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Geometric Aspects of Quantum Hall States

A Dissertation Presented

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Andrey Gromov

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Abstract of the Dissertation

Geometric Aspects of Quantum Hall States

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Explanation of the quantization of the Hall conductance at low temperatures in strong magnetic field is one of the greatest accomplishments of theoretical physics of the end of the 20th century. Since the publication of the Laughlin's charge pumping argument condensed matter theorists have come a long way to topological insulators, classification of noninteracting (and sometimes interacting) topological phases of matter, non-abelian statistics, Majorana zero modes in topological superconductors and topological quantum computation - the framework for "error-free" quantum computation. While topology was very important in these developments, geometry has largely been neglected.

We explore the role of space-time symmetries in topological phases of matter. Such symmetries are responsible for the conservation of energy, momentum and angular momentum. We will show that if these symmetries are maintained (at least on average) then in addition to Hall conductance there are other, in principle, measurable transport coefficients that are quantized and sensitive to topological phase transition. Among these coefficients are non-

dissipative viscosity of quantum fluids, known as Hall viscosity; thermal Hall conductance, and a recently discovered coefficient - orbital spin variance. All of these coefficients can be computed as linear responses to variations of geometry of a physical sample. We will show how to compute these coefficients for a variety of abelian and non-abelian quantum Hall states using various analytical tools: from RPA-type perturbation theory to non-abelian Chern-Simons-Witten effective topological quantum field theory.

We will explain how non-Riemannian geometry known as Newton-Cartan (NC) geometry arises in the computation of momentum and energy transport in non-relativistic gapped systems. We use this geometry to derive a number of thermodynamic relations and stress the non-relativistic nature of condensed matter systems. NC geometry is also useful in the study of Galilean invariant systems in manifestly coordinate independent form. We study the Ward identities of the Galilean symmetry and find new relations between universal, quantized transport coefficients and long-wave corrections there of.

Publications

1. A. Gromov, A. Abanov "Induced action for non-interacting fermions in magnetic field: perturbative computation"
   (In preparation)

2. A. Gromov, K. Jensen, A. Abanov "Boundary effective action for quantum Hall states"
   (arXiv:1506.07171)

3. B. Bradlyn, A. Gromov "Supersymmetric waves in Bose-Fermi mixtures"
   (arXiv:1504.08019, submitted to PRL)

4. F. Franchini, A. Gromov, M. Kulkarni, A. Trombettoni "Universal dynamics of a soliton after a quantum quench"
   J. Phys. A: Math. Theor. 48 (2015) 28FT01

5. A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, E. Fradkin "Framing Anomaly in the Effective Theory of Fractional Quantum Hall Effect"
   Phys. Rev. Lett. 114, 016805 (Editor's Suggestion)

6. A. Gromov, A. G. Abanov "Thermal Hall and Geometry with Torsion"
   Phys. Rev. Lett. 114, 016802

7. A. Gromov, A. G. Abanov "Density-curvature response and gravitational anomaly"
   Phys. Rev. Lett. 113, 266802

8. A. Gromov, R. Santos "Entanglement Entropy in 2D Non-abelian gauge theory"
   Physics Letters B 737 (2014) 60-64

9. A. G. Abanov, A. Gromov "Electromagnetic and gravitational responses of two-dimensional noninteracting electrons in constant magnetic field"
   Phys. Rev. B 90, 014435

10. A. G. Abanov, A. Gromov, M. Kulkarni "Soliton solutions of a Calogero model in Harmonic potential"
    J. Phys. A: Math. Theor. 44 (2011) 295203 (21pp)

To my family.

Contents

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Chapter 1

Introduction

1.1 Observation of integer quantum Hall effect

Around 35 years ago, in High Magnetic Field Laboratory in Grenoble, in the middle of the night Klaus von Klitzing observed strange behavior of the Hall conductance in a quasi $2D$ layer of metal-oxide-semi-conductor field-effect transistor or MOSFET [1]. This strange behavior is depicted in Fig. 1.1.2.

The classical electromagnetism predicts that in strong magnetic field in a $2D$ material there will be a current transverse to external magnetic field and to the potential difference. This current was first observed by Edwin Hall. The resistance of the material is classically given by

$$R_H = \frac{B}{e\rho} = \frac{1}{e\nu}\,, \tag{1.1}$$

where we have introduced the carrier density ρ, magnetic field B and filling fraction ν defined as ration of density to magnetic field. Classically, ν can take any value.

In reality at certain values of ν the longitudinal resistance would vanish (at low temperature) and the material would turn into a perfect insulator. In this insulating state the material will not conduct current along the potential difference thus there will be no dissipation. Instead, the material allows for a *non-dissipative* current in the direction transverse to the potential difference. In this state the conductance (or resistance) is precisely quantized in the units $\frac{e^2}{h}$ ($\frac{h}{e^2}$) with accuracy of one part in a billion. This effect of quantization of Hall conductance is called Integer Quantum Hall Effect (IQHE).

In the original work [1] it was suggested that such an accurate measurement

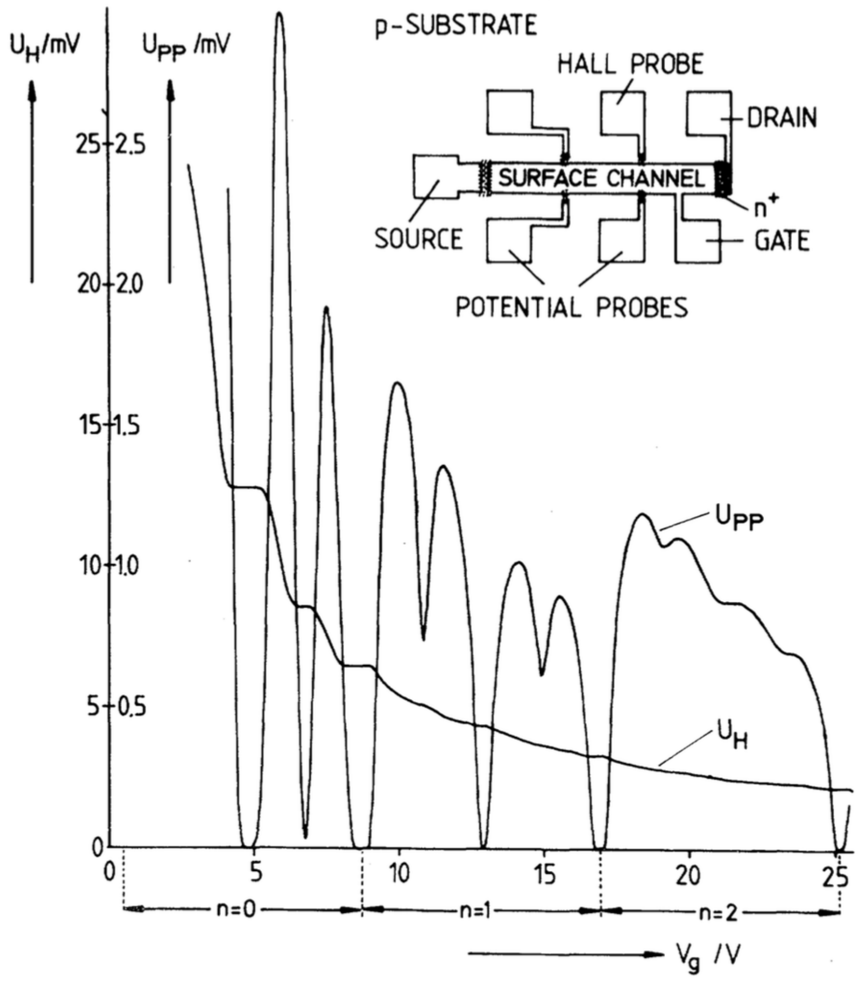

Figure 1.1: This is a plot obtained by von Klitzing [1]. It shows the Hall voltage plotted against the voltage drop between the potential probes. Notice that for special values of the filling factor n there are plateaus in the dependence. These plateaus contradict the classical e/m prediction. The resistance on these plateaus is quantized precisely in the units of $\frac{e^2}{h}$ as $1, \frac{1}{2}, \frac{1}{3}, \ldots$. The formation of the plateaus is called integer quantum Hall effect of IQHE.

of resistance can provide the most accurate procedure to measure the fine structure constant (which, unfortunately, did not happen for the reasons). To this day the Hall resistance quantization serves a standard unit of resistance. The unit Ohm is defined through the von Klitzing constant $R_{K-90} = 25812.807$, which is the Hall resistance at filling fraction $\nu = 1$. Von Klitzing constant *does not depend on material properties, concentration of impurities or sample geometry*: it is truly a stunning consequence of coherent collective behavior of electrons at low temperatures and strong magnetic field.

1.1.1 Laughlin charge pumping argument

It took theorists about a year to explain this precise quantization. It was clear that there must be a truly fundamental (not depending on microscopic detailes) principle at work. The explanation was given in an ingenious 2 page paper by Robert Laughlin [5]. The fundamental principle turned out to be the charge conservation or, more formally, gauge invariance.

Laughlin considered a sample of cylindrical shape with external magnetic field perpendicular to the surface of the cylinder. He assumed that the chemical potential lies in the mobility gap (or, in clean case, between the Landau levels) then the density of the conducting states will be small an longitudinal conductance will vanish. Now consider and adiabatic threading of one quantum of magnetic flux Φ_0 through the cylinder then the net effect of the flux threading would be transfer of one unit of charge from one edge to the other. The flux threading is equivalent to a gauge transformation and therefore in the end of the adiabatic process the quantum state of the system would not change. However, during the adiabatic flux threading by the Faradey's law there was a current $I \sim \frac{\partial U}{\partial \Phi}$ around the cylinder, where U is the electron energy. If the potential difference between the edges of a cylinder is V then the change in electron energy is $\partial U = e \times V$ whereas the flux quantum is $\Phi_0 = \frac{h}{e} = \partial\Phi$. The ratio gives

$$I = \frac{\partial U}{\partial \Phi} = \frac{e^2}{h} V \tag{1.2}$$

thus concluding that Hall resistance is $R_H = \frac{h}{e^2}$.

When the physical system consists of dirty, weakly interacting electrons the argument still holds as long as there are extended states in the bulk that will carry the charge. The brilliance of this argument is that it relies on the fact that even in a dirty system, at finite (but small) temperature the gauge invariance or charge conservation is still an exact symmetry and therefore after a flux insertion one electron can only travel from one edge to the other. It could not disappear or split into several "quarks" etc.

Around the same time it was realized [6] that (integer) Hall conductance can be understood as a topological invariant of the $U(1)$ bundle over a Brillouin zone, thus giving a very strong argument for the *topological* protection of the value of Hall conductance. Topological invariants cannot change by a small amount under any continuous deformations of, say, band structure or external (random) potential.

1.1.2 Gapless edge states

In a subsequent work another important observation was made by Halperin [2]. An important part of the Laughlin argument was existence of the extended states in the bulk, which at the time was a controversial topic. Halperin has shown that even when the bulk of the quantum Hall system is insulating there are always edged states that are localized in the direction transverse to the edge, but are extended in the direction along the edge. These extended edge states are stable against disorder and carry *part* of the Hall current.

Later on the picture painted by Halperin was formalized by Wen [7] who proposed the generalization of the gapless edge to interacting quantum Hall systems. In that case the edge states are described by a chiral WZW model. We will have more to say about the edge physics later.

1.2 Fractional quantum Hall effect

Around two years after von Klitzing's observation another great breakthrough has happened Tsui, Stormer and Gossard observed a formation of a plateau at the filling factor $\nu = \frac{1}{3}$ in $GaAs$ heterostructure [3]. This effect was called fractional quantum Hall effect or FQHE.

This was very puzzling at the time, because all of the theoretic understanding was based on, roughly speaking, adding disorder to a free electron problem. In order to study the free problem analytically one had to place the chemical potential outside of Landau level. If the chemical potential is inside a Landau level (which is equivalent to saying that the filling factor is less than one) then the problem becomes extremely degenerate and the linear response cannot be done in a familiar manner.

One of the resolutions would be to include interactions into the picture of IQHE. Unfortunately, this is easier said than done, because the interactions in such systems are usually very strong and analytical treatment is unimaginable. Nonetheless, some unorthodox treatments were suggested.

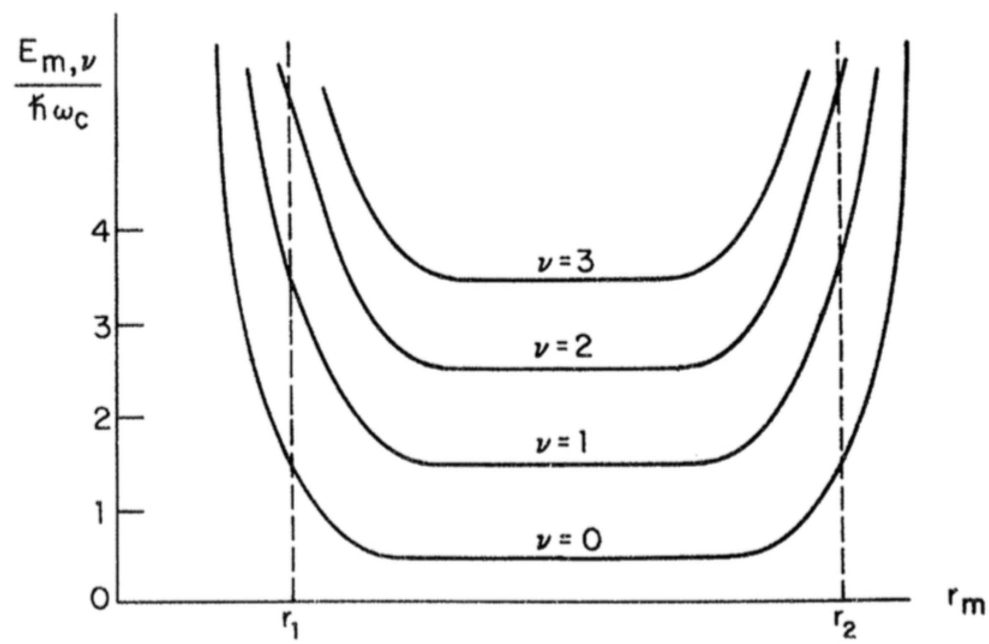

Figure 1.2: The energy levels in a finite size sample taken from the original work of Halperin [2]. One can see that even though in the bulk the gap is well defined, no matter what the value of the Fermi level is, there are always states available at the edge of a sample.

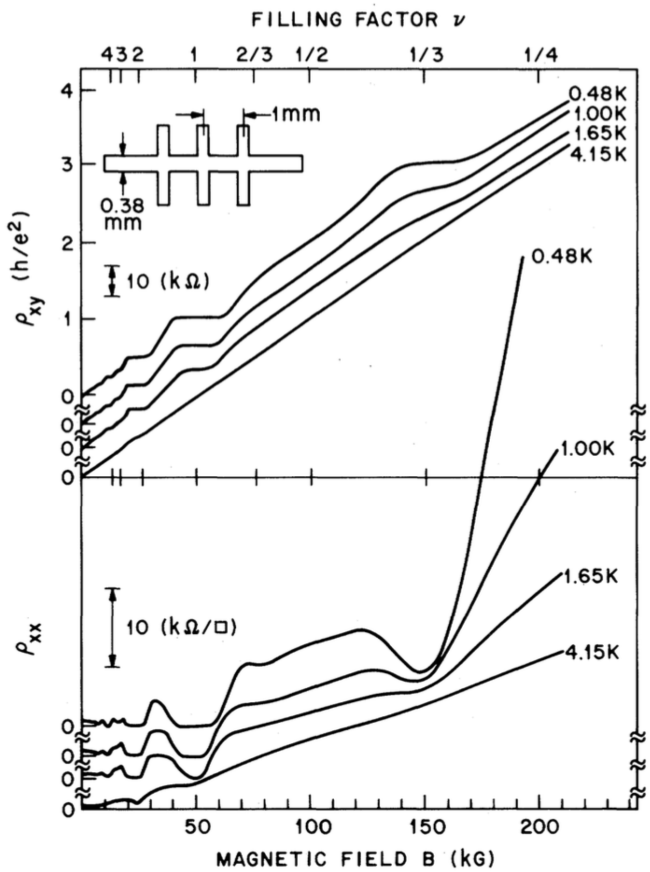

Figure 1.3: A plot from the original work of Tsui [3]. We see a similar picture to Fig. 1.1.2: at the filling $\nu = \frac{1}{3}$ a plateau is formed and longitudinal resistivity vanished.

1.2.1 Laughlin function: rise of the first quantized approach

One year after the experimental observation of fractional Hall conductance was made Laughlin had another brilliant insight [8]. Since the interacting, disordered system is intractable, why not guess a ground state at least approximately? Perhaps, there is some universality in fractional quantum Hall systems that on average can be described by a representative, a "trial" wavefunction that is easy to write down from some general principles and experimental facts? Laughlin also immediately realized that: "The ground state is a new state of matter, a quantum fluid the elementary excitations of which, the quasielectrons and quasiholes, are fractionally charged.". This is still exactly the way we think about FQHE today. This insight and this way of reasoning led to the field of topological phases of matter as it is today (mostly general principles and only a few experimental facts).

To guess the ground state wavefunction Laughlin realized that on the lowest Landau level the wave function must have a form (in symmetric gauge, with magnetic length $l = 1$)

$$\psi \sim \prod_{j \neq k} f(z_j - z_k) e^{-\sum_i \frac{|z_i|^2}{4}} , \tag{1.3}$$

where $z = x+iy$ is a complex coordinate in the $x-y$ plane, $f(z)$ some unknown function of only holomorphic coordinate z and not $\bar{z}$. Since z_k are the electron coordinates $f(z)$ must be anti-symmetric and, in order to conserve angular momentum, $f(z)$ must be a homogeneous polynomial. With this information Laughlin concluded that $f(z) = z^m$, where m is an odd integer. Now, the only parameter left in the problem is m.

What is the relation between m and the filling factor? To answer this question Laughlin used another great insight that is now known as the "plasma mapping". He wrote the square modulus of the wave function as

$$|\psi|^2 = \Big| \prod_{j \neq k} (z_j - z_k)^m e^{-\sum_l \frac{|z_l|^2}{4}} \Big|^2 = e^{-\beta H} , \tag{1.4}$$

where H is the energy of a classical "plasma" of particles of charge m interacting with each other classically with $2D$ Coulomb potential

$$H = -\sum_{j<k} 2m^2 \ln |z_j - z_k| + \frac{1}{2} m \sum_l |z_l|^2 \tag{1.5}$$

The reason plasma mapping is useful is that it allows one to estimate the (uniform) electron density in the state. Plasma wants to be neutral on average (there is screening). The term in the potential energy $\frac{1}{2}m\sum_l |z_l|^2$ provides a neutralizing background charge. This charge is smeared over the whole complex plane and its density is $n = \frac{1}{2\pi}$. Due to screening the plasma charge density and the density of the neutralizing background must be equal. The plasma charge density ρ equals m times the electron (or charge 1 particle density). It follows then that

$$\frac{\rho}{n} = me = \nu^{-1} \tag{1.6}$$

So the plasma mapping helped us to understand that the Laughlin wave function describes the state with homogeneous density with filing fraction $\nu = \frac{1}{m}$.

It is also easy to write the wavefunction of an excited quasihole state. Even though the state is excited, it properties are really the properties of the ground state wave function. In the integer case a hole is created by inserting a thin solenoid tube into the Hall fluid at some position z_0 and adiabatically threading a quantum of flux through the solenoid. The wave function of such an (integer) hole state is

$$\psi_h = \prod_j (z_j - z_0) \prod_{j\neq k} (z_j - z_k) e^{-\frac{|z|^2}{4}} \,. \tag{1.7}$$

Laughlin guessed that in the fractional case this ansatz should be replaced by

$$\psi_{qh} = \prod_j (z_j - z_0) \prod_{j\neq k} (z_j - z_k)^m e^{-\frac{|z|^2}{4}} \,. \tag{1.8}$$

It is obvious that multiplying by a factor $\prod_j (z_j - z_0)^m$ simply adds one more electron into the fluid. In view of (1.8) we see that inserting an electron is the same as inserting m quasiholes. Since electron has electric charge e the quasihole has charge $\frac{e}{m}$. This is an example of a *fractionalisation* of charge. This effect became a benchmark for topological phases in condensed matter physics. The particles with fractional charge also often have fractional statistics. We will say a few words about these aspects in Chapter 4.

Finally repeating the charge pumping argument we find that the Hall conductance must be equal to

$$\sigma_H = \frac{e^* e}{h} = \frac{e^2}{h} \times \frac{1}{m} \,, \tag{1.9}$$

where e^* is the smallest charge of a quasiparticle, which we have found to be $\frac{e}{m}$.

1.3 Geometric response

Time has passed and we have learned that there is a swarm of different fractional quantum Hall states. More surprisingly, we have learned that there are *different* quantum Hall states that can exist at the same filling fraction. What is different about them? We have just discussed that FQH states support fractional quasi particles. Depending on the structure of a state it can support many different quasi particles. There are many examples of states that despite occurring at the same filling fraction have different quasi particle content. The simples example is $\nu = \frac{1}{2}$ bosonic Laughlin state and $\nu = \frac{1}{2}$ bosonic Moore-Read state [9]. The latter supports neutral excitations with non-abelian statistics, whereas the former does not support any neutral excitations at all. In fact, Laughlin charge pumping argument is not sensitive to any kind of neutral excitations!

With this in mind, it is very reasonable to ask: are there other transport experiments one could perform on a quantum Hall state that will give additional information about neutral excitations? Fortunately, the answer to this question is yes: there are at least two more transport coefficients one could try to measure. These transport coefficients have one important thing in common: they characterize the linear response of a system perturbations of geometry. Clearly, neutral excitations cannot be accessed by perturbing the electromagnetic field, so the next "easiest" thing to do is to apply stress, shear, shear rate and temperature gradient to a sample. Since FQH forms an incompressible fluid the responses to the first two perturbations vanish, but responses to the last two perturbations do not. The temperature gradient can also be thought of in geometric terms as we will explain in Chapter 6.

1.3.1 Hall viscosity

Hall viscosity was introduced by Avron *et. al.* [10] and Levay [11]. This is "mechanical" transport coefficient in a sense that it is proportional to a two-point correlation function of stress tensors

$$\langle T_{11}(0,\omega) T_{12}(0,-\omega)\rangle \sim i\omega\eta_H \,. \tag{1.10}$$

One can get some intuition about how dissipationless viscosity is possible by examining Figure (1.3.1). The trick is that when parity is broken the viscous

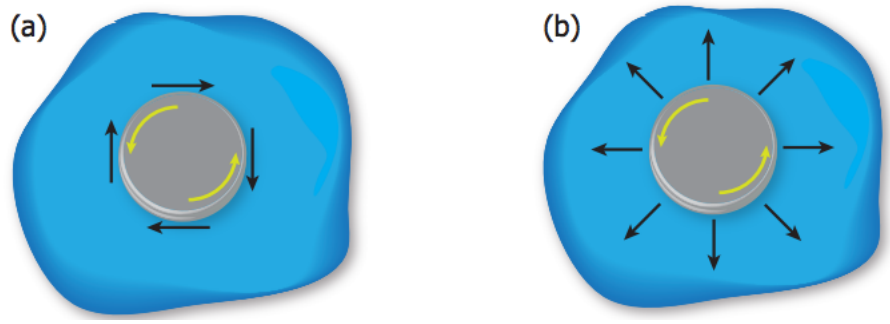

Figure 1.4: (a) Illustrates usual viscosity: when a disc rotating anti-clockwise submerged into a viscous fluid the viscous force will act in the direction, opposite to the velocity, thus slowing down the rotation and dissipating energy. (b) Illustrates Hall viscosity: when a disc rotating anti-clockwise submerged into a viscous fluid the Hall viscous force will act in the direction, *perpendicular* to the velocity. This force does not cause dissipation. The picture is taken from [4]

force can choose to act perpendicular to the velocity, so that a pair of vectors (force, velocity) form either left or right pair. In a parity-invariant system such transport coefficient is not possible. It was found by Read [12] that in general Hall viscosity is given by

$$\eta_H = \frac{\bar{s}}{2}\rho\,, \tag{1.11}$$

where $\bar{s}$ is the "orbital spin"- measure of average angular momentum per particle. For the Read-Rezayi $\mathbb{Z}_k$ parafermion states it is given by

$$\bar{s} = \frac{\nu^{-1}}{2} + \frac{1}{k}\,, \tag{1.12}$$

where the integer k contains information about the "neutral sector". In overwhelming majority of cases $\bar{s}$ is an integer or half integer (although see [13]). Thus we see that it precisely does the job that we set out in the last Section.

Hall viscosity and the orbital spin will be one of the main topics of this Thesis.

1.3.2 Thermal Hall effect

Another linear response occurs when a temperature gradient is applied to a Hall sample. This response was first analyzed by Kane and Fisher [14]. This response is also quantized and also introduces a new topological quantum number. This number is called chiral central charge and it roughly characterizes the number of degrees of freedom on the edge of the sample (including the neutral ones). The precise expression is [15]

$$K_H = \frac{\pi k_B^2 T}{6} c\,, \tag{1.13}$$

where c is the chiral central charge. This quantity can easily distinguish between $\nu = \frac{1}{2}$ bosonic Laughlin state and $\nu = \frac{1}{2}$ bosonic Moore-Read state. In the former case $c = 1$ and in the latter $c = \frac{3}{2}$. In this Thesis we will study the chiral central charge in great detail and we will find that it is somewhat obscured in the bulk and can be measured (even in principle) with certainty only at the edge.

One main result of the Thesis is the relation between these two responses that can be seen in curved space

$$\eta_H = \frac{\bar{s}}{2}\rho + \left(\frac{\nu}{2} \cdot \mathrm{v}ars - \frac{c}{24}\right) \frac{R}{4\pi}\,, \tag{1.14}$$

where vars is the *orbital spin variance* - another topological number that characterizes an FQH state and it was introduced by the author in [16]. The relation (1.14) first appeared in [17].

1.3.3 Disclaimer about disorder

Formation of quantum Hall plateaus would not be possible in a clean system. The Laughlin charge pumping would not really work as electron or quasiparticle could not travel across the gapped bulk without any extended states that would transmit it. We do understand and appreciate this important fact. Nonetheless, in the bulk of the Thesis we will carefully avoid discussion of the influence of the disorder on the geometric response.

Instead of speculating about the influence of disorder on our results we simply leave it for the future work and only state that in ideal, clean systems both viscosity and thermal transport should be present and quantized.

1.4 Plan of the thesis

The Thesis is organized as follows: in Chapter 2 we will review the main technical tool that will be used throughout the Thesis. This tool is known under many names: induced action, effective action, generating functional of correlation functions, etc. In the Chapter 3 we will derive the geometric response of IQH state in a perturbative computation. While most of the results are not new, the general equations for the gradient corrections to linear response are novel. Using the general relations we corrected a mistake in older literature on geometric response. In the Chapter 4 we will extend our results to FQH states and use the full power of the effective field theory, topological quantum field theory and discover that a very abstract effect known as framing anomaly contributes to the linear response and is (in principle) observable. In Chapter 5 we will discuss the additional restrictions on the linear response and induced action imposed by the local Galilean symmetry. The new result of Chapter 5 is the relation between chiral central charge and a correction to density due to gradients of curvature of the sample. In Chapter 6 we will look at the finite temperature physics of FQH. We will explain how to use geometry in non-relativistic system and what kind of geometry is related to Luttinger's theory of thermoelectric transport. In this Chapter we will find some inconsistencies of modern literature on the subject and explain how to fix them. In Chapter 7 we will look at the edge physics and discuss the edge consequences of the bulk Hall viscosity. We will find that unlike Hall conductivity and thermal Hall conductivity Hall viscosity is not related to quantum anomalies of the edge theory and is not "carried" by the edge modes in the same way as Hall current or thermal Hall current. In Chapter 8 we will discuss the problems that are not touched by the Thesis and the likely research direction one could take in the field. Finally, in the Appendix we will present various technicalities that did not find a logical place in the main text.

Chapter 2

Induced Action

The induced action is an extremely powerful formalism that allows one to build in the Ward identities of continuous and discrete symmetries as well as quantum anomalies into a response theory. Here we will define the induced action and explain how to compute the response functions. Before going into details we give a disclaimer: the induced action is designed to work in a clean system at zero temperature or at thermal equilibrium at finite temperature. While it is conceivable that out-of-equilibrium systems can be described by some sort of generating functional it is beyond the scope of this Thesis. We also will carefully distinguish the notion of *induced* action from the notion of *effective* action. The former is completely classical object defined below, whereas the latter is a quantum field theory describing the dynamics (or absence there of) of the low energy degrees of freedom.

2.1 Definition of the induced action

We now turn to the definition of the induced action. Given a quantum field theory of matter fields $\{\psi\}$ coupled to various external fields $A_\mu, g_{ij}, \ldots$ described by an action $S[\{\psi\}, A_\mu, g_{ij}, \ldots]$ one defines the *induced* action (or generating functional) as

$$W[A_\mu, g_{ij}, \ldots] = -i \ln \int D(g^{\frac{1}{4}}\psi)\, e^{iS[\psi, A_\mu, g_{ij}, \ldots]}\,. \tag{2.1}$$

The functional W encodes various multipoint correlation functions of the operators conjugate to the external fields $A_\mu, g_{ij}, \ldots$. The external fields are conjugate to operators in the quantum field theory. The local symmetries of W ensure that the correlation functions of these operators satisfy appropriate Ward identities. If the microscopic theory is gapped W is a local functional of external fields and can be understood as an expansion in gradients of external fields.

The observable quantities of a quantum field theory are the correlation functions of various local operators. These correlation function can be related to more familiar transport coefficients. If we are interested in a correlation function of an operator O defined by

$$\langle O \rangle \equiv \frac{\int D(g^{\frac{1}{4}}\psi)D(g^{\frac{1}{4}}\psi^\dagger)e^{\frac{i}{\hbar}S}O}{\int D(g^{\frac{1}{4}}\psi)D(g^{\frac{1}{4}}\psi^\dagger)e^{\frac{i}{\hbar}S}} \tag{2.2}$$

we simply need to execute the following three-step program. First, we introduce a field f conjugate to an operator O into the action

$$S[\psi, f] = S[\psi, f = 0] + fO \tag{2.3}$$

Second, we compute the induced action according to (2.1). Third, we compute the variational derivaitve

$$\frac{\delta W[f]}{\delta f(x)} = \frac{\int D(g^{\frac{1}{4}}\psi)D(g^{\frac{1}{4}}\psi^\dagger)e^{\frac{i}{\hbar}S}O}{\int D(g^{\frac{1}{4}}\psi)D(g^{\frac{1}{4}}\psi^\dagger)e^{\frac{i}{\hbar}S}} = \langle O(x) \rangle \tag{2.4}$$

The notation $Dg^{\frac{1}{4}}\psi$ means that the region of integration in the functional integral is the space of functions $\psi(x)$ equipped with invariant scalar product given by

$$(\psi, \phi) \equiv \int dx \sqrt{g}\psi^\dagger\phi\,. \tag{2.5}$$

2.2 Electro-magnetic response functions

When the matter fields ψ are charged it is useful to introduce a source for the current operator. This source is traditionally called vector potential and is denoted A_μ. If the quantum field theory conserves the electric charge then the external vector potential satisfies the Ward identity

$$\partial_\mu \langle J^\mu \rangle = 0\,. \tag{2.6}$$

In order to ensure that this Ward identity holds we impose the local $U(1)$ gauge symmetry as follows. First, we demand that the vector potential transforms like a connection (*i.e.* in the adjoint representation of the gauge group). In the abelian case it amounts to

$$A_\mu \to A_\mu + e^{-\theta}\partial_\mu e^\theta \approx A_\mu + \partial_\mu \theta\,, \tag{2.7}$$

where we have expanded in θ in the last step. If this symmetry is imposed the correlation functions of the current defined as

$$\langle\rho(x)\rangle = \frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_0(x)} \tag{2.8}$$

$$\langle J^i(x)\rangle = \frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_i(x)}. \tag{2.9}$$

will automatically satisfy the Ward identity (2.6). We have also included the factor $\frac{1}{\sqrt{g}}$ into the definition in order to ensure that current J^μ is a true vector (and not a vector density). We will discuss this in great detail later on.

The Ward identity can easily be derived as follows

$$\delta W = W[A_\mu + \delta A_\mu] - W[A_\mu] = \int d^d x \frac{\delta S}{\delta A_\mu}\delta A_\mu = \int d^d x \alpha \left(\partial_\mu \frac{\delta S}{\delta A_\mu}\right) = 0\,, \tag{2.10}$$

since the last equality must hold for any α we have

$$\partial_\mu \frac{\delta W}{\delta A_\mu} = \partial_\mu \langle J^\mu\rangle = 0\,. \tag{2.11}$$

Multi-point Ward identities can be obtained by taking the variational derivatives of this Ward identity.

2.3 Stress, strain and curved space

The visco-elastic responses are encoded in the stress tensor T^{ij}. In the theory of elasticity the stress tensor is defined in terms of total force acting on a macroscopic element of a fluid or a solid. In this Section we explain give a brief introduction to the subject. We will follow [18]

Consider an undeformed solid, it is intuitively clear that under the influence of external force the solid will change shape and deform. Let's say that the coordinate of a given point in a body before deformation was x_i and after a small deformation it became x'_i. We define the *displacement field* u^i

$$u_i(x) = x'_i - x_i\,. \tag{2.12}$$

When a solid is deformed the relative distances between points are changed. We are going to interpret strain as change of the *geometry* of the solid. Consider a small deformation described by the distortion field u^i so that

$$dx'^i = dx^i + du^i \,. \tag{2.13}$$

The distance between points before deformation was

$$(ds^2) = dx_i dx^i \tag{2.14}$$

and after deformation it became

$$(ds')^2 = dx'_i dx'^i \tag{2.15}$$

We can express $(ds')^2$ in terms of the displacement field u^i. We have

$$(ds')^2 = \left(\delta_{ij} + \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\right) dx^i dx^j \,. \tag{2.16}$$

This equation can be interpreted as a length element in a slightly curved space with the metric given by

$$g_{ij} = \delta_{ij} + \delta g_{ij} = \delta_{ij} + \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} = \delta_{ij} + 2u_{ij} \,, \tag{2.17}$$

where we have defined a strain tensor $u_{ik} = \frac{1}{2}\delta g_{ik}$. In the linear approximation the strain tesnor is give by

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) \,. \tag{2.18}$$

Thus in the *linear* approximation the variation of metric is given by *twice* the strain tensor. Notice that in general the relationship between metric and the displacement field is not linear. In the incompressible fluids the stress is sometimes created not by the strain, but by the strain rate. This phenomenon is known is viscosity. The strain rate is given by

$$v_{ik} = \dot{u}_{ik} = \frac{1}{2}\dot{g}_{ik} \,. \tag{2.19}$$

2.4 Stress tensor in the theory of elasticity

Here we will define the stress tensor from very general considerations. Later on we will relate this definition with quantum field theory definition. We will follow [18]. Consider a solid body in thermal and mechanical equilibrium. If a body is deformed and the relative position of small macroscopic constituents of the body is changed then the internal forces that try to bring the body back into equilibrium will appear. Lets consider a total force $\mathcal{F}^i$ acting in a volume of a slightly deformed solid body. This force is given by a volume integral

$$\mathcal{F}^i = \int dV F^i \,. \tag{2.20}$$

The forces inside the volume must cancel each other due to the third Newton's law. Therefore, the total force acting on a volume is concentrated on the surface of the volume. This is equivalent to saying that the above integral can be re-written as a surface integral. This is only possible if the force F_i is a total divergence of rank 2 tensor

$$F^i = \partial_k T^{ik} \,. \tag{2.21}$$

This tensor known as the *stress tensor*. The total force acting on a volume element is then given by

$$\mathcal{F}^i = \int dV F^i = \int dV \partial_k T^{ik} = \oint dS_k T^{ik} \,, \tag{2.22}$$

where S_k is a surface element directed along the normal to the surface (pointing inwards). A component of the stress tensor T^{ik} describes the i-th component of a force acting on a unit surface element perpendicular to the k-th coordinate axis. If a body is in mechanical equilibrium then both total force and its density vanish. Then the stress tensor satisfies the conservation law

$$\partial_k T^{ik} = 0 \,. \tag{2.23}$$

Consider total moment $\mathcal{M}^{ik}$ of the force $\mathcal{F}^i$ acting on a volume of a sightly deformed solid body. It is given by a surface integral

$$\begin{aligned} \mathcal{M}^{ik} &= \int dV (x^i F^k - x^k F^i) = \int dV \left(x^i \partial_l T^{kl} - x^k \partial_l T^{il} \right) \\ &= \int dV \left(T^{ki} - T^{ik} \right) + \oint dS_l \left(x^i T^{kl} - x^k T^{il} \right) \,. \end{aligned} \tag{2.24}$$

Just like the total force the total moment has to be written as a surface integral since the moments inside the volume must cancel. This is only possible if the anti-symmetric part of the stress tensor can be represented as a total divergence.

$$T^{ik} - T^{ki} = 2\partial_j \varphi^{ikj} \,. \tag{2.25}$$

The stress tensor can *always* be brought to a symmetric form. This can be done as follows. Notice that the definition (2.21) of the stress tensor is inherently ambiguous. The force (which is an observable) will not change if the stress tensor is redefined by a total divergence

$$T'^{ik} = T^{ik} + \partial_l \chi^{ikl}\,, \qquad with \qquad \chi^{ikl} = -\chi^{ilk}\,. \tag{2.26}$$

Using this freedom one can always cancel the right hand side of (2.25) by and appropriate redefinition of stress tensor. In particular, this can be accomplished by choosing

$$\chi^{ikl} = \varphi^{kli} + \varphi^{ikl} - \varphi^{ikl}\,. \tag{2.27}$$

To summarise, we have defined a rank 2 symmetric stress tensor that satisfies the conservation law (2.23). In the next section we will relate the stress tensor with the strain tensor and explain how to derive the stress tensor from the action principle.

2.5 Visco-elastic response

Hooke's law is a linear relation between *stress* in a solid or a fluid and applied *strain*. It is given by

$$T_{ij} = \Lambda_{ijkl} u_{kl} + \eta_{ijkl} v_{kl} = \frac{1}{2}\Lambda_{ijkl} g_{kl} + \frac{1}{2}\eta_{ijkl}\dot{g}_{kl}\,, \tag{2.28}$$

where Λ_{ijkl} and η_{ijkl} are the rank 4 tensors known as tensor of elastic moduli and viscosity tensor. We have to mention here that in the most general case the stress tensor can also depend on the *anti-symmetric* part of $\partial_i u_k$, but this happens when the solid or a fluid does not have local rotational invariance and possesses local degrees of freedom such as spin.

In the following we will be interested in incompressible, ideal fluids in 2+1D. Incompressible fluid is a state of matter for which stress tensor T_{ij} does not depend on the displacement field from an "undeformed" configuration [1]. With these assumptions we can parametrize the stress tensor as follows

$$\begin{aligned} T_{ik} &= \zeta_{bulk}\delta_{ik}v_{nn} + 2\zeta_{shear}(v_{ik} - \frac{1}{2}\delta_{ik}v_{nn}) + \eta_H(\epsilon_{in}v_{nk} + \epsilon_{kn}v_{ni}) && (2.29)\\ &= \frac{1}{2}\zeta_{bulk}\delta_{ik}\dot{g}_{nn} + \zeta_{shear}(\dot{g}_{ik} - \frac{1}{2}\delta_{ik}\dot{g}_{nn}) + \frac{1}{2}\eta_H(\epsilon_{in}\dot{g}_{nk} + \epsilon_{kn}\dot{g}_{ni}) && (2.30) \end{aligned}$$

[1] When the strain is inhomogeneous the elastic moduli require a redefinition to ensure that they remain vanishing.

where we have defined three kinetic coefficients known as bulk viscosity ζ_{bulk}, shear viscosity ζ_{shear} and Hall viscosity η_H (also known as Odd viscosity or Lorentz shear modulus) [10, 11, 19]. We have also used completely anti-symmetric Levi-Civita symbol ϵ^{ij} defined as

$$\epsilon^{ij} = -\epsilon^{ji}\,, \qquad \epsilon^{12} = 1\,. \tag{2.31}$$

If the fluid is *ideal* (there is no dissipation), then first two coefficients must vanish. This can be easily seen from the local version of the second law of thermodynamics. The entropy (or heat) production is given by [20]

$$\dot{s} + \partial_i j^i_Q = \frac{1}{T}\eta_{ijkl}v^{ij}v^{kl}\,, \tag{2.32}$$

where T is the temperature (in the units where the Boltzmann constant is $k_B = 1$). In order for $\dot{s} = 0$ it is necessary and sufficient to impose the condition on the viscosity tensor η_{ijkl}

$$\eta_{ijkl} = -\eta_{klij}\,, \tag{2.33}$$

that is the viscosity tensor is anti-symmetric with respect to exchange of the first and second pairs of indices. It is easy to see that only the third term in (2.28) satisfies this condition. Thus, η_H is a *non-dissipative* viscosity. This type of viscosity is not possible in 3 dimensional, isotropic fluids, but if the isotropy is broken by, say, a large magnetic field then this transport coefficient can appear.

Hall viscosity carries a strong resemblance to the Hall conductivity. First, as we have just established it is a non-dissipative transport coefficient that contributes to the transverse transport of momentum. Second, this coefficient is only possible in a system with broken parity. This can easily be seen from (2.28) as follows. Apply parity transformation to both sides of (2.28). Since the stress tensor is parity even and the ϵ-tensor changes sign under parity we conclude that η_H must be parity odd. Analogously to the Hall conductivity, Hall viscosity can be viewed as a response to a gravitational version of electric field, defined in terms of the strain rate (more details below). In the IQH states Hall viscosity is quantized in the units of density times $\hbar$. For an IQH states with filling factor N we have

$$\eta_H = \hbar\frac{N}{2} \times \frac{N}{2\pi l^2} = \hbar\frac{N}{2} \times \rho\,, \tag{2.34}$$

where $l^2 = \frac{\hbar}{B}$ is square of the magnetic length (in the units $e = c = 1$). We will discuss the derivation of this fact as well as the value of the Hall viscosity for many other quantum Hall states in great detail later on.

2.6 Stress tensor in quantum field theory

We have learned in the previous section that a deformation of a solid or a fluid can be viewed as a change in geometry described by the metric g_{ij} and that the Hooke's laws states that the stress tensor is linear in metric and its time derivatives. In this Section we will explain how to derive the stress tensor from the Lagrangian formalism.

We start at a somewhat unexpected point. Consider an action for matter coupled coupled to the gravitational field

$$S_{tot}[\psi, g^{\mu\nu}] = S_{gr}[g^{\mu\nu}] + S_{matter}[\psi, g^{\mu\nu}]\,, \tag{2.35}$$

where $g^{\mu\nu}$ is the space-time metric. The difference between greek and latin indices in this and previous section is that the greek indices run through both space and time, whereas latin indices run only through space. The gravitational action is given by, say, Einstein-Hilbert action.

$$S_{gr}[g^{\mu\nu}] = \int dV R\,, \tag{2.36}$$

where R is the Ricci scalar and dV. The equations of motion of General Relativity or Einstein equations (in Euclidean space) are

$$R^{\mu\nu} - \frac{1}{d}\delta^{\mu\nu} R = T^{\mu\nu}\,, \tag{2.37}$$

where $R^{\mu\nu}$ is the Ricci tensor and $T^{\mu\nu}$ is the *stress-energy tensor*. In addition to the stress tensor the stress-energy tensor includes momentum T^{i0}, energy current $T^{0i} = T^{i0}$ and energy density T^{00}. The combination $R^{\mu\nu} - \frac{1}{d}\delta^{\mu\nu} R = G^{\mu\nu}$ is known as Einstein tensor.

If we instead attempt to compute the equation of motion from (2.35) we will find

$$R^{\mu\nu} - \frac{1}{d}\delta^{\mu\nu} R = \frac{2}{\sqrt{g}} \frac{\delta S_{matter}}{\delta g_{\mu\nu}}\,. \tag{2.38}$$

Comparing (2.37) and (2.38) we discover that

$$\frac{2}{\sqrt{g}} \frac{\delta S_{matter}}{\delta g_{\mu\nu}} = T^{\mu\nu}\,. \tag{2.39}$$

This computation is a formal trick to derive the expression for the stress-energy tensor. Gravitational field was an intermediate step and can be turned off in the end of the computation.

We have to pause here for an extremely important comment. The above trick silently assumed the Lorentz symmetry and therefore is not useful in deriving either energy current or momentum or energy density of a non-relativistic system (such as quantum Hall system). We will introduce a procedure for deriving these quantities later in the text. Despite this fact, the outlined trick does give a correct expression for the *stress tensor* (*i.e.* the space-space components of the stress-energy tensor). If one wishes to retain the metric dependence of the stress tensor, one has to set metric $g^{\mu\nu}$ to

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & g^{ij} \end{pmatrix} \tag{2.40}$$

in the final expression.

With the expression (2.39) at hand it is very easy to check that stress tensor satisfies the equilibrium conservation law (2.23). Since the physics has to be independent of the choice of (spatial) coordinates the action must be a scalar under a coordinate transformation. Under a diffeomorphism parametrized by an infinitesimal vector ξ^j (that is $x^j \longrightarrow x^j + \xi^j(x)$) we have

$$\delta g_{ij} = -\xi^k \partial_k g_{ij} - \xi^k \partial_j g_{ik} - \xi^k \partial_i g_{jk} = D_i \xi_j + D_j \xi_i \,, \tag{2.41}$$

where D_j is the covariant derivative. The parameter of a diffeomorphism ξ^j plays a role of the displacement vector u^j. Under this transformation the action transforms as

$$\begin{aligned} \delta S &= S[\psi, g_{ij} + \delta g_{ij}] - S[\psi, g_{ij}] = \int d^d x \sqrt{g} \frac{\delta S}{\delta g_{ij}} \delta g_{ij} \\ &= \int d^d x \sqrt{g} \xi_j D_i \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ij}} = 0 \,, \end{aligned} \tag{2.42}$$

since the last equality must hold for *any* ξ_j we have derived a conservation law

$$D_i \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ij}} = D_i T^{ij} = 0 \,. \tag{2.43}$$

It is worth noting that had we demanded the full coordinate invariance, including the time dependent diffeomorphism we would have obtained the Ward identity

$$D_\mu T^{\mu\nu} = 0 \,, \tag{2.44}$$

but as we have mentioned before this Ward identity is not suitable for non-relativistic systems. Later on we will use the conservation laws to define the momentum, energy and energy current responses of a non-relativistic systems.

2.7 Momentum, energy and energy current

In the seminal work of 1964 Luttinger developed a linear response theory for thermoelectric transport [21]. An essential part of his approach is the coupling of the many body system to an auxiliary external "gravitational potential" conjugated to the energy density. The evolution of the energy density is defined by the divergence of energy current, the latter is a fundamental object in the theory of thermal transport. In this section we identify the appropriate sources of the momentum, energy, and energy current in *non-relativistic* systems. We will use the developed general formalism later on.

In *relativistic* systems the energy density and the corresponding current are naturally combined into a stress-energy tensor $T^{\mu\nu}$ coupled to an external gravitational field described by the spacetime metric. The energy-momentum and charge conservation laws can be written as

$$\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \qquad \partial_\mu J^\mu = 0, \tag{2.45}$$

where $T^{\mu\nu}$ is a stress-energy tensor defined as a response to the external metric $g_{\mu\nu}$. Here, we introduced an electric current J^μ and an external electromagnetic field $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$. Given a matter action S we can compute the energy-momentum tensor and the electric current as

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}, \qquad J^\mu = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A_\mu}\,. \tag{2.46}$$

In the absence of the external sources the first equation in (2.45) encodes two conservation laws: conservation of momentum and conservation of energy

$$\dot{P}^j + \partial_i T^{ij} = 0, \qquad \dot{\varepsilon} + \partial_i J^i_E = 0, \tag{2.47}$$

where we introduced momentum, energy and energy current as $P^j \equiv T^{0j}$, $\varepsilon = T^{00}$ and $J^i_E = T^{i0}$. These notations will be very natural later on. In *relativistic* systems the stress-energy tensor $T^{\mu\nu}$ (being defined as response to the external space-time metric) is symmetric. This implies equality of momentum and energy current $P^i = J^i_E$.

In *non-relativistic systems* this equality no longer holds. For example, for a single massive non-relativistic particle with mass m moving with velocity v^i

we have $P^i = mv^i$ and $J^i_E = \frac{mv^2}{2}v^i$.

In the next Section we will explain how to introduce the appropriate sources for the momentum, energy and energy current. We will introduce a non-relativistic analogue of (2.46). This is achieved by replacing the space-time metric $g_{\mu\nu}$ by a different geometric data known as Newton-Cartan (NC) geometry with *torsion*. We explain how to couple a given non-relativistic system to the NC geometry. Our analysis does *not* assume Galilean symmetry and is valid in systems without boost symmetry. The NC geometry has appeared in the context of Quantum Hall effect [22], non-relativistic (Lifshitz) Holography [23] and fluid dynamics [24]. The relation between NC geometry and quantum transport in non-relativistic physics is one of the new results of this Thesis.

2.8 Construction of the NC geometry

Here we review the construction of NC geometry data from the familiar Einstein-Cartan (EC) geometry (also known as first order formalism or triad formalism). The NC geometry can be understood as a generalization of the latter for the cases where Lorentz symmetry is absent.

The geometric data of EC geometry consists of four objects: vielbeins (also known as frame fields) e^a_μ and their inverse E^μ_a, spin connection $\omega_\mu{}^a{}_b$ and torsion $T^a_{\mu\nu}$ [25].

Vielbeins satisfy the following relations

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \quad g^{\mu\nu} = \eta^{ab} E^\mu_a E^\nu_b, \quad \delta^\mu_\nu = \delta^a_b E^\mu_a e^b_\nu\,, \tag{2.48}$$

where $g_{\mu\nu}$ is space-time metric and η_{ab} is a flat metric in tangent space. The geometric data satisfies the Cartan structure equations [25].

$$de^a + \omega^a{}_b \wedge e^b = T^a\,. \tag{2.49}$$

The Eq. (2.49) is written in the form notations. For example the torsion T^a in the right hand side is given by $T^a = T^a_{\mu\nu} dx^\mu \wedge dx^\nu$ and $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. We impose constraints on these equations and obtain the essential ingredients of NC geometry.

First, we split (2.49) into temporal and spatial parts and impose the *non-relativistic* constraint

$$\omega^A{}_0 = 0\,. \tag{2.50}$$

This constraint has a simple physical meaning: the part of the spin connection, responsible for Lorentz boosts vanishes identically.

In order to simplify the discussion we also impose ${\omega^0}_A = 0$ and $T^A = 0$ [2]. Then (2.49) takes form

$$de^0 = T^0 \equiv \mathcal{T}, \qquad de^A + {\omega^A}_B \wedge e^B = 0\,. \tag{2.51}$$

Notice, that while these equations are still covariant in space-time, the tangent space has lost the Lorentz symmetry. From the objects that appear in (2.51) together with relations (2.48) we can construct all of the NC geometry data. In particular, the Eq. (2.51) clarifies why we refer to $\mathcal{T}$ as temporal torsion.

After the constraint (2.50) is imposed we define a *degenerate* "metric" $h^{\mu\nu} = \delta^{AB} E^\mu_A E^\nu_B$, 1-form $n_\mu = e^0_\mu$ and a vector $v^\mu = E^\mu_0$. Notice, that the spatial part of the metric h^{ij} is a (inverse) metric on a fixed time slice, it is symmetric and invertible. We have denoted its determinant $\det(h^{ij}) = h^{-1}$. The introduced objects are not independent, but obey the relations

$$v^\mu n_\mu = 1, \quad h^{\mu\nu} n_\nu = 0. \tag{2.52}$$

These are precisely the conditions satisfied by the NC geometry data [22, 26][3]. Some detailed discussion of the first order (*i.e.* using the vielbeins) formulation of the NC geometry can be found in [27, 28].

Introduction of the NC geometry allows to write non-relativistic actions and equations of motion in arbitrary coordinate system. The invariant volume element is $dV = e dt d^2x$ with $e = \sqrt{\det(e^a_\mu e^a_\nu)}$. If the underlying physical system was spatially isotropic then vielbeins naturally combine into the degenerate metric $h^{\mu\nu}$. Similarly, the temporal components of vielbeins (denoted v^μ and n_μ) will appear independently of their spatial counterpart thus explicitly breaking the (local) Lorentz symmetry down to $SO(2)$.

To couple a generic matter action to the NC geometry one has to proceed as follows. One should modify the space and time derivatives according to

$$\partial_A \to E^\mu_A \partial_\mu, \quad \partial_0 \to E^\mu_0 \partial_\mu\,. \tag{2.53}$$

Then the objects v^μ, n_μ and $h^{\mu\nu}$ (NC data) will naturally arise (we assume spatial isotropy from now on). When the 1-form n_μ is not closed we define the Newton-Cartan *temporal torsion* 2-form as

$$\mathcal{T}_{\mu\nu} = \partial_\mu n_\nu - \partial_\nu n_\mu. \tag{2.54}$$

[2] These fields in general do not have to be set zero, but in this Thesis we will only consider the backgrounds that satisfy these constraints

[3] It is often convenient to define the "inverse metric" $h_{\mu\nu} = e^A_\mu e^A_\nu$. It satisfies $h^{\mu\nu} h_{\nu\rho} = \delta^\mu_\rho - v^\mu n_\rho$ and $h_{\mu\nu} v^\mu = 0$ and is fully determined by v^μ, n_ν and h^{ij}.

If the physical system were anisotropic the replacement (6.25) would still make sense, but one would have to treat each vielbein as an independent object, *i.e.* not constrained by any local symmetries of the tangent space. The NC geometry also provides a natural definition of a covariant derivative that satisfies

$$D_\lambda n_\mu = 0\,, \qquad D_\lambda h^{\mu\nu} = 0\,. \tag{2.55}$$

These conditions fix the Christofell symbols as

$$\Gamma^{\mu}{}_{\nu,\rho} = v^\mu \partial_\rho n_\nu + \frac{1}{2} h^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}). \tag{2.56}$$

Temporal torsion can also be pulled down from the tangent space. According to our general rule we replace the 0 tangent space index by v^μ. We have

$$T^\rho_{\mu\nu} = v^\rho(\partial_\mu n_\nu - \partial_\nu n_\mu) \tag{2.57}$$

In practice, it is convenient to use a particular parametrization of the NC background fields. Let us specify the spatial part h^{ij} of the degenerate metric and assume that $n_\mu = (n_0, n_i)$ and $v^\mu = (v^0, v^i)$ are also specified and satisfy the first relation in (6.27). Then we find from other relations in (6.27) $h^{\mu\nu} = \begin{pmatrix} \frac{n^2}{n_0^2} & -\frac{n^i}{n_0} \\ -\frac{n^i}{n_0} & h^{ij} \end{pmatrix}$, where we defined $n^i = h^{ij} n_j$, $n^2 = n_i n_j h^{ij}$. In this parametrization the invariant volume element is given by $dV = \sqrt{h} n_0 dt d^2x$.

2.9 Conserved currents

Here we explain how to obtain the momentum, energy and energy current from the action. We will use the parametrization described in the previous Section.

We now consider an induced action (or generating functional) that depends only on the external fields $W = W[n, v, h^{\mu\nu}]$. We will derive two (local) conservation laws that follow from (local) space and time translation invariance. In order to simplify the expressions we will turn off the fields v^i and n_i and set the metric flat after all variations are taken (see below).

The passive action of the (local) space translation with parameter $\xi^i(x,t)$ on the external fields is

$$\delta v^\nu = v^\mu \partial_\mu \xi^i \delta_i^\nu\,, \tag{2.58}$$

$$\delta n_\mu = -n_i \partial_\mu \xi^i\,, \tag{2.59}$$

$$\delta h^{ij} = h^{jk}\partial_k j \xi^i + h^{ik}\partial_k \xi^j - \frac{n^i}{n_0}\dot{\xi}^j - \frac{n^j}{n_0}\dot{\xi}^i\,. \tag{2.60}$$

We demand that under these variations $\delta W = 0$, *i.e.* the induced action is diffeomorphism invariant. Explicitly we have

$$\begin{aligned}\delta W &= \int dV \left[\frac{\delta W}{\delta v^\mu}\delta v^\mu + \frac{\delta W}{\delta n_\mu}\delta n_\mu + \frac{\delta W}{\delta h^{ij}}\delta h^{ij}\right] \\ &= -\int dV \xi^i \left[\partial_0\left(v^0\frac{\delta W}{\delta v^i}\right) + \partial^j\left(2\frac{\delta W}{\delta h^{ij}}\right)\right] = 0 \end{aligned} \tag{2.61}$$

for any $\xi^i(x,t)$. On the other hand space translation invariance implies conservation of the momentum in the form

$$\dot{P}^i + \partial_j T^{ij} = 0\,. \tag{2.62}$$

Comparing the equation (2.63) with (2.61) we identify the momentum vector and stress tensor as (in flat spce)

$$P_i = -\frac{\delta W}{\delta v^i}\,, \qquad T_{ij} = -2\frac{\delta W}{\delta h^{ij}}\,. \tag{2.63}$$

The action of the (local) time translation with parameter $\zeta(x,t)$ on the external fields is

$$\delta v^\mu = \delta^\mu_0 v^\nu \partial_\nu \zeta\,, \tag{2.64}$$
$$\delta n_\mu = -n_0 \partial_\mu \zeta\,, \tag{2.65}$$
$$\delta h^{ij} = 0\,. \tag{2.66}$$

We again demand that under these variations $\delta S = 0$. Explicitly we have

$$\begin{aligned}\delta W &= \int dV \left[\frac{\delta W}{\delta v^\mu}\delta v^\mu + \frac{\delta W}{\delta n_\mu}\delta n_\mu\right] \\ &= \int dV \zeta \left[\partial_0\left(-\frac{\delta W}{\delta v^0}v^0 + \frac{\delta W}{\delta n_0}n_0\right) + \partial_i\left(n_0\frac{\delta W}{\delta n_i}\right)\right] = 0\,. \end{aligned} \tag{2.67}$$

On the other hand the time translation invariance implies conservation of energy.

$$\dot{\varepsilon} + \partial_i J^i_E = 0\,. \tag{2.68}$$

Comparing the equation (2.69) with (2.67) we identify energy and energy current as

$$\varepsilon = \frac{\delta W}{\delta v^0}v^0 - \frac{\delta W}{\delta n_0}n_0\,, \qquad J^i_E = -n_0\frac{\delta W}{\delta n_i}\,. \tag{2.69}$$

When $n_i = 0$ and $v^0 = \frac{1}{n_0}$ we restore the Luttinger's expression for energy.

$$\varepsilon = -2\frac{\delta W}{\delta n_0} n_0 \,. \tag{2.70}$$

2.10 Examples of coupling to Newton-Cartan geometry

Let us illustrate how one can derive expressions for conserved currents using the coupling to NC geometry on two examples of physical systems.

The first example is the system of free fermions (studied in the next Chapter). The action describing free fermions coupled to external Newton-Cartan geometry is given by

$$S = \int dV \left[\frac{i}{2} v^\mu (\Psi^\dagger \partial_\mu \Psi - \partial_\mu \Psi^\dagger \Psi) - \frac{h^{\mu\nu}}{2m} \partial_\mu \Psi^\dagger \partial_\nu \Psi \right] , \tag{2.71}$$

Applying (2.63) and (2.69) together with (5.17), using equations of motion to exclude time derivatives, and turning off NC fields after the variations we obtain the familiar expressions for energy and energy current

$$\varepsilon = -\frac{1}{2m} (\partial_i \Psi)^\dagger (\partial_i \Psi) \,, \tag{2.72}$$

$$J_i^E = \frac{i}{4m^2} \left(\partial^2 \Psi^\dagger \partial_i \Psi - \partial_i \Psi^\dagger \partial^2 \Psi \right) \,. \tag{2.73}$$

These are the familiar expressions for the energy density and energy current.

As another example we consider the action for the non-relativistic electrodynamics, *i.e.* electrodynamics in a medium. The action in the flat background is given by

$$S = \int d^2x dt \left(\frac{\epsilon}{8\pi} E^2 - \frac{\mu^{-1}}{8\pi} B^2 \right) \,. \tag{2.74}$$

Replacing $\partial_0 \to v^\mu \partial_\mu$ and using $h^{\mu\nu}$ instead of contracting spatial indices we obtain from (2.74)

$$S = \int d^2x dt \sqrt{h} n_0 \; h^{\mu\lambda} \left(\frac{\epsilon}{8\pi} v^\nu v^\rho - \frac{\mu^{-1}}{8\pi} h^{\nu\rho} \right) F_{\mu\nu} F_{\lambda\rho} \,, \tag{2.75}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. Applying (2.63) and (2.69) together with (5.17), and turning off the NC fields after the variations are taken we obtain the familiar expressions for energy, energy current and momentum of electromagnetic field

$$
\begin{aligned}
\varepsilon &= \frac{\epsilon}{8\pi}E^2 + \frac{\mu^{-1}}{8\pi}B^2\,, && (2.76)\\
J_i^E &= \frac{1}{2\pi}\mu^{-1}\epsilon_{ij}E_jB = \frac{1}{2\pi}\mathbf{E}\times\frac{1}{\mu}\mathbf{B}\,, && (2.77)\\
P^i &= \frac{1}{2\pi}\epsilon\mathbf{E}\times\mathbf{B} = \frac{1}{v^2}J_E^i\,. && (2.78)
\end{aligned}
$$

Here $v = (\epsilon\mu)^{-1/2}$ is the speed of electromagnetic waves in the medium. One can easily recognize the Poynting vector J_E^i and the momentum density P^i of the electromagnetic field.

2.11 Outlook

In this Chapter we have learned how to compute the induced action (at least in principle) and how to extract transport coefficient from it. In order to successfully study the stress response and momentum transport we had to introduce the curved background geometry. To study energy transport in non-relativistic systems we had to abandon the Riemannian geometry and, instead, we introduced the Newton-Cartan geometry with torsion [22, 24, 29, 30].

There are three ways one can use the induced action. First, one can honestly compute it and this program will be carried out in the next Chapter for free electrons in an IQH state. In the computation of the induced action one can either use a microscopic theory or long wave effective theory. When the underlying microscopic system is interacting (such as FQH system) we first have to replace it by a tractable effective action and then integrate out the effective long-wave degrees of freedom. This program will be carried out in Chapter 4.

Alternatively, one could ask a question: what kind of terms are allowed in the induced action, based on symmetries alone? The answer to this question allows one to distill the terms appearing with the *dimentionless* transport coefficients. These coefficients are the best candidates to be universal characteristic of a phase of matter. Hall conductance, Hall viscosity and chiral central charge are among these properties (see Chapter 4). One also could demand some additional symmetries from the induced action, such as Galilean symmetry. Additional symmetries will impose additional constrains on the action. This program will be carried out in Chapter 5.

Finally, one can change the topology of space where the microscopic theory "lives". Then it turns out that the induced action will give a hint about the structure of boundary excitations. This will be discussed in Chapter 7.

Chapter 3

Induced Action for Integer Quanum Hall States

Recent interest to the Hall viscosity in the theory of Fractional Quantum Hall effect (FQHE) and the interest to the interplay of defects and mechanical stresses with electromagnetic properties of materials motivates studies of gravitational, electromagnetic and mixed responses in condensed matter physics. Gravitational field, as was explained in the previous Chapter, is simply a trick to represent deformational strains present in the material under consideration and a technical tool allowing to extract correlation functions involving stress tensor components.

It is always important to have a simple model system for which such responses can be calculated exactly. For the quantum Hall effect one can consider two-dimensional electron gas in a constant magnetic field (2DEGM) as such a model. When the density of fermions is commensurate with magnetic field the integer number of Landau levels is filled and one expects local and computable response to weak external fields. This model is as important starting point of analysis for quantum Hall systems as a free electron gas for the theory of metals. However, while some electromagnetic responses for 2DEGM can be found in literature we were not able to find the complete results for mixed and gravitational linear responses. The goal of this Chapter is to compute these responses providing the analogue of Lindhard [31] function, both e/m and gravitational, for 2DEGM. We compute the effective action encoding linear responses in the presence of external inhomogeneous, time-dependent, slowly changing electro-magnetic and gravitational fields.

We compare and find an agreement of the obtained responses with known e/m responses [32–35] and with known results for Hall viscosity at integer fillings [36, 37]. In addition we find the stress, charge and current densities induced by perturbations of spatial geometry. Another point of comparison is given by phenomenological hydrodynamic models for FQHE [38–43] and Ward identities following from the exact local Galilean symmetry (also known as non-relativistic diffeomorphism) of the model [44, 45].

The main results of this Chapter are presented in the eqs.(3.174),(3.169). The former states that the low energy effective action for the integer quantum Hall system is *not* completely captured by the Wen-Zee arguments [46] and the correct coefficient in front of the gravitational Chern-Simons term is *not* completely determined by the orbital spin and the filling fraction, but, in addition, requires the knowledge of the chiral central charge. The latter states that the chiral central charge manifests itself in a curved space and shifts the value of the Hall viscosity. In particular, (3.169) implies that one could determine the chiral central charge and, therefore, thermal Hall conductivity [14, 47] from the Hall viscosity computed on a curved space.

3.1 The Model

Our starting point is the system of two-dimensional free non-relativistic fermions interacting with an external gauge A_μ and spatial metric g_{ij} fields (in an attempt for brevity we postpone the discussion of general NC geometry). We assume that the spatial metric can depend on time. The action has a form

$$S = \int d^2x dt \sqrt{g} \left[\frac{i}{2}\hbar \psi^\dagger \partial_0 \psi - \frac{i}{2}\hbar (\partial_0 \psi^\dagger)\psi + \right. \\ \left. + eA_0 \psi^\dagger \psi - \frac{\hbar^2}{2m} g^{ij} (D_i \psi)^\dagger D_j \psi + \frac{g_s B}{4m} \psi^\dagger \psi \right] . \quad (3.1)$$

We assume that the fermions are spin polarized and treat ψ field as a complex grassman scalar. We have also added Zeeman term with the g-factor g_s. For the case of electrons in vacuum $g_s = 2$, but it is convenient to keep it arbitrary for potential condensed matter applications. The covariant derivative $D_i = \partial_i - i\frac{e}{\hbar c}(\bar{A}_i + A_i)$ and includes both vector potential of the constant background magnetic field $\bar{B} = \partial_1 \bar{A}_2 - \partial_2 \bar{A}_1$ and a weak perturbation. In the curved background magnetic field is defined as $B = \frac{1}{\sqrt{g}} \left(\partial_1 \bar{A}_2 - \partial_2 \bar{A}_1 + \partial_1 A_2 - \partial_2 A_1 \right)$, so it transforms as a (pseudo)scalar under coordinate transformations. We separate it into constant part and perturbation as $B = \bar{B} + b$. In this Chapter we use the expression linear in fields $b = B - \bar{B} \approx \partial_1 A_2 - \partial_2 A_1 - \frac{1}{2}\delta g_{ii}$. Here δg_{ij} is a deviation from the flat background $g_{ij} = \delta_{ij} + \delta g_{ij}$.

We omit the chemical potential term in (3.1) for brevity, but assume throughout the paper that the lowest N Landau levels are completely filled in the ground state. We use conventional notations for metric fields so that

g_{ij} and g^{ij} are reciprocal matrices and an invariant spatial volume is given by $\sqrt{g}\, d^2x$ with $g = \det(g_{ij})$.

The equation of motion

$$i\left[\hbar\partial_0 - ieA_0 + \frac{1}{2}\partial_0 \ln\sqrt{g}\right]\psi + \frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}D_i\left[\sqrt{g}g^{ij}D_j\psi\right] = 0\,. \tag{3.2}$$

3.2 Hawking's field redefinition

In order to find the linear responses of the system (3.1) with respect to perturbations in gauge and metric fields we compute the induced action of the theory in quadratic (aka RPA) approximation. The induced action W is defined as a path integral over the fermionic fields

$$e^{\frac{i}{\hbar}W[A_\mu, g_{ij}]} \equiv \int D(g^{\frac{1}{4}}\psi)D(g^{\frac{1}{4}}\psi^\dagger)e^{\frac{i}{\hbar}S}\,. \tag{3.3}$$

We stress that only a *finite* number of the Landau levels is filled, therefore, only a finite number of eigenmodes contributes to the fluctuation determinant. There are no divergencies and no renormalization is required.

The notation [48, 49] $D(g^{\frac{1}{4}}\psi)$ means that the actual integration variables are the weight $\frac{1}{2}$ fields Ψ given by

$$\Psi = g^{\frac{1}{4}}\psi \tag{3.4}$$

$$\Psi^\dagger = g^{\frac{1}{4}}\psi\,. \tag{3.5}$$

The reason for seemingly awkward notation is hiding in the functional integral measure. The measure on a functional space in general depends on the scalar product on the functional space. If the functions are defined on a curved manifold then their scalar product is defined by

$$(\psi, \phi) \equiv \int d^2x\, \sqrt{g}\psi^\dagger\phi\,. \tag{3.6}$$

That is, it depends on the external metric on the manifold. Through the scalar product the metric dependence spreads into the measure of the functional integral. For this reason the computation of the stress tensor will become cumbersome: *a priori* we do not know how to differentiate the functional integral measure with respect to the metric. Fortunately, this disease can be cured by the field redefinition (3.4). The scalar product in the space of Ψ is now given by

$$(\Psi, \Phi) \equiv \int d^2x \, \Psi^\dagger \Phi \tag{3.7}$$

and the functional measure $D(\Psi)D(\Psi^\dagger)$ does not depend on metric.

The new field Ψ transforms with the weight $\frac{1}{2}$

$$\delta\Psi = -\xi^k \partial_k \Psi - \frac{1}{2}\Psi \partial_k \xi^k \, . \tag{3.8}$$

In these new variables the action has form

$$\begin{aligned} S &= \int d^2x dt \Big[i\Psi^\dagger (\hbar\partial_0 - ieA_0)\Psi \\ &- \frac{\hbar^2}{2m} g^{ij} \left((D_i - \frac{1}{4}\partial_i \ln g)\Psi \right)^\dagger (D_j - \frac{1}{4}\partial_j \ln g)\Psi \\ &+ \frac{g_s B}{4m} \Psi^\dagger \Psi \Big] \end{aligned} \tag{3.9}$$

the classical current, density and stress tensor are given by

$$\rho(x) \equiv J^0 = e\Psi^\dagger \Psi \tag{3.10}$$

$$J^i(x) = \frac{e\hbar}{2mi} g^{ij} \left[\Psi^\dagger D_j \Psi - (D_j \Psi)^\dagger \Psi \right] + \frac{g_s}{4m} \epsilon^{ij} \partial_j \rho \tag{3.11}$$

$$\begin{aligned} T^{ij}(x) &= -\frac{\hbar^2}{m} (D^i \Psi)^\dagger D^j \Psi + \\ &+ \frac{\hbar^2}{2m} \left[\partial^i \ln \sqrt{g} \Psi^\dagger D^j \Psi + (D^i \Psi)^\dagger \partial^j \ln \sqrt{g} \Psi - \frac{1}{2} \partial^i \ln \sqrt{g} \partial^j \ln \sqrt{g} \Psi^\dagger \Psi \right] - \\ &- \frac{\hbar^2}{2m} g_{kl} \partial^k g^{ij} \left[\Psi^\dagger D^l \Psi + (D^l \Psi)^\dagger \Psi - 2\partial^l \ln \sqrt{g} \Psi^\dagger \Psi \right] \end{aligned} \tag{3.12}$$

Notice, that the stress tensor does not include time derivatives. Usually in the computation of stress tensor by varying metric one gets a contribution that contains time derivative of the fermionic field ψ. Then the equations of motion for ψ are used to exclude it. In our case the time derivative was excluded by the virtue of the field redefinition.

For simplicity we will set $g_s = 0$ in the following. In the very end we will restore g_s.

3.3 Symmetries of the action

Action (3.9) possesses a few interesting symmetries. First of all it is gauge invariant, provided that the field Ψ transforms according to

$$\Psi' = e^{i\alpha}\Psi \qquad and \qquad A'_\mu = A_\mu + \partial_\mu \alpha \tag{3.13}$$

This symmetry implies the familiar local charge conservation law. In addition, the action is invariant with respect to spatial diffeomorphisms (or, simply, coordinate transformations). This symmetry implies the conservation of stress tensor in the form (with background fields turned off).

$$\partial_k T^{ik} = \epsilon^{kl} J_k B_0 \tag{3.14}$$

There are two less obvious symmetries. One of them is non-relativistic scaling

$$x \longrightarrow \lambda x \tag{3.15}$$
$$t \longrightarrow \lambda^2 t \tag{3.16}$$
$$\Psi \longrightarrow \lambda^{-1}\Psi \tag{3.17}$$

This symmetry can easily be seen in the flat space

$$S = \int d^2x'dt'\Big[i\Psi'\dot{\Psi}' - \frac{\hbar^2}{2m}|D'\Psi'|^2\Big] = \int d^2x dt \lambda^4 \times \lambda^{-4}\Big[i\Psi\dot{\Psi} - \frac{\hbar^2}{2m}|D\Psi|^2\Big] \tag{3.18}$$

This is a non-relativistic version of the Weyl symmetry and it leads to the Ward identity

$$T^{00} = \frac{1}{2}T^i{}_i \tag{3.19}$$

or, simply, energy equals to half the pressure. This implies that in order to compute the energy density in the free system we simply need to compute the response to $g^i i$. Any reasonable interaction will break the symmetry.

Finally, there is local Galilean invariance [50]. This symmetry is quite non-trivial and is described by the transformation laws.

$$\begin{aligned}
\delta A_i &= -\xi^k F_{ki} - m g_{ik}\dot{\xi}^k - \partial_i(\alpha + A^k\xi_k)\,, \\
\delta A_0 &= -\xi^k F_{k0} - \partial_0(\alpha + A^k\xi_k) + \frac{g_s}{4}\frac{\epsilon^{ij}}{\sqrt{g}}\partial_i(g_{jk}\dot{\xi}^k)\,, \\
\delta g_{mn} &= -\xi^k\partial_k g_{mn} - g_{mk}\partial_n\xi^k - g_{nk}\partial_m\xi^k\,,
\end{aligned} \tag{3.20}$$

This symmetry implies an additional Ward identity (c.f. (3.11)).

$$J^i = \frac{e}{m} P^i + \frac{g_s}{4m} \epsilon^{ij} \partial_j \rho \,, \tag{3.21}$$

where P^i is the momentum of the electron fluid. This symmetry is extremely powerful as it allows to relate different orders in the gradient expansion of the linear response functions. We will study this symmetry in detail in the Chapter 5.

3.4 Computation of the induced action

We will compute the induced action as a gradient expansions in the external fields. Throughout the computation we will only keep the terms quadratic in the external fields, but to arbitrary order in the gradients. The expansion will be well defined due to the presence of the gap between the ground state and excited states. One can view the gradient expansion as the expansion in either inverse gap or in the powers of the magnetic length l which is small compared to any other scale in the problem.

We start with rewriting the action as differential operator sandwiched between the fermionic fields.

$$S = \int d^2x dt \Psi^\dagger G^{-1} \Psi \,, \tag{3.22}$$

where G^{-1} is the differential operator obtained by integrating by parts the derivatives acting on $\Psi^\dagger$. Since we will be assuming the perturbations of external fields to be small we can write

$$G^{-1} = G_0^{-1} + V \,, \tag{3.23}$$

where G_0^{-1} is the unperturbed action given by

$$G_0^{-1} = i\partial_0 - \frac{\hbar^2}{2m} |D|^2 + \frac{g_s \bar{B}}{4m} \tag{3.24}$$

and V encodes the terms at least linear in the perturbations of the external fields.

$$V = \left[A_0 + \frac{\hbar^2}{2m} \left(D_i + \frac{1}{4} \partial_i \ln g \right) g^{ij} \left(D_j - \frac{1}{4} \partial_j \ln g \right) + \frac{g_s B}{4m} \right] \tag{3.25}$$

Since the functional integral is quadratic in the external fields it can be formally

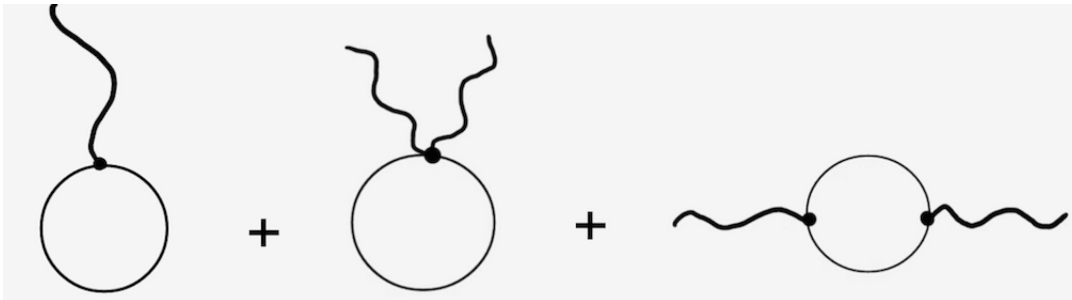

Figure 3.1: The first diagram corresponds to the linear part of the induced action $W^{(1)}$, the second diagram contains the so-called contact terms $W^{(2)}_{cont}$ and the last diagram contains the remainder of the quadratic induced action $W^{(2)}$.

computed as a determinant of the differential operator acting on the fermions.

$$\begin{aligned} W &= \frac{\hbar}{i}\ln\det[G^{-1}] = \frac{\hbar}{i}\ln\det[G_0^{-1}+V] = \frac{\hbar}{i}\mathrm{Tr}\,(\ln(1+G_0V)-\ln G_0) \\ &= -\frac{\hbar}{i}\ln G_0 - \frac{\hbar}{i}\Sigma_{n\geq 1}\frac{(-1)^n}{n!}\mathrm{Tr}\left(G_0V\right)^n \\ &= -\frac{\hbar}{i}\ln G_0 + \frac{\hbar}{i}\mathrm{Tr}\,(G_0V) - \frac{1}{2}\frac{\hbar}{i}\mathrm{Tr}\,(G_0VG_0V)+\dots\,, \end{aligned} \tag{3.26}$$

where in the last line we kept only the terms that will participate in the computation. We can also disregard the first term in the last line since it will not contribute to the linear response because it does not depend on the external fields. Thus the object we are interested in is only given by

$$W = \frac{\hbar}{i}\mathrm{Tr}\,(G_0V) - \frac{1}{2}\frac{\hbar}{i}\mathrm{Tr}\,(G_0VG_0V)+\dots = W^{(1)}+W^{(2)}+W^{(2)}_{cont}+\dots\,, \tag{3.27}$$

where $W^{(2)}_{cont}$ denotes the contact terms (explained below).

On the formal level this is the end of the procedure. We now only need to compute G_0, V and the functional traces. In the diagrammatic notations G_0's are the lines and the V's are the vertices. Each V carries an external field with some derivatives acting on it. The first term in the sum contains the so-called contact terms.

The terms in the induced action can be illustrated as Feynman diagrams presented in Fig. 3.4.

3.5 Complex notations

Before diving into the perturbative computations we pause for another formality. We will use the following notation very often in this Chapter. For any spatial 1-form A_i we introduce the complex form

$$A_z = A_1 - iA_2 \tag{3.28}$$
$$A_{\bar{z}} = A_1 + iA_2 \tag{3.29}$$

Then $A^2 = A_i A^i = \frac{1}{2}(A_z A_{\bar{z}} + A_{\bar{z}} A_z)$. This is equivalent to introducing a new complex (flat) metric

$$g^{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{3.30}$$

so that the length element is given by

$$ds^2 = \frac{1}{2}(dzd\bar{z} + d\bar{z}dz) = dzd\bar{z} = dx^2 + dy^2 \tag{3.31}$$

For the components of a tensor $\tau_{\alpha\beta}$ we have

$$\tau_{zz} = \tau_{11} - \tau_{22} - i(\tau_{12} + \tau_{21}) \tag{3.32}$$
$$\tau_{\bar{z}\bar{z}} = \tau_{11} - \tau_{22} + i(\tau_{12} - \tau_{21}) \tag{3.33}$$
$$\tau_{\bar{z}z} = \tau_{11} + \tau_{22} - i(\tau_{12} - \tau_{21}) \tag{3.34}$$
$$\tau_{z\bar{z}} = \tau_{11} + \tau_{22} + i(\tau_{12} - \tau_{21}) \tag{3.35}$$

We stress that direct consequence of these notations is that the derivative ∂_z is given by

$$\partial_z = \partial_1 - i\partial_2\,, \tag{3.36}$$

and, therefore, does not act exactly the way on would expect

$$\partial_z z = 2\,, \qquad \partial_z \bar{z} = 0 \tag{3.37}$$

Despite this unpleasant relations, we believe that the uniform notations will be very useful in what comes next.

3.6 General structure of the quadratic induced action

In this section we set up general notations that will slightly simplify our computations and will allow to present an answer in a tractable form. First of all, the induced action is Taylor expanded in external fields and, therefore, all of the symmetries we have discussed before are not going to be manifest and it will be a hard technical exercise to check the Ward identites.

The induced action is a quadratic form in external fields. In general it can be written as follows

$$W = \int d^2x dt \Big[\bar{\rho} A_0 + \bar{p} g^i{}_i + \frac{1}{2}\left(A_\mu \Pi^{\mu\nu} A_\nu + A_\mu \Theta^\mu_{ij} \delta g^{ij} + \delta g^{ij} \Lambda_{ijkl} \delta g^{kl}\right)\Big], \quad (3.38)$$

where Π, Θ and Λ are matrices of differential operators acting, say, on the right. It will turn out to be much more convenient to re-write the action in momentum space, so that the differential operators will turn into polynomials in momentum and frequency.

We define a vector

$$v_I(k,\omega) = \begin{pmatrix} A_0(k,\omega) \\ A_z(k,\omega) \\ A_{\bar{z}}(k,\omega) \\ g_{zz}(k,\omega) \\ g_{\bar{z}\bar{z}}(k,\omega) \\ g_{z\bar{z}}(k,\omega) \end{pmatrix}, \quad (3.39)$$

the index I runs from 0 to 5 and another vector $w_I = (\bar{\rho}, 0, 0, 0, 0, \bar{p})^T$, so that

$$v_I w_I = \bar{\rho} A_0 + \bar{p} g^i{}_i \quad (3.40)$$

We also combine the Fourier images of Π, Θ and Λ into one 6 by 6 matrix $W_{IJ}(k,\omega)$ so that the induced action takes form

$$W = \frac{1}{(2\pi)^3} \int d^2k d\omega \left(v_I(k,\omega) w_I + \frac{1}{2} v_I(-k,-\omega) W_{IJ}(k,\omega) v_J(k,\omega) + \dots \right) \quad (3.41)$$

In these notations all we need to do is to compute is the vector w_I and the matrix W_{IJ}. We term the matrix W_{IJ} a generalized polarization operator.

3.7 Fock basis in the Hilbert space and G_0

Non-interacting electrons in quantizing magnetic field at finite and fixed chemical potential μ so that they fill precisely an integer number of the Landau levels is effectively a zero-dimensional quantum mechanical problem. In order to make this manifest we will use Fock representation for the basis states instead of the coordinate representation.

Define the creation and annihilation operators

$$a = \frac{i}{\sqrt{2}}\frac{l}{\hbar}D_{\bar{z}} = \frac{i}{\sqrt{2}}\frac{l}{\hbar}(D_1 + iD_2) \tag{3.42}$$

and

$$a^\dagger = \frac{i}{\sqrt{2}}\frac{l}{\hbar}D_z = \frac{i}{\sqrt{2}}\frac{l}{\hbar}(D_1 - iD_2) \tag{3.43}$$

So that $[a, a^\dagger] = 1$. The inverse relations are

$$D_z = -i\sqrt{2}\frac{\hbar}{l}a^\dagger \tag{3.44}$$

$$D_{\bar{z}} = -i\sqrt{2}\frac{\hbar}{l}a \tag{3.45}$$

In terms of these operators the Hamiltonian takes form

$$\begin{aligned} G_0^{-1} &= \left(i\hbar\partial_0 + \frac{1}{2m}D_iD_i + \frac{g_s}{4m}\right) = i\hbar\partial_0 - \hbar\omega_c\left(a^\dagger a + \frac{1}{2} + \frac{g_s}{4m}\right) \\ &= i\hbar\partial_0 - H_0\,, \end{aligned} \tag{3.46}$$

We introduce another set of generators of the Fock space.

$$b^\dagger = \frac{1}{\sqrt{2}}\frac{l}{\hbar}(D - \frac{\hbar}{l^2}z) = -a + \frac{i}{\sqrt{2}l}z \tag{3.47}$$

and

$$b = \frac{1}{\sqrt{2}}\frac{l}{\hbar}(\bar{D} - \frac{\hbar}{l^2}z) = -a^\dagger - \frac{i}{\sqrt{2}l}\bar{z} \tag{3.48}$$

so that $[b, b^\dagger] = 1$ and all a's commute with all b's. These two pairs of the operators generate the entire Hilbert space of the problem. Using this relation we can express the coordinates themselves (viewed as operators on the Hilbert space) in terms of a and b. We have

$$z = -\sqrt{2}li(b^\dagger + a) \tag{3.49}$$

$$\bar{z} = \sqrt{2}li(a^\dagger + b) \tag{3.50}$$

More formally the complete basis in the Hilbert space is

$$|nm\rangle = |n\rangle \otimes |m\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\frac{(b^\dagger)^m}{\sqrt{m!}}|0\rangle \otimes |0\rangle \tag{3.51}$$

The a-operators move us between the Landau levels, whereas operators b explore the available states within each Landau level since

$$[H_0, b] = [H_0, b^\dagger] = 0\,. \tag{3.52}$$

Each Landau level is infinitely degenerate and every eigen space of the Hamiltonian is spanned by the operators $b^\dagger$ acting on the vacuum. The bare Green's function is then given by

$$G_0 = \int \frac{d\omega}{2\pi} \sum_{nm} e^{-i\omega t} \frac{|nm\rangle\langle nm|}{\hbar\omega - E_n}\,, \tag{3.53}$$

where $E_n = (n + \frac{1}{2})\hbar\omega_c + \frac{g_s}{4m}$ is the spectrum of the unperturbed Hamiltonian $\hbar\omega_c(a^\dagger a + \frac{1}{2} + \frac{g_s}{4m})$.

It is easy to check that $G_0^{-1} G_0 = \delta(t) \times \sum_{m,n} |nm\rangle\langle nm| = \mathbf{1}$.

Thus we have derived an expression for G_0 and also were able to define the trace in the Hilbert space as

$$\mathrm{Tr}\,(X) = \sum_{m,n,t} \langle mnt|X|tmn\rangle = \int dt \sum_{m,n} \langle mn|X|mn\rangle\,. \tag{3.54}$$

3.8 Vertices V

In this section we will derive the expression for various vertices that will appear in our gradient expansion. First we have to expand the classical action to the second order in external fields on top of the background of flat space an constant magnetic field. Then we will read off the vertices that contain to every external field one by one. The unperturbed action is given by

$$S^{(0)} = \int d^2x dt \Psi^\dagger \left[i\hbar\partial_0 - \hbar\omega_c \left(a^\dagger a + \frac{1}{2} + \frac{g_s}{4ml^2} \right) \right] \Psi = \int d^2x dt \Psi^\dagger G_0^{-1} \Psi \tag{3.55}$$

The part of the action linear in external fields is given by

$$\begin{aligned} S^{(1)} &= \int d^2x dt \Psi^\dagger \Big[+ eA_0 - \frac{e\hbar}{2\sqrt{2}ml} (\{a^\dagger, A_{\bar z}\} + \{a, A_z\}) \\ &- \frac{\hbar^2}{4ml^2} \left(a(g_{zz} a) + a^\dagger (g_{\bar z\bar z} a^\dagger) + \frac{1}{2} l^2 (\partial\bar\partial g_{z\bar z}) + a^\dagger (g_{z\bar z} a) + a(g_{z\bar z} a^\dagger) \right) \Big] \Psi \\ &= \int d^2x dt \Psi^\dagger V^{(1)} \Psi \end{aligned} \tag{3.56}$$

The part of the action quadratic in external fields is given by

$$\int d^2xdt\Psi^\dagger \left[-\frac{e^2}{2m}|A|^2 - \frac{\hbar^2}{32m}(\partial \ln g)^2 - \frac{\hbar}{8m}\partial_j(g_{ij}\partial_i \ln g) + i\frac{e}{2m}\partial_j(A_i g_{ij})\right]\Psi\,, \tag{3.57}$$

although the only terms that will give a non-trivial contribution to the quadratic induced action are

$$S^{(2)} = \int d^2xdt\Psi^\dagger \left[-\frac{e^2}{2m}A_z A_{\bar{z}} - \frac{\hbar^2}{32m}\partial g_{z\bar{z}}\bar{\partial} g_{z\bar{z}}\right]\Psi = \int d^2xdt\Psi^\dagger V^{(2)}\Psi \tag{3.58}$$

Total vertex function consists of the terms linear and quadratic in fields

$$V = V(x,t) = V^{(1)}(x,t) + V^{(2)}(x,t) \tag{3.59}$$

Keeping in mind that in the previous section we have identified x with an operator on a Hilbert space, V is also an operator (or infinite matrix) on a Hilbert space.

In accordance with the Section 3.5 we wish to re-write all the vertices in Fourier space and introduce a vector $V_I^{(1)}(k,\omega)$ so that

$$V^{(1)} = \mathcal{V}_I^{(1)}(k,\omega)v_I(k,\omega)\,, \tag{3.60}$$

this is always possible since $V^{(1)}$ is linear in external fields by definition. To do this consider, say, the terms in V linear in $A_{\bar{z}}$. We have

$$V(x,t)\Big|_{A_{\bar{z}}} = \frac{\hbar}{2\sqrt{2}ml}\{a^\dagger, A_{\bar{z}}\} \tag{3.61}$$

Its Fourier transform is given by

$$\tilde{V}(k,\omega)\Big|_{A_{\bar{z}}} = e^{-i\omega t}\frac{\hbar}{2\sqrt{2}ml}\{a^\dagger, e^{ikx}\}A_{\bar{z}}(k,\omega) \equiv \mathcal{V}_2^{(1)}(k,\omega)v_2(k,\omega) \tag{3.62}$$

Since x is expressible in terms of creation and annihilation operators we have

$$\exp i\vec{k}\cdot\vec{x} = \exp\frac{i}{2}(k\bar{z})\exp\frac{i}{2}(\bar{k}z) = e^{-\frac{kl}{\sqrt{2}}a^\dagger}e^{\frac{\bar{k}l}{\sqrt{2}}a}e^{-\frac{kl}{\sqrt{2}}b}e^{\frac{\bar{k}l}{\sqrt{2}}b^\dagger} \tag{3.63}$$

Thus putting everything together and using that a's and b's commute with each other we have

$$\mathcal{V}_2^{(1)}(k,\omega) = e^{-i\omega t}e^{-\frac{kl}{\sqrt{2}}b}e^{\frac{\bar{k}l}{\sqrt{2}}b^\dagger}\frac{\hbar}{2\sqrt{2}ml}\left\{a^\dagger, e^{-\frac{kl}{\sqrt{2}}a^\dagger}e^{\frac{\bar{k}l}{\sqrt{2}}a}\right\}. \tag{3.64}$$

Expressions for the other vertext operators can be derived in the same way. We list them all

$$\begin{aligned}
\mathcal{V}_0^{(1)} v_0 &= e^{-i\omega t} e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} \times A_0(k,\omega) && (3.65)\\
\mathcal{V}_1^{(1)} v_1 &= -e^{-i\omega t} e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} \frac{e\hbar}{2\sqrt{2}ml} \{a^\dagger, e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a}\} \times A_z(k,\omega) && (3.66)\\
\mathcal{V}_2^{(1)} v_2 &= -e^{-i\omega t} e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} \frac{e\hbar}{2\sqrt{2}ml} \{a, e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a}\} \times A_{\bar{z}}(k,\omega) && (3.67)\\
\mathcal{V}_3^{(1)} v_3 &= e^{-i\omega t} e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} \frac{\hbar^2}{4ml^2} a e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a \times g_{\bar{z}\bar{z}}(k,\omega) && (3.68)\\
\mathcal{V}_4^{(1)} v_4 &= e^{-i\omega t} e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} \frac{\hbar^2}{4ml^2} a^\dagger e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a^\dagger \times g_{zz}(k,\omega) && (3.69)\\
\mathcal{V}_5^{(1)} v_5 &= e^{-i\omega t} e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} \frac{\hbar^2}{4ml^2} \Big(\frac{|kl|^2}{2} e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} + a^\dagger e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a \\
&+ a e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a^\dagger \Big) \times g_{z\bar{z}}(k,\omega) && (3.70)
\end{aligned}$$

Notice that part of the operators that depends on both time and b's has completely factorized and is the same for all vertices. We will be able to use this fact to integrate over time and to trace over the Fock space spanned by b before tracing over the Fock space spanned by a. It is the trace over a where all of the complexity is concentrated. For this reason it will be convenient to introduce another notation for part of the vertex insertions that act in a-space.

$$\mathcal{V}_I^{(1)} = e^{-i\omega t} e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} \hat{V}_I \qquad (3.71)$$

Our final result will be written in terms of V_I so we find it important to write them out.

$$\begin{aligned}
\hat{V}_0 &= 1 && (3.72)\\
\hat{V}_1 &= -\frac{e\hbar}{2\sqrt{2}ml} \{a^\dagger, e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a}\} && (3.73)\\
\hat{V}_2 &= -\frac{e\hbar}{2\sqrt{2}ml} \{a, e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a}\} && (3.74)\\
\hat{V}_3 &= -\frac{\hbar^2}{4ml^2} a e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a && (3.75)\\
\hat{V}_4 &= -\frac{\hbar^2}{4ml^2} a^\dagger e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a^\dagger && (3.76)\\
\hat{V}_5 &= -\frac{\hbar^2}{4ml^2} \left(\frac{|kl|^2}{2} e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} + a^\dagger e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a + a e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} a^\dagger \right) && (3.77)
\end{aligned}$$

3.9 Coherent states and Laguerre polynomials

In the next Section we will have to simplify the expressions that involve the expectation values and amplitudes of (3.65)-(3.70). The following two important relations will play the central role in the computations.

$$[b, f(b^\dagger)] = f'(b^\dagger) \tag{3.78}$$

$$e^{Qb} f(b^\dagger) = f(b^\dagger + Q) e^{Qb} \tag{3.79}$$

Also, using this relations and elementary properties of the oscillator algebra we derive an important formula for this Chapter. We have

$$\begin{aligned} \langle n| e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} |m\rangle &= \sqrt{\frac{n!}{m!}} \left(\frac{\bar{k}l}{\sqrt{2}} \right)^{m-n} L_n^{m-n} \left(\frac{|kl|^2}{2} \right) & (3.80) \\ &= \sqrt{\frac{m!}{n!}} \left(\frac{-kl}{\sqrt{2}} \right)^{n-m} L_m^{n-m} \left(\frac{|kl|^2}{2} \right) . & (3.81) \end{aligned}$$

Similar equations can be found in [51].

In the same manner we simplify

$$\begin{aligned} \langle n| e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} |m\rangle &= e^{-\frac{|kl|^2}{2}} \sqrt{\frac{n!}{m!}} \left(\frac{\bar{k}l}{\sqrt{2}} \right)^{m-n} L_n^{m-n} \left(\frac{|kl|^2}{2} \right) & (3.82) \\ &= e^{-\frac{|kl|^2}{2}} \sqrt{\frac{m!}{n!}} \left(\frac{-kl}{\sqrt{2}} \right)^{n-m} L_m^{n-m} \left(\frac{|kl|^2}{2} \right) . & (3.83) \end{aligned}$$

3.10 Induced action to the first order

First, we want to compute the vector w_I defined in Section 3.5. It comes from the term

$$\frac{\hbar}{i} \mathrm{Tr}\, G_0 V^{(1)} \tag{3.84}$$

Before moving forward we notice that Geen's function G_0 is diagonal in frequency space and the vertex insertion V is diagonal in time, *i.e.* we have (omitting the Fock space parts)

$$\begin{aligned} \langle t|V|t'\rangle &= \langle t|V|t\rangle \sim \delta(t-t')V(t) & (3.85) \\ \langle \omega|G_0|\omega'\rangle &= \langle \omega|G_0|\omega\rangle = \frac{1}{\hbar\omega - E_n} & (3.86) \end{aligned}$$

These relations will be important in the following computations. To evaluate the trace we insert the resolution of unity between G_0 and $V^{(1)}$.

$$\begin{aligned}
\mathrm{Tr}\, G_0 V^{(1)} &\equiv \sum_{n,m,t} \langle nmt|G_0 V^{(1)}|nmt\rangle \\
&= \sum_{n,m,t,}\sum_{n',m',t} \langle nmt|G_0|n'm't'\rangle\langle n'm't'|V^{(1)}|nmt\rangle \\
&= \sum_{n,m,t,}\sum_{n',m',t} \langle nmt|\sum_{\omega}|\omega\rangle\langle\omega|G_0\sum_{\omega'}|\omega'\rangle\langle\omega'|n'm't'\rangle\langle n'm't'|V^{(1)}|nmt\rangle \\
&= \sum_{n,m,t,\omega}\sum_{n',m',\omega',t'} \langle nm\omega|G_0|\omega' n'm'\rangle\langle n'm't|V^{(1)}|t'nm\rangle\langle t'|\omega\rangle\langle\omega'|t\rangle \\
&= \sum_{n,m,t,\omega}\sum_{n',m',\omega,t} \langle nm\omega|G_0|\omega n'm'\rangle\langle n'm't|V^{(1)}|tnm\rangle \\
&= \int dt \int \frac{d\omega}{2\pi}\sum_{n,m}\frac{1}{\hbar\omega - E_n}\langle nm|V^{(1)}(x,t)|nm\rangle \\
&= \int dt \sum_{n,m}\theta(N-n)\langle nm|V^{(1)}(x,t)|nm\rangle \\
&= \int dt \int \frac{d\omega}{2\pi}\int\frac{d^2k}{(2\pi)^2}\sum_{n,m}\theta(N-n)\langle nm|\mathcal{V}_I^{(1)}(k,\omega)|nm\rangle v_I(k,\omega)\,,
\end{aligned}$$

where we have used the definition of the trace in the first line, insertion of unity in the second, (3.85) and (3.86) in the fifth and sixth and, finally, we have included only first N poles into the frequency integration in the seventh line and we took the Fourier transform and re-expressed e^{ikx} according to (3.63). The first N poles correspond to N filled Landau levels. In a sense, the integral

$$\frac{\hbar}{2\pi i}\int d\omega \frac{1}{\hbar\omega - E_n} = \theta(N-n) \tag{3.87}$$

is a projector on the first N Landau levels.

Now we have to plug in a particular expression for $\mathcal{V}_I^{(1)}$. There are no currents and torques in ground states, so the terms with the index $I = 1, 2, 3, 4$ (*i.e.* terms proportional to A_z, $A_{\bar z}$, g_{zz}, $g_{\bar z\bar z}$) do not lead to a non-vanishing answer. The only non-trivial contributions come from $I = 0$ and $I = 5$ describing the density and pressure in the ground state correspondingly.

We will treat these terms one by one. We start with $I = 0$

$$
\begin{aligned}
\int dV w_0 v_0 &= \int dt \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \sum_{n,m} \theta(N-n) \langle nm | \mathcal{V}_0^{(1)}(k,\omega) | nm \rangle v_0(k,\omega) \\
&= \int \frac{d^2k}{(2\pi)^2} \sum_{n,m} \theta(N-n) \langle nm | e^{-\frac{kl}{\sqrt{2}} b} e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} | nm \rangle A_0(k,0) \\
&= \int \frac{d^2k}{(2\pi)^2} e^{-\frac{|kl|^2}{2}} \sum_{n,m} \theta(N-n) \langle nm | e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} e^{-\frac{kl}{\sqrt{2}} b} e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} | nm \rangle A_0(k,0) \\
&= \int \frac{d^2k}{(2\pi)^2} e^{-\frac{|kl|^2}{2}} \sum_{n,m} \theta(N-n) \langle m | e^{\frac{\bar{k}l}{\sqrt{2}} b^\dagger} e^{-\frac{kl}{\sqrt{2}} b} | m \rangle \cdot \langle n | e^{-\frac{kl}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}} a} | n \rangle A_0(k,0) \\
&= \int \frac{d^2k}{(2\pi)^2} \sum_{n,m} \theta(N-n) e^{-\frac{|kl|^2}{2}} L_m^0\left(\frac{|kl|^2}{2}\right) \cdot L_n^0\left(\frac{|kl|^2}{2}\right) A_0(k,0) \,, \qquad (3.88)
\end{aligned}
$$

The summation over m gives the momentum conservation delta-function $\delta^{(2)}(\vec{k})$. This is very important fact: even though the original problem is *not* translation invariant, the completely filled Landau level is translation invariant. Technically this happens after the summation over states within every Landau level (*i.e.* over the b-space). We will encounter the same effect on the quadratic level as well. We explain the technical side of this summation in the Appendix E.

We proceed with computation of the linear order of the induced action.

$$
\begin{aligned}
\int dV w_0 v_0 &= \frac{1}{2\pi l^2} \sum_n \theta(N-n) A_0(k=0,\omega=0) = \frac{N}{2\pi l^2} A_0(k=0,\omega=0) \\
&= \int dt d^2x \frac{N}{2\pi l^2} A_0(x,t) \,, \qquad (3.89)
\end{aligned}
$$

where we have used the property of Fourier transform

$$
A_0(k=0,\omega=0) = \int d^2x dt A_0(x,t) \qquad (3.90)
$$

thus we derived $w_0 = \bar{\rho} = \frac{N}{2\pi l^2}$. Perhaps this is not the easiest way to derive the background density for non-interacting electrons in external magnetic field, but the methods we used will be important at the quadratic level.

We turn to the computation of $\bar{p} = w_5$. The difference now is that $\mathcal{V}_5^{(1)}$ is more complicated. Going through the same steps and using the expression for

$\mathcal{V}_5^{(1)}$ we get instead of (3.88)

$$\begin{aligned}
\int dV w_5 v_5 &= \frac{\hbar\omega_c}{4}\int\frac{d^2k}{(2\pi)^2}\sum_{n,m}\theta(N-n)e^{-\frac{|kl|^2}{2}}L_m^0\left(\frac{|kl|^2}{2}\right)\\
&\cdot\left(\frac{|kl|^2}{2}L_n^0\left(\frac{|kl|^2}{2}\right)+nL_{n-1}^0\left(\frac{|kl|^2}{2}\right)+(n+1)L_{n+1}^0\left(\frac{|kl|^2}{2}\right)\right)A_0(k,0)\\
&= \sum_n\theta(N-n)\frac{\hbar\omega_c}{2}\left(n+\frac{1}{2}\right)g_{z\bar{z}}(k=0,\omega=0) \qquad (3.91)\\
&= \int dtd^2x\frac{N}{2\pi l^2}\frac{\hbar\omega_c}{2}Ng_{z\bar{z}}(x,t)\,.
\end{aligned}$$

Thus we have found $\bar{p} = \bar{\rho}\frac{N\hbar\omega_c}{2}$. Notice that the total energy of N filled Landau levels equals to $\frac{N^2\hbar\omega_c}{2}$, thus we obtained that energy density equals to the pressure density. This was indeed expected since the model was Weyl invariant.

To summarize the induced action in the linear order is

$$W^{(1)} = \int dtd^2x\left(\frac{N}{2\pi l^2}A_0+\frac{N}{2\pi l^2}\frac{N\hbar\omega_c}{2}g_{z\bar{z}}\right) \qquad (3.92)$$

Before moving to the tedious quadratic terms computations we make a fun remark: both terms can be written covariantly as follows

$$W^{(1)} = \frac{N}{2\pi}\int dtd^2x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu\bar{A}_\rho+\frac{N}{2\pi l^2}\frac{N\hbar\omega_c}{2}\int dtd^2x\sqrt{g} \qquad (3.93)$$

The first term is the expected Chern-Simons term. Its coefficient also gives the Hall conductance $\sigma_H = \frac{N}{2\pi}$. The re-writting is useful, because it actually has predictive power. Since the backround vector potential $\bar{A}_\mu$ always enters in a sum with A_μ we actually can restore one of the terms of the quadratic induced action! We will derive the term in any case, but it is a nice check.

Remarkably, the last term (when written covariantly) is the "cosmological constant" term in general relativity with the coefficient given by the energy density, as we just argued. Thus "cosmological constant" of a quantum Hall problem is given by the total energy density in the system. This is not too surprising since there is only one energy scale in the problem. When g_s is present the energy density is shifted as

$$\frac{N}{2\pi l^2}\frac{N\hbar\omega_c}{2}\longrightarrow\frac{N}{2\pi l^2}\hbar\omega_c\frac{2N-g_s}{4}\,. \qquad (3.94)$$

3.11 Induced action to the second order

This is the central section of the Chapter. We will perform the *exact* computation of the entire quadratic induced action including *all* gradient corrections. There will be two contributions to the induced action. One contribution comes from the so-called contact terms. These are generated from plugging $V^{(2)}$ into

$$\frac{\hbar}{i}\mathrm{Tr}\, G_0 V \tag{3.95}$$

These contributions are always zero momentum and zero frequency corrections. In fact, these terms can be restored simply from analyzing at the Ward identities of various symmetries. We will compute these terms later.

The main contribution, however, comes from

$$\frac{1}{2}\frac{\hbar}{i} Tr G_0 V^{(1)} G_0 V^{(1)}\,. \tag{3.96}$$

First we will trace over time and frequency, then we will trace over the b-Fock space and in the end we will be left with irreducible expression for the trace in a-space. This is the general strategy.

We start with trace over time and frequency

$$\begin{aligned}
&\mathrm{Tr}\, G_0 V^{(1)} G_0 V^{(1)} = \sum_t \langle t|G_0 V^{(1)} G_0 V^{(1)}|t\rangle \\
=& \sum_{t,\omega}\sum_{t',\omega'} \langle t|\omega\rangle\langle\omega|G_0|\omega\rangle\langle\omega|t'\rangle\langle t'|V^{(1)}|t'\rangle\langle t'|\omega'\rangle\langle\omega'|G_0|\omega'\rangle\langle\omega'|t\rangle\langle t|V^{(1)}|t\rangle \\
=& \sum_{n,n'}\sum_{t,\omega}\sum_{t',\omega'} e^{it(\omega-\omega')}e^{-it'(\omega-\omega')}\frac{1}{\hbar\omega-E_n}V^{(1)}(t)\frac{1}{\hbar\omega'-E_{n'}}V^{(1)}(t') \\
=& \sum_{n,n'}\sum_{t,\omega,\Omega}\sum_{t',\omega',\Omega} e^{it(\omega-\omega'-\Omega)}e^{-it'(\omega-\omega'-\Omega')}\frac{1}{\hbar\omega-E_n}V^{(1)}(\Omega)\frac{1}{\hbar\omega'-E_{n'}}V^{(1)}(\Omega') \\
=& \sum_{n,n'}\sum_{\omega,\Omega}\sum_{\omega',\Omega} \delta(\omega-\omega'-\Omega)\delta(\omega-\omega'-\Omega')\frac{1}{\hbar\omega-E_n}V^{(1)}(\Omega)\frac{1}{\hbar\omega'-E_{n'}}V^{(1)}(\Omega') \\
=& \sum_{n,n'}\sum_{\omega,\Omega} \frac{1}{\hbar(\omega+\Omega)-E_n}V^{(1)}(\Omega)\frac{1}{\hbar\omega-E_{n'}}V^{(1)}(-\Omega) \\
=& \sum_{n,n'}\int\frac{d\Omega}{2\pi}\frac{d\omega}{2\pi}\frac{1}{\hbar(\omega+\Omega)-E_n}\frac{1}{\hbar\omega-E_{n'}}V^{(1)}(\Omega)V^{(1)}(-\Omega)
\end{aligned}$$

We perform the frequency integration by re-writting the fraction as a sum

$$\frac{1}{\hbar(\omega+\Omega)-E_n}\frac{1}{\hbar\omega-E_{n'}}=\left(\frac{1}{\hbar(\omega+\Omega)-E_n}-\frac{1}{\hbar\omega-E_{n'}}\right)\frac{-1}{\hbar\Omega-(E_n-E_n')} \quad (3.97)$$

and taking only first N poles in the integral over ω. This integration (as we pointed out before) will project onto Hilbert space of the first N Landau levels. When this is done we have

$$\begin{aligned}
&\int\frac{d\Omega}{2\pi}\left(\sum_{n,n'}\frac{\theta(N-n)}{E_{n'}-E_n-\hbar\Omega}-\frac{\theta(N-n')}{E_{n'}-E_n-\hbar\Omega}\right)V^{(1)}(\Omega)V^{(1)}(-\Omega)\\
=&\int\frac{d\Omega}{2\pi}\sum_{n\leq N,n'>N}\left(\frac{1}{E_{n'}-E_n-\hbar\Omega}+\frac{1}{E_n-E_{n'}+\hbar\Omega}\right)V^{(1)}(\Omega)V^{(1)}(-\Omega)\\
=&\frac{\hbar}{i}TrG_0V^{(1)}G_0V^{(1)}\Big|_{\omega} \quad (3.98)
\end{aligned}$$

This is the final outcome of the computation. Notice that in the notations we have suppressed the matrix elements in a and b Fock spaces.

The next easiest thing to do is to perform the summation over the Fock space generated by b operator. This is possible because the b operators completely factorize out, due to the fact that even the perturbed action does not depend on b. We compute the trace over the Fock spaces now (suppressing the frequency integrations that we have already performed)

$$\begin{aligned}
&\sum_{n,n',m,m'}\langle nm|G_0|nm\rangle\langle nm|V^{(1)}|n'm'\rangle\langle n'm'|G_0|n'm'\rangle\langle n'm'|V^{(1)}|nm\rangle\\
=&\frac{1}{\hbar\omega-E_n}\frac{1}{\hbar\omega'-E_{n'}}\langle nm|V^{(1)}|n'm'\rangle\langle n'm'|V^{(1)}|nm\rangle \quad (3.99)\\
=&TrG_0V^{(1)}G_0V^{(1)}\Big|_{a,b} \quad (3.100)
\end{aligned}$$

The matrix elements $\langle n'm'|V^{(1)}|nm\rangle$ factorize as

$$\langle n'm'|V^{(1)}|nm\rangle=\langle m'|e^{-\frac{kl}{\sqrt{2}}b}e^{\frac{\bar{k}l}{\sqrt{2}}b^\dagger}|m\rangle\Big|_b\cdot\langle n'|\mathcal{V}^{(1)}|n\rangle\Big|_a \quad (3.101)$$

because a commutes with b. In Eq. (3.101) $\langle m|X|m'\rangle\Big|_b$ means that the average value of operator X is computed in the Fock space generated by the b operators.

With this in mind we continue the computation.

$$
\begin{aligned}
TrG_0V^{(1)}G_0V^{(1)}\Big|_{a,b} &= \sum_{n,n',m,m',k,q} \frac{1}{\hbar\omega - E_n}\frac{1}{\hbar\omega' - E_{n'}}\langle m|e^{-\frac{kl}{\sqrt{2}}b}e^{\frac{\bar{k}l}{\sqrt{2}}b^\dagger}|m'\rangle\Big|_b \\
\times \quad & \langle m'|e^{-\frac{ql}{\sqrt{2}}b}e^{\frac{\bar{q}l}{\sqrt{2}}b^\dagger}|m\rangle\Big|_b\langle n'|V_I^{(1)}|n\rangle\Big|_a\langle n'|V_J^{(1)}|n\rangle\Big|_a v_I(k)v_J(q) \\
= \quad & \sum_{n,n',m,k,q}\frac{1}{\hbar\omega - E_n}\frac{1}{\hbar\omega' - E_{n'}}\langle m|e^{-\frac{kl}{\sqrt{2}}b}e^{\frac{\bar{k}l}{\sqrt{2}}b^\dagger}e^{-\frac{ql}{\sqrt{2}}b}e^{\frac{\bar{q}l}{\sqrt{2}}b^\dagger}|m\rangle\Big|_b \\
\times \quad & \langle n'|V_I^{(1)}|n\rangle\Big|_a\langle n'|V_J^{(1)}|n\rangle\Big|_a v_I(k)v_J(q)\,, \qquad (3.102)
\end{aligned}
$$

where in the last lime we have used the fact that $|m'\rangle\langle m'|$ is an identity operator in the Fock space spanned by b operators. We have, thus, established that in all of the components of the generalized polarization operator the summation over m can be done explicitly and amounts to the computation of the sum

$$
\begin{aligned}
\sum_m\langle m|e^{-\frac{kl}{\sqrt{2}}b}e^{\frac{\bar{k}l}{\sqrt{2}}b^\dagger}e^{-\frac{ql}{\sqrt{2}}b}e^{\frac{\bar{q}l}{\sqrt{2}}b^\dagger}|m\rangle &= \frac{1}{\pi}\int d\alpha e^{|\alpha|^2}\langle 0|e^{\alpha b}e^{-\frac{kl}{\sqrt{2}}b}e^{\frac{\bar{k}l}{\sqrt{2}}b^\dagger}e^{-\frac{ql}{\sqrt{2}}b}e^{\frac{\bar{q}l}{\sqrt{2}}b^\dagger}e^{\bar{\alpha}b^\dagger}|0\rangle \\
&= \frac{2\pi}{l^2}e^{-\frac{|kl|^2}{2}}\delta^{(2)}(k+q) \qquad (3.103)
\end{aligned}
$$

In the first line we replaced the summation in m with integration over the coherent states (we explain how to do such trick in the Appendix E).

In resume: for any component of the generalized polarization tensor summation over m can be replaced by $\frac{2\pi}{l^2}e^{-\frac{|kl|^2}{2}}\delta^{(2)}(k+q)$. This delta function allows to remove the integration over q leaving us with translationally invariant induced action.

It's time to put everything together. For the trace $TrG_0V^{(1)}G_0V^{(1)}$ we have

$$
\begin{aligned}
& \frac{1}{(4\pi l^2)}\int\frac{d^2kd\Omega}{(2\pi)^3}e^{-\frac{|kl|^2}{2}}\sum_{n'<N,n\geq N}\left(\frac{\langle n|V_I^{(1)}(k)|n'\rangle\langle n'|V_J^{(1)}(-k)|n\rangle}{E_n - E_{n'} - \hbar\Omega}\right. \\
+ \quad & \left.\frac{\langle n'|V_I^{(1)}(k)|n\rangle\langle n|V_J^{(1)}(-k)|n'\rangle}{E_n - E_{n'} - \hbar\Omega}\right)v_I(k,\Omega)v_J(-k,-\Omega) \\
= \quad & TrG_0V^{(1)}G_0V^{(1)} \qquad (3.104)
\end{aligned}
$$

Finally, we introduce the notation

$$
\Gamma^I_{nn'}(k,\Omega) = \langle n|V_I^{(1)}(k)|n'\rangle \qquad (3.105)
$$

and the final expression for the quadratic part of the induced action (minus the contact terms) is

$$W^{(2)} = \frac{i}{4\pi l^2 \hbar^2 \omega_c} \int \frac{d^2\mathbf{k} d\omega}{(2\pi)^3} e^{-\frac{|kl|^2}{2}} \sum_{n\geq N, m\leq N} \frac{\Gamma^I_{nm}(k)\Gamma^J_{mn}(-k) + \Gamma^J_{nm}(k)\Gamma^I_{mn}(-k)}{n-m-\omega} v_I v_J \tag{3.106}$$

and the generalized polarization operator (defined in (6.51)) is given by

$$W_{IJ}(k,\Omega) = \frac{i}{\hbar}\frac{1}{(4\pi l^2)}\frac{1}{\hbar\omega_c} e^{-\frac{|kl|^2}{2}} \sum_{n\geq N, m\leq N} \frac{\Gamma^I_{nm}(k)\Gamma^J_{mn}(-k) + \Gamma^J_{nm}(k)\Gamma^I_{mn}(-k)}{n-m-\omega} \tag{3.107}$$

This is the main result of this Chapter. In the following we will massage this expression to a form that can be easily treated on a computer and use it to derive linear response function of free fermions.

3.12 The generating function

While (3.107) is indeed the final expression that cannot be reduced to anything nicer, it is not very convenient to work with since one has to use various complicated expressions for the vertex insertions. In this Section we will introduce a trick that will allow to express *all* of the components of the generalized polarization operator in terms of derivatives of a certain master function.

The master function is constructed out of W_{00}. Consider the following expression (not to be confused with the Green's function)

$$G(k,q;N) = \sum_{n\geq N, m<N} \left(\frac{\Gamma^0_{nm}(k)\Gamma^0_{mn}(q)}{n-m-\omega} + \frac{\Gamma^0_{nm}(q)\Gamma^0_{mn}(k)}{n-m+\omega} \right)$$

Using the equation (3.80) we have

$$\Gamma^0_{nm}(k) = \sqrt{\frac{n!}{m!}} \left(\frac{\bar{k}l}{\sqrt{2}}\right)^{m-n} L_n^{m-n}\left(\frac{|kl|^2}{2}\right) = \sqrt{\frac{m!}{n!}} \left(\frac{-kl}{\sqrt{2}}\right)^{n-m} L_m^{n-m}\left(\frac{|kl|^2}{2}\right) \tag{3.108}$$

we can write an explicit expression for G to

$$G(k,q;N) = \sum_{n\geq N, m<N} \left(-\frac{l^2}{2}\right)^{n-m} \frac{m!}{n!} \left(\frac{(k\bar{q})^{n-m}}{n-m-\omega} + \frac{(\bar{k}q)^{n-m}}{n-m+\omega} \right) \tag{3.109}$$

$$\times\ L_m^{n-m}\left(\frac{|lk|^2}{2}\right) L_m^{n-m}\left(\frac{|lq|^2}{2}\right) \tag{3.110}$$

To illustrate the formula notice that

$$W_{00} = \frac{1}{(4\pi)}\frac{m}{\hbar^2}e^{-\frac{|kl|^2}{2}}G(k,-k;N) \tag{3.111}$$

We used above the following identity

$$\frac{1}{\hbar\omega_c l^2} = \frac{m}{\hbar^2} \tag{3.112}$$

What about other components of W_{IJ}? It turns out that they all can be expressed as derivatives of $G(k,q,N)$. To see this we use the definitions and identities

$$\begin{aligned}
k &= k_1 + ik_2 &(3.113)\\
\partial_k &= \frac{1}{2}(\partial_{k_1} - i\partial_{k_2}) &(3.114)\\
\partial_k k &= 1 &(3.115)\\
\partial_{\bar{k}}\bar{k} &= 1 &(3.116)\\
e^{-ka^\dagger}e^{\bar{k}a}a^\dagger &= (a^\dagger + \bar{k})e^{-ka^\dagger}e^{\bar{k}a} = (-\partial_k + \bar{k})e^{-ka^\dagger}e^{\bar{k}a} &(3.117)\\
ae^{-ka^\dagger}e^{\bar{k}a} &= e^{-ka^\dagger}e^{\bar{k}a}(a-k) = (\partial_{\bar{k}} - k)e^{-ka^\dagger}e^{\bar{k}a}. &(3.118)
\end{aligned}$$

These identities allow us to re-write the vertex insertions (3.73)-(3.77) in terms of derivatives with respect to momentum as follows (indices correspond to our previous $6D$ notations)

$$\begin{aligned}
\hat{V}_0(k) &= 1 &(3.119)\\
\hat{V}_1(k) &= -\frac{e\hbar}{2\sqrt{2}lm}(-\frac{2\sqrt{2}}{l}\partial_k + \frac{l}{\sqrt{2}}\bar{k}) &(3.120)\\
\hat{V}_2(k) &= -\frac{e\hbar}{2\sqrt{2}lm}(\frac{2\sqrt{2}}{l}\partial_{\bar{k}} - \frac{l}{\sqrt{2}}k) &(3.121)\\
\hat{V}_3(k) &= -\frac{\hbar^2}{4ml^2}\frac{2}{l^2}\partial_{\bar{k}}(\partial_{\bar{k}} - \frac{l}{\sqrt{2}}k) &(3.122)\\
\hat{V}_4(k) &= -\frac{\hbar^2}{4ml^2}\frac{2}{l^2}\partial_k(\partial_k - \frac{l^2}{2}\bar{k}) &(3.123)\\
\hat{V}_5(k) &= -\frac{\hbar^2}{4ml^2}\left(1 - |kl|^2 - \frac{4}{l^2}\partial_k\partial_{\bar{k}} + (\bar{k}\partial_{\bar{k}} + k\partial_k)\right) &(3.124)
\end{aligned}$$

Then an arbitrary element of the genralized polarization operator is given by

$$W_{IJ}(\omega,k)=\frac{1}{(4\pi)}\frac{m}{\hbar^2}e^{-\frac{|kl|^2}{2}}\lim_{q\to -k}\hat{V}_I(k)\hat{V}_J(q)G(k,q;N) \tag{3.125}$$

This expression is the one we will use for practical computations.

3.13 Contact terms

The contact terms are obtained from $\mathrm{Tr}\,G_0V^{(2)}$. These terms are quadratic in external fields, but do not contain an infinite series. Schematically the computation is similar to the computation of the linear order terms. We remind the reader the relevant part of the action

$$S^{(2)}=\int d^2xdt\Psi^\dagger\left[-\frac{e^2}{2m}A_zA_{\bar{z}}-\frac{\hbar^2}{32m}\partial g_{z\bar{z}}\bar{\partial}g_{z\bar{z}}\right]\Psi=\int d^2xdt\Psi^\dagger V^{(2)}\Psi \tag{3.126}$$

We start with the contact term involving the vector potential. Using the same methods as in previous Sections we calculate

$$\begin{aligned}\frac{\hbar}{i}\mathrm{Tr}\,G_0V^{(2)} &= \frac{1}{2m}\frac{\hbar}{i}\sum_{n\le N}\sum_{m}\int\frac{d^2q}{(2\pi)^2}\frac{d^2k}{(2\pi)^2}\langle nm|e^{i(k+q)x}|nm\rangle A_z(k,0)A_{\bar{z}}(q,0)\\ &= \frac{1}{4\pi}\frac{1}{ml^2}\int\frac{d^2k}{(2\pi)^2}A_z(k,0)A_z(-k,0)\,,\end{aligned} \tag{3.127}$$

other contact terms are given by (these terms are evaluated at zero frequency $\omega=0$ as it always happens for the contact terms).

$$\begin{aligned}W^{(2)}_{cont} &= \frac{1}{4\pi}\frac{1}{ml^2}\int\frac{d^2k}{2\pi}\Big[\frac{|k|^2}{4}g_{z\bar{z}}(k,0)g_{z\bar{z}}(-k,0)\\ &+ \frac{g_s}{4}\left(\frac{\bar{k}}{2}A_z(k,0)g_{z\bar{z}}(k,0)-\frac{k}{2}A_{\bar{z}}(k,0)g_{z\bar{z}}(k,0)\right) \qquad (3.128)\\ &+ \frac{g_s}{4}\frac{1}{8}g_{zz}(k,0)g_{\bar{z}\bar{z}}(k,0)\Big]\,. \qquad (3.129)\end{aligned}$$

These terms are all obtained in the same fashion.

3.14 Induced action: final expression

For conveniece we list here our results in terms of dimensionless momentum $q = \frac{kl}{\sqrt{2}}$.

The generating function is given by

$$\begin{aligned} G(k,q;N) &= \sum_{n\geq N, m<N} (-1)^{n-m} \frac{m!}{n!} \left(\frac{(k\bar{q})^{n-m}}{n-m-\omega} + \frac{(\bar{k}q)^{n-m}}{n-m+\omega} \right) && (3.130) \\ &\cdot \ L_m^{n-m}\left(\frac{|k|^2}{2}\right) L_m^{n-m}\left(\frac{|q|^2}{2}\right) && (3.131) \end{aligned}$$

The vertices are given by

$$\begin{aligned} \hat{V}_0(q) &= 1 && (3.132) \\ \hat{V}_1(q) &= -\frac{\hbar}{2\sqrt{2}lm}\left(2\partial_{\bar{q}} - \left(1+\frac{g_s}{2}\right) q\right) && (3.133) \\ \hat{V}_2(q) &= -\frac{\hbar}{2\sqrt{2}lm}\left(-2\partial_q + \left(1+\frac{g_s}{2}\right) \bar{q}\right) && (3.134) \\ \hat{V}_3(k) &= -\frac{\hbar^2}{4ml^2}\partial_{\bar{q}}(\partial_{\bar{q}} - q) && (3.135) \\ \hat{V}_4(k) &= -\frac{\hbar^2}{4ml^2}\partial_q(\partial_q - \bar{q}) && (3.136) \\ \hat{V}_5(k) &= -\frac{\hbar^2}{4ml^2}\left(\left(1+\frac{g_s}{2}\right) - 2|q|^2 - 2\partial_q\partial_{\bar{q}} + (\bar{q}\partial_{\bar{q}} + q\partial_q)\right), && (3.137) \end{aligned}$$

where we have also added the dependence on the g_s-factor that describes the non-minimal coupling of the electrons to the magnetic field due to the intrinsic magnetic moment. The generalized polarization operator is given by

$$W_{IJ} = \frac{1}{(4\pi)}\frac{m}{\hbar^2}e^{-|k|^2}\lim_{q\to -k}\hat{V}_i(k)\hat{V}_j(q)G(k,q;N) \qquad (3.138)$$

The induced action in momentum space is given by

$$W = \int \frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\left(w_I v_I + \frac{1}{2}W_{IJ}(k,\omega)v_I(k,\omega)v_J(k,\omega)\right) + W^{(2)}_{cont} \qquad (3.139)$$

Since it is quadratic in external fields it is straightforward to go back to coordinate space.

3.15 Spin connection

Before presenting the explicit answer for the induced action we introduce a geometric object that will be useful to write down the answer and to restore non-linear contributions to the induced action using the diffeomorphism invariance.

In the external magnetic field the electron is moving along a circular orbit. There is an orbital spin $\bar{s}$ associated to this motion. The orbital spin is not a part of the original action (3.1), but an *emergent* phenomenon [36] that appears after the averaging over the "fast" cyclotron motion of the electron. The *emergent* orbital spin couples to the $SO(2)$ spin connection just like electric charge couples to the vector potential.

The Levi-Civita $SO(2)$ spin connection can be expressed in terms of the vielbeins as [44]

$$\omega_0 = -\frac{1}{2}\epsilon^{ab}e^{aj}\partial_0 e^b_j \,, \tag{3.140}$$

$$\omega_i = -\frac{1}{2}\epsilon^{ab}e^{aj}\partial_i e^b_j - \frac{1}{2\sqrt{g}}\epsilon^{jk}\partial_j g_{ik} \,, \tag{3.141}$$

This connection transforms as an abelian gauge field under the local $SO(2)$ rotations [44]. While more covariant formulas can be written using the Newton-Cartan geometry, they will not be practical for our purposes.

This spin connection can be used to construct the gravi-magnetic and gravi-electric fields

$$\mathcal{E}_i = \partial_i\omega_0 - \partial_0\omega_i \tag{3.142}$$

$$\frac{R}{2} = \epsilon^{ij}\partial_i\omega_j \,, \tag{3.143}$$

where gravi-magnetic field coincides with the Gaussian curvature or *half* the Ricci curvature. There are several general arguments that explain why the spin connection has to be a part of the low energy description of the FQH states [22, 46, 52]. Nevertheless, there is a confusion in the literature about the Chern-Simons term that can be constructed from ω_μ. Methods based on the local Galilean invariance cannot say anything about the term or the coefficient in front of it because it is too far in the gradient expansion. Methods of [46] give the wrong prediction for the coefficient in front of the gravitational Chern-Simons (gCS) term. The major result of this Thesis is the direct computation of the coefficient. We notice that the mismatch between our computation and the result of [46] is captured *precisely* by the gravitational anomaly of the edge theory.

3.16 Quadratic induced action in coordinate space.

The induced action defined in (3.3) was computed as a regular expansion in background fields $A_\mu(x,t)$ and $g_{ij}(x,t)$ and their gradients. In the following we expand the effective action to quadratic order in the external fields. It is convenient to separate it as

$$W = W^{(1)} + W^{(geom)} + W^{(2)} . \tag{3.144}$$

The first contribution is given by

$$W^{(1)} = \int d^2x dt \sqrt{g} \left[-\epsilon_0 + \rho_0 A_0 + s_0 \omega_0 \right] , \tag{3.145}$$

where ω_0 is the time component of the spin connection and ϵ_0, ρ_0, and s_0 are the energy density, density, and the orbital spin density in the ground state. They are given respectively by

$$\rho_0 = \frac{N}{2\pi l^2} , \quad \epsilon_0 = \rho_0 \, \hbar\omega_c \frac{2N - g_s}{4} , \quad s_0 = \rho_0 \hbar \frac{N}{2} . \tag{3.146}$$

Magnetic length and cyclotron frequency are given in term of the constant part of the background magnetic field B_0 as

$$l^2 = \frac{\hbar c}{e\bar{B}} , \qquad \omega_c = \frac{e\bar{B}}{mc} . \tag{3.147}$$

We notice here that although (3.145) includes all terms linear in A_μ, g_{ij} they also contain several quadratic terms. Indeed, the expansion of the $\sqrt{g}$ in terms of the deviations from the flat background is

$$\sqrt{g} = 1 + \frac{1}{2}\delta g_{ii} - \frac{1}{8} \left[(\delta g_{11} - \delta g_{22})^2 + 4\delta g_{12}\delta g_{21} \right] + \dots \tag{3.148}$$

and (3.145) should be re-expanded and truncated to the terms up to the second order in fields.

The second term in (3.144) contains the topological and *geometrical* contributions to the effective action (with $\hbar = c = e = 1$)

$$W^{(geom)} = \frac{N}{4\pi} \int \left(AdA + NAd\omega + \frac{2N^2 - 1}{6} \omega d\omega \right) , \tag{3.149}$$

where we used the "form notation" $\int AdA \equiv \int d^2x dt \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$. The coefficients of the three terms in (3.149) give, respectively the Hall conductivity $\sigma_H = \frac{N}{2\pi}$, the average orbital spin per particle $\bar{s} = \frac{N}{2}$ (corresponding to the Wen-Zee shift $\mathcal{S} = N$), and the gCS coefficient $\frac{N(2N^2-1)}{24\pi}$.

The following comment is in order. The action (3.1) is written in terms of the gauge potential A_μ and metric g_{ij}. It does not require spin connection ω_μ as it is already covariant due to the fact that ψ is a scalar field. Thus, the $W^{(geom)}$ should also depend solely on the vector potential and metric. It is, however, instructive to write $W^{(geom)}$ in terms of A_μ and ω_μ as we did in (3.149). With the accuracy used in this paper the dependence on ω_μ can be restored with the help of linearized version of (3.140), *i.e.* $\omega_i \leftrightarrow -\frac{1}{2}\epsilon^{jk}\partial_j\delta g_{ik}$ and $\omega_0 \leftrightarrow \frac{1}{2}\epsilon^{jk}\delta g_{ij}\delta\dot{g}_{ik}$.

It is illuminating to present (3.149) as an explicit sum over Landau levels

$$W^{(geom)} = \sum_{n=1}^{N} \int \left[\frac{1}{4\pi}(A + \bar{s}_n\omega)d(A + \bar{s}_n\omega) - \frac{1}{48\pi}\omega d\omega\right], \qquad (3.150)$$

where $\bar{s}_n = \frac{2n-1}{2}$ is the orbital spin per particle on the n-th Landau level and the last term is an anomalous gCS contribution. It is the same for all Landau levels. It is equal to the non-relativistic limit of the well-known relativistic gCS term [53]. The latter is related to the gravitational anomaly via the usual anomaly inflow arguments. Its presence shows that the spin connection *does not* simply combine with vector potential in the effective action as suggested in [46, 52]. We speculate that the offset is related to the gravitational anomaly experienced by the chiral edge modes in the curved space. In the next chapters we will show the precise relation.

The physical meaning of Chern-Simons and Wen-Zee terms have been extensively discussed in literature. The relativistic gCS term is usually related to the transverse heat transport via Luttinger's argument [15, 54, 55]. It was shown in [53] that this relations is misleading. We will elaborate on this issue in the Chapter 6. The last term in (3.144) gives the remaining second order terms

$$\begin{aligned} W^{(2)} &= \int d^2x dt\, \mathcal{L}^{(2)}\,, \\ \mathcal{L}^{(2)} &= \frac{1}{2}\left(A_\mu \Pi^{\mu\nu} A_\nu + A_\mu \Theta^{\mu}_{ij}\delta g^{ij} + \delta g^{ij}\Lambda_{ijkl}\delta g^{kl}\right)\,, \end{aligned} \qquad (3.151)$$

where differential operators Π, Θ, Λ encode electro-magnetic, mixed, and gravitational responses, respectively. These operators can be computed exactly as infinite series in time and spatial derivatives or as series in frequency and wavevectors in Fourier representation.

$$
\begin{aligned}
\frac{4\pi}{N}\mathcal{L}^{(2)} &= ml^2E_i^2 - \frac{N}{m}b^2 - \frac{3N}{2}l^2b(\partial_iE_i) - \frac{2N^2-1}{4m}bR\,, \\
&+ \frac{2N^2-1}{6}l^2R(\partial_iE_i) + \frac{N(N^2-1)}{8m}R^2 + \dots\,,
\end{aligned} \tag{3.152}
$$

where R is the scalar curvature given by $R = \partial_i\partial_j\delta g_{ij} - \Delta\delta g_{ii}$ after linearization. While the first three terms of the expansion (3.152) can be found in literature [32] the other terms were computed in this Thesis for the first time [17].

The induced action presented above is the most compact way to summarize linear responses. However, we find it convenient to have direct formulas for observables such as charges, currents and stresses in a dynamic and inhomogeneous background. We present the explicit expressions and their physical meaning for linear responses in next sections. For the illustration purposes and to lighten up the equations in the following we consider only the lowest Landau level filled, i.e. $N = 1$.

3.17 Response functions.

Below we will compute various response functions using the techniques developed above.

3.17.1 Density.

The expectation value of the electric charge density is given by the variational derivatives of the action (3.3) with respect to the scalar potential

$$
\rho(x) \equiv \frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_0(x)} = \langle\psi^\dagger\psi\rangle\,. \tag{3.153}
$$

In the curved background the density has to be understood as number of particles per invariant volume element

$$
\begin{aligned}
\rho - \rho_0 = \frac{1}{2\pi}&\left(1 + \frac{3-g_s}{4}l^2\Delta\right)b \\
&+ \frac{1}{8\pi}\left(1 + \frac{1}{3}l^2\Delta\right)R + \frac{ml^2}{2\pi}\left(1 + \frac{3}{8}l^2\Delta\right)(\partial_iE_i)\,,
\end{aligned} \tag{3.154}
$$

where Δ is the Laplacian.

Integrating (3.154) over a closed manifold we obtain the shift of the degeneracy of the lowest Landau level due to topology of the manifold

$$Q = \int d^2x\sqrt{g}\rho = \int d^2x\sqrt{g}\left(\frac{B}{2\pi} + \frac{R}{8\pi}\right) = N_\phi + \frac{1}{2}\chi\,, \qquad (3.155)$$

where N_ϕ is the total magnetic flux and χ is the Euler characteristics of the manifold [46]. The correction to the density due to curvature gradients in (3.154) is in agreement with Refs.[43, 56]. Extending (3.155) to the case of an isolated conic singularity with the deficit angle θ we find

$$\delta Q = \int d^2x\sqrt{g}\,(\rho - \rho_0) = \frac{1}{8\pi}\int d^2x\sqrt{g}R = \frac{1}{4\pi}\theta\,. \qquad (3.156)$$

The points of higher positive curvature attract particles in and increase local density. Although the derivation presented here cannot be rigorously applied to the case of conic singularity where curvature $R = 2\theta\delta(x)$ is highly singular, the integral formula (3.156) is exact and can be checked by direct computation of the density on a surface of the cone.

In the following we illustrate some structures arising as the time dependence is introduced.

In the flat background and for $N = 1$, $g_s = 0$ we have

$$\begin{aligned}\frac{\rho(\omega)}{\rho_0} &= \frac{1}{1-\omega^2}\Big(1 + l^2 b + ml^4\partial_i E_i \\ &- \frac{3}{2}l^2\Delta\frac{2l^2 b + ml^4\partial_i E_i}{4-\omega^2} + \dots\Big)\,,\end{aligned}$$

where ω is measured in units of ω_c. The overall pole at $\omega = 1$ is expected even in the presence of interactions as a consequence of the Kohn's theorem. The poles at $\omega = n$, $n = 2, 3, \dots$ corresponding to transitions between different Landau levels occur in the next terms of gradient expansion.

Expanding in frequency and including the gravitational perturbations we have the leading term (first order in time derivative)

$$\rho(x,t) = \rho(x,0) + \frac{3}{16\pi ml^2}\epsilon_{ij}\partial_i\partial_k\dot{g}_{ik} \qquad (3.157)$$

with $\rho(x,0)$ given by (3.154).

3.17.2 Electric current

The expectation value of the electric current density is given by the variational derivative of the action (3.3) with respect to the vector potential

$$J^i(x) \equiv \frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_i(x)} = \left\langle \frac{g^{ij}}{2mi}\left[\psi^\dagger D_j\psi - (D_j\psi)^\dagger\psi\right]\right\rangle . \tag{3.158}$$

We find

$$\langle J_i\rangle = \epsilon^{ij}\left(\sigma_H E_j + \frac{2-g_s}{4\pi m}\partial_j\left(b+\frac{R}{8}\right)\right) , \tag{3.159}$$

where the wavevector dependent Hall conductivity is given by

$$\sigma_H(k) = \frac{1}{2\pi}\left(1 - \frac{3-g_s}{4}|kl|^2 + \frac{22-9g_s}{96}|kl|^4\right) . \tag{3.160}$$

The correction of the order of k^2 is in full agreement with general results for Galilean invariant systems [37, 44]. The k^4 term calculated here is new.

The second term in (3.159) is another new result of this work. It shows that in low orders of gradient expansion the gradient of magnetic field and curvature affect current similarly to the electric field. We also point out that in agreement with [22] $m \to 0$ limit is regular for $g_s = 2$.

3.17.3 Stress tensor

The expectation value of the stress tensor is given by

$$\begin{aligned} T_{ij} \equiv -\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{ij}(x)} &= \frac{1}{2m}\Big\langle \left(D_i\psi)^\dagger D_j\psi + (D_j\psi)^\dagger D_i\psi\right) \Big\rangle \\ &- \frac{1}{4m} g_{ij}(\Delta_g + g_s B)\left\langle \psi^\dagger\psi\right\rangle . \end{aligned} \tag{3.161}$$

Here Δ_g is the Laplace-Beltrami operator defined as $\Delta_g\rho = \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j\rho)$ [1].

Using (3.161) we find the stress tensor in the leading order in gradients

$$\begin{aligned} T_{ij} &= \frac{1}{8\pi}\left(\partial_i E_j + \partial_j E_i\right) \\ &+ \delta_{ij}\left(\epsilon_0 - \frac{4-g_s}{8\pi}\partial_k E_k + \frac{2-g_s}{8\pi m l^2}\left(b+\frac{R}{8}\right)\right) . \end{aligned} \tag{3.162}$$

[1] We remark here that the term $-\frac{1}{4m}g_{ij}\Delta_g\rho$ in (3.161) comes from the path integral measure (3.3) while the rest of (3.161) can be obtained in conventional way from the variation of (3.1) over the metric.

The stress tensor has a regular limit $m \to 0$ limit for $g_s = 2$.

The action (3.1) is Weyl-invariant. The Weyl symmetry implies a relation between one point correlation functions of the energy density and pressure (3.19)

$$\epsilon = \frac{1}{2} T_i^{\ i} \,, \tag{3.163}$$

so (3.162) can be used to extract the energy density in the ground state in the presence of external fields. Keeping only the lower gradients we obtain the correction to the energy density

$$\epsilon - \epsilon_0 = -\frac{4 - g_s}{8\pi} \partial_i E_i + \frac{2 - g_s}{8\pi m l^2} \left(b + \frac{R}{8} \right) . \tag{3.164}$$

In the case of an isolated conic singularity we get a contribution to the total energy $\frac{\delta E}{E_0} = \frac{\theta}{8\pi}$ per singularity [2].

3.17.4 Hall viscosity of free fermions

Time-dependent part of the stress tensor is related to another quantity of great interest: the Hall viscosity. We are looking for the parity odd terms in the stress tensor containing no more than two spatial derivatives.

$$\begin{aligned} T_{ij}^{odd} &= \frac{1}{2} \eta_H (\epsilon_{ik} \dot{g}_{kj} + \epsilon_{jk} \dot{g}_{ki}) \\ &+ \frac{1}{2} \eta_H^{(2)} l^2 \left[\epsilon_{il} \partial_l \partial_j + \epsilon_{jl} \partial_l \partial_i \right] \dot{g}_{kk} \end{aligned} \tag{3.165}$$

where $\eta_H = \eta_H(\omega, k)$ is a generalization of the Hall viscosity to finite wave number and frequency (here $N = 1$ and we measure ω in units of ω_c)

$$\frac{\eta_H(\omega, k)}{\eta_H^{(0)}} = \frac{4}{4 - \omega^2} + |kl|^2 \left(\frac{1}{1 - \omega^2} - \frac{6}{4 - \omega^2} + \frac{6}{9 - \omega^2} \right) .$$

Here the conventional Hall viscosity

$$\eta_H(\omega = 0, k = 0) \equiv \eta_H^{(0)} = \frac{1}{2} \rho_0 \bar{s} \,. \tag{3.166}$$

At the zero wavevector $\eta_H(\omega, k = 0)/\eta_H^{(0)} = 4/(4 - \omega^2)$ is in full agreement with Ref. [37]. For the coefficient in front of the second tensor (second line of

[2] $E_0 = \epsilon_0 / \rho_0$ is the energy per particle in the unperturbed state.

Eq. 3.165) we have

$$\eta_H^{(2)} \;=\; \frac{1}{8}\rho_0 \left(\frac{2}{1-\omega^2} - \frac{4}{4-\omega^2} \right) . \tag{3.167}$$

In the static limit and for general N we rewrite the expression for the Hall viscosity as a sum over Landau levels

$$\eta_H(k,\omega=0) = \frac{1}{2\pi l^2} \sum_{n=1}^{N} \left(\frac{\bar{s}_n}{2} + \frac{1}{4}\left[\bar{s}_n^2 - \frac{c}{12} \right] |kl|^2 \right) . \tag{3.168}$$

One has to recall that $c = 1$ and that the orbital spin per particle at the n-th Landau level $\bar{s}_n = \frac{2n-1}{2}$. We remark here that the gCS term gives a long wave k^2 correction to the Hall viscosity in a fashion similar to how the Wen-Zee term produces the long wave correction to the Hall conductivity [44]. In fact, the k^2 correction to the Hall viscosity (3.168) comes solely from the gCS term.

We note, that the gCS term also corrects the value of the Hall viscosity in the presence of constant background curvature R_0. Indeed, the gCS term gives a contribution $\sqrt{g}\frac{1}{2}R_0\omega_0$ to the induced action, which results in $\delta\eta_H = \frac{N(2N^2-1)}{96\pi}R_0$. Then the total value of the Hall viscosity (for $N = 1$) is given by

$$\eta_H = \frac{\bar{s}}{2}\rho_0 - \frac{c}{24}\frac{R}{4\pi} \tag{3.169}$$

The second term gives the correction due to the curvature of the background and should be compared to (3.168). If the coefficient c is indeed the chiral central charge then (3.169) suggests very non-trivial relation. One could determine the chiral central charge (and, therefore, thermal Hall conductivity) simply measuring the Hall viscosity on a curved sample.

Equation (3.169) is also in (somewhat surprising) correspondence with Ref. [57], where the same (for $N = 1$) curvature-induced shift of the relativistic version of the Hall viscosity was calculated.

3.17.5 Hall viscosity and Berry curvature

It was shown by Avron *et. al.* [10] and Levay [11] that Hall viscosity (at the time [10] called it Odd viscosity) as related to the Berry curvature on the space of moduli of a torus. We will briefly pause to remind this result.

One considers a Schroedinger equation on a tours

$$\frac{\hbar^2}{2m} g^{ij}(\partial_i - iA_i)(\partial_j - iA_j)\psi = E\psi\,, \tag{3.170}$$

where metric is chosen to be flat (that is Ricci curvature of the metric-compatible affine connection vanishes). On a torus such metric is parametrized by as single complex number τ known as modulus.

$$g^{ij} = \begin{pmatrix} \frac{1}{\tau_2} & \frac{\tau_1}{\tau_2} \\ \frac{\tau_1}{\tau_2} & \frac{|\tau|^2}{\tau_2} \end{pmatrix} , \tag{3.171}$$

The Berry curvature of the $U(1)$ line bundle (described by ground state of (3.170)) on the space of τ is given by

$$\Omega = \frac{\nu \bar{s}}{2} N_\phi \frac{d\tau_1 \wedge d\tau_2}{\tau_2^2} = \frac{1}{4} N_\phi \frac{d\tau_1 \wedge d\tau_2}{\tau_2^2} . \tag{3.172}$$

The Chern class of this connection is a fractional number and is given by the integrated curvature

$$\frac{1}{2\pi} \int \Omega = \frac{N_\phi}{24} . \tag{3.173}$$

The Berry phase computation proves that in principle the (average) Hall viscosity is topologically protected since it is a Chern number of a moduli space of a torus. This is similar to Hall conductance that is also a Chern number of a certain $U(1)$ bundle over a torus.

This computation must be contrasted to the computation we did in this Chapter, where we have computed the local response function, while the Berry phase computation gives *average* viscosity. Only the latter is topologically protected and these quantities coincide for the integer quantum Hall states.

3.18 Outlook

In this Chapter we have developed techniques to derive the linear visco-elastic and electo-magnetic response. The response is summarized in the induced action, that is a gauge invariant functional of external fields. We have derived this functional to quadratic order in external fields and to arbitrary order in gradients of the external fields. With the functional at hand we have derived the gradient corrections to various transport coefficients, that is corrections to transport due to inhomogeneity of the external fields.

We believe that the *geometric* part of the induced action is a universal characteristic of a topological phase, because the coefficients do not depend on any energy scale in the problem and are given by *dimensionless* numbers.

Thus we regard the equation

$$
\begin{aligned}
W^{(geom)} &= \sum_{n=1}^{N} \int \left[\frac{1}{4\pi}(A+\bar{s}_n\omega)d(A+\bar{s}_n\omega) - \frac{1}{48\pi}\omega d\omega\right] \qquad (3.174)\\
&= \frac{\nu}{4\pi}\int AdA + 2\bar{s}Ad\omega + \overline{s^2}\omega d\omega - \frac{c}{48\pi}\int \mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right),
\end{aligned}
$$

as a major new result of this Chapter. Notice, that non-minimal couplings of the original theory do not contribute to these terms (*i.e.* nothing depends on g_s). The second line is the most general induced action (keeping only the terms with dimensionless coefficients) that one can write based on the symmetries of the problem: charge and angular momentum conservation. For IQHE the coefficients are given by

$$\nu = N, \qquad (3.175)$$

$$\bar{s} = \frac{N}{2}, \qquad (3.176)$$

$$\overline{s^2} = \frac{N(4N^2-1)}{12}, \qquad (3.177)$$

$$\mathrm{v}ars = \nu^{-1}\overline{s^2} - \bar{s}^2 = \frac{N(N^2-1)}{12}, \qquad (3.178)$$

$$c = N \qquad (3.179)$$

where the last three coefficients were for the first time computed in this Thesis. This is a full set of numbers that characterizes rotationally invariant gapped topological phase with no other symmetry restrictions. To an extent the remainder of the Thesis will focus on investigation of the origin of these dimensionless quantities. There are many interesting questions related to these topological numbers: Are they topologically protected? Are they related to anomalies of the edge gapless theory? Did we miss any important numbers? Are these numbers quantized for a generic interacting system?...

The results related to the gravitational Chern-Simons term obtained in this Chapter disagree with previous results derived from the effective Chern-Simons TQFT by Wen and Zee [46]. In the next Chapter we will carefully analyze the Wen and Zee derivation and find a slight oversight. We will explain how to fix the oversight and derive the induced action for a variety of quantum Hall states admitting effective description in terms of Chern-Simons topological quantum field theory.

Chapter 4

Induced Action for Fractional Quantum Hall States

In this Chapter we will derive (3.175)-(3.179) for a number of *interacting*, fractional quantum Hall states. The main tool will be the effective Chern-Simons quantum field theory, which we will also derive from an interacting quantum field theory using various tools such as flux attachment and parton construction. We will also clean up the old flat space versions of these constructions.

In the following we will study the interacting, non-relativistic, quantum field theories. This view on a quantum Hall effect is complementary to the trial wavefunctions [9] and some properties of the trail wave functions can be derived directly from a quantum field theory via an appropriate mean field approximation.

4.1 FQHE as a non-relativistic interacting quantum field theory

The prototypical example of a quantum field theory we have in mind is

$$S = S_0 + S_{int}\,, \tag{4.1}$$

where S_0 is the kinetic part and is given by the familiar expression

$$S_0 = \int d^2x dt \sqrt{g}\left[\frac{i}{2}\hbar\psi^\dagger\partial_0\psi - \frac{i}{2}\hbar(\partial_0\psi^\dagger)\psi + \right.$$
$$\left. + eA_0\psi^\dagger\psi - \frac{\hbar^2}{2m}g^{ij}(D_i\psi)^\dagger D_j\psi\right]\,. \tag{4.2}$$

and where S_{int} is the interaction term, new to this Chapter

$$S_{int} = \int dtd^2xd^2y\sqrt{g(x)}\sqrt{g(y)}\psi^\dagger(x)\psi(x)V(d(x,y))\psi^\dagger(y)\psi(y) \quad (4.3)$$

$$= \int dtd^2xd^2y\sqrt{g(x)}\sqrt{g(y)}\rho(x)V(d(x,y))\rho(y)\,, \quad (4.4)$$

where $V(x,y)$ is the interaction potential and $d(x,y)$ is the shortest geodesic distance between x and y. We will only consider small deviations from flat, topologically trivial, space. In this case $d(x,y)$ should be a well defined, singlevalued function.

4.2 Flux attachment

In general the theory (4.1) is completely non-tractable: when the interaction is strong we cannot use the perturbation theory; we cannot easily use the mean field theory, since we do not know the state that will serve as a base for the mean field. No controlled set of approximations is available. These obstructions were the driving force behind the use of first quantized approach and the trial wave functions.

After influential work of Zhang-Hansson-Kivelson [40] and Read [39] the Landau-Ginzburg-Chern-Simons theory was developed. This theory provided a way to guess a state around which the mean field theory is done. Later on Lopez and Fradkin [58], using Polyakov's [59, 60] statistics transmutation, carefully explained how Chern-Simons theory can be derived on the second-quantized language. We start by reviewing the flux attachment procedure and explain how to derive the effective and induced actions from it.

The starting point is to consider a formally equivalent theory described by the action

$$S' = S[A+a] + \frac{1}{4\pi}\frac{1}{2n}\int ada\,. \quad (4.5)$$

First, we want to argue that this modification of the action leaves all of the physical quantities unchanged. That is, the partition functions before and after the flux attachment coincide, at least in flat space. We have

$$Z[A] = \int D\psi D\psi^\dagger e^{\frac{i}{\hbar}S[\psi;A]} = \int D\psi D\psi^\dagger Da e^{\frac{i}{\hbar}S'[\psi;A,a]} = Z'[A] \quad (4.6)$$

To be perfectly fair, the action (4.5) is not even well defined, since the *level* of the Chern-Simons theory is not integer (we assume the field a is compact gauge field) and equals $\frac{1}{k}$. To circumvent this problem we re-write the last term in (4.5) as

$$\int Da \exp i \left[\frac{1}{4\pi n}\int a\wedge da\right] = \int DaDb \exp i\left[-\frac{2n}{4\pi}\int b\wedge db + \frac{1}{2\pi}\int b\wedge da\right] \tag{4.7}$$

The total action then reads

$$S' = S[\psi; A+a] - \frac{2n}{4\pi}\int b\wedge db - \frac{1}{2\pi}\int a\wedge db\,, \tag{4.8}$$

where we have integrated by parts in the last term.

$$S = S_0[\psi; A] + \int d^3x J^\mu a_\mu - \frac{2n}{4\pi}\int b\wedge db - \frac{1}{2\pi}\int a\wedge db \tag{4.9}$$

Taking integral over a_μ gives a delta function condition $J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu b_\rho$ or, more invariantly, $J = \frac{1}{2\pi} * db$. This relation can be inverted as

$$b_\rho = -2\pi\epsilon_{\alpha\beta\rho}\partial^\alpha J^\beta \tag{4.10}$$

Then as a result of the integration over a_μ and b_μ we get

$$S = S_0[\psi; A] + 2\pi n\int \epsilon_{\mu\nu\rho}J^\mu\frac{1}{\Delta}\partial^\nu J^\rho\,, \tag{4.11}$$

Thus, the path integral of a_μ is equivalent to multiplication of the partition function by a phase

$$Z' = Z\times\exp\left(2\pi i\times\frac{2n}{2}\times\left[\int\epsilon_{\mu\nu\rho}J^\mu\frac{1}{\Delta}\partial^\nu J^\rho\right]\right) = Z. \tag{4.12}$$

Thus we have $Z' = Z$, since the object in the brackets is the integer Gauss linking number $\int\epsilon_{\mu\nu\rho}J^\mu\frac{1}{\Delta}\partial^\nu J^\rho$.

4.3 Flux attachment in curved space

This derivation works only naively. In fact, the integral $\int\epsilon_{\mu\nu\rho}J^\mu\frac{1}{\Delta}\partial^\nu J^\rho$ equals to the so-called writhing number W that equals to $L-T$, where L is the Gauss linking number of the currents and T is the torsion of the curve; it is not a topological invariant and only contributes if the currents J carry particles with spin. It was shown [61] that given a framing of the ambient space e^A_μ the curve can inherit this framing and then the torsion T can be expressed in terms of the spin connection of the framing e_μ^A [61] .

$$T = \frac{1}{2\pi}\int dx^i\omega_i = \frac{1}{2\pi}\int d^3xJ^i\omega_i \tag{4.13}$$

The only covariant version of this formula is

$$T = \frac{1}{2\pi}\int d^3xJ^\mu\omega_\mu \tag{4.14}$$

Thus we obtain the "flux attachment identity"

$$Z' = Z\times\exp 2\pi i\left(\frac{2n}{2}\frac{1}{2\pi}\int d^3xJ^\mu\omega_\mu\right) \tag{4.15}$$

In order to remove the phase factor we change the coupling to spin connection as

$$S' = S[\psi; A+a+n\omega] - \frac{2n}{4\pi}\int b\wedge db - \frac{1}{2\pi}\int a\wedge db \tag{4.16}$$

or This coupling introduces an additional factor of $\exp 2\pi i\left(\frac{2n}{2}\frac{1}{2\pi}\int d^3xJ^\mu\omega_\mu\right)$ to Z'. This can be seen as follows

$$S'[\psi; A+a+n\omega] \approx S'[\psi; A+a] + n\int d^3xJ^\mu\omega_\mu \tag{4.17}$$

plugging this into the functional integral we obtain the required phase factor.

In conclusion, we have shown that going from S to S' (with simultaneous coupling to spin connection) is a "trivial" operation, in a sense that it the same as to do nothing. On a more detailed level it multiplies all of the correlation functions by a factor of $e^{2\pi iL}$, where L is an integer number. Nevertheless, the action S' is a suitable starting point to build the mean field theory.

4.4 Mean field theory

First, we are going to describe the mean field theory in the flat space. The starting point is the flux attached action $S'[\psi, A, a]$. We are going to reduce the interacting problem to a non-interacting one by a clever choice of the background value of the *statistical* gauge field a_μ. We start by examining the stationary point of the action S' with respect to a_0

$$\bar\rho + \frac{1}{4\pi n}\epsilon^{ij}\partial_i a_j \equiv \bar\rho + \frac{1}{4\pi n}\mathcal{B} = 0\,, \tag{4.18}$$

we assume that the electrons have formed an incompressible state state at filling fraction ν. Then the density is given by

$$\bar{\rho} = \frac{\nu}{2\pi l^2} = \frac{\nu \bar{B}}{2\pi} \,. \tag{4.19}$$

Combining these two relations we get

$$\mathcal{B} = -n\nu\bar{B} \tag{4.20}$$

We choose the background value of a_μ so that the relation (4.20) is satisfied. Now since electrons feel the total gauge field $A_\mu + a_\mu$ the total magnetic field felt by electrons is

$$B_{total} = \bar{B} + \mathcal{B} = (1 - 2n\nu)\bar{B} = \nu\bar{B} \,, \tag{4.21}$$

depending on the sign of $2n$ the statistical magnetic field will either increase or reduce the value of the real magnetic field. This equation illustrates the meaning of the flux attachment. The statistical gauge field binds n fluxes of the statistical gauge field to every electron, the composite object "electron + $2n$ fluxes" is called composite fermion. These fluxes change the statistics by $e^{2\pi i n}$ and, therefore, the topological spin is changed by n, hence the coupling to spin connection. The physical picture of the flux attachment is very transparent.

We can demand that the electrons will form an integer quantum Hall state with respect to the new total magnetic field B_{tot}. This is the mean field we were looking for

$$\bar{\rho} = \frac{1}{2\pi} B_{total} = \frac{1 - 2n\nu}{2\pi}\bar{B} = \frac{\nu}{2\pi}\bar{B} \tag{4.22}$$

this is only possible when

$$\nu = \frac{1}{2n+1} \,, \tag{4.23}$$

which is precisely the Laughlin series! As a side note, we can also map IQHE at $\nu = 1$ into IQHE at $\nu = -1$ by choosing the statistical magnetic field to be $\mathcal{B} = -2\bar{B}$ (or, simply for $n = -1$) . This mapping will be important later on.

When the total magnetic field is adjusted so that composite fermions fill precisely 1 Landau level the interaction do not matter and one can integrate them out using the methods of the previous chapter. Indeed, using the induced action (3.174) we obtain an *effective theory* of the Laughlin state (in flat space)

$$S_{eff} = \frac{1}{4\pi} (A + a) \wedge d\,(A + a) - \frac{2n}{4\pi} b \wedge db - \frac{1}{2\pi} a \wedge db \tag{4.24}$$

Integrating out the statistical gauge fields a and b we obtain the induced action.

$$W = \frac{1}{4\pi}\frac{1}{2n+1}\int AdA\,, \tag{4.25}$$

this give the Hall conductance

$$\sigma_H = \frac{1}{2n+1}\frac{1}{2\pi}\,, \tag{4.26}$$

which is the correct expression. Similar procedure can be done by filling N Landau levels with the composite fermions. In this case one would obtain the so-called Jain states.

There is, however, a problem when one attempts to implement the outlined program in the curved space.

4.5 Inconsistency of flux attachment in curved space

Here we consider a trivial example: we take IQHE with 1 filled Landau level. We do not really need the meanfield or flux attachment to compute the induced action. The answer is given by (3.174)

$$W_0 = \frac{1}{4\pi}\int A\wedge dA + \frac{1}{4\pi}\int A\wedge d\omega + \frac{1}{4\pi}\left(\frac{1}{4}-\frac{1}{12}\right)\int \omega\wedge d\omega \tag{4.27}$$

Nonetheless, as a sanity check it is useful to see if we can get the same answer from the flux attachment. More precisely, we will be able to check perturbatively whether attaching fluxes is the same thing as "doing nothing". Thus we attach -2 fluxes to every electron. This turns $\nu = 1$ IQHE into $\nu = -1$ IQHE (as we have discussed before).

When the flux attachment is done we can integrate out the fermions using the parity conjugate of (3.174). We have

$$\begin{aligned} W &= -\frac{1}{4\pi}\left(A+a+\omega-\frac{1}{2}\omega\right)\wedge d\left(A+a+\omega-\frac{1}{2}\omega\right)+\frac{1}{48\pi}\omega\wedge d\omega \\ &- \frac{2}{4\pi}b\wedge db - \frac{1}{2\pi}a\wedge db\,, \end{aligned} \tag{4.28}$$

where the first line comes from (3.174) and the rest of the terms are doing the flux attachment according to (4.8). We shift the variable $a \to a + A + \frac{1}{2}\omega$.

$$W = -\frac{1}{4\pi}a\wedge da - \frac{1}{2\pi}a\wedge db - \frac{2}{4\pi}b\wedge db + \frac{1}{2\pi}\left(A + \frac{1}{2}\omega\right)\wedge db + \frac{1}{48\pi}\omega\wedge d\omega \quad (4.29)$$

Integration over a yields

$$\begin{aligned} W &= \frac{1}{4\pi}b\wedge db - \frac{2}{4\pi}b\wedge db + \frac{1}{2\pi}\left(A + \frac{1}{2}\omega\right)\wedge db + \frac{1}{48\pi}\omega\wedge d\omega & (4.30) \\ &= -\frac{1}{4\pi}b\wedge db + \frac{1}{2\pi}\left(A + \frac{1}{2}\omega\right)\wedge db + \frac{1}{48\pi}\omega\wedge d\omega & (4.31) \end{aligned}$$

Finally performing the b integral we get

$$W = \frac{1}{4\pi}\left(A + \frac{1}{2}\omega\right)\wedge d\left(A + \frac{1}{2}\omega\right) + \frac{1}{48\pi}\omega\wedge d\omega \quad (4.32)$$

This is the result for the induced action. A wary reader might notice that the last term has wrong sign. It is not a typo or a sign error. There is a genuine problem in the flux attachment. This is, in fact, the same problem that lead to an error in the work of Wen and Zee [46]. The resolution of this problem is another major result of this Thesis.

4.6 Framing anomaly

We give a very brief review of the framing anomaly tailored to our purposes. The integration over the hydrodynamic Chern-Simons gauge field in the action of the type Eq.(4.8) is done by substituting the solutions of equations of motion back into the action. While it is true that stationary phase approximation for the gaussian integral is exact there is a subtlety that arises when Chern-Simons theory is defined on a curved space.

It is well known that the Chern-Simons theory is *topological* at the classical level, *i.e.* it does not depend on the metric and has vanishing stress-energy tensor. However, this is not true for the full quantum theory [62, 63]. The reason is that while the action is metric-independent, the path integral measure does depend on metric in a non-trivial way. Indeed, the definition of the path integral measure $\mathcal{DA}$ requires gauge fixing, which should be defined in a covariant way to avoid dependence of the partition function on the choice of coordinates. For example, the gauge fixing can be done by including an additional gauge fixing term into the action

$$S_\phi = \int dV \phi D^\mu \mathcal{A}_\mu \,, \tag{4.33}$$

with the integration over the auxiliary field ϕ included in the path integral. The term Eq.(4.33) depends on the *geometry* of the manifold through both covariant derivative D^μ and the invariant space-time integration measure dV. The term of Eq.(4.33) is understood as a part of the definition of the integration measure $\mathcal{D}\mathcal{A}_\mu$.

The dependence of the full partition function Z on the metric of the manifold can be quantified [62, 63]. Consider the partition function of the Chern-Simons theory with arbitrary compact, semi-simple group G at level k. Its partition function is given by [62]

$$\begin{aligned} Z &= \int \mathcal{D}\mathcal{A}\mathcal{D}\phi \, \exp\left\{-i\frac{k}{4\pi}\int_M \mathrm{Tr}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) - iS_\phi\right\} \\ &= \tau \exp\left\{-i\frac{c}{96\pi}\int_M \mathrm{Tr}\left(\Omega d\Omega + \frac{2}{3}\Omega^3\right)\right\}\,, \end{aligned} \tag{4.34}$$

where τ is the Ray-Singer analytic torsion [64]. The latter is a topological invariant and is not important for the upcoming discussion. The phase of the partition function Z is given by the framing anomaly and c is the chiral central charge given by

$$c_G = \frac{k\,\dim(G)}{k+h} \tag{4.35}$$

In Eq.(4.34) $\Omega^a{}_{b,\mu}$ is the Levi-Civita $SO(1,2)$ spin connection [65]. We denote it by Ω to avoid the confusion with the $SO(2)$ spin connection ω (see below). In this work we are interested in quantum Hall states, which are inherently non-relativistic systems. For this reason we turn off the temporal components of the spin connection $\Omega^a{}_{0,\mu} = \Omega^0{}_{b,\mu} = 0$ because non-relativistic physical systems generally do not couple to these components. With this choice the $SO(2)$ component of the spin connection $\omega_\mu \equiv \Omega^1{}_{2,\mu}$ is precisely the one used in Eq.(3.174). Then, we obtain

$$\frac{c}{96\pi}\int_M \mathrm{Tr}\left(\Omega d\Omega + \frac{2}{3}\Omega^3\right) = \frac{c}{48\pi}\int_M \omega d\omega \,. \tag{4.36}$$

4.6.1 Relation to the gravitational anomaly.

Here we emphasize the relation of the framing anomaly to the edge theory of FQHE. The edge theory has a contribution from the gravitational anomaly [66, 67] which can be related to the bulk gravitational Chern-Simons term in the following way. First, let us rewrite the gravitational Chern-Simons term Eq.(4.36) replacing the $SO(1,2)$ spin connection Ω by Christoffel symbols as [68]

$$\begin{aligned}\frac{c}{96\pi}\int \mathrm{Tr}\left(\Omega d\Omega + \frac{2}{3}\Omega^3\right) &= \frac{c}{96\pi}\int \mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right) \\ &- \frac{c}{288\pi}\int \mathrm{Tr}\,(e^{-1}de)^3\,, \qquad (4.37)\end{aligned}$$

The last term in this relation describes the winding number of the dreibeins e and is irrelevant here since the variations of this term on a closed manifold vanish [69].

The gravitational Chern-Simons term written in terms of Christoffel symbols $\Gamma^{\mu}{}_{\nu,\rho}$ is not invariant with respect to changes of coordinates in the presence of a boundary and induces the gravitational anomaly of the edge theory [70]. Thus, in general expressions such as Eq.(3.174), we present the contributions of the framing anomaly in terms of Christoffel symbols to emphasize the relation to the gravitational anomaly and, in turn, to the thermal Hall effect [15].

In summary, every integration over a statistical Chern-Simons field must be accompanied by adding a gravitational CS term with the coefficient given by the chiral central charge of the Kac-Moody algebra of the gauge group.

4.7 Integer quantum Hall state

Here we consider a trivial example: we take IQHE with 1 filled Landau level. We do not really need the meanfield or flux attachment to compute the induced action. The answer is given by (3.174)

Ab before we attach -2 fluxes to every electron. This turns $\nu = 1$ IQHE into $\nu = -1$ IQHE (as we have discussed before).

When the flux attachment is done we can integrate out the fermions using the parity conjugate of (3.174). We still have have

$$
\begin{aligned}
W &= -\frac{1}{4\pi}\left(A+a+\omega-\frac{1}{2}\omega\right)\wedge d\left(A+a+\omega-\frac{1}{2}\omega\right)+\frac{1}{48\pi}\omega\wedge d\omega \\
&- \frac{2}{4\pi}b\wedge db-\frac{1}{2\pi}a\wedge db\,,
\end{aligned} \tag{4.38}
$$

where the first line comes from (3.174) and the rest of the terms are doing the flux attachment according to (4.8). We shift the variable $a \to a + A + \frac{1}{2}\omega$.

$$
W=-\frac{1}{4\pi}a\wedge da-\frac{1}{2\pi}a\wedge db-\frac{2}{4\pi}b\wedge db+\frac{1}{2\pi}\left(A+\frac{1}{2}\omega\right)\wedge db+\frac{1}{48\pi}\omega\wedge d\omega \tag{4.39}
$$

This is where things start to change. Integration over a yields

$$
\begin{aligned}
W &= \frac{1}{4\pi}b\wedge db-\frac{2}{4\pi}b\wedge db+\frac{1}{2\pi}\left(A+\frac{1}{2}\omega\right)\wedge db \\
&+ \frac{1}{48\pi}\omega\wedge d\omega-\frac{1}{48\pi}\omega\wedge d\omega && (4.40)\\
&= -\frac{1}{4\pi}b\wedge db+\frac{1}{2\pi}\left(A+\frac{1}{2}\omega\right)\wedge db\,, && (4.41)
\end{aligned}
$$

where the last term in the first line came from the framing anomaly.

Finally performing the b integral we get

$$
W=\frac{1}{4\pi}\left(A+\frac{1}{2}\omega\right)\wedge d\left(A+\frac{1}{2}\omega\right)-\frac{1}{48\pi}\omega\wedge d\omega\,, \tag{4.42}
$$

where, again, the last term came from the framing anomaly. This is the correct result for the induced action. While it might seem that the framing anomaly is tailored to fix only IQHE problem, it actually fixes all such problems we have encountered in all of the states we have considered.

4.8 Laughlin states

We solve the flux constraint equation by demanding that the composite fermions fill 1 Landau level, *i.e.*

$$
B_{total}=2\pi\bar{\rho}=\nu\bar{B}=\bar{B}(1-2p\nu) \tag{4.43}
$$

This is solved by $\nu=\frac{1}{2p+1}$. This is Laughlin series.

Alternatively we could have asked that the composite fermions fill -1 Landau levels. Then We solve this equation by demanding that the composite fermions fill 1 Landau level, *i.e.*

$$B_{total} = -2\pi\bar{\rho} = \nu\bar{B} = \bar{B}(1-2p\nu) \qquad (4.44)$$

This is solved by $\nu = \frac{1}{2p-1}$. This is again Laughlin series. Let us emphasize that these are two *different* meanfields for the *same* Laughlin state. So the induced action should coincide (at least in the lowest orders).

In this Section we will derive the induced action for using both types of the flux attachment and then show that the induced actions again coincide because of the contribution of the framing anomaly.

4.8.1 Meanfield around $\nu_{eff} = -1$

We start with the meanfield around $\nu_{eff} = -1$ as it is very similar to the calculation of the previous section. We integrate out the composite fermions to get

$$\begin{aligned} W &= -\frac{1}{4\pi}\left(A+a+p\omega-\frac{1}{2}\omega\right)\wedge d\left(A+a+p\omega-\frac{1}{2}\omega\right)+\frac{1}{48\pi}\omega\wedge d\omega \\ &- \frac{2p}{4\pi}b\wedge db-\frac{1}{2\pi}a\wedge db\,, \end{aligned} \qquad (4.45)$$

notice that the difference now in the coefficient of b-field Chern-Simons action. After the shift of variables we have

$$W = -\frac{1}{4\pi}a\wedge da-\frac{2p}{4\pi}b\wedge db-\frac{1}{2\pi}a\wedge db+\frac{1}{2\pi}\left(A+p\omega-\frac{1}{2}\omega\right)\wedge db+\frac{1}{48\pi}\omega\wedge d\omega \qquad (4.46)$$

Integration over a gives

$$\begin{aligned} W &= \frac{1}{4\pi}b\wedge db-\frac{2p}{4\pi}b\wedge db+\frac{1}{2\pi}\left(A+p\omega-\frac{1}{2}\omega\right)\wedge db+\frac{1-c_{-1}}{48\pi}\omega\wedge d\omega \\ &= -\frac{2p-1}{4\pi}b\wedge db+\frac{1}{2\pi}\left(A+p\omega-\frac{1}{2}\omega\right)\wedge db+\frac{1-c_{-1}}{48\pi}\omega\wedge d\omega \end{aligned} \qquad (4.47)$$

Integration over b gives expected answer (for $c_{-1}=1$).

$$W = \frac{1}{4\pi}\frac{1}{2p-1}\left(A+\frac{2p-1}{2}\omega\right)\wedge d\left(A+\frac{2p-1}{2}\omega\right)+\frac{1-2c_{-1}}{48\pi}\omega\wedge d\omega\,, \qquad (4.48)$$

here we kept c_{-1} before replacing it by 1. This gives the expected value of the chiral central charge for the Laughlin state.

4.8.2 Meanfield around $\nu_{eff} = 1$

We now choose $\nu = 1$. In this background the fermionic determinant gives

$$\begin{aligned} W &= \frac{1}{4\pi}\left(A + a + p\omega + \frac{1}{2}\omega\right) \wedge d\left(A + a + p\omega + \frac{1}{2}\omega\right) - \frac{1}{48\pi}\omega \wedge d\omega \\ &- \frac{2p}{4\pi} b \wedge db - \frac{1}{2\pi} a \wedge db \end{aligned} \tag{4.49}$$

Shifting the variables

$$W = \frac{1}{4\pi} a \wedge da - \frac{1}{2\pi} a \wedge db - \frac{2p}{4\pi} b \wedge db + \frac{1}{2\pi}\left(A + p\omega + \frac{1}{2}\omega\right) \wedge db - \frac{1}{48\pi}\omega \wedge d\omega \tag{4.50}$$

Integrating over a we have

$$\begin{aligned} W &= -\frac{1}{4\pi} b \wedge db - \frac{2p}{4\pi} b \wedge db + \frac{1}{2\pi}\left(A + p\omega + \frac{1}{2}\omega\right) \wedge db - \frac{1 + c_1}{48\pi}\omega \wedge d\omega \\ &= -\frac{2p+1}{4\pi} b \wedge db + \frac{1}{2\pi}\left(A + p\omega + \frac{1}{2}\omega\right) \wedge db - \frac{1 + c_1}{48\pi}\omega \wedge d\omega \end{aligned} \tag{4.51}$$

Finally integrating out b

$$W = \frac{1}{4\pi}\frac{1}{2p+1}\left(A + \frac{2p+1}{2}\omega\right) \wedge d\left(A + \frac{2p+1}{2}\omega\right) - \frac{1 + c_1 + c_{-1}}{48\pi}\omega \wedge d\omega \tag{4.52}$$

In the last equation c_1 and c_{-1} cancel each other. Also, fixing $\nu = \frac{1}{2m+1}$ we get different values for p depending on the sign of ν_{eff}. For $\nu_{eff} = -1$ we have $p = m + 1$ and for $\nu_{eff} = 1$ we have $p = m$. Thus (4.52) and (4.48) are the same and given by

$$W = \frac{1}{4\pi}\frac{1}{2p+1}\left(A + \frac{2p+1}{2}\omega\right) \wedge d\left(A + \frac{2p+1}{2}\omega\right) - \frac{1}{48\pi}\omega \wedge d\omega \tag{4.53}$$

Again, we see that framing anomaly ensures the self-consistency of the calculation.

To summarize the important numbers for the Laughlin series are

$$\nu = \frac{1}{2p+1}, \tag{4.54}$$

$$\bar{s} = \frac{2p+1}{2}, \tag{4.55}$$

$$\overline{s^2} = \left(\frac{2p+1}{2}\right)^2, \tag{4.56}$$

$$\mathrm{v}ars = 0, \tag{4.57}$$

$$c = 1. \tag{4.58}$$

These results are in agreement with the Berry phase computation of [71].

It was shown by Bradlyn and Read that in general conformal block trial wave functions that come from what they call a "diagonal CFT" [71, 72]

$$\mathrm{v}ars = 0. \tag{4.59}$$

4.9 Jain states

The Jain states we reintroduced in [73] and were exceptionally successfull in explaining the observed quantum Hall plateaus at fractional fillings. Countrary to Laughlin states we cannot perform the self-consistency checks anymore because the different mean fields will describe genuinely different Jain sequences.

From the flux attachment perspective the Jain sequences are obtained by either filling N or $-N$ Landau levels with composite fermions.

4.9.1 "Quiver" Chern-Simons gauge theory

"Quiver" gauge theory is a fancy term for a gauge theory with a gauge group of the form

$$U(1) \times U(1) \times \ldots \times U(1). \tag{4.60}$$

This kind of Chern-Simons theory allows to encode any *abelian* FQH state. We will explain how to integrate out a "quiver" Chern-Simons given by an arbitrary, integer valued, K-matrix. Consider an action

$$S_K = \int d^3x \left[\frac{1}{4\pi} K_{ij}\alpha_i \wedge d\alpha_j + \frac{1}{2\pi}\alpha_i \wedge t_i d\mathcal{A}\right], \tag{4.61}$$

where $\mathcal{A}$ is an arbitrary abelian external field and α_i is a compact $U(1)$ gauge field. The EoM give

$$K_{ij}\alpha_j = -t_i\mathcal{A}\,, \qquad or \qquad \alpha_j = -K^{-1}_{nj}t_n\mathcal{A} \tag{4.62}$$

Plugging this back we have

$$W_K = \int d^3x \left[-\frac{1}{4\pi}t_iK^{-1}_{ij}t_j\mathcal{A}\wedge d\mathcal{A} + \frac{sign(K)}{48\pi}\omega\wedge d\omega\right]\,, \tag{4.63}$$

where $sign(K)$ stands for the *signature* of the K-matrix, that is number of positive eigenvalues minus the number of negative eigenvalues.

4.9.2 $\nu = \frac{N}{2pN+1}$ Jain sequence.

We now proceed with the Jain states that are obtained by filling N Landau levels of composite fermions. They correspond to taking $\nu = \frac{N}{2pN+1}$. This time we take fermionic functional integral over the electrons that fill N Landau levels. The answer is given by (3.174)

$$\begin{aligned} W &= \frac{1}{4\pi}\sum_{n=1}^{N}\Big[\left(A + a + p\omega + \frac{2n-1}{2}\omega\right)\wedge d\left(A + a + p\omega + \frac{2n-1}{2}\omega\right) \\ &- \frac{1}{48\pi}\omega\wedge d\omega\Big] - \frac{2p}{4\pi}b\wedge db - \frac{1}{2\pi}a\wedge db \end{aligned} \tag{4.64}$$

We proceed with re-writing (4.64) as follows

$$\begin{aligned} W &= \frac{1}{4\pi}\left(Na\wedge da - 2a\wedge db - 2pb\wedge db\right) - \frac{1}{2\pi}b\wedge d\left(A + \frac{N+2p}{2}\omega\right) \\ &+ \frac{1}{4\pi}\left(NA\wedge dA + 2\sum_{k=1}^{N}\frac{2k-1+2p}{2}A\wedge d\omega + \sum_{k=1}^{N}\left(\frac{2k-1+2p}{2}\right)^2\omega\wedge d\omega\right) \\ &- \frac{1}{4\pi}\left(NA\wedge dA + 2\sum_{k=1}^{N}\frac{2k-1+2p}{2}A\wedge d\omega + \frac{1}{N}\left(\sum_{k=1}^{N}\frac{2k-1+2p}{2}\right)^2\omega\wedge d\omega\right) \\ &- \frac{N}{48\pi}\omega\wedge d\omega\,, \end{aligned} \tag{4.65}$$

where we have used

$$\begin{aligned}
\frac{N}{4\pi} a \wedge da &+ \frac{N}{4\pi} a \wedge d\left(A + \frac{N+2p}{2}\omega\right) = \\
&= \frac{N}{4\pi}\left[(a + \left(A + \frac{N+2p}{2}\right)\right] \wedge d\left[a\left(A + \frac{N+2p}{2}\right)\right] \quad (4.66) \\
&- \frac{1}{4\pi}\left(NA \wedge dA + 2\sum_{k=1}^{N}\frac{2k-1+2p}{2} A \wedge d\omega\right. \\
&+ \left.\frac{1}{N}\left(\sum_{k=1}^{N}\frac{2k-1+2p}{2}\right)^2 \omega \wedge d\omega\right) \quad (4.67)
\end{aligned}$$

We integrate (4.65) over a and b using (4.63) with $\mathcal{A} = A + \frac{N+2p}{2}\omega$. The "$K$-matrix" is given by

$$K = \begin{pmatrix} N & -1 \\ -1 & -2p \end{pmatrix} \quad and \quad sign(K) = 0 \quad and \quad t_i K^{-1}_{ij} t_j = -\frac{N}{2pN+1} \quad (4.68)$$

We have then

$$\begin{aligned}
W &= \frac{1}{4\pi}\frac{N}{2pN+1}\left(A + \frac{N+2p}{2}\omega\right) \wedge d\left(A + \frac{N+2p}{2}\omega\right) \\
&+ \frac{1}{48\pi}N(N^2-1)\omega \wedge d\omega - \frac{N}{48\pi}\omega \wedge d\omega\,, \quad (4.69)
\end{aligned}$$

where we have used a marvelous identity

$$\sum_{k=1}^{N}\left(\frac{2k-1+2p}{2}\right)^2 - \frac{1}{N}\left(\sum_{k=1}^{N}\frac{2k-1+2p}{2}\right)^2 = \frac{N(N^2-1)}{12} \quad (4.70)$$

The equation (4.69) is the induced action for the positive Jain series. We again emphasize the topological numbers

$$\begin{aligned}
\nu &= \frac{N}{2pN+1}\,, && (4.71)\\
\bar{s} &= \frac{N+2p}{2}\,, && (4.72)\\
\overline{s^2} &= \frac{N}{12}\left(N^2-1+\frac{3(N+2p)^2}{1+2Np}\right)\,, && (4.73)\\
\mathrm{var}s &= \frac{N(N^2-1)}{12}\,, && (4.74)\\
c &= N\,. && (4.75)
\end{aligned}$$

We emphasize that *orbital spin variance* $\mathrm{var}s$ does not vanish. This was expected since Jain series cannot be written as conformal blocks in any CFT.

4.9.3 $\nu = \frac{N}{2pN-1}$ Jain sequence.

We demand that composite fermions fill N Landau levels of opposite chirality. That is we direct the magnetic field into an opposite way. We have

$$-\frac{\nu}{N}B_0 = B_0(1-2p\nu) \qquad (4.76)$$

This is solved by $\nu = \frac{N}{2pN-1}$. This series is more interesting since it has $\nu = \frac{2}{3}, \frac{3}{5}$ in it for which there is a violation of the Wiedeman-Franz law found by Kane and Fisher [14]. Integrating out the composite fermions gives us (notice the signs)

$$\begin{aligned}
W &= -\frac{1}{4\pi}\sum_{n=1}^{N}\left[\left(A+a+p\omega-\frac{2n-1}{2}\omega\right)\wedge d\left(A+a+p\omega-\frac{2n-1}{2}\omega\right)\right.\\
&+ \left.\frac{1}{48\pi}\omega\wedge d\omega\right] - \frac{2p}{4\pi}b\wedge db - \frac{1}{2\pi}a\wedge db \qquad (4.77)
\end{aligned}$$

Going through the same steps we get to the following result

$$\begin{aligned}
W &= \frac{1}{4\pi}\frac{N}{2pN-1}\left(A+\frac{-N+2p}{2}\omega\right)\wedge d\left(A+\frac{-N+2p}{2}\omega\right)\\
&- \frac{1}{48\pi}N(N^2-1)\omega\wedge d\omega + \frac{(N+sign(K))}{48\pi}\omega\wedge d\omega\,, \qquad (4.78)
\end{aligned}$$

with

$$K = \begin{pmatrix} -N & -1 \\ -1 & -2p \end{pmatrix} \quad and \qquad sign(K) = \pm 2 \quad and \quad t_i K^{-1}_{ij} t_j = -\frac{N}{2pN-1} \tag{4.79}$$

The equation (4.78) is the induced action for the negative Jain series. We again emphasize the topological numbers

$$\nu = \frac{N}{2pN-1}\,, \tag{4.80}$$

$$\bar{s} = \frac{-N+2p}{2}\,, \tag{4.81}$$

$$\overline{s^2} = -\frac{N}{12}\left(N^2 - 1 + \frac{3(-N+2p)^2}{1-2Np}\right)\,, \tag{4.82}$$

$$var s = -\frac{N(N^2-1)}{12}\,, \tag{4.83}$$

$$c = 2 - N\,. \tag{4.84}$$

We emphasize that *orbital spin variance* $var s$ does not vanish. This was expected since Jain series cannot be written as conformal blocks in any CFT.

4.9.4 $\nu = 2/3$ state

The equation (4.78) is quite powerful. Consider $\nu = \frac{2}{3}$. It is described by $N = 2$ and $p = 1$. For these values K-matrix has eigenvalues $-1, -3$ and signature $sign(K) = -2$. Then we get

$$W = \frac{1}{4\pi}\frac{2}{3} A \wedge dA - \frac{1}{16\pi}\omega \wedge d\omega + \frac{0}{48\pi}\omega \wedge d\omega \tag{4.85}$$

So assuming that the last gravitational Chern-Simons term is responsible for the gravitational anomaly we have $c = 0$ and vanishing thermal conductivity - exactly what was predicted by Kane and Fisher.

4.9.5 $\nu = 3/5$ state

The equation (4.78) is quite powerful. Consider $\nu = \frac{3}{5}$. It is described by $N = 3$ and $p = 1$. For these values K-matrix has eigenvalues $-1, -3$ and signature $sign(K) = -2$. Then we get

$$W = \frac{1}{4\pi}\frac{3}{5}\left(A + \frac{-1}{2}\omega\right) \wedge d\left(A + \frac{-1}{2}\omega\right) - \frac{1}{2\pi}\omega \wedge d\omega + \frac{1}{48\pi}\omega \wedge d\omega\,, \tag{4.86}$$

Again, assuming that the last gravitational Chern-Simons term is the one responsible for the gravitational anomaly we have $c = -1$ and the thermal conductivity $K_H = -\frac{\pi^2}{3}\frac{k_B^2}{h}T$ - exactly what was predicted by Kane and Fisher.

We will talk about the thermal transport at great length in the next chapter.

4.10 Composite Boson theory

We obtain the consistent gravitaional Chern-Simons term from composite boson theory in the presence of the curved background geometry [61] for a few Abelian FQH states, which can be easily generalized to any heirarchy state.

We start with the Laughlin state at the filling $\nu = \frac{1}{m}$, $m \in 2\mathbb{Z}+1$. In the composite boson theory, we attach m-flux quanta to the electron to form a composite boson. After flux attachment, we perform the average flux approximation in which we smear out the flux attached to the composite boson onto the two dimensional space uniformly. Then it is clear, in this approximation, that the composite boson sees no background magnetic field in average and therefore condenses into a superfluid state (more precisely, into the Higgs phase of the fluctuating internal gauge fields). After integrating out the phase and density fluctuation of the boson field, we arrive with the hydrodynamical theory of the Laughlin state,

$$\mathcal{L} = -\frac{m}{4\pi}bdb - \frac{1}{2\pi}adb + \frac{1}{2\pi}(A + a + \frac{m}{2}\omega)d\alpha. \tag{4.87}$$

Here α field is introduced as the hydrodynamic gauge field to parametrize the conserved electronic current,

$$J = \frac{1}{2\pi}d\alpha. \tag{4.88}$$

Integrating out the gauge fields $\{a_\mu, b_\mu, \alpha_\mu\}$ and taking care of the framing anomaly associated with the Chern-Simons terms of $\{a_\mu, b_\mu, \alpha_\mu\}$, we obtain,

$$\mathcal{L} = \frac{1}{4m\pi}(A + \frac{m}{2}\omega)d(A + \frac{m}{2}\omega) - \frac{1}{48\pi}\omega d\omega. \tag{4.89}$$

This is the result for the Laughlin state at $\nu = \frac{1}{m}$. We can obtain the heirarchy state by condensing quasiparticles on top of the Laughlint states.

In the Laughlin state, the quasiparticle current j^{qp}_μ couples to the hydrodynamical gauge field as $\mathcal{L}_{\text{coupling}} = t\alpha_\mu j^{qp}_\mu$, $t = \pm 1$ for quasiparticle and quasihole currents. We apply flux attachment procedure to the quasiparticle currents. Suppose we attach n-flux quantum ($n \in 2\mathbb{Z}$) to the quasiparticle, the composite quasiparticle condenses into a superfluid state. This generically gives rise a heirarchy state. We present a few cases below.

4.10.1 $\nu = 2/5$ state

We consider $t = 1, n = 2, m = 3$ which describes the FQH state at the filling $\frac{1}{3-1/2} = \frac{2}{5}$. Following the above discussions and references, we find the effective theory

$$\begin{aligned}
\mathcal{L} &= -\frac{3}{4\pi}\alpha d\alpha - \frac{1}{2\pi}(A + \frac{3}{2}\omega)d\alpha - \frac{2}{4\pi}cdc + \frac{1}{2\pi}(\alpha - \frac{2}{2}\omega)dc, \\
&= -\frac{1}{4\pi}a^I K_{IJ} da^J - \frac{1}{2\pi}At^J da^J - \frac{1}{2\pi}\omega s^J da^J, \\
K &= \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}, a^i = (\alpha, c), t^i = (1, 0), s^i = (3/2, 1).
\end{aligned} \tag{4.90}$$

Here c is introduced as the hydrodynamic gauge field for the conserved quasi-particle current. The signature of the K-matrix is positive and thus we finally obtain by taking the framing anomaly into account,

$$\begin{aligned}
\mathcal{L} &= \frac{2}{5} \times \frac{1}{4\pi}(A + 2\omega)d(A + 2\omega) + \frac{1}{8\pi}\omega d\omega - \frac{2}{48\pi}\omega d\omega, \\
&= \sum_{l=1}^{2} \frac{1}{4\pi}(A + \omega + \frac{(2l-1)}{2}\omega)d(A + \omega + \frac{(2l-1)}{2}\omega) \\
&- \frac{1}{4\pi(2 + 1/2)}(\sum_{l=1}^{2}(A + \omega + \frac{(2l-1)}{2}\omega))d(\sum_{l=1}^{q}(A + \omega + \frac{(2l-1)}{2}\omega)) - \frac{2}{48\pi}\omega d\omega.
\end{aligned}$$

4.10.2 $\nu = 2/3$ state

In the same way, if we choose $t = -1, n = 2, m = 1$, the theory describes a first hierarchy state at the filling fraction $\nu = \frac{1}{1+1/2} = \frac{2}{3}$

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4\pi}\alpha d\alpha - \frac{1}{2\pi}(A + \frac{1}{2}\omega)d\alpha + \frac{2}{4\pi}cdc - \frac{1}{2\pi}(\alpha - \frac{2}{2}\omega)dc, \\
&= -\frac{1}{4\pi}a^J K_{IJ} da^J - \frac{1}{2\pi}At^J da^J - \frac{1}{2\pi}s^J \omega da^J, \\
K &= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, a^I = (\alpha, c), t^I = (1, 0), s^I = (1/2, -1).
\end{aligned} \tag{4.91}$$

This K matrix has no net chirality, i.e., the signature is zero. Hence, integrating out the gauge fields, we obtain

$$\begin{aligned}\mathcal{L} &= \frac{2}{12\pi} AdA - \frac{1}{8\pi}\omega d\omega, \\ &= \frac{1}{4\pi}\left(A + \frac{1}{2}\omega\right) d\left(A + \frac{1}{2}\omega\right) - \frac{1}{12\pi}\left(A + \frac{3}{2}\omega\right) d\left(A + \frac{3}{2}\omega\right), \end{aligned} \tag{4.92}$$

which is consistent with the composite fermion description.

4.11 Arbitrary abelian states

In the general case an abelian FQH state is described by an effective quiver Chern-Simons TQFT copled to the space curvature and electromagnetic field via minimal couplings to the vector potential and $SO(2)$ spin connection

$$S_{eff} = \int d^3x \left[\frac{1}{4\pi} K_{ij}\alpha_i \wedge d\alpha_j + \frac{q_i}{2\pi}\alpha_i \wedge dA + \frac{s_i}{2\pi}\alpha_i \wedge d\omega\right], \tag{4.93}$$

Integrating out statistical gauge fields and adding the framing anomaly we obtain the induced action is given by

$$W = W_K + W_{\text{anom}}, \tag{4.94}$$

$$W_K = \frac{1}{4\pi}\int \left(\boldsymbol{q}^T A + \boldsymbol{s}^T \omega\right) \boldsymbol{K}^{-1} d\left(\boldsymbol{q}A + \boldsymbol{s}\omega\right), \tag{4.95}$$

$$W_{\text{anom}} = -\frac{c}{96\pi}\int \text{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right), \tag{4.96}$$

From this action we read off the topological numbers

$$\nu = \boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{q} \tag{4.97}$$

$$\bar{s} = \nu^{-1}\boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{s} = \frac{\boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{s}}{\boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{q}} \tag{4.98}$$

$$\overline{s^2} = \nu^{-1}\boldsymbol{s}^T \boldsymbol{K}^{-1}\boldsymbol{s} = \frac{\boldsymbol{s}^T \boldsymbol{K}^{-1}\boldsymbol{s}}{\boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{q}}, \tag{4.99}$$

$$\text{v}ars = \frac{\boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{q} \cdot \boldsymbol{s}^T \boldsymbol{K}^{-1}\boldsymbol{s} - (\boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{s})^2}{(\boldsymbol{q}^T \boldsymbol{K}^{-1}\boldsymbol{q})^2}, \tag{4.100}$$

$$c = \text{s}ign\boldsymbol{K} \tag{4.101}$$

There is a good reason to believe that every possible *abelian* quantum Hall state can be described by an abelian Chern-Simons theory. If we accept this conjecture then we have solved the problem of computing these topological numbers for all abelian quantum Hall states. We have to emphasize that the number q_i and t_i have to be computed from a microscopic model. In the previous sections we have used the flux attachment combined with the mean field theory to compute them.

Sadly, these tool are not sufficient to even scratch the surface of the world of *nonabelian* quantum Hall states. We turn to these states in the next section.

4.12 What is non-abelian quantum Hall state?

We are not going to indulge in the definition of fractional statistics too much, but we will say a few general words about statistics of excitations. One of the advantages of the first-quantized approach to FQHE is that it allows to construct wavefunctions of "collective", gapped excitations (we explained this point in the introduction). The properties of the excitations are particularly transparent if the wave function is written as a conformal block in a rational conformal field theory.

For example, Laughlin state at filling $\frac{1}{q}$ can be written as a conformal block in $c = 1$ chiral boson theory

$$\Psi(\{z_i\}) = \left\langle \prod_{i=1}^{N} e^{i\sqrt{q}\phi(z_i)} \times e^{-\frac{i}{2\pi\sqrt{q}}\int d^2z\phi(z)} \right\rangle \quad (4.102)$$

$$= \prod_{i=1}^{N}(z_i - z_j)^q e^{-\frac{1}{4}\sum_i |z_i|^2} \quad (4.103)$$

where ϕ is free scalar field with logarithmic propagator.

In order to construct excitations one has to insert a vertex operator $e^{i\frac{r\phi(w)}{\sqrt{q}}}$ into the correlation function. This operator is a primary field for the values $r = 0, 1, \ldots, q-1$. With this insertion the wavefunction takes form

$$\Psi(w, \{z_i\})_{q.h.} = \left\langle e^{i\frac{r\phi(w)}{\sqrt{q}}} \prod_{i=1}^{N} e^{i\sqrt{q}\phi(z_i)} \times e^{-\frac{i}{2\pi\sqrt{q}}\int d^2z\phi(z)} \right\rangle \quad (4.104)$$

$$= \prod_{i}(z_i - w)^r \prod_{i,j=1}^{N}(z_i - z_j)^q e^{-\frac{1}{4}\sum_i |z_i|^2 - \frac{r}{4q}|w|^2} \quad (4.105)$$

and similarly for several quasi-holes

$$\Psi(\{w_k\},\{z_i\})_{q.h.} = \left\langle \prod_k e^{i\frac{r\phi(w_k)}{\sqrt{q}}} \prod_{i=1}^{N} e^{i\sqrt{q}\phi(z_i)} \times e^{-\frac{i}{2\pi\sqrt{q}}\int d^2z\phi(z)} \right\rangle \tag{4.106}$$

exchanging the position of two quasi-holes multiplies the wavefunction by a phase factor after subtraction of Aharonov-Bohm effect

$$e^{2\pi i\frac{1}{q}}\,, \tag{4.107}$$

as can be deduced from the properties of the primary fields in free boson CFT. Thus quasi holes are abelian anyons, that is quasi-particles with abelian statistics that is nor fermion nor boson. Possibility of the existence of such statistics was first pointed out by Leinaas and Myrheim in 1976 in [74], only 5 years before the discovery of integer quantum Hall effect and only 7 years before Laughlin wrote a wavefunction that supported fractional excitations. In the case of IQHE $q = 1$ and the quasi holes are real holes with charge $+1$ and fermionic statistics.

Moore and Read suggested that the correspondance between trial states and CFT correlators should be taken very seriously and trial wave functions can be guessed by studying various rational conformal field theories (RCFT). In [9] they suggested a construction of the "pfaffian" state (now called Moore-Read state). To construct this state one has to consider the correlation function in conformal field theory $U(1) \times Ising$, where $U(1)$ denotes the chiral boson theory. More explicitly the correlation function is given by

$$\Psi_{MR} = \left\langle \prod_{i=1}^{N} \psi(z_i) \right\rangle_{Ising} \times \left\langle \prod_{i=1}^{N} e^{i\sqrt{q}}\phi(z_i) \times e^{-\frac{i}{2\pi\sqrt{q}}\int d^2z\phi(z)} \right\rangle_{boson} \tag{4.108}$$

similarly one can construct a wave function with local excitations. These excitations can only appear in pairs due to flux quantization (their charge is half the minimal charge allowed in the theory)

$$\Psi_{MR} = \left\langle \prod_{i=1}^{N} \psi(z_i) \right\rangle_{Ising} \times \left\langle e^{i\frac{r\phi(w_1)}{\sqrt{q}}} \sigma(w_1) e^{i\frac{r\phi(w_2)}{\sqrt{q}}} \sigma(w_2) \prod_{i=1}^{N} e^{i\sqrt{q}}\phi(z_i) \times e^{-\frac{i}{2\pi\sqrt{q}}\int d^2z\phi(z)} \right\rangle_{boson} \tag{4.109}$$

Moore and Read have shown that two pairs of such excitations have non-abelian statistics [9]. The reason it is possible is that the state with 2 pairs of such excitations is doubly degenerate and a clever exchange of the excitations induces a non-abelian transformation in the degenerate space.

In general, in $2D$ the (quasi-)particle statistics is a representation of a braid group. If the braids live in any dimension except 2 the representation is either trivial or "anti-trivial" which corresponds to bosons and fermions respectively. In $2D$ the braid group is extremely rich and possesses many inequivalent abelian and non-abelian irreducible representations. In higher dimensions it is still possible to have non-trivial statistics, but one has to consider extended objects like strings, loops, and branes.

4.13 Parton construction

In order to find a way to do a mean field theory with non-abelian states Wen [75] has introduced a parton construction. In this Section, we introduce the projective parton construction [75, 76] which is by now standard, for Laughlin states and $\mathbb{Z}_k$ non-Abelian FQH states at filling $\nu = \frac{k}{k+2}$ (which can be generalized to the states at the filling $\nu = \frac{k}{Mk+2}$ following [76]). Furthermore, we will also show that the geometric responses can be derived for the FQH states from the projective parton constructions as shown in [61].

To demonstrate how the projective parton method works for the Laughlin state, we first consider an example of the bosonic Laughlin state at filling $\nu = 1/2$. For this state, we fractionalize a bosonic field b into the two fermionic partons $\psi_i, i = 1, 2$ carrying $\frac{1}{2}$ electric charge.

$$b(z) = \psi_1(z)\psi_2(z) \tag{4.110}$$

The Hilbert space of the partons ψ_i has unphysical states, and we need to project out those unphysical states by requiring that $\rho_b = \langle b^\dagger b\rangle$ and $\rho_j^\psi = \langle \psi_j^\dagger \psi_j\rangle, j = 1, 2$ are the same, *i.e.* $\rho_b = \rho_j^\psi, j = 1, 2$. This projection can be implemented by introducing an internal $U(1)$ gauge field a_μ. Under the $U(1)$ gauge field, ψ_1 and ψ_2 are oppositely charged because the fundamental boson b should be invariant under the $U(1)$ gauge transformation. To describe the Laughlin state, we choose the mean field state where the fermionic partons ψ_i are in $\nu = 1$ state. Furthermore, the partons are scalars and thus do not minimally couple with the spin connection.

$$\mathcal{L} = \sum_{j=1}^{2} \sqrt{g}\left[\frac{i}{2}\left(\left(D_0^j\psi_j(x)\right)^\dagger \psi_j - \psi_j^\dagger(x)\left(D_0^j\psi_j(x)\right)\right) - \frac{1}{2}(D_a^j\psi_j(x))^\dagger g^{ab}(D_b^j\psi_j(x))\right] \tag{4.111}$$

in which $D_\mu^j = d_\mu + i\frac{1}{2}\bar{A}_\mu + i\frac{1}{2}\delta A_\mu \pm i a_\mu$ are the covariant derivatives of the fermionic partons $\psi_j, j = 1, 2$ ($+ia_\mu$ for ψ_1 and $-ia_\mu$ for ψ_2), and $\bar{A}$ (δA) is the average gauge field parametrizing the uniform magnetic field (the probe field). We integrate out the partons and obtain the effective theory

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^{2}\Big[\frac{1}{4\pi}\Big(\frac{1}{2}A+\frac{1}{2}\omega\pm a\Big)d\Big(\frac{1}{2}A+\frac{1}{2}\omega\pm a\Big)-\frac{1}{48\pi}\omega d\omega\Big] \\ &= \frac{2}{4\pi}\Big(\frac{1}{2}A+\frac{1}{2}\omega\Big)d\Big(\frac{1}{2}A+\frac{1}{2}\omega\Big)+\frac{2}{4\pi}ada-\frac{2}{48\pi}\omega d\omega, \qquad (4.112)\end{aligned}$$

in which the first line is obtained by integrating out the partons $\psi_j, j=1,2$ at filling $\nu=1$. Then the second line follows by identifying $\rho_j^{\psi}=\rho_b, j=1,2$, a constraint of the partons in construction (4.110), and expand the Chern-Simons terms, the first term in the second sum in the first line. Notice that there is no coupling between the internal gauge field a_μ and the "probe" fields $\{A_\mu,\omega_\mu\}$. Now we integrate out the internal gauge field a_μ and, following the prescription of Witten on the frame anomaly, we find

$$\begin{aligned}\mathcal{L} &= \frac{2}{4\pi}\Big(\frac{1}{2}\delta A+\frac{1}{2}\omega\Big)d\Big(\frac{1}{2}\delta A+\frac{1}{2}\omega\Big)-\frac{2}{48\pi}\omega d\omega+\frac{1}{48\pi}\omega d\omega, \\ &= +\frac{2}{4\pi}\Big(\frac{1}{2}A+\frac{1}{2}\omega\Big)d\Big(\frac{1}{2}A+\frac{1}{2}\omega\Big)-\frac{1}{48\pi}\omega d\omega \\ &= +\frac{1}{4\pi}\frac{1}{2}\Big(A+\omega\Big)d\Big(A+\omega\Big)-\frac{1}{48\pi}\omega d\omega. \qquad (4.113)\end{aligned}$$

The last term in the first line, Chern-Simons term of the spin connection, emerges from the frame anomaly of integrating out the gauge field a_μ. The second and third lines follow by the elementary algebra. Now we compare with the effective actions of general quantum Hall states to identify the important topological quantum numbers of the fluid

$$\mathcal{L}=\frac{\nu}{4\pi}(A+\bar{s}\omega)d(A+\bar{s}\omega)-\frac{c}{48\pi}\omega d\omega. \qquad (4.114)$$

in which ν is the filling factor, $\bar{s}$ is the orbital spin, and c is the chiral central charge of the edge state of the quantum Hall state.

Now by comparing Eq.(4.113) and Eq.(4.114), we notice that the orbital spin $\bar{s}=1$, with the known value of the spin of the Laughlin state, and $\nu=\frac{1}{2}$, and the central charge $c=1$ are correctly reproduced.

4.13.1 Laughlin series

for a general Laughlin state at filling $1/k$,

$$\Psi=f_1f_2\cdots f_k, \qquad (4.115)$$

in which $k \in 2\mathbb{Z}+1$, then the fundamental particle Ψ is fermionic, and if $k \in 2\mathbb{Z}$, then the particle is bosonic. The partons $f_j, j = 1\cdots k$ are fermionic and carry the electric charge $1/k$. All the partons are in the integer quanutm Hall state at the filling $\nu = 1$ for the Laughlin state. Formally, the mean-field state is invariant (or can be extended to be invariant) under $SU(k)$ gauge group, but instead of using the full SU(k) invariance, we use $U^{k-1}(1) = U(1) \times U(1) \cdots U(1)$ gauge group, which is enough for the Laughlin state. We assume that the projective partons are spinless (in that the partons do not couple minimally to the spin connection) and see the same metric as the fundamental electron. Then integrating out the fermionic partons, we can obtain the effective theory in terms of the geometric deformations and gauge fields.

To demonstrate how this can be done explicitly, we introduce the internal gauge fields as $\{a_{1,\mu}, a_{2,\mu}, \cdots a_{k-1,\mu}\}$ in the gauge group $U^{k-1}(1) = U(1) \times U(1) \cdots U(1)$. Then the parton f_i couples to the gauge field

$$\beta_{i,\mu} = a_{i,\mu} - a_{i+1,\mu} + \frac{1}{k}\delta A_\mu, \text{ for } i = 1, 2, \cdots k-1, \tag{4.116}$$

and $\beta_{k,\mu} = a_{k,\mu} + \frac{1}{k}\delta A_\mu$ for $i = k$. With this at hand, it is clear that, the gauge transformation $a_{j,\mu}$. Then, integrating out the partons, we find the effective theory

$$\mathcal{L} = \sum_{i=1}^{k} \Big[\Big(\frac{1}{4\pi}\Big(\beta_i + \frac{1}{2}\omega\Big) d\Big(\beta_i + \frac{1}{2}\omega\Big) - \frac{1}{48\pi}\omega d\omega \Big) \Big] \tag{4.117}$$

Finally, we get

$$\mathcal{L} = \sum_{i=1}^{k} \Big(\frac{1}{4\pi}\Big(\beta_i + \frac{1}{2}\omega\Big) d\Big(\beta_i + \frac{1}{2}\omega\Big) - \frac{1}{48\pi}\omega d\omega \Big). \tag{4.118}$$

Now we proceed to the hydrodynamic description. For this, we need to introduce $b_i, i = 1, 2, \cdots k$. Then we find

$$\mathcal{L} = \sum_{i=1}^{k} \Big[-\frac{1}{4\pi} b_i db_i + \frac{1}{2\pi} b_i d\Big(\beta_i + \frac{1}{2}\omega\Big) \Big] \tag{4.119}$$

in which one can check that integrating out b_i fields gives the expression Eq.(3). Notice that there is no gravitational CS term in the hydrodynamic theory. Now we integrate out a_i fields, and this generates all the $b_1 = b_2 = \cdots b_k$ which we will call as b. Then we can perform sum over i to find

$$\mathcal{L} = \Big[-\frac{k}{4\pi}bdb + \frac{1}{2\pi}bd\Big(A + \frac{k}{2}\omega\Big)\Big] \tag{4.120}$$

Now we perform integration over b_μ, taking care of the frame anomaly associated with the Chern-Simons term of b_μ, and find

$$\mathcal{L} = \frac{1}{4\pi k}\Big(A + \frac{k}{2}\omega\Big)d\Big(A + \frac{k}{2}\omega\Big) - \frac{1}{48\pi}\omega d\omega \tag{4.121}$$

in which every coefficient $\{\bar{s}, \sigma_H, c\}$ in Eq.(4.114) for the Laughlin state is correctly reproduced. The discussion here can be generalized to the non-Abelian states including $\mathbb{Z}_k$ parafermion states which are explicitly demonstrated in the main text (and in [61] in which the framing anomaly has not been included).

In the following we will use the results of [61] to guess the correct bulk effective theory for Read-Rezayi states and derive the induced action.

4.14 Effective and induced actions for Read-Rezayi states

In the following we will derive the effective action for the non-abelian $\mathbb{Z}_k$ Read-Rezayi (RR) parafermion states [77] at filling $\nu = \frac{k}{Mk+2}$. While the problem of deriving the bulk effective theory for a generic non-abelian gapped FQH state is not solved, the answer for a variety of different states can be obtained through the parton construction [75, 76]. The effective bulk theory for the non-abelian $\mathbb{Z}_k$ Read-Rezayi parafermion states at filling $\nu = \frac{k}{Mk+2}$ is given by the $(U(M) \times Sp(2k))_1$ Chern-Simons theory [76] and $U(1)_1^{2k+M}$ Abelian theory.

$$\begin{aligned} S &= \frac{1}{4\pi}\int \mathrm{Tr}\left[ada + \frac{2}{3}a^3 + \omega da\right] \\ &- \frac{1}{4\pi}\int \mathrm{Tr}\Big[bdb + 2(QA + S\omega)db\Big], \end{aligned} \tag{4.122}$$

where $Q = \frac{1}{kM+2}\mathrm{diag}\,(1_{2k}, k \times 1_M)$ and $S = \frac{1}{2}1_{2k+M}$ are $(2k+M) \times (2k+M)$ charge and spin matrices. There are $2k+M$ hydrodynamic $U(1)$ gauge fields b and one non-abelian $U(M)\times Sp(2k)$ field a. In the second line of Eq. (4.122) we have coupled the bulk theory to external electromagnetic field and geometry (see [61]). In Eq.(4.122) we have essentially used the coset construction of [78]. Note that the introduction of the abelian fields b does not change the degeneracy on the higher genus surfaces because the corresponding K-matrix is unity.

Integration over the low energy degrees of freedom implies the universal effective action Eq.(3.174) with the filling factor, the average orbital spin, and the orbital spin variance given by

$$\nu = \mathrm{Tr}\, Q^2 = \frac{k}{Mk+2}\,, \tag{4.123}$$

$$\bar{s} = \nu^{-1}\mathrm{Tr}\, QS = \frac{M+2}{2}\,, \tag{4.124}$$

$$\mathrm{v}ars = \nu^{-1}\mathrm{Tr}\, S^2 - \bar{s}^2 = 0\,. \tag{4.125}$$

The chiral central charge c of the boundary $U(1)_1^{2k+M}/(U(M)\times Sp(2k))_1$ coset CFT is given by

$$c = c_{U(1)_1^{2k+M}} - c_{U(M)_1} - c_{Sp(2k)_1} = \frac{3k}{k+2}\,, \tag{4.126}$$

which is the correct value of the central charge of the edge states of the RR parafermion states.

4.15 Outlook

In this Chapter we have derived the induced action for a variety of the fractional quantum Hall states. The presented derivation actively used Chern-Simons topological quantum field theory as well as various mean field theories that allowed to derive the details of the Chern-Simons theory such as gauge group, level, spin and charge vector or matrix, etc. This is, in principle additional information that comes from the UV details and (at least in principle) could depend on the UV completion of the effective description.

In this Section we want to offer an opposite point of view. Suppose we are given only induced action and no other information whatsoever. To an extent an experimentalist performing *only* transport measurements only has access to induced action, so the question we are raising is relevant at least in principle. Let's fix the space-time to be a closed manifold and investigate the generic induced action

$$\begin{aligned} W &= \frac{\nu}{4\pi}\int\Big((A+\bar{s}\omega)d(A+\bar{s}\omega) + \mathrm{v}ars\cdot\omega d\omega\Big) \\ &- \frac{c}{96\pi}\int \mathrm{Tr}\,\left[\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right]\,, \end{aligned} \tag{4.127}$$

As we have mentioned before on a closed manifold (that is, if experimentalist cannot access the edge physics) the affine gravitational Chern-Simons term can be traded with $SO(2)$ gravitational Chern-Simons term without any consequence. One can even view this trade of as a $GL(3,\mathbb{R})$-valued gauge transformation

$$\omega_\mu = e^{-1}\Gamma_\mu e + e^{-1}\partial_\mu e \tag{4.128}$$

Thus a priori we do not know how to split the coefficient in front of the gravitational CS term into $\bar{s}^2$, vars and c. In fact, the notion of c is somewhat meaningless since we do not know what CFT or TQFT this induced action came from. The best we can do is to compute the linear response functions. All of the information is contained in two transport coefficients

$$\sigma_H = \frac{\nu}{4\pi} \tag{4.129}$$

$$\eta_H = \frac{\bar{s}}{2}\rho + \left(\frac{\nu}{2}\text{v}ars - \frac{c}{24}\right)\frac{R}{4\pi} \tag{4.130}$$

Measuring the Hall conductance we can find ν, measuring the Hall viscosity on a flat space (say, torus) we can find $\bar{s}$, finally measuring the Hall viscosity on a sphere or any other curved space we can find $\frac{\nu}{2}\text{v}ars - \frac{c}{24}$. If we do not have any additional information we cannot distill the chiral central charge c from this combination. If, however, we have access to the edge theory we can measure the thermal Hall conductance that will only be sensitive to c.

In the next two Chapters we will subject the quantum Hall states to additional perturbations and restrictions, such as extra Galilean symmetry and finite temperature in an attempt to get more information from the bulk physics, but will only confirm the general conclusions presented in this discussion.

Chapter 5

Galilean Invariant Induced Action

We have mentioned before that the induced action can also be investigated from the general symmetry principles. The more symmetry is imposed on a physical system the more restricted the induced action will be. In this Chapter we will study an additional symmetry that is only possible in non-relativistic world with $z = 2$ dynamical scaling exponent. This symmetry generalizes the familiar symmetry of the Newton's laws

$$x \longrightarrow x - vt \tag{5.1}$$

to a local symmetry that is termed local Galilean symmetry or non-relativistic diffeomorphism symmetry. The pioneering works of Son and Wingate [45] in cold gas physics and Son and Hoyos [50] in quantum Hall physics shown that this symmetry is useful in restricting the induced action, but at the same time it remains realistic symmetry to be likely implemented on the Lowest Landau Level.

In this Chapter we will construct the quadratic induced action that is invariant under local Galilean symmetry to any order in gradient expansion and illustrate the interesting relations that follow. In particular, we will discover that Laughlin and Moore-Read wave functions satisfy the relations we derive, thus giving a strong argument in support of the local Galilean symmetry as a universal dynamic symmetry of the Lowest Landau Level.

5.1 Galilean symmetry in free fermions

In this Section we will derive an unexpected symmetry of the non-interacting non-relativistic theory (3.1). The action is given by

$$S_0 = \int d^2x dt \sqrt{g}\left[\frac{i}{2}\hbar\psi^\dagger\partial_0\psi - \frac{i}{2}\hbar(\partial_0\psi^\dagger)\psi + \right. \\ \left. + eA_0\psi^\dagger\psi - \frac{1}{2m}g^{ij}((\partial_i - iA_i)\psi)^\dagger(\partial_j - iA_j)\psi - \frac{g_s}{4m}B\psi^\dagger\psi\right]. \quad (5.2)$$

We assume that all of the external fields depend on time. It is a matter of an unpleasant computation to verify that the following set of infinitesimal transformations is indeed a symmetry of the action

$$\begin{aligned} \delta A_i &= -\xi^k F_{ki} - m g_{ik}\dot{\xi}^k - \partial_i(\alpha + A^k\xi_k)\,, \\ \delta A_0 &= -\xi^k F_{k0} - \partial_0(\alpha + A^k\xi_k) + \frac{g_s}{4}\frac{\epsilon^{ij}}{\sqrt{g}}\partial_i(g_{jk}\dot{\xi}^k)\,, \\ \delta g_{mn} &= -\xi^k\partial_k g_{mn} - g_{mk}\partial_n\xi^k - g_{nk}\partial_m\xi^k\,, \\ \delta g^{mn} &= -\xi^k\partial_k g^{mn} + g^{mk}\partial_k\xi^n + g^{nk}\partial_k\xi^m\,. \end{aligned} \quad (5.3)$$

These transformations combine a local version of Galilean transformations parameterized by $\xi^k(x,t)$ and gauge transformations $\alpha(x,t)$. In the following we use Galilean transformations accompanied by a particular gauge transformation $\alpha = -A_k\xi^k$, so that (5.3) have an explicitly gauge invariant form.

Conventional (global) Galilean transformations corresponding to a constant velocity v^k are given by $\xi^k(x,t) = v^k t$. Under this transformation we have (in the integrated form)

$$B' = B \quad (5.4)$$

$$E'_{||} = E_{||} \quad (5.5)$$

$$E'_\perp = E_\perp - vB \quad (5.6)$$

$$g'_{ij} = g_{ij} \quad (5.7)$$

as it should be. The fermionic field transforms according to

$$\psi'(x,t) = e^{imv^i x_i}\psi(x - vt, t) \quad (5.8)$$

We will later show that this symmetry is a field-theoretic way to impose the relation between momentum P^i and and electric current J^i (see e.g., [79])

$$J^i = \frac{e}{m} P^i + \frac{g_s}{4m} \epsilon^{ij} \partial_j \rho \, . \tag{5.9}$$

5.2 Galilean invariant interactions

It is possible to introduce interactions that respect the local Galilean symmetry (5.3). A rule of thumb is that the interaction should not involve time derivatives. For example, the action [45]

$$S = S_0 + \int dV \left(\lambda \psi^\dagger \psi \sigma - \frac{1}{2} g^{ij} \partial_i \sigma \partial_j \sigma - \frac{\sigma^2}{2r_0^2} \right) \tag{5.10}$$

If the field σ transforms as a scalar

$$\delta\sigma = -\xi^k \partial_k \sigma \tag{5.11}$$

the action S is Galilean invariant. This action describes non-relativistic fields that interact with an instantaneous interaction

$$V(r) = -\frac{\lambda^2}{4\pi r} e^{-\frac{r}{r_0}} \, . \tag{5.12}$$

This can be obtained by integration out the field σ, which is easy to do since it is non-dynamical.

Coulomb potential can be introduced as well, but one has to introduce an additional spatial dimension, transverse to the quantum Hall plane. The action is given by [50]

$$S = S_0 + \int d^2x dt \sqrt{g} a_0 (\psi^\dagger \psi - \bar{\rho}) + 2\pi\varepsilon \int dt s^2 x dz \sqrt{g} \left((\partial_z a_0)^2 + g^{ij} \partial_i a_0 \partial_j a_0 \right) \tag{5.13}$$

integrating out the slave field a_0 ond obtains a theory of non-relativistic fields interacting with the coulomb potential

$$V(r) \sim \frac{\varepsilon}{r} \tag{5.14}$$

The action is again Galilean invariant if a_0 transforms as a scalar. Thus we have established that there is a number of very reasonable interactions that are allowed by the symmetry.

5.2.1 Galilean symmetry from large c limit

Son and Wingate have also given a beautiful construction of Galilean invariant actions starting from a relativistic quantum field theory coupled to a relativistic gravity. Consider a relativistic massive scalar coupled to *spacetime* metric

$$S = -\int d^3x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\Psi^*\partial_\nu\Psi + m^2c^2\Psi^*\Psi\right) \tag{5.15}$$

under space-time diffemorphisms Ψ transforms as a scalar and $g_{\mu\nu}$ transforms as a tensor according to

$$\delta g_{\mu\nu} = -\xi^\lambda\partial_\lambda g_{\mu\nu} - g_{\lambda\nu}\partial_\mu\xi^\lambda - g_{\mu\lambda}\partial_\nu\xi^\lambda \tag{5.16}$$

We make an ansatz for $\Psi = e^{-imc^2t}\frac{\psi}{\sqrt{mc}}$ and the metric

$$g_{\mu\nu} = \begin{pmatrix} -1-\frac{2A_0}{mc^2} & -\frac{A_i}{mc} \\ -\frac{A_i}{mc} & g_{ij} \end{pmatrix}, \tag{5.17}$$

plugging this into (5.15) and keeping the leading terms in $\frac{1}{c}$ we get back to action (5.2) at $g_s = 0$. The diffeomorphism of in the time direction $\xi^0 = \frac{\alpha}{mc}$ turns into a gauge transformation.

Using this limiting procedure it is possible to construct a number of Galilean invariant induced actions starting from relativistic gravitational actions. We will give a couple of examples later on.

5.3 Galilean transformations in constant magnetic field

The constant part of the external magnetic field B_0 is a parameter of the macroscopic theory and will enter the coefficients in the gradient expansion of the effective action. We do not transform it under Galilean transformations, but instead absorb the corresponding part into the transformation laws of the vector potential A_i (compare to eq. 5.3) as

$$\delta A_i \;\; = \;\; -\xi^k\overline{F}_{ki} - \xi^k F_{ki} - mg_{ik}\dot{\xi}^k\,. \tag{5.18}$$

The external metric is a small perturbation over flat background $g_{ik} = \delta_{ik} + \delta g_{ik}$.

We first find the 0-th order piece of these transformations, because we are going to require symmetry in the first order in fields.

$$\delta^{(0)} A_i = -\epsilon_{ki}\xi^k B - m\dot{\xi}_i \tag{5.19}$$
$$\delta^{(0)} A_0 = 0 \tag{5.20}$$
$$\delta^{(0)} g^{mn} = \partial^n \xi^m + \partial^m \xi^n \tag{5.21}$$

We will impose the symmetry in Fourier space. The 0-th order transformation in Fourier space the transformations take form

$$\delta^{(0)} A_z = im(\omega + \omega_c)\xi^{\bar{z}} \tag{5.22}$$
$$\delta^{(0)} A_{\bar{z}} = im(\omega - \omega_c)\xi^{z} \tag{5.23}$$
$$\delta^{(0)} g^{zz} = 2ik^z \xi^z \tag{5.24}$$
$$\delta^{(0)} g^{\bar{z}\bar{z}} = 2ik^{\bar{z}} \xi^{\bar{z}} \tag{5.25}$$
$$\delta^{(0)} g^{\bar{z}z}(\omega, k) = i(k^z \xi^{\bar{z}} + k^{\bar{z}} \xi^z) \tag{5.26}$$
$$ \tag{5.27}$$

and the 1-st order piece

$$\delta^{(1)} A_0 = -\frac{i}{2}(k_{\bar{z}} A_0 + \omega A_{\bar{z}})\xi^{\bar{z}} + c.c. \tag{5.28}$$
$$\delta^{(1)} g^{z\bar{z}} = \frac{i}{2} k_z g^{zz} \xi^{\bar{z}} + c.c. \tag{5.29}$$

5.4 Building blocks for quadratic induced action

To restrict the form of the induced action we use the rotational invariance, locality, gauge invariance and similarities between electro-magnetism and gravity.

The gauge invariance requires that the effective action depends on the vector potential A_μ only through electric field E_i and magnetic field B. The only exception is the Chern-Simons term which is gauge invariant only up to boundary terms. We also assume that the system under consideration is gapped. Therefore, linear response functions are local, i.e., can be written as Taylor series in frequency and momentum, so that the quadratic effective action is constructed as an expansion in derivatives. As transformations (5.3) mix *different* orders in the gradient expansion we expect nontrivial relations between the universal response coefficients and higher order gradient corrections thereof.

We analyze the gravitational terms in a similar way by introducing an

Abelian gauge field that encodes coupling to the background curvature. This field is a non-relativistic spin-connection [50] $\omega_0 = -\frac{1}{2}\epsilon^{ab}e^{aj}\dot{e}^b_j$, $\omega_i = -\frac{1}{2}\epsilon^{ab}e^{aj}\partial_i e^b_j - \frac{1}{2\sqrt{g}}\epsilon^{jk}\partial_j g_{ik}$ where e^a_j are the time-dependent zweibeins [65]. The spin connection depends only on the metric and transforms as an Abelian gauge field under local $SO(2)$ spatial rotations $\omega_\mu \to \omega_\mu + \partial_\mu \alpha$.

With the spin connection at hand we construct the gravi-electric $\mathcal{E}_i = \dot{\omega}_i - \partial_i \omega_0$ and gravi-magnetic $\frac{1}{2}\sqrt{g}R = \partial_1\omega_2 - \partial_2\omega_1$ fields which are explicitly invariant under the local $SO(2)$ rotations. Notice, that the parity properties of e/m fields and their elastic cousins are different: R is a scalar while B is a pseudo-scalar and $\mathcal{E}_i$ is an axial vector.

In the linear order in deviations from the flat background we have explicitly

$$R \approx \partial_i\partial_j g_{ij} - \Delta g^i_i\,, \quad \mathcal{E}_i \approx -\frac{1}{2}\epsilon^{jk}\partial_j \dot{g}_{ik}\,, \tag{5.30}$$

where Δ is the flat space Laplace operator.

The spin connection ω can be expressed in terms of perturbations of the metric as follows.

$$\omega_0 \;=\; \frac{1}{2}\epsilon^{jk}\delta g_{ij}\dot{g}_{ik} \qquad \omega_i = -\frac{1}{2}\epsilon^{jk}\partial_j \delta g_{ik}\,. \tag{5.31}$$

There is an additional building block describing dilatations - the trace of the metric which we denote as

$$G \equiv \delta g^i_i\,. \tag{5.32}$$

5.5 Induced action

In the following we present the quadratic induced action as a sum

$$W = W^{(1)} + W^{(\eta)} + W^{(geom)} + W^{(em)} + W^{(g)} + W^{(mix)}\,. \tag{5.33}$$

The first contribution collects all "linear" terms

$$W^{(1)} \;=\; \int d^2x dt \sqrt{g}\left(-\epsilon_0 + \rho_0 A_0\right)\,. \tag{5.34}$$

Notice, that although (5.34) is linear in A_0 it also contains (through $\sqrt{g}$) terms quadratic in deviations from the constant background. This term encodes the properties of the unperturbed ground state: energy density ϵ_0 and density $\rho_0 = \frac{\nu}{2\pi l^2}$, where $l^2 = 1/B_0$ is the magnetic length and ν is the filling fraction.

The coefficients in (5.34) and below generally depend on the external magnetic field B_0, filling fraction ν and other microscopic parameters of the system such as the Coulomb gap, cyclotron mass, etc.

The next term has a form

$$W^{(\eta)} = \int d^2x dt \, \eta_H \epsilon^{jk} g_{ij} \dot{g}_{ik} \,, \tag{5.35}$$

where η_H (in Fourier space) is a function of frequency. One can think about $\eta_H(\omega)$ as of frequency dependent Hall viscosity. We notice comparing to (5.31) that the term (5.35) at zero frequency has a form $2\eta_H(0)\omega_0$ which allows to identify $2\eta_H(0)$ as the orbital spin density and $\bar{s} = 2\eta_H(0)/\rho_0$ as an average orbital spin per particle. For the conformal block states [9] the latter is given by $2\bar{s} = \nu^{-1} + 2h_\psi$, where h_ψ is the conformal weight of the electron operator in the "neutral" sector of the conformal field theory [13, 36].

The next contribution contains topological and *geometric* terms

$$W^{(geom)} = \int \Big(\frac{\sigma_H}{2} AdA + SAd\omega + C\omega d\omega \Big) \,, \tag{5.36}$$

known as the Chern-Simons, Wen-Zee [46] and the gravitational Chern-Simons terms. These terms are special as they are invariant with respect to gauge transformations and local rotations only up to full derivatives. In the presence of the boundary they are related to the boundary theory and are the natural candidates for encoding universal properties. It is convenient to allow η_H, σ_H, S, and C in (5.36) to depend on frequency, so that they coincide with their conventional values at zero frequency. In the following expressions (5.37-5.39) the coefficients $\epsilon, \sigma, \mu, \ldots$ depend on both frequency and momentum [1].

The electro-magnetic response is represented by

$$W^{(em)} = \int d^2x dt \left(\epsilon E^2 + \sigma(\partial_i E_i) B - \mu^{-1} B^2 \right) \,. \tag{5.37}$$

Here ϵ and μ are electromagnetic susceptibilities and σ encodes the gradient corrections to the Hall conductivity.

[1] We stress that since we are interested in the long wave corrections to the linear response functions the coefficients in (5.34) and below are Taylor series in ∂_t and (in (5.37-5.39) Δ, therefore in the Fourier space these coefficients are frequency and momentum dependent. For example, ϵE^2 of (5.37) should be understood as $E(x,t)\epsilon(\partial_t, \Delta)E(x,t)$. To make the equations more readable we suppress the arguments and ordering

Analogously, we write down gravitational and mixed terms

$$
\begin{aligned}
W^{(g)} &= \int d^2xdt\Big(\epsilon_g\mathcal{E}^2+\sigma_g(\partial_i\mathcal{E}_i)R-\frac{1}{\mu_g}R^2 \\
&+ \zeta_3 GR+\zeta_4 G(\partial_i\mathcal{E}_i)+\zeta_5 G^2\Big)\,, \qquad (5.38)\\
W^{(mix)} &= \int d^2xdt\Big(\epsilon_m(E_i\mathcal{E}_i)+\sigma_{m1}(\partial_iE_i)R-\frac{1}{\mu_m}BR \\
&+ \sigma_{m2}(\partial_i\mathcal{E}_i)B+\zeta_1 G(\partial_iE_i)+\zeta_2 GB\Big)\,. \qquad (5.39)
\end{aligned}
$$

Eqs. (5.33-5.39) give the effective action expanded to the second order in fields and to an arbitrary order in gradients.

Equations (5.33-5.39) contain all possible combinations that can enter real, rotationally, gauge and PT invariant quadratic effective action of a gapped system in transverse constant magnetic field. They define 19 different response coefficients $\eta_H, \sigma_H, S, C, \epsilon, \ldots$. The coefficients in W encode *all* possible two point correlation functions of electric charge density, electric current, and stress tensor at finite frequency ω and momentum k. Imposing the LGI (5.3) will give additional relations between the coefficients.

The next step is to derive the Ward identities of LGI. We apply the transformations (5.3) to W and demand the invariance of the full effective action under these transformations up to the terms quadratic in fields. This requirement imposes constraints on the linear response functions in *all* orders of the gradient expansion in a form of a system of linear (in response functions) equations. In full generality these relations are not enlightening and we present only several particular relations.

We write the induced action in the notations of Chapter 3.

$$
W=\int\frac{d\Omega}{2\pi}\frac{d^2k}{(2\pi)^2}\left(w_Iv_I+\frac{1}{2}v_I(k)W_{IJ}(k)v_J(-k)\right) \qquad (5.40)
$$

linear term has form

$$
w_iv_i=\bar\rho A_0(k)-\bar\epsilon g_{z\bar z}(k) \qquad (5.41)
$$

where $\bar\rho$ and $\bar p$ are average density and pressure in the ground state and W_{IJ} is the generalized polarization operator.

Using (5.22)-(5.26) we can write the 0-th order transformation laws of $v_I(k)$

$$\delta^{(0)}_{Gal} v_i(k) = i \begin{pmatrix} 0 \\ m\Omega^+ \\ 0 \\ 2k^{\bar z} \\ 0 \\ k^z \end{pmatrix} \xi^{\bar z} + i \begin{pmatrix} 0 \\ 0 \\ m\Omega^- \\ 0 \\ 2k^z \\ k^{\bar z} \end{pmatrix} \xi^z \,, \tag{5.42}$$

where $\Omega^\pm = \omega \pm \omega_c$.

Now taking the variation of the linear term in the induced action we find

$$\delta_{Gal}(w_i v_i) = (\delta^{(1)}\bar n A_0^- - \bar\epsilon \delta^{(1)} g^{-,zz}) = -\frac{i}{2}\left(k_{\bar z} n A_0^- + \omega n A_{\bar z}^- + k_z \epsilon g^{-,zz}\right)\xi^{\bar z} \tag{5.43}$$

Variation of the effective action is given by

$$\delta_{Gal}\left[w_I v_I + \frac{1}{2} v_I(-k) W_{IJ}(k) v_J(k)\right] = c_I(k) v_I(-k) + v_I(-k) W_{IJ} \delta^{(0)}_{Gal} v_I(k) = 0 \tag{5.44}$$

We introduce a notation c_I

$$c_I(k) = -\frac{i}{2} \begin{pmatrix} \bar n k_{\bar z} \\ 0 \\ \bar n \omega \\ \bar\epsilon k_z \\ 0 \\ 0 \end{pmatrix} \xi^{\bar z} - \frac{i}{2} \begin{pmatrix} \bar n k_z \\ \bar n \omega \\ 0 \\ 0 \\ \bar\epsilon k_{\bar z} \\ 0 \end{pmatrix} \xi^z \tag{5.45}$$

So the requirement that W is invariant w.r.t. Galilean and gauge $U(1)$ transformations can be written simply as a system of linear equations

$$W_{IJ}(k)\delta^{(0)}_{Gal} v_J(k) = c_I(k) \tag{5.46}$$

$$W_{IJ}(k)\delta_{gauge} v_J(k) = 0 \,, \tag{5.47}$$

where

$$\delta_{gauge} v(k) = i \begin{pmatrix} \omega \\ -k_z \\ k_{\bar z} \\ 0 \\ 0 \\ 0 \end{pmatrix} \alpha \tag{5.48}$$

The linear equations (5.46) encode infinite number of the relations because the left hans side must be understood as an expansion in momentum and frequency. Coefficient in front of every monomial in k and ω must vanish independently. Also, notice that the linear order term contributes contact-like terms into the Ward identites.

These are the main results of the Chapter in the implicit form. One needs to impose both of these constraints on the effective action and obtain the relations between responses. We point out that constraints from Galilean invariance mix the electromagnetic response with the "mixed" or "gravi-electro-magnetic" responses and the latter with purely gravitational (or visco-elastic) responses.

5.6 Ward Identities

In this section we will write out the constraints (5.46) in glory details. We will find many interesting relations between the response functions: some of these relations will be old and some will be new.

5.6.1 Hall conductivity and orbital spin

We start with the following relations

$$\sigma_H = \frac{\nu}{2\pi}\frac{\omega_c^2}{\omega_c^2-\omega^2}\,, \quad S = 2\eta_H l^2\frac{\omega_c^2}{\omega_c^2-\omega^2}\,, \tag{5.49}$$

where $\omega_c = B_0/m$ is the cyclotron frequency. These are the familiar relations for the Hall conductivity and the Wen-Zee shift [37]. Integrating the charge density ρ over the curved manifold and using (5.49) we obtain that the shift in the total charge on the curved manifold of the Euler character χ is given by

$$Q = \nu N_\phi + \nu\bar{s}\chi\,. \tag{5.50}$$

5.6.2 Zero momentum relations

Here we present the Ward identities at zero momentum $k = 0$. In order to lighten up the notations we suppress the dependence on frequency. We stress that all response function below are evaluated at finite frequency ω and $k = 0$.

We start with relations

$$\epsilon(\omega) = \frac{\nu}{4\pi}\frac{\omega_c}{\omega_c^2-\omega^2}\,, \tag{5.51}$$

$$\epsilon_m(\omega) = \eta_H l^2\frac{\omega_c}{\omega_c^2-\omega^2}\,. \tag{5.52}$$

The first relation determines the homogeneous dielectric response function $\epsilon(\omega, k=0)$ completely and the pole at ω_c reflects the Kohn's theorem. The second relation is an elastic analogue of Kohn's theorem.

The next relation is the finite frequency version of the Hall viscosity-conductivity relation [50]

$$\frac{\sigma}{l^2} = \frac{\omega_c^2(\omega_c^2+\omega^2)}{(\omega_c^2-\omega^2)^2}\left(\eta_H l^2 - \frac{\nu g_s}{16\pi}\right) - \frac{\omega_c^2}{\omega_c^2-\omega^2}\frac{\mu^{-1}}{\omega_c l^2}. \tag{5.53}$$

Here we slightly generalized the relation obtained in [37] by including an arbitrary g_s-factor.

We also find two elastic analogues of (5.71)

$$\frac{\mu_m^{-1}}{\omega_c l^2} = \frac{C}{2} - \frac{g_s}{4}\eta_H l^2 \frac{\omega_c^2+\omega^2}{\omega_c^2-\omega^2} - \frac{\sigma_{m1}}{l^2} + \frac{\epsilon_m^{(1)}\omega^2}{\omega_c}, \tag{5.54}$$

$$\frac{\sigma_{m2}}{l^2} = \frac{g_s}{2}\eta_H l^2 \frac{\omega_c^2}{\omega_c^2-\omega^2} + \left(2\epsilon_m^{(1)} - \epsilon_g\right)\omega_c\,, \tag{5.55}$$

where we introduced $(kl)^2\epsilon_m^{(1)} = \epsilon_m(k,\omega) - \epsilon_m(0,\omega)$.

The coefficients $\zeta_1, \ldots, \zeta_5$ are completely fixed by the Galilean invariance in terms of other coefficients. Their expansions start with ω^2 and we do not list them here.

5.7 Regularity of the limit $m \to 0$

Let us consider the static limit $\omega = 0$ of (5.54)

$$m\mu_m^{-1}(0) = \frac{C}{2} - \frac{\nu \bar{s} g_s}{16\pi} - \frac{1}{l^2}\sigma_{m1}(0)\,. \tag{5.56}$$

The coefficient $\sigma_{m1}(0)$ describes the contribution to the expectation value of the density proportional to the Laplacian of curvature ΔR. We introduce $b = -8\pi\sigma_{m1}(0)/l^2$ defined as a coefficient in the gradient expansion for the static density-curvature response[2]

$$\delta\rho = \frac{\nu\bar{s}}{4\pi}R + \frac{b}{8\pi}l^2\Delta R + \ldots\,. \tag{5.57}$$

For $g_s = 2$ the ground state of noninteracting electrons is degenerate even

[2]The subleading coefficient b was introduced in [80] and shown to be related to $(kl)^6$ coefficient in the static structure factor.

in the presence of inhomogenious background fields and it is expected that the limit $m \to 0$ (i.e., $\omega_c \to \infty$) is regular for $\nu \leq 1$ [22, 81]. Therefore, $\mu_m^{-1}(0)$ is finite in the limit $m \to 0$ at $g_s = 2$.

We take the limit $m \to 0$ of (5.56) at $g_s = 2$. The left hand side vanishes and we find a relation between the coefficients of the Wen-Zee and gCS terms (5.36) and the coefficient b

$$C = \frac{S}{2} - \frac{b}{4\pi}\,. \tag{5.58}$$

This relation is obtained for $g_s = 2$. However, b is a response of the density to curvature and cannot depend on g_s, neither can the coefficients C or S. Therefore, the relation (5.58) is valid for general g_s.

5.8 Chiral central charge

We split the geometric part of the effective action (5.36) as

$$W^{(geom)} = \int \frac{\nu}{4\pi}\,(A + \bar{s}\omega)\,d(A + \bar{s}\omega) - \frac{c}{48\pi}\omega d\omega\,. \tag{5.59}$$

Here we used (5.49) at zero frequency. The first contribution in (5.59) reflects the Wen-Zee construction [46] (see also [22]) stating that every electron carries not only charge, but also intrinsic orbital spin $\bar{s}$ that couples to the curvature. Thus, in any transport process the electric current will be accompanied by the "spin current". Formally, this amounts to changing the vector potential as $A_i \to A_i + \bar{s}\omega_i$. We have noted in [17] however, that even in the noninteracting case with $\nu = 1$ there is an additional contribution to gCS term represented by the second term in (5.59). Comparing (5.36) with (5.59) we identify $C = \frac{\nu}{4\pi}\bar{s}^2 - \frac{c}{48\pi}$ and rewrite (5.58) as

$$b = \nu\bar{s}(1 - \bar{s}) + c/12\,. \tag{5.60}$$

This equation relates the coefficients of geometric terms with the static bulk density-curvature response. A relation of this kind appeared recently in [80].

We refer to c as to the chiral central charge. In relativistic physics c is related to the gravitational anomaly at the boundary [70].

Let us consider the relation (5.60) for few cases where b has been computed independently. The first such case is non-interacting fermions filling the lowest Landau level $\nu = 1$. It was found in [17] that in this case $\nu = 1$, $\bar{s} = \frac{1}{2}$, and $b = 8\pi\sigma_{m1}(0)/l^2 = 1/3$. Then (5.60) gives $c = 1$ corresponding to $C = \frac{1}{24\pi}$ and is in agreement with the straightforward calculation of [17]. The coefficient b was also computed in [82] from the Bergman kernel expansion for free fermions.

For Laughlin states $\nu\bar{s} = 1/2$ and using $b = \frac{1}{3} + \frac{\nu-1}{4\nu}$ calculated in [80] we predict using (5.60) and assuming that the results of [80] are compatible with Galilean invariance

$$C = \frac{1}{8\pi} - \frac{1}{4\pi}b = \frac{1}{24\pi} + \frac{1}{2\pi}\frac{\nu^{-1}-1}{8} \tag{5.61}$$

again corresponding to $c = 1$.

In both cases the boundary theory is the chiral boson $c = 1$ and the results given by (5.60) are in agreement with our expectations for the (chiral) central charge. Therefore, we conjecture that c in (5.59) coincides with the central charge of boundary theory for all other states of FQHE hierarchy.

Note that the relation (5.60) was derived using regularity conditions at $g_s = 2$ specific for $\nu \leq 1$ and is not supposed to hold for $\nu > 1$. However, for non-interacting case with $\nu = N$ we found using the results of [17] that (5.60) can still be written as a sum over filled Landau levels

$$b = \sum_{n=1}^{N} \left(\nu_n \bar{s}_n (1 - \bar{s}_n) + \frac{c_n}{12} \right) . \tag{5.62}$$

Here $\bar{s}_n = \frac{2n-1}{2}$, $\nu_n = 1$ and $c_n = 1$ for the n-th Landau level.

The significance of the equations (5.59,5.60) is that in the non-relativistic case, the averaging over the microscopic degrees of freedom produces two gCS terms. One originates from the coupling of the orbital spin to the curvature and the other one is related to the gravitational anomaly of the boundary.

5.9 Abelian quantum Hall states

For general Abelian states we re-write the geometric action (5.59) as

$$W^{(geom)} = \frac{1}{4\pi} \int (t_i A + \bar{s}_i \omega) \, K^{-1}_{ij} d(t_j A + \bar{s}_j \omega) - \frac{c}{12} \omega d\omega \, , \tag{5.63}$$

where K-matrix, charge vector t_i and spin vector $\bar{s}_i$ characterize the state [83]. Then (5.58) takes the form (in matrix notations)

$$\frac{c}{12} = (\bar{s} - t)^t K^{-1} \bar{s} + b \tag{5.64}$$

generalizing (5.60) to more general Abelian Quantum Hall states. Here the parameter c counts the number of chiral propagating modes and is equal to $c = n_+ - n_-$, where $n_\pm$ is the number of positive/negative eigenvalues of

K-matrix, respectively.

We conclude this section with few examples of applications of (5.64) to some well-known FQHE states. For the Laughlin's state $\nu = \frac{1}{m}$, $K = (m)$, $t = 1$, $\bar{s} = m/2$, $c = 1$ and we obtain $b = \frac{1}{3} - \frac{m-1}{4}$. For the corresponding particle-hole conjugated state $\nu = 1 - 1/m$, $K = \begin{pmatrix} 1 & 1 \\ 1 & 1-m \end{pmatrix}$, $t = (1, 0)$, $\bar{s} = (\frac{1}{2}, \frac{1-m}{2})$, and $c = 0$ [83]. The relation (5.64) gives $b = \frac{m-1}{4}$.

As an example of non-Abelian state we consider the fermionic Pfaffian state [9] with $\nu = 1/2$, $t = (-1, -2)$, $\bar{s} = (-3/2, -3)$, $c = 3/2$, and $K = \begin{pmatrix} 3 & 4 \\ 4 & 8 \end{pmatrix}$ [83, 84]. We obtain $b = -1/4$.

5.9.1 Thermal Hall effect

It has been demonstrated that the thermal Hall current (the Leduc-Righi effect) is related to the chiral central charge of edge modes via the relation [14, 15, 47]

$$K_H = \frac{\partial J_H}{\partial T} = \frac{\pi k_B^2 T}{6} c \,. \tag{5.65}$$

We use (5.64) in order to express the thermal Hall conductivity through other response functions.

$$\frac{K_H}{2\pi k_B^2 T} = (\bar{s} - t)^t K^{-1} \bar{s} + b \,. \tag{5.66}$$

An important remark is in order. Eq. (5.66) allows to obtain the thermal Hall response in terms of the bulk quantities. Of course, "measuring" b involves gradients of curvature or "tidal forces" (c.f., Ref. [53]). We will have more to say about the thermal Hall effect in the next chapter.

5.10 Non-relativistic limits of gravitational actions

In this Section we will illustrate what happens if on takes the standard gravitational actions and takes the "non-relativistic limit" described above [85].

First consider Einstein-Hilbert action

$$S_{EH} = \int d^3x \sqrt{g^{(3)}} R^{(3)} \,, \tag{5.67}$$

where $d^3x\sqrt{g^{(3)}}$ is $2+1D$ invariant volume element of spacetime and $R^{(3)}$ is the $2+1D$ Ricci scalar. Substituting (5.17) into the action we get

$$\begin{aligned} S_{EH} &= mc^2 \int dt \int d^2x \sqrt{g} R \\ &- 2\int dt d^2x \sqrt{g} \left(\frac{m}{8} \dot{g}^{ij} \dot{g}_{ij} + \frac{m}{8} (g^{ij} \dot{g}_{ij})^2 + A d\omega - \frac{1}{4m} B^2 \, , \right) \end{aligned} \quad (5.68)$$

notice that the first term is the Euler character of a time slice. If a sample does not abruptly change a topology it is simply a constant and can be removed from the action.

Consider now the gravitational Chern-Simons action

$$S_{gCS} = \int d^3x \mathrm{Tr} \left(\Gamma d\Gamma + \frac{2}{3} \Gamma^3 \right) \, , \quad (5.69)$$

where Γ is the Christoffel connection (see Appendix). The same limiting prcedure will give

$$S_{gCS} = \int dt d^2x \left(\omega d\omega - \frac{1}{2m} \sqrt{g} R B \right) \, . \quad (5.70)$$

Both actions S_{EH} and S_{gCS} are by construction Galilean invariant. Unfortunately, it is not the case that all Galilean invariant actions come as non-relativistic limit if some gravitational relativistic action. In fact it was shown [86] that some of the non-relativistic actions are not obtained by a large c limit of any relativistic action!

5.11 Outlook

Galilean invariance helped us to get additional insight about gradient corrections to the linear response function. We have derived a very important relation between the gradient correction to Hall conductivity and Hall viscosity, that was first derived by Son and Hoyos

$$\sigma_H^{(2)} = \eta_H l^4 - \frac{\mu^{-1}}{\omega_c}. \quad (5.71)$$

This relation, in principle, provides an alternative method of measuring the Hall viscosity: one has to measure the long wave correction to the Hall conductivity and magnetic susceptibility, then one can extract η_H from (5.71). What's important is that mechanical experiment is thus reduced to an electromagnetic one. We will come back to the issue of measuring the Hall viscosity in the Discussion Chapter. We have also derived a relation that expresses a chiral central charge (perhaps shifted by the orbital spin variance) in terms of a response of density to the gradients curvature

$$b = \nu\bar{s}(1-\bar{s}) - \left(\nu \cdot \mathrm{v}ars - \frac{c}{12}\right) \, , \tag{5.72}$$

If vars vanishes then one can extract the chiral central charge and thermal Hall conductance from (5.72).

Another interesting issue is of more abstract nature. In this Chapter we attempted to derive the most general Galilean invariant action. To some extent we did not succeed completely, because we still do not know how to generate an induced action that would be invariant including the non-linear terms. The reason is that the transformations (5.3) are utterly complicated and non-covariant. There was some progress in understanding the covariant formulation of the Galilean symmetry [30], but even in the covariant formulation the symmetry is very unnatural (from the geometric point of view) and the construction of an invariant unduced action seems prohibitively complicated. We will indulge in the discussion of the formal matters in the Appendix.

Chapter 6

Induced Action in Thermal Equilibrium

In the previous Chapters we have mentioned a few times the relation between chiral central charge and thermal Hall conductivity. In this Chapter we will investigate the finite temperature physics of quantum Hall states using the induced action. We will discover that the whole literature on the subject is extremely confused and often incorrect. For example, it was believed for a while that there exists a thermal Hall transport in the bulk and on the level of the induced action it is described by the gravitational Chern-Simons term.

We will show here (with the help of [87] and [88] as well as our own work) that thermal Hall conductance vanishes in a bulk of a quantum Hall system and is solely an edge effect. We will explain the subtle relation between thermal Hall effect and gravitational anomaly. Finally, we will also show what one can learn by studying local thermodynamics of a quantum Hall system in local thermal equilibrium.

6.1 Short review of Cooper-Halperin-Ruzin transport theory

Following the pioneering work of Luttinger of 1964 [21], Cooper, Halperin and Ruzin (CHR) developed a theory of linear thermoelectric response. In this section we will briefly summarize their results [88].

We consider a physical system that is subject to external electric potential A_0 and "mechanical" gravitational field ψ that couples to energy density (much like A_0 couples to charge density). The geometric meaning of ψ will be explained below. We define a set of response functions L^n_{ij} independent of momentum or frequency so that

$$J^i = -L^{(1)}_{ij}\partial_j A_0 - L^{(2)}_{ij}\partial_j\psi\,, \tag{6.1}$$

$$J^i_E = -L^{(3)}_{ij}\partial_j A_0 - L^{(4)}_{ij}\partial_j\psi\,, \tag{6.2}$$

where J^i is the local electric current and J^i_E is local energy current. In a system placed in external strong magnetic field there are also magnetization currents

$$J^i_{mag} = \epsilon^{ij}\partial_j M \tag{6.3}$$

$$J^i_{E,mag} = \epsilon^{ij}\partial_j M_E\,, \tag{6.4}$$

Total local currents can be separated into two contributions: transport currents and magnetization currents. At this point it is sufficient to say that transport currents are defined as total currents minus magnetizations currents (although there are subtleties)

$$J^i_{tr} = -L^{(1)}_{ij}\partial_j A_0 - \left(L^{(2)}_{ij} + M\right)\partial_j\psi \tag{6.5}$$

$$J^i_{E,tr} = -\left(L^{(3)}_{ij} + M\epsilon^{ij}\right)\partial_j A_0 - \left(L^{(4)}_{ij} + 2M_E\epsilon^{ij}\right)\partial_j\psi \tag{6.6}$$

The transport currents are the currents measured by an ampere-meter in a transport experiment.

In order to relate the linear response to fields that we can apply in an experiment we use the Einstein relations that state that close to equilibrium we can replace $\partial\psi \leftrightarrow \frac{\partial_i T}{T}$ and $\partial\xi + T\partial\frac{\mu}{T}$, where ξ is the electro-chemical potential (the number measured by an ideal voltmeter). Then

$$J^i_{tr} = -N^1_{ij}\partial_j\xi - \frac{1}{T}N^{(2)}_{ij}\partial_j T \tag{6.7}$$

$$J^i_{Q,tr} = -N^{(3)}_{ij}\partial_j\xi - \frac{1}{T}N^{(4)}_{ij}\partial_j T \tag{6.8}$$

where we have also defined the thermal current

$$J^i_{Q,tr} = J^i_{E,tr} - \mu J^i_{tr}\,. \tag{6.9}$$

The new *transport* $N_{ij}^{(n)}$ coefficients are given by

$$N_{ij}^{(1)} = L_{ij}^{(1)} \tag{6.10}$$
$$N_{ij}^{(2)} = L_{ij}^{(2)} - \mu L_{ij}^{(1)} - M\epsilon_{ij} \tag{6.11}$$
$$N_{ij}^{(3)} = L_{ij}^{(3)} - \mu L_{ij}^{(1)} - M\epsilon_{ij} \tag{6.12}$$
$$N_{ij}^{(4)} = L_{ij}^{(4)} - \mu(N_{ij}^{(2)} + N_{ij}^{(3)}) - \mu^2 L_{ij}^{(1)} - 2M_E\epsilon_{ij} \tag{6.13}$$

The coefficients $N_{ij}^{(1)}$ and $N_{ij}^{(4)}$ are related to Hall conductance and thermal Hall conductance. CHR proceed with derivation of the Kubo formula for the transport coefficients [88]. We will not need those relations as we will attempt to understand the thermoelectric transport on the language of Newton-Cartan geometry and the induced action. To our surprise we will find that Luttinger's "fictitious gravitational field" ψ is the zero component of the Newton-Cartan "clock form" n_μ.

6.1.1 Kane-Fisher computation of thermal Hall conductivity

Before diving into the induced action business we briefly pause and recall the derivation of the thermal Hall conductivity formula by Kane and Fisher [14]. Their main result was a relation between the number of chiral edge modes (or chiral central charge) and thermal Hall conductance K_H. Let's see how to derive it.

We start with the definition of thermal current. Each edge mode is characterized by its velocity v_i (the velocity does not equal to speed of light and depends on microscopic details), its direction η_i, energy $E_i(q) = \hbar v_i q$, where q is the momentum. The modes are distributed according to Bose distribution $b(\frac{E}{k_B T})$ (the modes are chiral bosons).

$$J_Q = \sum_i \eta_i v_i n_{Q,i}\,, \tag{6.14}$$

where $n_{Q,i}$ is the energy density of the i-th channel. It is given by

$$n_{Q,i} = \int \frac{dq}{2\pi} E_i(q) b\left(\frac{E_i(q)}{k_B T}\right) = \frac{1}{v_i}\frac{\pi^2}{6}\frac{k_B^2}{h}T^2 \tag{6.15}$$

Combining these relations we have

$$J_Q = \left(\sum_i \eta_i\right) \frac{\pi^2}{6} \frac{k_B^2}{h} T^2 \,, \tag{6.16}$$

it is extremely important that the velocities cancel in this derivation. From here we get the thermal Hall conductance

$$K_H = \frac{\partial J_Q}{\partial T} = c \frac{\pi^2}{3} \frac{k_B^2}{h} T \,, \tag{6.17}$$

where c is the chiral central charge and is given by $c = \sum_i \eta_i$.

Several comments are in order.

First, this derivation did not appeal to any notion of conformal field theory or central charge, but when the edge theory is indeed conformal Read and Green argued that K_H is indeed given by (6.17) with c being the chiral central charge of the edge theory [15].

Second, since K_H is proportional to c severe deviation from Wiedeman-Franz law are possible: the ratio

$$\frac{K_H}{\sigma_H} = \frac{c}{\nu} \frac{\pi^2}{3} \frac{k_B^2}{e^2} \tag{6.18}$$

can be either positive, negative or zero. Indeed, for the $\nu = \frac{2}{3}$ Jain state the ratio is zero and for $\nu = \frac{3}{5}$ Jain state the ratio is negative.

6.1.2 Gravitational Chern-Simons and thermal Hall conductivity

In the view of the previous discussion relating the thermal Hall conductance to chiral central charge there appeared a belief that the gravitational Chern-Simons term

$$W[g] = \frac{c}{96\pi} \int \mathrm{Tr} \left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right) \tag{6.19}$$

is somehow related to energy or thermal transport.

The discussions of this matter were extremely vague [15, 54]: it was not even clear how to define an energy current as a variational derivative of the induced action. Additionally, the fact that condensed matter systems are not relativistically invariant and do not couple to the usual metric compatible affine connection Γ or space-time metric $g_{\mu\nu}$.

Even if one could assume Lorentz invariance then the relation

$$K_H = \frac{\partial J_Q}{\partial T} = c\frac{\pi^2}{3}\frac{k_B^2}{h}T\,, \tag{6.20}$$

is simply arithmetically impossible to derive from the Eq. (6.19), even using the Tolman-Ehrenfest effect that states that in thermal equilibrium gravitational potential acts as temperature gradient

$$\frac{\partial T}{T} \sim \partial g_{00}\,. \tag{6.21}$$

The reason very simple: gravitational Chern-Simons term simply contains too many derivatives for this relation to follow. The stress energy tensor that is obtained from (6.19) is known as Cotton tensor and is given by

$$T^{\mu\nu} = -\frac{c}{48\pi}\frac{1}{2\sqrt{g}}(\epsilon^{\rho\sigma\mu}D_\rho R^\nu_\sigma + \epsilon^{\rho\sigma\nu}D_\rho R^\mu_\sigma) \tag{6.22}$$

The energy current in of a Lorentz invariant system is given by

$$J^\mu_E = T^{\mu 0} = T^{0\mu} \tag{6.23}$$

Form (6.22) it is immediately clear that the energy current is proportional to the *third* derivative of metric. So the gravitational Chern-Simons cannot possibly be related to the thermal Hall effect the same way as electromagnetic Chern-Simons related to the electromagnetic quantum Hall effect.

6.2 Coupling matter to Newton-Cartan geometry

Conservation laws of energy and momentum (2.47) follow from the space and time translation symmetries. Gauging these symmetries allowed us to introduce external fields that naturally couple to momentum, energy and energy current. We have already discussed the Newton-Cartan geometry in Chapter 2, so in this Section we briefly explain how to couple free fermions to the NC geometry.

Before going to general formulations we consider an example of free fermions. The action is given by

$$S = \int dt d^2x \left(i\Psi^\dagger \partial_0 \Psi - \frac{1}{2m}(\partial_A\Psi)^\dagger(\partial_A\Psi)\right)\,. \tag{6.24}$$

In order to make this action coordinate independent, *i.e.* gauge the time and space translations we introduce frame fields (or vielbeins) E_a^μ and replace the derivatives in (6.24) as follows

$$\partial_A \to E_A^\mu \partial_\mu , \quad \partial_0 \to E_0^\mu \partial_\mu \,. \tag{6.25}$$

The second replacement can be understood as a material derivative so that the vielbein E_0^μ is the velocity field. Then the action (6.24) takes the form

$$\begin{aligned} S &= \int dt d^2 x e \mathcal{L} \,, \\ \mathcal{L} &= \left(\frac{i}{2} v^\mu (\Psi^\dagger \partial_\mu \Psi - \partial_\mu \Psi^\dagger \Psi) - \frac{h^{\mu\nu}}{2m} \partial_\mu \Psi^\dagger \partial_\nu \Psi \right) . \end{aligned} \tag{6.26}$$

Our conventions $a, b, \ldots = 0, 1, 2$ and $\mu, \nu, \ldots = 0, 1, 2$, also $A, B, \ldots = 1, 2$ and $i, j, \ldots = 1, 2$. General coordinate transformations act on the greek indices and local frame transformations act on the latin $a, b, \ldots$ indices.

We have defined a *degenerate* metric $h^{\mu\nu} = \delta^{AB} E_A^\mu E_B^\nu$, 1-form $n_\mu = e_\mu^0$ and a vector $v^\mu = E_0^\mu$. Notice, that the spatial part of the metric h^{ij} is a (inverse) metric on a fixed time slice, it is symmetric and invertible. We have denoted its determinant $\det(h^{ij}) = h^{-1}$. The introduced objects are not independent, but obey the relations

$$v^\mu n_\mu = 1, \quad h^{\mu\nu} n_\nu = 0. \tag{6.27}$$

These are precisely the conditions satisfied by the NC geometry data [22, 26][1]. Some detailed discussion of the first order (*i.e.* using the vielbeins) formulation of the NC geometry can be found in [27, 28].

The action (6.26) can be viewed as an action (6.24) written in an arbitrary coordinate system. The invariant volume element is $dV = edtd^2x$ with $e = \det(e_\mu^a e_\nu^a)$. Due to the spatial isotropy of (6.24) the vielbeins naturally combine into the degenerate metric $h^{\mu\nu}$. Similarly, the temporal components of vielbeins (denoted v^μ and n_μ) stand aside in (6.26) explicitly breaking the (local) Lorentz symmetry down to $SO(2)$. If the physical system was anisotropic the replacement (6.25) would still make sense, but one would have to treat each vielbein as an independent object, *i.e.* not constrained by any local symmetries of the tangent space.

To couple a generic matter action to the NC geometry one has to proceed in the same way as for the example considered above. Namely, one should modify the derivatives according to (6.25). Then the objects v^μ, n_μ and $h^{\mu\nu}$ (NC data)

[1] It is often convenient to define the "inverse metric" $h_{\mu\nu} = e_\mu^A e_\nu^A$. It satisfies $h^{\mu\nu} h_{\nu\rho} = \delta_\rho^\mu - v^\mu n_\rho$ and $h_{\mu\nu} v^\mu = 0$ and is fully determined by v^μ, n_ν and h^{ij}.

will naturally arise (we assume spatial isotropy from now on). When the 1-form n_μ is not closed we define the Newton-Cartan *temporal torsion* 2-form as

$$\mathcal{T}_{\mu\nu} = \partial_\mu n_\nu - \partial_\nu n_\mu . \tag{6.28}$$

In practice, it is convenient to use a particular parametrization of the NC background fields. Let us specify the spatial part h^{ij} of the degenerate metric and assume that $n_\mu = (n_0, n_i)$ and $v^\mu = (v^0, v^i)$ are also specified and satisfy the first relation in (6.27). Then we find from other relations in (6.27) $h^{\mu\nu} = \begin{pmatrix} \frac{n^2}{n_0^2} & -\frac{n^i}{n_0} \\ -\frac{n^i}{n_0} & h^{ij} \end{pmatrix}$, where we defined $n^i = h^{ij} n_j$, $n^2 = n_i n_j h^{ij}$. In this parametrization the invariant volume element is given by $dV = \sqrt{h} n_0 dt d^2x$.

The momentum, stress, energy and energy current are identified as responses to the NC geometry as follows

$$P_i = \frac{v^0}{\sqrt{h} n_0} \frac{\delta S}{\delta v^i}, \quad T_{ij} = -\frac{2}{\sqrt{h} n_0} \frac{\delta S}{\delta h^{ij}} , \tag{6.29}$$

$$\varepsilon = -\frac{1}{\sqrt{h} n_0} \left(n_0 \frac{\delta S}{\delta n_0} - v^0 \frac{\delta S}{\delta v^0} \right) , \tag{6.30}$$

$$J^i_E = -\frac{1}{\sqrt{h} n_0} \left(n_0 \frac{\delta S}{\delta n_i} - v^i \frac{\delta S}{\delta v^0} \right) , \tag{6.31}$$

where we turn off the fields n_i after the variation is taken. The introduced NC geometry is general and reduces to some cases considered in literature. For example, the choice $n_\mu = (1, 0, 0)$, $v = (1, v^i)$ corresponds to the *torsionless* NC background which turned out to be convenient in studying Galilean invariant actions [17, 22, 50, 85, 89].

Another particular limit is given by $n_\mu = (e^\psi, 0, 0)$, $v^\nu = (e^{-\psi}, 0, 0)$. This is an example of the NC geometry with *temporal torsion*. The torsion is given by

$$\mathcal{T} = e^\psi (\partial_i \psi) dx^i \wedge dt . \tag{6.32}$$

In this case the only non-vanishing component of the torsion tensor is $\mathcal{T}_{0i}$. This NC geometry essentially appeared in the procedure introduced by Luttinger [21, 88]. The field ψ is precisely the "gravitational potential" introduced in [21]. The disadvantage of this choice of geometry is the absence of the field n_i that couples to the energy current.

In the following we consider a general case keeping all of the components of NC geometry turned on.

6.3 Induced action for thermal transport and absence of bulk thermal Hall conductivity

While our main focus will be on equilibrium physics, we will briefly discuss the thermal transport. according to the logic we have been following in this Thesis, in order to study the linear response we have to write down the induced action that satisfies some set of symmetries. This time we have additional fields in our disposal. These fields are the Newton-Cartan data.

The induced action is given by

$$W = \frac{\alpha}{4\pi}\int ndA + \frac{\beta}{4\pi}\int ndn + \frac{\nu}{4\pi}\int AdA \equiv W^{(loc)} + \frac{\nu}{4\pi}\int AdA\,, \tag{6.33}$$

where α and β are *dimensionful* phenomenological coefficients. Notice that despite the fact that first two terms look like Chern-Simons terms and indeed do not depend on external space metric, these terms are *invariant* with respect to all symmetries of the problem. Finally, the coefficients α and β can be made spacedependent. Already these observations scream that α and β are not universal coefficeints!

Electric and energy currents are given by

$$J^i = \frac{\nu}{2\pi}\epsilon^{ij}E_j - \frac{\alpha}{4\pi}\epsilon^{ij}\partial_j n_0 \tag{6.34}$$

$$J^i_E = \frac{\alpha}{4\pi}\epsilon^{ij}E_j - \frac{\beta}{2\pi}\epsilon^{ij}\partial_j n_0 \tag{6.35}$$

From here we can read off the L coefficients

$$L^{(1)}_{ij} = \frac{\nu}{2\pi}\epsilon_{ij} \tag{6.36}$$

$$L^{(2)}_{ij} = L^{(3)}_{ij} = \frac{\alpha}{4\pi}\epsilon^{ij} \tag{6.37}$$

$$L^{(4)}_{ij} = \frac{\beta}{4\pi}\epsilon^{ij}\,, \tag{6.38}$$

notice that $L^{(4)}$ is not related to chiral central charge or gravitational Chern-Simons in any way. We also compute the magnetizations

$$M = \frac{\delta W^{(loc)}}{\delta \bar{B}} = \frac{\alpha}{4\pi}n_0 \tag{6.39}$$

$$M_E = \frac{\delta W^{(loc)}}{\delta \epsilon^{ij}\partial_i n_j} = \frac{\alpha}{2\pi}A_0 + \frac{\beta}{2\pi}n_0 \tag{6.40}$$

This implies that the currents are given by

$$J^i = \frac{\nu}{2\pi}\epsilon^{ij}E_j - \epsilon^{ij}\partial_j M \quad (6.41)$$

$$J^i_E = \epsilon^{ij}\partial_j M_E \quad (6.42)$$

Thus the energy current is *purely magnetization current* and the only part of the local current densities that will contribute to transport is electric current J^i.

From this reasoning we find

$$N^{(1)}_{ij} = \frac{\nu}{2\pi}\epsilon_{ij} \quad (6.43)$$

$$N^{(2)}_{ij} = N^{(3)}_{ij} = 0 \quad (6.44)$$

$$N^{(4)}_{ij} = 0\,, \quad (6.45)$$

These relations were written in [87].

When an edge is introduced, the chemical potential will indeed make a difference, but all of the thermoelectric transport will happen at the edge. Also when an edge is introduced, an edge computation shows that [87]

$$N^{(4)}_{ij} = c\frac{\pi}{6}\frac{k_B T^2}{h}\,. \quad (6.46)$$

We have to notice here that the definition of the transport currents is very subtle. The transport current densities are defined as integrated along a crosssection of a sample of local current densities divided by the volume of the system. This integration usually kills all of the gradient corrections.

6.4 Thermal equilibrium

We construct a partition function, consistent with time independent, local space and time translations and gauge symmetries. The partition function can be written as a Euclidian functional integral

$$W = -\ln \mathrm{Tr}\, \exp\left\{-\frac{H-\bar{\mu}N}{\bar{T}}\right\} = -\ln\int D\Psi D\Psi^\dagger e^{-S_E}\,, \quad (6.47)$$

where we introduced a Euclidean action

$$S_E[\Psi,\Psi^\dagger; A_\mu, n_\mu, v^\mu, h^{ij}] = \int d^2x\sqrt{h}\oint_0^{1/\bar{T}} d\tau n_0 \mathcal{L}_E\,. \quad (6.48)$$

This action is coupled to the NC geometry as explained in the previous section. The time-independent field n_0 can be viewed as an inhomogeneous temperature $T(x)$ defined according to

$$\oint_0^{1/\bar{T}} d\tau n_0 \to \oint_0^{1/T(x)} d\tau' , \quad \frac{1}{T(x)} = \frac{n_0}{\bar{T}} . \tag{6.49}$$

The NC geometry allows to introduce spatial variations in the size of the compact imaginary time direction.

It is easy to see via usual scaling arguments [90] that the Euclidean action has the following functional form

$$S_E = S_E \left[\Psi, \Psi^\dagger ; \frac{A_0}{\bar{T}}, \frac{n_0}{\bar{T}}, v^0 \bar{T}, A_i, \frac{n_i}{n_0} \bar{T}, v^i, h^{ij} \right] . \tag{6.50}$$

In (local) equilibrium external fields do not depend on Euclidean time. The generating functional W depends on the temperature T and external sources. We also assume that W can be written as an integral of a local density so that

$$W = \int d^2 x \sqrt{h} \frac{n_0}{\bar{T}} \mathcal{P} \left(\frac{A_0}{\bar{T}}, \frac{n_0}{\bar{T}}, v^0 \bar{T}, A_i, \frac{n_i}{n_0} \bar{T}, v^i, h^{ij} \right) , \tag{6.51}$$

where we have already replaced the integral over Euclidean time by the overall factor $1/\bar{T}$. It is worth noting that results derived from the Euclidean generating functional can be used to obtain the zero frequency correlation functions in real time upon a Wick rotation.

6.4.1 Local time shifts

We are mainly interested in the thermal transport, so from now on we set the external field $v^i = 0$ and parametrize $v^0 = \frac{1}{n_0} \equiv e^{-\psi}$ in order to satisfy (6.27). This field configuration is preserved by the symmetries.

The transformation law of the external field n_i under a local time shift $t \to t + \zeta(x)$ takes form

$$\delta(e^{-\psi} n_i) = -\partial_i \zeta , \tag{6.52}$$

i.e. the field $e^{-\psi} n_i$ transforms like a $U(1)$ gauge field under a local time shift. This field can be regarded as a connection on an S^1 bundle over the base manifold, where S^1 is the thermal circle. The field strength is related to the NC temporal torsion.

It is convenient to introduce $\mathcal{A}_i = A_i - A_0 e^{-\psi} n_i$. This field transform like a gauge field under electro-magnetic gauge transformations and it is *invariant* under local time shifts.

The symmetry (6.52) implies a local conservation law of the *thermal current*

$$J^i_Q = -\frac{\bar{T}}{\sqrt{h}}\left(\frac{\delta W}{\delta e^{-\psi}n_i} + A_0\frac{\delta W}{\delta A_i}\right) = J^i_E - A_0 J^i\,. \qquad (6.53)$$

This current is conserved

$$\nabla_i J^i_Q = 0\,, \qquad (6.54)$$

where $\nabla_i X^i = \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\,X^i\right)$ is the covariant divergence.

6.5 Equilibrium generating functional

We present the partition function as an expansion in derivatives of the external fields. We consider the following generating functional

$$W = \int d^2x\sqrt{h}\frac{1}{T}\mathcal{P}\left(\mu, T, \mathcal{B}, B_E\right)\,, \qquad (6.55)$$

where we made the identifications

$$\frac{1}{T(x)} = \frac{e^{\psi}}{\bar{T}}, \quad \mu(x) = e^{-\psi}A_0(x)\,, \qquad (6.56)$$

and defined gauge invariant (pseudo) scalars

$$\mathcal{B} = \epsilon^{ij}\partial_i\mathcal{A}_j, \quad B_E = \epsilon^{ij}\partial_i(e^{-\psi}n_j)\,. \qquad (6.57)$$

Writing (6.55) we assumed that both $\mathcal{B}$ and B might be large, while their derivatives are small and can be neglected. We also assumed that gradients of both μ and T are small.

The generating functional (6.55) encodes various local thermodynamic quantities and relations. For example, the energy (in flat space) can be found with the help of (6.30), appropriately modified for the presence of the gauge field

$$\begin{aligned}\varepsilon &= \bar{T}\frac{\delta W}{\delta e^{\psi}} + TA_0\frac{\delta W}{\delta A_0} = \frac{\partial(\mathcal{P}/T)}{\partial(1/T)} - \mu\frac{\partial\mathcal{P}}{\partial\mu}\\ &= \mathcal{P} + sT + n\mu\,,\end{aligned} \qquad (6.58)$$

where we made the identifications

$$n(x) = \bar{T}\frac{\delta W}{\delta A_0} = -\frac{\partial\mathcal{P}}{\partial\mu} \qquad (6.59)$$

and

$$s(x) = -\frac{\partial \mathcal{P}}{\partial T}. \tag{6.60}$$

The relation (6.58) suggests that $\mathcal{P}(\mu, T, \mathcal{B}, B_E)$ is the density of the grand thermodynamic potential (in the presence of external fields) and that (6.58) is the local version of the known thermodynamic relation $\mathcal{P} = E - \bar{T}S - \bar{\mu}N$.

It is instructive to find the pressure in the presence of external fields, also known as *internal* pressure

$$P_{int} = \bar{T}\frac{\delta W}{\delta h^i{}_i} = P_{(0)} - M\mathcal{B} - M_E B_E\,, \tag{6.61}$$

where we have introduced the magnetization $M = e^{\psi}\frac{\partial \mathcal{P}}{\partial \mathcal{B}}$, the energy magnetization $M_E = e^{\psi}\frac{\partial \mathcal{P}}{\partial B_E}$ and $P_{(0)}$ is the pressure at zero magnetic field.

The additional contribution to the pressure given by the second term in (6.61) comes from the Lorentz force acting on magnetization currents. The last term of (6.61) gives a similar contribution present in non-vanishing background field B_E.

6.6 Magnetization currents

While all transport currents vanish in thermal equilibrium, there are still *magnetization* currents flowing in a material even at equilibrium. These currents cannot be measured in transport experiments [88]. However, e.g., the electric magnetization current can be in principle observed in spectroscopy experiments or by measuring the magnetic field created by moving charges. The energy current can (at least in principle) be observed by the frame drag [54] due to distortions in the gravitational field created by the flow of energy. In the presence of the inhomogeneous external fields magnetization currents can flow in the bulk of the material, otherwise they are concentrated on the boundary of the sample.

Knowing magnetization currents is important as this knowledge can be used to separate transport currents from the magnetization ones for systems driven out of equilibrium [88]. Also, for a particular case of the chemical potential lying in the excitation gap the magnetization currents are the only currents responsible for the Hall effect [91].

In the following we consider both electric and thermal magnetization currents. They are given, respectively, by

$$
\begin{aligned}
J^i &= \bar{T}\frac{\delta W}{\delta A_i} = \epsilon^{ij}\partial_j M\,, && (6.62)\\
J^i_Q &= \epsilon^{ij}\partial_j M_E\,. && (6.63)
\end{aligned}
$$

The currents (6.62) and (6.63) are conserved in the presence of arbitrary temperature profile $T(x)$ set by (6.56) and coincide with the ones found in [88, 92, 93] at the level of linear response.

We note here that usually the energy magnetization M_E is *defined* by the Eq. (6.63) while the NC "magnetic field" B_E (usually denoted as B_g and referred to as gravimagnetic field) is defined as a quantity thermodynamically conjugated to M_E. In this work we clarified how one can systematically introduce external fields n_i in *non-relativistic* system and couple the system to B_E (6.57). Previous approaches explicitly used the presence of Lorentz symmetry [54, 93] and cannot be applied in majority of condensed matter systems.

6.6.1 Streda formulas

It is possible to express the Hall conductivity and other parity odd responses purely in terms of derivatives of thermodynamic quantities. We define electric and thermal conductivities as

$$
\begin{aligned}
J^i &= \epsilon^{ij}\left(\sigma_H\partial_i\mu + \sigma_H^T\partial_i T\right)\,, && (6.64)\\
J^i_E &= \epsilon^{ij}\left(\kappa_H^\mu\partial_i\mu + \kappa_H\partial_i T\right)\,. && (6.65)
\end{aligned}
$$

Comparing with (6.62-6.63) we obtain using the Maxwell's relations [2]

$$
\begin{aligned}
\sigma_H &= \left(\frac{\partial M}{\partial\mu}\right)_{T,\mathcal{B},B_E} = \left(\frac{\partial n}{\partial\mathcal{B}}\right)_{T,\mu,B_E}\,, && (6.66)\\
\sigma_H^T &= \left(\frac{\partial M}{\partial T}\right)_{\mu,\mathcal{B},B_E} = \left(\frac{\partial s}{\partial\mathcal{B}}\right)_{T,\mu,B_E}\,, && (6.67)\\
\kappa_H^\mu &= \left(\frac{\partial M_E}{\partial\mu}\right)_{T,\mathcal{B},B_E} = \left(\frac{\partial n}{\partial B_E}\right)_{T,\mu,\mathcal{B}}\,, && (6.68)\\
\kappa_H &= \left(\frac{\partial M_E}{\partial T}\right)_{\mu,\mathcal{B},B_E} = \left(\frac{\partial s}{\partial B_E}\right)_{T,\mu,\mathcal{B}}\,. && (6.69)
\end{aligned}
$$

These are thermodynamic Streda-type formulas [94, 95] for the response coefficients.

[2] As $d\mathcal{P} = -sdT - nd\mu - Md\mathcal{B} - M_E d\Omega$ we have $\partial M/\partial\mu = \partial n/\partial B$ etc.

6.7 Galilean vs. Lorentz symmetries

So far we assumed that the (un-perturbed) system under consideration is gauge invariant, spatially isotropic and homogeneous, and time translation invariant. In this general case there are no additional relations between electric current, momentum and energy current. Several new relations appear if additional symmetries are present. For simplicity, we assume below that the underlying microscopic system consists of charged particles of a single species or several species with the same e/m ratio.

If the system is *Galilean* invariant the electric current is proportional to the momentum $J^i = \frac{e}{m}P^i$, therefore, the magnetization density is proportional to the density of the angular momentum $M = \frac{e}{m}L_z$.

Then from (6.66) we have

$$\sigma_H = \frac{e}{m}\left(\frac{\partial L_z}{\partial \mu}\right)_{T,\mathcal{B},B_E}, \tag{6.70}$$

that is Hall conductivity can be expressed in terms of derivatives of the angular momentum.

If the system is *Lorentz* invariant then there is an additional equality between momentum and energy current as we pointed out in the introduction $J^i_E = P^i$ and, therefore, $M_E = L_z$. Therefore, we have another version of Streda formula for thermal Hall conductivity [93]

$$\kappa_H = \left(\frac{\partial L_z}{\partial T}\right)_{\mu,\mathcal{B},B_E}. \tag{6.71}$$

In general case, when no additional symmetries are present the angular momentum is *not* related to either electric or thermal magnetization and the relations (6.70)-(6.71) do not hold.

6.8 Outlook

In this Chapter we have investigated the physics of quantum Hall states at finite temperature. We understood that thermal Hall effect is impossible in the bulk and, therefore, is purely an edge effect. This should not come as a surprise, since at low temperatures the thermal effects in the bulk should be exponentially suppressed. Despite this simple logic, there was a large confusion in the literature about relationship between the gravitational Chern-Simons term and thermal effects.

We have shown that the relevant term in the induced action is related to Newton-Cartan torsion in the temporal direction and has a form

$$\frac{\alpha}{4\pi}\int n dn\,, \tag{6.72}$$

where n is the Newton-Cartan "clock form" and dn is the temporal torsion with α being a dimensionful coefficient. This term is not universal and is not related to Leduc-Righi or thermal Hall effect. In fact, this terms describes only the magnetization currents.

Comparing this Chapter with the previous ones we can't help, but to notice an unfortunate trade of we have to make in the study of transport phenomena at finite temperature. We either can compute finite frequency response functions at zero temperature or we can compute equilibrium properties at finite temperature. True frequency dependent out-of-equilibrium physics is not accessible in either formalism. It seems to be extremely useful to have an analogue of induced action for non-equilibrium phenomena. We feel that such an object must exist and be of fundamental importance in non-equilibrium physics. We will focus on this problem in the future research.

Chapter 7

Induced Action at the Edge

We have mentioned in the previous Chapters that edge contains additional information that is necessary to distinguish the chiral central charge from the orbital spin variance. We have also learned that thermal Hall transport is purely an edge effect and, therefore, it seems impossible to even define a notion of central charge in the bulk just in terms of the induced action.

In this Chapter we will look at the boundary from a different perspective. It is well known that Chern-Simons term are not invariant with respect to their local symmetries if a boundary is introduced. This effect is known as Callan-Harvey anomaly inflow [70] that we will discuss in some detail below. Wen-Zee term studied in the previous Chapters does look like a type of Chern-Simons term. Is there a corresponding anomaly? Could it be that without Lorentz symmetry the spectrum of anomalies became bigger? How do the edge modes know about Hall viscosity and do they know about it at all? We will be able to answer these question using the induced action on a manifold with a boundary.

7.1 Induced action on a closed manifold

In the following we will study the induced action that contains all possible Chern-Simons-type terms written in terms of the external gauge field A_μ,

$SO(2)$ spin connection ω_μ and the Levi-Civita affine connection $\Gamma^\sigma_{\mu\rho}$ [1]

$$\begin{aligned} W &= \frac{\nu}{4\pi}\int AdA + 2\bar{s}Ad\omega + \overline{s^2}\,\omega d\omega \\ &+ \frac{c}{96\pi}\int \mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right) + \dots . \end{aligned} \tag{7.1}$$

All four coefficients $\nu, \bar{s}, \overline{s^2}$ and c are dimensionless and are known as "filling factor", mean orbital spin per particle, mean orbital spin squared per particle, and chiral central charge respectively. The corresponding terms of (7.1) encode (in flat space) the Hall conductivity $\sigma_H = \frac{\nu}{2\pi}$ [2], Wen-Zee shift $\mathcal{S} = 2\bar{s}$ (in the presence of finite magnetic field B it is also related to the Hall viscosity $\eta_H = \frac{\nu\bar{s}}{2}\frac{B}{2\pi}$), and thermal Hall conductivity $\kappa_H = c\frac{\pi}{3}k_B T$. The coefficient $\overline{s^2}$ contributes to the Hall viscosity in the background with finite Ricci curvature R: $\eta_H = \frac{\nu\bar{s}}{2}\frac{B}{2\pi} + \left(\frac{\nu\overline{s^2}}{2} - \frac{c}{24}\right)\frac{R}{4\pi}$. This formula can be re-written in terms of the charge density $\eta_H = \frac{\bar{s}}{2}\rho + \left(\frac{\nu}{2}\beta - \frac{c}{24}\right)\frac{R}{4\pi}$ where we introduced the *orbital spin variance* $\beta = \overline{s^2} - \bar{s}^2$. All these four coefficients have been computed for the integer quantum Hall states in [96] and for various model fractional quantum Hall states in [16, 61, 72].

The terms of the induced action (7.1) are the only terms having dimensionless coefficients which are required to be space and time independent by local symmetry requirements. Therefore, these coefficients are natural candidates for being the universal properties distinguishing topological phases of matter. The ellipsis in (7.1) denote the higher gradient terms with dimension-full coefficients depending on various scales in the problem (e.g, energy gap or impurity concentration). In the following we will be only interested in "geometrical" terms written explicitly in (7.1).

Computation of the induced action for a generic gapped interacting system is not a tractable problem. However, it is often possible to significantly constraint W by symmetries of the problem. For example, for topological phases in 2 spatial dimensions with various symmetries this was done in [50, 87, 97–99]. In particular, the induced action (7.1) is the most general functional of the gauge field and metric having local $U(1)$ charge conservation, local $SO(2)$ rotational invariance and invariance with respect to *spatial* coordinate transformations [3].

[1] Here we use the conventional differential form and matrix notations so that $AdA = \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\, d^3x$, $\mathrm{Tr}\,\Gamma d\Gamma = \epsilon^{\mu\nu\lambda}\Gamma^\rho_{\mu\sigma}\partial_\nu\Gamma^\sigma_{\lambda\rho}\, d^3x$ etc.

[2] Here and in the following we use the units in which $e^2/\hbar = 1$.

[3] The geometric background in (7.1) is assumed to be torsionless. For a discussion of the role of torsion see [87, 97, 99, 100].

The action (7.1) is invariant on a closed manifold but it loses its symmetries if considered on a manifold with the boundary. To retain these symmetries the induced action (7.1) must be supplemented with appropriate boundary terms. This boundary action is well known in Lorentz-invariant case [7, 62, 78, 101]. However, (7.1) contains $SO(2)$ spin connection and is explicitly *not* Lorentz invariant. In the following we will construct the induced boundary action appropriate for this case.

7.2 Chern-Simons - Wess-Zumino-Witten correspondance

We have already seen that both topological quantum field theory and conformal field theory play central role in the description of FQH physics. It is not of course a surprise that these objects are related to each other [78]. In this Section we will work out the simplest relation between these quantum theories. Before we proceed we have to warn reader that the relation goes much deeper than our presentation or understanding.

Here we will consider an effective action on a $3D$ manifold $M = \mathbb{R}\times\Sigma$ with boundary $\partial M = \mathbb{R}\times\partial\Sigma$. For simplicity we will assume that time slices Σ are topologically equivalent to a disc D. In other words, we will be studying an effective theory of a quantum Hall droplet. Finally we only allow the gauge transformations that smoothly go to 1 near the boundary.

To simplify the equations we will consider only coupling of the effective theory to the external e/m field A. Thus our starting point is

$$\mathcal{L} = -\left[\frac{k}{4\pi}ada - \frac{q}{2\pi}adA\right] \tag{7.2}$$

Integrating by parts it can be brought to the form

$$S = -\int_M a_0\left(\frac{k}{4\pi}\epsilon^{ij}\partial_i a_j - \frac{q}{4\pi}\epsilon^{ij}\partial_i A_j\right) - \frac{k}{4\pi}\epsilon^{ij}a_i\dot{a}_j \tag{7.3}$$

$$- \frac{q}{2\pi}\epsilon^{ij}\left(a_i\dot{A}_j + a_i\partial_j A_0\right) - \frac{k}{4\pi}\int_{\partial M} a_0 a_1 \tag{7.4}$$

Integration over a_0 in the path integral leads to

$$a_i = \partial_i\phi + \frac{q}{k}A_i \tag{7.5}$$

Making the variable change from a_i to $\partial_i\phi$ and assuming there is not Jacobian we have

$$S=\frac{1}{4\pi}\frac{q^2}{k}\int_M AdA+\frac{k}{4\pi}\int_{\partial M}\dot{\phi}\phi'-a_0a_1+\frac{q}{k}A_0\phi'+2\frac{q^2}{k}A_0A_1 \tag{7.6}$$

The boson ϕ is charged under the electromagnetic $U(1)$ symmetry, so it is more convenient (and esthetically pleasing) to re-write the action in a manifestly gauge invariant way. We have

$$S=\frac{\nu}{4\pi}\int_M AdA+\frac{\nu}{4\pi}\int_{\partial M}D_0\phi D_1\phi+E\phi-a_0a_1\,, \tag{7.7}$$

where $D_\mu\phi=\partial_\mu\phi+A_\mu$, where we have also rescaled ϕ by $\frac{q^2}{k}$ and denoted $\frac{q^2}{k}=\nu$.

Under a gauge transformation ϕ transforms as a phase (after rescaling)

$$\delta\phi=-\alpha \tag{7.8}$$

This is as far as we can go without any additional input about the edge physics. This additional input will enter through the boundary conditions for the gauge field a.

7.2.1 Elitzur, et. al. boundary conditions

The simplest (and traditional) choice of the boundary conditions ensures the vanishing of the last term in (7.7). We set $a_0=0$ and get

$$S=\frac{\nu}{4\pi}\int_M AdA+\frac{\nu}{4\pi}\int_{\partial M}D_0\phi D_1\phi+E\phi \tag{7.9}$$

This is the chiral boson without dynamics coupled to external gauge field.

We stress that we have made a choice of boundary conditions and not a gauge choice. As we have mentioned in the beginning of the Section: the group of gauge transformations consists only of those transformations that smoothly go to 1 near the boundary.

7.2.2 Holomorphic boundary conditions

Another way to fix boundary conditions is to preserve the invariance under anti-holomorphic gauge transformations on the boundary (*i.e.* $\bar{\partial}\alpha=0$).

$$a_z=0 \quad or \quad a_0=va_1\,, \tag{7.10}$$

for some constant v. This boundary condition leads to action

$$\begin{aligned} S &= \frac{\nu}{4\pi}\int_M AdA + \frac{\nu}{4\pi}\int_{\partial M} D_z\phi D_1\phi + E\phi & (7.11)\\ &= \frac{\nu}{4\pi}\int_M AdA + \frac{\nu}{4\pi}\int_{\partial M} (D_0 - vD_1)\phi D_1\phi + E\phi & (7.12)\end{aligned}$$

In order to get some feeling for this theory we compute the density and current of the gauged chiral boson. We have

$$\begin{aligned} J^0 &= \frac{\delta S}{\delta A_0} = \frac{\nu}{4\pi}\left(2\phi' + A_1\right) & (7.13)\\ J^1 &= \frac{\delta S}{\delta A_1} = \frac{\nu}{4\pi}\left(-vD_1\phi + (D_0 - vD_1)\phi - \dot{\phi}\right) = -v\frac{\nu}{2\pi}D_1\phi & (7.14)\end{aligned}$$

Notice that the total number of the bosons is given by the winding of the compact field ϕ.

7.2.3 Covariant boundary conditions

Finally one can choose the boundary conditions that depend on the induced geometry of the boundary. Given a boudnary vielbein E^μ_z. We have

$$E^\mu_z a_\mu = 0 \qquad or \qquad a_0 = \frac{E^1_z}{E^0_z}a_1 \equiv K a_1 \tag{7.15}$$

This condition corresponds to choosing velocity $v = K$ in (7.11). This is exactly the coupling obtained by [102]. This coupling indeed leads to the gravitational anomaly discussed below [103].

7.2.4 Non-abelian CS-WZW correspondance

In the non-abelian case the Chern-Simons action is given by

$$S = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) \tag{7.16}$$

The solution of equations of motion (in the temporal gauge $A_0 = 0$) is

$$A = U^{-1}dU\,, \tag{7.17}$$

where U is an element of the group of gauge transformations. We choose only the transformations that equal to 1 at the edge. Plugging this relation back in we obtain.

$$S = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}\,\left(U^{-1}\partial_x U U^{-1}\partial_0 U\right) + \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{Tr}\,\left(U^{-1}dU\right)^3 . \tag{7.18}$$

This action is the chiral Wess-Zumino-Witten theory. It was obtained from the Chern-Simons action in [101].

7.3 Callan-Harvey anomaly inflow

We will demonstrate the Callan-Harvey mechanism in $2+1D$ for a system of Dirac fermions coupled to an axion string (or a boundary) that extends it $1+1D$. Dirac fermions are free of anomalies in odd dimensions, but the degrees of freedom that “live” on the string are chiral (in a sense that they propagate along the string in one direction only) and therefore experience gauge and gravitational anomalies. For the illustrative purpose we will focus on gauge anomaly. The gauge anomaly is characterized by the failure of the electric current to be conserved

$$\partial_i J^i = \frac{1}{4\pi} F_{ij}\epsilon^{ij}\,, \qquad i = 0, 1\,. \tag{7.19}$$

At the same time in the $2+1$ theory the current must be conserved since the theory is non-anomalous [104]. Therefore there must be something outside of the string that cancels the anomaly. In fact, what happens is that charge from the surrounding space flows onto the string (or boundary). The way it happens formally is relatively simple. The induced action of the fermions in $2+1D$ is given in the leading order by a parity violating radiative correction

$$W = -\frac{1}{4\pi}\int A\wedge F = -\frac{1}{4\pi}\int A\wedge dA\,, \tag{7.20}$$

The gauge variation of the induced action W is

$$\delta W = \frac{1}{4\pi}\int d(F\alpha) = -\frac{1}{4\pi}\int d^2x\epsilon^{ij}F_{ij}\alpha\,, \tag{7.21}$$

where α is the parameter of the gauge variation. This gauge variation precisely matches the anomaly of the chiral degrees of freedom. Similar arguments can be made for the gravitational Chern-Simons theory.

7.3.1 Gravitational Chern-Simons and gravitational anomaly

We will show the anomaly inflow for the gravitational Chern-Simons term. Its diffeomorphism variation can easily be done after some preparation. We introduce the matrix valued one-form $\Gamma^{\mu}{}_{\nu}$. Then the curvature two-form is nicely written as

$$R = d\Gamma + \Gamma \wedge \Gamma \,, \tag{7.22}$$

where matrix product is understood. The gravitational Chern-Simons term is written as

$$\int_{\mathcal{M}} I_{gCS} = \int_{\mathcal{M}} \mathrm{Tr}\left(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma\right) = \int_{\mathcal{M}} \mathrm{Tr}\left(\Gamma \wedge R - \frac{1}{3}\Gamma \wedge \Gamma \wedge \Gamma\right) \tag{7.23}$$

The variation of the Christofell connection under a diffeomorphism ξ^{μ} is given by

$$\delta\Gamma^{\mu}{}_{\nu,\rho} = \mathcal{L}_{\xi}\Gamma^{\mu}{}_{\nu,\rho} + \partial_{\nu}\partial_{\rho}\xi^{\mu} \equiv \mathcal{L}_{\xi}\Gamma^{\mu}{}_{\nu,\rho} + \tilde{\delta}\Gamma^{\mu}{}_{\nu,\rho} \,, \tag{7.24}$$

where $\tilde{\delta}$ stands for the non-tensorial part of the variation. Obviously, I_4 is invariant under the tensorial Lie derivative part of the variation, so we will only care about non-tensorial part. On the form language we can re-write (7.24) as

$$\delta\Gamma = \mathcal{L}_{\xi}\Gamma + d(\partial\xi) \,, \tag{7.25}$$

where $\partial\xi$ is a matrix valued 0-form $\partial_{\mu}\xi^{\nu}$ and $d(\partial\xi)$ is a matrix valued 1-form $\partial_{\rho}(\partial_{\mu}\xi^{\nu})dx^{\rho}$. With this notation the variation of I_{gCS} is computed as follows

$$\begin{aligned}
\int_{\mathcal{M}} \delta I_{gCS} &= \int_{\mathcal{M}} \mathrm{Tr}\left(\tilde{\delta}\Gamma \wedge R - \tilde{\delta}\Gamma \wedge \Gamma \wedge \Gamma\right) \\
&= \int_{\mathcal{M}} \mathrm{Tr}\left(d(\partial\xi) \wedge d\Gamma\right) \\
&= \int_{\mathcal{M}} d\mathrm{Tr}\left[\partial\xi d\Gamma\right] = \int_{\partial\mathcal{M}} \mathrm{Tr}\left[\partial\xi d\Gamma\right] \,,
\end{aligned} \tag{7.26}$$

where we used $\tilde{\delta}R = 0$ and the cyclic property of the trace in the first step, definition of $\tilde{\delta}$ and (7.22) in the second step and Poincare lemma $d^2 = 0$ in the third, finally we used the Stokes theorem in the last step. In components we have (after integration by parts)

$$\delta \int_{\mathcal{M}} I_{gCS} = \int_{\partial\mathcal{M}} \xi^{\nu}\epsilon^{\lambda\rho}\partial_{\mu}\partial_{\lambda}\Gamma^{\mu}{}_{\nu,\rho} \tag{7.27}$$

This matches the consistent gravitational anomaly of the chiral edge theory as [104]

$$D_i T^i{}_j = \frac{1}{48\pi}\partial_k\partial_l \Gamma^k{}_{mj}\epsilon^{ml} \,. \tag{7.28}$$

7.4 Induced action at the boundary: gauge and gravitational anomalies

To construct an induced boundary action corresponding to (7.1) let us first consider the case when $\bar{s} = \overline{s^2} = 0$. In this case the construction is well known in relativistic physics (both CS and gCS terms are Lorentz-invariant). Namely, one needs to supplement the action (7.1) by boundary counterterms restoring the gauge and coordinate invariance. The corresponding boundary induced action can be formally written as

$$W^1_\partial = -\frac{1}{4\pi}\int dxdt \left[\nu E\frac{1}{\partial_-}A_- - \frac{c}{12}R\frac{1}{\partial_-}\Gamma_-\right] , \tag{7.29}$$

where $\Gamma_\alpha = \Gamma_{\gamma\beta,\alpha}\varepsilon^{\gamma\beta}$ is the abelianized affine connection defined on the boundary. Here we also defined "minus" components as $A_- = E^\mu_- A_\mu$ in terms of vielbeins $E^\mu_- = E^\mu_0 - E^\mu_1$ etc. [4] The notation $1/\partial_-$ is a notation for the Green's function of the operator ∂_-. We have to notice here that the non-local induced action is somewhat misleading as it misses the non-perturbative contributions. We will elaborate on this point later on in the Letter. The counter terms (7.29) are necessarily non-local reflecting gauge and gravitational anomalies of the boundary theory. It must be emphasized that (7.29) does not uniquely fix the boundary theory (additional local gauge and coordinate invariant boundary terms can be added) and only fixes anomalies. Although the boundary action (7.29) is nonlocal it can be obtained as an induced action of a local boundary theory (chiral CFT) as explained in the begining of the Chapter.

7.5 Extrinsic geometry of the boundary

In the next Section we will show that in contrast to CS and gCS terms, the gauge and general coordinate non-invariances of Wen-Zee terms of (7.1) *can* be fixed by *local* boundary counter terms. In this Section we briefly review the geometric data provided by the boundary necessary to construct the counter terms.

[4] In flat space we have $E^\mu = (1, v)$ and this becomes simply $A_- = A_0 - vA_1$.

A boundary can be described in a covariant way as a two-dimensional manifold embedded into the three-dimensional space time. Embedded manifolds are characterized by two types of geometry: intrinsic one that is described by the metric and extrinsic one described by *embedding functions* $x^\mu = x^\mu(\sigma, \tau)$ and the second fundamental form. A very nice exposition of these issues can be found for example in [105].

In particular, in 2 spatial dimensions the extrinsic geometry is very simple. Since we do not require the effective theory to be Lorentz invariant we restrict the embedding functions to be of the form $x^0 = t, x^i = x^i(x, t)$, where x is the "spatial" coordinate on the physical boundary and t is the global time. Given embedding functions one can construct tangent and normal vectors t^i and n^i that satisfy

$$n^i n_i = t^i t_i = 1\,, \qquad n_i t^i = 0\,. \tag{7.30}$$

Later on we will also need a one-form $\mathcal{K}_\alpha$ (related to the second fundamental form) defined on the boundary as

$$\mathcal{K}_0 = n^j \partial_0 t_j, \quad \mathcal{K}_1 = n^j t^i \nabla_i t_j. \tag{7.31}$$

The one-form $\mathcal{K}_\alpha$ has an interesting relation to the spin connection ω_μ evaluated on the boundary (in the bulk everything can be written only in terms of metric). If the spatial vielbeins E^i_A are aligned along the vectors t^i and n^i the the spin connection evaluated on the boundary will coincide with $-\mathcal{K}_\alpha$. Generally, at the boundary the difference $\omega + \mathcal{K}$ is a pure $SO(2)$ gauge

$$\omega_\alpha + \mathcal{K}_\alpha = \partial_\alpha \theta\,, \tag{7.32}$$

where θ is a rotation angle. In particular, integrating the spatial component of (7.32) along the one-dimensional boundary of a surface and using $R = 2(\partial_1\omega_2 - \partial_2\omega_1)$ and the Stokes theorem we obtain the Gauss-Bonnet theorem for a manifold with a boundary

$$\frac{1}{4\pi}\int_\Sigma dA\, R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, \mathcal{K}_1 = \chi\,. \tag{7.33}$$

Here dA and ds are invariant area and line element, respectively. The Euler characteristics χ is an integer-valued topological invariant equal to the winding number of the angle θ in (7.32)[5].

[5] For example, the Euler characteristics of a disc or a spherical cap is $\chi = 1$ while for a closed surface of genus g it is $\chi = 2 - 2g$.

7.6 Induced action at the boundary: Wen-Zee terms

Let us now consider the first Wen-Zee term written as $Ad\omega$ in Eq.(7.1). The form $d\omega$ can be viewed as a part of topologically conserved Euler current [98] that couples minimally to the vector potential A. This term is not gauge invariant in the presence of the boundary. Similarly, the second Wen-Zee term $\omega d\omega$ is not $SO(2)$ invariant in the presence of the boundary.

It is straightforward to check that the boundary term

$$W_\partial^2 = -\frac{\nu}{2\pi}\int dxdt\,\epsilon^{\alpha\beta}\left[\bar{s}A_\alpha\mathcal{K}_\beta + \frac{\overline{s^2}}{2}\omega_\alpha\mathcal{K}_\beta\right] \tag{7.34}$$

cures the non-invariances of both Wen-Zee terms. [6]

In contrast to (7.29) the boundary term (7.34) is *local.* This proves that *there are no anomalies related to the Wen-Zee terms.* This observation is the first main result of this Letter.

The total boundary induced action is

$$W_\partial = W_\partial^1 + W_\partial^2 + \dots\,, \tag{7.35}$$

where $W_\partial^1 + W_\partial^2$ is the part fixed by the bulk action (7.1) and ellipsis denote terms invariant under all symmetries.

7.7 Shift in the presence of the boundary

The bulk induced action (7.1) encodes local response functions which can be found by taking various variational derivatives of the induced action.

Density response is

$$\rho = \frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_0} = \frac{\nu}{2\pi}B + \frac{\nu\bar{s}}{4\pi}R + \dots\,. \tag{7.36}$$

This local relation means that electrons accumulate in the areas of higher magnetic field and curvature in the bulk. The relation (7.36) contains also higher gradient terms denoted by ellipsis. Therefore, it is more interesting to look at the integral version of this relation. The corrections to the density from higher order terms are necessarily full spatial derivatives and vanish upon

[6] The first term in (7.34) coincides with the non-relativistic limit of the counterterm needed to ensure the gauge invariance of the Euler current term [98].

integration. On an arbitrary manifold without boundary we have

$$N = \nu N_\phi + \nu \bar{s} \chi \,, \tag{7.37}$$

where χ is the Euler characteristics of a manifold defined by (7.33) and $N_\phi = \frac{1}{2\pi}\int_\Sigma B$ [7]. Relation (7.37) was known in FQH literature for closed manifolds. We will now generalize it to the systems with a physical edge.

The induced action $W_{tot} = W_\partial + W$, in principle, provides us with the means to compute the total number of electrons N in a given FQH state. In order to carefully evaluate the correction to the density coming from the non-local terms W^1_∂ one has to fix the boundary theory first. The contribution coming from the boundary is not topological and depends (to an extent) on the boundary conditions and the definition of the boundary on the scales comaparable with the magnetic length.

We are going to illustrate the answer on the example of IQHE. First, we adiabatically connect an IQHE state to a clean system of non-interacting electrons. This case is particularly well controlled because the computation of the total electron number can be done in two different ways. The first way is quite abstract and it relies on the Atiyah-Patodi-Singer (APS) index theorem [106]. The states at the lowest Landau level are the zero modes of the $\bar{\partial}$ operator and their number is determined by its *index* [8]. According to the APS theorem the index of $\bar{\partial}$ is given by

$$N = \mathrm{ind}\,(\bar{\partial}) = \frac{1}{2\pi}\int_\Sigma B + \frac{1}{2}\left(\frac{1}{4\pi}\int_\Sigma R + \frac{1}{2\pi}\int_{\partial\Sigma}\mathcal{K}_1\right) + \frac{1}{2}\eta \,. \tag{7.38}$$

Here $\mathrm{ind}\,(\bar{\partial})$ is the index of $\bar{\partial}$ operator and the η-invariant is defined by

$$\eta = \mathrm{s}ign\bar{\partial}|_{\partial\Sigma} \equiv \Sigma_\lambda \mathrm{s}ign\lambda \,, \tag{7.39}$$

where λ is an eigenvalue of operator $\bar{\partial}$ restricted to the boundary $\partial\Sigma$ [107].

We observe that the first terms of (7.38) and (7.37) at $\nu = 1$ and $\bar{s} = 1/2$ identically coincide. The last term in (7.38) is more subtle. Since the manifold Σ has a boundary and the sum of first three terms in (7.38) is not necessarily integer. In fact, the sum of first three terms is *half-integer* even if we choose to thread an integer flux N_ϕ through the disc. The role of the η-invariant is to make it integer. Indeed, for one dimensional operator $D = \bar{\partial}|_{\partial\Sigma} = -i\partial_x + A_x(x)$ we have $\eta(D) = 1 - 2\{N_\phi\}$, where $N_\phi = \frac{1}{2\pi}\int dx\, A_x$ and (7.38) becomes simply

[7] Strictly speaking, the formula (7.37) is valid only when its right hand side is integer so that there is no contribution from excitations.

[8] Generally $\mathrm{ind}(\bar{\partial}) = \dim \mathrm{Ker}\,\bar{\partial} - \dim \mathrm{Ker}\,\partial$ but the lowest Landau level is spanned only by holomorphic states and only first term contributes.

an integer part

$$N = [N_\phi] + \frac{\chi + 1}{2}\,, \tag{7.40}$$

where $[N_\phi]$ denotes the integer part of N_ϕ.

The applicability of the index theorem requires a specific choice of the boundary conditions on the wave-function and thus might seem unrealistic. The second way to compute the total charge N is more physical and it reproduces the results of the index theorem. We start from fixing the boundary theory to be a chiral fermion. Then the relation (7.37) has to be amended by the induced fermion number of the edge theory given by the $\eta-$invariant [108]. This again leads to (7.40).

Notice that the first term in (7.34) while not related to quantum anomalies is crucial to produce the correct expression for χ (7.33) entering (7.40).

7.8 Singular expansion of charge density

We find the corrections to the local formula for the equilibrium charge density (7.36) in the presence of the boundary. For simplicity we confine ourselves to the case of the flat domain Σ and constant magnetic field B. Then the variation of the induced action $W + W_\partial$ over A_0 gives

$$\rho = \frac{\nu}{2\pi} B\theta(\Sigma) + \frac{\nu\bar{s}}{4\pi}\mathcal{K}_1\delta(\partial\Sigma) + \frac{\zeta}{2\pi}\partial_n\delta(\partial\Sigma)\,. \tag{7.41}$$

Here $\theta(\Sigma)$ is an indicator function of the domain (equal to unity inside the domain and zero otherwise) and δ and $\partial_n\delta$ denote Dirac's delta function concentrated on the boundary $\partial\Sigma$ of the domain and its derivative normal to the boundary. The coefficient ζ describes the so-called "overshoot" phenomenon and in the case of the Laughlin function is related to the Hall viscosity.

A comment is in order, the last term in (7.41) gets contributions from *two* different terms: one in the bulk and one on the boundary. The relevant terms are

$$W_1 = \frac{\sigma_H^{(2)}}{2\pi}\int_M \nabla^i E_i B\,, \qquad W_2 = \frac{\xi}{2\pi}\int_{\partial M} n^i E_i\,, \tag{7.42}$$

where $\sigma_H^{(2)}$ is the longwave correction to the Hall conductance and ξ is a dimensionless parameter related to the total dipole moment at the edge. With these definitions we have $\zeta = l^2\sigma_H^{(2)} + \xi$. The first of these coefficients can be found purely in a bulk computation (one also has to substitute the background constant magnetic field B) and the other one is an edge effect. When the induced action is additionally restricted by the (local) Galilean invariance [50] the coefficient $\sigma_H^{(2)}$ gets a contribution from the Hall viscosity, thus relating the "overshoot" with η_H. In general, these phenomena are not related.

In order to illustrate our results we can compare them with the computation of the singular expansion of the charge density computed by Wiegmann and Zabrodin [109].

$$\rho = \frac{\nu}{2\pi}\Theta(R-r) + \frac{1}{2\pi}\delta(r^2-R^2) + \frac{1-2\nu}{2\pi}R^2\delta'(r^2-R^2)$$

This fixes $\zeta = 1-2\nu$.

7.9 Gibbons-Hawking-York boundary term.

Here we will demonstrate a curious observation regarding the Wen-Zee term: it can be obtained from the non-relativistic limit of Einstein-Hilbert term, whereas the boundary counter term (7.34) can be obtained from the non-relativistic limit of Gibbons-Hawking-York boundary term.

Consider an Einstein-Hilbert action written in first order formalism on a $2+1D$ manifold M.

$$S_{EH} = -\frac{1}{8\pi G}\int_M d^3x \epsilon_{abc}\varepsilon^{\mu\nu\rho} e^a_\mu R^{bc}{}_{\mu\nu}\,, \tag{7.43}$$

where G is the Newton's constant in $2+1D$.

If the manifold M has a boundary the action S should be supplemented with a boundary term in order for the variational principle to be well-defined. In the first order treatment this boundary term is given by

$$S_\partial = -\frac{1}{8\pi G}\int_{\partial M} d^2x\sqrt{g}\epsilon_{abc}\varepsilon^{\mu\nu} e^a_\mu \omega^{bc}_\nu \tag{7.44}$$

We want to stress that there are no symmetry reasons to add the term since S_{EH} is invariant even if there is a boundary.

There is a general procedure that allows to construct Galilean invariant theories from Lorentz invariant theories [45]. It amounts to placing a Lorentz invariant theory in a curved background with vielbeins given by

$$e^0_\mu = (1-\frac{A_0}{c}, -\frac{A_i}{c})\,, \quad e^A_\mu = (0, e^A_i)\,, \tag{7.45}$$

where A_μ will become a vector potential in the non-relativistic theory. It is easy to see that this substitution equates momentum to the electric current [45] which is equivalent to the statement of the Galilean invariance.

Substituting (7.45) into (7.43) and (7.44) we find

$$S_{EH} + S_\partial \approx \frac{1}{8\pi G}\int \epsilon^{\mu\nu\rho} A_\mu \partial_\nu \omega_\rho + \frac{1}{8\pi G}\int_{\partial M} \epsilon^{\alpha\beta} A_\alpha \omega_\beta + \dots\,, \tag{7.46}$$

where $+\dots$ means other terms fixed by the Galilean symmetry. We will not indulge in the computation of these terms.

7.10 Outlook

In this Chapter we have slightly touched upon the aspects of boundary physics of quantum Hall states. This topic is a gigantic field and we are certainly not competent enough to give a complete and wide discussion.

What we did accomplish however, is showing that Hall viscosity is not as strong property of a quantum Hall fluid as Hall conductance. That is, if we imagine a fluid with no Hall conductance, but with Hall viscosity only then there is no guarantee that there are gapless degrees of freedom if one attempts to introduce a boundary. At the same time, if there is a chiral central charge (or if there is Leduc-Righi effect on the boundary) then the only bulk response that will receive a contribution is the Hall viscosity.

We have also shown that despite some claims in literature, at a general level (*i.e.* without Galilean invariance) Hall viscosity is not related to the edge dipole moment. This happens because in the presence of the boundary it is possible to add manifestly invariant (with respect to all symmetries) boundary terms in the induced action. These terms are, however, restricted if the Galilean symmetry is enforced.

Chapter 8

Discussion and Perspectives

In this closing Chapter we will summarize the results obtained in the Thesis and will say a few words about the future directions of the research in the field of geometric responses.

8.1 Summary

In this Thesis we have studied both integer and fractional quantum Hall states in the background of the perturbed space geometry. This study allowed us to compute various dimensionless response functions such as Hall viscosity (in the units of density) and orbital spin variance. In all the cases these kinetic coefficients take rational values. It was found that when the computation of the induced action is done using the effective TQFT one has to add the "framing anomaly" contribution to the induced action. Only in that case various effective and mean field approaches to quantum Hall states are self-consistent in curved sapce.

We have investigated a class of gapped systems with local Galilean symmetry. For this class we have derived a number of relations (Ward Identities) between the aforementioned response functions and their gradient corrections. What seems to be interesting is that the response functions computed for the Laughlin function satisfy these Ward identities. We used these Ward identities to predict the values of gradient corrections to various kinetic coefficients.

We have investigated quantum Hall states in thermal equilibrium on a curved space. The developed formalism allowed to derive the local thermodynamic relations and a collection of Streda formulas for various thermodynamic quantities. We have emphasized that Lorentz invariance imposes additional constraints on these relations and in general case these constraints do not have to hold. It was found that in order to couple a non-relativistic system to curved space in coordinate reparametrisation invariant way one has to introduce additional external fields that form the Newton-Cartan geometry (a type of non-Riemannian geometry). This geometry was implicitly used in old Luttinger's work on thermoelectric transport. NC geometry is useful for computation of momentum and energy transport in non-relativistic systems.

We have studied quantum Hall states on a curved space with boundary. It was found that there are four independent "Chern-Simons-type" terms that one can construct in the bulk induced action. Two of these terms induce a quantum (gauge and gravitational) anomaly of the edge theory. These terms are related to Hall conductance and thermal Hall conductance correspondingly. The other two terms do not lead to an anomaly, but imply a certain boundary response. Thus, we have concluded that unlike Hall conductance, Hall viscosity does not imply existence of "protected" gapless edge modes.

8.2 Measurement of the Hall viscosity

The lead character of this exposition was the Hall viscosity. It is an *observable* transport coefficient. This kinetic coefficient manifests itself in a shear flow of a two dimensional fluid: it leads to a force, transverse, to the shear. Hall viscosity is different from zero only in systems where parity is broken either explicitly or spontaneously. For example, in quantum Hall states parity is broken by the magnetic field.

It is very difficult to access mechanical properties of an electron fluid. One promising possibility to access it is to reduce any mechanical experiment to an electro-magnetic one. The recent result of Hoyos and Son (5.71) is an example of such reduction. The difficulty with the proposal is that instead of Hall viscosity one has to measure a long wave correction to Hall conductivity and that quantity does not really contribute to any transport measurement either.

The problem of measuring (shear) viscosity in electron fluids is difficult even outside of the non-dissipatve quantum Hall regime. Recently there was a proposal of an apparatus that could measure the shear viscosity of electron fluid in Corbino disc geometry [110] and to the best of our knowledge it is the only one so far. It is very interesting to adopt the suggested method to measure the Hall viscosity.

Alternatively, one could try to obtain $\bar{s}$ directly from a measurement of the equilibrium density making use of the relation between the local density and local curvature [111]

$$\rho = \frac{\nu}{2\pi}B + \frac{\nu\bar{s}}{4\pi}R\,. \tag{8.1}$$

It is, in principle, possible to create ripples, cones and corners in, say, a graphene sample, but the difficulty with this approach is that there always will be a charge density accumulation around such defects and only a tiny contribution will come from the orbital spin.

It is not clear how robust the value of the Hall viscosity will be to disorder and other details of the system that violate rotational invariance. In the case of the Hall conductivity the robustness is justified by the Laughlin's argument that only appeals to the charge conservation, that will not be affected by disorder. There is a more precise argument of Pruisken [112] that appeals to the presence of a topological term in the effective non-linear sigma model that leads to precise quantization of Hall conductance in the presence of disorder. At the present time the relationship of Hall viscosity with disorder is not clear. Presumably, if disorder is sufficiently weak the orbital spin will not change after the averaging over disorder. It is extremely important to understand this question.

8.3 Gravitation as effective theory of FQH states

recently there was a lot of interest to hidden "geometric degree of freedom" in the effective theory of FQHE [113–116]. This degree of freedom should encode some universal information about dynamics and gapped collective excitations in FQHE. If this is indeed the case then the effective theory should contain some version of gravitation (perhaps, without Lorentz symmetry).

In a seemingly unrelated development [117–119] it was found that interactions can induce spontaneous breaking of $SO(2)$ rotational symmetry in quantum Hall state driving system to a so-called nematic phase. Hall viscosity changes its value in the transition. The (massive) fluctuations of the nematic order parameter on the isotropic side of the transition are identified with the Girvin-MacDonald-Platzman collective mode. The effective theory of the phase transition is written for the matrix $SO(2)$ local order parameter: that is one way to interpret the "geometric degree of freedom" mentioned before.

We believe that there is something to be understood about the role of $2+1$ dimensional gravity in the effective theory of FQHE. The relation seems to be possible particularly because $2+1D$ gravitation is a Chern-Simons theory in disguise (although with non-compact gauge group) [120]. It should describe the statistics, fusion and braiding of the space-time defects much like gauge Chern-Simons theory describes the statistics, fusion and braiding of quasiparticles. It is exciting to speculate that understanding (quantum) gravity as an effective theory for quantum Hall states might shed some new light on the fundamental problem of quantization of Einstein's theory in $3+1$ dimensions.

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Appendix A

Energy current

We will derive the expression for the energy current operator the way it is usually done in solid state literature. We will define the energy current J_i^E as the current, that participates in the local energy conservation law.

We start with the unperturbed action

$$S_0 = \int d^2xdt \left[i\hbar\psi^\dagger\dot{\psi} - \frac{\hbar^2}{2m} D_i^\dagger \psi^\dagger D_i \psi \right] \tag{A.1}$$

with $D_i = \partial_i - \frac{i}{\hbar} A_i^b$. The background vector potential A_i^b creates homogeneous magnetic field so that $\epsilon^{ij}\partial_i A_j^b = B_0 = const$ and we partially fix the gauge by choosing $\dot{A}_i^b = 0$. We chose the static external potential so it lightens the derivation, but should not affect the final conclusions.

Equations of motion are

$$i\dot{\psi} = -\frac{\hbar}{2m} D_i D^i \psi \tag{A.2}$$

$$i\dot{\psi}^\dagger = \frac{\hbar}{2m} D_i^\dagger D^{\dagger i} \psi^\dagger \tag{A.3}$$

The total energy H of the system is

$$H = \int d^2xdt \left[\frac{\hbar^2}{2m} D_i^\dagger \psi^\dagger D_i \psi \right] \equiv \int d^2xdt \times h(x,t) \tag{A.4}$$

We are looking for the continuity equation for the time derivative of h.

$$\dot{h} = \frac{\hbar^2}{2mi} \left(D_j^\dagger (i\dot{\psi}^\dagger) D_j \psi + D_j^\dagger \psi^\dagger D_j (i\dot{\psi}) \right) \tag{A.5}$$

Using the EoM we find.

$$\dot{h} = \frac{\hbar^3}{4m^2 i}\left(D_j^\dagger D_i^\dagger D^{\dagger i}\psi^\dagger D^j\psi - D_j^\dagger\psi^\dagger D^j D_i D^i\psi\right) \tag{A.6}$$

Using the integration by parts in the form $(D^\dagger f)g = -f(Dg) + \partial(fg)$. We apply it twice to the first term. The the send term will be canceled leaving us with a full derivative. As such:

$$\begin{aligned}(D_j^\dagger D_i^\dagger D^{\dagger i}\psi^\dagger)D^j\psi &= -(D_j^\dagger D^{\dagger j}\psi^\dagger)(D_i^\dagger D^i\psi) + \partial_j\left(D_i^\dagger D^{\dagger i}\psi^\dagger D^j\psi\right)\\ = \;& (D_j^\dagger\psi^\dagger)(D^j D_i D^i\psi) - \partial_j\left((D^{\dagger j}\psi^\dagger)(D_i D^i\psi)\right) + \partial_j\left((D_i^\dagger D^{\dagger i}\psi^\dagger)(D^j\psi)\right)\end{aligned}$$

First term cancels the second term in (A.6). We have shown that

$$\dot{h} = \frac{\hbar^3}{4m^2 i}\partial_j\left[-(D^{\dagger j}\psi^\dagger)(D_i D^i\psi) + (D_i^\dagger D^{\dagger i}\psi^\dagger)(D^j\psi)\right] \equiv \partial_j J_j^E \tag{A.7}$$

The final expression for energy current is given by

$$J_i^E = \frac{\hbar^3}{4m^2 i}\left[-(D_j^\dagger\psi^\dagger)D^2\psi + (D^2\psi)^\dagger(D_j\psi)\right] \tag{A.8}$$

We did not need to keep all of the external perturbations in the derivation since it is easy to embed this expression into the curved space, simply by demanding that it is a vector under spatial coordinate variations.

Appendix B

Free fermions in Newton-Cartan background

The action for the free fermions in general Newton-Catran background is given by (see Chapter 6)

$$
\begin{aligned}
S &= \int dtd^2xe\mathcal{L}\,,\\
\mathcal{L} &= \left(\frac{i}{2}v^{\mu}(\Psi^{\dagger}\partial_{\mu}\Psi - \partial_{\mu}\Psi^{\dagger}\Psi) - \frac{h^{\mu\nu}}{2m}\partial_{\mu}\Psi^{\dagger}\partial_{\nu}\Psi\right)\,. \qquad \text{(B.1)}
\end{aligned}
$$

As we have mentioned in Chapter 6, it is convenient to use a particular parametrization of the NC background fields. Let us specify the spatial part h^{ij} of the degenerate metric and assume that $n_{\mu} = (n_0, n_i)$ and $v^{\mu} = (v^0, v^i)$ are also specified and satisfy the first relation in (6.27). Then we find from other relations in (6.27) $h^{\mu\nu} = \begin{pmatrix} \frac{n^2}{n_0^2} & -\frac{n^i}{n_0} \\ -\frac{n^i}{n_0} & h^{ij} \end{pmatrix}$, where we defined $n^i = h^{ij}n_j$, $n^2 = n_i n_j h^{ij}$. In this parametrization the invariant volume element is given by $dV = \sqrt{h}n_0 dtd^2x$.

In this parametrization of the NC background the Lagrangian takes form

$$
\begin{aligned}
\mathcal{L} &= \left(\frac{i}{2}v^0(\Psi^{\dagger}\partial_0\Psi - \partial_0\Psi^{\dagger}\Psi) + \frac{i}{2}v^i(\Psi^{\dagger}D_i\Psi - D_i\Psi^{\dagger}\Psi) - \frac{h^{ij}}{2m}D_i\Psi^{\dagger}D_j\Psi\right)\\
&- \frac{n^i}{2mn_0}\left(D_i\Psi^{\dagger}D_0\Psi + D_0\Psi^{\dagger}D_i\Psi\right) + \frac{n^2}{n_0^2}D_0\Psi^{\dagger}D_0\Psi + A_0\Psi^{\dagger}\Psi \qquad \text{(B.2)}
\end{aligned}
$$

As promised in Chapter 6 the NC "clock form" n_{μ} couples to the energy and energy current. To see this one has to use the equations of motion in $\left(D_i\Psi^{\dagger}D_0\Psi + D_0\Psi^{\dagger}D_i\Psi\right)$.

Also we stress reader's attention that the spatial part of v^i couples to the matter the same way as spatial part of the vector potential A_i, apart from the A^2 term. This identical coupling is not accidental, it is consequence of the Galilean invariance and leads to the Ward identity

$$J^i = \frac{e}{m} P^i \tag{B.3}$$

Also notice that n_0 and $\sqrt{h}$ couple the same way to the energy. This is consequence of the non-relativistic version of Weyl symmetry that leads to equality of pressure and energy. Pretty much any interaction will destroy this symmetry. Thus, in order to restore n_0 in the calculation of Chapter 3 we only need to ensure that $\sqrt{g}$ and n_0 enter together.

We also have to point out that if the NC background does not satisfy the integrability condition $n \wedge dn = 0$ then the theory stops being casual. In order to avoid this problem we will only retain the terms linear in n_i in the induced action. This amounts to using n_i as a source for energy current and then setting n_i to zero in all 1-point functions.

B.1 Perturbation theory

We will only consider the perturbation by n_i since n_0 terms are easily restored as we pointed out in the last section.

The new contribution in the linear order is given by $n_i J^E_i$ (after using the equations of motion):

$$\delta S^{(1)} = \frac{\hbar^3}{4m^2 i} \int d^2x dt \left(n_i \left[-(D_i\Psi)^\dagger D^2\Psi + (D^2\Psi)^\dagger (D_i\Psi) \right] \right) \tag{B.4}$$

Integrating by parts enough times we get

$$\delta S^{(1)} = \frac{\hbar^3}{4m^2 i} \int d^2x dt \times \Psi^\dagger \left[D_i n_i D^2 + D^2 n_i D_i \right] \Psi, \tag{B.5}$$

where derivatives act all the way to the right. Now we go to complex basis.

$$\begin{aligned} \delta S^{(1)} &= \frac{\hbar^3}{16m^2 i} \int d^2x dt \times \Psi^\dagger \left[D_{\bar z} n_z (D_z D_{\bar z} + D_{\bar z} D_z) \right. \\ &+ \left. (D_z D_{\bar z} + D_{\bar z} D_z) D^2 n_z D_{\bar z} + c.c. \right] \Psi, \end{aligned} \tag{B.6}$$

Now we replace the derivatives with creation/annihilation operators as

$$D_z = \frac{\sqrt{2}\hbar}{li} a^\dagger \quad \text{(B.7)}$$

$$D_{\bar{z}} = \frac{\sqrt{2}\hbar}{li} a \quad \text{(B.8)}$$

So we get

$$\begin{aligned} \delta S^{(1)} &= \frac{\hbar^3}{4\sqrt{2}l^3 m^2} \int d^2x dt \psi^\dagger [a n_z (2a^\dagger a + 1) + (2a^\dagger a + 1) n_z a \\ &+ a^\dagger n_{\bar{z}} (2a^\dagger a + 1) + (2a^\dagger a + 1) n_{\bar{z}} a^\dagger] \psi \end{aligned} \quad \text{(B.9)}$$

The vertices are (up to an overall factor $= \frac{\hbar^3}{4\sqrt{2}l^3 m^2}$)

$$V_{n_z} = a e^{-qa^\dagger} e^{\bar{q}a} (2a^\dagger a + 1) + (2a^\dagger a + 1) e^{-qa^\dagger} e^{\bar{q}a} a, \quad \text{(B.10)}$$

$$V_{n_{\bar{z}}} = a^\dagger e^{-qa^\dagger} e^{\bar{q}a} (2a^\dagger a + 1) + (2a^\dagger a + 1) e^{-qa^\dagger} e^{\bar{q}a} a^\dagger, \quad \text{(B.11)}$$

where we introduced $q = \frac{kl}{\sqrt{2}}$. In order to simplify this we will use

$$e^{\bar{q}a} f(a^\dagger) = f(a^\dagger + \bar{q}) e^{\bar{q}a} \quad \text{(B.12)}$$

$$e^{qa^\dagger} f(a) = f(a - q) e^{qa^\dagger} \quad \text{(B.13)}$$

After some algebra

$$V_{n_z} = e^{-qa^\dagger} \left[4a^\dagger a^2 + 4q a^\dagger a + 4a + 2\bar{q} a^2 + 2|q|^2 a + q \right] e^{\bar{q}a} \quad \text{(B.14)}$$

$$V_{n_{\bar{z}}} = e^{-qa^\dagger} \left[4a^{\dagger 2} a + 4\bar{q} a^\dagger a + 4a^\dagger + 2q a^{\dagger 2} + 2|q|^2 a^\dagger + \bar{q} \right] e^{\bar{q}a} \quad \text{(B.15)}$$

Finally, replacing operators with momentum derivatives

$$V_{n_z} = \left[-4\partial_q \partial_{\bar{q}}^2 - 4q \partial_q \partial_{\bar{q}} + 4\partial_{\bar{q}} + 2\bar{q} \partial_{\bar{q}}^2 + 2|q|^2 \partial_{\bar{q}} + q \right] e^{-qa^\dagger} e^{\bar{q}a} \quad \text{(B.16)}$$

$$V_{n_{\bar{z}}} = \left[4\partial_q^2 \partial_{\bar{q}} - 4\bar{q} \partial_q \partial_{\bar{q}} - 4\partial_q + 2q \partial_q^2 - 2|q|^2 \partial_q + \bar{q} \right] e^{-qa^\dagger} e^{\bar{q}a} \quad \text{(B.17)}$$

And the corresponding differential operators, acting on the generating function are

$$\hat{V}_{n_z} = \frac{\hbar^3}{4\sqrt{2}l^3 m^2} \left[-4\partial_q \partial_{\bar{q}}^2 - 4q \partial_q \partial_{\bar{q}} + 4\partial_{\bar{q}} + 2\bar{q} \partial_{\bar{q}}^2 + 2|q|^2 \partial_{\bar{q}} + q \right] \quad \text{(B.18)}$$

$$\hat{V}_{n_{\bar{z}}} = \frac{\hbar^3}{4\sqrt{2}l^3 m^2} \left[4\partial_q^2 \partial_{\bar{q}} - 4\bar{q} \partial_q \partial_{\bar{q}} - 4\partial_q + 2q \partial_q^2 - 2|q|^2 \partial_q + \bar{q} \right] \quad \text{(B.19)}$$

B.2 Contact terms

To get the contact terms we expand the action to the second order.

$$\delta S = \frac{\hbar^3}{4m^2 i}\int d^2xdt\left(g^{ij}n_i\left[-(\nabla_j\Psi)^\dagger\nabla_g^2\Psi + (\nabla_g^2\Psi)^\dagger(\nabla_j\Psi)\right]\right)$$

We expand the differential operators to the linear order in fields

$$\nabla_i = D_i - \frac{i}{\hbar}A_i - \frac{1}{4}\partial_i g_{z\bar{z}} \tag{B.20}$$

$$\nabla_i^\dagger = D_i + \frac{i}{\hbar}A_i - \frac{1}{4}\partial_i g_{z\bar{z}} \tag{B.21}$$

$$\nabla_g^2 = D^2 + g^{ij}D_iD_j + (\partial_i g^{ij})D_j - \frac{1}{4}\Delta g_{z\bar{z}} - i\partial_i A_i - 2iA_iD_i \tag{B.22}$$

$$\nabla_g^2 = D^2 + g^{ij}D_iD_j + (\partial_i g^{ij})D_j - \frac{1}{4}\Delta g_{z\bar{z}} + i\partial_i A_i + 2iA_iD_i \tag{B.23}$$

Using these notations we can write down all of the quadratic contact terms. We do not list those here

B.3 The lowest order induced action

Finally we present a few interesting terms induced in the Newton-Cartan geometry. The terms of interest are the Chern-Simons-type parity odd terms

$$\delta W = \frac{\alpha}{4\pi}\int ndA + \frac{\beta}{4\pi}\int ndn \tag{B.24}$$

These terms contribute to the thermo-electric transport in the lowest order in gradients. These terms a *not universal* for the reasons discussed in Chapter 6. The coefficients are given by

$$\alpha = \frac{1}{ml^2} = 2\times\frac{\hbar\omega_c}{2}\,, \qquad \beta = \frac{1}{m^2l^4} = (\hbar\omega_c)^2 \tag{B.25}$$

We can easily check that at least α makes sense. If we plug in the background value for A. We get

$$W = \int d^2xdt\frac{\hbar\omega_c}{4\pi}n_0B = \int d^2xdtn_0\frac{\hbar\omega_c}{2}\frac{1}{2\pi l^2} = \int d^2xdtn_0\epsilon_0\,, \tag{B.26}$$

so n_0 is indeed coupled to the energy density. We do not have similar trick to check the sanity of the value of β except the dimensional analysis (that works our correctly). The value of β implies the transverse bulk energy current

$$J_E^i = \frac{(\hbar\omega_c)^2}{4\pi}\epsilon^{ij}\partial_j n_0\,. \tag{B.27}$$

This concludes our discussion of the free fermions in the general Newton-Cartan background.

Appendix C

Non-relativistic limit of gravitational Chern-Simons

Affine connection Γ^i_{jk} is defined such that covariant derivative has the right (covariant) transformation law. Covariant derivative of a vector and a co-vector are

$$D_i v_k = \partial_i v_k - \Gamma^{ik}_j v_j \tag{C.1}$$
$$D_i v^k = \partial_i v^k + \Gamma^k_{ij} v^j \tag{C.2}$$

Imposing the Levi-Cevita condition $D_i g_{jk} = 0$ we find the expression for the connection

$$\Gamma_{i,jk} = \frac{1}{2}\left(-\partial_i g_{jk} + \partial_j g_{ik} + \partial_k g_{ij}\right) \tag{C.3}$$
$$\Gamma^{jk}_i = g^{jm} g^{kn} \Gamma_{i,mn} \tag{C.4}$$

In order to take the non-relativistic limit we go to the co-moving frame where $g_{00} = 1$ and $g_{i0} = g_{0i} = 0$. We also note that $\frac{\partial}{\partial x_0} = \frac{1}{c}\frac{\partial}{\partial t}$. So that $\partial_0 f = \frac{1}{c}\dot{f}$. We will set $c = 1$ for now, but then restore it by counting the number of time derivatives. NR limit will be taken in 2 steps. First, we go to the co-moving frame and then we set $c \to \infty$. The measure will go to $d^3x \to c d^2x dt$. So, in the NR limit we must keep only $\frac{1}{c}$ terms.

The non-zero components of $\Gamma_{i,jk}$ are

$$\Gamma_{0,ij} = -\frac{1}{2}\dot{g}_{ij} \tag{C.5}$$
$$\Gamma_{i,0j} = \frac{1}{2}\dot{g}_{ij} = \Gamma_{i,j0} \tag{C.6}$$
$$\Gamma_{i,jk} \equiv \Gamma_{i,jk} \tag{C.7}$$

and the rest are zero.

The non-zero components of Γ_i^{jk} are

$$\Gamma_0^{ij} = -\frac{1}{2}\dot{g}^{ij}, \tag{C.8}$$

$$\Gamma_i^{0j} = \frac{1}{2}g^{jk}\dot{g}_{ki}, \tag{C.9}$$

$$\Gamma_i^{jk} \equiv \Gamma_i^{jk}. \tag{C.10}$$

C.1 Gravitational Chern-Simons in the NR limit

We will write the gCS term as

$$I[\Gamma] = \int d^3x \left[\epsilon^{\mu\nu\rho}\Gamma_{\mu,\alpha\beta}\partial_\nu\Gamma_\rho^{\alpha\beta} + \frac{2}{3}\Gamma^3\right] \tag{C.11}$$

Now we plug all that in, simplify and keep only quadratic order. After a tedious computation we obtain (restoring c)

$$\int d^3x\epsilon^{\mu\nu\rho}\Gamma_{\mu,\alpha\beta}\partial_\nu\Gamma_\rho^{\alpha\beta} \to \int d^2xdt \left[\epsilon^{ij}g_{in}\partial_m\partial_n\dot{g}_{jm}\right] \tag{C.12}$$

In the quadratic approximation we do not distinguish between upper and lower indices (up to an overall sign).

In the complex basis we have

$$I[\Gamma] = -\frac{1}{4}\int d^2xdt \left[\omega|k|^2 g_{zz}(-k)g_{\bar{z}\bar{z}}(k) + \omega\bar{k}^2 g_{z\bar{z}}(-k)g_{\bar{z}\bar{z}}(k) - \omega k^2 g_{z\bar{z}}(-k)g_{zz}(k)\right] \tag{C.13}$$

This is **twice** the gCS action that is obtained from Son's ω_μ.

C.2 Cartan equations and spin connection

We define frame 1-form as $e^a = e^a_\mu dx^\mu$. Driebeins satisfy $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$ and $e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}$. We define exterior derivative d acting on a 0-from as $df = (\partial_\mu f)dx^\mu$. Exterior derivative acting on a vector field is

$$dv^k = (\partial_\mu v^k)dx^\mu \tag{C.14}$$

finally, acting on a 1-form

$$d(f_\mu dx^\mu) = (\partial_\nu f_\mu)dx^\nu \wedge dx^\mu \tag{C.15}$$

We define torsion free spin connection matrix valued 1-form $\omega^a_{.b}$ by demanding $D_\mu e^a_\nu = 0$. This is equivalent to torsion free Cartan structure equations.

Cartan structure equations are

$$de^a + \omega^a_{.b} \wedge e^b = T^a \tag{C.16}$$

In components:

$$\left(D_\nu e^a_\mu + \omega^a_{\nu,b} e^b_\mu\right) dx^\nu \wedge dx^\mu = T^a_{\mu\nu} dx^\nu \wedge dx^\mu \tag{C.17}$$

In the presence of torsion one has to be careful with choosing the independent quantities. For now we set $T^a = 0$. In the NR limit we choose $e^0_0 = 1$ and $e^0_i = e^i_0 = 0$. This will determine the spin connection.

We solve the (C.17) in components since 2-forms $dx^\nu \wedge dx^\mu$ are linearly independent.

(0,0) gives

$$\omega^a_{0,0} = 0 = \omega^0_{0,a} \tag{C.18}$$

(0,i) combined with NR expressions form Γ gives

$$\omega_{0,ab} = e_{i,a}\partial_0 e^i_b + \frac{1}{2c} e^k_a e^l_b \dot{g}_{kl} \tag{C.19}$$

contraction with ϵ^{ab} gives then **twice** Son's expression.

$$\omega_0 = \epsilon^{ab} e_{i,a}\partial_0 e^i_b \tag{C.20}$$

(i,j) gives, combined with NR expressions form Γ and with the use of $e^l_a e^k_b \epsilon^{ab} = \frac{1}{\sqrt{g}}\epsilon^{lk}$

$$\omega_i = \epsilon^{ab} e_{i,a}\partial_0 e^i_b - \frac{1}{\sqrt{g}}\epsilon^{lk}\partial_l g_{ik} \tag{C.21}$$

which is again **twice** Son's expression. Thus we established that in order to compare NR limit of usual gCS with $\epsilon^{\mu\nu\rho}\omega_\mu\partial_\nu\omega_\rho$ one needs to add a factor of $\frac{1}{2}$ in front of the latter

C.3 Isothermal coordinates.

Here we derive an exact expression for the gCS term, but written in the isothermal coordinates. We choose $g_{ij} = g^{\frac{1}{2}}\delta_{ij}$ and $g^{ij} = g^{-\frac{1}{2}}\delta^{ij}$. Then affine connection has components The non-zero components of $\Gamma_{i,jk}$ are

$$\begin{aligned}
\Gamma_{0,ij} &= -\frac{1}{2}\dot{g}_{ij} = -\frac{1}{2}\sqrt{g}\partial_t \ln\sqrt{g} \times \delta_{ij} && \text{(C.22)}\\
\Gamma_{i,0j} &= \frac{1}{2}\dot{g}_{ij} = \Gamma_{i,j0} = -\frac{1}{2}\sqrt{g}\partial_t \ln\sqrt{g} \times \delta_{ij} && \text{(C.23)}\\
\Gamma_{i,jk} &\equiv \Gamma_{i,jk} = \frac{\sqrt{g}}{2}\left(-\partial_i \ln\sqrt{g}\delta_{jk} + \partial_j \ln\sqrt{g}\delta_{ik} + \partial_k \ln\sqrt{g}\delta_{ij}\right) && \text{(C.24)}
\end{aligned}$$

and the rest are zero.

The non-zero components of Γ_i^{jk} are

$$\begin{aligned}
\Gamma_0^{ij} &= -\frac{1}{2}g^{im}g^{jn}\dot{g}_{mn} = -\frac{1}{2}\frac{1}{\sqrt{g}}\partial_0 \ln\sqrt{g} \times \delta^{ij} && \text{(C.25)}\\
\Gamma_i^{0j} &= \frac{1}{2}g^{jk}\dot{g}_{ki} = \frac{1}{2}\partial_0 \ln\sqrt{g} \times \delta_i^j && \text{(C.26)}\\
\Gamma_i^{jk} &\equiv \Gamma_i^{jk} = \frac{1}{2\sqrt{g}}\left(-\partial_i \ln\sqrt{g}\delta^{jk} + \partial^j \ln\sqrt{g}\delta_i^k + \partial^k \ln\sqrt{g}\delta_i^j\right) && \text{(C.27)}
\end{aligned}$$

Define the gCS term as

$$\begin{aligned}
I[g] &= \int d^2x dt \left(\epsilon^{\alpha\beta\gamma}\Gamma_{\alpha,\mu\nu}\partial_\beta\Gamma_\gamma^{\mu\nu}\right) && \text{(C.28)}\\
&= \int d^2x dt \epsilon^{ij}\left(\Gamma_{0,mn}\partial_i\Gamma_j^{mn} + \Gamma_{i,\mu\nu}\left[\partial_0\Gamma_j^{\mu\nu} - \partial_j\Gamma_0^{\mu\nu}\right]\right) && \text{(C.29)}
\end{aligned}$$

After some algebra

$$I[g] = \int_M d^2x dt \epsilon^{ij}(\partial_i \ln\sqrt{g})(\partial_0\partial_j \ln\sqrt{g}) \qquad \text{(C.30)}$$

this looks more illuminating in form notation. Denoting $\det g_{ij} = e^{2\phi}$

$$I[g] = \int_M d\phi \wedge d\dot{\phi} = \int_M \left[-\dot{\phi}d^2\phi + d(\dot{\phi}d\phi)\right] \qquad \text{(C.31)}$$

The first term vanishes due to Poincare lemma and the second term goes to the boundary due to Stokes theorem

$$I[g] = \int_{\partial M} \dot{\phi}d\phi = \int_{\partial M} (\partial_0 \ln\sqrt{g})d\ln\sqrt{g} = \int dl_i(\partial_0 \ln\sqrt{g})\epsilon^{ij}\partial_i \ln\sqrt{g} \qquad \text{(C.32)}$$

Thus in isothermal coordinates $SO(2)$ gravitational Chern-Simons theory is a boundary term.

Appendix D

Gauge invariance of the induced action for IQHE

While the final expression (3.138) is useful for computational purposes all of the symmetries of the classical action are not manifest. In this Appendix we will show that (3.138) is gauge invariant. We note that it is sufficient to check that gauge variation of the quadratic part of the induced action vanishes.

$$\delta W^{(2)} \approx \int \frac{d^2\mathbf{k}d\omega}{(2\pi)^3} \left(e^{-\frac{|kl|^2}{2}} \left[\sum_{n\geq N, m\leq N} - \sum_{n\leq N, m\geq N} \right] \frac{\sum_{i=0}^{2} \delta\Gamma^i_{nm}(k)\Gamma^j_{mn}(-k)}{n-m-\omega} \right) = 0 \tag{D.1}$$

with

$$\delta\Gamma^i_{nm}(k) = \langle n|\delta\hat{V}_i(k)|m\rangle \tag{D.2}$$

We list transformations of vertices, with notation $\hat{\gamma}(k) = e^{-\frac{kl}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}l}{\sqrt{2}}a}$

$$\begin{aligned}
\delta\hat{V}_0 &= -i\omega\hat{\gamma}(k) && \text{(D.3)}\\
\delta\hat{V}_1 &= -i\frac{\hbar}{2\sqrt{2}ml}\{a^\dagger, \hat{\gamma}(k)\}k && \text{(D.4)}\\
\delta\hat{V}_2 &= -i\frac{\hbar}{2\sqrt{2}ml}\{a, \hat{\gamma}(k)\}\bar{k} && \text{(D.5)}\\
\delta\hat{V}_3 &= 0 && \text{(D.6)}\\
\delta\hat{V}_4 &= 0 && \text{(D.7)}\\
\delta\hat{V}_5 &= 0 && \text{(D.8)}
\end{aligned}$$

It is more convenient to use dimensionless momentum $q = \frac{kl}{\sqrt{2}}$ and frequency

$\omega \to \frac{\hbar\omega}{ml^2}$. Then

$$\delta\hat{V}_0 = -i\frac{\hbar\omega}{ml^2}\hat{\gamma}(q) \quad (D.9)$$
$$\delta\hat{V}_1 = -i\frac{\hbar}{2ml^2}\{a^\dagger, \hat{\gamma}(q)\}q \quad (D.10)$$
$$\delta\hat{V}_2 = -i\frac{\hbar}{2ml^2}\{a, \hat{\gamma}(q)\}\bar{q} \quad (D.11)$$
$$\delta\hat{V}_3 = 0 \quad (D.12)$$
$$\delta\hat{V}_4 = 0 \quad (D.13)$$
$$\delta\hat{V}_5 = 0 \quad (D.14)$$

Looking at (D.1) we see that we have to prove for $\forall\ j$

$$\left[\sum_{n\geq N, m\leq N} - \sum_{n\leq N, m\geq N}\right] \frac{\sum_{i=0}^{2}\langle n|\delta\hat{V}_i(k)|m\rangle}{n-m-\omega}\Gamma^j_{mn}(-k) =$$

$$\left[\sum_{n\geq N, m\leq N} - \sum_{n\leq N, m\geq N}\right] \left(\frac{\langle n|(n-m)\frac{\delta\hat{V}_0(k)}{\omega} + \delta\hat{V}_1(k) + \delta\hat{V}_2(k)|m\rangle}{n-m-\omega}\Gamma^j_{mn}(-k)\right.$$
$$\left. -i\langle n|\frac{\hbar}{ml^2}\hat{\gamma}(q)|m\rangle\Gamma^j_{mn}(-k)\right)$$

Close look at the last expression gives

$$\langle n|(n-m)\frac{\delta\hat{V}_0(k)}{\omega} + \delta\hat{V}_1(k) + \delta\hat{V}_2(k)|m\rangle = \langle n|[a^\dagger a, \frac{\delta\hat{V}_0(k)}{\omega}] + \delta\hat{V}_1(k) + \delta\hat{V}_2(k)|m\rangle \quad (D.15)$$

We move all creation/annihilation operators between exponents.

$$\delta\hat{V}_2 = -i\frac{\hbar}{2ml^2}\bar{q}e^{-qa^\dagger}(2a-q)e^{\bar{q}a} \quad (D.16)$$
$$\delta\hat{V}_1 = -i\frac{\hbar}{2ml^2}qe^{-qa^\dagger}(2a^\dagger+\bar{q})e^{\bar{q}a} \quad (D.17)$$
$$(D.18)$$

We also compute

$$[a^\dagger a, \frac{\delta\hat{V}_0(k)}{\omega}] = -i\frac{\hbar}{ml^2}\left(a^\dagger a e^{-qa^\dagger}e^{\bar{q}a} - e^{-qa^\dagger}e^{\bar{q}a}a^\dagger a\right) \tag{D.19}$$

$$= i\frac{\hbar}{ml^2}e^{-qa^\dagger}\left(qa^\dagger + \bar{q}a\right)e^{\bar{q}a} \tag{D.20}$$

So we find

$$[a^\dagger a, \frac{\delta\hat{V}_0(k)}{\omega}] + \delta\hat{V}_1(k) + \delta\hat{V}_2(k) = 0 \tag{D.21}$$

So gauge invariance holds for any Γ^j_{nm}.
Then

$$\delta W^{(2)} = -i\frac{\hbar}{ml^2}\left[\sum_{n\geq N, m\leq N} - \sum_{n\leq N, m\geq N}\right]\langle n|\hat{\gamma}(q)|m\rangle\Gamma^j_{mn}(-k)$$

$$= -i\frac{\hbar}{ml^2}\left[\sum_{n\geq N, m<N} - \sum_{n\leq N, m\geq N}\right]\Gamma^0_{nm}(k)\Gamma^j_{mn}(-k)$$

$$= -i\frac{\hbar}{ml^2}\sum_{n\geq N, m<N}\left(\Gamma^0_{nm}(k)\Gamma^j_{mn}(-k) - \Gamma^0_{mn}(k)\Gamma^j_{nm}(-k)\right)$$

$$= -i\frac{\hbar}{ml^2}\sum_{m<N}\sum_{n\geq 1}\left(\Gamma^0_{nm}(k)\Gamma^j_{mn}(-k) - \Gamma^0_{mn}(k)\Gamma^j_{nm}(-k)\right)$$

$$= -i\frac{\hbar}{ml^2}\sum_{m<N}\langle m|[\hat{V}_0(k), \hat{V}_j(-k)]|m\rangle$$

Finally, we compute the commutator

$$[\hat{V}_0(k), \hat{V}_j(-k)] = 0 \tag{D.22}$$

This computation shows that the induced action is gauge invariant. We have also checked the gauge invariance using the symbolic algebra tools in Mathematica and, of course, it also works.

Appendix E

Coherent states

Here we describe coherent states that will be useful for multiple calculations. Here we follow Perelomov [51], but customize the notations.

E.1 Heisenberg-Weyl group

We define Heisenberg-Weyl algebra via relations

$$[a, a^\dagger] = 1 \qquad [a, 1] = [a^\dagger, 1] = 0 \tag{E.1}$$

an arbitrary element of the algebra is given by a linear combination

$$W = is \cdot \mathbf{1} + qa^\dagger - \bar{q}a, \tag{E.2}$$

where s is real and q is complex.

We want to exponentiate the algebra to the group. Arbitrary Heisenbrg-Weyl group element is given by

$$e^W = e^{is} \cdot e^{qa^\dagger - \bar{q}a} = e^{is} e^{qa^\dagger} e^{-\bar{q}a} e^{-\frac{1}{2}[qa^\dagger, -\bar{q}a]} = e^{is} e^{-\frac{|q|^2}{2}} e^{qa^\dagger} e^{-\bar{q}a}, \tag{E.3}$$

where we have used $e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B$, which is true for linear combinations of creation/annihilation operators. We also denote $D(q) = e^{qa^\dagger - \bar{q}a}$ as these operators form a representation of the Heisenberg-Weyl group. Representations for different values of s are inequivalent. For fixed value of s all representations are unitary equivalent. So from now on we fix s and drop this factor.

We can freely switch between $D(q)$ and $e^{qa^\dagger} e^{-\bar{q}a}$ at the cost of an exponent, that is

$$D(q) = e^{-\frac{|q|^2}{2}} e^{qa^\dagger} e^{-\bar{q}a} \tag{E.4}$$

Operators D have the following multiplication relations.

$$D(q)D(k) = e^{i\mathrm{Im}(q\bar{k})}D(k+q) \tag{E.5}$$

This can be checked using the following simple rules. We can always choose representation in which a is a derivative and $a^\dagger$ is a variable or where $a^\dagger$ is **minus** derivative and a is a variable. In this case an operator $e^{c\cdot a}$ is an operator of shift by c and $e^{c\cdot a^\dagger}$ is an operator of shift by $-c$. This proves the following relations

$$\begin{aligned} e^{ca}f(a^\dagger) &= f(a^\dagger + c)e^{ca} & \text{(E.6)} \\ e^{ca^\dagger}f(a) &= f(a-c)e^{ca^\dagger} & \text{(E.7)} \end{aligned}$$

These relations can be used to prove the multiplication law. The latter can be obviously generalized as follows.

$$D(q_M)\cdot D(q_{M-1})\cdot\ldots\cdot D(q_1) = e^{i\sum_{i<j}\mathrm{Im}(q_j\bar{q}_i)}D(q_M + q_{M-1} + \ldots + q_1) \tag{E.8}$$

The multiplication law implies the permutation relations

$$D(q)D(k) = e^{2i\mathrm{Im}(q\bar{k})}D(k)D(q), \tag{E.9}$$

which is equivalent to (E.6)

E.2 Generalized coherent states

Operators a, $a^\dagger$ naturally generate Hilbert space $\mathcal{H}$. With a basis

$$|n\rangle = \frac{a^\dagger}{\sqrt{n!}}|0\rangle \tag{E.10}$$

and $|0\rangle$ is defined as $a|0\rangle = 0$. Consider an arbitrary state $|\Psi_0\rangle \in \mathcal{H}$. States of the form

$$D(q)|\Psi_0\rangle = |q\rangle \tag{E.11}$$

are generalized coherent states. One gets usual coherent states choosing $|\Psi_0\rangle = |0\rangle$. Most of relations for coherent sates hold for any $|\Psi_0\rangle$. The overlap of the coherent sates is

$$\langle q|k\rangle = e^{i\mathrm{Im}(k\bar{q})}\langle\Psi_0|D(k-q)|\Psi_0\rangle \qquad |\langle q|k\rangle|^2 \equiv \rho(k-q) \tag{E.12}$$

Also we have

$$D(k)|q\rangle = e^{i\text{Im}(k\bar{q})}|k+q\rangle \tag{E.13}$$

Since the hilbert space $\mathcal{H}$ is projected $D(k)$ acts on the q-plane by simple translations. Therefore an invariant (under the action of Heisenberg-Weyl group) measure is

$$d\mu(k) = c dk_1 dk_2 \qquad \text{with} \qquad k = k_1 + ik_2 \tag{E.14}$$

Consider an operator

$$A = \int d\mu(k)|k\rangle\langle k| \tag{E.15}$$

We find that for any k we have $[D(k), A] = 0$, thus $A = \lambda\hat{1}$ due to Schur's lemma. We also can always choose c form (E.14) to set $\lambda = 1$. Thus we have shown

$$\int \frac{dk_1 dk_2}{\pi}|k\rangle\langle k| = \hat{1} \tag{E.16}$$

We now present some relations that are valid for $|\Psi_0\rangle = |0\rangle$.

$$D^\dagger(q)aD(q) = a + q \tag{E.17}$$

and

$$a|k\rangle = k|k\rangle \tag{E.18}$$

Similarly we have

$$|k\rangle = D(k)|0\rangle = e^{-\frac{|k|^2}{2}}e^{ka^\dagger}|0\rangle = \sum_{n=0}^{\infty}\frac{k^n}{\sqrt{n!}}|n\rangle \tag{E.19}$$

Using the last relation we find

$$|\langle k|0\rangle|^2 = \rho(k) = e^{-|k|^2} \qquad |\langle k|q\rangle|^2 = \rho(q-k) = e^{-|k-q|^2} \tag{E.20}$$

Noticing that (E.16) is equivalent to $\int d\mu(k)\rho(k) = 1$ we find that $c = \frac{1}{\pi}$.

We want to be able to compute traces in $\mathcal{H}$. In the Fock basis we have

$$\begin{aligned}
\text{Tr}\, O &= \sum_n \langle n|O|n\rangle = \sum_n \int d\mu(k)d\mu(q)\langle q|n\rangle\langle n|k\rangle\langle k|O|q\rangle \\
&= \int d\mu(k)d\mu(q)\langle q|k\rangle\langle k|O|q\rangle
\end{aligned} \tag{E.21}$$

Contracting one more decomposition of unity

$$\text{Tr}\, O = \int d\mu(q) \langle q|O|q\rangle = \int d\mu(q) e^{-|q|^2} \langle 0|e^{\bar{q}a} O e^{qa^\dagger}|0\rangle \tag{E.22}$$

So we have derived

$$\text{Tr}\,_a \hat{O} = \frac{1}{\pi} \int dq_1 dq_2 \left(e^{-|q|^2} \langle 0|e^{\bar{q}a} \hat{O} e^{qa^\dagger}|0\rangle \right) \tag{E.23}$$

Consider a matrix element of $D(k)$

$$G(\bar{k}, q; p) \equiv e^{\frac{1}{2}\left(|k|^2+|q|^2\right)} \langle k|D(p)|q\rangle = e^{-\frac{|p|^2}{2}} e^{\bar{k}q+\bar{k}p-\bar{q}p} \tag{E.24}$$

Inserting resolution of unity in terms $|n\rangle$ we find

$$G(\bar{k}, q; p) = \sum_{m,n} \bar{u}_m(k) u_n(q) D_{mn}(p) \qquad \text{with} \qquad u_n(k) \equiv \langle n|k\rangle = \frac{k^m}{\sqrt{m!}} \tag{E.25}$$

G is a generating function of the matrix elements of $D_{mn}(p)$. The latter are obtain expanding (E.24) in series in k and q.

$$D_{nm}(p) = \sqrt{\frac{n!}{m!}} e^{-\frac{|p|^2}{2}} p^{m-n} L_n^{m-n}(|p|^2) \qquad m \geq n \tag{E.26}$$

$$D_{nm}(p) = \sqrt{\frac{m!}{n!}} e^{-\frac{|p|^2}{2}} p^{n-m} L_m^{n-m}(|p|^2) \qquad n \geq m \tag{E.27}$$

We can find trace of $\text{Tr}\, D(p) = \pi\delta^{(2)}(p)$ and $\text{Tr}\, [D(p)D^{-1}(q)] = \pi\delta^{(2)}(p-q)$

Another relevant object is the generating function of associated Laguerre polynomials

$$(1+t)^m e^{-tx} = \sum_n t^n L_n^{m-n}(x) \tag{E.28}$$

E.3 Application

We want to compute trace of a product

$$\text{Tr}\,_b \left[\prod_{i=1}^{M} O_i(x_i) \right] = \int [dk] \prod_{i=1}^{M} \left[\text{Tr}\,_b \left[e^{i\sum_i \mathbf{k}_i \cdot \mathbf{x}_i} \right] \tilde{O}_i(k_i) \right], \tag{E.29}$$

where we have introduce a shorthand notation $[dk] = \prod_{i=1}^{M} \frac{d^2\mathbf{k}_i}{(2\pi)^{2M}}$. Now we re-write the exponent in terms of a and b.

$$e^{i\mathbf{k}\cdot\mathbf{x}} = e^{\frac{\bar{k}l}{\sqrt{2}}a - \frac{kl}{\sqrt{2}}a^\dagger} e^{-\frac{\bar{k}l}{\sqrt{2}}b^\dagger + \frac{kl}{\sqrt{2}}b} = e^{\bar{q}a - qa^\dagger} e^{-\bar{q}b^\dagger + qb} \tag{E.30}$$

where we introduced $q = \frac{kl}{\sqrt{2}}$, so that $[dk] = \left(\frac{2}{l^2}\right)^M [dq]$. We have for the exponent

$$e^{i\mathbf{k}\cdot\mathbf{x}} = D_a(-q) e^{-\frac{|q|^2}{2}} e^{-\bar{q}b^\dagger} e^{qb} \tag{E.31}$$

Now, we plug this back into the trace

$$\left(\frac{2}{l^2}\right)^M \prod_{i=1}^{M} \int [dq] D_a(-q_i) \left[\mathrm{Tr}_b \left[e^{-\frac{|q_i|^2}{2}} e^{-\bar{q}_i b^\dagger} e^{q_i b}\right] \tilde{O}_i(q_i)\right] \tag{E.32}$$

In order to proceed we use (E.23) derived before.

$$\mathrm{Tr}_b \left[\prod_{i=1}^{M} e^{-\frac{|q_i|^2}{2}} e^{-\bar{q}_i b^\dagger} e^{q_i b}\right] = \frac{1}{\pi} e^{-\sum_i \frac{|q_i|^2}{2}} \int dp_1 dp_2 \left[e^{-|p|^2} \langle 0| e^{\bar{p}b} \prod_{i=1}^{M} e^{-\bar{q}_i b^\dagger} e^{q_i b} e^{pb^\dagger} |0\rangle\right] \tag{E.33}$$

We want to normal order the product. In order to do this we use permutation relations many times.

$$e^{\bar{p}b} \prod_{i=1}^{M} e^{-\bar{q}_i b^\dagger} e^{q_i b} e^{pb^\dagger} =: e^{\bar{p}b} \prod_{i=1}^{M} e^{-\bar{q}_i b^\dagger} e^{q_i b} e^{pb^\dagger} : e^{|p|^2} e^{\sum_{i>j} -\bar{q}_i q_j} e^{-\bar{p}\sum_i \bar{q}_i} e^{p \sum_i q_i} \tag{E.34}$$

Denoting $\sum_i q_i = Q$ and using $\langle 0| : e^{\bar{p}b} \prod_{i=1}^{M} e^{-\bar{q}_i b^\dagger} e^{q_i b} e^{pb^\dagger} : |0\rangle = 1$ we have

$$\mathrm{Tr}_b \left[\prod_{i=1}^{M} e^{-\frac{|q_i|^2}{2}} e^{-\bar{q}_i b^\dagger} e^{q_i b}\right] = \frac{1}{\pi} e^{-\sum_i \frac{|q_i|^2}{2}} e^{\sum_{i>j} -\bar{q}_i q_j} \int dp_1 dp_2 e^{-\bar{p}Q} e^{p\bar{Q}} \tag{E.35}$$

The latter integral is a δ-function

$$\frac{1}{\pi} \int dp_1 dp_2 e^{-\bar{p}Q} e^{p\bar{Q}} = \pi \delta^{(2)}(\mathbf{Q}) = \pi \int \frac{d^2\lambda}{(2\pi)^2} e^{i\lambda\cdot\mathbf{Q}} \tag{E.36}$$

We also use $\bar{q}_i q_j = \mathbf{q_i} \cdot \mathbf{q_j} + i\mathbf{q_i} \wedge \mathbf{q_j}$, where $\mathbf{a} \wedge \mathbf{b} = a_1 b_2 - a_2 b_1$. As well as

$$\frac{1}{2} \sum_i |q_i|^2 + \sum_{i<j} \mathbf{q_i} \cdot \mathbf{q_j} = \frac{1}{2} \mathbf{Q}^2 \tag{E.37}$$

then

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M} e^{-\frac{|q_i|^2}{2}} e^{-\bar{q}_i b^\dagger} e^{q_i b}\right] = \pi e^{-i\sum_{i>j}\mathbf{q_i}\wedge\mathbf{q_j}} \int \frac{d^2\lambda}{(2\pi)^2} e^{i\lambda\cdot\mathbf{Q}} \tag{E.38}$$

We have proven that

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M} O_i(x_i)\right] = \pi\left(\frac{2}{l^2}\right)^M \int \frac{d^2\lambda}{(2\pi)^2} \prod_{i=1}^{M}\int [dq]\left[D_a(-q_i)e^{i\lambda\cdot\mathbf{q_i}}\tilde{O}_i(q_i)\right] e^{-i\sum_{i>j}\mathbf{q_i}\wedge\mathbf{q_j}} \tag{E.39}$$

using relations derived above we have

$$D_a(-q_1)\cdot\ldots\cdot D_a(-q_M) = e^{i\sum_{i>j} q_i\wedge q_j} D_a(-q_1-\ldots-q_M) = e^{i\sum_{i>j} q_i\wedge q_j} D_a(-Q) \tag{E.40}$$

Plugging this back in

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M} O_i(x_i)\right] = \pi\left(\frac{2}{l^2}\right)^M \int \frac{d^2\lambda}{(2\pi)^2} \prod_{i=1}^{M}\int [dq]\left[e^{i\lambda\cdot\mathbf{q_i}}\tilde{O}_i(q_i)\right] \tag{E.41}$$

This is the generalization of the b-summation formula used in Chapter 2.

Appendix F

Laguerre polynomial identity

Here we derive a property of Laguerre polynomials used in the main text.

Functions $e^{-\frac{|kl|^2}{4}} L_m^0\left(\frac{|kl|^2}{2}\right)$ form a complete orthonormal basis in a functional space on a half-line with measure $d\frac{|kl|^2}{2}$. This implies (together with $L_m(0) = 1$).

$$\sum_m e^{-\frac{|kl|^2}{4}} L_m^0\left(\frac{|kl|^2}{2}\right) = \delta^{(1)}\left(\frac{|kl|^2}{2}\right), \tag{F.1}$$

Let's prove this. To simplify things let's say that functions $f_n(x)$ form a complete orthonormal ($\int dx f_n(x) f_m(x) = \delta_{n,m}$) basis and that $f_n(0) = 1$. Then consider a sum

$$\sum_n f_n(x) \tag{F.2}$$

as a generalized function. Let's integrate this functions against some trial continuous function that can also be expanded in the same basis $g(x) = \sum c_n f_n(x)$.

$$\int dx \sum_n f_n(x) g(x) = \sum_{n,m} c_n \int dx f_n(x) f_m(x) = \sum_n c_n \tag{F.3}$$

on the other hand lets integrate $g(x)$ against a delta function $\delta(x)$

$$\int dx \delta(x) g(x) = \sum_n c_n \int dx f_n(x) \delta(x) = \sum_n c_n \tag{F.4}$$

thus we conclude that is a sense of generalized functions

$$\sum_n f_n(x) = \delta(x). \tag{F.5}$$

Appendix G

Galilean symmetry in Newton-Cartan geometry

In this Appendix we briefly summarize the covariant description of Galilean symmetry given in [30]. Then Newton-Cartan structure on a manifold is a pair of $(n_\mu, h^{\mu\nu})$. Where n is a 1-form that specifies the time direction and $h^{\mu\nu}$ is degenerate "metric", it contains the information about metric on a spatial slice. These structures satisfy the relations

$$v^\mu n_\mu = 1, \qquad h^{\mu\nu} n_\mu = 0\,, \tag{G.1}$$

where v^μ is a vector field, dual to n_μ. We also define projector $P^\nu_\mu = \delta^\mu_\nu - v^\mu n_\mu$.

There is a field transformation that preserves these constraints called Milne transformation

$$v'^\mu = v^\mu + h^{\mu\nu}\psi_\nu \tag{G.2}$$

$$A'_\mu = A_\mu + P^\nu_\mu \psi_\nu - \frac{1}{2} n_\mu h^{\nu\rho}\psi_\nu\psi_\rho\,, \tag{G.3}$$

where ψ is the parameter of the transformation.

The action

$$S = \int dV \left[\frac{i}{2} v^\mu (\Psi^\dagger \partial_\mu \Psi - \partial_\mu \Psi^\dagger \Psi) - \frac{h^{\mu\nu}}{2m} \partial_\mu \Psi^\dagger \partial_\nu \Psi \right]\,, \tag{G.4}$$

is, in fact, Milne invariant.

To see how it works in detail we can re-write the action as

$$S = \int dV \Big(- \frac{m}{2} \left(h^{\mu\nu} A_\mu A_\nu - 2v^\mu A_\mu\right) \Psi^\dagger \Psi + \frac{i}{2}(v^\mu - h^{\mu\nu} A_\nu)\Psi^\dagger \partial_\mu \Psi - \frac{h^{\mu\nu}}{2m} \partial_\mu \Psi^\dagger \partial_\nu \Psi \Big) \tag{G.5}$$

The scalar $h^{\mu\nu} A_\mu A_\nu - 2v^\mu A_\mu$, vector $v^\mu - h^{\mu\nu} A_\nu$ and tensor $h^{\mu\nu}$ are explicitly Milne invariant (but not all gauge invariant).

It is an unsolved problem to construct an induced action invariant with respect to Milne transformations to arbitrary order in fields and gradients.

Appendix H

Chiral boson from gaussian free field

In the main text we have derived the chiral boson edge theory from the effective Chern-Simons theory. Alternatively, it is possible to derive the chiral boson theory starting from the non-chiral boson and restricting it to one of the sectors. We discuss the derivation in this Appendix.

H.1 Electro-mangetic field

The topological bulk action of the quantum Hall system is known to be the Chern-Simons theory.

$$S_M = \frac{\sigma_H}{2} \int dt d^2x\, A \wedge dA = \frac{\sigma_H}{2} \int dt d^2x \sqrt{g}\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{H.1}$$

where $\epsilon^{123} = \frac{1}{\sqrt{g}}$. On a closed manifold this action is gauge invariant. On a manifold with boundary this action is not invariant and requires a boundary theory that compensates the non-invariance. Let's choose the boundary to be a line $y = 0$. Then

$$\begin{aligned} \delta_\alpha S_M &= \frac{\sigma_H}{2} \int dt d^2x\; d(\alpha dA) = \frac{\sigma_H}{2} \int_{y=0} dt dx\; \alpha dA = \frac{\sigma_H}{2} \int_{y=0} dt dx \sqrt{g}\; \alpha \epsilon^{ij} \partial_i A_j \\ &= \frac{\sigma_H}{2} \int_{y=0} dt dx \sqrt{g} \alpha E \end{aligned} \tag{H.2}$$

or

$$\delta S_M = \frac{\sigma_H}{2} \int \alpha E, \tag{H.3}$$

where E is defined below.

We choose the following boundary theory:

$$S_{\partial M} = \frac{1}{\pi}\int dtdx\left(\frac{1}{2}(\partial_+\phi - qA_+)(\partial_-\phi - qA_-) + \gamma_l E_l\phi + \gamma_r E_r\phi\right) + I[A], \tag{H.4}$$

where we defined $E = \epsilon^{ij}\partial_i A_j = \partial_+ A_- - \partial_- A_+ = E_l + E_r = F_{01}$ and $x_+ = x_1 + x_2$ and $x_- = x_1 - x_2$ and $\partial_+ x_+ = \partial_- x_- = 1$. So that $2\partial_+ = \partial_1 - \partial_2$ and $2\partial_- = \partial_1 + \partial_2$. We also have $A_\pm = A_1 \mp A_2$. Also, $I[A]$ is a **local** counter term to be chosen later.

Under a gauge transformation we have $\delta E_l = \partial_+ A_- = \partial_+\partial_-\alpha = \partial^2\alpha$ and $\delta E_r = \partial_- A_+ = \partial_+\partial_-\alpha = \partial^2\alpha$. Also, the boson field ϕ transforms as a phase $\delta\phi = \alpha$.

We re-write the action as follows.

$$S_{\partial M} = \frac{1}{\pi}\int \frac{1}{2}\partial_+\phi\partial_-\phi + \left((\gamma_l + \frac{q}{2})E_l + (\gamma_r - \frac{q}{2})E_r)\right)\phi + \frac{q^2}{2}A_+A_- + I[A] \tag{H.5}$$

Now we perform a shift of the integration variable $\phi \to \phi + X$. We have

$$\begin{aligned} S_{\partial M} &= \frac{1}{\pi}\int \frac{1}{2}\partial_+\phi\partial_-\phi - \qquad \text{(H.6)}\\ &- \left(\partial^2 X\right)\phi + \left((\gamma_l + \frac{q}{2})E_l + (\gamma_r - \frac{q}{2})E_r)\right)\phi - \\ &- \frac{1}{2}X\partial^2 X + \left((\gamma_l + \frac{q}{2})E_l + (\gamma_r - \frac{q}{2})E_r)\right)X + \frac{q^2}{2}A_+A_- + I[A] \end{aligned}$$

Eliminating the linear term in ϕ we get

$$X = \frac{1}{\partial^2}\left((\gamma_l + \frac{q}{2})E_l + (\gamma_r - \frac{q}{2})E_r)\right) \tag{H.7}$$

With this condition we finally obtain

$$\begin{aligned} S_{\partial M} &= \frac{1}{\pi}\int \frac{1}{2}\partial_+\phi\partial_-\phi + \qquad \text{(H.8)}\\ &+ \frac{1}{2}\left[\left(\gamma_l + \frac{q}{2}\right)E_l + (\gamma_r - \frac{q}{2})E_r)\right]\frac{1}{\partial^2}\left[\left(\gamma_l + \frac{q}{2}\right)E_l + (\gamma_r - \frac{q}{2})E_r)\right] \\ &+ \frac{q^2}{2}A_+A_- + I[A] \qquad \text{(H.9)} \end{aligned}$$

Integrating out the bosonic field we get for the normalized $W_{\partial M} = -i \ln \int D\phi e^{iS_{\partial M}}$

$$\begin{aligned} W_{\partial M} &= \frac{1}{2\pi}\left(\gamma_l + \frac{q}{2}\right)^2 \int E_l \frac{1}{\partial^2} E_l + \frac{1}{2\pi}\left(\gamma_r - \frac{q}{2}\right)^2 \int E_r \frac{1}{\partial^2} E_r \quad \text{(H.10)} \\ &+ \frac{1}{\pi}\left(\gamma_r - \frac{q}{2}\right)\left(\gamma_l + \frac{q}{2}\right) \int E_l \frac{1}{\partial^2} E_r + \frac{q^2}{2\pi} A_+ A_- + I[A] \quad \text{(H.11)} \end{aligned}$$

In this form action is not satisfactory for our purposes. We add and subtract terms to convert $\int E_l \frac{1}{\partial^2} E_l$ into $\int E_l \frac{1}{\partial^2} E$. We have

$$\begin{aligned} W_{\partial M} &= \frac{1}{2\pi}\left(\gamma_l + \frac{q}{2}\right)^2 \int E_l \frac{1}{\partial^2} E + \frac{1}{2\pi}\left(\gamma_r - \frac{q}{2}\right)^2 \int E_r \frac{1}{\partial^2} E + \quad \text{(H.12)} \\ &+ \frac{1}{\pi}\left\{\left(\gamma_r - \frac{q}{2}\right)\left(\gamma_l + \frac{q}{2}\right) - \frac{1}{2}\left(\gamma_l + \frac{q}{2}\right)^2 - \frac{1}{2}\left(\gamma_r - \frac{q}{2}\right)^2\right\} \int E_l \frac{1}{\partial^2} E_r \\ &+ \frac{q^2}{2} A_+ A_- + I[A] \quad \text{(H.13)} \end{aligned}$$

Now we simplify the last line and find our counter term

$$\pi I[A] = (\gamma_r - \gamma_l)\left\{\frac{1}{2}(\gamma_r - \gamma_l) - q\right\} \int E_l \frac{1}{\partial^2} E_r \quad \text{(H.14)}$$

Also, upon taking $\gamma_r = \frac{q}{2}$ it takes the form

$$\pi I[A] = \left(\gamma_l^2 - \frac{q^2}{4}\right) \int E_l \frac{1}{\partial^2} E_r = \left(\gamma_l^2 - \frac{q^2}{4}\right) A_+ A_- \quad \text{(H.15)}$$

Plugging this back into the action we arrive at the final form of the action and effective action

$$\begin{aligned} S_{\partial M} &= \frac{1}{\pi} \int \frac{1}{2}\partial_+\phi\partial_-\phi + \left(\gamma_l + \frac{q}{2}\right) E_l \phi + \left(\gamma_l^2 + \frac{q^2}{4}\right) A_+ A_- \quad \text{(H.16)} \\ W_{\partial M} &= \frac{1}{2\pi}\left(\gamma_l + \frac{q}{2}\right)^2 \int E_l \frac{1}{\partial^2} E \quad \text{(H.17)} \end{aligned}$$

The corresponding Chern-Simons term in the bulk is

$$W_M = \frac{\sigma_H}{2} \int AdA \quad \text{(H.18)}$$

We demand

$$\delta(W_M + W_{\partial M}) = \frac{1}{2\pi}\left(\gamma_l + \frac{q}{2}\right)^2 \int \alpha E - \frac{\sigma_H}{2} \int \alpha E = 0 \quad \text{(H.19)}$$

or

$$\sigma_H = \frac{1}{\pi}\left(\gamma_l + \frac{q}{2}\right)^2 \tag{H.20}$$

We can choose $q = 0$ and get

$$\sigma_H = \frac{1}{\pi}\gamma_l^2 \tag{H.21}$$

For FQHE with ν^{-1} we have $\gamma_l = \sqrt{\frac{\nu^{-1}}{2}}$.

H.2 Turning on gravity

Just like before we start with the following boundary theory, but we set $q = 0$ from the start.

$$S_{\partial M} = \frac{1}{\pi}\int dtdx\left(\frac{1}{2}\nabla_+\phi\nabla_-\phi + \gamma_l E_l\phi + \gamma_r E_r\phi + \beta_l R_l\phi + \beta_r R_r\phi\right) + I[A,\omega], \tag{H.22}$$

where we have defined

$$\begin{aligned} \nabla_\pm &= E^\mu_\pm\partial_\mu + \omega_\pm\hat{J}, \quad \omega_\pm = \mp\frac{1}{e}\partial_\mu\left(eE^\mu_\pm\right), \quad E = (\nabla_+A_- - \nabla_-A+) = E_l + E_r \\ R &= \nabla_+\omega_- - \nabla_-\omega_+ = R_l + R_r, \end{aligned} \tag{H.23}$$

Then

$$\begin{aligned} S_{\partial M} &= \frac{1}{\pi}\int\frac{1}{2}\nabla_+\phi\nabla_-\phi - \\ &- \left(\nabla^2 X\right)\phi + \left((\gamma_l + \frac{q}{2})E_l + (\gamma_r - \frac{q}{2})E_r + \beta_l R_l + \beta_r R_r)\right)\phi - \\ &- \frac{1}{2}X\nabla^2 X + \left((\gamma_l + \frac{q}{2})E_l + (\gamma_r - \frac{q}{2})E_r) + \beta_l R_l + \beta_r R_r\right)X + \frac{q^2}{2}A_+A_- + I[A] \end{aligned} \tag{H.24}$$

we choose X in order to cancel the terms linear in ϕ.

$$X = \frac{1}{\nabla^2}\left((\gamma_l + \frac{q}{2})E_l + (\gamma_r - \frac{q}{2})E_r) + \beta_l R_l + \beta_r R_r\right) \tag{H.25}$$

Plugging this in we have

$$\begin{aligned} S_{\partial M} &= \frac{1}{\pi}\int \frac{1}{2}\nabla_+\phi\nabla_-\phi + \\ &+ \frac{1}{2}\left[\left(\gamma_l+\frac{q}{2}\right)E_l + \left(\gamma_r-\frac{q}{2}\right)E_r + \beta_l R_l + \beta_r R_r)\right]\frac{1}{\nabla^2} \\ &\quad \times\left[\left(\gamma_l+\frac{q}{2}\right)E_l + \left(\gamma_r-\frac{q}{2}\right)E_r + \beta_l R_l + \beta_r R_r\right] \\ &+ \frac{q^2}{2}A_+A_- + I[A] \end{aligned}$$

Integrating over ϕ we get the effective action

$$\begin{aligned} W_{\partial M} &= \frac{1}{2\pi}\left(\gamma_l+\frac{q}{2}\right)^2\int E_l\frac{1}{\nabla^2}E_l + \frac{1}{2\pi}\left(\gamma_r-\frac{q}{2}\right)^2\int E_r\frac{1}{\nabla^2}E_r + \quad \text{(H.26)} \\ &+ \frac{1}{2\pi}\left(\frac{1}{12}+\beta_l^2\right)\int R_l\frac{1}{\nabla^2}R_l + \frac{1}{2\pi}\left(\frac{1}{12}+\beta_r^2\right)\int R_r\frac{1}{\nabla^2}R_r + \\ &+ \frac{1}{\pi}\beta_l\left(\gamma_l+\frac{q}{2}\right)\int E_l\frac{1}{\nabla^2}R_l + \frac{1}{\pi}\beta_r\left(\gamma_r-\frac{q}{2}\right)\int E_r\frac{1}{\nabla^2}R_r \\ &+ \frac{1}{\pi}\beta_l\left(\gamma_r-\frac{q}{2}\right)\int E_r\frac{1}{\nabla^2}R_l + \beta_r\left(\gamma_l+\frac{q}{2}\right)\int E_l\frac{1}{\nabla^2}R_r \\ &+ \frac{1}{\pi}\left(\gamma_r-\frac{q}{2}\right)\left(\gamma_l+\frac{q}{2}\right)\int E_l\frac{1}{\nabla^2}E_r + \frac{1}{\pi}\beta_l\beta_r\int E_l\frac{1}{\nabla^2}E_r \quad \text{(H.27)} \\ &+ \frac{q^2}{2\pi}A_+A_- + I[A,\omega] \end{aligned}$$

We will ignore the counter terms for now and proceed with setting, $\beta_r^2 = -\frac{1}{12}$ and $\alpha_r = \frac{q}{2}$. Then

$$\begin{aligned} W_{\partial M} &= \frac{\left(\gamma_l+\frac{q}{2}\right)^2}{2\pi}\int E_l\frac{1}{\nabla^2}E + \frac{\beta_l\left(\gamma_l+\frac{q}{2}\right)}{2\pi}\int\left(E_l\frac{1}{\nabla^2}R + R_l\frac{1}{\nabla^2}E\right) \\ &+ \frac{1}{2\pi}\left(\frac{1}{12}+\beta_l^2\right)\int R_l\frac{1}{\nabla^2}R \end{aligned} \quad \text{(H.28)}$$

Finally, setting $\gamma_l = 0$ and $\beta_l = 0$. We also assumed that we chose a counter term $I[A,\omega]$ appropriately. Then the induced action takes form

$$W_{\partial M} = \frac{\nu}{4\pi}\int E_l\frac{1}{\Delta}E + \frac{1}{48\pi}\int R_l\frac{1}{\Delta}R \quad \text{(H.29)}$$

with $\nu = \frac{q^2}{2}$.

In the bulk we write

$$W_M = \frac{\nu}{4\pi}\int AdA - \frac{1}{96\pi}\int d^3x \mathrm{Tr}\left[\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right] \tag{H.30}$$

The total induced action is diffeomorphism and gauge invariant.