Aspects of Supersymmetric Field Theories and Complex Geometry

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Abstract of the Dissertation

Aspects of Supersymmetric Field Theories and Complex Geometry

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In this dissertation we study various aspects of Supersymmetric Quantum Field Theory and Complex Geometry. We focus on three main aspects. The first is general $\mathcal{N} = (2, 2)$ gauged linear sigma models involving semichiral fields. We show that integrating out the semichiral vector multiplet leads to the generalized potential for a hyperkähler manifold, providing a formulation of the hyperkähler quotient in a generalized setting. We then discuss a new quotient construction which leads to non-Kähler manifolds. The second problem we study is motivated by recent developments in the study of the Coulomb branch of supersymmetric theories with a hyperkähler moduli space. A crucial element in these developments is the expression for Darboux coordinates in the hyperkähler manifold. We give a simple derivation of this expression by using projective superspace techniques and we apply this to the study of the moduli space of theories with eight supercharges on $\mathbb{R}^3 \times S^1$ and $\mathbb{R}^3 \times T^2$. Finally, we study the partition function of three-dimensional Chern-Simons theories on S^3 with affine ADE quivers. We give a general formula for the partition function of affine D-type quivers in terms of the Chern-Simons levels, providing a prediction for the volume of an infinite family of tri-Sasaki Einstein manifolds corresponding to the gravitational duals of such field theories.

To my Parents, Bianca, and Irina.

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Chapter 1

Introduction

Quantum field theory (QFT) is a mathematical framework describing an enormous variety of physical phenomena. It was originally conceived as a marriage of special relativity and quantum mechanics to describe fundamental particles and their interactions. It has done so with an outstanding success, but in addition it has shed light on phenomena ranging from phase transitions, superconductivity, and superfluidity to the interior of stars, aspects of black holes, and cosmology.

Advances in physics and mathematics have often come together and QFT has been at the forefront of some modern developments in both fields. Being such an overarching framework, QFT has become a vast subject of study containing many sub-disciplines, each with interests and methods of its own.

One of the most important concepts in QFT (if not the most important) is the Renormalization Group and the idea of an effective action. A QFT is defined by its short-distance behavior. However, we are typically interested in the long-distance behavior, at scales at which we perform experiments. The effective action provides the relevant information for experiments performed at a certain scale, and it can be derived (in principle) from the microscopic theory by following the renormalization flow.

As mentioned above, each sub-discipline of QFT has developed methods of its own, depending on the problem of interest. Our main interest is the low energy behavior of certain gauge theories and our method will be supersymmetry. As we shall see, the concept of moduli space (the space of physical vacua of the theory) will play a fundamental role and we will encounter it in every chapter.

A technical advantage of supersymmetry is that, under certain circumstances, it allows for the computation of exact quantities that would not be otherwise possible, at least with currently known methods. In addition, it leads to deep connections with areas of pure mathematics, in particular Kähler and

Generalized Kähler Geometry which are interesting subjects in their own right. We clearly cannot give a full introduction to these subjects here, but let us give a brief review of some basic elements. A Kähler manifold is a complex manifold (*i.e.*, a manifold that admits an integrable complex structure J that squares to minus one) with a fundamental form ω defined by $\omega_{\mu\nu} = g_{\mu\rho}\, J^{\rho}_{\ \mu}$ that is closed: $d\omega = 0$. Thus, locally $\omega = i\partial\bar{\partial}K$, where K is called the Kähler potential. A hyperkähler manifold admits three complex structures I, J, K satisfying the quaternionic algebra and is Kähler with respect to each one of them. Other concepts, such as Generalized Kähler geometry and Sasaki-Einstein manifolds will be explained as needed in the main text and each chapter begins with an introduction to the main physical ideas discussed in that chapter.

This dissertation can be divided into two parts: Classical and Quantum. In Chapters 2 and 3 we study the classical moduli spaces of certain two-dimensional gauge theories and in Chapters 4 and 5 we venture into quantum aspects using non-perturbative techniques. Each chapter introduces new methods and we have tried to make each chapter as self-contained as possible. Where we have not succeeded, we encourage the reader to consult the references provided in each case.

A brief summary of the subject and the main results of each chapter is given below.

- In Chapter 2 we study the classical moduli space of certain two-dimensional gauged linear sigma models. We will prove that these spaces are hyperkähler. This leads to the generalized description of certain gravitational instantons. This chapter is based on [1].
- In Chapter 3 we consider gauged linear sigma models whose classical moduli space is non-Kähler, study their geometry, and make some comments about quantum corrections. This chapter is based on unpublished joint work with Martin Roček.
- In Chapter 4 we will use techniques of Projective Superspace to study the quantum moduli space of three-dimensional gauge theories that arise from compactifications of higher-dimensional theories with eight supercharges. This chapter is based on [2].
- In Chapter 5 we use non-perturbative techniques of localization in three dimensions to evaluate the partition function of certain superconformal Chern-Simons theories with AdS_4 duals. By the AdS/CFT correspon-

dence this leads to the prediction of the volume of certain tri-Sasaki-Einstein manifolds. This chapter is based on [3].

Chapter 2

Generalized $\mathcal{N} = (2,2)$ gauged linear sigma models[1]

Two-dimensional non-linear sigma models (NLSMs) are simple, yet extremely rich quantum field theories. As first pointed out by Polyakov [4], these simplified models share many features in common with four-dimensional non-abelian gauge theories describing the real world, which are much more resistant to analytical control than two-dimensional models. In fact, NLSMs exhibit asymptotic freedom, generation of a mass scale from strong coupling, solitons, confinement, a large-N expansion, all crucial features of four-dimensional Yang-Mills theories including QCD (see [5] for a classic review of these parallels). NLSMs were first proposed as an alternative description of spontaneous symmetry breaking and their name originates from one of the fields involved being the sigma meson [6]. Nowadays we study generalizations of this model.

There is yet another reason that makes these models so interesting. It is perhaps more abstract, but equally fascinating; supersymmetric NLSMs have surprising connections with complex geometry. This will be the main focus of this chapter.

2.1 Introduction

We begin by giving a review of basic elements of two-dimensional bosonic NLSMs and their supersymmetric versions. Following this, we will give an introduction to the main subject of the present chapter and Chapter 3: Generalized gauged linear sigma models.

In two dimensions, and in light-cone coordinates $x^{\pm}$, the general action for

[1]This chapter is based on the work [1].

the NLSM is:

$$S = -\frac{1}{2}\int d^2x(g_{\mu\nu}(\phi) + b_{\mu\nu}(\phi))\partial_{++}\phi^\mu\partial_{--}\phi^\nu\ , \tag{2.1.1}$$

where $g_{\mu\nu} = g_{\nu\mu}$, $b_{\mu\nu} = -b_{\nu\mu}$ and ϕ^μ are scalar fields and $\partial_{\pm\pm} = \frac{\partial}{\partial x^\pm}$ are spacetime derivatives. The richness of these models resides in the fact that g and b are general nonlinear functions of the fields ϕ. Since all the couplings are dimensionless, these models are renormalizable for any such functions. The term proportional to $b_{\mu\nu}$ in (2.1.1) is called a Wess-Zumino-Witten (WZW) term [7, 8]. We will have much more to say about this term shortly.

With the advent of string theory, two-dimensional NLSMs gained interest in their own right, beyond being toy models for four-dimensional gauge theories. They describe strings propagating in certain spacetime backgrounds, and are thus important tools in the development of string theory. In this context, the scalar fields ϕ^μ are interpreted as coordinates in spacetime, $g_{\mu\nu}(\phi)$ as the spacetime metric, and the WZW term corresponds to a background with the NS-NS two-form turned on. As a string theory, however, the bosonic model (2.1.1) is unsatisfactory. In part because upon quantization the spectrum contains a tachyonic particle (signaling an instability), but more importantly because the spectrum contains no fermionic particles, which are of course of fundamental importance in Nature. This naturally leads to the study of supersymmetric NLSMs, which solves both these problems in an elegant way. In addition, supersymmetric theories typically have a "milder" ultraviolet behavior than their non-supersymmetric counterparts and, under certain circumstances, allow for the computation of exact quantities. Thus, supersymmetric models are extremely attractive theoretical laboratories. (We will encounter some of these methods in Chapters 4 and 5.)

Two-dimensional models are special in that in 1+1 dimensions there is a Lorentz-invariant notion of left and right-moving massless modes. Thus, one can consider an independent number of left and right-moving supercharges. Let us consider $(1,1)$ SUSY, generated by one left-moving supercharge $\mathcal{Q}_+$ and one right-moving supercharge $\mathcal{Q}_-$[2]. The $(1,1)$ supersymmetric extension of the bosonic model (2.1.1) is straightforward. The action is most easily written in $(1,1)$ superspace as

$$\mathcal{L} = -\frac{1}{4}\int d\theta_+ d\theta_-\,(D_+\Phi^\mu)(D_-\Phi^\nu)\,(g_{\mu\nu}(\Phi) + b_{\mu\nu}(\Phi))\ , \tag{2.1.2}$$

[2] In general it is possible to construct SUSY algebras of type (p,q), generated by p right-handed Majorana-Weyl fermions and q left-handed ones [9]. For example $(0,2)$ or $(1,2)$ models are possible, but we will not consider these cases here.

where $\theta_\pm$ are real Grassmann coordinates, Φ^μ are $(1,1)$ superfields and $D_\pm$ are $\mathcal{N}=(1,1)$ supercovariant derivatives satisfying the algebra

$$\{D_\pm, D_\pm\} = 2\,i\,\partial_{\pm\pm}\,, \qquad \{D_+, D_-\} = 0\,. \tag{2.1.3}$$

It is easy to see that by performing the integral over the Grassmann variables in (2.1.2) (*i.e.*, reducing the action to components), leads to a bosonic sector as in (2.1.1), in addition of course to fermionic terms[3].

Geometry of $(2,2)$ sigma models

As we have just seen, it is possible to write a $(1,1)$ SUSY extension of the model (2.1.1) for any functions g and b. In other words, there is no obstruction to $(1,1)$ SUSY. However, as we shall review now, this is not the case for models with *extended* SUSY. In fact, what is required by the background geometry is that it describes a certain complex manifold.

The relation between SUSY sigma models and complex geometry was first noticed by the pioneering work of Zumino [10]. We will review shortly how this connection precisely arises. For pedagogical reasons, however, let us first give a specific example where $(2,2)$ SUSY imposes a restriction on the background; the more general case will be analyzed below. If we assume that the model has off-shell SUSY, we can write the action in $(2,2)$ superspace, which is parameterized by a total of four Grassmann coordinates: $\theta_\pm$ and their conjugates $\bar\theta_\pm$. Superfields in $(2,2)$ superspace have more degrees of freedom than $(1,1)$ fields. Thus, one imposes constraints. Chiral superfields Φ are constrained by imposing $\bar{\mathbb{D}}_\pm\Phi = 0$, where $\mathbb{D}_\pm$ are the $(2,2)$ supercovariant derivatives satisfying the algebra

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = 2\,i\,\partial_{\pm\pm}\,. \tag{2.1.4}$$

The general action is given by

$$\mathcal{L} = \int d^2\theta d^2\bar\theta K(\bar\Phi, \Phi)\,. \tag{2.1.5}$$

Note that no derivatives are needed in this expression since they are all hidden in the fermionic measure. Integrating over the full superspace measure (this time pushing in four D's) leads to a model of the form (2.1.2) with $g_{\bar\phi\phi} = \partial_{\bar\phi}\partial_\phi K$ and $b = 0$ (up to total derivatives). As mentioned in Chapter 1, this implies

[3]The easiest way to perform the Grassmann integration is by using $\int d^2\theta(...) = D_+D_-(...)|$, where $|$ means setting all the θ's to zero and pushing the $D_\pm$'s into the integrand. From here it is clear that the action contains terms with at most two spacetime derivatives.

that g is a metric on a Kähler manifold with Kähler potential K. Thus, we learn that $(2,2)$ models involving only chiral fields require the background geometry to be Kähler and that the Kähler potential is precisely the action in $(2,2)$ superspace. Imposing $(4,4)$ SUSY requires the background geometry to be hyperkähler [11].

Clearly, a natural question is how to extend this to the case $b \neq 0$. Namely, the question is whether it is possible for the model (2.1.2) to support $(2,2)$ SUSY with a non-trivial b-field. This was studied in [12] (see also [13–15]) and led to an extension of Kähler geometry whose mathematical structure has been elucidated in the last decade [16, 17]. We will now give an overview of these results, mostly following [18].

On dimensional grounds, the transformations for extended SUSY must be of the form

$$\delta\Phi^\mu = \varepsilon^+ D_+\Phi^\nu J^\mu_{+\nu}(\Phi) + \varepsilon^- D_-\Phi^\nu J^\mu_{-\nu}(\Phi)\,. \tag{2.1.6}$$

Acting with these transformations on (2.1.2), one finds that the action is invariant provided that

$$J^\mu_{\pm\rho} g_{\mu\nu} = -g_{\mu\rho} J^\mu_{\pm\nu} \tag{2.1.7}$$

and

$$\nabla^{(\pm)}_\rho J^\mu_{\pm\nu} = J^\mu_{\pm\nu,\rho} + \Gamma^{\pm\mu}_{\rho\sigma} J^\sigma_{\pm\nu} - \Gamma^{\pm\sigma}_{\rho\nu} J^\mu_{\pm\sigma} = 0\,, \tag{2.1.8}$$

where $\Gamma^{\pm\mu}_{\rho\nu}$ is a connection with torsion given by

$$\Gamma^{\pm\mu}_{\rho\nu} = \Gamma^\mu{}_{\rho\nu} \pm g^{\mu\sigma} H_{\sigma\rho\nu}\,, \quad H_{\mu\rho\sigma} = \frac{1}{2}(B_{\mu\rho,\sigma} + B_{\rho\sigma,\mu} + B_{\sigma\mu,\rho})\,, \tag{2.1.9}$$

where $\Gamma^\mu{}_{\rho\nu}$ is the Levi-Civita connection. Closure of the transformations (2.1.6) to the usual SUSY algebra requires that $J_\pm$ are integrable complex structures *i.e.*,

$$J^2_\pm = -1\,, \quad \mathcal{N}(J_\pm) = 0\,, \tag{2.1.10}$$

where $\mathcal{N}$ is the Nijenhuis tensor

$$\mathcal{N}^\rho_{\mu\nu}(J) \equiv J^\rho_\lambda \partial_{[\nu} J^\lambda_{\mu]} + \partial_\lambda J^\rho_{[\nu} J^\lambda_{\mu]}\,. \tag{2.1.11}$$

To summarize, associated to the $\mathcal{N} = (2,2)$ supersymmetry, there are two complex structures, $J_\pm$ (each defining a two-form $\omega_\pm$), and the metric is hermitean with respect to both. Furthermore, the presence of the b-field induces a connection with torsion (proportional to $H = db$) and the complex structures are covariantly constant with respect to this connection. Finally, one can see that

$$(d\omega_\pm)_{\mu\nu\rho} = \mp H_{\lambda\sigma\tau} J^\lambda_\mu J^\sigma_\nu J^\tau_\rho\,. \tag{2.1.12}$$

In the case $H = 0$ (and $J_+ = J_-$), this is precisely the definition of a Kähler manifold, but for $H \neq 0$ it is an extension of it.

Thus, imposing additional supersymmetries leads to the discovery [12] of a rich geometrical structure: bihermitean geometry, defined by the data $(M, g, J_\pm, H)$. The framework of Generalized Complex Geometry developed by Hitchin [16] and Gualtieri [17], describes this geometry as a generalized Kähler geometry and we will use these terms interchangeably. A summary of the relation between worldhseet SUSY and target space geometry is given in Table 2.1.

$\mathcal{N}$	g	g, b
$(0,0)$ or $(1,1)$	Riemannian	Riemannian
$(2,2)$	Kähler	Bihermitean
$(4,4)$	Hyperkähler	Bihypercomplex

Table 2.1: Summary of the requirement on the geometry depending on the amount of worldsheet SUSY and background fields g and b turned on.

Just as in the case of Kähler geometry, generalized Kähler geometry is locally completely determined by a single scalar function K called the generalized potential. As we shall see next, generalized models can be formulated off-shell and K is the action of the NLSM in $(2,2)$ superspace. See [18] for a comprehensive review.

Off-shell description

As disscused above, the class of models studied by Zumino admit a formulation in $(2,2)$ superspace. Thus, it is natural to wonder if models with a non-trivial b-field admit such a description as well. Clearly, this is not possible if only chiral fields are involved, as we have just seen that this leads to $b = 0$. In 1+1 dimensions one can impose chirality constraints on the left and right sector independently, leading to different multiplets. A well-known example is the twisted chiral multiplet, which is chiral in the left sector and anti-chiral in the right sector. Less-known, but equally important, are the right and left semichiral multiplets. These multiplets and the constraints are summarized in Table 2.2.

As shown in [18], the off-shell formulation of the most general $(2,2)$ NLSM is given by these fields and the action reads

Multiplet	Symbol	Left constraint	Right constraint
Chiral	Φ	$\bar{\mathbb{D}}_+\Phi = 0$	$\bar{\mathbb{D}}_-\Phi = 0$
Twisted-chiral	χ	$\bar{\mathbb{D}}_+\chi = 0$	$\mathbb{D}_-\chi = 0$
Left semichiral	$\mathbb{X}_L$	$\bar{\mathbb{D}}_+\mathbb{X}_L = 0$	$-$
Right semichiral	$\mathbb{X}_R$	$-$	$\bar{\mathbb{D}}_-\mathbb{X}_R = 0$

Table 2.2: General $\mathcal{N} = (2,2)$ multiplets and the supersymmetric constraints they satisfy.

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\, K(\Phi, \bar{\Phi}; \chi, \bar{\chi}; \mathbb{X}_L, \bar{\mathbb{X}}_L; \mathbb{X}_R, \bar{\mathbb{X}}_R)\,, \tag{2.1.13}$$

where K is the generalized potential for the generalized Kähler manifold. Note that the generalized potential K is defined up to generalized Kähler transformations $f(\phi, \chi, \mathbb{X}_L) + g(\phi, \bar{\chi}, \mathbb{X}_R) + \bar{f}(\bar{\phi}, \bar{\chi}, \bar{\mathbb{X}}_L) + \bar{g}(\bar{\phi}, \chi, \bar{\mathbb{X}}_R)$, since these vanish upon integration in superspace. As usual, performing the integral over the fermionic measure determines the metric and b-field in each sector in terms of second derivatives of K. We will give some explicit expressions as we need them in the main text.

One of the earliest examples of a generalized geometry with torsion is the $S^3 \times S^1$ WZW model, which can be described by chiral and twisted chiral fields [19]. Furthermore, this model falls outside the classification of [20], having $(4,4)$ SUSY and not being hyperkähler. (When the condition $K_{\bar{\Phi}\Phi} + K_{\bar{\chi}\chi} = 0$ is satisfied, the model has $(4,4)$ SUSY without necessarily being hyperkähler [12].)

A comment on notation: $\mathcal{N} = (2,2)$ spinor derivatives are denoted by $\mathbb{D}_\pm$ to distinguish them from the $\mathcal{N} = (1,1)$ derivatives $D_\pm$. We usually denote the lowest $\mathcal{N} = (1,1)$ components of chiral and twisted-chiral fields by the same letters as the $\mathcal{N} = (2,2)$ fields, whereas for semichiral fields we write $\mathbb{X}_{L,R}| = X_{L,R}$. When writing the metric and b-field, it should be understood that we are referring to the $\mathcal{N} = (1,1)$ components.

Gauged linear sigma models

Much of the success of NLSMs (for chiral fields) in the study of Kähler geometry (and in particular Calabi-Yau manifolds) stems from the fact that they can be realized as gauged linear sigma models (GLSMs). This fact was exploited with great success by Witten in [21]. As a typical example, consider a set of n chiral fields Φ_i with charges q_i under a $U(1)$ gauge superfield V. In $(2,2)$ superspace, the gauged action reads

$$\mathcal{L} = \int d^2\theta\, d^2\bar{\theta} \left(-\frac{1}{e^2}\bar{\Sigma}\Sigma + \sum_i \bar{\Phi}_i e^{q_i V} \Phi_i - r\, V \right) . \tag{2.1.14}$$

The last term is known as a Fayet–Iliopoulos (FI) term and $\Sigma = \bar{\mathbb{D}}_+ \mathbb{D}_- V$ is the field strength. Upon reduction to components, (and integrating out auxiliary fields) the action contains kinetic terms and a scalar potential given by

$$U = \sum_i |\sigma|^2 |\phi_i|^2 + \frac{e^2}{4} (\sum_i q_i |\phi_i|^2 - r\,)^2 \,, \tag{2.1.15}$$

where σ is the lowest component of Σ. Recall that in a supersymmetric theory, the energy of the vacuum vanishes. The vanishing of the kinetic terms is ensured by taking the fields to be constants: $\phi_i = \langle \phi_i \rangle$ and the vanishing of the potential imposes $\sigma = 0$ and the constraint $\sum_i q_i |\phi_i|^2 = r$. Thus, there is a whole space of vacua labeled by the VEVs of all the scalar fields present in the theory. Clearly, in the presence of a gauge symmetry, vacua that are related by a gauge transformation are physically equivalent. Thus, the space of physical vacua is defined by

$$\mathcal{M} = \{\phi^i \in \mathbb{C} : U = 0\} / U(1) \,. \tag{2.1.16}$$

This is known as the moduli space of the theory, a concept that we will encounter throughout this thesis. It is easy to see that due to the supersymmetric Higgs mechanism the vector multiplet acquires a mass $M^2 = e^2 r$ (see, *e.g.*, [22] for a standard textbook reference). Thus, in the IR limit $e \to \infty$ the massive modes can be neglected and only the massless modes are retained. Thus, in the IR limit the gauged linear sigma model is described by a NLSM on the moduli space $\mathcal{M}$.

This combined operation of setting $U = 0$ and dividing by $U(1)$ can be regarded as a Kähler quotient of the original manifold $\mathcal{M}_0$ (in this case $\mathbb{C}^n$) by $U(1)$, which is usually denoted by $\mathcal{M}_0//U(1)$ and reduces the complex dimension by 1. Similarly, one can also perform a hyperkähler quotient, denoted usually by $\mathcal{M}_0///U(1)$ which reduces the complex dimension by 2. See [11] for a full explanation of the relation between supersymmetry and quotients. We will often refer to the operation of integrating out the vector multiplet in the IR limit as performing a quotient.

An important aspect of this is that the geometry of $\mathcal{M}$ depends on the signs of the charges q_i and the FI parameters (there can be one FI parameter for every $U(1)$ factor in the gauge group). For example, if all the charges in (2.1.15) are positive, there is a sphere worth of classical vacua for $r > 0$ and no solution for $r < 0$. These are usually referred to as different phases [21].

Despite the great success of GLSMs in realizing NLSMs on Kähler and hyperkähler spaces, much less is known on how to engineer non-Kähler geometries with non-zero torsion using generalized $(2,2)$ GLSMs. Thus, our main goal is the expand the class of $(2,2)$ GLSMs considered so far, by considering the gauging of linear sigma models involving the most general set of fields of the form:

$$K = \sum_{a=1}^{d_c} \bar{\Phi}^a \Phi^a - \sum_{a'=1}^{d_t} \bar{\chi}^{a'} \chi^{a'} - \sum_{i=1}^{d_s} \left[\bar{\mathbb{X}}_L^i \mathbb{X}_L^i + \bar{\mathbb{X}}_R^i \mathbb{X}_R^i + \alpha(\bar{\mathbb{X}}_L^i \mathbb{X}_R^i + \bar{\mathbb{X}}_R^i \mathbb{X}_L^i) \right] .$$

The main reason why progress was hampered was due to the absence of vector multiplets that can that can gauge isometries in a general bihermitean geometry. This is simply because the gauge transformation of the gauge field is not always compatible with the chirality constraints on the matter fields. For example, a chiral vector multiplet cannot act on a twisted chiral field. (We will have more to say about this action and additional possible couplings among the fields in Chapter 3.)

As we shall review in the main text, this problem has been solved by the introduction of the semichiral vector and the large vector multiplets [23, 24] and we are now in a position to analyze the moduli spaces of general gauged linear sigma models. This is the main motivation for the work presented in this chapter as well as in Chapter 3.

2.2 Preliminaries

As mentioned earlier, in this chapter we will focus models involving semichiral exclusively. Thus, we begin by giving some elements of the geometry of semichiral models that and the semichiral vector multiplet [23, 24], one of the new vector multiplets which acts in these fields.

2.2.1 Geometry of semichiral sigma models

Consider a non-linear sigma model for a set of semichiral superfields $\mathbb{X}_L^a, \mathbb{X}_R^{a'}$, $a, a' = 1, ..., d_s$ with an action given by

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\mathbb{X}_L^a, \bar{\mathbb{X}}_L^a; \mathbb{X}_R^{a'}, \bar{\mathbb{X}}_R^{a'}) \,. \tag{2.2.1}$$

These models were first studied in [25], showing that upon reduction to $\mathcal{N} = (1,1)$, they lead to a general non-linear sigma model[4]. However, semichiral superfields are less constrained than chiral and twisted-chiral fields and contain auxiliary superfields which, when integrated out, induce non-linearities in the $\mathcal{N} = (1,1)$ action. As a consequence, the metric and b-field are non-linear functions of second derivatives of K. These can be written compactly [18, 26] in terms of the complex structures $J_\pm$ and a closed 2-form Ω as

$$g = \Omega\,[J_+, J_-], \qquad b = \Omega\,\{J_+, J_-\} \,. \tag{2.2.2}$$

The complex structures and Ω are completely determined by the generalized potential by

$$J_+ = \begin{pmatrix} J_s & 0 \\ \mathcal{K}_{RL}^{-1} C_{LL} & \mathcal{K}_{RL}^{-1} J_s \mathcal{K}_{LR} \end{pmatrix}, \qquad J_- = \begin{pmatrix} \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} & \mathcal{K}_{LR}^{-1} C_{RR} \\ 0 & J_s \end{pmatrix},$$

where J_s is a $2d_s$-dimensional matrix of the form $\mathrm{diag}(i, -i)$ and

$$\Omega = \begin{pmatrix} 0 & \mathcal{K}_{LR} \\ -(\mathcal{K}_{LR})^t & 0 \end{pmatrix}, \tag{2.2.3}$$

with

$$\mathcal{K}_{LL} = \begin{pmatrix} K_{LL} & K_{L\bar{L}} \\ K_{\bar{L}L} & K_{\bar{L}\bar{L}} \end{pmatrix}, \qquad \mathcal{K}_{LR} = \begin{pmatrix} K_{LR} & K_{L\bar{R}} \\ K_{\bar{L}R} & K_{\bar{L}\bar{R}} \end{pmatrix}, \tag{2.2.4}$$

where $K_{LR} \equiv \frac{\partial^2 K}{\partial X_L \partial X_R}$, etc. and $\mathcal{K}_{LR}^{-1} \equiv (\mathcal{K}_{RL})^{-1}$ and

$$C_{LL} = \begin{pmatrix} 0 & 2iK_{L\bar{L}} \\ -2iK_{L\bar{L}} & 0 \end{pmatrix} \tag{2.2.5}$$

and similarly C_{RR}.

In four dimensions (*i.e.*, $d_s = 1$) there is an additional structure, leading

[4]In fact, an equal number of left and right semichiral fields are needed to get a NLSM.

to the anti-commutator of the complex structures to being proportional to the identity, namely

$$\{J_+, J_-\} = c\,\mathbb{I}, \tag{2.2.6}$$

where c is a scalar function given by

$$c = -2\frac{|K_{LR}|^2 + |K_{L\bar{R}}|^2 - 2K_{L\bar{L}}K_{R\bar{R}}}{|K_{LR}|^2 - |K_{L\bar{R}}|^2}\,. \tag{2.2.7}$$

As we shall review next, it contains important information about the geometry; when c is a constant and $|c| < 2$, the manifold is hyperkähler.

2.2.2 Hyperkähler case

As shown in [18], a generalized Kähler manifold of $4N$ real dimesions, described in terms of semichiral superfields, is hyperkähler if $\{J_+, J_-\} = c\,\mathbb{I}$ with c a constant and $|c| < 2$ (see also [26] for the particular case $c = 0$). This is easy to see from the expression for the b-field in (2.2.2); since Ω is a closed 2-form, the torsion, $H = db = \Omega\, dc$, vanishes for constant c. If the manifold is hyperkähler, there must be three complex structures and, indeed, a third complex structure J_3 can be constructed from $J_\pm$ by

$$J_3 = \frac{1}{\sqrt{\left(\frac{2}{c}\right)^2 - 1}}\left(\mathbb{I} - \frac{2}{c}J_+J_-\right)\,. \tag{2.2.8}$$

A trivial example of a hyperkähler manifold (and one which will be used in what follows) is flat $\mathbb{R}^{4n}$ with a constant b-field. This is described by the generalized potential

$$K_{\mathbb{R}^{4n}} = \sum_{i=1}^{n}\left(\bar{\mathbb{X}}_L^i\mathbb{X}_L^i + \bar{\mathbb{X}}_R^i\mathbb{X}_R^i + \alpha(\bar{\mathbb{X}}_R^i\mathbb{X}_L^i + \bar{\mathbb{X}}_L^i\mathbb{X}_R^i)\right)\,. \tag{2.2.9}$$

From equations (2.2.2-2.2.4), one finds the (constant) metric, b-field, and complex structures satisfying

$$\{J_+, J_-\} = 2(1 - \frac{2}{\alpha^2})\mathbb{I}\,. \tag{2.2.10}$$

For the metric to be positive definite, $\alpha^2 > 1$ is required, which also ensures $|c| < 2$. For the special value $\alpha^2 = 2$, the b-field vanishes.

2.2.3 Semichiral vector multiplet

The semichiral vector multiplet [23, 24] was introduced to gauge isometries along semichiral directions, *e.g.*,

$$\delta \mathbb{X}_L = i\lambda, \qquad \delta \mathbb{X}_R = i\lambda\,. \tag{2.2.11}$$

It is described in terms of three real supervector fields $V^\alpha = (V_L, V_R, V')$, with gauge transformations

$$\delta V_L = i(\bar\Lambda_L - \Lambda_L)\,, \qquad \delta V_R = i(\bar\Lambda_R - \Lambda_R)\,, \qquad \delta V' = (\Lambda_R + \bar\Lambda_R - \Lambda_L - \bar\Lambda_L)\,. \tag{2.2.12}$$

It is convenient to introduce the complex combinations

$$\mathbb{V} = \frac{1}{2}(-V' + i(V_L - V_R))\,, \qquad \tilde{\mathbb{V}} = \frac{1}{2}(-V' + i(V_L + V_R))\,, \tag{2.2.13}$$

with gauge transformations

$$\delta \mathbb{V} = \Lambda_L - \Lambda_R\,, \qquad \delta\tilde{\mathbb{V}} = \Lambda_L - \bar\Lambda_R\,. \tag{2.2.14}$$

The corresponding chiral and twisted-chiral field strengths are

$$\mathbb{F} = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathbb{V}\,, \qquad \tilde{\mathbb{F}} = \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{\mathbb{V}}\,. \tag{2.2.15}$$

Thus, the nonvanishing commutation relations are [24]

$$\{\bar\nabla_\pm, \nabla_\pm\} = i\mathcal{D}_{\pm\pm}\,, \qquad i\{\bar\nabla_+, \bar\nabla_-\} = \mathbb{F}\,, \qquad i\{\bar\nabla_+, \nabla_-\} = \tilde{\mathbb{F}}\,,$$

where $\nabla_\pm$ are gauge-covariant superderivatives. The kinetic terms for the gauge fields are given by

$$\mathcal{L}_{gauge} = \int d^4\theta \frac{1}{e^2}(\mathbb{F}\bar{\mathbb{F}} - \tilde{\mathbb{F}}\bar{\tilde{\mathbb{F}}})\,. \tag{2.2.16}$$

It is also possible to add Fayet-Iliopoulos (FI) terms of the form

$$\mathcal{L}_{FI} = -\int d^4\theta\,(t\mathbb{V} + s\tilde{\mathbb{V}} + c.c.) = -\int d^4\theta\, t_\alpha V^\alpha\,, \tag{2.2.17}$$

where we defined $t_\alpha \equiv -\,(\mathrm{Im}(s+t), \mathrm{Im}(s-t), \mathrm{Re}(s+t))$. These will play an important role in what follows. Upon reduction to $\mathcal{N} = (1,1)$, (2.2.16) gives the usual kinetic terms. The only dimensionful scale is $[e] = 1$ and the low energy limit corresponds to taking $e \to \infty$. Therefore, the kinetic terms are irrelevant in the IR limit and the gauge fields V', V_L, V_R become non-dynamical and are integrated out. Thus, the gauged linear sigma model will flow in the IR to a non-linear sigma model given by a semichiral quotient, which we now describe.

2.3 The Semichiral Quotient

Here we describe what we refer to as the semichiral quotient. We consider a bihermitean manifold $\mathcal{M}$ of $d = 4(N+1)$ real dimensions, parameterized by semichiral coordinates $(\mathbb{X}_L^a, \mathbb{X}_R^{a'})$ with $a, a' = 1, ..., N+1$ and generalized potential $K(\mathbb{X}_L^a, \mathbb{X}_R^{a'})$. We assume the existence of a $U(1)$ Killing vector

$$k = k^a \partial_a + k^{\bar{a}} \partial_{\bar{a}} + k^{a'} \partial_{a'} + k^{\bar{a}'} \partial_{\bar{a}'} \,, \tag{2.3.1}$$

generating the isometry

$$\delta X = [\lambda k, X] \,, \tag{2.3.2}$$

where λ is the parameter of the transformation and X is any of the coordinates. We now choose coordinates $(\mathbb{X}_L^a, \mathbb{X}_R^{a'}) = (\mathbb{X}_L^i, \mathbb{X}_R^{i'}; \mathbb{X}_L, \mathbb{X}_R)$, with $i, i' = 1, ..., N$, which are adapted to the isometry and the Killing vector takes the form

$$k = i \left(\partial_L - \bar{\partial}_L + \partial_R - \bar{\partial}_R \right) \,. \tag{2.3.3}$$

In these adapted coordinates, the generalized potential depends explicitly on the $4N$ neutral coordinates $(\mathbb{X}_L^i, \mathbb{X}_R^{i'})$ and the 3 invariant combinations $\mathbb{X}^\alpha = (\mathbb{X}_L + \bar{\mathbb{X}}_L, \mathbb{X}_R + \bar{\mathbb{X}}_R, i(\mathbb{X}_R - \bar{\mathbb{X}}_R - \mathbb{X}_L + \bar{\mathbb{X}}_L))$. Now we proceed to gauge this isometry by promoting the parameter λ to a corresponding semichiral field and introducing a semichiral vector multiplet. Then, the function $\hat{K}$ is defined by

$$\hat{K}(\mathbb{X}_L^i, \mathbb{X}_R^{i'}) \;\; = \;\; K(\mathbb{X}_L^i, \mathbb{X}_R^{i'}; \mathbb{X}^\alpha + V^\alpha) - t_\alpha V^\alpha \,, \tag{2.3.4}$$

where $V^\alpha = V^\alpha(\mathbb{X}_{L,R}^i)$ is given by solving its equations of motion

$$\frac{\partial K(\mathbb{X}_L^i, \mathbb{X}_R^{i'}; \mathbb{X}^\alpha + V^\alpha)}{\partial V^\alpha} \;\; = \;\; t_\alpha \tag{2.3.5}$$

and choosing the gauge $X^\alpha = 0$. The new potential $\hat{K}$ depends on $4N$ coordinates (and three FI parameters t_α), and describes the quotient manifold $\hat{\mathcal{M}}$ of real dimension $4N$.

Now we state one of our main results. Assume that $\mathcal{M}$ is a hyperkähler manifold and therefore

$$\{J_+, J_-\} = c\,\mathbb{I}\,, \tag{2.3.6}$$

with c a constant, as discussed in Section 2.2.2. Then, the anticommutator of the complex structures on the quotient manifold $\hat{\mathcal{M}}$ is given by

$$\{\hat{J}_+, \hat{J}_-\} = c\,\mathbb{I} \tag{2.3.7}$$

(with the same c on the right-hand side). In particular, this implies that the quotient manifold is also hyperkähler. In the current setting, the proof of (2.3.7) requires some rather tedious algebra (see Appendix A.1), but is straightforward. Imposing (2.3.6) leads to the set of equations

$$\begin{aligned} \left\{\mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL}^{-1}, J_s\right\} &= 0\,, & (2.3.8)\\ J_s \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} + \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} J_s + \mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL}^{-1} C_{LL} &= c\,\mathbb{I}\,, & (2.3.9) \end{aligned}$$

and those which follow from these exchanging ($L \leftrightarrow R$). Using standard relations between second derivatives of Legendre-transformed functions, and identities for matrix inverses, we show that these equations also hold for $\hat{K}$, proving the assertion (2.3.7) (see Appendix A.1 for more details).

A brief comment is in order. In showing that the structure (2.3.6) is preserved by the quotient, we have actually not made use of the fact that c is a constant. Thus, one could in principle extend our results to bihermitean geometries satisfying (2.3.6), other than hyperkähler (with c an arbitrary function), if there are any such manifolds. This, however, is not the case due to the following result [27]. Although the set of equations (2.3.8, 2.3.9) are satisfied identically in four dimensions, they highly restrict the geometry in higher dimensions. So much indeed, that the only manifolds satisfying (2.3.6) in $d \geq 8$ are those with a constant c, *i.e.*, hyperkähler manifolds.

2.3.1 Geometrical interpretation

It might seem surprising at first that the semichiral quotient coincides with the hyperkähler quotient. However, this is clarified by the following geometrical interpretation [28]. The hyperkähler quotient [11, 29] is based on assuming the existence of three symplectic 2-forms ω^p, $p = 1, 2, 3$, and a triholomorphic Killing vector k, *i.e.*,

$$\mathcal{L}_k \omega^p = i_k d\omega^p + d(i_k \omega^p) = 0\,. \tag{2.3.10}$$

Since $d\omega^p = 0$, this implies the existence (locally) of the three moment maps, μ^p, such that

$$i_k \omega^p = d\mu^p \,. \tag{2.3.11}$$

Setting the moment maps to zero (and dividing by the isometry), leads to the hyperkähler quotient. The relation with the semichiral quotient is based on the observation that if $[J_+, J_-]$ is invertible (which requires the presence of only semichiral fields), the closed 2-form

$$\Omega = g\,[J_+, J_-]^{-1} \tag{2.3.12}$$

is well defined[5]. This symplectic form can be decomposed [26] into its holomorphic and anti-holomorphic part, with respect to both complex structures $J_\pm$, *i.e.*,

$$\Omega = \Omega_-^{(2,0)} + \bar{\Omega}_-^{(0,2)} = \Omega_+^{(2,0)} + \bar{\Omega}_+^{(0,2)} \tag{2.3.13}$$

and $d\Omega = 0$ implies

$$\partial \Omega_\pm^{(2,0)} = \bar{\partial} \Omega_\pm^{(2,0)} = 0 \,, \tag{2.3.14}$$

and the complex conjugates. This implies the existence of four moment maps $\mu_\pm, \bar{\mu}_\pm$, subject to the reality condition

$$\mu_- + \bar{\mu}_- = \mu_+ + \bar{\mu}_+ \,, \tag{2.3.15}$$

which follows from (2.3.13). Thus, there are three independent moment maps and the semichiral quotient coincides with the hyperkähler quotient.

It can also be understood [27] in these geometrical terms why only hyperkähler manifolds satisfy (2.3.6). In a generalized Kähler manifold, the 3-form $H = db$ has no $(3,0)$ or $(0,3)$ part (see, *e.g.*, [17]) with respect to both $J_\pm$, *i.e.*,

$$H = H_\pm^{(1,2)} + H_\pm^{(2,1)} \,. \tag{2.3.16}$$

Assuming (2.3.6), one has $H = \Omega\, dc$. Using (2.3.13) and $dc = \partial c + \bar{\partial} c$, one sees

[5] Even in the presence of only semichiral fields, $[J_+, J_-]$ can fail to be invertible at some points or loci in the manifold, leading to type change. We shall not consider this case here.

that $(3,0)$ and $(0,3)$ parts appear. The requirement that they vanish implies

$$\partial c = \bar{\partial} c = 0\,. \tag{2.3.17}$$

Thus, c is a constant and H vanishes completely.

2.3.2 Comment on more General Quotients

As we have just seen, quotients involving only semichiral fields will not lead to a non-trivial b-field. However, considering several types of fields typically does. Here we give a simple example. Consider a set of semichiral fields and a single chiral field Φ, gauged by the usual vector multiplet V, *i.e.*,

$$K = \bar{\mathbb{X}}_L e^V \mathbb{X}_L + \bar{\mathbb{X}}_R e^V \mathbb{X}_R + \alpha\left(\bar{\mathbb{X}}_R e^V \mathbb{X}_L + \bar{\mathbb{X}}_L e^V \mathbb{X}_R\right) + t\bar{\Phi} e^V \Phi - rV \tag{2.3.18}$$

Integrating out V (and choosing the gauge $\Phi = 1$) leads to

$$K = r \log\left(\bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R + \alpha\left(\bar{\mathbb{X}}_R \mathbb{X}_L + \bar{\mathbb{X}}_L \mathbb{X}_R\right) + t\right). \tag{2.3.19}$$

From (2.2.7) we find

$$c = -2 + \frac{4t\left(\alpha^2 - 1\right)}{\alpha(\alpha t - R)}\,, \tag{2.3.20}$$

where $R \equiv \bar{X}_R X_L + \bar{X}_L X_R + \alpha(\bar{X}_L X_L + \bar{X}_R X_R)$. Thus c is not a constant and there is a non-trivial b-field. Although we will not analyze this model in full detail here, we can already study some features. From (2.3.19), one sees that the limit $t \to \infty$ corresponds to flat space, while $t \to 0$ gives a singular metric. For finite t, the metric becomes singular ($c = \pm 2$) for $R = t/\alpha$ and $R \to \infty$.

2.4 T-Duality

A duality relation between hyperkähler manifolds, described in terms of semichiral superfields with $c = 0$, and $\mathcal{N} = (4,4)$ models with chiral/twisted-chiral fields was described in [26]. (For some early work on duality in $\mathcal{N} = (2,2)$ models see, *e.g.*, [30–32].) Actually, understanding this relation was one of the motivations for introducing the new vector multiplets and studying T-duality in [33]. In this Section we would like clarify the exact relation of the duality in [26] to T-duality and offer a geometrical interpretation, which also allows us to consider Kähler manifolds with $c \neq 0$ and even non-Kählerian manifolds (that may still have $\mathcal{N} = (4,4)$). As we shall see, this depends on the character of the isometry along which the duality is performed. We first discuss T-duality along a translational isometry, which leads to a hyperkähler manifold. Then, we discuss T-duality along a general isometry.

2.4.1 Translational isometry

The duality described in [26] involves two steps. Given a potential $\hat{F}(\Phi, \bar{\Phi}, \chi, \bar{\chi})$ satisfying the Laplace equation, one first constructs a potential $F(\Phi, \bar{\Phi}, \chi, \bar{\chi})$. Then, one performs a Legendre transformation to semichiral superfields. It is the first step which we reinterpret as a rotation of the $\mathcal{N} = (1, 1)$ components by a fixed angle. As we shall see below, considering an arbitrary[6] (constant) rotation by an angle ν leads to a non-zero (constant) c.

Consider a potential $\hat{F}(\phi, \bar{\phi}, \chi, \bar{\chi})$ and assume that there is a translational isometry, generated by the Killing vector

$$k = i(\partial_\phi - \partial_{\bar{\phi}} - \partial_\chi + \partial_{\bar{\chi}})\,. \tag{2.4.1}$$

Thus, in adapted coordinates

$$\hat{F} = \hat{F}\left(\phi + \bar{\phi}, \chi + \bar{\chi}, i(\phi - \bar{\phi} + \chi - \bar{\chi})\right)\,. \tag{2.4.2}$$

Assume now that the potential describes an $\mathcal{N} = (4, 4)$ model and, therefore, satisfies the Laplace equation

$$\hat{F}_{\phi\bar{\phi}} + \hat{F}_{\chi\bar{\chi}} = 0\,. \tag{2.4.3}$$

The important observation now is that a rotation among the $\mathcal{N} = (1, 1)$ fields, (ϕ, χ), is allowed and preserves the Laplace equation. Then, when integrating up to the $\mathcal{N} = (2, 2)$ potential, one must choose what to call a chiral or twisted-chiral field and we choose to take the rotated fields. That is, we consider the transformation

$$\phi \to \cos(\nu)\phi + \sin(\nu)\chi\,, \qquad \chi \to \cos(\nu)\chi - \sin(\nu)\phi\,. \tag{2.4.4}$$

For convenience, we introduce $\theta = \nu + \frac{\pi}{4}$ and *define* the potential $F(\Phi, \bar{\Phi}, \chi, \bar{\chi})$ by

$$F = \hat{F}\left(\Phi + \bar{\Phi}, \chi + \bar{\chi}, i(c(\Phi - \bar{\Phi}) + s(\chi - \bar{\chi}))\right)\,, \tag{2.4.5}$$

where we have abbreviated $\cos(\theta) = c, \sin(\theta) = s$. The Killing vector is now

[6] The author wishes to thank Martin Roček for this suggestion.

given by

$$k = i[(s(\partial_\Phi - \partial_{\bar{\Phi}}) - c(\partial_\chi - \partial_{\bar{\chi}})] \,, \tag{2.4.6}$$

which implies the transformations for the matter fields

$$\delta\Phi = is\lambda \,, \qquad \delta\chi = -ic\lambda \,. \tag{2.4.7}$$

This isometry can be gauged by the Large Vector Multiplet (LVM) [23, 24], defined similarly to the SVM by

$$\mathbb{V}_L = \frac{1}{2}\left(-V' + i(V^\phi - V^\chi)\right) \,, \qquad \mathbb{V}_R = \frac{1}{2}\left(-V' + i(V^\phi + V^\chi)\right) \,, \tag{2.4.8}$$

where the real vector fields $V^\alpha = (V^\phi, V^\chi, V')$ transform as

$$\delta V^\phi = i(\bar{\Lambda} - \Lambda) \,, \qquad \delta V^\chi = i(\bar{\tilde{\Lambda}} - \tilde{\Lambda}) \,, \qquad \delta V' = -(\Lambda + \bar{\Lambda}) + \tilde{\Lambda} + \bar{\tilde{\Lambda}} \,. \tag{2.4.9}$$

Following [33], we perform a T-duality to semichiral fields by defining

$$\begin{aligned} K(\mathbb{X}_L, \mathbb{X}_R) \;\; &= \;\; F\left(\Phi + \bar{\Phi} + sV^\phi, \chi + \bar{\chi} + cV^\chi, i(c(\Phi - \bar{\Phi}) + s(\chi - \bar{\chi})) - csV'\right) \\ &\quad - \left[\mathbb{X}_L \mathbb{V}_L + \mathbb{X}_R \mathbb{V}_R + c.c.\right] \,. \end{aligned} \tag{2.4.10}$$

In the gauge $\Phi = \chi = 0$, we have

$$\begin{aligned} K(\mathbb{X}_L, \mathbb{X}_R) =& F(sV^\phi, cV^\chi, -csV') - \frac{1}{2}\Big[iV^\phi(\mathbb{X}_L - \bar{\mathbb{X}}_L + \mathbb{X}_R - \bar{\mathbb{X}}_R) \\ & - iV^\chi(\mathbb{X}_L - \bar{\mathbb{X}}_L - \mathbb{X}_R + \bar{\mathbb{X}}_R) - V'(\mathbb{X}_L + \bar{\mathbb{X}}_L + \mathbb{X}_R + \bar{\mathbb{X}}_R)\Big] \,. \end{aligned}$$

Integrating out the LVM, *i.e.*, solving

$$\frac{\partial K}{\partial V^\alpha} = 0 \tag{2.4.11}$$

for the vector fields V^α leads to the semichiral potential. From the definition (2.2.7), and using standard implicit differentiation relations (see Appendix A.2 for more details), we find a non-zero c given by

$$c = -2\cos(2\theta) \,. \tag{2.4.12}$$

For the particular case $\theta = \pi/4$, this reduces to the duality described in [26].

A short observation that will be useful later[7] is that one may alternatively rescale the fields ϕ, χ in (2.4.6) to bring the Killing vector to its usual form. Then, the potential F will satisfy a *scaled* Laplace equation: If K describes a hyperkähler manifold with a constant $c = 2(1 - \frac{2}{\alpha^2})$, the dual potential satisfies

$$F_{\phi\bar{\phi}} + (\alpha^2 - 1)F_{\chi\bar{\chi}} = 0 \,. \tag{2.4.13}$$

2.4.2 General isometry

As we have just discussed, T-dualizing an $\mathcal{N} = (4,4)$ model along a translational isometry using the LVM leads to a hyperkähler manifold, described in terms of semichiral fields. In showing this, the form of the Killing vector was crucial. Indeed, if it acts by translation on Φ and χ by *equal* amounts, then $c = 0$, while if it acts by different amounts, it leads to a non-zero (but constant) c. We wish to investigate now what happens for a general isometry of the form $k = k^\Phi(\Phi)\partial_\Phi + k^\chi(\chi)\partial_\chi + c.c.$. If K is invariant under the isometry, the gauging along a general Killing vector is given by [33]

$$K^{(g)} = \exp\left(-\frac{1}{4}V^\phi \mathcal{L}_{(J_+ + J_-)k} - \frac{1}{4}V^\chi \mathcal{L}_{(J_+ - J_-)k} - \frac{1}{4}V^{'} \mathcal{L}_{J_+ J_- k}\right) K \,. \tag{2.4.14}$$

By implicit differentiation (again, see Appendix A.2 for details), we find

$$c = 2\left(\frac{|k^\Phi|^2 - |k^\chi|^2}{|k^\Phi|^2 + |k^\chi|^2}\right)\Bigg|_{\partial K/\partial V = 0} \,. \tag{2.4.15}$$

Note that although this expression does not depend on the potential explicitly, it does depend on it implicitly; to write the right-hand side in terms of semichiral coordinates, the relation of chiral/twisted-chiral fields to semichiral fields given by the Legendre transform is needed. In the case $k^\Phi = -k^{\bar{\Phi}} = i\cos(\theta)$ and $k^\chi = -k^{\bar{\chi}} = i\sin(\theta)$, we recover (2.4.12). We conclude from (2.4.15) that for a general isometry c will not be a constant and the dual geometry will not be hyperkähler, even if the T-duality preserves the supersymmetry (the isometries preserving $\mathcal{N} = (4,4)$ in this context are translational and rescaling [?]).

As an example, consider the gauging of the isometry along the S^1 in the $SU(2) \times U(1)$ WZW model, described in terms of chiral/twisted-chiral superfields [19], recently studied in [34]. The isometry in this case acts by a rescaling

[7] In coming Sections we will perform T-duality transformations in the other direction, namely from semichiral fields to chiral/twisted-chiral by the use of the semichiral vector multiplet. We expect, however, the same relations to hold.

of the fields, *i.e.*,

$$k = \Phi\partial_\Phi + \bar{\Phi}\partial_{\bar{\Phi}} + \chi\partial_\chi + \bar{\chi}\partial_{\bar{\chi}}\,. \tag{2.4.16}$$

T-dualizing along this direction, the dual potential again describes an $SU(2)\times U(1)$ WZW model, which is not hyperkähler. Indeed, from (2.4.15), one finds

$$c = \frac{2}{\sqrt{1-4e^{-X'}}}\,, \tag{2.4.17}$$

where $X' = X_L + \bar{X}_L + X_R + \bar{X}_R$. Since the isometry in the $SU(2)\times U(1)$ WZW model corresponds to a rescaling, $\mathcal{N}=(4,4)$ supersymmetry is preserved in the semichiral description.

2.5 Eguchi-Hanson

Here we give the first example of the semichiral quotient. We consider $\mathbb{R}^8 = \mathbb{R}^4\times\mathbb{R}^4$, described by two copies of a left and right semichiral field, $(\mathbb{X}_L^{(1)}, \mathbb{X}_R^{(1)})$ and $(\mathbb{X}_L^{(2)}, \mathbb{X}_R^{(2)})$, as discussed in Section 2.2.2. We assign equal[8] $U(1)$ charges $q_1 = q_2 = 1$ to both and proceed as described, defining

$$\hat{K} = \sum_{i=1,2}\left[\bar{\mathbb{X}}_L^{(i)} e^{V_L}\mathbb{X}_L^{(i)} + \bar{\mathbb{X}}_R^{(i)} e^{V_R}\mathbb{X}_R^{(i)} + \alpha(\bar{\mathbb{X}}_R^{(i)} e^{-i\tilde{\mathbb{V}}}\mathbb{X}_L^{(i)} + \bar{\mathbb{X}}_L^{(i)} e^{i\bar{\tilde{\mathbb{V}}}}\mathbb{X}_R^{(i)})\right] - t_\alpha V^\alpha\,. \tag{2.5.1}$$

Based on our results of Section 2.3, we know the resulting quotient manifold will be hyperkähler, with $c = 2(1-\frac{2}{\alpha^2})$. We show below that this is actually the well-known Eguchi-Hanson manifold. Before showing this explicitly, by computation of the quotient potential and metric, we show that this quotient construction actually reduces to the usual hyperkähler quotient construction of Eguchi-Hanson in terms of $\mathcal{N}=1$ fields.

2.5.1 Reduction to $\mathcal{N}=(1,1)$: Comparison to the hyperkähler quotient

The procedure to reduce to $\mathcal{N}=(1,1)$ is well known (see, *e.g.*, [18] for a review). One decomposes the $\mathcal{N}=(2,2)$ gauge-covariant (super)derivatives

[8] It is worth mentioning that one can invert the charge of one of the pairs, say $(\mathbb{X}_L^{(2)}, \mathbb{X}_R^{(2)})$, by dualizing to fields $\tilde{\mathbb{X}}_L, \tilde{\mathbb{X}}_R$ that impose the semichiral constraints on the original pair [?]. This duality is not based on an isometry and does not change the geometry. Hence, we expect the quotient involving two pairs of semis, either with charges $(+,+)$ or $(+,-)$, to lead to the same geometry.

into their real and imaginary part, namely

$$\nabla_{\pm} = \frac{1}{2}\left(\mathcal{D}_{\pm} - iQ_{\pm}\right), \qquad \bar{\nabla}_{\pm} = \frac{1}{2}\left(\mathcal{D}_{\pm} + iQ_{\pm}\right). \tag{2.5.2}$$

We perform the reduction of the matter fields $\mathbb{X}_L, \mathbb{X}_R$ in the covariant approach (see Appendix A.3 for more details), defining

$$\hat{\bar{\mathbb{X}}}_L = \bar{\mathbb{X}}_L e^{\mathbb{V}_L}, \quad \hat{\mathbb{X}}_L = \mathbb{X}_L, \quad \hat{\mathbb{X}}_R = e^{-\mathbb{V}_L} e^{i\bar{\tilde{\mathbb{V}}}} \mathbb{X}_R, \quad \hat{\bar{\mathbb{X}}}_R = \bar{\mathbb{X}}_R e^{-i\tilde{\mathbb{V}}},$$

in terms of which the Lagrangian (2.5.1) reads (relabeling the fields $\hat{\mathbb{X}}_{L,R} \to \mathbb{X}_{L,R}$)

$$\begin{aligned} \mathcal{L} &= \int d^2\theta Q_+ Q_- \left[\bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R + \alpha(\bar{\mathbb{X}}_L \mathbb{X}_R + \bar{\mathbb{X}}_R \mathbb{X}_L)\right] \\ &+ \int d^2\theta \left[t\mathbb{F} - s\tilde{\mathbb{F}} - c.c.\right], \end{aligned} \tag{2.5.3}$$

where $d^2\theta$ is the $\mathcal{N} = (1,1)$ measure and the relative minus sign between s and t comes from the ordering in the measure. Next, one imposes the fields to be gauge-covariantly semichiral and defines components with gauge-covariant $Q_\pm$'s, *i.e.*,

$$X_L = \mathbb{X}_L\big|, \quad Q_+ \mathbb{X}_L = i\mathcal{D}_+ \mathbb{X}_L, \quad Q_- \mathbb{X}_L\big| = \Psi_-, \tag{2.5.4}$$

$$X_R = \mathbb{X}_R\big|, \quad Q_- \mathbb{X}_R = i\mathcal{D}_- \mathbb{X}_R, \quad Q_+ \mathbb{X}_R\big| = \Psi_+. \tag{2.5.5}$$

The reduction of the semichiral vector multiplet is given by [23, 24]

$$f = -i\left(\mathbb{F} - \bar{\mathbb{F}} + \tilde{\mathbb{F}} - \bar{\tilde{\mathbb{F}}}\right)\Big|,$$

$$d^1 = \left(\mathbb{F} + \bar{\mathbb{F}}\right)\Big|, \quad d^2 = \left(\tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big|, \quad d^3 = i\left(\mathbb{F} - \bar{\mathbb{F}} - \tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big|.$$

Rescaling $X_L \to \alpha/(\sqrt{4-\alpha^2})X_L$ and writing

$$X_L = \frac{1}{4}\left(\frac{\phi_-}{\sqrt{\alpha+1}} - \frac{\bar{\phi}_+}{\sqrt{\alpha-1}}\right), \qquad X_R = \frac{1}{4}\left(\frac{\phi_-}{\sqrt{\alpha+1}} + \frac{\bar{\phi}_+}{\sqrt{\alpha-1}}\right),$$

the kinetic terms are diagonalized, *i.e.*,

$$\mathcal{L}_{kin.} \sim \int d^2\theta \left[\mathcal{D}_+\bar{\phi}_+\mathcal{D}_-\phi_+ + \mathcal{D}_+\bar{\phi}_-\mathcal{D}_-\phi_-\right] \tag{2.5.6}$$

and the constraints read

$$\begin{aligned} \bar{\phi}_+\phi_+ - \bar{\phi}_-\phi_- &= p\,, \\ \phi_+\phi_- + ib &= 0\,, \end{aligned} \tag{2.5.7}$$

where we have defined

$$r \equiv -2\mathrm{Re}[s+t]\,, \qquad q \equiv -2\mathrm{Im}[t]\,, \qquad p \equiv -2\mathrm{Im}[s]\,, \tag{2.5.8}$$

and $2b \equiv (r+iq)\sqrt{\alpha^2-1}$. The free action (2.5.6), subject to the constraints (2.5.7), is the usual hyperkähler quotient construction for Eguchi-Hanson [29] (see also, *e.g.*, [35, 36]). This is a specific example of our discussion in Section 2.3.1 of the semichiral quotient reducing to the hyperkähler quotient. Thus, performing the quotient at the $\mathcal{N}=(2,2)$ gives the generalized potential for Eguchi-Hanson.

2.5.2 Generalized Potential

We have learned that the semichiral quotient (2.5.1) coincides, in $\mathcal{N}=(1,1)$ language, to the hyperkähler construction of Eguchi-Hanson. Therefore, performing the quotient in terms of $\mathcal{N}=(2,2)$ superfields will lead us to the generalized description of this manifold. From (2.5.1), the equations of motion for the vector multiplet read

$$\begin{aligned} e^{V_L}\left(1+|\mathbb{X}_L|^2\right) + \frac{\alpha}{2}\left[e^{-i\tilde{V}}\left(1+\bar{\mathbb{X}}_R\mathbb{X}_L\right) + e^{i\bar{\tilde{V}}}\left(1+\bar{\mathbb{X}}_L\mathbb{X}_R\right)\right] - \frac{(p+q)}{2} &= 0\,, \\ e^{V_R}\left(1+|\mathbb{X}_R|^2\right) + \frac{\alpha}{2}\left[e^{-i\tilde{V}}\left(1+\bar{\mathbb{X}}_R\mathbb{X}_L\right) + e^{i\bar{\tilde{V}}}\left(1+\bar{\mathbb{X}}_L\mathbb{X}_R\right)\right] - \frac{(p-q)}{2} &= 0\,, \\ \frac{i\alpha}{2}\left[e^{-i\tilde{V}}\left(\bar{\mathbb{X}}_R\mathbb{X}_L+1\right) - e^{i\bar{\tilde{V}}}\left(\bar{\mathbb{X}}_L\mathbb{X}_R+1\right)\right] - \frac{r}{2} &= 0\,, \end{aligned}$$

where we have chosen the gauge $\mathbb{X}_L^{(2)} = \mathbb{X}_R^{(2)} = 1$ (and relabeled the remaining fields). These can be easily solved for V_L, V_R, V', leading to the quotient potential

$$\begin{aligned}\hat{K}_{EH} = &- \frac{p}{2}\log\left(\frac{-\left(q^2+r^2\right)\left(S^2-\alpha^2T^2\right)+p^2\left(S^2+T^2\alpha^2\right)-2ipQ}{\left(S^2-\alpha^2T^2\right)^2}\right)\\ &- \frac{q}{2}\log\left(\frac{(1+|\mathbb{X}_R|^2)^2\left(p^2S^2+r^2S^2-q^2\left(S^2-2T^2\alpha^2\right)+2iqQ\right)}{\left((p-q)^2+r^2\right)S^4}\right)\\ &- \frac{ir}{2}\log\left(\frac{(1+\bar{\mathbb{X}}_L\mathbb{X}_R)^2\left(-2r^2S^2+\left(p^2-q^2+r^2\right)T^2\alpha^2-2rQ\right)}{T^4\alpha^2}\right)\,,\end{aligned} \tag{2.5.9}$$

where we have defined

$$S^2 = (1+|\mathbb{X}_L|^2)(1+|\mathbb{X}_R|^2)\,, \qquad T^2 = (1+\bar{\mathbb{X}}_R\mathbb{X}_L)(1+\bar{\mathbb{X}}_L\mathbb{X}_R)\,, \tag{2.5.10}$$

and

$$Q^2 = r^2S^4 - (p^2-q^2+r^2)S^2T^2\alpha^2 - q^2T^4\alpha^4\,. \tag{2.5.11}$$

This quotient construction has been discussed to some extent in [37], where the authors suggest that this will lead to a non-trivial H-flux. Based on our result of Section 2.3, we know this is not the case. Instead, it must describe a hyperkähler manifold; in this case, Eguchi-Hanson. By explicit calculation, one can also verify that (2.5.9) satisfies the Monge-Ampere equation (2.2.7) with $c = 2(1-\frac{2}{\alpha^2})$, *i.e.*,

$$\{J_+, J_-\} = 2(1-\frac{2}{\alpha^2})\mathbb{I}\,. \tag{2.5.12}$$

To show explicitly that one can derive the standard metric for Eguchi-Hanson from this potential, we set the FI parameters to some convenient value for which the potential is simplified. The choice $r = q = 0$, for instance, leads to the left-right symmetric potential

$$K = p\log\left[S+\alpha T\right]\,, \tag{2.5.13}$$

while the choice $r = 0$, $p = -q$ leads to

$$K = p\log\left[\frac{S^2-\alpha^2T^2}{1+|\mathbb{X}_L|^2}\right]\,. \tag{2.5.14}$$

This form of the potential also coincides with that of [38], constructed by twistor methods. Note that these potentials are more compact than the usual

Kähler potential and contain no term with a square root outside the log. Working with the potential (2.5.14), we will explicitly construct the Eguchi-Hanson metric, but first we will study the $SU(2)$ symmetry of the problem.

2.5.3 $SU(2)$ symmetry

The action (2.5.1) is invariant under the global $SU(2)$ transformations which rotate $(\mathbb{X}^{(1)}, \mathbb{X}^{(2)})$, as well as under $U(1)$ gauge transformations. Recall that we have chosen the $U(1)$ gauge

$$\mathbb{X}_L^{(2)} = \mathbb{X}_R^{(2)} = 1\,, \tag{2.5.15}$$

which is not preserved by the $SU(2)$. Nevertheless, the $SU(2)$ symmetry can be realized non-linearly in the gauged action by introducing a compensating $U(1)$ transformation with parameter Λ_C, namely

$$\left(\begin{array}{c} \delta\mathbb{X}_L^{(1)} \\ \delta\mathbb{X}_L^{(2)} \end{array}\right) = i \left(\begin{array}{cc} \alpha & -i\beta \\ i\bar{\beta} & -\alpha \end{array}\right) \left(\begin{array}{c} \mathbb{X}_L^{(1)} \\ \mathbb{X}_L^{(2)} \end{array}\right) + i \left(\begin{array}{c} \Lambda_C\mathbb{X}_L^{(1)} \\ \Lambda_C\mathbb{X}_L^{(2)} \end{array}\right)\,, \tag{2.5.16}$$

and similarly for $\mathbb{X}_R$. Imposing that the transformation preserves the gauge (2.5.15), and relabelling $\mathbb{X}_{L,R}^{(1)} = \mathbb{X}_{L,R}$ henceforth, one finds

$$\delta\mathbb{X}_L = 2i\alpha\mathbb{X}_L + \bar{\beta}(\mathbb{X}_L)^2 + \beta\,, \qquad \delta\mathbb{X}_R = 2i\alpha\mathbb{X}_R + \bar{\beta}(\mathbb{X}_R)^2 + \beta\,. \tag{2.5.17}$$

The infinitesimal transformations are generated by the vector field

$$\xi = \delta\mathbb{X}_L\partial_L + \delta\mathbb{X}_R\partial_R + c.c. \tag{2.5.18}$$

and the finite transformations are given by the Möbius transformations

$$\mathbb{X}_L \to \frac{a\mathbb{X}_L + b}{\bar{a} - \bar{b}\mathbb{X}_L}\,, \qquad \mathbb{X}_R \to \frac{a\mathbb{X}_R + b}{\bar{a} - \bar{b}\mathbb{X}_R}\,, \tag{2.5.19}$$

with $|a|^2 + |b|^2 = 1$. Given the $SU(2)$ invariance of the potential (and therefore the metric), it will be convenient to find coordinates in which this symmetry is manifest. The first step in doing this is to note that a natural radial coordinate R is defined by the invariant cross-ratio

$$R^2 \equiv \frac{Z_{13}Z_{24}}{Z_{23}Z_{14}}\,, \tag{2.5.20}$$

where $Z_{ij} = Z_i - Z_j$. Since we have only two complex variables, namely X_L, X_R, there is only one, non-zero, independent cross ratio we can form. Taking $Z_1 = X_L, Z_2 = X_R, Z_3 = -1/\bar{X}_L$ and $Z_4 = -1/\bar{X}_R$ we have

$$R^2 = \frac{(1+|X_L|^2)(1+|X_R|^2)}{(1+\bar{X}_L X_R)(1+\bar{X}_R X_L)} = \frac{S^2}{T^2}\,. \tag{2.5.21}$$

One can easily verify that $\mathcal{L}_\xi R = \xi R = 0$. Therefore, one can reach every point (X_L, X_R), at a certain radius R, by choosing a point (X_L^0, X_R^0) on the sphere of that radius and acting by a $SU(2)$ transformation with parameters (a, b). Thus, we can parameterize any point (X_L, X_R) by a, b (subject to $|a|^2 + |b|^2 = 1$), and the radial coordinate R. Then, the natural remaining invariants are the Cartan 1-forms σ^i on the group manifold. As shown in Appendix A.4, this parameterization of the X_L, X_R coordinates leads to

$$\begin{aligned} dX_L &= \frac{1}{\bar{a}^2}(i\sigma^1 - \sigma^2)\,, \\ dX_R &= \frac{1}{(\bar{a}-\rho\bar{b})^2}\left[2i\rho\sigma^3 + i(1-\rho^2)\sigma^1 - (1+\rho^2)\sigma^2 + d\rho\right]\,, \end{aligned} \tag{2.5.22}$$

where $\rho^2 \equiv R^2 - 1$. As we shall see below, when writing the line element in this $SU(2)$ parameterization, all the dependence in a, b drops out as a consequence of the invariance of the metric. Also, one can see by explicit calculations of $J_\pm$ from the potential (2.5.13) that

$$\mathcal{L}_\xi J_\pm = 0\,. \tag{2.5.23}$$

That is, both complex structures, $J_\pm$ (and therefore the third one), are preserved by the $SU(2)$, which is an important property of Eguchi-Hanson (see Appendix A.4 for more details). To show explicitly that the potential (2.5.14) indeed describes this manifold, we compute the metric.

2.5.4 Metric

From the potential (2.5.14), and Eqs. (2.2.2)-(2.2.4), one finds the metric[9]

[9] For simplicity, we have taken $\alpha = \sqrt{2}$ here, but the final result (2.5.26) holds for any α, with appropriate redefinitions.

$$
\begin{aligned}
ds^2 =& \frac{F(R)(\bar{X}_L-\bar{X}_R)^2}{(1+\bar{X}_R X_L)^2(1+|X_L|^2)^2} dX_L dX_L + \frac{F(R)(\bar{X}_R-\bar{X}_L)^2}{(1+\bar{X}_L X_R)^2(1+|X_R|^2)^2} dX_R dX_R \\
&+ \frac{G(R)}{(1+|X_L|^2)^2} d\bar{X}_L dX_L + \frac{G(R)}{(1+|X_R|^2)^2} d\bar{X}_R dX_R \\
&+ \frac{H(R)(\bar{X}_L-\bar{X}_R)^2}{(1+|X_L|)^2(1+|X_R|^2)^2} dX_L dX_R + \frac{I(R)(1+\bar{X}_L X_R)^2}{(1+|X_L|)^2(1+|X_R|^2)^2} dX_L d\bar{X}_R \quad + c.c.,
\end{aligned}
$$

where

$$
\begin{aligned}
F(R) &= -\frac{16\,(2-2R^2+R^4)}{(-2+R^2)^3\,R^2}\,, \qquad G(R) = -\frac{8(2-2R^2+R^4)^2}{(-2+R^2)^3 R^2}\,, \\
H(R) &= \frac{4R^2\,(4-2R^2+R^4)}{(-2+R^2)^3}\,, \qquad I(R) = \frac{4R^2\,(4-6R^2+3R^4)}{(-2+R^2)^3}\,.
\end{aligned} \tag{2.5.24}
$$

Defining r through

$$
R^2 = \frac{2r^2}{r^2-1}\,, \tag{2.5.25}
$$

and using (2.5.22), after some algebra the line element reads

$$
\frac{1}{8}ds^2 = \frac{1}{1-\frac{1}{r^4}} dr^2 + r^2 \left(\sigma_1^2 + \sigma_2^2 + (1-\frac{1}{r^4})\sigma_3^2 \right)\,, \tag{2.5.26}
$$

which is the usual Eguchi-Hanson metric (see, *e.g.*, [39] for a review).

2.6 Taub-NUT

2.6.1 A gauged linear sigma model

Here we present a gauged linear sigma model in terms of semichiral superfields whose low-energy limit target space is Taub-NUT. Consider a gauged linear sigma model with two copies of semichiral superfields, just as the Eguchi-Hanson case, but with the difference that the isometry acts by translations on one of the pairs, *i.e.*,

$$
\begin{aligned}
K &= \bar{\mathbb{X}}_L^{(1)} e^{V_L} \mathbb{X}_L^{(1)} + \bar{\mathbb{X}}_R^{(1)} e^{V_R} \mathbb{X}_R^{(1)} + \alpha(\bar{\mathbb{X}}_R^{(1)} e^{-i\tilde{\mathbb{V}}} \mathbb{X}_L^{(1)} + \bar{\mathbb{X}}_L^{(1)} e^{i\bar{\tilde{\mathbb{V}}}} \mathbb{X}_R^{(1)}) \\
&+ \frac{1}{2}\left(\mathbb{X}_L^{(2)} + \bar{\mathbb{X}}_L^{(2)} + V_L\right)^2 + \frac{1}{2}\left(\mathbb{X}_R^{(2)} + \bar{\mathbb{X}}_R^{(2)} + V_R\right)^2 \\
&+ \frac{\alpha}{2}\left((\mathbb{X}_L^{(2)} + \bar{\mathbb{X}}_R^{(2)} - i\tilde{\mathbb{V}})^2 + (\mathbb{X}_R^{(2)} + \bar{\mathbb{X}}_L^{(2)} + i\bar{\tilde{\mathbb{V}}})^2\right) \\
&- (t\mathbb{V} + s\tilde{\mathbb{V}} + c.c.)\,.
\end{aligned}
\tag{2.6.1}
$$

It is known in general that such constructions (where the isometry acts transitively on some fields) lead to ALF (as opposed to ALE) spaces and we claim that performing the semichiral quotient in this way leads to the semichiral description of Taub-NUT. Although integrating out the vector field cannot be done explicitly, by implicit differentiation we could still compute the metric. Instead, we shall study the geometry of the T-dual theory.

2.6.2 T-dual

To perform a T-duality from the worldsheet perspective, one proceeds as according to [33, 40]. We introduce an additional vector multiplet U^α, which acts on the second pair and constrain its field strengths to be trivial by Lagrange multipliers Φ, χ, *i.e.*,

$$
\begin{aligned}
\tilde{K} &= \bar{\mathbb{X}}_L^{(1)} e^{V_L} \mathbb{X}_L^{(1)} + \bar{\mathbb{X}}_R^{(1)} e^{V_R} \mathbb{X}_R^{(1)} + \alpha\left(\bar{\mathbb{X}}_R^{(1)} e^{-i\tilde{\mathbb{V}}} \mathbb{X}_L^{(1)} + \bar{\mathbb{X}}_L^{(1)} e^{i\bar{\tilde{\mathbb{V}}}} \mathbb{X}_R^{(1)}\right) \\
&+ \frac{1}{2}\left(\mathbb{X}_L^{(2)} + \bar{\mathbb{X}}_L^{(2)} + U_L\right)^2 + \frac{1}{2}\left(\mathbb{X}_R^{(2)} + \bar{\mathbb{X}}_R^{(2)} + U_R\right)^2 \\
&+ \frac{\alpha}{2}\left((\mathbb{X}_L^{(2)} + \bar{\mathbb{X}}_R^{(2)} - i\tilde{\mathbb{U}})^2 + (\mathbb{X}_R^{(2)} + \bar{\mathbb{X}}_L^{(2)} + i\bar{\tilde{\mathbb{U}}})^2\right) \\
&- ((t+\Phi)\mathbb{V} + (s+\chi)\tilde{\mathbb{V}} + c.c.) + (\Phi\mathbb{U} + \chi\tilde{\mathbb{U}} + c.c.)\,,
\end{aligned}
\tag{2.6.2}
$$

were we have shifted $U^\alpha \to U^\alpha - V^\alpha$. Integrating out U^α yields the T-dual gauged linear sigma model

$$
\begin{aligned}
\tilde{K} &= \frac{1}{g^2}\left(-\frac{\bar{\chi}\chi}{\alpha^2 - 1} + \bar{\Phi}\Phi\right) + \bar{\mathbb{X}}_L e^{V_L} \mathbb{X}_L + \bar{\mathbb{X}}_R e^{V_R} \mathbb{X}_R + \alpha(\bar{\mathbb{X}}_R e^{-i\tilde{\mathbb{V}}} \mathbb{X}_L + \bar{\mathbb{X}}_L e^{i\bar{\tilde{\mathbb{V}}}} \mathbb{X}_R) \\
&- (\Phi\mathbb{V} + \chi\tilde{\mathbb{V}} + c.c)\,,
\end{aligned}
\tag{2.6.3}
$$

where we have shifted χ, ϕ to get rid of the FI parameters s and t, dropped terms that vanish upon integration in superspace (*i.e.*, generalized Kähler transformations) and rescaled the fields appropriately[10]. As we will see in Section 2.7, this gauged linear sigma model describes a smeared NS5-brane and, therefore, the original theory (2.6.1) is a gauged sigma model for Taub-NUT.

2.7 NS5-branes

It is well known that under type II string theory T-duality, Taub-NUT is mapped to an NS5-brane. A worldsheet discussion of such relation is given in [41]. There, a gauge theory description of NS5-branes involving a hypermultiplet, a twisted hypermultiplet, and a vector multiplet acting on the former is given and instanton corrections are discussed. We shall first show that the gauge theory (2.6.3), involving semichiral fields, also describes NS5-branes and we shall comment in Section 2.7.2 on instanton effects.

2.7.1 A gauged linear sigma model

Consider the action

$$\begin{aligned}\mathcal{L} \;=\; & \int d^4\theta \Big[\frac{1}{e^2}(\mathbb{F}\bar{\mathbb{F}} - \tilde{\mathbb{F}}\bar{\tilde{\mathbb{F}}}) + \frac{1}{g^2}(-\frac{\bar{\chi}\chi}{\alpha^2-1} + \bar{\Phi}\Phi) \\ & + \bar{\mathbb{X}}_L e^{V_L}\mathbb{X}_L + \bar{\mathbb{X}}_R e^{V_R}\mathbb{X}_R + \alpha(\bar{\mathbb{X}}_R e^{-i\tilde{\mathbb{V}}}\mathbb{X}_L + \bar{\mathbb{X}}_L e^{i\bar{\tilde{\mathbb{V}}}}\mathbb{X}_R) \\ & - (\Phi\mathbb{V} + \chi\tilde{\mathbb{V}} + c.c.)\Big]\,.\end{aligned} \tag{2.7.1}$$

In the IR limit ($e^2 \to \infty$), the equations of motion for the semichiral vector field are

$$\begin{aligned}\bar{\mathbb{X}}_L e^{V_L}\mathbb{X}_L + \frac{\alpha}{2}(\bar{\mathbb{X}}_R e^{-i\tilde{V}}\mathbb{X}_L + \bar{\mathbb{X}}_L e^{i\bar{\tilde{V}}}\mathbb{X}_R) - \frac{i}{2}(\Phi - \bar{\Phi} + \chi - \bar{\chi}) = 0\,,\\ \bar{\mathbb{X}}_R e^{V_R}\mathbb{X}_R + \frac{\alpha}{2}(\bar{\mathbb{X}}_R e^{-i\tilde{V}}\mathbb{X}_L + \bar{\mathbb{X}}_L e^{i\bar{\tilde{V}}}\mathbb{X}_R) - \frac{i}{2}(-\Phi + \bar{\Phi} + \chi - \bar{\chi}) = 0\,,\\ \alpha\frac{i}{2}(\bar{\mathbb{X}}_R e^{-i\tilde{V}}\mathbb{X}_L - \bar{\mathbb{X}}_L e^{i\bar{\tilde{V}}}\mathbb{X}_R) + \frac{1}{2}(\chi + \bar{\chi} + \Phi + \bar{\Phi}) = 0\,.\end{aligned} \tag{2.7.2}$$

[10] We have chosen to keep the kinetic terms of Φ with the usual normalization, leading to the $1/(\alpha^2-1)$ factor for χ. This relative coefficient, as we will see, is important to ensure the $\mathcal{N}=(4,4)$ symmetry of the quotient model, as is expected of a model which is dual in this manner to a model describing a hyperkähler manifold in terms of semichiral fields, as discussed in Section 2.4.

For simplicity, we gauge the semis to $\mathbb{X}_L = \mathbb{X}_R = 1$. Solving these equations leads to

$$K(\Phi,\chi) \;=\; \frac{1}{g^2}\Big(-\frac{\bar{\chi}\chi}{\alpha^2-1}+\bar{\Phi}\Phi\Big)+\Delta K(\Phi,\chi) \tag{2.7.3}$$

with

$$\begin{aligned}\Delta K(\Phi,\chi) \equiv \qquad & -i\chi\log\Big[i(\chi-\bar{\chi})\alpha^2+i(\chi+\bar{\chi}+\Phi+\bar{\Phi})(\alpha^2-1)-2R\Big] \\ & -i\Phi\log\Big[-\frac{i(\chi+\bar{\chi}+\Phi+\bar{\Phi})+i(\Phi-\bar{\Phi})\alpha^2+2R}{2i(\Phi+\bar{\chi})}\Big]+c.c.\,,\end{aligned} \tag{2.7.4}$$

where we have defined

$$R \equiv \frac{1}{2}\sqrt{(\chi+\bar{\chi}+\Phi+\bar{\Phi})^2(\alpha^2-1)-(\chi-\bar{\chi})^2\alpha^2-(\Phi-\bar{\Phi})^2\alpha^2(\alpha^2-1)}\,.$$

Note that $\alpha^2 \geq 1$ ensures the reality of R. From here we find

$$\begin{aligned} K_{\chi\bar{\chi}} &= -\frac{1}{\alpha^2-1}\Big(\frac{1}{g^2}+\frac{\alpha^2-1}{2R}\Big)\,, \quad K_{\Phi\bar{\Phi}}=\frac{1}{g^2}+\frac{\alpha^2-1}{2R}\,, \\ K_{\chi\bar{\Phi}} &= -\frac{1}{2R}\Big(\frac{(\alpha^2-1)(\bar{\Phi}+\chi)}{2iR-(\chi-\bar{\chi})-(\alpha^2-1)(\Phi-\bar{\Phi})}\Big)\,. \end{aligned} \tag{2.7.5}$$

Note that the $1/(\alpha^2-1)$ factor for $\bar{\chi}\chi$ in (2.7.1) is crucial for the potential to satisfy the scaled Laplace equation (2.4.13) (although in Section 2.4 we performed the duality in the other direction, one would expect the same relations to hold). After a trivial rescaling of the fields, the line element is given by

$$ds^2 = 2(K_{\Phi\bar{\Phi}}d\Phi d\bar{\Phi}-K_{\chi\bar{\chi}}d\chi d\bar{\chi}) = 2H(r)(d\Phi d\bar{\Phi}+d\chi d\bar{\chi}) \tag{2.7.6}$$

with

$$H(r) \equiv \Big(\frac{1}{g^2}+\frac{1}{2r}\Big)\,. \tag{2.7.7}$$

Defining

$$\chi| = \frac{(r_1+\theta)}{2}+i\frac{r_2}{\sqrt{2}}\,, \qquad \Phi| = \frac{(r_1-\theta)}{2}+i\frac{r_3}{\sqrt{2}}\,, \tag{2.7.8}$$

we finally have

$$ds^2 = H(r)(d\boldsymbol{r} \cdot d\boldsymbol{r} + d\theta^2)\,, \tag{2.7.9}$$

which is the metric for an NS5-brane, smeared along the θ direction.

2.7.2 Comment on instanton corrections

In [41] a gauge theory description of smeared NS5-branes and a worldsheet T-dual description of Taub-NUT was also given. It was argued that worldsheet instanton corrections to the effective action un-smear the NS5-brane, localizing it in the θ direction. (For a recent discussion of this phenomenon in the context of double field theory [42], see [43].) In two dimensions, instantons are Nielsen-Olesen vortices, which arise as BPS solutions to an abelian Higgs model contained in the gauge theory. Although our gauge theory construction is quite different (from the $\mathcal{N} = (2,2)$ point of view), the same arguments hold so we expect the same mechanism to be at work. Our construction does not add to the results of [41], but is consistent with it. This is more easily seen upon reduction of the gauge theory (2.7.1) to $\mathcal{N} = (1,1)$. Following [23], we find (see Appendix A.3 for details)

$$\begin{aligned}
\mathcal{L} \;=\; & \int d^2\theta \Big[\frac{1}{4e^2}(D_+ d^a)\,(D_- d^b)\, g_{ab} + \frac{1}{g^2}\left(D_+\bar\phi\, D_-\phi + D_+\bar\chi\, D_-\chi\right) \\
& +(\mathcal{D}_+ X^i)(\mathcal{D}_- X^j)E_{ij} + 2id^1(\bar X_L X_L - \bar X_R X_R - \frac{i}{8}(\phi - \bar\phi)) \\
& +d^3(\alpha(\bar X_R X_L - \bar X_L X_R) - \frac{i}{8}(\phi + \bar\phi + \chi + \bar\chi)) \\
& -2id^2(\bar X_L X_L + \bar X_R X_R + \alpha(\bar X_R X_L + \bar X_L X_R) \\
& -\frac{i}{8}(\chi - \bar\chi)) + if(\phi + \bar\phi - \chi - \bar\chi)\Big]\,,
\end{aligned}$$

where $d^a = (f, d^1, d^2, d^3)$, $X^i = (X_L, \bar X_L, X_R, \bar X_R)$ and $g_{ab} = diag(1,2,2,1)$. One can rewrite this in terms of the fields $\phi_\pm$ from Section 2.5.1 which diagonalize the kinetic terms for the semis. Then, following Tong, we allow only the lowest component of, say, ϕ_+ to vary over space and set all other fields to their classical expectation values. This results in an abelian Higgs model with a θ term for the gauge field, whose instanton solutions (in the limit $g^2 \to 0$) are conjectured to contribute to the low-energy effective action, effectively replacing

$$H(r) \to H(r,\theta) = \frac{1}{g^2} + \frac{1}{2r}\frac{\sinh r}{\cosh r - \cos\theta}\,, \qquad (2.7.10)$$

therefore unsmearing the NS5.

2.8 T-dual of Eguchi-Hanson

For completeness, we finally discuss the T-dual of Eguchi-Hanson. We can perform a T-duality before taking the quotient. As before, we introduce an additional semichiral vector multiplet U^α which acts only on the second pair $\mathbb{X}^{(2)}_{L,R}$, and defines

$$\begin{aligned} K &= \left(\bar{\mathbb{X}}^{(1)}_L\mathbb{X}^{(1)}_L + \bar{\mathbb{X}}^{(2)}_L\mathbb{X}^{(2)}_L e^{U_L}\right)e^{V_L} + \left(\bar{\mathbb{X}}^{(1)}_R\mathbb{X}^{(1)}_R + \bar{\mathbb{X}}^{(2)}_R\mathbb{X}^{(2)}_R e^{U_R}\right)e^{V_R} \\ &\quad +\alpha\left(\mathbb{X}^{(1)}_L\bar{\mathbb{X}}^{(1)}_R + \mathbb{X}^{(2)}_L\bar{\mathbb{X}}^{(2)}_R e^{-i\tilde{U}}\right)e^{-i\tilde{V}} + \alpha\left(\bar{\mathbb{X}}^{(1)}_L\mathbb{X}^{(1)}_R + \bar{\mathbb{X}}^{(2)}_L\mathbb{X}^{(2)}_R e^{i\bar{\tilde{U}}}\right)e^{i\bar{\tilde{V}}} \\ &\quad - \left[\Phi\mathbb{U} + \chi\tilde{\mathbb{U}} + c.c.\right]\,. \end{aligned}$$

Shifting $U^\alpha \to U^\alpha - V^\alpha$, the Lagrangian decouples and, gauging all the semis to 1, we have

$$K = K_1 + K_2\,, \qquad (2.8.1)$$

where

$$\begin{aligned} K_1 &= e^{U_L} + e^{U_R} + \alpha(e^{-i\tilde{\mathbb{U}}} + e^{i\bar{\tilde{\mathbb{U}}}}) + \Big(\Phi\,\mathbb{U} + \chi\,\tilde{\mathbb{U}} + c.c\Big)\,, \\ K_2 &= e^{V_L} + e^{V_R} + \alpha(e^{-i\tilde{\mathbb{V}}} + e^{i\bar{\tilde{\mathbb{V}}}}) - \Big((\Phi+t)\mathbb{V} + (\chi+s)\tilde{\mathbb{V}} + c.c\Big)\,. \end{aligned}$$

Thus, integrating out both U^α and V^α reduces to the case studied for NS5-branes with $K_1 = \Delta K(-\Phi,-\chi)$, $K_2 = \Delta K(\Phi+t,\chi+s)$ and therefore

$$\tilde{K} = \Delta K(-\Phi,-\chi) + \Delta K(\Phi+t,\chi+s)\,, \qquad (2.8.2)$$

with ΔK given in (2.7.4). Since the metric is linear in second derivatives of the potentials, we have

$$\tilde{K}_{\chi\bar{\chi}} = -\frac{1}{2}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)\,, \qquad \tilde{K}_{\Phi\bar{\Phi}} = \frac{\alpha^2-1}{2}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)\,, \qquad (2.8.3)$$

and similarly for the torsion terms. Again, this potential satisfies the scaled Laplace equation

$$\tilde{K}_{\Phi\bar{\Phi}} + (\alpha^2 - 1)\tilde{K}_{\chi\bar{\chi}} = 0\,, \tag{2.8.4}$$

in accordance with our results of Section 2.4. Note that changing the relative position of the mass-points corresponds to rotating the complex structures.

2.9 Summary and Conclusions

We have studied a supersymmetric quotient construction by the use of general $\mathcal{N} = (2,2)$ sigma models and the semichiral vector multiplet. We first restricted ourselves to the case in which only semichiral fields are involved. Due to the presence of a b-field in these models, one may naively think that a non-zero H-flux could be induced on the quotient manifold $\hat{\mathcal{M}}$, even if the original manifold $\mathcal{M}$ is hyperkähler. This, however, is prevented by our result of Section 2.3, asserting that the quotient of a hyperkähler manifold is hyperkähler, as in the usual hyperkähler quotient. Furthermore, the value of the anticommutator of the complex structures is preserved under the studied quotient. Thus, although the quotient manifold in general does have a b-field, its field strength $H = db$ vanishes. Nonetheless, the quotient provides a powerful method for constructing generalized potentials for hyperkähler manifolds, of which few explicit examples are known. We gave two examples of well-known hyperkähler manifolds, namely Eguchi-Hanson and Taub-NUT. We also used the SVM to perform T-duality transformations, giving a new $\mathcal{N} = (2,2)$ gauged linear sigma model description of (smeared) NS5-branes involving semichiral, chiral, and twisted-chiral superfields. This description is consistent with previous ones in that it contains an abelian Higgs model whose instanton solutions unsmear the NS5.

We have also clarified and extended some previous results on the duality relation of these semichiral models with $\mathcal{N} = (4,4)$ models with chiral/twisted-chiral fields. We showed that the T-dual of an $\mathcal{N} = (4,4)$ model with chiral and twisted-chiral fields, may or may not describe a hyperkähler manifold, depending on the character of the isometry along which the duality is performed. If the isometry is translational, the dual manifold is hyperkhäler. For a general isometry, however, the dual manifold is in general not hyperkähler, even if the $\mathcal{N} = (4,4)$ SUSY is preserved. This, for instance, is the case of the $SU(2) \times U(1)$ WZW model which was briefly discussed.

We also commented on more general quotients that can lead to manifolds with torsion, noting that this requires the presence of more than one type of

$\mathcal{N} = (2, 2)$ field and gave an example involving a chiral and a pair of semichiral fields.

In the next chapter we move to the study of certain GLSMs whose classical moduli spaces are non-Kähler.

Chapter 3

GLSMs with torsion[1]

In this chapter we describe some recent progress in the understanding of the classical moduli space of certain GLSMs that are non-Kähler. Although some aspects are still work in progress, we believe that the results are interesting enough to be included.

Recall from the previous chapter that the semichiral quotient of a hyperkähler manifold is hyperkähler. As we have discussed at length, this implies that the classical moduli space of GLSMs involving only semichiral fields coupled to the semichiral multiplet are hyperkähler and we have given two examples of gravitational instantons.

We will first give an overview of general GLSMs that can lead to non-Kähler geometries, but we will focus particularly on one example involving only semichiral fields. However, rather than coupling them to the semichiral vector multiplet (as we did in the previous chapter), we will couple them to the usual chiral vector multiplet V with gauge transformation $\delta_g V = i(\bar{\Lambda} - \Lambda)$ with Λ chiral. This is possible since $\delta_g \mathbb{X}_L = \delta_g \mathbb{X}_R = i\Lambda$ is compatible with the chirality constraints on semichiral fields. We will show that the classical moduli space of such GLSMs is non-Kähler and study the geometry

Recall that, as we reviewed in Chapter 2 for the case of chiral fields, the moduli space of the GLSM can be compact or non-compact, depending on the signs of the charges of the chiral fields. Rather surprisingly, this will not be the case here. In fact, we will see that the spaces obtained by this quotient are always non-compact, regardless of the signs of the charges of the $(2, 2)$ fields.

From the superspace point of view, however, this formulation is unsatisfactory for reasons to be discussed below. We will show that actually these models can be rewritten in terms of a *constrained* semichiral vector multiplet (the meaning of this will become clear below), thus resolving these issues and

[1]This chapter is based on unpublished joint work with Martin Rocek.

in addition clarifying the geometry of these spaces.

3.1 Preliminaries

As we have discussed, in full generality one can consider the gauging of linear sigma models with an action of the form

$$K=\sum_{a=1}^{d_c}\bar{\Phi}^a\Phi^a-\sum_{a'=1}^{d_t}\bar{\chi}^{a'}\chi^{a'}-\sum_{i=1}^{d_s}\left[\bar{\mathbb{X}}_L^i\mathbb{X}_L^i+\bar{\mathbb{X}}_R^i\mathbb{X}_R^i+\alpha(\bar{\mathbb{X}}_L^i\mathbb{X}_R^i+\bar{\mathbb{X}}_R^i\mathbb{X}_L^i)\right]\ . \tag{3.1.1}$$

The action is explicitly invariant under $\mathcal{N}=(2,2)$ SUSY and all these fields can be coupled to various gauge multiplets in $(2,2)$ superspace. Before discussing possible gaugings, let us make a few comments about this action. In principle, one can include terms of the form $\bar{\phi}^a\chi^{a'}+c.c.$, $\phi^a\chi^{a'}+c.c.$, or $\phi\,\mathbb{X}_L+c.c.$, etc. We have omitted these terms because they correspond to generalized Kähler transformations (*i.e.*, they integrate to zero over the full superspace measure). However, they might be important when coupling these fields to vector multiplets. In addition, there are possible terms which are not generalized Kähler transformations, namely

$$(L_{i\bar{a}}\mathbb{X}_L^i\bar{\Phi}^a+c.c.)+(R_{i\bar{a}}\mathbb{X}_R^i\bar{\Phi}^a+c.c.)+(\tilde{L}_{i\bar{a}'}\mathbb{X}_L^i\bar{\chi}^{a'}+c.c.)+(\tilde{R}_{i\bar{a}'}\mathbb{X}_R^i\chi^{a'}+c.c.)\,.$$

In full generality these terms should be included. Since we are thinking about coupling these models to vector fields, gauge invariance is the only principle determining the presence of such terms. Regarding the kinetic term for semichiral fields, terms such as $(\mathbb{X}_L^i\mathbb{X}_R^i+c.c.)$ are also allowed, but we are assuming that $\mathbb{X}_L^i$ and $\mathbb{X}_R^i$ have the same charge. As we shall review below, this is equivalent to the these fields having an opposite charge because a semichiral field of charge Q can be dualized to a semichiral field with charge $-Q$. This is in fact one of the many distinctive features about semichiral fields. One could also consider terms mixing different semichiral families, *i.e.*, $M_{ij}^{a\bar{b}}\mathbb{X}_a^i\bar{\mathbb{X}}_{\bar{b}}^i$ with M_{ij} not diagonal, but for simplicity we will assume that M_{ij} is diagonal. Regarding F-terms, it is also possible to add the usual superpotential and twisted superpotential terms. However, these cannot contain semichiral fields since this would break $\mathcal{N}=(2,2)$. The reason is simple: the SUSY variation of semichiral fields contains three superderivatives, which does not integrate to zero over a measure with only two θ's.

Let us review now possible gaugings of (3.1.1). The most general $(2,2)$ gauged linear sigma model requires chiral vector, twisted chiral vector, semichiral vector, and the large vector multiplet.

We are interested in studying the classical moduli space of these generalized GLSMs. As explained in the introduction to Chapter 2, this amounts to integrating out the vector multiplet and then setting the scalar potential $U(\phi)$ to zero. Actually, there is no need to go all the way down to components to identify the space of classical vacua. As we shall review below in Section 3.1.1, upon reduction to $(1,1)$ one can identify the moment map functions μ whose zero-level determines the classical moduli space.

Below we give a full review of the $(2,2)$ vector multiplets with their gauge transformations, couplings, and their $(1,1)$ components. This is summarized in Table 3.1.

Vector multiplet	Φ	χ	$\mathbb{X}_L, \mathbb{X}_R$	Field Strength	Prepotentials	μ
Chiral	✓	×	✓	χ	V_Φ	d
Twisted Chiral	×	✓	✓	Φ	V_χ	$\tilde{d}$
LVM	✓	✓	✓	$X_\pm$	$V_\Phi \oplus V_\chi \oplus V'$	$d_{1,2,3}$
SVM	×	×	✓	$\Phi \oplus \chi$	$V_L \oplus V_R \oplus V'$	$d_{1,2,3}$

Table 3.1: General $(2,2)$ vector multiplets, fields they can couple to, field strengths, prepotentials, and number of $(1,1)$ bosonic fields which yield moment map functions.

Chiral and twisted chiral vector multiplets

The chiral vector multiplet V is the best-known vector multiplet. The field strength is a twisted chiral superfield defined by

$$\Sigma \equiv \frac{1}{2}\{\bar{\nabla}_+, \nabla_-\}, \tag{3.1.2}$$

where $\nabla_\pm$ are $(2,2)$ gauge-covariant superderivatives and under gauge transformations

$$\delta V = -i(\Lambda - \bar{\Lambda}), \tag{3.1.3}$$

with $\bar{\mathbb{D}}_\pm \Lambda = 0$. The $(1,1)$ components are defined by

$$f = 2\,(\bar{\Sigma} + \Sigma)\big|, \qquad d = 2\,i\,(\bar{\Sigma} - \Sigma)\big|. \tag{3.1.4}$$

As shown in Table 3.1, this vector multiplet can act on semichiral fields since the gauge transformation (3.1.3) is compatible with the semichiral constraints.

Semichiral vector multiplet and Large vector multiplet

The semichiral multiplet (SVM) and the large vector multiplet (LVM) were discussed in Chapter 2. For completeness, we review the most important elements here. The semichiral multiplet is described by the three supervector fields (V_L, V_R, V'), with gauge transformations

$$\delta V_L = i(\bar{\Lambda}_L - \Lambda_L)\,, \qquad \delta V_R = i(\bar{\Lambda}_R - \Lambda_R)\,, \qquad \delta V' = (\Lambda_R + \bar{\Lambda}_R - \Lambda_L - \bar{\Lambda}_L)\,,$$

with $\bar{\mathbb{D}}_+\Lambda_L = \bar{\mathbb{D}}_-\Lambda_R = 0$. One can define two field strengths, one chiral and one twisted chiral, given by [24]

$$\mathbb{F} = i\{\bar{\nabla}_+, \bar{\nabla}_-\}\,, \qquad \tilde{\mathbb{F}} = i\{\bar{\nabla}_+, \nabla_-\}\,.$$

The $(1,1)$ components are defined by [23, 24]

$$\begin{gathered} f = -\,i\left(\mathbb{F} - \bar{\mathbb{F}} + \tilde{\mathbb{F}} - \bar{\tilde{\mathbb{F}}}\right)\Big|\,, \\ d^1 = \left(\mathbb{F} + \bar{\mathbb{F}}\right)\Big|\,, \quad d^2 = \left(\tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big|\,, \quad d^3 = i\left(\mathbb{F} - \bar{\mathbb{F}} - \tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\Big|\,. \end{gathered}$$

Thus, a quotient with the SVM leads to three moment maps, given by the field equations of each d_i. In other words, the quotient reduces the complex dimension by two because there is enough gauge symmetry to gauge away both $\mathbb{X}_L$ and $\mathbb{X}_R$.

The LVM has a similar structure [23, 24]. It is described by three supervector fields (V^ϕ, V^χ, V'), with gauge transformations

$$\delta V^\phi = i(\bar{\Lambda} - \Lambda)\,, \qquad \delta V^\chi = i(\bar{\tilde{\Lambda}} - \tilde{\Lambda})\,, \qquad \delta V' = -(\Lambda + \bar{\Lambda}) + \tilde{\Lambda} + \bar{\tilde{\Lambda}}\,,$$

with Λ chiral and $\tilde{\Lambda}$ twisted chiral. The reduction to $(1,1)$ contains three fields d_i which give rise to three moment maps. Again, there is enough gauge invariance to gauge away a chiral and a twisted chiral field and a quotient performed with the LVM reduces the complex dimension by 2.

3.1.1 Reduction of the action to $\mathcal{N}=(1,1)$ and moment maps

Here we give some details on the reduction to $(1,1)$ superspace. The $(2,2)$ superderivatives satisfy

$$\frac{1}{2}\{\nabla_\pm, \bar{\nabla}_\pm\} = i\mathcal{D}_{\pm\pm}\,, \tag{3.1.5}$$

where $\nabla_\pm$ are gauge-covariant superderivatives (other commutation relations depend on the gauge multiplet considered). To reduce to $(1,1)$ superspace, one writes

$$\nabla_\pm = \frac{1}{2}\left(\mathcal{D}_\pm - i\mathcal{Q}_\pm\right)\,, \qquad \bar{\nabla}_\pm = \frac{1}{2}\left(\mathcal{D}_\pm + i\mathcal{Q}_\pm\right)\,, \tag{3.1.6}$$

where $\mathcal{D}_\pm$ are gauge supercovariant $\mathcal{N}=(1,1)$ derivatives and $\mathcal{Q}_\pm$ are gauge-covariant generators of the additional SUSY and satisfy

$$\mathcal{D}_\pm^2 = 2\,i\,\mathcal{D}_{\pm\pm}\,, \quad \mathcal{Q}_\pm^2 = 2\,i\,\mathcal{D}_{\pm\pm}\,. \tag{3.1.7}$$

As discussed above, there are various quotients one can perform in this general setting. Generically, the action is given by

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_{gauge} + \int d^2\theta\, d^2\bar{\theta}\, K\left(\phi^a, \bar{\phi}^a; \chi^{a'}, \bar{\chi}^{a'}; \mathbb{X}_L^i, \bar{\mathbb{X}}_L^i, \mathbb{X}_R^i, \bar{\mathbb{X}}_R^i; V_I\right)\\ =& \mathcal{L}_{gauge} + \int d^2\,\theta\left[\mathcal{Q}_+\mathcal{Q}_- K\left(\Phi^a, \bar{\Phi}^a; \mathbb{X}_L^i, \bar{\mathbb{X}}_L^i, \mathbb{X}_R, \bar{\mathbb{X}}_R^i; V_I\right)\right]\Big|\,,\end{aligned}$$

where V_I collectively denotes all the gauge multiplets from Table 3.1 and K is any gauge invariant function (satisfying some mild conditions to ensure positivity of the metric), although here we will have in mind a gauged linear sigma model. In the second line, $d^2\theta = d\theta_+\, d\theta_-$ is the full $(1,1)$ superspace measure ($\theta_\pm$ are real here) and the integrand is the $(1,1)$ Lagrangian. Pushing in the $\mathcal{Q}_\pm$'s (and using the commutation relations (3.2.3)) one obtains the $(1,1)$ Lagrangian, which is always of the form

$$\mathcal{L} = \mathcal{L}_{gauge} + \int d^2\theta\left[E_{\mu\nu}(\mathcal{D}_+\phi^\mu)(\mathcal{D}_-\phi^\nu) + \sum_I d_I\,\mu^I(\phi)\right]\,, \tag{3.1.8}$$

where ϕ^μ are the $(1,1)$ components of all the fields (X_L, X_R, ϕ, χ) present in the theory and $\mu^I(\phi)$ are moment map functions and d^I components of the vector multiplet.

In the IR limit, the kinetic action for the gauge fields becomes irrelevant and the equation of motion for the d_I's becomes algebraic[2], and the submanifold is defined by the zero-level of the moment maps $\mu^I(\phi)$, which is preserved by the action of the Killing vector. As discussed above, the number of such moment maps depends on the type of gauging that was performed. The chiral and twisted chiral vector fields lead to one moment map and the quotient by the SVM and the LVM lead to three moment map equations.

We will give more details of the reduction in a particular model below.

In [44], the authors considered the conditions for a $(1,1)$ gauged linear sigma model to admit $(2,2)$ SUSY. Our models are the off-shell versions[3] with some minor differences. We have assumed that the Generalized Kähler potential is invariant under the action of the Killing vector ξ which is being gauged (and not simply invariant up to generalized Kähler transformations). Thus, there is no need for some additional terms that appear in the $(1,1)$ action in [44]. In this sense, our model is more restricted but can easily be generalized. On the other hand, we allow for the possibility of more than a single moment map function for each isometry due to the possibility of gauging with the LVM or the SVM.

Having off-shell $(2,2)$ SUSY might be an important aspect to perform localization along the lines of [46]. This will be studied elsewhere.

3.1.2 Duality between semichiral fields

Before going to a specific model, let us review the duality between semichiral fields of opposite charges. Consider relaxing the condition of semichirality on $\mathbb{X}_L$, imposing it by a (semichiral) Lagrange multiplier $\tilde{\mathbb{X}}_L$, *i.e.*,

$$K = \mathbb{X}_L^\dagger e^V \mathbb{X}_L + \mathbb{X}_R^\dagger e^V \mathbb{X}_R + \alpha \left(\mathbb{X}_L^\dagger e^V \mathbb{X}_R + \mathbb{X}_R^\dagger e^V \mathbb{X}_L \right) - \left(\mathbb{X}_L \tilde{\mathbb{X}}_L + \tilde{\mathbb{X}}_L^\dagger \mathbb{X}_L^\dagger \right) .$$

Integrating out $\mathbb{X}_L$ leads to

$$K = -\tilde{\mathbb{X}}_L^\dagger e^{-V} \tilde{\mathbb{X}}_L - (\alpha^2 - 1) \mathbb{X}_R^\dagger e^V \mathbb{X}_R + \alpha \left(\tilde{\mathbb{X}}_L \mathbb{X}_R + \mathbb{X}_R^\dagger \tilde{\mathbb{X}}_L^\dagger \right) .$$

Thus, $\tilde{\mathbb{X}}_L$ has opposite charge. One can see that this leads to a positive definite metric for $\alpha^2 > 1$. Note that there are no e^V factors in the cross terms as they are gauge invariant as they are. It is important to stress that this duality does

[2]It should be noted that the kinetic action for the chiral, twisted chiral, and semichiral vector multiplets have been constructed and are irrelevant in the IR. However, the kinetic action for the LVM is not fully understood.

[3]Some discussion of related issues can be found in [45].

not change the geometry since it is not based on an isometry. (Note that a duality like this for chiral fields is not possible because terms of the form $\Phi\tilde{\Phi}$, integrated over the full measure would vanish.)

3.2 Conifold with torsion

Here we describe a model which involves only semichiral fields, gauged by the action of the chiral vector multiplet. This quotient was mentioned briefly in [45, 47]. Here, we will analyze this model in more detail, showing that the GLSM always leads to non-compact spaces, regardless of the charge assignments of the superfields, quite unlike the usual models with only chiral superfields. Consider the linear sigma model for semichiral fields given by

$$\mathcal{L} = -\int d^2\theta\, d^2\bar{\theta} \sum_{i=1}^{2} \left[\bar{\mathbb{X}}^i_L \mathbb{X}^i_L + \bar{\mathbb{X}}^i_R \mathbb{X}^i_R + \alpha(\bar{\mathbb{X}}^i_L \mathbb{X}^i_R + \bar{\mathbb{X}}^i_R \mathbb{X}^i_L)\right] . \tag{3.2.1}$$

There is a global symmetry acting on all the fields with the same charge[4]. We can gauge this symmetry in the same way that it is done for chiral superfields, namely, introducing the vector multiplet V. In other words, we have identified a chiral symmetry, which we gauge by

$$\mathcal{L} = -\int d^2\theta\, d^2\bar{\theta} \left(\sum_{i=1,2} e^V \left[\bar{\mathbb{X}}^i_L \mathbb{X}^i_L + \bar{\mathbb{X}}^i_R \mathbb{X}^i_R + \alpha(\bar{\mathbb{X}}^i_L \mathbb{X}^i_R + \bar{\mathbb{X}}^i_R \mathbb{X}^i_L)\right] - tV\right) , \tag{3.2.2}$$

where t is an FI parameter that we will set to zero for now. As we shall show, in the low energy limit this GLSM is described by a NLSM on a non-compact manifold with torsion. To study the classical moduli space, we calculate the corresponding moment map by reducing the model to $(1,1)$ superspace, as discussed in Section 3.1.1.

3.2.1 Reduction to $(1,1)$ and Geometry

We define the $\mathcal{N} = (1,1)$ components in terms of gauge covariant derivatives. Semichiral fields contain a $(1,1)$ bosonic superfield and a $(1,1)$ fermionic aux-

[4] If we had assigned opposite charges to the fields, we would have instead terms of the form $\alpha(\mathbb{X}_L\mathbb{X}_R + c.c.)$.

iliary superfield:

$$
\begin{aligned}
X_L &= \mathbb{X}_L\big| \,, \qquad \mathcal{Q}_+\mathbb{X}_L = i\mathcal{D}_+\mathbb{X}_L \,, \qquad \mathcal{Q}_-\mathbb{X}_L\big| = \Psi_- \,, \\
X_R &= \mathbb{X}_R\big| \,, \qquad \mathcal{Q}_-\mathbb{X}_R = i\mathcal{D}_-\mathbb{X}_R \,, \qquad \mathcal{Q}_+\mathbb{X}_R\big| = \Psi_+ \,.
\end{aligned}
$$

The reduction of the vector multiplet was given in (3.1.4). From these definitions and the $(2,2)$ SUSY algebra we find

$$
\begin{aligned}
\{\mathcal{D}_+, \mathcal{D}_-\} &= f \,, \quad \{\mathcal{D}_+, \mathcal{Q}_-\} = -d \,, \\
\{\mathcal{Q}_+, \mathcal{Q}_-\} &= f \,, \quad \{\mathcal{D}_-, \mathcal{Q}_+\} = d \,.
\end{aligned} \tag{3.2.3}
$$

Using the commutation relations (3.2.3), and integrating out the auxiliary fields $\Psi_\pm$, the action reads

$$
\mathcal{L} = \int d^2\theta \left[E_{\mu\nu} (\mathcal{D}_+\phi^\mu)(\mathcal{D}_-\phi^\nu) + d\,\mu(X) \right] \,, \tag{3.2.4}
$$

with

$$
E_{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & -\frac{4}{\alpha} + 2\alpha \\ -2 & 0 & -\frac{4}{\alpha} + 2\alpha & 0 \\ 0 & -2\alpha & 0 & -2 \\ -2\alpha & 0 & -2 & 0 \end{pmatrix} \,, \tag{3.2.5}
$$

and

$$
\mu(X) = \sum_{i=1,2} \left[\bar{X}^i_L X^i_L + \bar{X}^i_R X^i_R + \alpha(\bar{X}^i_L X^i_R + \bar{X}^i_R X^i_L) \right] \,. \tag{3.2.6}
$$

It is convenient to define

$$
X^i_L = \frac{X_i}{\sqrt{\alpha - 1}} + \frac{Y_i}{\sqrt{\alpha + 1}} \,, \quad X^i_R = -\frac{X_i}{\sqrt{\alpha - 1}} + \frac{Y_i}{\sqrt{\alpha + 1}} \,. \tag{3.2.7}
$$

The classical moduli space is therefore given by

$$
\mathcal{M} = \Big\{ X_i, Y_i \in \mathbb{C} : |X_1|^2 + |X_2|^2 - |Y_1|^2 - |Y_2|^2 = 0 \Big\} / U(1) \,. \tag{3.2.8}
$$

As is well known, this corresponds to the usual description of the unresolved conifold, a six-dimensional complex space with a conical singularity at the origin (see, *e.g.*, [48]). Note that the reason one finds a non-compact manifold, despite all the $U(1)$ charges being positive, is due to the condition $\alpha^2 > 1$. (Alternatively, one can dualize, say, $\mathbb{X}_L$ to a semichiral field $\tilde{\mathbb{X}}_L$ with an opposite charge and then it is direct to see that the space is non-compact.)

3.2.2 UV metric and b-field

Rather than working in superspace, it is simpler in this case to use the usual quotient rules to determine the metric and b-field

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{B_{\theta\mu}B_{\theta\nu} - g_{\theta\mu}g_{\theta\nu}}{g_{\theta\theta}}\,, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{g_{\theta\mu}B_{\theta\nu} - B_{\theta\mu}g_{\theta\nu}}{g_{\theta\theta}}\,, \tag{3.2.9}$$

where Greek indices belong to the quotient manifold and θ is the direction parametrizing the isometry. A convenient set of coordiantes (which is adapted to solving the D-term constraint (3.2.8)) is

$$\begin{aligned} X_1 &= r\cos\frac{\theta_2}{2}\, e^{\frac{i}{4}(\psi_+ - \psi_- - 2\phi_2)}\,, \quad X_2 = r\sin\frac{\theta_2}{2}\, e^{\frac{i}{4}(\psi_+ - \psi_- + 2\phi_2)}\,, \\ Y_1 &= r\cos\frac{\theta_1}{2}\, e^{\frac{i}{4}(\psi_+ + \psi_- + 2\phi_1)}\,, \quad Y_2 = r\sin\frac{\theta_1}{2}\, e^{\frac{i}{4}(\psi_+ + \psi_- - 2\phi_1)}\,. \end{aligned}$$

The isometry acts only by shifting $\psi_+ \to \psi_+ + \beta$. From (3.2.9), we find

$$ds^2 = \frac{1}{\alpha}ds_0^2 + \frac{1}{16\alpha}(\alpha^2 - 1)A_{(1)}^2\,, \tag{3.2.10}$$

with

$$\begin{aligned} ds_0^2 = \; & dr^2 + r^2\Big[\frac{1}{8}(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2) \\ & + \frac{1}{16}\left(d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2\right)^2\Big] \end{aligned}$$

and

$$\begin{aligned} A_{(1)} = {} & \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\left(r\cos\frac{\phi+\psi}{2}(d\phi + d\psi) + 4\sin\frac{\phi+\psi}{2}dr\right) \\ & - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(r\cos\frac{\phi-\psi}{2}(d\phi - d\psi) + 4\sin\frac{\phi-\psi}{2}dr\right) \\ & - r\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(\sin\frac{\phi-\psi}{2}d\theta_1 + \sin\frac{\phi+\psi}{2}d\theta_2\right) \\ & - r\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\left(\sin\frac{\phi+\psi}{2}d\theta_1 + \sin\frac{\phi-\psi}{2}d\theta_2\right)\,, \end{aligned}$$

where $\phi \equiv \phi_1 + \phi_2$ and we relabeled $\psi_- \to \psi$. Computing $H = db$ we find:

$$H = \frac{2r^2\sqrt{\alpha^2-1}}{\alpha} \sin\theta_1 \Big[-\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\phi_1+\phi_2}{2}\right)\cos\left(\frac{\psi}{2}\right) + \cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\phi_1+\phi_2}{2}\right)\sin\left(\frac{\psi}{2}\right)\Big] d\theta_1\wedge d\phi_1 \wedge d\psi + ... \tag{3.2.11}$$

where for simplicity we have omitted additional terms.

Note that the metric does not have the cone form $ds^2 = dr^2 + r^2 ds_B^2$ for an arbitrary value of α (note nonetheless that for $r \to 0$, the cross terms $drd\theta_1$... dissapear). However, in the limit $\alpha \to 1$, the metric becomes $ds^2 \to ds_0^2$ and $b = 0$.

Recall that the topology of the conifold is $\mathbb{R}_+ \times S^2 \times S^3$ [49]. Thus, we can choose a representative 3-cycle by fixing r and θ_2, ϕ_2. Integrating over the remaining coordinates with ranges $\psi = (0, 4\pi]$, $\theta_1 \in (0,\pi)$ and $\phi_1 \in (0, 2\pi]$ gives the H-flux through the S^3. The only component of H we need to perform the integral is $H_{\theta_1\phi_1\psi}$, given in (3.2.11). Performing the integral, we find

$$\int_{S^3} H = 0\,. \tag{3.2.12}$$

Thus, although this model has a non-zero H-field, the flux is zero and the H-field is topologically trivial. It remains as an open question for the moment what is the IR geometry of this model. We will comment further on this in Section 3.4.

3.3 Relation to constrained semichiral quotient

We have shown above that a GLSM involving only semichiral fields coupled to the chiral vector multiplet realizes a non-compact generalized Kähler manifold with torsion, thus providing an example of a quotient which is not Kähler. However, from the superspace point of view the model is unsatisfactory. The reason is that in the Kähler quotient, the complexified action of the group allows one to choose a supersymmetric gauge, where a chiral field, say Φ^1, is gauged away completely. This is achieved by the gauge transformation $V \to V - \log(\bar\Phi^1\Phi^1)$ so that $\bar\Phi^1 e^V \Phi^1 \to e^V$ and terms such as $\bar\Phi^2 e^V\Phi^2 \to e^V$ transform into $\bar\Phi e^V \Phi \to e^V$, where $\Phi \equiv \Phi^2/\Phi^1$ is a chiral superfield corresponding to a projective coordinate on the manifold. This is not possible to do in the case of this quotient since a chiral gauge parameter does not contain enough degrees of freedom to gauge away a whole semichiral field.

In this section we describe an alternative point of view on this quotient which illuminates the superspace aspect of it. As we shall see below, this reformulation will also show us that these quotient manifolds can be thought of as a union of a continuous family of hyperkähler submanifolds along one (complex) dimension parameterized by a chiral superfield, making it clear that they are non-compact manifolds.

Let us go back to the GLSM

$$K = e^V \sum_{i=1,2} \left[\bar{\mathbb{X}}^i_L \mathbb{X}^i_L + \bar{\mathbb{X}}^i_R \mathbb{X}^i_R + \alpha(\bar{\mathbb{X}}^i_L \mathbb{X}^i_R + \bar{\mathbb{X}}^i_R \mathbb{X}^i_L)\right] - t\,V\,,$$

and shift $V \to V - \log \bar{\mathbb{X}}^1_L \mathbb{X}^1_L$. Defining

$$\mathbb{X}_L = \frac{\mathbb{X}^2_L}{\mathbb{X}^1_L}\,, \quad \mathbb{X}_R = \frac{\mathbb{X}^2_R}{\mathbb{X}^1_R}\,, \quad e^{\Sigma} = \frac{\mathbb{X}^1_R}{\mathbb{X}^1_L}\,, \tag{3.3.1}$$

where Σ is a complex linear superfield $\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\Sigma = 0$, we have

$$\begin{aligned} K =& e^V \left(1 + e^{\Sigma+\bar{\Sigma}} + \alpha(e^{\Sigma} + e^{\bar{\Sigma}})\right) \\ &+ e^V \left(\bar{\mathbb{X}}_L \mathbb{X}_L + e^{\Sigma+\bar{\Sigma}} \bar{\mathbb{X}}_R \mathbb{X}_R + \alpha(\bar{\mathbb{X}}_L \mathbb{X}_R e^{\Sigma} + \bar{\mathbb{X}}_R \mathbb{X}_L e^{\bar{\Sigma}})\right) - t\,V\,. \end{aligned}$$

In other words, while a similar shift of the gauge field for chiral fields amounts to choosing a gauge in which a whole chiral field is gauged away, here we have gauged away some parts of a left and right semichiral field and a complex linear superfield Σ remains. Note that Σ enters in the action as gauge fields usually do. However, unlike vector multiplets, Σ is a constrained superfield. We can relax this constraint and impose it by a chiral Lagrange multiplier Φ by

$$\begin{aligned} K =& e^V \left(1 + e^{\Sigma+\bar{\Sigma}} + \alpha(e^{\Sigma} + e^{\bar{\Sigma}})\right) \\ &+ e^V \left(\bar{\mathbb{X}}_L \mathbb{X}_L + e^{\Sigma+\bar{\Sigma}} \bar{\mathbb{X}}_R \mathbb{X}_R + \alpha(\bar{\mathbb{X}}_L \mathbb{X}_R e^{\Sigma} + \bar{\mathbb{X}}_R \mathbb{X}_L e^{\bar{\Sigma}})\right) - t\,V + i(\Phi\Sigma - \bar{\Phi}\bar{\Sigma})\,. \end{aligned}$$

In this formulation we see that we can identify Σ and V with the SVM studied in Chapter 2 by:

$$V = V_L\,, \quad \bar{\Sigma} + \Sigma = V_R - V_L\,, \quad \Sigma = i\bar{\tilde{\mathbb{V}}} - V_L = i\mathbb{V}\,. \tag{3.3.2}$$

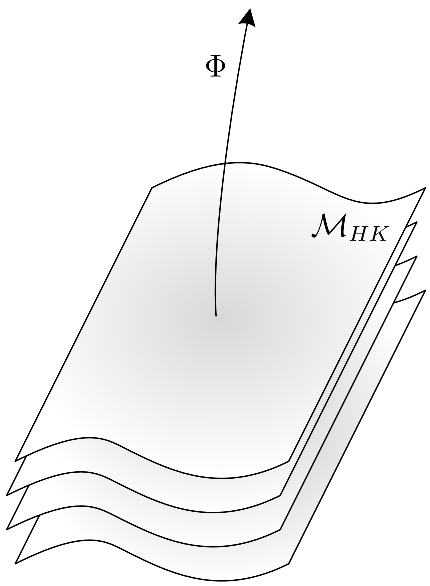

Figure 3.1: Slices of constant Φ in the quotient manifold correspond to Hyperkähler manifolds $\mathcal{M}_{HK}$ determined by a semichiral quotient.

Finally,

$$
\begin{aligned}
K = & \left(e^{V_L} + e^{V_R} + \alpha(e^{-i\tilde{\mathbb{V}}} + e^{i\bar{\tilde{\mathbb{V}}}})\right) \\
& + \left(\bar{\mathbb{X}}_L e^{V_L} \mathbb{X}_L + \bar{\mathbb{X}}_R e^{V_R} \mathbb{X}_R + \alpha(\bar{\mathbb{X}}_L e^{i\bar{\tilde{\mathbb{V}}}} \mathbb{X}_R + \bar{\mathbb{X}}_R e^{-i\tilde{\mathbb{V}}} \mathbb{X}_L)\right) \\
& - t\, V_L - (\Phi \mathbb{V} + \bar{\Phi} \bar{\mathbb{V}})\,.
\end{aligned}
$$

Thus, the chiral quotient of semichiral fields (3.2.2) can be reformulated as a semichiral quotient (where we have chosen a supersymmetric gauge in which a left and a right semichiral field have been gauged away, which is possible to do with the SVM) with the constraint that $\mathbb{V}$ is trivial (*i.e.*, $\mathbb{F} = 0$). Note that the sections of $\Phi =$ const. correspond to the (unconstrained) semichiral quotient of $\mathbb{C}^4$ and are thus hyperkähler submanifolds. In fact, they are Eguchi-Hanson manifolds, as discussed in Chapter 2 (compare to the GLSM in (2.5.1)). Thus, one can think of these manifolds as a union of hyperkähler submanifolds along the direction Φ, as shown in Figure 3.1. This also explains why these manifolds are always non-compact. The whole manifold $\mathcal{M}$ has a non-vanishing torsion and is a generalized Kähler manifold.

As a final observation, recall from Chapter 2, that one can perform a T-duality using the SVM by constraining both field strengths $\mathbb{F}$ and $\tilde{\mathbb{F}}$ to vanish by Lagrange multipliers $\Phi\mathbb{V}+\chi\tilde{\mathbb{V}}+c.c.$. Thus, this quotient can also be thought of as performing a T-duality of $\mathbb{C}^4$ (described in terms of semichiral fields) and taking the section $\chi = 0$.

3.4 Comments on quantum corrections

We have presented in this chapter the first thorough analysis of a generalized quotient that leads to a non-Kähler manifold. As a specific example, we discussed a generalized GLSM for the conifold. As we have discussed, there are many possible generalizations that need to be explored. In particular, the quotient by the LVM and the possible geometry that it leads to needs to be understood.

Our analysis thus far has been completely classical. An interesting aspect is of course quantum corrections, under which the metric renormalizes following the beta-function equations. Let us first recall the situation for chiral fields and, for the purpose of discussion, let us consider the example of the conifold. This model can be realized by a NLSM with four chiral fields Φ_a with charges $(1, 1, -1, -1)$. As is well-known, this space admits a Calabi-Yau metric [49]. However, at the classical level the quotient is not the Ricci-flat metric, but instead

$$\begin{aligned} ds^2 = \; & dr^2 + r^2 \Big[a(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2) \\ & \qquad + b\,(d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2)^2 \Big] , \end{aligned} \tag{3.4.1}$$

with $a = 1/8$ and $b = 1/16$ (which actually coincides with the metric (3.2.10) for $\alpha = 1$). The Ricci-flat metric has the same form but with $a = 1/6$ and $b = 1/9$. What is usually believed is that the metric (3.4.1) will follow the RG flow equation (which to 1-loop and for $H = 0$ is the Ricci-flow equation $\Lambda\partial_\Lambda g_{\mu\nu} = R_{\mu\nu}$) until it reaches the fixed point at the Calabi-Yau metric with $R_{\mu\nu} = 0$.

Similarly, we do not expect the quotient metric in our model to solve the beta function equations (even with a non-zero H), but rather to flow to a solution in the IR. The question thus is whether the fixed point of this flow is the same as in the torsionless case or if there is a fixed point with a non-zero H. This question remains unanswered for the moment, but we would like to make a comment. Recall that the H-flux generated in our example is topologically trivial, as implied by the value of the integral in (3.2.12). Since this integral is a topological quantity, it is invariant under RG flow. Thus, it is plausible that in fact our model flows to an IR fixed point with $H = 0$. It would be interesting to see if the LVM can lead to models with a topologically non-trival H-field, as these would necessarily flow in the IR to a non-Kähler NLSM. We plan to address these questions in future work.

In the next chapter we move to the study of the moduli space of higher-dimensional theories for which the full quantum moduli space metric can be determined exactly.

Chapter 4

Darboux Coordinates and Instanton Corrections in Projective Superspace[1]

In this chapter we move to the study of the moduli space of super Yang-Mills (SYM) theories with eight supercharges in four and five dimensions, and their hyperkähler structure.

Since classic work of Seiberg and Witten [50, 51], the structure of $\mathcal{N}=2$ theories in four dimensions has been extensively explored, leading to important insights into the dynamics of gauge theories. A recent area of progress in this field is the study of the Coulomb branch moduli space of $\mathcal{N}=2$ theories on $\mathbb{R}^3 \times S^1$, first analyzed in [52]. It has received renewed attention due its relation to the Kontsevich-Soibelman (KS) wall-crossing formula [53] for $\mathcal{N}=2$ theories in the work by Gaiotto, Moore, and Neitzke (GMN) [54]. As described by GMN, the KS formula ensures the continuity of the metric on the moduli space. Alternatively, demanding continuity of the metric provides a physical proof of the wall-crossing formula. The central idea in [54] was to find an efficient description of the moduli space metric and its corrections due to BPS instantons. Such a description was given in terms of holomorphic Darboux coordinates (η_e, η_m) by making crucial use of twistor techniques. The main goal of this chapter is to rederive these results using techniques of Projective Superspace.

As general background for the results in this chapter we begin by giving a review of basic elements of $\mathcal{N}=2$ SYM, its compactification to three dimen-

[1]This chapter is based on joint work with Dharmesh Jain in [2].

sions, and an account of the developments referred to above.

4.1 Introduction

Just as in the case of GLSMs, classical SYM has a space of inequivalent vacua determined by the vanishing of the classical potential:

$$U = \frac{1}{g^2}\mathrm{Tr}[\phi, \phi^\dagger]^2\,, \tag{4.1.1}$$

where ϕ is the scalar in the vector multiplet. For this to vanish, ϕ does not need to vanish, but just belong to the Cartan subalgebra. In this chapter we will consider only gauge group $SU(2)$ and therefore we can take $\phi = \frac{1}{2}\,a\,\sigma^3$, where a is a complex parameter labeling the vacua. Thus, the space of vacua is of complex dimension one. The space of *inequivalent* vacua is parametrized by the gauge-invariant coordinate $u = \mathrm{Tr}\phi^2$.

Note that in the case of GLSMs the moduli space was parametrized by the VEV of scalar fields in matter fields, while here it is parametrized by the VEV of the scalar field in the vector multiplet. This is usually referred to as the Coulomb branch of the moduli space. Since we are considering pure SYM there is only a Coulomb branch, but in the presence of hypermultiplets there can be a branch where these acquire a VEV, which is known as the Higgs branch.

The metric on the Coulomb branch is determined by the kinetic term of ϕ, which at the classical level is simply the flat-space metric. However, at the quantum level the action receives corrections, modifying the effective metric on the moduli space. Thus, at the quantum level the metric on the moduli space can be very different. In fact, it was shown by Seiberg and Witten that quantum corrections greatly modify the geometry and singularity structure of this moduli space, with important physical consequences. To determine the quantum-corrected metric one must study the low-energy effective action for the light fields, which in this case is a single vector multiplet W ($SU(2)$ has been spontaneously broken to $U(1)$). The only constraint on the low energy effective action for W is $\mathcal{N} = 2$ SUSY. In $\mathcal{N} = 2$ superspace the most general (two-derivative) action for the vector multiplet is given by

$$\frac{1}{16\pi}\int d^4x d^2\theta d^2\tilde{\theta}\mathcal{F}(W)\,, \tag{4.1.2}$$

where $\mathcal{F}$ is a holomorphic function called the prepotential. For an abelian theory, integrating over the Grassmann coordinates leads to the following bosonic terms:

$$S = \int d^4x \left(-\frac{\operatorname{Im}\mathcal{F}''}{4\pi} \left(F_{\mu\nu}F^{\mu\nu} + |\partial_\mu \phi|^2 \right) + \frac{\operatorname{Re}\mathcal{F}''}{4\pi} F_{\mu\nu}\tilde{F}^{\mu\nu} \right) , \tag{4.1.3}$$

in addition, of course, to fermionic terms which we omit for the purpose of our discussion. Thus, the metric in moduli space is given by

$$ds^2 = \operatorname{Im}\tau\, |da|^2 \,, \qquad \tau = \mathcal{F}''(a) \,. \tag{4.1.4}$$

At the classical level, $\mathcal{F}(W) \sim W^2$ and the metric is simply flat-space (except at the origin, where the full $SU(2)$ symmetry is restored). At the 1-loop level, however, the prepotential receives corrections and is given by [55]:

$$\mathcal{F}(W)_{1-loop} = \frac{i}{2\pi} W^2 \log \frac{W^2}{\Lambda^2} \,, \tag{4.1.5}$$

where Λ is the dynamically generated scale. Due to the amount of SUSY, this is the only perturbative correction to the prepotential, but there are also non-perturbative corrections due to four-dimensional instantons. The remarkable achievement of Seiberg and Witten [50, 51] was to determine the quantum prepotential exactly thereby determining the quantum moduli space completely, basing their analysis on symmetry and electromagnetic duality.

As we have just seen, the prepotential contains al the information about the metric. In addition, the prepotential also determines the mass M of BPS particles with a certain electric and magnetic charge at any point in the moduli space. Recall that for BPS particles $M^2 = 2|Z|^2$, where Z is the central charge given by

$$Z = a n_e + a_D n_m \,, \qquad a_D = \partial\mathcal{F}/\partial a \,, \tag{4.1.6}$$

where n_e, n_m are the electric and magnetic charges of the particle. Thus, given a lattice vector $\gamma = (n_e, n_m)$, the SW prepotential determines the mass of the BPS particle. However, except in the simplest cases, the question of *which* BPS particles are present in the theory at a certain point in the moduli space remained open. As was first observed in two dimensions [56, 57], the spectrum of BPS particles can change discontinuously as a curve in the moduli space is crossed. This in fact played a crucial in the development of SW theory. These are known as curves of marginal stability (or simply "walls") and separate the weak-coupling from the strong-coupling regions of moduli space. Kontsevich and Soibelman have now solved this problem, proposing a wall-crossing formula (WCF) that determines the spectrum of BPS particles and their decays as walls of marginal stability are crossed.

Recently, Gaiotto, Moore, and Neitzke [54] took a different point of view on the WCF formula, by studying instead what is at first a seemingly unrelated problem; the moduli space of the theory on $\mathbb{R}^3 \times S^1$. The critical observation is that the compactified theory receives contributions from all the existing BPS particles running around the S^1. Thus, the full moduli space metric g of the compactified theory must contain information about the BPS spectrum of the parent four-dimensional theory; perhaps this information can be "retrieved" from it. However, if this is the case, it raises a puzzle: while the BPS spectrum is discontinuous, the metric in moduli space is expected to be smooth. Before going into the resolution of this puzzle, let us review some basic elements about the moduli space of the compactified theory. As explained in the main text in Section 4.4, the moduli space is four-dimensional rather than two-dimensional. This space is parametrized by the complex coordinate a (corresponding to the complex scalar ϕ) and the real coordinates (θ_e, θ_m) (corresponding to the fourth component of the photon and the three-dimensional dual photon, respectively). Furthermore, due to the amount of SUSY this space must be hyperkähler [20].

The metric on this moduli space is as usual determined from the effective action for these fields. The leading behavior of the action at large R is determined by dimensionally reducing the four-dimensional action, assuming that fields are independent of the x^4-coordinate. From (4.1.3) one has

$$S = \int d^3x \left(-\frac{R}{2} \operatorname{Im}\tau \left(F_{ij}F^{ij} + |\partial_i \phi|^2 + \frac{1}{4\pi^2 R^2}(\partial_i \theta_e)^2 \right) + \frac{\operatorname{Re}\tau}{2\pi} \epsilon^{ijk} F_{ij} \partial_k \theta_e \right) .$$

As mentioned above, in three dimensions the photon can be dualized to a scalar field θ_m. Following standard methods for dualizing a vector field in three dimensions to a scalar $\theta_m = 2\pi R \tilde{A}_4$, the action reads (see [54] for details)

$$S = -\int d^3x \left(\frac{R}{2} \operatorname{Im}\tau |\partial_i \phi|^2 + \frac{1}{8\pi^2 R} \frac{1}{\operatorname{Im}\tau} |\partial_i \theta_m - \tau \partial_i \theta_e|^2 \right) , \tag{4.1.7}$$

which is in fact a three-dimensional NLSM with a four-dimensional target space with the metric

$$(ds^2)_{\text{sf}} = R \operatorname{Im}\tau \, |da|^2 + \frac{1}{4\pi^2 R} \frac{1}{\operatorname{Im}\tau} |d\theta_m - \tau d\theta_e|^2 . \tag{4.1.8}$$

This is known as the semiflat metric and will play an important role in this chapter. This, however is not the whole story. As discussed above, this metric receives corrections from BPS particles of the four-dimensional theory wrapping the S^1. Determining the full metric from first principles is a difficult problem and is not known.

The strategy taken by GMN was to first study a simplified case, $\mathcal{N} = 2$

SYM with $U(1)$ gauge group, coupled to a hypermultiplet of charge q. Since the gauge group is abelian, this does not contain magnetic monopoles and the three-dimensional theory does not contain instantons. Thus, the only contribution to the moduli space metric of the compactified theory comes from electric particles wrapping the S^1. This metric was studied in [58, 59] and is completely known. It can be written in the Gibbons-Hawking form:

$$ds^2 = \frac{1}{V(x)}\left(\frac{d\theta_m}{2\pi} + A(x)\right)^2 + V(x)dx^2\,, \tag{4.1.9}$$

where $F = \star dV = dA$ and

$$V = \frac{R}{4\pi}\sum_{k=-\infty}^{\infty}\left(\frac{1}{\sqrt{R^2|a|^2 + (\frac{\theta_e}{2\pi}+k)^2}} - \kappa_k\right)\,, \tag{4.1.10}$$

where κ_k is a regularization constant ensuring the convergence of the sum and $a = x^1 + ix^2, \theta_e = 2\pi R\, x^3$. By a Poisson resummation of (4.1.10), one finds

$$V = V^{sf} + V^{inst}\,, \tag{4.1.11}$$

where

$$V^{sf} = -\frac{R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right)\,, \quad V^{inst} = \frac{R}{2\pi}\sum_{n\neq 0} e^{in\theta_e}K_0(2\pi R|na|)\,, \tag{4.1.12}$$

and similarly for $A = A^{sf} + A^{inst}$. This metric is sometimes referred to as the Ooguri-Vafa or Seiberg-Shenker metric. Following the terminology of [60], we will refer to it as the periodic Taub-NUT (PTN) metric. In the absence of electric corrections, the metric reduces to the semiflat metric discussed above.

Note that the semiflat metric has two isometries, one corresponding to shifts of θ_e, and another one corresponding to shifts of θ_m. The PTN metric, however, has only an isometry in θ_m since the electric corrections have broken the isometry in θ_e. Thus, one expects (and this is indeed what happens) that magnetic corrections will break the only remaining isometry. At this point, it is not clear how to introduce corrections due to magnetic particles, but what is clear is that the metric cannot be of the Gibbons-Hawking form (4.1.9) because this metric always has an isometry in θ_m.

This is where hyperkähler geometry came into play. The essential insight of GMN was that by exploiting the hyperkähler structure of the metric (4.1.9), it is possible to identify an efficient way to incorporate the magnetic corrections. Furthermore, their proposal does so in such a way that the moduli space metric is ensured to be smooth. We will now give a brief account of this idea.

Recall that in a Kähler manifold there is symplectic form $\omega = Jg$, where J is a complex structure and g the metric. According to a theorem by Darboux, there is a set of coordinates (p, q) (known as Darboux coordinates) in which the symplectic form takes the canonical form $\omega = dp \wedge dq$. This is familiar from classical mechanics. In a Hyperkähler manifold there are three complex structures and therefore three symplectic forms. These can be combined into a single object

$$\varpi = \omega^{(2,0)} + \omega^{(1,1)}\zeta - \omega^{(0,2)}\zeta^2 \,, \tag{4.1.13}$$

where ζ is a coordinate in $\mathbb{CP}^1$ (the twistor sphere of complex structures). Being a symplectic form, according to Darboux, one can write

$$\varpi = i\zeta d\eta_e \wedge d\eta_m \,, \tag{4.1.14}$$

where η_e and η_m are Darboux coordinates for ϖ. We will refer to these coordinates as the electric and magnetic coordinates, respectively, for reasons that will become clear below.

With an ingenious use of twistor techniques, and representations of Bessel functions as contour integrals, it was shown by GMN that the Darboux coordinates for the PTN metric are given by[2]

$$\eta_e = \frac{a}{\zeta} + \theta_e - \bar{a}\zeta$$

and

$$\eta_m = \eta_m^{sf} + \frac{i}{2}\int_{l_+} \frac{d\zeta'}{2\pi i\zeta'}\frac{\zeta+\zeta'}{\zeta-\zeta'}\ln\left(1-e^{i\eta_e}\right) - \frac{i}{2}\int_{l_-} \frac{d\zeta'}{2\pi i\zeta'}\frac{\zeta+\zeta'}{\zeta-\zeta'}\ln\left(1-e^{-i\eta_e}\right), \tag{4.1.15}$$

where $l_\pm$ are certain rays in $\mathbb{CP}^1$ and η_m^{sf} will be given below. Performing these integrals, and evaluating the symplectic form, one can recover the Bessel function appearing in (4.1.12) and the η^{sf} part leads to the V^{sf} (see [54] for more details).

Thus far we have only described the PTN metric in a language that makes manifest its hyperkähler structure. This might seem like a convoluted form of describing a metric which was already known and we have not learned anything yet about the magnetic corrections. However, the essential insight of GMN is that from this equation one can propose an ansatz to include the magnetic corrections (namely, (4.4.2) below). This is an integral equation

[2] Here we have set the electric charge $q = 1$ and our conventions differ slightly from GMN. To go from our conventions to theirs, one must replace $a \to -i\pi Ra$ and $\eta \to -i\log\chi$.

for the Darboux coordinates whose solution determines the full metric with all the corrections due to electric as well as magnetic BPS particles. Furthermore, GMN showed that this construction leads to a smooth metric only if the WCF formula holds, ensuring that the corrections from single and multi-particle states combine in such a way that the metric is smooth. Thus, the WCF formula arises as a consistency condition ensuring that the metric constructed by this prescription is smooth.

As we have just explained, the crucial information in this approach is the expression (4.1.15) for the magnetic Darboux coordinate for the PTN metric. The derivation of this expression by GMN required some ingenious use of twistor techniques (which we did not review here and we refer the reader to [54] for details). The question we would like to address is whether one can give a simpler, systematic construction of the Darboux coordinates, not only for the PTN metric, but for any hyperkähler manifold.

This brings us to the main subject of this chapter. A natural physical context in which hyperkähler metrics and their twistor space occur is in Projective Superspace, where ζ is an additional bosonic coordinate. In this chapter we formulate the problem of describing Darboux coordinates in projective superspace. We will show that the Legendre transform construction of hyperkähler metrics (reviewed below) leads directly to the symplectic form ϖ and a consistency condition on the construction leads to a simple derivation of a general expression for the Darboux coordinates. Furthermore, we can easily derive by this method a generalization of this formula for any hyperkähler manifold described by $\mathcal{O}(2p)$ multiplets, with $p = 1$ corresponding to the case discussed above.

Before going into details, let us already give the answer. Our analysis will be based on the projective Legendre transform, which dualizes the $\mathcal{O}(2p)$ supermultiplet η_e to an "arctic" supermultiplet Υ. We will see that the magnetic coordinate η_m is the imaginary part of $\zeta^{p-1}\Upsilon$ and is given by

$$\eta_m = \frac{i}{2}\oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'}\frac{1}{\zeta-\zeta'}\left[\zeta\left(\frac{\zeta}{\zeta'}\right)^{p-1} + \zeta'\left(\frac{\zeta'}{\zeta}\right)^{p-1}\right]\frac{\partial f}{\partial \eta'_e}\,, \tag{4.1.16}$$

where f is the projective Lagrangian describing the manifold and C_0 is a contour around the origin. Such expression can also be obtained from gluing conditions for the Darboux coordinates, as done in [61–64] (see [65] for a recent review and references therein). Our derivation, however, is based on requiring the consistency of the Legendre transform by imposing the condition that Υ is regular at $\zeta = 0$. The kernel in (4.1.16) is understood as a projector ensuring this consistency condition.

In the specific case of the PTN metric we recover (4.1.15) and a natural generalization to incorporate mutually nonlocal corrections leads to the integral equation mentioned above. We will also apply this construction to the moduli space of five-dimensional SYM compactified on T^2 (considered in [66]) giving an explicit form for the Darboux coordinates including electric corrections, which we believe can serve as a starting point in understanding the full moduli space of this theory.

Let us begin by giving some necessary background on Projective Superspace and the construction of hyperkähler metrics. The reader familiar with this material may skip to Section 4.3.

4.2 Background

In this Section, we review some elements of $\mathcal{N} = 2$ projective superspace [67, 68] and the construction of hyperkähler metrics [11]. A recent review of essential aspects of the relation between projective superspace and hyperkähler manifolds can be found in [69].

4.2.1 Projective Superspace

The algebra of $d = 4$, $\mathcal{N} = 2$ supercovariant derivatives is

$$\{D_{i\alpha}, D_{j\beta}\} = 0\,, \qquad \{D_{i\alpha}, \bar{D}^j_{\dot{\beta}},\} = i\,\delta_i^j \partial_{\alpha\dot{\beta}}\,, \tag{4.2.1}$$

where $i, j = 1, 2$ are $SU(2)_R$ indices and $\alpha, \dot{\alpha}$ are spinor indices. Projective superspace is defined as the Abelian subspace parametrized by a coordinate $\zeta \in \mathbb{CP}^1$ and spanned by the combinations

$$\nabla_\alpha(\zeta) = D_{2\alpha} + \zeta D_{1\alpha}\,, \qquad \bar{\nabla}_{\dot{\alpha}}(\zeta) = \bar{D}^1_{\dot{\alpha}} - \zeta \bar{D}^2_{\dot{\alpha}}\,, \tag{4.2.2}$$

where $D_{1\alpha}$ and $\bar{D}^1_{\dot{\alpha}}$ are $\mathcal{N} = 1$ derivatives and $D_{2\alpha}$ and $\bar{D}^2_{\dot{\alpha}}$ are the generators of the extra supersymmetry. These combinations satisfy

$$\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_\alpha, \bar{\nabla}_{\dot{\beta}}\} = 0\,. \tag{4.2.3}$$

Projective superfields are then defined to satisfy the constraints:

$$\nabla_\alpha \Upsilon = \bar{\nabla}_{\dot{\alpha}} \Upsilon = 0\,. \tag{4.2.4}$$

There are several types of projective supermultiplets, characterized by their ζ-dependence. We shall mainly focus on two: real $\mathcal{O}(2p)$ and (ant)arctic supermultiplets. The first class of multiplets are polynomial in ζ, with its powers ranging from $-p$ to p, and real under the *bar* conjugation (complex conjugation composed with the antipodal map: $\zeta \to -1/\zeta$). In particular, the $\mathcal{O}(2)$ multiplet is defined by

$$\eta_e = \frac{a}{\zeta} + \theta_e - \bar{a}\zeta \,. \tag{4.2.5}$$

It follows from (4.2.4) that a and θ_e are $\mathcal{N}=1$ chiral and real linear superfields, respectively. The second class of multiplets are arctic and antarctic superfields, which are defined to be analytic around the north pole ($\zeta = 0$) and south pole ($\zeta = \infty$), respectively, *i.e.*,

$$\Upsilon = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n \,, \qquad \bar{\Upsilon} = \sum_{n=0}^{\infty} \bar{\Upsilon}_n \left(\frac{-1}{\zeta}\right)^n \,. \tag{4.2.6}$$

From (4.2.4), it follows that only the two lowest components of the arctic superfield are constrained $\mathcal{N}=1$ superfields (chiral and complex linear, respectively), while the remaining (infinite) components are auxiliary (unconstrained) complex superfields.

4.2.2 Hyperkähler Manifolds

Here we review the construction of hyperkähler metrics in projective superspace [11, 69–71]. Given an arbitrary analytic function $f(\eta_e;\zeta)$, one defines the function

$$F(a,\bar{a},\theta_e) \equiv \oint_C \frac{d\zeta}{2\pi i \zeta} f(\eta_e;\zeta) \,, \tag{4.2.7}$$

where C is an appropriately chosen contour, which typically depends on the choice of f (referred to as the projective Lagrangian henceforth). The Legendre transform of F serves as the Kähler potential K for a hyperkähler manifold, *i.e.*,

$$K(a,\bar{a},v+\bar{v}) = F(a,\bar{a},\theta_e) - (v+\bar{v})\,\theta_e \,, \qquad F_{\theta_e} = v + \bar{v} \,, \tag{4.2.8}$$

where v is an $\mathcal{N}=1$ chiral superfield. Note that Kähler metrics described in this way automatically have an isometry, associated to shifts of $\mathrm{Im}(v)$. The resulting metric is of the Gibbons-Hawking form

$$ds^2 = \frac{1}{V(x)}\left(d\theta_m + A\right)^2 + V(x) d\boldsymbol{x}\cdot d\boldsymbol{x} \,, \tag{4.2.9}$$

where $a = x^1 + ix^2$, $\theta_e = x^3$ and $dV = \star dA$, with

$$V = \oint_C \frac{d\zeta'}{2\pi i\zeta'} \frac{\partial^2 f}{\partial \eta_e'^2}\,, \qquad A = \frac{1}{2}\oint_C \frac{d\zeta'}{2\pi i\zeta'} \left(\frac{1}{\zeta'}da + \zeta' d\bar{a}\right) \frac{\partial^2 f}{\partial \eta_e'^2}\,. \qquad (4.2.10)$$

An important class of metrics are A_{N-1} ALE metrics and can be described in this way by taking

$$f(\eta_e) = \sum_k (\eta_e - \eta_k) \log(\eta_e - \eta_k)\ , \qquad (4.2.11)$$

where η_k are constant $\mathcal{O}(2)$ multiplets simply giving the position $\boldsymbol{x}_k$ of N mass points. For this Lagrangian, the contour in (4.2.7) is an 8-shaped contour $\tilde{C}$ enclosing the two roots of $\eta_e - \eta_k = 0$. Indeed, using (4.2.11) in (4.2.10) gives the harmonic function

$$V = \sum_k \oint_{\tilde{C}} \frac{d\zeta}{2\pi i\zeta} \frac{1}{\eta_e - \eta_k} = 2\sum_k \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_k|} \qquad (4.2.12)$$

and the corresponding A. Taking an infinite superposition of mass points along θ_e, *i.e.*, taking $\eta_k = k$ and $N \to \infty$, the metric becomes periodic along this direction[3]. This metric (commonly referred to as the Ooguri-Vafa metric) was discussed by Ooguri and Vafa in [58] and Seiberg and Shenker in [59]. Following the terminology of [60], we will refer to it as the periodic Taub-NUT (PTN) metric. We will refer to a PTN metric which is periodic along two directions as the doubly-periodic Taub-NUT (dPTN) metric.

As mentioned earlier, a hyperkähler manifold has three Kähler forms: $\omega^{(2,0)}$, $\omega^{(1,1)}$ and $\omega^{(0,2)}$, which can be conveniently organized into

$$\varpi = \omega^{(2,0)} + \omega^{(1,1)}\zeta - \omega^{(0,2)}\zeta^2\,. \qquad (4.2.13)$$

According to Darboux's theorem, there are certain coordinates (η_e, η_m) in which the symplectic form ϖ takes the canonical form:

$$\varpi = i\zeta\, d\eta_e \wedge d\eta_m\,. \qquad (4.2.14)$$

For Gibbons-Hawking metrics the differentials $d\eta_e$ and $d\eta_m$ are given by

$$d\eta_e = \frac{da}{\zeta} + d\theta_e - \zeta d\bar{a}\,, \qquad d\eta_m = d\theta_m + iA + \frac{iV}{2}\left(\frac{1}{\zeta}da + \zeta d\bar{a}\right)\,. \quad (4.2.15)$$

[3]Strictly speaking, V is logarithmically divergent and must be properly regularized. It should be understood that this has been done in what follows.

Note that $d\eta_e$ can easily be integrated, but this is not the case for $d\eta_m$. The main purpose of the coming sections is to find an explicit expression for η_m in terms of η_e and $f(\eta_e;\zeta)$.

4.2.3 Duality and Symplectic Form

One can alternatively describe these hyperkähler manifolds in terms of an arctic superfield Υ, rather than in terms of an $\mathcal{O}(2)$, by a duality relating these two multiplets [71, 72]. In terms of $\mathcal{N}=1$ components, this is based on the Legendre transform (4.2.8) exchanging a real linear superfield by a chiral superfield. This duality is described in terms of projective superfields as follows: One relaxes the condition of η_e being an $\mathcal{O}(2)$ multiplet, imposing this through a Lagrange multiplier $\Upsilon+\bar{\Upsilon}$. Integrating out Υ leads to the original description in terms of η_e, while integrating out η_e leads to a dual description in terms of Υ. That is, one defines

$$\tilde{f}(\Upsilon+\bar{\Upsilon};\zeta)=f(\eta_e;\zeta)-(\Upsilon+\bar{\Upsilon})\eta_e\,, \tag{4.2.16}$$

with the standard Legendre transform relations

$$\frac{\partial f}{\partial \eta_e}=\Upsilon+\bar{\Upsilon}\,,\qquad \frac{\partial \tilde{f}}{\partial \Upsilon}=-\eta_e\,. \tag{4.2.17}$$

The main advantage of this setup (for our purposes) is that one can define a holomorphic symplectic two-form that captures the essential aspects of the hyperkähler geometry [69] (see also [73] for related results). This is based on the observation that arctic superfields have infinitely many unconstrained $\mathcal{N}=1$ fields Υ_i, for $i\geq 2$, which must be integrated out. These equations of motion imply that

$$\tilde{\Upsilon}\equiv\zeta\frac{\partial\tilde{f}}{\partial\Upsilon}=-\zeta\eta_e \tag{4.2.18}$$

is also an arctic superfield. Thus, one can define a 2-form ϖ by

$$\varpi=d\Upsilon\wedge d\tilde{\Upsilon}=\omega^{(2,0)}+\omega^{(1,1)}\zeta-\omega^{(0,2)}\zeta^2\,. \tag{4.2.19}$$

In other words, Υ and $\tilde{\Upsilon}$ are (by construction) Darboux coordinates for the holomorphic symplectic form ϖ. Note that they are regular at $\zeta=0$, while $(\bar{\Upsilon},\bar{\tilde{\Upsilon}})$ are regular at $\zeta=\infty$, and

$$\varpi=-\zeta^2\overline{\varpi}=-\zeta^2 d\bar{\Upsilon}\wedge d\bar{\tilde{\Upsilon}}. \tag{4.2.20}$$

Thus, up to the twisting factor ζ^2, there is a symplectomorphism relating north pole and south pole coordinates and the generating function is precisely $\tilde{f}(\Upsilon+\bar{\Upsilon})$, giving a geometric interpretation to the $\mathcal{N}=2$ projective Lagrangian [69].

4.3 Darboux Coordinates

As seen in Section 4.2.3, the projective Legendre transform provides an expression for a set of Darboux coordinates, namely $(\Upsilon, \tilde{\Upsilon})$. The coordinate $\tilde{\Upsilon}$ is given by (4.2.18) whereas only the real part (under bar conjugation) of Υ is determined by (4.2.17), *i.e.*

$$\Upsilon = \frac{1}{2}\frac{\partial f}{\partial \eta_e} + i\eta_m \, , \tag{4.3.1}$$

where we have introduced $\eta_m = \bar{\eta}_m$ as the (undetermined) imaginary part of Υ. The crucial observation [74] is that Υ is actually completely determined by a consistency requirement on the whole construction. Recall that the constraint of η_e being an $\mathcal{O}(2)$ multiplet was imposed through a Lagrange multiplier, assuming that Υ was an arctic superfield. However, the first term on the r.h.s. of (4.3.1) contains negative powers of ζ and therefore, the consistency requirement is that these should be canceled by η_m. This η_m is precisely the magnetic coordinate we are after, since we find from (4.2.18) and (4.3.1) that

$$\varpi = d\Upsilon \wedge d\tilde{\Upsilon} = i\zeta \, d\eta_e \wedge d\eta_m \tag{4.3.2}$$

coincides with (4.2.14). To determine η_m, we introduce the antarctic projector

$$\Pi_N \equiv \oint_{C_0} \frac{d\zeta'}{2\pi i}\frac{1}{\zeta - \zeta'} \, , \qquad \Pi_N^2 = \Pi_N \, , \qquad \Pi_N \bar{\Pi}_N = 0 \, , \tag{4.3.3}$$

where C_0 is a closed contour around the origin (see Appendix C.1). This projector annihilates the non-negative powers of ζ. Thus, the consistency requirement is simply

$$\Pi_N \Upsilon = 0 \, . \tag{4.3.4}$$

It is easy to see that

$$\eta_m = \theta_m + \Big(i\Pi_N - i\bar{\Pi}_N\Big)\frac{1}{2}\frac{\partial f}{\partial \eta_e} \, , \tag{4.3.5}$$

with $\theta_m = \bar{\theta}_m$, solves the consistency condition[4]. We can rewrite (4.3.5) in a more familiar form. From (4.3.3), we see that the projectors combine into

$$i\Pi_N - i\bar{\Pi}_N = i \oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'} \frac{\zeta + \zeta'}{\zeta - \zeta'} \tag{4.3.6}$$

and hence

$$\eta_m = \theta_m + \frac{i}{2} \oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'} \frac{\zeta + \zeta'}{\zeta - \zeta'} \frac{\partial f}{\partial \eta'_e} \,, \tag{4.3.7}$$

recovering the expression obtained in [61–64]. The derivation of this expression, by ensuring and making manifest that Υ is arctic, is one of the main results of this chapter. This condition is enforced by the projector $(\zeta + \zeta')(\zeta - \zeta')^{-1}$ and will be extended below to include $\mathcal{O}(2p)$ multiplets.

We can easily check that from (4.3.7) we recover the expression (4.2.15) for Gibbons-Hawking metrics. Acting with d on η_m, we have

$$\begin{aligned} d\eta_m &= d\theta_m + \frac{i}{2} \oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'} \frac{\zeta + \zeta'}{\zeta - \zeta'} \frac{\partial^2 f}{\partial \eta'^2_e} \left(d\eta'_e - d\eta_e\right) \\ &= d\theta_m + \frac{i}{2} \oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'} \left[\left(\frac{1}{\zeta} + \frac{1}{\zeta'}\right) da + (\zeta + \zeta') d\bar{a}\right] \frac{\partial^2 f}{\partial \eta'^2_e} \\ &= d\theta_m + iA + \frac{iV}{2}\left(\frac{1}{\zeta} da + \zeta\, d\bar{a}\right) . \end{aligned} \tag{4.3.8}$$

In the first line, we have added a term proportional to $d\eta_e$, which gives no contribution to the symplectic form (4.3.2). In the last line, we have used the definitions (4.2.10), assuming that the contour giving the Kähler potential is C_0.

Although the derivation of (4.3.7) requires a contour enclosing only a singularity at the origin, note that choosing the contour to be the one defining the Kähler potential gives the correct symplectic form. This expression provides a systematic way of constructing Darboux coordinates for any hyperkähler manifold described by an $\mathcal{O}(2)$ multiplet η_e and projective Lagrangian f. We will use this in the following sections to describe instanton corrections to moduli spaces of SYM theories.

[4]Indeed, from (4.3.1), (4.3.5), and using the properties in (4.3.3), we see that

$$\Pi_N \Upsilon = \Pi_N \left(\frac{1}{2}\frac{\partial f}{\partial \eta_e} + i\left[\theta_m + \left(i\Pi_N - i\bar{\Pi}_N\right)\frac{1}{2}\frac{\partial f}{\partial \eta_e}\right]\right) = \Pi_N\left(\frac{1}{2}\frac{\partial f}{\partial \eta_e}\right) - \Pi_N\left(\frac{1}{2}\frac{\partial f}{\partial \eta_e}\right) = 0\,.$$

Semiflat Geometry and the c-map[5]

It is clear from (4.3.7) that, unlike η_e, the magnetic coordinate η_m will not be an $\mathcal{O}(2)$ in general, this depending on the singularity structure of $f(\eta_e;\zeta)$. A special case however is when the rigid c-map [75–77] (see Appendix C.2) can be applied. According to the c-map,

$$f^{sf}(\eta_e;\zeta) = -i\left(\frac{\mathcal{F}\left(\zeta\eta_e\right)}{\zeta^2} - \overline{\mathcal{F}}\left(-\frac{\eta_e}{\zeta}\right)\zeta^2\right), \tag{4.3.9}$$

where $\mathcal{F}(W)$ is the $\mathcal{N}=2$ holomorphic prepotential. The c-map gives the contribution from naïve dimensional reduction, without taking into account the effect of BPS particles. Thus, one expects η_m to be given by an $\mathcal{O}(2)$. However, by the direct substitution of (4.3.9) in (4.3.7), we see that this is not the case. This is resolved by recalling that the Darboux coordinates are defined up to terms that vanish in the symplectic form. In fact, we can add such a term to the definition of η_m that does lead to an $\mathcal{O}(2)$, namely

$$\Upsilon = \frac{1}{2}\frac{\partial f^{sf}}{\partial \eta_e} + i\left(\eta_m^{sf} - \frac{1}{2}\left(\frac{\mathcal{F}'}{\zeta} - \bar{\mathcal{F}}'\zeta\right)\right) \tag{4.3.10}$$

$$= -\frac{i\mathcal{F}'(\zeta\eta_e)}{\zeta} + i\eta_m^{sf}\,. \tag{4.3.11}$$

From the fact that $\mathcal{F}'(\zeta\eta_e) = \mathcal{F}'(a+\theta_e\zeta-\bar{a}\zeta^2)$ is regular at the origin, the condition that Υ in (4.3.11) is arctic is simply solved by

$$\eta_m^{sf} = \frac{\mathcal{F}'(a)}{\zeta} + \theta_m - \bar{\mathcal{F}}'(\bar{a})\,\zeta\,. \tag{4.3.12}$$

Therefore, naïve electric-magnetic duality $a \to a_D = \mathcal{F}'(a)$ holds. In general, dyonic multiplets have the form $\eta_\gamma^{sf} = \frac{Z_\gamma}{\zeta} + \theta_\gamma - \bar{Z}_\gamma\,\zeta$, where the central charge is $Z_\gamma = n_e a + n_m a_D$ with n_e and n_m being the electric and magnetic charges, respectively. Once BPS instanton corrections are included, the magnetic coordinate is no longer an $\mathcal{O}(2)$ since the total Lagrangian is

$$f = f^{sf} + f^{inst}\,,$$

[5] This section is based on [74].

where f^{inst} is *not* of the form (4.3.9). Thus, the full magnetic coordinate is given in general by

$$\eta_m = \eta_m^{sf} + \frac{i}{2}\oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'} \frac{\zeta+\zeta'}{\zeta-\zeta'} \frac{\partial f^{inst}}{\partial \eta'_e}\,, \tag{4.3.13}$$

where η_m^{sf} is given by (4.3.12).

Generalization to $\mathcal{O}(2p)$ Multiplets

Our construction so far includes only hyperkähler manifolds described by $\mathcal{O}(2)$ multiplets, but it can be easily extended to the case of $\mathcal{O}(2p)$ multiplets by a generalization of the Legendre transform relating Υ to an $\mathcal{O}(2p)$ multiplet η_e [72] . Additional factors of ζ have to be introduced in the Legendre transform to impose the corresponding constraint on η_e, namely

$$\tilde{f} = f - \left(\zeta^{p-1}\Upsilon + (-\zeta)^{-(p-1)}\,\bar{\Upsilon}\right)\eta_e \tag{4.3.14}$$

with the relations

$$\frac{\partial f}{\partial \eta_e} = \zeta^{p-1}\Upsilon + (-\zeta)^{-(p-1)}\,\bar{\Upsilon}\,, \qquad \tilde{\Upsilon} \equiv \zeta\frac{\partial \tilde{f}}{\partial \Upsilon} = -\zeta^p\eta_e\,. \tag{4.3.15}$$

Thus, we now have

$$\zeta^{p-1}\Upsilon = \frac{1}{2}\frac{\partial f}{\partial \eta_e} + i\eta_m\,, \tag{4.3.16}$$

and the symplectic form is still given by

$$\varpi = d\Upsilon \wedge d\tilde{\Upsilon} = i\zeta\, d\eta_e \wedge d\eta_m\,.$$

The magnetic coordinate η_m will again be determined by the requirement that the resulting superfield Υ is arctic. From (4.3.15) it follows that $\frac{\partial f}{\partial \eta_e}$ contains powers ζ^n with $|n| \geq (p-1)$ only. Thus, η_m in (4.3.16) is required to cancel the powers ζ^n with $n < -(p-1)$ of $\frac{\partial f}{\partial \eta_e}$ and we cannot add a ζ-independent term, contrary to the $\mathcal{O}(2)$ case. Using the corresponding projectors, we then find

$$\eta_m = \frac{i}{2}\oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'} \frac{1}{\zeta-\zeta'} \left[\zeta\left(\frac{\zeta}{\zeta'}\right)^{p-1} + \zeta'\left(\frac{\zeta'}{\zeta}\right)^{p-1}\right] \frac{\partial f}{\partial \eta'_e}\,. \tag{4.3.17}$$

The corresponding semiflat contribution can be determined using the c-map prescription for $\mathcal{O}(2p)$ multiplets given in [76].

A metric which is described, for example, by an $\mathcal{O}(4)$ multiplet is the Atiyah-Hitchin metric, characterizing the moduli space of two monopoles and the moduli space of three-dimensional SYM. It would be interesting to compare (4.3.17) to the Darboux coordinates given in [**?**]. In the remainder of the chapter, we will restrict ourselves to $\mathcal{O}(2)$ multiplets and apply our results to the study of moduli spaces of pure SYM theories with eight supercharges in $d = 4$ and $d = 5$.

4.4 $\mathcal{N} = 2$ SYM on $\mathbb{R}^3 \times S^1$

In this section, we apply our construction to the study of the Coulomb branch of pure $\mathcal{N} = 2$ SYM with gauge group $SU(2)$, first analyzed in [52]. The bosonic content of the four-dimensional theory consists of a complex scalar field a and a gauge field A_μ. Upon dimensional reduction on a circle S^1 of radius R (which we set to 1 in this section), the gauge field decomposes as $A_\mu \to (A_i, A_4)$, giving a three-dimensional photon and a real scalar field. Since in three dimensions the photon itself is dual to a scalar field, the moduli space of supersymmetric vacua is four-dimensional. Furthermore, due to the amount of supersymmetry it is hyperkähler. It can be parameterized by the VEV of the vector multiplet scalar field, a, in addition to the gauge-invariant electric and magnetic Wilson loops[6]

$$\theta_e \equiv \frac{1}{2\pi} \oint_{S^1_4} A_4 \,, \qquad \theta_m \equiv \frac{1}{2\pi} \oint_{S^1_4} A_{D,4} \,. \tag{4.4.1}$$

Naïve dimensional reduction of the 4D SYM action results in a 3D sigma model with a target space metric of Gibbons-Hawking form, specified by the "semiflat" potential $V^{sf} = \mathrm{Im}\tau$, where τ is the usual complexified 4D gauge coupling. However, the BPS particles from the four-dimensional theory can wrap the compactification circle S^1, generating instanton corrections to the semiflat metric in the compactified theory, which we discuss next.

4.4.1 Mutually Local Corrections

Following [54], we begin by assuming that all the BPS particles are mutually local and choose a duality frame in which there are no magnetically charged particles. This leads to a shift isometry in θ_m and therefore the space is

[6]We have normalized the angular variables $\theta_{e,m}$ to have period 1.

naturally described by the $\mathcal{O}(2)$ multiplet

$$\eta_e = \frac{a}{\zeta} + \theta_e - \bar{a}\zeta\,. \tag{4.4.2}$$

Integrating out a hypermultiplet of electric charge q (which we set to 1 here) leads to a Taub-NUT metric. Summing over the infinite tower of Kaluza-Klein momenta k along the S^1 turns it into the periodic Taub-NUT metric described in Section 4.2.2. Thus, the projective Lagrangian is given by

$$f(\eta_e) = \sum_{k=-\infty}^{\infty} (\eta_e - k)\log(\eta_e - k)\,. \tag{4.4.3}$$

Recall that here each term in the Lagrangian is to be integrated along an 8-figure contour around the roots of $\eta_e - k = 0$. To isolate instanton contributions, we perform a Poisson resummation. This is based on the indentity

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\,, \qquad \hat{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} f(x)\,.$$

Applying this to (4.4.3) we have

$$\begin{aligned} f &= f^{sf} + f^{inst}\,; \\ f^{sf} &= -i\left(\eta_e^2 \log\left(\frac{\zeta\eta_e}{\Lambda}\right) - \eta_e^2 \log\left(\frac{-\eta_e}{\zeta\bar{\Lambda}}\right)\right)\,, \qquad (4.4.4) \\ f^{inst} &= i\,s\sum_{n>0}\frac{1}{n^2}e^{in\eta_e}\theta(s) + i\,s\sum_{n<0}\frac{1}{n^2}e^{in\eta_e}\theta(-s)\,, \qquad (4.4.5) \end{aligned}$$

where Λ is the UV cutoff and $s \equiv \mathrm{sign}\,[\mathrm{Im}\,(\eta_e)]$ and we have omitted the divergent $n = 0$ term in (4.4.5).

The semiflat Lagrangian f^{sf} has been included using the c-map prescription described previously, with the 1-loop prepotential $\mathcal{F}(W) \sim W^2 \log W^2$. The full magnetic coordinate is then given by (4.3.13).

Note that since the Heaviside functions $\theta(\pm s)$ in f^{inst} contain ζ, they restrict the integration contour. Using the identity

$$\mathrm{Im}\,(\eta_e) = (1 + |\zeta|^2)\mathrm{Im}\left(\frac{a}{\zeta}\right)\,, \tag{4.4.6}$$

we see that $\theta(\pm s)$ imposes the BPS ray condition[7] $l_\pm = \left\{\zeta : \text{sign}\left[\text{Im}\left(\frac{a}{\zeta}\right)\right] = \pm 1\right\}$, leading to

$$\oint_{C_0} f^{inst}(\eta_e) = i\int_{l_+} \text{Li}_2\left(e^{i\eta_e}\right) - i\int_{l_-} \text{Li}_2\left(e^{-i\eta_e}\right) \,, \tag{4.4.7}$$

where we have used the series expansion for $\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Substituting (4.4.7) in (4.3.13) finally gives

$$\eta_m = \eta_m^{sf} + \frac{i}{2}\int_{l_+} \frac{d\zeta'}{2\pi i\zeta'}\frac{\zeta+\zeta'}{\zeta-\zeta'}\ln\left(1-e^{i\eta_e}\right) - \frac{i}{2}\int_{l_-} \frac{d\zeta'}{2\pi i\zeta'}\frac{\zeta+\zeta'}{\zeta-\zeta'}\ln\left(1-e^{-i\eta_e}\right), \tag{4.4.8}$$

where η_m^{sf} is given by (4.3.12). Thus, we have recovered GMN's result for the mutually local case. We now discuss the mutually nonlocal case.

4.4.2 Mutually Nonlocal Corrections

Inspired by the analytic and asymptotic properties of (4.4.8), an integral equation, of the form of a Thermodynamic Bethe Ansatz (TBA) equation, for the Darboux coordinates in the mutually nonlocal case was derived in [54]. The natural proposal to include dyonic multiplets is that each BPS particle of charge γ contributes independently to the projective instanton Lagrangian, with a weight given by the multiplicity of each state $\Omega(\gamma'; u)$, *i.e.*,

$$f^{inst} = i\sum_{\gamma'} \Omega(\gamma'; u)\,\text{Li}_2\left(\sigma(\gamma')e^{i\eta_{\gamma'}}\right)\,\theta(s_{\gamma'})\,. \tag{4.4.9}$$

Here $\gamma = (n_e, n_m)$ is a vector in the two-dimensional charge lattice with the antisymmetric product $\langle\gamma,\gamma'\rangle = n_e n'_m - n'_e n_m$, $\sigma(\gamma) = (-1)^{n_e n_m}$, $\eta_\gamma = n_e\eta_e + n_m\eta_m$, and $s_\gamma = \text{sign}\left[\text{Im}\left(\frac{Z_\gamma}{\zeta}\right)\right]$ that defines the BPS ray l_γ. From (4.3.13), it is natural to write the following integral equation for the dyonic coordinate

$$\eta_\gamma = \eta_\gamma^{sf} + \frac{i}{2}\sum_{\gamma'}\langle\gamma',\gamma\rangle\oint_{C_0}\frac{d\zeta'}{2\pi i\zeta'}\frac{\zeta+\zeta'}{\zeta-\zeta'}\frac{\partial f^{inst}}{\partial\eta'_{\gamma'}}\,. \tag{4.4.10}$$

[7] Our conventions in the definition of η_e differ by a factor i with those of GMN, and so does the definition of the BPS rays.

Inserting (4.4.9) above leads to

$$\eta_\gamma = \eta_\gamma^{sf} + \frac{i}{2}\sum_{\gamma'} \Omega(\gamma';u)\langle\gamma',\gamma\rangle \int_{l_{\gamma'}} \frac{d\zeta'}{2\pi i\zeta'}\frac{\zeta+\zeta'}{\zeta-\zeta'} \ln\left(1-\sigma(\gamma')e^{i\eta'_{\gamma'}}\right) , \quad (4.4.11)$$

corresponding to the TBA equation that determines the exact moduli space metric. Note that the Darboux coordinates played the central role in the analysis by GMN, being in some sense the fundamental objects. In the current setting, the fundamental object (which behaves additively and contains all the geometric information) is the projective Lagrangian f. The Darboux coordinates are determined by it through the integral equation (4.4.10).

4.5 $\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times T^2$

Minimally supersymmetric Yang-Mills in five dimensions has an interesting BPS spectrum, containing not only electrically charged particles, but also magnetically charged strings and dyonic instantons [78]. Since the theory is non-renormalizable by power-counting it should be viewed as a field theory with a cutoff. In this sense, one can still ask what are the quantum corrections to the moduli space. This was first studied in [79], where the exact Coulomb branch metric was determined. More recently, the compactification of this theory on T^2 was studied in [66], giving an important first step in analyzing the Coulomb branch metric of the compactified theory. Since dimensional reduction of this theory to four dimensions leads to the theory discussed in the previous section, compactification of the five-dimensional theory on T^2 gives a (two-parameter) generalization of the moduli space studied above.

The bosonic content of this theory consists of a real scalar σ and the gauge field $A_{\hat{\mu}}$. Upon dimensional reduction to three dimensions, the gauge field reduces according to $A_{\hat{\mu}} \to (A_i, A_4, A_5)$, leading again to a four-dimensional moduli space. The two electric coordinates φ_1, φ_2 and the "magnetic" coordinate λ are defined by

$$\varphi_1 \equiv \frac{1}{2\pi}\oint_{S^1_4} A_4\,, \qquad \varphi_2 \equiv \frac{1}{2\pi}\oint_{S^1_5} A_5\,, \qquad \lambda \equiv \int_{T^2} B\,, \qquad (4.5.1)$$

where $B_{\hat{\mu}\hat{\nu}}$ is the (2-form) dual of the photon $A_{\hat{\mu}}$. Under large gauge transformations, these variables are periodic and parameterize a torus T^2. Due to the electric particles running around these two compactified dimensions, the Coulomb branch metric inherits the modular properties of the torus and has an isometry in λ. A full analysis of the moduli space must include the effect of dyonic instantons, as well as the mutually nonlocal effect of magnetic strings wrapping the whole T^2, which will break the isometry in λ. In this chapter, we focus only on the projective description of the electric corrections to the moduli space metric, hoping that this will help in incorporating the effect of magnetic strings as well.

4.5.1 Electric Corrections

Here we apply the methods of Section 4.3 to find the corrections to η_m, due to electric particles running along the two compact directions. It is clear that the metric in this case is simply the dPTN metric. For simplicity, we discuss first the projective description of this metric in the case of a rectangular torus and then for a generic torus with complex structure τ.

Rectangular Torus

Consider a rectangular torus with radii R_1, R_2 and complex structure $\tau = i\frac{R_1}{R_2}$. We define the doubly periodic $\mathcal{O}(2)$ multiplet by

$$\eta_e = \frac{\sigma R_2 + i\varphi_2}{2R_2\zeta} + \frac{\varphi_1}{R_1} - \frac{(\sigma R_2 - i\varphi_2)}{2R_2}\zeta\,. \tag{4.5.2}$$

With this definition, the projective Lagrangian f for the dPTN metric has the form (4.2.11) with

$$\eta_k = \frac{1}{R_1}k_1 + \frac{i}{2R_2}\left(\frac{1}{\zeta} + \zeta\right)k_2 \;\equiv a_1k_1 + a_2k_2\,. \tag{4.5.3}$$

For convenience, rather than concentrating on the calculation of f, in this section we will focus on the Gibbons-Hawking potential V, given by

$$V = \sum_{\boldsymbol{k}} \oint_{\tilde{C}} \frac{d\zeta}{2\pi i\zeta}\,\frac{1}{\eta_e - a_1k_1 - a_2k_2}\,. \tag{4.5.4}$$

As before, $\tilde{C}$ is an 8−shaped contour enclosing the poles of the integrand for each $\boldsymbol{k}$, leading to a doubly periodic Gibbons-Hawking potential. This potential is linearly divergent and as in the PTN case should be understood to be properly regularized. We now perform a double Poisson resummation.

Resumming over k_1 first gives

$$V = V^{(0)} + V^{(1)}\,,$$

$$V^{(0)} = -R_1 \oint_{C_0} \frac{d\zeta}{2\pi i\zeta} \sum_{k_2} \log\left[\zeta R_1\left(\eta_e - a_2 k_2\right)\right] + c.c.\,, \tag{4.5.5}$$

$$V^{(1)} = -iR_1 \oint_{C_0} \frac{d\zeta}{2\pi i\zeta} \sum_{k_2} \sum_{n_1\neq 0} e^{in_1 R_1(\eta_e - a_2 k_2)} s\,\theta\left(n_1 s\right)\,, \tag{4.5.6}$$

where $s = \mathrm{sign}\left[\mathrm{Im}\left(\eta_e - a_2 k_2\right)\right]$. Here $V^{(0)}$ is a superposition of shifted semiflat potentials of Section 4.4. We now show that it leads to the effective gauge coupling $1/g_4(a)^2$ due to the dimensional reduction from 5D to 4D [80], and it reduces (after Poisson resummation) to the semiflat potential in the $R_2 \to 0$ limit. Performing the integral around the origin in (4.5.5) gives

$$V^{(0)} = -R_1 \sum_{k_2} \log\left(\frac{\sigma R_2 + i(\varphi_2 - k_2)}{2R_2}\right) + c.c. = R_1 \sum_{n_2\neq 0} \frac{1}{|n_2|} e^{-(R_2|n_2\sigma| - in_2\varphi_2)}\,, \tag{4.5.7}$$

where we performed a Poisson resummation for the second equality. This in fact matches the result in [80] (see also [66]). In the four-dimensional limit,

$$V^{(0)} \xrightarrow{R_2\to 0} V_{4D}^{sf} = R_1\left(\log a + \log \bar{a}\right)\,, \tag{4.5.8}$$

where $a = \frac{\sigma R_2 + i\varphi_2}{2R_2}$, which coincides with the potential derived from (4.4.4). The contribution to the magnetic coordinate is given by

$$\eta_m^{(0)} = \frac{\mathcal{F}'(a)}{\zeta} + \frac{\lambda}{R_1} - \bar{\mathcal{F}}'(\bar{a})\,\zeta\,,$$

$$\mathcal{F}(a) = \frac{1}{4R_2^2}\left[\mathrm{Li}_3\left(e^{2aR_2}\right)\theta(-\sigma) + \mathrm{Li}_3\left(e^{-2aR_2}\right)\theta(\sigma)\right]\,, \tag{4.5.9}$$

where we have integrated (4.5.7) twice with respect to a to determine $\mathcal{F}(a)$.

Now we turn to $V^{(1)}$, which in the $R_2 \to 0$ limit reduces to the instanton corrections in the four-dimensional theory. The contour in (4.5.6) splits into two rays $l_\pm$, and integration along these rays ensures that the limit $R_2 \to 0$ is well defined. In fact, in this limit the sum over k_2 is localized at $k_2 = 0$, *i.e.*,

$$V^{(1)} \xrightarrow{R_2\to 0} V_{4D}^{inst} = -iR_1 \oint_{C_0} \frac{d\zeta}{2\pi i\zeta} \sum_{n_1\neq 0} e^{in_1 R_1 \eta_e} s\,\theta\left(n_1 s\right)\,, \tag{4.5.10}$$

which is the Gibbons-Hawking potential one would obtain from (4.4.5). (One should rescale $a \to R_1 a$ in the four-dimensional case for comparison.)

For finite R_2, Poisson resumming (4.5.6) leads to[8]

$$V^{(1)} = -\oint_{C_0} \frac{d\zeta}{2\pi i\zeta} \sum_{\substack{n_1 \neq 0 \\ n_2 \in \mathbb{Z}}} \frac{e^{\frac{in_1\eta_e}{a_1}}}{a_2 n_1 - a_1 n_2}\,. \tag{4.5.11}$$

Note that after the double Poisson resummation, the contour in (4.5.11) remains a closed contour, enclosing only the essential singularity at the origin (and not the simple poles). By residue integration, we find

$$V^{(1)} = R_1 R_2 \sum_{\substack{n_1 \neq 0 \\ n_2 \in \mathbb{Z}}} \frac{1}{\sqrt{n_1^2 R_1^2 + n_2^2 R_2^2}} e^{i(n_1\varphi_1 + n_2\varphi_2) - |\sigma|\sqrt{n_1^2 R_1^2 + n_2^2 R_2^2}}\,. \tag{4.5.12}$$

Combining the two contributions, we have

$$V = V^{(0)} + V^{(1)} = R_1 R_2 \sum_{\boldsymbol{n} \in \mathbb{Z}^{2\prime}} \frac{1}{\sqrt{n_1^2 R_1^2 + n_2^2 R_2^2}} e^{i(n_1\varphi_1 + n_2\varphi_2) - |\sigma|\sqrt{n_1^2 R_1^2 + n_2^2 R_2^2}}\,, \tag{4.5.13}$$

which matches the expression for U_{1-loop} in [66]. Integrating twice with respect to η_e (and dropping a possible linear term, which does not contribute to η_m), we find

$$f^{(1)} = \sum_{\substack{n_1 \neq 0 \\ n_2 \in \mathbb{Z}}} \frac{a_1^2}{n_1^2\,(n_2 a_1 - n_1 a_2)}\, e^{\frac{in_1\eta_e}{a_1}}\,. \tag{4.5.14}$$

As explained in [66], the corrections due to $f^{(1)}$ to the Coulomb branch metric should coincide with the corrections to the hypermultiplet moduli space due to D1 instantons in type IIB theory. Indeed, we find that the projective Lagrangian $f^{(1)}$ matches with that given in [81]. Now, putting all the elements together, the magnetic coordinate for the dPTN metric finally reads

$$\eta_m = \eta_m^{(0)} + \frac{i}{2} \oint_{C_0} \frac{d\zeta'}{2\pi i\zeta'} \frac{\zeta + \zeta'}{\zeta - \zeta'} \frac{\partial f^{(1)}}{\partial \eta_e'}\,. \tag{4.5.15}$$

[8] Here we have dropped a term in the exponent

$$e^{in_1 R_1 \eta_e + i\,\mathrm{Im}(\eta_e)\frac{2R_1 R_2 |\zeta|^2}{(1+|\zeta|^2)\mathrm{Re}(\zeta)}\left[\frac{n_2}{R_1} - \frac{in_1}{2R_2}\left(\frac{1}{\zeta} + \zeta\right)\right]}\,,$$

because we choose the contour enclosing the origin along which $\mathrm{Im}(\eta_e) = \pm\epsilon$. In the limit $\epsilon \to 0$ this term does not contribute to the integral, which becomes simply an integral around the origin.

In summary, the magnetic coordinate contains two parts: the $\eta_m^{(0)}$ part from the naïve 5D to 4D reduction, which becomes η_m^{sf} in the 4D limit, and the remaining part, which reduces to η_m^{inst}.

Generic Torus

To consider a generic torus with complex structure τ, we perform a modular transformation from the rectangular case. Under the $SL(2,\mathbb{Z})$ symmetry group of the torus, the complex structure $\tau = \tau_1 + i\tau_2$ and the coordinates transform as

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix} \to \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix}\,, \tag{4.5.16}$$

with $ad - bc = 1$. The electric coordinate for a generic torus then becomes[9]

$$\eta_e = \frac{1}{2\zeta}\left(\sigma + i\sqrt{\frac{\tau_2}{\mathcal{V}}}\varphi_2\right) + \frac{\varphi_1 + \tau_1\varphi_2}{\sqrt{\mathcal{V}\tau_2}} - \frac{1}{2}\left(\sigma - i\sqrt{\frac{\tau_2}{\mathcal{V}}}\varphi_2\right)\zeta\,. \tag{4.5.17}$$

Here we also rescaled the φ_i's by the volume $\mathcal{V}$ of the torus. The Gibbons-Hawking potential is now given by (4.5.4) with

$$a_1 = \frac{1}{\sqrt{\mathcal{V}\,\tau_2}}\,, \qquad a_2 = \frac{1}{\sqrt{\mathcal{V}\tau_2}}\left[\tau_1 + i\tau_2\frac{1}{2}\left(\frac{1}{\zeta} + \zeta\right)\right]\,. \tag{4.5.18}$$

Upon Poisson resummation and contour integration, we find

$$V = \sqrt{\det g_{ij}} \sum_{\boldsymbol{n}\in\mathbb{Z}^{2\prime}} \frac{1}{\sqrt{n^i n^j g_{ij}}} e^{in^i\varphi_i - |\sigma|\sqrt{n^i n^j g_{ij}}}\,, \tag{4.5.19}$$

with the metric g on the torus given by

$$g_{ij} = \frac{\mathcal{V}}{\tau_2}\begin{pmatrix} 1 & -\tau_1 \\ -\tau_1 & |\tau|^2 \end{pmatrix}\,. \tag{4.5.20}$$

Finally, the magnetic coordinate is still given by (4.5.15), with the new definitions (4.5.17), (4.5.18), and the replacement $2a \to \left(\sigma + i\sqrt{\frac{\tau_2}{\mathcal{V}}}\varphi_2\right)$ in $\eta_m^{(0)}$.

---

[9] For a generic torus with complex structure $\tau = \tau_1 + i\tau_2$, we perform the $SL(2,\mathbb{Z})$ transformation $\varphi \to M^T\varphi$, where M is given by

$$M = \frac{1}{\sqrt{\tau_2}}\begin{pmatrix} \tau_2 & \tau_1 \\ 0 & 1 \end{pmatrix}\,.$$

The metric transforms according to $g \to (M^{-1})^T g\, M^{-1}$, leading to (4.5.20).

4.5.2 Dyonic Instanton Corrections

Dyonic instantons are particle-like objects which are the uplift of four-dimensional instantons to five dimensions. Due to the Chern-Simons term

$$\frac{\kappa}{24\pi^2} A \wedge F \wedge F \,, \tag{4.5.21}$$

they become electrically charged. Their central charge is given by

$$Z_I = \kappa\sigma|n_I| + \frac{|n_I|}{g_{5,0}^2} \,, \tag{4.5.22}$$

where $g_{5,0}$ is the five-dimensional gauge coupling and n_I is the four-dimensional instanton number. Since these particles are electrically charged, they contribute corrections to the metric preserving the isometry. Hence, their effect is incorporated easily by replacing $a \to a + Z_I$ in the definition of the $\mathcal{O}(2)$ multiplet.

A more interesting contribution to the metric will come from magnetic corrections. These are now given by magnetic strings and incorporating their effect will be studied elsewhere.

4.6 Summary and Outlook

We have presented a derivation of the expression for a set of Darboux coordinates on a hyperkähler manifold parameterized by $\mathcal{O}(2p)$ projective superfields. Our derivation relies on the projective Legendre transform construction of such manifolds and can be understood as enforcing a consistency condition. The application of our results to the PTN metric leads to the expression for the magnetic coordinate derived by GMN, describing the mutually local corrections to the moduli space metric of $\mathcal{N} = 2$ SYM on $\mathbb{R}^3 \times S^1$. Mutually nonlocal corrections can also be incorporated into the projective Lagrangian, leading to the TBA equation studied by GMN.

We also applied this method to the study of electric corrections to the moduli space of five-dimensional SYM compactified on T^2, providing a projective superspace description of the metric discussed in [66] and the corresponding Darboux coordinates. There are two contributions: an $\mathcal{O}(2)$ part determined by the five-dimensional perturbative prepotential, which reduces to the semi-flat part in the 4D limit; and the corrections due to electric particles, which reduce to the instanton corrections of the 4D theory.

There are several open questions which could be addressed within this formalism. For example, it could shed new light on the three-dimensional limit of GMN (recently analyzed in [82]), corresponding to the Atiyah-Hitchin metric. Regarding the five-dimensional theory, corrections due to magnetic strings could be incorporated in a form analogous to what was done in (4.4.9) for the four-dimensional case, leading to an integral equation for the Darboux coordinates.

In addition to Darboux coordinates, another important geometrical object is the hyperholomorphic connection (see for example [83]) and it would be interesting to investigate its description using the $\Upsilon \leftrightarrow \eta$ duality[10]. Finally, it would be quite interesting if this framework could yield any information about the six-dimensional SYM theory compactified on T^3, whose exact moduli space is K3.

In the next chapter we move to the study of three-dimensional CFTs with $AdS_4 \times Y_7$ duals where Y_7 is a tri-Sasaki Einstein manifold. Using localization techniques we will be able to evaluate the partition function of the CFT and through AdS/CFT predict the volume of an infinite family of tri-Sasaki Einstein manifolds.

[10]The authors wish to thank Greg Moore for this suggestion.

Chapter 5

AdS_4/CFT_3 and Matrix Models [1]

The AdS/CFT correspondence [84–86] is one of the most remarkable developments in modern theoretical physics. It asserts that a theory of quantum gravity in $d+1$ dimensions is equivalent to a non-gravitational conformal field theory (CFT) in d dimensions. One of the reasons it is remarkable is that the strongly-coupled quantum regime of the field theory is captured by the weakly-coupled classical regime of the gravitational theory, and vice versa. The best understood case, and the earliest example, is AdS_5/CFT_4. This duality has led to important insights into the properties of strongly-coupled four-dimensional gauge theories resembling the behavior of QCD such as confinement, meson spectrum, transport coefficients, etc. See [87–91] for reviews on AdS/CFT, [92–94] for some of these applications and, *e.g.*, [95] for a recent review on applications to heavy-ion physics.

Three-dimensional conformal field theories are also of great physical interest as they describe fixed points of condensed matter systems. However, despite the impressive success of AdS/CFT in illuminating aspects of four-dimensional theories, not much was known about three-dimensional theories. This situation has changed dramatically in the last couple of years with the discovery of a large number of precise AdS_4/CFT_3 dualities.

As mentioned above, the power of the AdS/CFT correspondence relies on its strong/weak coupling character. However, it is precisely this aspect that makes AdS/CFT difficult to test, since few strongly-coupled methods are known purely from the field theory side. However, in recent years there has been important progress developing tools of supersymmetric localization which allow testing certain AdS/CFT predictions.

In this chapter we use the technique of localization in three dimensions to

[1] This chapter is based on joint work with Dharmesh Jain and Chris Herzog [3].

study a certain (infinite) class of three-dimensional CFTs with AdS_4 duals. We compute the free energy of these theories which, through the AdS/CFT correspondence, leads to a prediction for the volume of certain tri-Sasaki Einstein manifolds.

Although the techniques used in this chapter are new, relative to those of previous chapters, we will see recurrent characters: the role of moduli spaces and of complex geometry. In fact, as we review below, the concept of moduli space plays a crucial role, leading the way to identify the dual pairs. Also, we encounter tri-Sasaki-Einsten manifolds, which are the odd-dimensional cousins of hyperkähler manifolds.

5.1 Introduction

We now give a brief account of the developments mentioned above, encouraging the reader to consult the bibliography provided for more details. The reader familiar with these developments can skip to Section 5.2. Although our main interest are the new $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities, we will give a brief review of the more familiar $\mathrm{AdS}_5/\mathrm{CFT}_4$ dualities first.

$\mathrm{AdS}_5/\mathrm{CFT}_4$ dualities

The earliest example of AdS/CFT originated from the study of N coincident D3-branes in a smooth manifold in type IIB supergravity. It was argued by Maldacena that in the 't Hooft large N limit the gauge theory living on the world-volume of D3-branes, namely $\mathcal{N} = 4$ SYM with gauge group $SU(N)$, is dual to Type IIB strings on $AdS_5 \times S^5$. Soon after Klebanov and Witten [48] constructed a simple generalization by considering D3-branes probing a conical singularity rather than a smooth space. (In fact, they considered as an example the conifold singularity, which we briefly encountered in Chapter 3 as a GLSM.) The effect of the singularity is to replace the internal space S^5 by the less-symmetric base of the conifold and, as a consequence, the dual theory is an $\mathcal{N} = 1$ theory, rather than an $\mathcal{N} = 4$ theory. For more general singularities, the near-horizon geometry of D3-branes is of the form $AdS_5 \times Y_5$ where Y_5 is a five-dimensional Sasaki-Einstein manifold [96]. A Sasaki-Einstein manifold is a manifold that is both Sasakian and Einstein. A Riemannian manifold $(\mathcal{M}, g)$ is Sasakian if and only if its metric cone $g_c = dr^2 + r^2 g$ with $r > 0$ is Kähler. Similarly, a manifold is tri-Sasakian if and only if its metric cone is hyperkähler. The metric g is usually refered to as the base of the cone g_c. See, *e.g.*, [97] for a nice review on Sasaki-Einstein manifolds. The five-sphere

S^5 and the base $T^{1,1}$ (of topology $S^2 \times S^3$) of the conifold are two simple examples of Sasaki-Einstein manifolds. With the discovery [98] of an infinite family of five-dimensional Sasaki-Einstein metrics, a plethora of $\mathrm{AdS}_5/\mathrm{CFT}_4$ became available for studying. The field theories were identified with the IR fixed points of quiver gauge theories with gauge group $SU(N)^{2p}$.

A similar sequence of developments has been followed by the study of M2-branes, and the $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities they lead to. We describe this next.

$\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities

Although it was understood early on that the study of M2-branes would lead to realizations of $\mathrm{AdS}_4/\mathrm{CFT}_3$, the world-volume gauge theories of M2-branes were much more elusive than their four-dimensional counterparts. This has finally been elucidated in [99] (inspired by the work [100–103]). The identification of the correct gauge theory in [99] was based on the observation that the moduli space of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity coincides with the moduli space of a certain three-dimensional superconformal Chern-Simons (CS) matter theory. This theory has a $U(N) \times U(N)$ gauge group at CS levels $(-k, k)$. In the $\mathcal{N} = 2$ superspace formulation the theory is coupled to bifundamental chiral superfields (A_1, A_2) and anti-bifundamental chiral superfields (B_1, B_2). This is now commonly referred to as the ABJM theory. The field content of the theory is usually summarized by a quiver diagram, as shown in Figure 5.1.

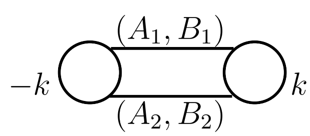

Figure 5.1: Quiver diagram for ABJM theory. Each node corresponds to a $U(N)$ gauge group at CS levels $-k$ and k. Each edge corresponds to a bifundamental chiral field and an anti-bifundamental chiral field.

The identification of the moduli spaces led the authors to conjecture that the CS matter theory describes the low energy limit of M2-branes probing the orbifold geometry. Since the near-horizon geometry of this system is $AdS_4 \times S^7/\mathbb{Z}_k$, it was conjectured that in the large N limit the superconformal CS field theory is dual to M-theory on this background.

Following the initial breakthrough of ABJM by now a plethora of $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities have been constructed [104]. Just as the case for D3-branes, these arise from considering M2-branes probing more general conical singularities, whose near-horizon geometry is $AdS_4 \times Y_7$, where Y_7 is a seven-dimensional tri-Sasaki-Einstein manifold. The dual field theories are simple extensions of ABJM corresponding to n-node quiver gauge theories with gauge groups $U(N_1) \times ... \times U(N_n)$ at CS levels $k_1, ..., k_n$ and bifundamental matter fields A_a and B_a with $a = 1, ..., n$. The moduli spaces of these theories are eight-dimensional manifolds given by toric hyperkähler quotients that are cones over Y_7, as explained in [104]. As discussed in previous chapters, these moduli spaces can be realized by GLSMs. As before, the field content of these theories is summarized by a quiver diagram that now consists of n nodes and edges connecting them according to the matter content. The ABJM theory corresponds to the simple case of a circular quiver with $n = 2$.

Having identified the dual field theories it became possible to use the gravitational properties of M2-branes to predict the strongly-coupled behavior of the field theories. A rather intriguing prediction [105] from gravitational considerations was the scaling with N of the free energy $F \sim N^{3/2}$. For an arbitrary compact space Y the gravitational free energy (in the large N limit) is given by [106, 107]

$$F = N^{3/2}\sqrt{\frac{2\pi^6}{27\,\mathrm{Vol}(Y_7)}} + o(N^{3/2})\,, \tag{5.1.1}$$

where $\mathrm{Vol}(Y_7)$ is the volume of the compact manifold Y_7. The geometry of this manifold depends on the quiver data, in particular the CS levels.

To test this prediction a non-perturbative method is needed. This is where the power of localization comes into play.

Localization

The idea of localization was introduced by Witten in [108]. It is based on the fact that under certain situations the semiclassical approximation to the path integral becomes exact. If the theory is invariant under the action of a certain operator $\mathcal{Q}$ which squares to a bosonic symmetry, the path integral receives contributions only from configurations which are annihilated by $\mathcal{Q}$. Recently, following the work of Pestun [109] there has been extensive work developing methods of localization in various dimensions [109–114]. Unlike the original localization procedure introduced by Witten, these new localization techniques are not based on a topological twist, but rather on placing the field theory on a curved space, typically the round sphere S^d (more general curved spaces can be considered; a systematic treatment is performed in [115]).

We are interested in the case of three dimensions to apply this method to the CS matter theories discussed above. Kapustin, Willett and Yaakov [110] applied localization in three dimensions to calculate the exact partition function on S^3 for theories with $\mathcal{N} \geq 2$ supersymmetry. By definition, the free energy is given by

$$F = -\log |Z_{S^3}| \,. \tag{5.1.2}$$

Thus, if the $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities described above are correct, (5.1.2) should match the gravitational expression (5.1.1). This brings us to the main subject of this chapter.

Volumes of Sasaki-Einstein manifolds

As we have mentioned above, one of the most intriguing predictions from gravitational considerations was the scaling with N of the free energy. One of the first applications [106] of localization in three dimensions was to reproduce this scaling for the ABJM theory and matching the coefficient predicted by (5.1.1). In the ABJM case, the volume of the Sasaki-Einstein manifold is simply $\mathrm{Vol}(Y_7) = \mathrm{Vol}(S^7/\mathbb{Z}^k) = \mathrm{Vol}(S^7)/k$. We wish to point out in this simple example that the volume of the Sasaki-Einstein manifold depends on the CS level.

If the field theory is described by a more complex quiver, the Sasaki-Einstein manifold is not a simple orbifold as in the case of ABJM and the computation of the volume might be more involved. In fact, in many cases these volume are not known. A simple class of theories that one can consider [107, 116–118] are circular quivers like the one in Figure 5.2.

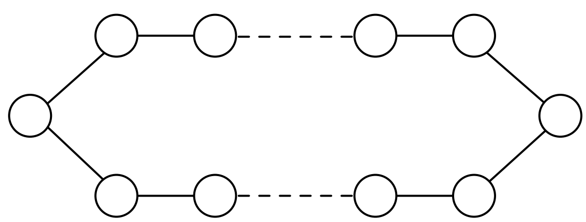

Figure 5.2: Circular quiver diagram. Each node 'a' corresponds to a $U(N_a)$ gauge group with CS level k_a and the edges to bifundamental matter.

The free energy for these quivers was computed in [107] as a function of the CS levels $k_a = q_{a+1} - q_a$ and by comparing it with the gravitational energy, it was shown that

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{(V,E)\in\mathcal{T}} \prod_{(a,b)\in E} |q_a - q_b|}{\prod_{a=1}^{p} \left[\sum_{b=1}^{p} |q_a - q_b|\right]} \,, \tag{5.1.3}$$

where the sum in the numerator is over the set $\mathcal{T}$ of all tree graphs with p nodes. This was corroborated in [119] by comparison with the explicit calculations of the volumes of toric Sasaki-Einstein manifolds [120] (see also [121] for a calculation in type-IIB supergravity).

As shown in [122], the circular quivers are actually an example of a more general class of quiver theories which have a nice large N limit, *i.e.*, long-range forces between eigenvalues in the matrix model cancel. In fact, quiver theories for which this happens are in one-to-one correspondence with the extended ADE Dynkin diagrams with circular quivers corresponding to the $\widehat{A}$-class.

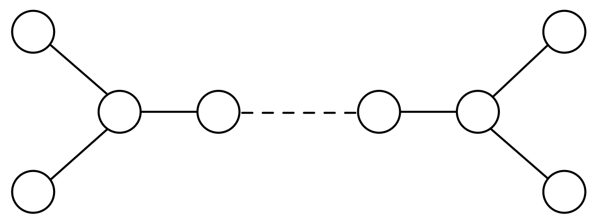

Figure 5.3: $\widehat{D}_n$ quiver diagram.

In this chapter we focus on theories with $\widehat{D}_n$ quivers, such as the one in Figure 5.3. By solving certain matrix models we compute the free energy leading us to conjecture that the volume of the Sasaki-Einstein manifolds is given by

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{\mathcal{R}_+} \det(\alpha_1 ... \alpha_n)^2 \prod_{a=1}^n |\alpha_a \cdot p|}{8(n-2)\left(\sum_{a=1}^n |p_n|\right) \prod_{a=1}^n \left[\sum_{b=1}^n \left(|p_a - p_b| + |p_a + p_b|\right) - 4|p_a|\right]}, \tag{5.1.4}$$

where $\mathcal{R}_+$ is an n-subset of positive roots α_a of D_n and the CS levels are $k_a = \alpha_a \cdot p$. Similarly to the corresponding formula (5.1.3) for circular quivers, we show that the numerator in (5.1.4) can be expressed as a sum over certain graphs known as *signed graphs*.

The relevant tri-Sasaki Einstein manifold Y is the base of the hyperkähler cone given by the $\mathbb{H}^{4n-8}///U(1)^{n-1} \times SU(2)^{n-3}$ hyperkähler quotient[2]. To the best of our knowledge, the volumes of these tri-Sasaki Einstein manifolds have not been computed independently.

Although we do not discuss exceptional quivers in full detail, we give the free energy for $\widehat{E}_6, \widehat{E}_7, \widehat{E}_8$ in Appendix D.3 for completeness.

[2]Note that the quaternionic dimension of this quotient is $(4n-8)-(n-1+3(n-3)) = 2$ as it should be.

5.2 Preliminaries

We will consider quiver Chern-Simons gauge theories involving products of unitary groups only, *i.e.*, $G = \otimes_a U(n_a N)$, coupled to bifundamental chiral superfields $(A_a,\, B_a)$. According to [110], the partition function of these theories on S^3 is localized on configurations where the auxiliary scalar fields σ_a in the $\mathcal{N}=2$ vector multiplets are constant $N \times N$ matrices. Thus, evaluating the free energy amounts to solving a matrix model.

Matrix Model

We denote the eigenvalues of σ_a in each vector multiplet by $\lambda_{a,i}$, $i = 1, ..., N_a$. The partition function is then given by [110]

$$Z = \int \left(\prod_{a,i} d\lambda_{a,i} \right) L_v(\{\lambda_{a,i}\}) L_m(\{\lambda_{a,i}\}) = \int \left(\prod_{a,i} d\lambda_{a,i} \right) \exp\left[-F(\{\lambda_{a,i}\})\right], \tag{5.2.1}$$

where the contribution from vector multiplets is

$$L_v = \prod_{a=1}^{d} \frac{1}{N_a!} \left(\prod_{i>j} 2\sinh[\pi(\lambda_{a,i} - \lambda_{a,j})] \right)^2 \exp\left(i\pi \sum_{a,j} k_a \lambda_{a,j}^2 \right),$$

and from matter multiplets is

$$L_m = \prod_{(a,b)\in E} \prod_{i,j} \frac{1}{2\cosh[\pi(\lambda_{a,i} - \lambda_{b,j})]} \prod_c \left(\prod_i \frac{1}{2\cosh[\pi \lambda_{c,i}]} \right)^{n_c^f}.$$

The first product in L_m is due to bifundamental fields while the second one is due to fundamental flavor fields, where n_c^f is the number of pairs of flavor fields at the node labeled by the index c.

Large N Limit and $\widehat{ADE}$ Classification

Following [107, 122], we assume that the eigenvalue distribution becomes dense in the large N limit, *i.e.*, $\lambda_{a,i} \to \lambda_a(x)$ with a certain density $\rho(x)$. In this limit the free energy becomes a 1-dimensional integral which we evaluate by saddle point approximation. We also assume that the eigenvalue distribution for a node with $N_a = n_a N$ is given by a collection of n_a curves in the complex plane labeled by $\lambda_{a,I}(x)$ with $I = 1, ..., n_a$ and write the ansatz

$$\lambda_{a,I}(x) = N^{\alpha} x + i\, y_{a,I}(x)\,. \tag{5.2.2}$$

The density $\rho(x)$ is assumed to be normalized, *i.e.*,

$$\int dx \rho(x) = 1\,, \tag{5.2.3}$$

which will be imposed through a Lagrange multiplier μ. As explained in [122], the leading order in N in the saddle point equation is proportional to the combination $2n_a - \sum_{b|(a,b)\in E} n_b$. The requirement that this term vanishes is equivalent to the quiver being in correspondence with simply laced extended Dynkin diagrams, leading to the ADE classification. To next order in N, the saddle point equation contains a tree-level contribution and a 1-loop contribution. Assuming that $\sum_a n_a k_a = 0$, the requirement that these two contributions are balanced leads to $\alpha = 1/2$, which is ultimately responsible for the $N^{3/2}$ scaling of the free energy[3]. Finally, the Lagrangian to be extremized reads

$$\begin{aligned} F = N^{3/2} \int dx \rho(x) \Bigg[& \pi n_F |x| + 2\pi x \sum_a \sum_{I=1}^{n_a} k_a y_{a,I}(x) \\ & + \frac{\rho(x)}{4\pi} \left(\sum_{a=1}^{d} \sum_{I=1}^{n_a} \sum_{J=1}^{n_a} \arg\left(e^{2\pi i (y_{a,I} - y_{a,J} - 1/2)}\right)^2 \right. \\ & \left. - \sum_{(a,b)\in E} \sum_{I=1}^{n_a} \sum_{J=1}^{n_b} \arg\left(e^{2\pi i (y_{a,I} - y_{b,J})}\right)^2 \right) \Bigg] - 2\pi\mu N^{3/2} \left(\int \rho(x)\, dx - 1 \right), \end{aligned} \tag{5.2.4}$$

where $n_F \equiv \sum_a n_a^f n_a$. Evaluating the free energy on-shell gives

$$F = \frac{4\pi N^{3/2}}{3} \mu\,, \tag{5.2.5}$$

which can be understood as a virial theorem [119]. Thus, the free energy is determined by μ, which in turn is determined as a function of the CS levels from the normalization condition (5.2.3). Note that from (5.1.1) and (5.2.5), it follows that

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{1}{8\,\mu^2}\,. \tag{5.2.6}$$

[3]Alternatively, one can assume that $\sum n_a k_a \neq 0$, and choose $\alpha = 1/3$, which leads to a massive IIA supergravity dual [116]. We will not consider this case here.

As mentioned earlier, theories with $\widehat{A}_{m-1}$ quiver diagrams have been extensively studied. Here, we wish to study theories with $\widehat{D}_n$ quivers as the one shown in Fig. 5.4. For now we will set $n_a^f = 0$, but we will reintroduce flavors in Section 5.5.

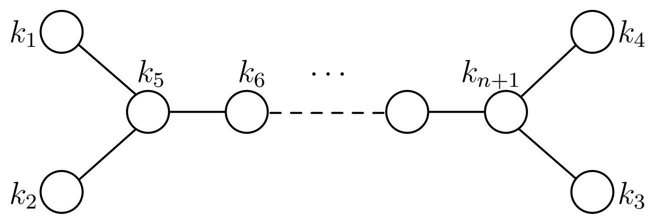

Figure 5.4: $\widehat{D}_n$ quiver diagram. Each node 'a' corresponds to a $U(n_a N)$ gauge group with CS level k_a, where n_a is the node's comark and we assume that $\sum_a n_a k_a = 0$.

It is convenient to relate the CS level $k_{(a)}$ at each node to a root α_a, by introducing a vector p and writing $k_{(a)} = \alpha_a \cdot p\,$. This way, the condition $\sum_a n_a k_a = 0$ is satisfied automatically. Choosing a basis for the roots of $\widehat{D}_n$ (see Appendix D.1 for conventions), we have

$$\begin{aligned} k_1 = -(p_1 + p_2)\,, \quad k_2 = p_1 - p_2\,, \quad k_3 = p_{n-1} - p_n\,, \quad k_4 = p_{n-1} + p_n\,, \\ k_i = p_{i-3} - p_{i-2}\,; \qquad i = 5, ..., n+1\,. \end{aligned} \tag{5.2.7}$$

In the next two sections we will solve the matrix models for various $\widehat{D}_n$ quivers and conjecture a general volume formula for arbitrary n.

5.3 Solving the Matrix Models

Here we describe the saddle point evaluation of the free energy (5.2.4). We show in detail the solution for $n = 5$, state the result for $n = 6$, and propose a general expression that we have checked for $n = 7, ..., 10$. Finally, we will relate this expression to the area of a certain polygon.

5.3.1 Explicit Solutions

Extremizing (5.2.4) (with respect to $y_{a,I}$ and ρ) requires an assumption on the branch of the arg functions. We will always take the principle value and therefore we assume that

$$|y_{a,I} - y_{a,J}| < 1\,; \qquad |y_{a,I} - y_{b,J}| < \frac{1}{2}\,, \quad \text{if } (a,b) \in E\,. \tag{5.3.1}$$

Based on numerical results [107, 122], we assume that the n_a curves for a given node initially coincide, *i.e.*, $|y_{a,I} - y_{a,J}| = 0$. Extremizing F under these assumptions, one finds that the solution is consistent only in a bounded region away from the origin. This is because as $|x|$ increases, the differences $|y_{a,I} - y_{b,J}|$ monotonically increase (or decrease), saturating one (or more) of the inequalities assumed in (5.3.1) at some point. The relation among the CS levels determines the sequence in which these inequalities saturate. This saturation will be maintained beyond this point, requiring the eigenvalue distribution involved either to bifurcate or develop a kink. As an example, consider the first plot in Fig. 5.5 where we show the eigenvalue distributions for the $\widehat{D}_5$ quiver[4]. It consists of seven regions determined by saturation of different inequalities. At the end of first region ($x = x_1$), one can see that $y_{1,1} - y_{5,2} = -1/2$ forcing $y_{5,1}$ and $y_{5,2}$ to bifurcate.

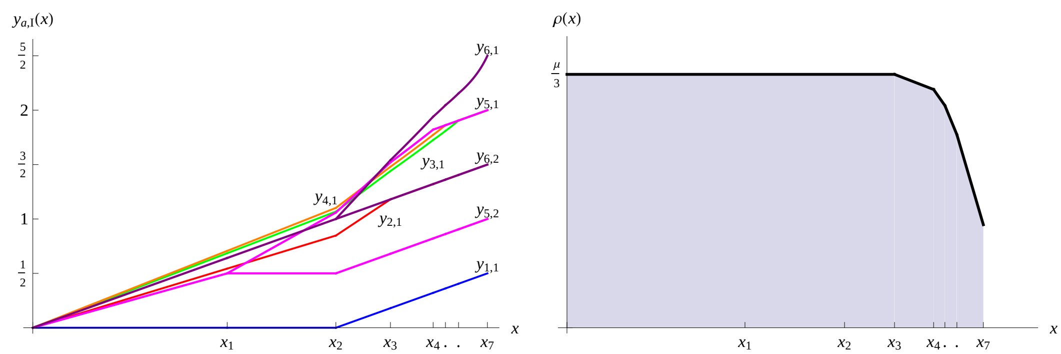

Figure 5.5: The eigenvalue distribution $y_{a,I}(x)$ (left) for all nodes and density $\rho(x)$ (right) for the $\widehat{D}_5$ quiver with CS levels: $(k_2, k_3, k_4, k_5, k_6) = (2, 2, 3, 4, 4)$.

After a saturation occurs, the total number of independent variables is reduced by one. Thus, at this point, we remove one variable from the Lagrangian, revise the inequalities and solve the equations of motion again until a new saturation is encountered. This process is iterated until all y_a's are related, determining a maximum of $(\sum_a n_a - 1)$ regions or until the eigenvalue distribution terminates, *i.e.*, $\rho(x) = 0$. Once the eigenvalue density $\rho(x)$ is determined in all regions, the value of μ (and therefore F) is found from the normalization condition (5.2.3).

The solution to the $\widehat{D}_4$ quiver consists of five regions and was solved in

[4]We have used the freedom to add an arbitrary function to the $y_{a,I}$ to set $y_{1,1}(x) = 0$ in the first region and we solve explicitly only for $x \geq 0$ since the eigenvalue distributions and density are even functions of x.

Region	x_i	$\delta y(x=x_i)$	$\rho_i(x)$
1	$\frac{\mu}{3(k_2+k_3+k_4+2k_5+2k_6)}$	$y_{1,1}-y_{5,2}=-\frac{1}{2}$	$\frac{1}{3}\mu$
2	$\frac{4\mu}{6k_2+9k_3+9k_4+12k_5+18k_6}$	$y_{5,2}-y_{6,2}=-\frac{1}{2}$	$\frac{1}{3}\mu$
3	$\frac{2\mu}{3(2k_2+k_3+k_4+2k_5+2k_6)}$	$y_{2,1}-y_{6,2}=0$	$\frac{1}{3}\mu$
4	$\frac{2\mu}{2k_2+3(k_3+k_4+2k_5+2k_6)}$	$y_{5,1}-y_{6,2}=\frac{1}{2}$	$\frac{1}{2}\mu+\frac{1}{4}x(k_1-k_2)$
5	$\frac{2\mu}{2k_2+3k_3+5k_4+4k_5+6k_6}$	$y_{4,1}-y_{6,2}=\frac{1}{2}$	$\mu+xk_1$
6	$\frac{2\mu}{2k_2+5k_3+3k_4+4k_5+6k_6}$	$y_{3,1}-y_{6,2}=\frac{1}{2}$	$\frac{3}{2}\mu+\frac{1}{4}x(6k_1-k_3-3k_4-2k_6)$
7	$\frac{2\mu}{2k_2+3k_3+3k_4+4k_5+6k_6}$	$y_{6,1}-y_{6,2}=1$	$2\mu+x(2k_1-k_3-k_4-k_6)$

Table 5.1: Key characteristics of the seven regions of the $\widehat{D}_5$ matrix model: their boundaries, the saturated inequalities and the eigenvalue densities, assuming $k_6 \geq k_5 \geq k_4 \geq k_3 \geq k_2 \geq 0$.

[122]. Assuming that $p_1 \geq p_2 \geq p_3 \geq p_4 \geq 0$, it was found that

$$\frac{1}{\mu^2} = -\frac{1}{4p_1} + \frac{2p_1+3p_2-p_3}{(p_1+p_2)^2} - \frac{1}{2(p_1+p_2+p_3-p_4)} - \frac{1}{2(p_1+p_2+p_3+p_4)} \,. \tag{5.3.2}$$

We now discuss the solution to the $\widehat{D}_5$ quiver, consisting of seven regions. We assume that $k_6 \geq k_5 \geq k_4 \geq k_3 \geq k_2 \geq 0$ with $k_1 = -(k_2+k_3+k_4+2k_5+2k_6)$ implying $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0$. The solution is summarized in Table 5.1 and Fig. 5.5 shows the eigenvalue distributions and density (further details are given in Appendix D.2). From the information given in Table 5.1 and (5.2.3), we find

$$\begin{aligned}\frac{1}{\mu^2} =& -\frac{1}{2k_2+5k_3+3k_4+4k_5+6k_6} - \frac{1}{2k_2+3k_3+5k_4+4k_5+6k_6} \\ &+ \frac{4(k_3+k_4+3k_6-2k_1)}{(2k_2+3k_3+3k_4+4k_5+6k_6)^2} \\ &- \frac{1}{9(2k_2+k_3+k_4+2k_5+2k_6)} - \frac{1}{2k_2+3k_3+3k_4+6k_5+6k_6}\,,\end{aligned}$$

which, using the relations in (5.2.7) gives

$$\begin{aligned}\frac{1}{\mu^2} =& -\frac{1}{18p_1} - \frac{1}{2(p_1+2p_2)} + \frac{(2p_1+2p_2+3p_3-p_4)}{(p_1+p_2+p_3)^2} \\ &- \frac{1}{2(p_1+p_2+p_3+p_4-p_5)} - \frac{1}{2(p_1+p_2+p_3+p_4+p_5)}\,.\end{aligned} \tag{5.3.3}$$

Similarly, solving the $\widehat{D}_6$ matrix model as described above leads to a total of nine regions and integrating the eigenvalue density gives

$$\begin{aligned}\frac{1}{\mu^2} =& -\frac{1}{48p_1} - \frac{1}{6(p_1+3p_2)} - \frac{1}{2(p_1+p_2+2p_3)} + \frac{2(p_1+p_2+p_3)+3p_4-p_5}{(p_1+p_2+p_3+p_4)^2} \\ & - \frac{1}{2(p_1+p_2+p_3+p_4+p_5-p_6)} - \frac{1}{2(p_1+p_2+p_3+p_4+p_5+p_6)}\,, \end{aligned} \tag{5.3.4}$$

for $p_1 \geq p_2 \geq ... \geq p_6 \geq 0$.

5.3.2 General Solution and Polygon Area

By comparing (5.3.2), (5.3.3) and (5.3.4), we propose that the free energy for $\widehat{D}_n$ quivers is determined by:

$$\begin{aligned}\frac{1}{\mu^2} =& \frac{1}{2}\sum_{a=1}^{n-3} \frac{c_a}{\sum_{b=1}^{a-1} p_b + (n-a-1)p_a} + \frac{2\sum_{b=1}^{n-3} p_b + 3p_{n-2} - p_{n-1}}{\left(\sum_{b=1}^{n-2} p_b\right)^2} \\ & - \frac{1}{2}\left(\frac{1}{\sum_{b=1}^{n-1} p_b - p_n} + \frac{1}{\sum_{b=1}^{n} p_b}\right)\,, \end{aligned} \tag{5.3.5}$$

with $c_a \equiv \frac{-2}{(n-a-1)(n-a-2)}$ and $p_1 \geq p_2 \geq ... \geq p_n > 0$. We have verified that this is correct for the $\widehat{D}_7, ..., \widehat{D}_{10}$ matrix models.

For $\widehat{A}$-quivers, it was shown in [119] that $\mathrm{Vol}(Y)$ can be interpreted as the area of a certain polygon. By rewriting (5.3.5) in a more suggestive form, we will show that there is a certain polygon (or rather a cone) whose area is related to $\mathrm{Vol}(Y)$ for $\widehat{D}$-quivers as well. This construction will be particularly useful in Sections 5.5 and 5.6. We start by observing that the denominators appearing in (5.3.5) can be written as

$$\begin{aligned}\bar{\sigma}_a &= \sum_{b=1}^{n} \left(|p_a - p_b| + |p_a + p_b|\right) - 4\,|p_a|\,; \quad a = 1, ..., n\,, \\ \bar{\sigma}_0 &= 2(n-2)\,, \qquad \bar{\sigma}_{n+1} = 2\sum_{b=1}^{n} |p_b|\,. \end{aligned} \tag{5.3.6}$$

The first step in rewriting (5.3.5) is to combine consecutive terms to get

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{1}{2}\left(\frac{1}{\bar{\sigma}_0\,\bar{\sigma}_1} + \sum_{a=1}^{n-1} \frac{p_a - p_{a+1}}{\bar{\sigma}_a\,\bar{\sigma}_{a+1}} + \frac{p_n}{\bar{\sigma}_n\,\bar{\sigma}_{n+1}}\right)\,, \tag{5.3.7}$$

where we have used the relation (5.2.6). The next step is to introduce the vectors $\beta_a = (1, p_a)$ together with $\beta_0 = (0, 1)$ and $\beta_{n+1} = (1, 0)$. Defining the wedge product $(a, b) \wedge (c, d) = (ad - bc)$, we can write all the $\bar{\sigma}_a$'s in (5.3.6) in terms of $\gamma_{a,b} \equiv |\beta_a \wedge \beta_b|$ as follows

$$\bar{\sigma}_a = \sum_{b=1}^{n} (\gamma_{a,b} + \gamma_{a,-b}) - 4\gamma_{a,n+1}\,; \quad a = 0, ..., n+1\,, \tag{5.3.8}$$

where we have also defined $\beta_{-a} \equiv (1, -p_a)$. This finally leads to

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{1}{2} \sum_{a=0}^{n} \frac{\gamma_{a,a+1}}{\bar{\sigma}_a\, \bar{\sigma}_{a+1}}\,. \tag{5.3.9}$$

Now, let us consider the vectors β_a, $a = 0, ..., n+1$ as defining a set of vertices v_a given by

$$v_a = v_0 + \textstyle\sum_{b=0}^{a-1} \beta_b\,,$$

where v_0 is a base point (undetermined for the moment). This set of vertices v_a in turn defines a new set of edges by the equations $v_a \wedge x = 1/2$. Then, the set of intersection points of consecutive edges, given by $w_a = \beta_a/(2\, v_a \wedge v_{a+1})$, together with the origin defines a cone $\mathcal{C}$ whose area is given by

$$\mathrm{Area}(\mathcal{C}) = \frac{1}{8} \sum_{a=0}^{n} \frac{\beta_{a+1} \wedge \beta_a}{(v_a \wedge v_{a+1})(v_{a+1} \wedge v_{a+2})}\,. \tag{5.3.10}$$

The denominators $v_a \wedge v_{a+1} = v_a \wedge (v_a + \beta_a) = v_a \wedge \beta_a = \big(v_0 + \sum_{b=0}^{a-1} \beta_b\big) \wedge \beta_a$ depend on the choice of base point v_0. Choosing $v_0 = (-n+2, -1)$, we can set $(v_a \wedge v_{a+1}) = -1/2\, \bar{\sigma}_a$ leading to

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \mathrm{Area}(\mathcal{C})\,. \tag{5.3.11}$$

We also note that by rescaling the cone $\mathcal{C}$ by a factor 2μ, we can actually interpret $\rho(x)$ as the height of the cone. In Fig. 5.6 we show the rescaled cone corresponding to the $\widehat{D}_5$ quiver. The x coordinates of the vertices of this cone correspond to the location of the kinks in $\rho(x)$ in Fig. 5.5. Thus, $1/2 = \int dx\, \rho(x) = 4\mu^2\, \mathrm{Area}(\mathcal{C})$, from where (5.3.11) follows immediately.

This construction is analogous to the polygon for the $\widehat{A}$-quiver [119]. The vectors β_a in that case correspond to the $(1, q_a)$ charges of five-branes involved in the brane description of the theory. The addition of the two extra vectors β_0 and β_{n+1} in the present case seems to suggest that one should also include

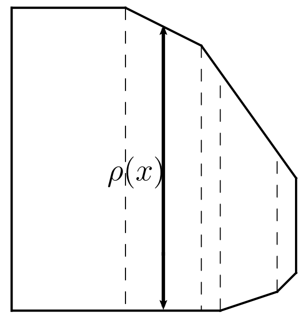

Figure 5.6: Schematic cone for the $\widehat{D}_5$-quiver. The height of the cone gives the density $\rho(x)$ in the regions defined by the x coordinates of the vertices w_a.

$(0,1)$ and $(1,0)$ branes in the description of these theories.

We would like to comment that solving the matrix model under a different ordering of the p's amounts to permuting them correspondingly in the expression (5.3.9). Moreover, regardless of the sign and ordering of p's, the denominators appearing in the expression for $\mathrm{Vol}(Y)$ are always given by the $\bar{\sigma}$'s in (5.3.8). In the next section we will propose a general expression, which is valid for any value of the CS levels and is explicitly invariant under Seiberg duality.

5.4 General Formula for $\widehat{D}_n$ Quivers

It was shown in [123, 124] that the free energy is invariant under a generalized Seiberg duality [125, 126]. For ADE quivers, Seiberg duality can be reinterpreted as the action of the Weyl group, which acts by permuting and changing the sign of an even number of p's in the case of $\widehat{D}$-quivers. Thus, we would like to have an expression for $\mathrm{Vol}(Y)$ that does not assume any particular relation among CS levels and is explicitly invariant under Seiberg duality. It was proposed in [122] that this can be written as a rational function whose numerator is given by $\sum_{\mathcal{R}_+} \det(\alpha_1 ... \alpha_n)^2 \prod_{a=1}^n |\alpha_a \cdot p|$, where $\mathcal{R}_+$ denotes all n-subsets of positive roots. Note that under Weyl transformations the $\bar{\sigma}_a$'s defined in (5.3.8) are simply shuffled among each other. Based on this, we propose that the general expression for the volume corresponding to $\widehat{D}_n$ quivers is given by

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{\mathcal{R}_+} \det(\alpha_1 ... \alpha_n)^2 \prod_{a=1}^n |\alpha_a \cdot p|}{2 \prod_{a=0}^{n+1} \bar{\sigma}_a} \,. \tag{5.4.1}$$

As we will prove below, (5.4.1) reduces to (5.3.9) when the CS levels are ordered.

We recall that in the corresponding formula for $\widehat{A}$-quivers, the numerator could be interpreted as a sum over tree graphs [107]. In a similar way, we will show now that the numerator of (5.4.1) can be interpreted as a sum over certain graphs known as *signed graphs* [127] (see also [128–130] and references therein). A graph $\Gamma = (V, E)$ consists of a set of vertices V and a set E of unordered pairs from V (the edges). A signed graph (Γ, σ) is a graph Γ with a signing $\sigma : E \to \{+1, -1\}$ associated to each edge. With these definitions, we can associate a signed graph to each term in the numerator of (5.4.1). Recall that the roots α_a for D_n are of the form $(e_i \pm e_j)$, where e_i are the canonical unit vectors of dimension n and $i \neq j$. To a root of the type $(e_i - e_j)$ we associate a positive edge ($\sigma = 1$) connecting the nodes i and j in the graph, and to a root of the type $(e_i + e_j)$ we associate a negative edge ($\sigma = -1$). Then, we think of the matrix $I = (\alpha_1 ... \alpha_n)$ as an incidence matrix for a diagram with n vertices and n edges[5]. Due to Euler's theorem, such graphs must contain loops. If the graph contains more than one loop then it must be disconnected. Loops are naturally associated a sign as well, given by the product of the signs of all the edges forming the loop. As we shall explain below, the determinant in (5.4.1) selects diagrams containing only negative loops. Some examples of diagrams contributing to the numerator for $\widehat{D}_4$ are shown in Fig. 5.7, where dashed lines represent negative edges and solid lines positive ones.

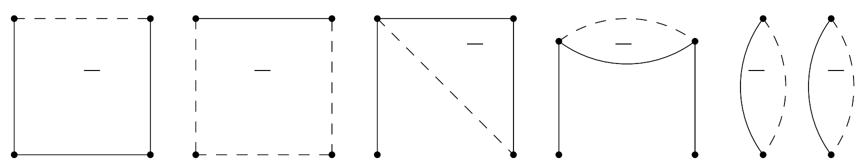

Figure 5.7: Some signed graphs contributing to the numerator for $\widehat{D}_4$. The first diagram, for example, contributes a term $4|(p_1+p_2)(p_2-p_3)(p_3-p_4)(p_4-p_1)|$.

To understand why the determinant vanishes for diagrams with positive loops, it is useful to introduce the operation acting on graphs called 'switching'. Switching is defined with respect to a vertex $v \in V$, and it acts by reversing the signs of all the edges connected to that vertex. This operation preserves the value of $(\det I)^2$ since it corresponds to multiplying some rows and columns of the incidence matrix I by -1. It is easy to see that by various switching operations one can turn any loop with an even number of negative edges into a loop made entirely of positive edges. Then, $\det I$ will vanish simply because the columns in I associated to these edges add up to zero. On the other hand,

[5]Note that due to the absence of roots of the form $2e_i$, one should not consider edges starting and ending on the same node.

if there are an odd number of negative edges in the loop, the above argument does not apply. In fact, one can easily check that $(\det I)^2 = 4$ for each negative loop. Thus, we can also write (5.4.1) as

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{(V,E,\sigma)\in\mathcal{T}^-} 4^{L_-} \prod_{(a,b)\in E} |p_a - \sigma p_b|}{2\prod_{a=0}^{n+1} \bar{\sigma}_a}\,, \tag{5.4.2}$$

where $\mathcal{T}^-$ denotes the set of signed diagrams with n vertices and n edges (connected or disconnected) and no positive loops, L_- is the number of negative loops in the diagram, and σ the sign of the corresponding edge. Using a generalized matrix-tree formula, we now show that (5.4.2) in fact reduces to (5.3.9) for $p_a > p_{a+1}$.

5.4.1 Generalized Matrix-tree Formula

We define the $n \times n$ adjacency matrix A for a signed graph by:

$$A_{aa} = \sum_{b=1}^{n-1} (\gamma_{a,b} + \gamma_{a,-b})\,, \qquad A_{ab} = -\gamma_{a,b} + \gamma_{a,-b}\,.$$

The generalized matrix-tree formula [130] states that

$$\det A = \textstyle\sum_{(V,E,\sigma)\in\mathcal{T}^-} 4^{L_-} \prod_{(a,b)\in E} |p_a - \sigma p_b|\,. \tag{5.4.3}$$

By row and column operations we can bring A into the tri-diagonal form:

$$A = \begin{pmatrix} \bar{\sigma}_1 + \bar{\sigma}_2 + 2\gamma_{12} & -\bar{\sigma}_2 & 0 & \cdots & \cdots & 0 \\ -\bar{\sigma}_2 & \bar{\sigma}_2 + \bar{\sigma}_3 + 2\gamma_{23} & -\bar{\sigma}_3 & \cdots & \cdots & \vdots \\ 0 & -\bar{\sigma}_3 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\bar{\sigma}_{n-1} & 0 \\ \vdots & \cdots & \cdots & -\bar{\sigma}_{n-1} & \bar{\sigma}_{n-1} + \bar{\sigma}_n + 2\gamma_{n-1,n} & -\bar{\sigma}_n \\ 0 & \cdots & \cdots & 0 & -\bar{\sigma}_n & \frac{1}{2}(\bar{\sigma}_n + \bar{\sigma}_{n+1}) \end{pmatrix}$$

Using the fact that the determinant of tri-diagonal matrices satisfies a recursion relation, we have

$$\det A = \frac{1}{2}(\bar{\sigma}_n + \bar{\sigma}_{n+1}) \det A_{n-1} - \bar{\sigma}_n^2 \det A_{n-2}\,, \tag{5.4.4}$$

$$\det A_a = (\bar{\sigma}_a + \bar{\sigma}_{a+1} + 2\gamma_{a,a+1}) \det A_{a-1} - \bar{\sigma}_a^2 \det A_{a-2}\,, \tag{5.4.5}$$

where A_a denotes the $a \times a$ sub-matrix of A for $a = 1, ..., n-1$. Then, using the identities: $\bar{\sigma}_{a+1} - \bar{\sigma}_a = -2(n-2-a)\gamma_{a,a+1}$ and

$$\sum_{d=0}^{a-1} \frac{(n-2-d)\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}} = \frac{1}{2\,\bar{\sigma}_a}\,,$$

we can show that the recursion relation (5.4.5) is solved by

$$\det A_{a-1} = \prod_{b=0}^{a} \bar{\sigma}_b \sum_{d=0}^{a-1} \frac{(a-d)\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}}\,. \tag{5.4.6}$$

Using (5.4.6) in (5.4.4), we have

$$\begin{aligned}
\det A &= \frac{1}{2} \prod_{b=0}^{n+1} \bar{\sigma}_b \left[\left(1 + \frac{\bar{\sigma}_n}{\sigma_{n+1}}\right) \sum_{d=0}^{n-1} \frac{(n-d)\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}} - \frac{2\bar{\sigma}_n}{\bar{\sigma}_{n+1}} \sum_{d=0}^{n-2} \frac{(n-1-d)\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}} \right] \\
&= \frac{1}{2} \prod_{b=0}^{n+1} \bar{\sigma}_b \sum_{d=0}^{n-1} \left[2\frac{\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}} + \left(\frac{\bar{\sigma}_{n+1} - \bar{\sigma}_n}{\bar{\sigma}_{n+1}} \right) \frac{(n-2-d)\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}} \right] \\
&= \prod_{b=0}^{n+1} \bar{\sigma}_b \left[\sum_{d=0}^{n-1} \frac{\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}} + \frac{1}{2} \frac{4\gamma_{n,n+1}}{\bar{\sigma}_{n+1}} \frac{1}{2\bar{\sigma}_n} \right] \\
&= \prod_{b=0}^{n+1} \bar{\sigma}_b \sum_{d=0}^{n} \frac{\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}}\,.
\end{aligned}$$

Finally, substituting (5.4.3) in (5.4.2) leads to

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\det A}{2 \prod_{b=0}^{n+1} \bar{\sigma}_b} = \frac{1}{2} \sum_{d=0}^{n} \frac{\gamma_{d,d+1}}{\bar{\sigma}_d\, \bar{\sigma}_{d+1}}\,,$$

recovering the expression (5.3.9).

5.5 Flavored $\widehat{D}_n$ Quivers and the F-theorem

The F-Theorem [116] states that the free energy (5.1.2) decreases along RG flows and is stationary at the RG fixed points of any three-dimensional field theory (supersymmetric or not). Thus, F gives a good measure of the number of degrees of freedom, in analogy with the c-function in two dimensions and the anomaly coefficient, a in four dimensions. This theorem was first tested in a variety of field theories [131–133] and recently it has been proven in [134, 135] for any three-dimensional field theory by relating F to the entanglement entropy of a disk-like region. Here we check that it holds for the the class of theories we have discussed. We trigger the RG flow by adding massive non-chiral fundamental flavors in the UV. By integrating out non-chiral flavor fields, there is no effective shift in the CS levels. Thus, we are interested in comparing $F(k_i; n_F)$ to $F(k_i; 0)$. The addition of $n_F \neq 0$ in (5.2.4) introduces no additional complications and the matrix model is solved as explained in section 5.3. We solved the flavored $\widehat{D}_n$ matrix model for $n = 4, ..., 9$ leading us to

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{1}{2}\left(\frac{\gamma_{01}}{\bar{\sigma}_0(\bar{\sigma}_1 + n_F)} + \sum_{a=1}^{n} \frac{\gamma_{a,a+1}}{(\bar{\sigma}_a + n_F)(\bar{\sigma}_{a+1} + n_F)}\right). \tag{5.5.1}$$

By comparing (5.5.1) with (5.3.9), it is clear that $F(k_i; n_F) \geq F(k_i; 0)$ verifying that

$$F_{UV} \geq F_{IR}\,,$$

in accordance with the F-theorem.

In terms of the polygon construction discussed in Section 5.3.2, adding flavor corresponds to adding the vector $\beta_F = (0, n_F/2)$ between β_0 and β_1. Then, (5.5.1) has the same form as (5.3.9) with $b = F, 1, ..., n$ in the definition (5.3.8).

5.6 Unfolding $\widehat{D}_n$ to $\widehat{A}_{2n-5}$

Here we provide a check of the formula (5.3.9), based on the folding/unfolding trick discussed in [136], which relates the free energy of various quiver gauge theories when some CS levels are identified. It can be used to change the gauge groups from unitary to orthosymplectic without changing the quiver or it can be used to change the quiver without changing the type of gauge group. Here we will deal with the latter use, as it relates the free energy of $\widehat{D}$-quivers to that of $\widehat{A}$-quivers.

When the external CS levels of a $\widehat{D}_n$ quiver are identified, it can be unfolded to an $\widehat{A}_{2n-5}$ quiver, as shown in Fig. 5.8. Each internal node in the $\widehat{D}$ quiver is duplicated to give two nodes with the same CS level, while the four external nodes combine to give two nodes with doubled CS levels. Each node in the $\widehat{A}$ quiver corresponds to a $U(2N)$ gauge group and the condition $\sum_a n_a k_a = 0$ is automatically satisfied in the unfolded quiver. Using this, it can be shown that in the large N limit, $Z_D = \sqrt{Z_A}$ and therefore the free energies are related by $F_D = \frac{1}{2} F_A$. Here we verify explicitly this proportionality by comparing the formula (5.3.9) to the corresponding formula for $\widehat{A}_{2n-5}$.

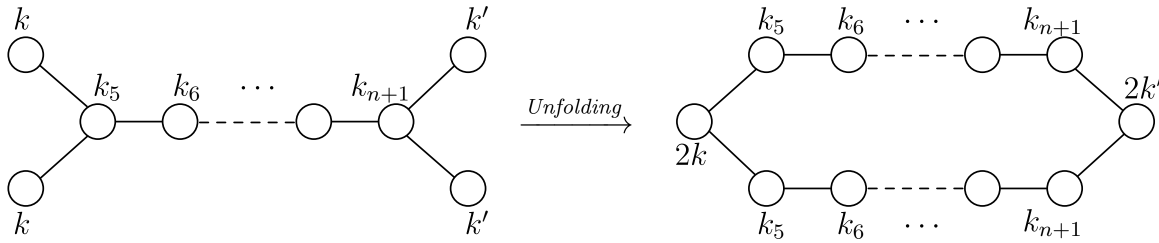

Figure 5.8: Unfolding $\widehat{D}_n$ to $\widehat{A}_{2n-5}$. Each node in the $\widehat{A}$ quiver corresponds to a $U(2N)$ gauge group.

Let us first look at the formula for the $\widehat{D}_n$ quiver when external CS levels are identified, *i.e.*, $k_1 = k_2 = k$ and $k_3 = k_4 = k'$. Due to the relations in (5.2.7), this is ensured by setting $p_1 = p_n = 0$. Thus, we need the solution to the matrix model with the ordering $p_2 \geq ... \geq p_{n-1} \geq p_n \geq p_1 \geq 0$. As mentioned at the end of Section 5.3, this is given by permuting the p's in (5.3.9) accordingly. Then, setting $p_1 = p_n = 0$ gives

$$\frac{\mathrm{Vol}(Y_D)}{\mathrm{Vol}(S^7)} = \frac{1}{2}\left(\frac{p_2}{\bar{\sigma}_2^2} + \sum_{a=2}^{n-2}\frac{\gamma_{a,a+1}}{\bar{\sigma}_a\,\bar{\sigma}_{a+1}} + \frac{p_{n-1}}{(\bar{\sigma}_{n-1})^2}\right). \tag{5.6.1}$$

Now we wish to compare this expression with the corresponding one for $\widehat{A}_{2n-5}$ [119], namely

$$\frac{\mathrm{Vol}(Y_A)}{\mathrm{Vol}(S^7)} = \frac{1}{2}\sum_{a=1}^{2n-4}\frac{\gamma_{a,a+1}}{\sigma_a\,\sigma_{a+1}}\,, \tag{5.6.2}$$

where $\sigma_a = \sum_{a=1}^{2n-4}|q_a - q_b|$, $\gamma_{a,b} = |q_a - q_b|$ and $\sum_{a=1}^{2n-4} q_a = 0$. The identification of opposite CS levels in the $\widehat{A}_{2n-4}$ quiver leads to $q_a = -q_{2n-3-a}$ (see Appendix D.1 for details). Then, we assume that $q_1 \geq ... \geq q_{n-2} \geq 0 \geq q_{n-1} \geq ... \geq q_{2n-4}$ and

$$\sigma_a = \sum_{b=1}^{n-2}|q_a - q_b| + \sum_{b=n-1}^{2n-4}|q_a - q_b| = \sum_{b=1}^{n-2}(|q_a - q_b| + |q_a + q_b|)\,; \quad a = 1, ..., n-2\,.$$

Noting that $\sigma_a = \bar{\sigma}_a$ and $q_a = p_{a+1}$ for $a = 1, ..., n-2$, we have

$$\begin{aligned}
\frac{\mathrm{Vol}(Y_A)}{\mathrm{Vol}(S^7)} &= \frac{1}{2}\left(\sum_{a=1}^{n-3}\frac{\gamma_{a,a+1}}{\sigma_a\,\sigma_{a+1}} + \frac{\gamma_{n-2,n-1}}{\sigma_{n-2}\,\sigma_{n-1}} + \sum_{a=n-1}^{2n-5}\frac{\gamma_{a,a+1}}{\sigma_a\,\sigma_{a+1}} + \frac{\gamma_{2n-4,2n-3}}{\sigma_{2n-4}\,\sigma_{2n-3}}\right) \\
&= \sum_{a=1}^{n-3}\frac{\gamma_{a,a+1}}{\bar{\sigma}_a\,\bar{\sigma}_{a+1}} + \frac{q_{n-2}}{(\bar{\sigma}_{n-2})^2} + \frac{q_1}{\bar{\sigma}_1^2} \\
&= \frac{p_2}{\bar{\sigma}_2^2} + \sum_{a=2}^{n-2}\frac{\gamma_{a,a+1}}{\bar{\sigma}_a\,\bar{\sigma}_{a+1}} + \frac{p_{n-1}}{(\bar{\sigma}_{n-1})^2}\,. \qquad (5.6.3)
\end{aligned}$$

Thus, comparing (5.6.3) to (5.6.1) we have

$$\mathrm{Vol}(Y_D) = \frac{1}{2}\,\mathrm{Vol}(Y_A)\,. \qquad (5.6.4)$$

This relation can also be seen clearly in terms of the areas of the corresponding polygons, as shown in Fig. 5.9 (the cone as defined in Section 5.3.2 has been doubled along the dotted line for visual clarity). The outer polygon corresponds to the $\widehat{A}$-quiver with opposite CS levels identified and the shaded region on the left represents the polygon corresponding to a general $\widehat{D}$-quiver; when $p_1 = p_n = 0$, this shaded region expands to fill half of the outer polygon on the right.

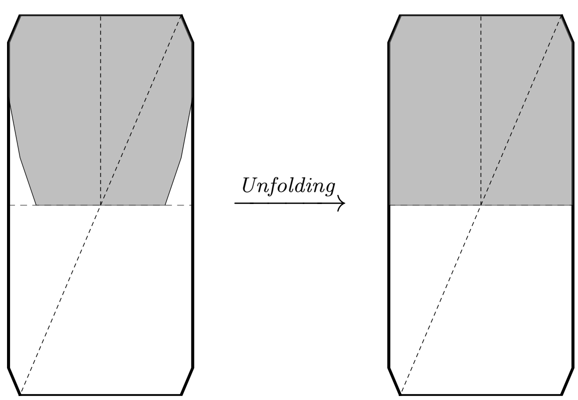

Figure 5.9: Polygons associated to the $\widehat{D}_4$ quiver (shaded region) and $\widehat{A}_3$ quiver (outer polygon). Upon unfolding, $\mathrm{Area}(\mathcal{P}_D) = 1/2\,\mathrm{Area}(\mathcal{P}_A)$.

Recalling that the nodes of the unfolded $\widehat{A}$-quiver correspond to $U(2N)$

gauge groups, we verify that

$$\frac{F_D}{F_A} = \frac{N^{3/2}}{(2N)^{3/2}} \sqrt{\frac{\mathrm{Vol}(Y_A)}{\mathrm{Vol}(Y_D)}} = \frac{1}{2}\,.$$

5.7 Discussion

In this chapter we have studied three-dimensional $\widehat{D}_n$ quiver Chern-Simons matter theories by using the localization method of Kapustin, Willet and Yaakov in the large N limit. These field theories are believed to be dual to M-theory on $AdS_4 \times Y$, where Y is a tri-Sasaki Einstein manifold. We have explicitly solved the corresponding matrix models for various values of n, leading us to conjecture a general expression for the free energy and therefore for the volume of the corresponding space Y given in (5.4.1). We have shown that the numerator of this expression can be interpreted as a sum over a class of graphs with edges that carry a sign, known as signed graphs. Using a generalized matrix-tree formula, we prove that for a particular ordering of CS levels, it can also be interpreted as the area of a certain polygon, given by (5.3.9). When external CS levels in the $\widehat{D}_n$ quiver are identified, the area of this polygon becomes half the area of the polygon corresponding to the $\widehat{A}_{2n-5}$ quiver, in accordance with the unfolding procedure. We have also studied the addition of massive flavor fields, showing that when they are integrated out, the area of the corresponding polygon always increases (thereby decreasing F), in accordance with the F-theorem.

The relevant tri-Sasaki Einstein space for a $\widehat{D}_n$ quiver is the base of the hyperkähler cone defined by the quotient $\mathbb{H}^{4n-8}///U(1)^{n-1}\times SU(2)^{n-3}$. To the best of our knowledge, the volumes of these spaces have not been computed. Thus, (5.4.1) can be considered as an AdS/CFT prediction for these volumes. A possible approach to proving the conjectured expression for the free energy would be to find the general solution to the matrix model, perhaps in terms of the polygon construction presented above, as it has been done for the $\widehat{A}$-quiver in [119]. Some questions which have not been addressed here are whether there is a group theory interpretation of the volume formula and whether its denominator can be written in a form that is universal for any ADE quiver.

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Appendix A

General $\mathcal{N} = (2, 2)$ GLSM's

A.1 Semichiral Quotient

Here we give the necessary elements and sketch the proof of (2.3.7). As mentioned in the text, the requirement

$$\{J_+, J_-\} = c\,\mathbb{I} \tag{A.1.1}$$

implies the set of equations

$$\begin{aligned}\left\{\mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL}^{-1}, J_s\right\} &= 0\,, &\text{(A.1.2)}\\ J_s \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} + \mathcal{K}_{LR}^{-1} J_s \mathcal{K}_{RL} J_s + \mathcal{K}_{LR}^{-1} C_{RR} \mathcal{K}_{RL}^{-1} C_{LL} &= c\,\mathbb{I}\,. &\text{(A.1.3)}\end{aligned}$$

We define the potential $\hat{K}$ by

$$\hat{K}(\mathbb{X}_l^i, \mathbb{X}_r^{i'}) = K(\mathbb{X}_l^i, \mathbb{X}_r^{i'}; \mathbb{X}^\alpha + V^\alpha) - t_\alpha V^\alpha\,, \tag{A.1.4}$$

from where the standard relation of second derivatives

$$\hat{K}_{\mu\nu} = K_{\mu\nu} - K_{\mu\alpha} K_{\beta\alpha}^{-1} K_{\beta\nu} \tag{A.1.5}$$

follows, where $\mu = (i, i', \bar{i}, \bar{i}')$ labels the $4N$ coordinates. From now on we suppress obvious indices, writing $(L, R) = (l, r, \alpha)$. Capital letters refer to the manifold $\mathcal{M}$, while lower-case are coordinates on $\hat{\mathcal{M}}$ and α labels coordinates which are gauged away. We decompose the relevant matrices as

$$\mathcal{K}_{LR} = \left(\begin{array}{c|c} K_{lr} & K_{l\alpha} \\ \hline K_{\beta r} & K_{\beta\alpha}\end{array}\right), \quad \mathcal{K}_{LL} = \left(\begin{array}{c|c} K_{ll} & K_{l\alpha} \\ \hline K_{\beta l} & K_{\beta\alpha}\end{array}\right), \quad C_{LL} = \left(\begin{array}{c|c} C_{ll} & C_{l\alpha} \\ \hline C_{\beta l} & C_{\beta\alpha}\end{array}\right), \quad J_s = \left(\begin{array}{c|c} \hat{J} & 0 \\ \hline 0 & j\end{array}\right),$$

with $\hat{J}^2 = -1$ and $j^2 = -1$ and

$$\begin{aligned} C_{ll} = [\hat{J}, K_{ll}], \qquad & C_{\beta\alpha} = [j, K_{\beta\alpha}]\,, \\ C_{l\alpha} = \hat{J}K_{l\alpha} - K_{l\alpha}j\,, \qquad & C_{\beta l} = jK_{\beta l} - K_{\beta l}\hat{J}, \end{aligned} \tag{A.1.6}$$

(and similarly for C_{RR}). The inverse matrices are given by

$$\mathcal{K}_{RL}^{-1} \equiv (\mathcal{K}_{LR})^{-1} = \left(\begin{array}{c|c} \hat{K}_{lr}^{-1} & -\hat{K}_{lr}^{-1}K_{l\alpha}K_{\beta\alpha}^{-1} \\ \hline -K_{\beta\alpha}^{-1}K_{\beta r}\hat{K}_{lr}^{-1} & T^{\alpha\beta} \end{array}\right) \tag{A.1.7}$$

and $\mathcal{K}_{LR}^{-1} = (\mathcal{K}_{RL}^{-1})^t$ and where

$$\hat{K}_{lr} = K_{lr} - K_{l\alpha}K_{\beta\alpha}^{-1}K_{\beta r}, \tag{A.1.8}$$

$$T^{\alpha\beta} = K_{\beta\alpha}^{-1} + K_{\delta\alpha}^{-1}K_{\delta r}\hat{K}_{lr}^{-1}K_{l\gamma}K_{\beta\gamma}^{-1}\,. \tag{A.1.9}$$

(Here we have changed the notation slightly to mean $\hat{K}_{lr}^{-1} = (\hat{K}_{lr})^{-1}$, $K_{\beta\alpha}^{-1} = (K^{-1})^{\alpha\beta}$, *etc.*). Similarly, we also have

$$\hat{C}_{rr} = C_{rr} - \left[\hat{J}, K_{r\beta}K_{\alpha\beta}^{-1}K_{\alpha r}\right], \qquad \hat{C}_{ll} = C_{ll} - \left[\hat{J}, K_{l\beta}K_{\alpha\beta}^{-1}K_{\alpha l}\right] \tag{A.1.10}$$

We would like to prove that the structure (A.1.1) is preserved under the quotient, namely that

$$\{\hat{J}_+, \hat{J}_-\} = c\mathbb{I}, \tag{A.1.11}$$

follows from (A.1.1) or, equivalently, that

$$\left\{\hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}, \hat{J}\right\} = 0, \tag{A.1.12}$$

$$\hat{J}\hat{K}_{rl}^{-1}\hat{J}\hat{K}_{rl} + \hat{K}_{rl}^{-1}\hat{J}\hat{K}_{rl}\hat{J} + \hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}\hat{C}_{ll} = c\,\mathbb{I}. \tag{A.1.13}$$

Since a bi-Hermitean manifold with c constant and $|c| < 2$ is a hyper-Kähler manifold, it follows as a corollary that the semi-chiral quotient of a hyper-Kähler manifold is hyper-Kähler.

We divide the calculation into two parts, first proving the off-diagonal equation (A.1.12) and then the diagonal equation (A.1.13).

First we prove that (2.3.8) leads to (A.1.12)

$$K_{LR}^{-1}C_{RR}K_{RL}^{-1} =$$

$$\left(\begin{array}{c|c} \hat{K}_{rl}^{-1} & -\hat{K}_{rl}^{-1}K_{r\beta}K_{\alpha\beta}^{-1} \\ \hline -K_{\alpha\beta}^{-1}K_{\alpha l}\hat{K}_{rl}^{-1} & T^{\beta\alpha} \end{array}\right)\left(\begin{array}{c|c} C_{rr} & C_{r\gamma} \\ \hline C_{\alpha r} & C_{\alpha\gamma} \end{array}\right)\left(\begin{array}{c|c} \hat{K}_{lr}^{-1} & -\hat{K}_{lr}^{-1}K_{l\delta}K_{\lambda\delta}^{-1} \\ \hline -K_{\lambda\gamma}^{-1}K_{\lambda r}\hat{K}_{lr}^{-1} & T^{\gamma\lambda} \end{array}\right)$$
$$= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{A.1.14}$$

where

$$A \equiv \hat{K}_{rl}^{-1}(C_{rr} - K_{r\beta}K_{\alpha\beta}^{-1}C_{\alpha r} - C_{r\gamma}K_{\lambda\gamma}^{-1}K_{\lambda r} + K_{r\beta}K_{\alpha\beta}^{-1}C_{\alpha\gamma}K_{\lambda\gamma}^{-1}K_{\lambda r})\hat{K}_{lr}^{-1}$$

On the other hand,

$$\{K_{LR}^{-1}C_{RR}K_{RL}^{-1}, J_s\} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} \hat{J} & 0 \\ 0 & j \end{pmatrix}\right\} = \begin{pmatrix} \{A, \hat{J}\} & Bj + \hat{J}B \\ C\hat{J} + jC & \{D, j\} \end{pmatrix} = 0$$

so we find

$$\left\{A, \hat{J}\right\} = \left\{\hat{K}_{rl}^{-1}\underbrace{(C_{rr} - K_{r\beta}K_{\alpha\beta}^{-1}C_{\alpha r} - C_{r\gamma}K_{\lambda\gamma}^{-1}K_{\lambda r} + K_{r\beta}K_{\alpha\beta}^{-1}C_{\alpha\gamma}K_{\lambda\gamma}^{-1}K_{\lambda r})}_{\equiv\bar{C}_{rr}}\hat{K}_{lr}^{-1}, \hat{J}\right\} = 0.$$

Using the definitions (A.1.6) after some cancellations we find

$$\bar{C}_{rr} = C_{rr} - \left[\hat{J}, K_{r\beta}K_{\alpha\beta}^{-1}K_{\alpha r}\right] = \hat{C}_{rr}$$

and therefore

$$\left\{\hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}, \hat{J}\right\} = 0 \tag{A.1.15}$$

as we wanted to prove. Now we show that (2.3.9) leads to (A.1.13). In (2.3.9) there are three terms.

The first one reads

$$
\begin{aligned}
J_s K_{LR}^{-1} J_s K_{RL} &= \left(\begin{array}{c|c} \hat{J} & 0 \\ \hline 0 & j \end{array}\right) \left(\begin{array}{c|c} \hat{K}_{rl}^{-1} & -\hat{K}_{rl}^{-1} K_{r\beta} K_{\alpha\beta}^{-1} \\ \hline -K_{\alpha\beta}^{-1} K_{\alpha l} \hat{K}_{rl}^{-1} & T^{\beta\alpha} \end{array}\right) \left(\begin{array}{c|c} \hat{J} & 0 \\ \hline 0 & j \end{array}\right) \left(\begin{array}{c|c} K_{rl} & K_{r\gamma} \\ \hline K_{\alpha l} & K_{\alpha\gamma} \end{array}\right) \\
&= \left(\begin{array}{c|c} \hat{J}\hat{K}_{rl}^{-1}(\hat{J}K_{rl} - K_{r\beta}K_{\alpha\beta}^{-1} j K_{\alpha l}) & \hat{J}\hat{K}_{rl}^{-1}(\hat{J}K_{r\gamma} - K_{r\beta}K_{\alpha\beta}^{-1} j K_{\alpha\gamma}) \\ \hline j(-K_{\alpha\beta}^{-1}K_{\alpha l}\hat{K}_{rl}^{-1}\hat{J}K_{rl} + T^{\beta\alpha} j K_{\alpha l}) & j(-K_{\alpha\beta}^{-1}K_{\alpha l}\hat{K}_{rl}^{-1}\hat{J}K_{r\gamma} + T^{\beta\alpha} j K_{\alpha\gamma}) \end{array}\right) \\
&\equiv \left(\begin{array}{c|c} a_1 & a_2 \\ \hline a_3 & a_4 \end{array}\right).
\end{aligned}
$$

The second term

$$
\begin{aligned}
K_{LR}^{-1} J_s K_{RL} J_s &= \left(\begin{array}{c|c} \hat{K}_{rl}^{-1} & -\hat{K}_{rl}^{-1} K_{r\beta} K_{\alpha\beta}^{-1} \\ \hline -K_{\alpha\beta}^{-1} K_{\alpha l} \hat{K}_{rl}^{-1} & T^{\beta\alpha} \end{array}\right) \left(\begin{array}{c|c} \hat{J} & 0 \\ \hline 0 & j \end{array}\right) \left(\begin{array}{c|c} K_{rl} & K_{r\gamma} \\ \hline K_{\alpha l} & K_{\alpha\gamma} \end{array}\right) \left(\begin{array}{c|c} \hat{J} & 0 \\ \hline 0 & j \end{array}\right) \\
&= \left(\begin{array}{c|c} \hat{K}_{rl}^{-1}(\hat{J}K_{rl}\hat{J} - K_{r\beta}K_{\alpha\beta}^{-1} j K_{\alpha l}\hat{J}) & \hat{K}_{rl}^{-1}(\hat{J}K_{r\gamma} j - K_{r\beta}K_{\alpha\beta}^{-1} j K_{\alpha\gamma} j) \\ \hline -K_{\alpha\beta}^{-1}K_{\alpha l}\hat{K}_{rl}^{-1}\hat{J}K_{rl}\hat{J} + T^{\beta\alpha} j K_{\alpha l}\hat{J} & -K_{\alpha\beta}^{-1}K_{\alpha l}\hat{K}_{rl}^{-1}\hat{J}K_{r\gamma} j + T^{\beta\alpha} j K_{\alpha\gamma} j \end{array}\right) \\
&\equiv \left(\begin{array}{c|c} b_1 & b_2 \\ \hline b_3 & b_4 \end{array}\right),
\end{aligned}
$$

and the third term

$$
\begin{aligned}
K_{LR}^{-1} C_{RR} K_{RL}^{-1} C_{LL} &= \\
& \left(\begin{array}{c|c} \hat{K}_{rl}^{-1} & -\hat{K}_{rl}^{-1} K_{r\beta} K_{\alpha\beta}^{-1} \\ \hline -K_{\alpha\beta}^{-1} K_{\alpha l} \hat{K}_{rl}^{-1} & T^{\beta\alpha} \end{array}\right) \left(\begin{array}{c|c} C_{rr} & C_{r\delta} \\ \hline C_{\alpha r} & C_{\alpha\delta} \end{array}\right) \\
& \times \left(\begin{array}{c|c} \hat{K}_{lr}^{-1} & -\hat{K}_{lr}^{-1} K_{l\epsilon} K_{\gamma\epsilon}^{-1} \\ \hline -K_{\epsilon\delta}^{-1} K_{\epsilon r} \hat{K}_{lr}^{-1} & T^{\delta\gamma} \end{array}\right) \left(\begin{array}{c|c} C_{ll} & C_{l\lambda} \\ \hline C_{\gamma l} & C_{\gamma\lambda} \end{array}\right) \\
&= \left(\begin{array}{c|c} \hat{K}_{rl}^{-1}(C_{rr} - K_{r\beta}K_{\alpha\beta}^{-1} C_{\alpha r}) & \hat{K}_{rl}^{-1}(C_{r\delta} - K_{r\beta}K_{\alpha\beta}^{-1} C_{\alpha\delta}) \\ \hline -\hat{K}_{\alpha\beta}^{-1}K_{\alpha l}\hat{K}_{rl}^{-1}C_{rr} + T^{\beta\alpha} C_{\alpha r} & -K_{\alpha\beta}^{-1}K_{\alpha l}\hat{K}_{rl}^{-1}C_{r\delta} + T^{\beta\alpha} C_{\alpha\delta} \end{array}\right) \\
& \times \left(\begin{array}{c|c} \hat{K}_{lr}^{-1}(C_{ll} - K_{l\epsilon}K_{\gamma\epsilon}^{-1} C_{\gamma l}) & \hat{K}_{lr}^{-1}(C_{l\lambda} - K_{l\epsilon}K_{\gamma\epsilon}^{-1} C_{\gamma\lambda}) \\ \hline -\hat{K}_{\epsilon\delta}^{-1}K_{\epsilon r}\hat{K}_{lr}^{-1}C_{ll} + T^{\delta\gamma} C_{\gamma l} & -K_{\epsilon\delta}^{-1}K_{\epsilon r}\hat{K}_{lr}^{-1}C_{l\lambda} + T^{\delta\gamma} C_{\gamma\lambda} \end{array}\right) \\
&\equiv \left(\begin{array}{c|c} c_1 & c_2 \\ \hline c_3 & c_4 \end{array}\right).
\end{aligned}
$$

We have in total four equations, but it is enough to focus only on the two

$$ a_1 + b_1 + c_1 = c\,, \tag{A.1.16} $$

$$ a_2 + b_2 + c_2 = 0\,. \tag{A.1.17} $$

We will show below that (A.1.13) follows from these two equations. Let us first simplify the expression for c_1,

$$\begin{aligned} c_1 &= \hat{K}^{-1}_{rl}(C_{rr} - K_{r\beta}K^{-1}_{\alpha\beta}C_{\alpha r})\hat{K}^{-1}_{lr}(C_{ll} - K_{l\epsilon}K^{-1}_{\gamma\epsilon}C_{\gamma l}) \\ &\quad + \hat{K}^{-1}_{rl}(C_{r\delta} - K_{r\beta}K^{-1}_{\alpha\beta}C_{\alpha\delta})(-\hat{K}^{-1}_{\epsilon\delta}K_{\epsilon r}\hat{K}^{-1}_{lr}C_{ll} + T^{\delta\gamma}C_{\gamma l})\,. \end{aligned} \tag{A.1.18}$$

Using (A.1.6) we have

$$\begin{aligned} c_1 &= \hat{K}^{-1}_{rl}([\hat{J}, K_{rr}] - K_{r\beta}K^{-1}_{\alpha\beta}(jK_{\alpha r} - K_{\alpha r}\hat{J}))\hat{K}^{-1}_{lr}([\hat{J}, K_{ll}] - K_{l\epsilon}K^{-1}_{\gamma\epsilon}(jK_{\gamma l} - K_{\gamma l}\hat{J})) \\ &+ \hat{K}^{-1}_{rl}(\hat{J}K_{r\delta} - K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\delta}) \\ &\quad \times(-\hat{K}^{-1}_{\epsilon\delta}K_{\epsilon r}\hat{K}^{-1}_{lr}[\hat{J}, K_{ll}] + (K^{-1}_{\gamma\delta} + K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha})(jK_{\gamma l} - K_{\gamma l}\hat{J}))\,. \end{aligned}$$

We omit in what follows all the terms involving C_{ll} and C_{rr} and consider them at the end. We have then

$$\begin{aligned} c'_1 &= \hat{K}^{-1}_{rl}(K_{r\beta}K^{-1}_{\alpha\beta}(jK_{\alpha r} - K_{\alpha r}\hat{J}))\hat{K}^{-1}_{lr}(K_{l\epsilon}K^{-1}_{\gamma\epsilon}(jK_{\gamma l} - K_{\gamma l}\hat{J})) \\ &+ \hat{K}^{-1}_{rl}(\hat{J}K_{r\delta} - K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\delta}) \\ &\quad \times((K^{-1}_{\gamma\delta} + K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha})(jK_{\gamma l} - K_{\gamma l}\hat{J}))\,, \end{aligned}$$

$$\begin{aligned} a_1 + b_1 + c'_1 &= \{\hat{J}, \hat{K}^{-1}_{rl}(\hat{J}K_{rl} - K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha l})\} \\ &\quad + \hat{K}^{-1}_{rl}(K_{r\beta}K^{-1}_{\alpha\beta}(jK_{\alpha r} - K_{\alpha r}\hat{J}))\hat{K}^{-1}_{lr}(K_{l\epsilon}K^{-1}_{\gamma\epsilon}(jK_{\gamma l} - K_{\gamma l}\hat{J})) \\ &\quad + \hat{K}^{-1}_{rl}(\hat{J}K_{r\delta} - K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\delta}) \\ &\quad \times(K^{-1}_{\gamma\delta} + K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha})(jK_{\gamma l} - K_{\gamma l}\hat{J})\,. \end{aligned}$$

We will simplify this expression considerably by grouping terms by the number of $j's$ and $\hat{J}'s$. First the terms involving only jj:

$$\begin{aligned} jj: \quad & \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha r}\hat{K}^{-1}_{lr}K_{l\epsilon}K^{-1}_{\gamma\epsilon}jK_{\gamma l} \\ & -\hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\delta}(K^{-1}_{\gamma\delta} + K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha})jK_{\gamma l}\,. \end{aligned}$$

The first and last terms cancel, leaving only

$$jj: \quad \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}K_{\alpha l}\,.$$

Now we look at terms with $\hat{J}\hat{J}$,

$$\begin{aligned}\hat{J}\hat{J}: \quad & \{\hat{J}, \hat{K}^{-1}_{rl}\hat{J}K_{rl}\} + \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}K_{\alpha r}\hat{J}\hat{K}^{-1}_{lr}K_{l\epsilon}K^{-1}_{\gamma\epsilon}K_{\gamma l}\hat{J} \\ & -\hat{K}^{-1}_{rl}\hat{J}K_{r\delta}(K^{-1}_{\gamma\delta} + K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha})K_{\gamma l}\hat{J}\end{aligned} \tag{A.1.19}$$

using (A.1.8) and after some manipulations, we are left with

$$\hat{J}\hat{J}: \quad \{\hat{J}, \hat{K}^{-1}_{rl}\hat{J}\hat{K}_{rl}\} + \hat{J}\hat{K}^{-1}_{rl}\hat{J}K_{r\alpha}K^{-1}_{\beta\alpha}K_{\beta l} \tag{A.1.20}$$

$$-\hat{K}^{-1}_{rl}[\hat{J}, K_{r\beta}K^{-1}_{\alpha\beta}K_{\alpha r}]\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha}K_{\gamma l}\hat{J}. \tag{A.1.21}$$

Now we look at terms involving $\hat{J}j + j\hat{J}$

$$\begin{aligned}\hat{J}j + j\hat{J}: & - \{\hat{J}, \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha l}\} - \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha r}\hat{K}^{-1}_{lr}K_{l\epsilon}K^{-1}_{\gamma\epsilon}K_{\gamma l}\hat{J} \\ & - \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}K_{\alpha r}\hat{J}\hat{K}^{-1}_{lr}K_{l\epsilon}K^{-1}_{\gamma\epsilon}jK_{\gamma l} \\ & + \hat{K}^{-1}_{rl}\hat{J}K_{r\delta}(K^{-1}_{\gamma\delta} + K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha})jK_{\gamma l} \\ & + \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\delta}(K^{-1}_{\gamma\delta} + K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha})K_{\gamma l}\hat{J}\end{aligned}$$

after some cancellations, we are left with

$$\hat{J}j + j\hat{J}: \quad -[\hat{J}, \hat{K}^{-1}_{rl}]K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha l} - \hat{K}^{-1}_{rl}[K_{r\beta}K^{-1}_{\alpha\beta}K_{\alpha r}, \hat{J}]\hat{K}^{-1}_{lr}K_{l\epsilon}K^{-1}_{\gamma\epsilon}jK_{\gamma l}\,.$$

Now we include the C_{ll} and C_{rr} terms we had left out before

$$C_{rr}: \quad -\hat{K}^{-1}_{rl}C_{rr}\hat{K}^{-1}_{lr}K_{l\epsilon}K^{-1}_{\gamma\epsilon}(jK_{\gamma l} - K_{\gamma l}\hat{J}) \tag{A.1.21}$$

This term, combined with the last $\hat{J}\hat{J}$ term and the last $\hat{J}j + j\hat{J}$ term gives

$$\hat{K}^{-1}_{rl}\hat{C}_{rr}\hat{K}^{-1}_{lr}K_{l\alpha}K^{-1}_{\gamma\alpha}(K_{\gamma l}\hat{J} - jK_{\gamma l}) \tag{A.1.22}$$

Now let us look at the C_{ll} term

$$\begin{aligned}C_{ll}: \quad & -\hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}(jK_{\alpha r} - K_{\alpha r}\hat{J})\hat{K}^{-1}_{lr}C_{ll} \\ & -\hat{K}^{-1}_{rl}(\hat{J}K_{r\delta} - K_{r\delta}j - K_{r\beta}K^{-1}_{\alpha\beta}[j, K_{\alpha\delta}])K^{-1}_{\epsilon\delta}K_{\epsilon r}\hat{K}^{-1}_{lr}C_{ll}\end{aligned} \tag{A.1.23}$$

which after some cancellations gives

$$C_{ll}: \quad \hat{K}^{-1}_{rl}[K_{r\beta}K^{-1}_{\alpha\beta}K_{\alpha r}, \hat{J}]\hat{K}^{-1}_{lr}C_{ll} \tag{A.1.23}$$

and the $C_{rr}C_{ll}$ term

$$C_{rr}C_{ll}: \quad \hat{K}^{-1}_{rl}C_{rr}\hat{K}^{-1}_{lr}C_{ll} \tag{A.1.24}$$

Combining these last two gives

$$\hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}C_{ll}\,. \tag{A.1.25}$$

Putting all the terms we have together, we get

$$\begin{aligned} a_1+b_1+c_1 &= \{\hat{J},\hat{K}_{rl}^{-1}\hat{J}K_{rl}\}+\hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}C_{ll} \\ &\quad +\hat{K}_{rl}^{-1}K_{r\beta}K_{\alpha\beta}^{-1}K_{\alpha l}+\hat{J}\hat{K}_{rl}^{-1}\hat{J}K_{r\alpha}K_{\beta\alpha}^{-1}K_{\beta l} \\ &\quad -[\hat{J},\hat{K}_{rl}^{-1}]K_{r\beta}K_{\alpha\beta}^{-1}jK_{\alpha l}+\hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}K_{l\alpha}K_{\gamma\alpha}^{-1}(K_{\gamma l}\hat{J}-jK_{\gamma l}) \end{aligned} \tag{A.1.26}$$

the first line in this equation starts looking like what we want to have. Using $C_{ll}=\hat{C}_{ll}+[\hat{J},K_{l\alpha}K_{\beta\alpha}^{-1}K_{\beta l}]$ we have

$$a_1+b_1+c_1 \;=\; \{\hat{J},\hat{K}_{rl}^{-1}\hat{J}K_{rl}\}+\hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}\hat{C}_{ll}+\Delta=c\,,\mathbb{I} \tag{A.1.25}$$

with

$$\begin{aligned} \Delta &= \hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}[\hat{J},K_{l\alpha}K_{\beta\alpha}^{-1}K_{\beta l}]+\hat{K}_{rl}^{-1}K_{r\beta}K_{\alpha\beta}^{-1}K_{\alpha l}+\hat{J}\hat{K}_{rl}^{-1}\hat{J}K_{r\alpha}K_{\beta\alpha}^{-1}K_{\beta l} \\ &\quad -[\hat{J},\hat{K}_{rl}^{-1}]K_{r\beta}K_{\alpha\beta}^{-1}jK_{\alpha l}+\hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}K_{l\alpha}K_{\gamma\alpha}^{-1}(K_{\gamma l}\hat{J}-jK_{\gamma l})\,. \end{aligned}$$

Thus, to prove (A.1.13), we must show that $\Delta=0$. This, we will show below, follows from (A.1.17), but first we simplify the expression for Δ. Part of the first term and the last term in this expression cancel and after some other manipulations, we have

$$\begin{aligned} \Delta = \hat{K}_{rl}^{-1}\hat{C}_{rr}\hat{K}_{lr}^{-1}(\hat{J}K_{l\alpha}K_{\beta\alpha}^{-1}-K_{l\alpha}K_{\beta\alpha}^{-1}j)K_{\beta l} \\ +[\hat{J},\hat{K}_{rl}^{-1}](\hat{J}K_{r\alpha}K_{\beta\alpha}^{-1}-K_{r\alpha}K_{\beta\alpha}^{-1}j)K_{\beta l}\,. \end{aligned} \tag{A.1.24}$$

Now, we see that

$$\begin{aligned} a_2 &= \hat{J}\hat{K}_{rl}^{-1}(\hat{J}K_{r\lambda}-K_{r\beta}K_{\alpha\beta}^{-1}jK_{\alpha\lambda})\,, \\ b_2 &= \hat{K}_{rl}^{-1}(\hat{J}K_{r\lambda}-K_{r\beta}K_{\alpha\beta}^{-1}jK_{\alpha\lambda})j \end{aligned}$$

while c_2 is given by

$$\begin{aligned} c_2 &= \hat{K}_{rl}^{-1}(C_{rr}-K_{r\beta}K_{\alpha\beta}^{-1}C_{\alpha r})\hat{K}_{lr}^{-1}(C_{l\lambda}-K_{l\epsilon}K_{\gamma\epsilon}^{-1}C_{\gamma\lambda}) \\ &\quad +\hat{K}_{rl}^{-1}(C_{r\delta}-K_{r\beta}K_{\alpha\beta}^{-1}C_{\alpha\delta})(-K_{\epsilon\delta}^{-1}K_{\epsilon r}\hat{K}_{lr}^{-1}C_{l\lambda}+T^{\delta\gamma}C_{\gamma\lambda}) \end{aligned}$$

and using (A.1.6), we find

$$\begin{aligned} c_2 = \quad & \hat{K}^{-1}_{rl}(C_{rr} - K_{r\beta}K^{-1}_{\alpha\beta}(jK_{\alpha r} - K_{\alpha r}\hat{J}))\hat{K}^{-1}_{lr}(\hat{J}K_{l\lambda} - K_{l\epsilon}K^{-1}_{\gamma\epsilon}jK_{\gamma\lambda}) \\ & +\hat{K}^{-1}_{rl}(\hat{J}K_{r\delta} - K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\delta})(K^{-1}_{\beta\delta}K_{\beta r}\hat{K}^{-1}_{lr}(K_{l\alpha}K^{-1}_{\gamma\alpha}jK_{\gamma\lambda} - \hat{J}K_{l\lambda}) \\ & \qquad +K^{-1}_{\gamma\delta}[j, K_{\gamma\lambda}])\,. \end{aligned}$$

Now we proceed similarly, grouping terms with $\hat{J}'s$ and $j's$ in $a_2 + b_2 + c_2$. After simplifications, the only terms remaining in jj are

$$jj: \qquad \hat{K}^{-1}_{rl}K_{r\beta}K^{-1}_{\alpha\beta}K_{\alpha\lambda} \tag{A.1.20}$$

while for $\hat{J}\hat{J}$ we are left with

$$\hat{J}\hat{J}: \qquad \hat{J}\hat{K}^{-1}_{rl}\hat{J}K_{r\lambda} + \hat{K}^{-1}_{rl}\hat{C}_{rr}\hat{K}^{-1}_{lr}\hat{J}K_{l\lambda} \tag{A.1.21}$$

The $\hat{J}j + j\hat{J}$ terms are

$$\hat{J}j + j\hat{J}: \qquad -[\hat{J}, \hat{K}^{-1}_{rl}](K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\lambda}) - \hat{K}^{-1}_{rl}\hat{C}_{rr}\hat{K}^{-1}_{lr}K_{l\epsilon}K^{-1}_{\gamma\epsilon}jK_{\gamma\lambda}$$

These constitute all the terms. Putting them all together, we arrive at

$$\begin{aligned} a_2 + b_2 + c_2 \;\; = \;\; & [\hat{J}, \hat{K}^{-1}_{rl}](\hat{J}K_{r\lambda} - K_{r\beta}K^{-1}_{\alpha\beta}jK_{\alpha\lambda}) \\ & +\hat{K}^{-1}_{rl}\hat{C}_{rr}\hat{K}^{-1}_{lr}(\hat{J}K_{l\lambda} - K_{l\epsilon}K^{-1}_{\gamma\epsilon}jK_{\gamma\lambda}) = 0\,. \end{aligned} \tag{A.1.21}$$

This is precisely the combination that appears in Δ and therefore

$$\Delta = 0$$

hence proving (A.1.13). Thus, finally

$$\{\hat{J}_+, \hat{J}_-\} = c\,\mathbb{I}\,, \tag{A.1.20}$$

as we wanted to prove.

A.2 T-duality

Here we give some of the details leading to (2.4.12) and (2.4.15). Writing the Legendre transform as

$$K(\mathbb{X}^i) = F(V^\alpha) - \frac{1}{2}V^\alpha\delta_{\alpha i}\mathbb{X}^i\,, \tag{A.2.1}$$

where we defined

$$\mathbb{X}^i \equiv \left(i(\mathbb{X}_L - \bar{\mathbb{X}}_L + \mathbb{X}_R - \bar{\mathbb{X}}_R), -i(\mathbb{X}_L - \bar{\mathbb{X}}_L - \mathbb{X}_R + \bar{\mathbb{X}}_R), -(\mathbb{X}_L + \bar{\mathbb{X}}_L + \mathbb{X}_R + \bar{\mathbb{X}}_R)\right) \,,$$

we find the standard relation of second derivatives

$$K_{ij} = -\frac{1}{2}\delta_{i\alpha}(F^{-1})^{\alpha\beta}\delta_{\beta j}\,. \tag{A.2.1}$$

Explicitly inverting the general 3×3 matrix $F_{\alpha\beta}$ and using these relations in the definition (2.2.7), one finds

$$c = \frac{2\left(F_{V^\phi V^\phi} + F_{V^\chi V^\chi} + 2F_{V'V'}\right)}{F_{V^\phi V^\phi} - F_{V^\chi V^\chi}}\,. \tag{A.2.2}$$

The important point now is that the Laplace equation ($F_{\phi\bar{\phi}} + F_{\chi\bar{\chi}} = 0$) translates into

$$\cos^2(\theta)F_{V^\phi V^\phi} + \sin^2(\theta)F_{V^\chi V^\chi} + F_{V'V'} = 0\,, \tag{A.2.3}$$

which is a direct consequence of how the gauging was performed in (2.4.10) (*i.e.*, the charges of the fields). Using (A.2.3) in (A.2.2) finally leads to

$$c = -2\cos(2\theta)\,. \tag{A.2.4}$$

To prove (2.4.15) it is more convenient to redefine the fields so that the Killing vector acts by translations. Note that this is allowed due to the chirality properties of the components of the Killing vector. This, of course, does not preserve the form of the Laplace equation, but instead turns into $\frac{1}{|k^\phi|^2}F_{\phi\bar{\phi}} + \frac{1}{|k^\chi|^2}F_{\chi\bar{\chi}} = 0$. Using this in (A.2.2) leads to (2.4.15).

A.3 Reduction to $\mathcal{N} = (1,1)$

To reduce to $\mathcal{N} = (1,1)$ (here we follow mostly [18, 23, 24]), one decomposes the $\mathcal{N} = (2,2)$ gauge covariant superderivatives into their real and imaginary part, namely

$$\nabla_\pm = \frac{1}{2}\left(\mathcal{D}_\pm - iQ_\pm\right), \qquad \bar{\nabla}_\pm = \frac{1}{2}\left(\mathcal{D}_\pm + iQ_\pm\right)\,. \tag{A.3.1}$$

Here $\mathcal{D}_\pm$ are $\mathcal{N} = (1,1)$ derivatives, which satisfy the algebra

$$\{\mathcal{D}_\pm, \mathcal{D}_\pm\} = i\mathcal{D}_{\pm\pm}\,, \tag{A.3.2}$$

where $\mathcal{D}_{\pm\pm}$ is the gauge-covariant space derivative and $Q_\pm$ generate the non-manifest supersymmetries. We perform the reduction of the matter fields $\mathbb{X}_L, \mathbb{X}_R$ in the covariant approach. That is, we define

$$
\begin{aligned}
\hat{\mathbb{X}}_R &= e^{-\mathbb{V}_L} e^{i\bar{\tilde{\mathbb{V}}}} \mathbb{X}_R\,, \\
\bar{\hat{\mathbb{X}}}_R &= \mathbb{X}_R^\dagger e^{-i\tilde{\mathbb{V}}}\,,
\end{aligned}
\tag{A.3.3}
$$

so that there are no factors e^V anywhere. For instance, $\bar{\hat{\mathbb{X}}}_R \hat{\mathbb{X}}_R = \mathbb{X}_R^\dagger e^{-i\tilde{\mathbb{V}}} e^{-\mathbb{V}_L} e^{i\bar{\tilde{\mathbb{V}}}} \mathbb{X}_R = \bar{\mathbb{X}}_R e^{V_R} \mathbb{X}_R$ and the Lagrangian is simply (dropping the hats)

$$
K = \bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R + \alpha(\bar{\mathbb{X}}_L \mathbb{X}_R + \bar{\mathbb{X}}_R \mathbb{X}_L)\,. \tag{A.3.3}
$$

Next, one imposes the fields to be gauge-covariantly semichiral and defines components with gauge-covariant $Q_\pm$'s, *i.e.*,

$$
X_L = \mathbb{X}_L\big|\,, \qquad Q_+ \mathbb{X}_L = i\mathcal{D}_+ \mathbb{X}_L\,, \qquad Q_- \mathbb{X}_L\big| = \Psi_-\,, \tag{A.3.4}
$$
$$
X_R = \mathbb{X}_R\big|\,, \qquad Q_- \mathbb{X}_R = i\mathcal{D}_- \mathbb{X}_R\,, \qquad Q_+ \mathbb{X}_R\big| = \Psi_+\,. \tag{A.3.5}
$$

The reduction of the semichiral vector multiplet is given by

$$
\begin{gathered}
d^1 = \left(\mathbb{F} + \bar{\mathbb{F}}\right)\big|\,, \quad d^2 = \left(\tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\big|\,, \quad d^3 = i\left(\mathbb{F} - \bar{\mathbb{F}} - \tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)\big|\,, \\
f = -\,i\left(\mathbb{F} - \bar{\mathbb{F}} + \tilde{\mathbb{F}} - \bar{\tilde{\mathbb{F}}}\right)\big|\,,
\end{gathered}
$$

from where

$$
\mathbb{F}\big| = \frac{1}{2}\left(d^1 + \frac{i}{2}(f - d^3)\right)\,, \qquad \tilde{\mathbb{F}}\big| = \frac{1}{2}\left(d^2 + \frac{i}{2}(f + d^3)\right)\,. \tag{A.3.6}
$$

From the definitions $\mathbb{F} = i\{\bar{\nabla}_+, \bar{\nabla}_-\}$ and $\tilde{\mathbb{F}} = i\{\bar{\nabla}_+, \nabla_-\}$, one can solve for the commutation relations

$$
\begin{aligned}
\{Q_+, \mathcal{D}_-\} &= \mp(d_1 + d_2), \\
\{\mathcal{D}_+, Q_-\} &= \mp(d_1 - d_2)\,, \\
\{Q_+, Q_-\} &= \pm d_3\,, \\
\{\mathcal{D}_+, \mathcal{D}_-\} &= f\,,
\end{aligned}
\tag{A.3.6}
$$

where the upper(lower) sign is chosen for positive(negative) charge. These are used repeatedly when reducing the matter fields, and the appropriate sign must be chosen depending on the charge of the field it acts on. Note that f is the usual field strength which, in two dimensions, is a total derivative giving the topological charge.

A.4 $SU(2)$ symmetry

As described in the text, the action (2.5.1) is invariant under the global $SU(2)$ transformations which rotate $(X^{(1)}, X^{(2)})$ and the cross-ratio (2.5.21) is a natural radial coordinate. At a fixed radius R, we can reach any point by a finite $SU(2)$ transformation from a single point X_L^0, X_R^0. We take $X_L^0 = 0$ and $X_R^0 = \sqrt{R^2-1}$. Thus, by acting with a finite $SU(2)$ transformation, an arbitrary point is parameterized as

$$X_L = \frac{b}{\bar{a}}\,, \qquad X_R = \frac{a\rho + b}{\bar{a} - \bar{b}\rho}\,, \tag{A.4.1}$$

where we have defined $\rho^2 \equiv R^2 - 1$. By means of this identification, the natural remaining invariants are given by the Cartan 1-forms on the group manifold. Consider a group element g of $SU(2)$,

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1\,. \tag{A.4.2}$$

The (real) invariant 1-forms σ^i are defined by

$$g^{-1}dg = i \begin{pmatrix} \sigma^3 & \sigma^1 + i\sigma^2 \\ \sigma^1 - i\sigma^2 & -\sigma^3 \end{pmatrix}. \tag{A.4.3}$$

In the parameterization (A.4.2), we have

$$\sigma^1 = \mathrm{Im}(\bar{a}db - bd\bar{a})\,, \;\; \sigma^2 = -\mathrm{Re}(\bar{a}db - bd\bar{a})\,, \;\; \sigma^3 = -i(\bar{a}da + bd\bar{b})\,. \tag{A.4.4}$$

The constraint $|a|^2 + |b|^2 = 1$ ensures the reality of σ^3. From (A.4.1) and (A.4.4) we find

$$\begin{aligned} dX_L &= \frac{1}{\bar{a}^2}(i\sigma^1 - \sigma^2)\,, \\ dX_R &= \frac{1}{(\bar{a} - \rho\bar{b})^2}\left[2i\rho\sigma^3 + i(1-\rho^2)\sigma^1 - (1+\rho^2)\sigma^2 + d\rho\right]. \end{aligned} \tag{A.4.4}$$

These are the expressions which allow us to rewrite the Eguchi-Hanson metric in an explicitly $SU(2)$-invariant form. Another well-known property of Eguchi-Hanson is that its complex structures are preserved by the $SU(2)$ (in the Taub-NUT case they form a triplet). The Lie derivative along ξ of a $(1,1)$ tensor such as a complex structure is given by

$$\mathcal{L}_\xi J_\pm = \xi J_\pm - [\partial \cdot \xi, J_\pm]\,, \qquad \partial \cdot \xi \equiv \begin{pmatrix} \partial_L \xi^L & 0 \\ 0 & \partial_R \xi^R \end{pmatrix}\,, \tag{A.4.5}$$

where

$$\partial_L \xi^L \equiv \begin{pmatrix} \partial_l \xi^l & 0 \\ 0 & \partial_{\bar{l}} \bar{\xi}^l \end{pmatrix}\,, \qquad \partial_R \xi^R \equiv \begin{pmatrix} \partial_r \xi^r & 0 \\ 0 & \partial_{\bar{r}} \bar{\xi}^r \end{pmatrix}\,. \tag{A.4.6}$$

The equations from $\mathcal{L}_\xi J_+ = 0$ read

$$\begin{aligned} \xi^\mu \partial_\mu (\mathcal{K}_{RL}^{-1} C_{LL}) - (\partial_R \xi^R \mathcal{K}_{RL}^{-1} C_{LL} - \mathcal{K}_{RL}^{-1} C_{LL} \partial_L \xi^L) &= 0\,, \\ \xi^\mu \partial_\mu (\mathcal{K}_{RL}^{-1} J_s \mathcal{K}_{LR}) - [\partial_R \xi^R, \mathcal{K}_{RL}^{-1} J_s \mathcal{K}_{LR}] &= 0\,, \end{aligned} \tag{A.4.6}$$

and similarly for J_-, exchanging R by L. We verified that these equations are satisfied by explicit calculations from the potential (2.5.13).

A.5 Constrained semichiral quotient

To understand the kind of geometry that such models lead to, we will reduce to $\mathcal{N} = (1, 1)$. We do this in the covariant formalism and we find:

$$\begin{aligned} \mathcal{Q}_+ \mathcal{Q}_- K_{matter} =& 2id \left[\bar{X}_L X_L + \bar{X}_R X_R + \alpha(\bar{X}_L X_R + \bar{X}_R X_L) + t\, \bar{\phi}\, \phi - s\right] \\ & - 2irf + \mathcal{L}_{\Psi,X} + \mathcal{L}_{\bar{\phi}\phi} \end{aligned}$$

with

$$\begin{aligned} \mathcal{L}_{\Psi,X} =& - 2i\bar{\Psi}_- \mathcal{D}_+ X_L + 2i\Psi_- \mathcal{D}_+ \bar{X}_L + 2i\bar{\Psi}_+ \mathcal{D}_- X_R - 2i\Psi_+ \mathcal{D}_- \bar{X}_R \\ & + \alpha \left(\bar{\Psi}_+ \Psi_- - \bar{\Psi}_- \Psi_+ + i\Psi_- \mathcal{D}_+ \bar{X}_R + i\bar{\Psi}_+ \mathcal{D}_- X_L - i\bar{\Psi}_- \mathcal{D}_+ X_R - i\Psi_+ \mathcal{D}_- \bar{X}_L\right) \\ & + \alpha \left(\mathcal{D}_+ \bar{X}_L \mathcal{D}_- X_R + \mathcal{D}_+ X_L \mathcal{D}_- \bar{X}_R\right) - \alpha f (\bar{X}_L X_R - \bar{X}_R X_L) \end{aligned}$$

and $\mathcal{L}_{\bar{\phi}\phi}$ the usual Lagrangian. From the equation of motion for d, we have the constraint

$$\bar{X}_L X_L + \bar{X}_R X_R + \alpha(\bar{X}_L X_R + \bar{X}_R X_L) + t\, \bar{\phi}\, \phi - s = 0\,. \tag{A.5.0}$$

To see what the geometry is, it is convenient to define

$$X_L = X + Y\,, \qquad X_R = -X + Y\,, \tag{A.5.0}$$

and then the vacuum manifold is given by

$$|X|^2(1-\alpha)+|Y|^2(1+\alpha)+t\,\bar{\phi}\phi-s=0\,, \tag{A.5.0}$$

which is non-compact (assuming that $\alpha^2>1$ as required for the original metric to be positive definite).

Let us look now at the kinetic terms. Integrating out the auxiliary spinor superfields $\Psi_\pm$, we find

$$\mathcal{L}_{\Psi(X),X}=-\frac{1}{4}\int d^2\theta E_{\mu\nu}(D_+X^\mu)(D_-X^\nu)\,, \tag{A.5.0}$$

where

$$E_{\mu\nu}=E^{(V=0)}_{\mu\nu}+\delta E_{\mu\nu}\,, \tag{A.5.0}$$

with

$$E^{(V=0)}_{\mu\nu}=-4\begin{pmatrix} 0 & -2 & 0 & -\frac{4}{\alpha}+\alpha \\ -2 & 0 & -\frac{4}{\alpha}+\alpha & 0 \\ 0 & -\alpha & 0 & -2 \\ -\alpha & 0 & -2 & 0 \end{pmatrix}\,,\quad \delta E_{\mu\nu}=-4\begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{pmatrix}$$

in the basis $X^\mu=(X_L,\bar{X}_L,X_R,\bar{X}_R)$. Note that there is an additional term with respect to what one would get for a flat ungauged potential, which only contributes to the b-field[1].

A.6 Quotient Rules

The quotient rules state that dualizing along a $U(1)$ isometry parameterized by θ, the dual metric and b-field are given by

$$\tilde{g}_{\theta\theta}=\frac{1}{g_{\theta\theta}}\,,\quad \tilde{g}_{\theta\mu}=\frac{B_{\theta\mu}}{g_{\theta\theta}}\,,\quad \tilde{g}_{\mu\nu}=g_{\mu\nu}+\frac{B_{\theta\mu}B_{\theta\nu}-g_{\theta\mu}g_{\theta\nu}}{g_{\theta\theta}}$$
$$\tilde{B}_{\theta\mu}=\frac{g_{\theta\mu}}{g_{\theta\theta}}\,,\quad \tilde{B}_{\mu\nu}=B_{\mu\nu}+\frac{g_{\theta\mu}B_{\theta\nu}-B_{\theta\mu}g_{\theta\nu}}{g_{\theta\theta}}$$

[1]Note also that $E^{(V=0)}$ is symmetric (*i.e.*, there is no b-field) when $\alpha^2=2$. However, adding the additional term, the b-field now vanishes for $\alpha^2=1$. This is the reason we will recover the usual metrics for $\alpha\to 1$.

As a simple exercise, let us consider the duality between the Taub-NUT metric and the smeared NS5 brane:

$$ds^2_{TN} = H(r)d\boldsymbol{r}\cdot d\boldsymbol{r} + \frac{1}{4H(r)}\left(d\theta + \boldsymbol{w}\cdot d\boldsymbol{r}\right)^2 \ , \quad B = 0\,, \tag{A.6.0}$$

where $H(r) = \frac{1}{g^2} + \frac{1}{2r}$. Dualizing along θ gives

$$\tilde{g}_{\theta\theta} = 4H(r)\,, \quad \tilde{g}_{\theta i} = 0\,, \quad \tilde{g}_{ij} = \left(H(r)\delta_{ij} + \frac{1}{4H(r)}w_i w_j\right) - \frac{1}{4H(r)}w_i w_j = H(r)\delta_{ij}$$
$$\tilde{B}_{\theta i} = w_i\,, \quad \tilde{B}_{ij} = 0.$$

which gives

$$ds^2_{\mathrm{NS5}} = H(r)(d\boldsymbol{r}\cdot d\boldsymbol{r} + 4\,d\theta^2)\,, \quad B_{\theta i} = w_i\,. \tag{A.6.0}$$

Appendix B

Useful Formulae

Here we collect some useful formulas to reduce the action of a GLSM involving the chiral and the vector multiplet to component fields in the covariant approach.

B.1 Covariant Approach

We define gauge covariant spinor derivatives satisfying

$$\{\nabla_\pm, \bar{\nabla}_\pm\} = 2i\mathcal{D}_{\pm\pm}\,, \tag{B.1.1}$$

$$[\nabla_\pm, \mathcal{D}_{\pm\pm}] = 0\,, \tag{B.1.2}$$

where $\mathcal{D}_{\pm\pm}$ is the space, gauge covariant, derivative. In these conventions $x_\pm = (x_0 \pm x_1)$ and $ds^2 = -\frac{1}{2}(dx_+dx_- + dx_-dx_+)$. We define the field strength:

$$\Sigma \equiv \frac{1}{2}\{\bar{\nabla}_+, \nabla_-\}\,. \tag{B.1.3}$$

From the Jacobi identities, it follows that

$$\Big[\nabla_-, 2i\mathcal{D}_{++}\Big] = -2\nabla_+\Sigma, \quad \Big[\nabla_+, 2i\mathcal{D}_{--}\Big] = -2\nabla_-\bar{\Sigma}\,, \tag{B.1.4}$$

$$\Big[\bar{\nabla}_-, 2i\mathcal{D}_{++}\Big] = -2\bar{\nabla}_+\bar{\Sigma}, \quad \Big[\bar{\nabla}_+, 2i\mathcal{D}_{--}\Big] = -2\bar{\nabla}_-\Sigma\,. \tag{B.1.5}$$

B.1.1 Conjugation rules

The operation of conjugation is denoted by $\bar{()}$ and defined by

$$\bar{\Phi} \equiv e^{2V}\Phi^\dagger\,, \tag{B.1.6}$$

and

$$\bar{\nabla}_{\pm} = e^{2V}\nabla_{\pm}^{\dagger}e^{-2V}\,. \tag{B.1.7}$$

Note that with this definition, if $\nabla_{\pm}\Phi = \lambda_{\pm}$ then (taking the hermitean conjugate and inserting e^{2V}'s) $\bar{\nabla}_{\pm}\bar{\Phi} = \bar{\lambda}_{\pm}$ with $\bar{\lambda}_{\pm} = e^{2V}\lambda_{\pm}^{\dagger}$, just as for Φ. Also, in the WZ gauge, this implies that the components of $\bar{\Phi}$ are the complex conjugates of the components of Φ. One can see easily that this conjugation satisfies $\overline{(\bar{\nabla}_{\pm})} = \nabla_{\pm}$. This in turn implies, from a Jacobi identity, that

$$\overline{(\,i\mathcal{D}_{\pm\pm})} = i\mathcal{D}_{\pm\pm}\,. \tag{B.1.8}$$

B.1.2 Vector multiplet

Note that

$$F_{-+} = \frac{1}{2i\sqrt{2}}(\bar{\nabla}_{+}\nabla_{-}\bar{\Sigma} - \bar{\nabla}_{-}\nabla_{+}\Sigma)\,, \tag{B.1.9}$$

and from the definition of Σ we have

$$F_{-+} = \frac{1}{4i\sqrt{2}}\Big(\bar{\nabla}_{+}\nabla_{-}\{\bar{\nabla}_{-},\nabla_{+}\} - \bar{\nabla}_{-}\nabla_{+}\{\bar{\nabla}_{+},\nabla_{-}\}\Big)\,. \tag{B.1.10}$$

Note that the second term in the parenthesis is the conjugate (in the sense of "bar" conjugation) of the first one. Lets focus on the first one for now. By virtue of a Jacobi identity, we have

$$[\nabla_{-},\{\bar{\nabla}_{-},\nabla_{+}\}] + [\bar{\nabla}_{-},\{\nabla_{+},\nabla_{-}\}] + [\nabla_{+},\{\nabla_{-},\bar{\nabla}_{-}\}] = 0\,, \tag{B.1.11}$$

and hence

$$[\nabla_{-},\{\bar{\nabla}_{-},\nabla_{+}\}] = -[\nabla_{+},2i\mathcal{D}_{--}]\,. \tag{B.1.12}$$

Thus,

$$\{\bar{\nabla}_{+},[\nabla_{-},\{\bar{\nabla}_{-},\nabla_{+}\}]\} = -\{\bar{\nabla}_{+},[\nabla_{+},2i\mathcal{D}_{--}]\} \tag{B.1.13}$$

and the Jacobi identities:

$$\begin{aligned}\{\bar\nabla_+, [\nabla_+, 2i\mathcal{D}_{--}]\} - \{\nabla_+, [2i\mathcal{D}_{--}, \bar\nabla_+]\} + [2i\mathcal{D}_{--}, \{\bar\nabla_+, \nabla_+\}] = 0\,,\\ \{\bar\nabla_+, [\nabla_+, 2i\mathcal{D}_{--}]\} - \{\nabla_+, [2i\mathcal{D}_{--}, \bar\nabla_+]\} - 4[\mathcal{D}_{--}, \mathcal{D}_{++}] = 0\,,\end{aligned}$$

imply

$$\{\bar\nabla_+, [\nabla_+, 2i\mathcal{D}_{--}]\} = \{\nabla_+, [2i\mathcal{D}_{--}, \bar\nabla_+]\} + 4[\mathcal{D}_{--}, \mathcal{D}_{++}]\,. \tag{B.1.12}$$

Note now the following: Taking the *bar* of the left expression

$$\overline{(\{\nabla_+, [\nabla_+, 2i\mathcal{D}_{--}]\})} = \{\nabla_+, [2i\mathcal{D}_{--}, \bar\nabla_+]\}\,, \tag{B.1.13}$$

which is exactly the first term on the right-hand side. Thus,

$$\{\bar\nabla_+, [\nabla_+, 2i\mathcal{D}_{--}]\} - \overline{(\{\nabla_+, [\nabla_+, 2i\mathcal{D}_{--}]\})} = 4[\mathcal{D}_{--}, \mathcal{D}_{++}]\,, \tag{B.1.14}$$

and finally

$$F_{-+} \;=\; \frac{1}{4i\sqrt{2}}\Big(- 4[\mathcal{D}_{--}, \mathcal{D}_{++}]\Big) = \frac{i}{\sqrt{2}}[\mathcal{D}_{--}, \mathcal{D}_{++}]\,. \tag{B.1.15}$$

which is the usual expression for the field strength and a formula we will need when computing the action to get the correct kinetic term for the matter fields.

B.1.3 Chiral multiplet

We define components using gauge covariant derivatives

$$\begin{aligned}\Sigma| &= \sigma, & \bar\Sigma| &= \sigma^* & \text{(B.1.16)}\\ \nabla_+\Sigma| &= -i\sqrt{2}\bar\lambda_+, & \bar\nabla_-\Sigma| &= -i\sqrt{2}\lambda_- & \text{(B.1.17)}\\ \nabla_-\bar\Sigma| &= i\sqrt{2}\bar\lambda_-, & \bar\nabla_+\bar\Sigma| &= i\sqrt{2}\lambda_+ & \text{(B.1.18)}\\ \bar\nabla_-\nabla_+\Sigma| &= \sqrt{2}(D^3 - iF_{-+}), & \bar\nabla_+\nabla_-\bar\Sigma| &= \sqrt{2}(D^3 + iF_{-+})\,. & \text{(B.1.19)}\end{aligned}$$

For Q we define

$$\bar Q \equiv Q^\dagger e^{2V}\,, \tag{B.1.20}$$

and components

$$
\begin{aligned}
Q| = q, &\qquad \bar{Q}| = q^* &\quad \text{(B.1.21)}\\
\nabla_+ Q| = \sqrt{2}\psi_+, &\quad \nabla_- Q| = \sqrt{2}\psi_- &\quad \text{(B.1.22)}\\
\bar{\nabla}_+ \bar{Q}| = \sqrt{2}\bar{\psi}_+, &\quad \bar{\nabla}_- \bar{Q}| = \sqrt{2}\bar{\psi}_- &\quad \text{(B.1.23)}\\
\nabla_-\nabla_+ Q| = 2F, &\quad \bar{\nabla}_+\bar{\nabla}_- \bar{Q}| = 2F^* &\quad \text{(B.1.24)}
\end{aligned}
$$

This way, in WZ gauge, the components of $\bar{Q}$ are the complex conjugates of the components of Q.

B.1.4 Action

$$
\begin{aligned}
-4\mathcal{L}_{\bar{Q}Q} =& \int \nabla_+\nabla_-\bar{\nabla}_+\bar{\nabla}_-(\bar{Q}Q) = \int \nabla_+\nabla_-(Q\bar{\nabla}_+\bar{\nabla}_-\bar{Q})\\
=& \int \nabla_+\Big[\nabla_- Q(\bar{\nabla}_+\bar{\nabla}_-\bar{Q}) + Q(\nabla_-\bar{\nabla}_+\bar{\nabla}_-\bar{Q})\Big]\\
=&(\nabla_+\nabla_- Q)(\bar{\nabla}_+\bar{\nabla}_-\bar{Q}) - (\nabla_- Q)(\nabla_+\bar{\nabla}_+\bar{\nabla}_-\bar{Q})\\
&(\nabla_+ Q)(\nabla_-\bar{\nabla}_+\bar{\nabla}_-\bar{Q}) + Q(\nabla_+\nabla_-\bar{\nabla}_+\bar{\nabla}_-\bar{Q})
\end{aligned}
$$

- 1: $(\nabla_+\nabla_- Q)(\bar{\nabla}_+\bar{\nabla}_-\bar{Q}) = -4|F|^2$
- 2: $-(\nabla_- Q)(\nabla_+\bar{\nabla}_+\bar{\nabla}_-\bar{Q})$

$$
\begin{aligned}
-(\nabla_- Q)(\nabla_+\bar{\nabla}_+\bar{\nabla}_-\bar{Q}) =& -(\nabla_- Q)\Big(\{\nabla_+, \bar{\nabla}_+\} - \bar{\nabla}_+\nabla_+\Big)\bar{\nabla}_-\bar{Q}\\
=& - 4i\psi_-\mathcal{D}_{++}\psi_- + \sqrt{2}\psi_-\bar{\nabla}_+(2\bar{\Sigma}\bar{Q})\\
=& - 4i\psi_-\mathcal{D}_{++}\bar{\psi}_- + 4i\psi_-\lambda_+ q^* + 4\psi_-\bar{\psi}_+\sigma^*
\end{aligned}
$$

- 3: $(\nabla_+ Q)(\nabla_-\bar{\nabla}_+\bar{\nabla}_-\bar{Q})$

$$
\begin{aligned}
(\nabla_+ Q)(\nabla_- \bar{\nabla}_+ \bar{\nabla}_- \bar{Q}) =&(\nabla_+ Q)(\{\nabla_-, \bar{\nabla}_+\} - \bar{\nabla}_+ \nabla_-)\bar{\nabla}_- \bar{Q} \\
=&(\nabla_+ Q)(2\Sigma)\bar{\nabla}_- \bar{Q} - (\nabla_+ Q)(\bar{\nabla}_+ \nabla_- \bar{\nabla}_- \bar{Q}) \\
=&(\nabla_+ Q)(2\Sigma)\bar{\nabla}_- \bar{Q} - (\nabla_+ Q)(\bar{\nabla}_+ 2i\mathcal{D}_{--} \bar{Q}) \\
=&(\nabla_+ Q)(2\Sigma)\bar{\nabla}_- \bar{Q} \\
&- (\nabla_+ Q)([\bar{\nabla}_+, 2i\mathcal{D}_{--}] + 2i\mathcal{D}_{--}\bar{\nabla}_+)\bar{Q} \\
=&(\nabla_+ Q)(2\Sigma)\bar{\nabla}_- \bar{Q} \\
&- (\nabla_+ Q)(-2\bar{\nabla}_- \Sigma + 2i\mathcal{D}_{--}\bar{\nabla}_+)\bar{Q} \\
=&4\psi_+ \sigma \bar{\psi}_- - \sqrt{2}\psi_+(-2(-i\sqrt{2}\lambda_-) + 2i\mathcal{D}_{--}\bar{\psi}_+) \\
\\
=&4\psi_+ \sigma \bar{\psi}_- - 4i\psi_+ \lambda_- q^* + 4i\psi_+ \mathcal{D}_{--}\bar{\psi}_+
\end{aligned}
$$

- 4: $Q(\nabla_+ \nabla_- \bar{\nabla}_+ \bar{\nabla}_- \bar{Q})$

$$
\begin{aligned}
\nabla_+ \nabla_- \bar{\nabla}_+ \bar{\nabla}_- \bar{Q} =&\nabla_+(\{\nabla_-, \bar{\nabla}_+\} - \bar{\nabla}_+ \nabla_-)\bar{\nabla}_- \bar{Q} \\
=&\nabla_+(2\Sigma - \bar{\nabla}_+ \nabla_-)\bar{\nabla}_- Q \\
=&\nabla_+(2\Sigma \bar{\nabla}_- \bar{Q}) - \nabla_+ \bar{\nabla}_+ \nabla_- \bar{\nabla}_- \bar{Q} \\
=&2(\nabla_+ \Sigma)(\bar{\nabla}_- \bar{Q}) + 2\Sigma(\nabla_+ \bar{\nabla}_- \bar{Q}) - \nabla_+ \bar{\nabla}_+ 2i\mathcal{D}_{--}\bar{Q}
\end{aligned}
$$

Now,

$$
\begin{aligned}
-\nabla_+ \bar{\nabla}_+ 2i\mathcal{D}_{--}\bar{Q} =& - \nabla_+([\bar{\nabla}_+, 2i\mathcal{D}_{--}] + 2iD_{--}\bar{\nabla}_+)\bar{Q} \\
=& - \nabla_+(-2\bar{\nabla}_- \Sigma + 2i\mathcal{D}_{--}\bar{\nabla}_+)\bar{Q} \\
=&2\nabla_+(\bar{\nabla}_- \Sigma)\bar{Q} - \nabla_+(2i\mathcal{D}_{--}\bar{\nabla}_+ \bar{Q}) \\
=&2(\nabla_+ \bar{\nabla}_- \Sigma)\bar{Q} - \nabla_+(2i\mathcal{D}_{--}\bar{\nabla}_+ \bar{Q})
\end{aligned}
$$

and

$$
\begin{aligned}
-\nabla_+(2i\mathcal{D}_{--}\bar{\nabla}_+ \bar{Q}) =& - ([\nabla_+, 2i\mathcal{D}_{--}] + 2i\mathcal{D}_{--}\nabla_+)\bar{\nabla}_+ \bar{Q} \\
=& - (-2\nabla_- \bar{\Sigma} + 2i\mathcal{D}_{--}\nabla_+)\bar{\nabla}_+ \bar{Q} \\
=&2(\nabla_- \bar{\Sigma})(\bar{\nabla}_+ \bar{Q}) + 4\mathcal{D}_{--}\mathcal{D}_{++}\bar{Q} \\
=&4i\bar{\lambda}_- \bar{\psi}_+ + 4\mathcal{D}_{--}\mathcal{D}_{++}\bar{Q}\,.
\end{aligned}
$$

Putting all these terms together, we have

$$Q\Big[2\nabla_+\Sigma\bar{\nabla}_-\bar{Q}+4\Sigma\bar{\Sigma}\bar{Q}+2(\nabla_+\bar{\nabla}_-\Sigma)\bar{Q} \tag{B.1.25}$$

$$+2(\nabla_-\bar{\Sigma})(\bar{\nabla}_+\bar{Q})+4\mathcal{D}_{--}\mathcal{D}_{++}\bar{Q}\Big]\,, \tag{B.1.26}$$

and projecting we have

$$4q\Big[-i\bar{\lambda}_+\bar{\psi}_-+|\sigma|^2q^*+i\bar{\lambda}_-\bar{\psi}_++\mathcal{D}_{--}\mathcal{D}_{++}q^*\Big]-2q\sqrt{2}(D^3-iF_{-+})q^*\,.$$

Now we use the expression for F_{-+} derived previously:

$$F_{-+}=\frac{i}{\sqrt{2}}[\mathcal{D}_{--},\mathcal{D}_{++}] \tag{B.1.27}$$

to write this term as:

$$\begin{aligned} 4q\mathcal{D}_{--}\mathcal{D}_{++}q^*+2iq\sqrt{2}F_{-+}q^* &= 4q\mathcal{D}_{--}\mathcal{D}_{++}q^*-2q[\mathcal{D}_{--},\mathcal{D}_{++}]q^* \\ &= 2q\Big[\mathcal{D}_{--}\mathcal{D}_{++}+\mathcal{D}_{++}\mathcal{D}_{--}\Big]q^* \end{aligned}$$

and so we have

$$4q\Big[-i\bar{\lambda}_+\bar{\psi}_-+|\sigma|^2q^*+i\bar{\lambda}_-\bar{\psi}_+\Big]+2q\Big[\mathcal{D}_{--}\mathcal{D}_{++}+\mathcal{D}_{++}\mathcal{D}_{--}\Big]q^*-2\sqrt{2}|q|^2D^3\,.$$

Putting all the terms together, the Lagrangian reads

$$\mathcal{L}_{kin}=-\left|\mathcal{D}q\right|^2+i(\bar{\psi}_-\mathcal{D}_{++}\psi_-+\bar{\psi}_+\mathcal{D}_{--}\psi_+)\,, \tag{B.1.28}$$

$$\mathcal{L}_{yuk} = -iq\Big[\bar{\lambda}_-\bar{\psi}_+-\bar{\lambda}_+\bar{\psi}_-\Big]+\sigma\bar{\psi}_-\psi_+ \tag{B.1.29}$$

$$+iq^*\Big[\psi_+\lambda_--\psi_-\lambda_+\Big]+\sigma^*\bar{\psi}_+\psi_-\,, \tag{B.1.30}$$

and

$$\mathcal{L}_{pot} = |F|^2 - |\sigma|^2|q|^2 + \frac{1}{\sqrt{2}}|q|^2 D^3\,. \tag{B.1.31}$$

Appendix C

Darboux Coordinates

C.1 Projectors

Here we give some details on the Antarctic (Π_N) and Arctic (Π_R) projectors. They are defined by

$$\Pi_N = \oint_{C_0} \frac{d\zeta'}{2\pi i} \frac{1}{\zeta - \zeta'}\,, \qquad \Pi_R = \oint_{C_0} \frac{d\zeta'}{2\pi i} \frac{1}{\zeta' - \zeta}\,,$$

where C_0 is a closed contour enclosing the origin. Consider the Laurent expansion around $\zeta = 0$ of the function $f(\zeta) = \sum_{m=-\infty}^{\infty} c_m \zeta^m$. Applying the projector Π_N, we will need to calculate

$$\oint_{C_0} \frac{d\zeta'}{2\pi i} \frac{\zeta'^m}{\zeta - \zeta'}\,.$$

Since there is a pole at $\zeta' = \zeta$, we avoid the singularity by moving the pole slightly *outwards* in the radial direction. This can be achieved by introducing the $\epsilon-$prescription

$$\oint_{C_0} \frac{d\zeta'}{2\pi i} \frac{\zeta'^m}{\zeta - \zeta' + \epsilon(\zeta + \zeta')}\,.$$

If $m \geq 0$, there are no singularities enclosed by the contour and the integral vanishes. If $m < 0$ the residue is simply ζ^m. Thus, only negative powers survive:

$$\oint_{C_0} \frac{d\zeta'}{2\pi i} \frac{\zeta'^m}{\zeta - \zeta'} = \begin{cases} 0 & \text{if } m \geq 0 \\ \zeta^m & \text{if } m < 0 \end{cases}\,,$$

or

$$\oint_{C_0} \frac{d\zeta'}{2\pi i} \frac{f(\zeta')}{\zeta - \zeta'} = \sum_{m=1}^{\infty} \frac{c_{-m}}{\zeta^m}\,. \qquad \text{(C.1.-2)}$$

Using the same $\epsilon-$prescription as above, the action of Π_R on $f\left(\zeta\right)$ is given by

$$\oint_{C_0} \frac{d\zeta'}{2\pi i} \frac{f(\zeta')}{\zeta' - \zeta} = \sum_{m=0}^{\infty} c_m \zeta^m \,. \tag{C.1.-2}$$

Thus, $\Pi_N + \Pi_R = 1$ as expected. In addition to these, we can construct other projectors by using appropriate powers of ζ/ζ'. An example of that is $\bar{\Pi}_N$, which annihilates the non-positive powers of ζ. Thus the combinations $\Pi_N \pm \bar{\Pi}_N$, annihilate only the ζ-independent term.

C.2 *c*-map

The c-map [75] relates classical hypermultiplet moduli spaces in compactifications of type II strings on a Calabi-Yau threefold to vector multiplet moduli spaces via a further compactification on a circle. In [76, 77], it was shown that the c-map has a natural description in projective superspace. It can be regarded as taking a vector multiplet from four to three dimensions and reinterpreting it as a tensor multiplet when returning to four dimensions. This is possible because in three-dimensions, a vector multiplet is equivalent to a tensor multiplet, which can then be dualized into a hypermultiplet in four dimensions.

This means that given an $\mathcal{N} = 2$ holomorphic prepotential $\mathcal{F}\left(W\right)$ describing a vector multiplet:

$$\mathcal{L}_v = -\mathrm{Im}\left[\int d^2\theta d^2\vartheta \mathcal{F}\left(W\right)\right] \,, \tag{C.2.0}$$

there is a corresponding dual projective hypermultiplet Lagrangian $\mathcal{G}$ describing a hyperkähler moduli space given by

$$\begin{aligned}\mathcal{L}_s = \int d^2\theta d^2\bar{\theta} \oint \frac{d\zeta}{2\pi i\zeta} \mathcal{G}(\zeta;\eta_{\rceil}) &= \int \lceil^{\in}\theta \lceil^{\in}\bar{\theta} \oint \frac{\lceil\zeta}{\in\pi\rangle\zeta} \overline{\mathrm{Im}} \left[\frac{\mathcal{F}(\zeta\eta_{\rceil})}{\zeta^{\in}}\right] \\ = -\, i \int d^2\theta d^2\bar{\theta} \oint \frac{d\zeta}{2\pi i\zeta} \left[\frac{\mathcal{F}\left(\zeta\eta_e\right)}{\zeta^2} - \overline{\mathcal{F}\left(\zeta\eta_e\right)}\, \zeta^2\right] \,.\end{aligned}$$

This expression determines the semiflat projective Lagrangian f^{sf} in (4.3.9).

Appendix D

Matrix Models

D.1 Roots of $\widehat{A}_{m-1}$ and $\widehat{D}_n$

Here we give some useful information about the roots for $\widehat{A}$ and $\widehat{D}$ Lie algebras. For $\widehat{A}_{m-1}$ we choose the following root basis

$$\tilde{\alpha}_a = e_a - e_{a+1}\,, \quad a = 1, ..., m-1\,; \qquad \tilde{\theta} = -e_1 + e_m\,,$$

where e_a are canonical unit vectors of dimension m. For $\widehat{D}_n$ we choose

$$\alpha_i = e_i - e_{i+1}\,, \quad i = 1, ..., n-1\,; \qquad \alpha_n = e_{n-1} + e_n\,, \qquad \theta = -(e_1 + e_2)\,,$$

where e_i are the unit vectors of dimension n.

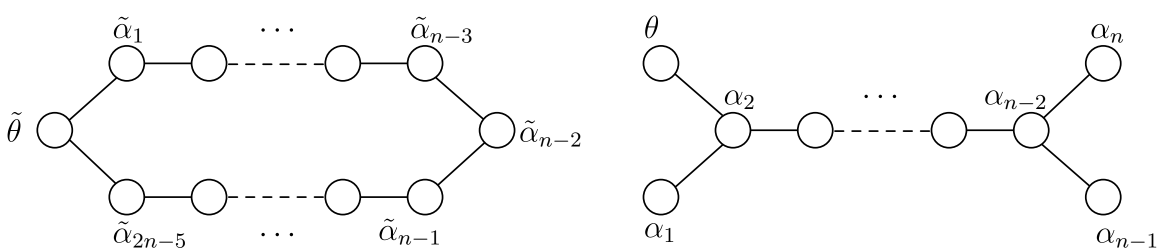

Figure D.1: Dynkin diagrams for $\widehat{A}_{2n-5}$ and $\widehat{D}_n$.

In Fig. D.1 we show the affine Dynkin diagrams for the $\widehat{A}$ and $\widehat{D}$ Lie algebras along with the roots associated with every node. At each node, the CS level is given by $\tilde{\alpha} \cdot q$ and $\alpha \cdot p$ for $\widehat{A}$ and $\widehat{D}$, respectively. The identification of opposite CS levels in the $\widehat{A}_{2n-5}$ quiver imposes $\tilde{\alpha}_a \cdot q = \tilde{\alpha}_{2n-4-a} \cdot q$ and hence $q_a = -q_{2n-3-a}$ for $a = 1, ..., n-2$. With these conventions, unfolding the $\widehat{D}$-quiver to the $\widehat{A}$-quiver relates $q_a = p_{a+1}$.

D.2 $\widehat{D}_5$

Here we give the detailed solution of the matrix model for the $\widehat{D}_5$ quiver gauge theory. As discussed in Section 5.3, there are 7 regions defining a generic solution of this model. To keep the notation simple, the second index for the four y's corresponding to the external nodes is suppressed.

Region 1: $0 \le x \le \frac{\mu}{3(k_2+k_3+k_4+2k_5+2k_6)}$

$$\begin{aligned}
\rho &= \frac{\mu}{3}\,; \\
y_1 - y_{6,2} &= \frac{(2k_1 - k_3 - k_4 - 2k_6)\,x}{4\rho}\,, \\
y_2 - y_{6,2} &= \frac{(2k_2 - k_3 - k_4 - 2k_6)\,x}{4\rho}\,, \\
y_3 - y_{6,2} &= \frac{k_3 x}{2\rho}\,, \\
y_4 - y_{6,2} &= \frac{k_4 x}{2\rho}\,, \\
y_{5,1} - y_{6,2} = y_{5,2} - y_{6,2} &= -\frac{(k_3 + k_4 + 2k_6)x}{4\rho}\,, \\
y_{6,1} - y_{6,2} &= 0\,.
\end{aligned}$$

Region 2: $\frac{\mu}{3(k_2+k_3+k_4+2k_5+2k_6)} \leq x \leq \frac{4\mu}{6k_2+9k_3+9k_4+12k_5+18k_6}$

$$\rho = \frac{\mu}{3}\,;$$
$$y_1 - y_{5,2} = -\frac{1}{2}\,,$$
$$y_2 - y_{6,2} = \frac{(2k_2 - k_3 - k_4 - 2k_6)\,x}{4\rho}\,,$$
$$y_3 - y_{6,2} = \frac{k_3 x}{2\rho}\,,$$
$$y_4 - y_{6,2} = \frac{k_4 x}{2\rho}\,,$$
$$y_{5,1} - y_{6,2} = -\frac{1}{2} - \frac{(2k_1 + k_3 + k_4 + 2k_6)\,x}{4\rho}\,,$$
$$y_{5,2} - y_{6,2} = \frac{1}{2} + \frac{(2k_1 - k_3 - k_4 - 2k_6)\,x}{4\rho}\,,$$
$$y_{6,1} - y_{6,2} = 0\,.$$

Region 3: $\frac{4\mu}{6k_2+9k_3+9k_4+12k_5+18k_6} \leq x \leq \frac{2\mu}{3(2k_2+k_3+k_4+2k_5+2k_6)}$

$$\rho = \frac{\mu}{3}\,;$$
$$y_1 - y_{5,2} = -\frac{1}{2}\,,$$
$$y_2 - y_{6,2} = -1 - \frac{(k_1 - k_2)\,x}{2\rho}\,,$$
$$y_3 - y_{6,2} = -1 - \frac{(2k_1 - 3k_3 - k_4 - 2k_6)\,x}{4\rho}\,,$$
$$y_4 - y_{6,2} = -1 - \frac{(2k_1 - k_3 - 3k_4 - 2k_6)\,x}{4\rho}\,,$$
$$y_{5,1} - y_{6,2} = -\frac{3}{2} - \frac{k_1 x}{\rho}\,,$$
$$y_{5,2} - y_{6,2} = -\frac{1}{2}\,,$$
$$y_{6,1} - y_{6,2} = -2 - \frac{(2k_1 - k_3 - k_4 - 2k_6)\,x}{2\rho}\,.$$

Region 4: $\frac{2\mu}{3(2k_2+k_3+k_4+2k_5+2k_6)} \leq x \leq \frac{2\mu}{2k_2+3(k_3+k_4+2k_5+2k_6)}$

$$\rho = \frac{\mu}{2} + \frac{x}{4}\left(k_1 - k_2\right)\,;$$
$$y_1 - y_{5,2} = -\frac{1}{2}\,,$$
$$y_2 - y_{6,2} = 0\,,$$
$$y_3 - y_{6,2} = -\frac{1}{2} + \frac{\left(2k_3 + k_4 + k_5 + 2k_6\right)x}{2\rho}\,,$$
$$y_4 - y_{6,2} = -\frac{1}{2} + \frac{\left(k_3 + 2k_4 + k_5 + 2k_6\right)x}{2\rho}\,,$$
$$y_{5,1} - y_{6,2} = -\frac{1}{2} - \frac{\left(k_1 - k_2\right)x}{2\rho}\,,$$
$$y_{5,2} - y_{6,2} = -\frac{1}{2}\,,$$
$$y_{6,1} - y_{6,2} = -1 + \frac{\left(k_3 + k_4 + k_5 + 2k_6\right)x}{\rho}\,.$$

Region 5: $\frac{2\mu}{2k_2+3(k_3+k_4+2k_5+2k_6)} \leq x \leq \frac{2\mu}{2k_2+3k_3+5k_4+4k_5+6k_6}$

$$\rho = \mu + xk_1\,;$$
$$y_1 - y_{5,2} = -\frac{1}{2}\,, \quad y_2 - y_{6,2} = 0\,,$$
$$y_3 - y_{6,2} = \frac{\left(3k_3 + k_4 + 2k_6\right)x}{4\rho}\,,$$
$$y_4 - y_{6,2} = \frac{\left(k_3 + 3k_4 + 2k_6\right)x}{4\rho}\,,$$
$$y_{5,1} - y_{6,2} = \frac{1}{2}\,,$$
$$y_{5,2} - y_{6,2} = -\frac{1}{2}\,,$$
$$y_{6,1} - y_{6,2} = \frac{\left(k_3 + k_4 + 2k_6\right)x}{2\rho}\,.$$

Region 6: $\frac{2\mu}{2k_2+3k_3+5k_4+4k_5+6k_6} \leq x \leq \frac{2\mu}{2k_2+5k_3+3k_4+4k_5+6k_6}$

$$\rho = \frac{3\mu}{2} + \frac{x}{4}\left(6k_1 - k_3 - 3k_4 - 2k_6\right) ;$$
$$y_1 - y_{5,2} = -\frac{1}{2} , \quad y_2 - y_{6,2} = 0 ,$$
$$y_3 - y_{6,2} = \frac{1}{6} + \frac{(2k_3 + k_6)\, x}{3\rho} ,$$
$$y_4 - y_{6,2} = \frac{1}{2} , \quad y_{5,1} - y_{6,2} = \frac{1}{2} , \quad y_{5,2} - y_{6,2} = -\frac{1}{2} , \quad y_{6,1} - y_{6,2} = \frac{1}{3} + \frac{(k_3 + 2k_6)\, x}{3\rho} .$$

Region 7: $\frac{2\mu}{2k_2+5k_3+3k_4+4k_5+6k_6} \leq x \leq \frac{2\mu}{2k_2+3k_3+3k_4+4k_5+6k_6}$

$$\rho = 2\mu + x\left(2k_1 - k_3 - k_4 - k_6\right) ;$$
$$y_1 - y_{5,2} = -\frac{1}{2} , \quad y_2 - y_{6,2} = 0 , \quad y_3 - y_{6,2} = \frac{1}{2} , \quad y_4 - y_{6,2} = \frac{1}{2} ,$$
$$y_{5,1} - y_{6,2} = \frac{1}{2} , \quad y_{5,2} - y_{6,2} = -\frac{1}{2} , \quad y_{6,1} - y_{6,2} = \frac{1}{2} + \frac{k_6 x}{2\rho} .$$

Finally, the last saturation occurs at the end of this region with $y_{6,1} = y_{6,2} + 1$.

D.3 Exceptional Quivers

We have also solved the matrix models for the exceptional quivers $\widehat{E}_6, \widehat{E}_7$ and $\widehat{E}_8$. They consist of eleven, seventeen and twenty-nine regions respectively. Here we give the corresponding free energies for a particular ordering of the CS levels. In Fig. D.2, we show our conventions in labeling the nodes.

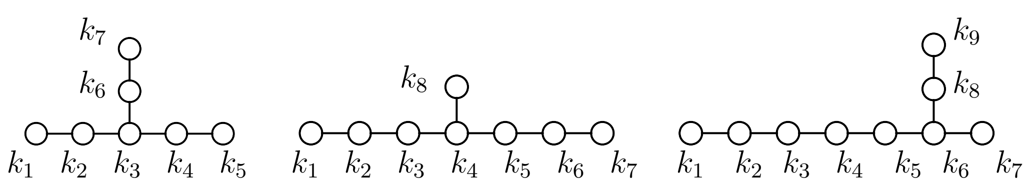

Figure D.2: Labeling of Chern-Simons levels for $\widehat{E}_6, \widehat{E}_7$ and $\widehat{E}_8$.

$\widehat{E}_6$

The matrix model for $\widehat{E}_6$ gives:

$$
\begin{aligned}
\frac{2}{\mu^2} =& \frac{2(4k_2 + 11k_3 + 8k_4 + 4k_5 + 6k_6 + 4k_7)}{(2k_2 + 5k_3 + 4k_4 + 2k_5 + 3k_6 + 2k_7)^2} \\
&- \frac{1}{42(3k_2 + 6k_3 + 4k_4 + 2k_5 + 5k_6 + k_7)} \\
&- \frac{1}{77(13k_2 + 12k_3 + 8k_4 + 4k_5 + 3k_6 + 2k_7)} \\
&- \frac{1}{3(3k_2 + 6k_3 + 4k_4 + 2k_5 + 5k_6 + 4k_7)} \\
&- \frac{9}{6k_2 + 14k_3 + 13k_4 + 6k_5 + 9k_6 + 6k_7} \\
&- \frac{9}{11(6k_2 + 14k_3 + 13k_4 + 12k_5 + 9k_6 + 6k_7)},
\end{aligned}
$$

under the assumptions that $k_6 \geq k_5 \geq k_4 \geq k_3 \geq k_2 \geq 0$ and $k_7 > 3k_2 + 6k_3 + 4k_4 + 2k_5 + 4k_6$.

$\widehat{E}_7$

The matrix model for $\widehat{E}_7$ gives:

$$
\begin{aligned}
\frac{2}{\mu^2} =& \frac{8k_2 + 24k_3 + 42k_4 + 4(8k_5 + 6k_6 + 3k_7 + 5k_8)}{(2k_2 + 6k_3 + 10k_4 + 8k_5 + 6k_6 + 3k_7 + 5k_8)^2} \\
&- \frac{1}{2k_2 + 7k_3 + 10k_4 + 8k_5 + 6k_6 + 3k_7 + 5k_8} \\
&- \frac{1}{2k_2 + 6k_3 + 10k_4 + 9k_5 + 6k_6 + 3k_7 + 5k_8} \\
&- \frac{1}{180(2k_2 + 3k_3 + 4k_4 + 3k_5 + 2k_6 + k_7 + 2k_8)} \\
&- \frac{4}{15(4k_2 + 11k_3 + 2(9k_4 + 8k_5 + 7k_6 + 6k_7) + 9k_8)} \\
&- \frac{27}{7(6k_2 + 17k_3 + 28k_4 + 24k_5 + 20k_6 + 9k_7 + 15k_8)} \\
&- \frac{32}{21(8k_2 + 25k_3 + 42k_4 + 32k_5 + 22k_6 + 12k_7 + 27k_8)},
\end{aligned}
$$

under the assumptions that $k_7 \geq k_6 \geq k_5 \geq k_4 \geq k_3 \geq k_2 \geq 0$ along with

$4k_3 + k_4 > 2k_5 + k_6$, $k_3 + 2k_4 + k_5 > k_7$, $k_4 + k_5 > k_6$ and $3k_8 > 6k_3 + 12k_4 + 15k_5 + 10k_6 + 5k_7$.

$\widehat{E}_8$

The solution of the matrix model for $\widehat{E}_8$ gives:

$$\begin{aligned}
\frac{2}{\mu^2} =& \frac{8k_2 + 24k_3 + 48k_4 + 74k_5 + 92k_6 + 48k_7 + 64k_8 + 32k_9}{(2k_2 + 6k_3 + 12k_4 + 18k_5 + 23k_6 + 12k_7 + 16k_8 + 8k_9)^2} \\
&- \frac{1}{3150(2k_2 + 3k_3 + 4k_4 + 5k_5 + 6k_6 + 3k_7 + 4k_8 + 2k_9)} \\
&- \frac{1}{2(k_2 + 3k_3 + 6k_4 + 9k_5 + 12k_6 + 6k_7 + 8k_8 + 4k_9)} \\
&- \frac{1}{2k_2 + 6k_3 + 13k_4 + 18k_5 + 23k_6 + 12k_7 + 16k_8 + 8k_9} \\
&- \frac{27}{7(6k_2 + 18k_3 + 35k_4 + 52k_5 + 69k_6 + 38k_7 + 48k_8 + 24k_9)} \\
&- \frac{108}{35(12k_2 + 36k_3 + 70k_4 + 104k_5 + 138k_6 + 69k_7 + 103k_8 + 48k_9)} \\
&- \frac{36}{55(12k_2 + 36k_3 + 70k_4 + 104k_5 + 138k_6 + 69k_7 + 103k_8 + 68k_9)} \\
&- \frac{9}{154(6k_2 + 17(3k_3 + 4k_4 + 5k_5 + 6k_6 + 3k_7 + 4k_8 + 2k_9))} \,,
\end{aligned}$$

assuming that $k_6 \geq k_5 \geq k_4 \geq k_3 \geq k_2 \geq 0$, $k_7 > 3k_4 + 6k_5 + 4k_6$, $2k_4 + 4k_5 + 6k_6 + 9k_7 > k_8$, $2k_3 + 4k_4 + 6k_5 + 8k_6 + 4k_7 + 6k_8 > k_9$ and $2k_9 > 6k_3 + 12k_4 + 18k_5 + 24k_6 + 16k_7 + 11k_8$.