Some Studies on Partition Functions
in Quantum Field Theory and Statistical Mechanics

A Dissertation presented

by

Jun Nian

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Physics

Stony Brook University

August 2015

Stony Brook University

The Graduate School

Jun Nian

We, the dissertation committe for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Martin Roček - Dissertation Advisor
Professor, C. N. Yang Institute for Theoretical Physics

Peter van Nieuwenhuizen - Chairperson of Defense
Distinguished Professor, C. N. Yang Institute for Theoretical Physics

Alexander Abanov
Associate Professor, Simons Center for Geometry and Physics

Dominik Schneble
Associate Professor, Department of Physics and Astronomy

Leon Takhtajan
Professor, Department of Mathematics

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

Some Studies on Partition Functions in Quantum Field Theory and Statistical Mechanics

by

Jun Nian

Doctor of Philosophy

in

Physics

Stony Brook University

2015

Quantum field theory and statistical mechanics share many common features in their formulations. A very important example is the similar form of the partition function. Exact computations of the partition function can help us understand non-perturbative physics and calculate many physical quantities. In this thesis, we study a number of examples in both quantum field theory and statistical mechanics, in which one can compute the partition function exactly.

Recently, people have found a consistent way of defining supersymmetric theories on curved backgrounds. This gives interesting deformations to the original supersymmetric theories defined on the flat spacetime. Using the supersymmetric localization method one is able to calculate the exact partition functions of some supersymmetric gauge theories on compact manifolds, which provides us with many more rigorous checks of dualities and many geometrical properties of the models. We discuss the localization of supersymmetric gauge theories on squashed S^3, round S^2 and T^2.

Entanglement entropy and Rényi entropy are key concepts in some branches of condensed matter physics, for instance, quantum phase transition, topological phase and quantum computation. They also play an increasingly important role in high energy physics and black-hole physics. A generalized and related concept is the supersymmetric Rényi entropy. In the the-

sis we review these concepts and their relation with the partition function on a sphere. We also consider the thermal correction to the Rényi entropy at finite temperature.

Partition functions can also be used to relate two apparently different theories. One example discussed in the thesis is the Gross-Pitaevskii equation and a string-like nonlinear sigma model. The Gross-Pitaevskii equation is known as a mean-field description of Bose-Einstein condensates. It has some nontrivial solutions like the vortex line and the dark soliton. We give a field theoretic derivation of these solutions, and discuss recent developments of the relation between the Gross-Pitaevskii equation and the Kardar-Parisi-Zhang equation. Moreover, a (2+1)-dimensional system consisting of only vortex solutions can be described by a statistical model called point-vortex model. We evaluate its partition function exactly, and find a phase transtition at negative temperature. The order parameter, the critical exponent and the correlation function are also discussed.

To the people I love

Contents

List of Figures

List of Tables

Acknowledgements

First, I would like to thank my advisor Prof. Martin Roček for his constant support, patience and always being available to help me and answer my questions.

I want to show my high respect and deep thank to Prof. Peter van Nieuwenhuizen, who has taught me much more than I could imagine and guided me into the way of research on theoretical physics.

I am also very grateful to Prof. Chris Herzog, who is a really nice collaborator and a good friend.

During my stay at Stony Brook, I benifited a lot from the courses given by many excellent professors, including Warren Siegel, Leonardo Rastelli, Sasha Abanov, Robert Shrock, Barry McCoy, Vladimir Korepin, Konstantin Likharev, Phil Allen, Lukasz Fidkowski, Ismail Zahed, Alexander Kirillov and Raanan Schul. I would like to thank all of them.

I also would like to thank Prof. Axel Drees, Prof. Dominik Schneble and Prof. Leon Takhtajan for being on my committee. I wish I could have chance to discuss with them about my work again in the future.

The quite intensive courses on elementary particle physics at SISSA were my first exposure to the wonderland of modern theoretical physics. I am very grateful to the professors who have taught me at SISSA, including Loriano Bonora, Roberto Iengo, Marco Serone, Matteo Bertolini, Giulio Bonelli, Sebastiano Sonego, Piero Ullio, Serguey Petcov, Andrea Romanino, Edi Gava and Sergio Cecotti.

Some visitors at Simons Center for Geometry and Physics generously shared their knowledge with me, and helped me resolve some technical problems in my research. Among them I especially would like to thank Sungjay Lee, Kazuo Hosomichi, Kentaro Hori and Michele Del Zotto.

I have also learned a lot from the conversation with many local postdocs, including Eoin Colgain, Kristan Jensen, Koushik Balasubramanian, Cyril Closset and Francesco Benini. I would like to thank them.

I really have fun in my research work thanks to my collaborators: Xinyu Zhang, Yachao Qian, Xiaoquan Yu, Antonio Muñoz, Stefano Giaccari, Yang Zhou, Peng Zhao and JP Ang.

Prof. George Sterman has done a wonderful job to make YITP a nice place to work in.

I would like to thank him and the people at administration: Betty, Helen, Ming, Sara and Diane.

I enjoy everyday at Stony Brook thanks to my YITP officemates including Yiming, Pin-Ju, Andrea, Tyler, Michael, Harri, Sujan, Matt, Mao, Madalena, Wolfger, Chia-Yi, Yiwen, Colin, Dharmesh and Ozan. Moreover, working out with Yachao, Scott and Saebyeok in the gym is also fun.

There are a lot more friends during my PhD study at Trieste and Stony Brook, which I could not mention in this little space, but I wish we could meet one day and hang out again to discuss physics, math, animation, music, history and philosophy.

Finally and most importantly, I would like to thank my family and the people I love for their persistent support, encouragement, stimulation and inspiration. Without them I would never be able to go so far.

Publications

1. *Localization of Supersymmetric Chern-Simons-Matter Theory on a Squashed* S^3 *with* $SU(2) \times U(1)$ *Isometry:*
   Jun Nian, 1309.3266, JHEP 1407 (2014) 126.
2. *Dynamics of Two-Dimensional* $\mathcal{N} = (2, 2)$ *Supersymmetric Theories with Semi-chiral Superfields I:*
   Jun Nian, Xinyu Zhang, 1411.4694.
3. *Thermal Corrections to Rényi Entropies for Conformal Field Theories:*
   Christopher Herzog, **Jun Nian**, 1411.6505, JHEP 1506 (2015) 009.
4. *Semichiral Fields on* S^2 *and Generalized Kähler Geometry:*
   Francesco Benini, P. Marcos Crichigno, Dharmesh Jain, **Jun Nian**, 1505.06207.
5. *Clustering Transitions of Two-Dimensional Vortices at Negative Temperature:*
   Xiaoquan Yu, **Jun Nian**, Thomas P. Billam, Matthew T. Reeves, Ashton Bradley, to appear soon.

Chapter 1

Introduction

As two cornerstones of modern physics, quantum theory (including quantum mechanics and quantum field theory) and statistical mechanics share many similarities. In this chapter, we recaptulate some important results along their developments and highlight some of their common features.

1.1 Historical Developments (before 1980)

1.1.1 Quantum Theory

In this and the next section, I will try to summarize some important developments that are relevant to our later discussions in this thesis. The list is according to my personal preference and far from complete.

Let us start with the discussion of the quantum theory. After the era of the old quantum theory opened by Max Planck, Niels Bohr, Albert Einstein and many others, in 1925 Werner Heisenberg (with Max Born and Pascual Jordan) and Erwin Schrödinger formulated two versions of quantum mechanics independently. In modern language, they worked in two different pictures. In the Heisenberg picture, the operator evolves with time while the wave function remains unchanged, and the evolution of the operator is governed by the Heisenberg equation:

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A] + \frac{\partial A}{\partial t}\,, \tag{1.1.1}$$

where A is an arbitrary operator, and H is the Hamiltonian of the system. In the Schrödinger picture, the operator remains the same, while the wave function evolves according to the

Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x)\,. \tag{1.1.2}$$

Later in 1928, Paul Dirac generalized the non-relativistic quantum mechanics into relativistic quantum mechanics. Based on some previous work done by Paul Dirac and John Wheeler, in 1948 Richard Feynman found an ingenious formulation of quantum mechanics using the path-integral. In this formulation, the transition matrix element is given by

$$\begin{aligned}
\langle q', t'|q, t\rangle &\equiv \langle q', t'|e^{-iH(t'-t)}|q, t\rangle \\
&= \lim_{n\to\infty}\left(\frac{m}{2\pi i\,\delta t}\right)^{n/2}\int\prod_{i=1}^{n-1}dq_i\exp\left[i\sum_{i=1}^{n}\delta t\left(\frac{m}{2}\left(\frac{q_i-q_{i-1}}{\delta t}\right)^2 - V\right)\right] \\
&= N\int[dq]\exp\left[i\int_t^{t'}d\tau\left(\frac{m}{2}\dot{q}^2 - V(q)\right)\right] \\
&= N\int[dq]\exp\left[i\int_t^{t'}d\tau\,L(q,\dot{q})\right]\,.
\end{aligned} \tag{1.1.3}$$

It turns out the path integral also provides an excellent framework to formulate quantum field theory. Up to now, there are still some open questions about the foundation of quantum mechanics. Some attempts to address these issues led to some other formulations of quantum mechanics, e.g. the de Broglie-Bohm theory, but I would stop our discussion on quantum mechanics for now and move on to the quantum field theory.

Unlike quantum mechanics, quantum field theory was founded by many people, and it is still developing and not in its final shape. As a natural generalization of quantum mechanics, there are actually various ways to start from classical mechanics and arrive at quantum field theory. In the following figure, we illustrate some possible paths. Different textbooks may follow different paths, for instance, Ref. [1] introduces quantum field theory by generalizing quantum mechanics to describe infinite many degrees of freedom, while Ref. [2] starts with classical field theory, e.g. the eletromagnetic field, and quantizes it to obtain quantum field theory.

Similar to quantum mechanics, quantum field theory also has different formulations. One is the canonical quantization, which implements the quantization condition by requiring the equal-time commutation relations:

$$\begin{aligned}
&[\pi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = -i\delta(\mathbf{x}-\mathbf{x}')\,, \\
&[\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = [\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = 0\,.
\end{aligned} \tag{1.1.4}$$

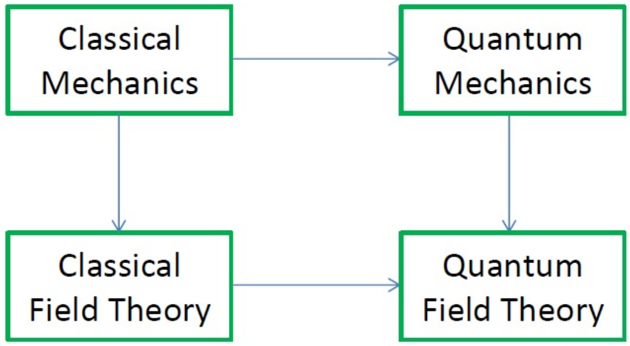


Figure 1.1: Different Paths from Classical Mechanics to Quantum Field Theory

Another way of quantizing a field is in the spirit of Feynman's path integral. The path integral (or the partition function) in this case is

$$Z = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]} , \tag{1.1.5}$$

where

$$S \equiv \int d\mathbf{x} \, dt \, \mathscr{L}(\phi(\mathbf{x}, t), \partial_\mu \phi(\mathbf{x}, t)) \tag{1.1.6}$$

is the action of the theory. The expectation value of an operator A is given by

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, A \, e^{\frac{i}{\hbar} S[\phi]} . \tag{1.1.7}$$

By coupling the field $\phi(\mathbf{x}, t)$ to a source $J(\mathbf{x}, t)$, one can define a generating functional

$$W[J] = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar} \int d\mathbf{x} \, dt \, \Big(\mathscr{L}(\phi(\mathbf{x}, t), \partial_\mu \phi(\mathbf{x}, t)) + J(\mathbf{x}, t) \, \phi(\mathbf{x}, t) \Big) \right] . \tag{1.1.8}$$

By differentiating $W[J]$ with respect to the source $J(\mathbf{x}, t)$, one is able to calculate some physical quantities, e.g. the correlation function $\langle \phi(\mathbf{x}, t) \phi(\mathbf{x}', t) \rangle$. Like for quantum mechanics, there are also other ways of quantizing a field theory, for instance the stochastic quantization invented by Edward Nelson and later developed by Giorgio Parisi and Yong-Shi Wu, which provide some other perspectives of quantum field theory. We do not discuss these new approaches in the thesis.

Now we can see some general features of a quantum field theory. The minimum action corresponds to the classical field theory, which provides the most dominant contribution to a quantum field theory. One can use the variational principle to minimize the action and obtain the classical solution. If we are able to calculate the partition function exactly, the result should incorporate all the contributions from the classical solution and the quantum corrections. If it is not possible to perform an exact computation, some approximations may give us useful information about the physics, for instance, one can do a pertubation expansion for a small coupling constant g or a WKB approximation which is essentially an expansion in $\hbar$.

In the above, we briefly discussed the general formalism of a quantum field theory. In fact, the quantum field theory finds its applications in almost all the branches of physics. We list a few of them only in high energy physics:

- Based on the work of Paul Dirac and Hans Bethe, during 1946-1950 several people including Sin-Itiro Tomonaga, Julian Schwinger, Richard Feynman and Freeman Dyson proved that Quantum Electrodynamics (QED) is a renormalizable quantum field theory.
- In 1954, Chen Ning Yang and Robert Mills found a new gauge theory with non-Abelian groups, as a generalization of an Abelian gauge theory that describes the electromagnetism.
- Many people made attemps to quantize the Yang-Mills theory. Today, a well-known way of quantizing the Yang-Mills theory is to add a gauge-fixing term to the theory and introduce additional ghost fields, which was formulated in 1967 by Ludvig Faddeev and Victor Popov. A modern way of quantizing a gauge theory is to perform a BRST quantization, which was invented by Becchi, Rouet, Stora and independently by Tyutin during 1974-1976. An even more general way is the Batalin-Vilkovisky formalism.
- In 1972, Gerard 't Hooft and Martinus Veltman proved that the Yang-Mills theory is a renormalizable quantum field theory.
- In 1973, Frank Wilczek, David Gross and David Politzer showed that the Yang-Mills theory has a property called asymptotic freedom.
- The theory for the electroweak interaction was formulated by Sheldon Glashow, Steven Weinberg and Abdus Salam in the 1960's.
- The theory for the strong interaction was suggested by Yoichiro Nambu and many other people in the 1960's.
- To explain the massive gauge boson, the so-called "Higgs mechanism" was proposed by Philip Warren Anderson, Robert Brout, Francois Englert, Peter Higgs, Gerald Guralnik, Carl Richard Hagen and Thomas Kibble during 1962-1964.
- SUSY: Based on the previouos work by Yu. A. Golfand and some other people, in 1974 Julius Wess and Bruno Zumino found some renormalizable 4-dimensional supersymmetric quantum field theories.

- SUGRA: In 1976, Daniel Freedman, Sergio Ferrara and Peter van Nieuwenhuizen discovered the supergravity theory.
- Soliton: In 1974, Gerard 't Hooft and Alexander Polyakov independently found a topological soliton solution to the Yang-Mills theory coupled to a Higgs field, which was later called the 't Hooft-Polyakov monopole.[1]
- Instanton: In 1975, Alexander Belavin, Alexander Polyakov, Albert S. Schwartz and Yu. S. Tyupkin found a classical solution to the self-dual Yang-Mills equation, which was later called the instanton.

The incomplete list above only selects some developments, which have something to do with the topics later discussed in this thesis, and it omits many important works especially in gravitational theory, e.g. Hawking's work on the black-hole radiation.

1.1.2 Statistical Mechanics

In contrast to quantum theory, statistical mechanics had a sophisticated formulation soon after it was founded, based on the seminal work of Ludwig Boltzmann and Josiah Willard Gibbs. Let us briefly review the ensembles often used in physical problems.

For a physical quantity A, its expectation value is given by

$$\langle A \rangle = \frac{\int dq\, dp\, A(q,p)\, \rho(q,p)}{\int dq\, dp\, \rho(q,p)} \tag{1.1.9}$$

where $\rho(q,p)$ is the probability density. The partition function of an ensemble is given by

$$Z = \int dq\, dp\, \rho(q,p)\,. \tag{1.1.10}$$

The microcanonical ensemble has a fixed number of particles N, a given volume V and the range of energy $[E, E+\Delta]$. The probability density of the microcanonical ensemble is

$$\rho(q,p) = \begin{cases} 1\,, & \text{if } E < H(q,p) < E + \Delta\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{1.1.11}$$

The canonical ensemble has a fixed number of particles N, a given volume V and the temperature of the system T. The probability density of the canonical ensemble is

$$\rho(q,p) = e^{-\beta\, H(q,p)}\,, \tag{1.1.12}$$

[1]The concept of soliton was introduced by Kruskal and Zabusky in 1965.

where $\beta \equiv 1/(k_B T)$. For the grand canonical ensemble, the number of particles can vary, and the corresponding probability density is given by

$$\rho(q, p, N) = \frac{1}{N!} z^N e^{-\beta H(q,p)}, \tag{1.1.13}$$

where $z \equiv e^{\beta\mu}$ with the fugacity μ. We see that the formalism of statistical mechanics is very similar to the one of quantum field theory.

Based on the powerful tools of the ensemble theory and the exact results of the partition functions, one is able to study many physical problems in a rigorous and quantitative way. The most famous and well-studied model in physics might be the Ising model:

$$\mathcal{H} = -\frac{J}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i \tag{1.1.14}$$

where $\sigma_i = \pm 1$ are the spins on the lattice, and $\langle i, j \rangle$ denotes the neighbor spins. The 1-dimensional case was studied by Ernst Ising in 1920s, and he published the results in 1925 based on his PhD study. He showed that the model has no phase transition in 1-dimension. Later, for the 2-dimensional case without the magnetic field people have found many ways to compute the partition function exactly, and confirmed the existence of a phase transition (see Ref. [3] for a summary). The interest to 3-dimensional Ising model has continued even till now.[2]

Just like in the d-dimensional Ising model ($d \geq 2$), phase transitions in other systems also attracted much attention, for instance in superconductivity and superfluidity. The exact and the approximate solutions in these models play crucial roles in the study of critical phenomena.

1.2 Recent Developments (after 1980)

In recent years, especially after the first string theory revolution in 1984, there have been many more new developments in quantum field theory, statistical mechanics and their connections. Let us list some of them in this section.

Again, we have to omit something. For example, the string theory, especially the topological string theory, is also a quantum theory, and it is very interesting on its own. In many cases, one is also able to compute the partition function of a string theory or a topological string theory exactly. However, since we will not study these theories later in the thesis, they are omitted in this section.

[2] For instance, people have tried to use the conformal bootstrap technique to solve the 3D Ising model [4].

1.2.1 Quantum Theory

One of the breakthroughs in theoretical physics during 1980s is the finding of the connection between certain topological quantum field theories and some topological invariants in mathematics. Among them the most important ones are probably the series of papers by Edward Witten, who used the equivariant localization technique which was known in mathematics long ago and will be briefly reviewed in Chapter 2, to find the exact solutions of the 2-dimensional Yang-Mills theory [5], and the exact solutions of 3-dimensional Chern-Simons theory on Seifert manifolds with Chris Beasley [6]. Moreover, he applied the topological twist to $\mathcal{N} = 2$ supersymmetric field theories to reproduce the Donaldson invariant for 4-dimensional manifolds [7], and later in 1994 Edward Witten and Nathan Seiberg found another supersymmetric theory, which can be solved exactly on the physics side and give a new topological invariant on the mathematics side [8, 9]. As an application, in 1994 Cumrun Vafa and Edward Witten used the partition function of $\mathcal{N} = 4$ topologically twisted supersymmetric Yang-Mills theory on 4-manifolds to test the strong-weak duality (S-duality) [10]. To compute the contribution of the instantons to the Seiberg-Witten theory, in 2002 Nikita Nekrasov introduced new techniques (Ω-deformation, noncommutative field theory) and obtain the exact results.

The next progress came from Vasily Pestun's work in 2007, in which he managed to define $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ supersymmetric Yang-Mills theories on S^4, and by using the supersymmetric localization technique he was able to compute their partition functions and the expectation values of the Wilson loop exactly [11]. Starting from Pestun's work in 2007, there has been a booming time for the study of supersymmetry on curved spacetime.[3] Many papers and results of supersymmetric theories on various compact manifolds had appeared, but a natural question is whether one can study these different backgrounds systematically. A paper by Seiberg and Festuccia first started to address this issue [12]. They started with an off-shell supergravity theory, and by fixing the metric and the background fields they were able to define a theory with rigid supersymmetry on a certain manifold.[4] In the following series of papers, they and their collaborators studied the problem systematically, and found some conditions on the 3- and 4-dimensional manifolds for the existence of certain amount of supersymmetries [14, 15, 16]. For instance, for a 4-dimensional manifold, in order to allow one supercharge the manifold has to be Hermitian. Since the results of supersymmetric localization are exact, one can use them to check some previously proposed conjectures in a more rigorous way. For example, Refs. [17, 18, 19, 20] tried to make some tests of the gauge gravity correspondence in this way.

As we discussed in the previous section, after the supergravity was founded, people paid

[3] In 1970s there were already some works of Killing spinors on some curved manifolds, e.g. the AdS space.
[4] The idea of this new method can be applied to more general superspaces, as discussed in Ref. [13].

more attention to the quantum perspective of the gravity theory. The breakthrough made by Juan Maldacena in 1997 connects certain gravity theories with gauge theories [21], and brings more ingredients to the study of the gravity as a quantum theory. Like the 2-dimensional Yang-Mills theory, the 3-dimensional pure gravity is also a topological theory, hence one can hope to evaluate its partition function exactly. This work started with E. Witten's paper, Ref. [22], and continued in Ref. [23]. The exact partition function was worked out in Refs. [24, 25, 26]. Also in the presence of supersymmetry, one may hope to apply the supersymemtric localization technique, which we willl discuss in detail in Chapter 2, to evaluate the partition function of some supergravity theories (see e.g. Refs. [27, 28]).

1.2.2 Statistical Mechanics

As we discussed in the previous section, using the exact results of the partition functions one can study the phenomena in many statistical models quantitatively, especially critical phenomena and phase transitions. At the critical point, a statistical system often preserves the conformal symmetry. Hence, the study of conformal field theory finds many applications in the real systems. In 1980s, some significant progress has been made in this direction. The most important ones are the solution of the 2-dimensional Ising model as a conformal field theory by Belavin, Polyakov and Zamolodchikov [29], and also the c-theorem in 2-dimensions by Zamolodchikov [30], which governs the renormalization group flow for an arbitrary conformal field theory.

Many exactly solvable statistical models have another property called integrability. For those models, one can find their soliton solutions using the Hamiltonian methods (see Ref. [31]), and for the second quantized integrable models one can also solve them using the Bethe-Ansatz equation (see Ref. [32]).

The partition function itself plays a crucial role in the recent study of the statistical models, especially the conformal field theory. An example is Ref. [33] by John Cardy, in which the exact partition function is evaluated. Together with the modular property, the author was able to find the spectrum of the 2-dimensional Ising model.

Another trend in the study of the statistical mechanics in recent years is that the connection between statistical models and quantum field theory is emphasized and paid more and more attention. For instance, after the Seiberg-Witten theory was founded in 1994, some people immediately observed that it can be related to some known solutions in integrable models [34]. In the last 20 years, some problems in this research field have been studied systematically, and accordingly there are a number of review articles trying to explain these significant progress in a padagogical way. Let us list them below:

- Starting from Ref. [35], there are a series of review articles explaining the integrability in the context of AdS/CFT correspondence. The basic idea is to map some gauge-invariant operators in $\mathcal{N}=4$ supersymmetric Yang-Mills theory into a spin chain model. Then the integrability of the statistical models can be studied in the framework of the string worldsheet model, i.e. a 2-dimensional nonlinear sigma model on a symmetric coset space.

- Starting from Ref. [36], there are a series of review articles on different aspects of the AGT correspondence [37], which states that the partition functions of some 4-dimensional supersymmetric gauge theories obtained using localization technique [11] are proportional to the correlation functions of the 2-dimensional Liouville theory, which is a conformal field theory given by

$$S_b^{\text{Liou}} = \frac{1}{4\pi}\int d^2z\,\left[(\partial_a\phi)^2 + 4\pi\mu e^{2b\phi}\right]. \tag{1.2.1}$$

To be more precise, the correspondence says that

$$\mathcal{Z}(a,m;\tau;\epsilon_1,\epsilon_2) \propto N_{21}(p)\,N_{43}(p)\,\mathcal{F}_p(q)\,, \tag{1.2.2}$$

where $\mathcal{Z}(a,m;\tau;\epsilon_1,\epsilon_2)$ is the instanton partition function appearing in the whole partition function

$$Z(m;\tau;\epsilon_1,\epsilon_2) = \int da\,|\mathcal{Z}(a,m;\tau;\epsilon_1,\epsilon_2)|^2\,, \tag{1.2.3}$$

while $\mathcal{F}_p(q)$ is the conformal block obtained from the correlation function

$$\langle e^{2\alpha_4\phi(\infty)}\,e^{2\alpha_3\phi(1)}\,e^{2\alpha_2\phi(q)}\,e^{2\alpha_1\phi(0)}\rangle_b^{\text{Liou}} = \int_{\mathbb{R}^+}\frac{dp}{2\pi}C_{21}(p)\,C_{43}(-p)\,\left|\mathcal{F}_p(q)\right|^2, \tag{1.2.4}$$

and $|N_{ij}(p)|^2 = C_{ij}(p)$. If we include the loop operators, e.g. the Wilson loop or the 't Hooft loop, the AGT correspondence states that

$$\langle\mathcal{W}\rangle_{S^4}^{\text{SYM}} = \langle \mathrm{L}_{\gamma_s}\rangle_b^{\text{Liou}}\,,\quad \langle\mathcal{T}\rangle_{S^4}^{\text{SYM}} = \langle \mathrm{L}_{\gamma_t}\rangle_b^{\text{Liou}}\,, \tag{1.2.5}$$

where $\langle\cdot\rangle$ denotes the normalized expectation value of an operator, and

$$\mathrm{L}_\gamma \equiv \mathrm{tr}\left[\mathcal{P}\exp\left(\int_\gamma \mathcal{A}\right)\right], \tag{1.2.6}$$

while γ_s and γ_t stand for the simple closed curves encircling the points $0,q$ and $1,q$ on $\mathbb{C}\setminus\{0,q,1\}$ respectively. Using the AGT correspondence, one can even check the S-duality conjecture:

$$\langle\mathcal{W}\rangle_{S_s} = \langle\mathcal{T}\rangle_{S_t}\,,\quad \langle\mathcal{T}\rangle_{S_s} = \langle\mathcal{W}\rangle_{S_t}\,. \tag{1.2.7}$$

- Some important work not reviewed in the articles mentioned above includes the recent progress made by Nikita Nekrasov and Samson Shatashvili [38, 39, 40, 41]. The basic idea is to identify the effective twisted superpotential $\widetilde{\mathcal{W}}$ of the supersymmetric gauge theory with the Yang-Yang function of a quantum integrable system. Then the vacuum equation of the supersymmetric gauge theory

$$\exp\left(\frac{\partial \widetilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i}\right) = 1 \tag{1.2.8}$$

  will correspond to the Bethe-Ansatz equation for the quantum integrable system.

Through the discussions above, we have seen that in the last century the two major branches of theoretical physics, quantum field theory and statistical mechanics, have developed in a quite parallel way, and their modern formulations are quite similar to each other. One can often introduce some techniques mutually between these two fields. The exchange of ideas and the attempt to connect ideas in both fields sometimes can lead to important breakthroughs. Today, one century after General Relativity was found, we believe that in the next century the fundamental study of quantum field theory, statistical mechanics and gravity will help us understand nature in a better way and unveil more secrets of the universe hidden in the dark.

1.3 Organization of the Thesis

This thesis is based on my research work during the last three years and some ongoing research projects. It covers quite diverse subjects in theoretical physics, including supersymmetric gauge theories, conformal field theory, AdS/CFT correspondence, Bose-Einstein condensation and quantum turbulence. A main theme connecting them is the concept of the partition function, i.e., the exact result of the partition function was obtained whenever possible. I summarize the different problems considered in each chapter in the following:

- **Chapter 2: SUSY Localization**
  We first review the general principle of localizatin from both physical aspects and mathematical aspects. After that we consider the localization of some supersymmetric gauge theories, including $\mathcal{N} = 2$ supersymmetric Chern-Simons theories with matter on a squahsed S^3 and $\mathcal{N} = (2, 2)$ supersymmetric theories on S^2 and T^2. We compute the exact partition function for each case and discuss their applications.

- **Chapter 3: Entanglement Entropy**
  The concepts entanglement entropy, Rényi entropy and supersymmetric Rényi entropy will be introduced, and their relation with the partition on a sphere will also be dis-

cussed. Moreover, we consider the thermal corrections to the Rényi entropy for a free scalar theory in higher dimensions.

- **Chapter 4: BEC, String Theory and KPZ Equation**
  The Gross-Pitaevskii equation provides a mean-field model for the Bose-Einstein condensate. In this chapter, we illustrate the relation between the Gross-Pitaevskii equation, a string theory-like nonlinear sigma model and the KPZ equation, which describes the growth of a random surface. Their equivalence in some limits can be shown by rewriting the partition function.

- **Chapter 5: 2-Dimensional Point-Vortex Model**
  The 2-dimensional point-vortex model is a statistical model in the study of quantum turbulence. Some numerical simulations suggest a phase transition in this model at negative temperature. We demonstrate the existence of the phase transition in two different ways: by the analysis of the bifurcation point of a partial differential equation and by the exact computation of the partition function.

Finally in Chapter 6, we summarize the thesis and discuss some open questions in the study of quantum field theory and statistical mechanics related to the computation of the partition function.

Chapter 2

Supersymmetric Localization

As we discussed in the introduction, many important recent developments are obtained using the supersymmetric (SUSY) localization technique. In this chapter, we first discuss the physical and the mathematical aspect of this technique. As some applications, we will see some concrete examples of the supersymmetric localization on the squashed S^3, the round S^2 and the torus T^2, based on my paper [42] and another paper with Xinyu Zhang [43] as well as some unpublished notes.

2.1 Introduction to Supersymmetric Localization

Localization is a technique which simplifies calculations in some quantum field theories, e.g., supersymmetric gauge theories, and can give non-perturbatively exact results. We follow Ref. [44] and briefly review the basic idea of localization. Suppose we have a supersymmetric Lagrangian, then adding a term that is a supersymmetric invariant does not spoil the supersymmetry of the new Lagrangian. If the additional term is also exact, i.e., it can be written as the supersymmetry transformation of some other terms, then the partition function remains the same after adding this additional term, provided that we do not change the behavior of the action functional at infinity in flat space. This can be shown as follows:

1. With a dimensionless coupling constant t, the new action with the additional term becomes $S + th$, where h is exact, i.e., $h = \delta g$. Provided that supersymmetry is not broken in the vacuum, i.e., the measure is supersymmetry, the partition function is independent of t:

$$\frac{\partial}{\partial t}\int d\Phi e^{-S-th} = \frac{\partial}{\partial t}\int d\Phi e^{-S-t\delta g} = \int d\Phi \delta g e^{-S-t\delta g} = \int d\Phi \delta \left(g e^{-S-t\delta g}\right) = 0\,. \tag{2.1.1}$$

Hence, we can evaluate the path integral with $S + th$ at any value of t, and the result should be the same.

2. Moreover, if we take the limit $t \to \infty$, then only $h = 0$ contributes to the partition function, or in other words, the theory localizes to the field configuration where $h = 0$. Solving the equation $h = 0$, we obtain the classical values of the fields.

3. Next, we can separate the classical value and the quantum fluctuation for each field, and only consider the classical and 1-loop contributions to the partition function. In principle, there should be contributions from higher orders, but in the limit $t \to \infty$, the result of the classical together with the 1-loop contributions becomes exact. Let us use Φ to denote all the fields, then

$$\Phi = \Phi_{cl} + \widetilde{\Phi}\,, \tag{2.1.2}$$

where Φ_{cl} and $\widetilde{\Phi}$ denote the classical value and the quantum fluctuation of the field respectively. The partition function can be expressed as

$$\begin{aligned}\int d\Phi e^{-S} &= \sum_{\text{BPS}} \int d\widetilde{\Phi} e^{-S(\Phi_{cl}) + t(\text{quadratic terms in } \widetilde{\Phi})} \\ &= \sum_{\text{BPS}} e^{-S(\Phi_{cl})} \frac{\det_{\text{ferm}}}{\det_{\text{bos}}}\,,\end{aligned} \tag{2.1.3}$$

where the sum is taken over the BPS eigenfunctions. By BPS we mean the field configurations where the theory is localized, i.e., all the possible solutions to the equation $h = 0$ mentioned above.

2.2 Mathematical Aspects

In this section we review the mathematical aspects of localization, especially the equivariant localization and the Duistermaat-Heckman formula, which is a special case of the more general Atiyah–Bott–Berline–Vergne localization formula. The readers who are more interested in physical aspects can skip this section. We closely follow the note by Antti J. Niemi in Ref. [45].

Let us start with a brief review of the symplectic geometry. Assume that $\mathcal{M}$ is a symplectic manifold. Instead of the variables p_a and q^a we adopt the local coordinates z^a $(a = 1, \cdots, 2n)$ on $\mathcal{M}$ with the Poisson brackets

$$\{z^a, z^b\} = \omega^{ab}(z)\,, \tag{2.2.1}$$

where ω^{ab} is nondegenerate and has the inverse

$$\omega^{ac}\,\omega_{cb} = \delta^a{}_b\,. \tag{2.2.2}$$

We define a symplectic two-form:

$$\omega \equiv \frac{1}{2}\omega_{ab}\, dz^a \wedge dz^b\,. \tag{2.2.3}$$

The Jacobi identity for Poisson brackets implies that the symplectic two-form is closed. Hence, locally it is also exact, i.e.,

$$\omega = d\vartheta = \partial_a \vartheta_b\, dz^a \wedge dz^b\,. \tag{2.2.4}$$

The canonical transformations are equivalent to

$$\vartheta \to \vartheta + d\psi\,. \tag{2.2.5}$$

A natural volume form on $\mathcal{M}$ is

$$\omega^n = \omega \wedge \cdots \wedge \omega\,, \tag{2.2.6}$$

and the classical partition function is then

$$Z = \int \omega^n\, e^{-\beta H}\,. \tag{2.2.7}$$

H is a Hamiltonian on $\mathcal{M}$, and its corresponding Hamiltonian vector field χ_H is defined by

$$\omega(\chi_H, \cdot) + dH = 0 \quad \Leftrightarrow \quad \chi_H^a = \omega^{ab}\,\partial_b H\,. \tag{2.2.8}$$

The Poisson bracket of two Hamiltonians has the expressions

$$\begin{aligned}\{H, G\} = \omega^{ab}\,\partial_a H\,\partial_b G = \chi_H^a\,\partial_a G = i_H\, dG \\ = \omega_{ab}\,\chi_H^a\,\chi_G^b = \omega(\chi_H, \chi_G)\,,\end{aligned} \tag{2.2.9}$$

where

$$i_H : \Lambda^k \to \Lambda^{k-1}\,, \quad (i_H)^{\,2} = 0\,. \tag{2.2.10}$$

The equivariant exterior derivative is defined as

$$d_H \equiv d + i_H\,, \tag{2.2.11}$$

which maps $\Lambda^k \to \Lambda^{k+1} \oplus \Lambda^{k-1}$. At this moment, one probably can realize the similarity between this mathematical formalism and the physical meaning of supersymmetry algebra. The Lie derivative is then given by

$$\mathcal{L}_H \equiv d_H\, i_H + i_H\, d_H = d_H^2\,. \tag{2.2.12}$$

Therefore, the Poisson bracket mentioned above is then

$$\{H, G\} = i_H\, dG = (d\, i_H + i_H\, d)G = \mathcal{L}_H\, G\,, \tag{2.2.13}$$

because G is a function and $i_H\, G = 0$.

Next, we move on to the discussion of the equivariant cohomology. Suppose that a compact group $\mathcal{G}$ is acting on the symplectic manifold $\mathcal{M}$:

$$\mathcal{G} \times \mathcal{M} \to \mathcal{M}\,. \tag{2.2.14}$$

If the action of $\mathcal{G}$ is free, then $\mathcal{M}/\mathcal{G}$ is a manifold, and the $\mathcal{G}$-equivariant cohomology is the ordinary cohomology

$$H^*_{\mathcal{G}} \sim H^*(\mathcal{M}/\mathcal{G})\,. \tag{2.2.15}$$

Assume that the Lie algebra of $\mathcal{G}$ is realized by the vector fields:

$$[\chi_\alpha, \chi_\beta] = f_{\alpha\beta\gamma}\, \chi_\gamma\,. \tag{2.2.16}$$

The Lie derivative $\mathcal{L}_\alpha = d\, i_\alpha + i_\alpha\, d$ also satisfies

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] = f_{\alpha\beta\gamma}\, \mathcal{L}_\gamma\,. \tag{2.2.17}$$

If the group action preserves the symplectic structure, i.e.,

$$\mathcal{L}_\alpha \omega = 0\,, \tag{2.2.18}$$

and we further assume that

$$H^1(\mathcal{M}, \mathbb{R}) = 0\,, \tag{2.2.19}$$

then $i_\alpha\, \omega$ is exact. Hence, we can define a moment map $H : \mathcal{M} \to g^*$, where g^* is the dual Lie algebra with a symplectic basis $\{\phi^\alpha\}$. Then

$$H = \phi^\alpha\, H_\alpha\,, \tag{2.2.20}$$

and

$$i_\alpha\, \omega = -dH_\alpha\,. \tag{2.2.21}$$

The H_α's satisfy

$$[H_\alpha, H_\beta] = f_{\alpha\beta\gamma}\, H_\gamma + \kappa_{\alpha\beta}\,, \tag{2.2.22}$$

where $\kappa_{\alpha\beta}$ is a two-cocycle. Take $\mathcal{G} = U(1)$ for example. In this case there is only one ϕ, which can be viewed as a real parameter. The equivariant exterior derivative is modified as

$$d_H = d + \phi\, i_H\,, \tag{2.2.23}$$

and

$$d_H^2 = \phi\, \mathcal{L}_H\,. \tag{2.2.24}$$

It is also easy to check that

$$d_H(\omega + \phi H) = 0\,. \tag{2.2.25}$$

To evaluate the classical partition function

$$Z = \int \omega^n \, e^{-\beta H} \,, \tag{2.2.26}$$

we apply the Duistermaat-Heckman formula. For simplicity, let us assume that the solutions to $dH = 0$ are isolated and nondegenerate points. Then the Duistermaat-Heckman formula states that only the critical points contribute to the partition function. More precisely,

$$Z = \sum_{dH=0} \exp\left[i\frac{\pi}{4}\eta_H\right] \frac{\sqrt{\det\|\omega_{ab}\|}}{\sqrt{\det\|\partial_{ab}H\|}} \exp(i\phi H) \,, \tag{2.2.27}$$

where

$$\eta_H \equiv \dim T_p^+ - \dim T_p^- \,, \tag{2.2.28}$$

and $T_p^{\pm}$ are the positive and the negative eigenspace of $\partial_{ab}H$ at the critical point p. In the following we will absorb the phase factor $\exp\left[i\frac{\pi}{4}\eta_H\right]$ into the definition of the determinants. The idea of proving this theorem is to show that

$$Z_\lambda \equiv \int \omega^n \exp\left[i\phi H + \lambda \, d_H \, \psi\right] \tag{2.2.29}$$

is independent of λ provided

$$\mathcal{L}_H \, \psi = 0 \,, \tag{2.2.30}$$

i.e.,

$$Z_\lambda = Z_{\lambda+\delta\lambda} \,. \tag{2.2.31}$$

When $\lambda \to 0$ the integral Z_λ reduces to the original partition function Z. We skip this proof, which can be found in Ref. [45], then we demonstrate how to obtain the Duistermaat-Heckman formula using this fact. First, the integration measure can be rewritten using some anticommuting variables $\mathbf{c}^a$:

$$\int \omega^n = \int d^{2n}z \, \sqrt{\det\|\omega_{ab}\|} = \int dz \, d\mathbf{c} \, \exp\left[\frac{1}{2}\mathbf{c}^a \, \omega_{ab} \, \mathbf{c}^b\right] \,. \tag{2.2.32}$$

We identify

$$\mathbf{c}^a \sim dz^a \tag{2.2.33}$$

and define

$$\mathbf{i}_a \, \mathbf{c}^b = \delta_a{}^b \,. \tag{2.2.34}$$

The equivariant exterior derivative and the Lie derivative become

$$d_H = \mathbf{c}^a \partial_a + \chi_H^a \, \mathbf{i}_a \,, \tag{2.2.35}$$

$$\mathcal{L}_H = \chi_H^a \partial_a + \mathbf{c}^a \partial_a \chi_H^b \, \mathbf{i}_b \,. \tag{2.2.36}$$

If we choose ψ in Z_λ to be

$$\psi = i_H\, g = g_{ab}\, \chi_H^a\, \mathbf{c}^b\,, \tag{2.2.37}$$

then

$$d_H\, \psi = g_{ab}\, \chi_H^a\, \chi_H^b + \frac{1}{2}\mathbf{c}^a\, \left[\partial_a(g_{bc}\, \chi_H^c) - \partial_b(g_{ac}\, \chi_H^c)\right]\, \mathbf{c}^b \equiv K + \Omega\,, \tag{2.2.38}$$

where we call the first and the second term K and Ω respectively. Therefore, Z_λ now becomes

$$Z_\lambda = \int dz\, d\mathbf{c}\, \exp\left[i\phi(H+\omega) - \frac{\lambda}{2}(K+\Omega)\right]\,. \tag{2.2.39}$$

Since we can also take the limit $\lambda \to \infty$ without violating $Z_\lambda = Z$, and in this limit we make use of the identity similar to

$$\delta(\alpha x) = \frac{1}{|\alpha|}\, \delta(x) = \lim_{\lambda\to\infty} \sqrt{\frac{\lambda}{2\pi}}\, \exp\left[-\frac{\lambda}{2}(\alpha x)^2\right]\,, \tag{2.2.40}$$

then the partition function becomes

$$Z = \int dz\, d\mathbf{c}\, \frac{\sqrt{\det\|\Omega_{ab}\|}}{\sqrt{\det\|g_{ab}\|}}\, \delta(\chi_H)\, e^{i\phi(H+\omega)} = \sum_{dH=0} \frac{\sqrt{\det\|\omega_{ab}\|}}{\sqrt{\det\|\partial_{ab} H\|}}\, \exp\left[i\phi H\right]\,. \tag{2.2.41}$$

Therefore, we obtain the Duistermaat-Heckman formula.

To generalize the above mentioned Duistermaat-Heckman formula to a path integral, one needs to consider the integration in a loop space. The partition function is

$$\begin{aligned} Z &= \mathrm{Tr}\left(e^{-iTH}\right) \\ &= \int_{PBC} [dz^a]\, [d\mathbf{c}^a]\, \exp\left[i\int_0^T \vartheta_a\, \dot{z}^a - H + \frac{1}{2}\mathbf{c}^a\, \omega_{ab}\, \mathbf{c}^b\right]\,, \end{aligned} \tag{2.2.42}$$

where T is the period of the loop, and PBC stands for the periodic boundary condition. The final result is

$$\begin{aligned} Z &= \int dz\, d\mathbf{c}\, \exp\left[-iTH + i\frac{T}{2}\mathbf{c}^a\, \omega_{ab}\, \mathbf{c}^b\right]\, \sqrt{\det\left[\frac{T(\Omega^a{}_b + R^a{}_b)/2}{\sinh[T(\Omega^a{}_b + R^a{}_b)]}\right]} \\ &= \int \mathrm{Ch}(H+\omega)\, \hat{A}(\Omega + R)\,, \end{aligned} \tag{2.2.43}$$

where

$$\Omega_{ab} \equiv \partial_b(g_{ac}\chi_H^c) - \partial_a(g_{bc}\chi_H^c) \tag{2.2.44}$$

is the Riemannian moment map, and

$$R^a{}_b \equiv R_{abcd}\, \mathbf{c}_0^c\, \mathbf{c}_0^d \tag{2.2.45}$$

with the constant modes $\mathbf{c}_0^a$, while Ch and $\hat{A}$ stand for the equivariant Chern class and the equivariant $\hat{A}$-genus respectively.

2.3 S^3 Localization

2.3.1 Review of Squashed S^3

In this section, we briefly discuss different squashed S^3's and the corresponding Killing spinors that one can define on them. By squashed S^3 we mean the continuous deformation of the round S^3 metric by some parameters without changing the global topology. When these small parameters become zero, the metric of the squashed S^3 returns to the one of the round S^3.

The metrics of squashed S^3 may have different isometry groups. As reviewed in Appendix B, the metric of round S^3 has $SU(2)_L \times SU(2)_R$ isometry. After squashing, the symmetry $SU(2)$ is reduced to some smaller group in the left-invariant frame or the right-invariant frame or both. Both Ref. [46] and Ref. [47] have discussed squashed S^3 with isometry group smaller than $SU(2)_L \times SU(2)_R$. We adapt their expressions a little according to our convention.

Ref. [46] introduced an example of squashed S^3 that preserves an $SU(2)_L \times U(1)_R$ isometry:

$$ds^2 = \tilde{\ell}^{\,2}\, \mu^1\mu^1 + \ell^2(\mu^2\mu^2 + \mu^3\mu^3)\,, \tag{2.3.1}$$

where in general the constant $\tilde{\ell}$ is different from the constant ℓ, and μ^a ($a = 1, 2, 3$) are the left-invariant forms which are defined by

$$2\mu^a T_a = g^{-1}dg\,, \quad g \in SU(2)\,. \tag{2.3.2}$$

In the frame

$$(e^1, e^2, e^3) = (\tilde{\ell}\mu^1, \ell\mu^2, \ell\mu^3) \tag{2.3.3}$$

the spin connections are

$$\omega^{23} = (2\tilde{\ell}^{\,-1} - f^{-1})e^1\,, \quad \omega^{31} = f^{-1}e^2\,, \quad \omega^{12} = f^{-1}e^3\,, \tag{2.3.4}$$

where $f \equiv \ell^2\tilde{\ell}^{\,-1}$. In this case, to define a Killing spinor, one has to turn on a background gauge field V. Then there can be two independent Killing spinors with opposite R-charges:

$$\nabla_m\epsilon = \frac{i}{2f}\gamma_m\epsilon + iV_m\epsilon\,, \quad D_m\bar{\epsilon} = \frac{i}{2f}\gamma_m\bar{\epsilon} - iV_m\bar{\epsilon}\,, \tag{2.3.5}$$

where

$$\nabla_m\epsilon \equiv \partial_m\epsilon + \frac{1}{4}\gamma_{ab}\omega_m^{ab}\epsilon\,, \quad \nabla_m\bar{\epsilon} \equiv \partial_m\bar{\epsilon} + \frac{1}{4}\gamma_{mn}\omega_m^{mn}\bar{\epsilon}\,, \tag{2.3.6}$$

and

$$V_m = e_m^1\left(\frac{1}{\tilde{\ell}} - \frac{1}{f}\right)\,, \qquad \epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \qquad \bar{\epsilon} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{2.3.7}$$

The same metric with $SU(2)_L \times U(1)_R$ isometry was also considered in Ref. [47]:

$$ds^2 = \ell^2 \left(\frac{1}{v^2} \mu^1 \mu^1 + \mu^2 \mu^2 + \mu^3 \mu^3 \right) , \tag{2.3.8}$$

where ℓ is a constant with dimension of length, and v is the constant squashing parameter. This metric is related to the previous case of squashed S^3 in the following way:

$$\frac{\ell}{v} = \tilde{\ell} \, . \tag{2.3.9}$$

The vielbeins and the spin connections are the same as in the previous case, i.e., they are still given by Eq. (2.3.3) and Eq. (2.3.4) respectively. However, Ref. [47] chose a different background gauge field V_m, and the Killing spinor equations are

$$\nabla_m \epsilon = -\frac{i}{2v\ell} \gamma_m \epsilon + \frac{u}{v\ell} V^n \gamma_{mn} \epsilon \, , \quad \nabla_m \bar{\epsilon} = -\frac{i}{2v\ell} \gamma_m \bar{\epsilon} - \frac{u}{v\ell} V^n \gamma_{mn} \bar{\epsilon} \, , \tag{2.3.10}$$

where again

$$\nabla_m \epsilon \equiv \partial_m \epsilon + \frac{1}{4} \gamma_{ab} \omega_m^{ab} \epsilon \, , \quad \nabla_m \bar{\epsilon} \equiv \partial_m \bar{\epsilon} + \frac{1}{4} \gamma_{ab} \omega_m^{ab} \bar{\epsilon} \, , \tag{2.3.11}$$

and u is defined by

$$v^2 = 1 + u^2 \, , \tag{2.3.12}$$

while the background gauge field is given by

$$V^m = e_1^m \, . \tag{2.3.13}$$

The Killing spinors in this case have the solution:

$$\epsilon = e^{\theta \frac{\sigma_3}{2i}} g^{-1} \epsilon_0 \, , \quad \bar{\epsilon} = e^{-\theta \frac{\sigma_3}{2i}} g^{-1} \bar{\epsilon}_0 \, , \tag{2.3.14}$$

where ϵ_0 and $\bar{\epsilon}_0$ are arbitrary constant spinors, and the angle θ is given by

$$e^{i\theta} = \frac{1 + iu}{v} \, . \tag{2.3.15}$$

Actually there is another example of squashed S^3 discussed in Ref. [46]:

$$ds^2 = \ell^2 ({dx_0}^2 + {dx_1}^2) + \tilde{\ell}^2 ({dx_2}^2 + {dx_3}^2) \, . \tag{2.3.16}$$

This metric preserves an $U(1) \times U(1)$ isometry. Transforming the coordinates (x_0, x_1, x_2, x_3) to $(\cos\theta \cos\varphi, \cos\theta \sin\varphi, \sin\theta \cos\chi, \sin\theta \sin\chi)$, we can rewrite the metric as:

$$ds^2 = f(\theta)^2 d\theta^2 + \ell^2 \cos^2\theta \, d\varphi^2 + \tilde{\ell}^2 \sin^2\theta \, d\chi^2 \, , \tag{2.3.17}$$

where

$$f(\theta) \equiv \sqrt{\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta} \, . \tag{2.3.18}$$

It is discussed in Ref. [46] in great detail. Since we focus on the one with $SU(2) \times U(1)$ isometry, we will not consider this case in the following.

2.3.2 Some Relevant Formulae

In this section we recall some relevant formulae from Ref. [16], which describes a set of generalized 3D Killing spinor equations. By adding a flat direction to a 3D manifold one obtains a 4D manifold, so in principle the 4D formalisms introduced in Refs. [14, 15] can also be applied to a squashed S^3. We will focus on the 3D formalism [16] in the following. The generalized Killing spinor equations discussed in Ref. [16] are:

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{1}{2}H\gamma_\mu\zeta - iV_\mu\zeta - \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta\,, \tag{2.3.19}$$

$$(\nabla_\mu + iA_\mu)\widetilde{\zeta} = -\frac{1}{2}H\gamma_\mu\widetilde{\zeta} + iV_\mu\widetilde{\zeta} + \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta}\,. \tag{2.3.20}$$

The Killing spinor equations in Refs. [46, 47] can be viewed as these generalized Killing spinor equations with special choices of the auxiliary fields A_m, V_m and H. If we choose the metric (C.0.18) in the left-invariant frame (C.0.19):

$$ds^2 = \frac{\ell^2}{v^2}\mu^1\mu^1 + \ell^2\mu^2\mu^2 + \ell^2\mu^3\mu^3\,, \tag{2.3.21}$$

$$e^1 = \frac{\ell}{v}\mu^1\,,\quad e^2 = \ell\mu^2\,,\quad e^3 = \ell\mu^3\,,$$

then the choice in Ref. [46] is

$$\begin{aligned} A_1 &= \frac{v}{\ell} - \frac{1}{v\ell}\,,\quad A_2 = A_3 = 0\,,\\ V_m &= 0\,,\quad (m = 1,\,2,\,3)\\ H &= -\frac{i}{v\ell}\,, \end{aligned} \tag{2.3.22}$$

while Ref. [47] chose

$$\begin{aligned} A_1 &= V_1 = -\frac{2iu}{v\ell}\,,\\ A_2 &= A_3 = V_2 = V_3 = 0\,,\\ H &= \frac{i}{v\ell}\,, \end{aligned} \tag{2.3.23}$$

where v is the constant squashing parameter, $u \equiv \sqrt{v^2-1}$, and ℓ denotes the length scale.

We make use of the formalism described in Ref. [16] to solve for the Killing spinors and the background auxiliary fields on the squashed S^3 discussed above. We expect that in some limits the results of Refs. [46, 47] can be reproduced within the framework of Ref. [16]. The following sketch illustrates the path of calculations:

$$\text{Define} \quad K_m \equiv \zeta\gamma_m\widetilde{\zeta}\,, \quad \eta_m \equiv \Omega^{-1}K_m\,, \quad \Phi^m{}_n \equiv \varepsilon^m{}_{np}\eta^p\,, \quad P_m \equiv \zeta\gamma_m\zeta \tag{2.3.24}$$

$$\Downarrow$$

$$\text{Define} \quad p \equiv P_{\bar{z}}\,, \quad s \equiv \frac{1}{\sqrt{2}}pg^{-\frac{1}{4}}\sqrt{\Omega}\,, \quad W_m \equiv -\frac{1}{4}\eta_m\varepsilon^{npq}\eta_n\partial_p\eta_q \tag{2.3.25}$$

$$V^m = \varepsilon^{mnp}\partial_n\eta_p + \kappa\eta^m \tag{2.3.26}$$

$$H = -\frac{1}{2}\nabla_m\eta^m + \frac{i}{2}\varepsilon^{mnp}\eta_m\partial_n\eta_p + i\kappa \tag{2.3.27}$$

$$\Downarrow$$

$$A_m = \frac{1}{8}\Phi_m{}^n\partial_n\text{log}g - \frac{i}{2}\partial_m\text{log}s + \frac{1}{2}(2\delta_m{}^n - i\Phi_m{}^n)V_n - \frac{i}{2}\eta_m H + W_m + \frac{3}{2}\kappa\eta_m \tag{2.3.28}$$

Finally, we obtain the auxiliary fields V_m, H and A_m. We should emphasize that the factor g appearing in the definition of s is the absolute value of the determinant of the metric with the form

$$ds^2 = \Omega^2(d\psi + adz + \bar{a}d\bar{z})^2 + c^2dzd\bar{z}\,, \tag{2.3.29}$$

and p is defined as the $\bar{z}$-component of P_m in this coordinate system.

As pointed out in Ref. [16], for the Killing spinors and the auxiliary fields satisfying the Killing spinor equations (2.3.19), one can shift the auxiliary fields while preserving the same Killing spinors:

$$\begin{aligned} V^\mu &\to V^\mu + \kappa\eta^\mu\,, \\ H &\to H + i\kappa\,, \\ A^\mu &\to A^\mu + \frac{3}{2}\kappa\eta^\mu\,, \end{aligned} \tag{2.3.30}$$

where κ satisfies

$$K^\mu\partial_\mu\kappa = 0\,. \tag{2.3.31}$$

It means that after obtaining a set of solutions of the auxiliary fields, one can always shift them to obtain new solutions without changing the Killing spinors, and the new auxiliary fields and the Killing spinors formally satisfy the same Killing spinor equations as before.

Let us apply the 3D formalism described in Ref. [16] to solve for the Killing spinors and the auxiliary fields for the new class of the squashed 3-spheres again. In this case, we work with the 3D metric given by

$$ds^2 = F^2(\theta)\,d\theta^2 + g^2(\theta)\,d\varphi^2 + h^2(\theta)\,d\chi^2\,, \tag{2.3.32}$$

where $F(\theta)$ is an arbitrary regular function with definite sign, while

$$g^2(\theta)\,u^2 + h^2(\theta) = A^2\,, \tag{2.3.33}$$

where u is a dimensionless positive constant which becomes 1 for the round S^3, and A is a positive constant with dimension of length. We can choose the frame

$$e^1 = g d\varphi\,, \quad e^2 = h d\chi\,, \quad e^3 = F d\theta\,, \tag{2.3.34}$$

and the Killing vector

$$K = u\partial_\varphi + \partial_\chi\,. \tag{2.3.35}$$

The Killing spinors have the form

$$\zeta_\alpha = \sqrt{\frac{A}{2}} \begin{pmatrix} e^{\frac{i}{2}(\varphi+\chi-\eta)} \\ e^{\frac{i}{2}(\varphi+\chi+\eta)} \end{pmatrix}\,, \qquad \widetilde{\zeta}^{\dot\alpha} = \sqrt{\frac{A}{2}} \begin{pmatrix} ie^{-\frac{i}{2}(\varphi+\chi+\eta)} \\ -ie^{-\frac{i}{2}(\varphi+\chi-\eta)} \end{pmatrix}\,. \tag{2.3.36}$$

Moreover, the auxiliary fields are given by

$$V^\theta = 0\,, \quad V^\varphi = -\frac{2u\eta'}{AF}\,, \quad V^\chi = -\frac{2\eta'}{AF}\,, \tag{2.3.37}$$

$$H = -\frac{i\eta'}{F}\,, \tag{2.3.38}$$

$$A_\theta = 0\,, \quad A_\varphi = \frac{1}{2} - \frac{5g^2u\eta'}{2AF} + \frac{h^2\eta'}{2AFu}\,, \quad A_\chi = \frac{1}{2} + \frac{g^2u^2\eta'}{2AF} - \frac{5h^2\eta'}{2AF}\,. \tag{2.3.39}$$

The Killing spinors and the auxiliary fields given above satisfy the Killing spinor equations:

$$(\nabla_m - iA_m)\zeta = -\frac{1}{2}H\gamma_m\zeta - iV_m\zeta - \frac{1}{2}\varepsilon_{mnp}V^n\gamma^p\zeta\,,$$

$$(\nabla_m + iA_m)\widetilde{\zeta} = -\frac{1}{2}H\gamma_m\widetilde{\zeta} + iV_m\widetilde{\zeta} + \frac{1}{2}\varepsilon_{mnp}V^n\gamma^p\widetilde{\zeta}\,.$$

These Killing spinor equations can be rewritten into more familiar forms as follows:

$$(\nabla_m - i\hat{A}_m)\zeta = \frac{i}{2}\sigma_m\xi\,, \tag{2.3.40}$$

$$(\nabla_m + i\hat{A}_m)\widetilde{\zeta} = \frac{i}{2}\sigma_m\widetilde{\xi}\,, \tag{2.3.41}$$

where

$$\hat{A}_m \equiv A_m - \frac{3}{2}V_m\,, \quad \xi \equiv iH\zeta + V^m\sigma_m\zeta\,, \quad \widetilde{\xi} \equiv iH\widetilde{\zeta} - V^m\sigma_m\widetilde{\zeta}\,. \tag{2.3.42}$$

In the frame given by Eq. (2.3.34), there is

$$V^m\sigma_m = -\frac{2\eta'}{F}\begin{pmatrix} 0 & e^{-i\eta} \\ e^{i\eta} & 0 \end{pmatrix}\,. \tag{2.3.43}$$

Hence, both ζ and $\widetilde{\zeta}$ are eigenvectors of $V^m\sigma_m$, then ξ and $\widetilde{\xi}$ can be computed explicitly and they have the form:

$$\xi = -\frac{\eta'}{F}\zeta = \frac{1}{f}\zeta\,, \quad \widetilde{\xi} = -\frac{\eta'}{F}\widetilde{\zeta} = \frac{1}{f}\widetilde{\zeta}\,, \tag{2.3.44}$$

where

$$f \equiv -\frac{F}{\eta'} \,. \tag{2.3.45}$$

Therefore, we obtain:

$$(\nabla_m - i\hat{A}_m)\zeta = \frac{i}{2f}\sigma_m\zeta \,, \tag{2.3.46}$$

$$(\nabla_m + i\hat{A}_m)\widetilde{\zeta} = \frac{i}{2f}\sigma_m\widetilde{\zeta} \,, \tag{2.3.47}$$

with

$$\hat{A}_\theta = 0 \,, \quad \hat{A}_\varphi = \frac{1}{2} + \frac{A\eta'}{2Fu} \,, \quad \hat{A}_\chi = \frac{1}{2} + \frac{A\eta'}{2F} \,. \tag{2.3.48}$$

We can also use the methods described in Refs. [14, 15], which are applicable for a 4D manifold, to solve for the Killing spinors and the auxiliary fields. For the squashed S^3, one has to add a flat direction to the 3D metric (2.3.32):

$$ds^2 = A^2 d\tau^2 + F(\theta)^2 d\theta^2 + g(\theta)^2 d\varphi^2 + h(\theta)^2 d\chi^2 \,. \tag{2.3.49}$$

Explicit calculations show that both methods give the same result as the one obtained by the 3D formalism.

2.3.3 Review of 3D $\mathcal{N} = 2$ Supersymmetry

In this section, we briefly review the theory and the corresponding supersymmetry transformations and algebra constructed in Ref. [16], then in the next section, we will try to localize this theory on a squashed S^3 with $SU(2) \times U(1)$ isometry.

As discussed in Ref. [16], the 3D $\mathcal{N} = 2$ vector multiplet in the Wess-Zumino gauge transforms in the following way:

$$\begin{aligned}
\delta a_\mu &= -i(\zeta\gamma_\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma_\mu\lambda) \,, \\
\delta\sigma &= -\zeta\widetilde{\lambda} + \widetilde{\zeta}\lambda \,, \\
\delta\lambda &= i\zeta(D + \sigma H) - \frac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\zeta f_{\mu\nu} - \gamma^\mu\,\zeta(i\partial_\mu\sigma - V_\mu\sigma) \,, \\
\delta\widetilde{\lambda} &= -i\widetilde{\zeta}(D + \sigma H) - \frac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\widetilde{\zeta} f_{\mu\nu} + \gamma^\mu\,\widetilde{\zeta}(i\partial_\mu\sigma + V_\mu\sigma) \,, \\
\delta D &= D_\mu(\zeta\gamma^\mu\widetilde{\lambda} - \widetilde{\zeta}\gamma^\mu\lambda) - iV_\mu(\zeta\gamma^\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma^\mu\lambda) - H(\zeta\widetilde{\lambda} - \widetilde{\zeta}\lambda) + \zeta[\widetilde{\lambda},\, \sigma] - \widetilde{\zeta}[\lambda,\, \sigma] \,.
\end{aligned} \tag{2.3.50}$$

The transformations of the chiral multiplet and the anti-chiral multiplet are given by

$$\begin{aligned}
\delta\phi &= \sqrt{2}\zeta\psi\,,\\
\delta\psi &= \sqrt{2}\zeta F - \sqrt{2}i(z - q\sigma - rH)\widetilde{\zeta}\phi - \sqrt{2}i\gamma^\mu\widetilde{\zeta}D_\mu\phi\,,\\
\delta F &= \sqrt{2}i(z - q\sigma - (r-2)H)\widetilde{\zeta}\psi + 2iq\phi\widetilde{\zeta}\widetilde{\lambda} - \sqrt{2}iD_\mu(\widetilde{\zeta}\gamma^\mu\psi)\,,\\
\delta\widetilde{\phi} &= -\sqrt{2}\widetilde{\zeta}\widetilde{\psi}\,,\\
\delta\widetilde{\psi} &= \sqrt{2}\widetilde{\zeta}\widetilde{F} + \sqrt{2}i(z - q\sigma - rH)\zeta\widetilde{\phi} + \sqrt{2}i\gamma^\mu\zeta D_\mu\widetilde{\phi}\,,\\
\delta\widetilde{F} &= \sqrt{2}i(z - q\sigma - (r-2)H)\zeta\widetilde{\psi} + 2iq\widetilde{\phi}\zeta\lambda - \sqrt{2}iD_\mu(\zeta\gamma^\mu\widetilde{\psi})\,,
\end{aligned} \tag{2.3.51}$$

where z, r and q denote the central charge, the R-charge and the charge under the gauge group of the chiral multiplet repectively, and

$$D_\mu \equiv \nabla_\mu - ir(A_\mu - \frac{1}{2}V_\mu) - izC_\mu - iq[a_\mu,\,\cdot\,]\,, \tag{2.3.52}$$

where C_μ satisfies

$$V^\mu = -i\varepsilon^{\mu\nu\rho}\partial_\nu C_\rho\,. \tag{2.3.53}$$

The transformation parameters ζ and $\widetilde{\zeta}$ satisfy the two Killing spinor equations (2.3.19) with opposite R-charges respectively. Suppose that ζ and η are two transformation parameters without tilde, and $\widetilde{\zeta}$ and $\widetilde{\eta}$ are two transformation parameters with tilde. It is checked in Ref. [16] that the transformations with only parameters with tilde and only parameters without tilde satisfy the algebra:

$$\begin{aligned}
\{\delta_\zeta,\,\delta_\eta\}\varphi &= 0\,,\\
\{\delta_{\widetilde{\zeta}},\,\delta_{\widetilde{\eta}}\}\varphi &= 0\,,\\
\{\delta_\zeta,\,\delta_{\widetilde{\zeta}}\}\varphi &= -2i\left(\mathcal{L}'_K\varphi + \zeta\widetilde{\zeta}(z - rH)\varphi\right)\,,
\end{aligned} \tag{2.3.54}$$

where φ denotes an arbitrary field, $K^\mu \equiv \zeta\gamma^\mu\widetilde{\zeta}$ and $\mathcal{L}'_K$ is a modified Lie derivative with the local R- and z-transformation

$$\mathcal{L}'_K\varphi \equiv \mathcal{L}_K\varphi - irK^\mu(A_\mu - \frac{1}{2}V_\mu)\varphi - izK^\mu C_\mu\varphi\,. \tag{2.3.55}$$

Under these transformations, the following Lagrangians are supersymmetry invariant:

1. Fayet-Iliopoulos Term (for $U(1)$-factors of the gauge group):

$$\mathscr{L}_{FI} = \xi(D - a_\mu V^\mu - \sigma H)\,. \tag{2.3.56}$$

2. Gauge-Gauge Chern-Simons Term:

$$\mathscr{L}_{gg} = \mathrm{Tr}\left[\frac{k_{gg}}{4\pi}(i\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - 2D\sigma + 2i\widetilde{\lambda}\lambda)\right]\,. \tag{2.3.57}$$

3. Gauge-R Chern-Simons Term (for $U(1)$-factors of the gauge group):

$$\mathscr{L}_{gr} = \frac{k_{gr}}{2\pi}\left(i\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu(A_\rho - \frac{1}{2}V_\rho) - DH + \frac{1}{4}\sigma(R - 2V^\mu V_\mu - 2H^2)\right). \tag{2.3.58}$$

4. Yang-Mills Term:

$$\begin{aligned}\mathscr{L}_{YM} =&\mathrm{Tr}\left[\frac{1}{4e^2}f^{\mu\nu}f_{\mu\nu} + \frac{1}{2e^2}\partial^\mu\sigma\partial_\mu\sigma - \frac{i}{e^2}\widetilde{\lambda}\gamma^\mu(D_\mu + \frac{i}{2}V_\mu)\lambda - \frac{i}{e^2}\widetilde{\lambda}[\sigma,\,\lambda]\right.\\ &\left.+\frac{i}{2e^2}\sigma\varepsilon^{\mu\nu\rho}V_\mu f_{\nu\rho} - \frac{1}{2e^2}V^\mu V_\mu\sigma^2 - \frac{1}{2e^2}(D+\sigma H)^2 + \frac{i}{2e^2}H\widetilde{\lambda}\lambda\right].\end{aligned} \tag{2.3.59}$$

5. Matter Term:

$$\begin{aligned}\mathscr{L}_{\mathrm{mat}} =&\mathscr{D}^\mu\widetilde{\phi}\mathscr{D}_\mu\phi - i\widetilde{\psi}\gamma^\mu\mathscr{D}_\mu\psi - \widetilde{F}F + q(D+\sigma H)\widetilde{\phi}\phi - 2(r-1)H(z-q\sigma)\widetilde{\phi}\phi\\ &\left((z-q\sigma)^2 - \frac{r}{4}R + \frac{1}{2}(r-\frac{1}{2})V^\mu V_\mu + r(r-\frac{1}{2})H^2\right)\widetilde{\phi}\phi\\ &\left(z - q\sigma(r-\frac{1}{2})H\right)i\widetilde{\psi}\psi + \sqrt{2}iq(\widetilde{\phi}\lambda\psi + \phi\widetilde{\lambda}\widetilde{\psi}),\end{aligned} \tag{2.3.60}$$

where

$$\mathscr{D}_\mu \equiv \nabla_\mu - ir(A_\mu - \frac{1}{2}V_\mu) + ir_0V_\mu - izC_\mu - iq[a_\mu,\,\cdot\,]\,. \tag{2.3.61}$$

In principle we could also add a superpotential term to the theory:

$$\int d^2\theta\, W + \int d^2\bar{\theta}\,\overline{W}\,, \tag{2.3.62}$$

which is δ-exact. The superpotential W should be gauge invariant and have R-charge 2, which imposes contraints on the fields and consequently affects the final result of the partition function. In this thesis, for simplicity we do not consider a superpotential term.

2.3.4 Squashed S^3 with $U(1)\times U(1)$ Isometry

Kapustin et al. first localize $\mathcal{N}=2$ superconformal Chern-Simons theory with matter on S^3 in Ref. [44]. The partition function of the theory can be expressed into the form of a matrix model. Then in Refs. [48, 49] similar results are obtained for $\mathcal{N}=2$ supersymmetric Chern-Simons theory with matter on S^3. The theory is localized to

$$A_\mu = \phi = 0\,,\quad \sigma = -\ell D = \text{constant}\,, \tag{2.3.63}$$

and the partition function is

$$Z = \frac{1}{|\mathcal{W}|}\int d^r\sigma Z_{\text{class}}\, Z^{\text{1-loop}}_{\text{gauge}}\, Z^{\text{1-loop}}_{\text{matter}}\,, \tag{2.3.64}$$

where $|\mathcal{W}|$ and r denote the order of the Weyl group and the rank of the gauge group respectively. Moreover, Z_{class} comes from the Chern-Simons term and the Fayet-Iliopoulos term

$$Z_{\text{class}} = \exp\left(\frac{ik}{4\pi}\int d^3x\sqrt{g}\mathcal{L}_{CS}\right)\cdot\exp\left(-\frac{i\xi}{\pi\ell}\int d^3x\sqrt{g}\mathcal{L}_{FI}\right) \tag{2.3.65}$$

$$= \exp(-ik\pi\ell^2\,\mathrm{Tr}(\sigma^2))\,\exp(4\pi i\xi\ell\,\mathrm{Tr}(\sigma))\,, \tag{2.3.66}$$

where k and ξ denote the Chern-Simons level and the Fayet-Iliopoulos coupling respectively, and for each $U(1)$ factor of the gauge group there exists a Fayet-Iliopoulos term. The contribution from the gauge multiplet is

$$Z^{\text{1-loop}}_{\text{gauge}} = \prod_{\alpha\in\Delta_+}(2\sinh(\pi\ell\alpha_i\sigma_i))^2\,, \tag{2.3.67}$$

where Δ_+ denotes the positive roots, and σ_i and α_i are given by

$$\sigma = \sum_{i=1}^{r}\sigma_i H_i\,,\quad [H_i, E_\alpha] = \alpha_i E_\alpha\,. \tag{2.3.68}$$

The contribution from one chiral multiplet of R-charge q in the representation R of the gauge group is

$$\prod_{\rho\in R} s_{b=1}(i - iq - \ell\rho_i\sigma_i)\,, \tag{2.3.69}$$

where ρ_i denote the weight vector, and $s_b(x)$ is the double sine function. Precisely speaking, the double sine function $s_b(x)$ is a special case of the so-called normalized multiple sine function $S_r(x,\omega)$ of period $\omega = (\omega_1,\cdots,\omega_r)$ with $\omega_1,\cdots,\omega_r > 0$ (Ref. [50]):

$$\begin{aligned} S_r(x,\omega) = &\left(\amalg^{\infty}_{n_1,\cdots,n_r=0}(n_1\omega_1+\cdots+n_r\omega_r+x)\right) \\ &\cdot\left(\amalg^{\infty}_{m_1,\cdots,m_r=0}(m_1\omega_1+\cdots+m_r\omega_r-x)\right)^{(-1)^{r-1}}\,, \end{aligned} \tag{2.3.70}$$

where $\amalg$ denotes the zeta regularized product. Hence, the double sine function $s_b(x)$ and the normalized multiple sine function $S_r(x,\omega)$ are related by

$$s_b(x) = S_2\left(-ix,\,(b,\,\frac{1}{b})\right)\,. \tag{2.3.71}$$

Some properties of the double sine function $s_b(x)$ are listed in Appendix A of Ref. [51]. Then $Z^{\text{1-loop}}_{\text{matter}}$ is just the product of the contributions from all the chiral multiplets, i.e.,

$$Z^{\text{1-loop}}_{\text{matter}} = \prod_{\Phi}\left(\prod_{\rho\in R} s_{b=1}(i - iq - \ell\rho_i\sigma_i)\right)\,. \tag{2.3.72}$$

For squashed S^3, Z_{class} in the partition function Z, i.e., the contributions from $\mathcal{L}_{CS}$ and $\mathcal{L}_{FI}$, remains almost the same. The only change is that the radius of round S^3, i.e., ℓ, is replaced by the factor f for the squashed S^3. Since f depends on the coordinates, we should calculate Z_{class} using Eq. (2.3.65). Unlike Z_{class}, the 1-loop determinants can change significantly for different squashings. As mentioned before, in Ref. [46] two different squashed 3-spheres are considered. For the one with $SU(2)_L \times U(1)_R$ isometry given by the metric:

$$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^{\,2}\mu^3\mu^3$$

Z_{class} becomes

$$Z_{\text{class}} = \exp\left(-ik\pi\tilde{\ell}^{\,2}\,\mathrm{Tr}(\sigma^2)\right)\,\exp\left(4\pi i\xi\frac{\tilde{\ell}^{\,2}}{\ell}\,\mathrm{Tr}(\sigma)\right)\,, \tag{2.3.73}$$

while the 1-loop determinants for the matter sector and the gauge sector are (after regularization)

$$Z^{\text{1-loop}}_{\text{matter}} = \prod_{\Phi}\left(\prod_{\rho\in R} s_{b=1}(i - iq - \tilde{\ell}\rho_i\sigma_i)\right) \tag{2.3.74}$$

and

$$Z^{\text{1-loop}}_{\text{gauge}} = \prod_{\alpha\in\Delta_+}\left(\frac{\sinh(\pi\tilde{\ell}\alpha_i\sigma_i)}{\pi\tilde{\ell}\alpha_i\sigma_i}\right)^2 \tag{2.3.75}$$

respectively, where Δ_+ denotes the positive roots. For another squashed S^3 with $U(1)\times U(1)$ isometry given by the metric:

$$ds^2 = \ell^2({dx_0}^2 + {dx_1}^2) + \tilde{\ell}^{\,2}({dx_2}^2 + {dx_3}^2) = f(\theta)^2 d\theta^2 + \ell^2\cos^2(\theta)d\varphi^2 + \tilde{\ell}^{\,2}\sin^2(\theta)d\chi^2$$

Z_{class} and the 1-loop determinants for the matter sector and the gauge sector are given by

$$Z_{\text{class}} = \exp\left(-ik\pi\ell\tilde{\ell}\,\mathrm{Tr}(\sigma^2)\right)\,\exp\left(4\pi i\xi\tilde{\ell}\,\mathrm{Tr}(\sigma)\right)\,, \tag{2.3.76}$$

$$Z^{\text{1-loop}}_{\text{matter}} = \prod_{\Phi}\left[\prod_{\rho\in R} s_b\left(\frac{iQ}{2}(1-q) - \rho_i\hat{\sigma}_i\right)\right]\,, \tag{2.3.77}$$

$$Z^{\text{1-loop}}_{\text{gauge}} = \prod_{\alpha\in\Delta_+}\frac{\sinh(\pi b\alpha_i\hat{\sigma}_i)\sinh(\pi b^{-1}\alpha_i\hat{\sigma}_i)}{(\pi\alpha_i\hat{\sigma}_i)^2}\,, \tag{2.3.78}$$

where

$$b \equiv \sqrt{\tilde{\ell}/\ell}\,,\quad Q \equiv b + b^{-1}\,,\quad \hat{\sigma} \equiv \sqrt{\ell\tilde{\ell}}\,\sigma\,.$$

We see that the results in two versions of squashed S^3 are quite similar, but the main difference is that in the second version of squashed S^3 the squashing parameter b can be any positive number, while in the first one b is fixed to be 1.

We want to emphasize that not every squashed S^3 with an $SU(2)_L \times U(1)_R$ isometry has the feature $b = 1$; for instance, Ref. [47] has shown that by choosing a different set of background auxiliary fields one can have a partition function of $\mathcal{N} = 2$ supersymmetric Chern-Simons theory with matter defined on a squashed S^3 with $SU(2)_L \times U(1)_R$ isometry depending on general values of b. The result of Ref. [47] appears to contradict the one of Ref. [46]. This puzzle is clarified by Imamura and Yokoyama in Ref. [52]: One can find supercharges Q_L and Q_R, and one has to preserve at least one of them to do the localization; the difference between Ref. [46] and Ref. [47] is that in the localization they have different choices of preserved supercharges. Ref. [46] preserves Q_R, which leaves $SU(2)_L$ unchanged, hence it is natural to expect a result similar to the one for the round S^3 given in Refs. [44, 48, 49], while Ref. [47] preserves Q_L, for which neither $SU(2)_L$ nor $SU(2)_R$ has degeneracy any more. Moreover, to preserve either Q_L or Q_R, one has to turn on some background auxiliary fields, which can modify the theory and appear in the final result. Not surprisingly, different choices of supercharges and background auxiliary fields can lead to different results. Therefore, there is no contradiction to have different partition functions for the same squashed S^3 with the isometry $SU(2)_L \times U(1)_R$. We will discuss this issue in more detail in the next subsection.

Now we return to the problem of localizing the $\mathcal{N} = 2$ supersymmetric Chern-Simons-Matter theory on the new class of squashed S^3. On the new squashed S^3, the Killing spinors are given by Eq. (2.3.36), which satisfy the 3D Killing spinor equations Eq. (2.3.46) and Eq. (2.3.47). The background gauge field is given by Eq. (2.3.48). Since the Killing spinors and the background field formally obey the same equations as in Ref. [46], they do not violate the supersymmetry of the original theory, we can use exactly the same method of Ref. [46] to calculate the partition function. To do it, we only need to consider the 1-loop determinants from the matter sector and the gauge sector, since the Chern-Simons term and the Fayet-Iliopoulos term have only contributions from the classical field configuration, which remain the same as before. There are various ways to compute the 1-loop determinants. One way is to find all the eigenmodes of the operators appearing in the quadratic terms of the expansion around the background, and then calculate the determinants explicitly. Another way, which is easier and used in Ref. [46], is to find the eigenmodes which are not paired by supersymmetry. As shown in Eq. (2.1.3), we are only interested in the quotient of two determinants. According to supersymmetry, most of the bosonic modes and the fermionic modes are in pairs, hence their contributions in the determinants cancel each other. The net effect is that only those modes not related by supersymmetry in pairs contribute to the final result of the 1-loop determinant. We only need to take these modes into account.

Before we calculate the 1-loop determinants for the matter sector and the gauge sector, let us first do some preparations. We rescale the Killing spinors (2.3.36) by some constant factors, and define

$$\epsilon \equiv \frac{1}{\sqrt{A}}\zeta_\alpha = \sqrt{\frac{1}{2}}\begin{pmatrix} e^{\frac{i}{2}(\varphi+\chi-\eta)} \\ e^{\frac{i}{2}(\varphi+\chi+\eta)} \end{pmatrix}, \quad \bar{\epsilon} \equiv -\frac{i}{\sqrt{A}}\widetilde{\zeta}^{\dot{\alpha}} = \sqrt{\frac{1}{2}}\begin{pmatrix} e^{-\frac{i}{2}(\varphi+\chi+\eta)} \\ -e^{-\frac{i}{2}(\varphi+\chi-\eta)} \end{pmatrix}. \tag{2.3.79}$$

The Killing spinors are Grassmann even, and we have:

$$\begin{aligned} \bar{\epsilon}\epsilon = 1\,, \quad \epsilon\bar{\epsilon} = -1\,, \\ v^a \equiv \bar{\epsilon}\gamma^a\epsilon = \epsilon\gamma^a\bar{\epsilon} = (\cos\eta,\, \sin\eta,\, 0)\,, \\ \epsilon\gamma^a\epsilon = (-i\sin\eta,\, i\cos\eta,\, -1)\, e^{i(\varphi+\chi)}\,, \\ \bar{\epsilon}\gamma^a\bar{\epsilon} = (-i\sin\eta,\, i\cos\eta,\, 1)\, e^{-i(\varphi+\chi)}\,. \end{aligned} \tag{2.3.80}$$

where $a = 1, 2, 3$ and $\gamma^a = \sigma^a$. Moreover, v_m $(m = \varphi,\, \chi,\, \theta)$ satisfies

$$v_m v^m = 1\,, \quad v_m\gamma^m\bar{\epsilon} = -\bar{\epsilon}\,, \quad D_m v^m = 0\,, \quad v_m\partial^m f = 0\,. \tag{2.3.81}$$

As discussed in Ref. [46], for simplicity we use a Lagrangian for the localization in the matter sector, which differs from $\mathcal{L}_m$ mentioned in the previous subsection by a supersymmetry exact term.

$$\begin{aligned} &\bar{\epsilon}\epsilon\cdot\mathcal{L}_{reg} = \delta_{\bar{\epsilon}}\delta_\epsilon\left(\bar{\psi}\psi - 2i\bar{\phi}\sigma\phi\right) \\ \Rightarrow\ &\mathcal{L}_{reg} = D_m\bar{\phi}D^m\phi + \frac{2i(q-1)}{f}v^m D_m\bar{\phi}\phi + \bar{\phi}\sigma^2\phi + i\bar{\phi}\left(\frac{\sigma}{f} + D\right)\phi + \frac{2q^2-3q}{2f^2}\bar{\phi}\phi + \frac{q}{4}R\bar{\phi}\phi \\ &\qquad - i\bar{\psi}\gamma^m D_m\psi + i\bar{\psi}\sigma\psi - \frac{1}{2f}\bar{\psi}\psi + \frac{q-1}{f}\bar{\psi}\gamma^m v_m\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi + \bar{F}F\,, \end{aligned} \tag{2.3.82}$$

where $v_m \equiv \bar{\epsilon}\gamma_m\epsilon$. Then for the matter sector the kinetic operators are

$$\Delta_\phi = -D_m D^m - \frac{2i(q-1)}{f}v^m D_m + \sigma^2 + \frac{2q^2-3q}{2f^2} + \frac{qR}{4}\,, \tag{2.3.83}$$

$$\Delta_\psi = -i\gamma^m D_m + i\sigma - \frac{1}{2f} + \frac{q-1}{f}\gamma^m v_m\,. \tag{2.3.84}$$

Moreover, it can be checked explicitly that

1. Suppose Ψ is a spinor eigenmode with eigenvalue M, then $\bar{\epsilon}\Psi$ is a scalar eigenmode with eigenvalue $M(M-2i\sigma)$.

2. Suppose Φ is a scalar eigenmode with eigenvalue $M(M-2i\sigma)$, then

$$\Psi_1 \equiv \epsilon\Phi \quad \text{and} \quad \Psi_2 \equiv i\gamma^m\epsilon D_m\Phi + i\epsilon\sigma\Phi - \frac{q}{f}\epsilon\Phi \tag{2.3.85}$$

form an invariant subspace, i.e.,

$$\begin{pmatrix} D_\psi\Psi_1 \\ D_\psi\Psi_2 \end{pmatrix} = \begin{pmatrix} 2i\sigma & -1 \\ -M(M-2i\sigma) & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Hence, Ψ_1 and Ψ_2 contribute a factor $M(2i\sigma - M)$ to the 1-loop determinant.

Hence, there are two kinds of modes which are not paired by supersymmetry. The first one is unpaired spinor eigenmodes, i.e., there are no corresponding physical scalar modes Φ with R-charge $-q$ as superpartners, instead the spinor eigenmodes Ψ with R-charge $1-q$ are paired with some auxiliary scalar modes with R-charge $2-q$. The second one is missing spinor eigenmodes, i.e., Ψ_1 and Ψ_2 shown above are not independent spinor eigenmodes, instead they are proportional to each other.

For the unpaired spinor eigenmodes with R-charge $1-q$ there is

$$\Psi = \bar{\epsilon} G \,, \tag{2.3.86}$$

where G is an auxiliary scalar with R-charge $2-q$ which is similar to the auxiliary field F. Then we can expand the equation

$$\Delta_\psi \Psi = M \Psi$$

as

$$(-i\gamma^m D_m + i\sigma - \frac{1}{2f} + \frac{q-1}{f}\gamma^m v_m)\bar{\epsilon} G = M \bar{\epsilon} G \,. \tag{2.3.87}$$

Multiplying ϵ and $\bar{\epsilon}$ from the left and using the relations (2.3.80) we obtain

$$\begin{cases} \cos\eta \, D_1 G + \sin\eta \, D_2 G = -i(M - i\sigma + \frac{q-2}{f}) G \,, \\ -i \sin\eta \, D_1 G + i \cos\eta \, D_2 G + D_3 G = 0 \,. \end{cases}$$

We choose the Ansatz

$$G = \widetilde{G}(\theta) e^{-im\varphi - in\chi} \,. \tag{2.3.88}$$

Then the first equation leads to

$$M = \frac{mu}{A} + \frac{n}{A} + \frac{2-q}{A}\left(\frac{1+u}{2}\right) + i\sigma \,, \tag{2.3.89}$$

where $A \equiv \sqrt{g^2 u^2 + h^2}$ is a positive constant. The second equation can be rewritten as

$$\frac{1}{f}\frac{d}{d\eta} G = -\frac{u \sin\eta}{A \cos\eta}\left[m + (2-q)\left(\frac{1}{2} + \frac{A\eta'}{2Fu}\right)\right] G + \frac{\cos\eta}{A \sin\eta}\left[n + (2-q)\left(\frac{1}{2} + \frac{A\eta'}{2F}\right)\right] G \,. \tag{2.3.90}$$

We do not need to solve this equation, instead we only need to know that near $\eta = 0$ or $\pi/2$ the solution behaves as

$$G \sim \cos^m\eta \, \sin^n\eta \,.$$

The regularity of G requires that there should be no singularity, i.e.,

$$m \geq 0 \,, \quad n \geq 0 \,. \tag{2.3.91}$$

For the missing spinor eigenmodes, due to Eq. (2.3.85) there is

$$\Psi_2 = M \Psi_1 \quad \Rightarrow \quad (i\gamma^m \epsilon D_m \Phi + i\epsilon\sigma\Phi - \frac{q}{f}\epsilon\Phi) = M\epsilon\Phi \,, \tag{2.3.92}$$

where Φ is a scalar with R-charge $-q$. Again, by multiplying $\bar{\epsilon}$ and ϵ on both sides and using the relations (2.3.80) we obtain

$$\begin{cases} (\cos\eta\, D_1\Phi + \sin\eta\, D_2\Phi) = i(i\sigma - \frac{q}{f} - M)\Phi\,, \\ -i\sin\eta\, D_1\Phi + i\cos\eta\, D_2\Phi - D_3\Phi = 0\,. \end{cases}$$

We use the Ansatz

$$\Phi = \widetilde{\Phi}(\theta)e^{-im\varphi - in\chi}\,. \tag{2.3.93}$$

The first equation leads to

$$M = \frac{mu}{A} + \frac{n}{A} - \frac{q}{A}\left(\frac{1+u}{2}\right) + i\sigma\,, \tag{2.3.94}$$

while the second equation can be brought into the form

$$\frac{1}{f}\frac{d}{d\eta}\Phi = \frac{u\sin\eta}{A\cos\eta}\left[m - q\left(\frac{1}{2} + \frac{A\eta'}{2Fu}\right)\right]\Phi - \frac{\cos\eta}{A\sin\eta}\left[n - q\left(\frac{1}{2} + \frac{A\eta'}{2F}\right)\right]\Phi\,. \tag{2.3.95}$$

The regularity of Φ requires that

$$m \leq 0\,, \quad n \leq 0\,. \tag{2.3.96}$$

I.e.,

$$M = \frac{-mu}{A} + \frac{-n}{A} - \frac{q}{A}\left(\frac{1+u}{2}\right) + i\sigma\,, \qquad (m \geq 0\,, \quad n \geq 0). \tag{2.3.97}$$

Combining Eq. (2.3.89) with Eq. (2.3.97) we obtain

$$\begin{aligned} \frac{\det\Delta_\psi}{\det\Delta_\phi} &= \prod_{m,n\geq 0} \frac{\frac{mu}{A} + \frac{n}{A} + \frac{2-q}{A}\left(\frac{1+u}{2}\right) + i\sigma}{-\frac{mu}{A} - \frac{n}{A} - \frac{q}{A}\left(\frac{1+u}{2}\right) + i\sigma} \\ &= -\prod_{m,n\geq 0} \frac{mu + n + \frac{1+u}{2} + \left((1-q)\left(\frac{1+u}{2}\right) + i\sigma A\right)}{mu + n + \frac{1+u}{2} - \left((1-q)\left(\frac{1+u}{2}\right) + i\sigma A\right)} \\ &= -\prod_{m,n\geq 0} \frac{m\sqrt{u} + \frac{n}{\sqrt{u}} + \frac{1+u}{2\sqrt{u}} + \left((1-q)\left(\frac{1+u}{2\sqrt{u}}\right) + \frac{i\sigma A}{\sqrt{u}}\right)}{m\sqrt{u} + \frac{n}{\sqrt{u}} + \frac{1+u}{2\sqrt{u}} - \left((1-q)\left(\frac{1+u}{2\sqrt{u}}\right) + \frac{i\sigma A}{\sqrt{u}}\right)} \\ &\sim -s_{b=1/\sqrt{u}}\left(i(1-q)\left(\frac{1+u}{2\sqrt{u}}\right) - \frac{\sigma A}{\sqrt{u}}\right)\,. \end{aligned} \tag{2.3.98}$$

In the last step, we have applied a regularization, and $s_b(x)$ is the double sine function.

For the gauge sector, to calculate the 1-loop determinant around the background we only need to keep the Lagrangian to the quadratic order, which is given by

$$\mathcal{L}_{YM} = \mathcal{L}_B + \mathcal{L}_F \,, \tag{2.3.99}$$

$$\mathcal{L}_B = \mathrm{Tr}\left(\frac{1}{4}\widetilde{F}_{mn}\widetilde{F}^{mn} + \frac{1}{2}\partial_m\widetilde{\varphi}\,\partial^m\widetilde{\varphi} - \frac{1}{2}[A_m,\,\sigma][A^m,\,\sigma] - i[A_m,\,\sigma]\partial^m\widetilde{\varphi}\right)\,, \tag{2.3.100}$$

$$\mathcal{L}_F = \mathrm{Tr}\left(\frac{i}{2}\bar{\lambda}\gamma^m D_m\lambda + \frac{i}{2}\bar{\lambda}[\sigma,\,\lambda] - \frac{1}{4f}\bar{\lambda}\lambda\right)\,, \tag{2.3.101}$$

where $\widetilde{F}_{mn} \equiv \partial_m A_n - \partial_n A_m$, while σ and $\widetilde{\varphi}$ denote the classical value and the quantum fluctuation of the scalar field in the chiral multiplet respectively. It is explained in Ref. [46] that the spinor eigenmodes Λ and the transverse vector eigenmodes $\mathcal{A}$ satisfy

$$M\Lambda = \left(i\gamma^m D_m + i\sigma\alpha - \frac{1}{2f}\right)\Lambda\,, \tag{2.3.102}$$

$$M\mathcal{A} = i\sigma\alpha\mathcal{A} - *d\mathcal{A}\,, \tag{2.3.103}$$

where $*$ denotes the Hodge star operator defined by

$$**=1\,,\quad *1 = e^1e^2e^3\,,\quad *e^1 = e^2e^3\,,\quad *e^2 = e^3e^1\,,\quad *e^3 = e^1e^2\,. \tag{2.3.104}$$

The spinor eigenmode and the transverse vector eigenmode are paired in the following way:

$$\mathcal{A} = d(\bar{\epsilon}\Lambda) + (iM + \sigma\alpha)\bar{\epsilon}\gamma_m\Lambda dx^m\,, \tag{2.3.105}$$

$$\Lambda = \gamma^m\epsilon\mathcal{A}_m\,. \tag{2.3.106}$$

According to Eq. (2.3.105) the unpaired spinor eigenmodes should satisfy

$$d(\bar{\epsilon}\Lambda) + (iM + \sigma\alpha)\bar{\epsilon}\gamma_m\Lambda dx^m = 0\,. \tag{2.3.107}$$

We use the Ansatz

$$\Lambda = \epsilon\Phi_0 + \bar{\epsilon}\Phi_2\,, \tag{2.3.108}$$

where Φ_0 and Φ_2 are scalars of R-charges 0 and 2. If we apply the Ansatz

$$\Phi_0 = \varphi_0(\theta)e^{-im\varphi - in\chi}\,,\quad \Phi_2 = \varphi_2(\theta)e^{-i(m-1)\varphi - i(n-1)\chi}\,, \tag{2.3.109}$$

and use the relations (2.3.80), then the θ-, φ- and χ-component of Eq. (2.3.107) become

$$\begin{cases} \partial_\theta\varphi_0 = -(iM + \sigma\alpha)F\varphi_2\,, \\ -im\varphi_0 + (iM + \sigma\alpha)(\frac{A}{u}\cos^2\eta\,\varphi_0 - i\frac{A}{u}\cos\eta\sin\eta\,\varphi_2) = 0\,, \\ -in\varphi_0 + (iM + \sigma\alpha)(A\sin^2\eta\,\varphi_0 + iA\sin\eta\cos\eta\,\varphi_2) = 0\,, \end{cases}$$

where F is a function of θ defined in the metric (2.3.32). In order that φ_0 and φ_2 have nonzero solutions, the second and the third equation lead to

$$(iM + \sigma\alpha)A\sin\eta\cos\eta\left(m + \frac{n}{u} + i(iM+\sigma\alpha)\frac{A}{u}\right) = 0\,. \tag{2.3.110}$$

Hence,

$$M = \frac{mu}{A} + \frac{n}{A} + i\sigma\alpha\,. \tag{2.3.111}$$

Inserting it back into the first equation above, we obtain

$$\partial_\theta\varphi_0 = -iF\left(\frac{mu}{A} + \frac{n}{A}\right)\varphi_2\,. \tag{2.3.112}$$

To constrain m and n by regularity, we multiply Eq. (2.3.102) by ϵ from the left. Using the relations (2.3.80) and the result of M in Eq. (2.3.111) we obtain

$$\left(\frac{1}{f}\frac{d}{d\eta} + \frac{mu}{A}\frac{\sin\eta}{\cos\eta} - \frac{n}{A}\frac{\cos\eta}{\sin\eta}\right)\varphi_0 = 0\,, \tag{2.3.113}$$

which implies that

$$m \geq 0\,, \quad n \geq 0\,. \tag{2.3.114}$$

But for $m = n = 0$ or $M = i\sigma\alpha$, φ_0 and consequently Φ_0 are unnormalizable, hence $m = n = 0$ should be excluded.

From Eq. (2.3.106) the missing spinor should satisfy

$$\mathcal{A}_a\gamma^a\epsilon = 0\,. \tag{2.3.115}$$

An Ansatz obeying this relation is

$$\mathcal{A}_1 = iY\sin\eta\,, \quad \mathcal{A}_2 = -iY\cos\eta\,, \quad \mathcal{A}_3 = Y\,, \tag{2.3.116}$$

where

$$Y = y(\theta)e^{-im\varphi - in\chi}\,. \tag{2.3.117}$$

Inserting this Ansatz into Eq. (2.3.103), we obtain for the e^3- and e^1-component:

$$MY = i\sigma\alpha Y + \frac{nY}{A} + \frac{muY}{A}\,, \tag{2.3.118}$$

$$\frac{1}{fy}\frac{dy}{d\eta} - \left(\frac{mu}{A} + \frac{1}{f}\right)\frac{\sin\eta}{\cos\eta} + \left(\frac{n}{A} + \frac{1}{f}\right)\frac{\cos\eta}{\sin\eta} = 0\,. \tag{2.3.119}$$

The first equation gives

$$M = \frac{mu}{A} + \frac{n}{A} + i\sigma\alpha\,, \tag{2.3.120}$$

while due to regularity the second equation determines

$$m \leq -1\,, \quad n \leq -1\,. \tag{2.3.121}$$

I.e.,

$$M = \frac{(-m-1)u}{A} + \frac{-n-1}{A} + i\sigma\alpha\,, \qquad (m \geq 0\,, \quad n \geq 0). \tag{2.3.122}$$

Taking both Eq. (2.3.111) and Eq. (2.3.122) into account, we obtain the 1-loop determinant for the gauge sector:

$$\begin{aligned} &\prod_{\alpha\in\Delta} \frac{1}{i\sigma\alpha} \prod_{m,n\geq 0} \frac{\frac{mu}{A} + \frac{n}{A} + i\sigma\alpha}{\frac{(-m-1)u}{A} + \frac{-n-1}{A} + i\sigma\alpha} \\ &= \prod_{\alpha\in\Delta_+} \prod_{n>0} \left(\frac{n^2u^2}{A^2} + (\sigma\alpha)^2\right)\left(\frac{n^2}{A^2} + (\sigma\alpha)^2\right) \\ &\sim \prod_{\alpha\in\Delta_+} \frac{u \sinh\left(\frac{\pi A\sigma\alpha}{u}\right)\, \sinh(\pi A\sigma\alpha)}{(\pi A\sigma\alpha)^2}\,, \end{aligned} \tag{2.3.123}$$

where in the last step we drop a constant factor and use the following formula:

$$\sinh(z) = z\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)\,. \tag{2.3.124}$$

The denominator in Eq. (2.3.123) will cancel the Vandermonde determinant from the integral measure in the partition function, as explained in Ref. [46], hence it does not appear in the final expression of the partition function.

Comparing Eq. (2.3.98) and Eq. (2.3.123) with the results in Ref. [46], we find similar features for the metric with $U(1)\times U(1)$ isometry. The double sine function $s_b(x)$ entering the 1-loop determinant of the matter sector has $b = \sqrt{u}$, which is in general not equal to 1. Similar modification happens also in the gauge sector. When $u \to 1$, $A \to L$, the results return to the ones for the round S^3, which are the same as the ones given in Refs. [48, 49].

Let us summarize our final results in following. We construct a new class of squashed S^3 with $U(1)_L \times U(1)_R$ isometry. The $\mathcal{N} = 2$ supersymmetric Chern-Simons theory with matter defined in Refs. [48, 46, 49] can be localized on this new class of squashed S^3, and the partition function is

$$Z = \frac{1}{|\mathcal{W}|}\int d^r\sigma Z_{\text{class}}\, Z^{\text{1-loop}}_{\text{gauge}}\, Z^{\text{1-loop}}_{\text{matter}}\,, \tag{2.3.125}$$

where

$$Z_{\text{class}} = \exp\left(-ik\pi\frac{A^2}{u}\,\text{Tr}(\sigma^2)\right)\, \exp\left(4\pi i\xi\frac{A}{u}\,\text{Tr}(\sigma)\right)\,, \tag{2.3.126}$$

$$Z_{\text{gauge}}^{\text{1-loop}} = \prod_{\alpha\in\Delta_+} u \sinh\left(\frac{\pi A\alpha_i\sigma_i}{u}\right) \sinh(\pi A\alpha_i\sigma_i)\,, \tag{2.3.127}$$

$$Z_{\text{matter}}^{\text{1-loop}} = \prod_{\Phi}\left[\prod_{\rho\in R} s_{b=1/\sqrt{u}}\left(i(1-q)\left(\frac{1+u}{2\sqrt{u}}\right) - \frac{A\rho_i\sigma_i}{\sqrt{u}}\right)\right]. \tag{2.3.128}$$

The advantage of our results is that one can construct plenty of new metrics that share the same structure of partition functions, and without further calculations one can immediately read off the parameters relevant to the partition functions directly from the metrics.

2.3.5 Squashed S^3 with $SU(2)\times U(1)$ Isometry

Solving for Killing Spinors and Auxiliary Fields

Following the path which is summarized in the previous subsection, we solve for the Killing spinors and the auxiliary fields for the squashed S^3 with $SU(2)\times U(1)$ isometry. Starting from the metric in the left-invariant frame (C.0.18)

$$ds^2 = \frac{\ell^2}{v^2}\mu^1\mu^1 + \ell^2\mu^2\mu^2 + \ell^2\mu^3\mu^3\,,$$

as discussed in Appendix C, we can first rewrite it into the form of Eq. (C.0.25):

$$ds^2 = \frac{1}{4v^2}(d\psi + a dz + \bar{a} d\bar{z})^2 + c^2 dz\, d\bar{z}\,,$$

where we omit the length scale ℓ for simplicity, and consequently it will be omitted in the auxiliary fields, but we will bring it back in the end. Comparing this expression with Eq. (2.3.29), we can read off

$$\Omega = \frac{1}{2v}\,. \tag{2.3.129}$$

We choose the Killing spinors to be

$$\zeta_\alpha = \sqrt{s}\begin{pmatrix}1\\0\end{pmatrix},\quad \widetilde{\zeta}_\alpha = \frac{\Omega}{\sqrt{s}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2v\sqrt{s}}\begin{pmatrix}0\\1\end{pmatrix}, \tag{2.3.130}$$

and use the matrix

$$\varepsilon^{\alpha\beta} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{2.3.131}$$

to raise the indices of ζ_α and $\widetilde{\zeta}_\alpha$:

$$\zeta^\alpha = \sqrt{s}\begin{pmatrix}0\\-1\end{pmatrix},\quad \widetilde{\zeta}^\alpha = \frac{1}{2v\sqrt{s}}\begin{pmatrix}1\\0\end{pmatrix}. \tag{2.3.132}$$

Next, we calculate K_m in the working frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ (C.0.26) (C.0.27). For practical reason, we will mainly work in this frame. Only in the end, we will bring the final results into the left-invariant frame (C.0.19). In the following, without special mentioning the index $m = 1, 2, 3$ denotes the frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ (C.0.26) (C.0.27).

$$
\begin{aligned}
K_1 &= \zeta\gamma_1\widetilde{\zeta} = \frac{1}{2v}\,, \\
K_2 &= \zeta\gamma_2\widetilde{\zeta} = 0\,, \\
K_3 &= \zeta\gamma_3\widetilde{\zeta} = 0\,.
\end{aligned} \tag{2.3.133}
$$

In the coordinates (X, Y, ψ) (C.0.13), K_m are given by

$$
\begin{aligned}
K_X &= -\frac{1}{4v^2}\cdot\frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{Y}{X^2+Y^2}\,, \\
K_Y &= \frac{1}{4v^2}\cdot\frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{X}{X^2+Y^2}\,, \\
K_\psi &= \frac{1}{4v^2}\,,
\end{aligned} \tag{2.3.134}
$$

while K^m have a relatively simple form:

$$
K^X = 0\,, \quad K^Y = 0\,, \quad K^\psi = 1\,. \tag{2.3.135}
$$

They satisfy

$$
K^m K_m = \frac{1}{4v^2} = \Omega^2\,. \tag{2.3.136}
$$

η_m can be obtained immediately

$$
\eta_m = \frac{1}{\Omega} K_m = 2v K_m\,, \tag{2.3.137}
$$

i.e.,

$$
\eta_1 = 1\,, \quad \eta_2 = \eta_3 = 0\,. \tag{2.3.138}
$$

Then

$$
\Phi^m{}_n \equiv \varepsilon^m{}_{np}\eta^p = \varepsilon^m{}_{n1}\eta^1 = \varepsilon^m{}_{n1}\,. \tag{2.3.139}
$$

Similarly,

$$
\begin{aligned}
P_1 &= \zeta\gamma_1\zeta = 0\,, \\
P_2 &= \zeta\gamma_2\zeta = s\,, \\
P_3 &= \zeta\gamma_3\zeta = -is\,,
\end{aligned} \tag{2.3.140}
$$

and

$$\begin{aligned}
P_X &= \frac{s}{1+X^2+Y^2}\,,\\
P_Y &= \frac{-is}{1+X^2+Y^2}\,,\\
P_\psi &= 0\,.
\end{aligned} \tag{2.3.141}$$

Since

$$P_z dz + P_{\bar{z}} d\bar{z} = P_z(dX+idY) + P_{\bar{z}}(dX-idY) = (P_z+P_{\bar{z}})dX + i(P_z-P_{\bar{z}})dY\,, \tag{2.3.142}$$

there is

$$P_X = P_z + P_{\bar{z}}\,,\quad P_Y = i(P_z - P_{\bar{z}})\,, \tag{2.3.143}$$

or equivalently

$$P_z = \frac{1}{2}(P_X - iP_Y)\,,\quad P_{\bar{z}} = \frac{1}{2}(P_X + iP_Y)\,. \tag{2.3.144}$$

Then in this case

$$p \equiv P_{\bar{z}} = \frac{1}{2}(P_X + iP_Y) = \frac{s}{1+X^2+Y^2}\,. \tag{2.3.145}$$

Plugging it into the definition of s given by Eq. (2.3.25), we obtain

$$\frac{1}{\sqrt{2}} p g^{-\frac{1}{4}} \sqrt{\Omega} = s\,. \tag{2.3.146}$$

Hence, the results are consistent, and we still have the freedom to choose the function s.

It is straightforward to calculate V_m, H and W_m:

$$V^1 = \frac{2}{v} + \kappa\,,\quad V^2 = 0\,,\quad V^3 = 0\,, \tag{2.3.147}$$

$$H = \frac{i}{v} + i\kappa\,, \tag{2.3.148}$$

$$W_1 = -\frac{1}{2v}\,,\quad W_2 = 0\,,\quad W_3 = 0\,. \tag{2.3.149}$$

To calculate A_m, we first calculate $\hat{A}_m$:

$$\hat{A}_m \equiv \frac{1}{8}\Phi_m{}^n \partial_n \mathrm{log} g - \frac{i}{2}\partial_m \mathrm{log} s\,, \tag{2.3.150}$$

which is valid only in the coordinates $(z,\, \bar{z},\, \psi)$ (C.0.15). Using the definition of $\Phi^m{}_n$ we obtain

$$\Phi^z{}_z = -i\,,\quad \Phi^{\bar{z}}{}_{\bar{z}} = i\,,\quad \Phi^z{}_{\bar{z}} = \Phi^{\bar{z}}{}_z = 0\,, \tag{2.3.151}$$

and

$$\Phi_n{}^m = -\Phi^m{}_n\,. \tag{2.3.152}$$

Therefore,

$$\begin{aligned}
\hat{A}_z &= -\frac{i}{2}\cdot\frac{\bar{z}}{1+z\bar{z}} - \frac{i}{2}\partial_z \log s\,,\\
\hat{A}_{\bar{z}} &= \frac{i}{2}\cdot\frac{z}{1+z\bar{z}} - \frac{i}{2}\partial_{\bar{z}} \log s\,,\\
\hat{A}_\psi &= -\frac{i}{2}\partial_\psi \log s\,.
\end{aligned} \tag{2.3.153}$$

Then we obtain

$$\begin{aligned}
\hat{A}_1 &= \hat{A}_\psi e_1^\psi = -iv\partial_\psi \log s\,,\\
\hat{A}_2 &= \hat{A}_z e_2^z + \hat{A}_{\bar{z}} e_2^{\bar{z}} + \hat{A}_\psi e_2^\psi\\
&= \frac{i}{2}(z-\bar{z})\left[1+\frac{i}{2}(\partial_\psi \log s)\frac{z\bar{z}-1}{z\bar{z}}\right] - \frac{i}{2}(1+z\bar{z})(\partial_z \log s + \partial_{\bar{z}} \log s)\\
&= \frac{i}{2}(z-\bar{z})\left[1+\frac{i}{2}(\partial_\psi \log s)\frac{z\bar{z}-1}{z\bar{z}}\right] - \frac{i}{2}(1+z\bar{z})\partial_X \log s\,,\\
\hat{A}_3 &= \frac{1}{2}(z+\bar{z})\left[1+\frac{i}{2}(\partial_\psi \log s)\frac{z\bar{z}-1}{z\bar{z}}\right] + \frac{1}{2}(1+z\bar{z})(\partial_z \log s - \partial_{\bar{z}} \log s)\\
&= \frac{1}{2}(z+\bar{z})\left[1+\frac{i}{2}(\partial_\psi \log s)\frac{z\bar{z}-1}{z\bar{z}}\right] - \frac{i}{2}(1+z\bar{z})\partial_Y \log s\,,
\end{aligned} \tag{2.3.154}$$

and

$$\begin{aligned}
A_1 &= \hat{A}_1 + V_1 - \frac{i}{2}\Phi_1{}^n V_n - \frac{i}{2}\eta_1 H + W_1 + \frac{3}{2}\kappa\eta_1\\
&= -iv\partial_\psi \log s + \frac{2}{v} - \frac{i}{2}\frac{i}{v} - \frac{1}{2v} + \frac{3}{2}\kappa\\
&= -iv\partial_\psi \log s + \frac{2}{v} + \frac{3}{2}\kappa\\
A_2 &= \hat{A}_2 + V_2 - \frac{i}{2}\Phi_2{}^n V_n - \frac{i}{2}\eta_2 H + W_2 + \frac{3}{2}\kappa\eta_2\\
&= \frac{i}{2}(z-\bar{z})\left[1+\frac{i}{2}(\partial_\psi \log s)\frac{z\bar{z}-1}{z\bar{z}}\right] - \frac{i}{2}(1+z\bar{z})\partial_X \log s\,,\\
A_3 &= \hat{A}_3 + V_3 - \frac{i}{2}\Phi_3{}^n V_n - \frac{i}{2}\eta_3 H + W_3 + \frac{3}{2}\kappa\eta_3\\
&= \frac{1}{2}(z+\bar{z})\left[1+\frac{i}{2}(\partial_\psi \log s)\frac{z\bar{z}-1}{z\bar{z}}\right] - \frac{i}{2}(1+z\bar{z})\partial_Y \log s\,,
\end{aligned} \tag{2.3.155}$$

where

$$V_1 = \frac{2}{v}\,,\quad H = \frac{i}{v}\,.$$

Our working frame ($\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$) (C.0.26) is not the left-invariant frame (e_1, e_2, e_3) (C.0.19). To transform between different frames, it is convenient to first consider the θ-, ϕ- and ψ-component of the fields, because different frames all have the same form of the metric (C.0.23).

Let us first calculate V_μ and A_μ ($\mu = \theta,\, \phi,\, \psi$), then transform them into other frames. V_μ can be obtained very easily:

$$
\begin{aligned}
V_\theta &= V_1 \hat{e}^1_\theta = 0\,,\\
V_\phi &= V_1 \hat{e}^1_\phi = \left(\frac{2}{v} + \kappa\right)\frac{1}{2v}\mathrm{cos}\theta\,,\\
V_\psi &= V_1 \hat{e}^1_\psi = \left(\frac{2}{v} + \kappa\right)\frac{1}{2v}\,.
\end{aligned} \tag{2.3.156}
$$

A_μ can also be calculated:

$$
\begin{aligned}
A_\theta &= \frac{i}{2\mathrm{sin}\theta}(X\partial_X \mathrm{log}s + Y\partial_Y \mathrm{log}s)\,,\\
A_\phi &= (\frac{1}{2} + \frac{1}{v^2} + \frac{3\kappa}{4v})\mathrm{cos}\theta + \frac{1}{2} + \frac{i}{2}(Y\partial_X \mathrm{log}s - X\partial_Y \mathrm{log}s)\,,\\
A_\psi &= -\frac{i}{2}\partial_\psi \mathrm{log}s + \frac{1}{v^2} + \frac{3\kappa}{4v}\,,
\end{aligned} \tag{2.3.157}
$$

where κ should satisfy

$$
K^m \partial_m \kappa = 0\,. \tag{2.3.158}
$$

Obeying this constraint it seems that we can choose any κ and s, but as in Refs. [46, 47] we do not want to turn on the 2- and 3-component of V_m and A_m in the left-invariant frame (C.0.19), because the deformation of the metric happens only in the 1-direction. For this reason we always set $A_\theta = 0$, because it is contributed only from A_2 and A_3 in the left-invariant frame (C.0.19). Hence, if $A_2 = A_3 = 0$, A_θ should also vanish.

$$
A_\theta = 0 \quad \Rightarrow \quad X\partial_X \mathrm{log}s + Y\partial_Y \mathrm{log}s = 0\,. \tag{2.3.159}
$$

The solution to this equation is still quite general, which is

$$
\mathrm{log}s = f(\psi)\cdot g\left(\frac{X}{Y}\right)\,, \tag{2.3.160}
$$

where $f(x)$ and $g(x)$ can be any regular functions. A possible solution to $A_2 = 0$ and $A_3 = 0$ is

$$
\mathrm{log}s = -i\,\mathrm{arctan}\left(\frac{Y}{X}\right) + i\psi \quad \Rightarrow \quad s = e^{i(\psi-\phi)}\,. \tag{2.3.161}
$$

With this choice there are

$$
\begin{aligned}
A_\theta &= 0\,,\\
A_\phi &= (\frac{1}{2} + \frac{1}{v^2} + \frac{3\kappa}{4v})\mathrm{cos}\theta\,,\\
A_\psi &= \frac{1}{2} + \frac{1}{v^2} + \frac{3\kappa}{4v}\,.
\end{aligned} \tag{2.3.162}
$$

Transforming V_m and A_m given above into the left-invariant frame (C.0.19), we obtain:

$$\begin{aligned} V_1 &= \frac{2}{v} + \kappa\,, \\ V_2 &= V_3 = 0\,, \\ A_1 &= v + \frac{2}{v} + \frac{3\kappa}{2}\,, \\ A_2 &= A_3 = 0\,, \end{aligned} \tag{2.3.163}$$

while H has the form:

$$H = \frac{i}{v} + i\kappa\,. \tag{2.3.164}$$

Now we can try to reproduce the choices of the auxiliary fields in Refs. [46, 47] using our results (2.3.163) (2.3.164) obtained above. Ref. [46] made a special choice

$$\kappa = -\frac{2}{v}\,, \tag{2.3.165}$$

hence setting $\ell = 1$ they had

$$A_1 = v - \frac{1}{v}\,, \tag{2.3.166}$$

or equivalently

$$A_\phi = (\frac{1}{2} - \frac{1}{2v^2})\cos\theta\,, \quad A_\psi = \frac{1}{2} - \frac{1}{2v^2}\,, \tag{2.3.167}$$

and all the other components of V_m and A_m vanish.

To reproduce the results in Ref. [47], things are a little involved, because there are no obvious solutions which can satisfy the conditions

$$H = \frac{i}{v}\,, \quad V_1 = A_1 = -\frac{2iu}{v}\,,$$

where $u \equiv \sqrt{v^2 - 1}$. We have to consider other freedom in the solution. The auxiliary fields are given by Eqs. (2.3.163)-(2.3.164) and the Killing spinor is given by Eq. (2.3.130):

$$\zeta_\alpha = \sqrt{s}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \quad \widetilde{\zeta}_\alpha = \frac{\Omega}{\sqrt{s}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\,,$$

where

$$s = e^{i(\psi - \phi)}\,, \quad \Omega = \frac{1}{2v}\,.$$

Suppose that we have obtained a set of solutions to the Killing spinor equations (2.3.19), i.e. Killing spinors and corresponding auxiliary fields. Then we can rotate the Killing spinors by a constant angle Θ in the following way:

$$\zeta \to e^{i\gamma_1\Theta}\zeta\,, \quad \widetilde{\zeta} \to e^{-i\gamma_1\Theta}\widetilde{\zeta}\,. \tag{2.3.168}$$

In order that the Killing spinor equations (2.3.19) still hold, the auxiliary fields have to be shifted correspondingly:

$$\begin{aligned}
H \to H' &= H\cos(2\Theta) - V_1\sin(2\Theta) - i\,\omega_2{}^{31}(1-\cos(2\Theta)) \\
&= \frac{i}{v} + i\kappa\cos(2\Theta) - (\frac{2}{v}+\kappa)\sin(2\Theta)\,, \\
V_1 \to V_1' &= V_1\cos(2\Theta) + H\sin(2\Theta) + i\,\omega_2{}^{31}\sin(2\Theta) \\
&= (\frac{2}{v}+\kappa)\cos(2\Theta) + i\kappa\sin(2\Theta)\,, \\
A_1 \to A_1' &= A_1 - \frac{i}{2}(H'-H) + (V_1'-V_1) \\
&= v + (\frac{2}{v}+\frac{3}{2}\kappa)\cos(2\Theta) + (\frac{i}{v}+\frac{3i}{2}\kappa)\sin(2\Theta)\,.
\end{aligned} \tag{2.3.169}$$

where $\omega_2{}^{31}$ is one of the spin connections (C.0.21) in the left-invariant frame (C.0.19). From the expressions above, we see that the effects of κ and Θ are not the same, i.e., in general one cannot always make $\Theta = 0$ by choosing an appropriate κ. So we still have the freedom to choose Θ and κ, where Θ is in general complex. Moreover, until now we have omitted the length scale ℓ, and actually rescaling ℓ also leaves the Killing spinor equations (2.3.19) invariant, hence it is a symmetry. Therefore, by choosing Θ, κ and ℓ we can make the conditions required by Ref. [47] valid simultaneously:

$$V_1' = A_1'\,, \tag{2.3.170}$$

$$\frac{V_1'}{H'} = -2u\,, \tag{2.3.171}$$

$$H' = \frac{i}{v\ell_0}\,, \tag{2.3.172}$$

where $u \equiv \sqrt{v^2-1}$. The constraints above have a solution:

$$\kappa = -5v + 7v^3 - 4v^5\,, \tag{2.3.173}$$

$$\ell = \ell_0(1-2v^2)\,, \tag{2.3.174}$$

$$\begin{aligned}
\Theta &= \arctan\left(\frac{2i + i\kappa v - 2\sqrt{-1-\kappa v - v^4}}{\kappa v - 2v^2}\right)\,, \quad \text{for } 1-2v^2 > 0\,; \\
&= \arctan\left(\frac{2i + i\kappa v + 2\sqrt{-1-\kappa v - v^4}}{\kappa v - 2v^2}\right)\,, \quad \text{for } 1-2v^2 < 0\,,
\end{aligned} \tag{2.3.175}$$

where ℓ is the length scale that appears in the solution from the formalism of Ref. [16], while ℓ_0 is the length scale in the final expression. Apparently, $v = \frac{\sqrt{2}}{2}$ could be a singularity, but actually the results can be continued analytically to $v = \frac{\sqrt{2}}{2}$. Hence, it is not a real singularity. With this choice of parameters, we find:

$$A_1 = V_1 = -\frac{2iu}{v\ell_0}\,, \tag{2.3.176}$$

$$H = \frac{i}{v\ell_0}\,, \tag{2.3.177}$$

where

$$u \equiv \sqrt{v^2-1}\,,\quad \ell = \ell_0(1-2v^2)\,. \tag{2.3.178}$$

This is exactly the choice of the background auxiliary fields in Ref. [47].

Localization

To preserve the supersymmetry given by Eq. (2.3.50) and Eq. (2.3.51), the following BPS equations should be satisfied:

$$Q\psi = 0\,,\quad Q\widetilde{\psi} = 0\,,\quad Q\lambda = 0\,,\quad Q\widetilde{\lambda} = 0\,. \tag{2.3.179}$$

In Appendix D, we show that these BPS equations lead to the classical solution

$$a_\mu = -\sigma C_\mu + a_\mu^{(0)}\,,\quad \partial_\mu\sigma = 0\,,\quad D = -\sigma H\,,\quad \text{all other fields} = 0\,, \tag{2.3.180}$$

where $a_\mu^{(0)}$ is a flat connection, and C_μ appears in the new minimal supergravity as an Abelian gauge field, which satisfies $V^\mu = -i\varepsilon^{\mu\nu\rho}\partial_\nu C_\rho$ and still has the background gauge symmetry in this case:

$$C_\mu \to C_\mu + \partial_\mu \Lambda^{(C)}\,. \tag{2.3.181}$$

On the squashed S^3, $a_\mu^{(0)}$ can be set to 0 by the gauge transformation. Moreover, since we have obtained V_μ before, we can also solve for C_μ, but the solution is not unique due to the background gauge symmetry (2.3.181). In the frame (C.0.19), we can set $C_1 = 0$ by a background gauge transformation (2.3.181). Hence, we have

$$\begin{aligned} a_1 = 0 \quad \Rightarrow \quad a_\mu V^\mu = 0 \quad \Rightarrow \quad & \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \propto \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu C_\rho \propto a_\mu V^\mu = 0\,, \\ & \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu \big(A_\rho - \frac{1}{2}V_\rho\big) \propto \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu \eta_\rho \propto a_\mu (V^\mu - \kappa\eta^\mu) = 0\,, \end{aligned} \tag{2.3.182}$$

which will be relevant for later computations. These classical solutions give classical contributions to the partition function. We also need to consider the quantum fluctuation around these classical solutions, which will give 1-loop determinants to the partition function.

The supersymmetry transformations introduced in the previous section is not nilpotent, and it is not obvious whether the supersymmetry invariant Lagrangians are also supersymmetry exact. It is more convenient to use a subset of the whole supersymmetry transformations to do the localization. In this thesis we choose the subset to be the transformations without tilde, i.e. δ_ζ-transformations. For the matter sector the δ_ζ transformations are

$$\begin{aligned}
Q\phi &\equiv \delta_\zeta \phi = \sqrt{2}\zeta\psi\,,\\
Q\psi &\equiv \delta_\zeta \psi = \sqrt{2}\zeta F\,,\\
QF &\equiv \delta_\zeta F = 0\,,\\
Q\widetilde{\phi} &\equiv \delta_\zeta \widetilde{\phi} = 0\,,\\
Q\widetilde{\psi} &\equiv \delta_\zeta \widetilde{\psi} = \sqrt{2}i(z - q\sigma - rH)\zeta\widetilde{\phi} + \sqrt{2}i\gamma^\mu\zeta D_\mu\widetilde{\phi}\,,\\
Q\widetilde{F} &\equiv \delta_\zeta \widetilde{F} = \sqrt{2}i(z - q\sigma - (r-2)H)\zeta\widetilde{\psi} + 2iq\zeta\lambda\widetilde{\phi} - \sqrt{2}iD_\mu(\zeta\gamma^\mu\widetilde{\psi})\,,
\end{aligned} \tag{2.3.183}$$

while for the gauge sector the δ_ζ transformations are

$$\begin{aligned}
Qa_\mu &\equiv \delta_\zeta a_\mu = -i\zeta\gamma_\mu\widetilde{\lambda}\,,\\
Q\sigma &\equiv \delta_\zeta \sigma = -\zeta\widetilde{\lambda}\,,\\
Q\lambda &\equiv \delta_\zeta \lambda = i\zeta(D + \sigma H) - \frac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\zeta f_{\mu\nu} - \gamma^\mu\zeta(i\partial_\mu\sigma - V_\mu\sigma)\,,\\
Q\widetilde{\lambda} &\equiv \delta_\zeta \widetilde{\lambda} = 0\,,\\
QD &\equiv \delta_\zeta D = \nabla_\mu(\zeta\gamma^\mu\widetilde{\lambda}) - iV_\mu(\zeta\gamma^\mu\widetilde{\lambda}) - H(\zeta\widetilde{\lambda}) + \zeta[\widetilde{\lambda},\, \sigma]\,.
\end{aligned} \tag{2.3.184}$$

From the supersymmetry algebra (2.3.54) we see that the δ_ζ-transformations are nilpotent. Then we can choose some δ_ζ-exact terms to localize the theory discussed in the previous section. For the matter sector we choose

$$\mathcal{V}_{\text{mat}} \equiv (Q\widetilde{\psi})^\dagger\widetilde{\psi} + (Q\psi)^\dagger\psi \tag{2.3.185}$$

$$\Rightarrow \quad Q\mathcal{V}_{\text{mat}} = (Q\widetilde{\psi})^\dagger(Q\widetilde{\psi}) + (Q\psi)^\dagger(Q\psi) - \widetilde{\psi}(Q(Q\widetilde{\psi})^\dagger) + Q(Q\psi)^\dagger\psi \tag{2.3.186}$$

$$\Rightarrow \quad Q(Q\mathcal{V}_{\text{mat}}) = 0\,. \tag{2.3.187}$$

For the gauge sector we choose

$$\mathcal{V}_g \equiv (Q\lambda)^\dagger\lambda + (Q\widetilde{\lambda})^\dagger\widetilde{\lambda} = (Q\lambda)^\dagger\lambda \tag{2.3.188}$$

$$\Rightarrow \quad Q\mathcal{V}_g = (Q\lambda)^\dagger(Q\lambda) + Q(Q\lambda)^\dagger\lambda \tag{2.3.189}$$

$$\Rightarrow \quad Q(Q\mathcal{V}_g) = 0\,. \tag{2.3.190}$$

Precisely speaking, both $Q\mathcal{V}_{\text{mat}}$ and $Q\mathcal{V}_g$ will appear in the Lagrangian of the theory as Q-exact terms, and both of them contain the bosonic part and the fermionic part, i.e.,

$$Q\mathcal{V}_{\text{mat}} = (Q\mathcal{V}_{\text{mat}})_B + (Q\mathcal{V}_{\text{mat}})_F\,, \quad Q\mathcal{V}_g = (Q\mathcal{V}_g)_B + (Q\mathcal{V}_g)_F\,, \tag{2.3.191}$$

where

$$
\begin{aligned}
(Q\mathcal{V}_{\text{mat}})_B &\equiv (Q\widetilde{\psi})^\dagger(Q\widetilde{\psi}) + (Q\psi)^\dagger(Q\psi)\,,\\
(Q\mathcal{V}_{\text{mat}})_F &\equiv -\widetilde{\psi}(Q(Q\widetilde{\psi})^\dagger) + Q(Q\psi)^\dagger\psi\,,\\
(Q\mathcal{V}_g)_B &\equiv (Q\lambda)^\dagger(Q\lambda)\,,\\
(Q\mathcal{V}_g)_F &\equiv Q(Q\lambda)^\dagger\lambda\,.
\end{aligned}
\tag{2.3.192}
$$

For later convenience, we employ a trick similar to Ref. [53] to rewrite the Q-transformations in the matter sector and $Q\mathcal{V}_{\text{mat}}$ in terms of a few operators. We will see that with the help of these operators the cancellation of the contributions from different modes to the partition function will be transparent.

$$
\begin{aligned}
Q\phi &= -\sqrt{2}S_1^{c*}\psi\,,\\
Q\psi &= \sqrt{2}S_1 F\,,\\
QF &= 0\,,\\
Q\widetilde{\phi} &= 0\,,\\
Q\widetilde{\psi} &= \sqrt{2}(S_2^c\phi)^\dagger\,,\\
Q\widetilde{F} &= \sqrt{2}(S_2^*\widetilde{\psi}^\dagger)^\dagger
\end{aligned}
\tag{2.3.193}
$$

where $\phi^\dagger \equiv \widetilde{\phi}$, and the operators S_1, S_2, S_1^c, S_2^c and their corresponding adjoint operators S_1^*, S_2^*, S_1^{c*}, S_2^{c*} are given by

$$
\begin{aligned}
S_1\Phi &\equiv \Phi\zeta\,,\\
S_2\Phi &\equiv -i\left[(z - q\sigma - (r-2)H)\Phi - \not{D}\Phi\right]\zeta\,,\\
S_1^c\Phi &\equiv \Phi\zeta^\dagger\,,\\
S_2^c\Phi &\equiv i\zeta^\dagger\left[(\bar{z} - q\bar{\sigma} - r\bar{H}) + \not{D}\right]\Phi\,,\\
S_1^*\Psi &\equiv \zeta^\dagger\Psi\,,\\
S_2^*\Psi &\equiv i\zeta^\dagger\left[(\bar{z} - q\bar{\sigma} - (r-\frac{1}{2})\bar{H}) - \frac{i}{2}V_\mu\gamma^\mu + \not{D}\right]\Psi\,,\\
S_1^{c*}\Psi &\equiv \zeta\Psi\,,\\
S_2^{c*}\Psi &\equiv -\left[i(z - q\sigma - (r-\frac{3}{2})H)\zeta + \frac{1}{2}V_\mu\gamma^\mu\zeta + i\zeta\not{D}\right]\Psi
\end{aligned}
\tag{2.3.194}
$$

where Φ denotes an arbitrary bosonic field, while Ψ denotes an arbitrary fermionic field. Direct computation shows that these operators satisfy the following orthogonality conditions:

$$
S_1^*S_1^c = 0 = S_1^{c*}S_1\,,\quad S_2^*S_2^c = 0 = S_2^{c*}S_2\,. \tag{2.3.195}
$$

The derivation of the second relation is given in Appendix E. Moreover, the following relation turns out to be crucial later:

$$
S_2S_1^* + S_2^cS_1^{c*} = S_1S_2^* + S_1^cS_2^{c*} - 2i\mathrm{Re}(z - q\sigma)e^{-2\mathrm{Im}\Theta}\Omega\,, \tag{2.3.196}
$$

where Θ is the angle that appears in the rotation of the Killing spinor (2.3.168). We prove the relation above in Appendix E. With these operators, one can show that

$$\begin{aligned}
\widetilde{\phi}\Delta_\phi\phi &\equiv (Q\mathcal{V}_{\text{mat}})_B = 2\widetilde{\phi}S_2^{c*}S_2^c\phi\,, \\
\widetilde{\psi}\Delta_\psi\psi &\equiv (Q\mathcal{V}_{\text{mat}})_F = 2\widetilde{\psi}(S_2S_1^* + S_2^cS_1^{c*})\psi\,.
\end{aligned} \tag{2.3.197}$$

Hence,

$$\Delta_\psi = 2(S_2S_1^* + S_2^cS_1^{c*})\,, \quad \Delta_\phi = 2S_2^{c*}S_2^c\,. \tag{2.3.198}$$

For the gauge sector, instead of defining operators as in the matter sector, we can do direct computations, and the results are

$$(Q\mathcal{V}_g)_B = e^{-2\text{Im}\Theta}\Omega\cdot\text{Tr}\left[-\frac{1}{2}f_{\mu\nu}f^{\mu\nu} - (D_\mu\sigma)(D^\mu\sigma) + (D+\sigma H)^2\right]\,, \tag{2.3.199}$$

$$(Q\mathcal{V}_g)_F = e^{-2\text{Im}\Theta}\Omega\cdot\text{Tr}\left[2i\widetilde{\lambda}\not{D}\lambda + 2i[\widetilde{\lambda},\sigma]\lambda - iH(\widetilde{\lambda}\lambda) - V_1(\widetilde{\lambda}\lambda) - 2V_\mu(\widetilde{\lambda}\gamma^\mu\lambda)\right]\,. \tag{2.3.200}$$

Classical Contribution

As discussed before, by inserting the classical solutions (2.3.180) of the localization condition (2.3.179) into the Lagrangians of the theory (2.3.56)-(2.3.60), one obtains the classical contributions to the partition function. One can see immediately that $\mathscr{L}_{YM}$ and $\mathscr{L}_{\text{mat}}$ do not have classical contributions to the partition function. Due to Eq. (2.3.182), the classical contributions from other Lagrangians also simplify to be

$$\begin{aligned}
\exp\left(i\int d^3x\sqrt{g}\mathscr{L}_{FI}\right) &= \exp\left(i\xi\int d^3x\sqrt{g}\,\text{Tr}[D-\sigma H]\right) \\
&= \exp\left(-\frac{4i\pi^2\xi\ell^3}{v}H\,\text{Tr}(\sigma)\right)\,,
\end{aligned} \tag{2.3.201}$$

$$\begin{aligned}
\exp\left(i\int d^3x\sqrt{g}\mathscr{L}_{gg}\right) &= \exp\left(\frac{ik_{gg}}{4\pi}\int d^3x\sqrt{g}\,\text{Tr}[-2D\sigma]\right) \\
&= \exp\left(\frac{i\pi k_{gg}\ell^3}{v}H\,\text{Tr}(\sigma^2)\right)\,,
\end{aligned} \tag{2.3.202}$$

$$\begin{aligned}
\exp\left(i\int d^3x\sqrt{g}\mathscr{L}_{gr}\right) &= \exp\left(\frac{ik_{gr}}{2\pi}\int d^3x\sqrt{g}\,\text{Tr}\left[-DH + \frac{1}{4}\sigma(R - 2V^\mu V_\mu - 2H^2)\right]\right) \\
&= \exp\left(\frac{i\pi k_{gr}\ell^3}{2v}(H^2 + \frac{1}{2}R - V_\mu V^\mu)\,\text{Tr}(\sigma)\right)\,,
\end{aligned} \tag{2.3.203}$$

where H and V_μ are in general the auxiliary fields after shifting which are given by Eq. (2.3.169), and $R = \frac{8}{\ell^2} - \frac{2}{\ell^2 v^2}$ is the Ricci scalar of the squashed S^3 considered in this thesis.

1-Loop Determinant for Matter Sector

The key step of localization is to calculate the 1-loop determinants for the gauge sector and the matter sector. There are a few methods available to do this step:

- Use the index theorem;
- Expand the Laplacians into spherical harmonics;
- Consider the modes that are not paired and consequently have net contributions to the partition function.

All three methods have been used in many papers. In our case, it is more convenient to use the third one, which originated in Ref. [46] and has been done in a more systematic way in Ref. [53]. In this and next subsection, I will follow closely the method in Ref. [53] and apply it to our case of interest.

The basic idea is to first find out how the modes are paired, since paired modes cancel out exactly and do not have contributions to the partition function. If in a pair the fermionic partner is missing, which is called missing spinor, then the bosonic partner has a net contribution to the denominator of the 1-loop determinant in the partition function. If in a pair the bosonic partner is missing, then the fermionic partner is called unpaired spinor, and has a net contribution to the numerator of the 1-loop determinant in the partition function.

Starting from Eq. (2.3.198), we assume that

$$\frac{1}{2}\Delta_\phi \Phi = S_2^{c*} S_2^c \Phi = \mu\Phi\,, \tag{2.3.204}$$

and define

$$\Psi_1 \equiv S_1^c \Phi\,, \quad \Psi_2 \equiv S_2^c \Phi\,, \tag{2.3.205}$$

then using Eq. (2.3.198) and Eq. (2.3.196) we obtain

$$\begin{aligned} \frac{1}{2}\Delta_\psi \Psi_1 &= (S_2 S_1^* + S_2^c S_1^{c*}) S_1^c \Phi \\ &= S_2^c S_1^{c*} S_1^c \Phi \\ &= -e^{-2\mathrm{Im}\Theta}\Omega\Psi_2\,, \end{aligned} \tag{2.3.206}$$

$$\begin{aligned} \frac{1}{2}\Delta_\psi \Psi_2 &= \left[S_1 S_2^* + S_1^c S_2^{c*} - 2i\mathrm{Re}(z - q\sigma) e^{-2\mathrm{Im}\Theta}\Omega\right] S_2^c \Phi \\ &= S_1^c \mu\Phi - 2i\mathrm{Re}(z - q\sigma) e^{-2\mathrm{Im}\Theta}\Omega S_2^c \Phi \\ &= \mu\Psi_1 - 2iAe^{-2\mathrm{Im}\Theta}\Psi_2\,, \end{aligned} \tag{2.3.207}$$

where $A \equiv \mathrm{Re}(z - q\sigma)\cdot\Omega$. I.e.,

$$\Delta_\psi \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 & -e^{-2\mathrm{Im}\Theta}\Omega \\ \mu & -2iAe^{-2\mathrm{Im}\Theta} \end{pmatrix} \cdot \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}\,. \tag{2.3.208}$$

The eigenvalues of Δ_ψ in this subspace are

$$\lambda_{1,2} = e^{-2\text{Im}\Theta}\left[-iA \pm \sqrt{-A^2 - \widetilde{\mu}}\right] , \tag{2.3.209}$$

where $\widetilde{\mu} \equiv \mu\Omega e^{2\text{Im}\Theta}$. Suppose that $\widetilde{\mu} = -M^2 + 2iAM$, then

$$\lambda_{1,2} = -e^{-2\text{Im}\Theta}M\,, \quad e^{-2\text{Im}\Theta}(M - 2iA)\,. \tag{2.3.210}$$

In other words, if there exists a bosonic mode Φ satisfying

$$\begin{aligned}\Delta_\phi\Phi = \mu\Phi &= \frac{1}{\Omega}e^{-2\text{Im}\Theta}\widetilde{\mu}\Phi \\ &= \frac{1}{\Omega}e^{-2\text{Im}\Theta}(-M)(M - 2iA)\Phi\,,\end{aligned} \tag{2.3.211}$$

there are corresponding fermionic modes Ψ_1 and Ψ_2, which span a subspace, and Δ_ψ has eigenvalues $\lambda_{1,2}$ with

$$\lambda_1 \cdot \lambda_2 = e^{-4\text{Im}\Theta}(-M)(M - 2iA)\,. \tag{2.3.212}$$

To make the modes paired, we can rescale the bosonic mode Φ appropriately, or equivalently define

$$\hat{\Delta}_\phi \equiv e^{-2\text{Im}\Theta}\Omega\Delta_\phi\,, \tag{2.3.213}$$

then

$$\hat{\Delta}_\phi\Phi = e^{-4\text{Im}\Theta}(-M)(M - 2iA)\Phi\,. \tag{2.3.214}$$

Therefore, if there are no missing spinors or unpaired spinors, the bosonic modes and the fermionic modes cancel out exactly.

Conversely, if there exists a fermionic mode Ψ satisfying

$$\Delta_\psi\Psi = -Me^{-2\text{Im}\Theta}\Psi\,, \tag{2.3.215}$$

then using Eq. (2.3.196) we can rewrite the condition above as

$$S_1 S_2^*\Psi + S_1^c S_2^{c*}\Psi = (2iA - M)e^{-2\text{Im}\Theta}\Psi\,, \tag{2.3.216}$$

where recall $A \equiv \text{Re}(z - q\sigma)\cdot\Omega$. Acting S_1^{c*} from the left and using the orthogonality condition (2.3.195), we obtain

$$S_2^{c*}\Psi = -\frac{1}{\Omega}(2iA - M)\Phi\,. \tag{2.3.217}$$

Acting S_2^{c*} from left on the equation

$$\Delta_\psi\Psi = (S_2 S_1^* + S_2^c S_1^{c*})\Psi = -Me^{-2\text{Im}\Theta}\Psi\,, \tag{2.3.218}$$

using the relation (2.3.217) we just obtained, we find

$$\Delta_\phi\Phi = S_2^{c*}S_2^c(S_1^{c*}\Psi) = -\frac{1}{\Omega}e^{-2\text{Im}\Theta}M(M - 2iA)\Phi\,. \tag{2.3.219}$$

I.e., for a fermionic mode Ψ, the corresponding bosonic mode can be constructed as $\Phi = S_1^{c*}\Psi$.

With the preparation above, we can consider the unpaired spinors and the missing spinors. For the unpaired spinor, there is no corresponding bosonic partner, i.e.,

$$\Phi = S_1^{c*}\Psi = 0\,.$$

Based on the orthogonality condition (2.3.195) there should be

$$\Psi = S_1 F\,. \tag{2.3.220}$$

Then

$$\begin{aligned}
&\Delta_\psi \Psi = S_2 S_1^* \Psi = M_\psi \Psi \\
\Rightarrow\quad & S_2 S_1^* S_1 F = M_\psi S_1 F \\
\Rightarrow\quad & e^{-2\mathrm{Im}\Theta}\Omega S_2 F = M_\psi S_1 F \\
\Rightarrow\quad & e^{-2\mathrm{Im}\Theta}\Omega S_1^* S_2 F = M_\psi S_1^* S_1 F \\
\Rightarrow\quad & M_\psi = i e^{-2\mathrm{Im}\Theta}\Omega[D_1 - z + q\sigma + (r-2)H]\,.
\end{aligned} \tag{2.3.221}$$

For the missing spinor, $\Psi_2 \propto \Psi_1$. Suppose that

$$\Psi_2 = a\Psi_1\,, \tag{2.3.222}$$

where a is a constant. Then from Eq. (2.3.208) we know that

$$\Delta_\psi \Psi_2 = a\Delta_\psi \Psi_1 = (\mu - 2iAe^{-2\mathrm{Im}\Theta}a)\Psi_1 \tag{2.3.223}$$

$$\Delta_\psi \Psi_1 = -e^{-2\mathrm{Im}\Theta}\Omega a \Psi_1 \tag{2.3.224}$$

$$\begin{aligned}
\Rightarrow\quad & \frac{\Delta_\psi \Psi_1}{\Psi_1} = \frac{\mu - 2iAe^{-2\mathrm{Im}\Theta}a}{a} = -e^{-2\mathrm{Im}\Theta}\Omega a \\
\Rightarrow\quad & \mu = 2iAe^{-2\mathrm{Im}\Theta}a - e^{-2\mathrm{Im}\Theta}\Omega a^2 = \frac{1}{\Omega}e^{-2\mathrm{Im}\Theta}(-M^2 + 2iAM) \\
\Rightarrow\quad & a = \frac{M}{\Omega}\,.
\end{aligned} \tag{2.3.225}$$

Hence,

$$\Psi_2 = \frac{M}{\Omega}\Psi_1\,, \tag{2.3.226}$$

i.e.,

$$S_2^c \Phi = \frac{M}{\Omega} S_1^c \Phi\,. \tag{2.3.227}$$

Acting S_1^{c*} from left, we obtain

$$\begin{aligned}
& S_1^{c*} S_2^c \Phi = \frac{M_\phi}{\Omega} S_1^{c*} S_1^c \Phi \\
\Rightarrow\quad & -i(\bar{z} - q\bar{\sigma} - r\bar{H})(\zeta^\dagger\zeta)\phi - i(\zeta^\dagger\gamma^\mu\zeta)D_\mu\phi = -\frac{M_\phi}{\Omega}(\zeta^\dagger\zeta)\phi \\
\Rightarrow\quad & M_\phi = \Omega[iD_1 + i(\bar{z} - q\bar{\sigma} - r\bar{H})]\,.
\end{aligned} \tag{2.3.228}$$

To proceed, we have to figure out the eigenvalues of the operator D_1 in M_ψ (2.3.221) and M_ϕ (2.3.228). Similar to Ref. [47], we use

$$|s,\, s_z\rangle \quad \text{with } s_z = -s,\, -s+1,\, \cdots,\, s-1,\, s$$

as the spin basis, which transforms in the $(0,\, s)$ representation of $SU(2)_L \times SU(2)_R$ and

$$|j,\, m',\, m\rangle = Y^j_{m',\, m}$$

as the orbital basis, which transforms in the $(j,\, j)$ representation of $SU(2)_L \times SU(2)_R$, where j is the azimuthal quantum number, while m' and m are the magnetic quantum numbers for $SU(2)_L$ and $SU(2)_R$ respectively, and they take values in the following ranges:

$$\begin{aligned} j &= 0,\, \frac{1}{2},\, 1,\, \cdots\, ; \\ m' &= -j,\, -j+1,\, \cdots,\, j-1,\, j\, ; \\ m &= -j,\, -j+1,\, \cdots,\, j-1,\, j\, . \end{aligned} \tag{2.3.229}$$

For a group element $g \in SU(2)$, a field $\Phi(g)$ can be expanded as

$$\Phi(g) = \sum_{j,\, m',\, m,\, s_z} \Phi^j_{m',\, m,\, s_z} |j,\, m',\, m\rangle \otimes |s,\, s_z\rangle\, . \tag{2.3.230}$$

The covariant derivative on S^3 can be written as

$$\nabla^{(0)} = \mu^1(2L^1 - S^1) + \mu^2(2L^2 - S^2) + \mu^3(2L^3 - S^3)\, , \tag{2.3.231}$$

where L_m denote the orbital angular momentum operators on $SU(2)_R$, while S_m are the spin operators. Similarly, it can also be written in the right-invariant frame as

$$\nabla^{(0)} = \widetilde{\mu}^{\,1}(2L'^1 + S^1) + \widetilde{\mu}^{\,2}(2L'^2 + S^2) + \widetilde{\mu}^{\,3}(2L'^3 + S^3)\, , \tag{2.3.232}$$

where L'_m denote the orbital angular momentum operators on $SU(2)_L$, while S_m are still the spin operators. More generally, we can write the covariant derivative $D^{(0)}$ as a combination of the expressions (2.3.231) and (2.3.232):

$$\begin{aligned} \nabla^{(0)} = \quad & a\left[\mu^1(2L^1 - S^1) + \mu^2(2L^2 - S^2) + \mu^3(2L^3 - S^3)\right] \\ & + (1-a)\left[\widetilde{\mu}^{\,1}(2L'^1 + S^1) + \widetilde{\mu}^{\,2}(2L'^2 + S^2) + \widetilde{\mu}^{\,3}(2L'^3 + S^3)\right]\, , \end{aligned} \tag{2.3.233}$$

where a is an arbitrary constant. For the squashed S^3 with $SU(2)\times U(1)$ isometry given by the metric (C.0.18), the covariant derivative also has different expressions as follows:

$$\begin{aligned}\nabla = \;& \mu^1\left(2L^1-(2-\frac{1}{v^2})S^1\right)+\mu^2(2L^2-\frac{1}{v}S^2)+\mu^3(2L^3-\frac{1}{v}S^3) && (2.3.234)\\ =\;& \widetilde{\mu}^1\left(2L'^1+(2-\frac{1}{v^2})S^1\right)+\widetilde{\mu}^2(2L'^2+\frac{1}{v}S^2)+\widetilde{\mu}^3(2L'^3+\frac{1}{v}S^3) && (2.3.235)\\ =\;& a\left[\mu^1\left(2L^1-(2-\frac{1}{v^2})S^1\right)+\mu^2(2L^2-S^2)+\mu^3(2L^3-S^3)\right]\\ &+(1-a)\left[\widetilde{\mu}^1\left(2L'^1+(2-\frac{1}{v^2})S^1\right)+\widetilde{\mu}^2(2L'^2+S^2)+\widetilde{\mu}^3(2L'^3+S^3)\right], && (2.3.236)\end{aligned}$$

where a again can be an arbitrary constant.

Since the squashed S^3 that we consider here has $SU(2)_L\times U(1)_R$ isometry, $L'_m L'_m$, L'_1 and L_1+S_1 should have well-defined eigenvalues as follows:

$$L'_m L'_m = -j(j+1)\,,\quad L'_1 = im'\,,\quad L_1+S_1 = im\,. \tag{2.3.237}$$

Knowing this, we can return to the discussion of the eigenvalues of the Laplacians Δ_ψ and Δ_ϕ (2.3.221) (2.3.228). Remember that both expressions are derived from some scalar modes, hence the spin $s=0$, i.e., S_1 has vanishing eigenvalues. Then the covariant derivative without the gauge connection, i.e. ∇_1, in both expressions has the form

$$\nabla_1 = \frac{v}{\ell}\cdot 2L_1 = \frac{v}{\ell}\cdot 2im \tag{2.3.238}$$

on the states $|j,\,m',\,m\rangle$ with $-j\leqslant m',\,m\leqslant j$ and $j=0,\,\frac{1}{2},\,1,\,\cdots$ of $SU(2)$. Hence, the eigenvalues of the Laplacians Δ_ψ and Δ_ϕ (2.3.221) (2.3.228) can be expressed as

$$\begin{aligned}M_\psi &= ie^{-2\mathrm{Im}\Theta}\Omega\left[\nabla_1-i(r-2)(A_1-\frac{1}{2}V_1)-i(z-q\sigma)C_1-(z-q\sigma)+(r-2)H\right]\\ &= ie^{-2\mathrm{Im}\Theta}\Omega\left[\frac{2iv}{\ell}m-i(r-2)(A_1-\frac{1}{2}V_1+iH)-(z-q\sigma)\right], && (2.3.239)\\ M_\phi &= i\Omega\left[\nabla_1-ir(A_1-\frac{1}{2}V_1)-i(z-q\sigma)C_1+(\bar{z}-q\bar{\sigma})+rH\right]\\ &= i\Omega\left[\frac{2iv}{\ell}m-ir(A_1-\frac{1}{2}V_1+iH)+(\bar{z}-q\bar{\sigma})\right], && (2.3.240)\end{aligned}$$

where we have used the background gauge symmetry (2.3.181) to set $C_1=0$, as mentioned before. In general, z and σ can be complex, which is crucial in some cases [16], in this thesis for simplicity we assume that

$$\bar{\sigma}=-\sigma\,,\quad \bar{z}=-z\,. \tag{2.3.241}$$

As a check, let us consider a few previously studied cases. For round S^3, there are

$$v = 1\,, \quad A_1 = V_1 = 0\,, \quad H = -\frac{i}{\ell}\,, \quad z = 0\,,$$

then the 1-loop determinant for the matter sector is

$$\begin{aligned} Z^{1-\text{loop}}_{\text{mat}} &= \prod_{\rho\in R}\prod_{j=0}^{\infty}\left(\prod_{m=-j}^{j}\frac{\frac{2im}{\ell} - i(r-2)\frac{1}{\ell} + q\rho(\sigma)}{\frac{2im}{\ell} - ir\frac{1}{\ell} + q\rho(\sigma)}\right)^{2j+1} \\ &= \prod_{\rho\in R}\prod_{n=0}^{\infty}\left(-\frac{\frac{n+1-r}{\ell} + i\rho(\sigma)}{\frac{n-1+r}{\ell} - i\rho(\sigma)}\right)^{n}\,. \end{aligned} \tag{2.3.242}$$

where ρ denotes the weights in the representation R, and we have set $q = -1$ and $n \equiv 2j+1$. This result is precisely the one obtained in Ref. [44]. Similarly, if we choose

$$A_1 = \frac{v}{\ell} - \frac{1}{v\ell}\,, \quad V_1 = 0\,, \quad H = -\frac{i}{v\ell}\,, \quad z = 0\,,$$

the 1-loop determinant for the matter sector becomes

$$\begin{aligned} Z^{1-\text{loop}}_{\text{mat}} &= \prod_{\rho\in R}\prod_{j=0}^{\infty}\left(\prod_{m=-j}^{j}\frac{\frac{2ivm}{\ell} - i(r-2)\frac{v}{\ell} + q\rho(\sigma)}{\frac{2ivm}{\ell} - ir\frac{v}{\ell} + q\rho(\sigma)}\right)^{2j+1} \\ &= \prod_{\rho\in R}\prod_{j=0}^{\infty}\left(-\frac{\frac{2j+2-r}{\tilde{\ell}} + i\rho(\sigma)}{\frac{2j+r}{\tilde{\ell}} - i\rho(\sigma)}\right)^{2j+1}\,, \end{aligned} \tag{2.3.243}$$

where $\tilde{\ell} \equiv \frac{\ell}{v}$, and this result is the same as the one with $SU(2)\times U(1)$ isometry in Ref. [46]. An immediate generalization is to shift the auxiliary fields by $\sim \kappa$ without rotating the Killing spinors, i.e., for

$$A_1 = \frac{v}{\ell} + \frac{2}{v\ell} + \frac{3\kappa}{2\ell}\,, \quad V_1 = \frac{2}{v\ell} + \frac{\kappa}{\ell}\,, \quad H = \frac{i}{v\ell} + i\frac{\kappa}{\ell}\,, \quad z = 0\,,$$

the partition function remains the same as Eq. (2.3.243), i.e., the one in Ref. [46]:

$$Z^{1-\text{loop}}_{\text{mat}} = \prod_{\rho\in R}\prod_{j=0}^{\infty}\left(-\frac{\frac{2j+2-r}{\tilde{\ell}} + i\rho(\sigma)}{\frac{2j+r}{\tilde{\ell}} - i\rho(\sigma)}\right)^{2j+1}\,. \tag{2.3.244}$$

At this point, we may conclude that the shift κ in the auxiliary fields does not affect the 1-loop determinant of the matter sector, and to obtain a nontrivial result like the one in Ref. [47] one has to consider the rotation of the Killing spinors (2.3.169), i.e., $\Theta \neq 0$. All the examples discussed above can be thought of to be $\Theta = 0$.

For the cases with $\Theta \neq 0$, e.g. the case discussed in Ref. [47], the main difference is that

$$A_1 - \frac{1}{2}V_1 + iH \neq \frac{v}{\ell}\,. \tag{2.3.245}$$

For Ref. [47] there is

$$A_1 - \frac{1}{2}V_1 + iH = -\frac{1}{\ell}\left(\frac{\sqrt{1-v^2}}{v} + \frac{1}{v}\right) = -\frac{1+iu}{v\ell}\,. \tag{2.3.246}$$

This change will affect the expressions of M_ψ (2.3.221) and M_ϕ (2.3.228). We can think of this effect as to use the background gauge fields to twist the connections in the covariant derivatives, i.e., we want to absorb the background gauge fields into the covariant derivatives. For $\Theta = 0$, there is always $A_1 - \frac{1}{2}V_1 + iH = \frac{v}{\ell}$. From Eq. (2.3.238) we can see that the background fields can be thought of to only twist the $SU(2)_L$ part of the connection, without affecting the $SU(2)_R$ part of the connection. For $\Theta \neq 0$, $A_1 - \frac{1}{2}V_1 + iH$ is in general not equal to $\frac{v}{\ell}$, hence cannot be absorbed only in the $SU(2)_L$ part of the connection, i.e., it has to twist also the $SU(2)_R$ part. We can use the expression (2.3.236) and figure out the coefficients in the linear combination. The guiding principle is that we still want to require that the background gauge fields can be absorbed only in the $SU(2)_L$ part of the connection. Hence, from Eq. (2.3.236) and Eq. (2.3.238) we see that on the states $|j,\, m',\, m\rangle$ with the spin $s = 0$:

$$\nabla_1 - 2i(A_1 - \frac{1}{2}V_1 + iH) = \left[a\frac{v}{\ell}2L^1 + (1-a)\frac{v}{\ell}2L'^1\right] - 2i\frac{v}{\ell_1} = \left[a\frac{v}{\ell}2im + (1-a)\frac{v}{\ell}2im'\right] - 2i\frac{v}{\ell_1}\,,$$

where

$$\frac{v}{\ell_1} \equiv \pm(A_1 - \frac{1}{2}V_1 + iH)\,. \tag{2.3.247}$$

Since $-j \leqslant m',\, m \leqslant j$, the choice of the sign does not change the final result. To combine $2i\frac{v}{\ell_1}$ with the term containing L^1 implies that

$$a\frac{v}{\ell} = \frac{v}{\ell_1} \quad \Rightarrow \quad a = \frac{\ell}{\ell_1}\,.$$

Then

$$\nabla_1 - 2i(A_1 - \frac{1}{2}V_1 + iH) = \left[\frac{v}{\ell_1}2L^1 + (\frac{v}{\ell} - \frac{v}{\ell_1})2L'^1\right] - 2i\frac{v}{\ell_1} = \left[\frac{v}{\ell_1}2im + (\frac{v}{\ell} - \frac{v}{\ell_1})2im'\right] - 2i\frac{v}{\ell_1}\,. \tag{2.3.248}$$

For the case discussed in Ref. [47] there is

$$\frac{v}{\ell_1} = \frac{1+iu}{v\ell} = \frac{v}{(1-iu)\ell} \tag{2.3.249}$$

$$\Rightarrow \quad \frac{v}{\ell} - \frac{v}{\ell_1} = \frac{v}{\ell} - \frac{v}{(1-iu)\ell} = \frac{-iu}{1-iu}\frac{v}{\ell} = -iu\frac{v}{\ell_1}\,. \tag{2.3.250}$$

Then the 1-loop determinant for the matter sector has the general expression

$$
\begin{aligned}
Z_{\text{mat}}^{1-\text{loop}} &= \prod_{\rho\in R}\prod_{j}\prod_{-j\leqslant m,\, m'\leqslant j} e^{-2\text{Im}\Theta}\, \frac{\frac{2iv}{\ell_1}(m-1)+2i\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)m'+ir\frac{v}{\ell_1}-(z-q\rho(\sigma))}{\frac{2iv}{\ell_1}m+2i\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)m'+ir\frac{v}{\ell_1}+(\bar{z}-q\rho(\bar{\sigma}_0))} \\
&= \prod_{\rho\in R}\prod_{j}\prod_{-j\leqslant m,\, m'\leqslant j} e^{-2\text{Im}\Theta}\, \frac{2i(m-1)+2um'+ir+\frac{q\ell\rho(\sigma)}{b}}{2im+2um'+ir+\frac{q\ell\rho(\sigma)}{b}} \\
&= \prod_{\rho\in R}\prod_{j}\prod_{-j\leqslant m'\leqslant j} e^{-2\text{Im}\Theta}\, \frac{2i(-j-1)+2um'+ir+\frac{q\ell\rho(\sigma)}{b}}{2ij+2um'+ir+\frac{q\ell\rho(\sigma)}{b}} \\
&= \prod_{\rho\in R}\prod_{j}\prod_{-j\leqslant m'\leqslant j} e^{-2\text{Im}\Theta}\left(-\frac{2j+2+2ium'-r+\frac{iq\ell\rho(\sigma)}{b}}{2j-2ium'+r-\frac{iq\ell\rho(\sigma)}{b}}\right),
\end{aligned}
\tag{2.3.251}
$$

where $b\equiv\frac{1+iu}{v}$, ρ again denotes the weights in the representation R, and we have assumed

$$
z=0\,,\quad \bar{\sigma}=-\sigma\,.
$$

If we identify r and $\frac{\ell}{b}$ with Δ and r in Ref. [47] respectively, and let $q=1$, then up to some constant this result is the same as the one in Ref. [47].

For the most general auxiliary fields (2.3.169) on a squashed S^3 with $SU(2)\times U(1)$ isometry, the 1-loop determinant for the matter sector is

$$
\begin{aligned}
Z_{\text{mat}}^{1-\text{loop}} &= \prod_{\rho\in R}\prod_{j}\prod_{-j\leqslant m,\, m'\leqslant j} e^{-2\text{Im}\Theta}\, \frac{2i(m-1)\frac{v}{\ell_1}+2im'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+ir\frac{v}{\ell_1}-(z-q\rho(\sigma))}{2im\frac{v}{\ell_1}+2im'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+ir\frac{v}{\ell_1}+(\bar{z}-q\rho(\bar{\sigma}_0))} \\
&= \prod_{\rho\in R}\prod_{j}\prod_{-j\leqslant m'\leqslant j} e^{-2\text{Im}\Theta}\left(-\frac{2j+2-2m'\frac{\frac{v}{\ell}-\frac{v}{\ell_1}}{\frac{v}{\ell_1}}-r-i\,\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}}{2j+2m'\frac{\frac{v}{\ell}-\frac{v}{\ell_1}}{\frac{v}{\ell_1}}+r+i\,\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}}\right) \\
&= \prod_{\rho\in R}\prod_{p,\,q=0}^{\infty} e^{-2\text{Im}\Theta}\left(-\frac{p+q+2-(p-q)W-r-i\,\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}}{p+q+(p-q)W+r+i\,\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}}\right) \\
&= \prod_{\rho\in R}\prod_{p,\,q=0}^{\infty} e^{-2\text{Im}\Theta}\left(-\frac{(1-W)p+(1+W)q+1-i\left(\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}-ir+i\right)}{(1+W)p+(1-W)q+1+i\left(\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}-ir+i\right)}\right) \\
&= \prod_{\rho\in R}\prod_{p,\,q=0}^{\infty} e^{-2\text{Im}\Theta}\left(-\frac{bp+b^{-1}q+\frac{b+b^{-1}}{2}-\frac{i(b+b^{-1})}{2}\left(\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}-ir+i\right)}{b^{-1}p+bq+\frac{b+b^{-1}}{2}+\frac{i(b+b^{-1})}{2}\left(\frac{z-q\rho(\sigma)}{\frac{v}{\ell_1}}-ir+i\right)}\right),
\end{aligned}
\tag{2.3.252}
$$

where

$$j = \frac{p+q}{2}\,, \quad m' = \frac{p-q}{2}\,, \tag{2.3.253}$$

$$W \equiv \frac{\frac{v}{\ell} - \frac{v}{\ell_1}}{\frac{v}{\ell_1}}\,, \quad b \equiv \frac{1-W}{\sqrt{1-W^2}} = \sqrt{\frac{1-W}{1+W}}\,, \tag{2.3.254}$$

and we have assumed that

$$\bar{z} = -z\,, \quad \bar{\sigma} = -\sigma\,.$$

Therefore, up to some constant the 1-loop determinant for the matter sector in the general background is

$$Z^{\text{1-loop}}_{\text{mat}} = \prod_\rho s_b\left(\frac{Q}{2}\left(\frac{z - q\rho(\sigma)}{\frac{v}{\ell_1}} - ir + i\right)\right)\,, \tag{2.3.255}$$

where

$$Q \equiv b + b^{-1}\,, \quad b \equiv \frac{1-W}{\sqrt{1-W^2}} = \sqrt{\frac{1-W}{1+W}}\,, \quad W \equiv \frac{\frac{v}{\ell} - \frac{v}{\ell_1}}{\frac{v}{\ell_1}}\,, \tag{2.3.256}$$

and $s_b(x)$ is the double-sine function, whose properties are discussed in Ref. [50] and Appendix A of Ref. [51]. For the general background auxiliary fields (2.3.169) there is

$$\frac{v}{\ell_1} = A_1 - \frac{1}{2}V_1 + iH = \frac{v}{\ell}\left(1 - \frac{2i}{v^2}\,\text{sin}\Theta\, e^{-i\Theta}\right)\,, \tag{2.3.257}$$

which leads to

$$W = \frac{2}{-2 + v^2 - iv^2\text{cot}\Theta}\,, \quad b = \sqrt{1 - \frac{2}{v^2}(1 - e^{-2i\Theta})}\,. \tag{2.3.258}$$

As a quick check, we see that for the round S^3 there is $v = 1$ and $\Theta = 0$, then as expected

$$W = 0\,, \quad b = 1\,.$$

Moreover, for $\Theta = 0$ there is always $b = 1$, and by choosing different v and ℓ in $\frac{v}{\ell_1}$ (2.3.257) one obtains the result for round S^3 [44] and the result in Ref. [46]. If we choose ℓ and Θ to be the ones given in Eqs. (2.3.173)-(2.3.175), and use the following identity for the double-sine function

$$s_b(x)\, s_b(-x) = 1\,, \tag{2.3.259}$$

we obtain the result in Ref. [47]. Hence, the result (2.3.255) incorporates all the previous results for a squashed S^3 with $SU(2) \times U(1)$ isometry.

1-Loop Determinant for Gauge Sector

In this section we discuss the 1-loop determinant for the gauge sector. The method is similar to the matter sector. We will see how the modes are paired, and how the missing spinors and the unpaired spinors give rise to the 1-loop determinant. First, the Lagrangians that are used to do the localization in the gauge sector, also have the bosonic part (2.3.199) and the fermionic part (2.3.200). We can rescale the fields in the vector multiplet appropriately, then the Lagrangians (2.3.199) (2.3.200) become

$$(Q\mathcal{V}_g)_B = \mathrm{Tr}\left[-\frac{1}{2}f_{\mu\nu}f^{\mu\nu} - (D_\mu\sigma)(D^\mu\sigma) + (D+\sigma H)^2\right]\,, \tag{2.3.260}$$

$$(Q\mathcal{V}_g)_F = \mathrm{Tr}\left[2i\widetilde{\lambda}\not{D}\lambda + 2i[\widetilde{\lambda},\,\sigma] - iH(\widetilde{\lambda}\lambda) - V_1(\widetilde{\lambda}\lambda) - 2V_\mu(\widetilde{\lambda}\gamma^\mu\lambda)\right]\,. \tag{2.3.261}$$

We follow the same procedure as in Ref. [44]. First, we add a gauge fixing term:

$$\mathscr{L}_{gf} = \mathrm{Tr}\left[\bar{c}\nabla^\mu\nabla_\mu c + b\nabla^\mu a_\mu\right]\,. \tag{2.3.262}$$

Integration over b will give the gauge fixing condition

$$\nabla^\mu a_\mu = 0\,. \tag{2.3.263}$$

If we decompose

$$a_\mu = \nabla_\mu\varphi + B_\mu \tag{2.3.264}$$

with

$$\nabla^\mu B_\mu = 0\,, \tag{2.3.265}$$

then the gauge fixing condition becomes

$$\nabla^\mu A_\mu = 0 \quad \Rightarrow \quad \nabla^\mu\nabla_\mu\varphi = 0\,, \tag{2.3.266}$$

which is equivalent to a δ-function in the Lagrangian:

$$\delta(\nabla^\mu A_\mu) = \delta(\nabla^2\varphi) = \frac{1}{\sqrt{\det(\nabla^2)}}\,\delta(\varphi)\,. \tag{2.3.267}$$

After Gaussian integration, c and $\bar{c}$ will contribute a factor to the partition function:

$$\det(\nabla^2)\,,$$

while integration over σ will contribute another factor

$$\frac{1}{\sqrt{\det(\nabla^2)}}\,.$$

Hence, the contributions from c, $\bar{c}$, σ and φ will cancel each other. Hence, around the classical solution (2.3.180), the bosonic part that contributes to the 1-loop determinant for the gauge sector, becomes

$$(Q\mathcal{V}_g)'_B = \mathrm{Tr}\left(B^\mu\hat{\Delta}_B B_\mu + [B_\mu,\,\sigma]^2\right)\,, \tag{2.3.268}$$

where $\hat{\Delta}_B B_\mu \equiv *d * dB_\mu$.

We see that the Lagrangians (2.3.261) and (2.3.268) are exactly the same as the ones in Ref. [53], therefore, we follow the same way as Ref. [53] to figure out the pairing of modes. Suppose that there is a fermionic mode Λ satisfying

$$\begin{aligned} \Delta_\lambda \Lambda &\equiv \Omega(i\gamma^\mu D_\mu + i\sigma\alpha - \frac{i}{2}H - \frac{1}{2}V_1 - V_\mu \gamma^\mu)\Lambda = M\Lambda \\ \Rightarrow \quad (iM + \sigma\alpha\Omega)\Lambda &= \Omega(-\gamma^\mu D_\mu + \frac{1}{2}H - \frac{i}{2}V_1 - iV_\mu\gamma^\mu)\Lambda\,. \end{aligned} \tag{2.3.269}$$

Similar to Ref. [53], we can use the equation above to prove the following important relation by direct computations:

$$B \equiv \Omega d(\widetilde{\zeta}\Lambda) + (iM + \sigma\alpha\Omega)(\widetilde{\zeta}\gamma_\mu\Lambda)d\xi^\mu = -i * \left(D(\widetilde{\zeta}\gamma_\mu\Lambda)d\xi^\mu\right)\,. \tag{2.3.270}$$

In our case Ω is a constant, and the details of deriving this relation are given in Appendix E. Then this relation leads to

$$\begin{aligned} B \equiv \Omega d(\widetilde{\zeta}\Lambda) + (iM + \sigma\alpha\Omega)(\widetilde{\zeta}\gamma_\mu\Lambda)d\xi^\mu &= -i * d\left(\frac{B - \Omega d(\widetilde{\zeta}\Lambda)}{iM + \sigma\alpha\Omega}\right) \\ \Rightarrow \quad (iM + \sigma\alpha\Omega)B &= -i * dB & (2.3.271) \\ \Rightarrow \quad \pm\sqrt{\hat{\Delta}_B}\; B = *dB &= -(M - i\sigma\alpha\Omega)B = -(M - i\sigma_0\alpha)B\,, & (2.3.272) \end{aligned}$$

with $\sigma_0 \equiv \sigma\Omega$. I.e., if there exists a fermionic mode Λ with eigenvalue M for Δ_λ, then there is a corresponding bosonic mode with eigenvalue $-(M - i\alpha(\sigma_0))$ for $*d$. Conversely, if the relation (2.3.272) is true, i.e., there is a bosonic mode with eigenvalue $-(M - i\alpha(\sigma_0))$ for $*d$, then the fermionic mode $\Lambda \equiv \gamma^\mu B_\mu \zeta$ satisfies

$$\Delta_\lambda \Lambda = M\Lambda\,. \tag{2.3.273}$$

In other words, the eigenmodes of $\sqrt{\hat{\Delta}_B}$ with eigenvalues $\pm(M - i\alpha(\sigma_0))$ are paired with the eigenmodes of Δ_λ with eigenvalues M, $-M + 2i\alpha(\sigma_0)$. On these paired bosonic modes $\Delta_B \equiv \hat{\Delta}_B + \alpha^2(\sigma_0)$ has the eigenvalue $M(M - 2i\alpha(\sigma_0))$. Hence,

$$\frac{\Delta_\lambda}{\Delta_B} = \frac{-M(M - 2i\alpha(\sigma_0))}{M(M - 2i\alpha(\sigma_0))}\,. \tag{2.3.274}$$

We see that up to some constant the paired modes cancel out exactly and do not have net contributions to the 1-loop determinant.

Similar to the matter sector, to calculate the 1-loop determinant of the gauge sector, we still consider the unpaired spinor and the missing spinor. For the unpaired spinor, there is no corresponding bosonic mode, i.e.,

$$\Omega\partial_\mu(\widetilde{\zeta}\Lambda) + (iM + \alpha(\sigma_0))\widetilde{\zeta}\gamma_\mu\Lambda = 0\,, \tag{2.3.275}$$

and now the fermionic mode Λ is

$$\Lambda = \zeta\Phi_0 + \zeta^c\Phi_2\,, \tag{2.3.276}$$

where Φ_0 and Φ_2 are bosonic fields with R-charges 0 and 2 respectively. Using the convention in Appendix A, we obtain for the first component of Eq. (2.3.275):

$$\begin{aligned} &\Omega^2\partial_1\Phi_0 + \Omega(iM_\Lambda + \alpha(\sigma_0))\Phi_0 = 0 \\ \Rightarrow\quad & M_\Lambda = i\alpha(\sigma_0) + i\Omega\partial_1 = i\Omega\alpha(\sigma) + i\Omega\partial_1\,. \end{aligned} \tag{2.3.277}$$

For the missing spinor there is

$$\Lambda = \gamma^\mu B_\mu\zeta = 0\,. \tag{2.3.278}$$

By multiplying $\zeta^\dagger$ and $\zeta^{c\dagger}$ from the left, we see that this relation implies that

$$B_1 = 0\,,\quad B_2 + iB_3 = 0 \;\Rightarrow\; B_3 = iB_2\,.$$

Then

$$*\,dB = (i\partial_2B_2 - \partial_3B_2)e^1 - i\Omega\partial_1B_2e^2 + \Omega\partial_1B_2e^3\,. \tag{2.3.279}$$

Combining Eq. (2.3.271) and Eq. (2.3.279), we obtain

$$\begin{aligned} &-i\Omega\partial_1B_2 = -(M - i\alpha(\sigma_0))iB_2 \\ \Rightarrow\quad & M_B = i\alpha(\sigma_0) + i\Omega\partial_1 = i\Omega\alpha(\sigma) + i\Omega\partial_1\,. \end{aligned} \tag{2.3.280}$$

Actually, as discussed in Refs. [46, 53], there are more conditions that can be deduced, but for the case studied in this thesis, the additional conditions are irrelevant. Moreover, it seems that the contributions from the unpaired spinors and the missing spinors (2.3.277) (2.3.280) cancel each other exactly. This is not the case, because the derivative ∂_1 acting on Φ_0 and B_2 gives the eigenvalues proportional to the third component of the orbital angular momentum, i.e. L_1, while Φ_0 and B_2 have spin 0 and 1 respectively. To know the precise form of the eigenvalues of ∂_1, i.e. the covariant derivative without spin and gauge connections, we use the same expression obtained in the matter sector (2.3.248):

$$\partial_1 = \frac{v}{\ell_1}2L^1 + (\frac{v}{\ell} - \frac{v}{\ell_1})2L'^1 = \frac{v}{\ell_1}2(im - S_1) + (\frac{v}{\ell} - \frac{v}{\ell_1})2im'\,, \tag{2.3.281}$$

where for the mode Φ_0 the eigenvalue of S_1 is 0, while for the mode B_2 the eigenvalue of S_1 can be $+1$ or -1, but since α runs over all the positive roots and negative roots, and m and m' run from $-j$ to j, the sign is actually irrelevant.

Considering all the modes contributing to $\frac{M_\Lambda}{M_B}$ in the most general background (2.3.169), we obtain the 1-loop determinant for the gauge sector:

$$
\begin{aligned}
Z_g^{1-\mathrm{loop}} &= \prod_{\alpha\in\Delta}\prod_j\prod_{-j\leqslant m,\, m'\leqslant j}\frac{-2\frac{v}{\ell_1}m-2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}{-2\frac{v}{\ell_1}(m-1)-2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}\\
&= \prod_{\alpha\in\Delta}\prod_j\prod_{-j\leqslant m'\leqslant j}\frac{-2\frac{v}{\ell_1}j-2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}{-2\frac{v}{\ell_1}(-j-1)-2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}\\
&= \prod_{\alpha\in\Delta}\prod_j\prod_{-j\leqslant m'\leqslant j}\left(-\frac{2j+2m'W-\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}}{2j+2-2m'W+\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}}\right)\\
&= \prod_{\alpha\in\Delta}\prod_{p,\,q=0}^{\infty}\left(-\frac{p+q+(p-q)W-\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}}{p+q+2-(p-q)W+\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}}\right)\\
&= \prod_{\alpha\in\Delta}\prod_{p,\,q=0}^{\infty}\left(-\frac{(1+W)p+(1-W)q+1-i\left(\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}-i\right)}{(1-W)p+(1+W)q+1+i\left(\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}-i\right)}\right)\\
&= \prod_{\alpha\in\Delta}\prod_{p,\,q=0}^{\infty}\left(-\frac{b^{-1}p+bq+\frac{Q}{2}-\frac{iQ}{2}\left(\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}-i\right)}{bp+b^{-1}q+\frac{Q}{2}+\frac{iQ}{2}\left(\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}-i\right)}\right)\,,
\end{aligned}
\tag{2.3.282}
$$

where α denotes the roots, and p, q, W, b and Q are defined in the same way as before (2.3.253) (2.3.254) (2.3.256). Up to some constant, the 1-loop determinant for the gauge sector can be written as

$$
Z_g^{1-\mathrm{loop}} = \prod_{\alpha\in\Delta} s_b\left(\frac{Q}{2}\left(\frac{i\alpha(\sigma)}{\frac{v}{\ell_1}}-i\right)\right)\,,
\tag{2.3.283}
$$

where $s_b(x)$ is the double-sine function, and $\frac{v}{\ell_1}$ is given by Eq. (2.3.257). By choosing appropriate parameters given by Eqs. (2.3.173)-(2.3.175), we obtain the result of Ref. [47] from this general result.

To see how the results of other cases emerge from the general one (2.3.283), we need to rewrite the expression at some intermediate step. If we define

$$
b_1+b_2\equiv -2\frac{v}{\ell_1}\,,\quad b_1-b_2\equiv -2\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)\,,
\tag{2.3.284}
$$

then

$$
\begin{aligned}
Z_g^{1-\text{loop}} &= \prod_{\alpha\in\Delta}\prod_j\prod_{-j\leqslant m'\leqslant j}\frac{-2\frac{v}{\ell_1}j-2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}{-2\frac{v}{\ell_1}(-j-1)-2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}\\
&= \prod_{\alpha\in\Delta}\prod_j\prod_{-j\leqslant m'\leqslant j}\frac{-2\frac{v}{\ell_1}j-2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}{-2\frac{v}{\ell_1}(-j-1)+2m'\left(\frac{v}{\ell}-\frac{v}{\ell_1}\right)+i\alpha(\sigma)}\\
&= \prod_{\alpha\in\Delta}\prod_j\prod_{-j\leqslant m'\leqslant j}\frac{(b_1+b_2)j+(b_1-b_2)m'+i\alpha(\sigma)}{-(b_1+b_2)(j+1)-(b_1-b_2)m'+i\alpha(\sigma)}\\
&= \prod_{\alpha\in\Delta}\prod_j\prod_{-j\leqslant m'\leqslant j}\frac{b_1(j+m')+b_2(j-m')+i\alpha(\sigma)}{-b_1(j+m'+1)-b_2(j-m'+1)+i\alpha(\sigma)}\\
&= \prod_{\alpha\in\Delta}\prod_j\prod_{p,\,q=0}^{\infty}\frac{b_1p+b_2q+i\alpha(\sigma)}{-b_1(p+1)-b_2(q+1)+i\alpha(\sigma)}\,,
\end{aligned}
\tag{2.3.285}
$$

which is exactly the result of the 1-loop determinant for the gauge sector in Ref. [53]. From this result, it is easy to obtain the results for other cases. To see it, we should rewrite the expression further and restrict α to be positive roots.

$$
\begin{aligned}
Z_g^{1-\text{loop}} &= \prod_{\alpha\in\Delta_+}\prod_j\prod_{p,\,q=0}^{\infty}\left(\frac{b_1p+b_2q+i\alpha(\sigma)}{-b_1(p+1)-b_2(q+1)+i\alpha(\sigma)}\cdot\frac{b_1p+b_2q-i\alpha(\sigma)}{-b_1(p+1)-b_2(q+1)-i\alpha(\sigma)}\right)\\
&= \prod_{\alpha\in\Delta_+}\prod_j\prod_{p,\,q=0}^{\infty}\left(\frac{b_1p+b_2q+i\alpha(\sigma)}{-b_1(p+1)-b_2(q+1)+i\alpha(\sigma)}\cdot\frac{-b_1p-b_2q+i\alpha(\sigma)}{b_1(p+1)+b_2(q+1)+i\alpha(\sigma)}\right)\\
&= \prod_{\alpha\in\Delta_+}\left(\prod_{m>0}(b_1m+i\alpha(\sigma))\cdot\prod_{n>0}(b_2n+i\alpha(\sigma))\cdot\prod_{m>0}(-b_1m+i\alpha(\sigma))\cdot\prod_{n>0}(-b_2n+i\alpha(\sigma))\right)\\
&= \prod_{\alpha\in\Delta_+}\prod_{m>0}(b_1^2m^2+\alpha^2(\sigma))\cdot\prod_{n>0}(b_2^2n^2+\alpha^2(\sigma))\\
&= \prod_{\alpha\in\Delta_+}\prod_{n>0}\left((b_1^2n^2+\alpha^2(\sigma))\cdot(b_2^2n^2+\alpha^2(\sigma))\right)\,,
\end{aligned}
\tag{2.3.286}
$$

where Δ_+ denotes the set of positive roots. For

$$b_1=b_2=\frac{1}{\ell}\,,$$

we obtain the result of round S^3 [44], while for

$$b_1=b_2=\frac{v}{\ell}=\frac{1}{\tilde{\ell}}\,,$$

the result becomes the one of the squashed S^3 with $SU(2)\times U(1)$ isometry discussed in Ref. [46]. Hence, like in the matter sector, the general result (2.3.283) also incorporates all the previous results on a squashed S^3 with $SU(2) \times U(1)$ isometry, and it does not depend on the shifts by $\sim \kappa$ of the auxiliary fields, instead the shifts induced by the rotation of the Killing spinors will affect the final result.

Finally, putting everything together (2.3.201)-(2.3.203) (2.3.255) (2.3.283), we obtain the results summarized in the introduction. As we emphasized there, an important feature is that the 1-loop determinants are independent of the shift κ, while only $\Theta \neq 0$ can give the results essentially different from the case of the round S^3.

2.4 S^2 Localization

2.4.1 2D $\mathcal{N} = (2,2)$ Supersymmetry

We begin by reviewing some basic aspects of $\mathcal{N} = (2,2)$ supersymmetry and defining our notation and conventions. We will then discuss gauge theories for semichiral fields and their non-linear sigma model (NLSM) description.

The algebra of $\mathcal{N} = (2,2)$ superderivatives is

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = \pm 2i\partial_{\pm\pm} \ , \tag{2.4.1}$$

where $\pm$ are spinor indices, $\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm$ are superderivatives and $\partial_{\pm\pm} = (\partial_1 \mp i\partial_2)/2$ are spacetime derivatives. The SUSY transformations are generated by

$$\delta = \bar{\epsilon}^+ \mathbb{Q}_+ + \bar{\epsilon}^- \mathbb{Q}_- + \epsilon^+ \bar{\mathbb{Q}}_+ + \epsilon^- \bar{\mathbb{Q}}_- \ , \tag{2.4.2}$$

where $\epsilon, \bar{\epsilon}$ are anticommuting Dirac spinors. The supercharges $\mathbb{Q}, \bar{\mathbb{Q}}$ satisfy the algebra $\{\mathbb{Q}_\pm, \bar{\mathbb{Q}}_\pm\} = \mp 2i\partial_{\pm\pm}$ and anticommute with the spinor derivatives: $\{\mathbb{Q}_\pm, \mathbb{D}_\pm\} = 0$, *etc.*

The basic matter supermultiplets are chiral, twisted chiral and semichiral fields. In *Lorentzian* signature these fields are defined by the set of constraints:

$$\begin{array}{lllll}
\text{Chiral}: & \bar{\mathbb{D}}_+\Phi = 0\ , & \bar{\mathbb{D}}_-\Phi = 0\ , & \mathbb{D}_+\bar{\Phi} = 0\ , & \mathbb{D}_-\bar{\Phi} = 0\ , \\
\text{Twisted Chiral}: & \bar{\mathbb{D}}_+\chi = 0\ , & \mathbb{D}_-\chi = 0\ , & \mathbb{D}_+\bar{\chi} = 0\ , & \bar{\mathbb{D}}_-\bar{\chi} = 0\ , \\
\text{Left semichiral}: & \bar{\mathbb{D}}_+\mathbb{X}_L = 0\ , & \mathbb{D}_+\bar{\mathbb{X}}_L = 0\ , & & \\
\text{Right semichiral}: & \bar{\mathbb{D}}_-\mathbb{X}_R = 0\ , & \mathbb{D}_-\bar{\mathbb{X}}_R = 0\ . & &
\end{array} \tag{2.4.3}$$

In Lorentzian signature, complex conjugation acts on superderivatives as $\mathbb{D}_\pm^\dagger = \bar{\mathbb{D}}_\pm$ and on superfields as $\mathbb{X}^\dagger = \bar{\mathbb{X}}$. Thus, the constraints (2.4.3) are compatible with complex conjugation.

In Euclidean signature, however, the conjugation of superderivatives changes helicity, namely $\mathbb{D}_\pm^\dagger = \bar{\mathbb{D}}_\mp$. As a consequence, taking the complex conjugate of the constraints (2.4.3) may lead to additional constraints. In the case of a twisted chiral field χ, for instance, this implies that the field be constant. The well-known resolution is to complexify the multiplet and consider χ and $\bar{\chi}$ as independent fields. Although this problem does not arise for semichiral fields, we nonetheless choose to complexify them.[1] That is, we will consider $\mathbb{X}_L$ a left semichiral field and $\bar{\mathbb{X}}_L$ an independent left anti-semichiral field, and similarly for $\mathbb{X}_R$ and $\bar{\mathbb{X}}_R$. Thus, the supersymmetric constraints (and their Euclidean conjugates) read:

$$\begin{array}{llll} \bar{\mathbb{D}}_+\mathbb{X}_L = 0 \,, & \mathbb{D}_+\bar{\mathbb{X}}_L = 0 \,, & \bar{\mathbb{D}}_-\mathbb{X}_R = 0 \,, & \mathbb{D}_-\bar{\mathbb{X}}_R = 0 \,, \\ \mathbb{D}_-\mathbb{X}_L^\dagger = 0 \,, & \bar{\mathbb{D}}_-\bar{\mathbb{X}}_L^\dagger = 0 \,, & \mathbb{D}_+\mathbb{X}_R^\dagger = 0 \,, & \bar{\mathbb{D}}_+\bar{\mathbb{X}}_R^\dagger = 0 \,. \end{array} \tag{2.4.4}$$

The target space geometry of these models is the complexification of the target space geometry of the corresponding models defined in Lorentzian signature. See [54] for a discussion of these issues.

Chiral and twisted chiral fields are well known. Semichiral fields, however, are less known so we review some of their elementary aspects here. They were originally introduced in [55] and their off-shell content is

$$\begin{array}{ll} \mathbb{X}_L: & (X_L, \psi_\pm^L, F_L, \bar{\chi}_-, M_{-+}, M_{--}, \bar{\eta}_-) \,, \\ \mathbb{X}_R: & (X_R, \psi_\pm^R, F_R, \bar{\chi}_+, M_{+-}, M_{++}, \bar{\eta}_+) \,, \end{array} \tag{2.4.5}$$

where $\psi_\alpha, \chi_\alpha, \eta_\alpha$ are fermionic and $X, F, M_{\alpha\beta}$ are bosonic fields, all valued in the same representation $\mathcal{R}$ of a gauge group G. The field content of antisemichiral fields $\bar{\mathbb{X}}_L, \bar{\mathbb{X}}_R$ is similarly given by

$$\begin{array}{ll} \bar{\mathbb{X}}_L: & (\bar{X}_L, \bar{\psi}_\pm^L, \bar{F}_L, \chi_-, \bar{M}_{-+}, \bar{M}_{--}, \eta_-) \,, \\ \bar{\mathbb{X}}_R: & (\bar{X}_R, \bar{\psi}_\pm^R, \bar{F}_R, \chi_+, \bar{M}_{+-}, \bar{M}_{++}, \eta_+) \,. \end{array} \tag{2.4.6}$$

Compared to the field content of chiral and twisted chiral fields, semichiral fields contain additional bosonic and fermionic components.

To treat left and right semichiral fields in a unified way, we consider a superfield $\mathbb{X}$ that satisfies at least one chiral constraint (either $\bar{\mathbb{D}}_+\mathbb{X} = 0$ or $\bar{\mathbb{D}}_-\mathbb{X} = 0$, or both), but we do not specify which one until the end of the calculation. Similarly, $\bar{\mathbb{X}}$ is an independent field that we take to satisfy at least one antichiral constraint (either $\mathbb{D}_+\bar{\mathbb{X}} = 0$ or $\mathbb{D}_-\bar{\mathbb{X}} = 0$, or both). The field content of the multiplet $\mathbb{X}$ is

$$\mathbb{X}: \quad (X, \psi_\alpha, F, \bar{\chi}_\alpha, M_{\alpha\beta}, \bar{\eta}_\alpha) \,, \qquad\qquad \alpha, \beta = \pm \,, \tag{2.4.7}$$

and similarly for $\bar{\mathbb{X}}$. By setting $\bar{\chi}_\alpha = M_{\alpha\beta} = \bar{\eta}_\alpha = 0$, the multiplet $\mathbb{X}$ describes a chiral field $\Phi : (X, \psi_\alpha, F)$. By setting $\bar{\chi}_+ = M_{+-} = M_{++} = \bar{\eta}_+ = 0$ it describes a left semichiral field,

[1] One may choose not to do so. Then, a left semichiral field $\mathbb{Y}_L$ satisfies $\bar{\mathbb{D}}_+\mathbb{Y}_L = 0$ and its (Euclidean) Hermitian conjugate $\bar{\mathbb{Y}}_L$ satisfies $\mathbb{D}_-\bar{\mathbb{Y}}_L = 0$, and similarly for a right semichiral field. However, the target space geometry of these models is not well understood. Since ultimately we are interested in learning about the target space geometry of models in Lorentzian signature, we choose to complexify semichiral fields.

and by setting $\bar{\chi}_- = M_{-+} = M_{--} = \bar{\eta}_- = 0$ it describes a right semichiral field, with the field content in (2.4.5). With appropriate identifications, it can also describe a twisted chiral field. However, since we are interested in minimally coupling $\mathbb{X}$ to the vector multiplet, we do not consider the latter case here.

The field content of the vector multiplet is $(A_\mu, \sigma_1, \sigma_2, \lambda_\pm, D)$, where σ_1, σ_2 are real in Lorentzian but complex in Euclidean signature. We will also use

$$\sigma = i\sigma_1 - \sigma_2 \, , \qquad \bar{\sigma} = -i\sigma_1 - \sigma_2 \, . \tag{2.4.8}$$

The SUSY transformation rules for the multiplet $\mathbb{X}$ coupled to the vector multiplet read:

$$\begin{aligned}
\delta X &= \bar{\epsilon}\psi + \epsilon\bar{\chi} \, , & \delta \bar{X} &= \epsilon\bar{\psi} + \bar{\epsilon}\chi \, , \\
\delta\psi_\alpha &= \left(\left[i\gamma^\mu D_\mu X + i\sigma_1 X + \sigma_2 X \gamma_3\right]\epsilon\right)_\alpha - \epsilon^\beta M_{\beta\alpha} + \bar{\epsilon}_\alpha F \, , \\
\delta\bar{\psi}_\alpha &= \left(\left[i\gamma^\mu D_\mu \bar{X} - i\sigma_1 \bar{X} + \sigma_2 \bar{X} \gamma_3\right]\bar{\epsilon}\right)_\alpha - \bar{\epsilon}^\beta \bar{M}_{\beta\alpha} + \epsilon_\alpha \bar{F} \, , \\
\delta F &= \left[-i\sigma_1\psi - \sigma_2\psi\gamma_3 - i\lambda X - i(D_\mu\psi)\gamma^\mu + \bar{\eta}\right]\epsilon \, , \\
\delta \bar{F} &= \left[i\sigma_1\bar{\psi} - \sigma_2\bar{\psi}\gamma_3 - i\bar{\lambda}\bar{X} - i(D_\mu\bar{\psi})\gamma^\mu + \eta\right]\bar{\epsilon} \, , \\
\delta\bar{\chi}_\alpha &= \bar{\epsilon}^\beta M_{\alpha\beta} \, , & \delta\chi_\alpha &= \epsilon^\beta \bar{M}_{\alpha\beta} \, , \\
\delta M_{\alpha\beta} &= -\bar{\eta}_\alpha\bar{\epsilon}_\beta - i\sigma_1\bar{\chi}_\alpha\epsilon_\beta - \sigma_2\bar{\chi}_\alpha\gamma_3\epsilon_\beta - i(D_\mu\bar{\chi}_\alpha)(\gamma^\mu\epsilon)_\beta \, , \\
\delta \bar{M}_{\alpha\beta} &= -\eta_\alpha\epsilon_\beta + i\sigma_1\chi_\alpha\bar{\epsilon}_\beta - \sigma_2\chi_\alpha\gamma_3\bar{\epsilon}_\beta - i(D_\mu\chi_\alpha)(\gamma^\mu\bar{\epsilon})_\beta \, , \\
\delta\bar{\eta}_\alpha &= -i(\epsilon\lambda)\bar{\chi}_\alpha + i(\epsilon\gamma^\mu)^\beta D_\mu M_{\alpha\beta} - i\sigma_1\epsilon^\beta M_{\alpha\beta} - \sigma_2(\gamma_3\epsilon)^\beta M_{\alpha\beta} \, , \\
\delta\eta_\alpha &= -i(\bar{\epsilon}\bar{\lambda})\chi_\alpha + i(\bar{\epsilon}\gamma^\mu)^\beta D_\mu \bar{M}_{\alpha\beta} + i\sigma_1\bar{\epsilon}^\beta \bar{M}_{\alpha\beta} - \sigma_2(\gamma_3\bar{\epsilon})^\beta \bar{M}_{\alpha\beta} \, ,
\end{aligned} \tag{2.4.9}$$

where $D_\mu = \partial_\mu - iA_\mu$ is the gauge-covariant derivative. To make the notation compact in what follows, it is convenient to introduce the operator

$$\mathcal{P}_{\alpha\beta} \equiv \begin{pmatrix} 2iD_{++} & \sigma \\ \bar{\sigma} & -2iD_{--} \end{pmatrix} .$$

The kinetic action for the field $\mathbb{X}$ in flat space is built out of terms of the form:

$$\begin{aligned}
\mathcal{L}_{\mathbb{X}}^{\mathbb{R}^2} &= \int d^4\theta \, \bar{\mathbb{X}} \mathbb{X} \\
&= D_\mu \bar{X} D^\mu X + \bar{X}\left(\sigma_1^2 + \sigma_2^2\right) X + i\bar{X} D X + \bar{F} F - \bar{M}_{\alpha\beta} M^{\beta\alpha} - \bar{X}\mathcal{P}_{\alpha\beta} M^{\alpha\beta} + \bar{M}^{\alpha\beta}\mathcal{P}_{\beta\alpha} X \\
&\quad - i\bar{\psi}\gamma^\mu D_\mu\psi + \bar{\psi}\left(i\sigma_1 - \gamma_3\sigma_2\right)\psi + i\bar{\psi}\lambda X - i\bar{X}\bar{\lambda}\psi - \eta\psi - \bar{\psi}\bar{\eta} \\
&\quad + i\bar{\chi}\gamma^\mu D_\mu\chi - \bar{\chi}\left(i\sigma_1 - \gamma_3\sigma_2\right)\chi + i\bar{X}\lambda\bar{\chi} - i\chi\bar{\lambda}X \, .
\end{aligned} \tag{2.4.10}$$

Setting $M_{\alpha\beta} = \bar{M}_{\alpha\beta} = \eta_\alpha = \bar{\eta}_\alpha = \chi_\alpha = \bar{\chi}_\alpha = 0$ reduces (2.4.10) to the usual action for a chiral multiplet. Setting only some of these fields to zero—according to the discussion below (2.4.7)—gives the action for the corresponding semichiral multiplet. If one considers a single semichiral field, though, this Lagrangian does not describe a sigma model with standard kinetic term. Taking, for instance, a neutral left semichiral field $\mathbb{X} = \mathbb{X}_L$, $\bar{\mathbb{X}} = \bar{\mathbb{X}}_L$, the equations of motion from (2.4.10) set $M_{-+} = \bar{M}_{-+} = 0$ and $\psi_+ = \bar{\psi}_+ = 0$ and

$$\begin{aligned} \partial_{++}X_L &= \partial_{++}\bar{X}_L = \partial_{++}M_{--} = \partial_{++}\bar{M}_{--} = 0\,, \\ \partial_{++}\psi_- &= \partial_{++}\bar{\psi}_- = \partial_{++}\chi_- = \partial_{++}\bar{\chi}_- = 0\,, \end{aligned}$$

which describe two left-moving bosonic and two left-moving fermionic modes. Although interesting, we leave the study of such Lagrangians to future work.

Traditional superpotential terms are not possible for semichiral fields. Fermionic superpotential terms (integrals over $d^3\theta$) may be possible if any fermionic semichiral multiplet is present, but this is not the case here.

To obtain sigma models with standard kinetic terms, we consider models with the same number of left and right semichiral fields and with an appropriate coupling between them. In such models, as we shall see, integrating out the auxiliary fields leads to standard kinetic terms as well as B-field-like couplings. If we restrict ourselves to linear models, the possible left and right mixing terms are of the form $\bar{\mathbb{X}}_L\mathbb{X}_R + c.c.$ or of the form $\mathbb{X}_L\mathbb{X}_R + c.c.$. One can choose either kind (and in the multiflavor case one may have both types of terms for different multiplets). However, for these terms to be gauge invariant, left and right semichiral fields must be in either the same or conjugate representations of the gauge group, accordingly. Thus, from now on we restrict ourselves to models containing pairs of semichiral fields $(\mathbb{X}_L, \mathbb{X}_R)$ either in a representation $(\mathcal{R}, \mathcal{R})$ or $(\mathcal{R}, \bar{\mathcal{R}})$ of the gauge group.

Consider a pair of semichiral fields $(\mathbb{X}_L, \mathbb{X}_R)$ in a representation $(\mathcal{R}, \mathcal{R})$ of a gauge group G. A gauge-invariant action is given by[2]

$$\mathcal{L}_{LR}^{\mathbb{R}^2} = -\int d^4\theta \left[\bar{\mathbb{X}}_L\mathbb{X}_L + \bar{\mathbb{X}}_R\mathbb{X}_R + \alpha\big(\bar{\mathbb{X}}_L\mathbb{X}_R + \bar{\mathbb{X}}_R\mathbb{X}_L\big)\right]\,, \tag{2.4.11}$$

where α is a real non-negative parameter. Notice that with this action the left and right semichiral fields must have the same charge under the R-symmetry. As we discussed above, the cross term proportional to α is crucial if one wishes to obtain a sigma model and from now on we assume $\alpha \neq 0$. Before gauging, the action (2.4.11) describes flat space with a constant B-field. For the flat-space metric to be positive definite, one finds that $\alpha > 1$. As

[2]One could consider a more general superspace Lagrangian of the form

$$K = \beta\bar{\mathbb{X}}_L\mathbb{X}_L + \gamma\bar{\mathbb{X}}_R\mathbb{X}_R + \alpha\bar{\mathbb{X}}_L\mathbb{X}_R + \alpha^*\bar{\mathbb{X}}_R\mathbb{X}_L\,,$$

where β, γ are real parameters and α is complex. If $\beta, \gamma \neq 0$, by rescaling the fields one can set $\beta = \pm 1$ and $\gamma = \pm 1$. By a further phase redefinition of the fields, α can be made real and non-negative. Requiring that the metric, after integrating out the auxiliary fields, be positive definite leads to $\beta = \gamma = -1$ and in addition imposes $\alpha > 1$.

we shall see below, after gauging and by an appropriate field redefinition, one may take the limit $\alpha \to 1$, which becomes a special case.

The general case with multiple pairs of semichiral fields is very similar. Under the assumption that the metric is positive definite, one can always reduce, with field redefinitions, to the case with multiple copies of (2.4.11).[3]

We now show that this gauge theory is a deformation of a gauge theory for chiral fields $(\Phi, \tilde{\Phi})$ in conjugate representations $(\mathcal{R}, \bar{\mathcal{R}})$ of the gauge group corresponding to turning on a gauged Wess-Zumino term controlled by the parameter α. To see this one must integrate out the auxiliary fields in the semichiral multiplets. Integrating out the auxiliary fields from (2.4.11) leads to a constant kinetic term that depends on α. To bring the metric into the canonical form one needs to introduce the new fields $X, \tilde{X}$ by

$$X_L = \sqrt{\tfrac{\alpha}{4(\alpha+1)}}\, X + \sqrt{\tfrac{\alpha}{4(\alpha-1)}}\, \bar{\tilde{X}}\ , \qquad X_R = \sqrt{\tfrac{\alpha}{4(\alpha+1)}}\, X - \sqrt{\tfrac{\alpha}{4(\alpha-1)}}\, \bar{\tilde{X}}\ ,$$

and similarly for the barred fields. Importantly, note that X and $\tilde{X}$ are in *conjugate* representations of the gauge group. Finally, one has

$$\mathcal{L}^{\mathbb{R}^2}_{LR} = \big(g_{ij} + b_{ij}\big)\, D_{++}X^i D_{--}X^j + \tfrac{\alpha}{2}\big(\bar{X}|\sigma|^2 X + \bar{\tilde{X}}|\sigma|^2\tilde{X}\big) + \tfrac{\alpha}{2}\big(\bar{X}DX + \bar{\tilde{X}}D\tilde{X}\big) + \text{fermions}\ , \tag{2.4.12}$$

where $X^i = (X, \bar{X}, \tilde{X}, \bar{\tilde{X}})$ and the metric and B-field are given by

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\ , \qquad b_{ij} = \sqrt{\alpha^2 - 1}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\ , \tag{2.4.13}$$

and we have omitted fermionic kinetic terms and Yukawa couplings. Note that at this point one may take the limit $\alpha \to 1$, for which $b_{ij} = 0$. In this limit, the Lagrangian (2.4.12) coincides with the component Lagrangian for chiral fields in conjugate representations. Thus, the parameter α controls a deformation of such model, corresponding to turning on a (gauged)

[3]Starting with the superspace Lagrangian

$$K = \beta_{ij}\bar{\mathbb{X}}^i_L\mathbb{X}^j_L + \gamma_{ij}\bar{\mathbb{X}}^i_R\mathbb{X}^j_R + \alpha_{ij}\bar{\mathbb{X}}^i_L\mathbb{X}^j_R + (\alpha^\dagger)_{ij}\bar{\mathbb{X}}^i_R\mathbb{X}^j_L$$

where β, γ are Hermitian matrices and α is a complex matrix. Again, by field redefinitions one can set β and γ to be diagonal with entries $\pm 1, 0$. Then requiring the metric to be positive definite leads to the following two conditions

$$-4(\alpha^\dagger)^{-1}\gamma\alpha^{-1} > 0 \quad \text{and} \quad 4\beta(\alpha^\dagger)^{-1}\gamma\alpha^{-1}\beta - 4\beta > 0.$$

These force $\beta = \gamma = -\mathbb{1}$ and by the singular value decomposition theorem we can, by further unitary field redefinitions, reduce α to a diagonal matrix with non-negative entries which now have to satisfy $\alpha_{ii} > 1$ as before.

B-field term. As we shall discuss next, these models give rise to NLSMs on Generalized Kähler manifolds. The H-field and metric in the target are controlled by the parameter α and for the special value $\alpha^2 = 1$, $H = 0$ and g becomes a Kähler metric, as one would expect.

Conjugate Representations

For a pair of semichiral fields $(\mathbb{X}_L, \mathbb{X}_R)$ in conjugate representations $(\mathcal{R}, \bar{\mathcal{R}})$ of a gauge group G, a gauge-invariant action is given by

$$\mathcal{L}_{LR}^{\mathbb{R}^2} = \int d^4\theta \left[\bar{\mathbb{X}}_L\mathbb{X}_L + \bar{\mathbb{X}}_R\mathbb{X}_R + \beta\big(\mathbb{X}_L\mathbb{X}_R + \bar{\mathbb{X}}_R\bar{\mathbb{X}}_L\big)\right] . \tag{2.4.14}$$

In this case the fields are forced to have opposite R-symmetry charges.

By reducing to components and integrating out the auxiliary fields, one sees once again that this is deformation of a gauge theory for chiral fields in a gauge representation $(\mathcal{R}, \bar{\mathcal{R}})$ by turning on a gauged Wess-Zumino term controlled by the parameter β. In the limit $\beta \to \infty$ this term vanishes. In fact the two gauge theories (2.4.11) (2.4.14) related by a simple field redefinition which corresponds to a change of coordinates on the target. Thus, without any loss of generality one may consider only one of these actions, choosing the most appropriate

As mentioned above, one of the main motivations for studying gauge theories for semichiral fields is that they realize NLSMs on Generalized Kähler manifolds, as opposed to Kähler manifolds when only chiral fields are present. Let us briefly illustrate this for the case of $G = U(1)$, although the discussion holds for an arbitrary gauge group.

Consider N_F pairs of semichiral fields $(\mathbb{X}_L^i, \mathbb{X}_R^i)$, $i = 1, .., N_F$, charged under a $U(1)$ vector multiplet with charges $(Q_i, -Q_i)$. A gauge invariant action is given by

$$\mathcal{L} = \mathcal{L}_{\text{VM}} + \sum_{i=1}^{N_F} \int d^4\theta \left[\bar{\mathbb{X}}_L^i\mathbb{X}_L^i + \bar{\mathbb{X}}_R^i\mathbb{X}_R^i + \big(\beta_{ij}\mathbb{X}_L^i\mathbb{X}_R^j + \beta_{ij}^\dagger\bar{\mathbb{X}}_R^j\bar{\mathbb{X}}_L^i\big)\right] + \frac{i}{2}\int d^2\tilde{\theta}\, t\,\Sigma + c.c.\,, \tag{2.4.15}$$

where

$$\mathcal{L}_{\text{VM}} = -\frac{1}{2e^2}\int d^4\theta\, \bar{\Sigma}\Sigma\,, \qquad \Sigma = \bar{\mathbb{D}}_+\mathbb{D}_- V\,, \tag{2.4.16}$$

and $t = i\xi + \frac{\theta}{2\pi}$ is the complexified Fayet-Iliopoulos (FI) parameter. The matrix β_{ij} is constrained by gauge invariance and must be invertible in order to obtain a non-trivial kinetic term. In addition, one may have any number of charged chiral multiplets with their own kinetic terms and couplings to the semichiral fields. In the case that only semichiral fields are present, a better formulation of these theories is discussed in [56].

Just as in the usual case of GLSMs for chiral fields [57], in the $e \to \infty$ limit the gauge fields become massive and can be integrated out. The effective theory is a sigma model on the space of vacua, which is determined by the vanishing of the scalar potential U, modulo the action of the gauge group. For the theory (2.4.15), the space of vacua is given by[4]

$$\mathcal{M} = \Big\{ \sum\nolimits_i Q_i \Big(\big\| X_L^i \big\|^2 - \big\| X_R^i \big\|^2 \Big) - \xi = 0 \Big\} \Big/ U(1) \,, \tag{2.4.17}$$

where the $U(1)$ acts by $X_L^i \to e^{i\alpha Q^i} X_L^i$, $X_R^i \to e^{-i\alpha Q^i} X_R^i$. The complex dimension of $\mathcal{M}$ is $2N_F - 1$. Topologically, this space coincides with the moduli space of a GLSM for the same number of pairs of chiral fields $(\Phi^i, \tilde{\Phi}^i)$, with charges $(Q_i, -Q_i)$. However, the geometric structure in the semichiral description is quite different: the space is endowed with a Generalized Kähler structure, rather than a Kähler structure. In other words, there are two complex structures $J_\pm$ and, due to the presence of semichiral fields, $[J_+, J_-] \neq 0$.

One may also consider a model with gauge charges (Q_i, Q_i). However, as discussed above, this is completely equivalent to the model with charges $(Q_i, -Q_i)$ and after a simple change of variables it is easy to see that the moduli space is also (2.4.17). In particular, the moduli spaces of these type of models are always non-compact, for any choice of gauge charges. The generalization of this discussion for a generic gauge group G is straightforward.

Before we proceed we would like to make a comment on couplings to other vector multiplets. Since semichiral fields are less constrained than chiral or twisted chiral fields, they admit minimal couplings to various vector multiplets. In addition to the usual vector multiplet, they can also couple minimally to the twisted vector multiplet, as well as to the Semichiral Vector Multiplet (SVM) introduced in [58, 59]. As shown in [60], gauge theories for semichiral fields coupled to the SVM realize NLSMs on hyper-kähler manifolds (examples are Eguchi-Hanson and Taub-NUT). Here we restrict ourselves to the coupling to the vector multiplet.

2.4.2 Semichiral Fields on S^2

The main goal of this section is to place the gauge theories (2.4.11) and (2.4.14) on the round sphere S^2 (neutral semichiral multiplets have already been studied in [61]). We will show that it is possible to construct such actions while preserving four supercharges, *i.e.*, a sort of $\mathcal{N} = (2,2)$ supersymmetry. Furthermore, these actions are not only $\mathcal{Q}$-closed but also $\mathcal{Q}$-exact, and therefore the partition function will not depend on the parameters therein.

[4] One way to see this is to compute the scalar potential explictly by going down to components and working, say, in Wess-Zumino gauge. Alternatively, one may work in superspace, by writing $\mathbb{X}^{(0)\,i} = \mathbb{X}^i$ and $\bar{\mathbb{X}}^{(0)\,i} = \bar{\mathbb{X}}^i e^{-Q_i V}$ in (2.4.15) to introduce the vector multiplet explicitly (here we are following similar notation to that in [57]). Then, the lowest component of the equation of motion for V leads to the constraint below. Note, in particular, that due to the absence of e^V terms in the cross terms, the matrix β_{ij} does not enter in the constraint.

We first determine the supersymmetry transformations on S^2. We will work in components, rather than in superspace.

One way to determine the supersymmetry transformations on S^2 is by first constructing the $\mathcal{N} = (2,2)$ superconformal transformations and then specializing the transformations to an $SU(2|1)$ sub-algebra, which is identified as the supersymmetry algebra on S^2. The superconformal transformations can be deduced from the $\mathcal{N} = (2,2)$ super Poincare transformations (2.4.9) by covariantizing them with respect to Weyl transformations, as we now explain.

Let the scalar X transform under Weyl transformations with a weight q, *i.e.* under an infinitesimal Weyl transformation $\delta X = -q\Omega\, X$. The supersymmetry transformations (2.4.9) are not covariant under Weyl transformations, but they can be covariantized by adding suitable terms proportional to $\nabla_\pm \epsilon$, as explained in [62]. Following this procedure, we find that the Weyl-covariant transformations for the superfields $\mathbb{X}$ and $\bar{\mathbb{X}}$ are:

$$
\begin{aligned}
\delta X &= \bar\epsilon\psi + \epsilon\bar\chi \;, \\
\delta \bar X &= \epsilon\bar\psi + \bar\epsilon\chi \;, \\
\delta\psi_\alpha &= \Big(\big[i(D_\mu X)\gamma^\mu + i\sigma_1 X + \sigma_2 X\gamma_3 + i\tfrac{q}{2}X\overleftarrow{\nabla\!\!\!/}\big]\,\epsilon\Big)_\alpha - \epsilon^\beta M_{\beta\alpha} + \bar\epsilon_\alpha F \;, \\
\delta\bar\psi_\alpha &= \Big(\big[i(D_\mu \bar X)\gamma^\mu - i\sigma_1 \bar X + \sigma_2 \bar X\gamma_3 + i\tfrac{q}{2}\bar X\overleftarrow{\nabla\!\!\!/}\big]\,\bar\epsilon\Big)_\alpha - \bar\epsilon^\beta \bar M_{\beta\alpha} + \epsilon_\alpha \bar F \;, \\
\delta F &= \big[-i\sigma_1\psi - \sigma_2\psi\gamma_3 - i\lambda X - i(D_\mu\psi)\gamma^\mu - i\tfrac{q}{2}\psi\overleftarrow{\nabla\!\!\!/} + \bar\eta\big]\,\epsilon \;, \\
\delta \bar F &= \big[i\sigma_1\bar\psi - \sigma_2\bar\psi\gamma_3 - i\bar\lambda\bar X - i(D_\mu\bar\psi)\gamma^\mu - i\tfrac{q}{2}\bar\psi\overleftarrow{\nabla\!\!\!/} + \eta\big]\,\bar\epsilon \;, \\
\delta\bar\chi_\alpha &= \bar\epsilon^\beta M_{\alpha\beta} \;, \\
\delta\chi_\alpha &= \epsilon^\beta \bar M_{\alpha\beta} \;, \\
\delta M_{\alpha\beta} &= -\bar\eta_\alpha\bar\epsilon_\beta - i\sigma_1\bar\chi_\alpha\epsilon_\beta - \sigma_2\bar\chi_\alpha(\gamma_3\epsilon)_\beta - i(D_\mu\bar\chi_\alpha)(\gamma^\mu\epsilon)_\beta \\
&\quad - i\tfrac{q+1}{2}\bar\chi_\alpha(\nabla\!\!\!/\epsilon)_\beta + \tfrac{i}{2}(\gamma_3\nabla\!\!\!/\epsilon)_\beta(\gamma_3\bar\chi)_\alpha \;, \\
\delta \bar M_{\alpha\beta} &= -\eta_\alpha\epsilon_\beta + i\sigma_1\chi_\alpha\bar\epsilon_\beta - \sigma_2\chi_\alpha(\gamma_3\bar\epsilon)_\beta - i(D_\mu\chi_\alpha)(\gamma^\mu\bar\epsilon)_\beta \\
&\quad - i\tfrac{q+1}{2}\chi_\alpha(\nabla\!\!\!/\bar\epsilon)_\beta + \tfrac{i}{2}(\gamma_3\nabla\!\!\!/\bar\epsilon)_\beta(\gamma_3\chi)_\alpha \;, \\
\delta\bar\eta_\alpha &= -i(\epsilon\lambda)\bar\chi_\alpha + i(\epsilon\gamma^\mu)^\beta D_\mu M_{\alpha\beta} - i\sigma_1\epsilon^\beta M_{\alpha\beta} - \sigma_2(\gamma_3\epsilon)^\beta M_{\alpha\beta} \\
&\quad + i\tfrac{q+1}{2}\left((\nabla_\mu\epsilon)\gamma^\mu\right)^\beta M_{\alpha\beta} + \tfrac{i}{2}(\nabla_\mu\epsilon\gamma_3\gamma^\mu)^\beta(\gamma_3)_\alpha{}^\rho M_{\rho\beta} \;, \\
\delta\eta_\alpha &= -i(\bar\epsilon\bar\lambda)\chi_\alpha + i(\bar\epsilon\gamma^\mu)^\beta D_\mu \bar M_{\alpha\beta} + i\sigma_1\bar\epsilon^\beta \bar M_{\alpha\beta} - \sigma_2(\gamma_3\bar\epsilon)^\beta \bar M_{\alpha\beta} \\
&\quad + i\tfrac{q+1}{2}\left((\nabla_\mu\bar\epsilon)\gamma^\mu\right)^\beta \bar M_{\alpha\beta} + \tfrac{i}{2}(\nabla_\mu\bar\epsilon\gamma_3\gamma^\mu)^\beta(\gamma_3)_\alpha{}^\rho \bar M_{\rho\beta} \;,
\end{aligned}
\tag{2.4.18}
$$

where $D_\mu = \nabla_\mu - iA_\mu$ is the gauge-covariant derivative on S^2. Splitting $\delta = \delta_\epsilon + \delta_{\bar\epsilon}$ and imposing the Killing spinor equations $\nabla_\mu\epsilon = \gamma_\mu\check\epsilon$, $\nabla_\mu\bar\epsilon = \gamma_\mu\check{\bar\epsilon}$ for some other spinors $\check\epsilon$, $\check{\bar\epsilon}$, one finds that the superconformal algebra is realized on semichiral fields as:

$$
\begin{aligned}
[\delta_\epsilon,\, \delta_{\bar\epsilon}]X &= \xi^\mu\partial_\mu X + i\Lambda X + \tfrac{q}{2}\rho X + iq\alpha X\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar X &= \xi^\mu\partial_\mu \bar X - i\Lambda \bar X + \tfrac{q}{2}\rho \bar X - iq\alpha \bar X\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\psi &= \xi^\mu\partial_\mu \psi + i\Lambda\psi + \tfrac{q+1}{2}\rho\psi + i(q-1)\alpha\psi + \tfrac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\psi + i\beta\gamma_3\psi\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar\psi &= \xi^\mu\partial_\mu \bar\psi - i\Lambda\bar\psi + \tfrac{q+1}{2}\rho\bar\psi - i(q-1)\alpha\bar\psi + \tfrac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\bar\psi - i\beta\gamma_3\bar\psi\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]F &= \xi^\mu\partial_\mu F + i\Lambda F + \tfrac{q+2}{2}\rho F + i(q-2)\alpha F\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar F &= \xi^\mu\partial_\mu \bar F - i\Lambda \bar F + \tfrac{q+2}{2}\rho \bar F - i(q-2)\alpha \bar F\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar\chi_\alpha &= \xi^\mu\partial_\mu \bar\chi_\alpha + i\Lambda\bar\chi_\alpha + \tfrac{q+1}{2}\rho\bar\chi_\alpha + i(q+1)\alpha\bar\chi_\alpha + \tfrac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\bar\chi_\alpha - i\beta(\gamma_3\bar\chi)_\alpha\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\chi_\alpha &= \xi^\mu\partial_\mu \chi_\alpha - i\Lambda\chi_\alpha + \tfrac{q+1}{2}\rho\chi_\alpha - i(q+1)\alpha\chi_\alpha + \tfrac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\chi_\alpha + i\beta(\gamma_3\chi)_\alpha\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]M_{\alpha\beta} &= \xi^\mu\partial_\mu M_{\alpha\beta} + i\Lambda M_{\alpha\beta} + \tfrac{q+2}{2}\rho M_{\alpha\beta} + iq\alpha M_{\alpha\beta} + \tfrac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\rho M_{\rho\beta} \\
&\quad + \tfrac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\beta{}^\rho M_{\alpha\rho} - i\beta(\gamma_3)_\alpha{}^\rho M_{\rho\beta} + i\beta(\gamma_3)_\beta{}^\rho M_{\alpha\rho}\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar M_{\alpha\beta} &= \xi^\mu\partial_\mu \bar M_{\alpha\beta} - i\Lambda \bar M_{\alpha\beta} + \tfrac{q+2}{2}\rho \bar M_{\alpha\beta} - iq\alpha \bar M_{\alpha\beta} + \tfrac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\rho \bar M_{\rho\beta} \\
&\quad + \tfrac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\beta{}^\rho \bar M_{\alpha\rho} + i\beta(\gamma_3)_\alpha{}^\rho \bar M_{\rho\beta} - i\beta(\gamma_3)_\beta{}^\rho \bar M_{\alpha\rho}\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar\eta_\alpha &= \xi^\mu\partial_\mu \bar\eta_\alpha + i\Lambda\bar\eta_\alpha + \tfrac{q+3}{2}\rho\bar\eta_\alpha + i(q-1)\alpha\bar\eta_\alpha + \tfrac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\bar\eta_\alpha - i\beta(\gamma_3\bar\eta)_\alpha\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\eta_\alpha &= \xi^\mu\partial_\mu \eta_\alpha - i\Lambda\eta_\alpha + \tfrac{q+3}{2}\rho\eta_\alpha - i(q-1)\alpha\eta_\alpha + \tfrac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\eta_\alpha + i\beta(\gamma_3\eta)_\alpha\,,
\end{aligned}
\tag{2.4.19}
$$

where the parameters are given by

$$
\begin{aligned}
\xi_\mu &\equiv i(\bar\epsilon\gamma_\mu\epsilon)\,, & \Lambda &\equiv -\xi^\mu A_\mu + (\bar\epsilon\epsilon)\sigma_1 - i(\bar\epsilon\gamma_3\epsilon)\sigma_2\,, \\
\Theta^{\mu\nu} &\equiv D^{[\mu}\xi^{\nu]} + \xi^\rho\omega_\rho{}^{\mu\nu}\,, & \rho &\equiv \tfrac{i}{2}(D_\mu\bar\epsilon\gamma^\mu\epsilon + \bar\epsilon\gamma^\mu D_\mu\epsilon) = \tfrac{1}{2}D_\mu\xi^\mu\,, \\
\alpha &\equiv -\tfrac{1}{4}(D_\mu\bar\epsilon\gamma^\mu\epsilon - \bar\epsilon\gamma^\mu D_\mu\epsilon)\,, & \beta &\equiv \tfrac{1}{4}(D_\mu\bar\epsilon\gamma_3\gamma^\mu\epsilon - \bar\epsilon\gamma_3\gamma^\mu D_\mu\epsilon)\,,
\end{aligned}
$$

and $\omega_\rho{}^{\mu\nu}$ is the spin connection. Here ξ_μ parameterizes translations, ρ is a parameter for dilations, and α, β parameterize vector and axial R-symmetry transformations, respectively. From here one reads off the charges of the fields under these transformations, which are summarized in Table 2.1. Note that the vector R-charge is twice the Weyl charge, as for chiral fields. All other commutators vanish, $[\delta_\epsilon, \delta_\epsilon] = [\delta_{\bar\epsilon}, \delta_{\bar\epsilon}] = 0$, if one imposes the extra condition $\Box\epsilon = h\epsilon$, $\Box\bar\epsilon = h\bar\epsilon$ with the same function h.

	$\mathbb{D}_+$	$\mathbb{D}_-$	$\bar{\mathbb{D}}_+$	$\bar{\mathbb{D}}_-$	X	ψ	$\bar\chi$	F	$M_{\alpha\beta}$	$\bar\eta$
ω	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{q}{2}$	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{q+2}{2}$	$\frac{q+2}{2}$	$\frac{q+3}{2}$
q_V	-1	-1	1	1	q	$q-1$	$q+1$	$q-2$	q	$q-1$
q_A	1	-1	-1	1	0	1	-1	0	$-2\epsilon_{\alpha\beta}$	-1

Table 2.1: Weyl charge, vector and axial R-charge for the component fields of the semichiral multiplet

There are four complex Killing spinors on S^2, satisfying $\nabla_\mu\epsilon = \pm\frac{i}{2r}\gamma_\mu\epsilon$. Restricting the transformations (2.4.18) to spinors $\epsilon, \bar{\epsilon}$ that satisfy

$$\nabla_\mu \epsilon = \tfrac{i}{2r}\gamma_\mu \epsilon \,, \qquad \nabla_\mu \bar{\epsilon} = \tfrac{i}{2r}\gamma_\mu \bar{\epsilon} \,, \tag{2.4.20}$$

one finds that the algebra (2.4.19) does not contain dilations nor axial R-rotations (*i.e.* $\rho = \beta = 0$). This is an $SU(2|1)$ subalgebra of the superconformal algebra that we identify as the $\mathcal{N} = (2,2)$ superalgebra on S^2, and we denote it by $\mathcal{Q}_A$. The transformations rules in (2.4.18) simplify to

$$\begin{aligned}
\delta X &= \bar{\epsilon}\psi + \epsilon\bar{\chi} \,, \\
\delta \bar{X} &= \epsilon\bar{\psi} + \bar{\epsilon}\chi \,, \\
\delta \psi_\alpha &= \epsilon^\beta(\mathcal{P}_{\beta\alpha}X - M_{\beta\alpha}) + \bar{\epsilon}_\alpha F - \tfrac{q}{2r}X\epsilon_\alpha \,, \\
\delta \bar{\psi}_\alpha &= \bar{\epsilon}^\beta(\mathcal{P}_{\alpha\beta}\bar{X} - \bar{M}_{\beta\alpha}) + \epsilon_\alpha \bar{F} - \tfrac{q}{2r}\bar{X}\bar{\epsilon}_\alpha \,, \\
\delta F &= \epsilon^\alpha \mathcal{P}_{\alpha\beta}\psi^\beta - i(\epsilon\lambda)X + \epsilon\bar{\eta} + \tfrac{q}{2r}\epsilon\psi \,, \\
\delta \bar{F} &= \bar{\epsilon}^\alpha \mathcal{P}_{\beta\alpha}\bar{\psi}^\beta - i(\bar{\epsilon}\bar{\lambda})\bar{X} + \bar{\epsilon}\eta + \tfrac{q}{2r}\bar{\epsilon}\bar{\psi} \,, \\
\delta \bar{\chi}_\alpha &= M_{\alpha\beta}\bar{\epsilon}^\beta \,, \\
\delta \chi_\alpha &= \bar{M}_{\alpha\beta}\epsilon^\beta \,, \\
\delta M_{\alpha\beta} &= \epsilon^\gamma \mathcal{P}_{\gamma\beta}\bar{\chi}_\alpha - \bar{\eta}_\alpha\bar{\epsilon}_\beta + \tfrac{q-2}{2r}\bar{\chi}_\alpha\epsilon_\beta + \tfrac{2}{r}\bar{\chi}_{(\alpha}\epsilon_{\beta)} \,, \\
\delta \bar{M}_{\alpha\beta} &= \bar{\epsilon}^\gamma \mathcal{P}_{\beta\gamma}\chi_\alpha - \eta_\alpha\epsilon_\beta + \tfrac{q-2}{2r}\chi_\alpha\bar{\epsilon}_\beta + \tfrac{2}{r}\chi_{(\alpha}\bar{\epsilon}_{\beta)} \,, \\
\delta \bar{\eta}_\alpha &= \epsilon^\kappa \mathcal{P}_{\kappa\gamma}M_{\alpha\beta}C^{\gamma\beta} - i(\epsilon\lambda)\bar{\chi}_\alpha + \tfrac{q}{2r}M_{\alpha\beta}\epsilon^\beta + \tfrac{2}{r}M_{[\alpha\beta]}\epsilon^\beta \,, \\
\delta \eta_\alpha &= \bar{\epsilon}^\kappa \mathcal{P}_{\gamma\kappa}\bar{M}_{\alpha\beta}C^{\gamma\beta} - i(\bar{\epsilon}\bar{\lambda})\chi_\alpha + \tfrac{q}{2r}\bar{M}_{\alpha\beta}\bar{\epsilon}^\beta + \tfrac{2}{r}\bar{M}_{[\alpha\beta]}\bar{\epsilon}^\beta \,,
\end{aligned} \tag{2.4.21}$$

where $C^{\alpha\beta}$ is the antisymmetric tensor with $C^{+-} = 1$ and $[\alpha\beta]$, $(\alpha\beta)$ denotes (anti-)symmetrization of indices, respectively[5]. Another way to derive these transformation rules is by coupling the theory to background supergravity, along the lines of [12]. Using this method, these transformations – and their generalization to any compact, orientable Riemann surface with no boundaries – were given in [61] for neutral semichiral fields.

The flat-space action (2.4.10) is not invariant under the curved-space transformations (2.4.21). However, it is possible to add suitable $\frac{1}{r}$ and $\frac{1}{r^2}$ terms to obtain an invariant Lagrangian:

$$\begin{aligned}
&\mathcal{L}^{S^2}_{\mathbb{X}} = \mathcal{L}^{\mathbb{R}^2}_{\mathbb{X}} + \delta\mathcal{L} \,, \\
&\delta\mathcal{L} = \tfrac{iq}{r}\,\bar{X}\sigma_1 X + \tfrac{q(2-q)}{4r^2}\,\bar{X}X - \tfrac{q}{2r}\,\bar{\psi}\psi - \tfrac{q}{2r}\,\chi\bar{\chi} - \tfrac{q}{2r}\left(\bar{X}C^{\alpha\beta}M_{\alpha\beta} + C^{\alpha\beta}\bar{M}_{\alpha\beta}X\right).
\end{aligned} \tag{2.4.22}$$

The first three terms in $\delta\mathcal{L}$ are the usual terms needed in the case of a chiral field and the last three terms are additional terms required for semichiral fields. In fact the action is not

[5]Here we have written the transformations using explicit representations for the gamma matrices and properties of spinors. We find this convenient for the calculations in the next sections.

only $\mathcal{Q}_A$-closed but also $\mathcal{Q}_A$-exact:

$$\bar{\epsilon}\epsilon \int d^2x\, \mathcal{L}_{\mathbb{X}}^{S^2} = \delta_\epsilon \delta_{\bar{\epsilon}} \int d^2x \left(\bar{\psi}\psi + \chi\bar{\chi} - 2i\bar{X}\sigma_1 X + \tfrac{q-1}{r}\,\bar{X}X + \bar{X}C^{\alpha\beta}M_{\alpha\beta} + C^{\alpha\beta}\bar{M}_{\alpha\beta}X \right). \tag{2.4.23}$$

Thus, one can use $\mathcal{L}_{\mathbb{X}}^{S^2}$ itself for localization, which is an important simplification to evaluate the one-loop determinant using spherical harmonics.[6]

Same Representations

Let us consider first the case of a pair of semichiral fields in a representation $(\mathcal{R}, \mathcal{R})$, with flat-space action (2.4.11). Since so far we treated $\bar{\mathbb{X}}$ and $\mathbb{X}$ as independent fields, we can use the result (2.4.22) for each individual term in (2.4.11) and the action on S^2 is given by

$$\begin{aligned}\mathcal{L}_{LR}^{S^2} &= D_\mu \bar{X}^i D^\mu X_i + \bar{X}^i \left(\sigma_1^2 + \sigma_2^2\right) X_i + i\bar{X}^i D X_i + \bar{F}^i F_i \\ &\quad - \bar{M}^i_{\alpha\beta} M_i^{\beta\alpha} - \bar{X}^i \mathcal{P}_{\alpha\beta} M_i^{\alpha\beta} + \bar{M}^{\alpha\beta, i}\mathcal{P}_{\beta\alpha} X_i \\ &\quad - i\bar{\psi}^i \gamma^\mu D_\mu \psi_i + \bar{\psi}^i \left(i\sigma_1 - \gamma_3\sigma_2\right)\psi_i + i\bar{\psi}^i \lambda X_i - i\bar{X}^i\bar{\lambda}\psi_i - \eta^i\psi_i - \bar{\psi}^i\bar{\eta}_i \\ &\quad + i\bar{\chi}_i \gamma^\mu D_\mu \chi^i - \bar{\chi}_i \left(i\sigma_1 - \gamma_3\sigma_2\right)\chi^i + i\bar{X}^i\lambda\bar{\chi}_i - i\chi^i\bar{\lambda}X_i \\ &\quad + \tfrac{iq}{r}\bar{X}^i\sigma_1 X_i + \tfrac{q(2-q)}{4r^2}\bar{X}^i X_i - \tfrac{q}{2r}\bar{\psi}^i\psi_i - \tfrac{q}{2r}\chi^i\bar{\chi}_i - \tfrac{q}{2r}\left(\bar{X}^i C^{\alpha\beta} M_{i\,\alpha\beta} + C^{\alpha\beta}\bar{M}^i_{\alpha\beta} X_i\right),\end{aligned} \tag{2.4.24}$$

where the flavor indices $i = (L, R)$ are contracted with

$$\mathcal{M}_{\bar{i}j} = -\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}. \tag{2.4.25}$$

Clearly the action (2.4.24) is also $\mathcal{Q}_A$-exact, being a sum of exact terms.

In this simple model with a single pair of semichiral fields the R-charge q is unphysical, in the sense that it can be set to the canonical value $q = 0$ by mixing the R-current with the gauge current (this is no longer true if we have multiple semichiral pairs charged under the same $U(1)$). However, we keep q for now and set it to zero only at the end of the calculation: as we shall see, this will reduce the number of BPS configurations to be taken into account in the localization.

[6] Strictly speaking, one should use a Q-exact action which is positive definite, so that one localizes to the zero-locus. $\mathcal{L}_{\mathbb{X}}^{S^2}$ has positive definite real part, provided that $0 \leq q \leq 2$.

Conjugate Representations

Let us move to the case of a pair of semichiral fields in conjugate representations, whose flat-space action includes the cross-terms of the form $\mathbb{X}_L\mathbb{X}_R$ appearing in (2.4.14):

$$\mathcal{L}_\beta^{\mathbb{R}^2} = \beta \int d^4\theta \left(\mathbb{X}_L\mathbb{X}_R + \bar{\mathbb{X}}_L\bar{\mathbb{X}}_R\right) = \beta \left(M^{\alpha\beta}M_{\alpha\beta} - \bar{\eta}\bar{\chi} + \bar{M}^{\alpha\beta}\bar{M}_{\alpha\beta} - \eta\chi\right). \tag{2.4.26}$$

This flat-space action is invariant under the S^2 SUSY transformations (2.4.21), with no need for $\frac{1}{r}$ improving terms, *i.e.*, $\mathcal{L}_\beta^{S^2} = \mathcal{L}_\beta^{\mathbb{R}^2}$. Furthermore, it is also $\mathcal{Q}_A$-exact:

$$\begin{aligned}\bar{\epsilon}\epsilon \int d^2x\, \mathcal{L}_\beta^{S^2} = \delta_\epsilon\delta_{\bar{\epsilon}} \int d^2x \Big(& X_L M_{+-} - X_R M_{-+} + \bar{\chi}_-\psi_+^R + \psi_-^L\bar{\chi}_+ \\ & + \bar{X}_L\bar{M}_{+-} - \bar{X}_R\bar{M}_{-+} + \chi_-\bar{\psi}_+^R + \bar{\psi}_-^L\chi_+ - \tfrac{1}{r}X_R X_L - \tfrac{1}{r}\bar{X}_L\bar{X}_R\Big)\,.\end{aligned} \tag{2.4.27}$$

Note that although $\frac{1}{r}$ terms appear inside the integral on the right-hand side, these are cancelled against $\frac{1}{r}$ terms coming from $\delta_\epsilon\delta_{\bar{\epsilon}}$. Summarizing, the generalization of (2.4.14) to S^2 is the sum of (2.4.24) where flavor indices are contracted with $\mathcal{M}_{\bar{i}j} = \delta_{\bar{i}j}$, and (2.4.26).

With the elements we have given here, one can explicitly write the general action on S^2 for any number of semichiral fields and general couplings between them. In the next section we focus on the simplest case of a single pair of semichiral fields and perform the localization on the Coulomb branch.

2.4.3 Localization on the Coulomb Branch

In this section, we compute the S^2 partition function of the gauge theories at hand by means of Coulomb branch localization.

We wish to compute the path integral

$$Z_{S^2} = \int \mathcal{D}\varphi\, e^{-S[\varphi]}\,,$$

where φ are all the fields in the gauge theory, including the vector multiplet, semichiral fields, and possibly chiral fields as well. The action is given by

$$S = \int d^2x \left(\mathcal{L}_{VM} + \mathcal{L}_{chiral} + \mathcal{L}_{semichiral} + \mathcal{L}_{FI}\right)\,, \tag{2.4.28}$$

where each term is the appropriate Lagrangian on S^2 and $\mathcal{L}_{FI} = -i\xi D + \frac{i\theta}{2\pi}F_{12}$ is the standard FI term (which needs no curvature couplings on S^2). We perform localization with respect to the supercharge $\mathcal{Q}_A$. Following the usual arguments [63, 64], the partition function localizes on the BPS configurations $\{\mathcal{Q}_A \cdot \text{fermions} = 0\}$ and is given exactly by the one-loop determinant around such configurations. The contribution from the vector and chiral multiplets were studied in [65, 62]. Here we study the contribution from semichiral fields.

We begin by studying the BPS equations. These follow from setting the variations of all fermions in (2.4.21) to zero. We show that for a generic value of $q \neq 0$ the only smooth solution is

$$X = \bar{X} = F = \bar{F} = M_{\alpha\beta} = \bar{M}_{\alpha\beta} = 0 \,. \tag{2.4.29}$$

Thus, like in the case of chiral multiplets, the BPS configuration for generic q is only the trivial one. The BPS configurations for the vector multiplet are given by [65, 62]

$$0 = F_{12} - \frac{\sigma_2}{r} = D + \frac{\sigma_1}{r} = D_\mu \sigma_1 = D_\mu \sigma_2 = [\sigma_1, \sigma_2] \,. \tag{2.4.30}$$

Flux quantization of F_{12} implies that $\sigma_2 = \mathfrak{m}/(2r)$, where $\mathfrak{m}$ is a co-weight, (*i.e.* $\mathfrak{m}$ belongs to the Cartan subalgebra of the gauge algebra and $\rho(\mathfrak{m}) \in \mathbb{Z}$ for any weight ρ of any representation $\mathcal{R}$ of the gauge group). Thus, the set of BPS configurations are parametrized by the continuous variable σ_1 and the discrete fluxes $\mathfrak{m}$.

As shown in the previous section, the kinetic actions for semichiral fields are $\mathcal{Q}_A$-exact. Thus, we can use the kinetic actions themselves as a deformation term for localization and we must compute the one-loop determinants arising from these actions. We now compute these determinants, both in the case in which the left and right semichiral fields are in the same representation of the gauge group, and in the case they are in conjugate representations. The determinants coincide, as they should since such theories are related by a simple change of variables.

Consider the action (2.4.24) and expand it at quadratic order around a BPS background. Let us look at bosonic fields first, using the basis

$$\mathcal{X} = \left(X^L, X^R, M^L_{-+}, M^L_{--}, M^R_{++}, M^R_{+-}, F^L, F^R\right)^T \,.$$

The bosonic part of the quadratic action is given by $\bar{\mathcal{X}} \mathcal{O}_B \mathcal{X}$, where $\mathcal{O}_B$ is the 8×8 operator

$$\mathcal{O}_B = \begin{pmatrix} \mathcal{O}_X & \alpha\,\mathcal{O}_X & \frac{q}{2r} - \sigma & 2iD_{++} & -\alpha\,2iD_{--} & -\alpha\left(\frac{q}{2r} + \bar{\sigma}\right) & 0 & 0 \\ \alpha\,\mathcal{O}_X & \mathcal{O}_X & \alpha\left(\frac{q}{2r} - \sigma\right) & \alpha\,2iD_{++} & -2iD_{--} & -\left(\frac{q}{2r} + \bar{\sigma}\right) & 0 & 0 \\ \alpha\left(-\frac{q}{2r} + \sigma\right) & -\frac{q}{2r} + \sigma & 0 & 0 & 0 & -1 & 0 & 0 \\ \alpha\,2iD_{--} & 2iD_{--} & 0 & \alpha & 0 & 0 & 0 & 0 \\ -2iD_{++} & -\alpha\,2iD_{++} & 0 & 0 & \alpha & 0 & 0 & 0 \\ \frac{q}{2r} + \bar{\sigma} & \alpha\left(\frac{q}{2r} + \bar{\sigma}\right) & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 \end{pmatrix} , \tag{2.4.31}$$

where

$$\mathcal{O}_X = -\Box + \sigma_1^2 + \sigma_2^2 + i\frac{(q-1)\sigma_1}{r} + \frac{q(2-q)}{4r^2} \,.$$

All the terms involving σ_1, σ_2 in this matrix (and all matrices below) are to be understood as $\rho(\sigma_1), \rho(\sigma_2)$, but we omit this to avoid cluttering. We will reinstate this in the expressions for the determinants. An analysis of the eigenvalues of (2.4.31) contains different cases, depending on the angular momentum j on S^2. Assuming $\alpha \neq 0$, and putting all cases together, the determinant in the bosonic sector is given by:

$$\begin{aligned}\text{Det}\,\mathcal{O}_B = \prod_{\rho\in\mathcal{R}} \frac{\alpha^{|\rho(\mathfrak{m})|-1}}{\alpha^{|\rho(\mathfrak{m})|+1}} \prod_{j=\frac{|\rho(\mathfrak{m})|}{2}}^{\infty} &\left[j^2 + (\alpha^2-1)\Big(\frac{\rho(\mathfrak{m})}{2}\Big)^2 - \alpha^2\Big(\frac{q}{2} - ir\rho(\sigma_1)\Big)^2 \right]^{2j+1} \times \\ &\left[(j+1)^2 + (\alpha^2-1)\Big(\frac{\rho(\mathfrak{m})}{2}\Big)^2 - \alpha^2\Big(\frac{q}{2} - ir\rho(\sigma_1)\Big)^2 \right]^{2j+1} \Big(\frac{(\alpha^2-1)^2}{r^4}\Big)^{2j+1} \,. \end{aligned} \tag{2.4.32}$$

Now we turn to the fermionic determinant. In the basis

$$\psi = \Big(\psi^{L+}, \psi^{L-}, \psi^{R+}, \psi^{R-}, \bar{\eta}^{L+}, \bar{\eta}^{R-}, \bar{\chi}^{L+}, \bar{\chi}^{R-}\Big)^T \,,$$

the fermionic quadratic action is given by $\bar{\psi}\mathcal{O}_F\psi$ where $\mathcal{O}_F$ reads

$$\begin{pmatrix} -\big(\frac{q}{2r}+\bar{\sigma}\big) & 2iD_{--} & -\alpha\big(\frac{q}{2r}+\bar{\sigma}\big) & 2i\alpha D_{--} & -1 & 0 & 0 & 0 \\ -2iD_{++} & \frac{q}{2r}-\sigma & -2i\alpha D_{++} & \alpha\left(\frac{q}{2r}-\sigma\right) & 0 & \alpha & 0 & 0 \\ -\alpha\big(\frac{q}{2r}+\bar{\sigma}\big) & 2i\alpha D_{--} & -\big(\frac{q}{2r}+\bar{\sigma}\big) & 2iD_{--} & -\alpha & 0 & 0 & 0 \\ -2i\alpha D_{++} & \alpha\big(\frac{q}{2r}-\sigma\big) & -2iD_{++} & \frac{q}{2r}-\sigma & 0 & 1 & 0 & 0 \\ -\alpha & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha\big(\frac{q}{2r}-\sigma\big) & -2iD_{--} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2iD_{++} & \alpha\big(\frac{q}{2r}+\bar{\sigma}\big) \end{pmatrix} . \tag{2.4.33}$$

The analysis of the eigenvalues of this operator gives the determinant

$$\begin{aligned}\text{Det}\,\mathcal{O}_F = \prod_{\rho\in\mathcal{R}} &\left(-\frac{(\alpha^2-1)\alpha^2}{r^2}\right)^{|\rho(\mathfrak{m})|} \left[\Big(\frac{\rho(\mathfrak{m})}{2}\Big)^2 - \Big(\frac{q}{2} - ir\rho(\sigma)\Big)^2\right]^{|\rho(\mathfrak{m})|} \times \\ \prod_{j=\frac{|\rho(\mathfrak{m})|}{2}+\frac{1}{2}}^{\infty} &\left(\frac{\alpha^2-1}{r^2}\right)^{4j+2} \left[\Big(j+\frac{1}{2}\Big)^2 + (\alpha^2-1)\Big(\frac{\rho(\mathfrak{m})}{2}\Big)^2 - \alpha^2\Big(\frac{q}{2} - ir\rho(\sigma_1)\Big)^2\right]^{4j+2} \,. \end{aligned} \tag{2.4.34}$$

Putting the bosonic and fermionic determinants together leads to many cancellations and the final result is[7]

$$Z_{LR} = \frac{\text{Det}\,\mathcal{O}_F}{\text{Det}\,\mathcal{O}_B} = \prod_{\rho\in\mathcal{R}} \frac{(-1)^{|\rho(\mathfrak{m})|}}{\frac{\rho(\mathfrak{m})^2}{4} - \big(\frac{q}{2} - ir\rho(\sigma_1)\big)^2} \,. \tag{2.4.35}$$

[7] Here we have ignored overall factors of r^2 and $(\alpha^2-1)^{-1}$.

This expression has simple poles at $\rho(\sigma_1) = -\frac{i}{2r}\left(q \pm \rho(\mathfrak{m})\right)$ for $\rho(\mathfrak{m}) \neq 0$, and a double pole at $\rho(\sigma_1) = -\frac{iq}{2r}$ for $\rho(\mathfrak{m}) = 0$. Using properties of the Γ-function, (2.4.35) can be written as

$$Z_{LR} = \prod_{\rho \in \mathcal{R}} \frac{\Gamma\left(\frac{q}{2} - ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)} \cdot \frac{\Gamma\left(-\frac{q}{2} + ir\rho(\sigma_1) + \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1 + \frac{q}{2} - ir\rho(\sigma_1) + \frac{\rho(\mathfrak{m})}{2}\right)} . \tag{2.4.36}$$

In fact (2.4.36) coincides with the one-loop determinant for two chiral fields in conjugate representations of the gauge group, opposite R-charges, and no twisted mass parameters turned on. Each Γ-function in the numerator has an infinite tower of poles, most of which though cancel against the poles of the denominator (such a cancellation does not happen for a pair of chiral multiplets with generic twisted masses and R-charges).

Note that the dependence on α has become an overall normalization, which we have omitted. This should be expected from the fact that this parameter appears in a $\mathcal{Q}_A$-exact term.

The full partition function requires the integration over σ_1 as well as a sum over the flux sectors $\mathfrak{m}$:

$$Z_{LR}^{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int \frac{d\sigma_1}{2\pi} e^{-4\pi i \xi \operatorname{Tr} \sigma_1 - i\theta \operatorname{Tr} \mathfrak{m}} Z_{\text{gauge}} Z_{LR} , \tag{2.4.37}$$

where Z_{LR} is given by (2.4.36), Z_{gauge} is the contribution from the vector multiplet given in [65, 62], and $|\mathcal{W}|$ is the order of the Weyl group. The integral is over some contour in the complex plane which needs to be specified, as we are going to discuss.

For concreteness, consider N_F pairs of semichiral fields coupled to a $U(1)$ gauge field, each pair having charges $(1, -1)$ and R-charges $(q, -q)$. From now on we set $r = 1$ to avoid cluttering the formulas. We can simply take the result for each pair and get

$$Z_{LR}^{S^2}(\xi, \theta) = \sum_m \int \frac{d\sigma_1}{2\pi} e^{-4\pi i \xi \sigma_1 - i\theta m} \left(\frac{\Gamma\left(\frac{q}{2} - i\sigma_1 - \frac{m}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + i\sigma_1 - \frac{m}{2}\right)} \cdot \frac{\Gamma\left(-\frac{q}{2} + i\sigma_1 + \frac{m}{2}\right)}{\Gamma\left(1 + \frac{q}{2} - i\sigma_1 + \frac{m}{2}\right)} \right)^{N_F} .$$

2.5 T^2 Localization

The elliptic genus can be computed both using the Hamiltonian formalism [66] and the path integral formalism [67, 68, 69]. In this section, we will compute the elliptic genus of the GLSM with semichiral superfields using both methods.

2.5.1 Hamiltonian formalism

The elliptic genus is defined in the Hamiltonian formalism as a refined Witten index,

$$Z = \mathrm{Tr}_{\mathrm{RR}}(-1)^F q^{H_L} \bar{q}^{H_R} y^J \prod_a x_a^{K_a} \,, \tag{2.5.1}$$

where the trace is taken in the RR sector, in which fermions have periodic boundary conditions, and F is the fermion number. In Euclidean signature, $H_L = \frac{1}{2}(H + iP)$ and $H_R = \frac{1}{2}(H - iP)$ are the left- and the right-moving Hamiltonians. J and K_a are the R-symmetry and the a-th flavor symmetry generators, respectively. It is standard to also define

$$q \equiv e^{2\pi i \tau} \,, \quad x_a \equiv e^{2\pi i u_a} \,, \quad y \equiv e^{2\pi i z} \,. \tag{2.5.2}$$

If $u_a = z = 0$ the elliptic genus reduces to the Witten index, and computes the Euler characteristic of the target space if there is a well-defined geometric description.

The contributions from different multiplets can be computed independently, and we will only consider the unexplored contribution from the semichiral multiplet. As we have seen in Appendix G, the physical component fields of the semichiral superfield $\mathbb{X}$ are two complex scalars X_L and X_R, and spinors $\psi'_\pm$, χ_-^L and χ_+^R. All fields have the same flavor symmetry charge Q. The R-charges of $(X_L, X_R, \psi'_+, \psi'_- \chi_-^L, \chi_+^R)$ are $(\frac{R}{2}, \frac{R}{2}, \frac{R}{2} - 1, \frac{R}{2}, \frac{R}{2}, \frac{R}{2} + 1)$.

Let us consider the fermionic zero modes first. We denote the zero modes of ψ'_+ and $\bar{\psi}'_+$ as $\psi'_{+,0}$ and $\bar{\psi}'_{+,0}$, respectively. They satisfy

$$\{\psi'_{+,0} \,, \bar{\psi}'_{+,0}\} = 1 \,, \tag{2.5.3}$$

which can be represented in the space spanned by $|\downarrow\rangle$ and $|\uparrow\rangle$ with

$$\psi'_{+,0} \,|\downarrow\rangle = |\uparrow\rangle \,, \quad \bar{\psi}'_{+,0} \,|\uparrow\rangle = |\downarrow\rangle \,. \tag{2.5.4}$$

One of $|\downarrow\rangle$ and $|\uparrow\rangle$ can be chosen to be bosonic, while the other is fermionic. Under the $U(1)_R$ the zero modes transform as

$$\psi'_{+,0} \to e^{-i\pi z(\frac{R}{2}-1)} \psi'_{-,0} \,, \quad \bar{\psi}'_{+,0} \to e^{i\pi z(\frac{R}{2}-1)} \bar{\psi}'_{-,0} \,, \tag{2.5.5}$$

while under $U(1)_f$ they transform as

$$\psi'_{+,0} \to e^{-i\pi u Q} \psi'_{-,0} \,, \quad \bar{\psi}'_{+,0} \to e^{i\pi u Q} \bar{\psi}'_{-,0} \,, \tag{2.5.6}$$

These two states contribute a factor

$$e^{-i\pi z(\frac{R}{2}-1)} \, e^{-i\pi u Q} - e^{i\pi z(\frac{R}{2}-1)} \, e^{i\pi u Q} \tag{2.5.7}$$

to the elliptic genus. Similarly, the contributions of the other zero modes are

$$\begin{aligned}
(\psi'_{-,0}, \bar{\psi}'_{-,0}) : &\quad e^{i\pi z\frac{R}{2}}\, e^{i\pi uQ} - e^{-i\pi z\frac{R}{2}}\, e^{-i\pi uQ}\,, \\
(\chi^L_{-,0}, \bar{\chi}^L_{-,0}) : &\quad e^{i\pi z\frac{R}{2}}\, e^{i\pi uQ} - e^{-i\pi z\frac{R}{2}}\, e^{-i\pi uQ}\,, \\
(\chi^R_{+,0}, \bar{\chi}^R_{+,0}) : &\quad e^{-i\pi z(\frac{R}{2}+1)}\, e^{-i\pi uQ} - e^{i\pi z(\frac{R}{2}+1)}\, e^{i\pi uQ}\,.
\end{aligned} \tag{2.5.8}$$

The contributions from the bosonic zero modes are relatively simple. They are

$$\frac{1}{\left[\left(1-e^{i\pi z\frac{R}{2}}\, e^{i\pi uQ}\right)\cdot\left(1-e^{-i\pi z\frac{R}{2}}\, e^{-i\pi uQ}\right)\right]^2}\,. \tag{2.5.9}$$

Bringing all the factors together, we obtain the zero mode part of the elliptic genus:

$$\frac{\left[1-e^{i\pi(R-2)z}\, e^{2i\pi uQ}\right]\cdot\left[1-e^{i\pi(R+2)z}\, e^{2i\pi uQ}\right]}{\left(1-e^{i\pi Rz}\, e^{2i\pi uQ}\right)^2} = \frac{\left(1-y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-y^{\frac{R}{2}+1}x^Q\right)}{(1-y^{\frac{R}{2}}x^Q)^2}\,. \tag{2.5.10}$$

We then consider the nonzero modes. The contribution from the fermionic sector $(\psi'_\pm, \chi^L_-, \chi^R_+)$ is

$$\begin{aligned}
\prod_{n=1}^{\infty} &\left(1-q^n e^{2i\pi z(\frac{R}{2}-1)}\, e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z(\frac{R}{2}-1)}\, e^{-2i\pi uQ}\right) \\
&\cdot\left(1-\bar{q}^n e^{2i\pi z\frac{R}{2}}\, e^{2i\pi uQ}\right)\cdot\left(1-\bar{q}^n e^{-2i\pi z\frac{R}{2}}\, e^{-2i\pi uQ}\right) \\
&\cdot\left(1-q^n e^{2i\pi z(\frac{R}{2}+1)}\, e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z(\frac{R}{2}+1)}\, e^{-2i\pi uQ}\right) \\
&\cdot\left(1-\bar{q}^n e^{2i\pi z\frac{R}{2}}\, e^{2i\pi uQ}\right)\cdot\left(1-\bar{q}^n e^{-2i\pi z\frac{R}{2}}\, e^{-2i\pi uQ}\right)\,,
\end{aligned} \tag{2.5.11}$$

while the contribution from the bosonic sector (X_L, X_R) is

$$\begin{aligned}
\prod_{n=1}^{\infty} &\frac{1}{\left(1-q^n e^{2i\pi z\frac{R}{2}}\, e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z\frac{R}{2}}\, e^{-2i\pi uQ}\right)} \\
&\cdot\frac{1}{\left(1-\bar{q}^n e^{2i\pi z\frac{R}{2}}\, e^{2i\pi uQ}\right)\cdot\left(1-\bar{q}^n e^{-2i\pi z\frac{R}{2}}\, e^{-2i\pi uQ}\right)} \\
&\cdot\frac{1}{\left(1-q^n e^{2i\pi z\frac{R}{2}}\, e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z\frac{R}{2}}\, e^{-2i\pi uQ}\right)} \\
&\cdot\frac{1}{\left(1-\bar{q}^n e^{2i\pi z\frac{R}{2}}\, e^{2i\pi uQ}\right)\cdot\left(1-\bar{q}^n e^{-2i\pi z\frac{R}{2}}\, e^{-2i\pi uQ}\right)}\,.
\end{aligned} \tag{2.5.12}$$

Hence, the nonzero modes contribute to the elliptic genus a factor

$$\prod_{n=1}^{\infty}\frac{\left(1-q^n e^{2i\pi z(\frac{R}{2}-1)}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z(\frac{R}{2}-1)}\,e^{-2i\pi uQ}\right)}{\left(1-q^n e^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}$$
$$\cdot\frac{\left(1-q^n e^{2i\pi z(\frac{R}{2}+1)}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z(\frac{R}{2}+1)}\,e^{-2i\pi uQ}\right)}{\left(1-q^n e^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}$$
$$=\prod_{n=1}^{\infty}\frac{\left(1-q^n y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}-1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}\cdot\frac{\left(1-q^n y^{\frac{R}{2}+1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}+1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}\,. \tag{2.5.13}$$

Taking both the zero modes (2.5.10) and the nonzero modes (2.5.13) into account, we obtain

$$\frac{\left(1-y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-y^{\frac{R}{2}+1}x^Q\right)}{(1-y^{\frac{R}{2}}x^Q)^2}\cdot\prod_{n=1}^{\infty}\frac{\left(1-q^n y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}-1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}$$
$$\cdot\prod_{n=1}^{\infty}\frac{\left(1-q^n y^{\frac{R}{2}+1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}+1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}\,. \tag{2.5.14}$$

Using the formula

$$\vartheta_1(\tau,z)=-iy^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-yq^{n+1})(1-y^{-1}q^n)\,, \tag{2.5.15}$$

where

$$q\equiv e^{2\pi i\tau}\,,\quad y\equiv e^{2\pi iz}\,, \tag{2.5.16}$$

we can rewrite (2.5.14) as

$$Z^{1-loop}(\tau,\,u,\,z)=\frac{\vartheta_1(\tau,\,z\left(\frac{R}{2}+1\right)+uQ)}{\vartheta_1(\tau,\,z\frac{R}{2}+uQ)}\cdot\frac{\vartheta_1(\tau,\,z\left(\frac{R}{2}-1\right)+uQ)}{\vartheta_1(q,\,z\frac{R}{2}+uQ)}\,. \tag{2.5.17}$$

Comparing to the contribution of a chiral superfield [67, 68], we see that the 1-loop determinant of the elliptic genus for one pair of semichiral superfields is equal to the product of the 1-loop determinants for two chiral superfields with the opposite R-charge and the opposite flavor charge, which is consistent with the result of the semichiral gauged linear sigma model localized on the two-sphere [70].

2.5.2 Path integral formalism

The elliptic genus can be equivalently described in the path integral formalism as a twisted partition function on the torus, we may apply the technique of localization to compute it.

Recall that the Witten index is expressed in the path integral formalism as the partition function of the theory on a torus, with periodic boundary conditions for both bosons and fermions. To deform the Witten index into the elliptic genus, we should specify twisted boundary conditions for all fields. Equivalently, we can keep the periodic boundary conditions and introduce background gauge fields A^R and $A^{f,a}$ for the R-symmetry and the a-th flavor-symmetry, respectively. They are related to the parameters in the definition of elliptic genus via

$$z \equiv \oint A_1^R \, dx_1 - \tau \oint A_2^R \, dx_2 \,, \quad u_a \equiv \oint A_1^{f,a} dx_1 - \tau \oint A_2^{f,a} dx_2 \,. \tag{2.5.18}$$

Following the general principle of localization, if we regard the background gauge fields as parameters in the theory, we only need the free part of the Lagrangian in order to compute the elliptic genus. The free part of the Lagrangian in the Euclidean signature is

$$\begin{aligned}\mathcal{L}^{\text{free}} = &D_\mu \bar{X}^I D^\mu X_I + i\bar{X}^I D X_I + \bar{F}^I F_I - \bar{M}^{++,I} M_{++,I} - \bar{M}^{--,I} M_{--,I} - \bar{M}^{+-,I} M_{-+,I} - \bar{M}^{-+,I} M_{+-,I} \\ &- \bar{M}^{++,I}(-2iD_+ X_I) - \bar{M}^{--,I} 2iD_- X_I + \bar{X}^I(-2iD_+ M_I^{++}) + \bar{X}^I(2iD_- M_I^{--}) \\ &- i\bar{\psi}^I \gamma^\mu D_\mu \psi_I - \bar{\eta}^I \psi_I - \bar{\psi}^I \eta_I + i\bar{\chi}^I \gamma^\mu D_\mu \chi_I \,,\end{aligned} \tag{2.5.19}$$

where the covariant derivative is defined as

$$D_\mu \equiv \partial_\mu - \hat{Q} u_\mu - \hat{R} z_\mu \,, \tag{2.5.20}$$

and the operators $\hat{Q}$ and $\hat{R}$ acting on different fields give their corresponding $U(1)_f$ and $U(1)_R$ charges as follows:

	X	ψ_+	ψ_-	F	χ_+	χ_-	M_{++}	M_{--}	M_{+-}	M_{-+}	η_+	η_-
$\hat{Q}$	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q
$\hat{R}$	$\frac{R}{2}$	$\frac{R}{2}-1$	$\frac{R}{2}$	$\frac{R}{2}-1$	$\frac{R}{2}+1$	$\frac{R}{2}$	$\frac{R}{2}$	$\frac{R}{2}$	$\frac{R}{2}+1$	$\frac{R}{2}-1$	$\frac{R}{2}$	$\frac{R}{2}-1$

Table 2.2: Charge assignments of the components of the semichiral multiplets

The BPS equations are obtained by setting the SUSY transformations of fermions to zero. The solutions to the BPS equtions provide the background that can perserve certain amount of supersymmetry. In this case, the BPS equations have only trivial solutions, i.e., all the fields in the semichiral multiplets are vanishing.

We adopt the metric on the torus

$$ds^2 = g_{ij}\, dx^i\, dx^j \,, \tag{2.5.21}$$

where

$$g_{ij} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \,, \tag{2.5.22}$$

and $\tau = \tau_1 + i\tau_2$ is the complex structure, and we expand all the fields in the modes

$$e^{2\pi i(nx_1 - mx_2)} \,,$$

where $n,\, m \in \mathbb{Z}$. Then we can integrate out the auxiliary fields, and calculate the 1-loop determinant of the free part of the Lagrangian on the torus. The result is

$$Z^{1-loop} = \prod_{m,\, n\in\mathbb{Z}} \frac{\big(m + n\tau - Qu - (\frac{R}{2}+1)z\big)\cdot\big(m + n\tau - Qu - (\frac{R}{2}-1)z\big)}{\big(m + n\tau - (Qu + \frac{R}{2}z)\big)\cdot\big(m + n\tau - (Qu + \frac{R}{2}z)\big)} \,. \tag{2.5.23}$$

After regularization, this expression can be written in terms of theta functions:

$$Z^{1-loop}(\tau,\, u,\, z) = \frac{\vartheta_1(\tau,\, z\left(\frac{R}{2}+1\right) + uQ)}{\vartheta_1(\tau,\, z\frac{R}{2} + uQ)} \cdot \frac{\vartheta_1(\tau,\, z\left(\frac{R}{2}-1\right) + uQ)}{\vartheta_1(q,\, z\frac{R}{2} + uQ)} \,. \tag{2.5.24}$$

The elliptic genus is

$$Z = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \text{JK-Res}_{u_*}(Q(u_*), \eta)\, Z_{1-loop}(u) \,, \tag{2.5.25}$$

where we obtain the Jeffrey-Kirwan residue (see also Appendix I), following the discussions in Refs. [69, 71].

2.5.3 Eguchi-Hanson space

Eguchi-Hanson space is the simplest example of the ALE spaces, and can be constructed via hyperkähler quotient in terms of semichiral superfields [60]:

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2e^2}\int d^4\theta(\bar{\tilde{\mathbb{F}}}\tilde{\mathbb{F}} - \bar{\mathbb{F}}\mathbb{F}) + \left(i\int d^2\theta\, \Phi\, \mathbb{F} + c.c.\right) + \left(i\int d^2\tilde{\theta}\, t\, \tilde{\mathbb{F}} + c.c.\right) \\ &- \int d^4\theta \left[\bar{\mathbb{X}}^L_i\, e^{Q_i V_L}\, \mathbb{X}^L_i + \bar{\mathbb{X}}^R_i\, e^{Q_i V_R}\, \mathbb{X}^R_i + \alpha(\bar{\mathbb{X}}^L_i\, e^{iQ_i \bar{\tilde{\mathbb{V}}}}\, \mathbb{X}^R_i + \bar{\mathbb{X}}^R_i\, e^{-iQ_i\tilde{\mathbb{V}}}\, \mathbb{X}^L_i)\right] \,, \end{aligned} \tag{2.5.26}$$

where $i = 1, 2$, and for simplicity we set $t = 0$.

The model (2.5.26) has $\mathcal{N} = (4, 4)$ supersymmetry, and the R-symmetry is $SO(4) \times SU(2) \cong SU(2)_1 \times SU(2)_2 \times SU(2)_3$ [71]. Hence, we can assign the R-charges (Q_1, Q_2, Q_R), where Q_R corresponds to the $U(1)_R$ charge that we discussed in the previous section. Similar to Ref. [71], we choose the supercharges $\mathcal{Q}_-$ and $\mathcal{Q}_+$ to be in the representation $(2,2,1)$ and $(2,1,2)$ respectively under the R-symmetry group. Moreover, the flavor symmetry Q_f now becomes $SU(2)_f$. In this case, the fields appearing in the model (2.5.26), which are relevant for the elliptic genus, have the following charge assignments:

	X_1^L	X_1^R	$\psi_{1+}^{(2)}$	$\psi_{1-}^{(2)}$	χ_{1+}^R	χ_{1-}^L	X_2^L	X_2^R	$\psi_{2+}^{(2)}$	$\psi_{2-}^{(2)}$	χ_{2+}^R	χ_{2-}^L
$Q_1 - Q_2$	-1	-1	0	-1	0	-1	-1	-1	0	-1	0	-1
Q_R	0	0	-1	0	1	0	0	0	-1	0	1	0
Q_f	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1

Table 2.3: Charges of the components of the semichiral multiplets under the R-symmetry

The components of the chiral and the twisted chiral field strength, $\mathbb{F}$ and $\tilde{\mathbb{F}}$, have the following charge assignments:

	$\tilde{\phi}$	$\tilde{\psi}_+$	$\tilde{\psi}_-$	σ	$\bar{\lambda}_+$	λ_-	A_μ
$Q_1 - Q_2$	1	2	1	-1	0	-1	0
Q_R	1	0	1	1	0	1	0
Q_f	0	0	0	0	0	0	0

Table 2.4: Charges of the components of the field strength superfields under the R-symmetry

As we discussed before, the constrained semichiral vector multiplet and the unconstrained semichiral vector multiplet differ by a F-term, which does not show up in the result of localization, hence we can make use of the 1-loop determinant from the previous section. Then for the GLSM given by Eq. (2.5.26), the 1-loop determinant is

$$Z^{EH}_{1-loop} = Z_{\tilde{\mathbb{F}},\mathbb{F}} \cdot Z_1^{L,R} \cdot Z_2^{L,R}\,, \tag{2.5.27}$$

where

$$\begin{aligned}
Z_{\tilde{\mathbb{F}},\mathbb{F}} &= \frac{i\eta(q)^3}{\vartheta_1(\tau,\xi_2 - z)} \cdot \frac{\vartheta_1(\tau, 2\xi_2)}{\vartheta_1(\tau,\xi_2 + z)}\,,\\
Z_1^{L,R} &= \frac{\vartheta_1(\tau, u+\xi_1 - z)}{\vartheta_1(\tau, u+\xi_1-\xi_2)} \cdot \frac{\vartheta_1(\tau, u+\xi_1+z)}{\vartheta_1(\tau, u+\xi_1+\xi_2)}\,,\\
Z_2^{L,R} &= \frac{\vartheta_1(\tau, u-\xi_1 - z)}{\vartheta_1(\tau, u-\xi_1-\xi_2)} \cdot \frac{\vartheta_1(\tau, u-\xi_1+z)}{\vartheta_1(\tau, u-\xi_1+\xi_2)}\,.
\end{aligned} \tag{2.5.28}$$

Then the elliptic genus is given by

$$Z^{EH}(\tau; z, \xi) = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \text{JK-Res}_{u_*}(Q(u_*), \eta)\, Z_{1-loop}(u)\,, \tag{2.5.29}$$

where "JK-Res" denotes the Jeffrey-Kirwan residue, which is discussed in detail in Refs. [69, 71] and also briefly reviewed in Appendix I. In practice, the Jeffrey-Kirwan residue can be calculated as follows:

$$Z = -\sum_{u_j \in \mathfrak{M}^+_{\text{sing}}} \oint_{u=u_j} du\, Z_{1-loop}\,. \tag{2.5.30}$$

The poles are at

$$Q_i u + \frac{R_i}{2} z + P_i(\xi) = 0 \quad (\text{mod } \mathbb{Z} + \tau\mathbb{Z})\,, \tag{2.5.31}$$

where

$$R_i \geq 0\,, \tag{2.5.32}$$

and the poles with $Q_i > 0$ and $Q_i < 0$ are grouped in to $\mathfrak{M}^+_{\text{sing}}$ and $\mathfrak{M}^-_{\text{sing}}$ respectively. In the Eguchi-Hanson case, for instance for the phase where the intersection of $H_X = \{u + \xi_1 - \xi_2 = 0\}$ and $H_Y = \{u - \xi_1 - \xi_2 = 0\}$ contributes,

$$\mathfrak{M}^+_{\text{sing}} = \{-\xi_1 + \xi_2,\, \xi_1 + \xi_2\}\,. \tag{2.5.33}$$

Hence, the elliptic genus equals

$$Z^{EH}(\tau; z, \xi) = \frac{\vartheta_1(\tau, -2\xi_1 + \xi_2 - z) \cdot \vartheta_1(\tau, 2\xi_1 - \xi_2 - z)}{\vartheta_1(\tau, -2\xi_1) \cdot \vartheta(\tau, 2\xi_1 - 2\xi_2)} + \frac{\vartheta_1(\tau, 2\xi_1 + \xi_2 - z) \cdot \vartheta_1(\tau, -2\xi_1 - \xi_2 - z)}{\vartheta_1(\tau, 2\xi_1) \cdot \vartheta_1(\tau, -2\xi_1 - 2\xi_2)}\,, \tag{2.5.34}$$

which is the same as the result obtained in Ref. [71].

From our construction of the ALE space using semichiral GLSM, it is also clear that the elliptic genus for the ALE space coincides with the one for the six-dimensional conifold space. The reason is following. As we discussed before, to obtain an ALE space through a semichiral GLSM we need the semichiral vector multiplet, which has three real components, while to construct a conifold (or resolved conifold when the FI parameter $t \neq 0$) one should use the constrained semichiral vector multiplet, which has only one real component. However, these two vector multiplets differ only by a superpotential term, which does not affect the result of the localization. Hence, the result that we obtained using localization give us the elliptic genus both for the ALE space and for the conifold.[8]

[8] We would like to thank P. Marcos Crichigno for discussing this.

2.5.4 Taub-NUT space

Taub-NUT space is an example of the ALF space, and can be constructed by semichiral GLSM as follows [60]:

$$
\begin{aligned}
\mathcal{L} = \int d^4\theta \Big[&- \frac{1}{2e^2}(\bar{\tilde{\mathbb{F}}}\tilde{\mathbb{F}} - \bar{\mathbb{F}}\mathbb{F}) + \bar{\mathbb{X}}_1^L\, e^{V_L}\, \mathbb{X}_1^L + \bar{\mathbb{X}}_1^R\, e^{V_R}\, \mathbb{X}_1^R + \alpha(\bar{\mathbb{X}}_1^L\, e^{i\bar{\tilde{\mathbb{V}}}}\, \mathbb{X}_1^R + \bar{\mathbb{X}}_1^R\, e^{-i\tilde{\mathbb{V}}}\, \mathbb{X}_1^L) \\
&+ \frac{1}{2}\left(\mathbb{X}_2^L + \bar{\mathbb{X}}_2^L + V_L\right)^2 + \frac{1}{2}\left(\mathbb{X}_2^R + \bar{\mathbb{X}}_2^R + V_R\right)^2 + \frac{\alpha}{2}\left(\mathbb{X}_2^L + \bar{\mathbb{X}}_2^R - i\tilde{\mathbb{V}}\right)^2 + \frac{\alpha}{2}\left(\mathbb{X}_2^R + \mathbb{X}_2^L + i\bar{\tilde{\mathbb{V}}}\right)^2 \Big] \\
&+ \left(\int d^2\theta\, \Phi\, \mathbb{F} + c.c.\right) - \left(\int d^2\tilde{\theta}\, t\, \tilde{\mathbb{F}} + c.c.\right), \qquad (2.5.35)
\end{aligned}
$$

where for simplicity we set $t = 0$.

Using the results from the previous section, and assigning the same R-symmetry and the flavor symmetry charges as in the Eguchi-Hanson case (2.5.27) (2.5.27), we can write down immediately the 1-loop contribution from the semichiral vector multiplet, $\tilde{\mathbb{F}}$ and $\mathbb{F}$, as well as the one from the semichiral multiplet, $\mathbb{X}_1^L$ and $\mathbb{X}_1^R$, of the model (2.5.35):

$$
Z_{\tilde{\mathbb{F}},\mathbb{F}} = \prod_{m,n\in\mathbb{Z}} \frac{n + \tau m - 2\xi_2}{(n + \tau m - \xi_2 + z)\cdot(n + \tau m - \xi_2 - z)} \cdot \prod_{(m,n)\neq(0,0)} (n + m\tau)\,, \qquad (2.5.36)
$$

$$
Z_1^{L,R} = \frac{\vartheta_1(\tau, u + \xi_1 - z)}{\vartheta_1(\tau, u + \xi_1 - \xi_2)} \cdot \frac{\vartheta_1(\tau, u + \xi_1 + z)}{\vartheta_1(\tau, u + \xi_1 + \xi_2)}\,. \qquad (2.5.37)
$$

However, to obtain the full 1-loop determinant, we still have to work out the part of the model from semichiral Stückelberg fields, and localize it to obtain its contribution to the 1-loop determinant. Let us start with the Lagrangian for the Stückelberg field in the superspace:

$$
\begin{aligned}
\mathcal{L}_{St} = \int d^4\theta \Big[&\frac{1}{2}\left(\mathbb{X}_2^L + \bar{\mathbb{X}}_2^L + V_L\right)^2 + \frac{1}{2}\left(\mathbb{X}_2^R + \bar{\mathbb{X}}_2^R + V_R\right)^2 \\
&+ \frac{\alpha}{2}\left(\mathbb{X}_2^L + \bar{\mathbb{X}}_2^R - i\tilde{\mathbb{V}}\right)^2 + \frac{\alpha}{2}\left(\mathbb{X}_2^R + \mathbb{X}_2^L + i\bar{\tilde{\mathbb{V}}}\right)^2 \Big]. \qquad (2.5.38)
\end{aligned}
$$

Expanding the Lagrangian into components and integrate out auxiliary fields (see Appendix H), we obtain

$$
\begin{aligned}
\mathcal{L}_{St} = &\frac{\alpha - 1}{\alpha}(\bar{r}_1\Box r_1 + \bar{\gamma}_1\Box\gamma_1) + \frac{\alpha + 1}{\alpha}(\bar{r}_2\Box r_2 + \bar{\gamma}_2\Box\gamma_2) \\
&+ \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}_+^2 D_-\psi_+^2 - \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}_-^2 D_+\psi_-^2 + \bar{\chi}_-^L 2iD_+\chi_-^L - \bar{\chi}_+^R 2iD_-\chi_+^R\,. \qquad (2.5.39)
\end{aligned}
$$

As discussed in Appendix H, among the real components $r_{1,2}$ and $\gamma_{1,2}$ only r_2 transforms under the gauge transformations. We can assign the following charges to the components of the Stückelberg field:

	r_1	r_2	γ_1	γ_2	ψ_+	ψ_-	χ_+	χ_-
$Q_1 - Q_2$	-2	0	-2	0	-1	-2	-1	0
Q_R	0	0	0	0	-1	0	1	0
Q_f	0	0	0	0	0	0	0	0

Table 2.5: Charges of the components of the Stückelberg field under the R-symmetry

Taking both the momentum and the winding modes into account, we obtain the contribution from the Stückelberg field to the 1-loop determinant

$$Z_{St} = \prod_{m,n\in\mathbb{Z}} \frac{(n+\tau m+\xi_2+z)\cdot(n+\tau m+\xi_2-z)}{n+\tau m+2\xi_2} \cdot \prod_{(m,n)\neq(0,0)} \frac{1}{n+m\tau} \cdot \sum_{v,w\in\mathbb{Z}} e^{-\frac{g^2\pi}{\tau_2}|u+v+\tau w|^2} . \tag{2.5.40}$$

Together with Eq. (2.5.36) and Eq. (2.5.37), we obtain the full 1-loop determinant of the elliptic genus for the Taub-NUT space

$$Z^{TN}_{1-loop} = \frac{\vartheta_1(\tau,u+\xi_1-z)}{\vartheta_1(\tau,u+\xi_1-\xi_2)} \cdot \frac{\vartheta_1(\tau,u+\xi_1+z)}{\vartheta_1(\tau,u+\xi_1+\xi_2)} \cdot \sum_{v,w\in\mathbb{Z}} e^{-\frac{g^2\pi}{\tau_2}|u+v+\tau w|^2} . \tag{2.5.41}$$

The elliptic genus for the Taub-NUT space is given by

$$Z^{TN} = g^2 \int_{E(\tau)} \frac{du\, d\bar{u}}{\tau_2} \frac{\vartheta_1(\tau,u+\xi_1-z)}{\vartheta_1(\tau,u+\xi_1-\xi_2)} \cdot \frac{\vartheta_1(\tau,u+\xi_1+z)}{\vartheta_1(\tau,u+\xi_1+\xi_2)} \cdot \sum_{v,w\in\mathbb{Z}} e^{-\frac{g^2\pi}{\tau_2}|u+v+\tau w|^2} , \tag{2.5.42}$$

where $E(\tau) = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. This result is the same as the one in Ref. [71] obtained from the chiral GLSM.

Similar to the ALE space, the elliptic genus for the ALF space should coincide with the one for some six-dimensional space. In semichiral GLSM language, one is obtained using the unconstrained semichiral vector multiplet, while the other is constructed using the constrained semichiral vector multiplet. However, as far as we know, this kind of six-dimensional space is not well studied in the literature as the conifold. We would like to investigate it in more detail in the future.

Chapter 3

Entanglement Entropy

Entanglement entropy plays an increasingly important role in different branches of physics. Proposed as a useful measure of the quantum entanglement of a system with its environment, entanglement entropy now features in discussions of black hole physics [72, 73], renormalization group flow [74, 75], and quantum phase transitions [76, 77]. A closely related set of quantities are the Rényi entropies and supersymmetric Rényi entropies. In this chapter, we review these concepts and their relations with the partition function on a sphere. The thermal corrections to Rényi entropies for conformal field theory (CFT) will also be discussed. This chapter is mainly based on my paper with Chris Herzog [78].

3.1 Introduction to Entanglement Entropy

We adopt the conventional definition of entanglement and Rényi entropy in this thesis. Suppose the space on which the theory is defined can be divided into a piece A and its complement $\bar{A} = B$, and correspondingly the Hilbert space factorizes into a tensor product. The density matrix over the whole Hilbert space is ρ; then the reduced density matrix is defined as

$$\rho_A \equiv \mathrm{tr}_B \rho \,. \tag{3.1.1}$$

The entanglement entropy is the von Neumann entropy of ρ_A,

$$S_E \equiv -\mathrm{tr}\rho_A \log \rho_A \,, \tag{3.1.2}$$

while the Rényi entropies are defined to be

$$S_n \equiv \frac{1}{1-n} \log \mathrm{tr}(\rho_A)^n \,. \tag{3.1.3}$$

Assuming a satisfactory analytic continuation of S_n can be obtained, the entanglement entropy can alternately be expressed as a limit of the Rényi entropies:

$$\lim_{n\to 1} S_n = S_E\,. \tag{3.1.4}$$

The Rényi entropy can also be calculated using the so-called "replica trick":

$$S_n = \frac{1}{1-n}\log\left(\frac{Z_n}{(Z_1)^n}\right)\,, \tag{3.1.5}$$

where Z_n is the Euclidean partition function on a n-covering space branched along A.

Both the entanglement entropy and the Rényi entropy can be used as the measure of the entanglement. Although the Rényi entropy can reproduce the entanglement entropy in some limit, it is not just a trick to calculate the entanglement entropy. One reason is that the parameter n in the definition (3.1.3) introduces a nontrivial deformation, which turns out in many cases to provide more information of the system.

Based on today's understanding of quantum measurement, one can only perform the measurement by applying a local unitary operator to the system. In that sense, neither the entanglement entropy nor the Rényi entropy can be measured directly, since they are defined in a nonlocal way. Nevertheless, one can still compute them and use them as an order parameter to distinguish different quantum phases.

Another related concept is the supersymmetric Rényi entropy, which was introduced in Ref. [79]. Let us take the theory on a 3-sphere for example. Similar to the "replica trick" for the Rényi entropy, the supersymmetric Rényi entropy can be defined as

$$S_q^{\mathrm{SUSY}} \equiv \frac{1}{1-q}\left[\log\left(\frac{Z_{\text{singular space}}(q)}{(Z_{S^3})^q}\right)\right]\,, \tag{3.1.6}$$

where Z_{S^3} is the partition function of a supersymmetric theory on S^3, while $Z_{\text{singular space}}(q)$ is the partition function on the q-branched sphere S_q^3 given by the metric

$$ds^2 = \ell^2(d\theta^2 + q^2\sin^2\theta\, d\tau^2 + \cos^2\theta\, d\phi^2) \tag{3.1.7}$$

with $\theta \in [0, \pi/2]$, $\tau \in [0, 2\pi q)$ and $\phi \in [0, 2\pi)$. For $q = 1$, the q-branched sphere returns to the round sphere.

In the last decade, people have calculated the entanglement entropy and the Rényi entropy for various cases. To calculate them, one can follow the definitions above and apply the replica trick for the Rényi entropy (see e.g. Ref. [80]), or use the heat kernel method (see e.g. Ref. [81]), or map the entanglement entropy to the modular Hamiltonian and further to the tt-component of the stress tensor (see e.g. Ref. [82]). Based on the AdS/CFT correspondence, Ref. [83] proposed a formula of calculating the entanglement entropy for a conformal field theory on $\mathbb{R}^{1,d}$ holographically:

$$S_A = \frac{\text{Area of } \gamma_A}{4\, G_N^{d+2}}\,, \tag{3.1.8}$$

where γ_A is the d-dimensional static minimal surface in AdS_{d+2} whose boundary is by ∂A, and G_N^{d+2} is the $(d+2)$-dimensional gravitational constant. This formula gives the right results for the leading terms in the entanglement entropy for many cases that have been checked, hence provides an alternative to calculate the entanglement entropy.

Independent of the concrete models, the entanglement entropy has the general form

$$S_A = \alpha|\partial A| - \gamma + \epsilon(|\partial A|^{-\beta})\,, \tag{3.1.9}$$

where α, β and γ are real numbers and $\beta > 0$, while ∂A stands for the boundary of the domain A. Usually one is interested in the dominant term $\alpha|\partial A|$, however, as pointed out by Kitaev and Preskill [84] as well as by Levin and Wen [85], the constant term $-\gamma$ is also of physical interest, which indicates the existence of the topological order in the system. Let us define

$$S_{\text{top}} = -\gamma\,. \tag{3.1.10}$$

If S_{top} is zero, there is no topological order. If S_{top} is nonzero, the system is necessarily topologically ordered.

Actually not just the term $\sim |\partial A|$ and the constant term in the entanglement entropy have physical meanings, it was pointed out in Refs. [86, 87] that for an even-dimensional spacetime the logarithmic term in the entanglement entropy is universial, whose coefficient is proportial to the A-type conformal anomaly. Let us briefly summarize the argument of Ref. [87] in the following. We know that in an even-dimensional spacetime the conformal anomaly of a conformal field theory can be written as

$$\langle T^\mu{}_\mu\rangle = \sum_n B_n I_n - 2(-)^{d/2} A E_d\,, \tag{3.1.11}$$

where E_d is the Euler density, and I_n are the independent Weyl invariants of weight $-d$. A d-dimensional Minkowski space can be mapped into a de Sitter space in the following way:

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2\, d\Omega_{d-2}^2 \\ &= \Omega^2 \left[-\cos^2\theta\, d\tau^2 + R^2(d\theta^2 + \sin^2\theta\, d\Omega_{d-2}^2)\right]\,, \end{aligned} \tag{3.1.12}$$

where the coordinate transformation is given by

$$\begin{aligned} t &= R\,\frac{\cos\theta \sinh(\tau/R)}{1+\cos\theta\cosh(\tau/R)}\,, \\ r &= R\,\frac{\sin\theta}{1+\cos\theta\cosh(\tau/R)}\,, \end{aligned} \tag{3.1.13}$$

and the conformal factor is

$$\Omega = \frac{1}{1+\cos\theta\cosh(\tau/R)}\,. \tag{3.1.14}$$

For a d-dimensional de Sitter space there is

$$\langle T^{\mu}{}_{\nu}\rangle = -2(-)^{d/2} A\,\frac{E_d}{d}\,\delta^{\mu}{}_{\nu}\,. \tag{3.1.15}$$

It is shown in Ref. [87] that the entanglement entropy for the sphere of radius R in flat space is equivalent to the thermodynamic entropy of the thermal state in de Sitter space. Hence, we consider the thermal density matrix in a de Sitter space

$$\rho = \frac{e^{\beta H_\tau}}{\mathrm{tr}\left(e^{-\beta H_\tau}\right)}\,, \tag{3.1.16}$$

which will lead to

$$S = -\mathrm{tr}(\rho\log\rho) = \beta\,\mathrm{tr}\rho\, H_\tau + \log\mathrm{tr}(e^{-\beta H_\tau}) = \beta E - W\,, \tag{3.1.17}$$

where E is the Killing energy, and $W = -\log Z$ with $Z = \mathrm{tr}\left(\exp[-2\pi R H_\tau]\right)$ is the free energy. After some analysis we find that the Killing energy is finite for the de Sitter space, hence only the free energy can contribute to the logarithmic part in the entanglement entropy. The free energy has the general expression

$$W = -\log Z = (\text{non-universal terms}) + a_{d+1}\log\delta + (\text{finite terms})\,, \tag{3.1.18}$$

where δ is a UV cut-off. Consider an infinitesimal transformation of the metric

$$g^{\mu\nu} \to (1-2\,\delta\lambda)g^{\mu\nu}\,. \tag{3.1.19}$$

Under the transformation we have

$$\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu}\rangle + (\text{divergent terms})\,, \tag{3.1.20}$$

and

$$\frac{\delta W}{\delta\lambda} = -\int d^d x\,\sqrt{g}\,\langle T^{\mu}{}_{\mu}\rangle + (\text{divergent terms})\,. \tag{3.1.21}$$

On the other hand, we can think of the transformation as a shift of the UV cut-off

$$\delta \to (1-\delta\lambda)\delta\,, \tag{3.1.22}$$

while leaving the metric unchanged, then we obtain

$$a_{d+1} = \int d^d x\,\sqrt{g}\,\langle T^{\mu}_{\mu}\rangle\,, \tag{3.1.23}$$

which is an integrated conformal anomaly. Since for the de Sitter space all the Weyl invariants vanish, i.e. $I_n = 0$, a_{d+1} has only contribution from the Eulder density, i.e. the A-type conformal anomaly. Finally, we obtain for a conformal field theory in even dimensions the universal contribution to the entanglement entropy

$$S_{\text{univ}} = (-1)^{\frac{d}{2}-1} 4A \log(R/\delta) \,. \tag{3.1.24}$$

To apply the theoretical results to a real system, it would be useful to know the thermal corrections to the entanglement entropy S_E and the Rényi entropy S_n. Ref. [88] found universal thermal corrections to both S_E and S_n for a CFT on $S^1 \times S^1$. The CFT is assumed to be gapped by having placed it on a spatial circle of circumference L, while the circumference of the second circle is the inverse temperature β. The results are

$$\delta S_n \equiv S_n(T) - S_n(0) = \frac{g}{1-n}\left[\frac{1}{n^{2\Delta-1}}\frac{\sin^{2\Delta}\left(\frac{\pi\ell}{L}\right)}{\sin^{2\Delta}\left(\frac{\pi\ell}{nL}\right)} - n\right] e^{-2\pi\beta\Delta/L} + o\left(e^{-2\pi\beta\Delta/L}\right)\,, \tag{3.1.25}$$

$$\delta S_E \equiv S_E(T) - S_E(0) = 2g\Delta\left[1 - \frac{\pi\ell}{L}\cot\left(\frac{\pi\ell}{L}\right)\right] e^{-2\pi\beta\Delta/L} + o\left(e^{-2\pi\beta\Delta/L}\right)\,, \tag{3.1.26}$$

where Δ is the smallest scaling dimension among the set of operators not equal to the identity and g is their degeneracy. The quantity ℓ is the length of the interval A.[1]

To generalize the results of Ref. [88] to higher dimensions, Ref. [82] considered thermal corrections to the entanglement entropy S_E on spheres. More precisely, a conformal field theory on $S^1 \times S^{d-1}$ is considered in Ref. [82], where the radius of S^1 and the one of S^{d-1} are $\beta/2\pi$ and R respectively. The region $A \subset S^{d-1}$ is chosen to be a cap with polar angle $\theta < \theta_0$. Then the thermal correction to the entanglement entropy S_E is

$$\delta S_E = g\Delta\, I_d(\theta_0)\, e^{-\beta\Delta/R} + o\left(e^{-\beta\Delta/R}\right)\,, \tag{3.1.27}$$

where

$$I_d(\theta_0) \equiv 2\pi\frac{\text{Vol}(S^{d-2})}{\text{Vol}(S^{d-1})}\int_0^{\theta_0} d\theta\, \frac{\cos\theta - \cos\theta_0}{\sin\theta_0}\sin^{d-2}\theta\,. \tag{3.1.28}$$

Ref. [82] noticed that this result is sensitive to boundary terms in the action. For a conformally coupled scalar, these boundary terms mean that the correction to entanglement entropy is given not by Eq. (3.1.27) but by Eq. (3.1.27) where $I_d(\theta_0)$ is replaced by $I_{d-2}(\theta_0)$.

A natural question is how to calculate the thermal corrections to the Rényi entropy in higher dimensions. We would like to address this issue in this chapter. Our main results are

[1] The fact that $S_E(T) - S_E(0) \sim e^{-2\pi\beta\Delta/L}$ is Boltzmann suppressed was conjectured more generally for gapped theories in Ref. [89]. That $S_E(T) - S_E(0)$ might have a universal form for 1+1 dimensional CFTs was suggested by the specific examples worked out in Refs. [90, 91, 92, 93]. See Ref. [94] for higher order temperature corrections when the first excited state is created by the stress tensor.

the following. The thermal correction to the Rényi entropy for a cap-like region with opening angle $2\theta_0$ on the sphere S^{d-1} in $\mathbb{R} \times S^{d-1}$ is given by

$$\delta S_n = \frac{n}{1-n} \left(\frac{\langle \psi(z)\, \psi(z') \rangle_n}{\langle \psi(z)\, \psi(z') \rangle_1} - 1 \right) e^{-\beta E_\psi} + o\left(e^{-\beta E_\psi} \right) , \tag{3.1.29}$$

where $\psi(z)$ is the operator that creates the first excited state of the CFT and E_ψ is its energy.[2] If we assume that $\psi(z)$ has scaling dimension Δ, then we know further that $E_\psi = \Delta/R$. The two point function $\langle \psi(z)\, \psi(z') \rangle_n$ is evaluated on an n-fold cover of $\mathbb{R} \times S^{d-1}$ that is branched over the cap of opening angle $2\theta_0$. Note the result (3.1.29) and the steps leading up to it are essentially identical to a calculation and intermediate result derived in Ref. [88] in 1+1 dimensions. The difference is that in 1+1 dimensions, the two-point function $\langle \psi(z)\, \psi(z') \rangle_n$ can be evaluated for a general CFT through an appropriate conformal transformation, while in higher dimensions we only know how to evaluate $\langle \psi(z)\, \psi(z') \rangle_n$ in some special cases.

In the case of free fields (and perhaps more generally) it makes sense to map this n-fold cover of the sphere to $\mathcal{C}_n \times \mathbb{R}^{d-2}$ where $\mathcal{C}_n$ is a two dimensional cone of opening angle $2\pi n$. In the case of a free theory, the two-point function $\langle \psi(y)\, \psi(y') \rangle_{1/m}$, where y, $y' \in \mathcal{C}_n \times \mathbb{R}^{d-2}$, can be evaluated by the method of images on a cone of opening angle $2\pi/m$ and then analytically continued to integer values of $1/m$. Ref. [95] made successful use of this trick to calculate a limit of the mutual information for conformally coupled scalars. We will use this same trick to look at thermal corrections to Rényi entropies for these scalars. Taking the $n \to 1$ limit, we find complete agreement with entanglement entropy corrections computed in Ref. [82]. (The method of images can also be used to study free fermions, but we leave such a calculation for future work.) We verify the Rényi entropy corrections numerically by putting the system on a lattice.

3.2 Thermal Corrections to Rényi Entropy

We start with the thermal density matrix:

$$\rho = \frac{|0\rangle\langle 0| + \sum_i |\psi_i\rangle\langle\psi_i|\, e^{-\beta E_\psi} + \cdots}{1 + g\, e^{-\beta E_\psi} + \cdots} , \tag{3.2.1}$$

where $|0\rangle$ stands for the ground state, while $|\psi_i\rangle$ $(i = 1, \cdots, g)$ denote the first excited states. For a conformal field theory on $\mathbb{R} \times S^{d-1}$,

$$E_\psi = \frac{\Delta}{R} , \tag{3.2.2}$$

[2] For simplicity, we have assumed that the first excited state is unique. For a degenerate first excited state, see the next section.

where Δ is the scaling dimension of the operators that create the states $|\psi_i\rangle$, and R is the radius of the sphere. From this expression one can calculate that

$$\begin{aligned}\operatorname{tr}(\rho_A)^n &= \left(\frac{1}{1+g\,e^{-\beta E_\psi}+\cdots}\right)^n \cdot \operatorname{tr}\left[\operatorname{tr}_B\left(|0\rangle\langle 0| + \sum_i |\psi_i\rangle\langle\psi_i| e^{-\beta E_\psi} + \cdots\right)\right]^n \\ &= \operatorname{tr}\left(\operatorname{tr}_B|0\rangle\langle 0|\right)^n \cdot \left[1 + \left(\frac{\operatorname{tr}\left[\operatorname{tr}_B \sum_i |\psi_i\rangle\langle\psi_i| \left(\operatorname{tr}_B|0\rangle\langle 0|\right)^{n-1}\right]}{\operatorname{tr}\left(\operatorname{tr}_B|0\rangle\langle 0|\right)^n} - g\right) n\, e^{-\beta E_\psi} + \cdots\right].\end{aligned} \tag{3.2.3}$$

Then the thermal correction to the Rényi entropy is

$$\begin{aligned}\delta S_n &\equiv S_n(T) - S_n(0) \\ &= \frac{n}{1-n}\sum_i \left(\frac{\operatorname{tr}\left[\operatorname{tr}_B|\psi_i\rangle\langle\psi_i| \left(\operatorname{tr}_B|0\rangle\langle 0|\right)^{n-1}\right]}{\operatorname{tr}\left(\operatorname{tr}_B|0\rangle\langle 0|\right)^n} - 1\right) e^{-\beta E_\psi} + o\left(e^{-\beta E_\psi}\right).\end{aligned} \tag{3.2.4}$$

Hence, the crucial step is to evaluate the expression

$$\frac{\operatorname{tr}\left[\operatorname{tr}_B|\psi_i\rangle\langle\psi_i| \left(\operatorname{tr}_B|0\rangle\langle 0|\right)^{n-1}\right]}{\operatorname{tr}\left(\operatorname{tr}_B|0\rangle\langle 0|\right)^n} = \frac{\langle\psi_i(z)\psi_i(z')\rangle_n}{\langle\psi_i(z)\psi_i(z')\rangle_1}, \tag{3.2.5}$$

which, using the operator-state correspondence, can be viewed as a two-point function on the n-fold covering of the space $\mathbb{R}\times S^{d-1}$. (Let z^μ be our coordinate system on $\mathbb{R}\times S^{d-1}$.) The n copies are glued sequentially together along A. Let τ be the time coordinate. To create the excited state, we insert the operator ψ_i in the far Euclidean past $\tau' = -i\infty$ of one of the copies of $\mathbb{R}\times S^{d-1}$. Similarly, $\langle\psi_i|$ is created by inserting ψ_i in the far future $\tau = i\infty$ of the same copy. The two-point function $\langle\psi_i(z)\psi_i(z')\rangle_1$ is needed in the denominator in order to insure that $\langle\psi_i|$ has the correct normalization relative to $|\psi_i\rangle$.

Our most general result is then

$$\delta S_n = \frac{n}{1-n}\sum_i \left(\frac{\langle\psi_i(z)\,\psi_i(z')\rangle_n}{\langle\psi_i(z)\,\psi_i(z')\rangle_1} - 1\right) e^{-\beta E_\psi} + o\left(e^{-\beta E_\psi}\right). \tag{3.2.6}$$

Following from the analytic continuation formula (3.1.4), the thermal correction to the entanglement entropy can be determined via

$$\delta S_E = \lim_{n\to 1} \delta S_n\,. \tag{3.2.7}$$

Evaluating the two-point function $\langle\psi_i(z)\psi_i(z')\rangle_n$ on an n-sheeted copy of $\mathbb{R}\times S^{d-1}$ is not simple for $n>1$. Using a trick of Ref. [95], we can evaluate $\langle\psi_i(z)\psi_i(z')\rangle_n$ for free CFTs.

The trick is to perform a conformal transformation that relates this two-point function to a two-point function on a certain conical space where the method of images can be employed. As interactions spoil the linearity of the theory and hence the principle of superposition, we expect this method will fail for interacting CFTs.

It is convenient to break the conformal transformation into two pieces. First, it is well known that $\mathbb{R}\times S^{d-1}$ is conformally related to Minkowski space (see the appendix of Ref. [96]):

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\Sigma^2 && (3.2.8) \\ &= \Omega^2(-d\tau^2 + d\theta^2 + \sin^2\theta d\Sigma^2) \ , && (3.2.9) \end{aligned}$$

where

$$\begin{aligned} t \pm r &= \tan\left(\frac{\tau \pm \theta}{2}\right) \ , && (3.2.10) \\ \Omega &= \frac{1}{2}\sec\left(\frac{\tau+\theta}{2}\right)\sec\left(\frac{\tau-\theta}{2}\right) \ , && (3.2.11) \end{aligned}$$

and $d\Sigma^2$ is a line element on a unit S^{d-2} sphere. Note that the surface $t=0$ gets mapped to $\tau=0$, and on this surface $r=\tan(\theta/2)$. Thus a cap on the sphere (at $\tau=0$) of opening angle 2θ is transformed into a ball inside $\mathbb{R}^{d-1}$ (at $t=0$) of radius $r=\tan(\theta/2)$. This coordinate transformation takes the operator insertion points $\tau=\pm i\infty$ in the far past and far future (with $\theta=0$) to $t=\pm i$ (and $r=0$).

Then we should employ the special conformal transformation

$$y^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2x^2} \ , \qquad (3.2.12)$$

$$ds^2 = dy^\mu dy^\nu \delta_{\mu\nu} = \frac{1}{(1-2b\cdot x+b^2x^2)^2} dx^\mu dx^\nu \delta_{\mu\nu} \ . \qquad (3.2.13)$$

We let x^0 and y^0 correspond to Euclidean times. We consider a sphere of radius r in the remaining $d-1$ dimensions, centered about the origin. If we set $b^1 = 1/r$ and the rest of the $b^\mu = 0$, this coordinate transformation will take a point on the sphere to infinity, specifically the point $x^\mu = (0, r, 0, \ldots, 0)$. The rest of the sphere will get mapped to a hyperplane with $y^1 = -r/2$. We can think of the total geometry as a cone in the (y^0, y^1) coordinates formed by gluing n-spaces together, successively, along the half plane $y^0 = 0$ and $y^1 < -r/2$. Let us introduce polar coordinates (ρ, ϕ) on the cone currently parametrized by (y^0, y^1). The tip of the cone $(y^0, y^1) = (0, -r/2)$ will correspond to $\rho = 0$. The insertion points $(\pm 1, 0, \ldots, 0)$ for the operator ψ_i get mapped to $(\pm 1, -1/r, 0, \ldots, 0)/(1+1/r^2)$. In polar coordinates, the insertion points of the ψ_i are at $(r/2, \pm\theta)$. By a further rescaling and rotation, we can put the insertion points at $(1, 2\theta, \vec{0})$ and $(1, 0, \vec{0})$.

For primary fields $\psi_i(x)$, the effect of a conformal transformation on the ratio (3.2.5) is particulary simple. Let us focus on one of the $\psi_i = \psi$ and assume that it is a primary scalar field. We have

$$\psi(x) = \left(\frac{1}{2} \sec\left(\frac{\tau+\theta}{2}\right) \sec\left(\frac{\tau-\theta}{2}\right) \right)^{-\Delta} \psi(z) \ ,$$
$$\psi(y) = (1 - 2b\cdot x + b^2 x^2)^{\Delta} \psi(x) \ .$$

We are interested in computing

$$\frac{\langle \psi(z)\psi(z')\rangle_n}{\langle \psi(z)\psi(z')\rangle_1} \ ,$$

where the subscript n indicates this n-fold covering of the sphere, glued along the boundary of A. In the ratio, the conformal factors relating the z coordinates to the x coordinates and the x coordinates to the y coordinates will drop out. All we need pay attention to is where y and y' are in the cone of opening angle $2\pi n$, which we have already done. For non-scalar and non-primary operators, the transformation rules are more involved.

For free CFTs, $\langle \psi(y)\, \psi(y')\rangle_{1/m}$ can be evaluated for $m = 1, 2, 3, \ldots$ by the method of images. For $n = 1/m$, the conical space has opening angle $2\pi/m$. Let us assume we know the two-point function on $\mathbb{R}^d$: $\langle \psi(y_1)\psi(y_2)\rangle_1 = f(y_{12}^2)$. Using the parametrization $y = (\rho, \theta, \vec{r})$, the square of the distance between the points is

$$y_{12}^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_{12}) + (\vec{r}_{12})^2 \ . \tag{3.2.14}$$

By the method of images,

$$\langle \psi(y_1)\psi(y_2)\rangle_{1/m} = \sum_{k=0}^{m-1} f\left(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_{12} + 2\pi k/m) + (\vec{r}_{12})^2\right) \ . \tag{3.2.15}$$

We are interested in two particular insertion points $y = (1, 2\theta, \vec{0})$ and $y' = (1, 0, \vec{0})$, for which the two point function reduces to

$$\langle \psi(y)\psi(y')\rangle_{1/m} = \sum_{k=0}^{m-1} f\left(2 - 2\cos(2\theta + 2\pi k/m)\right) \ . \tag{3.2.16}$$

Once we have obtained an analytic expression for all m, we can then evaluate it for integer $n = 1/m$.

3.2.1 The Free Scalar Case

We now specialize to the case of a free scalar, for which the scaling form of the Green's function in flat Euclidean space is $f(y^2) = y^{2-d}$. Our strategy will be to take advantage of recurrence relations that relate the Green's function in d dimensions to $d+2$ dimensions. Let us define

$$G^B_{(n,d)}(2\theta) \equiv \langle \psi(y)\psi(y')\rangle_n \ . \tag{3.2.17}$$

We need to compute the sum

$$G^B_{(1/m,d)}(2\theta) = \langle \psi(y)\,\psi(y')\rangle_{1/m} = \sum_{k=0}^{m-1} \frac{1}{\left[2 - 2\cos\left(2\theta + \frac{2\pi k}{m}\right)\right]^{\frac{d-2}{2}}} \,. \tag{3.2.18}$$

As can be straightforwardly checked, this sum obeys the recurrence relation

$$G^B_{(1/m,d+2)}(\theta) = \frac{1}{(d-2)(d-1)} \left[\left(\frac{d-2}{2}\right)^2 + \frac{\partial^2}{\partial\theta^2}\right] G^B_{(1/m,d)}(\theta)\,. \tag{3.2.19}$$

The most efficient computation strategy we found is to compute $G^B_{(n,d)}$ for $d=3$ and $d=4$ and then to use the recurrence relation to compute the two point function in $d>4$. (In $d=2$, the scalar is not gapped and there will be additional entanglement entropy associated with the degenerate ground state.)

To compute $G^B_{(n,4)}$, and more generally $G^B_{(n,d)}$ when d is even, we introduce the generalized sum

$$f_a(m,\theta,z,\bar z) \equiv \sum_{k=0}^{m-1} \frac{1}{|z - e^{i(\theta+2\pi k/m)}|^{2a}}\,, \tag{3.2.20}$$

With this definition, we have the restriction that

$$\lim_{z,\bar z\to 1} f_{(d-2)/2}(m,\theta,z,\bar z) = G^B_{(1/m,d)}(\theta)\ . \tag{3.2.21}$$

and the recurrence relation

$$\frac{\partial^2 f_a}{\partial z\,\partial \bar z} = a^2\, f_{a+1}(m,\theta,z,\bar z)\,. \tag{3.2.22}$$

In the case $d=4$, we find that

$$f_1(m,\theta,z,\bar z) = \frac{m}{|z|^2 - 1}\left[\frac{1}{1 - z^{-m}\,e^{im\theta}} + \frac{1}{1 - \bar z^{-m}\,e^{-im\theta}} - 1\right]. \tag{3.2.23}$$

The two-point function can be obtained from Eq. (3.2.23) by taking the limit $z, \bar z \to 1$:

$$G^B_{(n,4)}(\theta) = \lim_{z,\bar z\to 1} f_1\left(\frac{1}{n},\theta,z,\bar z\right) = \frac{1}{n^2\left[2 - 2\cos\left(\frac{\theta}{n}\right)\right]}\,. \tag{3.2.24}$$

For $d = 6$ dimensions the two-point function can be obtained by taking the $z, \bar{z} \to 1$ of $f_2(m, \theta, z, \bar{z})$:

$$G^B_{(n,6)}(\theta) = \lim_{z,\bar{z}\to 1} f_2\left(\frac{1}{n}, \theta, z, \bar{z}\right) = \frac{1 + \frac{2}{n^2} + (\frac{1}{n^2} - 1)\cos(\frac{\theta}{n})}{3n^2\left[2 - 2\cos(\frac{\theta}{n})\right]^2} . \tag{3.2.25}$$

Applying the recurrence relation (3.2.19) to the four dimensional result (3.2.24) yields the same answer. It is straightforward to calculate the Green's function in even $d > 6$.

For $d = 3$, we do not have as elegant expression for general n. Through a contour integral argument we will now discuss, for $n = 1$, 2, and 3 we obtain

$$G^B_{(1,3)}(\theta) = \frac{1}{2\sin\frac{\theta}{2}} , \tag{3.2.26}$$

$$G^B_{(2,3)}(\theta) = \frac{1 - \frac{\theta}{2\pi}}{2\sin\frac{\theta}{2}} , \tag{3.2.27}$$

$$G^B_{(3,3)}(\theta) = \frac{1}{2\sin\frac{\theta}{2}}\left[1 - \frac{2}{\sqrt{3}}\sin\frac{\theta}{6}\right] . \tag{3.2.28}$$

More general expressions for $G^B_{(n,d)}(\theta)$ with d odd can be found in the next section. Tables of thermal Rényi entropy corrections δS_n for some small d and n are in Appendix K.

3.2.2 Odd Dimensions and Contour Integrals

Following Ref. [95], for d an odd integer we express the Green's function in terms of an integral and evaluate it using the Cauchy residue theorem:

$$\begin{aligned}
G^B_{(1/m,d)}(\theta) &= \sum_{k=0}^{m-1} \frac{1}{\left[2\sin\left(\frac{\theta}{2} + \frac{\pi k}{m}\right)\right]^{d-2}} \\
&= \frac{1}{(2\pi)^{d-2}} \sum_{k=0}^{m-1} \left[\int_0^\infty dx \, \frac{x^{\frac{\theta}{2\pi} + \frac{k}{m} - 1}}{1+x}\right]^{d-2} \\
&= \frac{1}{(2\pi)^{d-2}} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{d-2} \left(\prod_{i=1}^{d-2} \frac{(x_i)^{\frac{\theta}{2\pi}-1}}{1+x_i}\right) \left[\sum_{k=0}^{m-1}\left(\prod_{i=1}^{d-2} x_i\right)^{\frac{k}{m}}\right] \\
&= \frac{1}{(2\pi)^{d-2}} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{d-2} \left(\prod_{i=1}^{d-2} \frac{(x_i)^{\frac{\theta}{2\pi}-1}}{1+x_i}\right) \left[\frac{1 - \prod_{i=1}^{d-2} x_i}{1 - \left(\prod_{i=1}^{d-2} x_i\right)^{\frac{1}{m}}}\right] .
\end{aligned} \tag{3.2.29}$$

Then $G^B_{(n,d)}(\theta)$ is obtained by replacing m with $\frac{1}{n}$. While this integral expression is valid for all integers d, even and odd, for the even integers d it is easier to evaluate the limit (3.2.21) or use the recurrence relation (3.2.19) with the (3+1) dimensional result (3.2.24).

Using the integral (3.2.29), the two-point function in $d = 3$ becomes

$$G^B_{(n,3)}(\theta) = \frac{1}{2\pi}\int_0^\infty \frac{x^{\frac{\theta}{2\pi}-1}(1-x)}{(1+x)(1-x^n)}dx\,, \tag{3.2.30}$$

This integral can be done analytically. Essentially it is a contour integral with a branch point at $z = 0$ and some poles on the unit circle. For convenience, we can choose a branch cut to be the positive real axis, and a contour shown in Fig. 1. For an even integer n, the poles are

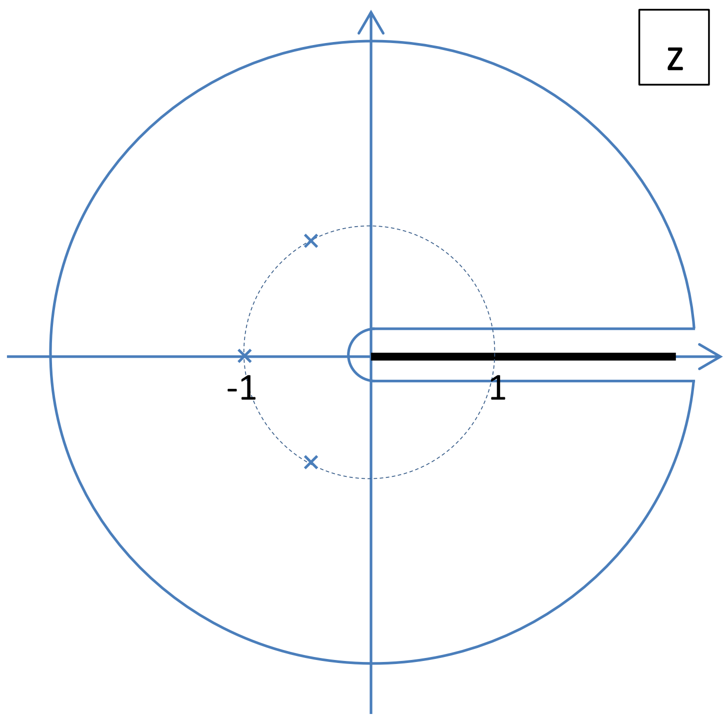


Figure 3.1: The contour for $d = 3$ dimensions and $n = 3$

$$z = -1 \text{ is a double pole,} \quad z = e^{2\pi i\frac{\ell}{n}}\ \left(\ell = 1, \cdots, \frac{n}{2}-1, \frac{n}{2}+1, \cdots, n-1\right) \text{ are simple poles.}$$

For an odd integer n, the poles are

$$z = -1 \text{ and } z = e^{2\pi i\frac{\ell}{n}}\ (\ell = 1, \cdots, n-1) \text{ are all simple poles.}$$

We emphasize that $z=1$ is not a pole. Then for an even integer n:

$$G^B_{(n,d)} = \frac{1}{2\pi}\int_0^\infty \frac{x^{\frac{\theta}{2\pi}-1}(1-x)}{(1+x)(1-x^n)}dx = \frac{i}{1-e^{2\pi i(\frac{\theta}{2\pi}-1)}}\sum_{\text{Poles}}\text{Res} \tag{3.2.31}$$
$$= \frac{i}{1-e^{2\pi i(\frac{\theta}{2\pi}-1)}}\left[-(-1)^{\frac{\theta}{2\pi}-1}\left(\frac{\theta}{n\pi}-1\right)+\sum_{\ell=1,\,\ell\neq\frac{n}{2}}^{n-1}\frac{e^{2\pi i\frac{\ell}{n}(\frac{\theta}{2\pi}-1)}}{1+e^{2\pi i\frac{\ell}{n}}}\left(\prod_{j=1,\,j\neq\ell}^{n-1}\frac{1}{e^{2\pi i\frac{\ell}{n}}-e^{2\pi i\frac{j}{n}}}\right)\right]$$
$$= \frac{1}{2\sin\frac{\theta}{2}}\left[1-\frac{\theta}{\pi n}-\frac{i}{n}\sum_{\ell=1,\ell\neq n/2}^{n-1}e^{i\theta(\frac{\ell}{n}-\frac{1}{2})}\tan\frac{\pi\ell}{n}\right],$$

while for an odd integer n:

$$G^B_{(n,d)} = \frac{1}{2\pi}\int_0^\infty \frac{x^{\frac{\theta}{2\pi}-1}(1-x)}{(1+x)(1-x^n)}dx = \frac{i}{1-e^{2\pi i(\frac{\theta}{2\pi}-1)}}\sum_{\text{Poles}}\text{Res} \tag{3.2.32}$$
$$= \frac{i}{1-e^{2\pi i(\frac{\theta}{2\pi}-1)}}\left[(-1)^{\frac{\theta}{2\pi}-1}+\sum_{\ell=1}^{n-1}\frac{e^{2\pi i\frac{\ell}{n}(\frac{\theta}{2\pi}-1)}}{1+e^{2\pi i\frac{\ell}{n}}}\left(\prod_{j=1,\,j\neq\ell}^{n-1}\frac{1}{e^{2\pi i\frac{\ell}{n}}-e^{2\pi i\frac{j}{n}}}\right)\right]$$
$$= \frac{1}{2\sin\frac{\theta}{2}}\left[1-\frac{i}{n}\sum_{\ell=1}^{n-1}e^{i\theta(\frac{\ell}{n}-\frac{1}{2})}\tan\frac{\pi\ell}{n}\right].$$

Therefore, for $d=3$ dimensions the results for $n=1,2,3$ are Eqs. (3.2.26)–(3.2.28).

Given the results for $d=3$ dimensions, the two-point functions for $d=5$ dimensions can be obtained by using the recurrence relation (3.2.19):

$$G^B_{(1,5)}(\theta) = \frac{1}{\left(2\sin\frac{\theta}{2}\right)^3}, \tag{3.2.33}$$
$$G^B_{(2,5)}(\theta) = \frac{2\pi-\theta+\sin\theta}{2\pi\left(2\sin\frac{\theta}{2}\right)^3}, \tag{3.2.34}$$
$$G^B_{(3,5)}(\theta) = \frac{1}{108\left(2\sin\frac{\theta}{2}\right)^3}\left[108-70\sqrt{3}\sin\left(\frac{\theta}{6}\right)+7\sqrt{3}\sin\left(\frac{5\theta}{6}\right)+5\sqrt{3}\sin\left(\frac{7\theta}{6}\right)\right]. \tag{3.2.35}$$

In Appendix J, we also compute the two-point function for $d=5$ dimensions and $n=1,2,3$ by directly evaluating the contour integral

$$G^B_{(n,5)}(\theta) = \frac{1}{(2\pi)^3}\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\,\frac{(xyz)^{\frac{\theta}{2\pi}-1}(1-xyz)}{(1+x)(1+y)(1+z)\left(1-(xyz)^n\right)}, \tag{3.2.36}$$

and the results are exactly the same.

3.3 Thermal Corrections to Entanglement Entropy

General results for thermal corrections to entanglement entropy were given in Ref. [82]. Here we will verify these general results in arbitrary dimension for the specific case of a conformally coupled scalar. To perform the check, we will use the fact that the $n \to 1$ limit of the Rényi entropies yields the entanglement entropy.

The Green's function $G^B_{(n,d)}(\theta)$ has an expansion near $n = 1$ of the form

$$G^B_{(n,d)}(\theta) = G^B_{(1,d)}(\theta) + (n-1)\delta G^B_{(d)}(\theta) + \mathcal{O}(n-1)^2 \ . \tag{3.3.1}$$

From the definition (3.2.17) and the main result (3.2.6), we have that

$$\delta S_E = -\frac{\delta G^B_{(d)}(2\theta)}{G^B_{(1,d)}(2\theta)} e^{-\beta E_\psi} + o(e^{-\beta E_\psi}) \ . \tag{3.3.2}$$

Note that $\delta G^B_{(d)}(\theta)$ will also satisfy the recurrence relation (3.2.19). Thus it is enough to figure out the thermal corrections for the smallest dimensions $d = 3$ and $d = 4$. The result in $d > 4$ will then follow from the recurrence.

Let us check that the expression (3.3.2) agrees with Ref. [82] in the cases $d = 3$ and $d = 4$. In the case $d = 3$, we can evaluate the relevant contour integral (3.2.30) in the limit $n \to 1$:

$$\begin{aligned} G^B_{(n,3)}(\theta) &= \frac{1}{2\pi}\int_0^\infty \frac{x^{\frac{\theta}{2\pi}-1}}{1+x} dx + \frac{n-1}{2\pi}\int_0^\infty \frac{x^{\frac{\theta}{2\pi}}\log x}{1-x^2} + \mathcal{O}(n-1)^2 \\ &= \frac{1}{2}\frac{1}{\sin\frac{\theta}{2}} - (n-1)\frac{\pi}{8}\frac{1}{\cos^2\frac{\theta}{4}} + \mathcal{O}(n-1)^2 \ . \end{aligned} \tag{3.3.3}$$

From Eqs. (3.3.2) and (3.3.3), we then have

$$\delta S_E = \frac{\pi}{2}\tan\left(\frac{\theta}{2}\right) e^{-\beta/2R} + o(e^{-\beta/2R}) \ . \tag{3.3.4}$$

For $d = 4$, we expand Eq. (3.2.24) near $n = 1$:

$$G^B_{(n,4)}(\theta) = \frac{1}{4\sin^2\frac{\theta}{2}}\left(1 + (n-1)\left(-2 + \theta\cot\frac{\theta}{2}\right) + \mathcal{O}(n-1)^2\right) \ . \tag{3.3.5}$$

We find from Eqs. (3.3.2) and (3.3.5) that

$$\delta S_E = 2(1 - \theta\cot\theta)e^{-\beta/R} + o(e^{-\beta/R}) \ . \tag{3.3.6}$$

The expressions (3.3.4) and (3.3.6) are precisely the results found for the conformally coupled scalar in Ref. [82] in $d = 3$ and $d = 4$ respectively.

Indeed, for general d, the result in Ref. [82] for the conformally coupled scalar is

$$\delta S_E = \frac{d-2}{2} I_{d-2}(\theta) e^{-\beta(d-2)/2R} + o(e^{-\beta(d-2)/2R}) \ . \tag{3.3.7}$$

where the definition (3.1.28) of $I_d(\theta)$ was given in the introduction. If our result (3.3.2) for the thermal correction is correct, we can relate $I_d(\theta)$ and $\delta G^B_{(d)}(\theta)$:

$$\delta G^B_{(d)}(2\theta) = -\frac{d-2}{2} (2\sin\theta)^{2-d} I_{d-2}(\theta) \ , \tag{3.3.8}$$

where we have used the fact that $G^B_{(1,d)}(2\theta) = (2\sin\theta)^{2-d}$.

To check that our thermal corrections are correct for general d, we will use a roundabout method. In Ref. [82], it was also found that the function $I_d(\theta)$ satisfies a recurrence relation

$$I_d(\theta) - I_{d-2}(\theta) = -2\pi \frac{\mathrm{Vol}(S^{d-2})}{\mathrm{Vol}(S^{d-1})} \frac{\sin^{d-2}\theta}{(d-1)(d-2)} \ . \tag{3.3.9}$$

We will use our recurrence relation (3.2.19) and the tentative identification (3.3.8) to replace $I_d(\theta)$ with $I_{d-2}(\theta)$ in the above expression:

$$\begin{aligned} I_d(\theta) &= -\frac{2}{d}(2\sin\theta)^d \, \delta G^B_{(d+2)}(2\theta) \\ &= -\frac{2(2\sin\theta)^d}{d(d-1)(d-2)} \left[\left(\frac{d-2}{2}\right)^2 + \frac{1}{4}\frac{\partial^2}{\partial\theta^2} \right] \delta G^B_{(d)}(2\theta) \\ &= \frac{(2\sin\theta)^d}{4d(d-1)} \left[(d-2)^2 + \frac{\partial^2}{\partial\theta^2} \right] (2\sin\theta)^{2-d} I_{d-2}(\theta) \ . \end{aligned} \tag{3.3.10}$$

Then we have checked that the resulting differential equation in $I_{d-2}(\theta)$ is solved by the integral formula (3.1.28).

3.4 Numerical Check

We check numerically the thermal Rényi entropy corrections obtained in section 3.2.1. The algorithm we use was described in detail in Ref. [82], so we shall be brief. (The method is essentially that of Ref. [73].) The action for a conformally coupled scalar on $\mathbb{R} \times S^{d-1}$ is

$$S = -\frac{1}{2} \int d^d x \sqrt{-g} \left[(\partial_\mu \phi)(\partial^\mu \phi) + \xi \, \mathcal{R} \, \phi^2 \right] \ , \tag{3.4.1}$$

where ξ is the conformal coupling $\xi = (d-2)/4(d-1)$ and $\mathcal{R}$ is the Ricci scalar curvature. Given that the region A can be characterized by the polar angle θ on S^{d-1}, we write the Hamiltonian as a sum $H = \sum_{\vec{l}} H_{\vec{l}}$, where we have replaced all the other angles on S^{d-1}

by corresponding angular momentum quantum numbers $|l_1| \le l_2 \le \cdots \le l_{d-2} \equiv m$. The individual Hamiltonians take the form

$$H_{\vec{l}} = \frac{1}{2R^2}\int_0^\pi \left\{ R^2 \Pi_{\vec{l}}^2 - \Phi_{\vec{l}} \partial_\theta^2 \Phi_{\vec{l}} + \frac{1}{4}(2m+d-2)(2m+d-4)\frac{\Phi_{\vec{l}}^2}{\sin^2\theta} \right\} d\theta \ . \tag{3.4.2}$$

It is convenient to discretize $H_{\vec{l}}$. In $d \ge 4$, we introduce a lattice in θ, while in $d=3$, a lattice in $\cos\theta$ appears to work better. The entanglement and Rényi entropies can then be expressed in terms of two-point functions restricted to the region A. In particular, the Rényi entropy can be expressed as

$$S_n(T) = S_n(0) + \sum_{m=1}^\infty \dim(m)\, S_n^{(m)} \ , \tag{3.4.3}$$

where

$$\dim(m) = \binom{d+m-2}{d-2} - \binom{d+m-4}{d-2} \ ,$$

and

$$S_n^{(m)} = \frac{1}{1-n} \log \operatorname{tr} \left[\left(C_m + \frac{1}{2} \right)^n - \left(C_m - \frac{1}{2} \right)^n \right] \ . \tag{3.4.4}$$

The matrix $C_m(\theta_1, \theta_2)$ has a continuum version

$$C_m(\theta_1,\theta_2)^2 = \int_0^{\theta_0} d\theta \, \langle \Phi_{\vec{l}}(\theta_1)\, \Phi_{\vec{l}}(\theta) \rangle \langle \Pi_{\vec{l}}(\theta)\, \Pi_{\vec{l}}(\theta_2) \rangle \ . \tag{3.4.5}$$

The thermal two-point functions have the following expressions:

$$\langle \Phi_{\vec{l}}(\theta)\, \Phi_{\vec{l}}(\theta') \rangle = \frac{1}{2} \sum_{l=m}^\infty U_l(\theta)\, \frac{1}{\omega_l} \coth \frac{\omega_l}{2T}\, U_l(\theta') \ , \tag{3.4.6}$$

$$\langle \Pi_{\vec{l}}(\theta)\, \Pi_{\vec{l}}(\theta') \rangle = \frac{1}{2} \sum_{l=m}^\infty U_l(\theta)\, \omega_l \coth \frac{\omega_l}{2T}\, U_l(\theta') \ , \tag{3.4.7}$$

where $\omega_l \equiv \frac{1}{R}\left(l + \frac{d-2}{2}\right)$. In the continuum limit, the matrix $U_l(\theta)$ is an orthogonal transformation involving associated Legendre functions whose explicit form is given in Ref. [82]. In practice, we use the discretized version of $U_l(\theta)$ that follows from the discretized $H_{\vec{l}}$.

As discussed in Ref. [88, 82], if the limit $\theta_0 \to \pi$ is taken first, the leading correction to δS_n comes from the thermal Rényi entropy instead of from the entanglement:

$$\delta S_n = \left[-g \frac{n}{1-n} + \mathcal{O}\left(1 - \frac{\theta_0}{\pi}\right)^{2\Delta} \right] e^{-\Delta/RT} + o\left(e^{-\Delta/RT}\right) \ . \tag{3.4.8}$$

Indeed, when $\pi - \theta$ is small compared to RT, the Rényi entropy looks like the thermal Rényi entropy and approaches it in the limit $\theta \to \pi$. To isolate the $e^{-\Delta/RT}$ dependence of δS_n

analytically, we can expand the coth-function in the thermal two-point functions (3.4.6) and (3.4.7). In principle, one can evaluate Eq. (3.4.3) to obtain $\delta S_n(T)$. Since we are interested in the low temperature limit, the contributions from $S_n^{(m)}$ $(m>0)$ to $\delta S_n(T)$ are exponentially suppressed compared with $S_n^{(0)}$. Therefore, in the limit of small T, we obtain the expansion of Eq. (3.4.3):

$$\delta S_n = \frac{n}{2(n-1)}\mathrm{tr}\left[\delta C_0\cdot C_0^{-1}\cdot\frac{(C_0+\frac{1}{2})^{n-1}-(C_0-\frac{1}{2})^{n-1}}{(C_0+\frac{1}{2})^{n}-(C_0-\frac{1}{2})^{n}}\right]e^{-\omega_0/T}+\cdots\,, \tag{3.4.9}$$

where

$$\delta C_m(\theta_1,\theta_2)\equiv\int_0^{\theta_0}d\theta\,\left[\langle\Phi_{\vec{l}}(\theta_1)\,\Phi_{\vec{l}}(\theta)\rangle\,\delta\Pi_m(\theta,\theta_2)+\delta\Phi_m(\theta_1,\theta)\,\langle\Pi_{\vec{l}}(\theta)\,\Pi_{\vec{l}}(\theta_2)\rangle\right]\,, \tag{3.4.10}$$

$$\delta\Phi_m(\theta,\theta')\equiv U_m(\theta)\,\frac{1}{\omega_m}\,U_m(\theta')\,, \tag{3.4.11}$$

$$\delta\Pi_m(\theta,\theta')\equiv U_m(\theta)\,\omega_m\,U_m(\theta')\,. \tag{3.4.12}$$

Some results of δS_n in different dimensions are shown in Figs. 2 – 5. To diagonalize the matrices with enough accuracy, high precision arithmetic is required.

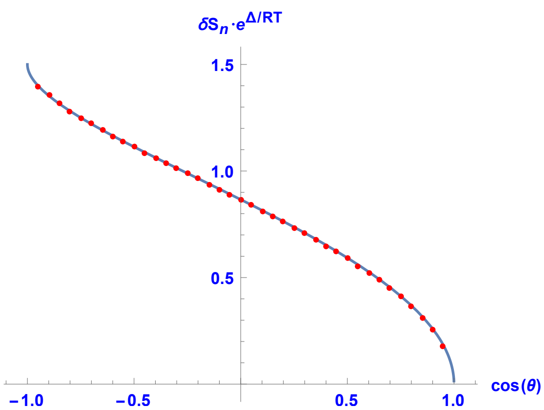


Figure 3.2: $\delta S_{n=3}$ in $(2+1)$ D, 400 grid points

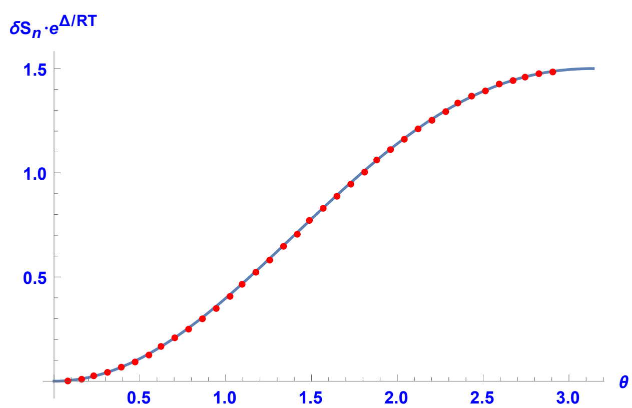


Figure 3.3: $\delta S_{n=3}$ in $(3+1)$ D, 400 grid points

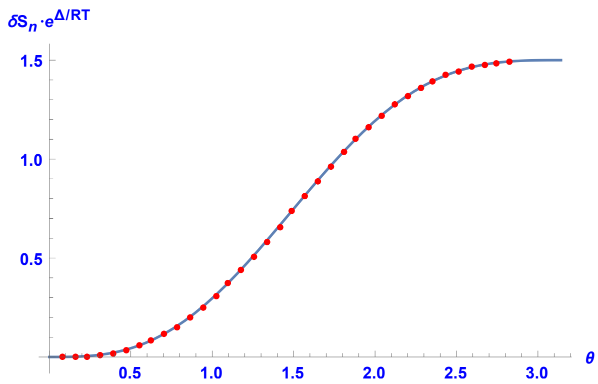


Figure 3.4: $\delta S_{n=3}$ in $(4+1)$ D, 400 grid points

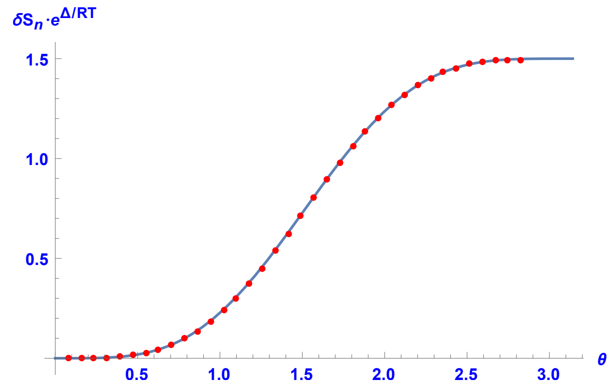


Figure 3.5: $\delta S_{n=3}$ in $(5+1)$ D, 400 grid points

3.5 Discussion

Our main result provides a way to calculate the leading thermal correction to a specific kind of Rényi entropy for a CFT. In particular, the CFT should live on $\mathbb{R} \times S^{d-1}$, and the region is a cap on the sphere with opening angle 2θ. We demonstrated that this correction is equivalent to knowing the two-point function on a certain conical space of the operator that creates the first excited state. In the case of a conformally coupled free scalar, the scalar field itself creates the first excited state, and the two point function can be computed by the method of images. In the $n \to 1$ limit, the Rényi entropy becomes entanglement entropy, and we were able to show that our results agree with Ref. [82]. We were also able to check our thermal corrections for $n > 1$ numerically, using a method based on Ref. [73].

We would like to make two observations about our results. The first is that our thermal Rényi entropy corrections are often but not always invariant under the replacement $\theta \to 2\pi n - \theta$. (The exceptions are δS_n for even n and odd d.) A similar observation was made in Ref. [88] in the 1+1 dimensional case. There, the invariance could be explained by moving twist operators around the torus (or cylinder). The branch cut joining two twist operators is the same cut along which the different copies of the torus are glued together. By moving a twist operator n times around the torus, n branch cuts are equivalent to nothing while $n-1$ branch cuts are equivalent to a single branch cut that moves one down a sheet rather than up a sheet. Perhaps in higher dimensions the invariance can be explained in terms of surface operators that glue the n copies of $S^1 \times S^{d-1}$ together. It is not clear to us how to generalize the argument. It is tempting to speculate that the invariance is spoiled in odd dimensions (even dimensional spheres) because only $(2n+1)$-dimensional spheres ($n \geqslant 1$) are Hopf fibrations over projective space.

The second observation is that the leading corrections to δS_n for small caps $\theta \ll 1$ have a power series expansion that starts with the terms $a\theta^{d-2} + b\theta^d + \ldots$. In 1+1 dimensions, the power series starts with θ^2 [88]. When we bring two twist operators together, the twist operators can be replaced by their operator product expansion, a leading term of which is the stress tensor. The θ^2 term in δS_n comes from a three point function of the stress tensor with the operators that create and annihilate the first excited state. The two in the exponent of θ^2 comes from the scaling dimension of the stress tensor, and the coefficient of the θ^2 can be related to the scaling dimension of the twist operators [88]. In our higher dimensional case, we can replace the surface operator along the boundary of the cap by an operator product expansion at a point. Because of Wick's theorem, the leading operator that can contribute to δS_n will be ϕ^2 which has dimension $d-2$. The subleading θ^d term may come from the stress tensor and descendants of ϕ^2. A more detailed analysis might shed some light on the structure of these surface operators.[3]

In addition to developing the above observations, we give a couple of projects for future research. One would be to compute these thermal corrections for free fermions. The two point function on this conical space can quite likely be computed. It would be interesting to see how the results compare to the scalar. Given the importance of boundary terms for the scalar, it would also be nice to get further confirmation of the general story for thermal corrections to entanglement entropy presented in Ref. [82].

Another interesting project would be to see how to obtain these results holographically. As the corrections are subleading in a large central charge (or equivalently large N) expansion, they would not be captured by the Ryu-Takayanagi formula [83]. However, it may be possible to generalize the computation in $d = 2$ [92] to $d > 2$. Finally, it would be interesting to see what can be said about negativity in higher dimensions. See Refs. [100, 101] for the two dimensional case.

[3] See Refs. [95, 97, 98, 99] for related work on higher dimensional analogs of twist operators.

Chapter 4

BEC, String Theory and KPZ Equation

The Bose-Einstein condensation is a phenomenon in cold atom systems, for instance, a dilute gaseous system of ^{87}Rb or ^{23}Na atoms at ~ 100 nK. It is a consequence of the Bose-Einstein statistics, and happens in a system consisting of only bosons at very low temperature when all the particles accumulate in the ground state. The theory of the Bose-Einstein condensation can also be applied to systems like superfluid Helium, superconductors and quasiparticles (magnons, excitons, polaritons etc.).

The Gross-Pitaevskii equation is known as a mean-field description of the Bose-Einstein condensates. This equation is also known as the nonlinear Schrödinger equation in mathematical physics, which is an integrable model in (1+1)D and has been well studied in the literature. One can find the soliton solutions to the nonlinear Schrödinger equation using the Hamiltonian methods [31]. If one treats the equation as an operator equation and performs the second quantization in the beginning, it can be solved by the Bethe Ansatz equation, and consequently the ground state and the excited states can also be found [32].

The Gross-Pitaevskii equation has some nontrivial solutions, for instance, the vortex line solution and the dark soliton solution. They behave like the open string and the D-brane in string theory in the sense that the end of the vortex line can attach to the dark soliton to form a stable configuration. In Ref. [102], A. Zee made the first attempt to understand this similarity. He mapped the (2+1)-dimensional Gross-Pitaevskii theory into a nonlinear sigma model which is quite close to the standard bosonic string theory action. This approach can easily be generalized to other dimensions, for instance, Ref. [103] mapped the (3+1)-dimensional Gross-Pitaevskii theory to a string-like nonlinear sigma model. Recently, some people also found the relation between the Gross-Pitaevskii equation and the so-called Kardar-Parisi-Zhang equation (KPZ equation) [104], which describes the growth of a random surface.

In this chapter, we disucss the connections between the Gross-Pitaevskii equation, the string-like nonlinear sigma model and the KPZ equation. We will see that they have some deep connections, and there are still many open questions which can possibly lead to some exciting discoveries. For instance, if we could understand the dualities among these theories in a better way, we would be able to calculate physical quantities with higher precision even at strong coupling and make some interesting quantitative predictions such as the Hawking radiation and the Unruh effect in the BEC system, which can possibly be found in experiments. The discussion in this chapter is mainly based on Ref. [105] and some of my unpublished notes.

4.1 Gross-Pitaevskii Equation

Let us briefly review the Gross-Pitaevskii equation in this section. The discussion is mainly based on Ref. [106]. The equation is following

$$i\hbar\frac{\partial}{\partial t}\Psi_0(\mathbf{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ext}}(\mathbf{r},t)+g\,|\Psi_0(\mathbf{r},t)|^2\right)\,\Psi_0(\mathbf{r},t)\,, \tag{4.1.1}$$

where

$$g\equiv\int d\mathbf{r}\,V_{\mathrm{eff}}(\mathbf{r})\,. \tag{4.1.2}$$

It is the nonlinear Schrödinger equation known in mathematical physics. For simplicity, let us consider only the stationary case, and we also introduce the chemical potential μ, then the Gross-Pitaevskii equation becomes

$$\left(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ext}}(\mathbf{r})-\mu+g|\Psi_0(\mathbf{r})|^2\right)\,\Psi_0(\mathbf{r})=0\,. \tag{4.1.3}$$

It still has various nontrivial solutions. Let us list them in the following.

The so-called vortex line solution emerges in the problem of a gas system confined in a cylindrical vessel of radius R and length L. Due to the symmetry we can employ the Ansatz

$$\Psi_0(\mathbf{r})=e^{is\varphi}\,|\Psi_0(r)|\,,\quad |\Psi_0|=\sqrt{n}\,f(\eta)\,, \tag{4.1.4}$$

where

$$\eta\equiv\frac{r}{\xi}\,,\quad \xi\equiv\frac{\hbar}{\sqrt{2mgn}}\,. \tag{4.1.5}$$

The stationary Gross-Pitaevskii equation (4.1.3) in this case reduces to

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{df}{d\eta}\right)+\left(1-\frac{s^2}{\eta^2}\right)f-f^3=0 \tag{4.1.6}$$

with the constraint $f(\infty) = 1$. When $\eta \to 0$, f behaves as $f \sim \eta^{|s|}$. One can solve this equation numerically, and the solution gives the profile of the vortex line. When the two end points of the vortex line attach to each other to form a circular ring, this solution is called the vortex ring.

Besides the vortex line solution, there are also some soliton solutions to the Gross-Pitaevskii equation or the nonlinear Schrödinger equation. There is a systematical way to obtain these soliton solutions, which is called the Hamiltonian method and discussed in great detail in Ref. [31]. In this section, we just follow Ref. [106] to find the soliton solution using an appropriate Ansatz, while in the next section we will rederive these solutions using the BPS procedure in field theory. Let us assume

$$\Psi_0 = \sqrt{n}\, f(\eta, \zeta)\, e^{-i\mu t/\hbar}\,, \tag{4.1.7}$$

where

$$\eta \equiv \frac{r}{\xi}\,, \quad \zeta \equiv \frac{z - vt}{\xi}\,, \tag{4.1.8}$$

and v is the velocity of the center of the soliton. Plugging this Ansatz into the Gross-Pitaevskii equation (4.1.1), we obtain

$$2iU \frac{\partial f}{\partial \zeta} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial f}{\partial \eta}\right) + \frac{\partial^2 f}{\partial \zeta^2} + f(1 - |f|^2)\,, \tag{4.1.9}$$

where

$$U \equiv \frac{mv\xi}{\hbar} = \frac{v}{c\sqrt{2}}\,, \tag{4.1.10}$$

and c is the speed of sound. Since we are looking for some 1-dimensional solution like a domain-wall, we assume that the solution is independent of the coordinate r or η, then the equation above becomes

$$2iU \frac{df}{d\zeta} = \frac{d^2 f}{d\zeta^2} + f(1 - |f|^2)\,. \tag{4.1.11}$$

The physical boundary conditions are

$$|f| \to 1\,, \quad \frac{df}{d\zeta} \to 0\,, \tag{4.1.12}$$

when $\zeta \to \pm\infty$. We separate the real and the imaginary part of f as $f = f_1 + i f_2$, then the imaginary part of Eq. (4.1.11) becomes

$$2U \frac{df_1}{d\zeta} = \frac{d^2 f_2}{d\zeta^2} + f_2(1 - f_1^2 - f_2^2)\,. \tag{4.1.13}$$

After some consistency analysis we find that $f_2 = \sqrt{2}U = \frac{v}{c}$, then the equation above becomes

$$\sqrt{2}\frac{df_1}{d\zeta} = 1 - \frac{v^2}{c^2} - f_1^2\,. \tag{4.1.14}$$

It has the solutions

$$\Psi_0(z-vt)=\sqrt{n}\left(i\frac{v}{c}+\sqrt{1-\frac{v^2}{c^2}}\tanh\left[\frac{z-vt}{\sqrt{2}\xi}\sqrt{1-\frac{v^2}{c^2}}\right]\right). \tag{4.1.15}$$

The density profile $n(z-vt)=|\Psi_0|^2$ has the minimum value $n(0)=nv^2/c^2$. For $v=0$ the density profile is equal to zero at the center of the soliton, i.e., it looks "dark" in real experiments. Hence, the solution with $v=0$ is called "dark soliton", while a solution with $v\neq 0$ is called "grey soliton".

The soliton solutions discussed above are obtained for the repulsive interaction, i.e. $g>0$. Actually, for the attractive interaction ($g>0$) there is also a kind of soliton solution, which is given by

$$\Psi_0(z)=\Psi_0(0)\frac{1}{\cosh(z/\sqrt{2}\xi)}. \tag{4.1.16}$$

One can check that it satisfies the stationary Gross-Pitaevskii equation (4.1.3). Since this soliton solution has the maximum value of the density profile at its center, it is called the "bright soliton".

4.2 BEC and String Theory

4.2.1 Derivation of the String Action

In this section we show how to derive the string action from the Gross-Pitaevskii equation in (1+1)-dimensions. The steps are similar to (2+1)-dimensions [105] or (3+1)-dimensions [103]. We choose the coordinates (t,z).

Let us start with the Gross-Pitaevskii Lagrangian:

$$\mathcal{L}_{GP}=i\phi^\dagger\partial_t\phi-\frac{1}{2m}(\partial_z\phi^\dagger)(\partial_z\phi)-\frac{g}{2}(|\phi|^2-\rho_0)^2. \tag{4.2.1}$$

Varying it with respect to $\phi^\dagger$, we obtain a version of the Gross-Pitaevskii equation:

$$i\,\partial_t\phi+\frac{1}{2m}\partial_z^2\phi-g\left(|\phi|^2-\rho_0\right)\phi=0. \tag{4.2.2}$$

We parametrize ϕ as

$$\phi=\sqrt{\rho}\,e^{i\eta}, \tag{4.2.3}$$

where η is the Goldstone boson, and ρ can be thought of as the Higgs boson. It is easy to derive

$$\begin{aligned}
\phi^\dagger &= \sqrt{\rho}\, e^{-i\eta}\,, \\
\partial_t\phi &= \frac{\dot{\rho}}{2\sqrt{\rho}}\, e^{i\eta} + \sqrt{\rho}\, i\, e^{i\eta}\dot{\eta}\,, \\
\partial_z\phi &= \frac{1}{2\sqrt{\rho}}\, e^{i\eta}\,(\partial_z\rho) + \sqrt{\rho}\, i\, e^{i\eta}(\partial_z\eta)\,.
\end{aligned} \tag{4.2.4}$$

Hence, the original Gross-Pitaevskii Lagrangian (4.2.1) becomes

$$\mathcal{L} = \frac{i\dot{\rho}}{2} - \rho\dot{\eta} - \frac{\rho}{2m}(\partial_z\eta)^2 - \frac{(\partial_z\rho)^2}{8m\rho} - \frac{g}{2}(\rho-\rho_0)^2\,. \tag{4.2.5}$$

If we drop out the first term as a total derivative, and define

$$\mathcal{L}_1 \equiv -\rho\dot{\eta} - \frac{\rho}{2m}(\partial_z\eta)^2\,, \tag{4.2.6}$$

$$\mathcal{L}_2 \equiv -\frac{(\partial_z\rho)^2}{8m\rho} - \frac{g}{2}(\rho-\rho_0)^2\,, \tag{4.2.7}$$

then the Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2\,. \tag{4.2.8}$$

Pay attention to that in Eq. (4.2.3) the field η takes values in $\mathbb{R}/2\pi\mathbb{Z}$, but now we temporarily release this condition, and it has values in $\mathbb{R}$. The constraint will be imposed later, which will give another piece to the theory.

We see that the term $-\rho\dot{\eta}$ in $\mathcal{L}_1$ is not written in a Lorentz-invariant way. We can complete ρ to a two-vector $f^\mu = (\rho, f)$, where ρ is its zeroth component, and f is an auxiliary field. Then one can show the following identity:

$$\mathcal{L}_1 + \frac{m}{2\rho}\left(f - \frac{\rho}{m}\partial_z\eta\right)^2 = -\rho\dot{\eta} + \frac{m}{2\rho}f^2 - f\partial_z\eta = -f^\mu\partial_\mu\eta + \frac{m}{2\rho}f^2\,, \tag{4.2.9}$$

where

$$f^\mu = (\rho, f)\,, \tag{4.2.10}$$

$$\partial_\mu\eta = (\dot{\eta}, \partial_z\eta)\,. \tag{4.2.11}$$

If we define the path integral measure to be

$$\int \mathcal{D}f \exp\left[i\int d^2x\, \frac{m}{2\rho}f^2\right] = 1\,, \tag{4.2.12}$$

then

$$e^{i\int d^2x\,\mathcal{L}_1} = \int \mathcal{D}f \exp\left[i\int d^2x\,\left(\mathcal{L}_1 + \frac{m}{2\rho}\left(f - \frac{\rho}{m}\partial_z\eta\right)^2\right)\right]$$
$$= \int \mathcal{D}\vec{f} \exp\left[i\int d^2x\,\left(-f^\mu\partial_\mu\eta + \frac{m}{2\rho}f^2\right)\right]. \tag{4.2.13}$$

Integrating η out, we obtain

$$\partial_\mu f^\mu = 0\,, \tag{4.2.14}$$

which can be solved locally by

$$f^\mu = \epsilon^{\mu\nu}H_\nu \tag{4.2.15}$$

with $H = dB$. Using this expression of f^μ we obtain

$$\frac{m}{2\rho}f^2 = \frac{m}{2\rho}H_0^2\,. \tag{4.2.16}$$

To rewrite $\mathcal{L}_2$ we first split B into the background part and the fluctuation part:

$$B = B^{(0)} + b\,, \tag{4.2.17}$$

and correspondingly,

$$H_\nu = H_\nu^{(0)} + h_\nu\,. \tag{4.2.18}$$

Since

$$\rho = f^0 = \epsilon^{01}H_1 = H_1\,, \tag{4.2.19}$$

we can rewrite $\mathcal{L}_2$ as

$$\mathcal{L}_2 = -\frac{(\partial_z h_1)^2}{8m\rho} - \frac{g}{2}h_1^2\,. \tag{4.2.20}$$

Consequently,

$$\int \mathcal{D}\rho\mathcal{D}\eta \exp\left[i\int d^2x\,(\mathcal{L}_1 + \mathcal{L}_2)\right]$$
$$= \int \mathcal{D}B \exp\left[i\int d^2x\,\left(-\frac{g}{2}\eta^{\mu\nu}h_\mu h_\nu - \frac{(\partial_z h_1)^2}{8m\rho}\right)\right], \tag{4.2.21}$$

where

$$\eta^{\mu\nu} \equiv \mathrm{diag}\left(\frac{m}{\rho g}, 1\right). \tag{4.2.22}$$

For the BEC system that we are interested in, we can drop the term $\sim (\partial_z h_1)^2$, because it contributes to the dispersion relation only in the UV:

$$\omega^2 = c_s^2 k^2 + \frac{\sim k^4}{m^2}\,. \tag{4.2.23}$$

If we are only interested in IR physics, we can drop the higher-order terms.

Now we return to the point that the theory should be invariant under $\eta \to \eta + 2\pi$, which we have not taken into account so far. The difference comes in Eq. (4.2.13). Now we cannot simply integrate out η, instead

$$- f^\mu \partial_\mu \eta = -f^\mu \partial_\mu \eta_{\text{vortex}} - f^\mu \partial_\mu \eta_{\text{smooth}} \,, \tag{4.2.24}$$

where we can only integrate out η_{smooth}, which still induces the constraint

$$\partial_\mu f^\mu = 0 \,. \tag{4.2.25}$$

For η_{vortex} there is

$$\begin{aligned} -f^\mu \partial_\mu \eta_{\text{vortex}} &= -\epsilon^{\mu\nu} \partial_\nu B \, \partial_\mu \eta_{\text{vortex}} \\ &= B \, \epsilon^{\mu\nu} \partial_\mu \partial_\nu \eta_{\text{vortex}} \\ &= 2\pi B \, \delta^2 \left(X^\mu - X^\mu(\tau, \sigma) \right) \,, \end{aligned} \tag{4.2.26}$$

where X^μ denote the position of the vortex. Formally, one can write its integral as

$$- \int d^2x \, f^\mu \partial_\mu \eta_{\text{vortex}} = \mu_1 \int d^2x \, B \, \delta^2(X^\mu - X^\mu(\tau, \sigma)) = \mu_1 \int_{\Sigma_\alpha} B \,. \tag{4.2.27}$$

with $\mu_1 = 2\pi$. Therefore, the theory can be written as

$$\int \mathcal{D}B \exp \left[i \int d^2x \left(-\frac{g}{2} \eta^{\mu\nu} h_\mu h_\nu \right) + i \mu_1 \int_{\Sigma_\alpha} B \right] \,. \tag{4.2.28}$$

Finally, as in Ref. [103], we add a term by hand to take into account the string tension induced by the vortices.

$$S_{\text{eff}} = \sum_\alpha \left[-c_s \tau_{1,\text{bare}} \int_{\Sigma_\alpha} dt \, d\theta \, |\partial_\theta \vec{X}_\alpha| + \mu_1 \int_{\Sigma_\alpha} B \right] - \int d^2x \, \frac{g}{2} h^2 \,, \tag{4.2.29}$$

where α runs over the several separate vortices. One can see that we obtain a string-like action from the Gross-Pitaevskii equation. The next question is how to find D-brane solution and study its dynamics of this nonlinear sigma model. This is one of my current research projects.

The Gross-Pitaevskii equation, or the nonlinear Schödinger equation, is an integrable model in (1+1)D. Hence, we expect the integrability also on the nonlinear sigma model side.[1] The proof will be presented in a paper appearing soon.

[1] I would like to thank Pedro Vieira for discussions on this point.

4.2.2 Derrick's Theorem, 1D and 2D Solutions

In this section, we would like to see the soliton solutions from purely field theoretical point of view. Before we discuss the explicit solutions, let us first make some general considerations.

In field theory, there is a well-known so called Derrick's Theorem, which tells us the existence of the soliton solution based on very simple arguments. The simplest version is to consider the scalar theory with a potential

$$\mathcal{L} = -(\partial^\mu \varphi)(\partial_\mu \varphi) - V(\varphi) \tag{4.2.30}$$

with

$$V(\varphi) \geq 0\,. \tag{4.2.31}$$

The theorem says that there is no stable soliton solution for the dimensions $D \geq 3$, while for $D = 2$ the soliton solution exists only when $V(\varphi) = 0$, and for $D = 1$ the soliton solution always exists. The Gross-Pitaevskii equation is a theory of this kind, hence the soliton-like solutions only exist for $D = 1$ or 2.

One may find some dark soliton solutions to the Gross-Pitaevskii equation in 3D, but according to the Derrick's theorem they must be quasi 1D or 2D. Therefore, we only need to focus on these two cases. For 2D in order to have a soliton solution, $V(\varphi)$ has to vanish, then the theory becomes a free theory:

$$\rho = \rho_0\,, \quad \mathcal{L} = -\rho_0 \dot{\eta} - \frac{\rho_0}{2m} (\nabla \eta)(\nabla \eta)\,, \tag{4.2.32}$$

where ρ and η are defined as

$$\varphi = \sqrt{\rho}\, e^{i\eta}\,. \tag{4.2.33}$$

According to Chapter 19 of Ref. [107], although this model is quite simple, it indeed has a D-brane solution on $\mathbb{R} \times S^1$.

Next, we want to use the BPS procedure to find the soliton solutions in 1D. The BPS procedure can be summarized as follows. Suppose that the energy of the system is given by

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_x \phi)^2 + (W')^2 \right]\,, \tag{4.2.34}$$

where W is a functional of the field ϕ, and

$$W' \equiv \frac{\partial W}{\partial \phi}\,. \tag{4.2.35}$$

Then

$$\begin{aligned}
E &= \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{\sqrt{2}} \partial_x \phi - W' \right)^2 + \sqrt{2}\, W' \partial_x \phi \right] \\
&= \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{\sqrt{2}} \partial_x \phi - W' \right)^2 + \sqrt{2}\, \frac{\partial W}{\partial x} \right] \\
&= \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{\sqrt{2}} \partial_x \phi - W' \right)^2 \right] + \sqrt{2} \left[W(+\infty) - W(-\infty) \right] . \qquad (4.2.36)
\end{aligned}$$

If $W(+\infty)$ and $W(-\infty)$ correspond to different vacua, the soliton solution is then given by the solution of the equation

$$\partial_x \phi = \sqrt{2} W' . \qquad (4.2.37)$$

A scalar field theory is in general given by

$$\mathcal{L} = -\frac{1}{2} (\partial_x \phi)^2 - V(\phi) , \qquad (4.2.38)$$

which implies the field equation

$$\partial_x^2 \phi - V'(\phi) = 0 . \qquad (4.2.39)$$

If there is the relation

$$V = (W')^2 , \qquad (4.2.40)$$

then the BPS equation (4.2.37) implies the field equation (4.2.39), since

$$\partial_x^2 \phi = \partial_x (\sqrt{2}\, W') = \sqrt{2}\, W'' \frac{\partial \phi}{\partial x} = \sqrt{2}\, W'' \sqrt{2}\, W' = 2 W' W'' , \qquad (4.2.41)$$

which is exactly the field equation

$$\partial_x^2 \phi = V' = 2 W' W'' . \qquad (4.2.42)$$

For the Gross-Pitaevskii equation, the energy is given by Eq. (5.58) in the book "Bose-Einstein Condensation" by Pitaevskii and Stringari:

$$E = \int_{-\infty}^{\infty} dz \left[\frac{\hbar^2}{2m} \left| \frac{d\Psi_0}{dz} \right|^2 + \frac{g}{2} \left(|\Psi_0|^2 - n \right)^2 \right] , \qquad (4.2.43)$$

where

$$\Psi_0 = \sqrt{n}\, f \exp\left[-\frac{i \mu t}{\hbar} \right] , \qquad (4.2.44)$$

where f is in general complex

$$f = f_1 + i f_2 , \qquad (4.2.45)$$

and we can choose $f_2 = \frac{v}{c}$. We also define

$$\phi \equiv \frac{\hbar}{\sqrt{m}} \Psi_0 \,, \tag{4.2.46}$$

then the energy becomes

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dz \left[\frac{1}{2} \left| \frac{d\phi}{dz} \right|^2 + \frac{g}{2} \left(\frac{m}{\hbar^2} |\phi|^2 - n \right)^2 \right] \\ &= \int_{-\infty}^{\infty} dz \left[\frac{1}{2} \left(\frac{d|\phi|}{dz} \right)^2 + \frac{g}{2} \left(\frac{m}{\hbar^2} |\phi|^2 - n \right)^2 \right] \,. \end{aligned} \tag{4.2.47}$$

According to what we discussed before, we can immediately write down the BPS equation for the soliton:

$$\frac{d|\phi|}{dz} = \sqrt{g} \left(n - \frac{m}{\hbar^2} |\phi|^2 \right) \quad \text{or} \quad \frac{d|\phi|}{dz} = \sqrt{g} \left(\frac{m}{\hbar^2} |\phi|^2 - n \right) \,. \tag{4.2.48}$$

The solutions to these two equations only differ by a minus sign. Let us consider the first equation, which is equivalent to

$$\begin{aligned} & \frac{\hbar}{\sqrt{m}} \frac{d|\Psi_0|}{dz} = \sqrt{g} \left(n - |\Psi_0|^2 \right) \\ \Rightarrow \quad & \frac{\hbar}{\sqrt{m}} \frac{df}{dz} = \sqrt{gn} (1 - |f|^2) \\ \Rightarrow \quad & \frac{\hbar}{\sqrt{m}} \frac{df_1}{dz} = \sqrt{gn} (1 - \frac{v^2}{c^2} - {f_1}^2) \,, \quad \frac{\hbar}{\sqrt{m}} \frac{i \, df_2}{dz} = 0 \,. \end{aligned} \tag{4.2.49}$$

For $v = 0$ the equation above simplifies to

$$\sqrt{2} \xi \, \frac{df_1}{dz} = 1 - {f_1}^2 \,, \tag{4.2.50}$$

which is exactly the dark soliton solution for $v = 0$ discussed in Ref. [106]. The solution to this equation is the dark soliton:

$$\Psi_0(z) = \sqrt{n} \tanh \left[\frac{z}{\sqrt{2} \xi} \right] \,. \tag{4.2.51}$$

If we perform a Galilean boost to the first one of Eq. (4.2.49) using the method described in Ref. [108], then it becomes

$$\sqrt{2} \xi \, \frac{df_1}{dz'} = 1 - \frac{v^2}{c^2} - {f_1}^2 \,, \tag{4.2.52}$$

where $z' \equiv z - vt$. This new equation is exactly the same as Eq. (4.1.14) for an arbitrary constant v that we quoted from Ref. [106], and its solution is

$$\Psi_0(z - vt) = \sqrt{n} \left(i \frac{v}{c} + \sqrt{1 - \frac{v^2}{c^2}} \tanh \left[\frac{z - vt}{\sqrt{2} \xi} \sqrt{1 - \frac{v^2}{c^2}} \right] \right) \,, \tag{4.2.53}$$

which includes both the dark soliton solution and the grey soliton solution.

On top of the soliton solution, one can do linear perturbations, as Eqs. (5.65) and (5.66) in the book by Pitaevskii and Stringari:

$$\Psi_0(r,t) = \left[\Psi_0(r) + \vartheta(r,t)\right]\, e^{-i\mu t/\hbar}\,, \tag{4.2.54}$$

$$\vartheta(r,t) = \sum_i \left[u_i(r)\, e^{-i\omega_i t} + v_i^*(r)\, e^{i\omega_i t}\right]\,, \tag{4.2.55}$$

which will consequently lead to the Bogoliubov - de Gennes equations, which we omit for the moment.

4.3 KPZ Equation

4.3.1 Review of the KPZ Equation

The Kardar-Parisi-Zhang equation (KPZ equation) was first introduced in Ref. [109]. Recently, people also found that it can be related to the Gross-Pitaevskii equation [104]. In this section, we review the KPZ equation and its relation with the Gross-Pitaevskii equation, and some comments about the possible relation between the KPZ equation and the string action will also be made. We follow closely the discussion about the KPZ equation in Ref. [105].

For a D-dimensional space with coordinates $\vec{x}$, a random surface grows on top of it with the height $h(\vec{x},t)$. Then the growth of the surface is governed by the Kardar-Parisi-Zhang equation

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\vec{x},t)\,, \tag{4.3.1}$$

where the term $\nu\nabla^2 h$ is similar to a diffusion equation, which tends to smooth out the surface, and the term $\frac{\lambda}{2}(\nabla h)^2$ makes the equation nonlinear, while $\eta(\vec{x},t)$ is a random variable. For a Gaussian random variable, the probability distribution for $\eta(\vec{x},t)$ is

$$P(\eta) \propto e^{-\frac{1}{2\sigma^2}\int d^D x\, dt\, \eta(\vec{x},t)^2}\,. \tag{4.3.2}$$

In Ref. [105] it is proven that the KPZ equation can be mapped into a quantum field theory. It works as follows:

$$\begin{aligned} Z &\equiv \int \mathcal{D}h \int \mathcal{D}\eta\, e^{-\frac{1}{2\sigma^2}\int d^D x\, dt\, \eta(\vec{x},t)^2}\, \delta\left[\frac{\partial h}{\partial t} - \nu\nabla^2 h - \frac{\lambda}{2}(\nabla h)^2 - \eta(\vec{x},t)\right] \\ &= \int \mathcal{D}h\, e^{-S(h)}\,, \end{aligned} \tag{4.3.3}$$

where

$$S(h) = \frac{1}{2\sigma^2}\int d^D\vec{x}\, dt\, \left[\frac{\partial h}{\partial t} - \nu\nabla^2 h - \frac{\lambda}{2}(\nabla h)^2\right]^2\,. \tag{4.3.4}$$

By rescaling $t \to t/\nu$ and $h \to \sqrt{\sigma^2/\nu}\, h$, we obtain

$$S(h) = \frac{1}{2} \int d^D \vec{x}\, dt \left[\left(\frac{\partial}{\partial t} - \nabla^2 \right) h - \frac{g}{2} (\nabla h)^2 \right]^2 , \tag{4.3.5}$$

where

$$g^2 \equiv \frac{\lambda^2 \sigma^2}{\nu^3} . \tag{4.3.6}$$

We can also expand the action into the powers of h and rewrite it as

$$S(h) = \frac{1}{2} \int d^D \vec{x}\, dt \left[\left(\left(\frac{\partial}{\partial t} - \nabla^2 \right) h \right)^2 - g (\nabla h)^2 \left(\frac{\partial}{\partial t} - \nabla^2 \right) h + \frac{g^2}{4} (\nabla h)^4 \right] . \tag{4.3.7}$$

To calculate some physical quantities like $\langle [h(\vec{x}, t) - h(\vec{x}', t)]^2 \rangle$, we should couple the field h to a source and compute the generating functional

$$Z[J] = \int \mathcal{D}h\, e^{-S(h) + \int d^D x\, dt\, J(x,t)\, h(x,t)} . \tag{4.3.8}$$

If we define a new variable

$$U \equiv e^{\frac{1}{2} g h} , \tag{4.3.9}$$

we can even rewrite the action (4.3.5) into a nonlinear sigma model

$$S = \frac{2}{g^2} \int d^D \vec{x}\, dt \left(U^{-1} \frac{\partial}{\partial t} U - U^{-1} \nabla^2 U \right)^2 . \tag{4.3.10}$$

4.3.2 KPZ Equation and GP Equation

Recently, Ref. [104] discussed how to map the Gross-Pitaevskii equation (GP equation) to the KPZ equation. Strictly speaking, one does not map the GP equation directly to the KPZ equation, instead one maps the conservation law of the GP equation plus some random variable (noise) to the KPZ equation. Let us review the approach of Ref. [104] in the following.

Let us recall the Gross-Pitaevskii equation

$$i \partial_t \psi = -\frac{1}{2m} \partial_x^2 \psi + g |\psi|^2 \psi \tag{4.3.11}$$

and the parametrization

$$\psi(x, t) = \sqrt{\rho(x, t)}\, e^{i\theta(x,t)} . \tag{4.3.12}$$

The velocity is given by

$$v(x, t) = \frac{1}{m} \frac{\partial \theta(x, t)}{\partial x} . \tag{4.3.13}$$

The continuity equation and the Euler equation of the system are

$$\partial_t \rho + \partial_x(\rho v) = 0\,, \tag{4.3.14}$$

$$\partial_t v + \partial_x \left(\frac{v^2}{2} + \frac{g}{m}\rho\right) = 0\,, \tag{4.3.15}$$

or equivalently written in a compact form as

$$\partial_t \vec{u} + \partial_x A\vec{u} = 0\,, \tag{4.3.16}$$

where

$$\vec{u} = \begin{pmatrix} \rho \\ v \end{pmatrix}\,, \quad A = \begin{pmatrix} 0 & \rho_0 \\ \frac{g}{m} & 0 \end{pmatrix}\,. \tag{4.3.17}$$

Next, we add the diffusion and the noise to the equation above, and obtain

$$\partial_t \vec{u} + \partial_x \left[A\vec{u} + \frac{1}{2}\sum_{\alpha,\beta=1}^{2} \vec{H}_{\alpha,\beta}\, u_\alpha\, u_\beta - \partial_x(D\vec{u}) + B\vec{\xi}\right] = 0\,, \tag{4.3.18}$$

where D and B are diffusion and noise matrix respectively, while

$$H^\gamma_{\alpha,\beta} \equiv \partial_{u_\alpha}\partial_{u_\beta}\, j^\gamma \quad \text{with} \quad \vec{j} = (\rho v, \frac{1}{2}v^2)\,. \tag{4.3.19}$$

By rotating the vector $\vec{u}$ in the following way:

$$\begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} = R\begin{pmatrix} \rho \\ v \end{pmatrix} \quad \text{with} \quad R = \frac{1}{c\sqrt{2c_1}}\begin{pmatrix} -c & \rho_0 \\ c & \rho_0 \end{pmatrix}\,, \tag{4.3.20}$$

where

$$c \equiv \sqrt{\frac{g\rho_0}{m}}\,, \quad c_1 \equiv \int dx\, \left(\langle \rho(x,0)\,\rho(0,0)\rangle - \rho_0^2\right)\,, \tag{4.3.21}$$

we obtain

$$\partial_t \phi_x \left[\sigma c \phi_\sigma + \langle \vec{\phi}, G^\sigma\, \vec{\phi} - \partial_x(D_{\text{rot}}\phi)_\sigma + (B_{\text{rot}}\xi)_\sigma\right] = 0\,, \tag{4.3.22}$$

where $\sigma = \pm$,

$$D_{\text{rot}} = RDR\,, \quad B_{\text{rot}} = RB\,, \tag{4.3.23}$$

and

$$G^- = \frac{c}{2\rho_0}\sqrt{\frac{c_1}{2}}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\,, \quad G^+ = \frac{c}{2\rho_0}\sqrt{\frac{c_1}{2}}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\,. \tag{4.3.24}$$

Since in practice the terms $\phi_\sigma\phi_{-\sigma}$ and $(\phi_{-\sigma})^2$ are negligible compared to $(\phi_\sigma)^2$, we can drop them and the remaining equation becomes

$$\partial_t \phi_\sigma + \partial_x \left[\sigma c \phi_\sigma + G^\sigma_{\sigma\sigma}\phi_\sigma^2 - \partial_x(D_{\text{rot}}\phi)_\sigma + (B_{\text{rot}}\xi)_\sigma\right] = 0\,, \tag{4.3.25}$$

which is known as the stochastic Burgers equation, or a KPZ equation with $h = \partial_x\phi_\sigma$. So finally we have mapped the Gross-Pitaevskii equation into a KPZ equation.

4.3.3 KPZ Equation and String Theory

We have seen in the previous sections that a Gross-Pitaevskii equation can be mapped into a string theory-like nonlinear sigma model and also into a KPZ equation at least in some limits. A natural question is if one can map the string theory-like nonlinear sigma model directly into a KPZ equation, and an even further question is if one can understand the string worldsheet theory as the growth of a random surface. It turns out that for the first question at least in some special limits the answer is yes. We demonstrate how to perform this mapping in the following.

Let us recall the intermediate expression of the string action (4.2.21):

$$Z = \int \mathcal{D}B \exp\left[i \int d^2x \left(-\frac{g}{2}\eta^{\mu\nu} h_\mu h_\nu - \frac{(\partial_z h_1)^2}{8m\rho} \right) \right] \tag{4.3.26}$$

$$= \int \mathcal{D}B \exp\left[i \int d^2x \left(-\frac{g}{2}\left[-\frac{m}{\rho g}(\partial_t B)(\partial_t B) + (\partial_z B)(\partial_z B) \right] - \frac{(\partial_z^2 B)^2}{8m\rho} \right) \right] . \tag{4.3.27}$$

Applying the classical equation of motion $\partial_t B \sim \partial_z^2 B$, we can rewrite the term $(\partial_z B)(\partial_z B)$ as a total derivative and hence drop it.

In the KPZ action (4.3.3), if we set the nonlinear term $\frac{\lambda}{2}(\nabla h)^2$ to zero, and apply the classical equation of motion without the random variable and the nonlinear term, i.e.,

$$\frac{\partial}{\partial t} h = \nu \nabla^2 h \,, \tag{4.3.28}$$

we also obtain a theory given by

$$Z = \int \mathcal{D}h \, e^{-S(h)} \tag{4.3.29}$$

with

$$S(h) = \frac{1}{2\sigma^2} \int d^D\vec{x}\, dt \left[\left(\frac{\partial h}{\partial t} \right)^2 - \left(\nu \nabla^2 h \right)^2 \right] . \tag{4.3.30}$$

Hence, for the limit $\lambda \to 0$ the string theory-like nonlinear sigma model is equivalent to the KPZ theory at classical level.

Chapter 5

2-Dimensional Point-Vortex Model

In recent years, the interest in two-dimensional quantum turbulence has revived [110, 111], partly because of the developments in the experimental technique for the Bose-Einstein condensates (BEC).

On the theoretical side, the Gross-Pitaevskii equation provides an effective mean-field theory describing BEC systems. As we reviewed in the previous chapter, it has various nontrivial solutions, e.g. vortex line and dark soliton. If we focus on a system consisting of only vortex solutions, it can be well described by a statistical model called point-vortex model. Joyce and Montgomery showed that this model exhibits a negative temperature phase [112]. This model was further studied by Edwards and Taylor in Ref. [113], and they found the clustering of vortices at negative temperature. Smith and O'Neil investigated the single-charge case and pointed out that the transition between the symmetric and the asymmetric phase resembles a second-order phase transition [114, 115]. Recent results from Monte Carlo simulations [116] suggest that for the two-charge point-vortex system, there is also a second-order phase transition at a negative temperature.

Besides the study of the model itself, in mathematical physics the so-called sinh-Poisson equation or elliptic sinh-Gordon equation, which lies at the center of the model, was studied by different groups [117, 118, 119], and they found some nontrivial solutions at negative temperature.

In this chapter, we study the mean-field theory of the phase transition at negative temperature of the two-charge point-vortex model. We first briefly review the original model discussed by Joyce and Montgomery in Ref. [112]. Then we extend the method introduced by Ref. [115] to study the phase transition of the two-charge system at negative temperature. For the two-charge system in a square box, we also confirm the phase transition in the spirit of Ref. [113]. This chapter is mainly based on some of my unpublished notes and a paper by me, X. Yu, T. Billam, M. Reeves and A. Bradley [120], which will appear soon.

5.1 Review of the Model

In this section we review the point-vortex model first set up in Ref. [112]. We consider the following Hamiltonian

$$H = \frac{1}{2}\sum_{i\neq j}(N_i^+ - N_i^-)\phi_{ij}(N_j^+ - N_j^-)\,, \tag{5.1.1}$$

where

$$\phi_{ij} = -\frac{2\,e^2}{\ell}\ln|\vec{r}_i - \vec{r}_j|, \tag{5.1.2}$$

and ℓ is a length scale in the model, while $N_i^{(\pm)}$ is the number of positive (negative) vortices in a cell respectively. The particle numbers are fixed

$$\sum_i N_i^+ = N\,, \quad \sum_i N_i^- = N\,. \tag{5.1.3}$$

The number of states is given by

$$W = \left[N!\prod_i \frac{\Delta^{N_i^+}}{N_i^+!}\right]\cdot\left[N!\prod_i \frac{\Delta^{N_i^-}}{N_i^-!}\right]\,, \tag{5.1.4}$$

where Δ is the cell area. The entropy is $S = \log W$.

The typical state is obtained by maximizing $S[\{N_i^+, N_i^-\}]$ with the fixed number of vortices N and the total energy E. The target function is

$$\log W + \alpha^+\left(N - \sum_i N_i^+\right) + \alpha^-\left(N - \sum_i N_i^-\right) + \beta\left(E - \frac{1}{2}\sum_{i\neq j}(N_i^+ - N_i^-)\phi_{ij}(N_j^+ - N_j^-)\right)\,, \tag{5.1.5}$$

where $\alpha^\pm$ and β are Lagrange multipliers. In the large N limit, we apply Sterling formula to the leading order

$$\log N! \sim N\ln N - N + \cdots\,. \tag{5.1.6}$$

Varying the expression (5.1.5) with respect to N_i^+ and N_i^- gives us

$$\ln N_i^+ + \alpha^+ + \beta\sum_j(N_j^+ - N_j^-)\phi_{ij} = 0\,, \tag{5.1.7}$$

$$\ln N_i^- + \alpha^- - \beta\sum_j(N_j^+ - N_j^-)\phi_{ij} = 0\,. \tag{5.1.8}$$

Taking the sum and the difference of the two equations above, we obtain

$$N_i^+ \, N_i^- = e^{-\alpha^+ - \alpha^-} = \text{const} \quad \text{for all } i \,, \tag{5.1.9}$$

and

$$N_i^+ - N_i^- = e^{-\alpha^+ - \beta \sum_j \phi_{ij} (N_j^+ - N_j^-)} - e^{-\alpha^- + \beta \sum_j \phi_{ij} (N_j^+ - N_j^-)} \,. \tag{5.1.10}$$

One can then take the continuous limit

$$\begin{aligned} \Delta &\to 0 \,, \\ N_i^+ - N_i^- &\to \sigma(\vec{r})\Delta \,, \\ e^{-\alpha^\pm/\Delta} &\to n^0_\mp \,, \\ \pm\beta \sum_j \phi_{ij}(N_j^+ - N_j^-) &\to \pm\beta \int d\vec{r}' \, \phi(\vec{r} - \vec{r}') \, \sigma(\vec{r}') \,, \end{aligned} \tag{5.1.11}$$

where

$$\phi(\vec{r} - \vec{r}') = -\frac{2e^2}{\ell} \ln |\vec{r} - \vec{r}'|. \tag{5.1.12}$$

In this limit, Eq. (5.1.10) becomes an integral equation

$$\sigma(\vec{r}) = n^0_+ \, e^{-\beta \int d\vec{r}_1 \, \phi(\vec{r} - \vec{r}_1) \, \sigma(\vec{r}_1)} - n^0_- \, e^{+\beta \int d\vec{r}_1 \, \phi(\vec{r} - \vec{r}_1) \, \sigma(\vec{r}_1)} = n^+(\vec{r}) - n^-(\vec{r}). \tag{5.1.13}$$

To obtain a differential equation, we define

$$\psi(\vec{r}) \equiv \int d\vec{r}_1 \, \phi(\vec{r} - \vec{r}_1) \, \sigma(\vec{r}_1) \,, \tag{5.1.14}$$

which satisfies the Poisson equation

$$\nabla^2 \psi = -\frac{4\pi e^2}{\ell} \sigma(\vec{r}) = -\frac{4\pi e^2}{\ell} \left[n^0_+ \, e^{-\beta\psi} - n^0_- \, e^{\beta\psi} \right] \,. \tag{5.1.15}$$

The constants $n^0_\pm$ are determined by the constraints

$$\int d\vec{r} \, n_+ \equiv n^0_+ \int d\vec{r} \, e^{-\beta\psi} = N \,, \tag{5.1.16}$$

$$\int d\vec{r} \, n_- \equiv n^0_- \int d\vec{r} \, e^{+\beta\psi} = N \,, \tag{5.1.17}$$

while the constant β, which can be interpreted as the inverse temperature, is determined by

$$\ell \int d\vec{r} \, \frac{(\nabla\psi)^2}{8\pi e^2} = E \,. \tag{5.1.18}$$

From the last expression we see that the energy of the system is positive definite.

In the limit $\beta^2\psi^2 \ll 1$, $n_+ \approx n_- = n_0 = \frac{N}{V}$, and Eq. (5.1.13) becomes a well-known equation in mathematical physics, which is called the sinh-Poisson equation or the elliptic sinh-Gordon equation. In some cases, it can be solved exactly [117, 118, 119]. Up to leading order in β, we have

$$\sigma(\vec{r}) = -2n_0\beta \int d\vec{r}_1\, \phi(\vec{r}-\vec{r}_1)\, \sigma(\vec{r}_1) \tag{5.1.19}$$

$$\Rightarrow \quad \beta = -\frac{1}{2n_0} \frac{\int d\vec{r}\, \sigma^2(\vec{r})}{\int d\vec{r} \int d\vec{r}_1\, \sigma(\vec{r})\, \phi(\vec{r}-\vec{r}_1)\, \sigma(\vec{r}_1)}\,, \tag{5.1.20}$$

which indicates that the temperature is negative.

5.2 System in a Disc

In this section, we extend the method developed in Ref. [115] to analyze the two-charge point-vortex system in a disc.

5.2.1 Review of the Approach by Smith and O'Neil

Onset of ordered phase

R is the radius of the disc. We rescale the length $r \to rR$, the density $n_\pm$ by $\frac{N}{R^2}$, the potential ψ by $\frac{Ne^2}{\ell}$ and the energy E by $\frac{N^2e^2}{\ell}$. Then the radial coordinate r takes values in $[0, 1]$. The constraints (5.1.16) - (5.1.18) are normalized to be

$$1 = \int d^2r\, n_+\,, \quad 1 = \int d^2r\, n_-\,, \quad E = \frac{1}{2}\int d^2r\, (n_+ - n_-)\psi\,. \tag{5.2.1}$$

The Poisson equation becomes

$$\nabla^2\psi = -4\pi\sigma(\vec{r})\,, \tag{5.2.2}$$

with $\sigma(\vec{r}) = n_+(\vec{r}) - n_-(\vec{r})$ and $n_\pm(\vec{r}) = n_\pm^0 \exp\left[\mp\beta\psi(\vec{r})\right] = \exp\left[\mp\beta\psi(\vec{r}) + \gamma_\pm\right]$, where $n_\pm^0 = \exp(\gamma_\pm)$ are constants. Eq. (5.2.2) is a nonlinear equation of $\psi(\vec{r})$. We expect that there is a bifurcation point at certain energy. Before the bifurcation point, Eq.(5.2.2) only has axisymmetric solutions and after the bifurcation point Eq.(5.2.2) has nonaxisymmetric solutions which have higher entropy. We therefore associate this bifurcation point as a phase transition point.

In the following we are going to locate the bifurcation point. We start at one axisymmetric solution of Eq.(5.2.2) at energy E and consider a nearby solution $n_\pm + \delta n_\pm$ at $E + \delta E$, and there are the following relations:

$$\delta n_+ = n_+(-\delta\beta\,\psi - \beta\,\delta\psi + \delta\gamma_+) + \mathcal{O}(\delta E^2)\,, \tag{5.2.3}$$

$$\delta n_- = n_-(\delta\beta\,\psi + \beta\,\delta\psi + \delta\gamma_-) + \mathcal{O}(\delta E^2)\,. \tag{5.2.4}$$

$\delta\gamma_-, \delta\gamma_+, \delta\beta$ are obtained from the constraint Eq. (5.2.1)

$$0 = \int d^2r\,\delta n_+\,, \tag{5.2.5}$$

$$0 = \int d^2r\,\delta n_-\,, \tag{5.2.6}$$

$$\delta E = \int d^2r\,(\delta n_+ - \delta n_-)\psi + \mathcal{O}(\delta E^2)\,. \tag{5.2.7}$$

Plugging Eq. (5.2.3) and Eq. (5.2.4) into Eqs. (5.2.5)-(5.2.7), we obtain

$$\begin{pmatrix} 0 \\ 0 \\ \delta E \end{pmatrix} + \beta \int d^2r\,\delta\psi \begin{pmatrix} n_+ \\ n_- \\ (n_+ + n_-)\psi \end{pmatrix} = -M \begin{pmatrix} \delta\beta \\ \delta\gamma_+ \\ \delta\gamma_- \end{pmatrix}\,, \tag{5.2.8}$$

where

$$M \equiv \begin{pmatrix} n_+\psi & -n_+ & 0 \\ n_-\psi & 0 & n_- \\ (n_+ + n_-)\psi^2 & -n_+\psi & n_-\psi \end{pmatrix}\,. \tag{5.2.9}$$

Then

$$\begin{aligned} \begin{pmatrix} \delta\beta \\ \delta\gamma_+ \\ \delta\gamma_- \end{pmatrix} &= -M^{-1} \begin{pmatrix} 0 \\ 0 \\ \delta E \end{pmatrix} - \beta M^{-1} \int d^2r\,\delta\psi \begin{pmatrix} n_+ \\ n_- \\ (n_+ + n_-)\psi \end{pmatrix} \\ &= -M^{-1} \begin{pmatrix} 0 \\ 0 \\ \delta E \end{pmatrix} + \beta \begin{pmatrix} \mathscr{L}_\beta(\delta\psi) \\ \mathscr{L}_{\gamma_+}(\delta\psi) \\ \mathscr{L}_{\gamma_-}(\delta\psi) \end{pmatrix}\,, \end{aligned} \tag{5.2.10}$$

where

$$\begin{pmatrix} \mathscr{L}_\beta(\delta\psi) \\ \mathscr{L}_{\gamma_+}(\delta\psi) \\ \mathscr{L}_{\gamma_-}(\delta\psi) \end{pmatrix} \equiv -M^{-1} \int d^2r\,\delta\psi \begin{pmatrix} n_+ \\ n_- \\ (n_+ + n_-)\psi \end{pmatrix}\,. \tag{5.2.11}$$

Written in an equivalent expression,

$$\delta\beta = -(M^{-1})_{13}\,\delta E + \beta\mathscr{L}_\beta(\delta\psi)\,, \tag{5.2.12}$$

$$\delta\gamma_+ = -(M^{-1})_{23}\,\delta E + \beta\mathscr{L}_{\gamma_+}(\delta\psi)\,, \tag{5.2.13}$$

$$\delta\gamma_- = -(M^{-1})_{33}\,\delta E + \beta\mathscr{L}_{\gamma_-}(\delta\psi)\,. \tag{5.2.14}$$

Let us recall Eq. (5.2.2):

$$\nabla^2\psi = -4\pi\sigma(\vec{r}) = -4\pi\left[n_+(\vec{r}) - n_-(\vec{r})\right] .$$

Variation of this equation gives us

$$\begin{aligned}\nabla^2\delta\psi &= -4\pi\left[\delta n_+(\vec{r}) - \delta n_-(\vec{r})\right]\\ &= 4\pi(n_+ + n_-)\delta\beta\,\psi + 4\pi(n_+ + n_-)\beta\,\delta\psi - 4\pi n_+\,\delta\gamma_+ + 4\pi n_-\,\delta\gamma_- .\end{aligned} \tag{5.2.15}$$

Combining the expression above with Eqs. (5.2.12)-(5.2.14), we obtain

$$\begin{aligned}\left[\nabla^2 - 4\pi(n_+ + n_-)\beta\right]\delta\psi &= 4\pi(n_+ + n_-)\delta\beta\,\psi - 4\pi n_+\,\delta\gamma_+ + 4\pi n_-\,\delta\gamma_-\\ &= 4\pi(n_+ + n_-)\psi\left[-(M^{-1})_{13}\,\delta E + \beta\mathscr{L}_\beta(\delta\psi)\right]\\ &\quad - 4\pi n_+\left[-(M^{-1})_{23}\,\delta E + \beta\mathscr{L}_{\gamma_+}(\delta\psi)\right]\\ &\quad + 4\pi n_-\left[-(M^{-1})_{33}\,\delta E + \beta\mathscr{L}_{\gamma_-}(\delta\psi)\right] .\end{aligned} \tag{5.2.16}$$

$$\begin{aligned}\Rightarrow\quad &\left[\nabla^2 - 4\pi(n_+ + n_-)\beta - 4\pi(n_+ + n_-)\beta\psi\mathscr{L}_\beta + 4\pi n_+\beta\mathscr{L}_{\gamma_+} - 4\pi n_-\beta\mathscr{L}_{\gamma_-}\right]\delta\psi\\ &= 4\pi\left[-(n_+ + n_-)\psi(M^{-1})_{13} + n_+(M^{-1})_{23} - n_-(M^{-1})_{33}\right]\delta E .\end{aligned} \tag{5.2.17}$$

Let us define

$$\mathscr{L} \equiv \nabla^2 - 4\pi(n_+ + n_-)\beta - 4\pi(n_+ + n_-)\beta\psi\mathscr{L}_\beta + 4\pi n_+\beta\mathscr{L}_{\gamma_+} - 4\pi n_-\beta\mathscr{L}_{\gamma_-} , \tag{5.2.18}$$

$$A(\vec{r}) \equiv -(n_+ + n_-)\psi(M^{-1})_{13} + n_+(M^{-1})_{23} - n_-(M^{-1})_{33} , \tag{5.2.19}$$

then the equation above can be written as

$$\mathscr{L}\,\delta\psi = 4\pi A(\vec{r})\,\delta E . \tag{5.2.20}$$

Now the problem becomes a linear differential equation system. In our case, we know a priori that at the positive temperature the solution is axisymmetric, while at large negative temperature a nonaxisymmetric solution is expected. Hence, we anticipate that a bifurcation point appears in the solution of the differential equation above. The condition of the onset of nonaxisymmetric solutions is given by the zero mode of the operator $\mathscr{L}$, namely,

$$\mathscr{L}\,\delta\psi = 0 \tag{5.2.21}$$

on a unit disk D with the boundary condition $\delta\psi(\partial D) = 0$. We can decompose Eq. (5.2.21) in azimuthal Fourier harmonics with the mode number l. Then a nonaxisymmetric solution exists only when the linear equation

$$\mathscr{L}_l\,\delta\psi = 0 \quad (l > 0) \tag{5.2.22}$$

has a nonzero solution. Moreover, we can drop $\mathscr{L}_\beta$, $\mathscr{L}_{\gamma_+}$ and $\mathscr{L}_{\gamma_-}$ in the operator $\mathscr{L}$, because they are proportional to $\delta\beta$, $\delta\gamma_+$ and $\delta\gamma_-$ respectively, which are higher-order variations as we will see in the next section. Therefore, to confirm the existence of the bifurcation point, we only need to analyze the following differential equation:

$$\left[\nabla^2 - 4\pi(n_+ + n_-)\beta\right]|\delta\psi| = \left[\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{l^2}{r^2}\right) - 4\pi(n_+ + n_-)\beta\right]|\delta\psi| = 0\,, \qquad (5.2.23)$$

satisfying the boundary condition

$$\delta\psi(r=1) = 0\,, \quad \delta\psi(r=0) = 0\,, \qquad (5.2.24)$$

where

$$\delta\psi = |\delta\psi|\, e^{i\theta}\,. \qquad (5.2.25)$$

To estimate the critical temperature T_c for the system in a disc, we assume that at the critical point the condition $n_+ + n_- = 2\,n_0$ holds. This assumption is not true as we will see from the numerical results, but as a first estimate it still gives us an answer quite close to the accurate result. Under this assumption, Eq. (5.2.23) becomes essentially an eigenvalue problem, namely,

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{l^2}{r^2}\right)|\delta\psi| = -\lambda\,|\delta\psi| \qquad (5.2.26)$$

with the boundary condition $|\delta\psi(r=1)| = 0$. We only consider $l = 1$ due to the thermodynamic stability discussed in Ref. [115]. For $l = 1$ we have $\lambda = j_{1,1}^2$, and the corresponding eigenfunction is $J_1(\sqrt{\lambda}r)$, where $j_{1,1}$ is the first positive zero of of the Bessel function $J_1(r)$. The other zeros $j_{1,n}$ $(n > 1)$ also satisfy the eigenvalue equation and the boundary condition, but since $-\Delta|\delta\psi| \sim E|\delta\psi|$, we only choose the lowest energy state, which corresponds to $j_{1,1}$. Therefore,

$$|\delta\psi(r)| \sim J_1(\sqrt{\lambda}r)\,, \qquad (5.2.27)$$

and the instability condition of the disordered phase reads

$$-\lambda = 8\pi n_0\,\beta_c\,, \qquad (5.2.28)$$

i.e., the bifurcation point is at the temperature given by

$$T_c = -\frac{8\pi n_0}{\lambda}\,. \qquad (5.2.29)$$

To compare our result with the one in Ref. [116], we identify the units used in this chapter with the one adopted in Ref. [116]:

$$T_0 \equiv \frac{2Ne^2}{\ell} = \frac{\rho_s \kappa^2 N}{4\pi k_B}\,. \qquad (5.2.30)$$

Then in this unit our result has the value

$$T_c = -\frac{4\pi}{\lambda V} T_0 = -\frac{4\pi}{j_{1,1}^2 \pi} T_0 \approx -0.272\, T_0\,, \tag{5.2.31}$$

which is very close to the value $-0.25\, T_0$ obtained from simulation in Ref. [116].

A more accurate way of determining T_c is to solve Eq. (5.2.23) for $l = 1$ numerically. Eq. (5.2.23) should have a nontrivial solution only at the bifurcation point, while zero solutions elsewhere. By analyzing the numerical results, we can determine the critical temperature

$$T_c \approx -0.267\, T_0\,, \tag{5.2.32}$$

which takes place at

$$E_c = 0.153\, E_0\,, \tag{5.2.33}$$

with $E_0 \equiv \frac{Ne^2}{\ell}$. Fig. 5.1 shows the numerical result of $\beta(E)$, and the dashed line indicates the branch of the nontrivial solutions.

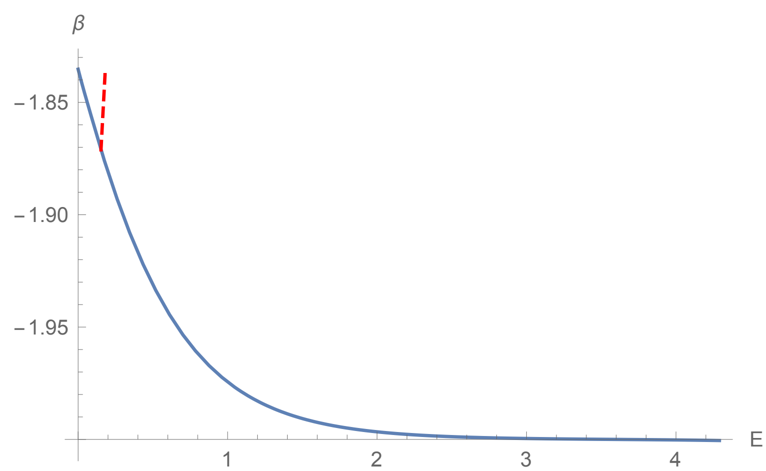


Figure 5.1: The β-E curve for the system in a disc

Before the bifurcation point, the system follows the solid line as increasing the energy, and then follow the dashed line after the bifurcation point. Hence, the bifurcation point is the lowest point of the β-E curve. In canonical ensemble, such a point corresponds to the divergence of specific heat, which is an indication of a phase transition. However, for the micro-canonical ensemble that we are working with, the physics meaning of the divergence of specific heat is not clear. Nevertheless, the behavior of the system on the two sides of the bifurcation point are quite different.

Higher-Order Analysis

To analyze the branch of the nontrivial solutions deviated from the bifurcation point, we need to consider higher-order terms in the variation of the Poisson equation (5.2.2) and the constraints (5.2.1).

We apply the method introduced in Appendix A of Ref. [115] to our case, i.e. the case with two kinds of charges and without the angular momentum conservation constraint. We still consider a system in a disc. Recall the Poisson equation (5.2.2)

$$\nabla^2 \psi = -4\pi\sigma\,, \tag{5.2.34}$$

and its variation

$$\nabla^2 \delta\psi = -4\pi\delta\sigma\,, \tag{5.2.35}$$

where

$$\sigma(\vec{r}) = n_+(\vec{r}) - n_-(\vec{r}) \tag{5.2.36}$$

with $n_\pm = e^{\mp\beta\psi+\gamma_\pm}$.

Suppose that

$$\delta\psi = x\psi_1 + f^{(2)} + f^{(3)}\,, \tag{5.2.37}$$

where x is a small parameter, and $f^{(i)}$ $(i = 2, 3)$ is of the order $\mathcal{O}(x^i)$. Then

$$1 + \frac{\delta n_+}{n_+} = e^{-\delta\beta\,\psi - \beta\,\delta\psi - \delta\beta\,\delta\psi + \delta\gamma_+}\,, \tag{5.2.38}$$

$$1 + \frac{\delta n_-}{n_-} = e^{\delta\beta\,\psi + \beta\,\delta\psi + \delta\beta\,\delta\psi + \delta\gamma_-}\,, \tag{5.2.39}$$

where $\delta\beta$ and $\delta\gamma_\pm$ are of the order $\mathcal{O}(x^2)$ or higher.

At the order $\mathcal{O}(x)$, Eq. (5.2.35) becomes

$$\nabla^2(x\psi_1) = -4\pi(-n_+\beta x\psi_1 - n_-\beta x\psi_1) \tag{5.2.40}$$

$$\Rightarrow \quad \left[\nabla^2 - 4\pi(n_+ + n_-)\beta\right]\psi_1 = 0\,, \tag{5.2.41}$$

which is exactly the equation that the bifurcation point should satisfy.

At the order $\mathcal{O}(x^2)$, Eq. (5.2.35) becomes

$$\begin{aligned}\nabla^2 f^{(2)} &= -4\pi\left[n_+(-\delta\beta\,\psi + \frac{1}{2}\beta^2 x^2 {\psi_1}^2 - \beta f^{(2)} + \delta\gamma_+) - n_-(\delta\beta\,\psi + \frac{1}{2}\beta^2 x^2 {\psi_1}^2 + \beta f^{(2)} + \delta\gamma_-)\right]\\ &= -4\pi\left[-(n_+ + n_-)(\delta\beta\,\psi + \beta f^{(2)}) + (n_+ - n_-)\frac{1}{2}\beta^2 x^2 {\psi_1}^2 + n_+\,\delta\gamma_+ - n_-\,\delta\gamma_-\right]\end{aligned} \tag{5.2.42}$$

$$\Rightarrow \left[\nabla^2 - 4\pi(n_+ + n_-)\beta\right] f^{(2)} = -4\pi \left[-(n_+ + n_-)\delta\beta\,\psi + (n_+ - n_-)\frac{1}{2}\beta^2 x^2 {\psi_1}^2 + n_+\,\delta\gamma_+ - n_-\,\delta\gamma_-\right] . \tag{5.2.43}$$

Suppose

$$\psi_1(r,\theta) = \hat{\psi}_1(r)\, e^{i\theta} . \tag{5.2.44}$$

The right-hand side of the equation above has only the zeroth and the second power of $e^{i\theta}$, hence the Fredholm solubility condition is trivially true.

Next, let us consider the constraints:

$$1 = \int d^2r\, n_+ , \tag{5.2.45}$$

$$1 = \int d^2r\, n_- , \tag{5.2.46}$$

$$E = \frac{1}{2}\int d^2r\, (n_+ - n_-)\psi . \tag{5.2.47}$$

Their variations are

$$0 = \int d^2r\, \delta n_+ , \tag{5.2.48}$$

$$0 = \int d^2r\, \delta n_- , \tag{5.2.49}$$

$$\delta E = \frac{1}{2}\int d^2r\, \left[(\delta n_+ - \delta n_-)\psi + (n_+ - n_-)\delta\psi + (\delta n_+ - \delta n_-)\delta\psi\right] . \tag{5.2.50}$$

Combining the first two constraints above, we obtain two equivalent conditions:

$$0 = \int d^2r\, \delta(n_+ - n_-) = \int d^2r\, \delta\sigma , \tag{5.2.51}$$

$$0 = \int d^2r\, (\delta n_+ + \delta n_-) . \tag{5.2.52}$$

At the order $\mathcal{O}(x^2)$, the constraint (5.2.51) becomes

$$0 = \int d^2r \left(-\frac{1}{4\pi}\right) \nabla^2\delta\psi = -\frac{1}{4\pi}\int d^2r\, \nabla^2 f^{(2)} \tag{5.2.53}$$

$$\Rightarrow \quad 0 = \int d^2r \nabla^2 f^{(2)} = \int_0^{2\pi} d\theta \int_0^1 dr\, \left(\frac{d}{dr} r \frac{d}{dr} f^{(2)}\right) = \int_0^{2\pi} d\theta \left[r\frac{d}{dr}f^{(2)}\right]\Bigg|_0^1 \tag{5.2.54}$$

$$\Rightarrow \quad 0 = \int_0^{2\pi} d\theta (f^{(2)})'(1) . \tag{5.2.55}$$

At the order $\mathcal{O}(x^2)$, the constraint (5.2.50) becomes

$$\begin{aligned}
\delta E &= \frac{1}{2}\int d^2r\left[-\frac{1}{4\pi}(\nabla^2\delta\psi)\psi + (n_+ - n_-)f^{(2)} + \big(n_+(-\beta x\psi_1) - n_-(\beta x\psi_1)\big)(x\psi_1)\right] \\
&= \frac{1}{2}\int d^2r\left[-\frac{1}{4\pi}\delta\psi(\nabla^2\psi) + \sigma f^{(2)} - (n_+ + n_-)\beta x^2{\psi_1}^2\right] \\
&= \frac{1}{2}\int d^2r\left[\delta\psi\,\sigma + \sigma f^{(2)} - (n_+ + n_-)\beta x^2{\psi_1}^2\right] \\
&= \frac{1}{2}\int d^2r\left[2\sigma f^{(2)} - (n_+ + n_-)\beta x^2{\psi_1}^2\right] \\
&= \int d^2r\,\sigma f^{(2)} - \frac{1}{2}x^2\beta\int d^2r\,(n_+ + n_-){\psi_1}^2 \,. \qquad (5.2.56)
\end{aligned}$$

Finally, we consider Eq. (5.2.35) at the order $\mathcal{O}(x^3)$:

$$\begin{aligned}
\nabla^2 f^{(3)} = &-4\pi n_+\left[(-\delta\beta\,\psi - \beta f^{(2)} + \delta\gamma_+)(-\beta x\psi_1) - \delta\beta\,x\psi_1 - \frac{1}{3!}\beta^3x^3{\psi_1}^3 - \beta f^{(3)}\right] \\
&+ 4\pi n_-\left[(\delta\beta\,\psi + \beta f^{(2)} + \delta\gamma_-)(\beta x\psi_1) + \delta\beta\,x\psi_1 + \frac{1}{3!}\beta^3x^3{\psi_1}^3 + \beta f^{(3)}\right]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad &\left[\nabla^2 - 4\pi(n_+ + n_-)\beta\right]f^{(3)} \\
&= 4\pi x\psi_1\left[-(n_+ - n_-)(\delta\beta\,\psi\beta + \beta^2 f^{(2)}) + (n_+ + n_-)(\delta\beta + \frac{1}{6}\beta^3x^2{\psi_1}^2) + \delta\gamma_+ n_+\beta + \delta\gamma_- n_-\beta\right]. \qquad (5.2.57)
\end{aligned}$$

The Fredholm solubility condition is then

$$0 = \int d^2r\,{\psi_1}^2\left[-(n_+ - n_-)(\delta\beta\,\psi\beta + \beta^2 f^{(2)}) + (n_+ + n_-)(\delta\beta + \frac{1}{6}\beta^3x^2{\psi_1}^2) + \delta\gamma_+ n_+\beta + \delta\gamma_- n_-\beta\right]. \qquad (5.2.58)$$

To rewrite the constraints and the solubility condition, we first define

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - 4\pi(n_+ + n_-)\beta\right)\chi_1 = 4\pi(n_+ + n_-)\psi\,, \qquad (5.2.59)$$

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - 4\pi(n_+ + n_-)\beta\right)\chi_2 = 4\pi n_+\,, \qquad (5.2.60)$$

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - 4\pi(n_+ + n_-)\beta\right)\chi_3 = 4\pi n_-\,, \qquad (5.2.61)$$

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - 4\pi(n_+ + n_-)\beta\right)\chi_4 = 4\pi(n_+ - n_-)\beta^2{\hat\psi_1}^2\,, \qquad (5.2.62)$$

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{4}{r^2} - 4\pi(n_+ + n_-)\beta\right)\chi_5 = 4\pi(n_+ - n_-)\beta^2{\hat\psi_1}^2\,, \qquad (5.2.63)$$

where again

$$\psi_1(r,\theta) = \hat{\psi}_1(r)\cos\theta\,, \tag{5.2.64}$$

and all the χ_i's satisfy the boundary condition

$$\chi_i(r=1) = 0\,, \quad \chi_i'(r=0) = 0\,. \tag{5.2.65}$$

Then $f^{(2)}$ can be expressed as

$$f^{(2)} = \chi_1\,\delta\beta - \chi_2\,\delta\gamma_+ + \chi_3\,\delta\gamma_- - \frac{1}{4}x^2\,\chi_4 - \frac{1}{4}x^2\,\chi_5\cos(2\theta)\,. \tag{5.2.66}$$

Since

$$\int_0^{2\pi} d\theta\cos(2\theta) = 0\,, \tag{5.2.67}$$

we can rewrite Eq. (5.2.55) as

$$\begin{aligned} 0 &= \int_0^{2\pi} d\theta\,\left[\chi_1'(1)\,\delta\beta - \chi_2'(1)\,\delta\gamma_+ + \chi_3'(1)\,\delta\gamma_- - \frac{1}{4}x^2\,\chi_4'(1)\right] \\ &= 2\pi\chi_1'(1)\,\delta\beta - 2\pi\chi_2'(1)\,\delta\gamma_+ + 2\pi\chi_3'(1)\,\delta\gamma_- - 2\pi\frac{1}{4}x^2\,\chi_4'(1) \end{aligned} \tag{5.2.68}$$

$$\begin{aligned} \Rightarrow\quad 0 &= \chi_1'(1)\,\delta\beta - \chi_2'(1)\,\delta\gamma_+ + \chi_3'(1)\,\delta\gamma_- - \frac{1}{4}x^2\,\chi_4'(1) \\ &= \begin{pmatrix}\chi_1'(1) & \chi_2'(1) & \chi_3'(1) & \chi_4'(1)\end{pmatrix}\cdot\begin{pmatrix}\delta\beta \\ -\delta\gamma_+ \\ \delta\gamma_- \\ -\frac{1}{4}x^2\end{pmatrix}\,. \end{aligned} \tag{5.2.69}$$

Eq. (5.2.56) can be written as

$$\begin{aligned} \delta E &= \int_0^{2\pi} d\theta\,\int_0^1 dr\,r\sigma\left(\chi_1\,\delta\beta - \chi_2\,\delta\gamma_+ + \chi_3\,\delta\gamma_- - \frac{1}{4}x^2\,\chi_4\right) \\ &\quad - \frac{1}{2}x^2\beta\int_0^{2\pi} d\theta\,\int_0^1 dr\,r(n_+ + n_-)\hat{\psi}_1{}^2\cos^2\theta \\ &= \int_0^1 dr\,r\sigma\left(2\pi\chi_1\,\delta\beta - 2\pi\chi_2\,\delta\gamma_+ + 2\pi\chi_3\,\delta\gamma_- - 2\pi\frac{1}{4}x^2\,\chi_4\right) \\ &\quad - \pi\frac{1}{2}x^2\beta\int_0^1 dr\,r(n_+ + n_-)\hat{\psi}_1{}^2 \end{aligned} \tag{5.2.70}$$

$$\begin{aligned} \Rightarrow\quad \frac{\delta E}{2\pi} &= \int_0^1 dr\,r\sigma\left(\chi_1\,\delta\beta - \chi_2\,\delta\gamma_+ + \chi_3\,\delta\gamma_- - \frac{1}{4}x^2\,\chi_4\right) - \frac{1}{4}x^2\beta\int_0^1 dr\,r(n_+ + n_-)\hat{\psi}_1{}^2 \\ &= \left(\int dr\,r\sigma\chi_1 \quad \int dr\,r\sigma\chi_2 \quad \int dr\,r\sigma\chi_3 \quad \int dr\,r\sigma\chi_4 + \beta\int dr\,r(n_+ + n_-)\hat{\psi}_1{}^2\right)\cdot\begin{pmatrix}\delta\beta \\ -\delta\gamma_+ \\ \delta\gamma_- \\ -\frac{1}{4}x^2\end{pmatrix}\,. \end{aligned} \tag{5.2.71}$$

At the order $\mathcal{O}(x^2)$, the constraint (5.2.52) becomes

$$
\begin{aligned}
0 &= \int d^2r\,(\delta n_+ + \delta n_-) \\
&= \int d^2r\,\left[n_+(-\delta\beta\,\psi - \beta f^{(2)} + \delta\gamma_+ + \frac{1}{2}\beta^2x^2\psi_1^2) + n_-(\delta\beta\,\psi + \beta f^{(2)} + \delta\gamma_- + \frac{1}{2}\beta^2x^2\psi_1^2)\right] \\
&= \int d^2r\,\left[-(n_+ - n_-)\delta\beta\,\psi - (n_+ - n_-)\beta f^{(2)} + n_+\,\delta\gamma_+ + n_-\,\delta\gamma_- + \frac{1}{2}(n_+ + n_-)\beta^2x^2\psi_1^2\right] \\
&= \int d^2r\,\Big[-\sigma\,\delta\beta\,\psi - \sigma\beta\left(\chi_1\,\delta\beta - \chi_2\,\delta\gamma_+ + \chi_3\,\delta\gamma_- - \frac{1}{4}x^2\,\chi_4 - \frac{1}{4}x^2\,\chi_5\cos(2\theta)\right) \\
&\qquad\qquad + n_+\delta\gamma_+ + n_-\delta\gamma_- + \frac{1}{2}(n_+ + n_-)\beta^2x^2\hat{\psi}_1^2\cos^2\theta\Big] \\
&= \int_0^1 dr\,r\Big[-2\pi\sigma(\psi + \beta\chi_1)\delta\beta + 2\pi(n_+ + \sigma\beta\chi_2)\delta\gamma_+ + 2\pi(n_- - \sigma\beta\chi_3)\delta\gamma_- + 2\pi\frac{1}{4}x^2\sigma\beta\chi_4 \\
&\qquad\qquad + \frac{\pi}{2}(n_+ + n_-)\beta^2x^2\hat{\psi}_1^2\Big]
\end{aligned}
\tag{5.2.72}
$$

$$
\begin{aligned}
\Rightarrow\quad 0 &= \int_0^1 dr\,r\left[-\sigma(\psi + \beta\chi_1)\delta\beta + (n_+ + \sigma\beta\chi_2)\delta\gamma_+ + (n_- - \sigma\beta\chi_3)\delta\gamma_- + \frac{1}{4}x^2\sigma\beta\chi_4\right] \\
&= \Big(-\int dr\,r\sigma(\psi + \beta\chi_1) \quad -\int dr\,r(n_+ + \sigma\beta\chi_2) \\
&\qquad \int dr\,r(n_- - \sigma\beta\chi_3) \quad -\int dr\,r\sigma\beta\chi_4 - \int dr\,r(n_+ + n_-)\beta^2\hat{\psi}_1^2\Big)\cdot\begin{pmatrix}\delta\beta \\ -\delta\gamma_+ \\ \delta\gamma_- \\ -\frac{1}{4}x^2\end{pmatrix}.
\end{aligned}
\tag{5.2.73}
$$

Eq. (5.2.58) leads to

$$
\begin{aligned}
0 &= \int d^2r\,{\psi_1}^2\left[(n_+ - n_-)\left(\delta\beta\,\psi + \beta f^{(2)}\right) - (n_+ + n_-)\left(\beta^{-1}\delta\beta + \frac{1}{6}\beta^2x^2{\psi_1}^2\right) - n_+\delta\gamma_+ - n_-\delta\gamma_-\right] \\
&= \int_0^1 dr\,r{\hat{\psi}_1}^2\Big[\pi(n_+ - n_-)\,(\delta\beta\,\psi) + \pi(n_+ - n_-)\beta\left(\chi_1\,\delta\beta - \chi_2\,\delta\gamma_+ + \chi_3\,\delta\gamma_- - \frac{1}{4}x^2\,(\chi_4 + \frac{1}{2}\chi_5)\right) \\
&\qquad\qquad - \pi(n_+ + n_-)\beta^{-1}\delta\beta - \pi n_+\delta\gamma_+ - \pi n_-\delta\gamma_-\Big] \\
&\quad - \frac{1}{6}x^2\beta^2\frac{3\pi}{4}\int_0^1 dr\,r{\hat{\psi}_1}^4(n_+ + n_-)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad 0 = & \int_0^1 dr\, r\hat{\psi}_1{}^2 \Big[(n_+ - n_-) \Big((\psi + \beta\chi_1)\delta\beta - \beta\chi_2\,\delta\gamma_+ + \beta\chi_3\,\delta\gamma_- - \frac{1}{4}\beta x^2 \,(\chi_4 + \frac{1}{2}\chi_5) \Big) \\
& \qquad - (n_+ + n_-)\beta^{-1}\delta\beta - n_+\delta\gamma_+ - n_-\delta\gamma_- \Big] \\
& - \frac{1}{8}x^2\beta^2 \int_0^1 dr\, r\hat{\psi}_1{}^4 (n_+ + n_-) \\
= & \Big(\int dr\, r\hat{\psi}_1{}^2 \left(\sigma(\psi + \beta\chi_1) - \rho\beta^{-1}\right) \quad \int dr\, r\hat{\psi}_1{}^2 (\sigma\beta\chi_2 + n_+) \\
& \int dr\, r\hat{\psi}_1{}^2 (\sigma\beta\chi_3 - n_-) \quad \int dr\, r\hat{\psi}_1{}^2 (\sigma\beta(\chi_4 + \frac{1}{2}\chi_5) + \frac{1}{2}\rho\beta^2\hat{\psi}_1{}^2) \Big) \cdot \begin{pmatrix} \delta\beta \\ -\delta\gamma_+ \\ \delta\gamma_- \\ -\frac{1}{4}x^2 \end{pmatrix} .
\end{aligned}
\tag{5.2.74}
$$

To summarize, we can rewrite the constraints (5.2.69) (5.2.71) (5.2.73) and the solubility condition (5.2.74) into a compact form

$$
\mathbf{Q} \cdot \begin{pmatrix} \delta\beta \\ -\delta\gamma_+ \\ \delta\gamma_- \\ -\frac{x^2}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\delta E}{2\pi} \\ 0 \\ 0 \end{pmatrix} , \tag{5.2.75}
$$

where the matrix elements of $\mathbf{Q}$ are given by

$$\mathbf{Q}_{1j} = \chi_j'(1)\,, \quad j = 1, \cdots, 4\,, \tag{5.2.76}$$

$$\mathbf{Q}_{2j} = \int_0^1 dr\, r\sigma\chi_j\,, \quad j = 1, 2, 3\,, \tag{5.2.77}$$

$$\mathbf{Q}_{24} = \int_0^1 dr\, r\sigma\chi_4 + \beta \int_0^1 dr\, r(n_+ + n_-)\hat{\psi}_1{}^2\,, \tag{5.2.78}$$

$$\mathbf{Q}_{31} = -\int dr\, r\sigma(\psi + \beta\chi_1)\,, \tag{5.2.79}$$

$$\mathbf{Q}_{32} = -\int dr\, r(n_+ + \sigma\beta\chi_2)\,, \tag{5.2.80}$$

$$\mathbf{Q}_{33} = \int dr\, r(n_- - \sigma\beta\chi_3)\,, \tag{5.2.81}$$

$$\mathbf{Q}_{34} = -\int dr\, r\sigma\beta\chi_4 - \int dr\, r(n_+ + n_-)\beta^2\hat{\psi}_1^2\,, \tag{5.2.82}$$

$$\mathbf{Q}_{41} = \int dr\, r\hat{\psi}_1{}^2(\sigma(\psi + \beta\chi_1) - \rho\beta^{-1})\,, \tag{5.2.83}$$

$$\mathbf{Q}_{42} = \int dr\, r\hat{\psi}_1{}^2(\sigma\beta\chi_2 + n_+)\,, \tag{5.2.84}$$

$$\mathbf{Q}_{43} = \int dr\, r\hat{\psi}_1{}^2(\sigma\beta\chi_3 - n_-)\,, \tag{5.2.85}$$

$$\mathbf{Q}_{44} = \int dr\, r\hat{\psi}_1{}^2(\sigma\beta(\chi_4 + \frac{1}{2}\chi_5) + \frac{1}{2}\rho\beta^2\hat{\psi}_1{}^2)\,. \tag{5.2.86}$$

Therefore, we have

$$\begin{pmatrix} \delta\beta \\ -\delta\gamma_+ \\ \delta\gamma_- \\ -\frac{x^2}{4} \end{pmatrix} = \mathbf{Q}^{-1} \cdot \begin{pmatrix} 0 \\ \frac{\delta E}{2\pi} \\ 0 \\ 0 \end{pmatrix}\,. \tag{5.2.87}$$

$\mathbf{Q^{-1}}$ needs to be evaluated numerically. The equation shows that for a given energy change δE, we can compute the changes in $\delta\beta$, $\delta\gamma_\pm$ and x, and consequently $\delta n_\pm$, $\delta\sigma$ and $\delta\psi$. In this way, we can construct the branch of nonaxisymmetric solutions and compute some physical quantities along this branch. Note that we construct the nonaxisymmetric solutions from the bifurcation point, where we assume

$$\psi_1(r,\theta) = \hat{\psi}_1(r)\cos\theta$$

Therefore the solutions constructed using this method only valid when close to the bifurcation point.

5.2.2 Results for a System in Disc

In this section, we use the formalism developed in the previous section to calculate some physical quantities of interest.

S-E Curve

To make sure the branch of the nontrivial solutions has larger entropy than the branch of the trivial solutions, we plot the entropy vs. energy in Fig. 5.2, where the entropy is defined as

$$S = -\int d^2r\, n_+ \ln n_+ - \int d^2r\, n_- \ln n_- \,. \tag{5.2.88}$$

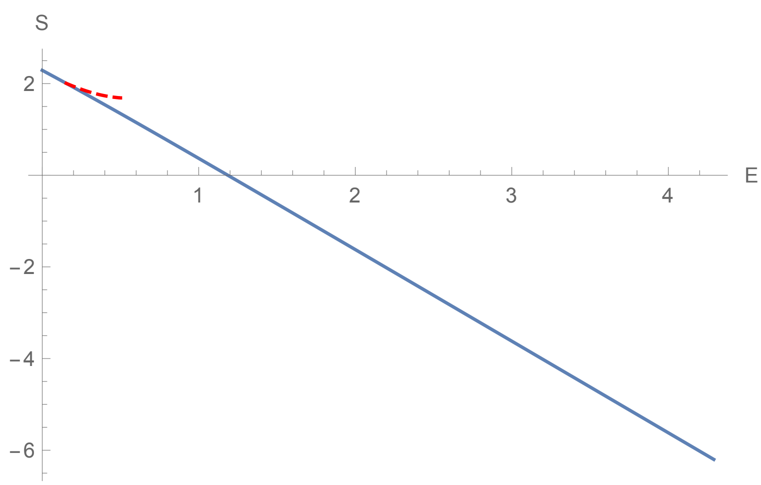


Figure 5.2: The S-E curve for the system in a disc. The dashed line shows the branch of the nontrivial solutions.

Density Profiles

As an example, we plot the solution preserving $U(1)$ rotational symmetry at the bifurcation point $E_c = 0.153\, E_0$ in Fig. 5.3 - Fig. 5.6. This solution shows that the positive charges are relatively concentrated in the center, while the negative charges accumulate close to the edge of the disc. In fact, due to the symmetry between the positive and the negative charge, interchanging the signs of the charges will give us another possible solution with the same energy and entropy.

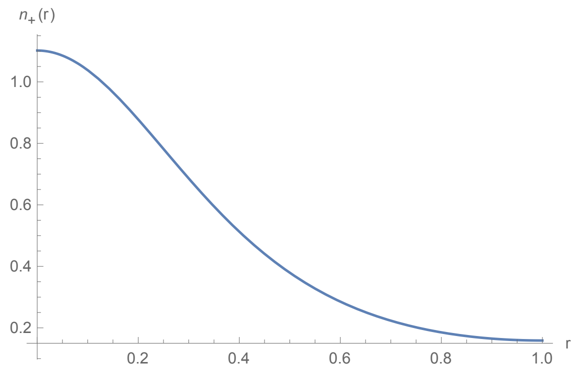


Figure 5.3: The profile of $n_+(r)$

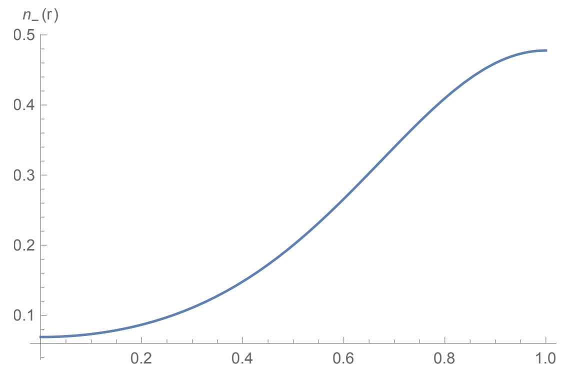


Figure 5.4: The profile of $n_-(r)$

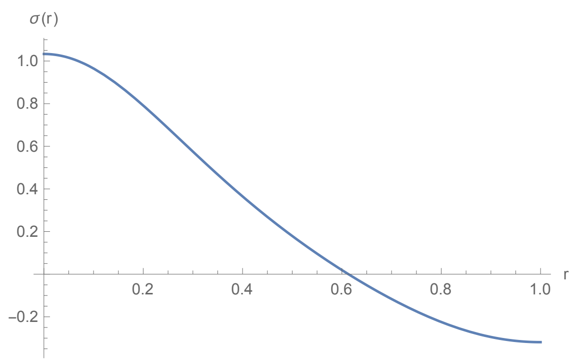


Figure 5.5: The profile of $\sigma(r)$

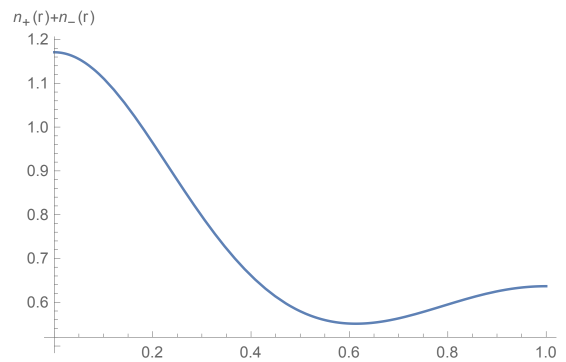


Figure 5.6: The profile of $n_+(r) + n_-(r)$

Above E_c two branches of solutions are developed. One preserves the $U(1)$ rotational symmetry, while the other breaks the $U(1)$ symmetry. The two solutions at $E \approx 0.24\, E_0$ are shown in Fig. 5.7 and Fig. 5.8.

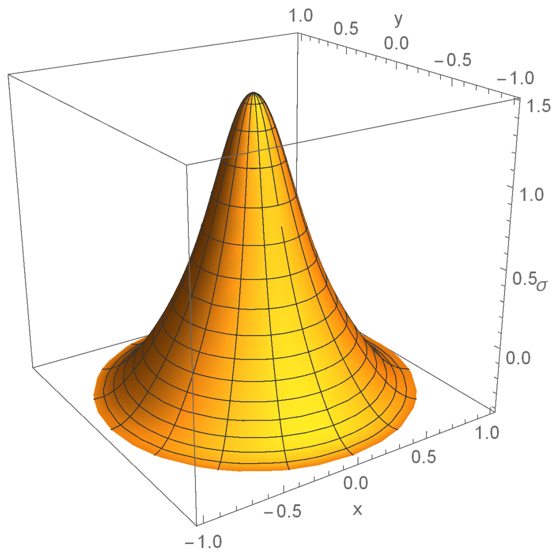


Figure 5.7: The solution preserving $U(1)$

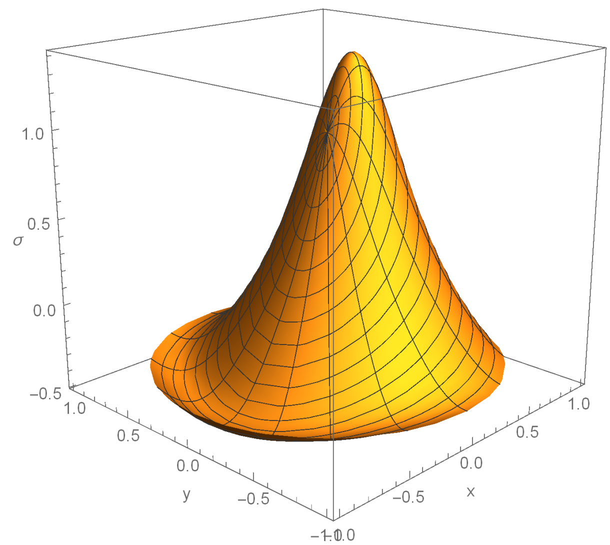


Figure 5.8: The solution breaking $U(1)$

Order parameter

The total dipole moment of the system plays the role of the order parameter of the phase transition, which can be expressed as

$$\langle D\rangle = |\vec{D}| = \sqrt{D_x^2 + D_y^2}\,, \tag{5.2.89}$$

where

$$D_x \equiv \int d^2r\, x\sigma(\vec{r})\,, \quad D_y \equiv \int d^2r\, y\sigma(\vec{r})\,, \tag{5.2.90}$$

and $\sigma(\vec{r}) = n_+(\vec{r}) - n_-(\vec{r})$ is obtained from the branch of the nontrivial solutions. Fig. 5.9 shows the dipole moment $\langle D\rangle$ as a function of E.

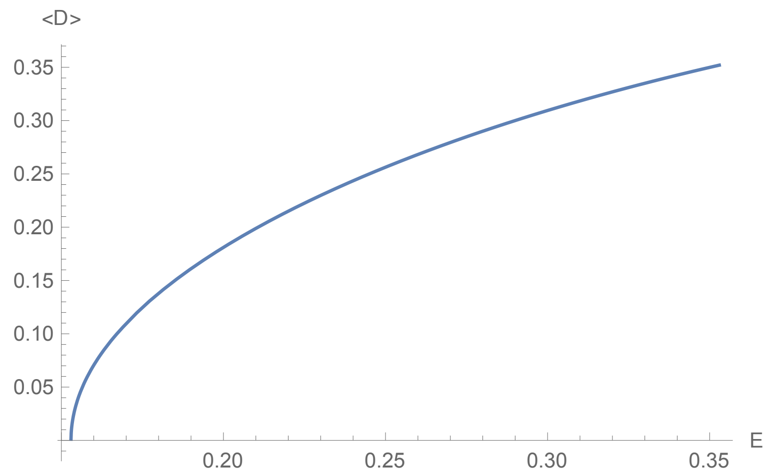


Figure 5.9: The $\langle D\rangle$-E curve for the system in a disc.

Close to the transition point ($E_c \simeq 0.153$), we find

$$\langle D\rangle = A\,(E - E_c)^{\mu}\,, \tag{5.2.91}$$

with fitting parameter $A = 0.840$ and $\mu \approx 0.5$.

5.3 System in a Box

In this section, we consider the system in a square box. We mainly follow the formalism developed in Ref. [113]. As advertised in the introduction, we will evaluate the partition function of the two-charge point-vortex model exactly in this case. From it one can further calculate many physical quantities and once again recover the phase transition at negative temperature.

5.3.1 Review of the Approach by Edwards and Taylor

Let us first review the formalism introduced in Ref. [113]. The Hamiltonian of the system is

$$H = -\sum_{i<j} \frac{2e_i\, e_j}{\ell} \ln|\vec{r}_i - \vec{r}_j|\,. \tag{5.3.1}$$

The Fourier transforms of the positive and the negative charge are

$$p_k = \frac{e}{V}\sum_{+} e^{i\vec{k}\cdot\vec{r}_i}\,,\quad q_k = \frac{e}{V}\sum_{-} e^{i\vec{k}\cdot\vec{r}_j}\,. \tag{5.3.2}$$

Defining

$$\rho_k \equiv p_k - q_k\,,\quad \eta_k \equiv p_k + q_k\,, \tag{5.3.3}$$

we obtain

$$H = 2\pi V \sum_k \left(|\rho_k|^2 - \frac{4\pi N e^2}{V^2} \right) \frac{1}{k^2}\,, \tag{5.3.4}$$

where $k = \frac{2\pi n}{L}$. If we drop the second term in the brackets of Eq. (5.3.4), the energy is always positive, and it returns to the Hamiltonian (5.1.1).

In the micro-canonical ensemble, for a given energy E, the statistical weight is

$$\Omega = \int \frac{d\lambda}{2\pi} \prod_i d\vec{r}_i \, e^{i\lambda(E-H)}\,, \tag{5.3.5}$$

where λ is a Lagrange multiplier. Changing from the coordinates $\{\vec{r}_i, \vec{r}_j\}$ to $\{\rho_k, \eta_k\}$, and taking into account the corresponding Jacobian, one obtains

$$\Omega = \frac{V^{2N}}{2\pi} \int d\lambda \prod_k dr_k^2 \left[\frac{V^2}{2Ne^2} \, e^{-\frac{V^2}{2Ne^2} r_k^2} \right] e^{i\lambda E} \, e^{2\pi i \lambda V \sum_k \left(r_k^2 - \frac{4\pi N e^2}{V^2} \right) \frac{1}{k^2}}\,, \tag{5.3.6}$$

where r_k is given by

$$\rho_k = r_k \, e^{i\phi_k}\,. \tag{5.3.7}$$

Integrating out r_k gives us

$$\Omega = \frac{V^{2N}}{2\pi} \int d\lambda \, e^{i\lambda E - \sum_k \left[\ln\left(1 + \frac{i\alpha^2 \lambda}{k^2}\right) - \frac{i\alpha^2\lambda}{k^2} \right]}\,. \tag{5.3.8}$$

After we introduce some new variables

$$\epsilon \equiv \frac{E}{Ne^2}\,, \quad z \equiv Ne^2 \lambda\,, \quad k^2 \equiv \frac{4\pi^2}{V} \kappa^2\,, \tag{5.3.9}$$

Ω becomes

$$\Omega = \frac{V^{2N}}{Ne^2} \int \frac{dz}{2\pi} \exp\left[iz\epsilon - \sum_\kappa \left(\ln\left(1 + \frac{iz}{\pi\kappa^2}\right) - \frac{iz}{\pi\kappa^2} \right) \right]\,. \tag{5.3.10}$$

When replacing the sum by the integration, we have to introduce a lower cutoff in the momentum:

$$\begin{aligned}
&\int d^2\kappa \left[\ln\left(1 + \frac{iz}{\pi\kappa^2}\right) - \frac{iz}{\pi\kappa^2} \right] \\
&= 2\pi \int_b^\infty \kappa d\kappa \left[\ln\left(1 + \frac{iz}{\pi\kappa^2}\right) - \frac{iz}{\pi\kappa^2} \right] \\
&= \pi \int_{b^2}^\infty d\kappa^2 \left[\ln\left(1 + \frac{iz}{\pi\kappa^2}\right) - \frac{iz}{\pi\kappa^2} \right] \\
&= \pi \left(\frac{iz}{\pi} + \kappa^2 \right) \ln\left(1 + \frac{iz}{\pi\kappa^2}\right) \Bigg|_{\kappa^2 = b^2}^\infty \\
&= iz - \pi b^2 \left(1 + \frac{iz}{\pi b^2}\right) \ln\left(1 + \frac{iz}{\pi b^2}\right)\,.
\end{aligned} \tag{5.3.11}$$

Therefore,

$$\Omega = \frac{V^{2N}}{Ne^2} \int \frac{dz}{2\pi} \exp\left[iz\epsilon - iz + \pi b^2 \left(1 + \frac{iz}{\pi b^2}\right) \ln\left(1 + \frac{iz}{\pi b^2}\right)\right] . \tag{5.3.12}$$

A natural choice for the lower cutoff b is

$$\pi b^2 = 1 , \tag{5.3.13}$$

i.e., we neglect the modes with

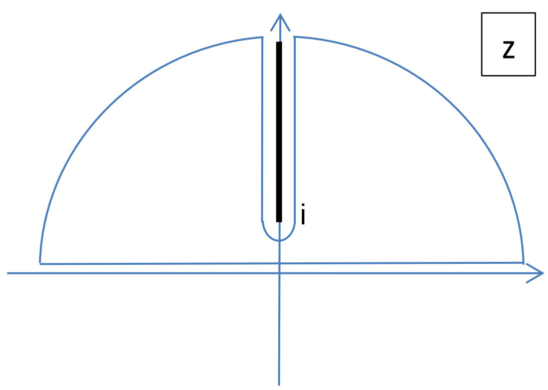
$$\frac{V}{4\pi^2} k^2 < \frac{1}{\pi} \quad \Longleftrightarrow \quad \frac{L}{\lambda} < \frac{1}{\sqrt{\pi}} , \tag{5.3.14}$$

which means that we do not consider the modes whose wavelengths are larger than the system size. Finally, after introducing the lower cutoff in momentum (5.3.13), we obtain

$$\Omega = \frac{V^{2N}}{Ne^2} \int \frac{dz}{2\pi} \exp\left[iz\epsilon - iz + (1+iz)\ln(1+iz)\right] = \frac{V^{2N}}{Ne^2}\, g(\epsilon) , \tag{5.3.15}$$

where

$$g(\epsilon) \equiv \int \frac{dz}{2\pi} \exp\left[iz\epsilon + (1+iz)\ln(1+iz) - iz\right] . \tag{5.3.16}$$



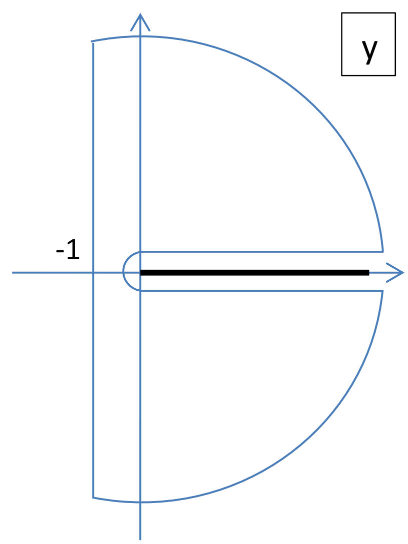


Figure 5.10: The contour for the variable z Figure 5.11: The contour for the variable y

The entropy is given by

$$S = \ln \Omega = 2N \ln V + \ln\big(g(\epsilon)\big) . \tag{5.3.17}$$

To evaluate $g(\epsilon)$, we observe that it has no poles, but it has a branch point at i. As shown in Fig. 5.10, we can choose a branch cut from i to $i\infty$, and always close the contour in the upper half plane, no matter what the value of ϵ is. Only the two sides of the branch cut contribute to the contour integral. Let us define

$$ix \equiv 1 + iz\,, \tag{5.3.18}$$

which corresponds to shifting the complex plane by i. The integral becomes

$$g(\epsilon) = \int \frac{dx}{2\pi} \exp\big[(ix-1)(\epsilon-1) + ix\ln(ix)\big]\,. \tag{5.3.19}$$

Now $x = 0$ becomes the branch point, and the branch cut is $(0, i\infty)$. If we further define

$$x \equiv iy \quad \Rightarrow \quad y = -ix\,, \tag{5.3.20}$$

then as shown in Fig. 5.11, $y = 0$ is still the branch point, but the branch cut becomes $(0,\infty)$, i.e., for $y \in \mathbb{R}$ the function $\ln(y)$ is multiple-valued, and its value depends on which side of the real axis y lies. The contour integral is now

$$g(\epsilon) = -\int_0^\infty \frac{i\,dy}{2\pi} \exp\big[(-y-1)(\epsilon-1)\big]\,\exp\big[-y\ln(-y)\big] - \int_\infty^0 \frac{i\,dy}{2\pi} \exp\big[(-y-1)(\epsilon-1)\big]\,\exp\big[-y\ln(-y)\big]\,. \tag{5.3.21}$$

We have to treat the term $\ln(-y)$ carefully:

$$\ln(-y) = \ln(|y|) + i\arg(-y)\,. \tag{5.3.22}$$

For y slightly above the positive real axis $\arg(-y)$ takes the value $-i\pi$, while for y slightly below the positive real axis $\arg(-y)$ takes the value $i\pi$. Then

$$\begin{aligned} g(\epsilon) &= -\int_0^\infty \frac{i\,dy}{2\pi} \exp\big[(-y-1)(\epsilon-1)\big]\cdot\exp\left(-y\ln y\right)\cdot\big[\exp(-y(-i\pi)) - \exp(-y(i\pi))\big] \\ &= -\int_0^\infty \frac{i\,dy}{2\pi}\, e^{1-\epsilon}\cdot e^{-y\epsilon+y}\cdot e^{-y\ln y}\cdot\big(2i\sin(\pi y)\big) \\ &= \frac{1}{\pi}e^{1-\epsilon}\int_0^\infty dy\,\sin(\pi y)\,e^{y-y\ln y-\epsilon y}\,. \end{aligned} \tag{5.3.23}$$

Next, we consider the correlation function $\langle|\rho_k^2|\rangle$, which can be expressed as

$$\begin{aligned} \langle|\rho_k^2|\rangle &= \frac{2Ne^2}{V}\frac{1}{\Omega}\int dz\,\frac{k^2V}{k^2V+4\pi iz}\prod_\kappa\left[\left(1+\frac{iz}{\pi\kappa^2}\right)^{-1}\right]\exp\left(iz\epsilon + iz\sum_\kappa\frac{1}{\pi\kappa^2}\right) \\ &= \frac{2Ne^2}{V}\frac{1}{\Omega}\int dz\,\frac{k^2V}{k^2V+4\pi iz}\exp\left[iz\epsilon - \sum_\kappa\left(\ln\left(1+\frac{iz}{\pi\kappa^2}\right) - iz\frac{1}{\pi\kappa^2}\right)\right]\,, \end{aligned} \tag{5.3.24}$$

where again

$$\epsilon \equiv \frac{E}{Ne^2}\,,\quad z \equiv Ne^2\lambda\,,\quad k^2 \equiv \frac{4\pi^2}{V}\kappa^2\,. \tag{5.3.25}$$

One can replace $\sum_\kappa$ by $\frac{V}{(2\pi)^2}\int d^2k$, but since in this case the error is not small when $V \to \infty$, one has to introduce a lower cut-off b in the κ integration as before. By taking $\pi b^2 = 1$, one obtains

$$\begin{aligned} \langle|\rho_k^2|\rangle &= \frac{2Ne^2}{V}\frac{1}{\Omega}\int dz\,\frac{k^2V}{k^2V+4\pi iz}\exp\left[iz\epsilon+(1+iz)\ln(1+iz)-iz\right] \\ &= \frac{2Ne^2}{V}\frac{1}{\Omega}\int dz\,\frac{-ia}{z-ia}\exp\left[(1+iz)\ln(1+iz)\right]\,\exp\left[iz(\epsilon-1)\right]\,, \end{aligned} \tag{5.3.26}$$

where $a \equiv \frac{k^2V}{4\pi}$. We see that the contour integral above has a simple pole at $z_0 = ia$ and a branch cut at $z = i$. Since we have taken a lower cutoff for κ, which is given by

$$\frac{V}{4\pi^2}k^2 = b^2 = \frac{1}{\pi} \tag{5.3.27}$$

i.e.

$$a = \frac{k^2V}{4\pi} > 1\,. \tag{5.3.28}$$

We can choose the branch cut to be from i to $i\infty$. The pole $z_0 = ia$ must lie above the branch point $z = i$, hence it lies on the branch cut (see Fig. 5.12).

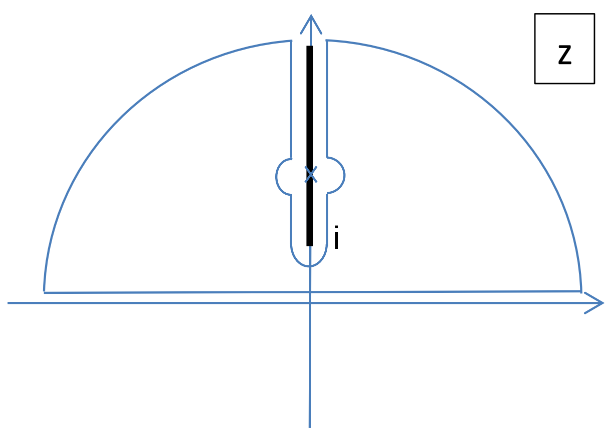


Figure 5.12: The new contour for the variable z

Using the residue theorem, one can evaluate the contour integral (5.3.26) explicitly, and the result is following:

$$\langle|\rho_k^2|\rangle = \frac{4\pi Ne^2}{V\Omega}a\cos[\pi(a-1)]\,(a-1)^{-(a-1)}\,e^{-a(\epsilon-1)} + \frac{4Ne^2}{V\Omega}e^{1-\epsilon}\,\mathcal{P}\widetilde{F}(a,\epsilon)\,, \tag{5.3.29}$$

where

$$\widetilde{F}(a,\epsilon) \equiv \int_0^\infty dx\ \sin(\pi x) \frac{-a}{x-(a-1)} e^{x - x\ln x - \epsilon x}\,, \tag{5.3.30}$$

and $\mathcal{P}\widetilde{F}$ is the principal value of $\widetilde{F}$. Since the inverse Fourier transform of $\widetilde{F}(k^2,s)$ is

$$F(s,\epsilon) \equiv \int d^2k\, \widetilde{F}(k^2,s)\, e^{i\vec{k}\cdot\vec{s}}, \tag{5.3.31}$$

we can Fourier transform $\langle|\rho_k^2|\rangle$ to obtain $\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle$:

$$\begin{aligned}
&\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle \\
=&\frac{1}{\pi}\frac{4\pi N e^2}{V\Omega}\int_1^\infty \tilde{k}\, d\tilde{k} \int_0^{2\pi} d\varphi\, \tilde{k}^2 \cos[\pi(\tilde{k}^2-1)]\,(\tilde{k}^2-1)^{-(\tilde{k}^2-1)}\, e^{-\tilde{k}^2(\epsilon-1)}\, e^{i\tilde{k}\tilde{s}\cos\varphi} \\
&+\frac{1}{\pi}\frac{4Ne^2}{V\Omega}e^{1-\epsilon}\int_0^\infty dx\ \sin(\pi x)\left[\mathcal{P}\int d^2\tilde{k}\,\frac{\tilde{k}^2}{\tilde{k}^2-(x+1)}e^{i\vec{\tilde{k}}\cdot\vec{\tilde{s}}} - \int_0^1 \tilde{k}d\tilde{k}\,\frac{\tilde{k}^2}{\tilde{k}^2-(x+1)}2\pi J_0(\tilde{k}\tilde{s})\right] e^{x-x\ln x-\epsilon x}\,,
\end{aligned} \tag{5.3.32}$$

where

$$\tilde{k} \equiv \sqrt{\frac{V}{4\pi}}k = \sqrt{a}\,, \quad \tilde{s} \equiv \sqrt{\frac{4\pi}{V}}s\,. \tag{5.3.33}$$

Let us evaluate the expression (5.3.32). The principal value part in the second term can be evaluated analytically.

$$\begin{aligned}
\mathcal{P}\left[\int d^2\tilde{k}\,\frac{\tilde{k}^2}{\tilde{k}^2-(x+1)}e^{i\vec{\tilde{k}}\cdot\vec{\tilde{s}}}\right] &= (2\pi)^2\delta(\vec{\tilde{s}}) + \mathcal{P}\left[\int_0^\infty d\tilde{k}\,\frac{\tilde{k}(x+1)}{\tilde{k}^2-(x+1)}2\pi J_0(\tilde{k}\tilde{s})\right] \\
&= (2\pi)^2\delta(\vec{\tilde{s}}) - \pi^2(x+1)\,Y_0(\sqrt{x+1}\,\tilde{s})\,.
\end{aligned} \tag{5.3.34}$$

Then $\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle$ becomes

$$\begin{aligned}
\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle =& \frac{8\pi N e^2}{V\Omega}\int_1^\infty d\tilde{k}\,\tilde{k}^3\cos[\pi(\tilde{k}^2-1)]\,(\tilde{k}^2-1)^{-(\tilde{k}^2-1)}\,e^{-\tilde{k}^2(\epsilon-1)}\,J_0(\tilde{k}\tilde{s}) \\
&+\frac{16\pi N e^2}{V\Omega}\delta(\vec{\tilde{s}})\,e^{1-\epsilon}\int_0^\infty dx\,\sin(\pi x)\,x^{-x}\,e^{-x(\epsilon-1)} \\
&-\frac{4\pi N e^2}{V\Omega}\,e^{1-\epsilon}\int_0^\infty dx\,\sin(\pi x)(x+1)\,Y_0(\sqrt{x+1}\,\tilde{s})\,x^{-x}\,e^{-x(\epsilon-1)} \\
&-\frac{8Ne^2}{V\Omega}e^{1-\epsilon}\int_0^\infty dx\,\sin(\pi x)\int_0^1 d\tilde{k}\,\frac{\tilde{k}^3}{\tilde{k}^2-(x+1)}J_0(\tilde{k}\tilde{s})x^{-x}e^{-x(\epsilon-1)}\,.
\end{aligned} \tag{5.3.35}$$

The first term in Eq. (5.3.35) can be rewritten as

$$\begin{aligned}
&\frac{8\pi N e^2}{V\Omega}\int_1^\infty d\tilde{k}\,\tilde{k}^3\cos[\pi(\tilde{k}^2-1)]\,(\tilde{k}^2-1)^{-(\tilde{k}^2-1)}\,e^{-\tilde{k}^2(\epsilon-1)}\,J_0(\tilde{k}\tilde{s})\\
&=\frac{8\pi N e^2}{V\Omega}\int_1^\infty \frac{1}{2}d\tilde{k}^2\,\tilde{k}^2\cos[\pi(\tilde{k}^2-1)]\,(\tilde{k}^2-1)^{-(\tilde{k}^2-1)}\,e^{-\tilde{k}^2(\epsilon-1)}\,J_0(\tilde{k}\tilde{s})\\
&=\frac{4\pi N e^2}{V\Omega}\int_0^\infty d(\tilde{k}^2-1)\,\tilde{k}^2\cos[\pi(\tilde{k}^2-1)]\,(\tilde{k}^2-1)^{-(\tilde{k}^2-1)}\,e^{-\tilde{k}^2(\epsilon-1)}\,J_0(\tilde{k}\tilde{s})\\
&=\frac{4\pi N e^2}{V\Omega}\int_0^\infty dx\,(x+1)\cos(\pi x)\,x^{-x}\,e^{-(x+1)(\epsilon-1)}\,J_0(\sqrt{x+1}\tilde{s})\\
&=\frac{4\pi N e^2}{V\Omega}e^{1-\epsilon}\int_0^\infty dx\,(x+1)\cos(\pi x)\,x^{-x}\,e^{-x(\epsilon-1)}\,J_0(\sqrt{x+1}\tilde{s})\,.
\end{aligned} \tag{5.3.36}$$

Therefore, the correlation $\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle$ can be written as

$$\begin{aligned}
\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle &= \frac{4\pi N e^2}{V\Omega}\,e^{1-\epsilon}\int_0^\infty dx\,(x+1)\,x^{-x}\,e^{-x(\epsilon-1)}\Big[\cos(\pi x)\,J_0(\sqrt{x+1}\,\tilde{s})-\sin(\pi x)\,Y_0(\sqrt{x+1}\,\tilde{s})\Big]\\
&\quad+\frac{16\pi N e^2}{V\Omega}\delta(\vec{\tilde{s}})\,e^{1-\epsilon}\int_0^\infty dx\sin(\pi x)\,x^{-x}\,e^{-x(\epsilon-1)}\\
&\quad-\frac{8Ne^2}{V\Omega}e^{1-\epsilon}\int_0^\infty dx\sin(\pi x)\int_0^1 d\tilde{k}\,\frac{\tilde{k}^3}{\tilde{k}^2-(x+1)}J_0(\tilde{k}\tilde{s})x^{-x}e^{-x(\epsilon-1)}\,.
\end{aligned} \tag{5.3.37}$$

For $\tilde{s}\neq 0$ we can evaluate the expression (5.3.35) numerically. We find that for a fixed value of ϵ, the correlation $\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle$ diverges logarithmically when $s\to 0$.

5.3.2 Results for a System in a Box

T_c, S-E and β-E Curve

First, we can still estimate the critical temperature using the method applied in Section 5.2 for the system in a disc. Now suppose that the region is $[0,L]\times[0,L]$, and we still assume that $n_+ + n_- = 2n_0$ is constant, then Eq. (5.2.23) can be written as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{8\pi n_0 e^2}{\ell}\beta\right)|\delta\psi| = 0\,. \tag{5.3.38}$$

To satisfy the Dirichlet boundary conditon

$$|\delta\psi|(0,y) = |\delta\psi|(L,y) = 0 \quad \text{for } y\in[0,\pi]\,, \tag{5.3.39}$$

$$|\delta\psi|(x,0) = |\delta\psi|(x,L) = 0 \quad \text{for } x\in[0,\pi]\,, \tag{5.3.40}$$

$|\delta\psi|$ can take the form

$$|\delta\psi|(x,y) \sim \sin\left(\frac{\pi m}{L}x\right)\cdot\sin\left(\frac{\pi n}{L}y\right), \tag{5.3.41}$$

where $m, n \in \mathbb{Z}$. Therefore,

$$\Delta|\delta\psi| = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\delta\psi| = -\frac{\pi^2}{L^2}(m^2+n^2)\,. \tag{5.3.42}$$

The nonzero $|\delta\psi|$ with the lowest eigenvalue for $-\Delta$ is given by $m = n = \pm 1$. For this solution, Eq. (5.3.38) becomes

$$\left(-\frac{2\pi^2}{L^2} - \frac{8\pi n_0 e^2}{\ell}\beta\right)|\delta\psi| = 0\,. \tag{5.3.43}$$

Therefore,

$$T_c = -\frac{4Ne^2}{\pi\ell} \approx -1.273\,\frac{Ne^2}{\ell} \approx -0.637\,T_0\,. \tag{5.3.44}$$

The more accurate way of determining T_c is to evaluate Eq. (5.3.17) and Eq.(5.3.23) numerically. Then the temperature is defined as

$$\beta = \frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N}\,. \tag{5.3.45}$$

The S-E and the β-E curve are shown in Fig. 5.13 and Fig. 5.14 respectively.

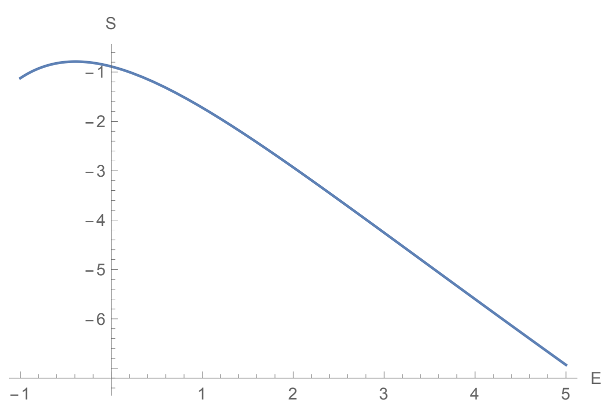


Figure 5.13: S-E for the system in a box

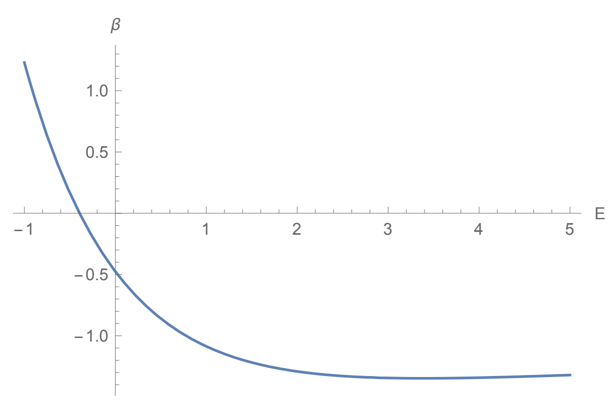


Figure 5.14: β-E for the system in a box

From Fig. 5.14 we see that at $E \approx -0.393 E_0$ the temperature of the system changes from positive to negative, while at $E \approx 3.383\,E_0$ there is

$$\frac{\partial\beta}{\partial E}(E \approx 3.383\,E_0) = 0\,. \tag{5.3.46}$$

Similar to the system in a disc, we identify this point as the critical point. Hence, we can read off from it

$$T_c \approx -0.742 \frac{Ne^2}{\ell} = -0.371\, T_0 \,, \tag{5.3.47}$$

which is not far from the estimate we did before.

Correlation Function

We evaluate Eq. (5.3.37) numerically. Fig. 5.15 shows $\langle \sigma(\vec{r})\, \sigma(\vec{r} + \vec{s}) \rangle$ as a function of $s = |\vec{s}|$ for $E = 3.2\, E_0 \simeq E_c$.

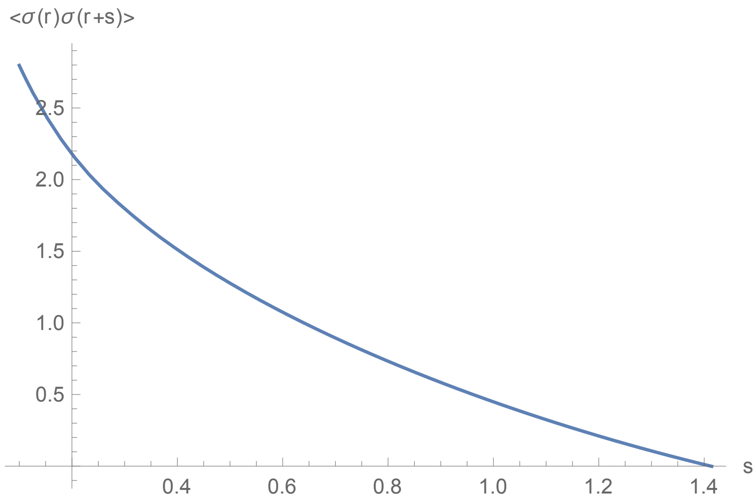


Figure 5.15: The correlation function $\langle \sigma(\vec{r})\, \sigma(\vec{r} + \vec{s}) \rangle$ as a function of $|\vec{s}|$ for $E = 3.2\, E_0$

We find that the correlation function is positive in the ordered phase. In the disordered phase (but still negative temperature), the correlation function changes sign.

On the other hand, when $E < -0.393\, E_0$ the temperature of the system becomes positive. This change can also be seen from the correlation function. Fig. 5.16 shows $\langle \sigma(\vec{r})\, \sigma(\vec{r} + \vec{s}) \rangle$ as a function of s for $E = -0.4\, E_0$, which has different behavior for $s \to 0$ compared to $E > -0.393\, E_0$.

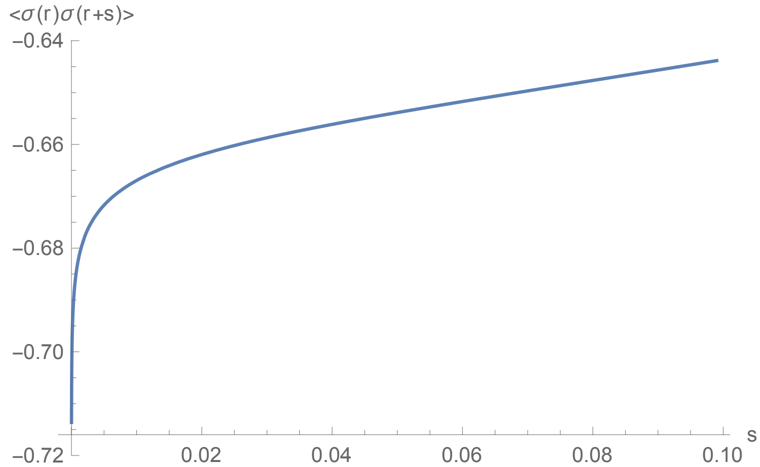


Figure 5.16: The correlation function $\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle$ as a function of $|\vec{s}|$ for $E = -0.4\,E_0$

$\beta\langle D^2\rangle$-E **Curve**

From the correlation function $\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle$, we can calculate $\langle D^2\rangle$ in the following way:

$$
\begin{aligned}
\langle D^2\rangle &= \int d^2r\, d^2s\, \vec{r}\cdot(\vec{r}+\vec{s})\,\langle\sigma(\vec{r})\,\sigma(\vec{r}+\vec{s})\rangle \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \int_{-x-\frac{L}{2}}^{-x+\frac{L}{2}} ds_x \int_{-y-\frac{L}{2}}^{-y+\frac{L}{2}} ds_y \left[x(x+s_x)+y(y+s_y)\right]\,\langle\sigma(x,y)\,\sigma(x+s_x,y+s_y)\rangle \\
&= L^8 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tilde{x} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tilde{y} \int_{-\tilde{x}-\frac{1}{2}}^{-\tilde{x}+\frac{1}{2}} \frac{d\tilde{s}}{2\sqrt{\pi}} \int_{-\tilde{y}-\frac{1}{2}}^{-\tilde{y}+\frac{1}{2}} \frac{d\tilde{s}}{2\sqrt{\pi}} \left[\tilde{x}\left(\tilde{x}+\frac{\tilde{s}_x}{2\sqrt{\pi}}\right)+\tilde{y}\left(\tilde{y}+\frac{\tilde{s}_y}{2\sqrt{\pi}}\right)\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\langle\sigma(x,y)\,\sigma(x+s_x,y+s_y)\rangle \\
&= 4L^8 \int_0^1 \frac{d\tilde{s}_x}{2\sqrt{\pi}} \int_0^1 \frac{d\tilde{s}_y}{2\sqrt{\pi}} \int_{-\frac{1}{2}}^{-\tilde{s}_x+\frac{1}{2}} d\tilde{x} \int_{-\frac{1}{2}}^{-\tilde{s}_y+\frac{1}{2}} d\tilde{y} \left[\tilde{x}\left(\tilde{x}+\frac{\tilde{s}_x}{2\sqrt{\pi}}\right)+\tilde{y}\left(\tilde{y}+\frac{\tilde{s}_y}{2\sqrt{\pi}}\right)\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\langle\sigma(x,y)\,\sigma(x+s_x,y+s_y)\rangle \\
&= \frac{4V^3Ne^2}{3\sqrt{\pi}\Omega}\, e^{1-\epsilon} \int_0^1 \frac{d\tilde{s}_x}{2\sqrt{\pi}} \int_0^1 \frac{d\tilde{s}_y}{2\sqrt{\pi}} (1-\tilde{s}_x)(1-\tilde{s}_y)\left[2\sqrt{\pi}-2\sqrt{\pi}(\tilde{s}_x+\tilde{s}_y)+(4\sqrt{\pi}-3)(\tilde{s}_x^2+\tilde{s}_y^2)\right] \\
&\quad \cdot\left[\int_0^\infty dx\,(x+1)\,x^{-x}\,e^{-x(\epsilon-1)}\left[\cos(\pi x)\,J_0(\sqrt{x+1}\,\tilde{s})-\sin(\pi x)\,Y_0(\sqrt{x+1}\,\tilde{s})\right]\right. \\
&\qquad + \delta(\vec{\tilde{s}}) \int_0^\infty dx\,\sin(\pi x)\,x^{-x}\,e^{-x(\epsilon-1)} \\
&\qquad \left. - \frac{2}{\pi}\int_0^\infty dx\,\sin(\pi x)\int_0^1 d\tilde{k}\,\frac{\tilde{k}^3}{\tilde{k}^2-(x+1)}J_0(\tilde{k}\tilde{s})x^{-x}e^{-x(\epsilon-1)}\right]
\end{aligned}
$$

$$= \frac{V^3 N e^2}{3\pi^{\frac{3}{2}}\Omega}\, e^{1-\epsilon} \int_0^1 d\tilde{s}_x \int_0^1 d\tilde{s}_y\, (1-\tilde{s}_x)(1-\tilde{s}_y) \left[2\sqrt{\pi} - 2\sqrt{\pi}(\tilde{s}_x+\tilde{s}_y) + (4\sqrt{\pi}-3)(\tilde{s}_x^2+\tilde{s}_y^2)\right]$$

$$\cdot \left[\int_0^\infty dx\, (x+1)\, x^{-x}\, e^{-x(\epsilon-1)} \left[\cos(\pi x)\, J_0(\sqrt{x+1}\,\tilde{s}) - \sin(\pi x)\, Y_0(\sqrt{x+1}\,\tilde{s})\right] \right.$$

$$\left. - \frac{2}{\pi} \int_0^\infty dx \sin(\pi x) \int_0^1 d\tilde{k}\, \frac{\tilde{k}^3}{\tilde{k}^2 - (x+1)} J_0(\tilde{k}\tilde{s}) x^{-x} e^{-x(\epsilon-1)} \right]$$

$$+ \frac{V^3 N e^2}{3\pi^{\frac{3}{2}}\Omega}\, 2\sqrt{\pi}\, e^{1-\epsilon} \int_0^\infty dx \sin(\pi x)\, x^{-x}\, e^{-x(\epsilon-1)}\,, \tag{5.3.48}$$

where Ω is obtained by combining Eq. (5.3.15) and Eq. (5.3.23):

$$\Omega = \frac{V^{2N}}{\pi N e^2}\, e^{1-\epsilon} \int_0^\infty dy \sin(\pi y)\, e^{y - y\ln y - \epsilon y}\,. \tag{5.3.49}$$

We plot $|\beta|\langle D^2\rangle$ as a function of E in Fig. 5.17.

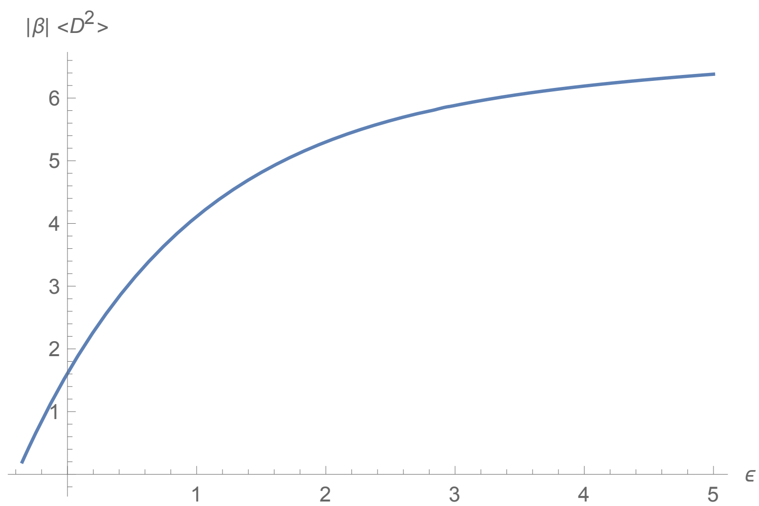


Figure 5.17: The $\beta\langle D^2\rangle$-E curve

5.4 Discussion

In previous sections, we calculated some physical quantities for the system in a disc or in a square box at negative temperature. They unambiguously confirm the existence of a phase transition. We would like to address some issues, that we have not discussed in the text.

There is the well-known Mermin-Wagner-Hohenberg theorem, which says that continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions in dimensions $d \leq 2$. One may worry about that the $U(1)$ rotational symmetry for the system in a disc cannot be broken, hence there would not be a bifurcation point. However, the Mermin-Wagner theorem requires short-range interactions. In the model discussed in this thesis, we are interested in the Coulomb interaction, which is a long-range interaction. Therefore, for the system in a disc there can be a bifurcation point without violating the Mermin-Wagner theorem.

We have discussed in the text the system in a disc and in a square box. In both cases we have observed phase transitions at negative temperature. An interesting question is how the geometry affects the physical phenomena. For a two-dimensional domain with some symmetry, e.g. the $U(1)$ symmetry for the disc and the $\mathbb{Z}_4$ symmetry for the square box, in the new phase after the bifurcation the symmetry will be broken. For a domain without any symmetry, the phase transition still can happen, since the bifurcation point in this model is determined by an eigenvalue problem for the Laplacian operator, which always has solutions for a simply-connected domain in two dimensions. Hence, we anticipate a phase transition without symmetry breaking for a domain without any symmetry.

Besides the number of vortex species, there is another difference between the model considered in this chapter and the one in Ref. [115]. In Ref. [115] there is one more constraint, which requires the totoal angular momentum to be constant. We have not imposed this constraint. It would be interesting to impose this additional constraint and see how it affects the result. We would like to postpone this problem to the future research.

Finally, we would like to emphasize that the results of Section 3 is essentially the mean-field result, which can be seen from the critical exponent, which is exactly $\frac{1}{2}$ (5.2.91). It is worth studying the phase transition for the point-vortex model at negative temperature more carefully. For instance, one can use the renormalization group method to find more accurate values of the critical exponents.

Chapter 6

Conclusion and Prospect

In the previous chapters, we have studied the partition functions in many concrete models, including supersymmetric gauge theory, conformal field theory, nonlinear sigma model and some statistical models in the quantum fluid. In all these cases, we see that the partition function provides us with a powerful tool to investigate the quantum properties of the models and the dualities among different models. People have learned a lot of techniques in recent years, but still not completely understood the fascinating but at the same time mysterious quantum world. Let us list a few important open problems related to the study of the partition function in the below. I believe that progress on any of these problems can possibly lead to the next major breakthrough.

- **Pure Yang-Mills Theory:**

  The problems from the pure Yang-Mills theory always lie at the center of modern theoretical physics, including the mass gap problem, the confinement problem, the strong CP problem and some problems related to the Higgs mechanism, e.g. the naturalness problem. If one day these problems could be resolved, one would first expect the exact or at least a good approximate expression of the partition function of the pure d-dimensional Yang-Mills theory ($d > 2$).

- **Chern-Simons Theory and Related Theories:**

  The Chern-Simons theory has abundant applications in mathematics, especially in knot theory. On the physics side, it is known as a topological field theory and provides an effective theory for the quantum Hall effect. Starting from late 1980s through 1990s, people found more and deep connections between the Chern-Simons theory and other theories. First, the quantum Chern-Simons theory was found to be equivalent to the WZW model [121, 122], which is a conformal field theory. Furthermore, Edward Witten found that a free fermion theory is also equivalent to the WZW model under the concept non-Abelian bosonization [123]. Some good reviews on these subjects and their connection with topological quantum computation are Refs. [124, 125]. The connection between these different aspects can somehow be understood in the quantum group theory, which is reviewed in the note by D. Freed in Ref. [45]. A recent trend of the Chern-Simons theory is to study its implication in the topological phases, which probably started with Ref. [126]. After the studies I have done in this thesis, a natural question I would like to ask is whether we can study the topological phases in the presence of quantum turbulence.

- **Pure Gravity:**

  As we discussed in the introduction, although we have not touched the gravity in this thesis, some studies on the quantum properties of the gravity using the partition function have appeared, like the 3-dimensional gravity or some supergravities. The next question for the future study is if we can obtain more exact results, which probably can give us some hints of finding a consistent theory of quantum gravity. Also, there are some recent works relating the entanglement entropy and the black-hole, e.g. Ref. [127]. Hopefully, these works can help us address the information paradox problem.

- **Statistical Mechanics:**

  Of course, as the origin of the concept partition function, the statistical mechanics is still worth studying. More importantly, the central problem is still there, i.e. the 3-dimensional Ising model. We discussed before, Ref. [33] made some attempts to solve the 3-dimensional Ising model by evaluating the partition function exactly plus generalizing the concept modularity to higher dimensions, but a naive generalization did not work. Can we give another try?

- **String Theory (or Topological String Theory):**

  String theory is claimed to be the best candidate for quantum gravity and unification, however, there are still many issues that are not well understood in nonperturbative string theory. One example is the $M5$-brane and its low energy theory, the 6-dimensional $(2,0)$ theory. Can the study in the partition function help us understand these nonperturbative aspects of string theory?

Because of these big unsolved problems, I am quite optimistic about the future development in theoretical physics and willing to actively participate in it. I would like to quote a comment by Arthur Jaffe and Edward Witten in Ref. [128] as the closing remark of this thesis:

> *" ... one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition of quantum gauge*

theory in four dimensions. Will this change in the 21st century? We hope so! "

Appendix A

Convention in S^3 Localization

In this appendix we review our convention and some identities used in the thesis. We mainly follow the convention of Ref. [16]. The 3D γ-matrices are chosen to be

$$\gamma_1 = \sigma_3\,, \quad \gamma_2 = -\sigma_1\,, \quad \gamma_3 = -\sigma_2\,, \tag{A.0.1}$$

where σ_i are the Pauli matrices. They still satisfy

$$[\gamma_m,\, \gamma_n] = 2i\varepsilon_{mnp}\gamma^p\,. \tag{A.0.2}$$

This will consequently affect the eigenvalues of spherical harmonics defined on the squashed S^3. The main difference is that for a spin-0 field instead of L_3 now L_1 has the eigenvalues

$$im \quad \text{with} \quad -j \leqslant m \leqslant j, \quad j = 0,\, \frac{1}{2},\, 1,\, \cdots$$

In this thesis, we use commuting spinors. The product of two spinors are defined as

$$\psi\chi = \psi^\alpha C_{\alpha\beta}\chi^\beta\,, \quad \psi\gamma_\mu\chi = \psi^\alpha (C\gamma_\mu)_{\alpha\beta}\chi^\beta\,, \tag{A.0.3}$$

where the indices can be raised and lowered using

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the charge conjugation matrix. The spinor bilinears of commuting spinors satisfy

$$\psi\chi = -\chi\psi\,, \quad \psi\gamma_\mu\chi = \chi\gamma_\mu\psi\,. \tag{A.0.4}$$

The Fierz identity for commuting spinors is

$$(\psi_1\chi_1)(\psi_2\chi_2) = \frac{1}{2}(\psi_1\chi_2)(\psi_2\chi_1) + \frac{1}{2}(\psi_1\gamma^\mu\chi_2)(\psi_2\gamma_\mu\chi_1)\,. \tag{A.0.5}$$

The Hermitian conjugate of a spinor is given by

$$(\psi^\dagger)^\alpha \equiv \overline{(\psi_\alpha)}\,, \tag{A.0.6}$$

where $^-$ denotes the complex conjugate. The charge conjugate of a spinor is defined as

$$\chi^c \equiv \sigma_2 \overline{\chi}\,. \tag{A.0.7}$$

We use ψ and $\widetilde{\psi}$ to denote the spinors that are Hermitian conjugate to each other in Lorentzian signature, while independent in Euclidean signature. In particular, ζ and $\widetilde{\zeta}$ are such a kind of spinor pair. They satisfy the generalized Killing spinor equations (2.3.19). Although in Euclidean signature they are independent, one can prove that the charge conjugate of ζ, i.e., ζ^c satisfies the same Killing spinor equation as $\widetilde{\zeta}$. Hence,

$$\zeta^c \propto \widetilde{\zeta}\,. \tag{A.0.8}$$

Appendix B

S^3 as an $SU(2)$-Group Manifold

A convenient way to discuss different ways of squashing is to introduce the left-invariant and the right-invariant frame. A group element in $SU(2)$ is given by

$$g \equiv i x_\mu \sigma^\mu = \begin{pmatrix} x_0 + i x_3 & x_2 + i x_1 \\ -x_2 + i x_1 & x_0 - i x_3 \end{pmatrix}, \tag{B.0.1}$$

where $\sigma^\mu = (-iI, \vec{\sigma})$. Then

$$\mu \equiv g^{-1} dg \quad \text{and} \quad \widetilde{\mu} \equiv dg\, g^{-1} \tag{B.0.2}$$

are left-invariant and right-invariant 1-form respectively, i.e., they are invariant under the transformations with a constant matrix h

$$g \to h\, g \quad \text{and} \quad g \to g\, h$$

respectively. It can be checked explicitly that the metric of S^3 can be written as

$$ds^2 = \frac{\ell^2}{2} \mathrm{tr}(dg\, dg^{-1}) = \ell^2 \mu^m \mu^m = \ell^2 \widetilde{\mu}^{\,m} \widetilde{\mu}^{\,m} = \frac{\ell^2}{2} (\mu^m \mu^m + \widetilde{\mu}^{\,m} \widetilde{\mu}^{\,m}), \tag{B.0.3}$$

with $m = 1, 2, 3$, if we impose the constraint

$$\det g = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 = 1, \tag{B.0.4}$$

where

$$\mu^m = \frac{i}{2} \mathrm{tr}(\mu \gamma^m), \quad \widetilde{\mu}^{\,m} = \frac{i}{2} \mathrm{tr}(\widetilde{\mu}\, \gamma^m), \tag{B.0.5}$$

or equivalently,

$$-2\mu^m T_m = \mu = g^{-1} dg, \quad T_m \equiv i \frac{\gamma_m}{2}, \; g \in SU(2). \tag{B.0.6}$$

Hence, the metric of S^3 is both left-invariant and right-invariant.

A 3D Killing spinor is a spinor that satisfies

$$D\epsilon \equiv d\epsilon + \frac{1}{4}\gamma_{mn}\omega^{mn}\epsilon = e^m\gamma_m\widetilde{\epsilon}\,, \tag{B.0.7}$$

where $\gamma_{mn} \equiv \frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m)$, and the choice of γ_m is given in Appendix A, while ω^{mn} is the spin connection, and $\widetilde{\epsilon}$ is another spinor which in general can be different from ϵ. In this thesis, sometimes we try to bring the spinors satisfying generalized Killing spinor equations back into this simple form. Moreover, we can define two Killing vector fields by their group actions on a general group element g in $SU(2)$:

$$\mathcal{L}^m g = i\gamma^m g\,, \qquad \mathcal{R}^m g = ig\gamma^m\,. \tag{B.0.8}$$

As discussed before, the metric of S^3 is both left-invariant and right-invariant, hence it is also invariant under the actions of $\mathcal{L}^m$ and $\mathcal{R}^m$ given above. By definition, a Killing vector field is a vector field that preserves the metric. Therefore, $\mathcal{L}^m$ and $\mathcal{R}^m$ are Killing vector fields. Since the actions of $\mathcal{L}^m$ and $\mathcal{R}^m$ are equivalent to multiplications of $i\gamma^m$ from the left and from the right respectively, after some rescaling they are the same as the generators of $SU(2)$ algebra, i.e., $\{\frac{1}{2i}\mathcal{L}^m\}$ and $\{-\frac{1}{2i}\mathcal{R}^m\}$ both satisfy the commutation relation of the $SU(2)$ algebra.

In the above, we define $\mathcal{L}^m$ and $\mathcal{R}^m$ as group actions. Sometimes we also use them to denote the variations caused by the group actions. In this sense, there are

$$\mathcal{L}^m(g^{-1}dg) = 0 \quad \Rightarrow \quad \mathcal{L}^m\mu^n = 0 \tag{B.0.9}$$

and

$$\begin{aligned}\mathcal{L}^m\widetilde{\mu}^{\,n} &= \mathcal{L}^m\frac{i}{2}\mathrm{tr}(\widetilde{\mu}\gamma^n) = \mathcal{L}^m\frac{i}{2}\mathrm{tr}(dg\,g^{-1}\gamma^n)\\ &= \frac{i}{2}\left[\mathrm{tr}(i\gamma^m\widetilde{\mu}\gamma^n) + \mathrm{tr}(\widetilde{\mu}(-i\gamma^m)\gamma^n)\right]\\ &= 2\varepsilon^{mnp}\,\widetilde{\mu}^{\,p}\,.\end{aligned} \tag{B.0.10}$$

Similarly, there are

$$\mathcal{R}^m\,\widetilde{\mu}^{\,n} = 0 \quad \text{and} \quad \mathcal{R}^m\mu^n = -2\varepsilon^{mnp}\mu^p\,. \tag{B.0.11}$$

In other words, $\mathcal{L}^m$ acts only on the right-invariant frames, while $\mathcal{R}^m$ acts only on the left-invariant frames. Hence, the round S^3 has an $SU(2)_L \times SU(2)_R$ isometry.

Appendix C

Different Metrics of Squashed S^3

There are different expressions of S^3 and squashed S^3. In this appendix we review some relevant ones for this thesis. A more thorough discussion can be found in Appendix A of Ref. [129].

As discussed in Appendix B, the metric of S^3 can be written as (B.0.3):

$$ds^2 = \ell^2 \mu^m \mu^m = \ell^2 \widetilde{\mu}^m \widetilde{\mu}^m \,, \tag{C.0.1}$$

where

$$g^{-1} dg = -i \mu^m \gamma^m \,, \quad dg' \, g'^{-1} = -i \widetilde{\mu}^m \gamma^m \,. \tag{C.0.2}$$

In general the $SU(2)$ group elements g and g' can be different. If we choose on S^3

$$g = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \cdot \exp\left(i \frac{\phi+\psi}{2}\right) & \sin\left(\frac{\theta}{2}\right) \cdot \exp\left(i \frac{\phi-\psi}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \cdot \exp\left(-i \frac{\phi-\psi}{2}\right) & \cos\left(\frac{\theta}{2}\right) \cdot \exp\left(-i \frac{\phi+\psi}{2}\right) \end{pmatrix} ,$$

$$g' = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \cdot \exp\left(i \frac{\phi+\psi}{2}\right) & \sin\left(\frac{\theta}{2}\right) \cdot \exp\left(-i \frac{\phi-\psi}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \cdot \exp\left(i \frac{\phi-\psi}{2}\right) & \cos\left(\frac{\theta}{2}\right) \cdot \exp\left(-i \frac{\phi+\psi}{2}\right) \end{pmatrix} ,$$

then the vielbeins in the left-invariant frame and in the right-invariant frame are given by:

$$\begin{aligned} e_1^{(0)} &\equiv \ell \mu^1 = -\frac{\ell}{2}(d\psi + \cos\theta \, d\phi) \,, \\ e_2^{(0)} &\equiv \ell \mu^2 = -\frac{\ell}{2}(\sin\psi \, d\theta - \sin\theta \cos\psi \, d\phi) \,, \\ e_3^{(0)} &\equiv \ell \mu^3 = \frac{\ell}{2}(\cos\psi \, d\theta + \sin\theta \sin\psi \, d\phi) \,. \end{aligned} \tag{C.0.3}$$

$$\widetilde{e}_1^{(0)} \equiv \ell\, \widetilde{\mu}^{\,1} = -\frac{\ell}{2}(d\psi + \cos\theta\, d\phi)\,,$$
$$\widetilde{e}_2^{(0)} \equiv \ell\, \widetilde{\mu}^{\,2} = \frac{\ell}{2}(\sin\psi\, d\theta - \sin\theta \cos\psi\, d\phi)\,,$$
$$\widetilde{e}_3^{(0)} \equiv \ell\, \widetilde{\mu}^{\,3} = \frac{\ell}{2}(\cos\psi\, d\theta + \sin\theta \sin\psi\, d\phi)\,. \tag{C.0.4}$$

They satisfy

$$de^a_{(0)} + \frac{1}{\ell}\varepsilon^{abc} e^b_{(0)} \wedge e^c_{(0)} = 0\,, \quad d\,\widetilde{e}^{\,a}_{(0)} - \frac{1}{\ell}\varepsilon^{abc}\,\widetilde{e}^{\,b}_{(0)} \wedge \widetilde{e}^{\,c}_{(0)} = 0\,, \tag{C.0.5}$$

i.e., the spin connections are

$$\omega^{ab}_{(0)} = -\frac{1}{\ell}\varepsilon^{abc} e^c_{(0)}\,, \quad \widetilde{\omega}^{\,ab}_{(0)} = \frac{1}{\ell}\varepsilon^{abc}\widetilde{e}^{\,c}_{(0)}\,. \tag{C.0.6}$$

We see explicitly that for this choice of g and g' there are

$$e^1_{(0)} = \widetilde{e}^{\,1}_{(0)} = -\frac{\ell}{2}(d\psi + \cos\theta\, d\phi)\,, \quad (e^2_{(0)})^2 + (e^3_{(0)})^2 = (\widetilde{e}^{\,2}_{(0)})^2 + (\widetilde{e}^{\,3}_{(0)})^2 = \frac{\ell^2}{4}(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{C.0.7}$$

That is the reason why in the thesis we can occasionally interchange between the left-invariant frame and the right-invariant frame.

Besides the form (B.0.3), the metric of S^3 can also be written as a Hopf fibration or a torus fibration. Both the left-invariant frame (C.0.3) and the right-invariant frame (C.0.4) can give the Hopf fibration of S^3:

$$ds^2 = \frac{\ell^2}{4}(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta\, d\phi)^2)\,, \tag{C.0.8}$$

where

$$0 \leqslant \theta \leqslant \pi\,, \quad 0 \leqslant \phi \leqslant 2\pi\,, \quad 0 \leqslant \psi \leqslant 4\pi\,. \tag{C.0.9}$$

The torus fibration of S^3 is

$$ds^2 = \ell^2(d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2)\,, \tag{C.0.10}$$

where

$$0 \leqslant \vartheta \leqslant \frac{\pi}{2}\,, \quad 0 \leqslant \varphi_1\,, \varphi_2 \leqslant 2\pi\,. \tag{C.0.11}$$

The following conditions relate the coordiantes of the Hopf fibration and the torus fibration of S^3:

$$\theta = 2\vartheta\,, \quad \phi = \varphi_2 - \varphi_1\,, \quad \psi = \varphi_1 + \varphi_2\,. \tag{C.0.12}$$

To apply the method introduced in Ref. [16], we have to rewrite the metric of S^3 as a Hopf fibration (C.0.8) further. Using the stereographic projection

$$X \equiv \cot\frac{\theta}{2}\cos\phi\,, \quad Y \equiv \cot\frac{\theta}{2}\sin\phi \tag{C.0.13}$$

we can rewrite the metric of S^2 as

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2\theta d\phi^2 \\ &= \frac{4}{(1+X^2+Y^2)^2}(dX^2+dY^2) \\ &= \frac{4}{(1+z\bar{z})^2}dz\, d\bar{z}\,, \end{aligned} \tag{C.0.14}$$

where

$$z \equiv X + iY\,, \quad \bar{z} \equiv X - iY\,. \tag{C.0.15}$$

Consequently, the metric of S^3 as a Hopf fibration (C.0.8) has the following forms, if we set $\ell = 1$:

$$\begin{aligned} ds^2 &= \frac{1}{4}(d\psi + \cos\theta\, d\phi)^2 + \frac{1}{4}d\theta^2 + \frac{1}{4}\sin^2\theta\, d\phi^2 \\ &= \frac{1}{4}\left[d\psi - \frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{Y}{X^2+Y^2}\,dX + \frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{X}{X^2+Y^2}\,dY\right]^2 \\ &\quad + \frac{1}{(1+X^2+Y^2)^2}(dX^2+dY^2) \\ &= \frac{1}{4}(d\psi + a dz + \bar{a}d\bar{z})^2 + c^2 dz\, d\bar{z}\,, \end{aligned} \tag{C.0.16}$$

where

$$a \equiv -\frac{i}{2z}\cdot\frac{z\bar{z}-1}{z\bar{z}+1}\,, \quad c \equiv \frac{1}{1+z\bar{z}}\,. \tag{C.0.17}$$

Based on our choice of left-invariant frame (C.0.3) and the right-invariant frame (C.0.4), the metric of the squashed S^3 with $SU(2)\times U(1)$ isometry has the following expression:

$$ds^2 = \frac{\ell^2}{v^2}\mu^1\mu^1 + \ell^2\mu^2\mu^2 + \ell^2\mu^3\mu^3 = \frac{\ell^2}{v^2}\widetilde{\mu}^1\widetilde{\mu}^1 + \ell^2\widetilde{\mu}^2\widetilde{\mu}^2 + \ell^2\widetilde{\mu}^3\widetilde{\mu}^3\,, \tag{C.0.18}$$

where v is a constant squashing parameter. For this squashed S^3, we choose the left-invariant frame and the right-invariant frame to be

$$\begin{aligned} e_1 &\equiv \frac{\ell}{v}\mu^1 = -\frac{\ell}{2v}(d\psi + \cos\theta\, d\phi)\,, \\ e_2 &\equiv \ell\mu^2 = -\frac{\ell}{2}(\sin\psi\, d\theta - \sin\theta\cos\psi\, d\phi)\,, \\ e_3 &\equiv \ell\mu^3 = \frac{\ell}{2}(\cos\psi\, d\theta + \sin\theta\sin\psi\, d\phi)\,. \end{aligned} \tag{C.0.19}$$

$$\begin{aligned} \widetilde{e}_1 &\equiv \frac{\ell}{v}\,\widetilde{\mu}^1 = -\frac{\ell}{2v}(d\psi + \cos\theta\, d\phi)\,, \\ \widetilde{e}_2 &\equiv \ell\,\widetilde{\mu}^2 = \frac{\ell}{2}(\sin\psi\, d\theta - \sin\theta\cos\psi\, d\phi)\,, \\ \widetilde{e}_3 &\equiv \ell\,\widetilde{\mu}^3 = \frac{\ell}{2}(\cos\psi\, d\theta + \sin\theta\sin\psi\, d\phi)\,. \end{aligned} \tag{C.0.20}$$

The corresponding spin connections are

$$\omega^{23} = -(2 - \frac{1}{v^2})\,\mu^1\,, \quad \omega^{31} = -\frac{1}{v}\,\mu^2\,, \quad \omega^{12} = -\frac{1}{v}\,\mu^3 \tag{C.0.21}$$

for the left-invariant frame, and

$$\widetilde{\omega}^{\,23} = (2 - \frac{1}{v^2})\,\widetilde{\mu}^{\,1}\,, \quad \widetilde{\omega}^{\,31} = \frac{1}{v}\,\widetilde{\mu}^{\,2}\,, \quad \widetilde{\omega}^{\,12} = \frac{1}{v}\,\widetilde{\mu}^{\,3} \tag{C.0.22}$$

for the right-invariant frame. As for S^3, the metric (C.0.18) can also be written in some other coordinates:

$$\begin{aligned} ds^2 &= \frac{1}{4v^2}(d\psi + \cos\theta\, d\phi)^2 + \frac{1}{4}d\theta^2 + \frac{1}{4}\sin^2\theta\, d\phi^2 && \text{(C.0.23)} \\ &= \frac{1}{4v^2}\left[d\psi - \frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{Y}{X^2+Y^2}\,dX + \frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{X}{X^2+Y^2}\,dY\right]^2 \\ &\quad + \frac{1}{(1+X^2+Y^2)^2}(dX^2+dY^2) && \text{(C.0.24)} \\ &= \frac{1}{4v^2}(d\psi + a dz + \bar{a} d\bar{z})^2 + c^2 dz\, d\bar{z}\,, && \text{(C.0.25)} \end{aligned}$$

In practice it is more convenient to choose a frame different from the right-invariant frame (C.0.20), which is given by

$$\begin{aligned} \hat{e}^1 &= \frac{1}{2v}\left[d\psi - \frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{Y}{X^2+Y^2}\,dX + \frac{X^2+Y^2-1}{X^2+Y^2+1}\cdot\frac{X}{X^2+Y^2}\,dY\right]\,, \\ \hat{e}^2 &= \frac{1}{1+X^2+Y^2}dX\,, \\ \hat{e}^3 &= \frac{1}{1+X^2+Y^2}dY\,. \end{aligned} \tag{C.0.26}$$

In the coordinates $(\theta,\, \phi,\, \psi)$ and $(z,\, \bar{z},\, \psi)$ the vielbeins look like

$$\begin{aligned} \hat{e}^1 &= \frac{1}{2v}d\psi + \frac{1}{2v}\cos\theta\, d\phi = \frac{1}{2v}(d\psi + a dz + \bar{a} d\bar{z})\,, \\ \hat{e}^2 &= -\frac{1}{2}\cos\phi\, d\theta - \frac{1}{2}\sin\theta\,\sin\phi\, d\phi = c\frac{dz + d\bar{z}}{2}\,, \\ \hat{e}^3 &= -\frac{1}{2}\sin\phi\, d\theta + \frac{1}{2}\sin\theta\,\cos\phi\, d\phi = c\frac{dz - d\bar{z}}{2i}\,. \end{aligned} \tag{C.0.27}$$

Appendix D

BPS Solutions in S^3 Localization

As we discussed in the text, to preserve the supersymmetry given by Eq. (2.3.50) and Eq. (2.3.51), the BPS equations (2.3.179) should be satisfied:

$$Q\psi = 0\,, \quad Q\widetilde{\psi} = 0\,, \quad Q\lambda = 0\,, \quad Q\widetilde{\lambda} = 0\,, \tag{D.0.1}$$

or in explicit form

$$0 = \sqrt{2}\zeta F - \sqrt{2}i(z - q\sigma - rH)\widetilde{\zeta}\phi - \sqrt{2}i\gamma^\mu\widetilde{\zeta}D_\mu\phi\,, \tag{D.0.2}$$

$$0 = \sqrt{2}\widetilde{\zeta}\widetilde{F} + \sqrt{2}i(z - q\sigma - rH)\zeta\widetilde{\phi} + \sqrt{2}i\gamma^\mu\zeta D_\mu\widetilde{\phi}\,, \tag{D.0.3}$$

$$0 = i\zeta(D + \sigma H) - \frac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\zeta f_{\mu\nu} - \gamma^\mu\zeta(i\partial_\mu\sigma - V_\mu\sigma)\,, \tag{D.0.4}$$

$$0 = -i\widetilde{\zeta}(D + \sigma H) - \frac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\widetilde{\zeta} f_{\mu\nu} + \gamma^\mu\widetilde{\zeta}(i\partial_\mu\sigma + V_\mu\sigma)\,. \tag{D.0.5}$$

Using the solutions of the Killing spinor equations (2.3.132)

$$\zeta^\alpha = \sqrt{s}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\,, \quad \widetilde{\zeta}^\alpha = \frac{1}{2v\sqrt{s}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{D.0.6}$$

and choosing the frame given by (C.0.26) (C.0.27), we obtain for commuting spinors

$$\zeta\zeta = 0\,, \quad \widetilde{\zeta}\widetilde{\zeta} = 0\,, \quad \zeta\widetilde{\zeta} = -\widetilde{\zeta}\zeta = -\frac{1}{2v}\,, \tag{D.0.7}$$

$$\zeta\gamma_m\zeta = (0, s, -is)\,, \quad \widetilde{\zeta}\gamma_m\widetilde{\zeta} = (0, -\frac{1}{4sv^2}, -\frac{i}{4sv^2})\,, \tag{D.0.8}$$

$$\zeta\gamma_m\widetilde{\zeta} = (\frac{1}{2v}, 0, 0)\,, \quad \widetilde{\zeta}\gamma_m\zeta = (\frac{1}{2v}, 0, 0)\,, \tag{D.0.9}$$

where $m = 1, 2, 3$.

Contracting Eq. (D.0.2) with $\widetilde{\zeta}$ and Eq. (D.0.3) with ζ from the left, one obtains

$$F = 0\,, \quad \widetilde{F} = 0\,. \tag{D.0.10}$$

Plugging these solutions into Eq. (D.0.2) and Eq. (D.0.3), can contracting them with ζ and $\widetilde{\zeta}$ respectively from the left, one has

$$\frac{\sqrt{2}i}{2v}(z - q\sigma - rH)\phi - \frac{\sqrt{2}i}{2v}D_1\phi = 0\,, \tag{D.0.11}$$

$$\frac{\sqrt{2}i}{2v}(z - q\sigma - rH)\widetilde{\phi} + \frac{\sqrt{2}i}{2v}D_1\widetilde{\phi} = 0\,. \tag{D.0.12}$$

Since $\widetilde{\phi} = \phi^\dagger$, for generic values of $(z - q\sigma - rH)$ the equations above do not have nontrivial solutions. Hence,

$$\phi = \widetilde{\phi} = 0\,. \tag{D.0.13}$$

In the gauge sector, we can contract Eq. (D.0.4) and Eq. (D.0.5) with ζ and $\widetilde{\zeta}$ respectively from the left, then we obtain

$$(\zeta\gamma_\mu\zeta)\left(-\frac{i}{2}\varepsilon^{\rho\sigma\mu}f_{\rho\sigma} - i\partial^\mu\sigma + V^\mu\sigma\right) = 0\,, \tag{D.0.14}$$

$$(\widetilde{\zeta}\gamma_\mu\widetilde{\zeta})\left(-\frac{i}{2}\varepsilon^{\rho\sigma\mu}f_{\rho\sigma} + i\partial^\mu\sigma + V^\mu\sigma\right) = 0\,. \tag{D.0.15}$$

Taking into account that

$$\zeta\gamma_2\zeta \neq 0\,, \quad \zeta\gamma_3\zeta \neq 0\,, \quad \widetilde{\zeta}\gamma_2\widetilde{\zeta} \neq 0\,, \quad \widetilde{\zeta}\gamma_3\widetilde{\zeta} \neq 0\,,$$

we obtain that

$$\partial_2\sigma = \partial_3\sigma = 0\,. \tag{D.0.16}$$

Similarly, contracting Eq. (D.0.4) and Eq. (D.0.5) with $\widetilde{\zeta}$ and ζ respectively from the left will give

$$\frac{i}{2v}(D + \sigma H) + (\widetilde{\zeta}\gamma_\mu\zeta)\left(-\frac{i}{2}\varepsilon^{\rho\sigma\mu}f_{\rho\sigma} - i\partial^\mu\sigma + V^\mu\sigma\right) = 0\,, \tag{D.0.17}$$

$$\frac{i}{2v}(D + \sigma H) + (\zeta\gamma_\mu\widetilde{\zeta})\left(-\frac{i}{2}\varepsilon^{\rho\sigma\mu}f_{\rho\sigma} + i\partial^\mu\sigma + V^\mu\sigma\right) = 0\,. \tag{D.0.18}$$

Then

$$\widetilde{\zeta}\gamma_1\zeta = \zeta\gamma_1\widetilde{\zeta} \neq 0$$

implies that

$$\partial_1\sigma = 0\,. \tag{D.0.19}$$

Therefore, we can conclude that

$$\partial_\mu\sigma = 0\,, \tag{D.0.20}$$

i.e., σ is constant. In the above, we prove this condition in a special frame (C.0.26) (C.0.27), but actually the equations Eqs. (D.0.14) $\sim$ (D.0.18) are frame-independent, i.e., they are valid for an arbitrary frame. Hence, in general Eq. (D.0.14) and Eq. (D.0.15) imply that

$$-\frac{i}{2}\varepsilon^{\rho\sigma\mu}f_{\rho\sigma}+V^{\mu}\sigma=0\,,\tag{D.0.21}$$

which leads to

$$a_{\mu}=-\sigma C_{\mu}+a_{\mu}^{(0)}\,,\tag{D.0.22}$$

where $a_{\mu}^{(0)}$ is a flat connection, and C_{μ} is an Abelian gauge field satisfying

$$V^{\mu}=-i\varepsilon^{\mu\nu\rho}\partial_{\nu}C_{\rho}\,.\tag{D.0.23}$$

Under the condition (D.0.21), Eq. (D.0.17) and Eq. (D.0.18) give us

$$D=-\sigma H\,.\tag{D.0.24}$$

We do not want fermionic background, hence the classical solutions of fermions are zero. To summarize, the classical solutions to the BPS equations in our model are

$$a_{\mu}=-\sigma C_{\mu}+a_{\mu}^{(0)}\,,\quad \partial_{\mu}\sigma=0\,,\quad D=-\sigma H\,,\quad \text{all other fields}=0\,.\tag{D.0.25}$$

Appendix E

Some Important Relations in S^3 Localization

In this appendix, we prove a few crucial relations in our calculations. The relations include the second equation of Eq. (2.3.195), Eq. (2.3.196) and Eq. (2.3.270).

To prove the second equation of Eq. (2.3.195), we first observe that

$$\langle S_2^* S_2^c \Phi_1,\, \Phi_2 \rangle = \langle \Phi_1,\, S_2^{c*} S_2 \Phi_2 \rangle\,. \tag{E.0.1}$$

Hence, we only need to prove

$$S_2^* S_2^c \Phi = 0\,.$$

Plugging in the definitions (2.3.194), we can find the relation above by a long but direct calculation. In the intermediate steps we made use of the following relations:

$$\zeta^\dagger \zeta^\dagger = 0\,; \tag{E.0.2}$$

$$V_\mu (\zeta^\dagger \gamma^\mu \zeta^\dagger) = 0 \quad \text{since} \quad V_2 = V_3 = 0 \text{ and } \zeta^\dagger \gamma_1 \zeta^\dagger = 0\,; \tag{E.0.3}$$

$$D_\mu \zeta^\dagger = -\frac{1}{2} H \gamma_\mu \zeta^\dagger + \frac{i}{2} V_\mu \zeta^\dagger + \frac{1}{2} \varepsilon_{\mu\nu\rho} V^\nu \gamma^\rho \zeta^\dagger\,; \tag{E.0.4}$$

$$\gamma^\mu D_\mu \zeta^\dagger = -\frac{3}{2} H \zeta^\dagger - \frac{i}{2} \gamma^\mu V_\mu \zeta^\dagger\,; \tag{E.0.5}$$

$$\varepsilon^{\mu\nu\rho} (\zeta^\dagger \gamma_\rho \zeta^\dagger) V_{\mu\nu} = \varepsilon^{\mu\nu\rho} (\zeta^\dagger \gamma_\rho \zeta^\dagger) A_{\mu\nu} = 0\,, \tag{E.0.6}$$

where

$$A_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu\,, \quad V_{\mu\nu} \equiv \nabla_\mu V_\nu - \nabla_\nu V_\mu\,. \tag{E.0.7}$$

Next, to prove Eq. (2.3.196), we just calculate

$$\widetilde{\Psi} (S_2 S_1^* + S_2^c S_1^{c*}) \Psi$$

and

$$\widetilde{\Psi}(S_1 S_2^* + S_1^c S_2^{c*})\Psi\,,$$

and then compare them to figure out their difference. The results are

$$\begin{aligned}\widetilde{\Psi}(S_2 S_1^* + S_2^c S_1^{c*})\Psi = &-i\left(z - q\sigma - (r-\frac{1}{2})H\right)e^{-2\mathrm{Im}\Theta}\Omega(\widetilde{\Psi}P_+\Psi)\\ &-i\left(\bar{z} - q\bar{\sigma} + (r-\frac{3}{2})H\right)e^{-2\mathrm{Im}\Theta}\Omega(\widetilde{\Psi}P_-\Psi)\\ &+\frac{1}{2}e^{-2\mathrm{Im}\Theta}\Omega V_1(\widetilde{\Psi}\Psi) + i(\zeta^\dagger\gamma^\mu D_\mu\Psi)(\widetilde{\Psi}\zeta) - i(\zeta\gamma^\mu D_\mu\Psi)(\widetilde{\Psi}\zeta^\dagger)\,, \quad \text{(E.0.8)}\end{aligned}$$

$$\begin{aligned}\widetilde{\Psi}(S_1 S_2^* + S_1^c S_2^{c*})\Psi = i&\left(\bar{z} - q\bar{\sigma} + (r-\frac{1}{2})H\right)e^{-2\mathrm{Im}\Theta}\Omega(\widetilde{\Psi}P_+\Psi)\\ &+i\left(z - q\sigma - (r-\frac{3}{2})\right)e^{-2\mathrm{Im}\Theta}\Omega(\widetilde{\Psi}P_-\Psi)\\ &+\frac{1}{2}e^{-2\mathrm{Im}\Theta}\Omega V_1(\widetilde{\Psi}\Psi) + i(\zeta^\dagger\gamma^\mu D_\mu\Psi)(\widetilde{\Psi}\zeta) - i(\zeta\gamma^\mu D_\mu\Psi)(\widetilde{\Psi}\zeta^\dagger)\,, \quad \text{(E.0.9)}\end{aligned}$$

where

$$P_\pm \equiv \frac{1}{2}(1\pm\gamma_1)\,. \tag{E.0.10}$$

Hence,

$$\begin{aligned}(S_2 S_1^* + S_2^c S_1^{c*}) - (S_1 S_2^* + S_1^c S_2^{c*}) &= -2i\mathrm{Re}(z-q\sigma)e^{-2\mathrm{Im}\Theta}\Omega P_+ - 2i\mathrm{Re}(z-q\sigma)e^{-2\mathrm{Im}\Theta}\Omega P_-\\ &= -2i\mathrm{Re}(z-q\sigma)e^{-2\mathrm{Im}\Theta}\Omega\,, \qquad \text{(E.0.11)}\end{aligned}$$

which proves Eq. (2.3.196).

Finally, let us show how to prove Eq. (2.3.270). From Eq. (2.3.269) we see that

$$(iM + \sigma\alpha\Omega)\Lambda = \Omega(-\gamma^\mu D_\mu + \frac{1}{2}H - \frac{i}{2}V_1 - iV_\mu\gamma^\mu)\Lambda\,.$$

Using this relation, one can gradually prove that

$$\begin{aligned}&\Omega d(\widetilde{\zeta}\Lambda) + (iM + \sigma\alpha\Omega)(\widetilde{\zeta}\gamma_\mu\Lambda)d\xi^\mu\\ &= \Omega H(\widetilde{\zeta}\gamma_\mu\Lambda)d\xi^\mu - \frac{i}{2}\Omega V_1(\widetilde{\zeta}\gamma_\mu\Lambda)d\xi^\mu - \frac{i}{2}\Omega V_\mu(\widetilde{\zeta}\Lambda)d\xi^\mu\\ &\quad + \frac{1}{2}\Omega\varepsilon_{\mu\nu\rho}V^\nu(\widetilde{\zeta}\gamma^\rho\Lambda)d\xi^\mu - i\varepsilon_{\mu\nu\rho}\Omega(\widetilde{\zeta}\gamma^\rho D^\nu\Lambda)d\xi^\mu\,. \qquad \text{(E.0.12)}\end{aligned}$$

On the other hand, there is

$$-i*\left(D(\widetilde{\zeta}\gamma_\mu\Lambda)d\xi^\mu\right) = -i\varepsilon_{\mu\nu\rho}\Omega D^\nu(\widetilde{\zeta}\gamma^\rho\Lambda)d\xi^\mu\,. \tag{E.0.13}$$

After some steps it becomes

$$-i\varepsilon_{\mu\nu\rho}\Omega D^{\nu}(\widetilde{\zeta}\gamma^{\rho}\Lambda)d\xi^{\mu}=\Omega H(\widetilde{\zeta}\gamma_{\mu}\Lambda)d\xi^{\mu}-i\Omega V_{\mu}(\widetilde{\zeta}\Lambda)d\xi^{\mu}-i\varepsilon_{\mu\nu\rho}\Omega(\widetilde{\zeta}\gamma^{\rho}D^{\nu}\Lambda)d\xi^{\mu}\,. \quad \text{(E.0.14)}$$

Then we only need to prove that the expressions in Eq. (E.0.12) and Eq. (E.0.14) are equal, or equivalently their difference vanishes, i.e.,

$$-\frac{i}{2}\Omega V_1(\widetilde{\zeta}\gamma_{\mu}\Lambda)d\xi^{\mu}+\frac{i}{2}\Omega V_1(\widetilde{\zeta}\Lambda)d\xi^{1}-\frac{1}{2}\Omega\varepsilon_{1\mu\nu}V^{1}(\widetilde{\zeta}\gamma^{\nu}\Lambda)d\xi^{\mu}=0\,, \quad \text{(E.0.15)}$$

where we have used the fact that V_m has only the 1-component non-vanishing. The last expression can be checked explicitly by using

$$\begin{aligned} & \widetilde{\zeta}\propto(1\quad 0) \\ \Rightarrow\quad & \widetilde{\zeta}\gamma_1=\widetilde{\zeta}\,,\quad i\widetilde{\zeta}\gamma_2=-\widetilde{\zeta}\gamma_3\,,\quad i\widetilde{\zeta}\gamma_3=\widetilde{\zeta}\gamma_2\,, \end{aligned} \quad \text{(E.0.16)}$$

where recall our convention

$$\gamma_1=\sigma_3\,,\quad \gamma_2=-\sigma_1\,,\quad \gamma_3=-\sigma_2\,.$$

Appendix F

2-Dimensional $\mathcal{N} = (2, 2)$ Superspace

The bosonic coordinates of the superspace are $x^\mu, \mu = 0, 1$. We take the flat Minkowski metric to be $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$. The fermionic coordinates of the superspace are θ^+, θ^-, $\bar{\theta}^+$ and $\bar{\theta}^-$, with the complex conjugation relation $(\theta^\pm)^* = \bar{\theta}^\pm$. The indices $\pm$ stand for the chirality under a Lorentz transformation. To raise or lower the spinor index, we use

$$\psi_\alpha = \epsilon_{\alpha\beta}\,\psi^\beta\,, \quad \psi^\alpha = \epsilon^{\alpha\beta}\,\psi_\beta\,, \tag{F.0.1}$$

where

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha, \beta = -, +\,. \tag{F.0.2}$$

Hence, we have $\psi_+ = \psi^-$, $\psi_- = -\psi^+$.

The supercharges and the supercovariant derivative operators are

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm\,, \quad \bar{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm\,, \tag{F.0.3}$$

$$\mathbb{D}_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm\,, \quad \bar{\mathbb{D}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm\,, \tag{F.0.4}$$

where

$$\partial_\pm \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)\,. \tag{F.0.5}$$

They satisfy the anti-commutation relations

$$\{Q_\pm, \bar{Q}_\pm\} = -2i\partial_\pm\,, \quad \{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = 2i\partial_\pm\,, \tag{F.0.6}$$

with all the other anti-commutators vanishing. In particular,

$$\{Q_\pm, \mathbb{D}_\pm\} = 0\,. \tag{F.0.7}$$

Appendix G

Gauged Linear Sigma Model with Semichiral Superfields in Components

If we expand the theory in components, we obtain the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{SC} = {} & \bar{X}^L 2iD_- 2iD_+ X^L - \bar{X}^L(\sigma_1^2 + \sigma_2^2)X^L + \bar{X}^L D X^L + \bar{F}^L F^L \\
& - \bar{M}^L_{-+} M^L_{-+} - \bar{M}^L_{--} 2iD_+ X^L - \bar{X}^L 2iD_+ M^L_{--} + \bar{M}^L_{-+} \bar{\hat{\sigma}} X^L + \bar{X}^L \hat{\sigma} M^L_{-+} \\
& + \bar{\psi}^L_- 2iD_+ \psi^L_- + \bar{\psi}^L_+ 2iD_- \psi^L_+ - \bar{\psi}^L_- \hat{\sigma} \psi^L_+ - \bar{\psi}^L_+ \bar{\hat{\sigma}} \psi^L_- \\
& + \bar{X}^L i\lambda_+ \psi^L_- - \bar{X}^L i\lambda_- \psi^L_+ + \bar{\psi}^L_+ i\bar{\lambda}_- X^L - \bar{\psi}^L_- i\bar{\lambda}_+ X^L - \bar{\eta}^L_- \psi^L_+ - \bar{\psi}^L_+ \eta^L_- \\
& - \bar{\chi}^L_- 2iD_+ \chi^L_- + \bar{X}^L i\bar{\lambda}_+ \chi^L_- - \bar{\chi}^L_- i\lambda_+ X^L \\
& + \bar{X}^R 2iD_- 2iD_+ X^R - \bar{X}^R(\sigma_1^2 + \sigma_2^2)X^R + \bar{X}^R D X^R + \bar{F}^R F^R \\
& - \bar{M}^R_{+-} M^R_{+-} - \bar{M}_{++} 2iD_- X^R - \bar{X}^R 2iD_- M^R_{++} + \bar{M}^R_{+-} \hat{\sigma} X^R + \bar{X}^R \bar{\hat{\sigma}} M^R_{+-} \\
& + \bar{\psi}^R_- 2iD_+ \psi^R_- + \bar{\psi}^R_+ 2iD_- \psi^R_+ - \bar{\psi}^R_- \hat{\sigma} \psi^R_+ - \bar{\psi}^R_+ \bar{\hat{\sigma}} \psi^R_- \\
& + \bar{X}^R i\lambda_+ \psi^R_- - \bar{X}^R i\lambda_- \psi^R_+ + \bar{\psi}^R_+ i\bar{\lambda}_- X^R - \bar{\psi}^R_- i\bar{\lambda}_+ X^R + \bar{\eta}^R_+ \psi^R_- + \bar{\psi}^R_- \eta^R_+ \\
& - \bar{\chi}^R_+ 2iD_- \chi^R_+ - \bar{X}^R i\bar{\lambda}_- \chi^R_+ + \bar{\chi}^R_+ i\lambda_- X^R \\
& + \alpha \bar{X}^L 2iD_- 2iD_+ X^R - \alpha \bar{X}^L(\sigma_1^2 + \sigma_2^2)X^R + \alpha \bar{X}^L D X^R + \alpha \bar{F}^L F^R \\
& + \alpha \bar{M}^L_{--} M^R_{++} - \alpha \bar{M}^L_{--} 2iD_+ X^R - \alpha \bar{X}^L 2iD_- M^R_{++} + \alpha \bar{M}^L_{-+} \bar{\hat{\sigma}} X^R + \alpha \bar{X}^L \bar{\hat{\sigma}} M^R_{+-} \\
& + \alpha \bar{\psi}^L_- 2iD_+ \psi^R_- + \alpha \bar{\psi}^L_+ 2iD_- \psi^R_+ - \alpha \bar{\psi}^L_- \hat{\sigma} \psi^R_+ - \alpha \bar{\psi}^L_+ \bar{\hat{\sigma}} \psi^R_- \\
& + \alpha \bar{X}^L i\lambda_+ \psi^R_- - \alpha \bar{X}^L i\lambda_- \psi^R_+ + \alpha \bar{\psi}^L_+ i\bar{\lambda}_- X^R - \alpha \bar{\psi}^L_- i\bar{\lambda}_+ X^R - \alpha \bar{\eta}^L_- \psi^R_+ + \alpha \bar{\psi}^L_- \eta^R_+ \\
& + \alpha \bar{\chi}^L_- \bar{\hat{\sigma}} \chi^R_+ - \alpha \bar{X}^L i\bar{\lambda}_- \chi^R_+ - \alpha \bar{\chi}^L_- i\lambda_+ X^R \\
& + \alpha \bar{X}^R 2iD_- 2iD_+ X^L - \alpha \bar{X}^R(\sigma_1^2 + \sigma_2^2)X^L + \alpha \bar{X}^R D X^L + \alpha \bar{F}^R F^L \\
& + \alpha \bar{M}^R_{++} M^L_{--} - \alpha \bar{M}^R_{++} 2iD_- X^L - \alpha \bar{X}^R 2iD_+ M^L_{--} + \alpha \bar{M}^R_{+-} \hat{\sigma} X^L + \alpha \bar{X}^R \hat{\sigma} M^L_{-+} \\
& + \alpha \bar{\psi}^R_- 2iD_+ \psi^L_- + \alpha \bar{\psi}^R_+ 2iD_- \psi^L_+ - \alpha \bar{\psi}^R_- \hat{\sigma} \psi^L_+ - \alpha \bar{\psi}^R_+ \bar{\hat{\sigma}} \psi^L_- \\
& + \alpha \bar{X}^R i\lambda_+ \psi^L_- - \alpha \bar{X}^R i\lambda_- \psi^L_+ + \alpha \bar{\psi}^R_+ i\bar{\lambda}_- X^L - \alpha \bar{\psi}^R_- i\bar{\lambda}_+ X^L + \alpha \bar{\eta}^R_+ \psi^L_- - \alpha \bar{\psi}^R_+ \eta^L_- \\
& + \alpha \bar{\chi}^R_+ \hat{\sigma} \chi^L_- + \alpha \bar{X}^R i\bar{\lambda}_+ \chi^L_- + \alpha \bar{\chi}^R_+ i\lambda_- X^L \,.
\end{aligned}
\tag{G.0.1}
$$

The supersymmetry transformation laws for the abelian vector multiplet are

$$\begin{aligned}
\delta A_\mu &= \frac{i}{2}\epsilon\sigma_\mu\bar{\lambda} + \frac{i}{2}\bar{\epsilon}\sigma_\mu\lambda\,,\\
\delta\hat{\sigma} &= -i\epsilon_-\bar{\lambda}_+ - i\bar{\epsilon}_+\lambda_-\,,\\
\delta\bar{\hat{\sigma}} &= -i\epsilon_+\bar{\lambda}_- - i\bar{\epsilon}_-\lambda_+\,,\\
\delta\lambda_+ &= 2\epsilon_-\partial_+\bar{\hat{\sigma}} + i\epsilon_+ D - \epsilon_+ F_{01}\,,\\
\delta\lambda_- &= 2\epsilon_+\partial_-\hat{\sigma} + i\epsilon_- D + \epsilon_- F_{01}\,,\\
\delta\bar{\lambda}_+ &= 2\bar{\epsilon}_-\partial_+\hat{\sigma} - i\bar{\epsilon}_+ D - \bar{\epsilon}_+ F_{01}\,,\\
\delta\bar{\lambda}_- &= 2\bar{\epsilon}_+\partial_-\bar{\hat{\sigma}} - i\bar{\epsilon}_- D + \bar{\epsilon}_- F_{01}\,,\\
\delta D &= \epsilon_+\partial_-\bar{\lambda}_+ + \epsilon_-\partial_+\bar{\lambda}_- - \bar{\epsilon}_+\partial_-\lambda_+ - \bar{\epsilon}_-\partial_+\bar{\lambda}_-\,,
\end{aligned} \tag{G.0.2}$$

where $F_{01} = \partial_0 A_1 - \partial_1 A_0$, and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{G.0.3}$$

The supersymmetry transformations for the components of semichiral multiplets $\mathbb{X}$ are

$$\begin{aligned}
\delta X &= \epsilon\psi + \bar{\epsilon}\chi\,,\\
\delta\psi_+ &= -\epsilon_+ F - \bar{\epsilon}_+\bar{\hat{\sigma}}X + \bar{\epsilon}_+ M_{-+} + \bar{\epsilon}_- 2iD_+ X - \bar{\epsilon}_- M_{++}\,,\\
\delta\psi_- &= -\epsilon_- F - \bar{\epsilon}_+ 2iD_- X + \bar{\epsilon}_+ M_{--} + \bar{\epsilon}_-\hat{\sigma}X - \bar{\epsilon}_- M_{+-}\,,\\
\delta F &= \bar{\epsilon}_+ 2iD_-\psi_+ + \bar{\epsilon}_- 2iD_+\psi_- - \bar{\epsilon}_+\eta_- + \bar{\epsilon}_-\eta_+ - \bar{\epsilon}_+\bar{\hat{\sigma}}\psi_- - \bar{\epsilon}_-\hat{\sigma}\psi_+ + i\bar{\epsilon}_+\bar{\lambda}_- X - i\bar{\epsilon}_-\bar{\lambda}_+ X\,,\\
\delta\chi_+ &= -\epsilon_- M_{++} + \epsilon_+ M_{+-}\,,\\
\delta\chi_- &= -\epsilon_- M_{-+} + \epsilon_+ M_{--}\,,\\
\delta M_{+-} &= -\epsilon_-\eta_+ + \bar{\epsilon}_-\hat{\sigma}\chi_+ - \bar{\epsilon}_+ 2iD_-\chi_+\,,\\
\delta M_{-+} &= -\epsilon_+\eta_- + \bar{\epsilon}_- 2iD_+\chi_- - \bar{\epsilon}_+\bar{\hat{\sigma}}\chi_-\,,\\
\delta M_{++} &= -\epsilon_+\eta_+ + \bar{\epsilon}_- 2iD_+\chi_+ - \bar{\epsilon}_+\bar{\hat{\sigma}}\chi_+\,,\\
\delta M_{--} &= -\epsilon_-\eta_- + \bar{\epsilon}_-\hat{\sigma}\chi_- - \bar{\epsilon}_+ 2iD_-\chi_-\,,\\
\delta\eta_+ &= \bar{\epsilon}_- 2iD_+ M_{+-} - \bar{\epsilon}_- i\bar{\lambda}_+\chi_+ - \bar{\epsilon}_-\hat{\sigma}M_{++} - \bar{\epsilon}_+\bar{\hat{\sigma}}M_{+-} + \bar{\epsilon}_+ i\bar{\lambda}_-\chi_+ + \bar{\epsilon}_+ 2iD_- M_{++}\,,\\
\delta\eta_- &= \bar{\epsilon}_- 2iD_+ M_{--} - \bar{\epsilon}_- i\bar{\lambda}_+\chi_- - \bar{\epsilon}_-\hat{\sigma}M_{-+} - \bar{\epsilon}_+\bar{\hat{\sigma}}M_{--} + \bar{\epsilon}_+ i\bar{\lambda}_-\chi_- + \bar{\epsilon}_+ 2iD_- M_{-+}\,,
\end{aligned} \tag{G.0.4}$$

and similarly for $\bar{\mathbb{X}}$

$$
\begin{aligned}
\delta\bar{X} &= \epsilon\bar{\chi} + \bar{\epsilon}\bar{\psi}\,, \\
\delta\bar{\psi}_+ &= \epsilon_- 2iD_+\bar{X} + \epsilon_-\bar{M}_{++} - \epsilon_+\hat{\sigma}\bar{X} - \epsilon_+\bar{M}_{-+} + \bar{\epsilon}_+\bar{F}\,, \\
\delta\bar{\psi}_- &= \epsilon_-\bar{\hat{\sigma}}\bar{X} + \epsilon_-\bar{M}_{+-} - \epsilon_+ 2iD_-\bar{X} - \epsilon_+\bar{M}_{--} + \bar{\epsilon}_-\bar{F}\,, \\
\delta\bar{F} &= \epsilon_+ 2iD_-\bar{\psi}_+ + \epsilon_- 2iD_+\bar{\psi}_- + \epsilon_+\bar{\eta}_- - \epsilon_-\bar{\eta}_+ - \epsilon_+\hat{\sigma}\bar{\psi}_- - \epsilon_-\bar{\hat{\sigma}}\bar{\psi}_+ - \epsilon_+ i\lambda_-\bar{X} + \epsilon_- i\lambda_+\bar{X}\,, \\
\delta\bar{\chi}_+ &= \bar{\epsilon}_-\bar{M}_{++} - \bar{\epsilon}_+\bar{M}_{+-}\,, \\
\delta\bar{\chi}_- &= \bar{\epsilon}_-\bar{M}_{-+} - \bar{\epsilon}_+\bar{M}_{--}\,, \\
\delta\bar{M}_{+-} &= \epsilon_-\bar{\hat{\sigma}}\bar{\chi}_+ - \epsilon_+ 2iD_-\bar{\chi}_+ + \bar{\epsilon}_-\bar{\eta}_+\,, \\
\delta\bar{M}_{-+} &= \epsilon_- 2iD_+\bar{\chi}_- - \epsilon_+\hat{\sigma}\bar{\chi}_- + \bar{\epsilon}_+\bar{\eta}_-\,, \\
\delta\bar{M}_{++} &= \epsilon_- 2iD_+\bar{\chi}_+ - \epsilon_+\hat{\sigma}\bar{\chi}_+ + \bar{\epsilon}_+\bar{\eta}_+\,, \\
\delta\bar{M}_{--} &= \epsilon_-\bar{\hat{\sigma}}\bar{\chi}_- - \epsilon_+ 2iD_-\bar{\chi}_- + \bar{\epsilon}_-\bar{\eta}_-\,, \\
\delta\bar{\eta}_+ &= \epsilon_- 2iD_+\bar{M}_{+-} - \epsilon_- i\bar{\lambda}_+\bar{\chi}_+ - \epsilon_-\bar{\hat{\sigma}}\bar{M}_{++} - \epsilon_+\hat{\sigma}\bar{M}_{+-} + \epsilon_+ i\bar{\lambda}_-\bar{\chi}_+ + \epsilon_+ 2iD_-\bar{M}_{++}\,, \\
\delta\bar{\eta}_- &= \epsilon_- 2iD_+\bar{M}_{--} - \epsilon_- i\bar{\lambda}_+\bar{\chi}_- - \epsilon_-\bar{\hat{\sigma}}\bar{M}_{-+} - \epsilon_+\hat{\sigma}\bar{M}_{--} + \epsilon_+ i\bar{\lambda}_-\bar{\chi}_- + \epsilon_+ 2iD_-\bar{M}_{-+}\,.
\end{aligned}
\tag{G.0.5}
$$

The transformation laws are written in the general form, and one should set some fields to be zero after imposing the constraints.

Varying the fields M^L_{--}, M^L_{-+}, M^R_{++}, M^R_{+-}, $\bar{M}^L_{--}$, $\bar{M}^L_{-+}$, $\bar{M}^R_{++}$ and $\bar{M}^R_{+-}$, we obtain

$$0 = \alpha\bar{M}^R_{++} + 2iD_+\bar{X}^L + \alpha 2iD_+\bar{X}^R\,, \tag{G.0.6}$$
$$0 = -\bar{M}^L_{-+} + \bar{X}^L\hat{\sigma} + \alpha\bar{X}^R\hat{\sigma}\,, \tag{G.0.7}$$
$$0 = \alpha\bar{M}^L_{--} + 2iD_-\bar{X}^R + \alpha 2iD_-\bar{X}^L\,, \tag{G.0.8}$$
$$0 = -\bar{M}^R_{+-} + \bar{X}^R\bar{\hat{\sigma}} + \alpha\bar{X}^L\bar{\hat{\sigma}}\,, \tag{G.0.9}$$
$$0 = \alpha M^R_{++} - 2iD_+X^L - \alpha 2iD_+X^R\,, \tag{G.0.10}$$
$$0 = -M^L_{-+} + \bar{\hat{\sigma}}X^L + \alpha\bar{\hat{\sigma}}X^R\,, \tag{G.0.11}$$
$$0 = \alpha M^L_{--} - 2iD_-X^R - \alpha 2iD_-X^L\,, \tag{G.0.12}$$
$$0 = -M^R_{+-} + \hat{\sigma}X^R + \alpha\hat{\sigma}X^L\,. \tag{G.0.13}$$

Similarly, varying the fields η^L_-, η^R_+, $\bar{\eta}^L_-$ and $\bar{\eta}^R_+$, we obtain

$$0 = -\bar{\psi}^L_+ - \alpha\bar{\psi}^R_+ \equiv -\sqrt{\alpha^2+1}\,\bar{\psi}^1_+\,, \tag{G.0.14}$$
$$0 = \bar{\psi}^R_- + \alpha\bar{\psi}^L_- \equiv \sqrt{\alpha^2+1}\,\bar{\psi}^1_-\,, \tag{G.0.15}$$
$$0 = -\psi^L_+ - \alpha\psi^R_+ \equiv -\sqrt{\alpha^2+1}\,\psi^1_+\,, \tag{G.0.16}$$
$$0 = \psi^R_- + \alpha\psi^L_- \equiv \sqrt{\alpha^2+1}\,\psi^1_-\,. \tag{G.0.17}$$

Orthogonal to these fields, we can define

$$\bar\psi^2_+ \equiv \frac{1}{\sqrt{\alpha^2+1}}(\alpha\bar\psi^L_+ - \bar\psi^R_+)\,, \tag{G.0.18}$$

$$\bar\psi^2_- \equiv \frac{1}{\sqrt{\alpha^2+1}}(\bar\psi^L_- - \alpha\bar\psi^R_-)\,, \tag{G.0.19}$$

$$\psi^2_+ \equiv \frac{1}{\sqrt{\alpha^2+1}}(\alpha\psi^L_+ - \psi^R_+)\,, \tag{G.0.20}$$

$$\psi^2_- \equiv \frac{1}{\sqrt{\alpha^2+1}}(\psi^L_- - \alpha\psi^R_-)\,. \tag{G.0.21}$$

We can regard them as the physical fermionic fields. Let us call them $\psi'_\pm$ and $\bar\psi'_\pm$.

Integrating out these auxiliary fields will give us the on-shell Lagrangian consisting of three parts, the kinetic terms for the bosons and fermions, and their interaction,

$$\begin{aligned}\mathcal{L}_{\text{bos}} = &\begin{pmatrix}\bar X^L & \bar X^R\end{pmatrix}\cdot\begin{pmatrix}\Box + D + \alpha^2|\hat\sigma|^2 & \frac{1}{\alpha}\Box + \alpha D + \alpha|\hat\sigma|^2 \\ \frac{1}{\alpha}\Box + \alpha D + \alpha|\hat\sigma|^2 & \Box + D + \alpha^2|\hat\sigma|^2\end{pmatrix}\cdot\begin{pmatrix}X^L \\ X^R\end{pmatrix} \\ &+ \bar F^L F^L + \bar F^R F^R + \alpha\bar F^L F^R + \alpha\bar F^R F^L\,,\end{aligned} \tag{G.0.22}$$

$$\mathcal{L}_{\text{ferm}} = -\frac{\alpha^2-1}{\alpha^2+1}\bar\psi'_- 2iD_+\psi'_- - \frac{\alpha^2-1}{\alpha^2+1}\bar\psi'_+ 2iD_-\psi'_+ - \chi^L_- 2iD_+\bar\chi^L_- - \chi^R_+ 2iD_-\bar\chi^R_+\,, \tag{G.0.23}$$

$$\begin{aligned}\mathcal{L}_{\text{int}} = &-\bar\psi^L_-\hat\sigma\psi^L_+ - \bar\psi^L_+\bar{\hat\sigma}\psi^L_- + \bar X^L i(\lambda\psi^L) - i(\bar\psi^L\bar\lambda)X^L + \bar X^L i\bar\lambda_+\chi^L_- - \bar\chi^L_- i\lambda_+ X^L \\ &- \bar\psi^R_-\hat\sigma\psi^R_+ - \bar\psi^R_+\bar{\hat\sigma}\psi^R_- + \bar X^R i(\lambda\psi^R) - i(\bar\psi^R\bar\lambda)X^R - \bar X^R i\bar\lambda_-\chi^R_+ + \bar\chi^R_+ i\lambda_- X^R \\ &- \alpha\bar\psi^L_-\hat\sigma\psi^R_+ - \alpha\bar\psi^L_+\bar{\hat\sigma}\psi^R_- + \alpha\bar X^L i(\lambda\psi^R) - \alpha i(\bar\psi^L\bar\lambda)X^R + \alpha\bar\chi^L_-\bar{\hat\sigma}\chi^R_+ - \alpha\bar X^L i\bar\lambda_-\chi^R_+ - \alpha\bar\chi^L_- i\lambda_+ X^R \\ &- \alpha\bar\psi^R_-\hat\sigma\psi^L_+ - \alpha\bar\psi^R_+\bar{\hat\sigma}\psi^L_- + \alpha\bar X^R i(\lambda\psi^L) - \alpha i(\bar\psi^R\bar\lambda)X^L + \alpha\bar\chi^R_+\hat\sigma\chi^L_- + \alpha\bar X^R i\bar\lambda_+\chi^L_- + \alpha\bar\chi^R_+ i\lambda_- X^L\,.\end{aligned} \tag{G.0.24}$$

Appendix H

Semichiral Stückelberg Field

Expanding the Lagrangian of the semichiral Stückelberg field (2.5.38) in components, we obtain

$$
\begin{aligned}
\mathcal{L}_{St} = &-4(D_+D_-X_L)(X_L+\bar{X}_L) - 4(D_-X_L)(D_+X_L) - \bar{M}^L_{-+}M^L_{-+} + F_L\bar{F}_L \\
&+ 2i(D_+M^L_{--})(X_L+\bar{X}_L) + M^L_{--}2i(D_+X_L) - \bar{M}^L_{--}2i(D_+X_L) \\
&+ \bar{\psi}^L_+ 2i(D_-\psi^L_+) - \bar{\psi}^L_- 2i(D_+\psi^L_-) + \chi^L_- 2i(D_+\bar{\chi}^L_-) - \bar{\eta}^L_-\bar{\psi}^L_+ - \eta^L_-\psi^L_+ \\
&+ 2iD^0 r^0_L + i\lambda^0_-\bar{\psi}^{L0}_+ - i\bar{\lambda}^0_-\psi^{L0}_+ \\
&- 4(D_-D_+X_R)(X_R+\bar{X}_R) - 4(D_-X_R)(D_+X_R) - \bar{M}^R_{+-}M^R_{+-} + F_R\bar{F}_R \\
&- 2i(D_-M^R_{++})(X_R+\bar{X}_R) - 2i(D_-X_R)M^R_{++} + 2i(D_-X_R)\bar{M}^R_{++} \\
&- \bar{\psi}^R_- 2i(D_+\psi^R_-) + \bar{\psi}^R_+ 2i(D_-\psi^R_+) - \chi^R_+ 2i(D_-\bar{\chi}^R_+) + \bar{\eta}^R_+\bar{\psi}^R_- + \eta^R_+\psi^R_- \\
&+ 2iD^0 r^0_R + i\lambda^0_-\bar{\psi}^{R0}_+ - i\bar{\lambda}^0_-\psi^{R0}_+ \\
&- 4\alpha(D_+D_-X_L)(X_L+\bar{X}_R) + \alpha(2iD_-X_L + M^L_{--})(2iD_+X_L + \bar{M}^R_{++}) \\
&+ 2i\alpha(D_+M^L_{--})(X_L+\bar{X}_R) + \alpha F_L F_R \\
&+ \alpha\bar{\psi}^R_+ 2i(D_-\psi^L_+) - \alpha\bar{\psi}^R_- 2i(D_+\psi^L_-) - \alpha\bar{\eta}^L_-\bar{\psi}^R_+ - \alpha\psi^L_-\eta^R_+ \\
&+ i\alpha D^0(X_L+\bar{X}_R)^0 + i\alpha\lambda^0_-\bar{\psi}^{R0}_+ \\
&- 4\alpha(D_-D_+X_R)(X_R+\bar{X}_L) + \alpha(2iD_-X_R - \bar{M}^L_{--})(2iD_+X_R - M^R_{++}) \\
&- 2i\alpha(D_-M^R_{++})(X_R+\bar{X}_L) + \alpha F_R\bar{F}_L \\
&- \alpha\bar{\psi}^L_- 2i(D_+\psi^R) + \alpha\bar{\psi}^L_+ 2i(D_-\psi^R_+) - \alpha\bar{\psi}^L_-\bar{\eta}^R_+ - \alpha\eta^L_-\psi^R_+ \\
&+ i\alpha D^0(X_R+\bar{X}_L)^0 + i\alpha\lambda^0_-\bar{\psi}^{L0}_+ \,.
\end{aligned} \tag{H.0.1}
$$

where $r_{L,R}$ stand for the real part of $\mathbb{X}_2^{L,R}$, and the upper index 0 denotes the zero mode. Varying the fields M^L_{--}, $\bar{M}^L_{--}$, M^R_{++} and $\bar{M}^R_{++}$, we obtain

$$\begin{aligned}
0 &= -2iD_+\bar{X}_L - 2i\alpha D_+\bar{X}_R + \alpha\bar{M}^R_{++}\,, \\
0 &= -2iD_+X_L - 2i\alpha D_+X_R + \alpha M^R_{++}\,, \\
0 &= 2iD_-\bar{X}_R + 2i\alpha D_-\bar{X}_L + \alpha\bar{M}^L_{--}\,, \\
0 &= 2iD_-X_R + 2i\alpha D_-X_L + \alpha M^L_{--}\,.
\end{aligned} \tag{H.0.2}$$

Similarly, varying the fields η^L_-, $\bar{\eta}^L_-$, η^R_+ and $\bar{\eta}^R_+$ will give us

$$\begin{aligned}
0 &= -\psi^L_+ - \alpha\psi^R_+ \equiv -\sqrt{1+\alpha^2}\psi^1_+\,, \\
0 &= -\bar{\psi}^L_+ - \alpha\bar{\psi}^R_+ \equiv -\sqrt{1+\alpha^2}\bar{\psi}^1_+\,, \\
0 &= -\psi^R_- - \alpha\psi^L_- \equiv -\sqrt{1+\alpha^2}\psi^1_-\,, \\
0 &= \bar{\psi}^R_- + \alpha\bar{\psi}^L_- \equiv -\sqrt{1+\alpha^2}\bar{\psi}^1_-\,.
\end{aligned} \tag{H.0.3}$$

We can define

$$\begin{aligned}
\psi^2_+ &\equiv \frac{1}{\sqrt{1+\alpha^2}}\psi^L_+ - \alpha\psi^R_+\,, \\
\bar{\psi}^2_+ &\equiv \frac{1}{\sqrt{1+\alpha^2}}\bar{\psi}^L_+ - \alpha\bar{\psi}^R_+\,, \\
\psi^2_- &\equiv \frac{1}{\sqrt{1+\alpha^2}}\psi^R_- - \alpha\psi^L_-\,, \\
\bar{\psi}^2_- &\equiv \frac{1}{\sqrt{1+\alpha^2}}\bar{\psi}^R_- - \alpha\bar{\psi}^L_-\,.
\end{aligned} \tag{H.0.4}$$

Integrating out the auxiliary fields, we obtain

$$\begin{aligned}
\mathcal{L}_{St} &= \begin{pmatrix} \bar{X}_L & \bar{X}_R \end{pmatrix} \begin{pmatrix} \Box & \frac{1}{\alpha}\Box \\ \frac{1}{\alpha}\Box & \Box \end{pmatrix} \begin{pmatrix} X_L \\ X_R \end{pmatrix} \\
&\quad + \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}^2_+ D_-\psi^2_+ - \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}^2_- D_+\psi^2_- + \bar{\chi}^L_- 2iD_+\chi^L_- - \bar{\chi}^R_+ 2iD_-\chi^R_+ \\
&= \frac{\alpha-1}{\alpha}\,\bar{X}_1\Box X_1 + \frac{\alpha+1}{\alpha}\,\bar{X}_2\Box X_2 \\
&\quad + \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}^2_+ D_-\psi^2_+ - \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}^2_- D_+\psi^2_- + \bar{\chi}^L_- 2iD_+\chi^L_- - \bar{\chi}^R_+ 2iD_-\chi^R_+ \\
&= \frac{\alpha-1}{\alpha}\,(\bar{r}_1\Box r_1 + \bar{\gamma}_1\Box\gamma_1) + \frac{\alpha+1}{\alpha}\,(\bar{r}_2\Box r_2 + \bar{\gamma}_2\Box\gamma_2) \\
&\quad + \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}^2_+ D_-\psi^2_+ - \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\bar{\psi}^2_- D_+\psi^2_- + \bar{\chi}^L_- 2iD_+\chi^L_- - \bar{\chi}^R_+ 2iD_-\chi^R_+\,,
\end{aligned} \tag{H.0.5}$$

where

$$X_1 \equiv \frac{-X_L + X_R}{\sqrt{2}}\,, \quad X_2 \equiv \frac{X_L + X_R}{\sqrt{2}}\,, \tag{H.0.6}$$

while $r_{1,2}$ and $\gamma_{1,2}$ denote the real parts and the imaginary parts of $X_{1,2}$ respectively. Among these real components only one of them, r_2, transforms under the gauge transformations

Appendix I

Jeffrey-Kirwan Residue

In the computation of section 2.5, we need the Jeffrey-Kirwan residue. Here we give a brief discussion following [69, 71] and the references therein.

Suppose n hyperplanes intersect at $u_* = 0 \in \mathbb{C}^r$, which are given by

$$H_i = \{u \in \mathbb{C}^r | Q_i(u) = 0\} \,, \tag{I.0.1}$$

where $i = 1, \cdots, n$ and $Q_i \in (\mathbb{R}^r)^*$. In the GLSM, Q_i correspond to the charges, and they define the hyperplanes as well as their orientations. Then for a vector $\eta \in (\mathbb{R}^r)^*$, the Jeffrey-Kirwan residue is defined as

$$\text{JK-Res}_{u=0}(Q_*, \eta)\, \frac{dQ_{j_1}(u)}{Q_{j_1}(u)} \wedge \cdots \wedge \frac{dQ_{j_r}(u)}{Q_{j_r}(u)} = \begin{cases} \text{sign}\det(Q_{j_1} \cdots Q_{j_r})\,, & \text{if } \eta \in \text{Cone}(Q_{j_1} \cdots Q_{j_r}) \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{I.0.2}$$

where $Q_* = Q(u_*)$, and $\text{Cone}(Q_{j_1} \cdots Q_{j_r})$ denotes the cone spanned by the vectors $Q_{j_1}, \cdots, Q_{j_r}$. For instance, for the case $r = 1$,

$$\text{JK-Res}_{u=0}(\{q\}, \eta)\, \frac{du}{u} = \begin{cases} \text{sign}(q)\,, & \text{if } \eta q > 0\,, \\ 0\,, & \text{if } \eta q < 0\,. \end{cases} \tag{I.0.3}$$

To obtain the elliptic genus, we still have to evaluate the contour integral over u. Since in the thesis we often encounter the function $\vartheta_1(\tau, u)$, its residue is very useful in practice:

$$\frac{1}{2\pi i} \oint_{u=a+b\tau} du\, \frac{1}{\vartheta_1(\tau, u)} = \frac{(-1)^{a+b}\, e^{i\pi b^2 \tau}}{2\pi\, \eta(q)^3}\,, \tag{I.0.4}$$

where $q = e^{2\pi i \tau}$. This relation can be derived by combining the properties

$$\vartheta_1'(\tau, 0) = 2\pi\, \eta(q)^3\,, \tag{I.0.5}$$

and

$$\vartheta_1(\tau, u + a + b\tau) = (-1)^{a+b}\, e^{-2\pi i b u - i\pi b^2 \tau} \vartheta_1(\tau, u) \tag{I.0.6}$$

for $a, b \in \mathbb{Z}$ and the fact that $\vartheta_1(\tau, u)$ has only simple zeros at $u = \mathbb{Z} + \tau\mathbb{Z}$ but no poles.

Appendix J

Two-Point Functions in $d = (4+1)$ Dimensions

In this appendix, we compute the two-point function for $d = (4+1)$ dimensions given by Eq. (3.2.36). In contrast to Eq. (3.2.30), Eq. (3.2.36) is a multi-variable contour integral and we need to do some changes of variables first. The procedure used here can be applied in higher dimensions and for any integer $n \geq 1$. In $d = 5$, we find

$$
\begin{aligned}
G^B_{(n,5)}(\theta) &= \frac{1}{(2\pi)^3}\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz\, \frac{(xyz)^{\frac{\theta}{2\pi}-1}(1-xyz)}{(1+x)(1+y)(1+z)\left(1-(xyz)^n\right)} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty dx \int_0^\infty dy \int_0^\infty \frac{1}{xy} dz'\, \frac{z'^{\frac{\theta}{2\pi}-1}(1-z')}{(1+x)(1+y)(1+\frac{z'}{xy})(1-z'^n)} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz'\, \frac{z'^{\frac{\theta}{2\pi}-1}(1-z')}{(1+x)(1+y)(xy+z')(1-z'^n)} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty dx \int_0^\infty \frac{1}{x} dy' \int_0^\infty dz'\, \frac{z'^{\frac{\theta}{2\pi}-1}(1-z')}{(1+x)(1+\frac{y'}{x})(y'+z')(1-z'^n)} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty dx \int_0^\infty dy' \int_0^\infty dz'\, \frac{z'^{\frac{\theta}{2\pi}-1}(1-z')}{(1+x)(x+y')(y'+z')(1-z'^n)} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz\, \frac{z^{\frac{\theta}{2\pi}-1}(1-z)}{(1+x)(x+y)(y+z)(1-z^n)}\,, \qquad \text{(J.0.1)}
\end{aligned}
$$

where

$$
z' \equiv xyz\,, \quad y' \equiv xy\,, \qquad \text{(J.0.2)}
$$

and we drop the $'$ in the last line. Performing the integration over x and y in Eq. (J.0.1), we obtain

$$
G^B_{(n,5)}(\theta) = \frac{1}{2(2\pi)^3}\int_0^\infty dz\, \frac{z\left(\pi^2+(\log z)^2\right)(z-1)}{(1+z)(z^n-1)}\,. \qquad \text{(J.0.3)}
$$

This integral can be done analytically by choosing the same branch cut and contour used in the $d = (2+1)$ dimensional case discussed in Section 3.2.2; the poles are exactly the same. The result for $n = 1$ is Eq. (3.2.33). For $n = 2$ the result is Eq. (3.2.34). To obtain this result, one needs the following intermediate results

$$\int_0^\infty dz\, \frac{z(z-1)}{(1+z)(z^2-1)} = \frac{\pi(\frac{\theta}{2\pi}-1)}{\sin\left[\pi\left(\frac{\theta}{2\pi}-1\right)\right]}\,, \tag{J.0.4}$$

$$\int_0^\infty dz\, \frac{z\log z\,(z-1)}{(1+z)(z^2-1)} = \frac{\pi\left[1-\pi(\frac{\theta}{2\pi}-1)\cot\left(\pi(\frac{\theta}{2\pi}-1)\right)\right]}{\sin\left(\pi(\frac{\theta}{2\pi}-1)\right)}\,. \tag{J.0.5}$$

Similarly, for $n = 3$ one can follow exactly the same procedure and find Eq. (3.2.35). Again, one needs some intermediate steps:

$$\int_0^\infty dz\, \frac{z(z-1)}{(1+z)(z^3-1)} = \frac{\pi\left[\sqrt{3}\cos\left(\frac{\pi}{6}\left(\frac{\theta}{2\pi}-1\right)\right)-3\sin\left(\frac{\pi}{6}\left(\frac{\theta}{2\pi}-1\right)\right)\right]}{3\left[\cos\left(\frac{\pi}{6}\left(\frac{\theta}{2\pi}-1\right)\right)+\cos\left(\frac{\pi}{2}\left(\frac{\theta}{2\pi}-1\right)\right)+\cos\left(\frac{5\pi}{6}\left(\frac{\theta}{2\pi}-1\right)\right)\right]}\,, \tag{J.0.6}$$

$$\begin{aligned}\int_0^\infty dz\, \frac{z\log z\,(z-1)}{(1+z)(z^3-1)} = {}& \frac{\pi^2}{18\left[\cos\left(\frac{1}{6}\pi(\frac{\theta}{2\pi}-1)\right)+\cos\left(\frac{1}{2}\pi(\frac{\theta}{2\pi}-1)\right)+\cos\left(\frac{5}{6}\pi(\frac{\theta}{2\pi}-1)\right)\right]^2} \\ & \cdot\Bigg[-6\cos\left(\frac{1}{3}\pi(\frac{\theta}{2\pi}-1)\right)-6\cos\left(\frac{2}{3}\pi(\frac{\theta}{2\pi}-1)\right)+6\cos\left(\pi(\frac{\theta}{2\pi}-1)\right) \\ & +2\sqrt{3}\sin\left(\frac{1}{3}\pi(\frac{\theta}{2\pi}-1)\right)+4\sqrt{3}\sin\left(\frac{2}{3}\pi(\frac{\theta}{2\pi}-1)\right)+2\sqrt{3}\sin\left(\pi(\frac{\theta}{2\pi}-1)\right) \\ & -3\Bigg]\,.\end{aligned} \tag{J.0.7}$$

Appendix K

Examples of Thermal Corrections to Rényi Entropies

In this appendix we summarize the thermal corrections to the nth Rényi entropy for the conformally coupled scalar. The Rényi entropy is calculated with respect to a cap of opening angle 2θ on S^{d-1} for small values of d and n. Define the coefficient $f(\theta)$ such that $\delta S_n = S_n(T) - S_n(0)$ has the form

$$\delta S_n = f(\theta)\, e^{-\beta\Delta/R} + o(e^{-\beta\Delta/R})\,, \tag{K.0.1}$$

where $\Delta = \frac{d-2}{2}$ is the scaling dimension of the free scalar and R is the radius of S^{d-1}. The following tables give the form of $f(\theta)$. (We also give results for the entanglement entropy, denoted EE.)

For $(2+1)$ dimensions:

EE	$\frac{\pi}{2}\tan\left(\frac{\theta}{2}\right)$
$n=2$	$\frac{2\theta}{\pi}$
$n=3$	$\sqrt{3}\sin\left(\frac{\theta}{3}\right)$

For $(3+1)$ dimensions:

EE	$2 - 2\,\theta\cot(\theta)$
$n=2$	$1-\cos(\theta)$
$n=3$	$\frac{4}{3}\left[2+\cos\left(\frac{2\theta}{3}\right)\right]\sin^2\left(\frac{\theta}{3}\right)$

For $(4+1)$ dimensions:

EE	$3\pi \csc(\theta)\sin^4\left(\frac{\theta}{2}\right)$
$n=2$	$\frac{1}{\pi}\left[2\theta - \sin(2\theta)\right]$
$n=3$	$\frac{1}{6\sqrt{3}}\left[51 + 44\cos\left(\frac{2\theta}{3}\right) + 10\cos\left(\frac{4\theta}{3}\right)\right]\ \sin^3\left(\frac{\theta}{3}\right)$

For $(5+1)$ dimensions:

EE	$\frac{2}{3}\left[5 + \cos(2\theta) - 6\,\theta\cot(\theta)\right]$
$n=2$	$2\left[2 + \cos(\theta)\right]\ \sin^4\left(\frac{\theta}{2}\right)$
$n=3$	$\frac{16}{81}\left[50 + 60\cos\left(\frac{2\theta}{3}\right) + 21\cos\left(\frac{4\theta}{3}\right) + 4\cos(2\theta)\right]\ \sin^4\left(\frac{\theta}{3}\right)$

Bibliography

[1] K. Huang, *Quantum field theory: From operators to path integrals*. 1998.

[2] L. Landau and E. Lifshitz, eds., *The Classical Theory of Fields (Fourth Edition)*, vol. 2 of *Course of Theoretical Physics*. Pergamon, Amsterdam, fourth edition ed., 1975.

[3] B. McCoy, *Advanced Statistical Mechanics*. 2009.

[4] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, *et al.*, "Solving the 3D Ising Model with the Conformal Bootstrap," *Phys.Rev.* **D86** (2012) 025022, `arXiv:1203.6064 [hep-th]`.

[5] E. Witten, "Two-dimensional gauge theories revisited," *J.Geom.Phys.* **9** (1992) 303–368, `arXiv:hep-th/9204083 [hep-th]`.

[6] C. Beasley and E. Witten, "Non-Abelian localization for Chern-Simons theory," *J.Diff.Geom.* **70** (2005) 183–323, `arXiv:hep-th/0503126 [hep-th]`.

[7] E. Witten, "Topological Quantum Field Theory," *Commun.Math.Phys.* **117** (1988) 353.

[8] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl.Phys.* **B426** (1994) 19–52, `arXiv:hep-th/9407087 [hep-th]`.

[9] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," *Nucl.Phys.* **B431** (1994) 484–550, `arXiv:hep-th/9408099 [hep-th]`.

[10] C. Vafa and E. Witten, "A Strong coupling test of S duality," *Nucl.Phys.* **B431** (1994) 3–77, `arXiv:hep-th/9408074 [hep-th]`.

[11] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Commun.Math.Phys.* **313** (2012) 71–129, `arXiv:0712.2824 [hep-th]`.

[12] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," *JHEP* **1106** (2011) 114, arXiv:1105.0689 [hep-th].

[13] S. M. Kuzenko, "Symmetries of curved superspace," *JHEP* **1303** (2013) 024, arXiv:1212.6179 [hep-th].

[14] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, "Exploring Curved Superspace," *JHEP* **1208** (2012) 141, arXiv:1205.1115 [hep-th].

[15] T. T. Dumitrescu and G. Festuccia, "Exploring Curved Superspace (II)," *JHEP* **1301** (2013) 072, arXiv:1209.5408 [hep-th].

[16] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, "Supersymmetric Field Theories on Three-Manifolds," *JHEP* **1305** (2013) 017, arXiv:1212.3388 [hep-th].

[17] A. Buchel, J. G. Russo, and K. Zarembo, "Rigorous Test of Non-conformal Holography: Wilson Loops in N=2* Theory," *JHEP* **1303** (2013) 062, arXiv:1301.1597 [hep-th].

[18] D. Z. Freedman and S. S. Pufu, "The holography of F-maximization," *JHEP* **1403** (2014) 135, arXiv:1302.7310 [hep-th].

[19] N. Bobev, H. Elvang, D. Z. Freedman, and S. S. Pufu, "Holography for $N = 2^*$ on S^4," *JHEP* **1407** (2014) 001, arXiv:1311.1508 [hep-th].

[20] D. Martelli and J. Sparks, "The gravity dual of supersymmetric gauge theories on a biaxially squashed three-sphere," *Nucl.Phys.* **B866** (2013) 72–85, arXiv:1111.6930 [hep-th].

[21] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int.J.Theor.Phys.* **38** (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th].

[22] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," *Nucl.Phys.* **B311** (1988) 46.

[23] E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th].

[24] A. Maloney and E. Witten, "Quantum Gravity Partition Functions in Three Dimensions," *JHEP* **1002** (2010) 029, arXiv:0712.0155 [hep-th].

[25] X. Yin, "Partition Functions of Three-Dimensional Pure Gravity," *Commun.Num.Theor.Phys.* **2** (2008) 285–324, arXiv:0710.2129 [hep-th].

[26] S. Giombi, A. Maloney, and X. Yin, "One-loop Partition Functions of 3D Gravity," *JHEP* **0808** (2008) 007, arXiv:0804.1773 [hep-th].

[27] A. Dabholkar, J. Gomes, and S. Murthy, "Quantum black holes, localization and the topological string," *JHEP* **1106** (2011) 019, arXiv:1012.0265 [hep-th].

[28] A. Dabholkar, N. Drukker, and J. Gomes, "Localization in supergravity and quantum AdS_4/CFT_3 holography," *JHEP* **1410** (2014) 90, arXiv:1406.0505 [hep-th].

[29] A. Belavin, A. M. Polyakov, and A. Zamolodchikov, "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory," *Nucl.Phys.* **B241** (1984) 333–380.

[30] A. Zamolodchikov, "Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory," *JETP Lett.* **43** (1986) 730–732.

[31] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons.* 1987.

[32] V. Korepin, N. Bogoliubov, and A. Izergin, *Quantum Inverse Scattering Method and Correlation Functions.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1997.

[33] J. L. Cardy, "Operator content and modular properties of higher dimensional conformal field theories," *Nucl.Phys.* **B366** (1991) 403–419.

[34] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov, and A. Morozov, "Integrability and Seiberg-Witten exact solution," *Phys.Lett.* **B355** (1995) 466–474, arXiv:hep-th/9505035 [hep-th].

[35] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et al.*, "Review of AdS/CFT Integrability: An Overview," *Lett.Math.Phys.* **99** (2012) 3–32, arXiv:1012.3982 [hep-th].

[36] J. Teschner, "Exact results on N=2 supersymmetric gauge theories," arXiv:1412.7145 [hep-th].

[37] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," *Lett.Math.Phys.* **91** (2010) 167–197, arXiv:0906.3219 [hep-th].

[38] N. A. Nekrasov and S. L. Shatashvili, "Supersymmetric vacua and Bethe ansatz," *Nucl.Phys.Proc.Suppl.* **192-193** (2009) 91–112, arXiv:0901.4744 [hep-th].

[39] N. A. Nekrasov and S. L. Shatashvili, "Quantum integrability and supersymmetric vacua," *Prog.Theor.Phys.Suppl.* **177** (2009) 105–119, arXiv:0901.4748 [hep-th].

[40] N. A. Nekrasov and S. L. Shatashvili, "Quantization of Integrable Systems and Four Dimensional Gauge Theories," arXiv:0908.4052 [hep-th].

[41] N. A. Nekrasov and S. L. Shatashvili, "Bethe/Gauge correspondence on curved spaces," *JHEP* **1501** (2015) 100, arXiv:1405.6046 [hep-th].

[42] J. Nian, "Localization of Supersymmetric Chern-Simons-Matter Theory on a Squashed S^3 with $SU(2) \times U(1)$ Isometry," *JHEP* **1407** (2014) 126, arXiv:1309.3266 [hep-th].

[43] J. Nian and X. Zhang, "Dynamics of Two-Dimensional $\mathcal{N} = (2,2)$ Supersymmetric Theories with Semichiral Superfields I," arXiv:1411.4694 [hep-th].

[44] A. Kapustin, B. Willett, and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," *JHEP* **1003** (2010) 089, arXiv:0909.4559 [hep-th].

[45] G. Semenoff and L. Vinet, *Particles and Fields*. CRM Series in Mathematical Physics. Springer New York, 1999.

[46] N. Hama, K. Hosomichi, and S. Lee, "SUSY Gauge Theories on Squashed Three-Spheres," *JHEP* **1105** (2011) 014, arXiv:1102.4716 [hep-th].

[47] Y. Imamura and D. Yokoyama, "N=2 supersymmetric theories on squashed three-sphere," *Phys.Rev.* **D85** (2012) 025015, arXiv:1109.4734 [hep-th].

[48] N. Hama, K. Hosomichi, and S. Lee, "Notes on SUSY Gauge Theories on Three-Sphere," *JHEP* **1103** (2011) 127, arXiv:1012.3512 [hep-th].

[49] D. L. Jafferis, "The Exact Superconformal R-Symmetry Extremizes Z," *JHEP* **1205** (2012) 159, arXiv:1012.3210 [hep-th].

[50] S.-y. Koyama and N. Kurokawa, "Values of the Double Sine Function," *Journal of Number Theory* **123** (2007) 204–223.

[51] S. Pasquetti, "Factorisation of N = 2 Theories on the Squashed 3-Sphere," *JHEP* **1204** (2012) 120, arXiv:1111.6905 [hep-th].

[52] Y. Imamura and D. Yokoyama, "$\mathcal{N} = 2$ supersymmetric theories on squashed three-sphere," *Int.J.Mod.Phys.Conf.Ser.* **21** (2013) 171–172.

[53] L. F. Alday, D. Martelli, P. Richmond, and J. Sparks, "Localization on Three-Manifolds," *JHEP* **1310** (2013) 095, arXiv:1307.6848 [hep-th].

[54] C. Hull, U. Lindstrom, L. Melo dos Santos, R. von Unge, and M. Zabzine, "Euclidean Supersymmetry, Twisting and Topological Sigma Models," *JHEP* **0806** (2008) 031, arXiv:0805.3321 [hep-th].

[55] T. Buscher, U. Lindstrom, and M. Rocek, "New Supersymmetric σ Models With Wess-Zumino Terms," *Phys.Lett.* **B202** (1988) 94.

[56] P. M. Crichigno and M. Roček, "On gauged linear sigma models with torsion," arXiv:1506.00335 [hep-th].

[57] E. Witten, "Phases of N=2 theories in two-dimensions," *Nucl.Phys.* **B403** (1993) 159–222, arXiv:hep-th/9301042 [hep-th].

[58] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge, and M. Zabzine, "New N = (2,2) vector multiplets," *JHEP* **0708** (2007) 008, arXiv:0705.3201 [hep-th].

[59] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge, and M. Zabzine, "Nonabelian Generalized Gauge Multiplets," *JHEP* **0902** (2009) 020, arXiv:0808.1535 [hep-th].

[60] P. M. Crichigno, "The Semi-Chiral Quotient, Hyperkahler Manifolds and T-Duality," *JHEP* **1210** (2012) 046, arXiv:1112.1952 [hep-th].

[61] C. Closset and S. Cremonesi, "Comments on $\mathcal{N}$ = (2, 2) supersymmetry on two-manifolds," *JHEP* **1407** (2014) 075, arXiv:1404.2636 [hep-th].

[62] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, "Exact Results in D=2 Supersymmetric Gauge Theories," *JHEP* **1305** (2013) 093, arXiv:1206.2606 [hep-th].

[63] E. Witten, "Topological Sigma Models," *Commun.Math.Phys.* **118** (1988) 411.

[64] E. Witten, "Mirror manifolds and topological field theory," arXiv:hep-th/9112056 [hep-th].

[65] F. Benini and S. Cremonesi, "Partition Functions of $\mathcal{N} = (2, 2)$ Gauge Theories on S^2 and Vortices," *Commun.Math.Phys.* **334** no. 3, (2015) 1483–1527, arXiv:1206.2356 [hep-th].

[66] E. Witten, "On the Landau-Ginzburg description of N=2 minimal models," *Int.J.Mod.Phys.* **A9** (1994) 4783–4800, arXiv:hep-th/9304026 [hep-th].

[67] A. Gadde and S. Gukov, "2d Index and Surface operators," *JHEP* **1403** (2014) 080, arXiv:1305.0266 [hep-th].

[68] F. Benini, R. Eager, K. Hori, and Y. Tachikawa, "Elliptic genera of two-dimensional N=2 gauge theories with rank-one gauge groups," *Lett.Math.Phys.* **104** (2014) 465–493, arXiv:1305.0533 [hep-th].

[69] F. Benini, R. Eager, K. Hori, and Y. Tachikawa, "Elliptic Genera of 2d $\mathcal{N}$ = 2 Gauge Theories," *Commun.Math.Phys.* **333** no. 3, (2015) 1241–1286, arXiv:1308.4896 [hep-th].

[70] F. Benini, P. M. Crichigno, D. Jain, and J. Nian, "Semichiral Fields on S^2 and Generalized Kähler Geometry," arXiv:1505.06207 [hep-th].

[71] J. A. Harvey, S. Lee, and S. Murthy, "Elliptic genera of ALE and ALF manifolds from gauged linear sigma models," *JHEP* **1502** (2015) 110, arXiv:1406.6342 [hep-th].

[72] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, "A Quantum Source of Entropy for Black Holes," *Phys.Rev.* **D34** (1986) 373–383.

[73] M. Srednicki, "Entropy and area," *Phys.Rev.Lett.* **71** (1993) 666–669, `arXiv:hep-th/9303048 [hep-th]`.

[74] H. Casini and M. Huerta, "A c-theorem for the entanglement entropy," *J.Phys.* **A40** (2007) 7031–7036, `arXiv:cond-mat/0610375 [cond-mat]`.

[75] H. Casini and M. Huerta, "On the RG running of the entanglement entropy of a circle," *Phys.Rev.* **D85** (2012) 125016, `arXiv:1202.5650 [hep-th]`.

[76] T. J. Osborne and M. A. Nielsen, "Entanglement in a simple quantum phase transition," *Phys.Rev.A* **66** (2002) 032110, `arXiv:quant-ph/0202162 [quant-ph]`.

[77] G. Vidal, J. Latorre, E. Rico, and A. Kitaev, "Entanglement in quantum critical phenomena," *Phys.Rev.Lett.* **90** (2003) 227902, `arXiv:quant-ph/0211074 [quant-ph]`.

[78] C. P. Herzog and J. Nian, "Thermal corrections to Rényi entropies for conformal field theories," *JHEP* **1506** (2015) 009, `arXiv:1411.6505 [hep-th]`.

[79] T. Nishioka and I. Yaakov, "Supersymmetric Rényi Entropy," *JHEP* **1310** (2013) 155, `arXiv:1306.2958 [hep-th]`.

[80] P. Calabrese and J. L. Cardy, "Entanglement entropy and quantum field theory," *J.Stat.Mech.* **0406** (2004) P06002, `arXiv:hep-th/0405152 [hep-th]`.

[81] H. Casini and M. Huerta, "Entanglement entropy for the n-sphere," *Phys.Lett.* **B694** (2010) 167–171, `arXiv:1007.1813 [hep-th]`.

[82] C. P. Herzog, "Universal Thermal Corrections to Entanglement Entropy for Conformal Field Theories on Spheres," *JHEP* **1410** (2014) 28, `arXiv:1407.1358 [hep-th]`.

[83] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," *Phys.Rev.Lett.* **96** (2006) 181602, `arXiv:hep-th/0603001 [hep-th]`.

[84] A. Kitaev and J. Preskill, "Topological entanglement entropy," *Phys.Rev.Lett.* **96** (2006) 110404, `arXiv:hep-th/0510092 [hep-th]`.

[85] M. Levin and X.-G. Wen, "Detecting Topological Order in a Ground State Wave Function," *Phys.Rev.Lett.* **96** (2006) 110405.

[86] S. N. Solodukhin, "Entanglement entropy, conformal invariance and extrinsic geometry," *Phys.Lett.* **B665** (2008) 305–309, `arXiv:0802.3117 [hep-th]`.

[87] H. Casini, M. Huerta, and R. C. Myers, "Towards a derivation of holographic entanglement entropy," *JHEP* **1105** (2011) 036, arXiv:1102.0440 [hep-th].

[88] J. Cardy and C. P. Herzog, "Universal Thermal Corrections to Single Interval Entanglement Entropy for Conformal Field Theories," *Phys.Rev.Lett.* **112** (2014) 171603, arXiv:1403.0578 [hep-th].

[89] C. P. Herzog and M. Spillane, "Tracing Through Scalar Entanglement," *Phys.Rev.* **D87** (2013) 025012, arXiv:1209.6368 [hep-th].

[90] T. Azeyanagi, T. Nishioka, and T. Takayanagi, "Near Extremal Black Hole Entropy as Entanglement Entropy via AdS(2)/CFT(1)," *Phys.Rev.* **D77** (2008) 064005, arXiv:0710.2956 [hep-th].

[91] C. P. Herzog and T. Nishioka, "Entanglement Entropy of a Massive Fermion on a Torus," *JHEP* **1303** (2013) 077, arXiv:1301.0336 [hep-th].

[92] T. Barrella, X. Dong, S. A. Hartnoll, and V. L. Martin, "Holographic entanglement beyond classical gravity," *JHEP* **1309** (2013) 109, arXiv:1306.4682 [hep-th].

[93] S. Datta and J. R. David, "Rényi entropies of free bosons on the torus and holography," *JHEP* **1404** (2014) 081, arXiv:1311.1218 [hep-th].

[94] B. Chen and J.-q. Wu, "Single interval Renyi entropy at low temperature," *JHEP* **1408** (2014) 032, arXiv:1405.6254 [hep-th].

[95] J. Cardy, "Some results on the mutual information of disjoint regions in higher dimensions," *J.Phys.* **A46** (2013) 285402, arXiv:1304.7985 [hep-th].

[96] P. Candelas and J. Dowker, "Field Theories on Conformally Related Space-Times: Some Global Considerations," *Phys.Rev.* **D19** (1979) 2902.

[97] N. Shiba, "Entanglement Entropy of Two Spheres," *JHEP* **1207** (2012) 100, arXiv:1201.4865 [hep-th].

[98] L.-Y. Hung, R. C. Myers, and M. Smolkin, "Twist operators in higher dimensions," *JHEP* **1410** (2014) 178, arXiv:1407.6429 [hep-th].

[99] H. Casini and M. Huerta, "Remarks on the entanglement entropy for disconnected regions," *JHEP* **0903** (2009) 048, arXiv:0812.1773 [hep-th].

[100] V. Eisler and Z. Zimborás, "Entanglement negativity in the harmonic chain out of equilibrium," *New Journal of Physics* **16** no. 12, (2014) 123020.

[101] P. Calabrese, J. Cardy, and E. Tonni, "Finite temperature entanglement negativity in conformal field theory," *J.Phys.* **A48** no. 1, (2015) 015006, arXiv:1408.3043 [cond-mat.stat-mech].

[102] A. Zee, "Vortex strings and the antisymmetric gauge potential," *Nucl.Phys.* **B421** (1994) 111–124.

[103] S. S. Gubser, R. Nayar, and S. Parikh, "Strings, vortex rings, and modes of instability," *Nucl.Phys.* **B892** (2015) 156–180, `arXiv:1408.2246 [hep-th]`.

[104] M. Kulkarni, D. A. Huse, and H. Spohn, "Fluctuating hydrodynamics for a discrete Gross-Pitaevskii equation: mapping to Kardar-Parisi-Zhang universality class," *ArXiv e-prints* (Feb., 2015) , `arXiv:1502.06661 [cond-mat.quant-gas]`.

[105] A. Zee, *Quantum field theory in a nutshell.* 2003.

[106] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation.* International Series of Monographs on Physics. Clarendon Press, 2003.

[107] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, *et al.*, "Mirror symmetry,".

[108] H. Padmanabhan and T. Padmanabhan, "Non-relativistic limit of quantum field theory in inertial and non-inertial frames and the Principle of Equivalence," *Phys.Rev.* **D84** no. 8, (2011) 085018, `arXiv:1110.1314 [gr-qc]`.

[109] M. Kardar, G. Parisi, and Y.-C. Zhang, "Dynamic Scaling of Growing Interfaces," *Phys.Rev.Lett.* **56** (1986) 889.

[110] M. T. Reeves, T. P. Billam, B. P. Anderson, and A. S. Bradley, "Inverse energy cascade in forced two-dimensional quantum turbulence," *Phys. Rev. Lett.* **110** (Mar, 2013) 104501, `1209.5824`.

[111] T. P. Billam, M. T. Reeves, B. P. Anderson, and A. S. Bradley, "Onsager-kraichnan condensation in decaying two-dimensional quantum turbulence," *Phys. Rev. Lett.* **112** (Apr, 2014) 145301, `1307.6374`.

[112] G. Joyce and D. Montgomery, "Negative temperature states for the two-dimensional guiding-centre plasma," *Journal of Plasma Physics* **10** (8, 1973) 107–121.

[113] S. F. Edwards and J. B. Taylor, "Negative temperature states of two-dimensional plasmas and vortex fluids," *Proc. R. Soc. Lond. A* **336** (2, 1974) 257–271.

[114] R. A. Smith, "Phase-transition behavior in a negative-temperature guiding-center plasma," *Phys. Rev. Lett.* **63** (Oct, 1989) 1479–1482.

[115] R. A. Smith and T. M. O'Neil, "Nonaxisymmetric thermal equilibria of a cylindrically bounded guiding-center plasma or discrete vortex system," *Physics of Fluids B: Plasma Physics* **2** (7, 1990) 2961–2975.

[116] T. Simula, M. J. Davis, and K. Helmerson, "Emergence of order from turbulence in an isolated planar superfluid," *Phys. Rev. Lett.* **113** (Oct, 2014) 165302, `1405.3399`.

[117] "Exact solutions of a nonlinear boundary value problem: The vortices of the two-dimensional sinh-poisson equation," *Physica D: Nonlinear Phenomena* **26** no. 1–3, (1987) 37 – 66.

[118] E. S. Gutshabash, V. Lipovskii, and S. Nikulichev, "(2+ 0)-dimensional integrable equations and exact solutions," *arXiv preprint nlin/0001012* (2000) .

[119] D. Gurarie and K. W. Chow, "Vortex arrays for sinh-poisson equation of two-dimensional fluids: Equilibria and stability," *Physics of Fluids (1994-present)* **16** no. 9, (2004) .

[120] X. Yu, J. Nian, T. P. Billam, M. T. Reeves, and A. S. Bradley, "Clustering transitions of two-dimensional vortices at negative temperature," *to appear* .

[121] J. Wess and B. Zumino, "Consequences of anomalous Ward identities," *Phys.Lett.* **B37** (1971) 95.

[122] E. Witten, "Global Aspects of Current Algebra," *Nucl.Phys.* **B223** (1983) 422–432.

[123] E. Witten, "Nonabelian Bosonization in Two-Dimensions," *Commun.Math.Phys.* **92** (1984) 455–472.

[124] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, "Non-Abelian anyons and topological quantum computation," *Rev.Mod.Phys.* **80** (2008) 1083–1159.

[125] B. Bakalov and A. Kirillov, *Lectures on Tensor Categories and Modular Functors*. Translations of Mathematical Monographs. American Mathematical Soc., 2001.

[126] S. Zhang, T. Hansson, and S. Kivelson, "An effective field theory model for the fractional quantum hall effect," *Phys.Rev.Lett.* **62** (1988) 82–85.

[127] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," *Fortsch.Phys.* **61** (2013) 781–811, `arXiv:1306.0533 [hep-th]`.

[128] A. M. Jaffe and E. Witten, "Quantum Yang-Mills theory,".

[129] N. Drukker, T. Okuda, and F. Passerini, "Exact results for vortex loop operators in 3d supersymmetric theories," *JHEP* **1407** (2014) 137, `arXiv:1211.3409 [hep-th]`.