, \tfrac{1}{P_{\tilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\tilde{\lambda}} \\
& - (\gamma_{\mathbf{b}})_{\alpha'\,\nu}\, (\widetilde{\mathit{\Gamma}}_5)^{\nu\,\lambda}\, \tfrac{1}{2\,(r_{AdS})^3}\, \tfrac{1}{P_{\tilde{-}}\, P_{-}\, P_{\tilde{-}}}\, H_{+\mathbf{a}\,\tilde{\lambda}}
\end{aligned}
\tag{A.38}
$$

We repeat the first important relation we derived by the above analysis from (17.13) where we add results from (A.16) and (A.37):

$$D_{\alpha'}\, H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \;=\; -\,\frac{1}{g}\,\frac{1}{(r_{AdS})\,P_{\tilde{-}}}\,(\gamma_{\mathbf{b}})_{\alpha'\,\sigma}\,(\widetilde{\varGamma}_5)^{\sigma\,\beta}\,H_{\widetilde{\beta}\,+\mathbf{a}} + \frac{1}{2\,g}\,(\,1\,-\,\frac{1}{f}\,\frac{1}{(r_{AdS})^2\,P_{\tilde{-}}\,P_{-}}\,)(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta}\,H_{\beta\,\widetilde{+\mathbf{b}}} + \frac{1}{f\,g}\,\frac{1}{2\,(r_{AdS})^3\,P_{-}\,(P_{\tilde{-}})^2}\,(\gamma_{\mathbf{a}})_{\alpha'\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\beta}\,H_{\widetilde{\beta}\,+\mathbf{b}} \tag{A.39}$$

where f and g are defined as follows:

$$f \;:=\; (\,1\,-\,\frac{1}{2\,(r_{AdS})^2\,P_{\tilde{-}}\,P_{-}}\,) \qquad g \;:=\; (\,1\,-\,\frac{1}{f}\,\frac{1}{2\,(r_{AdS})^2\,P_{\tilde{-}}\,P_{-}}\,) \tag{A.40}$$

A.3 The $H_{D\,\widetilde{D}}$

To obtain the AdS equation (18.3), we need to fix $H_{\alpha'\,\beta}$. This term is fixed by the zero dimensional torsion $T_{\widetilde{P}\,D\,D} \equiv T_{\tilde{-}\,\alpha'\,\beta} = 0$:

$$T_{\tilde{-}\,\alpha'\,\beta} = 0 \;=\; P_{[\tilde{-}}\,H_{\alpha'\,\beta)} + H_{[\tilde{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,\beta)\,\mathcal{N}} \;=\; P_{\tilde{-}}\,H_{\alpha'\,\beta} + H_{(\beta\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha')\,\tilde{-}\,\mathcal{N}} \tag{A.41}$$

$$\Rightarrow \qquad H_{\alpha'\,\beta} \;=\; -\,f_{\tilde{-}\,(\,\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,H_{\beta\,)\,\mathcal{N}} \;\rightsquigarrow\; 0 \tag{A.42}$$

In (A.41) we again used the mixed light-cone gauge. In the flat case the mixed f terms are zero so is $H_{\alpha'\,\beta}$. In the AdS case (after evaluation), term proportional to $f_{\alpha'\,\tilde{-}\,\mathcal{N}}$ is zero because of $(\gamma_{-})_{\alpha'\,\beta'} = 0$. But the term proportional to $f_{\beta\,\tilde{-}\,\mathcal{N}}$ is nonzero. Luckily for us the $f_{\beta\,\tilde{-}\,\mathcal{N}} \propto \frac{1}{r_{AdS}}\,(\gamma_{-})_{\beta\,\sigma}\,(\widetilde{\varGamma}_5)^{\sigma\,\nu}\,\eta_{\widetilde{\nu}\,\mathcal{N}}$. That structure constant just eats up the β index and returns $\widetilde{\nu}$ index with some fixed constant dependence. The torsion constraint $T_{P\,\widetilde{D}\,D} \equiv T_{-\,\widetilde{\sigma}\,\alpha'} = 0$ relates $H_{\widetilde{\sigma}\,\alpha'}$ back to $H_{\sigma\,\alpha'}$ (after the evaluation). From that and assuming some wider invertibility (P_{-} and $P_{\tilde{-}}$ are bigger than some constant lower bound in AdS) we get also in the AdS space $H_{\alpha'\,\beta} \rightsquigarrow 0$ (after the evaluation).

We apply $S_{\widetilde{+\mathbf{a}}}$ on the result of non-evaluated (A.42), thus get:

$$S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta} \;=\; -\,\eta_{\tilde{-}\,\tilde{+}}\,f_{\widetilde{\mathbf{a}}\,(\,\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\tilde{-}}}\,H_{\beta\,)\,\mathcal{N}} - f_{\tilde{-}\,(\,\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{a}}}\,\frac{1}{P_{\tilde{-}}}\,H_{\beta\,)\,\mathcal{N}} \tag{A.43}$$

$$S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta} \;\rightsquigarrow\; -\,(\gamma_{-})_{\beta\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\sigma}\,\frac{1}{r_{AdS}}\,S_{\widetilde{+\mathbf{a}}}\,\frac{1}{P_{\tilde{-}}}\,H_{\alpha'\,\widetilde{\sigma}} + (\gamma_{\mathbf{a}})_{\alpha'\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\sigma}\,\frac{1}{r_{AdS}}\,\frac{1}{P_{\tilde{-}}}\,H_{\beta\,\widetilde{\sigma}} \tag{A.44}$$

The last term in (A.44) does not bother us too much (it will be a part of the pre-potential), the first term in (A.44) is actually something we need to evaluate. For that we need to fix $H_{\alpha'\widetilde{\sigma}}$. That could be done by the torsion constraint $T_{P\widetilde{D}D} \equiv T_{-\alpha'\widetilde{\sigma}} = 0$:

$$
\begin{aligned}
T_{-\alpha'\widetilde{\sigma}} = 0 \quad &= \quad P_{[-}\,H_{\alpha'\widetilde{\sigma})} + H_{[-|\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\widetilde{\sigma})\mathcal{N}} &\text{(A.45)}\\
&= \quad P_{-}\,H_{\alpha'\widetilde{\sigma}} - D_{\widetilde{\sigma}}\,H_{-\alpha'} + H_{-\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\widetilde{\sigma}\mathcal{N}} &\text{(A.46)}\\
&\quad\; + H_{\alpha'\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{\sigma}-\mathcal{N}}
\end{aligned}
$$

moreover the $H_{-\alpha'}$ has been fixed in (A.28), plugging that into (A.46) we get fixing of $H_{\alpha'\widetilde{\sigma}}$:

$$
\begin{aligned}
H_{\alpha'\widetilde{\sigma}} \quad = \quad & f_{-\widetilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_-}\,D_{\widetilde{\sigma}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{\alpha'\mathcal{N}} - f_{\alpha'\widetilde{\sigma}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_-}\,H_{-\mathcal{N}} &\text{(A.47)}\\
& + f_{\widetilde{\sigma}-\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_-}\,H_{\alpha'\mathcal{N}}
\end{aligned}
$$

We are ready to calculate $S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\widetilde{\sigma}}$ i.e. the term needed in (A.44):

$$
\begin{aligned}
S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\widetilde{\sigma}} \quad = \quad & \eta_{\widetilde{-}\,\widetilde{+}}\,f_{-\widetilde{\mathbf{a}}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_-}\,D_{\widetilde{\sigma}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{\alpha'\mathcal{N}} &\text{(A.48)}\\
& + f_{-\widetilde{-}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{a}}}\,\tfrac{1}{P_-}\,D_{\widetilde{\sigma}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{\alpha'\mathcal{N}}\\
& + \tfrac{1}{2}\,(\gamma_{+\mathbf{a}})_{\sigma}{}^{\nu'}\,f_{\alpha'\widetilde{\nu}'\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_-}\,H_{-\mathcal{N}} - f_{\alpha'\widetilde{\sigma}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{a}}}\,\tfrac{1}{P_-}\,H_{-\mathcal{N}}\\
& - \tfrac{1}{2}\,(\gamma_{+\mathbf{a}})_{\sigma}{}^{\nu'}\,f_{\widetilde{\nu}'-\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\tfrac{1}{P_-}\,H_{\alpha'\mathcal{N}} + f_{\widetilde{\sigma}-\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{a}}}\,\tfrac{1}{P_-}\,H_{\alpha'\mathcal{N}}
\end{aligned}
$$

Now, we can evaluate (A.48), for clearness we include terms that we already know are evaluated to zero or are zero by the mixed light-cone gauge:

$$
\begin{aligned}
S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\widetilde{\sigma}} \quad \rightsquigarrow \quad & \tfrac{1}{2\,(r_{AdS})^2}\,\tfrac{1}{P_-}\,D_{\widetilde{\sigma}}\,\tfrac{1}{P_{\widetilde{-}}}\,(\,H_{\widetilde{\alpha}'+\mathbf{a}} &\text{(A.49)}\\
& + H_{\widetilde{\alpha}'\,\widetilde{+\mathbf{a}}} \pm H_{\widetilde{\alpha}'-\mathbf{a}} \pm H_{\widetilde{\alpha}'\,\widetilde{-\mathbf{a}}}\,)\\
& (\pm)\,\tfrac{1}{2}\,\tfrac{1}{r_{AdS}}\,(\gamma_{+\mathbf{a}})_{\sigma}{}^{\nu'}\,(\gamma^{\mathbf{cd}})_{\nu'}{}^{\rho'}\,(\widetilde{\varGamma}_5)_{\rho'\alpha'}\,\tfrac{1}{2\,P_-}\,(\,H_{-\mathbf{cd}} &\text{(A.50)}\\
& + H_{-\,\widetilde{\mathbf{cd}}}\,)\\
& -(\pm)\,\tfrac{1}{r_{AdS}}\,(\gamma^{\mathbf{cd}})_{\sigma}{}^{\rho'}\,(\widetilde{\varGamma}_5)_{\rho'\alpha'}\,S_{\widetilde{+\mathbf{a}}}\,\tfrac{1}{2\,P_-}\,(\,H_{-\mathbf{cd}} &\text{(A.51)}\\
& + H_{-\,\widetilde{\mathbf{cd}}}\,)\\
& - \tfrac{1}{r_{AdS}}\,(\gamma_-)_{\sigma\nu}\,(\widetilde{\varGamma}_5)^{\nu\rho}\,S_{\widetilde{+\mathbf{a}}}\,\tfrac{1}{P_-}\,H_{\alpha'\rho} &\text{(A.52)}
\end{aligned}
$$

Note that in the lines (A.50) and (A.51) we have the $\pm$ symbol. It comes from the mixed structure constant $f_{D\,\widetilde{D}\,\Sigma^{\mathbf{d}}} \equiv f_{\underline{\alpha}\,\underline{\widetilde{\beta}}}{}^{\mathbf{cd}}$, where underline indices are now (and just now) the $SO(10)$ chiral indices (for the left and right algebra), and $\Sigma^{\mathbf{d}}$ is the Σ index for the $SO(5) \otimes SO(5)$ diagonal subgroup of the original $SO(10) \otimes SO(10)$ group. The $(\pm)$ symbol determines to which $SO(5)$ of the diagonal subgroup given $\Sigma^{\mathbf{d}}$ belongs. This mixed structure constant could be written without the $\pm$ symbols as $f_{\underline{\alpha}\,\underline{\widetilde{\beta}}}{}^{\mathbf{cd}} = \frac{1}{r_{AdS}} (\gamma^{[\mathbf{c}})_{\sigma\,\rho} (\widetilde{\Gamma}_5)^{\rho\,\nu} (\gamma^{\mathbf{d}]})_{\nu\,\alpha'}$. The $\pm$ then comes from the fact that by the construction $\widetilde{\Gamma}_5$ commutes with $\mathbf{a} \in \{10, 1 \ldots 4\}$ and anti-commutes with $\mathbf{a} \in \{5, \ldots 9\}$. We used the prior definition in this section for some convenience. In the final expressions we will always use the definition without the $\pm$ symbol.

Let's evaluate expressions (A.49), (A.50), (A.51) and (A.52). The line (A.49) is evaluated to 0 by the table (10). We note very important property in the lines (A.50) and (A.51). The summation over the **cd** indices is really just a summation over the $SO(5) \otimes SO(5)$ diagonal subgroup of the full $SO(10) \otimes SO(10)$. The line (A.50) is evaluated to 0 by the mixed light-cone gauge (second term) and by the following fixing of the $H_{-\,\mathbf{cd}}$ (coming from torsion constraint $T_{\widetilde{P}\,P\,S} \equiv T_{\widetilde{-}\,-\,\mathbf{ab}} = 0$):

$$H_{-\,\mathbf{ab}} \;=\; f_{-\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \frac{1}{P_{\widetilde{-}}}\, H_{\mathbf{ab}\,\mathcal{N}} \;\Rightarrow\; H_{-\,\mathbf{ab}} \rightsquigarrow 0 \tag{A.53}$$

The line (A.52) has an action $S_{\widetilde{+\mathbf{a}}}\, H_{\alpha'\,\rho}$ that is exactly what we want to determine. The line (A.51) is fixed as follows. The vielbein $H_{-\,\widetilde{\mathbf{cd}}} = 0$ by the mixed light-cone gauge. The action $S_{\widetilde{+\mathbf{a}}}\, H_{-\,\mathbf{cd}}$ is however nontrivial. We should take fixing (A.53) and apply $S_{\widetilde{+\mathbf{a}}}$:

$$\begin{aligned} S_{\widetilde{+\mathbf{a}}}\, H_{-\,\mathbf{cd}} \;&=\; -\, f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, \frac{1}{P_{\widetilde{-}}}\, H_{\mathbf{cd}\,\mathcal{N}} \\ &\quad +\, f_{-\,\widetilde{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}}\, S_{+\mathbf{a}}\, \frac{1}{P_{\widetilde{-}}}\, H_{\mathbf{cd}\,\mathcal{N}} \end{aligned} \tag{A.54}$$

$$\begin{aligned} S_{\widetilde{+\mathbf{a}}}\, H_{-\,\mathbf{cd}} \;&\rightsquigarrow\; \frac{1}{2\,(r_{AdS})^2}\, \frac{1}{P_{\widetilde{-}}}\, (\, H_{\mathbf{cd}\,+\mathbf{a}} + H_{\mathbf{cd}\,\widetilde{+\mathbf{a}}} \\ &\quad \pm H_{\mathbf{cd}\,-\mathbf{a}} \pm H_{\mathbf{cd}\,\widetilde{-\mathbf{a}}}\,) \end{aligned} \tag{A.55}$$

By the table (9) the only nonzero term in (A.55) is $H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}}$ thus we get:

$$S_{\widetilde{+\mathbf{a}}}\, H_{-\,+\mathbf{d}} \;\rightsquigarrow\; \frac{1}{2\,(r_{AdS})^2}\, \frac{1}{P_{\widetilde{-}}}\, H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}} \tag{A.56}$$

Then finally the equation (A.49 till A.52) is evaluated to:

$$\begin{aligned} S_{\widetilde{+\mathbf{a}}}\, H_{\alpha'\,\widetilde{\sigma}} \;&\rightsquigarrow\; \pm\, \frac{(-1)}{4\,(r_{AdS})^3\, P_-\, P_{\widetilde{-}}}\, (\gamma^{+\mathbf{d}})_{\sigma}{}^{\rho'}\, (\widetilde{\Gamma}_5)_{\rho'\,\alpha'}\, H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}} \\ &\quad -\, \frac{1}{(r_{AdS})\, P_-}\, (\gamma_-)_{\sigma\,\nu}\, (\widetilde{\Gamma}_5)^{\nu\,\rho}\, S_{\widetilde{+\mathbf{a}}}\, H_{\alpha'\,\rho} \end{aligned} \tag{A.57}$$

Combining (A.57) and (A.44) we will get the following relation for the evaluated action of $S_{\widetilde{+\mathbf{a}}}\, H_{\alpha'\,\beta}$:

$$
\begin{aligned}
& (\,1 \,-\, \tfrac{1}{P_-\,P_{\widetilde{-}}\,(r_{AdS})^2}\,)\, S_{\widetilde{+\mathbf{a}}}\, H_{\alpha'\,\beta} = \\
&= -\,\tfrac{1}{4\,(r_{AdS})^4\,P_-\,P_{\widetilde{-}}{}^2}\,(\widetilde{\Gamma}_5)_{\beta\,\nu}\,(\gamma^{\mathbf{d}})^{\nu\,\rho'}\,(\widetilde{\Gamma}_5)_{\rho'\,\alpha'}\,(\widetilde{\Gamma}_5)_{\mathbf{d}}{}^{\mathbf{g}}\,H_{+\mathbf{g}\,\widetilde{+\mathbf{a}}} \\
&\quad + \tfrac{1}{(r_{AdS})\,P_{\widetilde{-}}}\,(\gamma_{\mathbf{a}})_{\alpha'\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,H_{\beta\,\widetilde{\sigma}} \qquad \text{(A.58)}
\end{aligned}
$$

where we simplified (A.57) by using the explicit property of γ_- and γ_+ being the unit or zero matrix (depending on specific indices), see (14.14). We used this simplification in another equations as well (for example in equation (17.23)). In the equation (A.58) we also used new matrix $(\widetilde{\Gamma}_5)_{\mathbf{d}}{}^{\mathbf{g}}$, that was be introduced in (18.28). We also used the identity (18.26) to simplify (A.58). Plugging the evaluated expression (A.58) into (18.3) we will get the action of $D_{\alpha'}\, H_{\beta\,\widetilde{+\mathbf{a}}}$:

$$
\begin{aligned}
0 \;=\; & D_{\alpha'}\, H_{\beta\,\widetilde{+\mathbf{a}}} \,-\, \tfrac{1}{h\,4\,(r_{AdS})^4\,P_-\,P_{\widetilde{-}}{}^2}\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta}\,H_{+\mathbf{c}\,\widetilde{+\mathbf{a}}} \qquad \text{(A.59)} \\
& +2\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta}\,H_{\widetilde{+\mathbf{a}\,\mathbf{c}}} \,+\, \tfrac{1}{h\,(r_{AdS})\,P_{\widetilde{-}}}\,(\gamma_{\mathbf{a}})_{\alpha'\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,H_{\beta\,\widetilde{\sigma}} \\
& \text{left} \,\leftrightarrow\, \text{right} \qquad \text{(A.60)}
\end{aligned}
$$

where h is defined as:

$$
h \,:=\, (\,1 \,-\, \tfrac{1}{P_-\,P_{\widetilde{-}}\,(r_{AdS})^2}\,) \qquad \text{(A.61)}
$$

The equations (A.59) and (A.60) are very interesting since after applying the $D_{\alpha'}$ (or $D_{\widetilde{\alpha}'}$) derivatives we are getting terms like $H_{\widetilde{+\mathbf{a}\,\mathbf{c}}}$ that is basically our original $H_{\widetilde{+\mathbf{a}}\,+\mathbf{c}}$, see (17.6). Moreover we got also term $H_{\beta\,\widetilde{\sigma}}$ that is a new term and was important in chapters where we constructed the pre-potential.

Another important derivatives are $D_{\widetilde{\alpha}'}$ on $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $D_{\alpha'}$ on $H_{\widetilde{\beta}\,+\mathbf{a}}$. We will look at those closer:

$$
\begin{aligned}
T_{\widetilde{D}\,D\,\widetilde{S}} \,\equiv\, T_{\widetilde{\alpha}'\,\beta\,\widetilde{+\mathbf{a}}} \,=\, 0 \;&=\; D_{[\widetilde{\alpha}'}\, H_{\beta\,\widetilde{+\mathbf{a}})} \,+\, H_{[\widetilde{\alpha}'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}})\,\mathcal{N}} \qquad \text{(A.62)} \\
&=\; D_{\widetilde{\alpha}'}\, H_{\beta\,\widetilde{+\mathbf{a}}} \,+\, S_{\widetilde{+\mathbf{a}}}\, H_{\widetilde{\alpha}'\,\beta} \,-\, D_{\beta}\, H_{\widetilde{+\mathbf{a}}\,\widetilde{\alpha}'} \qquad \text{(A.63)} \\
&\quad +H_{\widetilde{\alpha}'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}}\,\mathcal{N}} \\
&\quad +H_{\widetilde{+\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{\alpha}'\,\beta\,\mathcal{N}} \,-\, H_{\beta\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{+\mathbf{a}}\,\widetilde{\alpha}'\,\mathcal{N}}
\end{aligned}
$$

The mixed f terms are zero in the flat superspace. In the AdS case the $f_{\widetilde{\alpha}'\,\beta\,\mathcal{N}} \neq 0$ and there is also AdS contribution coming from $S_{\widetilde{+\mathbf{a}}}\, H_{\widetilde{\alpha}'\,\beta}$. This contribution can be calculated by analogy with the equations (A.49), (A.50), (A.51), (A.52) and (A.58). Thus getting evaluated action $S_{\widetilde{+\mathbf{a}}}\, H_{\widetilde{\alpha}'\,\beta}$:

$$S_{\widetilde{+\mathbf{a}}}\, H_{\widetilde{\alpha}'\,\sigma} \;=\; \pm\, \frac{(-1)}{h\,4\,(r_{AdS})^3\, P_{\widetilde{-}}\, P_{-}}\, (\gamma^{+\mathbf{d}})_{\alpha'}{}^{\nu}\, (\widetilde{\Gamma}_5)_{\nu\,\sigma}\, H_{\widetilde{+\mathbf{d}}\,+\mathbf{a}} + \frac{1}{h\,(r_{AdS})^2\, P_{\widetilde{-}}\, P_{-}}\, (\gamma_{-})_{\sigma\,\nu}\, (\widetilde{\Gamma}_5)^{\nu\,\rho}\, H_{\widetilde{\rho}\,\beta}\, (\widetilde{\Gamma}_5)^{\beta\,\epsilon}\, (\gamma_{\mathbf{a}})_{\epsilon\,\alpha'} \tag{A.64}$$

where h was defined in (A.61). The (A.63) mixed structure constant $f_{\beta\,\widetilde{+\mathbf{a}}\,\mathcal{N}} = 0$ and the $f_{\widetilde{\alpha}'\,\beta\,\mathcal{N}}$ has been discussed before (see equations (A.48) and (A.49) till (A.52)). Moreover, the vielbein $H_{\widetilde{+\mathbf{a}}\,\widetilde{\alpha}'}$ is evaluated to 0, see table (10). Evaluating everything in (A.64) we get:

$$0 \;=\; D_{\widetilde{\alpha}'}\, H_{\beta\,\widetilde{+\mathbf{a}}} - \frac{1}{h\,4\,(r_{AdS})^3\, P_{\widetilde{-}}\, P_{-}}\, (\gamma^{\mathbf{d}})_{\alpha'\,\epsilon}\, (\widetilde{\Gamma}_5)^{\epsilon\,\sigma}\, \gamma^{+}{}_{\sigma\,\beta}\, H_{\widetilde{+\mathbf{d}}\,+\mathbf{a}} + \frac{1}{r_{AdS}}\, (\gamma^{\mathbf{d}})_{\alpha'\,\epsilon}\, (\widetilde{\Gamma}_5)^{\epsilon\,\sigma}\, (\gamma^{+})_{\sigma\,\beta}\, H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}} + \frac{1}{h\,(r_{AdS})^2\, P_{\widetilde{-}}\, P_{-}}\, (\gamma_{-})_{\beta\,\nu}\, (\widetilde{\Gamma}_5)^{\nu\,\rho}\, H_{\widetilde{\rho}\,\lambda}\, (\widetilde{\Gamma}_5)^{\lambda\,\epsilon}\, (\gamma_{\mathbf{a}})_{\epsilon\,\alpha'} - \frac{1}{2}\, (\gamma_{+\mathbf{a}})_{\alpha'}{}^{\nu}\, H_{\beta\,\widetilde{\nu}} \tag{A.65}$$

$$\text{left} \leftrightarrow \text{right} \tag{A.66}$$