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Eikonal Scattering at Strong Coupling

A Dissertation Presented

by

Melvin Eloy Irizarry-Gelpí

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Abstract of the Dissertation

Eikonal Scattering at Strong Coupling

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The scattering of subatomic particles is a source of important physical phenomena. Decades of work have yielded many techniques for the computation of scattering amplitudes. Most of these techniques involve perturbative quantum field theory and thus apply only at weak coupling. Complementary to scattering is the formation of bound states, which are intrinsically nonperturbative. Regge theory arose in the late 1950s as an attempt to describe, with a single framework, both scattering and the formation of bound states. In Regge theory one obtains an amplitude with bound state poles after analytic continuation of a nonperturbative scattering amplitude, corresponding to a sum of an infinite number of Feynman diagrams at large energy and fixed momentum transfer (but with crossed kinematics). Thus, in order to obtain bound states at fixed energy, one computes an amplitude at large momentum transfer.

In this dissertation we calculate amplitudes with bound states in the regime of fixed energy and small momentum transfer. We formulate the elastic scattering problem in terms of many-body path integrals, familiar from quantum mechanics. Then we invoke

the semiclassical JWKB approximation, where the path integral is dominated by classical paths. The dynamics in the semiclassical regime are strongly coupled, as found by Halpern and Siegel. When the momentum transfer is small, the classical paths are simple straight lines and the resulting semiclassical amplitudes display a spectrum of bound states that agrees with the spectrum found by solving wave equations with potentials. In this work we study the bound states of matter particles with various types of interactions, including electromagnetic and gravitational interactions. Our work has many analogies with the work started by Alday and Maldacena, who computed scattering amplitudes of gluons at strong coupling with semiclassical quantum mechanics of strings in anti de-Sitter spacetime. We hope that in the future we can apply our methods to nonabelian matter and better understand bound states in quantum chromodynamics.

A mi familia y mis amistades, por el interminable apoyo y cariño.

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Chapter 1

Introduction

One of the main goals in physics is to understand the interactions between physical objects. Before the 20th century, all theories in physics involve classical objects. The motion of any classical object is, in principle, determined by the solution of its classical equation of motion. The solution of this problem yields a deterministic description of the motion: given initial information about the position and velocity of the object, in principle, its later position and velocity can be predicted with complete certainty[1].

But the set of physical objects was drastically enlarged at the beginning of the 20th century with the discovery of quantum theory. It was quickly realized that the atomic and subatomic "quantum objects" could not be described with the classical, deterministic theories developed earlier. A new description, based on a probabilistic interpretation, emerged after the work of Bohr, Schrödinger, Heisenberg, Dirac, and many others. Later, Dirac and Feynman developed a formulation of quantum theory that is based on functional integration. In this approach, the probability amplitude for a particle to travel from a point with position $\mathbf{x}_I$ to a point with position $\mathbf{x}_O$ involves computing a certain phase factor for a given path $\mathbf{q}(t)$ that connects the two points, and then adding the contributions from all possible paths. This so-called "path integral" formulation is intuitive since it "builds up" the quantum problem with classical ingredients (like paths in space).

The quantum analog of scattering phenomena offers an important link between theory and experiment. Both the experimental and the theoretical study of scattering are quite challenging tasks. In this dissertation we will only follow the theoretical route[2].

In a theoretical scattering event, an *incoming* set of quanta is made to

[1] Of course, as long as chaotic dynamics are not involved.

[2] There are many exciting experiments currently taking place. A big thanks to all experimentalists, for performing such an excellent job.

interact and produce an *outgoing* set of quanta. The incoming set starts out isolated, and, after waiting a very long time, the outgoing set also becomes isolated. These two sets of external quanta are, in principle, different. That is, we can have an event like

$$a(p_1) + b(p_2) \longrightarrow c(p_3) + d(p_4) \tag{1.1}$$

where an a quantum[3] with momentum p_1, and a b quantum with momentum p_2 interact and scatter into a c quantum with momentum p_3, and a d quantum with momentum p_4. We can have any number of incoming quanta, and any number of outgoing quanta.

Bound state phenomena are complementary to scattering phenomena. One might think (incorrectly) that the physics of bound states has nothing to do with the physics of scattering. Bound states can form outside of the region of values that momenta can take in a scattering experiment. A bound state certainly does not satisfy the definition of a scattering event: the incoming or outgoing quanta are certainly not isolated! But, starting with the work of Regge [1, 2], it was realized that under the right conditions, one could study both scattering phenomena *and* bound state phenomena with the same framework.

The scattering of quantum matter provides a theoretical arena where many physicists (and some mathematicians) have done battle with the difficulties inherent of the quantum theory. In order to make progress, like in most problems, approximations have to be made. For example, in Regge theory (see §3.1), one keeps the momentum transfer fixed while taking the center-of-momentum energy to be very large in the crossed channel. In this dissertation we will use a different set of approximations.

We consider *four-point elastic* scattering, where we have *two incoming* quanta, and *two outgoing* quanta of the same kind as the incoming quanta. That is, we only consider scattering events of the form

$$a(p_1) + b(p_2) \longrightarrow a(p_3) + b(p_4) \tag{1.2}$$

This type of scattering events have another interpretation: the propagation of a *bound state*,

$$ab \longrightarrow ab \tag{1.3}$$

We use a relativistic, two-body quantum-mechanical path integral $\mathcal{F}$ to de-

[3]By an x quantum, we mean a unit of matter of type x. Quantum matter has particle-like and wave-like behavior.

scribe the elastic event (1.2):

$$\mathcal{F}(3,4|1,2) = \int_{x_1}^{x_3} \mathrm{D}q_a(\tau) \int_{x_2}^{x_4} \mathrm{D}q_b(\sigma) \exp\left(-iS[q_a, q_b]\right) \tag{1.4}$$

That is, the a and b quanta behave like *particles*. Quantum theory requires us to consider *all* possible spacetime paths $q_a(\tau)$ that connect the a quantum at position x_1 to the a quantum at position x_3, and also *all* possible spacetime paths $q_b(\sigma)$ that connect the b quantum at position x_2 to the b quantum at position x_4.

The first approximation that we use is the *semiclassical approximation*, where we extract from $\mathcal{F}$ the contribution from the pair of classical paths $\bar{q}_a(\tau)$ and $\bar{q}_b(\sigma)$:

$$\mathcal{F}(3,4|1,2) \quad \xrightarrow[\text{approximation}]{\text{semiclassical}} \quad \mathcal{V}(3,4|1,2) = \sqrt{-\det\left(V\right)} \exp\left(-i\Sigma\right) \tag{1.5}$$

Here $\Sigma = S[\bar{q}_a, \bar{q}_b]$ and the matrix V is a 2×2 array of spacetime matrices,

$$V = \begin{pmatrix} V_{13} & V_{23} \\ V_{14} & V_{24} \end{pmatrix}, \qquad (V_{jk})_{mn} = -i \frac{\partial^2 \Sigma}{\partial x_j^m \partial x_k^n} \tag{1.6}$$

This is the familiar JWKB approximation. As we discuss in chapter 6, the semiclassical approximation in the quantum-mechanical path integral leads to a *strong-coupling expansion*, in the sense that the result at the end of the calculation does not correspond to a specific perturbative order. This is already a promising sign that we are *en ruta* to study bound states.

In order to find $\mathcal{V}$, we must find $\bar{q}_a(\tau)$ and $\bar{q}_b(\sigma)$ by solving the classical equations of motion obtained from the action functional S that appears in $\mathcal{F}$. The classical solution is elusive for the kind of systems that we consider, so we use another approximation: we restrict the kinematical regime to *small-angle scattering*. In this regime, the classical paths of the particles can be nicely approximated by the eikonal paths, $e_a(\tau)$ and $e_b(\sigma)$, which describe straight paths in spacetime. Thus,

$$\mathcal{V}(3,4|1,2) \quad \xrightarrow[\text{scattering}]{\text{small-angle}} \quad \mathcal{E}(3,4|1,2) = \sqrt{-\det\left(V_{\text{eik}}\right)} \exp\left(-i\Sigma_{\text{eik}}\right) \tag{1.7}$$

where $\Sigma_{\text{eik}} = S[e_a, e_b]$ and V_{eik} is defined in the same way as V, but with derivatives of Σ_{eik} instead of Σ.

To summarize, we use two approximations in order to evaluate the quantum path integral,

$$\mathcal{F}(3,4|1,2) \quad \xrightarrow[\text{approximation}]{\text{semiclassical}} \quad \mathcal{V}(3,4|1,2) \quad \xrightarrow[\text{scattering}]{\text{small-angle}} \quad \mathcal{E}(3,4|1,2) \tag{1.8}$$

In four-dimensional Minkowski spacetime, these approximations yield a result for the scattering amplitude that contains all perturbative orders in the coupling parameter, and have an infinite number of singularities. After analytic continuation of the momenta, from the physical scattering region to the bound state region, these singularities can be identified with bound states. The form of the scattering amplitude thus obtained exhibit Regge behavior, but it should be noted that we do not need to take the high-energy limit. That is, our results arrive at Regge amplitudes *without* taking the Regge limit.

Every scattering event that we consider has *massive* external particles. The interactions between the particles are analogous to exchanging massless or massive quanta. We consider three types of massless exchanges: scalar, vector (photons) and tensor (linearized gravitons). In $D = 4$, the result for the nonperturbative scattering amplitude with any massless exchange have the same general form:

$$\mathcal{A}_{\text{tree}}(s,t)\exp\left[\alpha\Gamma(\epsilon)\rho(s)\right]\frac{\Gamma[1-\alpha\rho(s)]}{\Gamma[1+\alpha\rho(s)]}\left(-\frac{t}{2\mu^2}\right)^{\alpha\rho(s)} \qquad \epsilon = \frac{D-4}{2} \tag{1.9}$$

Here $\mathcal{A}_{\text{tree}}$ is the tree level amplitude, due to the exchange of a single massless quantum; α is a coupling parameter, and ρ is a function of the center-of-mass energy s that is imaginary inside the physical scattering region. The particular form of ρ depends on the nature of the massless exchange quanta. We also consider the exchange of a massive scalar. The results with the massive scalar do not exhibit bound states in $D = 4$, but in $D = 3$ the result contains an Euler Gamma function whose singularities correspond to the multi-particle branch points in the t-channel.

The use of the quantum-mechanical semiclassical approximation to obtain nonperturbative scattering amplitudes is not new. In its most recent application, it can be found in the program started by Alday & Maldacena [3] who computed a four-point amplitude of gluons in $\mathcal{N} = 4$ super Yang-Mills theory at strong-coupling by considering semiclassical strings in five-dimensional anti de-Sitter spacetime. This is another example of using semiclassical *mechanical* objects (the strings) to obtain a nonperturbative amplitude. It is somewhat comforting to find that Nature has allowed classical methods to still play a useful role in the description of its deepest quantum secrets.

Chapter 2

Dissertation Outline

The main results of this work are scattering amplitudes for elastic events obtained after using the relativistic eikonal JWKB approximation. These results are contained in chapter 8. We consider scattering due to scalar, electromagnetic and gravitational interactions.

Before we get there, first we briefly review nonrelativistic quantum mechanics, the semiclassical (JWKB) approximation and the semiclassical eikonal approximation, which as we shall see in chapter 4, is a special case of the semiclassical approximation.

After reviewing the nonrelativistic theory, we present an example calculation in chapter 5 where we compute a four-point scattering amplitude for matter particles interacting via the exchange of an instantaneous scalar wave. This system is equivalent to the problem of Coulomb scattering. Our result for the amplitude will exhibit an infinite set of singularities that correspond to two-body bound states. The steps we follow will generalize later to the relativistic theory. Indeed, this calculation will serve as a preview of the difficulties that we will encounter in the relativistic calculation.

Then we will boost into the relativistic theory. In chapter 6 we begin by reviewing the action functionals for relativistic particles, and discuss the couplings of particles to different kinds of fields. We will also discuss the difference between coupling fields to other fields, and coupling fields to particles.

After this, in chapter 7, we introduce the relativistic analogs of the concepts reviewed in chapter 4. This chapter introduces all the ingredients that are used in chapter 8.

There are many different directions along which our work can be continued further. We discuss some of these in chapter 9.

We also include many appendices. In appendix A we discuss the quantum and semiclassical kernels for free scalar particles. We discuss both massive and massless cases. Although free particles are not very interesting, this discussion will allow us to understand some of the differences between the quantum and the semiclassical kernels for massive particles. The calculations in this appendix are a good test of the validity of the tools introduced in chapter 7.

In appendix B we review the different momentum invariants that can be constructed with the external momentum vectors. This discussion is brief and general, but in appendix C we present a more detailed discussion relevant to the case with four external states (and specific to elastic scattering). In that appendix we also discuss different kinematical regimes, including the Regge limit and small-angle scattering.

Appendices D and E consist of a collection of well-known results involving the Euler Gamma and Beta functions, the Riemann zeta function, binomial combinatorics and Fourier transforms in arbitrary dimensions. Some of these results are used in the main body of work, but most are included for fun.

But before any of this, we begin with a historical overview in chapter 3.

Chapter 3

Historical Overview

In this chapter we first give a brief overview of scattering at high-energy. We mention the observation, first made by Regge, that the high-energy behavior of the theory in a particular scattering channel determines the bound state spectrum of the theory in a crossed channel. Then we discuss briefly a formalism that uses first-quantization to obtain perturbative scattering amplitudes. We also include a short discussion of the work started by Alday & Maldacena, involving the use of the AdS/CFT correspondence for the computation of scattering amplitudes at strong coupling.

3.1 Regge Scattering

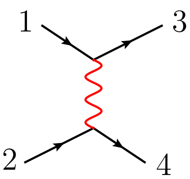
In order to be concrete, we will consider the elastic event

$$a(p_1) + b(p_2) \longrightarrow a(p_3) + b(p_4) \tag{3.1}$$

For simplicity, we will assume that a massless scalar field is being exchanged by the massive a and b quanta. The traditional way to study scattering of relativistic matter is to use perturbative quantum field theory[1]. In this approach, one assumes that the coupling parameter that describes the strength of the interaction is *small*. This allows us to break the event (3.1) into perturbative contributions, where at lowest order in perturbations one has the least amount of interactions. As the order of perturbation increases, one consider more complicated events with greater number of interactions. With the aid of Feynman graphs, one can assign a picture to each of these perturbative contributions.

[1]See [4, 5, 6] for some textbooks on the subject.

At lowest order in perturbations, the only connected Feynman graph is



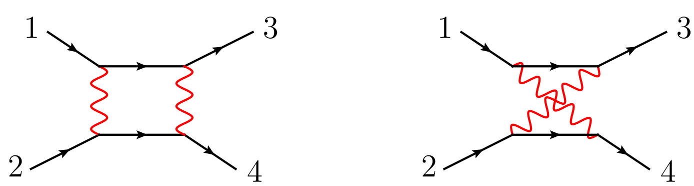
(3.2)

Since this graph has no loops, it is referred to a the tree graph. The tree graph translates to the expression

$$-\frac{\alpha}{t}\delta(p_1 + p_2 - p_3 - p_4) \tag{3.3}$$

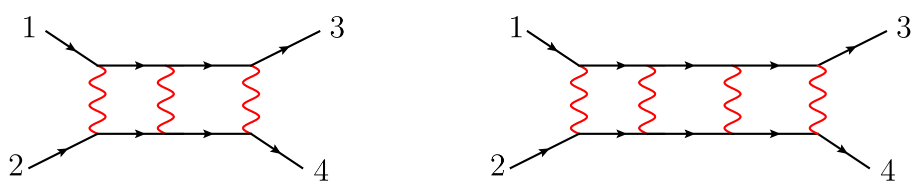
Here α is the coupling parameter and $t = -(p_1 - p_3)^2$ parametrizes the momentum transfer between the two external states. The Dirac delta is in charge of making sure that the total momentum that is incoming (given by $p_1 + p_2$) equals the total momentum that is outgoing (given by $p_3 + p_4$).

The next order in perturbations involve many contributions. For our discussion, the relevant contributions come from the box graph and the crossed box graph:



(3.4)

These graphs are examples of ladder graphs. Other examples include



(3.5)

Each of the graphs in (3.4) has one closed loop, so they are referred to as one-loop graphs. The box graph translates to the expression

$$\alpha^2 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathrm{d}f_{31}\mathrm{d}f_{42}\mathrm{d}f_{12}\mathrm{d}f_{34} \left[\frac{\delta(1 - f_{12} - f_{34} - f_{31} - f_{42})}{B(s, t|f_{ij})}\right] \tag{3.6}$$

with

$$B(s, t|f_{ij}) = m_a^2 f_{31}^2 + m_b^2 f_{42}^2 + (m_a^2 + m_b^2 - s) f_{31} f_{42} - t f_{12} f_{34} \tag{3.7}$$

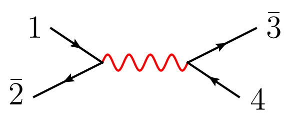
Here $s = -(p_1 + p_2)^2$ parametrizes the squared center-of-momentum energy. Expression (3.6) is valid for *generic* values of the kinematical variables s and t, as long as they are *inside* of the physical scattering region (see appendix C). However, the contribution from the box graph involves a nontrivial integration. With considerable effort, the integrals can be evaluated exactly [7, 8].

At higher-orders in perturbations, the challenges are much greater. In the face of inherent mathematical difficulties, one can always use approximations. One type of approximation involves restricting to a certain kinematical regime. For example, we can study the limit $t \to \infty$ while keeping s, and the masses m_a and m_b, fixed. At first glance, this limit appears to be in the wrong direction: t is negative inside of the physical scattering region! If we want to study scattering phenomena, it makes a lot of sense to stay inside of the physical scattering region. But the $t \to \infty$ limit allows us to study another kind of phenomena.

Besides the scattering event (3.1), we can also have the event

$$a(p_1) + \bar{a}(\bar{p}_2) \longrightarrow \bar{b}(\bar{p}_3) + b(p_4) \tag{3.8}$$

The Feynman graphs for this event are related to the graphs for event (3.1) by a rotation and a relabelling of the external states:



(3.9)

Indeed, event (3.8) follows from event (3.1) after crossing the incoming b state with the outgoing a state, and setting

$$\bar{p}_2 = -p_3, \qquad \bar{p}_3 = -p_2 \tag{3.10}$$

For event (3.8), the center-of-momentum energy is

$$\bar{s} = -(p_1 + \bar{p}_2)^2 = -(p_1 - p_3)^2 = t \tag{3.11}$$

and the momentum transfer is

$$\bar{t} = -(p_1 - \bar{p}_3)^2 = -(p_1 + p_2)^2 = s \tag{3.12}$$

Thus, we can use the same kinematical variables (the four momentum vectors p_j) to describe both events (3.1) and (3.8). However, the physical scattering region for each event do not overlap: $\bar{s} > 0$ implies $t > 0$ in event (3.8), but $t < 0$ in event (3.1). In particular, the high-energy limit $\bar{s} \to \infty$ corresponds to the unphysical limit $t \to \infty$ that we mentioned earlier. The cosine of the scattering angle in event (3.1) is

$$z_s \equiv \cos(\theta_s) = 1 + \frac{2st}{[s-(m_a-m_b)^2][s-(m_a+m_b)^2]} \tag{3.13}$$

Thus, $z_s \to \infty$ as $t \to \infty$. This suggest that the scattering angle θ_s becomes *complex*, which leads one to believe that the corresponding conjugate variable to the scattering angle, the angular momentum, also becomes complex. This argument lead Regge to promote the orbital angular momentum in quantum mechanics to a complex variable [1, 2].

In the $\bar{s} \to \infty$ limit, one can use asymptotic methods to evaluate some of the integrals in (3.6):

$$\frac{\alpha}{t}[\alpha\rho(\bar{t})]\log\left(\frac{\bar{s}}{2\mu^2}\right) \tag{3.14}$$

where

$$\rho(\bar{t}) = \int_0^1 \frac{\mathrm{d}f}{m_b^2 + (m_a^2 - m_b^2 - \bar{t})f + \bar{t}f^2} \tag{3.15}$$

(This result is only valid in $D = 4$.) Indeed, (3.14) agrees with the *exact* result for the scalar box with elastic kinematics, which can be found in [9]. For longer ladder graphs, the $\bar{s} \to \infty$ limit no longer yields the exact answer, but one can still obtain the leading behavior in the $\bar{s} \to \infty$ limit. After adding all ladder contributions [10], one obtains

$$\alpha\Gamma[-R(\bar{t})]\left(\frac{\bar{s}}{2\mu^2}\right)^{R(\bar{t})}, \qquad R(\bar{t}) = -1 + \alpha\rho(\bar{t}) \tag{3.16}$$

with $R(\bar{t})$ being complex. When an amplitude takes this form, it is said to show Regge behavior. This result corresponds to the *high-energy* ($\bar{s} \to \infty$) behavior of the ladder series for the event (3.8). But, by analytic continuation to the event (3.1), the same result corresponds to an amplitude outside of the physical scattering region ($t \to \infty$, or $z_s \to \infty$). In this region, the singularities of the Euler Gamma function are accessible, and thus correspond to physical states in the theory. These singularities are identified as bound states. Thus, the high-energy behavior of the scattering amplitude in one scattering event is responsible for the formation of bound states in another scattering event, related to the former by crossing. This is the main idea behind Regge theory.

The previous discussion is meant as a quick introduction to Regge scattering and the Regge limit. Since Regge theory yields information about the spectrum of bound states, it has direct phenomenological and experimental relevance. The topic is vast, and we do not have the time or the space to cover it properly. The curious reader may consult textbooks on Regge theory [11, 12], textbooks on high-energy scattering [13, 14, 15] and reviews [16, 17, 18, 19, 20] for more details.

3.2 Perturbative First-Quantization

We use first-quantized path integrals (i.e. functional integrals over mechanical variables, like paths) to compute nonperturbative scattering amplitudes. In this section we briefly discuss a formalism that uses first-quantization to compute *perturbative* scattering amplitudes.

Let us consider the following event: a massive scalar particle a with incoming momentum p_I and outgoing momentum p_O moves in spacetime and emits N massless scalar quanta. The n-th emitted massless quanta carries momentum k_n. At tree level, we can describe this event with the quantum mechanical amplitude

$$\mathcal{S}_T(O|I) = g^N \langle p_O|V(k_N)\cdots V(k_2)V(k_1)|p_I\rangle \tag{3.17}$$

Here $V(k_n)$ is a vertex operator that describes the emission of the n-th massless quanta and g is the coupling parameter. This amplitude can be rewritten as a path integral,

$$\mathcal{S}_T(O|I) = g^N \int\int \mathrm{d}x_I \mathrm{d}x_O \, \overline{\mathcal{W}}_O \mathcal{W}_I \int_{x_I}^{x_O} \mathrm{D}q_a(\tau)\, \exp\left(-iS_J[q_a]\right) \tag{3.18}$$

where

$$\mathcal{W}_I \equiv \exp\left(ix_I \cdot p_I\right) \qquad \overline{\mathcal{W}}_O \equiv \exp\left(-ix_O \cdot p_O\right) \tag{3.19}$$

and the action functional S_J is given by

$$S_J[q_a] = \int \mathrm{d}\tau \left[-\frac{1}{2}\dot{q}_a^2 + \frac{1}{2}m_a^2 - J\cdot q_a\right] \tag{3.20}$$

with the source J given by

$$J(\tau) = -\sum_{n=1}^{N} k_n \delta(\tau - \tau_n) \tag{3.21}$$

Note that

$$\tau_I < \tau_1 < \tau_2 < \ldots < \tau_N < \tau_O \tag{3.22}$$

The path integral in (3.18) can be evaluated *exactly* by using the semiclassical approximation. This involves finding the classical solution $q_*(\tau)$ by solving

$$\ddot{q}_* = J, \qquad q_*(\tau_I) = x_I, \qquad q_*(\tau_O) = x_O \tag{3.23}$$

and evaluating the action functional S_J at this classical solution. After obtaining the semiclassical path integral, and integrating over the $N+1$ moduli

$$T_{1I} = \tau_1 - \tau_I, \qquad T_{21} = \tau_2 - \tau_1, \qquad \dots \qquad T_{ON} = \tau_O - \tau_N \tag{3.24}$$

one recovers, after truncating two external a propagators, the familiar expression for an $(N+2)$-point tree level scattering amplitude with $N-1$ simple poles at m_a^2. For external scalars, this procedure might seem like a complicated way to obtain a tree level amplitude. The formalism can be applied to external states with spin and yields tree level amplitudes with arbitrary number of external states [21]. Indeed, this formalism is inspired by the methods used in string theory to compute scattering amplitudes [22, 23, 4]. Similar ideas can be used to compute one-loop amplitudes [24, 25, 26, 27, 28, 29].

While this formalism allows efficient *exact* computations, the results are still *perturbative*. One can argue that this formalism uses a classical solution to compute perturbative scattering amplitudes, and thus the semiclassical approximation does not necessarily leads to strong-coupling dynamics. This is somewhat misleading: the path integral (3.18) is Gaussian and hence the exact value happens to coincide with the semiclassical path integral.

3.3 Alday-Maldacena Theory

The AdS/CFT correspondence [30] provides a relation between two different theories: the four-dimensional conformal field theory $\mathcal{N}=4$ super Yang-Mills theory with gauge group $SU(N_c)$, and the ten-dimensional type IIB superstring theory in $AdS_5 \times S^5$ with N_c units of Ramond-Ramond five-flux (see [31] for a review). This relation is a "duality": it relates the *planar strong-coupling* regime of the CFT to the *semiclassical* regime of the superstring theory.

Alday & Maldacena [3] proposed a way to compute four-point scattering amplitudes of gluons in $\mathcal{N}=4$ SYM at strong-coupling, by using classical solutions of strings in AdS_5. As a string moves in AdS_5, it traces a surface in spacetime. Thus, solving the classical equations of motion for a string involves searching for a surface of minimal area in AdS_5. The scattering regime that Alday & Maldacena consider is *fixed-angle scattering*, where one takes all the kinematical variables to be very large while keeping all the ratios fixed (and thus, keeping the scattering angle fixed). Indeed, this calculation in AdS_5 is a generalization of a calculation done much earlier in flat spacetime [32, 33, 34].

As usual in string theory, the kinematical information of the external states is encoded in the boundary conditions of the classical solution. In the naive formulation of the problem, it seems hopeless to find a minimal surface in AdS_5 with the required boundary conditions (four punctures where external momentum is inserted into the worldsheet).

But Alday & Maldacena realized that after a change of variables (analogous to a noncompact T-duality), one could re-formulate the problem in terms of different boundary conditions. In terms of the new variables, the minimal surface ends on a particular polygon in spacetime with four lightlike edges, each corresponding to the external momentum of a gluon. This lightlike polygon was later identified with a Wilson loop living on the boundary of AdS_5. The classical string solution was found, and, after introducing a regulator to deal with infrared divergences, the area of the classical string solution (i.e. the minimal action) was shown to agree with a previous ansatz of Bern, Dixon & Smirnov [35] for the planar MHV scattering amplitude with four gluons.

The work of Alday & Maldacena uncovered a connection between planar MHV amplitudes and expectation values of Wilson loops (see [36, 37] for reviews). The change of variables that leads to the formulation of the problem in terms of the lightlike polygon signalled the existence of another copy of conformal symmetry, now referred to as *dual conformal symmetry*. The invariants of this symmetry are conformal ratios that involve momentum variables. This in turn lead to the realization that planar scattering amplitudes in $\mathcal{N} = 4$ SYM enjoy an infinite-dimensional symmetry in the guise of a Yangian [38, 39].

Further work includes higher-point amplitudes [40, 41], where it was found that the BDS ansatz was incomplete, since it failed to account for dual conformal symmetry. Other work makes use of integrable systems on the worldsheet to study amplitudes with any number of external gluons [42, 43, 44], the development of an operator product expansion for the perturbative study of lightlike Wilson loops [45, 46, 47], form factors at strong-coupling [48], and a study of the cusp anomalous dimension [49, 50, 51].

Quark scattering was also studied, both massless [52, 53] and massive [54]. In light of the results presented in this dissertation, the amplitude with quarks might be more relevant than the amplitude with gluons. The work in this dissertation started as an attempt to generalize Alday-Maldacena theory to other cases. We are still nowhere ready to do that. Indeed, we might even be late to the party [55]. Earlier work on scattering amplitudes via AdS/CFT include [56, 57, 58]. These references might be important and/or useful in developing a more general formalism that does not rely on the specific details of the Alday-Maldacena setup.

Chapter 4

Nonrelativistic Path Integrals

Before we discuss the relativistic theory, it will be fruitful to briefly venture into the nonrelativistic realm where we will find familiar results. The relativistic theory in later chapters is constructed in close analogy with the nonrelativistic topics discussed in this chapter.

4.1 Quantum Kernels

In nonrelativistic quantum mechanics we describe the state ψ_t of a system under consideration at time t as a vector[1] $|\psi_t\rangle$ in a Hilbert space. This means that we are working in the Heisenberg picture, where state kets do not carry the explicit time dependence. One of the main objects of interest is the amplitude for a state ψ_I at time $t = t_I$ (the "in" state) to transition to a state ψ_O at a later time $t = t_O > t_I$ (the "out" state). This amplitude is given by

$$A_{IO} = \frac{\langle\psi_O|\psi_I\rangle}{\sqrt{\langle\psi_O|\psi_O\rangle}\sqrt{\langle\psi_I|\psi_I\rangle}} \tag{4.1}$$

The denominator is required in order for

$$P_{IO} \equiv |A_{IO}|^2 \tag{4.2}$$

to have a probability interpretation with the correct normalization

$$0 \leq P_{IO} \leq 1 \tag{4.3}$$

After properly normalizing the state kets we can set this denominator to unity.

[1] Also referred to as a "state ket".

In practice we decompose a state vector into components along a convenient complete basis. The basis of choice is the position basis,

$$|\psi_t\rangle = \int \mathrm{d}x\, \psi(\mathbf{x},t)|\mathbf{x},t\rangle, \qquad \psi(\mathbf{x},t) \equiv \langle \mathbf{x},t|\psi_t\rangle \tag{4.4}$$

The position eigenkets are orthogonal

$$\langle \mathbf{x},t|\mathbf{y},t\rangle = \delta(\mathbf{x}-\mathbf{y}) \tag{4.5}$$

and form a complete set,

$$\int \mathrm{d}x\, |\mathbf{x},t\rangle\langle \mathbf{x},t| = 1 \tag{4.6}$$

Note that we have adopted the normalization conventions

$$\delta(\mathbf{x}) \equiv (2\pi)^{d/2}\, \delta^d(\mathbf{x}), \qquad \int \mathrm{d}x \equiv \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}} \tag{4.7}$$

where d is the number of spatial dimensions. Another useful basis is the momentum basis,

$$|\psi_t\rangle = \int \mathrm{d}p\, \hat{\psi}(\mathbf{p},t)|\mathbf{p},t\rangle, \qquad \hat{\psi}(\mathbf{p},t) \equiv \langle \mathbf{p},t|\psi_t\rangle \tag{4.8}$$

The momentum eigenkets are also orthogonal and form a complete set,

$$\langle \mathbf{p},t|\mathbf{q},t\rangle = \delta(\mathbf{p}-\mathbf{q}), \qquad \int \mathrm{d}p\, |\mathbf{p},t\rangle\langle \mathbf{p},t| = 1 \tag{4.9}$$

However, the normalization involves $\hbar$:

$$\delta(\mathbf{p}) \equiv \left(2\pi\hbar^2\right)^{d/2} \delta^d(\mathbf{p}), \qquad \int \mathrm{d}p \equiv \int \frac{\mathrm{d}^d p}{(2\pi\hbar^2)^{d/2}} \tag{4.10}$$

Position and momentum are conjugate quantities. This means that we can switch between them via a Fourier transform:

$$|\mathbf{p},t\rangle = \int \mathrm{d}x\, \exp\left(\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{p}\right)|\mathbf{x},t\rangle \tag{4.11}$$

Thus,

$$\langle \mathbf{x}, t | \mathbf{p}, t \rangle = \exp\left(\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{p}\right), \qquad \langle \mathbf{p}, t | \mathbf{x}, t \rangle = \exp\left(-\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{p}\right) \tag{4.12}$$

Note that these relations hold when the eigenkets are at the same instant in time.

After inserting a complete set of position eigenkets at time $t = t_I$ and $t = t_O$, we can write the inner product as

$$\langle \psi_O | \psi_I \rangle = \int\int \mathrm{d}x_I \mathrm{d}x_O \, \psi_O^*(\mathbf{x}_O, t_O)\mathcal{F}(O|I)\psi_I(\mathbf{x}_I, t_I) \tag{4.13}$$

where we have introduced the position basis **quantum kernel** $\mathcal{F}$, defined as

$$\mathcal{F}(O|I) \equiv \langle \mathbf{x}_O, t_O | \mathbf{x}_I, t_I \rangle \tag{4.14}$$

Here I and O are labels that denote the set of "in" and "out" variables. One can equivalently insert a complete set of momentum eigenkets at time $t = t_I$ and $t = t_O$,

$$\langle \psi_O | \psi_I \rangle = \int\int \mathrm{d}p_I \mathrm{d}p_O \, \hat{\psi}_O^*(\mathbf{p}_O, t_O)\widehat{\mathcal{F}}(O|I)\hat{\psi}_I(\mathbf{p}_I, t_I) \tag{4.15}$$

which leads to the momentum basis quantum kernel $\widehat{\mathcal{F}}$,

$$\widehat{\mathcal{F}}(O|I) \equiv \langle \mathbf{p}_O, t_O | \mathbf{p}_I, t_I \rangle \tag{4.16}$$

Due to the Fourier-Heisenberg conjugacy of the position and momentum bases, we have

$$\widehat{\mathcal{F}}(O|I) = \int\int \mathrm{d}x_I \mathrm{d}x_O \mathcal{K}(O|I)\mathcal{F}(O|I) \tag{4.17}$$

with

$$\mathcal{K}(O|I) = \exp\left(\frac{i}{\hbar}\mathbf{x}_I\cdot\mathbf{p}_I - \frac{i}{\hbar}\mathbf{x}_O\cdot\mathbf{p}_O\right) \tag{4.18}$$

Note that $\mathcal{K}$ does not depend explicitly on t_I or t_O.

Besides the Heisenberg picture, we can also work in the Schrödinger picture, where operators (and also their eigenvalues) carry no time dependence. In the position basis, the quantum kernel becomes

$$\mathcal{F}(O|I) = \langle \mathbf{x}_O | U(t_O, t_I) | \mathbf{x}_I \rangle \tag{4.19}$$

where U is the time evolution operator. Similarly, in the momentum basis,

$$\widehat{\mathcal{F}}(O|I) = \langle \mathbf{p}_O | U(t_O, t_I) | \mathbf{p}_I \rangle \tag{4.20}$$

The Schrödinger picture will facilitate the discussion of the S-matrix.

4.2 S-Matrix

The **S-matrix** $\mathcal{S}$ can be defined in the momentum basis as

$$\mathcal{S}(O|I) \equiv \langle \mathbf{p}_O | U_0^\dagger(t_O) U(t_O, t_I) U_0(t_I) | \mathbf{p}_I \rangle \tag{4.21}$$

where U_0 is the *free* time evolution operator. Using

$$U_0(t)|\mathbf{p}\rangle = \exp\left(-\frac{it}{2m\hbar}\mathbf{p}^2\right)|\mathbf{p}\rangle, \qquad \langle \mathbf{p}|U_0^\dagger(t) = \langle \mathbf{p}| \exp\left(\frac{it}{2m\hbar}\mathbf{p}^2\right) \tag{4.22}$$

leads to

$$\mathcal{S}(O|I) = \int\int \mathrm{d}x_I \mathrm{d}x_O \overline{\mathcal{U}}_O(O)\mathcal{U}_I(I)\mathcal{F}(O|I) \tag{4.23}$$

which is analogous to (4.17), except that instead of $\mathcal{K}$ we have

$$\mathcal{U}_I(I) \equiv \langle \mathbf{x}_I | U_0(t_I) | \mathbf{p}_I \rangle = \exp\left(\frac{i}{\hbar}\mathbf{x}_I \cdot \mathbf{p}_I - \frac{it_I}{2m\hbar}\mathbf{p}_I^2\right) \tag{4.24}$$

$$\overline{\mathcal{U}}_O(O) \equiv \langle \mathbf{p}_O | U_0^\dagger(t_O) | \mathbf{x}_O \rangle = \exp\left(-\frac{i}{\hbar}\mathbf{x}_O \cdot \mathbf{p}_O + \frac{it_O}{2m\hbar}\mathbf{p}_O^2\right) \tag{4.25}$$

These factors will be generalized accordingly when we consider the relativistic theory.

In practice it is more appropriate to consider the **asymptotic S-matrix**,

$$\mathcal{A}(O|I) \equiv \left[\lim_{T\to\infty}\right]\left[\lim_{t_O\to+T/2}\right]\left[\lim_{t_I\to-T/2}\right]\mathcal{S}(O|I) \tag{4.26}$$

From right to left, the first two limits make the time interval symmetric with duration T and centered at the origin. The third limit makes the time interval very long.

4.3 Semiclassical Kernels

The quantum kernel $\mathcal{F}$ can be written as a functional integral,

$$\mathcal{F}(O|I) = \int_{\mathbf{x}_I}^{\mathbf{x}_O} \mathrm{D}\mathbf{q}(t) \exp\left(-\frac{i}{\hbar} S[\mathbf{q}]\right) \tag{4.27}$$

where the functional integration is over all path configurations $\mathbf{q}(t)$ with boundary conditions

$$\mathbf{q}(t_I) = \mathbf{x}_I \text{ and } \mathbf{q}(t_O) = \mathbf{x}_O \tag{4.28}$$

This form of the quantum kernel is known as the Feynman path integral. From now on, we will work almost exclusively with the path integral formulation of the kernel.

The meaning of the quantum kernel can be extracted from the formulation in (4.27). The path integral corresponds to summing over *all* possible paths with appropriate boundary conditions. Each path $\mathbf{q}(t)$ contributes a factor of the form

$$\exp\left(-\frac{i}{\hbar} S[\mathbf{q}]\right) \tag{4.29}$$

where S is the action functional, familiar from classical mechanics. The motion of classical objects is specified by the classical path configuration $\bar{\mathbf{q}}(t)$. This path makes the action functional S stationary. Looking at (4.27), one can see that in the $\hbar \to 0$ limit, the path integral is dominated by the path that makes the action stationary, the classical path. But $\hbar$ is a fixed (dimensionful) constant in Nature, so $\hbar \to 0$ means that quantities with units of $\hbar$ are very large compared to $\hbar$. One quantity with units of $\hbar$ is angular momentum, and since angular momentum is quantized in the quantum theory, the $\hbar \to 0$ limit corresponds to the regime of *large quantum numbers*. More details about the regime when a classical path is dominant will be discussed later when we work with the relativistic theory (see §6.3).

The $\hbar \to 0$ limit is also known as the *semiclassical approximation* or the *JWKB approximation*. In the wavefunction formulation of quantum mechanics, the semiclassical approximation is valid when the de Broglie wavelength

$$\lambda_{dB} = \frac{2\pi\hbar}{|\mathbf{p}|} \tag{4.30}$$

is small compared to the distance over the which the interaction potential varies. Classically, we have

$$|\mathbf{p}| = \sqrt{2m[E - V(\mathbf{x})]} \tag{4.31}$$

Thus, small λ_{dB} leads to large $|\mathbf{p}|$, which in turn leads to the condition $E \gg |V(\mathbf{x})|$. In this sense, the nonrelativistic semiclassical approximation is a *large-energy approximation.* This is a kinematical consequence of the semiclassical approximation. As we will see later, the semiclassical approximation has dynamical consequences that sometimes lead to strong-coupling.

In the semiclassical approximation, the quantum kernel becomes the **semiclassical kernel** $\mathcal{V}$, which takes the form

$$\mathcal{V}(O|I) = \sqrt{\det(\mathbf{V})} \exp\left(-\frac{i}{\hbar}\Sigma\right) \tag{4.32}$$

where the **Van Vleck function** Σ is the value of the action functional at the classical path $\bar{\mathbf{q}}(t)$,

$$\Sigma \equiv S[\bar{\mathbf{q}}] \tag{4.33}$$

and the **Van Vleck matrix** $\mathbf{V}$ is given by

$$\mathbf{V} \equiv -\frac{i}{\hbar}\frac{\partial^2\Sigma}{\partial\mathbf{x}_I\partial\mathbf{x}_O} \tag{4.34}$$

The classical path $\bar{\mathbf{q}}(t)$ is a function of the boundary values $\mathbf{x}_I$ and $\mathbf{x}_O$. Thus, Σ and $\mathbf{V}$ are also functions of the boundary values. In order to compute the semiclassical kernel, we must know the classical path.

At first glance, the form of (4.32) might seem odd. In the $\hbar \to 0$ limit we use the functional analog of the stationary phase approximation to perform the functional integration over $\mathbf{q}(t)$. Thus, we expect something of the form

$$\frac{1}{\sqrt{\det(\mathbf{M})}} \exp\left(-\frac{i}{\hbar}\Sigma\right), \qquad \mathbf{M} \equiv \frac{i}{\hbar}\frac{\delta^2 S}{\delta\mathbf{q}(t)\delta\mathbf{q}(s)} \tag{4.35}$$

with the determinant[2] appearing with a different power than in (4.32). However, (4.32) is correct, and it can be derived in many ways.

The simplest way to derive (4.32) involves using the fact that in the Schrödinger picture, time evolution acts as a unitary transformation on states,

$$|\psi_O\rangle = U(t_O, t_I)|\psi_I\rangle \tag{4.36}$$

[2]Note that the determinant here is a *functional* determinant, unlike in (4.32) where we have a *traditional* determinant.

Thus,

$$\psi_O(\mathbf{x}_O, t_O) = \int \mathrm{d}\mathbf{x}_I \, \mathcal{F}(O|I)\psi_I(\mathbf{x}_I, t_I) \tag{4.37}$$

This means that the spatial and temporal dependence of ψ_O is governed by the quantum kernel $\mathcal{F}$. So, if ψ_O satisfies the Schrödinger equation, then so does the quantum kernel. Recall the "out" Schrödinger equation,

$$H_O \mathcal{F} = i\hbar \frac{\partial \mathcal{F}}{\partial t_O} \tag{4.38}$$

where H_O is the "out" quantum Hamiltonian operator. We will assume that H_O is a generic function of the "out" position and "out" momentum operators, and also is time-dependent,

$$H_O(\mathbf{Q}_O, \mathbf{P}_O, t_O) \tag{4.39}$$

When $\hbar \approx 0$ we Taylor-expand H_O around the classical values. That is, we expand in powers of

$$[\langle \mathbf{i}|\mathbf{Q}_O|\mathbf{j}\rangle - \mathbf{x}_O\langle \mathbf{i}|\mathbf{j}\rangle] \quad \text{and} \quad [\langle \mathbf{i}|\mathbf{P}_O|\mathbf{j}\rangle - \mathbf{p}_O\langle \mathbf{i}|\mathbf{j}\rangle] \tag{4.40}$$

where $\mathbf{x}_O$ and $\mathbf{p}_O$ are commuting numbers (i.e. not operators). Here $\langle \mathbf{i}|$ and $|\mathbf{j}\rangle$ belong to an arbitrary complete basis that we have used to write the matrix elements of the operators $\mathbf{Q}_O$ and $\mathbf{P}_O$. In order to avoid any ordering issues we work with a symmetric expansion,

$$H_O = H_0 + H_1 + \ldots \tag{4.41}$$

where

$$H_0 = H_O(\mathbf{x}_O, \mathbf{p}_O, t_O) \qquad H_1 = H_Q + H_P \tag{4.42}$$

with

$$H_Q = \frac{1}{2}\left[(\mathbf{Q}_O - \mathbf{x}_O) \cdot \frac{\partial H_0}{\partial \mathbf{x}_O} + \frac{\partial H_0}{\partial \mathbf{x}_O} \cdot (\mathbf{Q}_O - \mathbf{x}_O)\right] \tag{4.43}$$

$$H_P = \frac{1}{2}\left[(\mathbf{P}_O - \mathbf{p}_O) \cdot \frac{\partial H_0}{\partial \mathbf{p}_O} + \frac{\partial H_0}{\partial \mathbf{p}_O} \cdot (\mathbf{P}_O - \mathbf{p}_O)\right] \tag{4.44}$$

We *define* the semiclassical kernel $\mathcal{V}$ by the equation

$$(H_O + H_Q + H_P)\,\mathcal{V} = i\hbar \frac{\partial \mathcal{V}}{\partial t_O} \tag{4.45}$$

This amounts to keeping only H_0 and H_1 in the expansion of the "out" quantum Hamiltonian H_O.

Although we have used an arbitrary complete basis to expand the matrix elements of our operators, in practice we work with the **coordinate basis**:

$$\mathbf{Q}_O \to \mathbf{q}_O, \qquad \mathbf{P}_O \to -i\hbar\frac{\partial}{\partial \mathbf{q}_O} \tag{4.46}$$

Then it follows that

$$H_Q\mathcal{V} = 0 \tag{4.47}$$

Furthermore,

$$H_P\mathcal{V} = -i\hbar\frac{\partial H_0}{\partial \mathbf{p}_O}\cdot\frac{\partial \mathcal{V}}{\partial \mathbf{x}_O} - \mathbf{p}_O\cdot\frac{\partial H_0}{\partial \mathbf{p}_O}\mathcal{V} - \frac{i\hbar}{2}\frac{\partial}{\partial \mathbf{x}_O}\cdot\left(\frac{\partial H_0}{\partial \mathbf{p}_O}\right)\mathcal{V} \tag{4.48}$$

We start with the ansatz

$$\mathcal{V} = \sqrt{\rho}\exp\left(-\frac{i}{\hbar}\Sigma\right) \tag{4.49}$$

Taking a time derivative yields

$$i\hbar\frac{\partial \mathcal{V}}{\partial t_O} = \left[i\hbar\frac{1}{2\rho}\frac{\partial \rho}{\partial t_O} + \frac{\partial \Sigma}{\partial t_O}\right]\mathcal{V} \tag{4.50}$$

Similarly, taking a spatial derivative yields

$$-i\hbar\frac{\partial \mathcal{V}}{\partial \mathbf{x}_O} = \left[-i\hbar\frac{1}{2\rho}\frac{\partial \rho}{\partial \mathbf{x}_O} - \frac{\partial \Sigma}{\partial \mathbf{x}_O}\right]\mathcal{V} \tag{4.51}$$

So then, (4.45) becomes

$$\begin{aligned}&\left[H_0 - \left(\mathbf{p}_O + \frac{\partial \Sigma}{\partial \mathbf{x}_O}\right)\cdot\frac{\partial H_0}{\partial \mathbf{p}_O} - \frac{\partial \Sigma}{\partial t_0}\right]\mathcal{V}\\ &-\frac{i\hbar}{2\rho}\left[\rho\frac{\partial}{\partial \mathbf{x}_O}\cdot\left(\frac{\partial H_0}{\partial \mathbf{p}_O}\right) + \frac{\partial \rho}{\partial \mathbf{x}_O}\cdot\frac{\partial H_0}{\partial \mathbf{p}_O} + \frac{\partial \rho}{\partial t_O}\right]\mathcal{V} = 0\end{aligned} \tag{4.52}$$

Note that the first line is of order-zero in $\hbar$ and the second line is of order-one in $\hbar$. We will solve this equation by setting each term equal to zero. Thus, we find two equations which can be used to solve for ρ and Σ:

$$H_0 - \left(\mathbf{p}_O + \frac{\partial \Sigma}{\partial \mathbf{x}_O}\right) \cdot \frac{\partial H_0}{\partial \mathbf{p}_O} - \frac{\partial \Sigma}{\partial t_0} = 0 \tag{4.53}$$

$$\rho \frac{\partial}{\partial \mathbf{x}_O} \cdot \left(\frac{\partial H_0}{\partial \mathbf{p}_O}\right) + \frac{\partial \rho}{\partial \mathbf{x}_O} \cdot \frac{\partial H_0}{\partial \mathbf{p}_O} + \frac{\partial \rho}{\partial t_O} = 0 \tag{4.54}$$

Equation (4.54) can be written in the form of a continuity equation,

$$\frac{\partial \rho}{\partial t_O} + \frac{\partial}{\partial \mathbf{x}_O} \cdot \left(\rho \frac{\partial H_0}{\partial \mathbf{p}_O}\right) = 0 \tag{4.55}$$

Instead of (4.53), we will consider a more *restrictive* case,

$$\mathbf{p}_O = -\frac{\partial \Sigma}{\partial \mathbf{x}_O} \tag{4.56}$$

Combining this with (4.53) leads to a set of equations that have the same form as the Hamilton-Jacobi equations,

$$H_0 = \frac{\partial \Sigma}{\partial t_0}, \qquad \mathbf{p}_O = -\frac{\partial \Sigma}{\partial \mathbf{x}_O} \tag{4.57}$$

where H_0 plays the role of the classical Hamiltonian, and Σ plays the role of the classical Hamilton function, which is related to the value of the action functional at the classical path.

Before we move forward, we must address an apparent inconsistency. The classical Hamiltonian H_0 is a function of the "out" position $\mathbf{x}_O$ and the "out" momentum $\mathbf{p}_O$. But we expect $\mathcal{V}$ to be a function of the "out" position and the "in" position $\mathbf{x}_I$. So we should make a change of variables

$$\mathbf{p}_O \longrightarrow \mathbf{x}_I \tag{4.58}$$

This change of variable leads to a Jacobian matrix

$$\mathbf{J} \equiv \frac{\partial \mathbf{p}_O}{\partial \mathbf{x}_I} = -\frac{\partial^2 \Sigma}{\partial \mathbf{x}_I \partial \mathbf{x}_O} \tag{4.59}$$

So then

$$\frac{\partial H_0}{\partial \mathbf{p}_O} = \mathbf{J}^{-1} \cdot \frac{\partial^2 \Sigma}{\partial \mathbf{x}_I \partial t_O} \tag{4.60}$$

and the continuity equation (4.55) becomes

$$\frac{\partial \rho}{\partial t_O} + \frac{\partial}{\partial \mathbf{x}_O} \cdot \left(\rho \mathbf{J}^{-1} \cdot \frac{\partial^2 \Sigma}{\partial \mathbf{x}_I \partial t_O} \right) = 0 \tag{4.61}$$

Expanding the second term yields

$$\frac{\partial \rho}{\partial t_O} - \rho \,\mathrm{tr} \left(\mathbf{J}^{-1} \cdot \frac{\partial \mathbf{J}}{\partial t_O} \right) = - \left(\frac{\partial \rho}{\partial \mathbf{x}_O} \cdot \mathbf{J}^{-1} + \rho \frac{\partial}{\partial \mathbf{x}_O} \cdot \mathbf{J}^{-1} \right) \cdot \frac{\partial^2 \Sigma}{\partial \mathbf{x}_I \partial t_O} \tag{4.62}$$

Now recall some properties of the determinant and the inverse of a matrix. Consider an $n \times n$ matrix $\mathbf{M}$ that is a function of an n-dimensional vector parameter $\mathbf{x}$ and a scalar parameter t. We denote the inverse of $\mathbf{M}$ by $\mathbf{W}$. The determinant of $\mathbf{M}$ satisfies

$$\frac{\partial}{\partial t} [\det (\mathbf{M})] = \det (\mathbf{M}) \left[W_i{}^j \frac{\partial M_j{}^i}{\partial t} \right] \tag{4.63}$$

and

$$\frac{\partial}{\partial x_k} [\det (\mathbf{M})] = \det (\mathbf{M}) \left[W_i{}^j \frac{\partial M_j{}^i}{\partial x_k} \right] \tag{4.64}$$

The inverse of $\mathbf{M}$ satisfies

$$\frac{\partial W_i{}^j}{\partial x_k} = -W_i{}^m \frac{\partial M_m{}^n}{\partial x_k} W_n{}^j \tag{4.65}$$

Consider the case when $\mathbf{M}$ has the "Jacobian" form

$$M_i^j = \frac{\partial y_i}{\partial x_j} \tag{4.66}$$

Then

$$T_i^{jk} \equiv \frac{\partial M_i^j}{\partial x_k} = \frac{\partial^2 y_i}{\partial x_j \partial x_k} \tag{4.67}$$

is symmetric in the upper indices. One can check that when $\mathbf{M}$ has the form (4.66) then

$$\frac{\partial}{\partial x_k} [\det (\mathbf{M})] W_k{}^l + [\det (\mathbf{M})] \frac{\partial W_k{}^l}{\partial x_k} = 0 \tag{4.68}$$

because of the symmetry of T_i^{jk}. One can use these identities to check that

$$\rho = \det (k\mathbf{J}) \tag{4.69}$$

with k a constant satisfies (4.62): The left side of (4.62) vanishes due to (4.63) and the pre-factor on the right side vanishes due to (4.68). Hence, we have found that the semiclassical kernel is

$$\mathcal{V}(O|I) = \sqrt{\det(\mathbf{V})} \exp\left(-\frac{i}{\hbar}\Sigma\right), \qquad \mathbf{V} \equiv k\mathbf{J} = -k\frac{\partial^2\Sigma}{\partial \mathbf{x}_I \partial \mathbf{x}_O} \tag{4.70}$$

The constant k is fixed after appropriate normalization. One finds $k = i/\hbar$.

The derivation of the semiclassical kernel presented here is based on exercise VA2.1 from [4]. A more rigorous derivation, using functional methods, can be found in [59] and references therein. Indeed, the semiclassical kernel is also known as the Van Vleck-Morette kernel, after C. DeWitt-Morette, who developed a very rigorous approach to the semiclassical limit of the Feynman path integral. The semiclassical (JWKB) approximation in quantum mechanics was developed separately by Jeffreys [60], Wentzel [61], Kramers [62] and Brillouin [63]. The role of classical Hamilton-Jacobi theory was pointed out by Van Vleck [64].

4.3.1 Semiclassical S-Matrix

In equation (4.23) we wrote the S-matrix in terms of the quantum kernel. After using the semiclassical approximation, the quantum kernel $\mathcal{F}$ becomes the semiclassical kernel $\mathcal{V}$. So in the semiclassical approximation, we define the **semiclassical S-matrix** as

$$\mathcal{S}(O|I) \approx \int\int \mathrm{d}x_I \mathrm{d}x_O \overline{\mathcal{U}}_O(O)\mathcal{U}_I(I)\mathcal{V}(O|I) \tag{4.71}$$

which is analogous to (4.23). In the same way, one can also define the **semiclassical asymptotic S-matrix**.

4.4 Semiclassical Eikonal Kernels

In order to find the semiclassical kernel $\mathcal{V}$ we must first solve the classical equations of motions and find the classical path $\bar{\mathbf{q}}(t)$. For most interacting systems the classical path is either very complicated or elusive, so the semiclassical approximation appears to have a limited scope. We follow a different approach: We adopt a path $\mathbf{f}(t)$ as the *de facto* classical path and compute the semiclassical kernel with $\mathbf{f}(t)$ instead of $\bar{\mathbf{q}}(t)$. This approach is valid as long as the path $\mathbf{f}(t)$ approximates the classical path in some particular regime.

The simplest path between two points $\mathbf{x}_I$ and $\mathbf{x}_O$ is the **eikonal path**,

$$\mathbf{e}(t) = \frac{\mathbf{x}_I + \mathbf{x}_O}{2} + (\mathbf{x}_O - \mathbf{x}_I)\left(\frac{t}{\Delta t}\right), \tag{4.72}$$

where the range of the time parameter t is

$$-\frac{\Delta t}{2} < t < \frac{\Delta t}{2}, \qquad \Delta t = t_O - t_I > 0 \tag{4.73}$$

The eikonal[3] path describes a spatial trajectory where the motion has *fixed direction* and *fixed speed*:

$$\dot{\mathbf{e}}(t) = \frac{\mathbf{x}_O - \mathbf{x}_I}{\Delta t} \tag{4.74}$$

In many-body systems, if the path of each body is approximated by an eikonal path, then $\mathcal{E}$ is a good approximation in the regime of **small momentum transfer** or, equivalently, **small-angle scattering**. By Fourier-Heisenberg conjugacy, the small momentum transfer regime is the same as the regime where the separation between each body is kept very large. Since in the nonrelativistic semiclassical approximation each body has an energy that is much greater than the interaction energy, the *nonrelativistic eikonal JWKB approximation* corresponds to *large-energies* and *small momentum transfer*. This approximation is discussed in the lectures by Glauber [65].

We define the **semiclassical eikonal kernel** $\mathcal{E}$ as

$$\mathcal{E}(O|I) \equiv \sqrt{\det\left(\mathbf{V}_{\text{eik}}\right)} \exp\left(-\frac{i}{\hbar}\Sigma_{\text{eik}}\right) \tag{4.75}$$

where the **eikonal Van Vleck function** Σ_{eik} is

$$\Sigma_{\text{eik}} \equiv S[\mathbf{e}] \tag{4.76}$$

and the **eikonal Van Vleck matrix** $\mathbf{V}_{\text{eik}}$ is

$$\mathbf{V}_{\text{eik}} \equiv -\frac{i}{\hbar}\frac{\partial \Sigma_{\text{eik}}}{\partial \mathbf{x}_I \partial \mathbf{x}_O} \tag{4.77}$$

The definition of $\mathcal{E}$ is analogous to (4.32), but with the eikonal path instead of the true classical path.

[3] The word "eikonal" here is meant as an adjective. Eikonal comes from the Greek word for image. In this work it is meant as a reference to Geometric Optics, where one works with light rays that move along straight paths. We will comment on other uses of the word "eikonal" later.

4.4.1 Semiclassical Eikonal S-Matrix

After introducing the semiclassical eikonal kernel $\mathcal{E}$, we can also introduce the corresponding S-matrix, the **semiclassical eikonal S-matrix**:

$$\mathcal{S}(O|I) \approx \int\int \mathrm{d}x_I \mathrm{d}x_O \overline{\mathcal{U}}_O(O)\mathcal{U}_I(I)\mathcal{E}(O|I) \tag{4.78}$$

Similarly, by analogy with (4.26), we introduce the **asymptotic semiclassical eikonal S-matrix**. Indeed, this is the *only* version of the S-matrix that we will use through this work.

4.5 Many-body Systems

The simplest way to generalize all the results that we have collected so far for single-body systems is to work with the quantum kernel as a path integral. For example, the two-body quantum kernel $\mathcal{F}_2$ can be written as a double path integral:

$$\mathcal{F}_2(3,4|1,2) = \int_{\mathbf{x}_1}^{\mathbf{x}_3} \mathrm{D}\mathbf{q}_a(t) \int_{\mathbf{x}_2}^{\mathbf{x}_4} \mathrm{D}\mathbf{q}_b(t)\, \exp\left(-\frac{i}{\hbar}S[\mathbf{q}_a,\mathbf{q}_b]\right) \tag{4.79}$$

where $\mathbf{q}_a(t)$ and $\mathbf{q}_b(t)$ are the paths for bodies a and b, respectively. Note that in the nonrelativistic theory there is a universal time parameter t, and thus both "in" boundary conditions are defined at time $t = t_I$:

$$\mathbf{q}_a(t_I) = \mathbf{x}_1 \text{ and } \mathbf{q}_b(t_I) = \mathbf{x}_2 \tag{4.80}$$

and similarly for both of the "out" boundary conditions, which are defined at time $t = t_O$:

$$\mathbf{q}_a(t_O) = \mathbf{x}_3 \text{ and } \mathbf{q}_b(t_O) = \mathbf{x}_4 \tag{4.81}$$

This feature will change in the relativistic theory.

Chapter 5

Eikonal Coulomb Scattering

As an application of the nonrelativistic eikonal JWKB approximation, in this chapter we compute the four-point scattering amplitude for a two-body system of particles that are coupled to an instantaneous scalar wave field U. This system is analogous to two particles interacting via a Coulomb force. In principle, this problem is separable into two single-body problems. We will keep the "two-body-ness" explicit, as the methods we use will generalize to the relativistic theory. Coulomb scattering via the semiclassical eikonal approximation is treated (with less detail) in section 9.6.2 of [66]. This problem is also considered in [65].

We start with the two-body path integral,

$$\mathcal{G}[U] = \int_{\mathbf{x}_1}^{\mathbf{x}_3} \mathrm{D}\mathbf{q}_a(t) \int_{\mathbf{x}_2}^{\mathbf{x}_4} \mathrm{D}\mathbf{q}_b(t) \, \exp\left(-\frac{i}{\hbar} S_P[\mathbf{q}_a, \mathbf{q}_b, U]\right) \tag{5.1}$$

This is a functional of the scalar wave field U. The particle action functional S_P is

$$S_P[\mathbf{q}_a, \mathbf{q}_b, U] = S_0[\mathbf{q}_a, \mathbf{q}_b] + S_{\text{int}}[\mathbf{q}_a, \mathbf{q}_b, U] \tag{5.2}$$

with the free term,

$$S_0[\mathbf{q}_a, \mathbf{q}_b] = \int \mathrm{d}t \left[-\frac{m_a}{2}\dot{\mathbf{q}}_a^2 - \frac{m_b}{2}\dot{\mathbf{q}}_b^2\right] \tag{5.3}$$

and the term with the couplings to the field U,

$$S_{\text{int}}[\mathbf{q}_a, \mathbf{q}_b, U] = \int \mathrm{d}t \, (Z_a U[\mathbf{q}_a(t)] + Z_b U[\mathbf{q}_b(t)]) \tag{5.4}$$

Here Z_a and Z_b are dimensionless charges for particles a and b, respectively.

We integrate over the field U to obtain the effective two-body quantum kernel:

$$\mathcal{F}(3,4|1,2) \equiv \int \mathrm{D}U(\mathbf{x})\,\mathcal{G}[U]\exp\left(-\frac{i}{\hbar}S_{\mathrm{kin}}[U]\right) \tag{5.5}$$

Here, the functional S_{kin} acts as a kinetic term for U,

$$S_{\mathrm{kin}}[U] = \frac{1}{2g^2}\int\int \mathrm{d}t_x \mathrm{d}t_y \int\int \mathrm{d}x\mathrm{d}y\,[U(\mathbf{x})K(\mathbf{x},t_x|\mathbf{y},t_y)U(\mathbf{y})] \tag{5.6}$$

where K is the kinetic operator for an *instantaneous* scalar wave,

$$K(\mathbf{x},t_x|\mathbf{y},t_y) \equiv \delta(t_x - t_y)\delta(\mathbf{x}-\mathbf{y})\left[-\frac{1}{2}\left(\frac{\partial}{\partial\mathbf{x}}\right)^2\right] \tag{5.7}$$

and g is a dimensionful coupling parameter. The functional measure in (5.5) is normalized such that

$$\int \mathrm{D}U(\mathbf{x})\ \exp\left(-\frac{i}{\hbar}S_{\mathrm{kin}}[U]\right) = 1 \tag{5.8}$$

In the presence of $\mathcal{G}$, the functional integral over U can be done exactly. Let

$$J(\mathbf{x},t) \equiv Z_a\delta[\mathbf{x}-\mathbf{q}_a(t)] + Z_b\delta[\mathbf{x}-\mathbf{q}_b(t)] \tag{5.9}$$

such that we can rewrite S_{int} as

$$S_{\mathrm{int}}[\mathbf{q}_a,\mathbf{q}_b,U] = \int \mathrm{d}t \int \mathrm{d}x\ J(\mathbf{x},t)U(\mathbf{x}) \tag{5.10}$$

After integration over U, one obtains

$$\mathcal{F}(3,4|1,2) = \int_{\mathbf{x}_1}^{\mathbf{x}_3} \mathrm{D}\mathbf{q}_a(t) \int_{\mathbf{x}_2}^{\mathbf{x}_4} \mathrm{D}\mathbf{q}_b(t)\ \exp\left(-\frac{i}{\hbar}S_U[\mathbf{q}_a,\mathbf{q}_b]\right) \tag{5.11}$$

where the effective particle action functional S_U is

$$\begin{aligned} S_U[q_a,q_b] &= S_0[\mathbf{q}_a,\mathbf{q}_b] \\ &\quad - \frac{g^2}{2}\int \mathrm{d}t_x\mathrm{d}t_y \int \mathrm{d}x\mathrm{d}y\,[J(\mathbf{x},t_x)G(\mathbf{x},t_x|\mathbf{y},t_y)J(\mathbf{y},t_y)] \end{aligned} \tag{5.12}$$

Here G is the Green function for an *instantaneous* scalar wave in d spatial dimensions:

$$G(\mathbf{x}, t_x|\mathbf{y}, t_y) = \delta(t_x - t_y)\Gamma\left(\frac{d-2}{2}\right)\left(\frac{2}{(\mathbf{x}-\mathbf{y})^2}\right)^{(d-2)/2} \tag{5.13}$$

Using the explicit form of J, we find

$$S_U[\mathbf{q}_a, \mathbf{q}_b] = S_0[\mathbf{q}_a, \mathbf{q}_b] - S_1[\mathbf{q}_a, \mathbf{q}_b] - S_2[\mathbf{q}_a, \mathbf{q}_b] \tag{5.14}$$

where S_1 contains (divergent) one-body self-interactions and S_2 contains a two-body interaction. The contributions from the self-interactions are divergent because the require evaluating the Green function at the same spatial point. We will *ignore* these contributions. Explicitly, S_2 is given by

$$S_2[\mathbf{q}_a, \mathbf{q}_b] = g^2 Z_a Z_b \Gamma\left(\frac{d-2}{2}\right)\int \mathrm{d}t\left(\frac{2}{[\mathbf{q}_a(t) - \mathbf{q}_b(t)]^2}\right)^{(d-2)/2} \tag{5.15}$$

We have essentially derived the action functional for a two-body system of charged particles interacting via a Coulomb-like potential.

We now perform a little dimensional analysis. Note that we have kept $\hbar$ and the speed of light[1] dimensionful. All action functionals have units of $\hbar$. From (5.4) we find that the field U has units of energy. Then, from (5.6) we find that the coupling parameter g has units

$$[g] = \frac{1}{2}[\hbar] + \frac{1}{2}[c] + \left(\frac{d-3}{2}\right)[\text{length}] \tag{5.16}$$

Thus, for $d = 3$ the coupling parameter g has units

$$[g] = \frac{1}{2}[\hbar] + \frac{1}{2}[c] \tag{5.17}$$

We introduce a *dimensionless* coupling parameter α via the equation

$$g^2 = \frac{\hbar c \alpha}{\sqrt{2\pi}} L^{(3-d)} \tag{5.18}$$

where L is a constant with units of length and the numerical factor in the denominator is for later convenience. Note that in $d = 3$ we recover the familiar dimensionless coupling, the fine-structure constant:

$$\alpha \sim \frac{g^2}{\hbar c} \tag{5.19}$$

[1]Strictly speaking, the speed of light is infinite in the nonrelativistic theory.

If we keep g^2 fixed and take the limit $\hbar \to 0$ we find $\alpha \to \infty$. In other words, the semiclassical approximation is a *strong-coupling* approximation. However, in the nonrelativistic theory we take the limit $c \to \infty$, which if we keep g^2 fixed leads us to $\alpha \to 0$. We will argue that, in principle, α is not well-defined in the nonrelativistic semiclassical theory. We will still use α (and c) as a formal symbol. This issue is not present in the semiclassical relativistic theory.

5.1 Eikonal Kernel

In the eikonal JWKB approximation, the quantum kernel (5.11) becomes the semiclassical eikonal kernel:

$$\mathcal{F}(3,4|1,2) \longrightarrow \mathcal{E}(3,4|1,2) = \sqrt{\det(\mathbf{V})}\exp\left(-\frac{i}{\hbar}\Sigma\right) \tag{5.20}$$

We have dropped the "eik" labels from the two-body eikonal Van Vleck function and the two-body eikonal Van Vleck matrix. The two-body eikonal Van Vleck function is

$$\Sigma = S_U[\mathbf{e}_a, \mathbf{e}_b] \tag{5.21}$$

and the two-body eikonal Van Vleck matrix is a 2×2 array of single-body eikonal Van Vleck matrices

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{13} & \mathbf{V}_{23} \\ \mathbf{V}_{14} & \mathbf{V}_{24} \end{pmatrix}, \qquad \mathbf{V}_{jk} = -\frac{i}{\hbar}\frac{\partial^2\Sigma}{\partial\mathbf{x}_j\partial\mathbf{x}_k} \tag{5.22}$$

In the two-body problem, the eikonal paths are

$$\begin{aligned} \mathbf{e}_a(t) &= \frac{\mathbf{x}_1 + \mathbf{x}_3}{2} + (\mathbf{x}_3 - \mathbf{x}_1)\left(\frac{t}{\Delta t}\right) \\ \mathbf{e}_b(t) &= \frac{\mathbf{x}_2 + \mathbf{x}_4}{2} + (\mathbf{x}_4 - \mathbf{x}_2)\left(\frac{t}{\Delta t}\right) \end{aligned} \tag{5.23}$$

For convenience, the range of the time parameter t is

$$-\frac{\Delta t}{2} < t < \frac{\Delta t}{2}, \qquad \Delta t = t_O - t_I > 0 \tag{5.24}$$

We now determine Σ and $\mathbf{V}$.

5.1.1 Eikonal Van Vleck Function

At the eikonal paths (5.23), the free term in the action functional becomes

$$\begin{aligned}\Sigma_0 &\equiv S_0\left[\mathbf{e}_a, \mathbf{e}_b\right] \\ &= -\frac{m_a}{2\Delta t}\left(\mathbf{x}_3 - \mathbf{x}_1\right)^2 - \frac{m_b}{2\Delta t}\left(\mathbf{x}_4 - \mathbf{x}_2\right)^2\end{aligned} \tag{5.25}$$

Similarly, the two-body interaction term becomes

$$\begin{aligned}\Sigma_2 &\equiv S_2\left[\mathbf{e}_a, \mathbf{e}_b\right] \\ &= g^2 Z_a Z_b \Gamma\left(\frac{d-2}{2}\right)\int \mathrm{d}t\left(\frac{2}{[\mathbf{e}_a(t) - \mathbf{e}_b(t)]^2}\right)^{(d-2)/2}\end{aligned} \tag{5.26}$$

With the eikonal paths (5.23), we have

$$\mathbf{e}_a(t) - \mathbf{e}_b(t) = \mathbf{X} - \mathbf{x}\left(\frac{t}{\Delta t}\right) \tag{5.27}$$

and thus

$$[\mathbf{e}_a(t) - \mathbf{e}_b(t)]^2 = \mathbf{X}^2 - 2(\mathbf{X}\cdot\mathbf{x})\left(\frac{t}{\Delta t}\right) + \mathbf{x}^2\left(\frac{t}{\Delta t}\right)^2 \tag{5.28}$$

where we have introduced the vectors

$$\mathbf{X} \equiv \frac{\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_4}{2}, \qquad \mathbf{x} \equiv \mathbf{x}_4 - \mathbf{x}_2 - \mathbf{x}_3 + \mathbf{x}_1 \tag{5.29}$$

The vector $\mathbf{X}$ corresponds to the *vector average* of the initial separation of the particles (given by $\mathbf{x}_1 - \mathbf{x}_2$) and the final separation (given by $\mathbf{x}_3 - \mathbf{x}_4$).

In order to evaluate the integral over t in (5.26), we first introduce a Schwinger variable ω and write

$$\Sigma_2 = \frac{\hbar c\alpha Z_a Z_b L}{\sqrt{2\pi}}\int_0^\infty \mathrm{d}\omega\left(\frac{1}{\omega}\right)^{d/2}\int \mathrm{d}t\ \exp\left(-\frac{1}{2\omega L^2}[\mathbf{e}_a(t) - \mathbf{e}_b(t)]^2\right) \tag{5.30}$$

This way, the integrand becomes a Gaussian. In the eikonal approximation, the momentum transfer between the particles is small when compared to other momenta in the problem. The Fourier-Heisenberg conjugate of this statement is that the separation between the particles is always large when compared to other distances in the problem. Thus,

$$\frac{1}{2L^2}[\mathbf{e}_a(t) - \mathbf{e}_b(t)]^2 \gg 1 \tag{5.31}$$

In this regime, the integral over t can be evaluated with stationary methods. The stationary point is

$$t_* = \Delta t \left(\frac{\mathbf{X} \cdot \mathbf{x}}{\mathbf{x}^2} \right) \tag{5.32}$$

This value of the time parameter yields the minimum separation between the particles,

$$\mathbf{B} \equiv \mathbf{e}_a(t_*) - \mathbf{e}_b(t_*) = \mathbf{X} - \left(\frac{\mathbf{X} \cdot \mathbf{x}}{\mathbf{x}^2} \right) \mathbf{x} \tag{5.33}$$

as long as

$$-\frac{1}{2} < \frac{\mathbf{X} \cdot \mathbf{x}}{\mathbf{x}^2} < \frac{1}{2} \tag{5.34}$$

Note that the vector $\mathbf{B}$ is orthogonal to $\mathbf{x}$, so it only has $d-1$ independent components.

After dealing with the integration over t, we find

$$\Sigma_2 \approx \hbar\alpha \left[\frac{Z_a Z_b c \Delta t}{|\mathbf{x}|} \right] \int_0^\infty \mathrm{d}\omega \left(\frac{1}{\omega} \right)^{(d-1)/2} \exp\left(-\frac{1}{2\omega L^2} \mathbf{B}^2 \right) \tag{5.35}$$

which, after integration over ω, yields

$$\Sigma_2 \approx \hbar\alpha \left[\frac{Z_a Z_b c \Delta t}{|\mathbf{x}|} \right] \Gamma\left(\frac{d-3}{2} \right) \left(\frac{2L^2}{\mathbf{B}^2} \right)^{(d-3)/2} \tag{5.36}$$

This result is divergent in $d = 3$, which happens to be the case of most relevance. We will use dimensional regularization by setting $d = 3 + 2\varepsilon$ with $\varepsilon > 0$. After introducing

$$\rho \equiv \frac{Z_a Z_b c \Delta t}{|\mathbf{x}|} \tag{5.37}$$

we can write

$$\Sigma_2 \approx \hbar\alpha\rho\Gamma(\varepsilon) \left(\frac{2L^2}{\mathbf{B}^2} \right)^\varepsilon \tag{5.38}$$

We will find similar expressions in the relativistic theory.

The divergence in Σ_2 is somewhat troubling. One could think that this divergence follows as a consequence of using stationary methods for evaluating the integral over t. In principle, we can evaluate the integral over t in $d = 3$

exactly and obtain a result that does not exhibit explicit divergences. However, when we use the resulting semiclassical eikonal kernel to obtain the asymptotic S-matrix, we need to take the large Δt limit. The result from the exact integral is divergent in this limit, so a divergence is re-introduced into the problem (similar issues are encountered in [66]). Our result (5.38) has an implicit Δt hidden in ρ, but we will see that the particular combination in ρ is finite.

5.1.2 Eikonal Van Vleck Matrix

The two-body eikonal Van Vleck matrix has four blocks:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{13} & \mathbf{V}_{23} \\ \mathbf{V}_{14} & \mathbf{V}_{24} \end{pmatrix}, \qquad \mathbf{V}_{jk} = -\frac{i}{\hbar} \frac{\partial^2 \Sigma}{\partial \mathbf{x}_j \partial \mathbf{x}_k} \tag{5.39}$$

Since Σ has the form $\Sigma_0 - \Sigma_2$, we write

$$\mathbf{V} = \mathbf{V}_0 - \mathbf{V}_2 = (\mathbf{I} - \mathbf{W}) \cdot \mathbf{V}_0, \qquad \mathbf{W} \equiv \mathbf{V}_2 \cdot (\mathbf{V}_0)^{-1} \tag{5.40}$$

Taking the determinant gives

$$\det(\mathbf{V}) = \det(\mathbf{I} - \mathbf{W}) \det(\mathbf{V}_0) \tag{5.41}$$

Now, recall the identity

$$\det(\mathbf{I} - \mathbf{W}) = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}(\mathbf{W}^n) \right] \tag{5.42}$$

Thus,

$$\sqrt{\det(\mathbf{V})} = \sqrt{\det(\mathbf{V}_0)} \exp\left[-\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}(\mathbf{W}^n) \right] \tag{5.43}$$

The free part $\mathbf{V}_0$ is easy to obtain. We have

$$\mathbf{V}_0 = \begin{pmatrix} \mathbf{u}_{13} & \mathbf{u}_{23} \\ \mathbf{u}_{14} & \mathbf{u}_{24} \end{pmatrix}, \qquad \mathbf{u}_{jk} = -\frac{i}{\hbar} \frac{\partial^2 \Sigma_0}{\partial \mathbf{x}_j \partial \mathbf{x}_k} \tag{5.44}$$

With (5.25), we find

$$\mathbf{u}_{13} = \left(-\frac{i m_a}{\hbar \Delta t} \right) \mathbf{I}, \qquad \mathbf{u}_{23} = \mathbf{u}_{14} = \mathbf{0}, \qquad \mathbf{u}_{24} = \left(-\frac{i m_b}{\hbar \Delta t} \right) \mathbf{I} \tag{5.45}$$

Hence, $\mathbf{V}_0$ is invertible and thus $\mathbf{W}$ is well-defined. However, as part of our approximations, we will neglect all the contributions to the determinant from

$\mathbf{W}$. In principle, these contributions are very interesting, since they involve the coupling parameter α. In practice, all of these contributions are subleading in powers of $\mathbf{B}^2$, and we only keep the leading contribution that comes from Σ_2. Note that the determinant of $\mathbf{V}$ corresponds to the order-zero in $\hbar$ correction to the Van Vleck function. We have found that this $\hbar$-correction is nonperturbative in α since it involves an infinite number of contributions (coming from $\mathbf{W}$). This is a nice example of how the semiclassical approximation is nonperturbative.

5.2 Semiclassical Eikonal S-Matrix

With Σ and $\mathbf{V}$ we can build the semiclassical eikonal kernel:

$$\mathcal{E}(3,4|1,2) = \left(-\frac{im_a}{\hbar\Delta t}\right)^{d/2}\left(-\frac{im_b}{\hbar\Delta t}\right)^{d/2}\exp\left(-\frac{i}{\hbar}\Sigma_0 + \frac{i}{\hbar}\Sigma_2\right) \tag{5.46}$$

We are interested in the three-dimensional theory, so we expand near $\varepsilon = 0$:

$$\frac{i}{\hbar}\Sigma_2 \approx i\alpha\rho\left[\Gamma(\varepsilon) + \log\left(\frac{2L^2}{\mathbf{B}^2}\right)\right] \tag{5.47}$$

Thus

$$\exp\left(\frac{i}{\hbar}\Sigma_2\right) \approx \left(\frac{2L^2}{\mathbf{B}^2}\right)^{i\alpha\rho}\exp\left(\Lambda_\varepsilon\right) \tag{5.48}$$

with

$$\Lambda_\varepsilon \equiv i\alpha\rho\Gamma(\varepsilon) \tag{5.49}$$

The semiclassical eikonal S-matrix is

$$\mathcal{S}(3,4|1,2) = \int\int\int\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\,\overline{\mathcal{U}}_O(3,4)\mathcal{U}_I(1,2)\mathcal{E}(3,4|1,2) \tag{5.50}$$

where

$$\mathcal{U}_I(1,2) = \exp\left[\frac{i}{\hbar}\left(\mathbf{x}_1\cdot\mathbf{p}_1 + \mathbf{x}_2\cdot\mathbf{p}_2\right) - \frac{it_I}{\hbar}\left(\frac{\mathbf{p}_1^2}{2m_a} + \frac{\mathbf{p}_2^2}{2m_b}\right)\right] \tag{5.51}$$

$$\overline{\mathcal{U}}_O(3,4) = \exp\left[-\frac{i}{\hbar}\left(\mathbf{x}_3\cdot\mathbf{p}_3 + \mathbf{x}_4\cdot\mathbf{p}_4\right) + \frac{it_O}{\hbar}\left(\frac{\mathbf{p}_3^2}{2m_a} + \frac{\mathbf{p}_4^2}{2m_b}\right)\right] \tag{5.52}$$

With $\mathcal{S}$, we compute the asymptotic semiclassical eikonal S-matrix:

$$\mathcal{A}(3,4|1,2) = \left[\lim_{T\to\infty}\right]\left[\lim_{t_O\to+T/2}\right]\left[\lim_{t_I\to-T/2}\right]\mathcal{S}(3,4|1,2) \tag{5.53}$$

We will apply these three limits in two stages. First, we apply the last two limits: these just set $t_I = -T/2$ and $t_F = T/2$. Hence,

$$\Delta t = t_F - t_I = T \tag{5.54}$$

The third limit will be taken later.

In order to perform the integration to obtain $\mathcal{S}$ we first make a change of variables in the position basis and introduce the corresponding conjugate momenta:

$$\begin{aligned}
\mathbf{R} &\equiv \frac{\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4}{4} & \mathbf{K} &\equiv \mathbf{p}_4 + \mathbf{p}_3 - \mathbf{p}_2 - \mathbf{p}_1 \\
\mathbf{X} &\equiv \frac{\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_4}{2} & \mathbf{P} &\equiv \frac{\mathbf{p}_3 - \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_4}{2} \\
\mathbf{x}_{31} &\equiv \mathbf{x}_3 - \mathbf{x}_1 & \mathbf{p}_{31} &\equiv \frac{\mathbf{p}_1 + \mathbf{p}_3}{2} \\
\mathbf{x}_{42} &\equiv \mathbf{x}_4 - \mathbf{x}_2 & \mathbf{p}_{42} &\equiv \frac{\mathbf{p}_2 + \mathbf{p}_4}{2}
\end{aligned} \tag{5.55}$$

such that

$$\mathbf{x}_1 \cdot \mathbf{p}_1 + \mathbf{x}_2 \cdot \mathbf{p}_2 - \mathbf{x}_3 \cdot \mathbf{p}_3 - \mathbf{x}_4 \cdot \mathbf{p}_4 = -\mathbf{R} \cdot \mathbf{K} - \mathbf{X} \cdot \mathbf{P} - \mathbf{x}_{31} \cdot \mathbf{p}_{31} - \mathbf{x}_{42} \cdot \mathbf{p}_{42} \tag{5.56}$$

Note that

$$\mathbf{p}_1^2 + \mathbf{p}_3^2 = \frac{1}{8} (\mathbf{K} + 2\mathbf{P})^2 + 2\mathbf{p}_{31}^2 \tag{5.57}$$

and

$$\mathbf{p}_2^2 + \mathbf{p}_4^2 = \frac{1}{8} (\mathbf{K} - 2\mathbf{P})^2 + 2\mathbf{p}_{42}^2 \tag{5.58}$$

These two identities allow us to write

$$\begin{aligned}
\overline{\mathcal{U}}_O \mathcal{U}_I = \; & \exp\left[-\frac{i}{\hbar} \mathbf{x}_{31} \cdot \mathbf{p}_{31} - \frac{i}{\hbar} \mathbf{x}_{42} \cdot \mathbf{p}_{42} - \frac{i}{\hbar} (\mathbf{R} \cdot \mathbf{K} + \mathbf{X} \cdot \mathbf{P}) \right] \\
& \times \exp\left(\frac{iT}{2m_a \hbar} \mathbf{p}_{31}^2 + \frac{iT}{2m_b \hbar} \mathbf{p}_{42}^2 \right) \\
& \times \exp\left(\frac{iT}{8\hbar} \left[\frac{1}{4m_a} (\mathbf{K} + 2\mathbf{P})^2 + \frac{1}{4m_b} (\mathbf{K} - 2\mathbf{P})^2 \right] \right)
\end{aligned} \tag{5.59}$$

Since $\mathcal{E}$ has no dependence on $\mathbf{R}$, the integral yields a Dirac delta:

$$\int \mathrm{d}R \, \exp\left(-\frac{i}{\hbar} \mathbf{R} \cdot \mathbf{K} \right) = \hbar^d \delta(\mathbf{K}) \tag{5.60}$$

This Dirac delta imposes the constraint $\mathbf{K} = 0$ which leads to

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 \tag{5.61}$$

This is nothing more than the conservation of the total external momentum. After this constraint is enforced, we have

$$\mathbf{P} = \mathbf{p}_3 - \mathbf{p}_1 = \mathbf{p}_2 - \mathbf{p}_4 \tag{5.62}$$

That is, $\mathbf{P}$ measures the momentum transfer between the particles. The eikonal paths (5.23) are valid in the regime where the incoming and outgoing momenta of each particle is much larger than the momentum transfer between the particles. That is,

$$\mathbf{p}_1^2 \gg \mathbf{P}^2 \qquad \mathbf{p}_2^2 \gg \mathbf{P}^2 \qquad \mathbf{p}_3^2 \gg \mathbf{P}^2 \qquad \mathbf{p}_4^2 \gg \mathbf{P}^2 \tag{5.63}$$

It then follows that

$$\mathbf{p}_{31}^2 \gg \mathbf{P}^2 \qquad \mathbf{p}_{42}^2 \gg \mathbf{P}^2 \tag{5.64}$$

This argument enables us to neglect the third line in (5.59). Thus, after the integration over $\mathbf{R}$ and using the eikonal approximation, we obtain

$$\begin{aligned} \overline{\mathcal{U}}_O(3,4)\mathcal{U}_I(1,2) = {} & \exp\left[-\frac{i}{\hbar}\mathbf{x}_{31}\cdot\mathbf{p}_{31} - \frac{i}{\hbar}\mathbf{x}_{42}\cdot\mathbf{p}_{42} - \frac{i}{\hbar}\mathbf{X}\cdot\mathbf{P}\right] \\ & \times \exp\left[\frac{iT}{2m_a\hbar}\mathbf{p}_{31}^2 + \frac{iT}{2m_b\hbar}\mathbf{p}_{42}^2\right] \end{aligned} \tag{5.65}$$

Next, we tackle the integration over $\mathbf{x}_{31}$ and $\mathbf{x}_{42}$. The exact integration is nontrivial since ρ is a function of $\mathbf{x} = \mathbf{x}_{42} - \mathbf{x}_{31}$. We make another change of variables:

$$\mathbf{x}_{31} = \frac{T}{m_a}\mathbf{k}_{31}, \qquad \mathbf{x}_{42} = \frac{T}{m_b}\mathbf{k}_{42} \tag{5.66}$$

Then

$$\frac{\mathbf{x}}{cT} = \frac{\mathbf{k}_{42}}{m_b c} - \frac{\mathbf{k}_{31}}{m_a c} = \left(\frac{m_a + m_b}{m_a m_b c}\right)\left(\frac{m_a\mathbf{k}_{24} - m_b\mathbf{k}_{31}}{m_a + m_b}\right) \equiv \frac{\mathbf{k}}{mc} \tag{5.67}$$

where m is the reduced mass,

$$m \equiv \frac{m_a m_b}{m_a + m_b} \tag{5.68}$$

Thus, ρ has no explicit dependence on T when written in terms of $\mathbf{k}_{31}$ and

$\mathbf{k}_{42}$. We now have

$$\overline{\mathcal{U}}_O \mathcal{U}_I \exp\left(-\frac{i}{\hbar}\Sigma_0\right) = \exp\left[\frac{iT}{2m_a\hbar}\left(\mathbf{k}_{31}-\mathbf{p}_{31}\right)^2 + \frac{iT}{2m_b\hbar}\left(\mathbf{k}_{42}-\mathbf{p}_{42}\right)^2\right] \times \exp\left[-\frac{i}{\hbar}\mathbf{X}\cdot\mathbf{P}\right] \tag{5.69}$$

In the limit $T \to \infty$ the integral over $\mathbf{k}_{31}$ and $\mathbf{k}_{42}$ is dominated by a stationary point:

$$\bar{\mathbf{k}}_{31} = \mathbf{p}_{31}, \qquad \bar{\mathbf{k}}_{42} = \mathbf{p}_{42} \tag{5.70}$$

At this stationary point, we have

$$\bar{\mathbf{k}} = \frac{m_a\mathbf{p}_{42} - m_b\mathbf{p}_{31}}{m_a+m_b} = \frac{1}{2}\left(\frac{m_a\mathbf{p}_2 - m_b\mathbf{p}_1}{m_a+m_b}\right) + \frac{1}{2}\left(\frac{m_a\mathbf{p}_4 - m_b\mathbf{p}_3}{m_a+m_b}\right) \tag{5.71}$$

Note that $\bar{\mathbf{k}}$ corresponds to the *average* of the incoming and outgoing reduced momenta.

After this integration is done, we remain with the integral over $\mathbf{X}$:

$$\mathcal{A}(3,4|1,2) \approx \lim_{T\to\infty} \hbar^d \delta(\mathbf{K}) \int \mathrm{d}X \left(\frac{2L^2}{\mathbf{B}^2}\right)^{i\alpha\rho} \exp\left[-\frac{i}{\hbar}\mathbf{X}\cdot\mathbf{P} + \Lambda_\varepsilon\right] \tag{5.72}$$

Earlier, we defined $\mathbf{B}$ as the component of $\mathbf{X}$ that is orthogonal to $\mathbf{x}$:

$$\mathbf{X} = \mathbf{B} + b\mathbf{x} \tag{5.73}$$

After integrating over $\mathbf{x}_{31}$ and $\mathbf{x}_{42}$, this becomes

$$\mathbf{X} = \mathbf{B} + \frac{bT}{m}\bar{\mathbf{k}} \tag{5.74}$$

The integration measure becomes

$$\mathrm{d}X = \frac{T|\bar{\mathbf{k}}|}{m}\mathrm{d}B\mathrm{d}b \tag{5.75}$$

Note that

$$\mathbf{X}\cdot\mathbf{P} = \mathbf{B}\cdot\mathbf{P} + \frac{Tb}{m}(\bar{\mathbf{k}}\cdot\mathbf{P}) \tag{5.76}$$

Since the integrand has no dependence on b, integration yields a Dirac delta:

$$\int \mathrm{d}b \exp\left[-\frac{iT(\bar{\mathbf{k}}\cdot\mathbf{P})}{m\hbar}b\right] = \frac{\hbar m}{T}\delta(\bar{\mathbf{k}}\cdot\mathbf{P}) \tag{5.77}$$

This one-dimensional Dirac delta imposes the constraint $\bar{\mathbf{k}} \cdot \mathbf{P} = 0$, which leads to

$$\frac{\mathbf{p}_{42} \cdot \mathbf{P}}{m_b} - \frac{\mathbf{p}_{31} \cdot \mathbf{P}}{m_a} = 0 \quad \Longrightarrow \quad \frac{\mathbf{p}_1^2}{2m_a} + \frac{\mathbf{p}_2^2}{2m_b} = \frac{\mathbf{p}_3^2}{2m_a} + \frac{\mathbf{p}_4^2}{2m_b} \tag{5.78}$$

This corresponds to conservation of the total kinetic energy.

Finally, we need to integrate over $\mathbf{B}$:

$$\begin{aligned} \mathcal{A}(3,4|1,2) \approx\ & i\alpha m c \hbar^4 \delta(\mathbf{K}) \delta\left(\bar{\mathbf{k}} \cdot \mathbf{P}\right) \\ & \times \left(\frac{|\bar{\mathbf{k}}|}{i\alpha m c}\right) \int \mathrm{d}B \left(\frac{2L^2}{\mathbf{B}^2}\right)^{i\alpha\rho} \exp\left[-\frac{i}{\hbar}\mathbf{B} \cdot \mathbf{P} + \Lambda_\varepsilon\right] \end{aligned} \tag{5.79}$$

In $d \approx 3$ dimensions the integral over $\mathbf{B}$ is over a $d-1 \approx 2$ dimensional volume. Integration gives

$$\begin{aligned} \mathcal{A}(3,4|1,2) \approx\ & i\alpha m c \hbar^4 \delta(\mathbf{K}) \delta\left(\bar{\mathbf{k}} \cdot \mathbf{P}\right) \\ & \times \left(\frac{\hbar}{\mu c}\right)^2 \left[\frac{\Gamma(1 - i\alpha\rho)}{\Gamma(1 + i\alpha\rho)}\right] \left(\frac{2\mu^2 c^2}{\mathbf{P}^2}\right)^{1-i\alpha\rho} \exp\left(\Lambda_\varepsilon\right) \end{aligned} \tag{5.80}$$

where μ is a constant with units of mass, related to L:

$$L = \frac{\hbar}{\mu c} \tag{5.81}$$

The appearance of the Euler Gamma function suggests an infinite number of singularities, satisfying the equation

$$1 - i\alpha\rho = -J, \qquad J = 0, 1, 2, \ldots \tag{5.82}$$

In order to make sense of these singularities, we introduce the reduced energy E and the reduced generalized Rydberg energy E_R

$$E \equiv \frac{\bar{\mathbf{k}}^2}{2m} \quad \Longrightarrow \quad |\bar{\mathbf{k}}| = \sqrt{2mE}, \qquad E_R \equiv \frac{1}{2}\alpha^2 m c^2 \tag{5.83}$$

These allow us to write

$$\alpha\rho = \frac{Z_a Z_b \alpha m c}{|\bar{\mathbf{k}}|} = Z_a Z_b \sqrt{\frac{E_R}{E}} \tag{5.84}$$

Thus, the singularities satisfy

$$1 - i Z_a Z_b \sqrt{\frac{E_R}{E_J}} = -J, \qquad J = 0, 1, 2, \ldots \tag{5.85}$$

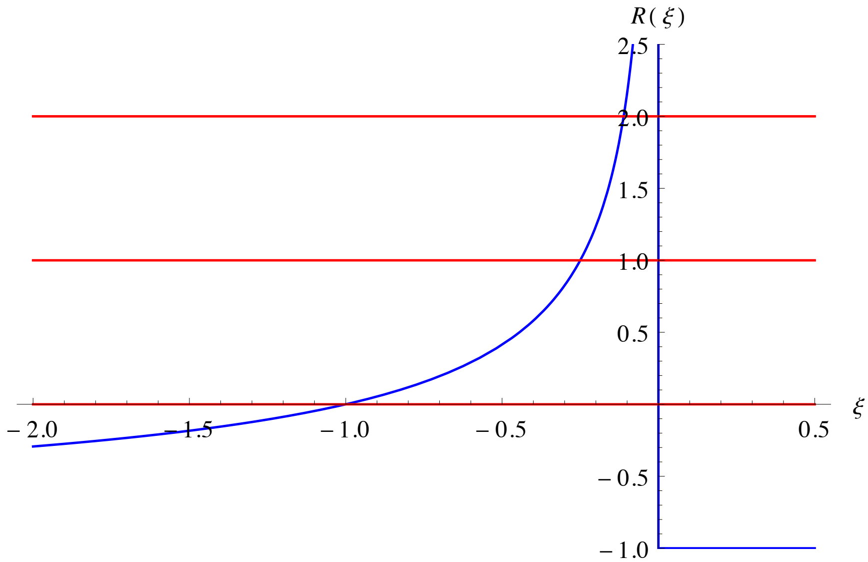


Figure 5.1: Real part of R as a function of $\xi \equiv E/E_R$. The red lines correspond to $R = 0, 1, 2$. We have used $Z_a Z_b = -1$.

Solving for E_J yields

$$E_J = -\frac{Z_a^2 Z_b^2 E_R}{(J+1)^2} \tag{5.86}$$

which is the familiar Coulomb spectrum. Note that we must require $Z_a Z_b < 0$.

The amplitude (5.80) exhibits Regge behavior with leading Regge trajectory function $R(E)$ given by

$$R(E) = -1 + iZ_a Z_b \sqrt{\frac{E_R}{E}} \tag{5.87}$$

Figure 5.1 has a plot of the real part of (5.87).

Chapter 6

Classical Relativistic Particles

We begin our approach towards the relativistic theory with the classical description of relativistic particles. Scalar particles do not have any intrinsic spin degrees of freedom. We will only consider massive scalar particles.

6.1 Free System

The simplest case is the free system. In order to describe a relativistic particle in a Lorentz-covariant way, we must treat the time parameter t in the same way that we treat the d spatial coordinates $\mathbf{x}$. This is best accomplished by working in D-dimensional spacetime, with $D = d + 1$. The D-position vector q has components

$$q = (t, \mathbf{x}) \tag{6.1}$$

Of course, we will use the "mostly plus" signature.

The relativistic dynamics of a scalar particle can be described by a path $q(\tau)$ in spacetime (the worldline). In order to respect Lorentz covariance, we parametrize both the spatial coordinates *and* the time parameter with the same parameter τ. This parameter should not have a particular meaning, so the formalism must be valid for any choice of parametrization of the path. In the end, we always resort to describing the evolution of the spatial coordinates along time, so we expect the spacetime description to have some redundancy.

In Hamiltonian form, the action functional for a relativistic **free massive particle** is

$$S_0[q, p, v] = \int \mathrm{d}\tau \left[-\dot{q} \cdot p + v \left(\frac{p^2 + m^2}{2} \right) \right] \tag{6.2}$$

where $q(\tau)$ is the D-position, $p(\tau)$ is the classical conjugate D-momentum and $v(\tau)$ acts as the worldline metric. This action functional is similar to the nonrelativistic one, with the main difference that the mass m only appears once.

Let us first consider the equation of motion for $v(\tau)$:

$$p^2 + m^2 = 0 \tag{6.3}$$

This is a constraint that relates the components of the D-momentum. Since this constraint follows from an equation of motion, it must be satisfied by classical particles. We will leave p unconstrained for the moment.

The equation of motion for $p(\tau)$ gives

$$-\dot{q} + vp = 0 \Longrightarrow p(\tau) = \frac{1}{v(\tau)}\dot{q} \tag{6.4}$$

Using this solution for p in the action functional yields

$$S_0[q, v] = \int \mathrm{d}\tau \left[-\frac{1}{2v}\dot{q}^2 + \frac{v}{2}m^2\right] \tag{6.5}$$

This is the Lagrangian form of the action functional. We can eliminate v by solving its equation of motion. This leads to an action that depends only on the D-position q. However, the resulting functional is not very practical.

The action (6.5) is invariant under reparametrizations of τ. This is a gauge symmetry. If we want to work with (6.5) we need to perform a gauge-fixing procedure. The particular details of this are not relevant for what follows and can be found in many references (see section III.B.1 of [4]). After gauge-fixing we find

$$S_0[q, T] = \int \mathrm{d}\tau \left[-\frac{1}{2}\dot{q}^2 + \frac{1}{2}m^2\right], \qquad 0 < \tau < T, \qquad T > 0 \tag{6.6}$$

This choice of gauge-fixing effectively amounts to setting $v = 1$. The constant parameter T is a leftover from the gauge invariance. We will refer to T as the **worldline modulus**.

In the relativistic theory we can set $c = 1$. This means that time intervals and spatial distances have the same units, and similarly for momentum, energy and mass. Action functionals have units of $\hbar$, which effectively correspond to units of mass multiplied by length. The spacetime D-position has units of length. Thus, the worldline parameter τ and the modulus T have units

$$[\tau] = [T] = [\hbar] - 2[\text{mass}] \tag{6.7}$$

In the next section we discuss different terms that can be added to the free gauge-fixed action (6.6) in order to incorporate interactions. The gauge-fixing procedure has already been dealt with for each one of these terms (otherwise, there would be some dependence on v).

6.2 Coupling to External Fields

In nonrelativistic classical mechanics one can study many-body systems where the constituents interact via arbitrary interaction potentials (although solvability is another issue). The relativistic theory requires us to only consider local interactions. The safest way to guarantee locality is to introduce an external mediating agent that couples to each body separately. One adds a term in the action $S_{\text{int}}[q, F]$ that accounts for the coupling of the particle described by q to a *fixed* external mediating agent F. The mediating agent F is made dynamical after including an appropriate kinetic term $S_{\text{kin}}[F]$ in the action functional of the system. The result is a system of classical particles interacting via a dynamical classical field.

Scalar Field

The coupling to a scalar field ϕ can be accomplish by adding a term of the form

$$S_{\text{int}}[q, \phi] = \int \mathrm{d}\tau \, \phi[q(\tau)] \tag{6.8}$$

The field ϕ has units

$$[\phi] = 2[\text{mass}] \tag{6.9}$$

in *any* number of dimensions. This coupling term is analogous to the terms that appear in the nonrelativistic theory in (5.4), except that the time integral there is replaced here with the integral over the worldline parameter. We do not need to include any charges.

Vector Field

The coupling to a vector field A_m can be described with a term of the form

$$S_{\text{int}}[q, A] = Z \int \mathrm{d}\tau \, \dot{q}^m A_m[q(\tau)] \tag{6.10}$$

where Z is a dimensionless charge. Note that the field A_m has units of mass in *any* number of dimensions.

Symmetric Tensor Field

Finally, we can couple a massive scalar particle to a symmetric tensor field h_{mn} by including a term of the form

$$S_{\text{int}}[q,h] = \frac{1}{2}\int \mathrm{d}\tau\, \dot{q}^m \dot{q}^n h_{mn}[q(\tau)] \tag{6.11}$$

This term describes the coupling to linearized gravity. The field h_{mn} is dimensionless in *any* number of dimensions.

Higher-Spin Fields

All three previous cases can be viewed as particular examples of the coupling term

$$S_{\text{int}}[q,H] = \frac{1}{\Gamma(N+1)}\int \mathrm{d}\tau\, \dot{q}^{m_1}\cdots\dot{q}^{m_N} H_{m_1\cdots m_N}[q(\tau)] \qquad N \geq 0 \tag{6.12}$$

with H totally symmetric in the spacetime indices. The field H has units of

$$[H] = (2-N)[\text{mass}] \tag{6.13}$$

in *any* number of dimensions.

The external scalar is obtained with $N=0$, the external vector with $N=1$, and the external symmetric tensor with $N=2$. The next natural step is $N=3$:

$$S_{\text{int}}[q,W] = \frac{1}{6}\int \mathrm{d}\tau\, \dot{q}^m \dot{q}^n \dot{q}^l W_{mnl}[q(\tau)] \tag{6.14}$$

The nice thing about the external scalar, the external vector and the external tensor is that the coupling terms are *at most quadratic* in $\dot{q}$. With $N=3$, one finds a coupling term that is *cubic* in $\dot{q}$. In any case, the physical meaning of dynamical massless fields with spin larger than 2 is notoriously obscure. We will not consider couplings to such fields.

6.3 Semiclassical Dimensional Analysis

We have already mentioned the units of some of the quantities that appear in the action functional for a particle. The external fields that appear have the familiar mass-dimension for a mediating field. We did not include the coupling parameter in these interaction terms. The coupling parameter will appear in the kinetic term for the mediating field, as is customary in Yang-Mills theory.

Since the external mediating fields that we consider have the usual mass-dimension, we could expect (incorrectly) that the interactions mediated by a field between *matter particles* are very similar to the interactions mediated by a field between *matter fields*. In this section we use dimensional analysis to show that, in the semiclassical approximation, these two types of theories can behave very differently.

We start with the kinetic term for a *matter* field φ:

$$S_{\text{kin}}[\varphi] = \frac{1}{2}\int\int \mathrm{d}x\mathrm{d}y\,[\varphi(x)\cdot K_\varphi(x|y)\cdot\varphi(y)] \tag{6.15}$$

(i.e. a kinetic term with no coupling parameter). The kinetic operator K_φ has units

$$\begin{aligned}[K_\varphi] &= -(D+2)[\text{length}] \\ &= -(D+2)[\hbar] + (D+2)[\text{mass}]\end{aligned} \tag{6.16}$$

The action has units of $\hbar$, so we find that the *matter* field φ has units

$$[\varphi] = \left(\frac{3-D}{2}\right)[\hbar] + \left(\frac{D-2}{2}\right)[\text{mass}] \tag{6.17}$$

The mass-dimension is familiar, but the $\hbar$-dimension is seldom mentioned, since one typically sets $\hbar = 1$ (i.e. $[\hbar] = 0$). We will see that the $\hbar$-dimension allows us to understand better the dynamical consequences of the semiclassical approximation.

In this dissertation we consider theories of matter quanta interacting via the exchange of mediating quanta (the force carriers). The matter quanta is described in terms of *particles*. For comparison, let us briefly consider the theory of a matter field φ interacting with a massless scalar field ϕ_f via a term of the form

$$S_n[\varphi,\phi_f] = \int \mathrm{d}x\,\varphi_a^2(\phi_f)^n \tag{6.18}$$

In Feynman graphs, this term leads to an interaction vertex of degree $n+2$. Since φ is a matter field, it has units given by (6.17). Since S_n has units of $\hbar$, with the units φ we can find the units of ϕ_f,

$$[\phi_f] = -\frac{2}{n}[\text{length}] = -\frac{2}{n}[\hbar] + \frac{2}{n}[\text{mass}] \tag{6.19}$$

Finally, the kinetic term for the field ϕ_f is

$$\frac{1}{2(f_n)^{2/n}} \int\int \mathrm{d}x\mathrm{d}y \, [\phi_f(x) K_\phi(x|y) \phi_f(y)] \tag{6.20}$$

It follows that the coupling parameter f_n has units

$$[f_n] = \left(\frac{nD - 3n - 4}{2} \right) [\hbar] + \left(\frac{4 + 2n - nD}{2} \right) [\text{mass}] \tag{6.21}$$

When $n = 1$, we have

$$[f_1] = \left(\frac{D - 7}{2} \right) [\hbar] + \left(\frac{6 - D}{2} \right) [\text{mass}] \tag{6.22}$$

Other interesting cases are $n = 2$,

$$[f_2] = (D - 5) \, [\hbar] + (4 - D) \, [\text{mass}] \tag{6.23}$$

and $n = 4$,

$$[f_4] = 2 \, (D - 4) \, [\hbar] + 2 \, (3 - D) \, [\text{mass}] \tag{6.24}$$

The case $n = 1$ corresponds to a cubic interaction. The mass-dimension is familiar, but again, the $\hbar$-dimension is seldom mentioned. In practice, we expect the coupling parameter to appear in the form of a dimensionless combination α_f given by

$$\alpha_f = k_f f_1^2 \hbar^{(7-D)} \mu^{(D-6)} \tag{6.25}$$

where k_f is a numerical constant and μ has units of mass. Note the power of $\hbar$ in this expression. If $D < 7$ and we let $\hbar \to 0$ while keeping f_1^2 fixed, then $\alpha_f \to 0$. Thus, in $D < 7$ (which includes $D = 4$), the semiclassical approximation of the theory with ϕ_f yields a *weak-coupling* expansion.

The coupling term (6.8) for a scalar particle to an external scalar field also corresponds to a cubic interaction. We will denote the field that appears there by ϕ_p since it couples to a *particle*. We already found that ϕ_p has units

$$[\phi_p] = 2[\text{mass}] \tag{6.26}$$

The kinetic term for ϕ_p is similar to (6.20),

$$\frac{1}{2g_0^2} \int\int \mathrm{d}x\mathrm{d}y \, [\phi_p(x) K_\phi(x|y) \phi_p(y)] \tag{6.27}$$

but we expect the coupling parameter g_0 to have different units. Indeed,

$$[g_0] = \left(\frac{D-3}{2}\right)[\hbar] + \left(\frac{6-D}{2}\right)[\text{mass}] \tag{6.28}$$

The corresponding dimensionless combination α_p is now given by

$$\alpha_p = k_p g_0^2 \hbar^{(3-D)} \mu^{(D-6)} \tag{6.29}$$

where k_p is a numerical constant and μ has units of mass. If $D > 3$ and we set $\hbar \to 0$ while keeping g_0^2 fixed, then $\alpha_p \to \infty$. That is, in $D > 3$ (which includes $D = 4$), the semiclassical approximation of the theory with ϕ_p yields a *strong-coupling* expansion!

The previous example illustrates that the interaction between fields, and the interaction between fields and particles, are different, at least in the semiclassical approximation.

We can generalize the particle coupling to a scalar in (6.8) to

$$S_{\text{int}}[q, \phi_p] = \int \mathrm{d}\tau\, \phi_p^n[q(\tau)], \qquad n \geq 1 \tag{6.30}$$

Now the field ϕ_p has units

$$[\phi_p] = \frac{2}{n}[\text{mass}] \tag{6.31}$$

Hence, from the kinetic term for the field ϕ_p,

$$\frac{1}{2(g_p)^{2/n}} \int\int \mathrm{d}x \mathrm{d}y\, [\phi_p(x) K_\phi(x|y) \phi_p(y)] \tag{6.32}$$

we find that the coupling parameter g_p has units

$$[g_p] = \left(\frac{nD-3n}{2}\right)[\hbar] + \left(\frac{4+2n-nD}{2}\right)[\text{mass}] \tag{6.33}$$

Thus, as long as $D > 3$, we find that the semiclassical approximation leads to a strong-coupling expansion.

A similar outcome follows for the interaction mediated by a massless vector field. Consider a massless vector field A_f coupled to matter. The interaction is introduced via the covariant derivative,

$$\partial \longrightarrow \nabla = \partial + iA_f \tag{6.34}$$

Thus, the field A_f has units

$$[A_f] = -[\text{length}] = -[\hbar] + [\text{mass}] \tag{6.35}$$

From the kinetic term

$$\frac{1}{g_f^2}\int\int \mathrm{d}x\mathrm{d}y \left[\frac{1}{2}A_f(x)\cdot K_A(x|y)\cdot A_f(y)\right] \tag{6.36}$$

we find that the coupling parameter g_f has units

$$[g_f] = \left(\frac{D-5}{2}\right)[\hbar] + \left(\frac{4-D}{2}\right)[\text{mass}] \tag{6.37}$$

The dimensionless combination α_f is

$$\alpha_f = k_f g_f^2 \hbar^{(5-D)}\mu^{(D-4)} \tag{6.38}$$

Like in the theory with ϕ_f, if $D < 5$ and we let $\hbar \to 0$ while keeping g_f^2 fixed, then $\alpha_f \to 0$. Thus, in $D < 5$ (which includes $D = 4$), the semiclassical approximation of the theory with A_f yields a *weak-coupling* expansion.

In the particle coupling (6.10), the field A_p has units of mass. Thus, from the kinetic term

$$\frac{1}{g_1^2}\int\int \mathrm{d}x\mathrm{d}y \left[\frac{1}{2}A_p(x)\cdot K_A(x|y)\cdot A_p(y)\right] \tag{6.39}$$

we find that the coupling parameter g_1 has units

$$[g_1] = \left(\frac{D-3}{2}\right)[\hbar] + \left(\frac{4-D}{2}\right)[\text{mass}] \tag{6.40}$$

The mass-dimension agrees with the expected result, but the $\hbar$-dimension is different from (6.37). The dimensionless combination α_p is

$$\alpha_p = k_p g_1^2 \hbar^{(3-D)}\mu^{(D-4)} \tag{6.41}$$

Just like in the theory with ϕ_p, if $D > 3$ and we set $\hbar \to 0$ while keeping g_1^2 fixed, then $\alpha_p \to \infty$. So again, we find that in $D > 3$ (which includes $D = 4$), the semiclassical approximation of the theory with A_p yields a *strong-coupling* expansion.

The last example is the gravitational interaction, where the field h_f is a massless symmetric tensor. Since the field corresponds to the linearized metric tensor, it is dimensionless:

$$[h_f] = 0 \tag{6.42}$$

From the kinetic term, we find that the coupling g_f has units

$$[g_f] = \left(\frac{D-3}{2}\right)[\hbar] + \left(\frac{2-D}{2}\right)[\text{mass}] \tag{6.43}$$

The dimensionless combination α_f is

$$\alpha_f = k_f g_f^2 \hbar^{(3-D)} \mu^{(D-2)} \tag{6.44}$$

The semiclassical approximation yields a weak-coupling expansion when $D < 3$ and a strong-coupling expansion when $D > 3$ (which includes $D = 4$). In the particle coupling (6.11), the field h_p is also dimensionless, and thus we find the same feature.

Indeed, if we consider the particle coupling to a spin N field as in (6.12), we would find that the coupling parameter has units

$$[g_N] = \left(\frac{D-3}{2}\right)[\hbar] + \left(\frac{6-2N-D}{2}\right)[\text{mass}] \tag{6.45}$$

which can be compared to the coupling parameter for the interaction between fields,

$$[f_N] = \left(\frac{D+2N-7}{2}\right)[\hbar] + \left(\frac{6-2N-D}{2}\right)[\text{mass}] \tag{6.46}$$

We see that, in the semiclassical approximation, the particle coupling parameter g_N becomes very large for any value of N when $D > 3$.

The moral of this discussion is that there are different semiclassical approximations: the semiclassical approximation for fields mediating the interactions between matter *particles*, and the semiclassical approximation for fields mediating the interaction between matter *fields*. More details about the relation between the semiclassical approximation in field theory and particle theory can be found in [67].

We have found that the semiclassical approximation has dynamical consequences. In theories with particle couplings, it leads to a strong-coupling expansion in $D = 4$, and in theories with field couplings, it sometimes leads to a weak-coupling expansion in $D = 4$. Since we are going to work with particles in $D = 4$, we expect the semiclassical approximation to yield nonperturbative results. As we will see, the results will be nonperturbative in the sense that they correspond to all orders in perturbation theory. It is due to this nonperturbative nature that we will be able to find bound states.

Chapter 7

Relativistic Path Integrals

In this chapter we introduce the relativistic analogs of the quantum and semiclassical kernels that were introduced in chapter 4.

Note

Starting in this chapter, and unless otherwise specified, we set $\hbar = 1$.

7.1 Quantum Kernels

In nonrelativistic quantum mechanics we first introduced the quantum kernel as a sort of "metric tensor" in the inner product between two state vectors in the Hilbert space (see section 4.1). There are different ways to describe the Hilbert space, according to whether the state vectors carry the time dependence (the Schrödinger picture), or the operators do (the Heisenberg picture). The culmination is either the Schrödinger equation for the wavefunction, or the Heisenberg equation for the operators. Both of these relate the time evolution to the spatial evolution. But this description is, of course, not compatible with Lorentz symmetry.

The other formulation of the nonrelativistic quantum kernel is as a path integral. The classical description of the nonrelativistic problem is key for constructing this path integral, since the classical action functional appears in it. Classically, we describe a relativistic system with parametrized spacetime variables. Instead of developing the relativistic analogs of the Schödinger and Heisenberg pictures (along with all the acrobatics in the Hilbert space), we will generalize the path integral formulation to accommodate the relativistic theory.

In Lagrangian form, the action for a free massive scalar particle is

$$S_0[q,v] = \int \mathrm{d}\tau \left[-\frac{1}{2v}\dot{q}^2 + \frac{v}{2}m^2 \right] \tag{7.1}$$

Since we have two functional variables ($q(\tau)$ and $v(\tau)$), the quantum kernel should involve a functional integration over both variables. We *define* the **quantum kernel** for a (not necessarily free) massive particle as

$$\mathcal{F}(O|I) \equiv \int \mathrm{D}v(\tau) \int_{x_I}^{x_O} \mathrm{D}q(\tau) \exp\left(-iS[q,v]\right) \tag{7.2}$$

Integrating over v is analogous to using the equation of motion to solve for v in terms of q. As we already mentioned, we are not going to do this. We will instead perform a gauge-fixing that essentially amounts to setting $v = 1$. This procedure does not remove v entirely, but leaves the "global" part. After gauge-fixing, the relativistic path integral becomes

$$\mathcal{F}(O|I) = \int_0^\infty \mathrm{d}T \int_{x_I}^{x_O} \mathrm{D}q(\tau) \exp\left(-iS[q,T]\right) \tag{7.3}$$

where the integral over the modulus T is a traditional integral (i.e. not functional). In contrast with (4.27), the main difference between the relativistic and nonrelativistic path integrals is the integration over the modulus. For this reason, we introduce the **un-integrated quantum kernel** $\mathcal{F}_T$

$$\mathcal{F}(O|I) = \int_0^\infty \mathrm{d}T \mathcal{F}_T(O|I) \tag{7.4}$$

That is,

$$\mathcal{F}_T(O|I) \equiv \int_{x_I}^{x_O} \mathrm{D}q(\tau) \exp\left(-iS[q,T]\right) \tag{7.5}$$

which is completely analogous to (4.27), albeit modulus-dependent. We will exploit this analogy and use $\mathcal{F}_T$ to define relativistic analogs of all the tools introduced in chapter 4. Working with un-integrated kernels is not a problem as long as we remember that the true quantum description is obtained after integrating over the modulus.

7.2 S-Matrix

In what follows we will assume that the proper gauge-fixing procedure has already been carried out. By analogy with (4.23), we *define* the **un-integrated S-matrix** by

$$\mathcal{S}_T(O|I) = \int\int \mathrm{d}x_I \mathrm{d}x_O \overline{\mathcal{W}}_O(O)\mathcal{W}_I(I)\mathcal{F}_T(O|I) \tag{7.6}$$

where $\overline{\mathcal{W}}_O$ and $\mathcal{W}_I$ are the relativistic analogs of $\overline{\mathcal{U}}_O$ and $\mathcal{U}_I$:

$$\mathcal{W}_I(I) = \exp\left[\frac{iT}{4}\left(p_I^2 + m_I^2\right) + ix_I \cdot p_I\right] \tag{7.7}$$

$$\overline{\mathcal{W}}_O(O) = \exp\left[\frac{iT}{4}\left(p_O^2 + m_O^2\right) - ix_O \cdot p_O\right] \tag{7.8}$$

Before integrating over the modulus T, the masses of the "in" and "out" quanta are different from the particle mass m that appears in the action functional. Constraints will result from the integration over the modulus that relate the external masses m_I and m_O to the internal mass m. This sounds a bit odd, but it works. Part of the truncation will involve removing these constraints.

The nonrelativistic asymptotic S-matrix is replaced by the **integrated S-matrix**,

$$\mathcal{A}(O|I) \equiv \int_0^\infty \mathrm{d}T \mathcal{S}_T(O|I) \tag{7.9}$$

At this stage, the external momenta are still off-shell. Before we can put them on-shell, we have to perform a truncation. We will see how this works experimentally in chapter 8.

7.3 Semiclassical Kernels

After gauge-fixing at the level of the classical theory, we go from an action functional $S[q, v]$ to an action functional $S[q, T]$. In practice, this later functional is the one that is used to find classical paths. We *define* the integrated **semiclassical kernel** $\mathcal{V}$ by

$$\mathcal{V}(O|I) \equiv \int_0^\infty \mathrm{d}T \sqrt{-\det(V)} \exp(-i\Sigma) \tag{7.10}$$

The sign with the determinant follows from working in Minkowski signature. Here, just like before, the **Van Vleck function** Σ corresponds to the value of the action functional at the classical path, and the **Van Vleck matrix** V is defined by

$$V = -i\frac{\partial^2 \Sigma}{\partial x_I \partial x_O} \tag{7.11}$$

In practice it is more convenient to work with the **un-integrated semiclassical kernel** $\mathcal{V}_T$,

$$\mathcal{V}_T(O|I) \equiv \sqrt{-\det{(V)}} \exp{(-i\Sigma)} \tag{7.12}$$

which is completely analogous to (4.32). However, we will not derive (7.12). Since our definition of the relativistic semiclassical kernel is based on the relativistic path integral, a proper way to derive (7.12) should rely on functional methods. In principle, after gauge-fixing $v = 1$, we can formulate a "Schrödinger equation" with the worldline parameter playing the role of time. Then one could "derive" (7.12) in the same way the nonrelativistic case was derived in section 4.3. This method does not generalize to many-body systems, since one has multiple worldlines and thus multiple worldline parameters.

The relativistic analog of the de Broglie wavelength is the Compton wavelength,

$$\lambda_C = \frac{2\pi\hbar}{mc} \tag{7.13}$$

In the relativistic semiclassical approximation, λ_C is small compared to the other distances in the problem. This means that the mass m is very large compared to other mass scales. In contrast with the nonrelativistic semiclassical approximation, we argue that the relativistic semiclassical approximation is not necessarily a high-energy approximation. This will be more clear when we consider two-body systems.

7.3.1 Semiclassical S-Matrix

The **un-integrated semiclassical S-matrix** is defined in complete analogy with (4.71):

$$\mathcal{S}_T(p_O|p_I) \approx \int\int \mathrm{d}x_I \mathrm{d}x_O \overline{\mathcal{W}}_O(O)\mathcal{W}_I(I)\mathcal{V}_T(O|I) \tag{7.14}$$

Similarly, we can define the **integrated semiclassical S-matrix**.

7.4 Semiclassical Eikonal Kernels

In the relativistic theory, the **eikonal path** describes a line in *spacetime*,

$$e(\tau) = \frac{x_I + x_O}{2} + (x_O - x_I)\left(\frac{\tau}{T}\right) \tag{7.15}$$

Just like in the nonrelativistic theory, the range of the worldline parameter is

$$-\frac{T}{2} < \tau < \frac{T}{2}, \qquad T > 0 \tag{7.16}$$

This parametrization is convenient since it is symmetric with center at $\tau = 0$. This choice explains the appearance of T in $\overline{\mathcal{W}}_O$ and $\mathcal{W}_I$. As we will see later, this choice of parametrization makes clear which terms need to be truncated before putting the external states on the mass-shell.

The **semiclassical eikonal kernel** and the corresponding **semiclassical eikonal S-matrix** are defined in the analogous way.

7.5 Many-body Systems

One can study relativistic many-body systems by considering path integrals with many functional variables. The main difference is that in the relativistic theory each body is described by a different worldline. Each worldline has a different parametrization. Thus, after gauge fixing, we are left with more than one modulus.

For example, the integrated quantum kernel for a two-body system with particles a and b has the form

$$\mathcal{F}(3,4|1,2) = \int_0^\infty \mathrm{d}T_a \int_0^\infty \mathrm{d}T_b \, \mathcal{F}_T(3,4|1,2) \tag{7.17}$$

with the un-integrated quantum kernel given by

$$\mathcal{F}_T(3,4|1,2) = \int_{x_1}^{x_3} \mathrm{D}q_a(\tau) \int_{x_2}^{x_4} \mathrm{D}q_b(\sigma) \exp\left(-iS[q_a, q_b]\right) \tag{7.18}$$

The range of the worldline parameters τ and σ are

$$-\frac{T_a}{2} < \tau < \frac{T_a}{2} \qquad -\frac{T_b}{2} < \sigma < \frac{T_b}{2}, \qquad T_a > 0, \qquad T_b > 0 \tag{7.19}$$

Chapter 8

Four-Point Scattering of Scalars

In this chapter we compute four-point scattering amplitudes in the semiclassical eikonal approximation. We will study the *elastic* event

$$a(p_1) + b(p_2) \longrightarrow a(p_3) + b(p_4) \tag{8.1}$$

The external states are massive,

$$p_1^2 = p_3^2 = -m_a^2 \qquad p_2^2 = p_4^2 = -m_b^2 \tag{8.2}$$

The relativistic *semiclassical* approximation corresponds to large masses. We have the three Mandelstam invariants,

$$s = -(p_1 + p_2)^2 \qquad t = -(p_1 - p_3)^2 \qquad u = -(p_1 - p_4)^2 \tag{8.3}$$

with

$$s + t + u = 2m_a^2 + 2m_b^2 \tag{8.4}$$

Since the only open exchange channel (i.e. the kinematical variable that measures the energy-momentum of the exchanged particle) is t, we *define* the two-body semiclassical approximation as

$$\frac{t}{m_a^2} \to 0 \qquad \frac{t}{m_b^2} \to 0 \tag{8.5}$$

The *eikonal* approximation corresponds to the regime of small-angle scattering and fixed spatial velocity. The cosine of the scattering angle is

$$\cos(\theta_s) = 1 + \frac{2st}{[s - (m_a - m_b)^2][s - (m_a + m_b)^2]} \tag{8.6}$$

and the spatial velocities[1] are

$$|\mathbf{v}_1| = |\mathbf{v}_3| = \frac{\sqrt{[s-(m_a-m_b)^2][s-(m_a+m_b)^2]}}{s+m_a^2-m_b^2} \tag{8.7}$$

$$|\mathbf{v}_2| = |\mathbf{v}_4| = \frac{\sqrt{[s-(m_a-m_b)^2][s-(m_a+m_b)^2]}}{s-m_a^2+m_b^2} \tag{8.8}$$

We *define* the **two-body semiclassical eikonal approximation** as

$$\frac{t}{m_a m_b} \to 0 \qquad \frac{t}{s} \to 0 \qquad \text{fixed } \frac{s}{m_a m_b} \qquad \text{fixed } \frac{u}{m_a m_b} \qquad \text{fixed } \frac{m_a}{m_b} \tag{8.9}$$

These are the approximations that we use in this chapter. Note that, unlike in previous work on the eikonal approximation, we do not explicitly take $s \to \infty$.

First, in §8.1 we compute the amplitude for two particles exchanging massless scalar quanta. The bulk of this first computation is long, but it is meant to introduce the philosophy behind our methods. In §8.2 and §8.3 we consider particles exchanging massless quanta with spin: First the exchange of massless vector quanta (photons), and then the exchange of massless symmetric tensor quanta (linearized gravitons). The resulting amplitudes are very similar to those found for the massless scalar exchange. Finally, in §8.4 we consider particles exchanging massive scalar quanta. The results in this section are not as interesting as those in the previous three sections, but nevertheless, they point to some curious properties of scattering in three spacetime dimensions.

8.1 Massless Scalar Exchange

The starting point is a two-body path integral for two massive scalar particles a and b that are coupled to an external massless scalar field ϕ,

$$\mathcal{G}_\phi[\phi] = \int_{x_1}^{x_3} \mathrm{D}q_a(\tau) \int_{x_2}^{x_4} \mathrm{D}q_b(\sigma)\, \exp\left(-iS_P[q_a, q_b, \phi]\right) \tag{8.10}$$

The particle action functional S_P is

$$S_P[q_a, q_b, \phi] \equiv S_0[q_a, q_b] + S_{\text{int}}[q_a, q_b, \phi] \tag{8.11}$$

[1] See appendix C for more details.

with the free term S_0 given by

$$S_0\left[q_a,q_b\right] \equiv \int \mathrm{d}\tau \left[-\frac{1}{2}\dot{q}_a^2+\frac{1}{2}m_a^2\right]+\int \mathrm{d}\sigma \left[-\frac{1}{2}\dot{q}_b^2+\frac{1}{2}m_b^2\right] \tag{8.12}$$

and the term with the coupling to the external field ϕ is given by

$$S_{\mathrm{int}}[q_a,q_b,\phi] \equiv \int \mathrm{d}\tau\, \phi[q_a(\tau)]+\int \mathrm{d}\sigma\, \phi[q_b(\sigma)] \tag{8.13}$$

We integrate over the field ϕ in order to obtain the "effective" interacting two-body path integral, which we will call the **two-body quantum kernel**:

$$\mathcal{F}_\phi(3,4|1,2) \equiv \int \mathrm{D}\phi(x)\, \mathcal{G}_\phi[\phi] \exp\left(-iS_{\mathrm{kin}}[\phi]\right) \tag{8.14}$$

The functional measure $\mathrm{D}\phi(x)$ is normalized such that

$$\int \mathrm{D}\phi(x)\, \exp\left(-iS_{\mathrm{kin}}[\phi]\right)=1 \tag{8.15}$$

Since ϕ is a massless scalar field, the functional S_{kin} is given by

$$S_{\mathrm{kin}}[\phi] \equiv \frac{1}{2g_0^2}\int\int \mathrm{d}x\mathrm{d}y\left[\phi(x)K_0(x|y)\phi(y)\right] \tag{8.16}$$

with

$$K_0(x|y) \equiv \delta(x-y)\left(-\frac{1}{2}\partial^2\right) \tag{8.17}$$

and g_0 is a *dimensionful* coupling parameter. In order to integrate over ϕ, it is convenient to rewrite S_{int} as

$$S_{\mathrm{int}}[q_a,q_b,\phi]=\int \mathrm{d}x\, J(x)\phi(x) \tag{8.18}$$

with

$$J(x) \equiv \int \mathrm{d}\tau\, \delta[x-q_a(\tau)]+\int \mathrm{d}\sigma\, \delta[x-q_b(\sigma)] \tag{8.19}$$

After integrating over ϕ, we find

$$\mathcal{F}_\phi(3,4|1,2)=\int_{x_1}^{x_3} \mathrm{D}q_a(\tau)\int_{x_2}^{x_4} \mathrm{D}q_b(\sigma)\, \exp\left(-iS_\phi[q_a,q_b]\right) \tag{8.20}$$

with

$$S_\phi[q_a, q_b] \equiv S_0[q_a, q_b] - \frac{g_0^2}{2} \int\int \mathrm{d}x\mathrm{d}y \,[J(x)G_0(x|y)J(y)] \tag{8.21}$$

Here G_0 is the massless scalar Green function,

$$G_0(x|y) \equiv [K_0(x|y)]^{-1} \tag{8.22}$$

Using the explicit form of J we find

$$\frac{g_0^2}{2} \int\int \mathrm{d}x\mathrm{d}y \,[J(x)G_0(x|y)J(y)] = S_1^\phi[q_a, q_b] + S_2^\phi[q_a, q_b] \tag{8.23}$$

with S_1^ϕ containing terms that connect a particle worldline to itself,

$$\begin{aligned} S_1^\phi[q_a, q_b] \equiv \frac{g_0^2}{2} &\int\int \mathrm{d}\tau_1\mathrm{d}\tau_2 \, G_0\left[q_a(\tau_1)|q_a(\tau_2)\right] \\ &+ \frac{g_0^2}{2} \int\int \mathrm{d}\sigma_1\mathrm{d}\sigma_2 \, G_0\left[q_b(\sigma_1)|q_b(\sigma_2)\right] \end{aligned} \tag{8.24}$$

and S_2^ϕ containing a term that connects two different particle worldlines,

$$S_2^\phi[q_a, q_b] \equiv g_0^2 \int\int \mathrm{d}\tau\mathrm{d}\sigma \, G_0\left[q_a(\tau)|q_b(\sigma)\right] \tag{8.25}$$

Thus, after the functional integral over ϕ is done, we find the action functional (8.21) for a system of interacting massive particles with one-body and two-body interaction terms where the massless scalar Green function

$$G_0(x|y) = -i\Gamma\left(\frac{D-2}{2}\right)\left[\frac{2}{(x-y)^2}\right]^{(D-2)/2} \tag{8.26}$$

plays the role of an external potential. In what follows we will *ignore* the contributions from the self-interactions.

We set $\hbar = 1$ and $c = 1$. This means that action functionals are dimensionless. In §6.3 we found that the coupling g_0 has units

$$[g_0] = \left(\frac{6-D}{2}\right)[\text{mass}] \tag{8.27}$$

We are going to work in $D = 4$, where g_0 has units

$$D = 4: \qquad [g_0] = [\text{mass}] \tag{8.28}$$

and in $D = 3$, where g_0 has units

$$D = 3: \qquad [g_0] = \frac{3}{2}[\text{mass}] \tag{8.29}$$

In anticipation of infrared problems, we are going to use dimensional regularization. When we study the four-dimensional theory, we shall write

$$g_0^2 = \frac{\alpha_0}{2\pi}\mu^{(4-D)} \tag{8.30}$$

where both $\sqrt{\alpha_0}$ and μ have units of mass. Similarly, when we study the three-dimensional theory, we shall write

$$g_0^2 = \frac{\beta_0}{(2\pi)^{3/2}}\mu^{(3-D)} \tag{8.31}$$

where $\sqrt[3]{\beta_0}$ has units of mass.

In the eikonal JWKB approximation, the path integral in $\mathcal{F}_\phi$ is dominated by the eikonal paths

$$\begin{aligned} e_a(\tau) &= \frac{x_1 + x_3}{2} + \left(\frac{\tau}{T_a}\right)(x_3 - x_1), \qquad -\frac{T_a}{2} < \tau < \frac{T_a}{2} \\ e_b(\sigma) &= \frac{x_2 + x_4}{2} + \left(\frac{\sigma}{T_b}\right)(x_4 - x_2), \qquad -\frac{T_b}{2} < \sigma < \frac{T_b}{2} \end{aligned} \tag{8.32}$$

The two-body quantum kernel $\mathcal{F}_\phi$ becomes the **two-body semiclassical eikonal kernel**,

$$\mathcal{F}_\phi(3,4|1,2) \longrightarrow \mathcal{E}_\phi(3,4|1,2) = \sqrt{-\det(V_\phi)}\exp(-i\Sigma_\phi) \tag{8.33}$$

where Σ_ϕ is given by

$$\Sigma_\phi \equiv S_\phi[e_a, e_b] \tag{8.34}$$

and V_ϕ is

$$V_\phi = \begin{pmatrix} V_{13} & V_{23} \\ V_{14} & V_{24} \end{pmatrix}, \qquad V_{jk} \equiv -i\frac{\partial^2 \Sigma_\phi}{\partial x_j \partial x_k} \tag{8.35}$$

We now compute Σ_ϕ and V_ϕ.

8.1.1 Eikonal Van Vleck Function

The free part of the eikonal Van Vleck function is

$$\begin{aligned}\Sigma_0 &\equiv S_0\left[e_a, e_b\right] \\ &= -\frac{1}{2T_a}(x_3 - x_1)^2 + \frac{m_a^2 T_a}{2} - \frac{1}{2T_b}(x_4 - x_2)^2 + \frac{m_b^2 T_b}{2}\end{aligned} \tag{8.36}$$

After ignoring the self-interactions, the only contribution remaining is from the two-body interaction:

$$\begin{aligned}\Sigma_2^{\phi} &\equiv S_2^{\phi}\left[e_a, e_b\right] \\ &= g_0^2 \int\int \mathrm{d}\tau \mathrm{d}\sigma \, G_0\left[e_a(\tau)|e_b(\sigma)\right] \\ &= -\frac{i\alpha_0\mu^2}{2\pi}\Gamma\left(\frac{D-2}{2}\right)\int\int \mathrm{d}\tau\mathrm{d}\sigma\left(\frac{2}{\mu^2\left[e_a(\tau) - e_b(\sigma)\right]^2}\right)^{(D-2)/2} \\ &= -\frac{i\alpha_0\mu^2}{2\pi}\Upsilon\end{aligned} \tag{8.37}$$

where we have denoted

$$\Upsilon \equiv \Gamma\left(\frac{D-2}{2}\right)\int\int \mathrm{d}\tau\mathrm{d}\sigma\left(\frac{2}{\mu^2\left[e_a(\tau) - e_b(\sigma)\right]^2}\right)^{(D-2)/2} \tag{8.38}$$

We introduce a Schwinger parameter T_0 and rewrite Υ as

$$\Upsilon = \int\int \mathrm{d}\tau\mathrm{d}\sigma \int_0^{\infty} \mathrm{d}T_0 \left(\frac{1}{T_0}\right)^{D/2} \exp\left(-\frac{\mu^2}{2T_0}\left[e_a(\tau) - e_b(\sigma)\right]^2\right) \tag{8.39}$$

With the eikonal paths (8.32) we have

$$e_a(\tau) - e_b(\sigma) = X_{12} + \left(\frac{\tau}{T_a}\right)x_{31} - \left(\frac{\sigma}{T_b}\right)x_{42} \tag{8.40}$$

and thus

$$\begin{aligned}\left[e_a(\tau) - e_b(\sigma)\right]^2 = X_{12}^2 &+ 2\left(X_{12}\cdot x_{31}\right)\left(\frac{\tau}{T_a}\right) - 2\left(X_{12}\cdot x_{42}\right)\left(\frac{\sigma}{T_b}\right) \\ &+ x_{31}^2\left(\frac{\tau}{T_a}\right)^2 + x_{42}^2\left(\frac{\sigma}{T_b}\right)^2 - 2\left(x_{31}\cdot x_{42}\right)\left(\frac{\tau\sigma}{T_aT_b}\right)\end{aligned} \tag{8.41}$$

The variables X_{12}, x_{31} and x_{42} are defined below. The advantage of writing Σ_2^ϕ in the form (8.39) is that the integrand becomes Gaussian. In the eikonal JWKB approximation, the separation between the two particles is always very large, compared to other distances in the problem. That is,

$$[e_a(\tau) - e_b(\sigma)]^2 \gg 0 \tag{8.42}$$

and thus we can use stationary methods to evaluate the integral over (τ, σ). The stationary point is

$$\begin{aligned}\tau_* &= -T_a \left[\frac{x_{42}^2 (X_{12} \cdot x_{31}) - (X_{12} \cdot x_{42})(x_{31} \cdot x_{42})}{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}\right] \\ \sigma_* &= +T_b \left[\frac{x_{31}^2 (X_{12} \cdot x_{42}) - (X_{12} \cdot x_{31})(x_{31} \cdot x_{42})}{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}\right]\end{aligned} \tag{8.43}$$

At this stationary point, we find

$$\begin{aligned}B_{12} &\equiv e_a(\tau_*) - e_b(\sigma_*) \\ &= X_{12} - \left[\frac{x_{42}^2 (X_{12} \cdot x_{31}) - (X_{12} \cdot x_{42})(x_{31} \cdot x_{42})}{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}\right] x_{31} \\ &\quad - \left[\frac{x_{31}^2 (X_{12} \cdot x_{42}) - (X_{12} \cdot x_{31})(x_{31} \cdot x_{42})}{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}\right] x_{42}\end{aligned} \tag{8.44}$$

The scalar B_{12}^2 corresponds to the minimum of the squared separation between the particles. Even though it is the minimum separation, by virtue of the eikonal approximation, B_{12}^2 is large compared to other distances in the problem. Note that the vector B_{12} is orthogonal to any vector that is a linear combination of x_{31} and x_{42} (hence, B_{12} has $D-2$ independent components). After integrating over τ and σ, we find

$$\begin{aligned}\Upsilon &\approx \frac{2\pi}{\mu^2} \frac{T_a T_b}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}} \int_0^\infty \mathrm{d}T_0 \left(\frac{1}{T_0}\right)^{(D-2)/2} \exp\left(-\frac{\mu^2}{2T_0} B_{12}^2\right) \\ &= \frac{2\pi}{\mu^2} \frac{T_a T_b}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}} \Gamma\left(\frac{D-4}{2}\right) \left(\frac{2}{\mu^2 B_{12}^2}\right)^{(D-4)/2}\end{aligned} \tag{8.45}$$

Thus, Σ_2^ϕ gives

$$\Sigma_2^\phi \approx -i\alpha_0 \left[\frac{T_a T_b}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}}\right] \Gamma\left(\frac{D-4}{2}\right) \left(\frac{2}{\mu^2 B_{12}^2}\right)^{(D-4)/2} \tag{8.46}$$

Note that Σ_2^ϕ is proportional to a massless scalar propagator in $D-2$ dimensions, and thus has the familiar divergence when $D-2=2$, i.e. when $D=4$. In order to simplify the equations we work with $D=4+2\epsilon$ and introduce

$$\rho_0 \equiv \frac{T_a T_b}{\sqrt{x_{31}^2 x_{42}^2 - (x_{31}\cdot x_{42})^2}} \tag{8.47}$$

such that we can simply write

$$\Sigma_2^\phi \approx -i\alpha_0\rho_0\Gamma(\epsilon)\left(\frac{2}{\mu^2 B_{12}^2}\right)^\epsilon \tag{8.48}$$

The quantity ρ_0 has units of inverse mass squared and depends on the worldline moduli T_a and T_b. The product $\alpha_0\rho_0$ is dimensionless.

8.1.2 Eikonal Van Vleck Matrix

Since the Van Vleck function Σ_ϕ has the form $\Sigma_\phi = \Sigma_0 - \Sigma_2^\phi$, we can write the Van Vleck matrix V_ϕ as $V_\phi = V_0 - V_2^\phi$ with

$$V_0 = \begin{pmatrix} u_{13} & u_{23} \\ u_{14} & u_{24} \end{pmatrix}, \qquad u_{jk} \equiv -i\frac{\partial^2\Sigma_0}{\partial x_j \partial x_k} \tag{8.49}$$

and

$$V_2^\phi = \begin{pmatrix} v_{13} & v_{23} \\ v_{14} & v_{24} \end{pmatrix}, \qquad v_{jk} \equiv -i\frac{\partial^2\Sigma_2^\phi}{\partial x_j \partial x_k} \tag{8.50}$$

The determinant of V_ϕ can be written as

$$\det(V_\phi) = \det(V_0 - V_2^\phi) = \det(I-W)\det(V_0), \qquad W \equiv V_2^\phi (V_0)^{-1} \tag{8.51}$$

Recall that

$$\det(I-W) = \exp\left[-\sum_{n=1}^\infty \frac{1}{n}\operatorname{tr}(W^n)\right] \tag{8.52}$$

Hence, the square root of the Van Vleck determinant is

$$\sqrt{-\det(V_\phi)} = \sqrt{-\det(V_0)}\exp\left[-\sum_{n=1}^\infty \frac{1}{2n}\operatorname{tr}(W^n)\right] \tag{8.53}$$

Using (8.36) it is easy to show that

$$\begin{aligned} (u_{13})_{mn} &= \left(-\frac{i}{T_a}\right)\eta_{mn}, \qquad (u_{23})_{mn} = 0 \\ (u_{14})_{mn} &= 0, \qquad (u_{42})_{mn} = \left(-\frac{i}{T_b}\right)\eta_{mn} \end{aligned} \tag{8.54}$$

and thus

$$\sqrt{-\det\left(V_0\right)} = \left(-\frac{i}{T_a}\right)^{D/2}\left(-\frac{i}{T_b}\right)^{D/2} \tag{8.55}$$

Just like in the nonrelativistic case, we will *ignore* all the contributions from the matrix W. If we compute W, we find that all the contributions from the traces of powers of W involve powers of B_{12}^2 that are less (i.e. more negative) than the power in Σ_2^ϕ. Thus, in the eikonal JWKB approximation, the leading term with B_{12}^2 inside the exponential comes from Σ_2 and we can ignore all contributions from W. That is,

$$\sqrt{-\det\left(V_\phi\right)} \approx \sqrt{-\det\left(V_0\right)} \tag{8.56}$$

8.1.3 Eikonal S-Matrix

At this stage we have everything we need to build the two-body semiclassical eikonal kernel:

$$\mathcal{E}_\phi(3,4|1,2) = \sqrt{-\det\left(V_\phi\right)}\exp\left(-i\Sigma_\phi\right) \tag{8.57}$$

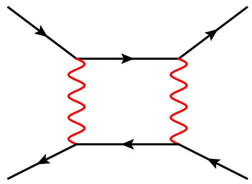
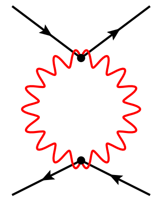
Using (8.36), (8.48), (8.56) and (8.55) we find

$$\mathcal{E}_\phi(3,4|1,2) \approx \left(-\frac{i}{T_a}\right)^{D/2}\left(-\frac{i}{T_b}\right)^{D/2}\left[-1+\exp\left(i\Sigma_2^\phi\right)\right]\exp\left(-i\Sigma_0\right) \tag{8.58}$$

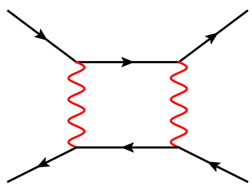
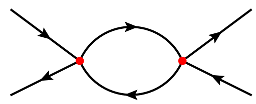
where we have subtracted the disconnected part. We write the factor inside square brackets as an infinite series:

$$-1+\exp\left(i\Sigma_2^\phi\right) = \sum_{l=1}^{\infty}\frac{(\alpha_0\rho_0)^l}{\Gamma(l+1)}[\Gamma(\epsilon)]^l\left(\frac{2}{\mu^2 B_{12}^2}\right)^{l\epsilon} \tag{8.59}$$

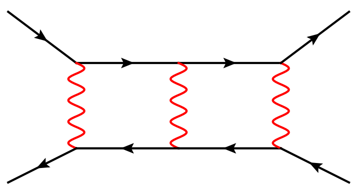
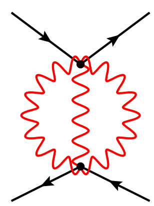
This expression provides a way to understand the meaning of the semiclassical eikonal kernel $\mathcal{E}_\phi$: The l-th term in the sum contains the product of l massless scalar propagators in $D-2$ dimensions, with each propagator connecting the *same* two points with separation vector B_{12}. The quantum kernel corresponds to the sum over ladder and crossed ladder contributions. We see that in the semiclassical JWKB approximation, the ladder and crossed ladder contributions are contracted along the s-channel direction and yield the semiclassical eikonal kernel. For example, at the 1-loop level we have the eikonal contraction of the box,

 $\longrightarrow$  (8.60)

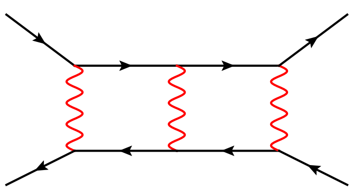
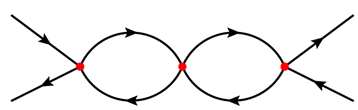
This is different to what happens in the Regge limit ($|t| \to \infty$), where the contraction is along the t-channel,

 $\longrightarrow$  (8.61)

Similarly, at the 2-loops level, the eikonal contraction of the double box is

 $\longrightarrow$  (8.62)

which is different from the Regge contraction,

 $\longrightarrow$  (8.63)

Of course, we do not expect agreement in the first place, since the eikonal approximation involves $t \to 0$, while the Regge limit involves $|t| \to \infty$.

With $\mathcal{E}_\phi$ in hand, we compute the **un-integrated semiclassical eikonal S-matrix**,

$$\mathcal{S}_\phi(3,4|1,2) \approx \int\int\int\int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \, \overline{\mathcal{W}}_O(3,4) \mathcal{W}_I(1,2) \mathcal{E}_\phi(3,4|1,2) \quad (8.64)$$

where, for the two-body system, we have

$$\mathcal{W}_I(1,2) = \exp\left[\frac{iT_a}{4}\left(p_1^2+m_1^2\right) + \frac{iT_b}{4}\left(p_2^2+m_2^2\right) + ix_1\cdot p_1 + ix_2\cdot p_2\right] \tag{8.65}$$

$$\overline{\mathcal{W}}_O(3,4) = \exp\left[\frac{iT_a}{4}\left(p_3^2+m_3^2\right) + \frac{iT_b}{4}\left(p_4^2+m_4^2\right) - ix_3\cdot p_3 - ix_4\cdot p_4\right] \tag{8.66}$$

Then we compute the **integrated semiclassical eikonal S-matrix**:

$$\mathcal{A}_\phi(3,4|1,2) \approx \int_0^\infty \mathrm{d}T_a \int_0^\infty \mathrm{d}T_b\, \mathcal{S}_\phi(3,4|1,2) \tag{8.67}$$

We will evaluate the integration over T_a and T_b later.

In order to perform the integration in (8.64) we first make a change of variables, and also introduce corresponding conjugate momenta:

$$\begin{aligned} X &\equiv \frac{x_1+x_2+x_3+x_4}{4} & P &\equiv p_4+p_3-p_2-p_1 \\ X_{12} &\equiv \frac{x_1-x_2+x_3-x_4}{2} & P_{12} &\equiv \frac{p_3-p_1+p_2-p_4}{2} \\ x_{31} &\equiv x_3-x_1 & p_{31} &\equiv \frac{p_1+p_3}{2} \\ x_{42} &\equiv x_4-x_2 & p_{42} &\equiv \frac{p_2+p_4}{2} \end{aligned} \tag{8.68}$$

such that

$$x_1\cdot p_1 + x_2\cdot p_2 - x_3\cdot p_3 - x_4\cdot p_4 = -X\cdot P - X_{12}\cdot P_{12} - x_{31}\cdot p_{31} - x_{42}\cdot p_{42} \tag{8.69}$$

Note that

$$p_1^2+p_3^2 = \frac{1}{8}\left(2P_{12}+P\right)^2 + 2p_{31}^2, \qquad p_2^2+p_4^2 = \frac{1}{8}\left(2P_{12}-P\right)^2 + 2p_{42}^2 \tag{8.70}$$

These two identities allow us to write

$$\begin{aligned} \overline{\mathcal{W}}_O\mathcal{W}_I =\ & \exp\left[-ix_{31}\cdot p_{31} - ix_{42}\cdot p_{42} - iX\cdot P - iX_{12}\cdot P_{12}\right] \\ & \times \exp\left[\frac{iT_a}{2}p_{31}^2 + \frac{iT_a}{32}\left(2P_{12}+P\right)^2 + \frac{iT_a}{4}(m_1^2+m_3^2)\right] \\ & \times \exp\left[\frac{iT_b}{2}p_{42}^2 + \frac{iT_b}{32}\left(2P_{12}-P\right)^2 + \frac{iT_b}{4}(m_2^2+m_4^2)\right] \end{aligned} \tag{8.71}$$

Since $\mathcal{E}_\phi$ has no dependence on X, integration yields a Dirac delta:

$$\int \mathrm{d}X \, \exp\left(-iX \cdot P\right) = \delta(P) \tag{8.72}$$

This Dirac delta imposes the constraint $P = 0$, which leads to

$$p_1 + p_2 = p_3 + p_4 \tag{8.73}$$

We find that the total external momentum is conserved. This is one of the requirements for the external momenta to be physical. Using $P = 0$, we find

$$P_{12} = p_3 - p_1 = p_2 - p_4 \tag{8.74}$$

That is, P_{12} measures the momentum transfer between the particles:

$$t = -P_{12}^2 = -(p_1 - p_3)^2 \tag{8.75}$$

Thus, after integrating over X, and using $P = 0$, we obtain

$$\begin{aligned} \overline{\mathcal{W}}_O \mathcal{W}_I = \; & \exp\left[-ix_{31} \cdot p_{31} - ix_{42} \cdot p_{42} - iX_{12} \cdot P_{12}\right] \\ & \times \exp\left[\frac{iT_a}{2} p_{31}^2 + \frac{iT_b}{2} p_{42}^2\right] \\ & \times \exp\left[\frac{iT_a}{4}\left(m_1^2 + m_3^2 - \frac{t}{2}\right) + \frac{iT_b}{4}\left(m_2^2 + m_4^2 - \frac{t}{2}\right)\right] \end{aligned} \tag{8.76}$$

Next we tackle the integration over x_{31} and x_{42}. The exact integration is non-trivial because ρ_0 is a function of x_{31} and x_{42}. We make another change of variables:

$$x_{31} = T_a k_{31}, \qquad x_{42} = T_b k_{42} \tag{8.77}$$

From dimensional analysis we expect k_{31} and k_{42} to have units of mass. In terms of k_{31} and k_{42}, we have

$$\rho_0 = \frac{1}{\sqrt{k_{31}^2 k_{42}^2 - (k_{31} \cdot k_{42})^2}} \tag{8.78}$$

which does not explicitly depend on the moduli T_a and T_b. In terms of these new variables we have

$$\begin{aligned}\overline{\mathcal{W}}_O \mathcal{W}_I \exp\left(-i\Sigma_0\right) = \; & \exp\left[\frac{iT_a}{2}\left(k_{31}-p_{31}\right)^2 + \frac{iT_b}{2}\left(k_{42}-p_{42}\right)^2\right] \\ & \times \exp\left[\frac{iT_a}{4}\left(m_1^2+m_3^2-2m_a^2-\frac{t}{2}\right)\right] \\ & \times \exp\left[\frac{iT_b}{4}\left(m_2^2+m_4^2-2m_b^2-\frac{t}{2}\right)\right] \\ & \times \exp\left(-iX_{12}\cdot P_{12}\right)\end{aligned} \tag{8.79}$$

The integrand in (k_{31}, k_{42}) is of the form "Gaussian $\times$ function". The Gaussian part is the first line in (8.79). At this stage we use stationary methods to evaluate the integral. The stationary point is

$$\bar{k}_{31} = p_{31}, \qquad \bar{k}_{42} = p_{42} \tag{8.80}$$

At this stationary point ρ_0 becomes a function of the external momenta,

$$\rho_0 = \frac{1}{\sqrt{p_{31}^2 p_{42}^2 - (p_{31}\cdot p_{42})^2}} \tag{8.81}$$

The integral over X_{12} remains:

$$\begin{aligned}\mathcal{A}_\phi = \delta(P)\int_0^\infty \mathrm{d}T_a \int_0^\infty \mathrm{d}T_b \int \mathrm{d}X_{12} & \left[\sum_{l=1}^{\infty}\frac{(\alpha_0\rho_0)^l}{\Gamma(l+1)}[\Gamma(\epsilon)]^l\left(\frac{2}{\mu^2 B_{12}^2}\right)^{l\epsilon}\right] \\ & \times \exp\left[\frac{iT_a}{4}\left(m_1^2+m_3^2-2m_a^2-\frac{t}{2}\right)\right] \\ & \times \exp\left[\frac{iT_b}{4}\left(m_2^2+m_4^2-2m_b^2-\frac{t}{2}\right)\right] \\ & \times \exp\left(-iX_{12}\cdot P_{12}\right)\end{aligned} \tag{8.82}$$

Earlier we defined B_{12} as the part of X_{12} that is orthogonal to any linear combination of the vectors x_{31} and x_{42}. But the net result of the integration over x_{31} and x_{42} was to replace x_{31} by $T_a p_{31}$ and x_{42} by $T_b p_{42}$. So we now write

$$X_{12} = B_{12} + T_a b_{31} p_{31} + T_b b_{42} p_{42} \qquad B_{12}\cdot p_{31} = 0 \qquad B_{12}\cdot p_{42} = 0 \tag{8.83}$$

The volume element becomes

$$\mathrm{d}X_{12} = T_a T_b \sqrt{p_{31}^2 p_{42}^2 - (p_{31}\cdot p_{42})^2}\,\mathrm{d}B_{12}\mathrm{d}b_{31}\mathrm{d}b_{42} \tag{8.84}$$

which can be rewritten in terms of ρ_0,

$$\mathrm{d}X_{12} = \alpha_0 T_a T_b \left(\frac{1}{\alpha_0 \rho_0}\right) \mathrm{d}B_{12}\mathrm{d}b_{31}\mathrm{d}b_{42} \tag{8.85}$$

Note that

$$X_{12} \cdot P_{12} = B_{12} \cdot P_{12} + T_a b_{31}(p_{31} \cdot P_{12}) + T_b b_{42}(p_{42} \cdot P_{12}) \tag{8.86}$$

Integration over b_{31} and b_{42} yields two Dirac deltas:

$$\int \mathrm{d}b_{31}\, \exp\left[-iT_a b_{31}(p_{31} \cdot P_{12})\right] = \frac{1}{T_a}\delta(p_{31} \cdot P_{12}) \tag{8.87}$$

$$\int \mathrm{d}b_{42}\, \exp\left[-iT_b b_{42}(p_{42} \cdot P_{12})\right] = \frac{1}{T_b}\delta(p_{42} \cdot P_{12}) \tag{8.88}$$

These two Dirac deltas impose constraints that will be discussed later. At this stage the only part of the amplitude that depends on T_a and T_b is the second and third exponential factors in (8.82). Performing the integration over T_a and T_b yields

$$\begin{aligned}
\int_0^\infty \mathrm{d}T_a\, \exp\left[\frac{iT_a}{4}\left(m_1^2 + m_3^2 - 2m_a^2 - \frac{t}{2}\right)\right] &= \frac{8i}{2m_1^2 + 2m_3^2 - 4m_a^2 - t} \\
\int_0^\infty \mathrm{d}T_b\, \exp\left[\frac{iT_b}{4}\left(m_2^2 + m_4^2 - 2m_b^2 - \frac{t}{2}\right)\right] &= \frac{8i}{2m_2^2 + 2m_4^2 - 4m_b^2 - t}
\end{aligned} \tag{8.89}$$

The last thing remaining is the integral over B_{12}:

$$\begin{aligned}
\mathcal{A}_\phi &= \alpha_0 \mathcal{N} \delta(P) \\
&\quad \times \int \mathrm{d}B_{12} \left[\sum_{l=1}^{\infty} \frac{(\alpha_0\rho_0)^{(l-1)}}{\Gamma(l+1)} [\Gamma(\epsilon)]^l \left(\frac{2}{\mu^2 B_{12}^2}\right)^{l\epsilon}\right] \exp\left(-iB_{12} \cdot P_{12}\right)
\end{aligned} \tag{8.90}$$

where we have collected many terms into

$$\mathcal{N} \equiv \frac{(8i)^2 \delta(p_{31} \cdot P_{12}) \delta(p_{42} \cdot P_{12})}{\left(2m_1^2 + 2m_3^2 - 4m_a^2 - t\right)\left(2m_2^2 + 2m_4^2 - 4m_b^2 - t\right)} \tag{8.91}$$

In $D = 4+2\epsilon$ dimensions the integral over B_{12} is over $D-2 = 2+2\epsilon$ dimensions.

After integrating over B_{12} we find

$$\mathcal{A}_\phi(3,4|1,2) = \alpha_0 \mathcal{N} \delta(P) \left(\frac{1}{\mu^2}\right)^{(1+\epsilon)} \times \sum_{l=1}^{\infty} \frac{(\alpha_0\rho_0)^{(l-1)}}{\Gamma(l+1)} \frac{[\Gamma(\epsilon)]^l \Gamma(1+\epsilon-l\epsilon)}{\Gamma(l\epsilon)} \left(\frac{2\mu^2}{P_{12}^2}\right)^{(1+\epsilon-l\epsilon)} \tag{8.92}$$

which can be written as

$$\mathcal{A}_\phi(3,4|1,2) = -\frac{2\alpha_0}{t} \mathcal{N} \delta(P) \left(\frac{1}{\mu^2}\right)^{\epsilon} \times \sum_{L=0}^{\infty} \frac{(\alpha_0\rho_0)^{L}}{\Gamma(L+1)} \frac{\Gamma(1+\epsilon)[\Gamma(\epsilon)]^L \Gamma(1-L\epsilon)}{\Gamma(1+\epsilon+L\epsilon)} \left(-\frac{t}{2\mu^2}\right)^{L\epsilon} \tag{8.93}$$

where we have used $t = -P_{12}^2$.

Before we put the external momenta on-shell (and thus make the external momenta *physical*), we need to truncate from $\mathcal{A}_\phi$ the part that is divergent on-shell. Traditionally, truncation involves multiplying the amplitude by a product of inverse propagators $(p_j^2 + m_j^2)$ and then taking the limit $p_j^2 \to -m_j^2$. Since we have four external states, we need to multiply $\mathcal{A}_\phi$ by four inverse propagators. The factor $\mathcal{N}$ is the product of four terms, each with units of inverse mass squared. On-shell, we must have

$$p_j^2 = -m_j^2, \qquad m_1 = m_3 = m_a, \qquad m_2 = m_4 = m_b \tag{8.94}$$

The two one-dimensional Dirac deltas in $\mathcal{N}$ impose the constraints

$$p_{31} \cdot P_{12} = \frac{p_3^2 - p_1^2}{2} = 0, \qquad p_{42} \cdot P_{12} = \frac{p_2^2 - p_4^2}{2} = 0 \tag{8.95}$$

These two constraints are satisfied on-shell. Thus, on-shell, we have

$$\delta(p_{31} \cdot P_{12}) \to \infty, \qquad \delta(p_{42} \cdot P_{12}) \to \infty \tag{8.96}$$

The two factors in the denominator of $\mathcal{N}$ vanish on-shell because, in the eikonal JWKB approximation, we have $m_a^2 \gg t$ and $m_b^2 \gg t$. Hence, we will argue that the net result of truncation and taking the on-shell limit is to eliminate

$\mathcal{N}$ from $\mathcal{A}_\phi$. Thus, the *truncated on-shell* scattering amplitude $\widehat{\mathcal{A}}_\phi$ is given by

$$\begin{aligned}\widehat{\mathcal{A}}_\phi &\equiv \left(\frac{\mu^{2\epsilon}}{\mathcal{N}}\right)\mathcal{A}_\phi(3,4|1,2) \\ &= -\frac{2\alpha_0}{t}\delta(P)\left[\sum_{L=0}^{\infty}\frac{(\alpha_0\rho_0)^L}{\Gamma(L+1)}\frac{\Gamma(1+\epsilon)[\Gamma(\epsilon)]^L\Gamma(1-L\epsilon)}{\Gamma(1+\epsilon+L\epsilon)}\left(-\frac{t}{2\mu^2}\right)^{L\epsilon}\right]\end{aligned} \tag{8.97}$$

We can recognize the pre-factor as the tree-level amplitude for the exchange of a massless scalar,

$$\mathcal{A}^\phi_{\text{tree}}(t) = -\frac{2\alpha_0}{t}\delta(P) \tag{8.98}$$

Before we move on to a more explicit discussion of the different terms in the sum in (8.97), we briefly discuss ρ_0. Recall that

$$\rho_0 = \frac{1}{\sqrt{p_{31}^2 p_{42}^2 - (p_{31}\cdot p_{42})^2}} \tag{8.99}$$

On-shell, we have

$$p_{31}^2 = \frac{t-4m_a^2}{4} \qquad p_{42}^2 = \frac{t-4m_b^2}{4} \qquad p_{31}\cdot p_{42} = \frac{m_a^2+m_b^2-s}{2} - \frac{t}{4} \tag{8.100}$$

Using

$$s+t+u = 2m_a^2 + 2m_b^2 \tag{8.101}$$

we can write

$$p_{31}^2 = \frac{2m_b^2 - 2m_a^2 - s - u}{4} \qquad p_{42}^2 = \frac{2m_a^2 - 2m_b^2 - s - u}{4} \tag{8.102}$$

and

$$p_{31}\cdot p_{42} = \frac{u-s}{4} \tag{8.103}$$

Thus,

$$\rho_0(s,u) = \frac{2}{\sqrt{su - (m_a+m_b)^2(m_a-m_b)^2}} \tag{8.104}$$

In terms of s and t, we have

$$\rho_0(s,t) = \frac{2}{\sqrt{[s-(m_a-m_b)^2][(m_a+m_b)^2-s] - st}} \tag{8.105}$$

or in terms of t and u,

$$\rho_0(t,u) = \frac{2}{\sqrt{[u-(m_a-m_b)^2][(m_a+m_b)^2-u]-tu}} \tag{8.106}$$

Note that s and u appear in the same way in (8.105) and (8.106). Also, (8.104) is symmetric under $s \longleftrightarrow u$.

8.1.4 Perturbative Amplitudes

Consider the ratio of the truncated on-shell amplitude $\widehat{\mathcal{A}}_\phi$ and the tree-level amplitude,

$$\mathcal{R}_\phi(s,t,u) \equiv \frac{\widehat{\mathcal{A}}_\phi(s,t,u)}{\mathcal{A}^\phi_{\text{tree}}(t)} \tag{8.107}$$

According to (8.97), $\mathcal{R}_\phi$ has the form

$$\mathcal{R}_\phi(s,t,u) = \sum_{L=0}^{\infty} \mathcal{R}^\phi_L(s,t,u) \tag{8.108}$$

with

$$\mathcal{R}^\phi_L(s,t,u) = \frac{[\alpha_0\rho_0(s,u)]^L}{\Gamma(L+1)} \frac{\Gamma(1+\epsilon)[\Gamma(\epsilon)]^L\Gamma(1-L\epsilon)}{\Gamma(1+\epsilon+L\epsilon)} \left(-\frac{t}{2\mu^2}\right)^{L\epsilon} \tag{8.109}$$

Recall that

$$\epsilon = \frac{D-4}{2} \tag{8.110}$$

Since (8.109) is a sum over powers of the coupling α_0, we can think of $\mathcal{R}^\phi_L$ as the L-loop perturbative amplitude. We see that, for $L > 0$, we *always* have a divergence when $\epsilon = 0$ (i.e. for $D = 4$). In integer number of dimensions, ϵ will be either an integer (when D is even), or half-integer (when D is odd). When ϵ is *any* positive integer, we will have a divergence from the $\Gamma(1-L\epsilon)$ term for *any* positive value of L. On the other hand, when ϵ is *any* positive half-integer, we will have a divergence from the $\Gamma(1-L\epsilon)$ term for *even* values of L. In $D = 4$, the divergence will involve poles in ϵ of at most order-L. For $D > 4$, the divergence is always a simple pole. This feature makes four spacetime dimensions special. Before we study $D = 4$ in more detail, we work with $D = 3$ where there are no divergences.

Three Dimensions

Setting $D = 3$ corresponds to $\epsilon = -1/2$. This leads to

$$\mathcal{R}_L^{\phi}(s,t,u) = (-1)^L \left[\frac{\beta_0 \rho_0(s,u)}{\sqrt{-t}}\right]^L \cos\left(\frac{L\pi}{2}\right) \tag{8.111}$$

Thus, when L is odd we obtain $\mathcal{R}_L^{\phi} = 0$, and when L is even we find

$$\mathcal{R}_{2n}^{\phi}(s,t,u) = \left[\frac{\beta_0^2 \rho_0^2(s,u)}{t}\right]^n, \qquad n > 0 \tag{8.112}$$

Summing over L gives

$$\mathcal{R}_{\phi}(s,t,u) = -\frac{t}{\beta_0^2 \rho_0^2(s,u) - t} \tag{8.113}$$

Hence, the scattering amplitude gives

$$\widehat{\mathcal{A}}_{\phi}(s,t,u) = \left[\frac{2\beta_0}{\beta_0^2 \rho_0^2(s,u) - t}\right] \delta(P) \tag{8.114}$$

This result has a simple pole in t. We can also interpret this as a singularity at $s = s_*$ and $u = u_*$ which satisfies

$$\beta_0^2 \rho_0^2(s_*, u_*) = t \tag{8.115}$$

Using (8.104), we find that the product $s_* u_*$ satisfies

$$s_* u_* = (m_a + m_b)^2 (m_a - m_b)^2 + \frac{4\beta_0^2}{t} \tag{8.116}$$

This means that s_* and u_* are *outside* of the physical scattering region if $t > 0$. Using

$$s_* + t + u_* = 2m_a^2 + 2m_b^2 \tag{8.117}$$

we find

$$s_* = m_a^2 + m_b^2 - \frac{t}{2} + 2m_a m_b \left[\left(1 - \frac{t}{4m_a^2}\right)\left(1 - \frac{t}{4m_b^2}\right) - \frac{\beta_0^2}{m_a^2 m_b^2 t}\right]^{1/2} \tag{8.118}$$

and

$$u_* = m_a^2 + m_b^2 - \frac{t}{2} - 2m_a m_b \left[\left(1 - \frac{t}{4m_a^2}\right)\left(1 - \frac{t}{4m_b^2}\right) - \frac{\beta_0^2}{m_a^2 m_b^2 t}\right]^{1/2} \tag{8.119}$$

In the two-body semiclassical eikonal approximation (8.9), we have

$$\frac{s_*}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} + 2\left(1 - \frac{\beta_0^2}{m_a^2 m_b^2 t}\right)^{1/2} \tag{8.120}$$

and

$$\frac{u_*}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} - 2\left(1 - \frac{\beta_0^2}{m_a^2 m_b^2 t}\right)^{1/2} \tag{8.121}$$

Note that *inside* of the physical scattering region we have $t < 0$ and thus s_* and u_* are real for any value of β_0. However, *outside* of the physical scattering region we have $t > 0$ and hence s_* and u_* are real as long as β_0 is bounded,

$$\beta_0^2 \leq m_a^2 m_b^2 t \tag{8.122}$$

This singularity outside of the physical scattering region can be interpreted as a bound state.

Four Dimensions

As we have already mentioned, in $D = 4$, every amplitude $\mathcal{R}_L^\phi$ is divergent when $L > 0$. Explicitly, at the one-loop level we have

$$\mathcal{R}_1^\phi(s,t,u) = [\alpha_0 \rho_0(s,u)] \left[\frac{\Gamma(1+\epsilon)\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+2\epsilon)}\right] \left(-\frac{t}{2\mu^2}\right)^\epsilon \tag{8.123}$$

Expanding near $\epsilon = 0$ gives

$$\mathcal{R}_1^\phi(s,t,u) \approx [\alpha_0 \rho_0(s,u)] \left[\frac{1}{\epsilon} + \gamma + \log\left(-\frac{t}{2\mu^2}\right)\right] \tag{8.124}$$

where γ is the Euler-Mascheroni constant. We can eliminate the term with γ by looking instead at

$$\bar{\mathcal{R}}_1^\phi(s,t,u) \equiv \Gamma(1+\epsilon)\mathcal{R}_1^\phi(s,t,u) \tag{8.125}$$

Near $\epsilon = 0$ this gives

$$\bar{\mathcal{R}}_1^\phi(s,t,u) \approx [\alpha_0 \rho_0(s,u)] \left[\frac{1}{\epsilon} + \log\left(-\frac{t}{2\mu^2}\right)\right] \tag{8.126}$$

Working with $\bar{\mathcal{R}}_1^\phi$ instead of $\mathcal{R}_1^\phi$ amounts to changing to a modified subtraction scheme, since it corresponds to scaling the coupling parameter α_0 by an ϵ-dependent constant.

Similarly, at the two-loops level we find

$$\mathcal{R}_2^{\phi}(s,t,u) = \frac{1}{2}\left[\alpha_0\rho_0(s,u)\right]^2 \left[\frac{\Gamma(1+\epsilon)[\Gamma(\epsilon)]^2\Gamma(1-2\epsilon)}{\Gamma(1+3\epsilon)}\right]\left(-\frac{t}{2\mu^2}\right)^{2\epsilon} \tag{8.127}$$

which after expanding near $\epsilon = 0$ yields

$$\begin{aligned}\mathcal{R}_2^{\phi}(s,t,u) \approx{}& \frac{1}{2}\left[\alpha_0\rho_0(s,u)\right]^2 \left[\frac{1}{\epsilon} + \gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^2 \\ &+ \frac{1}{2}\left[\alpha_0\rho_0(s,u)\right]^2 \left(\left[\gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^2 - \zeta(2)\right)\end{aligned} \tag{8.128}$$

If we consider instead

$$\bar{\mathcal{R}}_2^{\phi}(s,t,u) \equiv [\Gamma(1+\epsilon)]^2 \mathcal{R}_2^{\phi}(s,t,u) \tag{8.129}$$

then we can get rid off the terms with γ, and also the terms with $\zeta(2)$:

$$\bar{\mathcal{R}}_2^{\phi} \approx \frac{1}{2}\left[\alpha_0\rho_0(s,u)\right]^2 \left(\left[\frac{1}{\epsilon} + \log\left(-\frac{t}{2\mu^2}\right)\right]^2 + \left[\log\left(-\frac{t}{2\mu^2}\right)\right]^2\right) \tag{8.130}$$

Finally, the contribution at the three-loops level is

$$\mathcal{R}_3^{\phi}(s,t,u) = \frac{1}{6}\left[\alpha_0\rho_0(s,u)\right]^3 \left[\frac{\Gamma(1+\epsilon)[\Gamma(\epsilon)]^3\Gamma(1-3\epsilon)}{\Gamma(1+4\epsilon)}\right]\left(-\frac{t}{2\mu^2}\right)^{3\epsilon} \tag{8.131}$$

Near $\epsilon = 0$ we find

$$\begin{aligned}\mathcal{R}_3^{\phi}(s,t,u) \approx{}& \frac{1}{6}\left[\alpha_0\rho_0(s,u)\right]^3 \left[\frac{1}{\epsilon} + \gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^3 \\ &- \frac{1}{4\epsilon}\left[\alpha_0\rho_0(s,u)\right]^3 \left(\left[\gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^2 - \zeta(2)\right) \\ &- \frac{7}{12}\left[\alpha_0\rho_0(s,u)\right]^3 \left[\gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^3 - \frac{29}{6}\left[\alpha_0\rho_0(s,u)\right]^3 \zeta(3) \\ &+ \frac{3\zeta(2)}{4}\left[\alpha_0\rho_0(s,u)\right]^3 \left[\gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]\end{aligned} \tag{8.132}$$

Just as before, we can get rid off all the terms with γ and $\zeta(2)$ if we instead look at

$$\bar{\mathcal{R}}_3^{\phi}(s,t,u) \equiv [\Gamma(1+\epsilon)]^3 \mathcal{R}_3^{\phi}(s,t,u) \tag{8.133}$$

We find

$$\bar{\mathcal{R}}_3^{\phi}(s,t,u) \approx \frac{1}{6}\left[\alpha_0\rho_0(s,u)\right]^3\left[\frac{1}{\epsilon}+\log\left(-\frac{t}{2\mu^2}\right)\right]^3 + \frac{1}{4\epsilon}\left[\alpha_0\rho_0(s,u)\right]^3\log^2\left(-\frac{t}{2\mu^2}\right) + \frac{7}{12}\left[\alpha_0\rho_0(s,u)\right]^3\log^3\left(-\frac{t}{2\mu^2}\right)+\frac{14}{3}\left[\alpha_0\rho_0(s,u)\right]^3\zeta(3) \quad (8.134)$$

As is well-known, we cannot remove the term with $\zeta(3)$.

At this point, it is easy to think "and so what?"[2]. We have a formal answer for the all-order scattering amplitude in the eikonal JWKB approximation, but in $D = 4$ each term has many divergences. A closed form for the amplitude appears to be elusive. Furthermore, there are no signs of bound states. It is easy to think that we have gain nothing by using the eikonal JWKB approximation. If the reader feels this way, the following discussion aims at changing that particular mindset.

8.1.5 Bound States in Four Dimensions

Clearly, something special happens in $D = 4$. In the eikonal JWKB approximation we have t being very small. Thus, every logarithm term in the amplitudes (8.124), (8.128) and (8.132) will be very large. It would be convenient to have a way to keep the dominant contribution. Perhaps when this is done, the complete amplitude $\mathcal{R}_\phi$ takes a simpler form.

The increasing order of the poles at $\epsilon = 0$ is another issue. Indeed, the divergence at $\epsilon = 0$ first appears in an exponentiated way, since it is present in Σ_2^{ϕ} in (8.58). Recall that Σ_2^{ϕ} is

$$\Sigma_2^{\phi} \approx -i\alpha_0\rho_0\Gamma(\epsilon)\left(\frac{2}{\mu^2 B_{12}^2}\right)^{\epsilon} \quad (8.135)$$

Near $\epsilon = 0$ we find

$$\Sigma_2^{\phi} \approx -i\alpha_0\rho_0\left[\Gamma(\epsilon)+\log\left(\frac{2}{\mu^2 B_{12}^2}\right)\right] \quad (8.136)$$

So then

$$\exp\left(i\Sigma_2^{\phi}\right) \approx \left(\frac{2}{\mu^2 B_{12}^2}\right)^{\alpha_0\rho_0}\exp\left(\Theta_\epsilon\right) \quad (8.137)$$

[2]Or, like Professor J. R. López would frequently remark in Mayagüez, "¿Y a mí qué?".

where

$$\Theta_\epsilon \equiv \alpha_0\rho_0\Gamma(\epsilon) \tag{8.138}$$

In this way, the divergence at $\epsilon = 0$ remains exponentiated. Instead of (8.90), we now have

$$\begin{aligned}\mathcal{B}_\phi(3,4|1,2) &= \alpha_0\mathcal{N}\delta(P)\exp(\Theta_\epsilon) \\ &\quad\times\left(\frac{1}{\alpha_0\rho_0}\right)\int \mathrm{d}B_{12}\left(\frac{2}{\mu^2 B_{12}^2}\right)^{\alpha_0\rho_0}\exp(-iB_{12}\cdot P_{12})\end{aligned} \tag{8.139}$$

where we have dropped the -1 from the disconnected part. Integration yields

$$\begin{aligned}\mathcal{B}_\phi(3,4|1,2) &= \alpha_0\mathcal{N}\delta(P)\exp(\Theta_\epsilon) \\ &\quad\times\left(\frac{1}{\mu^2}\right)\left[\frac{\Gamma(1-\alpha_0\rho_0)}{\Gamma(1+\alpha_0\rho_0)}\right]\left(-\frac{2\mu^2}{t}\right)^{(1-\alpha_0\rho_0)}\end{aligned} \tag{8.140}$$

Thus, in $D = 4$, the truncated on-shell scattering amplitude is

$$\begin{aligned}\widehat{\mathcal{B}}_\phi(s,t,u) &\equiv \left(\frac{1}{\mathcal{N}}\right)\mathcal{B}_\phi(3,4|1,2) \\ &= \mathcal{A}^\phi_{\text{tree}}(t)\exp[\Theta_\epsilon(s,u)]\frac{\Gamma[1-\alpha_0\rho_0(s,u)]}{\Gamma[1+\alpha_0\rho_0(s,u)]}\left(-\frac{t}{2\mu^2}\right)^{\alpha_0\rho_0(s,u)}\end{aligned} \tag{8.141}$$

The infrared-divergent part has been isolated as an overall exponential factor:

$$\begin{aligned}\Theta_\epsilon(s,u) &= \alpha_0\rho_0(s,u)\Gamma(\epsilon) \\ &= \alpha_0\left[\frac{2}{\sqrt{su-(m_a+m_b)^2(m_a-m_b)^2}}\right]\Gamma(\epsilon)\end{aligned} \tag{8.142}$$

The argument of this exponential is *real* when the product su is in the range

$$su > (m_a+m_b)^2(m_a-m_b)^2 \tag{8.143}$$

which is *outside* of the physical scattering region (see §C.2.1). Thus, inside of the physical scattering region, $\Theta_\epsilon(s,u)$ is *imaginary* and the infrared divergence appears in the form of an overall *pure phase* factor.

Perturbative Comparison

The amplitude (8.141) is *not* equal to our previous result (8.97). To see this, consider the ratio

$$\mathcal{Q}_\phi(s,t,u) \equiv \frac{\widehat{\mathcal{B}}_\phi(s,t,u)}{\mathcal{A}^\phi_{\text{tree}}(t)} \tag{8.144}$$

In order to compare $\mathcal{Q}_\phi$ to $\mathcal{R}_\phi$ we expand $\mathcal{Q}_\phi$ in powers of α_0:

$$\mathcal{Q}_\phi(s,t,u) = \sum_{L=0}^{\infty} \mathcal{Q}^\phi_L(s,t,u) \tag{8.145}$$

At the one-loop level, we find

$$\mathcal{Q}^\phi_1(s,t,u) = [\alpha_0\rho_0(s,u)]\left[\Gamma(\epsilon) + 2\gamma + \log\left(-\frac{t}{2\mu^2}\right)\right] \tag{8.146}$$

which completely agrees with (8.124) after expanding near $\epsilon = 0$. However, beyond the one-loop level we find disagreement. For example, at the two-loops level we find

$$\mathcal{Q}^\phi_2(s,t,u) = \frac{1}{2}[\alpha_0\rho_0(s,u)]^2\left[\Gamma(\epsilon) + 2\gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^2 \tag{8.147}$$

Near $\epsilon = 0$ we have

$$\begin{aligned}\mathcal{Q}^\phi_2(s,t,u) \approx & \frac{1}{2}[\alpha_0\rho_0(s,u)]^2\left[\frac{1}{\epsilon} + \gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^2 \\ & + \frac{1}{2}[\alpha_0\rho_0(s,u)]^2\left[\gamma^2 + \zeta(2)\right]\end{aligned} \tag{8.148}$$

which does not contain all of the contributions in (8.128). Similarly, at the three-loops level we find

$$\mathcal{Q}^\phi_3(s,t,u) = \frac{1}{6}[\alpha_0\rho_0(s,u)]^3\left(\left[\Gamma(\epsilon) + 2\gamma + \log\left(-\frac{t}{2\mu^2}\right)\right]^3 + 4\zeta(3)\right) \tag{8.149}$$

which also disagrees with (8.132).

In principle, one can obtain $\mathcal{Q}^\phi_L$ from $\mathcal{R}^\phi_L$ after dropping some terms. One quickly learns that there are no simple criteria that decide what to keep and what to drop. The divergence in $\Theta_\epsilon(s,u)$ can be recognized as the divergence at one-loop level. Thus, the nice form of the amplitude (8.141) follows after *factorizing* the exponentiated one-loop divergence.

Bound State Spectrum

The amplitude (8.141) has an infinite number of singularities due to the Euler Gamma function in the numerator. These singularities satisfy

$$1 - \alpha_0 \rho_0(s_J, u_J) = -J, \qquad J = 0, 1, 2, \ldots \tag{8.150}$$

which leads to the relation

$$s_J u_J = (m_a + m_b)^2 (m_a - m_b)^2 + \frac{4\alpha_0^2}{(J+1)^2} \tag{8.151}$$

That is, s_J and u_J are *outside* of the physical scattering region. This is confirmation that these singularities correspond to bound states. Equation (8.151) is similar to the one we found in three dimensions. Indeed, (8.116) can be obtained from (8.151) with the replacement

$$\frac{\alpha_0^2}{(J+1)^2} \longrightarrow \frac{\beta_0^2}{t} \tag{8.152}$$

This replacement is valid only when $t > 0$, which is outside of the physical scattering region.

We can use

$$s_J + t + u_J = 2m_a^2 + 2m_b^2 \tag{8.153}$$

to find

$$\begin{aligned} s_J = {} & m_a^2 + m_b^2 - \frac{t}{2} \\ & + 2m_a m_b \left[\left(1 - \frac{t}{4m_a^2}\right)\left(1 - \frac{t}{4m_b^2}\right) - \frac{\alpha_0^2}{m_a^2 m_b^2 (J+1)^2} \right]^{1/2} \end{aligned} \tag{8.154}$$

and

$$\begin{aligned} u_J = {} & m_a^2 + m_b^2 - \frac{t}{2} \\ & - 2m_a m_b \left[\left(1 - \frac{t}{4m_a^2}\right)\left(1 - \frac{t}{4m_b^2}\right) - \frac{\alpha_0^2}{m_a^2 m_b^2 (J+1)^2} \right]^{1/2} \end{aligned} \tag{8.155}$$

However, in the two-body semiclassical eikonal approximation (8.9) we have

$$\frac{s_J}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} + 2\left[1 - \frac{\alpha_0^2}{m_a^2 m_b^2 (J+1)^2}\right]^{1/2} \tag{8.156}$$

and

$$\frac{u_J}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} - 2\left[1 - \frac{\alpha_0^2}{m_a^2 m_b^2 (J+1)^2}\right]^{1/2} \tag{8.157}$$

Since s_J in (8.156) lies below threshold, we can identify these singularities as bound states. Indeed, as J becomes very large, the s_J in (8.156) approaches the threshold value $(m_a + m_b)^2$. Note that (8.157) approaches the pseudo-threshold value $(m_a - m_b)^2$ when J becomes very large.

In order for the energies s_J and u_J to be real at a given value of J we must require the coupling α_0 to be bounded from above:

$$\alpha_0^2 \leq m_a^2 m_b^2 (J+1)^2 \tag{8.158}$$

If α_0 is larger than this value, then s_J and u_J will get an imaginary part and the bound state will become unstable.

We can understand these bound state singularities in a geometrical way. Recall the two-body Gram invariant (see appendix B),

$$G_{12}(s) = p_1^2 p_2^2 - (p_1 \cdot p_2)^2 = \frac{1}{4}[s - (m_a - m_b)^2][(m_a + m_b)^2 - s] \tag{8.159}$$

The square root of $G_{12}(s)$ corresponds to the "area" of the parallelogram made with the momentum vectors p_1 and p_2. In terms of $G_{12}(s)$, we have

$$\rho_0(s,t) \approx \frac{1}{\sqrt{G_{12}(s)}} \tag{8.160}$$

Thus, the singularity condition (8.150) can be understood as a quantization condition for an area A_J in momentum space,

$$A_J \equiv \sqrt{G_{12}(s_J)} = \frac{\alpha_0}{(J+1)} \tag{8.161}$$

One can check that for small values of the coupling α_0, we have

$$s_J \approx m_a^2 + m_b^2 + 2m_a m_b \left[1 - \frac{\alpha_0^2}{2m_a^2 m_b^2 (J+1)^2} + \ldots\right] \tag{8.162}$$

which agrees with the nonrelativistic Coulomb spectrum.

Regge Behavior

The amplitude (8.141) exhibits Regge behavior with leading Regge trajectory function $R_\phi(s,u)$ given by

$$\begin{aligned} R_\phi(s,u) &\equiv -1 + \alpha_0 \rho_0(s,u) \\ &= -1 + \frac{2\alpha_0}{\sqrt{su - (m_a+m_b)^2(m_a-m_b)^2}} \end{aligned} \tag{8.163}$$

Note the reality properties,

$$\begin{aligned} &\mathrm{Re}\,[R_\phi(s,u)] = -1 \text{ when } su < (m_a+m_b)^2(m_a-m_b)^2 \\ &\mathrm{Im}\,[R_\phi(s,u)] = 0 \text{ when } su > (m_a+m_b)^2(m_a-m_b)^2 \end{aligned} \tag{8.164}$$

When $su \ll (m_a+m_b)^2(m_a-m_b)^2$, the imaginary part of R_ϕ approaches zero.

We can write R_ϕ in terms of s and t:

$$R_\phi(s,t) \approx -1 + \frac{\alpha_0}{m_a m_b}\left[1 - \left(\frac{s - m_a^2 - m_b^2}{2m_a m_b}\right)^2\right]^{-1/2} \tag{8.165}$$

We have used the two-body semiclassical eikonal approximation (8.9). In order to visualize this function we introduce the dimensionless variable

$$\xi_s \equiv -\frac{p_1 \cdot p_2}{\sqrt{-p_1^2}\sqrt{-p_2^2}} = \frac{s - m_a^2 - m_b^2}{2m_a m_b} \tag{8.166}$$

At threshold $s = (m_a+m_b)^2$, we have $\xi_s = 1$. When $s = m_a^2 + m_b^2$ we have $\xi_s = 0$. At the pseudo-threshold $s = (m_a - m_b)^2$, we have $\xi_s = -1$. Finally, in order to demarcate when s becomes negative, we have that $s = 0$ leads to

$$\xi_s = -\frac{1}{2}\left(\frac{m_a}{m_b} + \frac{m_b}{m_a}\right) \leq -1 \tag{8.167}$$

In terms of ξ_s we have

$$R_\phi(\xi_s) = -1 + \frac{\alpha_0}{m_a m_b}\frac{1}{\sqrt{1-\xi_s^2}} \tag{8.168}$$

Figures 8.1 and 8.2 show the real and imaginary part of $R_\phi(\xi_s)$.

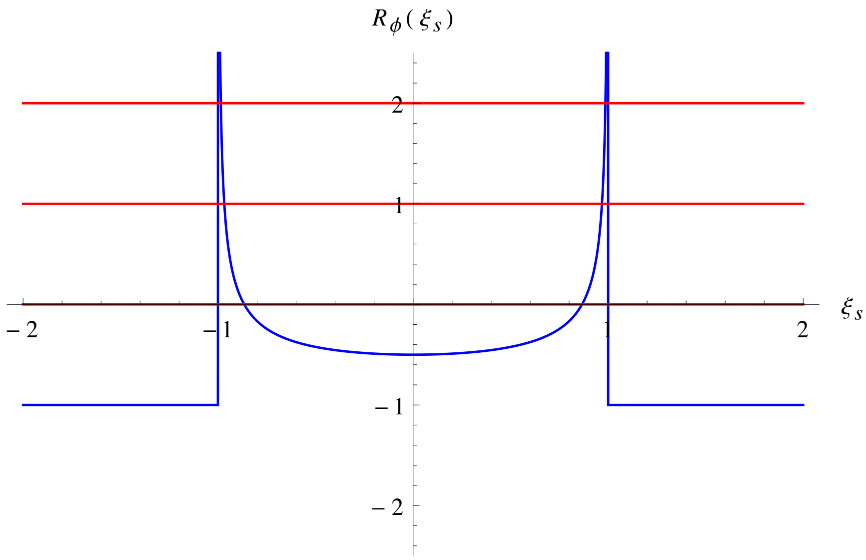


Figure 8.1: Real part of $R_\phi(\xi_s)$. The red lines correspond to $R_\phi = 0, 1, 2$. We have used $\alpha_0/m_a m_b = 0.5$.

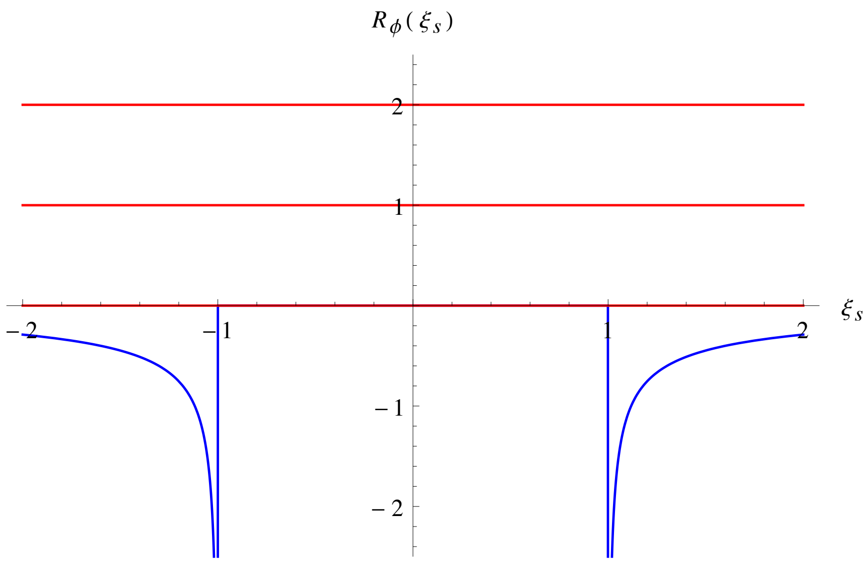


Figure 8.2: Imaginary part of $R_\phi(\xi_s)$. The red lines correspond to $R_\phi = 0, 1, 2$. We have used $\alpha_0/m_a m_b = 0.5$.

Crossing

Finally, a comment about crossing. The scattering event that we study is the elastic event

$$a(p_1) + b(p_2) \longrightarrow a(p_3) + b(p_4) \tag{8.169}$$

so the bound states are of the form ab. We can refer to (8.169) as the s-channel. If we cross the incoming b particle with the outgoing b particle, we find another elastic event

$$a(p_1) + \bar{b}(\bar{p}_2) \longrightarrow a(p_3) + \bar{b}(\bar{p}_4) \tag{8.170}$$

with

$$\bar{p}_2 = -p_4, \qquad \bar{p}_4 = -p_2 \tag{8.171}$$

We can refer to (8.170) as the u-channel. In the u-channel the bound states are of the form $a\bar{b}$. This crossing amounts to switching s with u. Note ρ_0 satisfies the functional relation $\rho_0(s,t) = \rho_0(u,t)$. That is, ρ_0 is invariant under crossing.

In the eikonal approximation we take $t \to 0$. Then s and u become eikonal-equivalent,

$$s + u = 2m_a^2 + 2m_b^2 \tag{8.172}$$

This relation allows us to perform an *eikonal crossing*, where one replaces s with $2m_a^2 + 2m_b^2 - u$ (instead of the usual crossing, where one replaces s with u). The eikonal crossing is what enabled us to find the sequence (8.157) from (8.156). Under the eikonal crossing we have

$$\xi_s = \frac{s - m_a^2 - m_b^2}{2m_a m_b} = \frac{m_a^2 + m_b^2 - u}{2m_a m_b} = -\xi_u \tag{8.173}$$

But under the usual crossing we have

$$\xi_s = \xi_u \tag{8.174}$$

Since the Regge trajectory function is parity-symmetric, the same singularity condition must determine both the spectrum in the s-channel and the spectrum in the u-channel:

$$R_\phi(\xi_s) = R_\phi(\xi_u) = R_\phi(-\xi_s) = R_\phi(-\xi_u) = J \tag{8.175}$$

This property accounts for the apparent degeneracy in figure 8.1.

8.2 Massless Vector Exchange

We now start with a two-boy path integral for scalar particles coupled to an external vector field A_m:

$$\mathcal{G}_A[A] = \int_{x_1}^{x_3} \mathrm{D}q_a(\tau) \int_{x_2}^{x_4} \mathrm{D}q_b(\sigma)\ \exp\left(-iS_0[q_a, q_b] - iS_{\mathrm{int}}[q_a, q_b, A]\right) \tag{8.176}$$

The free term S_0 is given by (8.12) and the interaction term is

$$S_{\mathrm{int}}[q_a, q_b, A] \equiv Z_a \int \mathrm{d}\tau\, (\dot{q}_a \cdot A[q_a(\tau)]) + Z_b \int \mathrm{d}\sigma\, (\dot{q}_b \cdot A[q_b(\sigma)]) \tag{8.177}$$

Here Z_a and Z_b are the dimensionless (electric) charges of particles a and b.

We integrate over the field A_m to obtain the "effective" interacting two-body path integral:

$$\mathcal{F}_A \equiv \int \mathrm{D}A_m(x)\, \mathcal{G}_A[A] \exp\left(-iS_{\mathrm{kin}}[A]\right) \tag{8.178}$$

where S_{kin} contains the gauge-fixed kinetic operator,

$$S_{\mathrm{kin}}[A] = \frac{1}{2g_1^2} \int\int \mathrm{d}x\mathrm{d}y\, [A_m(x)(K_1)^{mn}(x|y)A_n(y)] \tag{8.179}$$

with

$$(K_1)^{mn}(x|y) \equiv \delta(x-y)\left[-\frac{1}{2}\eta^{mn}\partial^2 + \frac{1}{2}\left(1 - \frac{1}{\xi_1}\right)\partial^m\partial^n\right] \tag{8.180}$$

In the Fermi-Feynman gauge ($\xi_1 = 1$) we have

$$(K_1)^{mn}(x|y) = \eta^{mn} K_0(x|y) \tag{8.181}$$

where K_0 is the massless scalar kinetic operator. We rewrite S_{int} as

$$S_{\mathrm{int}}[q_a, q_b, A] = \int \mathrm{d}x\, J^m(x) A_m(x) \tag{8.182}$$

with

$$J^m(x) \equiv Z_a \int \mathrm{d}\tau\, \dot{q}_a^m \delta[x - q_a(\tau)] + Z_b \int \mathrm{d}\sigma\, \dot{q}_b^m \delta[x - q_b(\sigma)] \tag{8.183}$$

After integrating over A_m, we find

$$\mathcal{F}_A = \int_{x_1}^{x_3} \mathrm{D}q_a(\tau) \int_{x_2}^{x_4} \mathrm{D}q_b(\sigma)\, \exp\left(-iS_A[q_a, q_b]\right) \tag{8.184}$$

where

$$S_A[q_a, q_b] \equiv S_0[q_a, q_b] - \frac{g_1^2}{2} \int\int \mathrm{d}x\mathrm{d}y\, [J^m(x)(G_1)_{mn}(x|y)J^n(y)] \tag{8.185}$$

In analogy with (8.23) we can use the explicit form of J^m to write

$$\frac{g_1^2}{2} \int\int \mathrm{d}x\mathrm{d}y\, [J^m(x)(G_1)_{mn}(x|y)J^n(y)] = S_1^A[q_a, q_b] + S_2^A[q_a, q_b] \tag{8.186}$$

with S_1^A containing self-interactions,

$$\begin{aligned} S_1^A[q_a, q_b] = \frac{Z_a^2 g_1^2}{2} &\int\int \mathrm{d}\tau_1\mathrm{d}\tau_2\, [\dot{q}_a(\tau_1)\cdot\dot{q}_a(\tau_2)]\, G_0[q_a(\tau_1)|q_a(\tau_2)] \\ + \frac{Z_b^2 g_1^2}{2} &\int\int \mathrm{d}\sigma_1\mathrm{d}\sigma_2\, [\dot{q}_b(\sigma_1)\cdot\dot{q}_b(\sigma_2)]\, G_0[q_b(\sigma_1)|q_b(\sigma_2)] \end{aligned} \tag{8.187}$$

and S_2^A containing two-body interactions,

$$S_2^A[q_a, q_b] = Z_a Z_b g_1^2 \int\int \mathrm{d}\tau\mathrm{d}\sigma\, [\dot{q}_a(\tau)\cdot\dot{q}_b(\sigma)]\, G_0\left[q_a(\tau)|q_b(\sigma)\right] \tag{8.188}$$

Just like in the previous section, we will *ignore* the contributions from S_1^A.

As we found in section 6.3, the coupling parameter g_1 has units

$$[g_1] = \left(\frac{4-D}{2}\right)[\text{mass}] \tag{8.189}$$

In $D = 4$, we have that g_1 is dimensionless. We write

$$g_1^2 = \frac{\alpha_1}{2\pi}\mu^{(4-D)} \tag{8.190}$$

where μ has units of mass, and α_1 is dimensionless. Similarly, in $D = 3$ we have that g_1^2 has units of mass. Thus, we write

$$g_1^2 = \frac{\beta_1}{(2\pi)^{3/2}}\mu^{(3-D)} \tag{8.191}$$

where μ and β_1 have units of mass.

8.2.1 Eikonal Van Vleck Function

Since the eikonal paths (8.32) have constant slope, the interaction part of the eikonal Van Vleck function is very similar to (8.37):

$$\begin{aligned}\Sigma_2^A &\equiv S_2^A\left[e_a, e_b\right] \\ &= Z_a Z_b g_1^2 \int\int \mathrm{d}\tau \mathrm{d}\sigma \left[\dot{e}_a(\tau)\cdot\dot{e}_b(\sigma)\right] G_0\left[e_a(\tau)|e_b(\sigma)\right] \\ &= -\frac{iZ_a Z_b \alpha_1 \mu^2}{2\pi}\frac{(x_{31}\cdot x_{42})}{T_a T_b}\Upsilon\end{aligned} \tag{8.192}$$

where Υ is the same integral that appeared in (8.38) for the massless scalar case. Using (8.45) we find

$$\Sigma_2^A \approx -i\alpha_1\left[\frac{Z_a Z_b(x_{31}\cdot x_{42})}{\sqrt{x_{31}^2 x_{42}^2-(x_{31}\cdot x_{42})^2}}\right]\Gamma\left(\frac{D-4}{2}\right)\left(\frac{2}{\mu^2 B_{12}^2}\right)^{(D-4)/2} \tag{8.193}$$

Just like (8.46), we find that Σ_2^A is proportional to a massless scalar propagator in $D-2$ dimensions. We work with $D=4+2\epsilon$ and introduce

$$\rho_1 \equiv \frac{Z_a Z_b(x_{31}\cdot x_{42})}{\sqrt{x_{31}^2 x_{42}^2-(x_{31}\cdot x_{42})^2}} \tag{8.194}$$

in order to write

$$\Sigma_2^A \approx -i\alpha_1\rho_1\Gamma(\epsilon)\left(\frac{2}{\mu^2 B_{12}^2}\right)^{\epsilon} \tag{8.195}$$

Note that unlike ρ_0 in (8.47), the dimensionless quantity ρ_1 does not depend on the worldline moduli.

8.2.2 Eikonal S-Matrix

Due to the similarity between Σ_2^A and Σ_2^ϕ, we expect the calculation of the scattering amplitude to follow §8.1.3 very closely. After repeating the same steps to do the integration over X, x_{31} and x_{42}, we arrive at

$$\begin{aligned}\mathcal{A}_A = \delta(P) \int_0^\infty \mathrm{d}T_a \int_0^\infty \mathrm{d}T_b \int \mathrm{d}X_{12} &\left[\sum_{l=1}^{\infty} \frac{(\alpha_1 \rho_1)^l}{\Gamma(l+1)} [\Gamma(\epsilon)]^l \left(\frac{2}{\mu^2 B_{12}^2}\right)^{l\epsilon}\right] \\ &\times \exp\left[\frac{iT_a}{4}\left(m_1^2 + m_3^2 - 2m_a^2 - \frac{t}{2}\right)\right] \\ &\times \exp\left[\frac{iT_b}{4}\left(m_2^2 + m_4^2 - 2m_b^2 - \frac{t}{2}\right)\right] \\ &\times \exp\left(-iX_{12} \cdot P_{12}\right)\end{aligned} \tag{8.196}$$

which has the same form as (8.82). After the integration over x_{31} and x_{42}, (8.194) becomes

$$\rho_1 = \frac{Z_a Z_b (p_{31} \cdot p_{42})}{\sqrt{p_{31}^2 p_{42}^2 - (p_{31} \cdot p_{42})^2}} \tag{8.197}$$

Thus, instead of (8.85), now the measure over X_{12} is

$$\mathrm{d}X_{12} = T_a T_b \sqrt{p_{31}^2 p_{42}^2 - (p_{31} \cdot p_{42})^2}\,\mathrm{d}B_{12}\mathrm{d}b_{31}\mathrm{d}b_{42} \tag{8.198}$$

which can be rewritten as

$$\mathrm{d}X_{12} = Z_a Z_b \alpha_1 T_a T_b \left(p_{31} \cdot p_{42}\right) \left(\frac{1}{\alpha_1 \rho_1}\right) \mathrm{d}B_{12}\mathrm{d}b_{31}\mathrm{d}b_{42} \tag{8.199}$$

such that, instead of (8.90), we now have

$$\begin{aligned}\mathcal{A}_A &= Z_a Z_b \alpha_1 \left(p_{31} \cdot p_{42}\right) \mathcal{N} \delta(P) \\ &\times \int \mathrm{d}B_{12} \left[\sum_{l=1}^{\infty} \frac{(\alpha_1 \rho_1)^{(l-1)}}{\Gamma(l+1)} [\Gamma(\epsilon)]^l \left(\frac{2}{\mu^2 B_{12}^2}\right)^{l\epsilon}\right] \exp\left(-iB_{12} \cdot P_{12}\right)\end{aligned} \tag{8.200}$$

which, after integrating over B_{12}, gives

$$\begin{aligned}\mathcal{A}_A = Z_a Z_b \alpha_1 \left(p_{31} \cdot p_{42}\right) \mathcal{N} \delta(P) &\left(\frac{1}{\mu^2}\right)^{(1+\epsilon)} \\ \times \sum_{l=1}^{\infty} \frac{(\alpha_1 \rho_1)^{(l-1)}}{\Gamma(l+1)} &\frac{[\Gamma(\epsilon)]^l \Gamma(1+\epsilon-l\epsilon)}{\Gamma(l\epsilon)} \left(\frac{2\mu^2}{P_{12}^2}\right)^{(1+\epsilon-l\epsilon)}\end{aligned} \tag{8.201}$$

After truncation, we obtain the truncated on-shell scattering amplitude

$$\begin{aligned}\widehat{\mathcal{A}}_A &\equiv \left(\frac{\mu^{2\epsilon}}{\mathcal{N}}\right)\mathcal{A}_A \\ &= \mathcal{A}^A_{\text{tree}}\left[\sum_{L=0}^{\infty}\frac{(\alpha_1\rho_1)^L}{\Gamma(L+1)}\frac{\Gamma(1+\epsilon)[\Gamma(\epsilon)]^L\Gamma(1-L\epsilon)}{\Gamma(1+\epsilon+L\epsilon)}\left(-\frac{t}{2\mu^2}\right)^{L\epsilon}\right]\end{aligned} \tag{8.202}$$

with the pre-factor now given by

$$\mathcal{A}^A_{\text{tree}}(s,t,u) = -\frac{2\alpha_1}{t}\left[\frac{Z_aZ_b(u-s)}{4}\right]\delta(P) \tag{8.203}$$

Except for the pre-factor, the general form of (8.202) agrees with what we found in (8.97) with α_0 replaced with α_1, and ρ_0 replaced with ρ_1.

After putting the external momenta on-shell, we have

$$\rho_1(s,u) = \frac{Z_aZ_b}{2}\left[\frac{u-s}{\sqrt{su-(m_a+m_b)^2(m_a-m_b)^2}}\right] \tag{8.204}$$

Using

$$s+t+u = 2m_a^2+2m_b^2 \tag{8.205}$$

we can write

$$\rho_1(s,t) = \frac{Z_aZ_b}{2}\left[\frac{2m_a^2+2m_b^2-2s-t}{\sqrt{[s-(m_a-m_b)^2][(m_a+m_b)^2-s]-st}}\right] \tag{8.206}$$

or equivalently

$$\rho_1(u,t) = \frac{Z_aZ_b}{2}\left[\frac{t+2u-2m_a^2-2m_b^2}{\sqrt{[u-(m_a-m_b)^2][(m_a+m_b)^2-u]-ut}}\right] \tag{8.207}$$

Three Dimensions

Since the amplitude (8.202) has the same form as the amplitude (8.97), in $D=3$ we find a similar result:

$$\widehat{\mathcal{A}}_A(s,t,u) = \left[\frac{2\beta_1}{\beta_1^2\rho_1^2(s,u)-t}\right]\left[\frac{Z_aZ_b(u-s)}{4}\right]\delta(P) \tag{8.208}$$

The singularity $s = s_*$ and $u = u_*$ now satisfies

$$\beta_1^2 \rho_1^2(s_*, u_*) = t \tag{8.209}$$

This leads to the relation

$$\frac{s_* u_* - (m_a + m_b)^2 (m_a - m_b)^2}{(u_* - s_*)^2} = \frac{Z_a^2 Z_b^2 \beta_1^2}{4t} \tag{8.210}$$

Using

$$s_* + t + u_* = 2m_a^2 + 2m_b^2 \tag{8.211}$$

we obtain

$$\begin{aligned} s_* = m_a^2 + m_b^2 - \frac{t}{2} \\ + 2m_a m_b \left(1 - \frac{t}{4m_a^2}\right)^{1/2} \left(1 - \frac{t}{4m_b^2}\right)^{1/2} \left(1 + \frac{Z_a^2 Z_b^2 \beta_1^2}{t}\right)^{-1/2} \end{aligned} \tag{8.212}$$

or, equivalently

$$\begin{aligned} u_* = m_a^2 + m_b^2 - \frac{t}{2} \\ - 2m_a m_b \left(1 - \frac{t}{4m_a^2}\right)^{1/2} \left(1 - \frac{t}{4m_b^2}\right)^{1/2} \left(1 + \frac{Z_a^2 Z_b^2 \beta_1^2}{t}\right)^{-1/2} \end{aligned} \tag{8.213}$$

In the two-body semiclassical eikonal approximation (8.9), we have

$$\frac{s_*}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} + 2\left(1 + \frac{Z_a^2 Z_b^2 \beta_1^2}{t}\right)^{-1/2} \tag{8.214}$$

and

$$\frac{u_*}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} - 2\left(1 + \frac{Z_a^2 Z_b^2 \beta_1^2}{t}\right)^{-1/2} \tag{8.215}$$

Note that if $t > 0$, then (8.214) and (8.215) are real for any value of β_1. On the other hand, if $t < 0$, then we must require

$$-t > Z_a^2 Z_b^2 \beta_1^2 \tag{8.216}$$

in order for (8.214) and (8.215) to be real. In the $(\beta_1^2/t) \to 0$ limit we have s_* in (8.214) approaching threshold and u_* in (8.215) approaching pseudo-threshold.

The product of (8.212) and (8.213) yields

$$s_* u_* = (m_a + m_b)^2 (m_a - m_b)^2 \\ + 4m_a^2 m_b^2 \left(1 - \frac{t}{4m_a^2}\right)\left(1 - \frac{t}{4m_b^2}\right)\left(\frac{Z_a^2 Z_b^2 \beta_1^2}{t + Z_a^2 Z_b^2 \beta_1^2}\right) \tag{8.217}$$

This suggests that s_* and u_* are bound states when $t > 0$.

8.2.3 Bound States in Four Dimensions

We can repeat the steps in §8.1.5 in order to obtain the part of the amplitude with bound states. The result is very similar to (8.141):

$$\widehat{\mathcal{B}}_A(s,t,u) = \mathcal{A}_{\text{tree}}^A \exp\left[\Xi_\epsilon(s,u)\right] \frac{\Gamma[1 - \alpha_1\rho_1(s,u)]}{\Gamma[1 + \alpha_1\rho_1(s,u)]}\left(-\frac{t}{2\mu^2}\right)^{\alpha_1\rho_1(s,u)} \tag{8.218}$$

where

$$\Xi_\epsilon(s,u) \equiv \alpha_1\rho_1(s,u)\Gamma(\epsilon) \tag{8.219}$$

Explicitly,

$$\Xi_\epsilon(s,u) = \frac{Z_a Z_b \alpha_1}{2}\left[\frac{u - s}{\sqrt{su - (m_a + m_b)^2 (m_a - m_b)^2}}\right]\Gamma(\epsilon) \tag{8.220}$$

In the two-body semiclassical eikonal approximation (8.9), this agrees with the exponentiated infrared divergence in QED proposed by Dalitz [68] and proved by Weinberg [69].

Bound State Spectrum

The amplitude (8.218) has an infinite number of singularities, satisfying

$$1 - \alpha_1\rho_1(s_J, u_J) = -J, \qquad J = 0, 1, 2, \ldots \tag{8.221}$$

This relation has the same form as (8.209). Indeed, from (8.221) we can obtain (8.209) by replacing

$$\frac{\alpha_1^2}{(J+1)^2} \longrightarrow \frac{\beta_1^2}{t} \tag{8.222}$$

Thus, we have the sequences

$$
\begin{aligned}
s_J = {} & m_a^2 + m_b^2 - \frac{t}{2} \\
& + 2m_a m_b \left(1 - \frac{t}{4m_a^2}\right)^{1/2} \left(1 - \frac{t}{4m_b^2}\right)^{1/2} \left(1 + \frac{Z_a^2 Z_b^2 \alpha_1^2}{(J+1)^2}\right)^{-1/2}
\end{aligned}
\tag{8.223}
$$

and

$$
\begin{aligned}
u_J = {} & m_a^2 + m_b^2 - \frac{t}{2} \\
& - 2m_a m_b \left(1 - \frac{t}{4m_a^2}\right)^{1/2} \left(1 - \frac{t}{4m_b^2}\right)^{1/2} \left(1 + \frac{Z_a^2 Z_b^2 \alpha_1^2}{(J+1)^2}\right)^{-1/2}
\end{aligned}
\tag{8.224}
$$

In the two-body semiclassical eikonal approximation (8.9), we find

$$
\frac{s_J}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} + 2\left(1 + \frac{Z_a^2 Z_b^2 \alpha_1^2}{(J+1)^2}\right)^{-1/2} \tag{8.225}
$$

and

$$
\frac{u_J}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} - 2\left(1 + \frac{Z_a^2 Z_b^2 \alpha_1^2}{(J+1)^2}\right)^{-1/2} \tag{8.226}
$$

The product of s_J in (8.225) and u_J in (8.226) yields

$$
s_J u_J \approx (m_a^2 - m_b^2)^2 + 4m_a^2 m_b^2 \left[\frac{Z_a^2 Z_b^2 \alpha_1^2}{(J+1)^2 + Z_a^2 Z_b^2 \alpha_1^2}\right] \tag{8.227}
$$

The values s_J in (8.225) and u_J in (8.226) lie outside of the physical scattering region, which confirm that they correspond to bound states. Note that, although the charges Z_a and Z_b appear squared in the spectrum, it is necessary that the product $Z_a Z_b$ be *negative* in order for the sequence s_J to approach the threshold $(m_a + m_b)^2$ as $J \to \infty$, and not the pseudo-threshold $(m_a - m_b)^2$.

For small values of the coupling α_1 we have

$$
s_J \approx m_a^2 + m_b^2 + 2m_a m_b \left[1 - \frac{Z_a^2 Z_b^2 \alpha_1^2}{2(J+1)^2} + \dots\right] \tag{8.228}
$$

However, unlike the case of the massless scalar exchange, the sequence (8.225) is well-defined for *any* real value of the coupling parameter. When α_1 is very large, we find

$$s_J \approx m_a^2 + m_b^2 + 2m_a m_b \left[1 - 1 + \frac{(J+1)}{|Z_a||Z_b|\alpha_1} + \dots\right] \tag{8.229}$$

which is suggestive of string-like behavior at strong coupling.

Regge Behavior

It should be no surprise that the result (8.218) exhibits Regge behavior with leading Regge trajectory function $R_A(s,u)$ given by

$$\begin{aligned} R_A(s,u) &\equiv -1 + \alpha_1 \rho_1(s,u) \\ &= -1 + \frac{Z_a Z_b \alpha_1}{2} \left[\frac{u-s}{\sqrt{su - (m_a+m_b)^2(m_a-m_b)^2}}\right] \end{aligned} \tag{8.230}$$

The function $R_A(s,u)$ has similar features with $R_\phi(s,u)$:

$$\begin{aligned} &\mathrm{Re}\,[R_A(s,u)] = -1 \text{ when } su < (m_a+m_b)^2(m_a-m_b)^2 \\ &\mathrm{Im}\,[R_A(s,u)] = 0 \text{ when } su > (m_a+m_b)^2(m_a-m_b)^2 \end{aligned} \tag{8.231}$$

In the two-body semiclassical eikonal approximation (8.9), we have

$$R_A(s,t) \approx -1 + Z_a Z_b \alpha_1 \left[\frac{m_a^2 + m_b^2 - s}{\sqrt{[s-(m_a-m_b)^2][(m_a+m_b)^2 - s]}}\right] \tag{8.232}$$

Figures 8.3 and 8.4 show the real and imaginary parts of $R_A(\xi_s)$. Note that we must require $Z_a Z_b < 0$ in order for bound states to form.

8.3 Massless Symmetric Tensor Exchange

The next natural step is the case when the external matter particles exchange massless symmetric tensor quanta. We consider a system that is equivalent to particles interacting via linearized gravity.

The interaction term for scalar particles coupled to a symmetric tensor field h_{mn} is

$$S_{\mathrm{int}}[q_a, q_b, h] \equiv \frac{1}{2}\int \mathrm{d}\tau\, \dot{q}_a^m \dot{q}_a^n h_{mn}[q_a(\tau)] + \frac{1}{2}\int \mathrm{d}\sigma\, \dot{q}_b^m \dot{q}_b^n h_{mn}[q_b(\sigma)] \tag{8.233}$$

We follow the same steps as in previous sections. Start with the two-body

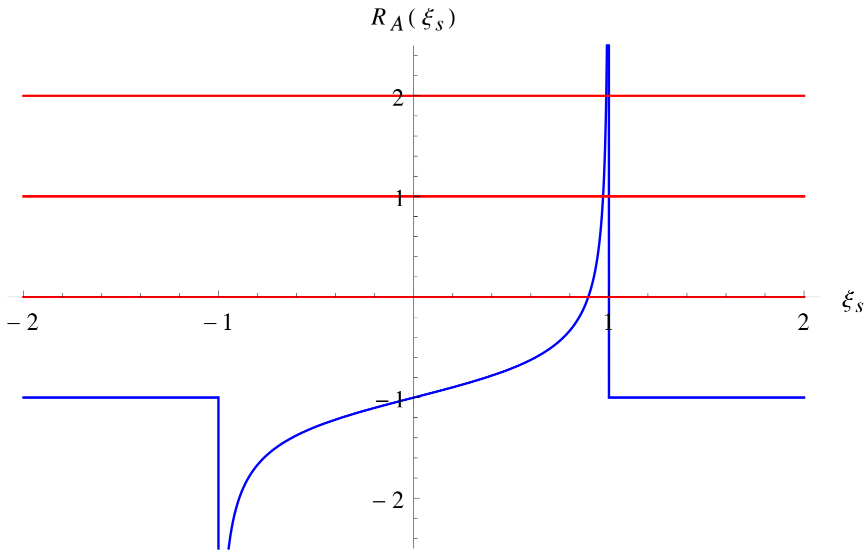


Figure 8.3: Real part of $R_A(\xi_s)$. The red lines correspond to $R_A = 0, 1, 2$. We have used $Z_a Z_b \alpha_1 = -0.5$.

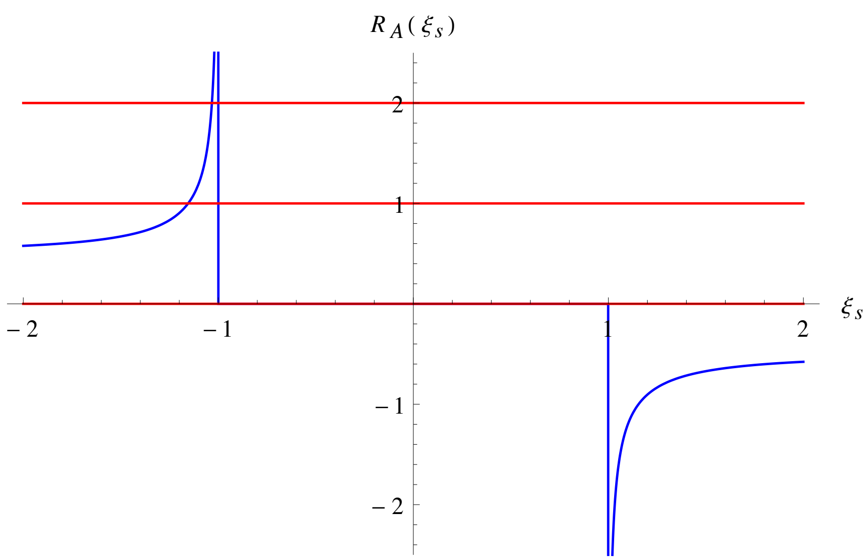


Figure 8.4: Imaginary part of $R_A(\xi_s)$. The red lines correspond to $R_A = 0, 1, 2$. We have used $Z_a Z_b \alpha_1 = -0.5$.

path integral

$$\mathcal{G}_h[h] \equiv \int\limits_{x_1}^{x_3} \mathrm{D}q_a(\tau) \int\limits_{x_2}^{x_4} \mathrm{D}q_b(\sigma)\ \exp\left(-iS_0[q_a, q_b] - iS_{\mathrm{int}}[q_a, q_b, h]\right) \tag{8.234}$$

and integrate over the field h to obtain the "effective" two-body path integral

$$\mathcal{F}_h(3,4|1,2) \equiv \int \mathrm{D}h_{mn}(x)\, \mathcal{G}_h[h] \exp\left(-iS_{\mathrm{kin}}[h]\right) \tag{8.235}$$

where S_{kin} is the (linearized) gauge-fixed kinetic term

$$S_{\mathrm{kin}}[h] = \frac{1}{2g_2^2} \int\int \mathrm{d}x\mathrm{d}y\, \left[h_{mp}(x)(K_2)^{mnpq} h_{nq}(y)\right] \tag{8.236}$$

with

$$\begin{aligned}
(K_2)^{mnpq}(x|y) \equiv{}& \delta(x-y) \\
& \times \left[\frac{1}{2}\eta^{mn}\eta^{pq} + \frac{1}{2}\eta^{mq}\eta^{np} - \frac{1}{2}\left(2 - \frac{1}{\xi_2}\right)\eta^{mp}\eta^{nq}\right] \\
& \times \left(-\frac{1}{2}\partial^2\right) \\
& + \delta(x-y)\left(1 - \frac{1}{\xi_2}\right) \\
& \times \left(\frac{1}{2}\eta^{mn}\partial^p\partial^q + \frac{1}{2}\eta^{mq}\partial^n\partial^q - \eta^{mp}\partial^n\partial^q\right)
\end{aligned} \tag{8.237}$$

In the Fermi-Feynman gauge ($\xi_2 = 1$) we have

$$(K_2)^{mnpq}(x|y) = \frac{1}{2}\left(\eta^{mn}\eta^{pq} + \eta^{mq}\eta^{np} - \eta^{mp}\eta^{nq}\right) K_0(x|y) \tag{8.238}$$

After integrating over h and ignoring the self-interactions, we find

$$\mathcal{F}_h(3,4|1,2) = \int\limits_{x_1}^{x_3} \mathrm{D}q_a(\tau) \int\limits_{x_2}^{x_4} \mathrm{D}q_b(\sigma)\ \exp\left(-iS_0[q_a, q_b] + iS_2^h[q_a, q_b]\right) \tag{8.239}$$

where the free term is given by (8.12) and the two-body interaction term is

$$S_2^h = \frac{g_2^2}{8} \int\int \mathrm{d}\tau\mathrm{d}\sigma\, \left[2\left[\dot{q}_a(\tau)\cdot\dot{q}_b(\sigma)\right]^2 - \dot{q}_a^2(\tau)\dot{q}_b^2(\sigma)\right] G_0\left[q_a(\tau)|q_b(\sigma)\right] \tag{8.240}$$

In section 6.3 we found that the coupling parameter g_2 has units

$$[g_2] = \left(\frac{2-D}{2}\right)[\text{mass}] \tag{8.241}$$

When $D = 4$, we have that g_2 has units of inverse mass (the Planck length). So we write

$$g_2^2 = \frac{8\alpha_2}{2\pi}\mu^{(4-D)} \tag{8.242}$$

where μ has units of mass and $\sqrt{\alpha_2}$ has units of inverse mass. Similarly, in $D = 3$ we have that g_2^2 has units of inverse mass. We write

$$g_2^2 = \frac{8\beta_2}{(2\pi)^{3/2}}\mu^{(3-D)} \tag{8.243}$$

where μ has units of mass and β_2 has units of inverse mass.

8.3.1 Eikonal Van Vleck Function

Since the pre-factor in (8.240) only depends on the slopes of the path functions, in the eikonal JWKB approximation we have a familiar situation. At the eikonal paths (8.32) we find

$$\begin{aligned}
\Sigma_2^h &\equiv S_2^h\left[e_a, e_b\right] \\
&= \frac{g_2^2}{8}\int\int \mathrm{d}\tau\mathrm{d}\sigma\left[2\left[\dot{e}_a(\tau)\cdot\dot{e}_b(\sigma)\right]^2 - \dot{e}_a^2(\tau)\dot{e}_b^2(\sigma)\right]G_0\left[e_a(\tau)|e_b(\sigma)\right] \\
&= -\frac{i\alpha_2}{2\pi}\left[\frac{2(x_{31}\cdot x_{42})^2 - x_{31}^2x_{42}^2}{T_a^2T_b^2}\right]\Upsilon
\end{aligned} \tag{8.244}$$

with Υ given by (8.38). In the eikonal JWKB approximation we use (8.45) and obtain

$$\begin{aligned}
\Sigma_2^h \approx -i\alpha_2&\left[\frac{2(x_{31}\cdot x_{42})^2 - x_{31}^2x_{42}^2}{T_aT_b\sqrt{x_{31}^2x_{42}^2 - (x_{31}\cdot x_{42})^2}}\right] \\
&\times\Gamma\left(\frac{D-4}{2}\right)\left(\frac{2}{\mu^2B_{12}^2}\right)^{(D-4)/2}
\end{aligned} \tag{8.245}$$

Using $D = 4 + 2\epsilon$ and introducing

$$\rho_2 \equiv \frac{2(x_{31}\cdot x_{42})^2 - x_{31}^2x_{42}^2}{T_aT_b\sqrt{x_{31}^2x_{42}^2 - (x_{31}\cdot x_{42})^2}} \tag{8.246}$$

we can write more compactly

$$\Sigma_2^h \approx -i\alpha_2\rho_2\Gamma(\epsilon)\left(\frac{2}{\mu^2 B_{12}^2}\right)^{\epsilon} \tag{8.247}$$

which has a by-now-familiar form. Note that, unlike ρ_1 (but similar to ρ_0), the quantity ρ_2 depends on the worldline moduli.

8.3.2 Eikonal S-Matrix

We follow the same steps as in §8.1.3 and §8.2.2. After integrating over X, x_{31} and x_{42}, the function ρ_2 becomes

$$\rho_2 = \frac{2(p_{31}\cdot p_{42})^2 - p_{31}^2 p_{42}^2}{\sqrt{p_{31}^2 p_{42}^2 - (p_{31}\cdot p_{42})^2}} \tag{8.248}$$

Similar considerations as before lead to the following truncated on-shell scattering amplitude

$$\begin{aligned}\widehat{\mathcal{A}}_h &= \mathcal{A}_{\text{tree}}^h(s,t,u) \\ &\quad\times \sum_{L=0}^{\infty} \frac{[\alpha_2\rho_2(s,u)]^L}{\Gamma(L+1)} \frac{\Gamma(1+\epsilon)[\Gamma(\epsilon)]^L\Gamma(1-L\epsilon)}{\Gamma(1+\epsilon+L\epsilon)}\left(-\frac{t}{2\mu^2}\right)^{L\epsilon}\end{aligned} \tag{8.249}$$

with

$$\rho_2(s,u) = \frac{s^2+u^2-6su+4(m_a+m_b)^2(m_a-m_b)^2}{8\sqrt{su-(m_a+m_b)^2(m_a-m_b)^2}} \tag{8.250}$$

and the pre-factor is now given by

$$\mathcal{A}_{\text{tree}}^h = -\frac{2\alpha_2}{t}\left[\frac{s^2+u^2-6su+4(m_a+m_b)^2(m_a-m_b)^2}{16}\right]\delta(P) \tag{8.251}$$

In the two-body semiclassical eikonal approximation (8.9), this pre-factor reduces to

$$\mathcal{A}_{\text{tree}}^h \approx -\frac{2\alpha_2}{t}\left[\frac{(s-m_a^2-m_b^2)^2-2m_a^2m_b^2}{2}\right]\delta(P) \tag{8.252}$$

which agrees with the tree level scattering amplitude for two scalar particles exchanging a linearized graviton.

8.3.3 Bound States in Four Dimensions

Since Σ_2^h has the same form as Σ_2^A and Σ_2^ϕ, near $\epsilon = 0$ we can repeat everything we did with (8.136) in §8.1.5. The result is analogous to (8.141) or (8.218):

$$\widehat{\mathcal{B}}_h(s,t,u) = \mathcal{A}^h_{\text{tree}} \exp\left[\Omega_\epsilon(s,u)\right] \frac{\Gamma[1-\alpha_2\rho_2(s,u)]}{\Gamma[1+\alpha_2\rho_2(s,u)]} \left(-\frac{t}{2\mu^2}\right)^{\alpha_2\rho_2(s,u)} \tag{8.253}$$

with

$$\Omega_\epsilon(s,u) \equiv \alpha_2\rho_2(s,u)\Gamma(\epsilon) \tag{8.254}$$

The leading Regge trajectory function $R_h(s)$ is now given by

$$\begin{aligned} R_h(s,u) &\equiv -1 + \alpha_2\rho_2(s,u) \\ &= -1 + \frac{\alpha_2}{8}\left[\frac{s^2+u^2-6su+4(m_a+m_b)^2(m_a-m_b)^2}{\sqrt{su-(m_a+m_b)^2(m_a-m_b)^2}}\right] \end{aligned} \tag{8.255}$$

In the two-body semiclassical eikonal approximation (8.9), this function reduces to

$$R_h(s) \approx -1 + \alpha_2\left[\frac{(s-m_a^2-m_b^2)^2 - 2m_a^2m_b^2}{\sqrt{[s-(m_a-m_b)^2][(m_a+m_b)^2-s]}}\right] \tag{8.256}$$

In terms of the dimensionless variable ξ_s defined in (8.166), we have

$$R_h(\xi_s) = -1 + m_a m_b \alpha_2 \left[\frac{2\xi_s^2-1}{\sqrt{1-\xi_s^2}}\right] \tag{8.257}$$

Figures 8.5 and 8.6 show the real and imaginary parts of $R_h(\xi_s)$.

Solving the singularity condition $J = R_h(s_J, u_J)$ leads to

$$\begin{aligned} s_J =& m_a^2 + m_b^2 - \frac{t}{2} \\ &+ 2m_a m_b \sqrt{\left(1-\frac{t}{4m_a^2}\right)\left(1-\frac{t}{4m_b^2}\right)\left[\frac{1}{2} + \left(1+\sqrt{1+\kappa}\right)^{-1}\right]} \end{aligned} \tag{8.258}$$

where

$$\kappa = \frac{8m_a^2m_b^2\alpha_2^2}{(J+1)^2}\left(1-\frac{t}{4m_a^2}\right)\left(1-\frac{t}{4m_b^2}\right) \tag{8.259}$$

Using

$$s_J + t + u_J = 2m_a^2 + 2m_b^2 \tag{8.260}$$

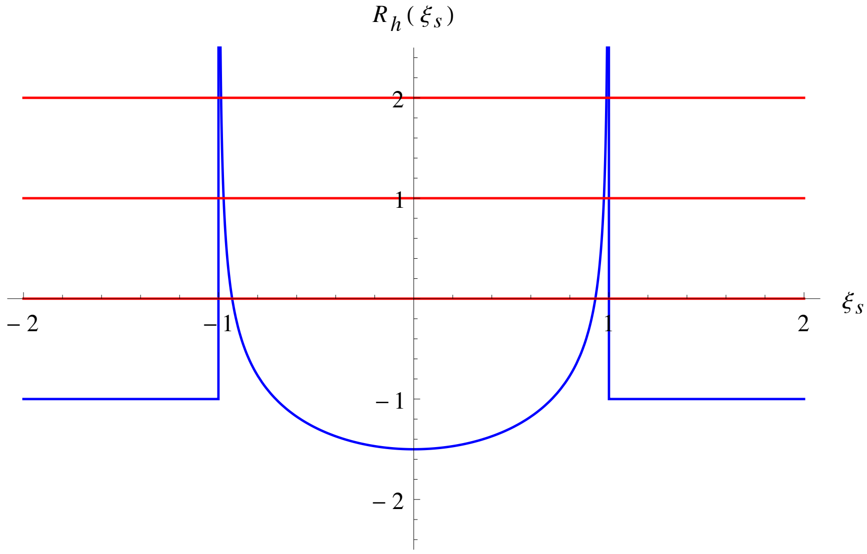


Figure 8.5: Real part of $R_h(\xi_s)$. The red lines correspond to $R_h = 0, 1, 2$. We have used $m_a m_b \alpha_2 = 0.5$.

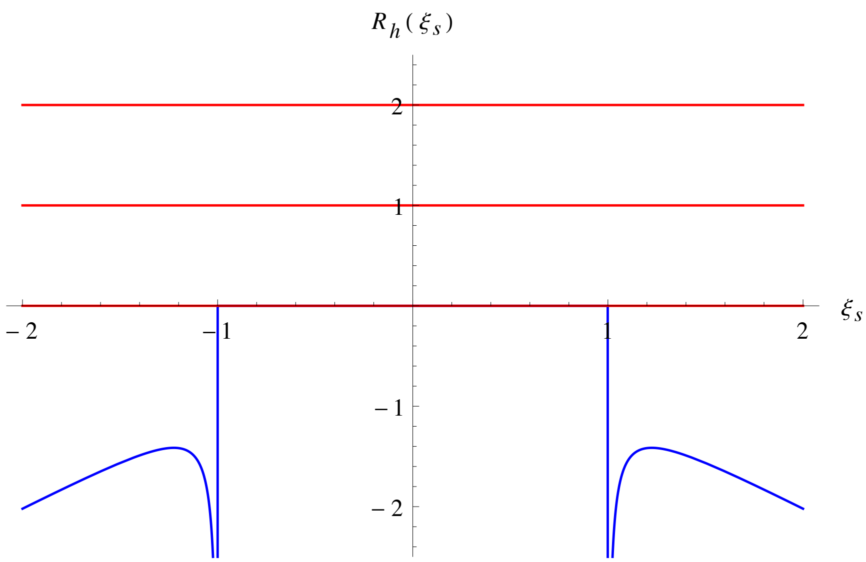


Figure 8.6: Imaginary part of $R_h(\xi_s)$. The red lines correspond to $R_h = 0, 1, 2$. We have used $m_a m_b \alpha_2 = 0.5$.

we find

$$
\begin{aligned}
u_J =& m_a^2 + m_b^2 - \frac{t}{2} \\
&- 2m_a m_b \sqrt{\left(1 - \frac{t}{4m_a^2}\right)\left(1 - \frac{t}{4m_b^2}\right)\left[\frac{1}{2} + \left(1 + \sqrt{1+\kappa}\right)^{-1}\right]}
\end{aligned}
\tag{8.261}
$$

In the two-body semiclassical eikonal approximation (8.9), we have

$$
\frac{s_J}{m_a m_b} \approx \frac{m_a^2 + m_b^2}{m_a m_b} + 2\left[\frac{1}{2} + \left(1 + \sqrt{1 + \frac{8m_a^2 m_b^2 \alpha_2^2}{(J+1)^2}}\right)^{-1}\right]^{1/2} \tag{8.262}
$$

Just like for the exchange of the massless vector, the expression for s_J in (8.262) is valid for any value of α_2. Near $\alpha_2 = 0$ we find

$$
s_J \approx m_a^2 + m_b^2 + 2m_a m_b \left[1 - \frac{m_a^2 m_b^2 \alpha_2^2}{2(J+1)^2} + \dots\right] \tag{8.263}
$$

On the other hand, near $\alpha_2 \to \infty$ we find

$$
s_J \approx m_a^2 + m_b^2 + 2m_a m_b \left[1 - \frac{\sqrt{2}-1}{\sqrt{2}} + \frac{(J+1)}{4m_a m_b \alpha_2} + \dots\right] \tag{8.264}
$$

which is analogous to (8.229).

8.4 Massive Scalar Exchange

The methods used in the previous three sections can be applied to problems where the exchange quanta is massive. In this section we study the case when the external particles exchange massive scalar quanta with mass M. In the eikonal approximation, the momentum transfer t is small compared to M^2. By Fourier-Heisenberg conjugacy, the separation between the particles is very large compared to $1/M$.

We follow similar steps as in section §8.1 to derive the two-body semiclassical eikonal kernel. The two-body action functional S_Φ is

$$
S_\Phi[q_a, q_b] = S_0[q_a, q_b] - S_2^\Phi[q_a, q_b] \tag{8.265}
$$

with the two-body interaction term S_2^Φ given by

$$S_2^\Phi [q_a, q_b] \equiv g_0^2 \int \int \mathrm{d}\tau \mathrm{d}\sigma \, G_M [q_a(\tau)|q_b(\sigma)] \tag{8.266}$$

Here G_M is the massive scalar Green function,

$$G_M(x|y) \equiv \int_0^\infty \mathrm{d}T_M \left(-\frac{i}{T_M}\right)^{D/2} \exp\left[\frac{i}{2T_M}(y-x)^2 - \frac{iM^2 T_M}{2}\right] \tag{8.267}$$

Just like in section §8.1, the coupling parameter g_0 is dimensionful. However, instead of (8.30), in $D = 4$ we now use

$$g_0^2 = \frac{\alpha_0}{(2\pi)^{3/2}} \mu^{(4-D)} \tag{8.268}$$

with μ and $\sqrt{\alpha_0}$ having units of mass. This choice of normalization is convenient to eliminate unwanted factors of 2π.

8.4.1 Eikonal Van Vleck Function

At the eikonal paths (8.32) we have

$$\begin{aligned}
\Sigma_2^\Phi &\equiv S_2^\Phi [e_a, e_b] \\
&= g_0^2 \int \int \mathrm{d}\tau \mathrm{d}\sigma \, G_M [e_a(\tau)|e_b(\sigma)] \\
&\approx \frac{\alpha_0 \rho_0 \mu^{(4-D)}}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}T_M \left(-\frac{i}{T_M}\right)^{(D-2)/2} \exp\left(\frac{i}{2T_M} B_{12}^2 - \frac{iM^2 T_M}{2}\right) \\
&\approx -i\alpha_0\rho_0 \left(\frac{M}{\mu}\right)^{(D-4)} \left(i\sqrt{-M^2 B_{12}^2}\right)^{(3-D)/2} \exp\left(-i\sqrt{-M^2 B_{12}^2}\right)
\end{aligned}$$

where ρ_0 is given by (8.47). To get the third line we integrated over τ and σ with stationary methods. To get the fourth line we integrated over T_M with stationary methods. Both of these steps are valid in the eikonal approximation.

8.4.2 Eikonal S-Matrix

With Σ_2^Φ, we have

$$-1+\exp\left(i\Sigma_2^\Phi\right)=\sum_{l=1}^{\infty}\frac{(\alpha_0\rho_0)^l}{\Gamma(l+1)}\left(\frac{M}{\mu}\right)^{l(D-4)}\left(i\sqrt{-M^2B_{12}^2}\right)^{l(3-D)/2}\times\exp\left(-il\sqrt{-M^2B_{12}^2}\right) \tag{8.269}$$

which can be written as a double sum,

$$-1+\exp\left(i\Sigma_2^\Phi\right)=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_0\rho_0)^l}{\Gamma(l+1)}\frac{(-l)^n}{\Gamma(n+1)}\left(\frac{M}{\mu}\right)^{l(D-4)}\times(-2)^{Y_{nl}}\left(-\frac{2}{M^2B_{12}^2}\right)^{Z_{nl}} \tag{8.270}$$

Here we have denoted

$$Y_{nl}\equiv\frac{l(3-D)}{4}+\frac{n}{2},\qquad Z_{nl}\equiv-Y_{nl}=\frac{l(D-3)}{4}-\frac{n}{2} \tag{8.271}$$

Before truncation, the scattering amplitude is

$$\mathcal{A}_\Phi=\frac{1}{\rho_0}\mathcal{N}\delta(P)\int \mathrm{d}B_{12}\left[-1+\exp\left(i\Sigma_2^\Phi\right)\right]\exp\left(-iB_{12}\cdot P_{12}\right) \tag{8.272}$$

After integrating over B_{12}, we find

$$\mathcal{A}_\Phi(3,4|1,2)=\alpha_0\mathcal{N}\delta(P)(iM)^{(2-D)}\left(\frac{M}{\mu}\right)^{(D-4)}\times\sum_{L=0}^{\infty}\frac{(\alpha_0\rho_0)^L}{\Gamma(L+2)}\left(\frac{M}{\mu}\right)^{L(D-4)}\mathcal{M}_L(t) \tag{8.273}$$

where $\mathcal{M}_L(t)$ is defined as

$$\mathcal{M}_L(t)\equiv\sum_{n=0}^{\infty}\frac{(-L-1)^n}{\Gamma(n+1)}(-2)^{Y(n,L)}\frac{\Gamma[W(n,L)]}{\Gamma[Z(n,L)]}\left(\frac{2M^2}{t}\right)^{W(n,L)} \tag{8.274}$$

Here we have introduced

$$Y(n, L) \equiv Y_0(L) + \frac{n}{2} \tag{8.275}$$

$$Z(n, L) \equiv Z_0(L) - \frac{n}{2} \tag{8.276}$$

$$W(n, L) \equiv W_0(L) + \frac{n}{2} \tag{8.277}$$

with

$$Y_0(L) \equiv \frac{(L+1)(3-D)}{4} \tag{8.278}$$

$$Z_0(L) \equiv \frac{(L+1)(D-3)}{4} \tag{8.279}$$

$$W_0(L) \equiv \frac{D-2}{2} - \frac{(L+1)(D-3)}{4} \tag{8.280}$$

We define the truncated on-shell scattering amplitude by

$$\widehat{\mathcal{A}}_\Phi(s, t, u) \equiv \left(\frac{\mu^{(D-4)}}{\mathcal{N}}\right) \mathcal{A}_\Phi(3, 4|1, 2) \tag{8.281}$$

Thus,

$$\widehat{\mathcal{A}}_\Phi(s, t, u) = \frac{\alpha_0}{M^2}\delta(P) i^{(2-D)} \sum_{L=0}^{\infty} \frac{[\alpha_0 \rho_0(s, u)]^L}{\Gamma(L+2)} \left(\frac{M}{\mu}\right)^{L(D-4)} \mathcal{M}_L(t) \tag{8.282}$$

The sum in $\mathcal{M}_L$ can be evaluated if we split it into the sum over even and odd integers:

$$\mathcal{M}_L(t) = \mathcal{E}_L(t) + \mathcal{O}_L(t) \tag{8.283}$$

Then each contribution can be evaluated separately, to give

$$\begin{aligned} \mathcal{E}_L(t) = {} & (-1)^{Y_0} 2^{(W_0+Y_0)} \frac{\Gamma(W_0)}{\Gamma(Z_0)} \left(\frac{M^2}{t}\right)^{W_0} \\ & \times {}_2F_1\left(W_0, 1-Z_0, \frac{1}{2}, \frac{(L+1)^2 M^2}{t}\right) \end{aligned} \tag{8.284}$$

$$\begin{aligned} \mathcal{O}_L(t) = {} & -(L+1)(-1)^{\tilde{Y}_0} 2^{(\tilde{W}_0+\tilde{Y}_0)} \frac{\Gamma(\tilde{W}_0)}{\Gamma(\tilde{Z}_0)} \left(\frac{M^2}{t}\right)^{\tilde{W}_0} \\ & \times {}_2F_1\left(\tilde{W}_0, 1-\tilde{Z}_0, \frac{3}{2}, \frac{(L+1)^2 M^2}{t}\right) \end{aligned} \tag{8.285}$$

where, for neatness, we have introduced

$$\tilde{Y}_0(L) \equiv Y_0(L) + \frac{1}{2} \quad \tilde{Z}_0(L) \equiv Z_0(L) - \frac{1}{2}, \quad \tilde{W}_0(L) \equiv W_0(L) + \frac{1}{2} \tag{8.286}$$

Here, ${}_2F_1$ is the Gauss hypergeometric function.

8.4.3 Perturbative Amplitudes

The result (8.282) for the scattering amplitude has form of a perturbative expansion, but each perturbative amplitudes appears to be quite complicated for generic spacetime dimension. We consider two specific cases.

Three Dimensions

With $D = 3$, we have

$$Y_0(L) = 0, \qquad Z_0(L) = 0, \qquad W_0(L) = \frac{1}{2} \tag{8.287}$$

and

$$\tilde{Y}_0(L) = \frac{1}{2}, \qquad \tilde{Z}_0(L) = -\frac{1}{2}, \qquad \tilde{W}_0(L) = 1 \tag{8.288}$$

The expressions for $\mathcal{E}_L$ and $\mathcal{O}_L$ simplify greatly:

$$\mathcal{E}_L(t) = 0, \qquad \mathcal{O}_L(t) = -i\sqrt{\frac{2}{\pi}}(L+1)\left[\frac{M^2}{(L+1)^2M^2 - t}\right] \tag{8.289}$$

The form of $\mathcal{O}_L$ is suggestive. At L-loops, $\mathcal{O}_L$ has a simple pole when

$$t = (L+1)^2M^2 = [(L+1)M]^2 \tag{8.290}$$

These singularities correspond to the usual multi-mass branch points. It is somewhat curious that instead of a branch cut (which is a continuum of singularities), we only get an isolated singularity. After summing over L, we find

$$\widehat{\mathcal{A}}_\Phi(s,t,u) = -\frac{\beta_0}{\sqrt{2\pi}}\frac{1}{M\sqrt{t}}\delta(P)\left[\mathcal{A}_-(s,t,u) + \mathcal{A}_+(s,t,u)\right] \tag{8.291}$$

where

$$\mathcal{A}_\mp \equiv \pm\left[-\frac{\beta_0\rho_0}{M}\right]^{R_\mp(t)}\left(\Gamma\left[-R_\mp(t), -\frac{\beta_0\rho_0}{M}\right] - \Gamma\left[-R_\mp(t)\right]\right) \tag{8.292}$$

with

$$R_{\mp}(t) \equiv -1 \mp \sqrt{\frac{t}{M^2}} \tag{8.293}$$

The form of (8.292) is also curious: It is similar to a Regge amplitude, with Regge trajectory function $R_{\mp}(t)$. However, the singularities from $R_{\mp}(t_J) = J$ are not bound states!

The features of the eikonal scattering amplitude in three dimensions are not expected in higher dimensions. The form of (8.289) can be explained as follows. In the eikonal JWKB approximation, the eikonal Van Vleck function for the massive exchange is again proportional to a propagator in $D-2$ dimensions. However, this lower-dimensional propagator is now massive. Thus, just like in the massless scalar case, considered in §8.1, the semiclassical eikonal kernel (8.270) corresponds to a sum over multiple exchanges between two points. In $D = 3$, the eikonal Van Vleck function is proportional to a massive scalar propagator in one spacetime dimension. Recall that in one dimension, the propagator has the form[3]

$$G_M(x) = -\frac{\sqrt{2\pi}i}{M} \exp\left[-i\sqrt{-M^2x^2}\right] \tag{8.294}$$

The term with $L+1$ exchanges between two points separated by x is proportional to the product of $L+1$ propagators,

$$[G_M(x)]^{(L+1)} = \left(-\frac{\sqrt{2\pi}i}{M}\right)^{(L+1)} \exp\left[-i\sqrt{-(L+1)^2M^2x^2}\right] \tag{8.295}$$

which in turn is proportional to a one-dimensional massive propagator with mass $(L+1)M$. Since the semiclassical propagator is exact in one dimension, the Fourier transform of (8.295) is proportional to a massive scalar Feynman propagator with a pole at $(L+1)^2M^2$, which is exactly what we get in (8.289).

[3]See appendix A for details.

Four Dimensions

The story is not very illuminating in $D = 4$. Just to be careful, we will work with $D = 4 - 2\epsilon$ and $\epsilon > 0$. Note the minus sign in front of ϵ. We have

$$Y_0(L) = \frac{(L+1)(2\epsilon - 1)}{4} \tag{8.296}$$

$$Z_0(L) = \frac{(L+1)(1 - 2\epsilon)}{4} \tag{8.297}$$

$$W_0(L) = \frac{3 - L + 2\epsilon(L-1)}{4} \tag{8.298}$$

and

$$\tilde{Y}_0(L) = \frac{2 + (L+1)(2\epsilon - 1)}{4} \tag{8.299}$$

$$\tilde{Z}_0(L) = \frac{L - 1 - 2\epsilon(L+1)}{4} \tag{8.300}$$

$$\tilde{W}_0(L) = \frac{5 - L + 2\epsilon(L-1)}{4} \tag{8.301}$$

At tree level (i.e. zero-loops), we have

$$\mathcal{E}_0(t) = (1-i)\left[\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]\left(\frac{M^2}{t}\right)^{3/4} {}_2F_1\left(\frac{3}{4}, \frac{3}{4}; \frac{1}{2}; \frac{M^2}{t}\right) \tag{8.302}$$

$$\mathcal{O}_0(t) = -2(1+i)\left[\frac{\Gamma(5/4)}{\Gamma(-1/4)}\right]\left(\frac{M^2}{t}\right)^{5/4} {}_2F_1\left(\frac{5}{4}, \frac{5}{4}; \frac{3}{2}; \frac{M^2}{t}\right) \tag{8.303}$$

where we have taken the $\epsilon \to 0$ limit. These expressions are much more complicated than a simple pole at $t = M^2$. Indeed, we have a cut at $t = 0$. Using the Euler identity (A.45), we can write

$${}_2F_1\left(\frac{3}{4}, \frac{3}{4}; \frac{1}{2}; \frac{M^2}{t}\right) = \left(\frac{t}{t - M^2}\right) {}_2F_1\left(-\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; \frac{M^2}{t}\right) \tag{8.304}$$

$${}_2F_1\left(\frac{5}{4}, \frac{5}{4}; \frac{3}{2}; \frac{M^2}{t}\right) = \left(\frac{t}{t - M^2}\right) {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{3}{2}; \frac{M^2}{t}\right) \tag{8.305}$$

So then,

$$\mathcal{M}_0(t) = -\left(\frac{2M^2}{M^2 - t}\right)\mathcal{P}_0(t) \tag{8.306}$$

with

$$\begin{aligned}\mathcal{P}_0(t) \equiv \left(\frac{1-i}{2}\right)\left[\frac{\Gamma\left(3/4\right)}{\Gamma\left(1/4\right)}\right]\left(\frac{M^2}{t}\right)^{-1/4} {}_2F_1\left(-\frac{1}{4},-\frac{1}{4};\frac{1}{2};\frac{M^2}{t}\right) \\ -\left(1+i\right)\left[\frac{\Gamma\left(5/4\right)}{\Gamma\left(-1/4\right)}\right]\left(\frac{M^2}{t}\right)^{1/4} {}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{3}{2};\frac{M^2}{t}\right)\end{aligned} \tag{8.307}$$

Thus, the tree level amplitude is

$$\widehat{\mathcal{A}}_0(t) = \mathcal{A}_{\text{tree}}(t)\mathcal{P}_0(t), \qquad \mathcal{A}_{\text{tree}}(t) \equiv \frac{2\alpha_0}{M^2-t}\delta(P) \tag{8.308}$$

Note that $\mathcal{P}_0$ is finite when $t = M^2$.

At one-loop level, we have

$$\mathcal{O}_1(t) = 0, \qquad \mathcal{E}_1(t) = -i\left(\frac{M^2}{t-4M^2}\right)^{1/2} \tag{8.309}$$

where we have taken the $\epsilon \to 0$ limit. This exhibits a cut at $t = (2M)^2$.

Similarly, after taking the $\epsilon \to 0$ limit, we find at two-loops level

$$\mathcal{E}_2(t) = -2(1+i)\left[\frac{\Gamma(5/4)}{\Gamma(3/4)}\right]\left(\frac{M^2}{t}\right)^{1/4} {}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};\frac{9M^2}{t}\right) \tag{8.310}$$

$$\mathcal{O}_2(t) = -3(1-i)\left[\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]\left(\frac{M^2}{t}\right)^{3/4} {}_2F_1\left(\frac{3}{4},\frac{3}{4};\frac{3}{2};\frac{9M^2}{t}\right) \tag{8.311}$$

However, at three-loops level, we find after taking the $\epsilon \to 0$ limit

$$\mathcal{O}_3(t) = i \arcsin\left(\sqrt{\frac{16M^2}{t}}\right) \tag{8.312}$$

but $\mathcal{E}_3$ is divergent when $\epsilon = 0$. At four-loops level, we find no problems when $\epsilon = 0$, but at five-loops level we find

$$\mathcal{E}_5(t) = -i\sqrt{\frac{t}{M^2}}\left[\sqrt{1-\frac{36M^2}{t}} + \sqrt{\frac{36M^2}{t}}\arcsin\left(\sqrt{\frac{36M^2}{t}}\right)\right] \tag{8.313}$$

but $\mathcal{O}_5$ is divergent when $\epsilon = 0$. Because of these divergences at odd number of loops, we cannot find a compact form for the scattering amplitude.

Chapter 9

Discussion and Outlook

In this last chapter, we offer some comments regarding the results obtained in this work and provide a list of items to be considered in future work.

9.1 Discussion

In this section we discuss our results and compare them with other results in the literature.

Regge Ladders

As we mentioned in chapter 3, the Regge limit is a powerful tool for extracting the asymptotic behavior of perturbative amplitudes in the high-energy regime. For the elastic scattering event

$$a(p_1) + b(p_2) \longrightarrow a(p_3) + b(p_4), \tag{9.1}$$

the Regge limit is

$$\frac{t}{m_a m_b} \to \infty \qquad \text{fixed } \frac{s}{m_a m_b} \qquad \text{fixed } \frac{m_a}{m_b} \tag{9.2}$$

Lee & Sawyer [10] used the Regge limit to evaluate the series of ladder diagrams in the Bethe-Salpeter equation. This approach relies on second-quantized methods. For scalar φ^3 theory they found Regge behavior, with leading Regge trajectory function $R_{LS}(s)$ given by

$$R_{LS}(s) = -1 + \frac{g^2}{8\pi^2} \int_{4m^2}^{\infty} \frac{\mathrm{d}\sigma}{(\sigma - s)\sqrt{\sigma(\sigma - 4m^2)}} \tag{9.3}$$

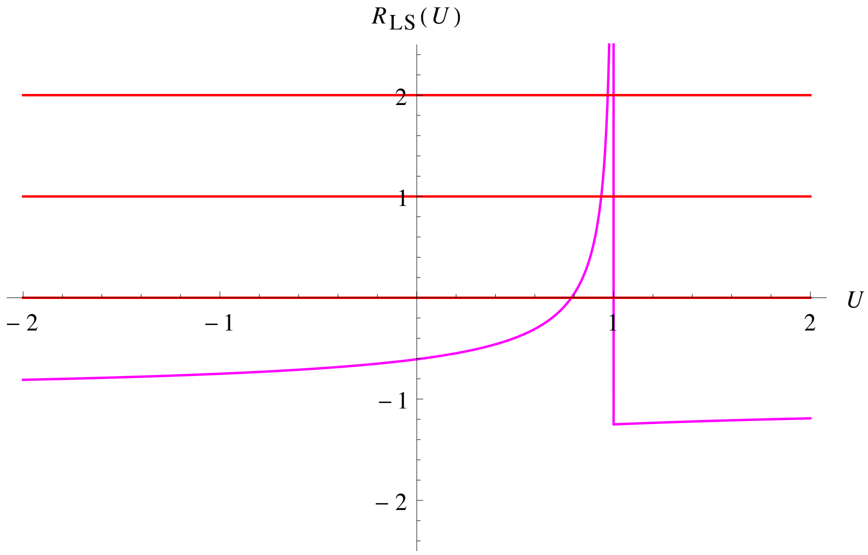


Figure 9.1: Real part of $R_{LS}(U)$. The red lines correspond to $R_{LS} = 0, 1, 2$. We have used $\alpha = 0.5$.

where we have kept their normalizations. After integrating over σ we find

$$R_{LS}(s) = -1 + \frac{g^2}{4\pi^2} \frac{1}{\sqrt{s(4m^2 - s)}} \arccos\left(\frac{2m^2 - s}{2m^2}\right) \tag{9.4}$$

In order to visualize this function, we introduce the dimensionless variables

$$U(s) \equiv \frac{s - 2m^2}{2m^2} \qquad \alpha \equiv \frac{g^2}{8\pi^2 m^2} \tag{9.5}$$

Then

$$R_{LS}(s) = -1 + \frac{\alpha}{2}\left[\frac{\arccos(-U)}{\sqrt{1 - U^2}}\right] \tag{9.6}$$

Figures 9.1 and 9.2 contains the plots of $R_{LS}(s)$ as a function of U. Unlike $R_\phi(s)$ in figures 8.1 and 8.2, the real part of $R_{LS}(s)$ is not constant in the region $s < (m_a - m_b)^2$.

The expression (9.3) can be generalize to the case of two different masses:

$$R_{LS}(s) = -1 + \frac{g^2}{8\pi^2} \int\limits_{(m_a+m_b)^2}^{\infty} \frac{\mathrm{d}\sigma}{(\sigma - s)\sqrt{[\sigma - (m_a - m_b)^2][\sigma - (m_a + m_b)^2]}} \tag{9.7}$$

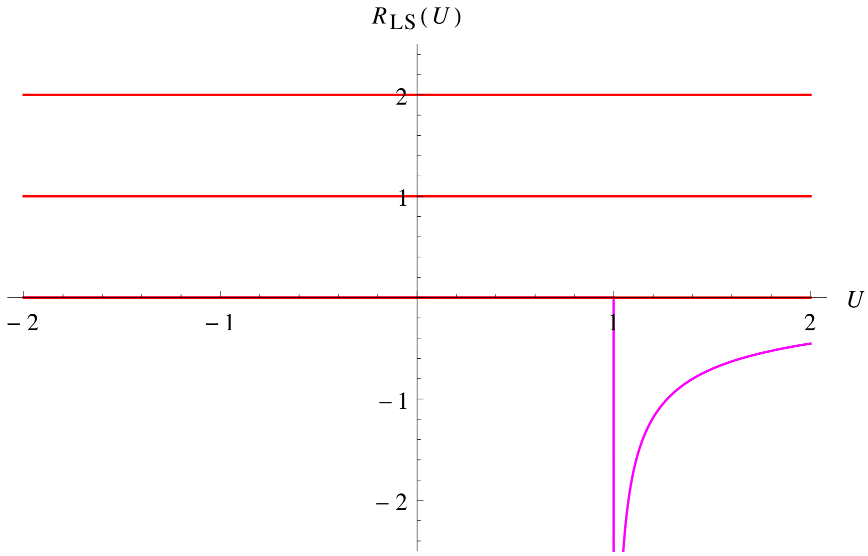


Figure 9.2: Imaginary part of $R_{LS}(U)$. The red lines correspond to $R_{LS} = 0, 1, 2$. We have used $\alpha = 0.5$.

After integration we find

$$R_{LS}(s) = -1 + \frac{\beta}{2}\left[\frac{\arccos\left(-\xi_s\right)}{\sqrt{1-\xi_s^2}}\right] \tag{9.8}$$

where

$$\xi_s \equiv \frac{s - m_a^2 - m_b^2}{2m_a m_b}, \qquad \beta \equiv \frac{g^2}{8\pi^2 m_a m_b} \tag{9.9}$$

The function $R_{LS}(s)$ shares the same asymptotic values with the trajectory function $R_\phi(s)$ that we found in §8.1,

$$R_\phi(s) = -1 + \frac{\alpha_0}{m_a m_b}\left[\frac{1}{\sqrt{1-\xi_s^2}}\right] \tag{9.10}$$

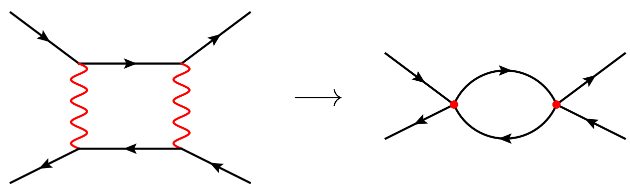
Namely,

$$\lim_{s\to\pm\infty} \mathrm{Re}\left[R_{LS}(s)\right] = -1, \qquad \lim_{s\to\pm\infty} \mathrm{Im}\left[R_{LS}(s)\right] = 0 \tag{9.11}$$

Just like R_ϕ, the function R_{LS} diverges at the threshold $s = (m_a + m_b)^2$. However, unlike R_ϕ, the function R_{LS} does not diverge at the pseudothreshold $s = (m_a - m_b)^2$. Indeed, R_{LS} is real in the region $s < (m_a + m_b)^2$, so for large values of the coupling some of the bound state energies will be real but negative. Unlike R_ϕ, which is real only in the interval $(m_a - m_b)^2 < s <$

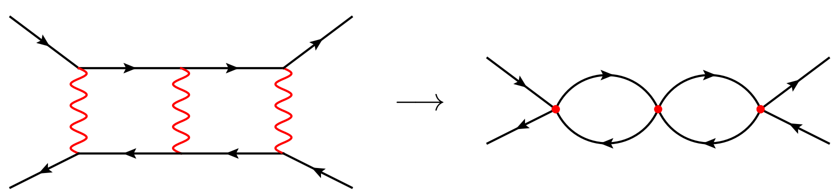
$(m_a + m_b)^2$, R_{LS} is "aware" of the instabilities in scalar φ^3 theory and can include tachyonic states in the spectrum.

If t is very large, then the conjugate length, the separation between the particles, is very small. Thus, the Regge limit (9.2) contracts ladder diagrams along the vertical direction (i.e. the t-channel). For example, at the one-loop level we have



(9.12)

with the *matter* propagators in the loop on the left living in D dimensions, but those on the right living in $D-2$ dimensions. The cubic interaction becomes a quartic interaction. Similarly, for the double box, we have



(9.13)

Indeed, the Regge trajectory R_{LS} can be extracted from the *exact* one-loop scalar box without taking the Regge limit. The exact one-loop amplitude can be found in [9] (see [7] for a more general result). It reads

$$\mathcal{A}_{\text{box}} \sim \left(-\frac{2\mu^2}{t}\right) H(s) \log\left(-\frac{t}{2\mu^2}\right) \tag{9.14}$$

where

$$H(s) \equiv \int\limits_{(m_a+m_b)^2}^{\infty} \frac{\mathrm{d}\sigma}{(\sigma - s)\sqrt{\Lambda(\sigma)}} \tag{9.15}$$

with

$$\Lambda(s) = [s - (m_a - m_b)^2][s - (m_a + m_b)^2] \tag{9.16}$$

The answer for the integral was given in [9] as

$$H(s) = \frac{1}{\sqrt{\Lambda(s)}} \log\left(\frac{(m_a + m_b)^2 - s + \sqrt{\Lambda(s)}}{(m_a + m_b)^2 - s - \sqrt{\Lambda(s)}}\right) \tag{9.17}$$

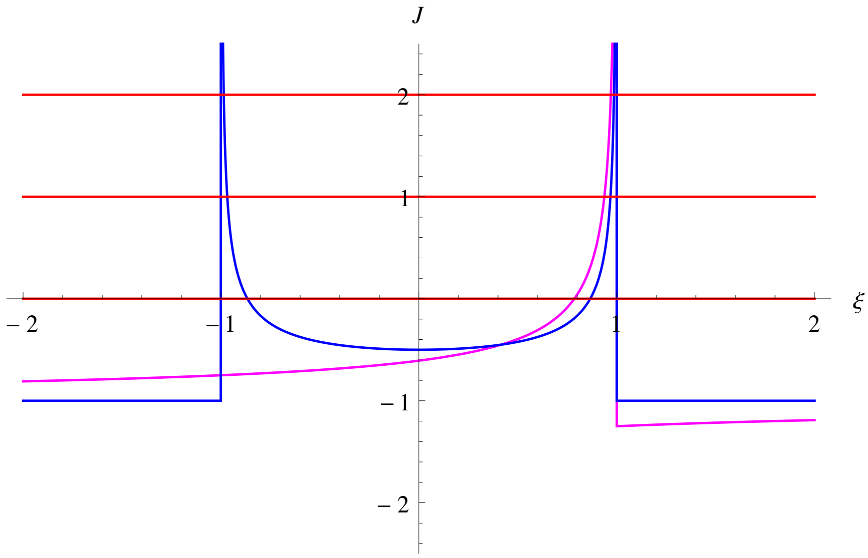


Figure 9.3: Real part of $R_{LS}(\xi)$ (magenta) and $R_\phi(\xi)$ (blue). The red lines correspond to $J = 0, 1, 2$. We have used $\alpha = 0.5$.

which can be written as

$$H(s) = \frac{1}{\sqrt{1-\xi^2}} \arccos\left(\sqrt{\frac{1-\xi}{2}}\right) = \frac{1}{2}\frac{\arccos(-\xi)}{\sqrt{1-\xi^2}} \tag{9.18}$$

Hence, $R_{LS}(s)+1$ is proportional to $H(s)$. The exact one-loop amplitude (9.14) has the same form as the one-loop amplitude (8.124) (modulo regularization), except that instead of $H(s)$ we have $\rho_0(s)$. Figures 9.3 and 9.4 have a graphical comparison.

Eikonal Ladders

All of our results were obtained from first-quantized path integrals in the eikonal JWKB approximation. Since we have ignored self-interactions, the only interaction terms in the (exact) quantum path integrals are two-body interactions. The quantum path integrals have a factor of the form

$$-1 + \exp\left(iS_2^F\right) = iS_2^F + \frac{1}{2!}(iS_2^F)^2 + \frac{1}{3!}(iS_2^F)^3 + \dots \tag{9.19}$$

with S_2^F being either of (8.25), (8.188), (8.240) or (8.266). In order to be explicit, let us consider the two-body interaction from the massless scalar

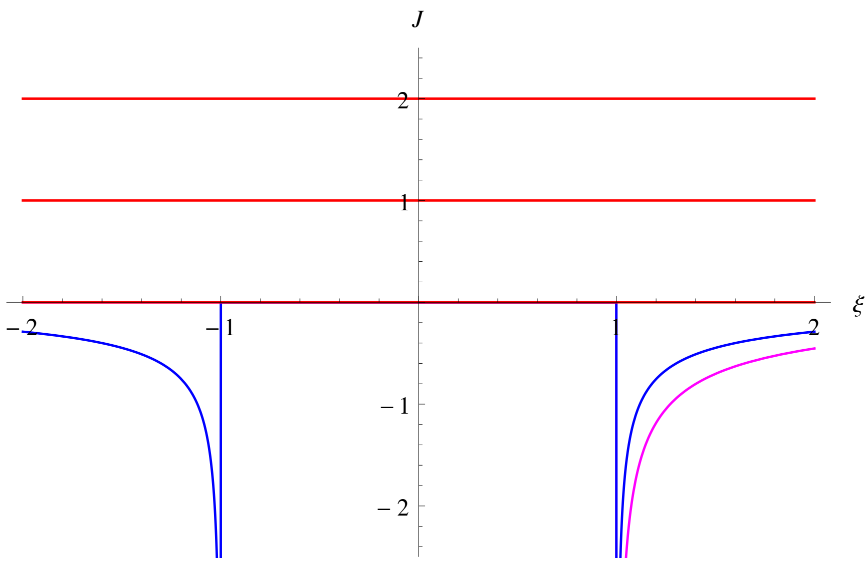


Figure 9.4: Imaginary part of $R_{LS}(\xi)$ (magenta) and $R_\phi(\xi)$ (blue). The red lines correspond to $J = 0, 1, 2$. We have used $\alpha = 0.5$.

exchange. At lowest order, we have

$$iS_2^\phi = ig_0^2 \int\int \mathrm{d}\tau \mathrm{d}\sigma \, G_0[q_a(\tau)|q_b(\sigma)] \tag{9.20}$$

This contribution can be interpreted as a tree level diagram: it corresponds to connecting a point in the worldline of particle a to a point in the worldline of particle b with a massless scalar propagator G_0. The integration over τ and σ corresponds to summing over all pairs of points. The next order is

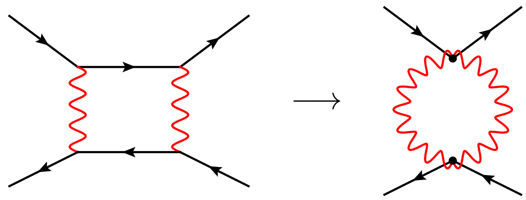
$$\begin{aligned} \frac{1}{2!}(iS_2^\phi)^2 = \frac{1}{2!}(ig_0^2)^2 \int\int \mathrm{d}\tau_1 \mathrm{d}\sigma_1 \int\int \mathrm{d}\tau_2 \mathrm{d}\sigma_2 \, G_0[q_a(\tau_1)|q_b(\sigma_1)] \\ \times G_0[q_a(\tau_2)|q_b(\sigma_2)] \end{aligned} \tag{9.21}$$

For some values of (τ_1, σ_1) and (τ_2, σ_2) this contribution can be interpreted as a one-loop box diagram, but for other values it can be interpreted as a one-loop *crossed* box diagram. Similar remarks hold for higher-order contributions. Thus, (9.19) can be interpreted as a first-quantized analog of the sum of ladder and crossed ladder diagrams.

In the eikonal JWKB approximation we evaluate S_2^ϕ and obtain Σ_2^ϕ,

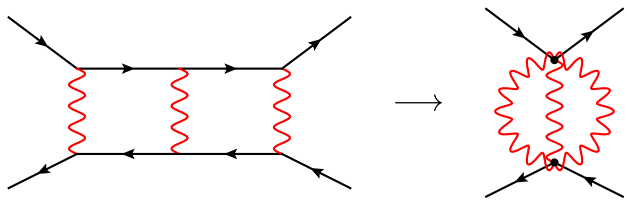
$$\Sigma_2^\phi \approx -i\alpha_0\rho_0\Gamma\left(\frac{D-4}{2}\right)\left(\frac{2}{\mu^2 B_{12}^2}\right)^{(D-4)/2} \tag{9.22}$$

which is proportional to a massless scalar propagator in $D - 2$ dimensions. Thus, instead of the contraction (9.12), we have



(9.23)

and similarly for the double box in (9.13):



(9.24)

We see that the eikonal JWKB approximation contracts the ladder diagrams along the horizontal direction (i.e. either the s-channel or u-channel, since they are eikonal-equivalent because of $s + u = 2m_a^2 + 2m_b^2$). However, these "ladder diagrams" are just schematic and do not correspond directly to (second-quantized) ladder diagrams since they come from a quantum mechanical path integral.

The work of Abarbanel & Itzykson [70] has the same spirit as our work. Their functional derivative technique is equivalent to coupling the particles to an external field, integrating over this field, and then dropping the self-interactions. However, they push their approximations too far into the high-energy regime and reach a result (their equation 11) that is equivalent to (8.218) in the $s \to \infty$ limit (but of no general validity).

Other work on the relativistic eikonal approximation include Cheng & Wu [71, 72, 73, 74, 75], Chang & Ma [76] and Lévy & Sucher [77]. A common theme of these papers is the relationship between the sum of (second-quantized) ladder and crossed ladder diagrams in the high-energy limit, and the "eikonal form" of the amplitude, where the amplitude is written as a two-dimensional integral:

$$\mathcal{A} \sim \int \mathrm{d}^2 B \left[-1 + \exp\left(\chi\right)\right] \exp\left(-iB \cdot P\right) \tag{9.25}$$

The "eikonal form" follows in a very straight-forward way from applying the JWKB approximation to (first-quantized) path integrals and restricting to small momentum transfer (what we called the eikonal JWKB approximation, which in the four-point process looks like (8.9)). Indeed, (8.90) and

(8.272) have the "eikonal form". Since the JWKB approximation is a strong-coupling approximation [67], we feel that it is more natural to work with a first-quantized approach and make no mention of second-quantized perturbative contributions.

Nonrelativistic Coulomb Spectrum

The first applications of both the eikonal approximation and the JWKB approximation were to potential scattering. In chapter 5, we used the path integral version of these approximations to obtain a nonrelativistic four-point amplitude which exhibits Regge behavior and an infinite number of singularities. These singularities agree with the Coulomb spectrum. Our calculation is a longer version of an example in [66]; it uses a two-body language that generalizes to the relativistic theory. The one-body "particle-in-a-potential" problem is recovered in the regime where one of the two particles becomes static (i.e. non-dynamical). The relativistic generalization of [66] appears to be in [78]. There, first-quantized methods were abandoned and second-quantized ladder and crossed ladder diagrams were summed. These authors used the high-energy version of the eikonal approximation (large s and fixed t), which is *not* the interpretation that we use in our version. However, their relativistic spectrum agrees with our result (8.225).

Relativistic Coulomb Spectrum

The amplitudes (8.141), (8.218) and (8.253) have the same form. In [79] one-body wave equations with Coulomb potentials were solved, without invoking any approximations. After summing over the partial wave expansion, the scattering amplitude was put in the form that almost agrees with our result (8.218) for the massless vector exchange. We rewrite our result as

$$\widehat{\mathcal{B}}_A(s,t) = \left(\frac{1}{\rho_0(s)\mu^2}\right)\delta(P)\exp\left(\Xi_\epsilon\right)\left(\frac{\Gamma[-R_A(s)]}{\Gamma[1+R_A(s)]}\right)\left(-\frac{t}{2\mu^2}\right)^{R_A(s)} \tag{9.26}$$

where we have used

$$\frac{1}{\rho_1(s)}\left(Z_a Z_b\frac{m_a^2+m_b^2-s}{2m_a m_b}\right) = \frac{1}{\rho_0(s)} \tag{9.27}$$

In the *static limit*, we take m_b to be very large. With the one-body energy E_a defined as

$$E_a \equiv \lim_{m_b\to\infty}\left(\frac{s-m_b^2}{2m_b}\right) \tag{9.28}$$

we find

$$\rho_0(s) \longrightarrow \rho_0(E_a) = \frac{m_a}{\sqrt{E_a^2 - m_a^2}} \tag{9.29}$$

The choice

$$\mu^2 = 2(E_a^2 - m_a^2) \tag{9.30}$$

reproduces the relativistic amplitude of [79],

$$\widehat{\mathcal{B}}_A(s,t) = \frac{\delta(P)\exp\left(\Xi_\epsilon\right)}{2m_a\sqrt{E_a^2 - m_a^2}} \left(\frac{\Gamma[-R_A(s)]}{\Gamma[1 + R_A(s)]}\right) \left(-\frac{t}{2\mu^2}\right)^{R_A(s)} \tag{9.31}$$

modulo the divergent phase. But we cannot just pick a convenient value for the arbitrary scale μ. In order to obtain (9.31) we have to incorporate effects that are outside of the eikonal JWKB regime.

Another set of similar results to (8.218) were obtained by Dittrich [80].

Eikonal Gravity

The system with matter coupled to the symmetric tensor describes two particles exchanging (linearized) gravitons. This problem was studied in [81] both from a second-quantized approach (via summing ladder and crossed ladder diagrams) and a first-quantized approach similar to that of [70]. Our results agree with those found by [81], who in turn agree with earlier work by 't Hooft [82, 83]. It is somewhat amusing that the intimidating problem of matter interacting via quantum gravity can be addressed with such simple methods.

9.2 Outlook

The are many directions in which our work can be generalized. Below is a fairly large amount of speculation. The options presented below were not explored fully due the short attention span of the author.

Spinning Matter

In this dissertation we only consider scalar matter. Spin is an important property that should not be ignored. In order to study particles with spin degrees of freedom, one adds anticommuting variables to the particle action functional [84]. For example, the action functional for a *massless* spinning particle is

$$S = \int \mathrm{d}\tau \left[-\dot{q}\cdot p - \frac{i}{2}\dot{\psi}\cdot\psi + \frac{v}{2}p^2 + i\chi\left(\psi\cdot p\right)\right] \tag{9.32}$$

where $\psi^m(\tau)$ is a spacetime vector which classically anticommutes,

$$\{\psi^m, \psi^n\} = 0 \tag{9.33}$$

and $\chi(\tau)$ is worldline Grassmann variable. The action for a *massive* spinning particle requires an extra Grassmann variable.

It should be straightforward to extend our methods to spinning matter. We must first fix the worldline parametrization invariance. This will involve setting v and χ to appropriate constants. Then one needs to solve the free classical equations of motion. This will yield the spinning generalization of the eikonal path. With the free solutions, we can obtain the semiclassical eikonal kernel, and then the semiclassical eikonal scattering amplitude. None of these steps present big challenges. The only thing needed is time.

Higher-point Scattering

Besides two-body path integrals, one can in general consider N-body path integrals. As one increases the number of external states, the combinatorics grow increasingly complicated. But six-point (three-body) and eight-point (four-body) scattering should be tractable. This raises the question of whether one can study three-body or four-body bound states, or interactions between two-body bound states. Both of these possibilities are very interesting.

Revisiting Alday-Maldacena Theory

From the string theory point of view, the Alday-Maldacena calculation in AdS_5 [3], like the Gross-Mende calculation in flat spacetime [32], describes the fixed-angle scattering regime of a *tree level* amplitude with *strings*. However, the calculation in AdS_5 corresponds to a nonperturbative calculation, due to the AdS/CFT correspondence. In some sense, Alday-Maldacena theory is the best of two worlds: it uses methods similar to those mentioned in §3.2 to compute a *tree level* amplitude with a classical solution for a first-quantized system, but because of AdS/CFT, it also makes partial contact with the BDS ansatz for the all-loop planar MHV scattering amplitude.

Indeed, the Alday-Maldacena solution does not include the gauge theory tree level amplitude pre-factor. This might not sound like a serious objection, considering the amount of insight that followed after [3], but we feel that this is an important point. The tree level pre-factor is important because it explicitly shows the nature of the external states that are scattering. With our methods, we obtained nonperturbative scattering amplitudes that include the appropriate tree level pre-factor. Of course, the problem that we study is *very* *different* from the problem that Alday & Maldacena studied. One cannot help but wonder, with perhaps a large dose of wishful thinking, if a more complete version of the Alday-Maldacena result can be obtained.

Dual Conformal Symmetry

One important feature of the Alday-Maldacena calculation is dual conformal symmetry. With massless particles, all four-point dual conformal invariants are fixed. But with massive particles, one can have one independent dual conformal invariant per planar class of Feynman graphs. However, in the two-body semiclassical eikonal approximation (8.9), the dual conformal invariants for the relevant planar classes either vanish or diverge. The remaining dual conformal invariant involves the product su. Outside of the physical region, in the eikonal approximation this product is *quantized*, so it raises the question of whether eikonal bound state singularities are dual conformal invariant. The problem is that the dual conformal invariant with su corresponds to a planar class that is forbidden in the elastic scattering event we consider. Analytic continuation could play a role. Higher-point amplitudes have a large number of dual conformal invariants, but the analysis might be intractable due to the combinatorics. If dual conformal symmetry has something to do with bound states, then it could explain why it remained hidden for so long.

Multiple Couplings

The systems that we study involve matter interacting via a single kind of interaction. One can also consider multiple couplings. For example, one can consider matter that couples to a massless real scalar field, and also to a massless vector field. The calculation of the scattering amplitude in the eikonal JWKB approximation remains almost unchanged. One finds a result with an infinite number of singularities satisfying

$$1 - \alpha_0 \rho_0(s) - \alpha_1 \rho_1(s) = -J, \qquad J = 0, 1, 2, \ldots \tag{9.34}$$

Solving this equation for s yields

$$s_J = m_a^2 + m_b^2 + 2m_a m_b \left[\frac{Z_a Z_b \alpha_0 \alpha_1 + (J+1)\sqrt{Z_a^2 Z_b^2 \alpha_1^2 + (J+1)^2 - \alpha_0^2}}{Z_a^2 Z_b^2 \alpha_1^2 + (J+1)^2} \right]$$

Since a real scalar cannot carry electric charge, the two mediating fields do not interact. We can consider other mixed couplings: scalar-graviton and vector-graviton. Since any field can couple to gravity, in both of these models one has the possibility of interactions that do not involve matter contributing to the bound state energies.

Anti-de Sitter Spacetime

Systems in anti-de Sitter spacetime are relevant in the study of conformal field theories (CFTs). Indeed, the concept of primary fields and descendants is somewhat analogous to bound states. A formalism for correlation functions in CFTs analogous to Regge theory was proposed in [85]. This aims to better understand correlation functions in the Mellin basis [86, 87]. Extending our methods to anti de-Sitter spacetime would involve promoting the eikonal path in flat spacetime to a geodesic path in AdS. If this is successful, it could provide another approach to study correlations functions. Some work on this subject has already been done in [88, 89, 90, 91, 92, 93].

Other Massive Exchanges

Our result for the exchange of the massive scalar in $D = 4$ is not as exciting as those for the massless exchanges. However, in $D = 3$ the result is quite odd. Perhaps one can also obtain similar results with the exchange of a massive vector or tensor.

Higher-spin Exchanges

The amplitudes we obtained for the exchange of massless scalars, vectors, or tensors have the same general form. Indeed, we can consider the exchange of massless quanta with arbitrary spin N by adding a coupling term to the free particle action of the form

$$\frac{1}{\Gamma(N+1)} \int \mathrm{d}\tau \, \dot{q}^{m_1} \cdots \dot{q}^{m_N} H_{m_1 \cdots m_N}[q(\tau)] \tag{9.35}$$

If we write down a kinetic term for the free massless higher-spin field H, fix the (higher) gauge symmetry in a way analogous to the Fermi-Feynman gauge-fixing (where the kinetic operator becomes the scalar term multiplying a polarization tensor), and integrate over the higher-spin field, then the two-body interaction term will have the form

$$\frac{g_N^2}{[\Gamma(N+1)]^2} \int \int \mathrm{d}\tau \mathrm{d}\sigma \, \mathcal{H}_N[q_a(\tau), q_b(\sigma)] G_0[q_a(\tau)|q_b(\sigma)] \tag{9.36}$$

where $\mathcal{H}_N$ involves a contraction of N factors of $\dot{q}_a$ and N factors of $\dot{q}_b$. For example, with $N = 3$ we expect

$$\mathcal{H}_3 = c_1 \dot{q}_a^2 \dot{q}_b^2 (\dot{q}_a \cdot \dot{q}_b) + c_2 (\dot{q}_a \cdot \dot{q}_b)^3 \tag{9.37}$$

with c_1 and c_2 numerical coefficients determined by the kinetic operator for the higher-spin field H. Thus, we expect ρ_3 to have the form

$$\rho_3 = \frac{c_1 x_{31}^2 x_{42}^2 (x_{31} \cdot x_{42}) + c_2 (x_{31} \cdot x_{42})^3}{T_a^2 T_b^2 \sqrt{x_{31}^2 x_{42}^2 - (x_{31} \cdot x_{42})^2}} \tag{9.38}$$

which would lead to a Regge trajectory of the form

$$R_H(\xi_s) = -1 + \alpha_3 m_a^2 m_b^2 \left[\frac{\xi_s (c_1 + c_2 \xi_s^2)}{\sqrt{1 - \xi_s^2}} \right] \qquad \xi_s \equiv \frac{s - m_a^2 - m_b^2}{2 m_a m_b} \tag{9.39}$$

Splines

The eikonal path is appropriate in the regime of small-angle scattering. If we wanted to move away from this regime, one would have to either solve the classical equations of motion to obtain the true classical path, or use another path as the *de facto* classical path. The eikonal path is the simplest example of a spline, a piece-wise continuous path made by concatenating different curves. We can imagine concatenating two straight paths, with different slopes. The change in the slope is a modulus of the spline. Similarly, the point along the worldline where the path changes direction is another modulus. Thus, we can view these type of splines as eikonal paths with many moduli. If we integrate over these moduli, we are considering paths with all possible changes in slope, and thus we would not necessarily be restricted to the regime of small momentum transfer.

Strings

Just like a particle couples to a one-form gauge field A, a string couples to a two-form gauge field B. We can consider a two-string system where each string has a different length (in analogy with two particles of different mass). One can add a coupling term to the B-field and then integrate over the B-field. The result would correspond to a two-string interaction term. A naive dimensional analysis indicates that the semiclassical eikonal two-string interaction term is infrared-divergent in $D = 6$. If this calculation can be made to yield sensible results, it could help understand the mysterious $(2, 0)$ theory, which is expected to contain string-like objects. This problem would require generalizing the eikonal path for a particle to an eikonal surface for a string [58].

Nonabelian coupling

And finally, we have to address the elephant in the room. In order to study bound states in quantum chromodynamics, one needs to couple matter to a nonabelian gauge field. Because of the nonabelian nature of the exchange quanta, the path integral for the matter particle involves a more complicated coupling term: a nonabelian Wilson line.

Perhaps something can be learned about the nonabelian interactions by studying a toy model with a scalar three-body force,

$$S_3[q_a, q_b, q_c] = g_0^3 \int\int\int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\rho \, \mathcal{Y}_3[q_a(\tau)|q_b(\sigma)|q_c(\rho)] \tag{9.40}$$

where $\mathcal{Y}_3$ is the un-truncated cubic vertex function,

$$\mathcal{Y}_3(x_a|x_b|x_c) \equiv \frac{f_3}{6} \int \mathrm{d}y \, G_0(x_a|y) G_0(x_b|y) G_0(x_c|y) \tag{9.41}$$

Here, f_3 is the dimensionful coupling constant that appears in the scalar cubic interaction vertex. Thus, f_3 has units

$$[f_3] = \left(\frac{D-7}{2}\right)[\hbar] + \left(\frac{6-D}{2}\right)[\text{mass}] \tag{9.42}$$

The dimensionless combinations involving these coupling parameters are

$$\alpha_0 = g_0^2 \hbar^{(3-D)} \mu^{(D-6)} \qquad \alpha_3 = f_3^2 \hbar^{(7-D)} \mu^{(D-6)} \tag{9.43}$$

In $D = 4$, the semiclassical approximation leads to $\alpha_0 \to \infty$ and $\alpha_3 \to 0$. The effective coupling parameter in (9.40) is $g_0^3 f_3$, which has units

$$[g_0^3 f_3] = 2(D-4)[\hbar] + 2(6-D)[\text{mass}] \tag{9.44}$$

so the dimensionless combination is

$$\beta_3 = g_0^3 f_3 \hbar^{2(4-D)} \mu^{2(D-6)} \tag{9.45}$$

Note that, if we keep g_0 and f_3 fixed in $D = 4$, the $\hbar \to 0$ limit keeps β_3 fixed.

One can also consider a scalar four-body force,

$$S_4[q_a, q_b, q_c, q_d] \equiv g_0^4 \int\int\int\int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\rho \mathrm{d}\omega\, \mathcal{Y}_4[q_a(\tau)|q_b(\sigma)|q_c(\rho)|q_d(\omega)] \quad (9.46)$$

with the un-truncated quartic vertex function

$$\mathcal{Y}_4(x_a|x_b|x_c|x_d) \equiv \frac{f_4}{24} \int \mathrm{d}y\, G_0(x_a|y)G_0(x_b|y)G_0(x_c|y)G_0(x_d|y) \quad (9.47)$$

Here the coupling parameter f_4 has units

$$[f_4] = (D-5)[\hbar] + (4-D)[\text{mass}] \quad (9.48)$$

The effective coupling parameter is now $g_0^4 f_4$, with units

$$[g_0^4 f_4] = (3D-11)[\hbar] + (16-3D)[\text{mass}] \quad (9.49)$$

and thus, the dimensionless combination is

$$\beta_4 = g_0^4 f_4 \hbar^{(11-3D)} \mu^{(3D-16)} \quad (9.50)$$

If we keep g_0 and f_4 in $D=4$, the $\hbar \to 0$ limit leads to $\beta_4 \to \infty$.

Both (9.40) and (9.46) are toy models for the gluon interaction vertices in Yang-Mills theory. In principle, one can evaluate them using the many-body eikonal JWKB approximation and obtain many-body eikonal kernels that might yield information about three- and four-body bound states.

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Appendix A

Relativistic Free Scalar Matter

In this appendix we discuss the kernels for relativistic free massless and massive scalar matter. Although free theories are somewhat trivial, these examples will allow us to discuss some of the differences between the quantum and the semiclassical kernels.

A.1 Massive Scalar Particle

After fixing the worldline reparametrization gauge symmetry with the choice $v = 1$, the action functional for a free massive scalar particle is

$$S[q] = \int \mathrm{d}\tau \left[-\frac{1}{2}\dot{q}^2 + \frac{1}{2}m^2 \right] \qquad m > 0 \tag{A.1}$$

where the range of the worldline parameter τ is

$$-\frac{T}{2} < \tau < \frac{T}{2} \tag{A.2}$$

with $T > 0$. The quantum kernel $\mathcal{F}$ is

$$\mathcal{F}(O|I) = \int_0^\infty \mathrm{d}T \int_{x_I}^{x_O} \mathrm{D}q(\tau) \exp\left(-iS[q]\right) \tag{A.3}$$

We will first calculate the semiclassical kernel, and then discuss the differences between the (exact) quantum kernel and the semiclassical kernel.

A.1.1 Semiclassical Kernel

In order to find the semiclassical kernel, we must first solve the classical equation of motion that follows from the action functional (A.1),

$$\ddot{q} = 0 \tag{A.4}$$

subject to the boundary conditions

$$q\left(-\frac{T}{2}\right) = x_I, \qquad q\left(\frac{T}{2}\right) = x_O \tag{A.5}$$

The classical path q_* is

$$q_*(\tau) = \frac{x_I + x_O}{2} + (x_O - x_I)\left(\frac{\tau}{T}\right) \tag{A.6}$$

Let $x \equiv x_O - x_I$. For a *physical* massive particle we expect

$$x^2 < 0 \tag{A.7}$$

That is, the separation between the "in" position and the "out" position is a time-like spacetime interval.

We define the *classical* conjugate momentum p_* as

$$p_*(\tau) \equiv \dot{q}_* = \frac{x}{T} \tag{A.8}$$

This is a *constant spacetime vector* that depends on the modulus T.

Evaluating the action functional (A.1) at the classical path yields the Van Vleck function,

$$\Sigma \equiv S[q_*] = -\frac{1}{2T}x^2 + \frac{m^2 T}{2} \tag{A.9}$$

With the Van Vleck function, we can obtain the Van Vleck matrix,

$$V_{mn} \equiv -i\frac{\partial^2 \Sigma}{\partial x_I^m \partial x_O^n} = \left(-\frac{i}{T}\right)\eta_{mn} \tag{A.10}$$

The semiclassical kernel $\mathcal{V}$ is

$$\mathcal{V}(O|I) = \int_0^\infty \mathrm{d}T\sqrt{-\det(V)}\exp(-i\Sigma) \tag{A.11}$$

After evaluating the determinant of V, we find

$$\mathcal{V}(O|I) = \int_0^\infty \mathrm{d}T \left(-\frac{i}{T}\right)^{D/2} \exp\left[\frac{i}{2T}x^2 - \frac{im^2T}{2}\right] \tag{A.12}$$

At first glance, this expression can be recognized as the *exact* quantum kernel. However, the semiclassical kernel is only valid in the $\hbar \to 0$ limit. We are using units where $\hbar = 1$. If we had kept $\hbar$ explicit, there would had been factor of $\hbar$ dividing the Van Vleck function in the semiclassical kernel. In order to understand the consequences of the $\hbar \to 0$ limit, we make a change of variables

$$T = \sqrt{-\frac{x^2}{m^2}}\omega \tag{A.13}$$

Then, the semiclassical kernel becomes

$$\begin{aligned} \mathcal{V} = -i &\left(-i\sqrt{-\frac{m^2}{x^2}}\right)^{(D-2)/2} \\ &\times \int_0^\infty \mathrm{d}\omega \left(\frac{1}{\omega}\right)^{D/2} \exp\left[-i\sqrt{-m^2x^2}\left(\frac{1}{2\omega} + \frac{\omega}{2}\right)\right] \end{aligned} \tag{A.14}$$

Thus, $\hbar \to 0$ is equivalent to the regime where $\sqrt{-m^2x^2} \to \infty$. Note that the latter limit allows two different cases,

$$-x^2 \to \infty, \qquad m^2 \text{ fixed} \tag{A.15}$$

or

$$-x^2 \text{ fixed}, \qquad m^2 \to \infty \tag{A.16}$$

The case (A.15) is consistent with the familiar intuition of having classical behavior (i.e. not quantum-mechanical) at long spacetime distances. Case (A.16) corresponds to a heavy scalar particle. In the semiclassical limit, we can integrate over ω with stationary methods. The stationary point is $\omega_* = 1$. At this stationary point, we have

$$T_* = \sqrt{-\frac{x^2}{m^2}} \tag{A.17}$$

Note that, at this value, the *classical* conjugate momentum becomes

$$p_*(\tau) = \frac{mx}{\sqrt{-x^2}} \quad \Longrightarrow \quad p_*^2 + m^2 = 0 \tag{A.18}$$

The "in" and "out" spacetime positions have components

$$x_I = (t_I, \mathbf{x}_I) \qquad x_O = (t_O, \mathbf{x}_O) \tag{A.19}$$

with $t_O > t_I$. Thus,

$$x^2 = (x_O - x_I)^2 = -(t_O - t_I)^2 + (\mathbf{x}_O - \mathbf{x}_I)^2 \tag{A.20}$$

We define the average spatial velocity vector $\mathbf{v}$ as

$$\mathbf{v} \equiv \frac{\mathbf{x}_O - \mathbf{x}_I}{t_O - t_I} \tag{A.21}$$

Thus, the components of the *classical* conjugate momentum,

$$p_* = (E, \mathbf{p}) \tag{A.22}$$

are given by

$$E = \frac{m}{\sqrt{1 - \mathbf{v}^2}} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \tag{A.23}$$

This are the familiar expressions for the energy E and translational momentum $\mathbf{p}$ of a free massive relativistic particle. We can think of T as parametrizing a family of worldlines. As part of the quantum theory, we must sum (i.e. integrate) over all possible values of T. Only the worldline with $T = T_*$ describes a truly classical (i.e. on-shell) relativistic particle.

After dealing with the integral over ω, we find

$$\mathcal{V} \approx -\sqrt{2\pi}i \left(-i\sqrt{-\frac{m^2}{x^2}} \right)^{(D-2)/2} \left(i\sqrt{-m^2x^2} \right)^{-1/2} \exp\left[-i\sqrt{-m^2x^2} \right] \tag{A.24}$$

which can be put in the form

$$\mathcal{V} \approx -\frac{\sqrt{2\pi}im^{(D-2)}}{\left(i\sqrt{-m^2x^2} \right)^{(D-1)/2}} \exp\left[-i\sqrt{-m^2x^2} \right] \tag{A.25}$$

Another useful form follows after we write the exponential as an infinite sum,

$$\mathcal{V} \approx -\sqrt{2\pi} i m^{(D-2)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} (-2)^{y_n} \left(-\frac{2}{m^2 x^2}\right)^{z_n} \tag{A.26}$$

where

$$y_n \equiv \frac{1-D}{4} + \frac{n}{2}, \qquad z_n \equiv -y_n = \frac{D-1}{4} - \frac{n}{2} \tag{A.27}$$

Equation (A.26) is our final result in the position basis.

Fourier Transform

Strictly speaking, the Fourier transform is a quantum device. By this, we mean that it involves $\hbar$. The $\hbar \to 0$ limit of the Fourier transform is the classical Legendre transform that switches between Hamiltonian and Lagrangian mechanics. This transform amounts to swapping $\dot{q}_*$ with p_* defined by (A.18). We found that p_* is constant in τ, so we have

$$p_I = p_O \tag{A.28}$$

We also found that p_* satisfies the on-shell condition at $T = T_*$. So, as expected, the *classical* momentum is conserved and on-shell.

However, nothing prevents us from taking a full Fourier transform of the semiclassical kernel $\mathcal{V}$. When we do this, the Fourier transform takes us to a *quantum* momentum basis, and thus, the momentum variables are *not* on-shell. The Fourier transform is

$$\widehat{\mathcal{V}}(O|I) = \int\int \mathrm{d}x_I \mathrm{d}x_O \, \mathcal{V}(O|I) \exp\left(i x_I \cdot p_I - i x_O \cdot p_O\right) \tag{A.29}$$

We first make a change of position variables,

$$X \equiv \frac{x_I + x_O}{2} \qquad x \equiv x_O - x_I \tag{A.30}$$

The corresponding conjugate momenta are

$$P \equiv p_O - p_I \qquad p \equiv \frac{p_I + p_O}{2} \tag{A.31}$$

such that

$$x_I \cdot p_I - x_O \cdot p_O = -X \cdot P - x \cdot p \tag{A.32}$$

Since $\mathcal{V}$ does not depend on X, the integral yields a Dirac delta:

$$\int \mathrm{d}X \, \exp\left(-iX \cdot P\right) = \delta(P) \tag{A.33}$$

The Dirac delta restricts to $P = 0$ or $p_O = p_I$. This means that the incoming momentum is equal to the outgoing momentum, a result expected from translation invariance. Then

$$p = p_I = p_O \tag{A.34}$$

We are left with the integral over x,

$$\widehat{\mathcal{V}}(p) = \delta(P) \int \mathrm{d}x \, \mathcal{V}(x) \exp\left(-ix \cdot p\right) \tag{A.35}$$

Using (A.26) and Fourier-transforming each term in the sum yields

$$\begin{aligned} \widehat{\mathcal{V}}(p) = -\sqrt{2\pi} i m^{D-2} \left(-\frac{1}{m^2}\right)^{D/2} \delta(P) \\ \times \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} (-2)^{y_n} \frac{\Gamma(w_n)}{\Gamma(z_n)} \left(-\frac{2m^2}{p^2}\right)^{w_n} \end{aligned} \tag{A.36}$$

with

$$w_n \equiv \frac{D}{2} - z_n = \frac{D+1}{4} + \frac{n}{2} \tag{A.37}$$

We separate the sum in (A.36) into the part with even n and the part with odd n,

$$\widehat{\mathcal{V}}(p) = \widehat{\mathcal{V}}_{\text{even}}(p) + \widehat{\mathcal{V}}_{\text{odd}}(p) \tag{A.38}$$

Each term can be evaluated separately,

$$\widehat{\mathcal{V}}_{\text{even}}(p) = \frac{E_D}{m^2} \delta(P) \left(-\frac{m^2}{p^2}\right)^{(1+D)/4} {}_2F_1\left(\frac{5-D}{4}, \frac{1+D}{4}; \frac{1}{2}; -\frac{m^2}{p^2}\right) \tag{A.39}$$

$$\widehat{\mathcal{V}}_{\text{odd}}(p) = \frac{O_D}{m^2} \delta(P) \left(-\frac{m^2}{p^2}\right)^{(3+D)/4} {}_2F_1\left(\frac{7-D}{4}, \frac{3+D}{4}; \frac{3}{2}; -\frac{m^2}{p^2}\right) \tag{A.40}$$

with

$$E_D \equiv -2\sqrt{\pi} i^{(D+1)} (-1)^{(1-D)/4} \frac{\Gamma\left(\frac{D+1}{4}\right)}{\Gamma\left(\frac{D-1}{4}\right)} \tag{A.41}$$

and

$$O_D \equiv -4\sqrt{\pi} i^D (-1)^{(1-D)/4} \frac{\Gamma\left(\frac{D+3}{4}\right)}{\Gamma\left(\frac{D-3}{4}\right)} \tag{A.42}$$

Recall the Pfaff identities,

$${}_2F_1(a,b;c;x) = \left(\frac{1}{1-x}\right)^a {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right) \tag{A.43}$$

$${}_2F_1(a,b;c;x) = \left(\frac{1}{1-x}\right)^b {}_2F_1\left(c-a, b; c; \frac{x}{x-1}\right) \tag{A.44}$$

Combining both of these yields the Euler identity,

$${}_2F_1(a,b;c;x) = \left(\frac{1}{1-x}\right)^{(a+b-c)} {}_2F_1\left(c-a, c-b; c; x\right) \tag{A.45}$$

Using this identity, we find

$${}_2F_1\left(\frac{5-D}{4}, \frac{1+D}{4}; \frac{1}{2}; -\frac{m^2}{p^2}\right) = \left(\frac{p^2}{m^2+p^2}\right) {}_2F_1\left(\frac{D-3}{4}, \frac{1-D}{4}; \frac{1}{2}; -\frac{m^2}{p^2}\right)$$

$${}_2F_1\left(\frac{7-D}{4}, \frac{3+D}{4}; \frac{3}{2}; -\frac{m^2}{p^2}\right) = \left(\frac{p^2}{m^2+p^2}\right) {}_2F_1\left(\frac{D-1}{4}, \frac{3-D}{4}; \frac{3}{2}; -\frac{m^2}{p^2}\right)$$

So then, (A.39) and (A.40) become

$$\widehat{\mathcal{V}}_{\text{even}} = -\frac{E_D}{m^2+p^2}\delta(P)\left(-\frac{m^2}{p^2}\right)^{(D-3)/4} {}_2F_1\left(\frac{D-3}{4}, \frac{1-D}{4}; \frac{1}{2}; -\frac{m^2}{p^2}\right) \tag{A.46}$$

$$\widehat{\mathcal{V}}_{\text{odd}} = -\frac{O_D}{m^2+p^2}\delta(P)\left(-\frac{m^2}{p^2}\right)^{(D-1)/4} {}_2F_1\left(\frac{D-1}{4}, \frac{3-D}{4}; \frac{3}{2}; -\frac{m^2}{p^2}\right) \tag{A.47}$$

In this form, the on-shell singularity at $p^2 + m^2 = 0$ appears as a common overall factor.

One-Dimensional Kernel

The one-dimensional kernel is somewhat trivial, but it will play a role later, so we discuss it now. In the position basis we have

$$\mathcal{V}(x) = -\frac{\sqrt{2\pi} i}{m} \exp\left[-i\sqrt{-m^2 x^2}\right] \tag{A.48}$$

All the x-dependence appears in the exponential. Note that the $m \to 0$ limit is not well-defined

Setting $D = 1$ yields

$$E_1 = 0 \qquad O_1 = 2i \tag{A.49}$$

We also have

$$\lim_{D\to 1}\left[\left(-\frac{m^2}{p^2}\right)^{(D-3)/4} {}_2F_1\left(\frac{D-3}{4}, \frac{1-D}{4}; \frac{1}{2}; -\frac{m^2}{p^2}\right)\right] = \sqrt{-\frac{p^2}{m^2}} \tag{A.50}$$

$$\lim_{D\to 1}\left[\left(-\frac{m^2}{p^2}\right)^{(D-1)/4} {}_2F_1\left(\frac{D-1}{4}, \frac{3-D}{4}; \frac{3}{2}; -\frac{m^2}{p^2}\right)\right] = 1 \tag{A.51}$$

Thus, in the momentum basis we have

$$\widehat{\mathcal{V}}(p) = \left[-\frac{2i}{m^2+p^2}\right]\delta(P) \tag{A.52}$$

which coincides with the familiar Feynman propagator for a massive scalar particle.

Three-Dimensional Kernel

The three-dimensional kernel will also play a role later. In the position basis we have

$$\mathcal{V}(x) = -\frac{\sqrt{2\pi}}{\sqrt{-x^2}}\exp\left[-i\sqrt{-m^2x^2}\right] \tag{A.53}$$

This is the Minkowski analog of the Yukawa potential. Note that the $m \to 0$ limit is well-defined,

$$\lim_{m\to 0}\mathcal{V}(x) = -\frac{\sqrt{2\pi}}{\sqrt{-x^2}} \tag{A.54}$$

Indeed, $D = 3$ is the *only* case for which the semiclassical kernel has a well-defined massless limit.

Setting $D = 3$ yields

$$E_3 = 2i \qquad O_3 = 0 \tag{A.55}$$

We also have

$$\lim_{D\to 3}\left[\left(-\frac{m^2}{p^2}\right)^{(D-3)/4} {}_2F_1\left(\frac{D-3}{4},\frac{1-D}{4};\frac{1}{2};-\frac{m^2}{p^2}\right)\right]=1 \tag{A.56}$$

$$\lim_{D\to 3}\left[\left(-\frac{m^2}{p^2}\right)^{(D-1)/4} {}_2F_1\left(\frac{D-1}{4},\frac{3-D}{4};\frac{3}{2};-\frac{m^2}{p^2}\right)\right]=\sqrt{-\frac{m^2}{p^2}} \tag{A.57}$$

Thus, in the momentum basis we have

$$\widehat{\mathcal{V}}(p)=\left[-\frac{2i}{m^2+p^2}\right]\delta(P) \tag{A.58}$$

which also coincides with the familiar Feynman propagator.

Four-Dimensional Kernel

In $D=1$ and $D=3$ we found that the semiclassical kernel in the momentum basis coincides with the traditional Feynman propagator. Indeed, these two cases are the only cases for which this is true. Since in this dissertation we will work with four-dimensional theories, we now turn to the case when $D=4$. In the position basis we have

$$\mathcal{V}(x)=-\frac{\sqrt{2\pi}im^2}{\left(i\sqrt{-m^2(x_O-x_I)^2}\right)^{3/2}}\exp\left[-i\sqrt{-m^2(x_O-x_I)^2}\right] \tag{A.59}$$

This corresponds to a higher-dimensional analog of the Yukawa potential.

Setting $D=4$ yields

$$E_4=2\sqrt{\pi}(-1)^{3/4}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\qquad O_4=4\sqrt{\pi}(-1)^{1/4}\frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \tag{A.60}$$

Unlike in $D=1$ or $D=3$, both the even and odd parts will contribute to the semiclassical kernel in the momentum basis:

$$\widehat{\mathcal{V}}_{\text{even}}(p)=-\frac{E_4}{m^2+p^2}\delta(P)\left(-\frac{m^2}{p^2}\right)^{1/4} {}_2F_1\left(\frac{1}{4},-\frac{3}{4};\frac{1}{2};-\frac{m^2}{p^2}\right) \tag{A.61}$$

$$\widehat{\mathcal{V}}_{\text{odd}}(p)=-\frac{O_4}{m^2+p^2}\delta(P)\left(-\frac{m^2}{p^2}\right)^{3/4} {}_2F_1\left(\frac{3}{4},-\frac{1}{4};\frac{3}{2};-\frac{m^2}{p^2}\right) \tag{A.62}$$

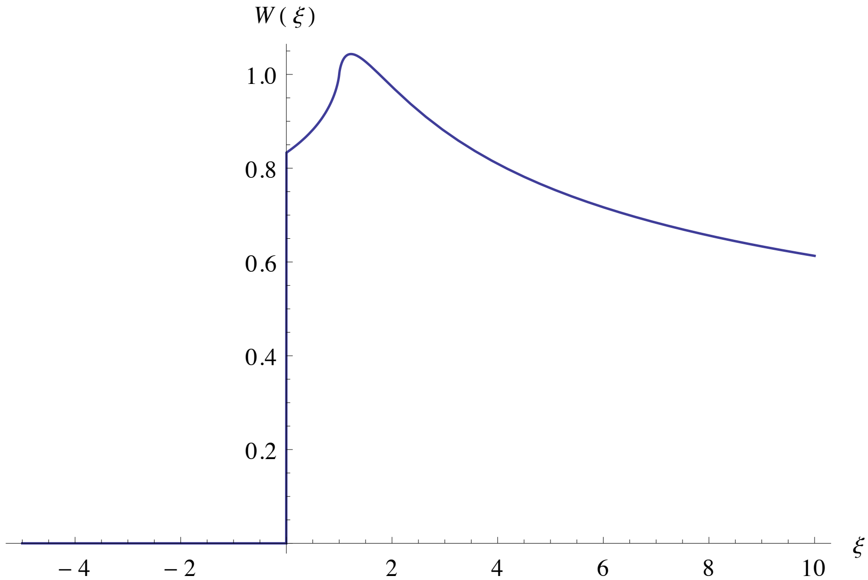

Figure A.1: Real part of $W(\xi)$.

We write

$$\widehat{\mathcal{V}}(p) = W(\xi)\left[-\frac{2i}{m^2+p^2}\right]\delta(P), \qquad \xi \equiv -\frac{p^2}{m^2} \tag{A.63}$$

with

$$W(\xi) = \frac{E_4}{2i}\left(\frac{1}{\xi}\right)^{1/4} {}_2F_1\left(\frac{1}{4}, -\frac{3}{4}; \frac{1}{2}; \frac{1}{\xi}\right) + \frac{O_4}{2i}\left(\frac{1}{\xi}\right)^{3/4} {}_2F_1\left(\frac{3}{4}, -\frac{1}{4}; \frac{3}{2}; \frac{1}{\xi}\right)$$

Figures A.1 and A.2 display the real and imaginary parts of $W(\xi)$.

Note that on-shell we have $\xi = 1$ and

$$\mathrm{Re}\,[W(1)] = 1 \qquad \mathrm{Im}\,[W(1)] = 0 \tag{A.64}$$

Far away from the origin we have

$$\mathrm{Re}\,[W(\infty)] = \mathrm{Im}\,[W(\infty)] = 0 \tag{A.65}$$

As we discussed earlier, there are two limits that are equivalent to the $\hbar \to 0$ limit. In limit (A.15), we have $-x^2 \to \infty$, which corresponds to $-p^2 \to 0$. On the other hand, in limit (A.16) we have $m^2 \to \infty$. Both of these limits are equivalent to the $\xi \to 0$ limit.

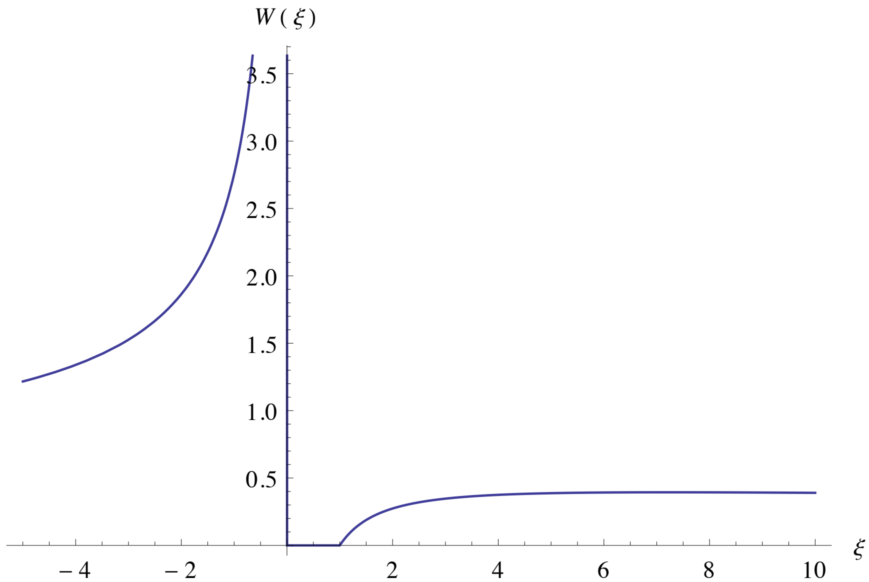

Figure A.2: Imaginary part of $W(\xi)$.

A.1.2 Quantum Kernel

The quantum kernel can be found by evaluating the integral in (A.14) with exact methods. The outcome involves a modified Bessel function of the second kind,

$$\mathcal{F}(x) = -2i\left(-i\sqrt{-\frac{m^2}{x^2}}\right)^{(D-2)/2} K_{(D-2)/2}\left[i\sqrt{-m^2x^2}\right] \tag{A.66}$$

This is the familiar Feynman propagator in the position basis.

Fourier Transform

To change to the momentum basis, we perform a Fourier transform,

$$\widehat{\mathcal{F}}(p) = \delta(P)\int \mathrm{d}x\,\mathcal{F}(x)\exp\left(-ix\cdot p\right) \tag{A.67}$$

Instead of using the complicated expression from above, we go back to (A.12). After integrating over x, we find

$$\widehat{\mathcal{F}}(p) = \delta(P) \int_0^\infty \mathrm{d}T \, \exp\left[-T\left(-\frac{p^2+m^2}{2i}\right)\right] \tag{A.68}$$

which, after integration over T, yields

$$\widehat{\mathcal{F}}(p) = \left[-\frac{2i}{m^2+p^2}\right]\delta(P) \tag{A.69}$$

Thus, the exact quantum kernel has the expected simple form in the momentum basis.

One-Dimensional Kernel

Setting $D = 1$ in (A.66) yields

$$\mathcal{F}(x) = -\frac{\sqrt{2\pi i}}{m} \exp\left[-i\sqrt{-m^2x^2}\right] \tag{A.70}$$

which coincides with the corresponding semiclassical kernel (A.48). This explains why in the momentum basis, the semiclassical kernel coincides with the Feynman propagator (A.69). We can say that in $D = 1$ the semiclassical kernel is *exact*.

Three-Dimensional Kernel

Setting $D = 3$ in (A.66) yields

$$\mathcal{F}(x) = -\frac{\sqrt{2\pi}}{\sqrt{-x^2}} \exp\left[-i\sqrt{-m^2x^2}\right] \tag{A.71}$$

which also coincides with the corresponding semiclassical kernel (A.53). This explains why in the momentum basis, the semiclassical kernel coincides with the Feynman propagator (A.69). This means that in $D = 3$ the semiclassical kernel is also *exact*. Indeed, $D = 1$ and $D = 3$ are the *only* cases for which this statement is true.

Four-Dimensional Kernel

Setting $D = 4$ in (A.66) yields

$$\mathcal{F}(x) = -\frac{2m}{\sqrt{-x^2}} K_1\left[i\sqrt{-m^2x^2}\right] \tag{A.72}$$

which differs considerably from the corresponding semiclassical kernel (A.59). However, as $\sqrt{-m^2x^2} \to \infty$ we can use the asymptotic expansion

$$K_\nu(z) \sim \frac{1}{2}\sqrt{\frac{2\pi}{z}} \exp\left(-z\right) \tag{A.73}$$

to obtain (A.59). So we find that in $D = 4$ the semiclassical kernel agrees with the quantum kernel in either the large spacetime distance or large mass limit. Because of Fourier-Heisenberg conjugacy, large $-x^2$ corresponds to small $-p^2$. Indeed, besides the simple pole when $p^2 + m^2 = 0$, the imaginary part of the semiclassical kernel (A.63) has a singularity when $p^2 = 0$.

A.2 Massless Scalar Particle

We cannot repeat the analysis from the previous section for the case of a free massless scalar particle and expect to obtain valid results. As we already saw, the semiclassical kernel in the position basis corresponds to either the large spacetime distance limit or the large mass limit. Both of these regimes are not valid for massless particles, which travel along null spacetime intervals ($x^2 = 0$) and are, well, massless ($m = 0$). For completeness we discuss the exact quantum kernel.

Setting $m = 0$ in (A.12) leads to

$$\begin{aligned}\mathcal{F}(x) &= \int_0^\infty \mathrm{d}T \left(-\frac{i}{T}\right)^{D/2} \exp\left[\frac{i}{2T}x^2\right] \\ &= -i\Gamma\left(\frac{D-2}{2}\right)\left(\frac{2}{x^2}\right)^{(D-2)/2}\end{aligned} \tag{A.74}$$

Note that for $D = 3$ this coincides with the massless limit of the massive quantum/semiclassical kernels obtained in (A.54).

Appendix B

Momentum Invariants

In this appendix we discuss some details on momentum invariants.

B.1 Mandelstam Invariants

The n-quanta Mandelstam invariants involve the squared magnitude of a particular linear combination of $n \geq 2$ momentum vectors. This linear combination contains incoming and/or outgoing momenta.

B.1.1 Two-quanta

We have two types of two-quanta Mandelstam invariants. One type is the s-type,

$$s_{ij} \equiv -(p_i + p_j)^2 \tag{B.1}$$

where either both p_i and p_j are incoming, or outgoing. The other is the t-type,

$$t_{ij} \equiv -(p_i - p_j)^2 \tag{B.2}$$

where either p_i is incoming and p_j is outgoing, or vice versa. Note that, by definition, the s-type invariants carry the information of two distinct bodies. The t-type invariants, on the other hand, can carry the information of a single body, or the incoming and outgoing information of two distinct bodies.

Let $p_i^2 = -m_i^2$ and $p_j^2 = -m_j^2$. If p_i and p_j are both incoming, or both outgoing, then we have

$$p_i \cdot p_j = \frac{m_i^2 + m_j^2 - s_{ij}}{2} \tag{B.3}$$

On the other hand, if p_i is incoming and p_j is outgoing or vice versa, then

$$p_i \cdot p_j = \frac{t_{ij} - m_i^2 - m_j^2}{2} \tag{B.4}$$

B.1.2 Three-quanta

There are also two types of three-quanta Mandelstam invariants. The s-type is analogous to (B.1),

$$s_{ijk} \equiv -(p_i + p_j + p_k)^2 \tag{B.5}$$

and the t-type is analogous to (B.2),

$$t_{ijk} \equiv -(p_i + p_j - p_k)^2 \tag{B.6}$$

For the s-type we must have all three vectors be either incoming, or outgoing. On the other hand, for the t-type we must have either p_i and p_j be incoming with p_k outgoing, or vice versa.

In principle, the three-quanta Mandelstam invariants do not provide any new information since they can always be written in terms of two-quanta Mandelstam invariants. For the s-type we have,

$$s_{ijk} = s_{ij} + s_{jk} + s_{ik} - m_i^2 - m_j^2 - m_k^2 \tag{B.7}$$

and similarly for the t-type,

$$t_{ijk} = s_{ij} + t_{jk} + t_{ik} - m_i^2 - m_j^2 - m_k^2 \tag{B.8}$$

However, in higher-point scattering events, it might prove useful to solve for some of the two-quanta invariants in terms of three-quanta invariants.

B.2 Gram Invariants

The n-quanta Gram invariants are defined as determinants of Gram matrices made with $n \geq 2$ momentum vectors. An $n \times n$ Gram matrix $\mathbf{G}_n$ is a matrix made with the n^2 inner products of n distinct vectors,

$$(\mathbf{G}_n)_{IJ} \equiv p_I \cdot p_J, \qquad I, J = 1, \ldots, n \tag{B.9}$$

The Gram invariants are sensitive to collinearity.

B.2.1 Two-quanta

With two distinct momentum vectors, the two-quanta Gram invariant is

$$G_{ij} \equiv \det \begin{pmatrix} p_i^2 & p_i \cdot p_j \\ p_i \cdot p_j & p_j^2 \end{pmatrix} = p_i^2 p_j^2 - (p_i \cdot p_j)^2 \tag{B.10}$$

We write G_{ij} as either

$$G_{ij} = \frac{1}{4}[s_{ij} - (m_i - m_j)^2][(m_i + m_j)^2 - s_{ij}] \tag{B.11}$$

or

$$G_{ij} = \frac{1}{4}[t_{ij} - (m_i - m_j)^2][(m_i + m_j)^2 - t_{ij}] \tag{B.12}$$

dependening on whether p_i and p_j form an s-type or t-type two-quanta Mandelstam invariant. Note that, if

$$c_i p_i + c_j p_j = 0 \tag{B.13}$$

with $c_i \neq 0$ and $c_j \neq 0$, then it follows that $G_{ij} = 0$.

The threshold value $(m_i + m_j)^2$ and the pseudo-threshold value $(m_i - m_j)^2$ mark the points where G_{ij} changes sign.

B.2.2 Three-quanta

With three distinct momentum vectors, the three-quanta Gram invariant is

$$G_{ijk} \equiv \det \begin{pmatrix} p_i^2 & p_i \cdot p_j & p_i \cdot p_k \\ p_i \cdot p_j & p_j^2 & p_j \cdot p_k \\ p_i \cdot p_k & p_j \cdot p_k & p_k^2 \end{pmatrix} \tag{B.14}$$

More explicitly,

$$G_{ijk} = 2(p_i \cdot p_j)(p_j \cdot p_k)(p_i \cdot p_k) + 2m_i^2 m_j^2 m_k^2 - m_i^2 G_{jk} - m_j^2 G_{ik} - m_k^2 G_{ij} \tag{B.15}$$

With special kinematics, this can be simplified further.

Note that $G_{ijk} = 0$ if any pair of the three momentum vectors are collinear.

Appendix C

Four-point Scalar Kinematics

In this appendix we record many results regarding four-point kinematics. We mostly consider elastic scattering events. An elastic scattering event is one where the incoming content is the same as the outgoing content. With four external quanta, a generic elastic scattering event has the form

$$a(p_1) + b(p_2) \longrightarrow a(p_3) + b(p_4) \tag{C.1}$$

We have external quanta of type a and b, with masses m_a and m_b. The total external momentum is conserved,

$$p_1 + p_2 = p_3 + p_4 \tag{C.2}$$

and each of the external momenta is on-shell,

$$p_1^2 = p_3^2 = -m_a^2 \qquad p_2^2 = p_4^2 = -m_b^2 \tag{C.3}$$

The constraints (C.2) and (C.3) are satisfied by *physical* momenta.

C.1 Momentum Invariants

We take stock of the different momentum invariants that are available to describe the event (C.1).

C.1.1 Mandelstam Invariants

We have one s-type two-quanta invariant,

$$s \equiv s_{12} = s_{34} = -(p_1 + p_2)^2 = -(p_3 + p_4)^2 \tag{C.4}$$

and two t-type two-quanta invariants,

$$t \equiv t_{13} = t_{24} = -(p_3 - p_1)^2 = -(p_2 - p_4)^2 \tag{C.5}$$
$$u \equiv t_{14} = t_{23} = -(p_4 - p_1)^2 = -(p_2 - p_3)^2 \tag{C.6}$$

Because of conservation of the total external momentum, the three Mandelstam invariants satisfy the constraint

$$s + t + u = 2m_a^2 + 2m_b^2 \tag{C.7}$$

Thus, we can always write one of the Mandelstam invariants (say, u) in terms of the other two. Note that s and u are two-body invariants, but t is a one-body invariant.

C.1.2 Gram Invariants

We have four two-quanta Gram invariants,

$$G_{12}(s) = G_{34}(s) = \frac{1}{4}[s - (m_a - m_b)^2][(m_a + m_b)^2 - s] \tag{C.8}$$
$$G_{14}(u) = G_{23}(u) = \frac{1}{4}[u - (m_a - m_b)^2][(m_a + m_b)^2 - u] \tag{C.9}$$
$$G_{13}(t) = \frac{1}{4}t(4m_a^2 - t) \tag{C.10}$$
$$G_{24}(t) = \frac{1}{4}t(4m_b^2 - t) \tag{C.11}$$

and one three-quanta Gram invariant,

$$\begin{aligned} G_{123}(s,t,u) &= G_{234}(s,t,u) = G_{341}(s,t,u) = G_{412}(s,t,u) \\ &= \frac{1}{4}t\left[(m_a + m_b)^2(m_a - m_b)^2 - su\right] \end{aligned} \tag{C.12}$$

It is useful to know the sign of the Gram invariants. For $G_{12}(s)$, we have

$$G_{12}(s) > 0 \text{ when } (m_a - m_b)^2 < s < (m_a + m_b)^2 \tag{C.13}$$
$$G_{12}(s) < 0 \text{ when } s < (m_a - m_b)^2 \text{ or } s > (m_a + m_b)^2 \tag{C.14}$$

Similarly, for $G_{14}(u)$:

$$G_{14}(u) > 0 \text{ when } (m_a - m_b)^2 < u < (m_a + m_b)^2 \tag{C.15}$$
$$G_{14}(u) < 0 \text{ when } u < (m_a - m_b)^2 \text{ or } u > (m_a + m_b)^2 \tag{C.16}$$

Note that $G_{12}(s)$ changes sign at the two-body threshold value $(m_a+m_b)^2$, and at the two-body pseudo-threshold value $(m_a-m_b)^2$. Similar remarks apply to $G_{14}(u)$ with s replaced by u. For $G_{13}(t)$, we have

$$G_{13}(t) > 0 \text{ when } 0 < t < 4m_a^2 \tag{C.17}$$

$$G_{13}(t) < 0 \text{ when } t < 0 \text{ or } t > 4m_a^2 \tag{C.18}$$

and similarly for $G_{24}(t)$:

$$G_{24}(t) > 0 \text{ when } 0 < t < 4m_b^2 \tag{C.19}$$

$$G_{24}(t) < 0 \text{ when } t < 0 \text{ or } t > 4m_b^2 \tag{C.20}$$

Note that $G_{13}(t)$ changes sign at $t = 0$, and at the two-particle threshold $t = (2m_a)^2$. Similar remarks apply to $G_{24}(t)$ with m_a replaced by m_b.

Finally, we consider the three-quanta Gram invariant $G_{123}(s,t,u)$. We have $G_{123}(s,t,u) > 0$ when either

$$t > 0 \text{ and } su < (m_a+m_b)^2(m_a-m_b)^2 \tag{C.21}$$

or

$$t < 0 \text{ and } su > (m_a+m_b)^2(m_a-m_b)^2 \tag{C.22}$$

Similarly, we have $G_{123}(s,t,u) < 0$ when either

$$t > 0 \text{ and } su > (m_a+m_b)^2(m_a-m_b)^2 \tag{C.23}$$

or

$$t < 0 \text{ and } su < (m_a+m_b)^2(m_a-m_b)^2 \tag{C.24}$$

Note that $G_{123}(s,t,u)$ changes sign at $t = 0$, and when

$$su = (m_a+m_b)^2(m_a-m_b)^2 \tag{C.25}$$

which defines a hyperbola in the (s,u) plane.

C.2 Center-of-Momentum Frame

In the center-of-momentum frame, each *incoming* quantum has spatial momentum with magnitude $|\mathbf{p}|$ but opposite direction, and each *outgoing* quantum has spatial momentum with magnitude $|\mathbf{q}|$ but opposite direction. That is,

$$p_1 = (E_1, \mathbf{p}) \qquad p_2 = (E_2, -\mathbf{p}) \qquad p_3 = (E_3, \mathbf{q}) \qquad p_4 = (E_4, -\mathbf{q}) \tag{C.26}$$

The on-shell constraints (C.3) fix the energy of each quantum in terms of its mass and the magnitude of its momentum,

$$\begin{aligned} E_1 &= \sqrt{m_a^2 + \mathbf{p}^2} \qquad & E_2 &= \sqrt{m_b^2 + \mathbf{p}^2} \\ E_3 &= \sqrt{m_a^2 + \mathbf{q}^2} \qquad & E_4 &= \sqrt{m_b^2 + \mathbf{q}^2} \end{aligned} \tag{C.27}$$

Thus,

$$E_1^2 - E_2^2 = m_a^2 - m_b^2 \qquad E_3^2 - E_4^2 = m_a^2 - m_b^2 \tag{C.28}$$

From these, it follows that

$$E_1 = \sqrt{m_a^2 - m_b^2 + E_2^2} \qquad E_3 = \sqrt{m_a^2 - m_b^2 + E_4^2} \tag{C.29}$$

Using the definition of s, we obtain an equation that relates the sum of the incoming energies, and the sum of the outgoing energies, to s:

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2 \tag{C.30}$$

After solving for some of the energies, we find

$$E_2 = E_4 = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}} \tag{C.31}$$

and thus

$$E_1 = E_3 = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \tag{C.32}$$

One can check that (C.31) and (C.32) satisfy

$$E_1 + E_2 = E_3 + E_4 \tag{C.33}$$

as is required by energy conservation.

The magnitude of the velocity $\mathbf{v}$ of a relativistic particle with energy E and mass m is

$$|\mathbf{v}| = \sqrt{1 - \frac{m^2}{E^2}} \tag{C.34}$$

Using (C.31) and (C.32), we find

$$|\mathbf{v}_1| = |\mathbf{v}_3| = \frac{\sqrt{-4G_{12}(s)}}{s + m_a^2 - m_b^2} \qquad |\mathbf{v}_2| = |\mathbf{v}_4| = \frac{\sqrt{-4G_{12}(s)}}{s - m_a^2 + m_b^2} \tag{C.35}$$

where

$$G_{12}(s) = \frac{1}{4}[s - (m_a - m_b)^2][(m_a + m_b)^2 - s] \tag{C.36}$$

Note that we can write the $|\mathbf{v}_j|$ in terms of dimensionless ratios (e.g. s/m_a^2 and m_a/m_b).

From (C.27), it follows that

$$\begin{aligned} (E_1E_2)^2 &= (m_a^2 + \mathbf{p}^2)(m_b^2 + \mathbf{p}^2) \\ (E_3E_4)^2 &= (m_a^2 + \mathbf{q}^2)(m_b^2 + \mathbf{q}^2) \end{aligned} \tag{C.37}$$

Using (C.31) and (C.32), we can solve for $|\mathbf{p}|$ and $|\mathbf{q}|$:

$$|\mathbf{p}| = |\mathbf{q}| = \sqrt{-\frac{G_{12}(s)}{s}} \tag{C.38}$$

The definitions of t and u give other relations:

$$t = 2m_a^2 - 2E_1E_3 + 2(\mathbf{p}\cdot\mathbf{q}) = 2m_b^2 - 2E_2E_4 + 2(\mathbf{p}\cdot\mathbf{q}) \tag{C.39}$$

$$u = m_a^2 + m_b^2 - 2E_1E_4 - 2(\mathbf{p}\cdot\mathbf{q}) = m_a^2 + m_b^2 - 2E_2E_3 - 2(\mathbf{p}\cdot\mathbf{q}) \tag{C.40}$$

The cosine of the scattering angle θ_s is defined as

$$z_s \equiv \cos(\theta_s) = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} \tag{C.41}$$

We can use either (C.39) or (C.40) to find

$$z_s = \frac{su - (m_a + m_b)^2(m_a - m_b)^2 - st}{4G_{12}(s)} \tag{C.42}$$

Using

$$\begin{aligned} 4G_{12}(s) &= [s - (m_a - m_b)^2][(m_a + m_b)^2 - s] \\ &= su - (m_a + m_b)^2(m_a - m_b)^2 + st \end{aligned} \tag{C.43}$$

we can write z_s as

$$z_s = \frac{su - (m_a + m_b)^2(m_a - m_b)^2 - st}{su - (m_a + m_b)^2(m_a - m_b)^2 + st} \tag{C.44}$$

If $m_a = m_b$ we recover the familiar

$$z_s = \frac{u - t}{u + t} \tag{C.45}$$

C.2.1 Physical Scattering Region

In order for the energies E_j in (C.31) and (C.32) to be *real* and *finite* we must require that $s > 0$. Similarly, in order for $|\mathbf{p}|$ and $|\mathbf{q}|$ in (C.38) to be *real* and *finite* we must require $G_{12}(s) < 0$. Finally, the scattering angle θ_s must be such that its cosine has the appropriate range,

$$-1 < z_s < 1 \tag{C.46}$$

Using (C.44) and $G_{12}(s) < 0$ on the lower limit gives

$$su < (m_a + m_b)^2(m_a - m_b)^2 \tag{C.47}$$

Similarly, the upper limit gives

$$st < 0 \quad \Longrightarrow \quad t < 0 \tag{C.48}$$

Hence, the **physical scattering region** is defined by

$$s > 0 \qquad G_{12}(s) < 0 \qquad t < 0 \qquad su < (m_a + m_b)^2(m_a - m_b)^2 \tag{C.49}$$

We can state these conditions in terms of the Gram invariants,

$$s > 0 \qquad G_{12}(s) < 0 \qquad G_{123}(s, t, u) < 0 \tag{C.50}$$

Outside of the physical scattering region we have the bonding region, where physical bound states exist.

C.3 Crossing

A crossing transformation amounts to switching an incoming quantum q_I with momentum p_I and electric charge Z_I, with an outgoing quantum q_O with momentum p_O and electric charge Z_O. That is, from the event

$$q_I(p_I, Z_I) + \ldots \longrightarrow q_O(p_O, Z_O) + \ldots \tag{C.51}$$

one obtains the event

$$\bar{q}_O(\bar{p}_I, \bar{Z}_I) + \ldots \longrightarrow \bar{q}_I(\bar{p}_O, \bar{Z}_O) + \ldots \tag{C.52}$$

with

$$\bar{p}_I = -p_O \qquad \bar{Z}_I = -Z_O \qquad \bar{p}_O = -p_I \qquad \bar{Z}_O = -Z_I \tag{C.53}$$

A *pure crossing* involves crossing an incoming quantum with an outgoing quantum of the *same* type. One can also perform a *mixed crossing*, which amounts to crossing an incoming quantum with an outgoing quantum of *different* type. If the starting event is elastic, then a pure crossing will yield another elastic event, but a mixed crossing will yield an inelastic event. We can act on event (C.1) with a pure crossing transformation and obtain another elastic scattering event.

For example, after crossing the incoming b quantum with the outgoing b quantum in (C.1), we obtain the elastic event

$$a(p_1) + \bar{b}(\bar{p}_2) \longrightarrow a(p_3) + \bar{b}(\bar{p}_4) \tag{C.54}$$

with

$$\bar{p}_2 = -p_4 \qquad \bar{p}_4 = -p_2 \tag{C.55}$$

For event (C.54), the center-of-momentum energy is

$$\bar{s} = -(p_1 + \bar{p}_2)^2 = -(p_1 - p_4)^2 = u \tag{C.56}$$

and the momentum transfer invariants are

$$\bar{t} = -(p_1 - p_3)^2 = t \tag{C.57}$$

$$\bar{u} = -(p_1 - \bar{p}_4)^2 = -(p_1 + p_2)^2 = s \tag{C.58}$$

Thus, we can use the Mandelstam invariants for event (C.1) to also describe event (C.54), with the caveat that s and u switch roles. The physical scattering region for event (C.54), in terms of the invariants for event (C.1), is given by

$$u > 0 \qquad t < 0 \qquad G_{14}(u) < 0 \qquad su < (m_a + m_b)^2(m_a - m_b)^2 \tag{C.59}$$

which is *different* from that of event (C.1). Note that the invariant t plays the same role in (C.49) and (C.59).

Similarly, we can cross the incoming a quantum with the outgoing a quantum in (C.1) to obtain the event

$$\bar{a}(\bar{p}_1) + b(p_2) \longrightarrow \bar{a}(\bar{p}_3) + b(p_4) \tag{C.60}$$

with

$$\bar{p}_1 = -p_3 \qquad \bar{p}_3 = -p_1 \tag{C.61}$$

It should be no surprise that the Mandelstam invariants for event (C.60) are the same as the invariants for event (C.54), since these two events are conjugate. Hence, events (C.54) and (C.60) share the same physical scattering region.

Doing both of the pure crossings mentioned above leads to the event

$$\bar{a}(\bar{p}_1) + \bar{b}(\bar{p}_2) \longrightarrow \bar{a}(\bar{p}_3) + \bar{b}(\bar{p}_4) \tag{C.62}$$

which is conjugate to event (C.1).

Although the scattering amplitude for event (C.1) can describe events (C.54), (C.60), and (C.62) after appropriate crossings, it is not true that all of these elastic events are physically equivalent. Event (C.1) has another interpretation: the propagation of the bound state ab. Similarly, event (C.54) can be interpreted as the propagation of the bound state $a\bar{b}$. Events (C.60) and (C.62) correspond to propagation of the antiparticles $b\bar{a}$ and $\bar{b}\bar{a}$, respectively. Thus, we have *two* distinct two-body bound states (ab and $a\bar{b}$) and the corresponding antiparticles. If a and b carry electric charge, then the bound states ab and $a\bar{b}$ have different electromagnetic properties. Indeed, if the product of the charges $Z_a Z_b$ is positive, then we only have the bound state $a\bar{b}$. On the other hand, if the product $Z_a Z_b$ is negative, then we only have the bound state ab. If the bonding is due to gravity, then both bound states are allowed.

Besides the two pure crossings, one can perform two mixed crossings on event (C.1). Both mixed crossings will yield an inelastic event. After crossing the incoming b quantum with the outgoing a quantum in event (C.1), we obtain the event

$$a(p_1) + \bar{a}(\bar{p}_2) \longrightarrow \bar{b}(\bar{p}_3) + b(p_4) \tag{C.63}$$

with

$$\bar{p}_2 = -p_3 \qquad \bar{p}_3 = -p_2 \tag{C.64}$$

For event (C.63), the center-of-momentum energy is

$$\bar{s} \equiv -(p_1 + \bar{p}_2)^2 = -(p_1 - p_3)^2 = t \tag{C.65}$$

and the momentum transfer invariants are

$$\bar{t} \equiv -(p_1 - \bar{p}_3)^2 = -(p_1 + p_2)^2 = s \tag{C.66}$$

$$\bar{u} \equiv -(p_1 - p_4)^2 = u \tag{C.67}$$

Similarly, crossing the incoming a quantum with the outgoing b quantum in event (C.1) leads to the event

$$\bar{b}(\bar{p}_1) + b(p_2) \longrightarrow a(p_3) + \bar{a}(\bar{p}_4) \tag{C.68}$$

with

$$\bar{p}_1 = -p_4 \qquad \bar{p}_4 = -p_1 \tag{C.69}$$

Event (C.68) is conjugate to event (C.63). Thus, these two events share the same Mandelstam invariants. In terms of the Mandelstam invariants for event (C.1), the physical scattering region is

$$t > 4m_a^2 \qquad t > 4m_b^2 \qquad su > (m_a + m_b)^2(m_a - m_b)^2 \tag{C.70}$$

with $s < 0$ and $u < 0$. The scattering angle is now given by

$$z_t \equiv \cos(\theta_t) = \frac{s - u}{\sqrt{(s+u)^2 - 4(m_a + m_b)^2(m_a - m_b)^2}} \tag{C.71}$$

We refer to the events (C.1) and (C.62) as the s-channel events, (C.63) and (C.68) as the t-channel events, and (C.54) and (C.60) as the u-channel events.

C.4 Types of Scattering Regimes

Some special types of scattering regimes are described below.

C.4.1 Forward Scattering

Forward scattering involves small scattering angle. In the s-channel, we have $z_s \to 1$. We first write z_s in terms of s and t:

$$z_s = 1 + \frac{2st}{[s - (m_a - m_b)^2][s - (m_a + m_b)^2]} \tag{C.72}$$

In terms of dimensionless ratios, we have

$$z_s = 1 + \frac{1}{2}\left(\frac{t}{m_a m_b}\right)\left(\frac{s}{m_a m_b}\right)\left[\left(\frac{s - m_a^2 - m_b^2}{2m_a m_b}\right)^2 - 1\right]^{-1} \tag{C.73}$$

Thus, the $z_s \to 1$ limit is equivalent to

$$\frac{t}{m_a m_b} \to 0 \qquad \text{fixed } \frac{s}{m_a m_b} \qquad \text{fixed } \frac{m_a}{m_b} \tag{C.74}$$

As a corollary, we have

$$\frac{t}{s} \to 0 \qquad \frac{t}{u} \to 0 \qquad \text{fixed } \frac{u}{m_a m_b} \tag{C.75}$$

Hence, forward scattering in the s-channel corresponds to the regime of small momentum transfer.

C.4.2 Backward Scattering

In the s-channel, backward scattering involves $z_s \to -1$. After writing z_s as

$$z_s = \frac{2(m_a + m_b)^2(m_a - m_b)^2 - 2su}{[s - (m_a - m_b)^2][s - (m_a + m_b)^2]} - 1 \tag{C.76}$$

one finds that backward scattering is equivalent to

$$su \to (m_a + m_b)^2(m_a - m_b)^2 \tag{C.77}$$

This regime can be stated as

$$\frac{u}{m_a m_b} \to \frac{m_a m_b}{s}\left(1 + \frac{m_b}{m_a}\right)^2\left(1 - \frac{m_a}{m_b}\right)^2 \quad \text{fixed } \frac{s}{m_a m_b} \quad \text{fixed } \frac{m_a}{m_b} \tag{C.78}$$

C.4.3 Fixed-angle Scattering

Fixed-angle scattering involves keeping z_s fixed in the regime where all Mandelstam invariants are large. That is,

$$\frac{m_a m_b}{t} \to 0 \qquad \text{fixed } \frac{s}{t} \qquad \text{fixed } \frac{m_a}{m_b} \tag{C.79}$$

As a corollary, we have

$$\frac{m_a m_b}{s} \to 0 \qquad \frac{m_a m_b}{u} \to 0 \qquad \text{fixed } \frac{u}{t} \tag{C.80}$$

C.4.4 Regge Scattering

Suppose we compute the scattering amplitude $\mathcal{A}_s$ for the s-channel event at large energy s and fixed transfer t. This amplitude can be analytically continued to describe the t-channel event, but with fixed energy $\bar{s} = t$ and large transfer $\bar{t} = s$. Similarly, if we compute the scattering amplitude $\mathcal{A}_t$ for the t-channel event at large energy $\bar{s} = t$ and fixed transfer $\bar{t} = s$, we can analytically continue and describe the s-channel event at fixed energy s and large momentum transfer t. This is the main motivation behind Regge scattering, which corresponds to the $z_s \to \infty$ limit. This is equivalent to

$$\frac{t}{m_a m_b} \to \infty \qquad \text{fixed } \frac{s}{m_a m_b} \qquad \text{fixed } \frac{m_a}{m_b} \tag{C.81}$$

This limit takes us outside of the physical scattering region. Thus, in this regime we have the possibility of dealing with bound states in some way.

Appendix D

Gamma, Beta, Zeta, Etcetera

In this appendix we list some properties of the Euler Gamma function, the Riemann zeta function and the binomial coefficient.

D.1 Riemann zeta

The Riemann zeta function $\zeta(s)$ is traditionally defined as an infinite series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{D.1}$$

This function also appears for special values of the polylogarithm $\mathrm{Li}_n(z)$

$$\mathrm{Li}_k(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^k} \tag{D.2}$$

That is,

$$\zeta(s) = \mathrm{Li}_s(1) \tag{D.3}$$

Some special values are

$$\zeta(0) = -\frac{1}{2} \qquad \zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90} \qquad \zeta(6) = \frac{\pi^6}{945} \tag{D.4}$$

D.2 Euler Gamma

The Euler Gamma function $\Gamma(z)$ can be defined as an integral

$$\Gamma(z) \equiv \int_0^\infty \mathrm{d}\omega \left(\frac{1}{\omega}\right)^{1-z} \exp\left(-\omega\right) \tag{D.5}$$

This function has singularities for $z = 0$ and $z = -n$ with n a positive integer. As a sum of simple poles:

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)(n+z)} + \int_1^\infty \mathrm{d}T \left(\frac{1}{T}\right)^{1-z} \exp\left(-T\right) \tag{D.6}$$

One can write $\Gamma(z)$ as an infinite product

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1} \tag{D.7}$$

or equivalently

$$\Gamma(z) = \frac{1}{z} \exp\left[z \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right) - \sum_{k=1}^{\infty} \log\left(1 + \frac{z}{k}\right)\right] \tag{D.8}$$

So then, the reciprocal of $\Gamma(z)$ gives

$$\frac{1}{\Gamma(z)} = z \exp\left[-z \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right) + \sum_{k=1}^{\infty} \log\left(1 + \frac{z}{k}\right)\right] \tag{D.9}$$

Thus

$$\begin{aligned} \frac{\Gamma(z)}{\Gamma(w)} &= \frac{w}{z} \exp\left[(z-w) \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right) - \sum_{k=1}^{\infty} \log\left(\frac{k+z}{k+w}\right)\right] \\ &= \exp\left[(z-w) \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right) - \sum_{k=0}^{\infty} \log\left(\frac{k+z}{k+w}\right)\right] \end{aligned} \tag{D.10}$$

In particular,

$$\frac{\Gamma(1-z)}{\Gamma(1+z)} = \exp\left[-2z \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right) - \sum_{k=0}^{\infty} \log\left(\frac{k+1-z}{k+1+z}\right)\right] \tag{D.11}$$

Other identities are

$$\frac{\Gamma(z+1)}{\Gamma(z-1)} = z(z-1) \qquad \frac{\Gamma(-z+1)}{\Gamma(-z-1)} = z(z+1) \tag{D.12}$$

and

$$\Gamma\left(z+\frac{1}{2}\right) = \left(z-\frac{1}{2}\right)\Gamma\left(z-\frac{1}{2}\right) \tag{D.13}$$

More general identities are

$$\begin{aligned}
\frac{\Gamma(z)}{\Gamma(z-n)} &= \prod_{k=1}^{n}(z-k), \qquad n = 1,2,3,\ldots \\
\frac{\Gamma(z+n)}{\Gamma(z)} &= \prod_{k=1}^{n}(z+k-1), \qquad n = 1,2,3,\ldots \\
\frac{\Gamma(z+n)}{\Gamma(z-n)} &= \prod_{k=1}^{n}(z+k-1)(z-k), \qquad n = 1,2,3,\ldots
\end{aligned} \tag{D.14}$$

and thus

$$\begin{aligned}
\frac{\Gamma(-z)}{\Gamma(-z-n)} &= (-1)^n \prod_{k=1}^{n}(z+k), \qquad n = 1,2,3,\ldots \\
\frac{\Gamma(-z+n)}{\Gamma(-z)} &= (-1)^n \prod_{k=1}^{n}(z-k+1), \qquad n = 1,2,3,\ldots \\
\frac{\Gamma(-z+n)}{\Gamma(-z-n)} &= \prod_{k=1}^{n}(z-k+1)(z+k), \qquad n = 1,2,3,\ldots
\end{aligned} \tag{D.15}$$

Hence

$$\frac{\Gamma(z)}{\Gamma(z-n)} = (-1)^n \frac{\Gamma(1-z+n)}{\Gamma(1-z)}, \qquad n = 0,1,2,\ldots \tag{D.16}$$

or equivalently,

$$\frac{\Gamma(z+1)}{\Gamma(z+1-n)} = (-1)^n \frac{\Gamma(n-z)}{\Gamma(-z)}, \qquad n = 0,1,2,\ldots \tag{D.17}$$

The Euler-Mascheroni constant γ can be written as the difference of two divergent series

$$\gamma = \sum_{k=1}^{\infty}\left[\frac{1}{k} - \log\left(1+\frac{1}{k}\right)\right] \tag{D.18}$$

One also has another expression for γ involving $\zeta(s)$

$$\gamma = \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} \tag{D.19}$$

which is a special case of

$$\log\left[\Gamma(z+1)\right] + \gamma z = \sum_{k=2}^{\infty} \frac{(-z)^k \zeta(k)}{k} \tag{D.20}$$

and thus

$$\frac{1}{\Gamma(z)} = z \exp\left[\gamma z - \sum_{k=2}^{\infty} \frac{(-z)^k \zeta(k)}{k}\right] \tag{D.21}$$

The Euler reflection formula,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{D.22}$$

and Euler's product for sine,

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{D.23}$$

lead to

$$\frac{1}{\Gamma(1+z)\Gamma(1-z)} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{D.24}$$

A result from Gauss is

$$\Gamma(nz) = \sqrt{\frac{2\pi}{n}} \left(\frac{n^z}{\sqrt{2\pi}}\right)^n \prod_{k=1}^{n} \Gamma\left(z + \frac{n-k}{n}\right), \qquad n = 1, 2, 3, \ldots \tag{D.25}$$

which leads to

$$\Gamma(1+nz) = z\sqrt{2\pi n} \left(\frac{n^z}{\sqrt{2\pi}}\right)^n \prod_{k=1}^{n} \Gamma\left(z + \frac{n-k}{n}\right), \qquad n = 1, 2, 3, \ldots \tag{D.26}$$

We can use the Euler Gamma to write the Riemann zeta as an integral. Using

$$\left(\frac{1}{\kappa}\right)^z = \frac{1}{\Gamma(z)} \int_0^{\infty} \mathrm{d}T \left(\frac{1}{T}\right)^{(1-z)} \exp(-\kappa T) \tag{D.27}$$

we find

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} \mathrm{d}T \left(\frac{1}{T}\right)^{(1-s)} \exp(-nT) \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}T \left(\frac{1}{T}\right)^{(1-s)} \frac{\exp(-T)}{1-\exp(-T)}
\end{aligned} \tag{D.28}$$

D.3 Euler Beta

The Euler Beta function $B(x, y)$ can be written in terms of the Euler Gamma function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{D.29}$$

This form makes it manifest that $B(x, y) = B(y, x)$. There are two useful integral representations. The first is

$$B(x, y) = \int_0^1 \mathrm{d}u \left(\frac{1}{u}\right)^{1-x} \left(\frac{1}{1-u}\right)^{1-y} \tag{D.30}$$

and the second is

$$B(x, y) = \int_0^{\infty} \mathrm{d}T \left(\frac{1}{T}\right)^{1-x} \left(\frac{1}{1+T}\right)^{x+y} \tag{D.31}$$

We can use the Euler Beta to write

$$\Gamma(x+y) = \frac{\Gamma(x)\Gamma(y)}{B(x, y)} \tag{D.32}$$

Taking the limit $y \to x$ yields

$$\Gamma(2x) = \frac{\Gamma^2(x)}{B(x, x)} \tag{D.33}$$

This identity relates $\Gamma(2x)$ to $\Gamma^2(x)$. We can use this to write

$$\Gamma(3x) = \frac{\Gamma(x)\Gamma(2x)}{B(x, 2x)} = \frac{\Gamma^3(x)}{B(x, x)B(x, 2x)} \tag{D.34}$$

For any integer $n > 1$ we have

$$\Gamma(nx) = \Gamma^n(x) \prod_{k=1}^{n-1} \frac{1}{B(x, kx)} \tag{D.35}$$

This identity allows us to write the ratio of $\Gamma^n(x)$ and $\Gamma(nx)$ in terms of Euler Beta functions:

$$\frac{\Gamma^n(x)}{\Gamma(nx)} = \prod_{k=1}^{n-1} B(x, kx) \tag{D.36}$$

Using $x = x_1$ and $y = x_2 + x_3$ in (D.32) gives

$$\Gamma(x_1 + x_2 + x_3) = \frac{\Gamma(x_1)\Gamma(x_2 + x_3)}{B(x_1, x_2 + x_3)} = \frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{B(x_1, x_2 + x_3)B(x_2, x_3)} \tag{D.37}$$

Equivalently, we could have used $x = x_2$ and $y = x_1 + x_3$ to get

$$\Gamma(x_1 + x_2 + x_3) = \frac{\Gamma(x_2)\Gamma(x_1 + x_3)}{B(x_2, x_1 + x_3)} = \frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{B(x_2, x_1 + x_3)B(x_1, x_3)} \tag{D.38}$$

There is still a third possibility: use $x = x_3$ and $y = x_1 + x_2$ to get

$$\Gamma(x_1 + x_2 + x_3) = \frac{\Gamma(x_3)\Gamma(x_1 + x_2)}{B(x_3, x_1 + x_2)} = \frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{B(x_3, x_1 + x_2)B(x_1, x_2)} \tag{D.39}$$

Thus, we find the relation

$$\begin{aligned} B(x_1, x_2 + x_3)B(x_2, x_3) &= B(x_2, x_1 + x_3)B(x_1, x_3) \\ &= B(x_3, x_1 + x_2)B(x_1, x_2) \end{aligned} \tag{D.40}$$

D.4 Binomial

Recall the binomial theorem,

$$(1 + x)^n = \sum_{k=0}^{n} \frac{1}{\Gamma(k + 1)} \frac{\Gamma(n + 1)}{\Gamma(n - k + 1)} x^k \tag{D.41}$$

This generalizes to any value z

$$(1 + x)^z = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + 1)} \frac{\Gamma(z + 1)}{\Gamma(z - k + 1)} x^k \tag{D.42}$$

Using the identity

$$\frac{\Gamma(z+1)}{\Gamma(z-k+1)} = (-1)^k \frac{\Gamma(k-z)}{\Gamma(-z)} \tag{D.43}$$

allows us to write

$$\left(\frac{1}{1-x}\right)^z = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} \frac{\Gamma(z+k)}{\Gamma(z)} x^k \tag{D.44}$$

In general, we have

$$(a+b)^n = \sum_{k=0}^{n} \frac{1}{\Gamma(k+1)} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} a^k b^{n-k} \tag{D.45}$$

Then it follows that

$$(a+b)^n - a^n = \sum_{k=0}^{n-1} \frac{1}{\Gamma(k+1)} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} a^k b^{n-k} \tag{D.46}$$

$$(a+b)^n - b^n = \sum_{k=1}^{n} \frac{1}{\Gamma(k+1)} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} a^k b^{n-k} \tag{D.47}$$

$$(a+b)^n - a^n - b^n = \sum_{k=1}^{n-1} \frac{1}{\Gamma(k+1)} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} a^k b^{n-k} \tag{D.48}$$

We can use the binomial expansion recursively:

$$\begin{aligned}(a+b+c)^n &= \sum_{k_1=0}^{n} \frac{1}{\Gamma(k_1+1)} \frac{\Gamma(n+1)}{\Gamma(n-k_1+1)} a^{k_1} (b+c)^{n-k_1} \\ &= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \frac{\Gamma(n+1)}{\Gamma(k_1+1)\Gamma(k_2+1)} \frac{a^{k_1} b^{k_2} c^{n-k_1-k_2}}{\Gamma(n-k_1-k_2+1)}\end{aligned} \tag{D.49}$$

Then it follows that

$$(a+b+c)^n - a^n - b^n - c^n \tag{D.50}$$

can be written as

$$\begin{aligned}&\sum_{k_1=1}^{n-1} \sum_{k_2=0}^{n-k_1} \frac{\Gamma(n+1)}{\Gamma(k_1+1)\Gamma(k_2+1)} \frac{a^{k_1} b^{k_2} c^{n-k_1-k_2}}{\Gamma(n-k_1-k_2+1)} \\ &+ \sum_{k_3=1}^{n-1} \frac{1}{\Gamma(k_3+1)} \frac{\Gamma(n+1)}{\Gamma(n-k_3+1)} b^{k_3} c^{n-k_3}\end{aligned} \tag{D.51}$$

Appendix E

Fourier Transforms

Given a function $f(x)$ in the position basis, we can find a corresponding function $\hat{f}(p)$ in the momentum basis by performing a Fourier transform:

$$\hat{f}(p) = \int \mathrm{d}x\, f(x) \exp\left(-ix \cdot p\right) \tag{E.1}$$

We work with units where $\hbar = 1$.

The simplest (and most useful) example is a Gaussian function,

$$\mathbb{G}(x|a) = \exp\left(\frac{i}{2a}\mu^2 x^2\right), \qquad a > 0 \tag{E.2}$$

Here a is dimensionless and μ is a constant with units of mass. The Fourier transform becomes the integral of a Gaussian,

$$\hat{\mathbb{G}}(p|a) = \int \mathrm{d}x\, \exp\left(\frac{i}{2a}\mu^2 x^2 - ix \cdot p\right) \tag{E.3}$$

This integral can be evaluated *exactly* with stationary methods. The stationary point is

$$x_* = \left(\frac{a}{\mu^2}\right) p \tag{E.4}$$

After integration, we find

$$\hat{\mathbb{G}}(p|a) = \left(\frac{i}{\mu^2}\right)^{D/2} \left(\frac{1}{a}\right)^{D/2} \exp\left(-\frac{ia}{2}\frac{p^2}{\mu^2}\right), \qquad a > 0 \tag{E.5}$$

which is a Gaussian function in the momentum basis.

Another simple example is an almost arbitrary power of x^2,

$$\mathbb{P}(x|z) = \left(\frac{2}{\mu^2 x^2}\right)^z, \qquad z \neq 0, -1, -2, \ldots \tag{E.6}$$

Using the Euler Gamma function, we can write

$$\left(\frac{2}{\mu^2 x^2}\right)^z = \frac{1}{\Gamma(z)} \int\limits_0^\infty \mathrm{d}\omega \left(\frac{1}{\omega}\right)^{1+z} \exp\left(-\frac{1}{2\omega}\mu^2 x^2\right) \tag{E.7}$$

This allows us to write $\mathbb{P}$ in terms of a Gaussian. Using the previous result for the Fourier transform of a Gaussian gives

$$\begin{aligned} \hat{\mathbb{P}}(p|z) &= \left(\frac{1}{\mu^2}\right)^{D/2} \frac{1}{\Gamma(z)} \int\limits_0^\infty \mathrm{d}\omega \left(\frac{1}{\omega}\right)^{1-w} \exp\left(-\frac{\omega}{2}\frac{p^2}{\mu^2}\right) \\ &= \left(\frac{1}{\mu^2}\right)^{D/2} \frac{\Gamma(w)}{\Gamma(z)} \left(\frac{2\mu^2}{p^2}\right)^w, \quad w \equiv \frac{D}{2} - z \end{aligned} \tag{E.8}$$