Stony Brook University

5d Supersymmetry and Contact Geometry

A Dissertation presented

by

Yiwen Pan

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Physics

Stony Brook University

August 2015

Stony Brook University

The Graduate School

Yiwen Pan

We, the dissertation committe for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Martin Roček - Dissertation Advisor
Professor, C.N. Yang Institute for Theoretical Physics

Peter van Nieuwenhuizen - Chairperson of Defense
Professor, C.N. Yang Institute for Theoretical Physics

Robert L. McCarthy
Professor, Department of Physics and Astronomy

Song Sun
Assistant Professor, Department of Mathematics

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

5d $\mathcal{N} = 1$ Supersymmetry and Contact Geometry

by

Yiwen Pan

Doctor of Philosophy

in

Physics

Stony Brook University

2015

Supersymmetry has been fruitful source of new Physics and Mathematics advancement. In particular, supersymmetric theories on curved manifolds often leads to very interesting connections between integrable geometry and supersymmetric physical quantities.

In this dissertation, we summarize the author's recent work on 5d $\mathcal{N} = 1$ supersymmetric theories on curved 5d Riemannian manifolds and its relation to contact geometry, which is the odd-dimensional counterpart of symplectic geometry in even dimension. We will discuss the geometric implications of the Killing spinor equations derived from the rigid limit of 5d $\mathcal{N} = 1$ supergravity. Combining with the dilatino equations, we see that a large class of supersymmetric backgrounds are transversal holomorphic foliations. With these, we go on to discuss the Higgs branch localization of $\mathcal{N} = 1$ theories on K-contact manifolds, in which case we discover that the BPS solutions are generalized Seiberg-Witten equations on K-contact manifolds. These solutions are in one-to-one correspondence with the poles of the Coulomb branch 1-loop determinant. Finally we will discuss the properties of contact instantons that arise from Coulomb branch localization of $\mathcal{N} = 1$ theories.

To my family and the universe.

Contents

Acknowledgements

First I would like to express my gratitude to my advisor Martin Roček, for helping and supporting me in my research. He guided me into the field of mathematical physics, from supersymmetric sigma models, generalized complex geometries, to gauge theories, supergravities. Every time I have a question, and no matter how elementary or stupid the question is, he always explains and shares his knowledge in great detail on blackboards.

I also want to thank Peter van Nieuwenhuizen, who has offered us a brilliant collection of classes. Those are the rare opportunities in which one can actually learn deep things in an organized and detailed manner. Without these classes, one can only learn those things by digging through a sea of literatures, during which one can easily get lost. In particular I gained most in the advanced QFT class on instantons, vortices and supersymmetry, which play important role in my study.

I also want to thank Maxim Zabzine from Uppsala University, for his pioneering papers, which are fruitful sources of inspiration for my study on 5d gauge theories, and also for many discussions on 5d geometry and gauge theories in the course of my study.

It is great experiences working with Wolfger Peelaers and Johannes Schmude on 4d and 5d projects. Johannes has insights on our subject that are keys to many progresses. Wolfger and I have a lot of discussions on our 4d project, from which I learned a lot about 3d and 4d gauge theories.

I also want to thank Francesco Benini, Cyril Closet for discussion and explaining to me their work on supersymmetry, vortices and geometry. Their work are excellent guideline for my study in 5d, and thanks to their detailed explanation I have better understanding of their work.

I also thank Jun Nian, Xinyu Zhang, Peng zhao for numerous helpful discussions on various topics in physics over the past few years.

I also thank my friends at Stony Brook, our friendship has greatly enriched my life and physics study. I thank my YITP officemates Madalena Lemos, Wolfger Peelaers, Mao Zeng, Jiayi Zhu, my friends in Physics, Andrea Massari, Jun Nian, Yachao Qian, Chengjian Wu, Youquan Zhang, Xinyu Zhang, Peng Zhao, Yiming Zhong, Zhedong Zhang. I also enjoy participating the Xchange-Coop graduate club and working with the club members, including Jun Duan, Kun Ma, Si Wen, Yuan Yao.

I am also grateful to my family, my father, mother, grandmother, who have been supporting me for all these years since I was born, without whom I would not have come this far.

Finally I thank Yuanyuan Deng for standing side by side with me these years, who has been supporting and caring for me.

This thesis is composed according to the following publications

1. Yiwen Pan, Rigid Supersymmetry on 5-dimensional Riemannian Manifolds and Contact Geometry, JHEP 1405 (2014) 041,

2. Yiwen Pan, 5d Higgs Branch Localization, Seiberg-Witten Equations and Contact Geometry, JHEP 1501 (2015) 145,

3. Yiwen Pan, Johannes Schmude, On rigid supersymmetry and notions of holomorphy in five dimensions, arXiv:1504.00321.

This thesis does NOT include the following paper

4. Yiwen Pan, Note on a Cohomological Theory of Contact-Instanton and Invariants of Contact Structures, arXiv:1401.5733.

Chapter 1

Introduction

1.1 Introduction

Quantum mechanics and the theory of relativity, the two greatest developments in physics in the early 20th century, gave birth to the most powerful tool ever invented by mankind to describe our universe, namely, Quantum Field Theory. It has enabled us to study our universe in unprecedented depth and detail, making amazing predictions of what the constituents of our world are and how they behave.

But the method of Quantum Field Theory does not stop at describing just our physical universe. In the past decades after its birth, it has gone much further. Quantum Field Theory has proven to be highly useful in revealing subtle mathematical structures and their relations to realistic or imaginary physical models. We have the example of quantum anomalies, in which case the quantum violation of classical conservation law reveals the cohomological structure of the group of gauge transformations. We also have the renowned example of Donaldson-Witten theory, where physical observables reveal the cohomological structure of the instanton moduli spaces, and the hidden smooth structures on four manifolds. The list goes on endlessly.

In many of these cases where Quantum Field Theory shines, the concept of supersymmetry plays an important role. The presence of supersymmetry in a theory often leads to interesting exactly computable physical quantities, thanks to the delicate cancellations between bosonic and fermionic degrees of freedom. On the other hand, the existence of a supersymmetric theory on a given manifold often implies that the manifold carries a certain geometric structure. What's more interesting is that, the above mentioned computable physical quantities and the underlying geometry are closely related, namely these physical quantities are actually invariants of the geometries. This can be viewed as a generalized notion of Donaldson-Witten theory: smooth structures are replaced by geometric structures, the $\mathcal{N} = 2$ $SU(2)$ theory is replaced by more general supersymmetric theories.

In this dissertation, we will focus on $\mathcal{N} = 1$ supersymmetric theories on Riemannian five-manifolds. We will explore the relation between the existence of $\mathcal{N} = 1$ supersymmetry and the contact or transversal holomorphic foliations on the five-manifolds. We will also discuss the notion of 5-dimensional Seiberg-Witten equations and their role in supersymmetric partition functions.

In Chapter [2], we will study the basic relation between 5d $\mathcal{N} = 1$ theories and almost contact geometries. We will explore the necessary geometric conditions for the existence of different number of supercharges.

In Chapter [3], we will further show that the existence of a large class of $\mathcal{N} = 1$ supersymmetry implies transversal holomorphic foliation on the five manifold.

In Chapter [4], we introduce the notion of 5d Seiberg-Witten equations, and its role in the Higgs branch localization. In particular, we will study the local behavior of Higgs branch BPS equations of $\mathcal{N} = 1$ theories around special circles, and use these information to match the poles in the integrand of the partition function.

Chapter 2

5d Rigid Supersymmetry and Contact Geometry

2.1 $\mathcal{N} = 1$ Minimal Off-shell Supergravity

5 dimensional minimal off-shell supergravity was studied by Zucker [1][1]. In his paper, the linearized supergravity multiplet and its SUSY transformation rules are obtained through coupling to the current multiplet of supersymmetric Maxwell multiplet. Then the linearized multiplet is covariantized (making the transformation local) and its supergravity transformation can be derived. In this section we summarize his work, and obtain the Killing spinor equation needed for the rigid limit.

The super-Maxwell multiplet consists of the field content $(\varphi, A_\mu, \lambda')$, where φ is a real scalar, A is a gravi-photon with field strength $f_{mn} = \partial_m A_n - \partial_n A_m$, and λ' is the gaugino, a complex 4-dimensional spinor.

The flat space Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} f_{mn} f^{mn} + \frac{1}{2}\partial_m \varphi \partial^m \phi + \frac{i}{2}\bar{\lambda}' \Gamma^m \partial_m \lambda'. \tag{2.1.1}$$

This Lagrangian is invariant under the on-shell supersymmetry transformation

$$\delta\varphi = i\bar{\epsilon}\lambda',\ \delta A_m = i\bar{\epsilon}\Gamma_m\lambda',\ \delta\lambda' = \frac{1}{2} f_{mn}\Gamma^{mn}\epsilon - \partial_m\phi\Gamma^m\epsilon, \tag{2.1.2}$$

which form a closed algebra modulo the equation of motion:

$$\Gamma^m \partial_m \lambda' = 0. \tag{2.1.3}$$

There are several symmetries of the theory:

- Spacetime symmetry, whose conserved current is the energy-momentum tensor

$$T_{mn} = -f_{mk} f_n{}^k + \frac{1}{4}\eta_{mn} f_{kl} f^{kl} + \partial_m\varphi\partial_n\varphi - \frac{1}{2}\eta_{mn}(\partial\varphi)^2 + \frac{i}{8}\bar{\lambda}' \left(\Gamma_m\partial_n + \Gamma_n\partial_m\right)\lambda'. \tag{2.1.4}$$

[1] It is called $\mathcal{N} = 2$ in [1], however, it actually has 8 supercharges following from the symplectic Majorana reality condition and it is more sensible to call it $\mathcal{N} = 1$

- Supersymmetry, whose conserved current is

$$J_I^m = \Gamma^n \Gamma^m \lambda'_I \partial_n \varphi + \frac{1}{2} f_{nl} \Gamma^{nl} \Gamma^m \lambda'_I \, . \tag{2.1.5}$$

- $SU(2)$ R-symmetry, whose the conserved R-current is

$$J_m^a = \bar{\lambda}'^I \tau^a \Gamma_m \lambda'_I . \tag{2.1.6}$$

These currents can form a supermultiplet if proper additional objects are added to close the algebra. The complete current multiplet consists of

$$\left(C, \zeta, X^a, w_{mn}, J_m^a, J_m, J_{(1)}^a, T^{mn} \right) . \tag{2.1.7}$$

Then one can couple this multiplet to linearized gravity. The bosonic components of the multiplet are $(h_{mn}, (A_m)^a, V_{mn}, \mathcal{A}_m, t, C)$, where a_m is $U(1)$ gauge field with field strength $F_{mn} = \partial_m \mathcal{A}_n - \partial_n \mathcal{A}_m$. The fermions are an auxiliary spinor λ of dimension $3/2$ (not to be confused with the gaugino λ_I of the $\mathcal{N} = 1$ vector multiplet in a later section) and the gravitino ψ_I^m

$$\mathcal{L} = \frac{1}{8} h_{mn} T^{mn} + \frac{i}{4} \bar{J}_I^m \psi_m^I - 4C'C - 2i\bar{\zeta}\lambda - \frac{1}{2} w_{mn} V^{mn} + X^a t^a + \frac{1}{2\sqrt{3}} \mathcal{A}_m J_{(1)}^m + \frac{1}{4} J_m^a A_a^m . \tag{2.1.8}$$

Requiring the Lagrangian to be supersymmetric, one obtains supergravity transformation (with parameter ξ_I which is a symplectic Majorana spinor) of the linearized multiplet. Further covariantizing the transformation gives the full Supergravity transformation (here we only list schematically first few lines and omit coefficients in front of each term)

$$\begin{cases} \delta e_m^i \sim \xi_I \Gamma^i \psi_m^I \\ \delta \mathcal{A}_m \sim \xi_I \psi_m^I \\ \delta \psi_{Im} \sim \mathcal{D}_m^{\hat{\omega}} \xi_I + \hat{F}_{mn} \Gamma^n \xi_I + V^{pq} \Gamma_{mpq} \xi_I + (A_m)_I{}^J \xi_J + t_I{}^J \xi_J + ... \\ \delta \lambda_I = \left(4 \nabla_m V^{mn} \Gamma_n + \mathcal{F}_{mn} \mathcal{F}_{kl} \Gamma^{mnkl} + C \right) \xi_I + 4 \left[\left(D_m t_I{}^J \right) \Gamma^m + t_I{}^J (\mathcal{F} + 2V)_{mn} \Gamma^{mn} \right] \xi_J + ... \end{cases} , \tag{2.1.9}$$

where ... in the third line denotes terms that will vanish when taking rigid limit. In the last line we schematically show a few terms involving V, and use ... to denote remaining complicated terms.

The rigid limit procedure sets fermions to zero, keeping only the bosonic fields (metric and other fields) to some background which needs to be determined. If such background is invariant under certain supergravity transformation, in particular, $\delta \psi = 0$, one obtain a rigid supersymmetric background with the resulting metric.

The condition $\delta \psi = 0$ reads, with some coefficients reinstalled without loss of generality,

$$\delta \psi_{mI} = \nabla_m \xi_I - t_I{}^J \Gamma_m \xi_J - \frac{1}{2} F_{mn} \Gamma^n \xi_I - \frac{1}{2} V^{pq} \Gamma_{mpq} \xi_I - (A_m)_I{}^J \xi_J = 0. \tag{2.1.10}$$

which is the Killing spinor equation we are going to analyze in the following sections.

In principle one needs to also solve the equation from $\delta \lambda = 0$ in taking the rigid limit. However, in this chapter we do not discuss this equation, but rather focus on the simpler yet important Killing spinor equation (2.1.10).

2.2 Symplectic Majorana spinors and bilinears

In this section, we review the properties of symplectic Majorana spinor and their bilinears. Note that we consider bosonic spinors in the following discussions. More detail can be found in the appendix [A]

On a 5-dimensional Riemannian manifold M, one can define Hermitian Gamma matrices, the charge conjugation matrix and $SU(2)$ symplectic Majorana spinors[2].

Hermitian Gamma matrices are denoted as Γ

$$\{\Gamma_m, \Gamma_n\} = 2g_{mn}, \tag{2.2.1}$$

and hermiticity implies

$$\overline{\Gamma_m} = (\Gamma_m)^T. \tag{2.2.2}$$

The Charge conjugation matrix is denoted as C,

$$C\Gamma^m C^{-1} = (\Gamma^m)^T = \overline{\Gamma_m}. \tag{2.2.3}$$

We also define the $SU(2)$-invariant tensor ϵ^{IJ} and ϵ_{IJ}

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = 1, \tag{2.2.4}$$

and raising and lowering convention

$$\epsilon_{IJ}X^J = X_I,\ \epsilon^{IJ}X_J = X^I. \tag{2.2.5}$$

With these quantities we define the symplectic-Majorana spinor condition as

$$\overline{\xi_I^\alpha} = \epsilon^{IJ}C_{\alpha\beta}\xi_J^\beta, \tag{2.2.6}$$

and a $\mathbb{C}$-valued product of any two spinors denoted by parenthesis ()

$$(\xi\eta) \equiv \xi^\alpha C_{\alpha\beta}\eta^\beta, \tag{2.2.7}$$

and further a positive-definite inner product (,) between symplectic Majorana spinors ξ, η

$$(\xi, \eta) \equiv \epsilon^{IJ}\left(\xi_I \eta_J\right). \tag{2.2.8}$$

2.2.1 Bilinears from 1 symplectic Majorana spinor

Now we're ready to define bilinears constructed from one symplectic Majorana spinor ξ_I.

(1) Function $s \in C^\infty(M)$:

$$s \equiv \epsilon^{IJ}\left(\xi_I\xi_J\right) = 2\left(\xi_1\xi_2\right). \tag{2.2.9}$$

[2] Note that ordinary Majorana condition cannot be defined in 5d.

Note that this function is strictly positive if ξ is nowhere-vanishing:

$$s = \epsilon^{IJ}\xi_I^\alpha C_{\alpha\beta}\xi_J^\beta = \sum_\alpha \xi_I^\alpha \overline{\xi_I^\alpha} > 0. \tag{2.2.10}$$

(2) Vector field $R \in \Gamma(TM)$:

$$R^m \equiv \epsilon^{IJ}\xi_I\Gamma^m\xi_J, \tag{2.2.11}$$

and the corresponding 1-form

$$\kappa_m \equiv g_{mn}R^n, \tag{2.2.12}$$

which implies, when acting on $\Omega^p(M)$

$$\iota_R \circ * = (-1)^p * \circ (\kappa\wedge)\,. \tag{2.2.13}$$

(3) 2-form[3]

$$\Theta^{IJ}_{mn} \equiv \left(\xi^I\Gamma_{mn}\xi^J\right), \tag{2.2.16}$$

with symmetry

$$\Theta^{IJ} = \Theta^{JI}. \tag{2.2.17}$$

Let t_{IJ} be an arbitrary triplet of functions, namely

$$t_{IJ} = t_{JI}, I = 1, 2; \tag{2.2.18}$$

then its contraction with Θ gives a real 2-form

$$(t\Theta) \equiv t^I{}_J\big(\Theta^J{}_I\big). \tag{2.2.19}$$

Using the Fierz identities one can derive useful relations between these quantities, which we list in appendix A.2.4.

Given the nowhere-vanishing 1-form κ and the vector field R, one can decompose the tangent bundle $TM = TM_H \oplus TM_V$, where at any point $p \in M$, $TM_H|_p$ is annihilated by κ, while TM_V is a trivial line bundle generated by R. Let's call TM_H, and similarly all tensors annihilated by κ (or R) "horizontal", while those in the orthogonal complement "vertical". In particular, one has decompositions

$$\Omega^2(M) = \Omega^2_V(M) \oplus \Omega^2_H(M) = \kappa \wedge \Omega^1_H(M) \oplus \Omega^2_H(M) \tag{2.2.20}$$

For an arbitrary nowhere-vanishing triplet of functions t_{IJ} with the property (readers may find conventions in Appendix [A])

$$t_{IJ} = t_{JI}, \;\; \overline{t_{IJ}} = \epsilon^{II'}\epsilon^{JJ'}t_{I'J'} \tag{2.2.21}$$

[3] One could of course go on defining higher forms $\Theta^{IJ}_{lmn} \equiv \xi^I\Gamma_{lmn}\xi^J$ and $\Theta^{IJ}_{mnpq} \equiv \xi^I\Gamma_{mnpq}\xi^J$, but duality of Gamma matrices gives

$$\Theta^{IJ}_{lmn} = -\frac{\sqrt{g}}{2!}\epsilon^{pq}{}_{lmn}\Theta^{IJ}_{pq}, \tag{2.2.14}$$

and

$$\Theta^{IJ}_{mnpq} = \sqrt{g}\epsilon^r{}_{mnpq}\Theta^{IJ}_r, \tag{2.2.15}$$

one can define a map $\varphi_t : \Gamma(TM) \to \Gamma(TM)$ as

$$(\varphi_t)_m{}^n \equiv \frac{1}{s}\sqrt{\frac{-2}{\operatorname{tr}(t^2)}}(t\Theta)_m{}^n. \tag{2.2.22}$$

Obviously, one has

$$\varphi_t \circ \varphi_t = -1 + s^{-2} R \otimes \kappa, \tag{2.2.23}$$

and when restricted on TM_H, φ_t is some sort of a"complex" structure:

$$\varphi_t \circ \varphi_t|_{TM_H} = -1. \tag{2.2.24}$$

Together with the vector field $s^{-1}R$ and 1-form $s^{-1}\kappa$, φ_λ defines an almost contact structure on M[2] (see also Appendix C).

Finally, let us comment on the "(anti)self-dual" horizontal forms. Define operator $*_H \equiv s^{-1}\iota_R *$, which is the hodge dual "within" horizontal hyperplanes. It is easy to verify that acting on any horizontal p-forms

$$*_H^2 = (-1)^p. \tag{2.2.25}$$

In particular, we decompose the horizontal 2-forms into 2 subspaces according to their eigenvalues of $*_H$

$$\Omega_H^2 = \Omega_H^{2+} \oplus \Omega_H^{2-}, \quad *_H\omega_H^{\pm} = \pm\omega_H^{\pm}, \forall\omega_H^{\pm} \in \Omega_H^{\pm}. \tag{2.2.26}$$

We call the horizontal forms in Ω_H^{2+} "self-dual", while the others "anti-self-dual". Clearly, these 2 notions are interchanged as one flips the sign of the vector field R, hence this notion of "self-duality" is not as intrinsic as the well-established notion of self-duality on 4-dimensional oriented manifolds.

Suppose Ω^+ is a self-dual 2-form. Then it satisfies, by definition,

$$\frac{\sqrt{g}}{2s}\epsilon^{pq}{}_{lmn}R^l\Omega^+_{pq} = \Omega^+_{mn}. \tag{2.2.27}$$

It follows immediately that

$$\boxed{\Omega^+_{mn}\Gamma^{mn}\xi_I = 0}, \tag{2.2.28}$$

using the fact that the inner product $(\psi,\psi) \equiv \epsilon^{IJ}(\psi_I\psi_J)$ is positive definite, and the action of Γ_{mn} preserve symplectic Majorana property.

2.2.2 Bilinears from 2 symplectic Majorana spinors

In this section, we consider the case when there are 2 symplectic Majorana spinors, and analyze their bilinears.

Denote the two spinors ξ_I and $\tilde{\xi}_I$. Obviously they each generates a set of quantities as we discussed in the previous sections: (s, R, κ, Θ) and $(\tilde{s}, \tilde{R}, \tilde{\kappa}, \tilde{\Theta})$.

In addition to these quantities, they form some new mixed bilinears. Conventions for IJ indices can be found in appendix A.

- Functions

$$u_{IJ} \equiv (\xi_I \tilde{\xi}_J), \tag{2.2.29}$$

with triplet-singlet decomposition

$$u_{IJ} = u_{(IJ)} + u_{[IJ]} = \hat{u}_{IJ} - \frac{1}{2}\epsilon_{IJ} u, \tag{2.2.30}$$

where

$$u \equiv \epsilon^{IJ} u_{IJ}. \tag{2.2.31}$$

Notice that

$$\overline{u_{IJ}} = \epsilon^{II'}\epsilon^{JJ'} u_{I'J'} \equiv u^{IJ}, \tag{2.2.32}$$

and in particular function u is real-valued

$$\overline{u} = u = \sum_I \xi_I^\alpha \overline{\tilde{\xi}_I^\alpha}, \tag{2.2.33}$$

which results in positivity

$$u_{IJ} u^{IJ} = \sum u_{IJ} \overline{u_{IJ}} = \frac{1}{2} u^2 + \hat{u}_{IJ} \hat{u}^{IJ} \geq 0. \tag{2.2.34}$$

- Vector fields Q_{IJ}

$$Q^m_{IJ} \equiv (\xi_I \Gamma^m \tilde{\xi}_J), \tag{2.2.35}$$

with a decomposition

$$Q_{IJ} = \hat{Q}_{IJ} - \frac{1}{2}\epsilon_{IJ} Q, \tag{2.2.36}$$

where

$$Q^m \equiv \epsilon^{IJ} (\xi_I \Gamma^m \tilde{\xi}_J). \tag{2.2.37}$$

Note that similar to the function case, we have

$$\overline{Q_{IJ}} = Q^{IJ}, \tag{2.2.38}$$

and in particular a real vector field

$$\overline{Q} = Q. \tag{2.2.39}$$

We denote corresponding 1-forms

$$\tau_{IJ} \equiv (Q_{IJ})_m dx^m = \hat{\tau}_{IJ} - \frac{1}{2}\epsilon_{IJ}\tau. \tag{2.2.40}$$

- Two forms

$$\chi^{IJ}_{mn} \equiv (\xi^I \Gamma_{mn} \tilde{\xi}^J). \tag{2.2.41}$$

Also we define

$$\chi \equiv \epsilon^{IJ} \chi_{IJ}, \quad \hat{\chi}_{IJ} = \chi_{(IJ)}. \tag{2.2.42}$$

These bilinears satisfy various algebraic relations. Here we list some relevant formulas.

- **Norms and inner products of vector fields**

(1)

$$R\cdot\tilde{R}=4u_{IJ}u^{IJ}-s\tilde{s}\Rightarrow\begin{cases}\left|\tilde{s}R+s\tilde{R}\right|^2=8s\tilde{s}u_{IJ}u^{IJ}\\ \left|\tilde{s}R-s\tilde{R}\right|^2=4s\tilde{s}\left(s\tilde{s}-2u_{IJ}u^{IJ}\right)\end{cases}\tag{2.2.43}$$

(2)

$$Q_{IJ}\cdot Q_{KL}=2u_{IL}u_{KJ}-u_{IJ}u_{KL}-\frac{1}{2}\epsilon_{IK}\epsilon_{LJ}s\tilde{s}\tag{2.2.44}$$

In particular

$$\begin{cases}\left|u^{IJ}Q_{IJ}\right|^2=\frac{1}{2}\left(u^{IJ}u_{IJ}\right)s\tilde{s}\\ |Q|^2=-2\hat{u}_{IJ}\hat{u}^{IJ}+s\tilde{s}\end{cases}\tag{2.2.45}$$

(3)

$$R\cdot Q_{IJ}=su_{IJ},\quad \tilde{R}\cdot Q_{IJ}=\tilde{s}u_{IJ}.\tag{2.2.46}$$

Positivity of the norms implies

$$s\tilde{s}\geq 2u_{IJ}u^{IJ}=2\hat{u}_{IJ}\hat{u}^{IJ}+u^2.\tag{2.2.47}$$

When $s\tilde{s}=2u_{IJ}u^{IJ}$, we have R and $\tilde{R}$ are parallel at such point, which in general we like to avoid.

(4) Using Fierz identity, one can shows

$$\tilde{s}R+s\tilde{R}=4u_{IJ}Q^{IJ}=2uQ+4\hat{u}_{IJ}\hat{Q}^{IJ},\tag{2.2.48}$$

$$\begin{cases}R_m\tilde{R}_n-R_n\tilde{R}_m=-4u_{IJ}\chi^{IJ}_{mn}\quad\Rightarrow\quad\kappa\wedge\tilde{\kappa}=-4u_{IJ}\chi^{IJ}\\ g_{mn}=-\dfrac{2s\tilde{s}}{\left|s\tilde{R}-\tilde{s}R\right|^2}\left[R_m\tilde{R}_n+R_n\tilde{R}_m-4(Q_{IJ})_m\left(Q^{IJ}\right)_n\right]\end{cases},\tag{2.2.49}$$

where the last equation tells us that the metric is completely determined by the bilinears constructed from 2 solution.

- **Contraction between the vectors and 2-forms**

$$\begin{cases}\iota_R\left(t\chi\right)=s\left(t\hat{\tau}\right)-\left(t\hat{u}\right)\kappa\\ \iota_Q(t\Theta)=\left(t\hat{u}\right)\kappa-s\left(t\hat{\tau}\right)\\ \iota_{\hat{u}\hat{Q}}\left(t\Theta\right)=\left(t\hat{u}\right)\left(u\kappa+s\tau\right)\\ \iota_R(t^{IJ}\tilde{\Theta}_{IJ})-\iota_{\tilde{R}}\left(t^{IJ}\Theta_{IJ}\right)=4t^{IJ}\left(u\hat{\tau}_{IJ}-\hat{u}_{IJ}\tau\right)\end{cases}\tag{2.2.50}$$

where again t_{IJ} is arbitrary triplet of functions.

2.3 Killing spinor equation

In this section we will discuss what constraints will be imposed on geometry of M when there exists different numbers of solutions to the Killing spinor equation (2.1.10). We focus on situations where there are 1, 2, and 4 pairs of solutions to the equation.

Recall that the Killing spinor equation required by rigid limit of supergravity is

$$\delta\psi_{mI} = \nabla_m \xi_I - \Gamma_m t_I{}^J \xi_J - \frac{1}{2} V^{pq} \Gamma_{mpq} \xi_I - \frac{1}{2} F_{mn} \Gamma^n \xi_I - (A_m)_I{}^J \xi_J = 0, \tag{2.3.1}$$

where t_{IJ} is a triplet of scalars (or more precisely, a global section of the $ad(P_{SU(2)})$ where $P_{SU(2)}$ is an underlying principal $SU(2)_{\mathcal{R}}$-bundle, with gauge field $(A_m)_I{}^J$), F is a closed 2-form, V is a 2-form.

The symplectic Majorana spinor ξ_I is a section of the $SU(2)_{\mathcal{R}}$ twisted spin bundle of M. In general the $SU(2)_{\mathcal{R}}$-bundle P is non-trivial. We define the gauge-covariant derivative on t_{IJ}

$$\nabla^A_m t_I{}^J \equiv \nabla_m t_I{}^J - (A_m)_I{}^K t_K{}^J + t_I{}^K (A_m)_K{}^J, \tag{2.3.2}$$

and curvature of A as

$$(W_{mn})_I{}^J \equiv \nabla_m (A_n)_I{}^J - \nabla_n (A_m)_I{}^J - \left[(A_m)_I{}^K (A_n)_K{}^J - (A_n)_I{}^K (A_m)_K{}^J\right]. \tag{2.3.3}$$

Note that the Killing spinor equation is $SU(2)$ gauge covariant. It is also invariant under complex conjugation, provided that the auxiliary fields satisfies reality conditions: F and V are real,

$$\overline{t_{IJ}} = \epsilon^{II'} \epsilon^{JJ'} t_{I'J'}, \tag{2.3.4}$$

and similar for A. The reality condition on t_{IJ} and A is just saying that they are linear combinations of Pauli matrices with imaginary coefficients.

Apart from the above obvious symmetries, the equation further enjoys a shifting symmetry and a Weyl symmetry.

- Shifting symmetry: The equation is invariant under the shifting transformation of auxiliary fields V and F

$$\begin{cases} V \to V + \Omega^+ \\ F \to F + 2\Omega^+ \end{cases}, \tag{2.3.5}$$

  where Ω^+ is any self-dual 2-form discussed in (2.2.26), following from the fact that

$$\Omega^+_{mn} \Gamma^{mn} \xi_I = 0. \tag{2.3.6}$$

- Weyl symmetry: after rescaling the metric $g \to e^{2\phi} g$, one can properly transform the auxiliary fields as well as the Killing spinor solution such that the Killing spinor equation is invariant. This can be seen by first rearranging the Killing spinor equation (2.1.10) into the form

$$\nabla_m \xi_I = \Gamma_m \tilde{\xi}_I + \frac{1}{2} P_{mn} \Gamma^n \xi_I, \tag{2.3.7}$$

where

$$\tilde{\xi}_I \equiv \left(t_I{}^J + \frac{1}{2} V_{pq} \Gamma^{pq} \delta_I^J \right) \xi_J, \quad P_{mn} \equiv F_{mn} - 2V_{mn}. \tag{2.3.8}$$

and we ignore the gauge field A_{IJ} for simplicity.

Focusing on (2.3.7) alone as an equation for pair $(\xi, \tilde{\xi})$ on any d-dimensional manifold, it is obvious that

$$\tilde{\xi}_I = \frac{1}{d} \Gamma^m \nabla_m \xi_I - \frac{1}{2d} P_{mn} \Gamma^{mn} \xi_I. \tag{2.3.9}$$

Substituting it back to (2.3.7), one obtains the equation

$$\mathcal{D}(g) \xi_I = \frac{1}{2d} P_{pq} \Gamma_m \Gamma^{pq} \xi_I + \frac{1}{2} P_{mn} \Gamma^n \xi_I \tag{2.3.10}$$

where the well-known differential operator $\mathcal{D}_g$ is defined as

$$\mathcal{D}(g) \equiv \nabla_m - \frac{1}{d} \Gamma_m \Gamma^n \nabla_n. \tag{2.3.11}$$

and depends on the metric g. It's easy to show that[4]

$$\mathcal{D}(e^{2\phi} g) e^{\phi/2} = e^{\phi/2} \mathcal{D}(g). \tag{2.3.14}$$

Hence, equation (2.3.7) is invariant under rescaling

$$g \to e^{2\phi} g, \quad P \to e^{\phi} P, ; \xi \to e^{\phi/2} \xi. \tag{2.3.15}$$

Now we return to the equation (2.1.10), and compute the transformation of auxiliary fields under Weyl rescaling. Suppose the scaling function ϕ is constant along vector field R:

$$R^m \nabla_m \phi = 0, \tag{2.3.16}$$

then one can see that the Killing spinor equation (2.1.10) is invariant under rescaling

$$\boxed{g \to e^{2\phi} g, \; t_{IJ} \to e^{-\phi} t_{IJ}, \; V \to e^{\phi} V - \frac{e^{\phi}}{2s} (\kappa \wedge d\phi), \; F \to e^{\phi} F - \frac{e^{\phi}}{s} (\kappa \wedge d\phi)}, \tag{2.3.17}$$

provided we also rescale $\xi \to e^{\phi/2} \xi$. Note that the Weyl rescaling only affects the vertical part of F and V. One can therefore use this rescaling symmetry with appropriate ϕ to make F horizontal, namely

$$\iota_R F = 0. \tag{2.3.18}$$

However, unless explicitly stated, in most of the following discussions, we will keep the general F without exploiting the Weyl symmetry.

[4]Under Weyl rescaling $g \to e^{2\phi} g$, the spin connection is shifted according to

$$\nabla_m^g \psi \to \nabla_m^{e^{2\phi} g} \psi = \nabla_m^g \psi + \frac{1}{2} (\nabla_n^g \phi) \Gamma_m{}^n \psi. \tag{2.3.12}$$

To prove the Weyl transformation rule for $\mathcal{D}(g)$, one just need to plug the above formula into

$$D\left(e^{2\phi} g\right) \left(e^{\phi/2} \psi\right) = \nabla_m^{e^{2\phi} g} \left(e^{\phi/2} \psi\right) - \frac{1}{d} \Gamma_m \Gamma^n \nabla_n^{e^{2\phi} g} \left(e^{\phi/2} \psi\right). \tag{2.3.13}$$

Let us comment on the reality condition defined earlier.

(1) In 5 dimension Euclidean signature, the spinors belong to 2^2 dimensional pseudoreal representation of Spin(5) $\sim Sp(2)$, spinor $(\psi^*)_\alpha$ and $(C\psi)_\alpha \equiv C_{\alpha\beta}\psi^\beta$ transform in the same way. It is impossible to impose the usual Majorana condition, but one can impose the symplectic Majorana condition on spinors. In this sense, 4 complex (8 real) supercharges correspond to unbroken supersymmetry, namely $\mathcal{N} = 1$.

The reality conditions introduced above are required by the supergravity that we started from, where one is interested in a real-valued action. However, it is fine to relax the reality condition on the Killing spinors and auxiliary fields, as long as one is only interested in a formally supersymmetric invariant theory. It makes perfect sense to consider complexified Killing spinor equation. In particular, the reality condition is not used in many of the following discussion, for instance, section 4.1 actually can be carried out without assuming the reality condition (except for the shifting symmetry of Ω^+ which requires positivity following from reality condition). One only needs to work with $\mathbb{C}$-valued differential forms. Also, when we compare our 5d Killing spinor equation to the 4d equations appearing in [3][4], we drop the reality requirement. However, in this paper we mainly restrict ourselves to the real case, and reality condition does helps simplify certain discussions.

(2) Solutions to equation (2.1.10) come in pairs. Suppose ξ is a solution, corresponding to one supercharge Q, then its complex conjugate ξ'

$$\xi_1' = \xi_2 = \overline{\xi_1}, \;\; \xi_2' = -\xi_1, \tag{2.3.19}$$

automatically satisfies (2.1.10) corresponding to the supercharge $\overline{Q}$. The pair of solutions ξ_I and ξ_I' define the same scalar function s and vector field R, but 2-forms Θ with different sign.

In view of such "pair-production" of solutions, we focus on finding different number of pairs of solutions to (2.1.10), and discuss them separately in the following subsections. When analyzing the case when M admits 1 and 2 pairs of solutions, we will select one representative solution from each pair, say, ξ and $\tilde{\xi}$, and study the relation between the bilinears that can be formed by these representing Killing spinors. Generically, the vector fields R and $\tilde{R}$ from separate pairs should not be parallel everywhere on M.

(3) One may worry about possible zeroes of Killing spinors. Similar to that in [4], the Killing spinor equations are a first order homogeneous differential equation system, whose set of solutions span a complex vector space $\mathbb{C}^{k\leq 4}$, with each solution completely specified by its value at a point $p \in M$. By the symplectic Majorana condition, $\xi_1(p) = 0$ implies $\xi_2(p) = 0$, and hence $\xi_I(\forall x \in M) = 0$. Therefore, any non-trivial solution of the Killing spinor equation must be nowhere-vanishing, which ensures that the many bilinears defined (especially the almost contact structure) will be global.

In some sense, our Killing spinor equation is a generalization of the well-known Killing spinor equation

$$\nabla_m \psi = \lambda \Gamma_m \psi, \tag{2.3.20}$$

The constant λ can be real, pure-imaginary or zero, and the equation is accordingly called real, imaginary Killing spinor equation and covariantly constant spinor equation. If a manifold admits a

Killing spinor, its Ricci curvature must take the form

$$Ric = 4\,(n-1)\,\lambda^2 g, \tag{2.3.21}$$

hence Einstein. For λ pure imaginary, Baum gave a classification in [5][6]. Prior to [7], manifolds with real Killing spinor are better known in low dimensions. For instance, 4-dimensional complete manifolds with real Killing spinor were shown to be isometric to the 4-sphere [8]. In 5-dimension, simply-connected manifolds with real Killing spinors were shown to be round S^5 or Sasaki-Einstein manifolds, with solutions coming down from covariantly constant spinors on their Calabi-Yau cone. In [7], these results were generalized to higher dimensions: in dimension $n = 4k+1$, only S^{4n+1} and Sasaki-Einstein manifolds admits real Killing spinors, while in $n = 4n+3 \geq 11$ dimension, only the round sphere, Sasaki-Einstein and 3-Sasakian manifolds admit real Killing spinors.

Our generalized Killing spinor equation has milder constraints on the geometry of manifold. We will see that the existence of one Killing spinor requires some soft geometry structure, one being an almost contact structure, similar to [9]. Of course, as the number of solutions increase, the geometry will be more constrained.

2.3.1 Manifolds admitting 1 pair of supercharges

General Result and ACMS structure

In this subsection we will analyze the case when there is one pair of solutions to the Killing spinor equation (2.1.10). We partially solve the auxiliary fields in terms of bilinears constructed, and rewrite the (2.1.10) into a simpler form. We will also briefly discuss 3 interesting cases with special auxiliary field configurations, which lead to geometrical restrictions of M being locally foliated by special manifolds, or dimensional reduction to known 4d equations.

By differentiating the bilinears and using (2.1.10), one arrives at the following differential constraints on the quantities:

- Derivative on real positive function s

$$ds = -\iota_R F. \tag{2.3.22}$$

- Derivative on real vector field R

$$\nabla_m R_n = 2(t\Theta)_{mn} - \sqrt{g}\epsilon^{rpq}{}_{nm} R_r V_{pq} + sF_{mn}. \tag{2.3.23}$$

- Derivative on the 2-form with any triplet r_{IJ}

$$\begin{aligned}\nabla_k\left(r^{IJ}\Theta_{IJ}\right)_{mn} = {} & \left(\nabla_k^A r^{IJ}\right)\left(\Theta_{IJ}\right)_{mn}\\ & +\mathrm{tr}\,(rt)\,(g_{nk}R_m - g_{mk}R_n) - 2r^{JI}t_I{}^K(*\Theta_{JK})_{kmn}\\ & +2\left[(*V)_{nk}{}^l r^{IJ}(\Theta_{IJ})_{ml} - (*V)_{mk}{}^l\left(r^{IJ}\Theta_{IJ}\right)_{nl}\right]\\ & -F_k{}^p r^{IJ}(*\Theta_{IJ})_{mnp}\end{aligned}\quad. \tag{2.3.24}$$

Let us comment on the above relations. The first equation implies $s = \text{const}$ and can be normalized to $s = 1$ when F is horizontal. Recall that one can always use the Weyl symmetry of the equation to achieve this, although we keep the general situation. The second implies that R is a Killing vector field:

$$\boxed{\nabla_m R_n + \nabla_n R_m = 0}. \tag{2.3.25}$$

The 3rd relation can be simplified as one puts in the solutions to F and V_H.

Using the 2nd and 3rd equation, one can solve (partially) the auxiliary fields in terms of the bilinears (field V is decomposed as $V = V_H + \kappa \wedge \eta$) :

$$\boxed{\begin{aligned} &F = (2s)^{-1} d\kappa + 2s^{-1}\Omega^- + 2s^{-1}\Omega^+ \\ &V_H = -s^{-1}(t\Theta) + s^{-1}\Omega^- + s^{-1}\Omega^+ \\ &\eta^m = \frac{1}{4s^3}\left(\Theta^{IJ}\right)^{mn}\nabla^k(\Theta_{IJ})_{nk} - \frac{3}{4}\left(\nabla^m s^{-1}\right) - \frac{1}{s^2}(A_n)_{IJ}\left(\Theta^{IJ}\right)^{nm} \end{aligned}}\,, \tag{2.3.26}$$

where $\Omega^\pm$ are self-dual (+) and anti-self-dual (−) 2-forms respectively, satisfying extra condition

$$\mathcal{L}_R \Omega^\pm = 0. \tag{2.3.27}$$

From previous discussions, we know that Ω^+ corresponds to the arbitrary shifting symmetry of Killing spinor equation, so we may simply consider $\Omega^+ = 0$.

Ω^- is in general non-zero. For instance, the well-known Killing spinor equation $\nabla_m \xi_I = t_I{}^J \Gamma_m \xi_J$ corresponds to

$$\Omega^- = -\frac{1}{4} d\kappa, \tag{2.3.28}$$

which is non-zero. Also, at the end of the paper we construct a supersymmetric theory for the $\mathcal{N} = 1$ vector multiplet using the Killing spinor equation corresponding to

$$\Omega^- = \frac{1}{4} d\kappa. \tag{2.3.29}$$

However, to highlight some interesting underlying geometry related to (2.1.10), we will consider

$$\Omega^- = 0, \tag{2.3.30}$$

in this section unless explicitly stated. It is straight forward to generalize to non-zero Ω^-, with sight modification to the following discussions.

Now that the auxiliary fields are partially solved, we can start simplifying the Killing spinor equation. As mentioned before, t_{IJ} is a global section of associate rank-3 vector bundle of $P_{SU(2)}$, it may have zeroes. Below we will focus on 2 cases corresponding to $t \neq 0$ and $t = 0$ everywhere on M.

First let us consider the case when $t_{IJ} \neq 0$.

(1) $\underline{t_{IJ} \neq 0}$

Notice that the quantities $(g, s^{-1}R, s^{-1}\kappa, \varphi_t)$ actually form an almost contact metric structure (abbreviated as ACMS). Using the ACMS, one can further rewrite the Killing spinor equation:

$$\boxed{\hat{\nabla}_m \hat{\xi}_I - (\hat{A}_m)_I{}^J \hat{\xi}_J = 0}, \tag{2.3.31}$$

where we rescaled ξ

$$\hat{\xi}_I \equiv (\sqrt{s})^{-1}\xi_I, \tag{2.3.32}$$

$$\begin{aligned}(\hat{A}_m)_I{}^J &\equiv (A_m)_I{}^J + \frac{1}{s}R_m t_I{}^J + \frac{1}{\mathrm{tr}\,(t^2)}\left(\nabla_m^A t^{JK}\right)t_{KI} + \eta \text{ terms} \\ &= \frac{1}{s}R_m t_I{}^J + \frac{1}{\mathrm{tr}\,(t^2)}\left(\nabla_m t^{JK}\right)t_{KI} + \eta \text{ terms},\end{aligned} \tag{2.3.33}$$

and $\hat{\nabla}$ being the compatible spin connection introduced in the appendix C.9.

$$\begin{aligned}\hat{\nabla}_m \xi_I = \nabla_m \xi_I &+ \frac{1}{\mathrm{tr}\,(t^2)}(T_m)^J{}_I \xi_J - \frac{1}{2s}\nabla_m R_n \Gamma^n \xi_I + \frac{1}{2}\left(\nabla_m \log s\right)\xi_I \\ &- \frac{1}{\mathrm{tr}\,(t^2)}\eta_q (t\Theta)^q{}_m t_I{}^J \xi_J + \frac{1}{2}\left(*V^V\right)_{mpq}\Gamma^{pq}\xi_I\end{aligned} .. \tag{2.3.34}$$

Notice that the new gauge connection is no longer $SU(2)$ connection, since the term

$$(T_m)_{IJ} \equiv \left(\nabla_m^A t_I{}^K\right)t_{KJ}, \tag{2.3.35}$$

might not be symmetric in I, J, but rather

$$T_m^{IJ} - T_m^{JI} = \frac{1}{2}\epsilon^{IJ}\nabla_m \mathrm{tr}\left(t^2\right), \tag{2.3.36}$$

which corresponds to an new extra $U(1)$ gauge field. Fortunately this extra $U(1)$ part is in pure gauge,

$$\hat{A}^{IJ}_{U(1)} \sim \epsilon^{IJ}\nabla \ln \mathrm{tr}\left(t^2\right), \tag{2.3.37}$$

and can be easily gauged away. Hence, let us choose a gauge

$$\nabla \mathrm{tr}\left(t^2\right) = 0. \tag{2.3.38}$$

Before moving to the $t \equiv 0$ case, let us make a few remarks.

(1) The appearing of ACMS has already been hinted in literatures . In [9], supersymmetric theory is obtained on any 3d almost contact metric manifold. [10] constructed twisted version of the super-Chern-Simons theory considered in [11] on any Seifert manifold M_3. Their twisted theory is defined with a choice of contact structure on M_3, with fermions replaced by differential forms. Note that the non-degenerate condition of a contact structure is crucial in defining the theory and the supersymmetry used for localization. Similar situations appear in [12][13], where the authors constructed twisted YM-CS theory on any 5d K-contact manifold M.

(2) There is an interesting configuration (among many similar ones). It corresponds to the case when

$$2V = F. \tag{2.3.39}$$

In such configuration,

$$d\kappa = -4t\Theta + 4\kappa \wedge \eta \Rightarrow \kappa \wedge d\kappa \wedge d\kappa \propto \kappa \wedge (t\Theta) \wedge (t\Theta) \neq 0, \tag{2.3.40}$$

which implies κ is a contact structure. To make things even simpler one can use the Weyl rescaling symmetry to make field F as well as V horizontal, and therefore $s = 1$:

$$F = \frac{1}{2}d\kappa + 2\Omega^-, \quad V = \frac{1}{4}d\kappa + \Omega^-, \tag{2.3.41}$$

where F, V, Ω^- are now all closed anti-self-dual 2-forms. The Killing spinor equation can be rewritten as

$$\nabla_m \xi_I = \Gamma_m \left(t_I{}^J + \frac{1}{4} F^{pq} \Gamma_{pq} \delta_I^J \right) \xi_J, \tag{2.3.42}$$

which takes the familiar form

$$\nabla_m \xi_I = \Gamma_m \tilde{\xi}_I, \tag{2.3.43}$$

with $\tilde{\xi}_I = (t_I{}^J + (1/4)F^{pq}\Gamma_{pq}\delta_I^J)\xi_J$. We will use this Killing spinor equation to construct a supersymmetric theory for the $\mathcal{N} = 1$ vector multiplet in section 2.4.

There are many examples of contact manifolds. For instance, any non-trivial $U(1)$-bundle over a 4d Hodge manifold, with unit Reeb vector field R pointing along the $U(1)$ fiber is a contact manifold. One should note that trivially fibered S^1-bundle, namely $M = S^1 \times N$ with Reeb vector field pointing along S^1 is not contact, because the non-degenerate condition cannot be satisfied. However, this type of manifold still serve as important examples admitting supersymmetry. Hence, we will have a brief discussion related to this type of manifold at the end of this section.

(2) $\underline{t_{IJ} \equiv 0}$.

There is no natural ACMS arises in this case (although, if possible, one could choose by hand a nowhere-vanishing section of $ad(P_{SU(2)})$ to play the role of t_{IJ}, and similar calculations goes through. In this paper we do not consider this approach). The auxiliary fields F and V read

$$\begin{cases} F_{mn} = (2s)^{-1} \left(\nabla_m R_n - \nabla_n R_m \right) \\ V_{mn} = R_m \eta_n - R_n \eta_m \end{cases}, \tag{2.3.44}$$

and the Killing spinor equation reads

$$\nabla_m \hat{\xi}_I + \left[-\frac{1}{4s^2} \left(R_l \nabla_m R_n - R_n \nabla_m R_l \right) + \frac{1}{2} (\iota_R * \eta)_{mnl} \right] \Gamma^{nl} \hat{\xi}_I = (A_m \hat{\xi})_I. \tag{2.3.45}$$

Similar to the previous discussion, we again have a new connection $\hat{\nabla}$ defined as

$$\hat{\Gamma}^l{}_{mn} = \Gamma^l{}_{mn} + \frac{1}{s^2} \left(R^l \nabla_m R_n - R_n \nabla_m R^l \right) - 2 (\iota_R * \eta)^l{}_{mn}, \tag{2.3.46}$$

satisfying

$$\hat{\nabla}_m \left(s^{-1} R^n \right) = 0, \tag{2.3.47}$$

although there is no obvious geometrical interpretation for this connection.

Again the Killing spinor equation can be rewritten as

$$\hat{\nabla}_m \hat{\xi}_I = (A_m)_I{}^J \hat{\xi}_J, \tag{2.3.48}$$

where $\hat{\xi} = \sqrt{s^{-1}}\xi$ has unit norm

To end this section, we discuss, in the following subsections, 3 special cases related to 5-manifolds of the form $M = S^1 \times M_4$, with the Reeb vector field R pointing along S^1. As we will see there are 2 cases corresponding to two different types of auxiliary field configurations: V horizontal, F vertical and V, F both vertical. The first configuration leads to geometric restrictions on the sub-manifold M_4, while the second corresponds to the dimensional-reduction of our 5d equation to 4d already discussed in the literatures.

For such product form (or foliation) to appear, one first needs the horizontal distribution TM_H to be integrable: the Frobenius integrability condition for κ reads

$$d\kappa \wedge \kappa = 0 \text{ , or equivalently } d\kappa = \kappa \wedge \lambda, \lambda \in \Omega^1_H(M). \tag{2.3.49}$$

Recall that $F \propto d\kappa$ (Ω^- is assumed to be 0), one sees that the Frobenius integrability condition requires vertical F

$$F = \kappa \wedge (...). \tag{2.3.50}$$

Special Manifold foliation

To proceed to the first class of special cases, let us define a local $SU(2)$ section of "almost complex structure":

$$J^a \equiv \frac{i}{s}(\sigma^a)^I{}_J \Theta^J{}_I, \tag{2.3.51}$$

satisfying

$$J^a J^b = \epsilon^{abc} J^c - \delta^{ab} I + \delta^{ab} s^{-1} R \otimes s^{-1}\kappa. \tag{2.3.52}$$

It is immediate that when restricted on TM_H,

$$\boxed{J^a J^b = \epsilon^{abc} J^c - \delta^{ab} I}. \tag{2.3.53}$$

Moreover, we have

$$\boxed{\hat{\nabla}_k (J^a)_{mn} = (\hat{A}_k)^a{}_b \Big(J^b\Big)_{mn}}, \tag{2.3.54}$$

where

$$(\hat{A}_m)^a{}_b \equiv (-i)^2 (\hat{A}_m)^I{}_K (\sigma^a)^J{}_I (\sigma_b)^K{}_J. \tag{2.3.55}$$

Note that we can solve the new connection in terms of "almost complex structures":

$$\boxed{(\hat{A}_k)^a{}_b = \frac{1}{4}(J_b)^{mn} \hat{\nabla}_k (J^a)_{mn}}, \tag{2.3.56}$$

which, depending on whether $t_{IJ} = 0$, provides constraints on t_{IJ} or A.

These equations closely resemble that of Quaternion-Kähler geometry, where one has on manifold M a $SU(2)$ bundle of local almost complex structure J^a satisfying

$$J^a J^b = \epsilon^{abc} J^c - \delta^{ab} I, \tag{2.3.57}$$

and is parallel with respect to the gauged connection

$$\nabla J^a = A^a{}_b J^b, \tag{2.3.58}$$

with the Levi-civita connection ∇ and a $SU(2)$ gauge connection A.

However the situation here is slightly different. We do not have actually a manifold but rather a horizontal part of tangent bundle TM_H of 5-fold M.

Let us assume V is horizontal:

$$\eta = 0. \tag{2.3.59}$$

The induced connection (for $t \neq 0$ case; $t = 0$ case goes through similarly and yields the same conclusion) on TM_H is

$$\hat{\nabla}_X Y = \nabla_X Y - g\left(s^{-1}R, \nabla_X Y\right) s^{-1}R - \frac{1}{\mathrm{tr}\,(t^2)}\left(\nabla^A_X t_I{}^K\right) t_{KJ} \Theta^{IJ}\left(Y\right), \ \forall X, Y \in TM_H \tag{2.3.60}$$

Consider the special case where the sub-bundle TM_H is integrable as the tangent bundle TM_4 of a co-dimension 1 sub-manifold M_4, then $\hat{\nabla}$ reduces to a connection on M_4. The first 2 terms of the connection combine to be the induced Levi-Civita connection ∇^{M_4} on M_4 ($s^{-1}R$ being the unit normal vector), while the third term add to it a torsion part:

$$\hat{\Gamma}^n{}_{mk} = \Gamma^n{}_{mk} + \gamma^n{}_{mk}, \tag{2.3.61}$$

where

$$\gamma^n{}_{mk} = -\frac{1}{\mathrm{tr}\,(t^2)}\left(\nabla^A_m t_I{}^K\right) t_{KJ} \left(\Theta^{IJ}\right)^n{}_k. \tag{2.3.62}$$

Rewrite the Quaternion-Kähler-like equation as

$$\hat{\nabla}^{M_4}_k J^a_{mn} = \nabla^{M_4}_k J^a_{mn} - \gamma^l{}_{km} J^a_{ln} - \gamma^l{}_{kn} J^a_{ml} = (\hat{A}_k)^a{}_b J^b_{mn}. \tag{2.3.63}$$

Now one can put back expression for both γ and J^a, and sees that the torsion terms gives

$$\gamma^l{}_{kn} J^a_{ml} - \gamma^l{}_{km} J^a_{nl} = \frac{1}{\mathrm{tr}\,(t^2)}\left(\nabla^A_k t_I{}^K\right) t_K{}^L (\sigma^a)_K{}^J (\sigma_b)^I{}_J \Big(J^b\Big)_{mn} \equiv (B_k)^a{}_b (J^b)_{mn}. \tag{2.3.64}$$

This implies that the Quaternion-Kähler-like equation, restricted on a horizontal sub-manifold M_4, actually reduces to Quaternion-Kähler equation (with newer version of gauge field $\hat{A} + B$)

$$\nabla^{M_4} J^a = (\hat{A} + B)^a{}_b J^b = \Big((A_k)_I{}^J + R_k t_I{}^J\Big) (\sigma^a)^I{}_K (\sigma_b)^K{}_J \Big(J^b\Big)_{mn}. \tag{2.3.65}$$

Thus, we see that for generic auxiliary fields t_{IJ} and A_m, provided that the horizontal distribution can be globally integrated to a sub-manifold M_4, M_4 is actually a Quaternion-Kähler manifold. Of course, there are special combinations of t_{IJ} and A such that $\hat{A}+B$ vanish. In such case, M_4 is a HyperKähler manifold.

With the integrability condition satisfied, we see that M is now locally foliated by Quaternion-Kähler (or HyperKähler in special case) manifold. In particular, compact manifold M could be a direct product

$$\boxed{M = S^1 \times M_4,\ M_4 \text{ is Quaternion Kahler}}. \tag{2.3.66}$$

In view of the fact that there are only 2 compact smooth Quaternion-Kähler manifolds in $4d$, possible examples are $M = S^1 \times \mathbb{C}P^2,\ S^1 \times S^4$, where the vector field R is chosen to be the unit vector field along S^1, with gauge field A turned on on $\mathbb{C}P^2$ and S^4. There are more examples when M_4 is allowed to be non-compact or orbifolds.

Normal ACMS, Cosymplectic manifold and Kähler foliation

As mentioned above, there are 2 ways to define ACMS structure on M using the data coming from Killing spinors: with the nowhere-vanishing auxiliary field t_{IJ} or some other nowhere-vanishing section of $ad(P)$. In general the ACMS structure so defined does not have nice differential property. However, when some (rather strong) conditions are satisfied, the ACMS will behave nicer.

Let us focus on the case $t \neq 0$ and $(s^{-1}R, s^{-1}\kappa, \varphi_t)$ define ACMS on M.

One obtains

$$\mathcal{L}_R t\Theta = \frac{1}{2}\left(\nabla^A_R t^{IJ}\right)(\Theta_{IJ}) + s\nabla^p\left(\frac{1}{s}R_m\right)(t\Theta)_{np} dx^m \wedge dx^n. \tag{2.3.67}$$

Setting

$$\nabla^A_R t = 0,\ \nabla_m\left(s^{-1}R_n\right) = 0 \Leftrightarrow \nabla_m R_n \propto F_{mn} = 0, \tag{2.3.68}$$

one has $\mathcal{L}_R t\Theta = 0$ and hence $\mathcal{L}_{s^{-1}R}\varphi_t = 0$.

If, in additional to the above, one further imposes V to be horizontal and $\nabla^A t = 0$, then it is easy to see that the ACMS satisfies

$$\nabla\varphi_t = 0, \tag{2.3.69}$$

and hence it is cosymplectic. In this case, the Levi-civita connection ∇ on M respects the ACMS, the restriction of ∇ on the horizontal distribution is automatically a connection on TM_H.

Note that $\nabla R = 0$ implies that the horizontal distribution is locally integrable. Therefore, restricted on the integral sub-manifold, ∇ is the induced Levi-civita connection, φ_t is an almost complex structure which can be shown to have vanishing Nijenhuis tensor and hence actually a complex structure. It is parallel with respect to induced Levi-civita connection, hence is Kähler.

In summary, we see that

$$\nabla^A t^{IJ} = 0,\ F = 0,\ V = V_H = -t\Theta, \tag{2.3.70}$$

implies a cosymplectic ACMS (namely $\nabla\varphi_t = 0$), and M is locally foliated by 4d Kähler manifold, with the Kähler structure provided by φ_t.

Recall that we had conclusion that M is locally foliated by Quaternion-Kähler manifold in the previous subsection, for configuration $F_H = 0$, $V = V_H$. Suppose $M = M_4 \times S^1$ with a Reeb vector field R from a Killing spinor pointing along S^1, then we see that M_4 must be Quaternion-Kähler as well as Kähler. If M_4 is a smooth compact manifold, then this leaves only one possibility:

$$M = \mathbb{CP}^2 \times S^1. \tag{2.3.71}$$

Of course, for more general Reeb vector field pointing along other directions, one could have other possibilities of M_4.

Reducing to 4d

Finally let us point out the reduction of (2.1.10) to 4d already discussed in literatures[4][4]. Consider $M = M_4 \times S^1$ with spinor ξ_I and auxiliary fields independent on the S^1 coordinate. The 4d part of the Killing spinor equation reads

$$\nabla_\mu \xi_I = t_I{}^J \gamma_\mu \xi_J + \frac{1}{2} F_{\mu 5} \gamma^5 \xi_I + \frac{1}{2} V^{\nu 5} \gamma_{\mu\nu 5} \xi_I + \frac{1}{2} V^{\lambda\rho} \gamma_{\mu\lambda\rho} \xi_I + \frac{1}{2} F_{\mu\nu} \gamma^\nu \xi_I + (A_\mu)_I{}^J \xi_J, \tag{2.3.72}$$

and the S^1 part serves as direct constraints on auxiliary fields

$$\partial_5 \xi_I = t_I{}^J \xi_J + \frac{1}{2} F_{5\mu} \gamma^\mu \xi_I + \frac{1}{2} V^{\mu\nu} \gamma_{\mu\nu} \gamma_5 \xi_I + (A_5)_I{}^J \xi_J = 0. \tag{2.3.73}$$

There are now 2 different ways to reduce the equation, each gives rise to the Killing equation discussed in [4][4]. The involved vertical condition $V_H = F_H = 0$ and requirement $t = 0$ or $t_{IJ} \propto \epsilon_{IJ}$ indeed imply the Frobenius Integrability condition

$$d\kappa \wedge \kappa = 0, \tag{2.3.74}$$

which is necessary for M to be a product.

I. Reduction to [4]

Setting $t = A = F_{\mu\nu} = V_{\mu\nu} = 0$, namely F and V are both vertical 2-forms, the equation simplifies to

$$\begin{cases} \nabla_\mu \xi_I = \frac{1}{2} F_{\mu 5} \gamma^5 \xi_I + \frac{1}{2} V^{\nu 5} \gamma_{\mu\nu 5} \xi_I \\ \partial_5 \xi_I = F_{5\mu} \gamma^\mu \xi_I = 0 \end{cases}, \tag{2.3.75}$$

or written in terms of Weyl components $\xi_I = (\zeta_I, \tilde{\zeta}_I)$,

$$\begin{cases} \nabla_\mu \zeta_I = \frac{1}{2} F_{\mu 5} \zeta_I + \frac{1}{2} V^{\nu 5} \sigma_{\mu\nu} \zeta_I \\ \nabla_\mu \tilde{\zeta}_I = -\frac{1}{2} F_{\mu 5} \tilde{\zeta}_I - \frac{1}{2} V^{\nu 5} \tilde{\sigma}_{\mu\nu} \tilde{\zeta}_I \end{cases}, \tag{2.3.76}$$

with constraint on $F_{\mu 5}$

$$F_{5\mu} \tilde{\sigma}^\mu \zeta_I = 0, \;\; F_{5\mu} \sigma^\mu \tilde{\zeta}_I = 0. \tag{2.3.77}$$

Suppose we relax the reality condition on ξ and also F and V, and define new complex auxiliary vector fields A and V

$$\begin{cases} 2iA_\mu \equiv F_{\mu 5} - V_{\mu 5} = \partial_\mu a_5 - V_{\mu 5} \\ -2iV_\mu \equiv V_{\mu 5} \end{cases}, \tag{2.3.78}$$

then the above equation takes a familiar form

$$\begin{cases} \nabla_\mu \zeta_I = -i\left(V_\mu - A_\mu\right)\zeta_I - iV^\nu \sigma_{\mu\nu}\zeta_I \\ \nabla_\mu \tilde{\zeta}_I = i\left(V_\mu - A_\mu\right)\tilde{\zeta}_I + iV^\nu \tilde{\sigma}_{\mu\nu}\tilde{\zeta}_I \end{cases}, \tag{2.3.79}$$

which is just the Killing equations discussed in [4] for 2 separate pairs of chiral spinors $(\zeta_1, \tilde{\zeta}_1)$ and $(\zeta_2, \tilde{\zeta}_2)$. $V_{\mu 5}$ has to satisfy conservation condition $\nabla_\mu V^{\mu 5} = 0$, and $F_{\mu 5}$ is holomorphic w.r.t $J^I_{\mu\nu}$ and $\tilde{J}^I_{\mu\nu}$ if any of them is non-zero. The conservation condition on $V_{\mu 5}$ is equivalent to d^*-closed condition on vertical 2-form V

$$\nabla_\mu V^{\mu 5} = 0 \Leftrightarrow \nabla^m V_{mn} = 0 \Leftrightarrow d * V = 0. \tag{2.3.80}$$

Now that we choose not to impose reality condition on auxiliary fields, it is also fine for ξ_I to be non-sympletic-Majorana, hence ξ_1 and ξ_2 are now unrelated complex spinors, and one of the two can vanish. This then leads to different numbers of Killing spinor solutions in 4d, ranging from 1 to 4. In [4], the cases when M_4 admits 1, 2 and 4 supercharges are discussed in detail. Here we list a few points and discuss their 5d interpretation. More results can be obtained similarly.

<u>(1) 2 supercharges of the form $(\zeta, 0)$ and $(\eta, 0)$</u>: then assuming M_4 is compact, M_4 has to be a Hyperhermitian manifold up to global conformal transformation. Moreover, the auxiliary fields satisfy

- a) $V_\mu - A_\mu$ is closed 1-form.
- b) $\partial_\mu V_\nu - \partial_\nu V_\mu$ is anti-self-dual 2-form.

Condition a) is obviously satisfied by definition: $V_\mu - A_\mu \sim \partial_\mu a_5$ is obviously closed. The condition b) reads in 5d point of view

$$\iota_R dV = - * dV, \tag{2.3.81}$$

<u>(2) 2 supercharges of the form $(\zeta, 0)$ and $(0, \tilde{\zeta})$</u>: there are 2 commuting Killing vector on M_4, and hence M_4 is locally T^2-fibration over Riemann surface Σ. The auxiliary fields $V_{\mu 5}$ and $F_{\mu 5}$ are given in terms of $J_{\mu\nu}$ and $\tilde{J}_{\mu\nu}$.

<u>II. Reduction to [4]</u>

Setting $A = F_{\mu\nu} = V_{\mu\nu} = 0$, $\frac{1}{2}F_{\mu 5} = \frac{1}{2}V_{\mu 5} = \frac{i}{3}b_\mu$, $t = (i/6)MI_{2\times 2}$, one similarly obtains

$$\begin{cases} \nabla_\mu \zeta_I = \frac{i}{6}M\tilde{\sigma}_\mu \tilde{\zeta}_I + \frac{i}{3}b_\mu \zeta_I + \frac{i}{3}b^\nu \sigma_{\mu\nu}\zeta_I \\ \nabla_\mu \tilde{\zeta}_I = \frac{i}{6}M\sigma_\mu \zeta_I - \frac{i}{3}b_\mu \tilde{\zeta}_I - \frac{i}{3}b^\nu \tilde{\sigma}_{\mu\nu}\tilde{\zeta}_I \end{cases}, \tag{2.3.82}$$

which is the Killing spinor equation for 2 pairs of spinor $(\zeta_1, \tilde{\zeta}_1)$ and $(\zeta_2, \tilde{\zeta}_2)$ discussed in [4] for but with condition $M = \tilde{M}$.

Again, ξ_I are no longer symplectic Majorana, and solution of the 5d Killing spinor equation leads to different number of solutions to 4d Killing spinor equation. Let us list a few examples from the detail discussion in [4]. Interested reader can refer to their paper for more results.

(1) 1 supercharge of the form $(\zeta, \tilde{\zeta})$: Any manifold (M_4, g) with a nowhere-vanishing complex Killing vector field K which squares to zero and commutes with its complex conjugate

$$K_\mu K^\mu = 0,\ [K, \bar{K}] = 0, \tag{2.3.83}$$

admits solution of the form $(\zeta, \tilde{\zeta})$ to the 4d Killing spinor equation. K and the metric can be used to build up a Hermitian structure on M_4.

(2) 2 supercharges of the form $(\zeta_1, 0)$ and $(\zeta_2, 0)$: M_4 is anti-self-dual with $V_{5\mu}$ and $F_{5\mu}$ closed 1-forms, and hence in 5d point of view, they are closed vertical 2-forms. Moreover, the form of solution requires $\tilde{M} = 0$, and according to our reduction, $M = \tilde{M} = 0$. If $F = V$ are exact, then M_4 is locally conformal to a Calabi-Yau 2-fold. Otherwise, M_4 is locally conformal to $\mathbb{H}^3 \times \mathbb{R}$.

(3) 2 supercharges of the form $(\zeta_1, 0)$ and $(0, \tilde{\zeta}_2)$: One must have $M = \tilde{M} = 0$. This is equivalent to M_4 having solution $(\zeta_1, \tilde{\zeta}_2)$ with $M = \tilde{M} = 0$.

2.3.2 Manifolds admitting 2 pairs of supercharges

In this section we consider the case when 2 pairs of solutions to the (2.1.10) exist. We see that when certain assumptions on vectors Q_{IJ} are made, and if the Killing vector fields form closed algebra, the geometry of M will be heavily constrained. And in particular, all the resulting geometries admit contact metric structures.

The spinors ξ and $\tilde{\xi}$ satisfy equations:

$$\begin{aligned} \nabla_m \xi_I &= t_I{}^J \Gamma_m \xi_J + \frac{1}{2} V^{pq} \Gamma_{mpq} \xi_I + \frac{1}{2} F_{mp} \Gamma^n \xi_I + (A_m)_I{}^J \xi_J \\ \nabla_m \tilde{\xi}_I &= t_I{}^J \Gamma_m \tilde{\xi}_J + \frac{1}{2} V^{pq} \Gamma_{mpq} \tilde{\xi}_I + \frac{1}{2} F_{mn} \Gamma^n \tilde{\xi}_I + (A_m)_I{}^J \tilde{\xi}_J \end{aligned} . \tag{2.3.84}$$

Similar to the previous section, we have

- Derivative on u_{IJ}

  (1)
$$u^{IJ} du_{IJ} = \hat{u}^{IJ} d\hat{u}_{IJ} + \frac{1}{2} u du = -2 t^{IJ} (\hat{u}\hat{\tau})_{IJ} - \iota_{(uQ)} F. \tag{2.3.85}$$

  (2)
$$du = -\iota_Q F. \tag{2.3.86}$$

- Derivative on Q_{IJ}

$$\boxed{\nabla_m Q_n + \nabla_n Q_m = 0}., \tag{2.3.87}$$

namely, Q is a Killing vector.

The derivative on u_{IJ} implies relation

$$2u_{IJ}u^{IJ} = s\tilde{s} + C, \tag{2.3.88}$$

where the function C is invariant along R and $\tilde{R}$. When $t_{IJ} = 0$, C reduces to constant. Notice that when $C = 0$,

$$s\tilde{R} = \tilde{s}R, \tag{2.3.89}$$

and when $C = -s\tilde{s}$

$$\tilde{s}R = -s\tilde{R}, \tag{2.3.90}$$

which are degenerate cases that we do not consider in the following.

- Commutator between R and $\tilde{R}$

$$K \equiv [R, \tilde{R}]^m = 8\,(t\hat{u})\,Q^m - 8u(t\hat{Q})^m - 4\big(\iota_R\iota_{\tilde{R}} * V\big)^m + \big(\tilde{s}\iota_R F - s\iota_{\tilde{R}}F\big)^m. \tag{2.3.91}$$

Recall that we now have several Killing vector fields, R, $\tilde{R}$, K and Q. If some of them form closed Lie algebra, the geometry of M will be heavily constrained. In the rest of this section, we discuss several simplest possibilities where they form 2 or 3 dimensional Lie algebras.

1. R and $\tilde{R}$ form 2-dimensional algebra

There exist only two 2-dimensional Lie algebras up to isomorphisms. One is the abelian algebra, the other is a unique non-abeilian algebra.

When R and $\tilde{R}$ commute, namely $K = 0$, one obtains the abelian algebra. If the orbits of R and $\tilde{R}$ are closed, then M is acted freely by T^2, and therefore M is a T^2-fibration.

The non-abelian algebra corresponds to $K \neq 0$. Assume K is a linear combination of R and $\tilde{R}$, then

$$[R, \tilde{R}] = aR + b\tilde{R}. \tag{2.3.92}$$

Contracting with R and $\tilde{R}$ it gives

$$\begin{cases} as^2 + b\,(s\tilde{s} + 2C) = s\iota_{\tilde{R}}\iota_R F \\ a\,(s\tilde{s} + 2C) + b\tilde{s}^2 = \tilde{s}\iota_{\tilde{R}}\iota_R F \end{cases}. \tag{2.3.93}$$

The determinant of the system is

$$\det = s^2\tilde{s}^2 - (s\tilde{s} + 2C)^2 = -4C\,(C + s\tilde{s})\,. \tag{2.3.94}$$

Notice that away from the degenerate cases when $C = 0$ and $C = -s\tilde{s}$, the determinate is non-zero. Therefore, when $\iota_R\iota_{\tilde{R}}F \neq 0$, the system allows solution (a, b)

$$\begin{cases} b = \dfrac{s\iota_{\tilde{R}}\iota_R F}{2\,(s\tilde{s} + C)} \\ a = \dfrac{\tilde{s}\iota_{\tilde{R}}\iota_R F}{2\,(s\tilde{s} + C)} \end{cases}. \tag{2.3.95}$$

Notice however that R, $\tilde{R}$ and their commutator are all Killing vectors, therefore the coefficients a and b must be constant. This implies

$$\frac{s}{\tilde{s}} = \text{const}, \tag{2.3.96}$$

and further

$$\mathcal{L}_R \tilde{s} = \mathcal{L}_{\tilde{R}} s = 0 \Rightarrow \iota_R \iota_{\tilde{R}} F = 0, \tag{2.3.97}$$

hence

$$a = b = 0. \tag{2.3.98}$$

To summarize, if R and $\tilde{R}$ form 2-dimensional algebra, it can only be trivial abelian algebra.

What remains is the Killing vector Q. Assume Q and the commuting R and $\tilde{R}$ form 3 dimensional algebra:

$$\begin{cases} [R, \tilde{R}] = 0 \\ [Q, R] = aR + b\tilde{R} + mQ \\ [Q, \tilde{R}] = cR + d\tilde{R} + nQ \end{cases}. \tag{2.3.99}$$

Let us make a Weyl rescaling to set $\iota_R F = 0$. Then it automatically implies $\iota_R \iota_Q F = \iota_{\tilde{R}} \iota_Q F = 0$ by previous arguments. Therefore,

$$\begin{cases} \mathcal{L}_R (u\tilde{s}) = \mathcal{L}_R(\tilde{R} \cdot Q) = \tilde{R} \cdot [R, Q] = 0 \\ \mathcal{L}_{\tilde{R}} (us) = \mathcal{L}_{\tilde{R}} (R \cdot Q) = R \cdot [\tilde{R}, Q] = 0 \end{cases}. \tag{2.3.100}$$

It is immediate to see that the determinant of the above linear system is

$$\det \propto |s\tilde{R} - \tilde{s}R|^2 |Q|^2, \tag{2.3.101}$$

and hence non-trivial solution requires $Q = 0$ or $\tilde{s}R = s\tilde{R}$, which we do not consider. Therefore, one has $Q, R, \tilde{R}$ forming abelian algebra, and M is a T^3-fibration over Riemann surface Σ. Up to an overall rescaling factor, the metric can be written as

$$ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + \sum_{i=1}^{3} (d\theta_i + \alpha_i(x))^2, \tag{2.3.102}$$

where θ_i are the periodic coordinates along R, $\tilde{R}$ and Q provided their orbits are closed, and α_i are 1-forms that determine the fibration.

2. R, $\tilde{R}$ and $[R, \tilde{R}]$ form 3-dimensional algebra

Assume that the algebra takes the form

$$\begin{cases} [R, \tilde{R}] = K \\ [R, K] = aR + b\tilde{R} + mK \\ [\tilde{R}, K] = cR + d\tilde{R} + nK \end{cases}. \tag{2.3.103}$$

In general, $\iota_R \iota_{\tilde{R}} F$ does not vanish. However, we can make a Weyl rescaling to make, for instance, $\iota_R F = 0$, and in particular, s is constant and $\iota_R \iota_{\tilde{R}} F = 0$. This implies

$$R \cdot K = \tilde{R} \cdot K = 0. \tag{2.3.104}$$

It is then easy to solve the coefficients in the above linear relation:

$$\begin{cases} a = -\dfrac{1}{4C}|K|^2 \dfrac{s\tilde{s} + 2C}{s\tilde{s} + C} \\ b = \dfrac{1}{4C}|K|^2 \dfrac{s^2}{s\tilde{s} + C} \end{cases}, \quad \begin{cases} c = -\dfrac{1}{4C}|K|^2 \dfrac{s^2}{s\tilde{s} + C} \\ d = \dfrac{1}{4C}|K|^2 \dfrac{s\tilde{s} + 2C}{s\tilde{s} + C} \end{cases}. \tag{2.3.105}$$

The fact that all coefficients must be constants implies

$$\frac{s}{\tilde{s}} = \text{const}, \quad \frac{s^2}{s\tilde{s} + 2C} = \text{const}, \tag{2.3.106}$$

and therefore both $\tilde{s}$ and C are constant as well.

It is then straight forward to renormalize and linearly recombine the vectors to form a standard $\mathfrak{su}(2)$ algebra. Therefore topologically M is a $SU(2)$-fibration over a Riemann surface Σ; however, there is no non-trivial $SU(2)$ bundle over a Riemann surface from the fact that the 3-skeleton of the classifying space $BSU(2)$ is a point), hence topologically $M = S^3 \times \Sigma$. Up to an overall scaling factor which was used to bring s to 1, the metric takes the form

$$ds_M^2 = ds_\Sigma^2 + ds_{S^3}^2 = h_{\alpha\beta}(x) dx^\alpha dx^\beta + \sum_{a=1}^{3} e^a e^a, \tag{2.3.107}$$

where $e^a = \kappa, \tilde{\kappa}, \gamma$ are $SU(2)$ invariant 1-forms on $SU(2)$. Note that $\iota_R F = \iota_{\tilde{R}} F = 0$ implies F is a form on Σ:

$$F = \frac{1}{2} F_{\alpha\beta}(x)\, dx^\alpha \wedge dx^\beta. \tag{2.3.108}$$

Recall that there is one more Killing vector field Q. The metric has isometry subgroup $SU(2)_L \times SU(2)_R$, which comes from the isometry of S^3. If $Q \notin \mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$, then Q must generate continuous isometry in Σ, which implies $\Sigma = T^2$ or S^2 if M is compact. In this case, by requiring Q commutes and being orthogonal to R, $\tilde{R}$ and K, one can derive new constraints on the auxiliary fields. For instance,

$$R \cdot Q = 0 \Leftrightarrow u = 0 \Leftrightarrow \iota_Q F = 0 \tag{2.3.109}$$

which, combining with the fact that F is a 2-form on Σ, implies actually $F = 0$.

2.3.3 Manifolds admitting 8 supercharges

In this section, we discuss the optimal case where the Killing spinor equation has full 4 complex dimensional space of solutions. This is done by taking the commutator of the ∇, applying Killing spinor equation and matching the Gamma matrix structure on both sides. We will see that there are 3 cases corresponding to the survival of only one of the 3 auxiliary fields (t, V, F), with the other two vanishes identically. Here we list main results that we will discuss in detail:

- $V \neq 0$: M is positively curved, with product structure $T^k \times G$ where G is a compact Lie group. The non-trivial example is then $T^2 \times SU(2)$ with standard bi-invariant metric.
- $F \neq 0$: M is locally of the form $M_3 \times \mathbb{H}^2$, where M_3 is a 3 dimensional flat manifold.
- $t \neq 0$: M is locally a space of constant curvature with positive scalar curvature, hence M is locally isometric to S^5.
- $t = V = F = 0$: M has zero curvature, hence is locally isometric to $\mathbb{R}^5$.

By explicitly writing down the commutator $[\nabla_m, \nabla_n]\xi_I$ using Killing spinor equation, one would obtain 2 immediate results:

- Terms independent of Γ.

$$\boxed{\left(W_I{}^J\right)_{mn} \equiv \nabla_m (A_n)_I{}^J - \nabla_n (A_m)_I{}^J + (A_n)_I{}^K (A_m)_K{}^J - (A_m)_I{}^K (A_n)_K{}^J = 0}. \tag{2.3.110}$$

  For simply-connected 5-manifolds, flat connections must be gauge equivalent to trivial connections.

- Terms linear in Γ.

$$\begin{aligned} 0 = &\left(\nabla_m t_I{}^J\right)\Gamma_n - \left[(A_m t)_I{}^J - (t A_m)_I{}^J\right]\Gamma_n \\ &+\frac{1}{2}(\nabla_m F_{np})\Gamma^p \delta_I{}^J - F_n{}^p (*V)_{mp}{}^q \Gamma_q \delta_I{}^J - 2t_I{}^J (*V)_{mn}{}^l \Gamma_l \ . \\ &-(m \leftrightarrow n) \end{aligned} \tag{2.3.111}$$

The solutions to the equation are:

Case 1

$$\begin{cases} t_{IJ} = 0 \\ F = 0 \end{cases} \tag{2.3.112}$$

Case 2

$$V = 0 \tag{2.3.113}$$

Now we study 2 cases separately.

<u>**Case 1**: Only $V \neq 0$.</u>

The solution $t^{IJ} = 0$ and $F = 0$ implies (2.3.111) vanishes identically, no further condition on V is required.

Combining with previous section, we know that

$$ds = 0, \tag{2.3.114}$$

and we conveniently set $s = 1$.

By identifying the terms quadratic in Γ matrices, one sees that the

- The curvature tensor satisfies a flat condition:

$$\hat{R}_{mnkl}(\hat{\nabla}) = 0, \tag{2.3.115}$$

where $\hat{R}$ is the curvature tensor of a metric connection with anti-symmetric torsion

$$\hat{\nabla}_m X^n = \nabla_m X^n + 2(*V)^n{}_{mk} X^k. \tag{2.3.116}$$

with ∇ the Levi-civita connection of g. This result is most easily understood by looking at the Killing spinor equation, where V can be absorbed into the metric connection as a totally anti-symmetric torsion.

- The Ricci curvature

$$Ric_{mn} = 4(*V)^{pq}{}_m (*V)_{pqn}. \tag{2.3.117}$$

- Scalar curvature

$$\mathcal{R} = +4(*V)^{kmn}(*V)_{kmn} \geq 0, \tag{2.3.118}$$

which indicates the manifold must have positive curvature. Moreover, compact manifolds admitting metric connection with anti-symmetric torsion are known to be products of $T^k \times G$ where G is a compact group. This leaves us only a few possibilities, the non-trivial one being

$$M = SU(2) \times T^2, \tag{2.3.119}$$

which has standard positive curvature.

<u>**Case 2**: $V = 0$</u>
Putting back $V = 0$ into (2.3.111), one has

$$\begin{cases} g_{nk}\left(\nabla^A_m t_I{}^J\right) - g_{mk}\left(\nabla^A_n t_I{}^J\right) = 0 \\ \left(\nabla_m F_{nk}\right) - \left(\nabla_n F_{mk}\right) = 0 \end{cases}. \tag{2.3.120}$$

These 2 condition implies covariant-constantness of t_{IJ} and F:

$$\boxed{\nabla^A_m t_I{}^J = 0,\ \nabla_k F_{mn} = 0}. \tag{2.3.121}$$

In particular,

$$d * F = 0, \quad dF = 0 \Leftrightarrow \Delta F = 0, \tag{2.3.122}$$

and 2nd/3rd Betti number is forced to be non-zero, if $F \neq 0$:

$$b^2 = b^3 \geq 1 \tag{2.3.123}$$

Compare the the terms quadratic in Γ, one obtains

$$\frac{1}{4} R_{mnpq} \Gamma^{pq} \delta_I{}^J = -2\left(t^2\right)_I{}^J \Gamma_{mn} - \frac{1}{2} F_{mp} F_{ns} \Gamma^{ps} \delta_I{}^J + \left[2 t_I{}^J F_{pm} \Gamma^p{}_n - (m \leftrightarrow n)\right]. \tag{2.3.124}$$

The solutions are

$$t_{IJ} = 0 \quad \text{or} \quad F = 0. \tag{2.3.125}$$

<u>i) $t = 0$ while $F \neq 0, t = 0$</u>:

- Riemann tensor

$$R_{mnkl} = F_{ml}F_{nk} - F_{mk}F_{nl}. \tag{2.3.126}$$

Note that the expression satisfies interchange symmetry automatically, while the 1st Bianchi identity implies

$$F_{m[l}F_{nk]} - F_{m[k}F_{nl]} = 0 \Rightarrow F \wedge F = 0. \tag{2.3.127}$$

- Ricci tensor

$$Ric_{mn} = F_{mk}F^{k}{}_{n}. \tag{2.3.128}$$

.

- Scalar curvature

$$\mathcal{R} = -F_{mn}F^{mn}, \tag{2.3.129}$$

which is negative definite if $F \neq 0$. Also note that F is covariantly constant, hence R_{mnnkl} is also covariantly constant.

Let's further constraint the form of curvature using the condition $F \wedge F = 0$. Noting that F_{mn} is a 5×5 antisymmetric matrix, we choose a coordinate where it takes block diagonal form:

$$F = F_{12}dx^1 \wedge dx^2 + F_{34}dx^3 \wedge dx^4. \tag{2.3.130}$$

Requiring that $F \wedge F = 0$ forces

$$F_{12}F_{34} = 0. \tag{2.3.131}$$

Assuming

$$F_{12} \neq 0, \tag{2.3.132}$$

with all other component zero, one arrives at a Riemann tensor with only one non-vanishing component:

$$R_{1212} = -(F_{12})^2 < 0. \tag{2.3.133}$$

Combining with the fact that F is parallel, this implies the 5-manifold M should locally be product manifold

$$M = T^3 \times \mathbb{H}^2, \tag{2.3.134}$$

where $F = F_{12}dx^1 \wedge dx^2$ serves as the volume form of $\mathbb{H}^2$.

The metric of M can be written as

$$ds^2 = ds^2_{T^3} + \frac{F_{12}}{y^2}\left(dx^2 + dy^2\right). \tag{2.3.135}$$

ii) The case where $t \neq 0, F = 0$

- Riemann tensor
$$R_{mnkl} = 2\mathrm{tr}\left(t^2\right)\left(g_{ml}g_{nk} - g_{mk}g_{nl}\right), \tag{2.3.136}$$
where interchange symmetry and first Bianchi identity are automatically satisfies.

  The second Bianchi identity forces $\mathrm{tr}(t^2)$ to be constant. The form of curvature implies that M is a space of constant curvature, and therefore it must be locally isometric to S^5. This corresponds to the well-known fact that maximal number of solutions to the well-known Killing spinor equation can only be achieved on round S^5.

2.4 Supersymmetric Theory for Vector Multiplet

In section 4, we analyzed many properties of the proposed Killing spinor equation (2.1.10 from supergravity, and discussed some necessary geometric conditions on the underlying manifold for solutions to exist.

In this section, we propose a slightly generalized version of the supersymmetric theory for $\mathcal{N} = 1$ vector multiplet. It is not the most general one, as there are other known examples (constructed by dimensional reduction from $6d$, for instance) in recent literatures that does not completely fit in the following discussion.

Let us consider a simplified Killing spinor equation, where we set $F = 2V \equiv \mathcal{F}$ in (2.1.10)
$$D_m\xi_I = t_I{}^J\Gamma_m\xi_J + \frac{1}{4}\mathcal{F}^{pq}\Gamma_{mpq}\xi_I + \frac{1}{2}\mathcal{F}_{mn}\Gamma^n\xi_I. \tag{2.4.1}$$
D_m contains Leve-civita connection, spin connection, gauge field A_m from the vector multiplet and background $SU(2)$-gauge field $A_I{}^J$, depending on the objects it acts on. The change of notation to $\mathcal{F}_{mn}$ is to avoid confusion with the field strength of $\mathcal{N} = 1$ gauge field
$$F_{mn} \equiv \nabla_m A_n - \nabla_n A_m - i\left[A_m, A_n\right]. \tag{2.4.2}$$
We propose a supersymmetry transformation of $\mathcal{N} = 1$ vector multiplet with parameter ξ being solution to the (2.4.1) is
$$\begin{cases} \delta_\xi A_m = i\epsilon^{IJ}\left(\xi_I\Gamma_m\lambda_J\right) \\ \delta_\xi\phi = i\epsilon^{IJ}\left(\xi_I\lambda_J\right) \\ \delta_\xi\lambda_I = -\frac{1}{2}F_{mn}\Gamma^{mn}\xi_I + \left(D_m\phi\right)\Gamma^m\xi_I + \epsilon^{JK}\xi_J D_{KI} + 2\phi t_I{}^J\xi_J + \frac{1}{2}\phi\mathcal{F}^{pq}\Gamma_{pq}\xi_I \\ \delta_\xi D_{IJ} = -i\left(\xi_I\Gamma^m D_m\lambda_J\right) + \left[\phi, \left(\xi_I\lambda_J\right)\right] + it_I{}^K\left(\xi_K\lambda_J\right) - \frac{i}{4}\mathcal{F}^{pq}\left(\xi_I\Gamma_{pq}\lambda_J\right) + \left(I \leftrightarrow J\right) \end{cases}. \tag{2.4.3}$$
Using previous results we obtain
$$d\kappa = -4\left(t\Theta\right) + 2sV_V,\ \mathcal{F} = -\frac{2}{s}\left(t\Theta\right) + \frac{2}{s}\Omega^- + V_V, \tag{2.4.4}$$
with V_V denoting the vertical part of field V.

As discussed in an earlier remark, the above equation implies that κ is a contact structure

$$\kappa \wedge d\kappa \wedge d\kappa \neq 0. \tag{2.4.5}$$

Applying Weyl rescaling symmetry, one can eliminate V_V and set $s = 1$. The Reeb vector field is then compatible with the contact structure κ:

$$\iota_R \kappa = 1, \iota_R d\kappa = 0. \tag{2.4.6}$$

Combining with the fact that R is a Killing vector field, the structure (κ, R, g) is actually a K-contact structure.

For simplicity let us consider a special case where

$$\mathcal{F} = d\kappa, \tag{2.4.7}$$

namely $\Omega^- = 1/4 d\kappa$.

Then it is straight forward to prove that the following Lagrangian $S(\kappa, g)$ is invariant under (2.4.3):

$$\begin{aligned} S = \int_M \operatorname{tr} \Big[& F \wedge *F - \kappa \wedge F \wedge F - d_A\phi \wedge *d_A\phi - \frac{1}{2} D_{IJ} D^{IJ} + i\lambda_I \not{D}_A \lambda^I - \lambda_I \left[\phi, \lambda^I\right] \\ & -it^{IJ} \left(\lambda_I \lambda_J\right) + 2\phi t^{IJ} D_{IJ} + \frac{i}{2} \nabla_m \kappa_n \left(\lambda_I \Gamma^{mn} \lambda^I\right) + 2\phi F \wedge *d\kappa + \frac{1}{4} \mathcal{R} \phi^2 \Big] \end{aligned} . \tag{2.4.8}$$

where $\mathcal{R}$ is the scalar curvature of the manifold.

As already mentioned, in the explicit form (2.4.8) we took the choice to assume $\Omega^- = (1/4)d\kappa$, which is in fact a special case of a large family of supersymmetric theories in the following sense.

Under supersymmetry (2.4.3) with ξ satisfying (2.4.1) without imposing $\Omega^- = (1/4)d\kappa$, the Lagrangian without $\kappa \wedge F \wedge F$ has variation

$$\frac{i}{2} \mathcal{F}_{mn} F_{pq} \left(\xi_I \Gamma^{mnpq} \lambda^I\right) \tag{2.4.9}$$

Such term can be identified in two ways. If we assume $\mathcal{F}$ is not only closed, but also exact

$$\mathcal{F} = d\mathcal{A} = \frac{1}{2} d\kappa + 2\Omega^- \tag{2.4.10}$$

for some 1-form $\mathcal{A}$, then the term can be identified as variation of

$$\mathcal{A} \wedge F \wedge F \tag{2.4.11}$$

In such case, the theory is specified by κ and $\mathcal{A}$.

However, if we do not assume anything of $\mathcal{F}$, then the term can also be identified as variation of

$$\mathcal{F} \wedge \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \tag{2.4.12}$$

In such case, the theory is specified by nowhere-vanishing 1-form κ and a closed anti-self-dual 2-form Ω^-, although the gauge invariance is not nicely manifested.

Following an analysis similar to that in [14], one can add to the Lagrangian (2.4.8) a δ-exact term $\delta_\xi V$ with

$$V = \mathrm{tr}\left((\delta_\xi \lambda)^\dagger \lambda\right). \tag{2.4.13}$$

Then the localization locus is

$$F_H^- = \phi d\kappa, \;\; \iota_R F = 0, \;\; d_A \phi = 0, \;\; D_{IJ} + 2\phi t_{IJ} = 0 \tag{2.4.14}$$

For general Ω^-, the first equation will take a more general form

$$F_H^- = \phi d\kappa + \phi \Omega_H^-. \tag{2.4.15}$$

This localization locus is a generalization of the contact instanton in [12].

It would be interesting to perform a complete localization for the theory (2.4.8) with the above localization locus, which we leave for future study.

2.5 Discussion

So far we have obtained many constraints on geometry of M imposed by the existence of supercharges. For 1 pair of supercharges, generically M must be almost contact manifold, and using the compatible connection, the Killing spinor equation can be simplified to a compact form. We also discussed a few interesting cases related to product manifold.d, which leads to special foliation and reduction to known 4-dimensional Killing spinor equations. The presence of 2 pairs supercharges with 2 additional assumptions restricts the isometry algebra of M, forcing M to be S^3 or T^3-fibration over Riemann surfaces. The presence of 4 pairs of supercharges allows for only 3 major possibilities, where the corresponding topologies and geometries are basically fixed.

There are several problems that are interesting to explore further.

(1) We obtained necessary conditions for supercharges to exist, but not sufficient conditions. In 3 dimension[9], the general solution to Killing spinor equation is obtained from the special coordinate, which requires some integrability of the almost contact structure. However, we do not have such integrability for the almost contact structure we defined, partly because the definition involves auxiliary field t_{IJ}, and the differential property of t_{IJ} is not known at priori. Moreover, in the extreme case where $t_{IJ} = 0$, it is not obvious that M is still a almost contact manifold. Perhaps it is possible to define almost contact structure of M without referring to t_{IJ}.

(2) We partially solved the auxiliary fields, but not all: gauge field A and t_{IJ} are entangled together. If t_{IJ} and A could be solved in terms of pure bilinears separately, the first problem above can also be solved.

(3) In the discussions, we made a few assumptions and simplifications. For examples, we did not study all possible bilinears formed by all solutions, but focused on those formed by the representatives from each pair. One should be able to obtain more information of M by taking into account all of them. Also, to simplify computation we assumed $\Omega^- = 0$ in some discussions. It is straight-forward and interesting to reinstate general Ω^-, and understand its role in the almost contact metric structure.

(4) We start from Zucker's off-shell supergravity[1]. However, it is not coupled to matter fields, and hence one would not automatically obtain any supersymmetric theory for matter multiplets. Our analysis, in this sense, is far from enough to obtain a complete picture. A next step one could try is to start from known 5-dimensional off-shell supergravity coupled with matter and take the rigid limit. For instance, one can start with $\mathcal{N}=1$ supergravity coupled to Yang-Mills matters in [15, 16], which was considered in [14]. After turning on auxiliary fields t_{IJ} and V_{mn}, the Killing spinor equation involved is then

$$\nabla_m \xi_I = t_I{}^J \Gamma_m \xi_J + \frac{1}{2} V_{mpq} \Gamma^{pq} \xi_I, \tag{2.5.1}$$

which is a special case of our more general equation.

2.6 Examples

In this section, we present simple explicit examples that illustrate some of the discussion before, by solving Killing spinor equations on selected manifolds and determining the auxiliary fields.

2.6.1 $M = S^1 \times S^4$

In earlier discussion, we discussed the possibility of having $M = S^1 \times N$ with N a 4d Quaternion-Kähler manifold. In this section, we consider the case where $N = S^4$.

Denote the coordinate along S^1 to be θ, x^μ are stereo-projection coordinates on S^4. The metric of $S^1 \times S^4$ is simply

$$ds^2 = d\theta^2 + \frac{\delta_{\mu\nu} dx^\mu dx^\nu}{(1+r^2)^2} \tag{2.6.1}$$

with function $r^2 = \sum_{\mu=1}^{4} (x^\mu)^2$

As discussed before, we partially fix the auxiliary fields

$$F = 0, \; V = t\Theta \tag{2.6.2}$$

However, non-zero $t\Theta$ will generate globally defined almost complex structure on S^4, which we already know does not exist, hence we can set $t = 0$ and $V = 0$. The only auxiliary fields allowed is thus $SU(2)$ gauge field A.

The Killing spinor equation (2.3.31) now reads

$$\begin{cases} \partial_\theta \xi_I = (\hat{A}_\theta)_I{}^J \xi_J \\ \nabla_\mu \xi_I = (\hat{A}_\mu)_I{}^J \xi_J \end{cases} \tag{2.6.3}$$

The gauge field A_μ is determined by the a choice of Quaternion-Kähler structure on S^4. Denoting

$$z_1 \equiv x^1 + ix^2, \; z_2 \equiv x^3 + ix^4 \tag{2.6.4}$$

one can define locally 3 almost complex structures as the basis,

$$\begin{cases} J^1 = \left(\frac{\partial}{\partial \overline{z_1}} \otimes dz_2 - \frac{\partial}{\partial \overline{z_2}} \otimes dz_1\right) + h.c. \\ J^2 = \frac{1}{i}\left(\frac{\partial}{\partial \overline{z_1}} \otimes dz_2 - \frac{\partial}{\partial \overline{z_2}} \otimes dz_1\right) + h.c. \\ J^3 = i\frac{\partial}{\partial z_i} \otimes dz_i - i\frac{\partial}{\partial \overline{z_i}} \otimes d\overline{z_i} \end{cases} \tag{2.6.5}$$

and determine the gauge field using (2.3.56).

We choose the Gamma matrices to be

$$\Gamma^i = \sigma^i \otimes \sigma^2,\ \Gamma^4 = I \otimes \sigma^1,\ \Gamma^5 = I \otimes \sigma_3,\ C = \Gamma^{13} \tag{2.6.6}$$

and the obvious vielbein

$$e^5 = -d\theta,\ e^a = \frac{1}{1+r^2}\delta^a_\mu dx^\mu \tag{2.6.7}$$

solution is given as

$$\xi_1 = e^{i\int A_\theta d\theta}\chi_+ \otimes \chi_-,\ \xi_2 = -e^{-i\int A_\theta d\theta}\chi_- \otimes \chi_- \tag{2.6.8}$$

2.6.2 $M = S^2 \times S^3$

Consider S^3 as a $U(1)$ bundle over S^2. Let S^3 be embedded into $\mathbb{C}^2$,

$$S^3 = \left\{|z_1|^2 + |z_2|^2 = 1|\,(z_i) \in \mathbb{C}^2\right\} \tag{2.6.9}$$

Similarly define

$$z_2 = \rho e^{i\theta},\ z \equiv \frac{z_1}{z_2} \Rightarrow \rho^2\big|_{S^3} = \frac{1}{1+|z|^2},\ z_1 = zz_2 = \rho e^{i\theta} z \tag{2.6.10}$$

and hence the induced round metric on S^3 can be written as

$$\begin{aligned} ds^2 = dz_1 d\overline{z_1} + dz_2 d\overline{z_2} &= \left[d\theta + i\frac{zd\bar{z} - \bar{z}dz}{2\left(1+|z|^2\right)}\right]^2 + \frac{dzd\bar{z}}{\left(1+|z|^2\right)^2} \\ = (d\theta + a)^2 + g^1 \end{aligned} \tag{2.6.11}$$

where

$$g^1 = \frac{dzd\bar{z}}{\left(1+|z|^2\right)^2} \tag{2.6.12}$$

is the metric on $\mathbb{C}P^1 = S^2$ with radius 1/2. In coordinate,

$$g^1_{z\bar{z}} = g^1_{\bar{z}z} = \frac{1}{2\left(1+|z|^2\right)^2} = \frac{1}{2}\partial_z\partial_{\bar{z}}\ln(1+|z|^2) \equiv \partial_z\partial_{\bar{z}}K \tag{2.6.13}$$

and

$$. \tag{2.6.14}$$

The vector field $R \equiv \partial_\theta$ is a Killing vector field, and its dual is $\kappa = d\theta + A$, such that $\iota_R \kappa = 1$.

Define the frame on S^3 to be

$$e^3 \equiv e^\theta = \kappa,\ e^1 = \frac{\mathrm{Re} dz}{1+|z|^2},\ \ e^2 = \frac{\mathrm{Im} dz}{1+|z|^2} \tag{2.6.15}$$

s.t.

$$g = e^\theta e^\theta + e^1 e^1 + e^2 e^2 \tag{2.6.16}$$

then it is obvious that

$$\omega_{\theta ab} = 0, a, b = \theta, 1, 2 \tag{2.6.17}$$

from the fact

$$de^\theta \sim \frac{idz \wedge d\bar{z}}{\left(1+|z|^2\right)^2} \tag{2.6.18}$$

The base manifold $S^2 \times S^2$ is complex, with natural complex structure and Kähler form. Setting the radius of the stand-alone S^2 to be l, with local complex coordinate w, the metric of $S^3 \times S^2$ reads

$$g = (d\theta + a)^2 + \frac{dzd\bar{z}}{\left(1+|z|^2\right)^2} + \frac{4l^2 dwd\bar{w}}{\left(1+|w|^2\right)^2} \tag{2.6.19}$$

with Kähler form on base manifold

$$\omega = \frac{idz \wedge d\bar{z}}{2\left(1+|z|^2\right)^2} + \frac{i4l^2 dw \wedge d\bar{w}}{2\left(1+|w|^2\right)^2} \tag{2.6.20}$$

or in components

$$\omega_{z\bar{z}} = -\omega_{\bar{z}z} = \frac{i}{2\left(1+|z|^2\right)^2},\ \omega_{w\bar{w}} = -\omega_{\bar{w}w} = ig_{w\bar{w}} = \frac{il}{2\left(1+|z)^2\right)^2} \tag{2.6.21}$$

The 2 complex structures on both $\mathbb{C}P^1$ can form linear combination

$$\varphi_\pm \equiv J_1 \pm J_2 \tag{2.6.22}$$

which satisfies

$$\varphi_\pm^2 = -1 + R \otimes \kappa \tag{2.6.23}$$

Let us now construct the auxiliary fields. We choose t_{IJ} such that $\mathrm{tr}\left(t^2\right) = -\frac{1}{2}$, and therefore

$$4(t\Theta)^2 \sim -1 + ... \tag{2.6.24}$$

We identify a combination of the 2 complex structures on 2 $\mathbb{C}P^1$ as $t\Theta$. Recall that $t\Theta$ also satisfies $\iota_R * (t\Theta) = -(t\Theta)$, hence we identify

$$\varphi_- \sim 2\,(t\Theta) \tag{2.6.25}$$

or a 2-form equation

$$2\,(t\Theta) = \frac{idz\wedge d\bar{z}}{2\Big(1+|z|^2\Big)^2} - \frac{i4l^2dw\wedge d\bar{w}}{2\Big(1+|w|^2\Big)^2} \tag{2.6.26}$$

Then we obtain F and V:

$$F = \frac{1}{2}d\kappa = \frac{idz\wedge d\bar{z}}{2\Big(1+|z|^2\Big)^2} \tag{2.6.27}$$

and

$$V = t\Theta = \frac{idz\wedge d\bar{z}}{4\Big(1+|z|^2\Big)^2} - \frac{il^2dw\wedge d\bar{w}}{\Big(1+|w|^2\Big)^2} \tag{2.6.28}$$

With these auxiliary fields, one can solve the Killing spinor equation

$$\hat{\nabla}_m\hat{\xi}_I = (\hat{A}_m\ )_I{}^J\hat{\xi}_J \tag{2.6.29}$$

Denote $\alpha = w,\bar{w}$, and $\mu,\nu = z,\bar{z}$, we have

$$\begin{cases} \nabla_\alpha\xi_I = (A_\alpha)_I{}^J\xi_J \\ \nabla_\mu\xi_I - \dfrac{1}{2}\left(\nabla_\mu R_\nu\right)\Gamma^\nu\xi_I = (\hat{A}_\mu)_I{}^J\xi_J \\ \nabla_\theta\xi_I = (\hat{A}_\theta)_I{}^J\xi_J \end{cases} \tag{2.6.30}$$

where

$$R_\theta = 1,\ R_z = \frac{1}{2}\frac{-i\bar{z}}{1+|z|^2} = -i\partial_z K,\ R_{\bar{z}} = \frac{1}{2}\frac{iz}{1+|z|^2} = i\bar{\partial}_{\bar{z}}K \tag{2.6.31}$$

and we used

$$\nabla_\mu R_\theta = \nabla_\theta R_\theta = 0 \tag{2.6.32}$$

Choosing gauge field to be $(A_m)_I^J = (A_m)(\sigma_3)_I^J$,

$$iA_z = \frac{\bar{z}}{4\left(1+|z|^2\right)},\ \ iA_{\bar{z}} = -\frac{z}{4\left(1+|z|^2\right)}, A_\theta = -\frac{1}{4} \tag{2.6.33}$$

and representation of Gamma matrices

$$\Gamma_{w,\bar{w}} \sim \sigma_{1,2}\otimes 1,\ \Gamma_{z,\bar{z},\theta} \sim \sigma_3\otimes\sigma_{1,2,3} \tag{2.6.34}$$

one obtains the chiral solution (ξ_2 is obtained from symplectic Majorana condition)

$$\xi_1 = e^{-\frac{i}{4}\theta}\chi_+\otimes\chi_+ \tag{2.6.35}$$

The calculation can be easily generalized to $M = S^3\times\Sigma$ for Riemann surface Σ.

Chapter 3

5d Supersymmetric Background and Transversal Holomorphic Structures

3.1 Introduction

This chapter focusses on the question that to what extend this notion and use of holomorphy can be extended to general five-dimensional backgrounds admitting rigid $\mathcal{N}=1$ supersymmetry. Our analysis is based on the gravitino and dilatino equations of [16, 15] which in our conventions and in Euclidean signature are

$$D_m \xi_I = t_I{}^J \Gamma_m \xi_J + \mathcal{F}_{mn} \Gamma^n \xi_I + \frac{1}{2} \mathcal{V}^{pq} \Gamma_{mpq} \xi_I \tag{3.1.1}$$

and

$$4\left[\left(D_m t_I{}^J\right)\Gamma^m + t_I{}^J (\mathcal{F}+2\mathcal{V})_{mn}\Gamma^{mn}\right]\xi_J + \Big(4\nabla_m \mathcal{V}^{mn}\Gamma_n + \mathcal{F}_{mn}\mathcal{F}_{kl}\Gamma^{mnkl} + C\Big)\xi_I = 0 \tag{3.1.2}$$

Here, $I = 1,2$ are indices for the fundamental representation of $SU(2)_\mathcal{R}$. $\mathcal{F} = d\mathcal{A}$ is a $U(1)$ field strength and $\mathcal{V}$ an antisymmetric tensor. The triplet $t_I{}^J$ is valued in the adjoint representation of $SU(2)_\mathcal{R}$. The covariant derivatives are $D_m\xi_I = \nabla_m \xi_I - A_{mI}{}^J \xi_J$ and $D_m t_I{}^J = \nabla_m t_I{}^J - [A_m, t]_I{}^J$. For later convenience, note that (3.1.1) can also be rewritten as

$$D_m \xi_I = \Gamma_m \tilde{\xi}_I + \frac{1}{2}\left(\mathcal{V}^{pq} - \mathcal{F}^{pq}\right)\Gamma_{mpq}\xi_I, \quad \tilde{\xi}_I = t_I{}^J \xi_J + \frac{1}{2}\mathcal{F}_{mn}\Gamma^{mn}\xi_I. \tag{3.1.3}$$

In the Lorentzian theory, the spinors ξ_I satisfy a symplectic Majorana condition (2.2.6). Transitioning to the Euclidean theory one usually drops such reality conditions and effectively doubles the degrees of freedom of all fields involved. In general, the spinor ξ_I defines a possibly complex vector R. Imposing the reality condition (2.2.6) for ξ_I it follows that R is real and non-vanishing and that the tangent space decomposes as in (C.3.1), which one refers to as an *almost Cauchy-Riemann (CR) structure (of hypersurface type)*.

In opposite to the familiar case of almost complex structures, the integrability condition for (C.3.1) is not unique. Indeed, there are two possibilities. To begin, there is the case of a *integrable*

CR structure,

$$[T^{1,0}, T^{1,0}] \subseteq T^{1,0}, \tag{3.1.4}$$

that defines a *CR manifold*. CR manifolds have previously appeared in the context of the rigid limit of new minimal supergravity with Lorentzian signature in [17]; there, the authors found fibrations of the real line over three dimensional CR manifolds. Alternatively, there is the condition

$$[T^{1,0} \oplus R, T^{1,0} \oplus R] \subseteq T^{1,0} \oplus R, \tag{3.1.5}$$

which defines a *transversally holomorphic foliation (THF)*.[1] The work of [3] relates rigid supersymmetry in three dimensions with the existence of a THF. Note that Sasakian manifolds fulfill both (3.1.4) and (3.1.5) as here $[R_{\text{Sasakian}}, T^{1,0}] \subseteq T^{1,0}$.

Naturally, the question whether solutions to the Killing spinor equations (3.1.1) and (3.1.2) admit integrable CR structures or THFs is closely related to the question whether a given five dimensional manifold $\mathcal{M}$ admits any solution in the first place. As we alluded above, this question was already addressed in [20] and [21], but not exhaustively answered. As we will see, existence of a solution to the Killing spinor equations that satisfies the symplectic Majorana condition implies the existence of a globally non-vanishing Killing vector field parallel to R. We will show that the existence of such a Killing vector field is not only necessary, but also sufficient. While we will do so by directly constructing a single solution and arguing that there are no topological obstructions, one can already give a short argument why one should be able to expect this result. The existence of a non-vanishing vector field implies that $\mathcal{M}$ admits an $SO(4)$ structure. Since the theory has an $SU(2)_\mathcal{R}$ symmetry, one can perform an operation akin to a Witten twist in four dimensions and identify the the $SU(2)_\mathcal{R}$ with an $SU(2)$ factor inside the structure group.

The structure of this note is as follows: The relation between the supersymmetry spinor ξ_I, almost CR-structures and almost contact structures is the topic of section 3.2. Then, we will discuss the integrability of the Killing spinor equations, possible obstructions and general differential properties of (3.1.1) and (3.1.2) in section 3.3. Section 3.4 is concerned with the implications for localization. We will argue that the results of [22, 21] can be generalized to CR-manifolds and THFs. Subsequently we discuss the existence of globally well-defined solutions (section 3.5) before concluding with some examples from the literature in section 3.6. Various appendices complement the discussion.

During the final stages of this project [23] appeared, which has some overlap with our work. There, the authors study rigid supersymmetry on Riemannian five-manifolds using a holographic approach.

3.2 Algebraic Properties

In this section we will further discuss the algebraic structures arrising from the existence of the spinors ξ_I.

[1] For some background material on transversely holomorphic foliations, see e.g. [18, 19].

3.2.1 The Almost Contact Structure

Recall that we have a set of bi-spinors that can be defined for any given ξ_I:

$$s \equiv \epsilon^{IJ}\left(\xi_I\xi_J\right), \qquad R^m \equiv -s^{-1}\epsilon^{IJ}\left(\xi_I\Gamma^m\xi_J\right) \equiv g^{mn}\kappa_n, \qquad (\Theta_{IJ})_{mn} \equiv \left(\xi_I\Gamma_{mn}\xi_J\right). \tag{3.2.1}$$

Let us emphasize that we have included a minus sign as well as the normalizing factor s^{-1} in the definition of R where we tacitly assume that $s \neq 0$. If one imposes the symplectic Majorana condition (2.2.6) one finds that s and R are a real function and a real vector field respectively. Moreover, $s \geq 0$ with equality if and only if $\xi_I = 0$. It follows that $s > 0$ everywhere on M since the gravitino equation is linear and of first order. Finally, the two forms Θ_{IJ} lie in the adjoint representation of $SU(2)_{\mathcal{R}}$.

Using Fierz-identities, one can show the following identities involving the bispinors:

$$1 = \iota_R\kappa, \qquad 0 = \iota_R\Theta_{IJ}, \qquad \iota_R * \Theta_{IJ} = \Theta_{IJ}, \qquad \star\Theta_{IJ} = \kappa\wedge\Theta_{IJ}, \qquad R^m\gamma_m\xi_I = -\xi_I. \tag{3.2.2}$$

Here, $*$ is the usual five-dimensional Hodge dual and ι_R denotes interior multiplication. The first of the above equations tells us that M carries an $SO(4)$ structure. This allows us to introduce a lot of structure that is familiar from four-dimensional geometry. As is usual, we will refer to vectors and forms parallel to R and κ respectively as vertical and their orthogonal complement as horizontal. I.e. forms can be decomposed as $\omega = \omega_H + \omega_V$. Then the Hodge dual defines the notion of self-dual and anti self-dual forms on the horizontal subspace, as discussed around 2.2.26. Since the Θ_{IJ} are both horizontal and self-dual, $\Theta_{IJ} = (\Theta_{IJ})^+$, they define an isomorphism between $su(2)_{\mathcal{R}}$ and the $su(2)_+$ factor in the typical $so(4) \cong su(2)_+ \times su(2)_-$ decomposition of the Lie algebra of the structure group. One can also verify some more involved identities involving Θ_{IJ}:

$$\begin{aligned}\Theta_{IJmp}\Theta_{KL}{}^{pn} &= -\frac{1}{4}s^2(\epsilon_{IK}\epsilon_{JL} + \epsilon_{IL}\epsilon_{JK})\Pi_m{}^n + \frac{1}{4}s(\epsilon_{JK}\Theta_{ILm}{}^n + \epsilon_{IK}\Theta_{JLm}{}^n + \epsilon_{JL}\Theta_{IKm}{}^n + \epsilon_{IL}\Theta_{JKm}{}^n),\\ s^{-2}\Theta_{IJkl}\Theta^{IJ}_{mn} &= \frac{1}{2}\left(\Pi_{km}\Pi_{ln} - \Pi_{kn}\Pi_{lm} + \epsilon_{klmnp}R^p\right).\end{aligned} \tag{3.2.3}$$

Here $\Pi_{mn} = g_{mn} - \kappa_m\kappa_n$ and thus the latter of these is a projection to horizontal, self-dual two-forms.

Suppose now that $\mathfrak{m}_{IJ}$ is an $SU(2)_{\mathcal{R}}$ triplet. Later we will show that $\mathfrak{m}_{IJ} = t_{IJ}$ emerges naturally when imposing integrability and we will refer to this as the canonical choice. Yet for now, we continue with a generic $\mathfrak{m}_{IJ}$ and define[2] $\det\mathfrak{m} \equiv -1/2\sum_{IJ}\mathfrak{m}_I{}^J\mathfrak{m}_J{}^I$. Once we impose the reality condition (3.2.10) for $\mathfrak{m}_{IJ}$, $\det\mathfrak{m}$ will be positive semi-definite. For now we proceed with the milder assumption $\det\mathfrak{m} \neq 0$ and define the following tensor

$$\Phi_{mn} = (\Phi[\mathfrak{m}])_{mn} \equiv s^{-1}\sqrt{\frac{1}{\det\mathfrak{m}}}\,\mathfrak{m}^{IJ}(\Theta_{IJ})_{mn}. \tag{3.2.4}$$

[2] Note that

$$\sum_{IJ}\mathfrak{m}_I{}^J\mathfrak{m}_J{}^I = \mathfrak{m}_1{}^1\mathfrak{m}_1{}^1 + \mathfrak{m}_1{}^2\mathfrak{m}_2{}^1 + \mathfrak{m}_2{}^1\mathfrak{m}_1{}^2 + \mathfrak{m}_2{}^2\mathfrak{m}_2{}^2 = -2\mathfrak{m}_{11}\mathfrak{m}_{22} + 2\mathfrak{m}_{12}\mathfrak{m}_{21} = -2\det\mathfrak{m}_{\bullet\bullet}.$$

As follows from (3.2.3), Φ satisfies the following condition:

$$\Phi^m{}_k \Phi^k{}_n = -\delta^m_n + R^m \kappa_n. \tag{3.2.5}$$

Mathematicians refer to a multiplet (κ, R, Φ) as an *almost contact structure* if

$$\kappa_m R^m = 1, \quad \Phi^m{}_k \Phi^k{}_n = -\delta^m_n + R^m \kappa_n, \quad \Phi^m{}_n R^n = \kappa_n \Phi^n{}_m = 0 \tag{3.2.6}$$

As we have shown, the quantities defined using ξ_I and a suitable $\mathfrak{m}_{IJ}$ satisfy these relations, and therefore define an almost contact structure. Note that Φ is invariant under $\mathfrak{m}_{IJ} \mapsto f\mathfrak{m}_{IJ}$ for any non-zero function f.

3.2.2 The Almost CR Structure

Equations (3.2.5) and (3.2.6) indicate that for each $\mathfrak{m}$, $\Phi[\mathfrak{m}]$ defines an almost CR structure. Indeed, each $\Phi[\mathfrak{m}]$ induces a decomposition of the complexified horizontal tangent bundle (almost CR structure) as in appendix [C.3] via

$$X \in T^{1,0} \quad \Leftrightarrow \quad \Phi X = \imath X. \tag{3.2.7}$$

The decomposition holds also for the exterior algebra and all horizontal n-forms $\omega = \omega_H$ can be decomposed into (p, q)-forms via

$$\omega = \sum_{p+q=n} \omega^{p,q}. \tag{3.2.8}$$

In this context Φ_{mn} is a horizontal $(1, 1)$-form. Similar to the case of four-dimensional Kähler manifolds, self-dual and anti-self-dual 2-forms have a simple (p, q)-decomposition,

$$\omega^+ = \omega^{2,0} + \omega^{0,2} + \omega|_\Phi, \qquad \omega^- = \omega^{1,1}, \tag{3.2.9}$$

with $\omega^{1,1}$ primitive and thus annihilated by contraction with Φ.

We continue by discussing the integrability of the almost CR structure. While this can be done using a direct analysis of the Niejenhuis tensor, we prefer to do a spinorial analysis in the spirit of [9].[3] This is computationally more straight forward, yet requires us to impose the reality condition

$$\overline{\mathfrak{m}_{IJ}} = \epsilon^{II'} \epsilon^{JJ'} \mathfrak{m}_{I'J'}. \tag{3.2.10}$$

for the triplet which we alluded to previously. The bar denotes complex conjugation. Let us emphasize that we are also using the symplectic Majorana condition since we assume R to be real.

In appendix E.1 we show that one can characterize elements of $T^{1,0}$ in terms of a spinorial equation:

$$X \in T^{1,0} \quad \Leftrightarrow \quad X^m H_I{}^J \Gamma_m \xi_J = 0, \tag{3.2.11}$$

where

$$H_I{}^J = H_I{}^J[\mathfrak{m}] = \sqrt{\frac{1}{\det \mathfrak{m}}} \mathfrak{m}_I{}^J - i\delta^J_I. \tag{3.2.12}$$

[3] For a third possibility using differential forms orthogonal to $T^{1,0}$ or $T^{1,0} \oplus \mathbb{R}$ respectively see [17].

Similarly, one can also characterize the tangent vectors in $T^{1,0} \oplus \mathbb{R}R$ by the spinorial equation

$$X \in T^{1,0} \oplus R \quad \Leftrightarrow \quad (\Pi^m{}_n X^n)\, H_I{}^J \Gamma_m \xi_J = 0. \tag{3.2.13}$$

Recall that $\Pi^m{}_n = \delta^m_n - R^m \kappa_n$ is a projection that maps a generic tangent vector to its horizontal component.

3.3 Differential Properties

We finally turn to the integrability conditions for the decomposition (C.3.1). To do so, we will first establish some useful identities involving the bispinors (3.2.1), the(refGamma-rep-2)) and dilatino (3.1.2) variations. Subsequently we consider the case of CR structures as a warm-up before studying the integrability conditions for THFs.

3.3.1 Supersymmetry variations and bispinors

Recall that by studying the gravitino variation (3.1.1), one finds that the scalar s satisfies $\nabla_n s = 2sR^m \mathcal{F}_{mn}$, from which it follows that $\mathcal{L}_R \mathcal{F} = \mathcal{L}_R s = 0$. Similarly, the non-normalized vector field sR is Killing:

$$\nabla_m (sR_n) = 2\big(t^{IJ}\Theta_{IJ}\big)_{mn} - 2s\mathcal{F}_{mn} - 2s(\iota_R \star \mathcal{V})_{mn}, \quad \nabla_m (sR_n) + \nabla_n (sR_m) = 0. \tag{3.3.1}$$

One also finds that $\iota_R d\kappa = -s^{-1} ds$ while $\iota_R d(s\kappa) = -2ds$. There is a more involved relation involving the two-form $t^{IJ}\Theta_{IJ}$:

$$\begin{aligned}\nabla_k \big(t^{IJ}\Theta_{IJ}\big)_{mn} = {} & D_k t^{IJ}(\Theta_{IJ})_{mn} + 2\det t\,(g_{nk}R_m - g_{mk}R_n) + 2\mathcal{F}_{kp} t^{IJ}\left(\xi_I \Gamma_{mn}{}^p \xi_J\right) \\ & + g_{nk}\mathcal{V}^{pq} t^{IJ}\left(\xi_I \Gamma_{mpq}\xi_J\right) - 2\mathcal{V}_n{}^q t^{IJ}\left(\xi_I \Gamma_{mkq}\xi_J\right) \\ & - g_{mk}\mathcal{V}^{pq} t^{IJ}\left(\xi_I \Gamma_{npq}\xi_J\right) + 2\mathcal{V}_m{}^q t^{IJ}\left(\xi_I \Gamma_{nkq}\xi_J\right).\end{aligned} \tag{3.3.2}$$

Similarly we are interested in the consequences of the dilatino equation (3.1.2) for bispinors and background fields. By contraction with $t^{IJ}\xi_I$ one finds that $R^m \nabla_m (t_{IJ}t^{IJ}) = 2\mathcal{L}_R \det t = 0$. Contraction with ξ^I on the other hand fixes the value of the scalar,

$$C = 4\kappa_n \nabla_m \mathcal{V}^{mn} - 4s^{-1}(\mathcal{F} + 2\mathcal{V})_{mn}\big(t^{IJ}\Theta_{IJ}\big)^{mn} + 2(\iota_R * \mathcal{F})^{mn}\mathcal{F}_{mn}. \tag{3.3.3}$$

We can extract additional information from the dilatino equation and start by projecting it onto its "chiral" components. Recalling the last identity in (3.2.2) we consider the projector $\frac{1}{2}(1 - R^m\Gamma_m)$. Acting on (3.1.2) and using (3.3.3), one finds

$$0 = D_R t_I{}^J \xi_J + t_I{}^J R^l (\mathcal{F} + 2\mathcal{V})^{mn}\Gamma_{lmn}\xi_J + s^{-1}(\mathcal{F} + 2\mathcal{V})^{mn}(t^{KL}\Theta_{KL})_{mn}\xi_I. \tag{3.3.4}$$

A related identity can be obtained by contracting (3.1.2) with $\xi^I \Gamma_{mn}$ and projecting onto the horizontal subspace:

$$\left(R^k D_k t^{IJ}\right)(\Theta_{IJ})_{mn} - 2\left[(\mathcal{F} + 2\mathcal{V})^H \times (t^{IJ}\Theta_{IJ})\right]_{mn} = 0. \tag{3.3.5}$$

where $(\eta \times \omega)_{mn} = \eta_m{}^p \omega_{pn} - \omega_m{}^p \eta_{pn}$. In passing, one needs to use the simple identity

$$\Omega_{mn} = \left(\omega^+\right)_{nk} \left(\omega^-\right)^k{}_n - \left(\omega^+\right)_{nk} \left(\omega^-\right)^k{}_m = 0. \tag{3.3.6}$$

As a point of consistency note that one can obtain the same result by contracting (3.3.4) with $\xi^I \Gamma_{mn}$ and again projecting onto the horizontal part.

3.3.2 Integrability

Cauchy-Riemann structures

Having established the existence of the almost CR structure C.3.1, it is natural to ask if it satisfies any integrability condition. As a warm-up to the integrability condition of a THF (3.1.5), we consider the slightly simpler case of a CR structure (3.1.4).

Thus we study the condition (3.2.11) for the commutator $[X, Y]$ for arbitrary X, $Y \in T^{1,0}$. I.e. by acting with $Y^n D_n$ on (3.2.11) and antisymmetrizing in X, Y, one finds that

$$[X, Y] \in T^{1,0} \quad \Leftrightarrow \quad 0 = X^{[m} Y^{n]} \left[D_m H_I{}^J \Gamma_n \xi_J + H_I{}^J \Gamma_n D_m \xi_J \right]. \tag{3.3.7}$$

This reduces quickly to

$$X^{[m} Y^{n]} \left[D_m H_I{}^J \Gamma_n \xi_J - [H, t]_I{}^J \Gamma_{mn} \xi_J + 2 H_I{}^J (\mathcal{F} + \mathcal{V})_{mn} \xi_J \right]. \tag{3.3.8}$$

Per usual, (3.3.8) can be mapped to two equations by suitable contractions.

To begin, we contract (3.3.8) with ξ^I and find that $([H, t]_I{}^J \Theta_J{}^I)^{2,0} = 0$. Due to the reality conditions for ξ_I, $\mathfrak{m}_I{}^J$ and $t_I{}^J$ this means that $[H, t]_I{}^J \Theta_J{}^I \in \Omega^{1,1}$. This in turn is equivalent to $[H, t]_I{}^J$ being proportional to $\mathfrak{m}_I{}^J$. However, $[H, t]_I{}^J$ is proportional to $[\mathfrak{m}, t]_I{}^J$ and thus the only solution is $\mathfrak{m}_I{}^J = f t_I{}^J$ for any non-zero function f.

Being rid of the commutator term, we consider the contraction with ξ_J symmetrized over $SU(2)_R$ indices. This leads to $s H_{IJ} [X^{[m} Y^{n]} (\mathcal{F} + \mathcal{V})_{mn}]$. The necessary vanishing of the expression in square brackets means that $(\mathcal{F} + \mathcal{V})^{2,0} = 0$.

Finally, we contract with $\xi_J \Gamma_k$:

$$X^{[m} Y^{n]} D_m H_I{}^K \left(\Theta_{JKkn} - \frac{1}{2} \epsilon_{JK} s g_{kn} \right). \tag{3.3.9}$$

By symmetrizing and antisymmetrizing over I and J, it is clear that both terms in parantheses have to vanish independently. It follows that $D_X H_I{}^J = 0$.

In summary, the almost CR-structure is integrable and the manifold is CR if and only if

$$\mathfrak{m}_I{}^J = f t_I{}^J, \qquad (\mathcal{F} + \mathcal{V})^{2,0} = 0, \qquad D_X \left(\frac{t_I{}^J}{\sqrt{\det t}} \right) = 0, \qquad \forall X \in T^{1,0}. \tag{3.3.10}$$

Note that due to our reality condition for $t_I{}^J$, the last statement is actually equivalent to

$$D_X \left(\frac{t_I{}^J}{\sqrt{\det t}} \right) = 0, \qquad \forall X \in TM_H, \tag{3.3.11}$$

where $TM_H = T^{1,0} \oplus T^{0,1}$.

Transversally holomorphic foliations

Having discussed integrable CR structures, we now turn to the integrability condition for transversal holomorphic foliations (3.1.5). Using identical arguments to those from the previous section, we note that the integrability condition is

$$[X,Y]\in T^{1,0}\oplus R \quad \Leftrightarrow \quad 0 = X^{[m}Y^{n]}\left[D_m H_I{}^J \Pi_n{}^k \Gamma_k \xi_J + \nabla_m \Pi_n{}^k H_I{}^J \Gamma_k \xi_J + H_I{}^J \Pi_n{}^k \Gamma_k D_m \xi_J\right]. \tag{3.3.12}$$

To begin, consider (3.3.12) for $X,Y\in T^{1,0}$. Direct substitution gives

$$X^{[m}Y^{n]}(D_m H_I{}^J\Gamma_n - [H,t]_I{}^J\Gamma_{mn} - 2\star\mathcal{V}_{mnk}(\Gamma^k + R^k)H_I{}^J)\xi_J \tag{3.3.13}$$

Now, since $X,Y\in T^{1,0}$, the only contributions to the last term arise from the components of $\star\mathcal{V}$ that lie in $\Omega^{2,1}\oplus\Omega^{2,0}\wedge R$. However, since $(\Gamma^k+R^k)\xi_I = \Pi^k{}_l\Gamma^l\xi_I$ the latter of these is annihilated by the projection while the former vanishes due to holomorphy — i.e. for any $\omega\in\Omega^{0,1}$, $H_I{}^J\omega_k\Gamma^k\xi_J = 0$. Thus we are left with

$$X^{[m}Y^{n]}(D_m H_I{}^J\Gamma_n - [H,t]_I{}^J\Gamma_{mn})\xi_J \tag{3.3.14}$$

Once again, contraction with ξ^I gives the first necessary condition, $([H,t]^{IJ}\Theta_{IJ})^{2,0}=0$, from which it follows once again that $\mathfrak{m}_I{}^J = f t_I{}^J$. Just as in the CR case the second condition is $D_X H_I{}^J = 0$, $\forall X\in T^{1,0}$.

We continue our analysis of (3.3.12) by considering $X\in T^{1,0}$ and $Y=R$. Using the results from the previous paragraph, one finds that the necessary and sufficient condition is the vanishing of

$$X^m[-D_R H_I{}^J\Gamma_m + 2(\mathcal{F}+2\iota_R\star\mathcal{V})_{mn}(\Gamma^n+R^n)H_I{}^J]\xi_J. \tag{3.3.15}$$

By inspection one finds that the only contributing terms including $\mathcal{F}$ or $\mathcal{V}$ lie in $\Omega^{2,0}$ — $(\Omega^{1,0}\oplus\Omega^{0,1})\wedge R$ as well as $\Omega^{0,2}$ components are projected to zero while those in $\Omega^{1,1}$ vanish due to holomorphy. The components in $\Omega^{2,0}$ are of course self-dual under $\iota_R\star$ so the above can be rewritten in terms of $\mathcal{F}+2\mathcal{V}$ instead of $\mathcal{F}+2\iota_R\star\mathcal{V}$.

To further simplify this, we consider the chiral projection of the dilatino equation (3.3.4). Acting with $X^m H_I{}^J\Gamma_m$ on (3.3.4) one finds that

$$H_I{}^J D_R t_J{}^K X^m\Gamma_m\xi_K = 4H_I{}^J t_J{}^K X^m \iota_R\star(\mathcal{F}+2\mathcal{V})_{mn}\Gamma^n\xi_K. \tag{3.3.16}$$

Now, we first note that $D_R t_I{}^J = \sqrt{\det t}\, D_R H_I{}^J$ as $D_R(\det t)=0$. Together with $H_I{}^K H_K{}^J = -2\imath H_I{}^J$ it follows that

$$D_R H_I{}^J X^m\Gamma_m\xi_K = 2\imath(\det t)^{-1/2}X^m H_I{}^J t_J{}^K\iota_R\star(\mathcal{F}+2\mathcal{V})_{mn}\Gamma^n\xi_K = 2X^m H_I{}^J\iota_R\star(\mathcal{F}+2\mathcal{V})_{mn}\Gamma^n\xi_K. \tag{3.3.17}$$

As before we argue that only the $\Omega^{2,0}$ and $\Omega^{0,2}$ terms contribute. Thus we find that (3.3.15) vanishes without any further conditions. In the end, the integrability conditions are

$$\mathfrak{m}_I{}^J = f t_I{}^J, \qquad D_X\left(\frac{t_I{}^J}{\sqrt{\det t}}\right) = 0, \qquad \forall X\in T^{1,0}. \tag{3.3.18}$$

As in the case of the CR structure the reality condition for ${t_I}^J$ implies that the last condition holds for all horizontal sections of the tangent bundle. By comparison with equation (3.3.10) it is clear that any solution defining a THF also defines an integrable CR structure while the converse is not the case.

3.4 Implications for Localization

3.4.1 The ∂_b and $\bar{\partial}_b$ operators

Suppose that our manifold satisfies either of the integrability conditions (3.3.10) or (3.3.18). Let us show one can define nilponent operators ∂_b and $\bar{\partial}_b$ similar to those on complex structures. To do so, consider a $(0,1)$-form $\alpha^{0,1}$. We can decompose its exterior derivative as

$$d\alpha^{0,1} = \pi_V\left(d\alpha^{0,1}\right) + \pi^{2,0}\left(d\alpha^{0,1}\right) + \pi^{1,1}\left(d\alpha^{0,1}\right) + \pi^{0,2}\left(d\alpha^{0,1}\right), \tag{3.4.1}$$

where π_V and $\pi^{p,q}$ are projectors to the vertical and (p,q) components. Since neither $[T^{1,0}, T^{1,0}]$ nor $[T^{1,0} \oplus R, T^{1,0} \oplus R]$ have a component in $T^{0,1}$ one finds that

$$d\alpha^{0,1}\left(X^{1,0}, Y^{1,0}\right) = X^{1,0}\left(\alpha^{0,1}\left(Y^{1,0}\right)\right) - Y^{1,0}\left(\alpha^{0,1}\left(X^{1,0}\right)\right) - \alpha^{0,1}\left(\left[X^{1,0}, Y^{1,0}\right]\right). \tag{3.4.2}$$

In other words, $\pi^{2,0}\left(d\alpha^{0,1}\right) = 0$, which allows us to define $(d_V, \partial_b, \bar{\partial}_b)$ via

$$d\alpha^{0,1} = \pi_V\left(d\alpha^{0,1}\right) + \pi^{1,1}\left(d\alpha^{0,1}\right) + \pi^{0,2}\left(d\alpha^{0,1}\right) \equiv d_V\alpha^{0,1} + \partial_b\alpha^{0,1} + \bar{\partial}_b\alpha^{0,1}. \tag{3.4.3}$$

From $d = \partial_b + \bar{\partial}_b + d_v$ and $d^2 = 0$ it follows directly that $\partial_b^2 = \bar{\partial}_b^2 = 0$ and one can define cohomology groups $H^{p,q}_{\bar{\partial}_b}$ via the exact sequence

$$\ldots \xrightarrow{\bar{\partial}_b} \Omega^{p,q-1} \xrightarrow{\bar{\partial}_b} \Omega^{p,q} \xrightarrow{\bar{\partial}_b} \Omega^{p,q+1} \xrightarrow{\bar{\partial}_b} \ldots \tag{3.4.4}$$

3.4.2 Mode counting and partition functions

As mentioned in the introduction, partition functions for supersymmetric gauge theories calculated in the context of topological field theories or localization simplify significantly on Kähler and Sasakian manifolds. The argument relies not only on the existence of the differential $\bar{\partial}_b$ ($\bar{\partial}$ in the Kähler case). Indeed, one also requires the compatibility of the decomposition C.3.1 with the action of the Lie derivative $\pounds_{sR}$. In this section we will go over this argument of [22, 24] in some detail and discuss under what circumstances it applies to the manifolds in question.

Consider a vector multiplet with Lie algebra $\mathfrak{g}$. The bosonic modes lie in $\Omega^1(\mathfrak{g}) \oplus H^0(\mathfrak{g}) \oplus H^0(\mathfrak{g})$, where $H^0(\mathfrak{g})$ denotes harmonic Lie algebra valued functions. Fermionic modes on the other hand can be mapped to $\Omega^+(\mathfrak{g}) \oplus \Omega^0(\mathfrak{g}) \oplus \Omega^0(\mathfrak{g})$. The one-loop contribution to the perturbative partition function is given by[4]

$$\sqrt{\frac{\det_{\text{fermions}} \pounds_{sR}}{\det_{\text{bosons}} \pounds_{sR}}}. \tag{3.4.5}$$

[4] This was shown to be true for generic Sasakian manifolds in [25]. Here we assume it to be true for five-dimensional Riemannian manifolds admitting a integrable CR-structure or THF.

If $\pounds_{sR}\Phi = \pounds_{sR}\kappa = 0$ we can calculate the determinants using the decomposition C.3.1. Clearly $\pounds_{sR}\kappa = 0$, so we need to evaluate $\pounds_{sR}\Phi = \iota_{sR}d\Phi$. Direct calculation using (3.3.2) yields

$$d\Phi = -s^{-1}ds\wedge\Phi + s^{-1}D\left(\frac{t^{IJ}}{\sqrt{\det t}}\right)\wedge\Theta_{IJ} + 2s^{-1}\left[\iota_R(\mathcal{F}+2\mathcal{V})\wedge\Phi - \kappa\wedge((\mathcal{F}+2\mathcal{V})\times\Phi)\right], \quad (3.4.6)$$

Thus

$$\pounds_{sR}\Phi = D_R\left(\frac{t^{IJ}}{\sqrt{\det t}}\right)\Theta_{IJ} - 2\left[(\mathcal{F}+2\mathcal{V})^H\times\Phi\right] = 0 \quad (3.4.7)$$

where we used (3.3.5). In conclusion we can rewrite (3.4.5) as

$$\sqrt{\frac{\det_{\pounds_{sR}}(\Omega^{2,0}\oplus\Omega^{0,0}\Phi\oplus\Omega^{0,2}\oplus\Omega^{0,0}\oplus\Omega^{0,0})}{\det_{\pounds_{sR}}(\Omega^{1,0}\oplus\Omega^{0,1}\oplus\Omega^{0,0}\kappa)}}\frac{1}{\det_{\pounds_{sR}}H^0}, \quad (3.4.8)$$

where we used the notation $\det_{\Omega^{p,q}}\pounds_{sR} = \det_{\pounds_{sR}}\Omega^{p,q}$ and dropped the various appearances of $\mathfrak{g}$ for readability. As $[\pounds_{sR},\bar\partial_b] = 0$ it follows that the above simplifies to

$$\sqrt{\frac{\det_{\pounds_{sR}}H^{0,2}_{\bar\partial_b}\det_{\pounds_{sR}}H^{0,0}_{\bar\partial_b}}{\det_{\pounds_{sR}}H^{0,1}_{\bar\partial_b}}}\sqrt{\frac{\det_{\pounds_{sR}}H^{2,0}_{\bar\partial_b}\det_{\pounds_{sR}}H^{0,0}_{\bar\partial_b}}{\det_{\pounds_{sR}}H^{1,0}_{\bar\partial_b}}}. \quad (3.4.9)$$

It is interesting to note that the above argument does not require a property akin to Lefschetz decomposition on Kähler manifolds. Recall that the Lefschetz theorem relates cohomology groups of the Dolbeault operator as $H^{0,0}_{\bar\partial} \cong H^{1,1}_{\bar\partial,\omega}$, where the subscript ω denotes forms parallel to the symplectic form ω. Such a decomposition, while true for e.g. Sasaki-Einstein manifolds does not hold in general for the operator Φ. That is, for $\alpha\in\Omega^{1,1}_\Phi$ one can write $\alpha = a\Phi$ for some scalar function a, yet $\bar\partial_b\alpha = 0$ is not in one-to-one correspondence with $\bar\partial_b a = 0$ since $\bar\partial_b\Phi$ does not vanish in the general case.

3.4.3 BPS equations on the Higgs branch

The nilponency of $\bar\partial_b$ has also immediate implications on the Higgs branch BPS equations of $\mathcal{N}=1$ theories. In [26] these were studied for supersymmetric backgrounds that are K-contact. Defining $\bar\partial_a \equiv \bar\partial_b - \imath a^{0,1}$ for a $U(1)$ connection a with field strength F_a, some of the relevant equations are

$$\bar\partial_a\alpha + \bar\partial_a^*\beta = 0, \qquad F_a^{0,2} = 2i\bar\alpha\beta, \qquad F_a^{d\kappa} = \frac{1}{2}\left(\zeta - |\alpha|^2 + |\beta|^2\right). \quad (3.4.10)$$

Here, α is a 0-form and β is a $(0,2)$-form; both are related to the scalar in the hypermultiplet. The superscript $d\kappa$ denotes the component along $d\kappa$. The BPS equations and the nilpotence then imply that $\bar\partial_a\bar\partial_a^*\beta = -\bar\partial_a\bar\partial_a\alpha = \imath F_a^{0,2}\alpha = -2|\alpha|^2\beta$. Thus $2\int|\alpha|^2|\beta|^2 + \int|\bar\partial_a^*\beta|^2 = 0$, and it follows that

$$\begin{cases}\beta = 0\\ \bar\partial_a\alpha = 0\end{cases}, \quad \text{or} \quad \begin{cases}\alpha = 0\\ \bar\partial_a^*\beta = 0\end{cases}. \quad (3.4.11)$$

In other words, similar to our discussion in the previous section we see that results for Sasaki (-Einstein) manifolds can be extended to geometries that are either THF or CR.

3.5 A Karlhede-Rocek-Witten twist in five dimensions

As discussed above in section 3.3.1 as well as in [20] a necessary condition for the existence of a solution of the background supergravity variations for supersymmetry spinors satisfying the symplectic Majorana condition is the existence of a Killing vector. Recall that the symplectic Majorana condition (2.2.6) implies that $s > 0$ from which it follows that v has no zeroes. In other words, the Killing vector is globally non-vanishing.[5] In this section we will show that the existence of a globally non-vanishing Killing vector is also sufficient for the manifold $\mathcal{M}$ to admit supersymmetry spinors that solve (3.1.1) and (3.1.2).[6] At the heart of the argument is the idea that the existence of the vector implies that the manifold supports an $SO(4)$ structure. This in turn allows us to do a standard Witten twist [27, 28]. Our strategy is to work in a patch using methods familiar from Kaluza-Klein reduction, yet show that we can write the overall result in terms of globally well-defined objects. In principal one should be able to make the same argument using the general, local solution of [20].

Given a manifold $\mathcal{M}$ with a Killing vector $\mathfrak{v} = \partial_\tau$ we can write the vielbein as[7]

$$\hat{e}_\mu{}^\alpha = \begin{pmatrix} e_m{}^a & k a_m \\ 0 & k \end{pmatrix}, \qquad \hat{E}_\alpha{}^\mu = \begin{pmatrix} E_a{}^m & -a_a \\ 0 & k^{-1} \end{pmatrix}. \tag{3.5.1}$$

I.e. the metric takes the form $ds^2 = g_{mn}dx^m dx^n + k^2(d\tau + a)^2$, where $\partial_\tau = \mathfrak{v}$. The spin connection is

$$\hat{\omega}_{abc} = \omega_{abc}, \quad \hat{\omega}_{ab5} = \frac{1}{2}k f_{ab}, \quad \hat{\omega}_{5bc} = -\frac{1}{2}k f_{bc}, \quad \hat{\omega}_{5b5} = -\partial_b \log k. \tag{3.5.2}$$

Here, $f = da$. Keeping in mind (3.2.2), we demand the spinor ξ_I to be anti-chiral. That is, $\Gamma^5\xi_I = -\xi_I$ which is why we write $\xi_I \equiv \xi_I^-$.

3.5.1 Gravitino Equation

One can then decompose the gravitino variation (3.1.1) into components along $a = 1, \ldots 4$, components along $a = 5$ as well as chiral and anti-chiral parts:

$$0 = D_a\xi_I^- - a_a(\partial_\tau\xi_I^- - A_{\tau I}{}^J\xi_J^-) + \mathcal{F}_{a5}\xi_I^- + \Gamma_{ab}\mathcal{V}^{b5}\xi_I^-, \tag{3.5.3}$$

$$0 = -\frac{1}{4}k f_{ab}\Gamma^b\xi_I^- - t_I{}^J\Gamma_a\xi_J^- - \mathcal{F}_{ab}\Gamma^b\xi_I^- - \frac{1}{2}\mathcal{V}^{bc}\Gamma_{abc}\xi_I^-, \tag{3.5.4}$$

$$0 = k^{-1}(\partial_\tau\xi_I^- - A_{\tau I}{}^J\xi_J^-) + t_I{}^J\xi_J^- - \frac{1}{8}k f_{bc}\Gamma^{bc}\xi_I^- + \frac{1}{2}\mathcal{V}^{bc}\Gamma_{bc}\xi_I^-, \tag{3.5.5}$$

$$0 = \frac{1}{2}\partial_b \log k\Gamma^b\xi_I^- + \mathcal{F}_{b5}\Gamma^b\xi_I^-. \tag{3.5.6}$$

[5] If one does not impose the symplectic Majorana condition, the situation is more complicated. I.e. both s and R are generally complex; it is also clear that the vector vanishes if the spinors are parallel. Moreover, note that R does not even vanish at a single point. Assume $\exists p \in \mathcal{M}$ such that $R|_p = 0$. It follows that $s(p) = 0$ and thus $\xi_I|_p = 0$. From the gravitino equation it follows immediately that ξ_I vanishes identically on $\mathcal{M}$.

[6] We would like to thank Diego Rodriguez-Gomez for many discussions and collaboration that lead to the approach used in this section.

[7] In this section, greek indices run from one to five while roman ones only run from one to four.

The last of these, (3.5.6), is solved by $\mathcal{A} = -\frac{1}{2}k^{-1}\mathfrak{v}$. It follows that

$$\mathcal{F}_{a5} = -\frac{1}{2}\partial_a \log k, \qquad \mathcal{F}_{ab} = -\frac{1}{2}k f_{ab}. \tag{3.5.7}$$

Equation (3.5.3) is solved by setting $A_{\tau I}{}^J = 0$, $\xi_I^- = \sqrt{k}\chi_I$, where $\chi_I\chi^I = 1$, and — more importantly — $D_a\chi_I = \nabla_a\chi_I - A_{aI}{}^J\chi_J = 0$. The possibility of finding a χ such that $\nabla_a\chi_I = A_{aI}{}^J\chi_J$ is of course at the heart of this argument. As long as $\Gamma^5\chi_I = -\chi_I$, it is possible to find such a spinor; explicit calculations can be done using 't Hooft matrices for example [29]. With all our previous assumptions and observations (3.5.4) becomes

$$4\mathcal{V}_{ab} = \frac{1}{2}k\epsilon_{abcd5}f^{cd} + 4s^{-1}\Theta^{IJ}_{ab}t_{IJ}. \tag{3.5.8}$$

Substituting this into (3.5.5) we find that $t_{IJ} = 0$ since

$$0 = \frac{1}{8}\left(kf_{ab} - 4\mathcal{V}_{ab}\right)\Theta^{ab}_{IJ} = \frac{1}{8}\left(kf_{ab} - \frac{1}{2}k\epsilon_{abcd5}f^{cd} + 4s^{-1}\Theta^{KL}_{ab}t_{KL}\right)\Theta^{ab}_{IJ} = \frac{1}{2}\Theta^{ab}_{IJ}\Theta^{KL}_{ab}t_{KL}. \tag{3.5.9}$$

In summary, the gravitino equation is fully solved by

$$\xi_I^- = \sqrt{k}\chi_I, \quad D_a\chi_I = 0, \quad \mathcal{A} = -\frac{1}{2}k^{-1}\mathfrak{v}, \quad t_{IJ} = 0, \quad 4\mathcal{V}_{ab} = \frac{1}{2}k\epsilon_{abcd5}f^{cd}, \quad \mathcal{V}_{a5} = 0. \tag{3.5.10}$$

By now it is clear that the spinor bilinears s, v coincide with the scalar and vector defined by the background, $k, \mathfrak{v}$, i.e. $s = k$, $v = \mathfrak{v}$, so we drop the distinction.

3.5.2 Dilatino Equation

Performing a similar decomposition of the Dilatino equation, one finds

$$0 = 4\hat{D}t_I{}^J\Gamma^a\xi_J^- - 8t_I{}^J(\mathcal{F}+2\mathcal{V})_{a5}\Gamma^a\xi_J^- + 4\hat{\nabla}_\alpha\mathcal{V}^{\alpha b}\Gamma_b\xi_I^- - 4\mathcal{F}_{ab}\mathcal{F}_{c5}\Gamma^{abc}\xi_I^-, \tag{3.5.11}$$
$$0 = -4\hat{D}_5t_I{}^J\xi_J^- + 4t_I{}^J(\mathcal{F}+2\mathcal{V})_{ab}\Gamma^{ab}\xi_I^- - 4\hat{\nabla}_\alpha\mathcal{V}^{\alpha 5}\xi_I^- + \mathcal{F}_{ab}\mathcal{F}_{cd}\Gamma^{abcd}\xi_I^- + C\xi_I^-. \tag{3.5.12}$$

Imposing the solution to the gravitino equations (3.5.10), this simplifies of course considerably. Also, note that

$$\hat{\nabla}_\alpha\mathcal{V}^{\alpha b} = \nabla_a\mathcal{V}^{ab} + \partial^a\log k\mathcal{V}_{ab}, \qquad \hat{\nabla}_\alpha\mathcal{V}^{\alpha 5} = -\frac{1}{2}kf_{ab}\mathcal{V}^{ab}. \tag{3.5.13}$$

Then, (3.5.12) is solved by $C = -\frac{1}{4}k^2 f_{ab}f_{cd}\epsilon^{abcd5}$. Since $4s\nabla_b\mathcal{V}^{ba} = -\frac{1}{2}s\epsilon^{abcd5}f_{bc}\partial_d k$, one finds that (3.5.11) is solved trivially.

3.5.3 Topological Issues

To conclude, we discuss whether the solution (3.5.10) is globally well-defined. Since $\mathcal{F}$ is globally exact we only have to worry about the $SU(2)_R$ field strength. Our strategy is to rewrite this in terms of the Riemann tensor. Thus we use the integrability condition for the spinor χ_I,

$$0 = [D_a, D_b]\chi_I = -F^{IJ}_{ab}\chi_J + \frac{1}{4}R_{ab\alpha\beta}\gamma^{\alpha\beta}\chi_I. \tag{3.5.14}$$

This implies $F^{IJ}_{ab} = -2s^{-1}R_{ab\kappa\lambda}\Theta^{IJ\kappa\lambda}$ from which it follows that we can express the $SU(2)_R$ connection in terms of a projection of the Riemann tensor. In summary, the two connections are

$$F^{IJ} = -2s^{-1}R^H_{\mu\nu}\Theta^{IJ\mu\nu}, \qquad \mathcal{F} = -\frac{1}{2}d(k^{-1}\mathfrak{v}). \tag{3.5.15}$$

where $R^H_{\mu\nu} = \Pi^\sigma_\kappa \Pi^\tau_\lambda R_{\sigma\tau\mu\nu} dx^\kappa \otimes dx^\lambda$ denotes the horizontal part of the curvature two-form. In both cases, all objects appearing on the right hand side are globally well defined. We proceed to consider characteristic classes defined by F^{IJ}. Using (3.2.3) one finds that

$$F_I{}^J \wedge F_J{}^I = -4R^H_{\kappa\lambda} \wedge R^H_{\mu\nu}\left(\Pi^{\kappa\mu}\Pi^{\lambda\nu} + \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu\rho}\kappa_\rho\right). \tag{3.5.16}$$

The expression is completely horizontal and since $\mathfrak{v}$ is Killing, $0 = \pounds_{\mathfrak{v}}(F_I{}^J \wedge F_J{}^I) = \iota_{\mathfrak{v}} d(F_I{}^J \wedge F_J{}^I)$ from which it follows that (3.5.16) is closed and defines thus an element of the de Rham cohomolgy group $H^4(\mathcal{M})$ as it should. Usually the next question would be whether this element is trivial and whether it might be an obstruction to the existence of the solution given by (3.5.10) and C. However, equation (3.5.16) clearly show that this class has a representative that is independent of our specific solution since it can be expressed in terms of $\mathfrak{v}$ and the Riemann tensor. Thus, in the case that the class is non-trivial, it is clear that the corresponding cycle in homology exists and vice versa.

One might worry about the f dependence of $\mathcal{V}$. In general, the manifolds are not bundles yet only foliations and one cannot necessarily think of f as the curvature of a connection. Yet as we saw above, f is a projection of $\mathcal{F}$ onto the horizontal space — $f = -2k^{-1}\mathcal{F}^H$. While one might not consider f globally as the curvature of a connection, it is well-defined as a two-form. Since it doesn't enter the solution directly yet only via $\mathcal{V}$, this is good enough and we conclude that any manifold $\mathcal{M}$ admits a solution to (3.1.1) and (3.1.2) with symplectic Majorana spinor if and only if there is a non-vanishing Killing vector $\mathfrak{v}$.

3.6 Examples

It follows from the previous section that any direct product $\mathbb{R} \times \mathcal{M}_4$ or $S^1 \times \mathcal{M}_4$ admits a solution to the Killing spinor equations and thus rigid supersymmetry. Similarly, it is clear that such manifolds do at least not trivially[8] admit an integrable CR-structure or a THF if $\mathcal{M}_4$ does not admit a complex structure — the example coming to mind here being $\mathbb{R} \times S^4$. See however the discussion in [23].

3.6.1 Sasakian manifolds

Sasakian manifolds are the odd-dimensional analogs of Kähler manifolds. They are either characterized by having Kähler metric cones, or by the existence of a Killing spinor satisfying

$$(\nabla_m - i\mathcal{A}_m)\,\xi = \frac{i}{2}\Gamma_m \xi. \tag{3.6.1}$$

[8] "Trivially" here means that one simply embeds the Killing vector in the obvious way. For a specific choice of $\mathcal{M}_4$ and Killing vector, this might change.

Here, $\mathcal{A}$ is the connection one-form associated to the Ricci-form on the metric cone. The equation and its complec conjugate corresponds to the special case of (3.1.1) with

$$\mathcal{F} = \mathcal{V} = 0, \qquad (\mathcal{A}_m)_I{}^J = \mathcal{A}_m(\sigma_3)_I{}^J, \qquad t_I{}^J = \frac{i}{2}(\sigma_3)_I{}^J. \tag{3.6.2}$$

Since both t and $\mathcal{A}$ have only components along σ_3 one finds that $\nabla_m t_{IJ} = 0$. The dilatino equation is solved by

$$C = 0. \tag{3.6.3}$$

Hence, $\mathcal{N} = 1$ supersymmetry can be defined on any 5-dimensional Sasakian structure as was first observed without resorting to supergravity [25].

Sasakian structures are examples of both Cauchy-Riemann or transversal-holomorphic structures, as follows from the fact that $\nabla_m t_{IJ} = \mathcal{F} = \mathcal{V} = 0$.

3.6.2 Squashed S^5 with $SU(3) \times U(1)$ Symmetry

Squashed five-spheres have appeared in various literatures. In particular, [20, 30] discussed a class of squashed S^5, defined by metric

$$ds^2_{S^5_b} = \frac{1}{b^2}(d\tau + h)^2 + d\sigma^2 + \frac{1}{4}\sin^2\!\sigma \left(d\theta^2 + \sin^2\!\theta d\varphi^2\right) + \frac{1}{16}\sin^2\!2\sigma(d\psi + \cos\theta d\varphi)^2. \tag{3.6.4}$$

Our discussion follows that of [30] closely. The real constant b is the squashing parameter, which gives a round sphere when $b = 1$, h is a 1-form defined as

$$h = -\frac{1}{2}\sin^2\!\sigma \left(d\psi + \cos\theta d\varphi\right). \tag{3.6.5}$$

where ω can be viewed as the Kahler form on $\mathbb{C}P^2$, satisfying $d\omega = 0$. The metric is written in a form adapted to the smooth $U(1)$-fibration over $\mathbb{C}P^2$, where $b^{-2}(d\tau + h)^2$ is the metric in the $U(1)$-fiber direction, and b is there to squash the radius. In this way it is easy to see the metric has $U(1) \times SU(3)$ symmetry, where $U(1)$ rotates the fiber, and $SU(3)$ is the isometry of $\mathbb{C}P^2$. The $\mathbb{C}P^2$ Kähler form is $\omega = \frac{1}{2}dh$. With the vielbein

$$e^1 = \frac{1}{2}\sin\sigma\cos\sigma\tau_3, \qquad e^2 = d\sigma, \qquad e^3 = \frac{1}{2}\sin\sigma\tau_2, \qquad e^4 = \frac{1}{2}\sin\sigma\tau_1, \qquad e^5 = b^{-1}(d\tau + h), \tag{3.6.6}$$

one finds

$$\omega = e^1 \wedge e^2 - e^3 \wedge e^4, \qquad \omega \wedge \omega = -2e^1 \wedge e^2 \wedge e^3 \wedge e^4, \qquad *(\omega \wedge \omega) = -2e^5, \tag{3.6.7}$$

where we have introduced the left-invariant one forms

$$\tau_1 + \imath\tau_2 = e^{-\imath\psi}(d\theta + \imath\sin\theta d\phi), \qquad \tau_3 = d\psi + \cos\theta d\phi. \tag{3.6.8}$$

This class of squashed sphere admits solutions to the Killing spinor equations

$$\nabla_m\xi_I + \frac{i}{2}(\mathcal{A}_m)_I{}^J\xi_J = -\frac{i}{2b}\left(1 + Q\sqrt{1-b^2}\right)(\sigma_3)_I{}^J\Gamma_m\xi_J + \frac{\sqrt{1-b^2}}{b}\omega_{mn}\Gamma^n\xi_I + \frac{1}{2}\frac{\sqrt{1-b^2}}{2b}\omega^{pq}\Gamma_{mpq}\xi_I. \tag{3.6.9}$$

where Q is a real parameter. And of course one can define bilinears as in (3.2.1). In terms of (A.1.11), the quarter BPS solution with $Q = -3$ is given by

$$\xi_1 = \frac{c_+}{\sqrt{2}} e^{-\frac{3\imath\tau}{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_2 = \frac{c_-}{\sqrt{2}} e^{\frac{3\imath\tau}{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.6.10}$$

The symplectic Majorana condition (2.2.6) corresponds to $(c_-)^* = c_+$. For more involved 3/4 BPS solutions refer to [30].

Comparing (3.1.1) with (3.6.9) one identifies

$$t_I{}^J = -\frac{i}{2b}\left(1 + Q\sqrt{1-b^2}\right)(\sigma_3)_I{}^J, \quad \mathcal{F} = \frac{\sqrt{1-b^2}}{b}\omega, \quad \mathcal{V} = \frac{\sqrt{1-b^2}}{2b}\omega, \quad (\mathcal{A}_m)_I{}^J = \frac{\left(1 + Q\sqrt{1-b^2}\right)\sqrt{1-b^2}}{b} e^5. \tag{3.6.11}$$

Note that $\kappa = -e^5$ and one finds that ω is horizontal and self-dual since $\star\omega = \kappa \wedge \omega$. Furthermore $d\kappa = -2b^{-1}\omega$ and $\nabla^m \omega_{mn} = 4b^{-1}\kappa_n$. Moreover

$$\omega_{mn}\omega_{kl}\Gamma^{mnkl}\xi_I = \omega_{mn}\omega_{kl}\epsilon^{mnkl}{}_r\Gamma^r\xi_I = 2\omega_{mn}(*\omega)^{mn}{}_r\Gamma^r\xi_I = 2\omega_{mn}\omega^{mn}\kappa_r\Gamma^r\xi_I = -8\xi_I. \tag{3.6.12}$$

Finally, substituting everything into the dilatino equation (3.1.2), one finds

$$0 = -\frac{4\imath}{b^2}(1 + Q\sqrt{1-b^2})\sqrt{1-b^2}(\sigma_3)_I{}^J\omega_{mn}\Gamma^{mn}\xi_J + 8\frac{\sqrt{1-b^2}}{b^2}\kappa_m\Gamma^m\xi_I - 8\frac{1-b^2}{b^2}\xi_I + C\xi_I. \tag{3.6.13}$$

From (3.3.3) it follows that

$$C = 8\frac{\sqrt{1-b^2}}{b^2} + 8\frac{1-b^2}{b^2} - 4\imath\frac{(1 + Q\sqrt{1-b^2})\sqrt{1-b^2}}{b^2 s}\omega^{mn}\Theta_{mnI}{}^J(\sigma_3)_J{}^I, \tag{3.6.14}$$

so the above simplifies to

$$0 = -4\imath\frac{(1 + Q\sqrt{1-b^2})\sqrt{1-b^2}}{b^2}\left[(\sigma_3)_I{}^J\omega_{mn}\Gamma^{mn}\xi_J + s^{-1}\omega^{mn}\Theta_{mnK}{}^L(\sigma_3)_L{}^K\xi_I\right], \tag{3.6.15}$$

which vanishes identically for the above solution.

Now compare the "algebraic equation" of [30]. Rewritten in our conventions, it is

$$0 = (1+Q)\sqrt{1-b^2}\xi_I - \frac{\imath}{2}\sqrt{1-b^2}(\sigma_3)_I{}^J\omega_{mn}\Gamma^{mn}\xi_J, \tag{3.6.16}$$

where we used (3.2.2). Contracting with ξ^I one finds $(\sigma_3)_I{}^J\omega^{mn}\Theta_{mnJ}{}^I = 2\imath s(1+Q)$. Substituting this into (3.6.15) yields (3.6.16), which tells us that the Dilatino equation and the "algebraic equation" are equivalent in the case of squashed S^5.

Comparing (3.6.11) with (3.3.10) and (3.3.18) it is clear that the squashing does not change the fact that S^5 admits both a CR-structure and a THF. In principle this is already clear from the form of the metric (3.6.4) since changes in the parameter b do not affect the $\mathbb{C}P^2$ base of the bundle.

Chapter 4

5d Higgs Branch Localization and Seiberg-Witten Equations

4.1 5-dimensional $\mathcal{N}=1$ Minimal Off-shell Supergravity

4.1.1 $\mathcal{N}=1$ Supergravity

In this subsection we briefly recall 5-dimensional minimal off-shell supergravity discussed in [1][31][16, 15] (see also literatures on superspace formalism [32][33]), and then extract the generalized Killing spinor equation by taking the rigid limit, following the idea of [34].

The Weyl multiplet contains the following bosonic field content (note that there is a curly $\mathcal{V}$ and straight V)

$$\mathcal{G}_{\text{Boson}} = \left\{ e^A_m, \quad \mathcal{A}_m, \quad \mathcal{V}_{mn}, \quad t_{IJ}, \quad C, \quad (V_m)_{IJ} \right\}. \tag{4.1.1}$$

Here $I, J = 1, 2$ are indices of $SU(2)_{\mathcal{R}}$ symmetry, $\mathcal{A}_m$ is the abelian gauge field corresponding to central charge with field strength $\mathcal{F} = d\mathcal{A}$, $\mathcal{V}$ is a 2-form, C is a scalar. Field t_{IJ} and V_{IJ} are both $SU(2)_{\mathcal{R}}$ triplet, meaning that

$$\overline{t_{IJ}} = \epsilon^{IK}\epsilon^{JL}t_{KL}. \tag{4.1.2}$$

and similarly for V_{IJ}. The fermionic field content contains

$$\mathcal{G}_{\text{Fermion}} = \{\psi_I, \quad \eta_I\}, \tag{4.1.3}$$

where ψ is the gravitino, η is the dilatino. Finally, the supergravity transformation δ_{Sugra} has symplectic-Majorana parameter ξ_I.

To obtain a supersymmetric theory of some matter multiplet on some manifold M, one can first couple it to the above Weyl multiplet $\mathcal{G}$, and then set all fields in $\mathcal{G}$ to some background values that is invariant under the supergravity transformation δ_{Sugra}. In particular, we set the fermions (ψ, η) to zero background, and requires two spinorial differential equations (with coefficients comprised with fields $\{V, \mathcal{V}, \mathcal{F}, t_{IJ}, C\}$)

$$\delta_{\text{Sugra}}\psi = 0, \qquad \delta_{\text{Sugra}}\eta = 0, \tag{4.1.4}$$

with transformation parameter ξ_I, and look for background values of $\{V, \mathcal{V}, \mathcal{F}, t_{IJ}, C\}$ that admit a solution ξ_I. The result of such procedure is [34, 4, 3, 35]:

- Supersymmetry transformation Q obtained from δ_{Sugra} by substituting in background values of $\{V, \mathcal{V}, \mathcal{F}, t_{IJ}, C\}$.
- A Q-invariant Lagrangian from the coupled supergravity Lagrangian, where all remaining bosonic fields from $\mathcal{G}$ are auxiliary background fields.
- Some geometric data, including metric g, p-forms and so forth, determined by combinations of $\{V, \mathcal{V}, \mathcal{F}, t_{IJ}, C\}$.

First of all, we focus on the equation $\delta_{\text{Sugra}}\psi = 0$, which we refer to as the *generalized Killing spinor equation* in the following discussion. The generalized Killing spinor equation reads

$$\nabla_m \xi_I = t_I{}^J \Gamma_m \xi_J + \mathcal{F}_{mn}\Gamma^n \xi_I + \frac{1}{2}\mathcal{V}^{pq}\Gamma_{mpq}\xi_I, \tag{4.1.5}$$

where ∇ contains the usual Levi-Civita spin connection as well as $SU(2)_{\mathcal{R}}$ gauge field V_m when acting on objects with I, J indices. Strictly speaking, ξ_I is a section of the bundle $S \otimes V$ where V is a $SU(2)_{\mathcal{R}}$-vector bundle on which $(V_M)_I{}^J$ is defined, therefore we should require M to be a spin manifold.

Equation (4.1.5) is studied in [21], where geometric restrictions imposed by different numbers of solutions is discussed. Subsequently, in [20] both differential equations $\delta\psi = \delta\eta = 0$ are solved in a coordinates patch. It is shown that, locally, deformations of auxiliary fields that preserves (4.1.5) and (4.1.6) can be realized as Q-exact deformation or gauge transformations. This suggests that path integrals of appropriate observables may be topological or geometrical invariants. For us, it is important to note that $\delta_{\text{Sugra}}\eta = 0$ implies (which we may call the *dilatino equation*)

$$4\left(\nabla_m t_I{}^J\right)\Gamma^m\xi_J + 4\nabla_m\mathcal{V}^{mn}\Gamma_n\xi_I + 4t_I{}^J\left(\mathcal{F}_{mn} + 2\mathcal{V}_{mn}\right)\Gamma^{mn}\xi_J + \mathcal{F}_{mn}\mathcal{F}_{kl}\Gamma^{mnkl}\xi_I = -C\xi_I \tag{4.1.6}$$

This will be used to ensure the closure of the rigid $\mathcal{N} = 1$ supersymmetry. Note that the field C can be solved using this equation in terms of $\{V, \mathcal{F}, \mathcal{V}, t_I{}^J\}$, by contracting both sides with ξ^I:

$$4R_n\nabla_m\mathcal{V}^{mn} - 4(\mathcal{F} + 2\mathcal{V})_{mn}\left(t^{IJ}\Theta_{IJ}\right)^{mn} + 2(\iota_R * \mathcal{F})^{mn}\mathcal{F}_{mn} = sC \tag{4.1.7}$$

where R, Θ and s are defined using ξ_I as explained earlier.

So to summarize, for the rigid limit to give rise to a rigid supersymmetry, we are required to study the Killing spinor equations and the dilatino equation

$$\begin{cases} \nabla_m \xi_I = t_I{}^J \Gamma_m \xi_J + \mathcal{F}_{mn}\Gamma^n \xi_I + \frac{1}{2}\mathcal{V}^{pq}\Gamma_{mpq}\xi_I \\ 4\left(\nabla_m t_I{}^J\right)\Gamma^m\xi_J + 4\nabla_m\mathcal{V}^{mn}\Gamma_n\xi_I + 4t_I{}^J\left(\mathcal{F}_{mn} + 2\mathcal{V}_{mn}\right)\Gamma^{mn}\xi_J + \mathcal{F}_{mn}\mathcal{F}_{kl}\Gamma^{mnkl}\xi_I = -C\xi_I \end{cases}$$

where one can immediately solve C in terms of other auxiliary fields using (4.1.7).

4.1.2 Generalized Killing Spinor Equation

In this subsection we will review some basic properties the Killing spinor equations that are relevant to later discussions. Some terminology in K-contact geometry will be reviewed in the following subsection.

As introduced in the previous subsection, the Killing spinor equation for symplectic-Majorana spinor ξ_I is

$$\boxed{\nabla_m \xi_I = t_I{}^J \Gamma_m \xi_J + \mathcal{F}_{mn}\Gamma^n \xi_I + \frac{1}{2}\mathcal{V}^{pq}\Gamma_{mpq}\xi_I} \,. \tag{4.1.8}$$

Recall that we have several background fields coming from the Weyl multiplet: $\mathcal{F}$ is a closed 2-form, and $\mathcal{V}$ is a usual 2-form as the field strength of $\mathcal{A}$, $t_I{}^J$ is a triplet of scalars. The connection ∇ contains the Levi-Civita spin connection and possibly a non-zero $SU(2)_\mathcal{R}$ background gauge field V_m acting on the I-indices. All these fields are from the Weyl multiplet $\mathcal{G}$ and we call them *auxiliary fields* below.

Equation (4.1.8) can also be written in a more convenient form

$$\boxed{\nabla_m \xi_I = \Gamma_m \tilde{\xi}_I + \frac{1}{2}\mathcal{P}^{pq}\Gamma_{mpq}\xi_I, \quad \tilde{\xi}_I \equiv t_I{}^J\xi_J + \frac{1}{2}\mathcal{F}_{mn}\Gamma^{mn}\xi_I, \quad \mathcal{P} \equiv \mathcal{V} - \mathcal{F}} \,. \tag{4.1.9}$$

1. Symmetries

The Killing spinor equation enjoys several symmetries that will help simplify later discussions.

- Background $SU(2)_\mathcal{R}$ symmetry, which acts on the I-index.
- Shifting symmetry: one can shift the auxiliary fields $\mathcal{F}$ and $\mathcal{V}$ by any anti-self-dual[1] 2-form Ω^-
$$\mathcal{F} \to \mathcal{F} + \Omega^-, \;\; \mathcal{V} \to \mathcal{V} + \Omega^-. \tag{4.1.10}$$
and the equation is invariant.
- Other symmetries related to the many degrees of freedoms discussed in [20]. We will come back to this shortly.

2. Solving the Killing spinor equation

Let ξ_I be a solution to the Killing spinor equation (4.1.8). Then one can construct bi-linears s, R^m, κ_m and Θ_{IJ} using ξ_I. By directly applying equation (4.1.8), one obtains several differential properties of these bi-linears:

- $\nabla_m s = 2R^n \mathcal{F}_{nm} \Leftrightarrow ds = 2\iota_R \mathcal{F}$ and therefore $\mathcal{L}_R s = 0, \;\; \mathcal{L}_R \mathcal{F} = 0$, where we have used the Bianchi identity $d\mathcal{F} = 0$.
- $\nabla_m R_n = 2t^{IJ}(\Theta_{IJ})_{mn} - 2s\mathcal{F}_{mn} - 2(\iota_R * \mathcal{V})_{mn}$, or equivalently,
$$d\kappa = 4\left(t^{IJ}\Theta_{IJ}\right) - 4s\mathcal{F} - 4\iota_R * \mathcal{V}, \qquad \mathcal{L}_R g = 0. \tag{4.1.11}$$

[1] Defined using $R^m \equiv -(\xi_I \Gamma^m \xi^I)$, and in the sense of general s as we remarked earlier.

Using the above basic properties, one can partially solve

$$\mathcal{F} = -\frac{d\kappa}{4s} - \frac{\Omega^+ + \Omega^-}{s}, \quad \mathcal{V}_H = s^{-1}\left(t^{IJ}\Theta_{IJ} + \Omega^+ - \Omega^-\right). \tag{4.1.12}$$

Recall that the Killing spinor equation enjoys a shifting symmetry, and therefore one can always set $\Omega^- = 0$ in the above solutions; so let us do this. Then we have

$$s\left(\mathcal{F}_H + \mathcal{V}_H\right) = -\frac{d\kappa_H}{4} + t^{IJ}\Theta_{IJ} \tag{4.1.13}$$

To further simplify later discussion, let us apply the results in [20]. The Killing spinor equation and the dilatino equation are solved locally, and it is shown that the auxiliary fields are highly unconstrained by the existence of solutions.

The freedom can be understood by looking at the Fierz identities. In some sense, solving the equations is just to properly match the "Γ-matrices structure" in (4.1.8) and (4.1.6). Note that one can use the Fierz-identities

$$-\frac{1}{4s}\lambda^{KL}(\Theta_{KL})_{mn}\Gamma^{mn}\xi_I = \lambda_I{}^J\xi_J, \quad \lambda^{KL}(\Theta_{KL})_{mn}\Gamma^n\xi_I = -\lambda_I{}^J\left(R_m + s\Gamma_m\right)\xi_J \tag{4.1.14}$$

to alter the Γ-structures. Hence one can adjust the $SU(2)_{\mathcal{R}}$-gauge field $(V_m)_{IJ}$ to cancel terms with Γ-matrices in (4.1.8), and consequently other auxiliary fields are left unconstrained.

We can use the local freedom in s and t_{IJ} to smoothly adjust them such that $s = 1$ and $\mathrm{tr}(t^2) \equiv t_I{}^J t_J{}^I = -1/2$ in a patch. Note that given a global Killing spinor solution, s and $\mathrm{tr}(t^2)$ should be patch-independent functions, and therefore, the adjustment can be made global. Therefore, let us deform the solution and auxiliary fields such that *globally* $s \equiv 1 \Rightarrow \iota_R\mathcal{F} = 0$ and $\mathrm{tr}(t^2) \equiv -1/2$. Furthermore, it is shown in [20] that resulting deformations in the actions are Q-exact, and therefore the above adjustment does not change the expectation values of BPS observables.

3. A special class of solutions

Equation (4.1.13) implies that it is interesting to look at a special class of solutions where the auxiliary fields $\mathcal{F}$ and $\mathcal{V}$ are such that

$$(\mathcal{F} + \mathcal{V}_H) = \Lambda d\kappa \Rightarrow d\kappa = \frac{4}{\Lambda + 1}t^{IJ}\Theta_{IJ}, \quad \iota_R\mathcal{F} = 0. \tag{4.1.15}$$

for some constant $\Lambda \in \mathbb{R}$. This implies κ is a contact 1-form, namely it satisfies (assuming $t_{IJ} \neq 0$)

$$\kappa \wedge d\kappa \wedge d\kappa \quad \propto \quad \kappa \wedge \left(t^{IJ}\Theta_{IJ}\right) \wedge \left(t^{IJ}\Theta_{IJ}\right) \neq 0. \tag{4.1.16}$$

4. Towards a K-contact structure

Now the bi-linears from the special class of solutions satisfy various conditions:

$$\begin{cases} \kappa \wedge d\kappa \wedge d\kappa \neq 0, \quad \kappa\left(R\right) = 1, \quad \iota_R d\kappa = 0 \\ (d\kappa)_{mn} = \dfrac{4}{1+\Lambda}(t\Theta)_{mn}, \quad \mathcal{L}_R g = 0, \quad \kappa_m = g_{mn}R^n \end{cases}. \tag{4.1.17}$$

The first row tells us that (κ, R) defines a contact structure, while the second row implies the contact structure closely resembles a K-contact structure. The only *violation* appears in

$$d\kappa = \frac{4}{1+\Lambda}(t\Theta)_{mn} = \left[\frac{1}{1+\Lambda}\right]\left(2g_{mk}\Phi^k{}_n\right), \quad \Phi^m{}_k\Phi^k{}_n = -\delta^m_n + R^m\kappa_n. \tag{4.1.18}$$

where we defined $\Phi = 2\left(t^{IJ}\Theta_{IJ}\right)$, instead of the standard form

$$(d\kappa)_{mn} = 2g_{mk}\Phi^k{}_n, \quad \Phi^m{}_k\Phi^k{}_n = -\delta^m_n + R^m\kappa_n \tag{4.1.19}$$

It is easy to bring the system to a standard *K-contact structure*. Let us use an adapted veilbein $\{e^A\}$ such that

$$g = \sum_a e^a e^a + \kappa \otimes \kappa, \quad e^5 = \kappa, \quad \iota_R e^{a=1,2,3,4} = 0, \quad \Phi(e^1) = e^2, \Phi(e^3) = e^4. \tag{4.1.20}$$

Define a function λ by $\lambda^2 \equiv (1+\Lambda)^{-1}$, and we rescale the horizontal piece of g by $g \to g' = \sum_a e'^a e'^a + \kappa \otimes \kappa$ with $e'^a = \lambda e^a$.

With the new metric, the quantities (κ, R, g', Φ) defines a standard K-contact structure on M:

$$\begin{cases} \kappa \wedge d\kappa \wedge d\kappa \neq 0, \quad \kappa(R) = 1, \;\; \iota_R d\kappa = 0 \\ (d\kappa)_{mn} = 2g'_{mk}\Phi^k{}_n, \quad \mathcal{L}_R g' = 0, \quad \kappa_m = g_{mn}R^n \end{cases} \tag{4.1.21}$$

Along with the change in metric, one needs to properly deform the auxiliary fields to preserve the equation (4.1.8). By explicitly working out the change in spin connection $\omega^A{}_B$, one can identify the required deformations in $\mathcal{F}$ and $\mathcal{V}$ (both are deformed by multiples of $d\kappa$), which indeed also preserve the condition (4.1.15), and therefore no inconsistency arises. Finally, since the deformed auxiliary fields are independent and unconstrained as shown in [20], the resulting deformations preserves the two equations (4.1.8) and (4.1.6) (and field C can be solved using (4.1.7)), and the actions are deformed by Q-exact, hence do not change the expectation values of BPS observables.

To summarize, any solution to (4.1.8) of the special class can be transformed into a standard one, such that the resulting set of geometric quantities (κ, R, g, Φ) form a K-contact structure. Later we will discuss BPS equations on K-contact and Sasakian backgrounds, where the equations are better behaved than on completely general supersymmetric backgrounds.

4.1.3 K-contact Geometry

In this subsection, we summarize most important aspects and formula of contact geometry that we will frequently use in later discussions. For more detail introduction, readers may refer to appendix [C].

<u>1. *Contact structure*</u>

A contact structure is most conveniently described in terms of a contact 1-form. A contact 1-form on a $2n+1$-manifold is a 1-form κ such that

$$\kappa \wedge (d\kappa)^n \neq 0. \tag{4.1.22}$$

This is analogous to the definition of a symplectic form on an even dimensional manifold.

We can associate quantities (R, g, Φ) to κ called a *contact metric structure*, such that

$$\kappa_m R^m = 1, \quad R^m d\kappa_{mn} = 0, \quad \Phi^m{}_k \Phi^k{}_n = -\delta^m_n + R^m \kappa_n, \quad (d\kappa)_{mn} = 2 g_{mk} \Phi^k{}_n \tag{4.1.23}$$

The vector field R is called the *Reeb* vector field, and Φ is like an almost complex structure in directions orthogonal to R.

On a contact metric 5-manifold, we will frequently use an *adapted vielbein* $\{e^A\}$, $\{e_A\}$, such that $e_5 = R$, $\Phi(e_1) = e_2$, $\Phi(e_3) = e_4$, and

$$d\kappa = 2\left(e^1 \wedge e^2 + e^3 \wedge e^4\right), \quad g = \sum_{a=1,2,3,4} e^a \otimes e^a + \kappa \otimes \kappa, \tag{4.1.24}$$

Note that the first equation implies $d\kappa$ is self-dual, namely $\iota_R * d\kappa = d\kappa$. We will also use the complexification of $\{e^A\}$:

$$\begin{cases} e^{z_i} \equiv e^{2i-1} + ie^{2i}, \quad e^{\bar{z}_i} \equiv e^{2i-1} - ie^{2i}, \quad e^5 = \kappa \\ e_{z_i} \equiv \dfrac{1}{2}\left(e_{2i-1} - ie_{2i}\right), \quad e_{\bar{z}_i} \equiv \dfrac{1}{2}\left(e_{2i-1} + ie_{2i}\right), \quad e_5 = R \end{cases} \tag{4.1.25}$$

so that $\left\{1, \dfrac{1}{\sqrt{2}} e^{\bar{z}_1}, \dfrac{1}{\sqrt{2}} e^{\bar{z}_2}, \dfrac{1}{2} e^{\bar{z}_1} \wedge e^{\bar{z}_2}\right\}$ are orthonormal.

2. *K-contact and Sasakian structure*

A *K-contact structure* is a contact structure κ and the associated (R, g, Φ), such that

$$\mathcal{L}_R g = 0 \quad \Leftrightarrow \quad \nabla_m R_n + \nabla_n R_m = 0 \tag{4.1.26}$$

Note that one immediately has $\mathcal{L}_R \Phi = 0$.

For a general contact structure, the integral curves of R, or equivalently, the 1-parameter diffeomorphisms $\varphi_R(t)$ (the *Reeb flow*) generated by R, can have three types of behavior. The *regular* or *quasi-regular* types are such that the flow are free or semi-free $U(1)$ action, respectively. The *irregular* type is such that the flow is not $U(1)$, and therefore the integral curves of R generally are *not* closed orbits.

Generic irregular Reeb flows are difficult to study, however, situation can be improved when the contact structure is K-contact. In this case, the *closure* of the Reeb flow (it preserves g by definition), viewed as a subgroup of the $\mathrm{Isom}(M, g)$, is a torus $T^k \subset \mathrm{Isom}(M, g)$; k is called the *rank* of the K-contact structure. On a K-contact 5-manifold, $1 \le k \le 3$.

Finally, a *Sasakian structure* is a K-contact structure with additional property

$$\nabla_m \Phi^k{}_n = g_{mn} R^k - \kappa_n \delta^k_m \tag{4.1.27}$$

Sasakian structures are the Kähler structures in the odd-dimensional world. They satisfies certain integrability condition, and all quantities discussed above, as well as some metric connections associated with g, live in great harmony. We will later see that on Sasakian structures, the Higgs branch BPS equations have very simple behavior, very much like Seiberg-Witten equations on Kähler manifolds.

To end this section, we tabulate the correspondence between the structures (including some we haven't mentioned) in even and odd dimensional worlds.

Even	Odd
Symplectic	Contact
Almost Hermitian	K-contact
Complex	Cauchy-Riemann
Kähler	Sasakian
Kähler-Einstein	Sasaki-Einstein
HyperKähler	3-Sasakian

4.2 Higgs Branch Localization and 5d Seiberg-Witten Equation

In this section, we begin by reviewing the 5-dimensional $\mathcal{N}=1$ vector multiplet and hypermultiplet. Then we consider deforming the theory with Q-exact terms to localize the path-integral. We discuss the deformed Coulomb branch solutions and the Higgs branch. We rewrite the Higgs branch equations and interpret them as 5-dimensional generalizations of Seiberg-Witten equations on symplectic 4-manifolds. We also discuss basic properties of solutions to the 5d Seiberg-Witten equations, including their local behavior near closed Reeb orbits.

4.2.1 Vector-multiplet and Hyper-multiplet

1. *Vector-multiplet*

The Grassman odd transformation Q of vector multiplet $(A_m, \sigma, \lambda_I, D_{IJ})$ can be obtained directly from $\mathcal{N}=1$ supersymmetry transformation, which can be obtained by taking the rigid limit of coupled supergravity in [16, 15]. Using a symplectic-Majorana spinor ξ_I satisfying Killing spinor equation (4.1.9), the transformation can be written as

$$\begin{cases} QA_m = i\epsilon^{IJ}\left(\xi_I\Gamma_m\lambda_J\right) \\ Q\sigma = i\epsilon^{IJ}\left(\xi_I\lambda_J\right) \\ Q\lambda_I = -\dfrac{1}{2}F_{mn}\Gamma^{mn}\xi_I + \left(D_m\sigma\right)\Gamma^m\xi_I + D_I{}^J\xi_J + 2\sigma\tilde{\xi}_I \\ QD_{IJ} = -i\left(\xi_I\Gamma^m D_m\lambda_J\right) + \left[\sigma, \left(\xi_I\lambda_J\right)\right] + i(\tilde{\xi}_I\lambda_J) - \dfrac{i}{2}\mathcal{P}_{mn}(\xi_I\Gamma^{mn}\lambda_J) + (I \leftrightarrow J) \end{cases}, \tag{4.2.1}$$

where $D_m(\cdot) = \nabla_m - i\left[A_m, \cdot\right]$, and $\tilde{\xi}_I$ is defined in (4.1.9). Here the spinor ξ_I is Grassman even. The transformation squares to

$$Q^2 = -i\mathcal{L}_R^A + \mathcal{G}_{s\sigma} + \mathcal{R}_{R_I{}^J} + L_\Lambda \tag{4.2.2}$$

where $\mathcal{G}$ is gauge transformation, $\mathcal{R}$ is $SU(2)_\mathcal{R}$ rotation acting on a generic field X_I as $\mathcal{R}_{R_I{}^J} X_I = R_I{}^J X_J$, and L is Lorentz rotation acting on spinors. The parameters are

$$\begin{cases} R^m = -(\xi_I \Gamma^m \xi^I) \\ s = (\xi_I \xi^I) \end{cases}, \quad \begin{cases} \Lambda_{mn} = (-2i)\left((\xi_J \Gamma_{mn} \tilde{\xi}^J) - s\left(\mathcal{P}^+_{mn} - \mathcal{P}^-_{mn}\right)\right) \\ R_I{}^J = 2i\left[3(\xi_I \tilde{\xi}^J) + \mathcal{P}^{mn}\left(\Theta_I{}^J\right)_{mn}\right] \end{cases} \tag{4.2.3}$$

and we used the vector field R^m to define self-duality $\Omega^\pm_H(M)$, see (2.2.26).

Note that, similar to [14], there is a term in $\delta^2 D_{IJ}$ that breaks the closure of the supersymmetry algebra, of the form

$$\delta^2 D_{IJ} = \ldots + \sigma\left[(\xi_I \Gamma^m \nabla_m \tilde{\xi}_J) + \frac{1}{2}\mathcal{P}^{mn}(\xi_I \Gamma_{mn} \tilde{\xi}_J) + (I \leftrightarrow J)\right]. \tag{4.2.4}$$

Such a term vanishes if there exists a function u and a vector field v_m such that

$$\nabla\!\!\!\!/\, \tilde{\xi}_I + \frac{1}{2}\mathcal{P}_{mn}\Gamma^{mn}\tilde{\xi}_I = u\xi_I + v_m \Gamma^m \xi_I \tag{4.2.5}$$

In the case of $\mathcal{P}_{mn} = 0$, one can show that $v = 0$ and the function u always exists and is proportional to the scalar curvature of the metric $(g, \nabla^{\mathrm{LC}})$. In the presence of $\mathcal{P}_{mn}$, by explicitly expanding every term, one can show that

$$\begin{aligned} &\nabla\!\!\!\!/\, \tilde{\xi}_I + \frac{1}{2}\mathcal{P}^{mn}\Gamma_{mn}\tilde{\xi}_I \\ =& \left(\nabla_m t_I{}^J\right)\Gamma^m \xi_J + \nabla_m \mathcal{V}^{mn}\Gamma_n \xi_I + t\left(\mathcal{F}_{mn} + 2\mathcal{V}_{mn}\right)\Gamma^{mn}\xi_I + \frac{1}{4}\mathcal{F}_{kl}\mathcal{F}_{mn}\Gamma^{mnkl}\xi \\ &+ \frac{3}{2}\mathcal{F}_{mn}\mathcal{F}^{mn}\xi - 2\mathcal{F}_{mn}\mathcal{V}^{mn}\xi + 5(t_I{}^K{}_K{}^J)\xi_J - \nabla_m\left(\mathcal{V}^{mn} - \mathcal{F}^{mn}\right)\Gamma_n \xi. \end{aligned} \tag{4.2.6}$$

We observe that the first row is just the left hand side of (4.1.6), and therefore, recalling $(t_I{}^K{}_K{}^J)\xi_J = 1/2\left(t_L{}^K t_K{}^L\right)\xi_I$,

$$\begin{aligned} &\nabla\!\!\!\!/\, \tilde{\xi}_I + \frac{1}{2}\mathcal{P}^{mn}\Gamma_{mn}\tilde{\xi}_I \\ =& \left[\frac{5}{2}\left(t_L{}^K t_K{}^L\right) - \frac{1}{4}C + \frac{3}{2}\mathcal{F}_{mn}\mathcal{F}^{mn} - 2\mathcal{F}_{mn}\mathcal{V}^{mn}\right]\xi_I - \nabla_m\left(\mathcal{V}^{mn} - \mathcal{F}^{mn}\right)\Gamma_n \xi. \end{aligned} \tag{4.2.7}$$

Namely, we found the required function and the vector field to be

$$\begin{cases} u = \frac{5}{2}\left(t_L{}^K t_K{}^L\right) - \frac{1}{4}C + \frac{3}{2}\mathcal{F}_{mn}\mathcal{F}^{mn} - 2\mathcal{F}_{mn}\mathcal{V}^{mn} \\ v^n = \nabla_m\left(\mathcal{F}^{mn} - \mathcal{V}^{mn}\right), \end{cases} \tag{4.2.8}$$

We therefore confirmed that the term (4.2.4) vanishes happily, thanks to (4.1.6). Finally, we point out that function u will appear in the supersymmetric Yang-Mills Lagrangian for the vector multiplet (which is denoted as P in [20]), in the form of

$$\mathcal{L}_{\mathrm{YM}} = \ldots - 4u\sigma^2 + 4i\sigma F_{mn}\mathcal{P}^{mn} - \mathcal{P}_{mn}\left(\lambda_I \Gamma^{mn}\lambda^I\right). \tag{4.2.9}$$

2. *Hypermultiplet*

A hypermultiplet in 5-dimension consists of a set of scalars ϕ_I^A, two spinors ψ^A and a set of auxiliary scalars $\Xi_{I'}^A$. Here $I, I' = 1,2$ are two *different* copies of $SU(2)$ indices (in particular, I corresponds to the $SU(2)_{\mathcal{R}}$-symmetry), while $A = 1,2$ is a separate $Sp(1)$ index. They satisfy reality conditions

$$\overline{\phi_I^A} = \epsilon^{IJ}\Omega_{AB}\phi_J^B, \quad \overline{\psi^{A\alpha}} = \Omega_{AB}C_{\alpha\beta}\psi^{B\beta}, \quad \overline{\Xi_{I'}^A} = \Omega_{AB}\epsilon^{I'J'}\Xi_{J'}^B. \tag{4.2.10}$$

In the above, Ω_{AB} is the invariant $Sp(1)$ tensor $\Omega_{12} = -\Omega_{21} = 1$.

The reality conditions reduces the independent components. The field ϕ_I^A can be represented by two complex scalar $\phi^{1,2}$

$$\phi_{I=1}^A = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}, \quad \phi_{I=2}^A = \frac{1}{\sqrt{2}}\begin{pmatrix} -\overline{\phi^2} \\ \overline{\phi^1} \end{pmatrix} \tag{4.2.11}$$

and similarly for the field $\Xi_{I'}^A$. The field ψ^A can be represented in terms of one spinor ψ

$$\psi^A = \begin{pmatrix} \psi \\ -C\bar{\psi} \end{pmatrix} \tag{4.2.12}$$

In the following, we couple the hypermultiplet to a $U(N_c)$ vector multiplet by setting the independent fields to be in appropriate representation of $U(N)$, for instance,

$$\phi^1 : \mathrm{N}, \qquad \phi^2 : \bar{\mathrm{N}}, \qquad \psi : \mathrm{N}, \qquad \bar{\psi} : \bar{N} \tag{4.2.13}$$

We define D_m on any field Φ in hypermultiplet as $D_m\Phi = \nabla_m\Phi - iA_m(\Phi)$, where ∇_m may contain spin connection and $SU(2)_{\mathcal{R}}$-the background gauge field $(V_m)_{IJ}$.

It is well-known that one cannot write down an off-shell supersymmetry transformation for a hypermultiplet with finitely many auxiliary fields. But it is possible to write down a Grassmann odd transformation Q which squares to bosonic symmetries. As transformation parameters, we use a symplectic-Majorana spinor ξ_I satisfying Killing spinor equation (4.1.9), and an additional $SU(2)'$-symplectic-Majorana spinor $\hat{\xi}_{I'}$, satisfying

$$(\hat{\xi}_I\hat{\xi}^I) = (\xi_I\xi^I) = s, \;\; (\xi_I\Gamma^m\xi^I) = -R^m = -(\hat{\xi}_I\Gamma^m\hat{\xi}^I), \;\; (\hat{\xi}_{I'}\xi_J) = 0. \tag{4.2.14}$$

One can view $\hat{\xi}_{I'}$ as a orthogonal complement of ξ_I in the spinor space, and therefore corresponds to anti-chiral spinors, in the sense that $\Gamma_C\xi_I = s\xi_I$, $\Gamma_C\hat{\xi}_{I'} = -s\hat{\xi}_{I'}$ where $\Gamma_C \equiv -R^m\Gamma_m$. Using the Fierz identities, one can show completeness relations for an arbitrary spinor ς (see appendix [A]):

$$\xi_I\left(\xi^I\varsigma\right) = -\frac{1}{4}\left(s + \Gamma_C\right)\varsigma \xrightarrow{s=1} -\frac{1}{2}P_+\varsigma, \quad \hat{\xi}_{I'}(\hat{\xi}^{I'}\varsigma) = -\frac{1}{4}\left(s - \Gamma_C\right)\varsigma \xrightarrow{s=1} -\frac{1}{2}P_-\varsigma. \tag{4.2.15}$$

The Grassman odd transformation Q is as follows:

$$\begin{cases} Q\phi_I^A = -2i\left(\xi_I\psi^A\right) \\ Q\psi^A = \epsilon^{IJ}\Gamma^m\xi_I D_m\phi_J^A + i\epsilon^{IJ}\xi_I\sigma\phi_J^A - 3\tilde{\xi}^I\phi_I^A + \mathcal{P}_{pq}\epsilon^{IJ}\Gamma^{pq}\xi_I\phi_J^A + \epsilon^{I'J'}\hat{\xi}_{I'}\Xi_{J'} \\ Q\Xi_{J'}{}^A = 2\hat{\xi}_{J'}\left(i\Gamma^m D_m\psi^A + \sigma\psi^A + \epsilon^{KL}\lambda_K\phi_L^A - \dfrac{i}{2}\mathcal{P}_{pq}\Gamma^{pq}\psi^A\right) \end{cases}. \tag{4.2.16}$$

The transformation squares to the bosonic symmetries

$$Q^2 = -i\mathcal{L}_R^A + \mathcal{G}_{s\sigma} + \mathcal{R}_{R_I{}^J} + \mathcal{R}_{\hat{R}_{I'}{}^{J'}} + L_\Lambda. \tag{4.2.17}$$

where $\mathcal{G}$ is the gauge transformation, $\mathcal{R}$ is $SU(2)$ rotations on I, J and I', J' indices, L is Lorentz rotation; the parameters are

$$\begin{cases} \Lambda_{mn} = (-2i)\left((\xi_J\Gamma_{mn}\tilde{\xi}^J) - s\left(\mathcal{P}^+_{mn} - \mathcal{P}^-_{mn}\right)\right) \\ R_I{}^J = 2i\left[3(\xi_I\tilde{\xi}^J) + \mathcal{P}^{mn}\left(\Theta_I{}^J\right)_{mn}\right] \\ \hat{R}_{I'}{}^{J'} = (-2i)\left[\left(\hat{\xi}_{I'}\Gamma^m\nabla_m\hat{\xi}^{J'}\right) - \frac{1}{2}\mathcal{P}_{mn}\left(\hat{\xi}_{I'}\Gamma^{mn}\hat{\xi}^{J'}\right)\right] \end{cases} . \tag{4.2.18}$$

As in previous sections we define the function $s \equiv (\xi_I\xi^I)$, and $\Omega^\pm_H(M)$ is defined with respect to the vector field $R^m \equiv -(\xi_I\Gamma\xi^I)$.

4.2.2 Twisting, Q-exact Deformations and Localization Locus

In this subsection, we first review a redefinition (the twisting) of field variables in vector multiplet and hypermultiplet. Then using the redefined variables, we introduce the Q-exact deformation terms and derive the localization locus. Here we explicitly used gauge group $U(N_c)$, but in general one can choose gauge groups with $U(1)$-components.

The twisting

First introduced in [12][13] in the context of Sasaki-Einstein backgrounds, all field variables with I or I' indices can be "twisted" (invertible using Fierz-identities (A.2.6)) using ξ_I and $\hat{\xi}_{I'}$. In our situation, assuming $s = 1$ and recalling (4.1.12), we define:

$$\begin{cases} \Psi_m \equiv \left(\xi_I\Gamma_m\lambda^I\right), \quad \chi_{mn} \equiv \left(\xi_I\Gamma_{mn}\lambda^I\right) + (\kappa_m\Psi_n - \kappa_n\Psi_m) \\ H = 2F_A^+ + D^{IJ}\Theta_{IJ} + \sigma\left(2t^{IJ}\Theta_{IJ} + d\kappa^+ + 4\Omega^+\right) \end{cases}, \quad \begin{cases} \phi^A_+ \equiv \epsilon^{IJ}\xi_I\phi^A_J \\ \Xi^A_- \equiv \epsilon^{I'J'}\hat{\xi}_{I'}\Xi^A_{J'} \end{cases} \tag{4.2.19}$$

After such redefinitions, χ and H are both horizontal self-dual two forms with respect to vector field R^m, ϕ^A_+ are chiral spinors[2] while $\Xi^{A=1,2}_-$ are anti-chiral.

In terms of these twisted field variables, the originally complicated BRST transformations can

[2] More explicitly, with the gauge index in place,

$$\left(\phi_+^{A=1}\right)^a = \xi^1\left(\phi^{A=1}\right)^a + \xi^2(-\overline{\phi^{A=2}})^a \tag{4.2.20}$$

.

be rewritten into very simple forms:

$$\begin{cases} QA = i\Psi \\ Q\sigma = -i\iota_R\Psi \\ Q\Psi = -\iota_R F_A + d_A\sigma \\ Q\chi = H \\ QH = -i\mathcal{L}_R^A\chi - [\sigma,\chi] \end{cases}, \quad \begin{cases} Q\phi_+^A = iP_+\psi^A \\ Q\psi^A = \not{D}\phi_+^A + i\sigma\phi_+^A + \frac{1}{8}(d\kappa)_{mn}\Gamma^{mn}\phi_+^A + \Xi_-^A \\ Q\Xi_-^A = -iP_-\not{D}\psi^A - \sigma P_-\psi^A - \Psi^m\left(\Gamma_m + R_m\right)\phi_+^A \end{cases}. \tag{4.2.21}$$

In order to derive $Q\psi^A$ and $Q\Xi_-^A$, one needs to use the symmetry $(\xi_I\tilde{\xi}_J) = (\xi_J\tilde{\xi}_I)$ and completeness relations (4.2.15). Also we will use $d\kappa \cdot \phi_+ \equiv 1/2(d\kappa)_{mn}\Gamma^{mn}\phi_+$ to simplify the notations in the following discussions.

For later convenience, we separate $Q\psi$ into chiral and anti-chiral part:

$$Q\psi_+^A = P_+\not{D}\phi_+^A + i\sigma\phi_+^A + \frac{1}{4}d\kappa\cdot\phi_+^A, \qquad Q\psi_-^A = P_-\not{D}\phi_+^A + \Xi_-^A, \tag{4.2.22}$$

which implies that

$$Q^2 = -i\left(R^mD_m + \frac{1}{4}d\kappa\cdot\right) - \sigma \tag{4.2.23}$$

Note that $d\kappa$ is horizontal, and therefore its Clifford multiplication does not change chirality, similar to that in 4-dimension. Also, the new spinorial variables have reality condition, for instance, where C is the charge conjugation matrix,

$$\overline{\phi_+^A} = \Omega_{AB}C\phi_+^B \tag{4.2.24}$$

Q-exact terms

We are now ready to introduce the Q-exact terms. There are three of them[3]

$$\begin{cases} QV_{\text{Vect}} = Q\int \operatorname{Tr}\left(\chi\wedge *\left(2F_A^+ - H\right) + \frac{1}{2}\Psi\wedge *Q\bar{\Psi}\right) \\ QV_{\text{Hyper}} = Q\int_M \Omega_{AB}\overline{Q\psi^A}\psi^B \\ QV_{\text{Mixed}} = Q\int_M \operatorname{Tr}\left[2\chi\wedge *h\left(\phi_+\right)\right] \end{cases}, \tag{4.2.25}$$

where h maps the "spinor" ϕ_+^A in the hypermultiplet to a *adjoint-valued* self-dual 2-form $h(\phi_+)$. Its explicit form will be given in

$$h\left(\phi\right) = \alpha\left(\phi\right) - \frac{\zeta}{2}d\kappa^+ - F^+_{A_0/2}, \tag{4.2.26}$$

where $\zeta \sim \zeta 1_{N_c\times N_c}$ is a "fake" FI-parameter taking value in the $\mathfrak{u}(1)$-component of the Lie-algebra $\mathfrak{u}(N_c)$, A_0 is a non-dynamical gauge field which we put in *by hand* for later computations, taking

[3] In the second line, expanding the terms and using the reality, one obtains, for instance the kinetic term $D_m\phi^{A=1,a}D^m\overline{\phi_a^{A=1}} + D_m\phi_a^{A=2}D_m\overline{\phi^{A=2,a}}$, where a is the gauge index that were suppressed.

value in the $\mathfrak{u}(1)$ in $\mathfrak{u}(N_c)$ with the property $\iota_R F_{A_0/2} = 0$ ($F_{A_0/2} = 1/2 dA_0$)[4]. α is an *adjoint-valued* bilinear map from chiral spinors to self-dual 2-forms, whose explicit form will be given in a spinor basis later on, schematically of the form

$$\alpha_{mn}(\phi)^a{}_b = (\phi_+^{A=1,a}\Gamma_{mn}\overline{\phi_{+,b}^{A=1}}), \tag{4.2.27}$$

Up to this point, other than $s = 1$, we make no assumption on the background geometry. Hence $d\kappa$ does not have to be self-dual; $d\kappa^+$ means we extract the self-dual part from $d\kappa$. To ensure positivity, we need to analytically continue $\sigma \to -i\sigma$, $\Xi_-^A \to i\Xi_-^A$.

Now one can expand all terms, and integrate out auxiliary field H, or equivalently, impose the field equation of H:

$$H = F_A^+ + h\left(\phi\right). \tag{4.2.28}$$

Then the bosonic Q-exact terms reads

$$\left(F_A^+ + h(\phi_+)\right)^2 + \frac{1}{2}(\iota_R F_A)^2 + (d_A\sigma)^2 + \left|D_A\phi_+ + \frac{1}{4}d\kappa\cdot\phi_+\right|^2 + \Xi_-^2 + |\sigma\phi|^2, \tag{4.2.29}$$

and therefore, we have the localization locus

$$\begin{cases} F_A^+ + h\left(\phi_+\right) = 0 \\ \not{D}_A\phi_+^A + \dfrac{1}{4}d\kappa\cdot\phi_+^A = 0 \end{cases}, \qquad \begin{cases} \iota_R F_A = 0 \\ d_A\sigma = 0 \\ \Xi_-^{A=1,2} = 0 \\ \sigma\left(\phi_+^A\right) = 0 \end{cases}. \tag{4.2.30}$$

Note that using the reality condition of ϕ_+^A, the second equation on the left is equivalent to that of one component $\phi_+ \equiv \phi_+^{A=1}$

$$\not{D}_A\phi_+ + \frac{1}{4}d\kappa\cdot\phi_+ = 0 \tag{4.2.31}$$

and similarly $\sigma\left(\phi_+\right) = 0 \Leftrightarrow \sigma\left(\phi_+^A\right) = 0$. Therefore, in the following, we will just ignore the index A, and regard ϕ_+ as in the fundamental representation of gauge group $G = U(N_c)$.

4.2.3 Deformed Coulomb Branch

The deformed Coulomb branch is the class of solutions to (4.2.30) such that $\phi_+^A = 0$. Then the equations reduces to

$$d_A\sigma = 0, \qquad F_A^+ - F_{A_0/2}^+ = \frac{\zeta}{2}d\kappa^+, \quad \iota_R F_A = 0 \tag{4.2.32}$$

This is a deformed version of the contact-instanton equation introduced in [13]. The undeformed version is later studied in [36, 37, 38, 39], in the context of κ being a contact structure. So in principle, there could be a tower of instantonic solutions, very much like the deformed instantons in 4d.

[4]It is straight-forward to generalize to other gauge groups with $U(1)$ components generated by h_a. There one picks $\zeta = \zeta^a h_a$, and A_0 takes value in the diagonal h_1 proportional to identity. For gauge groups without any $U(1)$-components, one cannot perform the Higgs branch localization described in this article.

To be more concrete, we consider the case when κ is a contact 1-form. Then $d\kappa^+ = d\kappa$, and one immediately has a most simple solution (assuming $\iota_R F_{A_0/2} = 0$)

$$A = \frac{\zeta}{2}\kappa + \frac{1}{2}A_0 \tag{4.2.33}$$

where σ takes constant value in the Lie-algebra $\mathfrak{g}$. On top of these simple solutions, one may have a lot of instantonic solutions.

When (κ, R, g, Φ) give rise to a Sasakian structure, the reference A_0 can be chosen to be the restriction on K_M of the Chern connection on $K_{C(M)}$, where $C(M)$ is the Kahler cone of Sasakian manifold M. In such case, one can show that $dA_0 \propto d\kappa$ and $\iota_R F_{A_0/2} = 0$.

4.2.4 5d Seiberg-Witten Equation

Let us consider other classes of solutions to (4.2.30), with non-vanishing ϕ_+. To be concrete in many statements, we will focus on the case where (κ, R, g, Φ) form a K-contact structure, or Sasakian structures to ensure concrete existence of solutions. This will allow us to rewrite the equations in a very geometric way that resembles the 4-dimensional Seiberg-Witten equation on symplectic manifolds. We will see that Sasakian structures serve as examples where Higgs vacua always exist, and other non-trivial solutions have nice behavior. We also discuss the case of general K-contact structures.

The algebraic equation

When we look for non-vanishing solution of ϕ_+, one of the non-trivial BPS equations is $(\sigma + m)(\phi_+) = 0$, where we have restored some masses for the hypermultplets by giving VEV to the scalars in the background vector multiplets that gauge the flavor symmetry. Let us consider gauge group $G = U(N_c)$ and N_f hypermultiplets, then we need to solve a matrix equation $\left(\sigma^a{}_b + m_i{}^j\right)\phi^b_j = 0$, where $a, b = 1, ..., N_c$ are gauge indices, while $i, j = 1, ..., N_f$ are flavor indices. After diagonalizing $m^i{}_j = \mathrm{diag}(m_1, ..., m_{N_f})$, one observes that, assuming $N_c \leq N_f$, any solution is determined by an ordered subset of integers $\{n_1, ..., n_{N_c}\}$ of size N_c

$$\sigma^a{}_b = -m_{n_a}\delta^a_b, \quad \phi^a_i \sim \delta_{i,n_a}, \quad \{n_1, ..., n_{N_c}\} \subset \{1, ..., N_f\}\,. \tag{4.2.34}$$

Therefore N_c among the N_f of ϕ's are selected to have non-zero values. The remaining $N_f - N_c$ of ϕ's are fixed to be zero, and trivially satisfy all other BPS equations. These vanishing components do not have further non-trivial solutions which we will discuss shortly. The 1-loop determinants for the trivial components will be the same as that in the Coulomb branch, with the argument σ replaced by solutions (4.2.34).

The selected N_c $(< N_f)$ non-zero components, on the other hand, requires extra care. First of all, given generic masses $\{m_{n_a} \neq m_{n_b}$ if $a \neq b\}$, equation $d_A\sigma = 0$ implies A is also completely diagonalized. Therefore, in such favorable situations, the gauge group $U(N_c)$ is completely broken to $U(1)^{N_c}$, which acts as phase rotations on the N_c non-zero components of ϕ. For each of these components, one only needs to consider a $U(1)$-gauge field, which we will assume from now on.

These non-zero components will have to satisfy the remaining BPS equations individually, to which we will discuss the solutions shortly. To do so, we will first rewrite the remaining BPS equations in a more familiar form.

Rewriting the localization locus

In the appendix [C][D], we review in detail Spin$^{\mathbb{C}}$ spinors and corresponding Dirac operators on any 5-dimensional K-contact structures. We summarize here several most relevant aspects:

- The spinor bundle S has a canonical Dirac operator $\nabla\!\!\!\!/^{\,\mathrm{TW}}$, induced from generalized Tanaka-Webster connection on TM for any given K-contact structure[40][41][42]. One can show that this Dirac operator can be written in terms of the Levi-Civita connection ∇^{LC}:

$$\nabla\!\!\!\!/^{\,\mathrm{TW}} = \nabla\!\!\!\!/^{\,\mathrm{LC}} + \frac{1}{8}(d\kappa)_{mn}\Gamma^{mn} \Rightarrow \begin{cases} P_- \nabla\!\!\!\!/^{\,\mathrm{TW}}\phi_+ = P_- \nabla\!\!\!\!/^{\,\mathrm{LC}}\phi_+ \\ P_+ \nabla\!\!\!\!/^{\,\mathrm{TW}}\phi_+ = P_+ \nabla\!\!\!\!/^{\,\mathrm{LC}}\phi_+ + \dfrac{1}{4}d\kappa\cdot\phi_+ \\ \qquad\qquad\qquad = -\left(\nabla^{\mathrm{LC}}_R\phi_+ + \dfrac{1}{4}d\kappa\cdot\phi_+\right) \end{cases} \tag{4.2.35}$$

  which are precisely the ones appearing in $Q\psi_\pm$ without the gauge field A.

- There exists a *canonical* Spin$^{\mathbb{C}}$-bundle $W^0 = T^{0,\bullet}M_H^*$, with chiral decomposition

$$W_+^0 = T^{0,0}M_H^* \oplus T^{0,2}M_H^*, \quad W_-^0 = T^{0,1}M_H^* \tag{4.2.36}$$

  and determinant line bundle $K_M \equiv T^{0,2}M_H^*$. Any other Spin$^{\mathbb{C}}$-bundle W can be written as $W = W^0 \otimes E$ for some $U(1)$-line bundle E. It is important to note that, when the manifold is spin, namely when the genuine spinor bundle exists, then S and W^0 is related by $S \otimes K_M^{1/2} = W^0 \Rightarrow S_+ = K_M^{-1/2} \otimes K_M^{1/2}$. Therefore W can also be written as $W = S \otimes \mathcal{L}$ where $\mathcal{L} = K_M^{1/2} \otimes E$.

- On K_M there exists a *canonical* $U(1)$ connection A_0, such that the Dirac operator (induced from ∇^{TW} on TM and $A_0/2$ on $K_M^{1/2}$) on the canonical Spin$^{\mathbb{C}}$-bundle W^0 satisfies the identity[5]

$$D^{\mathrm{TW}}_{A_0/2} = \mathcal{L}_R \oplus \sqrt{2}\left(\bar\partial + \bar\partial^*\right) : \Omega^{0,\mathrm{even}} \to \Omega^{0,\mathrm{even}} \oplus \Omega^{0,\mathrm{odd}} \tag{4.2.37}$$

Now we can include the gauge field A onto the stage. As discussed above, we only consider $G = U(1)$ and A is viewed as a $U(1)$-connection of certain line bundle $\mathcal{L}$. Therefore, ϕ_+ should be really considered as a section of $W_+ \equiv S_+ \otimes \mathcal{L}$. We decompose $\mathcal{L} = K_M^{1/2} \otimes E$ so that $S \otimes \mathcal{L} = W^0 \otimes E$, and we also decompose the gauge field A according to

$$\begin{array}{rcccccc} \phi_+ \in W_0^+ \otimes E = S_+ \otimes & K_M^{1/2} & \otimes & E & & \\ & A_0/2 & + & a & = & A. \end{array} \tag{4.2.38}$$

[5] It is the restriction onto K_M^{-1} of the Chern connection defined on $TC(M)$, where $C(M)$ is the almost hermitian cone over the K-contact 5-manifold M; however, there are other choices (induced by ∇^{TW} discussed in [41], for instance) of A_0 that leads to similar identification, with the only difference that $\mathcal{L}_R$ is replaced by $\mathcal{L}_R - ia_0(R)$ for some appropriate $U(1)$ gauge field a_0.

Therefore, the Dirac operator $\not{D}_A^{\mathrm{TW}}$ on $W_+ = W_0^+ \otimes E$ can be identified as

$$\not{D}_A + \frac{1}{8} d\kappa_{mn}\Gamma^{mn} = \not{D}_A^{\mathrm{TW}} = \mathcal{L}_R^a \oplus \sqrt{2}\left(\bar{\partial}_a + \bar{\partial}_a^*\right) : W_+ \to W_+ \oplus W_-. \tag{4.2.39}$$

where $\mathcal{L}_R^a = \mathcal{L}_R - ia\left(R\right)$, $\bar{\partial}_a = \bar{\partial} - ia^{0,1}$ and so forth.

With such identification in mind, one can rewrite the Dirac-like equation in (4.2.30)

$$\not{D}_A \phi_+ + \frac{1}{8} d\kappa_{mn}\Gamma^{mn}\phi_+ = \not{D}_A^{\mathrm{TW}}\phi_+ = 0 \Leftrightarrow \mathcal{L}_R^a\phi_+ = 0, \quad \left(\bar{\partial}_a + \bar{\partial}_a^*\right)\phi_+ = 0. \tag{4.2.40}$$

In particular, we write $\phi_+ = \alpha \oplus \beta \in \Omega^{0,0}\left(E\right) \oplus \Omega^{0,2}\left(E\right)$, and (4.2.30) can be written as

$$\begin{cases} F_a^{d\kappa} = \frac{1}{2}\left(\zeta - |\alpha|^2 + |\beta|^2\right) d\kappa \\ F_a^{0,2} = 2i\bar{\alpha}\beta \\ \bar{\partial}_a \alpha + \bar{\partial}_a^* \beta = 0 \\ \mathcal{L}_R^a \alpha = \mathcal{L}_R^a \beta = 0 \end{cases}, \qquad \begin{cases} \iota_R F_a + \iota_R F_{A_0/2} = 0 \\ d_A \sigma = 0 \\ \Xi_-^{A=1,2} = 0 \\ \sigma\left(\alpha\right) = \sigma\left(\beta\right) = 0 \end{cases} \tag{4.2.41}$$

where we have decompose $F_a^+ = F_a^{d\kappa} + F_a^{2,0} + F_a^{0,2}$, and the bilinear map $\alpha(\phi)$ is written more concretely as (see appendix [A, D] for choice of basis and matrix representation of Γ_{AB})

$$\alpha\left(\phi\right) \equiv \frac{1}{2}\left(|\alpha|^2 - |\beta|^2\right) d\kappa + 2i\left(\alpha\bar{\beta} - \bar{\alpha}\beta\right), \tag{4.2.42}$$

It is clear that the equations on the *left* take a similar form of ζ-perturbed Seiberg-Witten equations on a symplectic 4-manifold[43, 44, 45], and therefore we will call them the *5d Seiberg-Witten equations* in the following discussion.

Let us pause to remark that, the operator $\not{\nabla} + 1/8 d\kappa_{mn}\Gamma^{mn}$ is discussed in the context of Sasaki-Einstein manifold, and similar results were obtained in [25]. The unperturbed version of Seiberg-Witten-like equation on a contact metric manifold is also proposed in [42].

In the following we will focus on equations on the left in (4.2.41). They are a novel type of equations that awaits more study. Let us try to make a first step to understanding the solutions. As discussed earlier, we consider the gauge group $G = U(1)$, and therefore σ and ζ are just real constants.

A Higgs vacuum

First, we argue that the 5d Seiberg-Witten equations on Sasakian structures have one simple solution.

First of all, on any K-contact structure, $(\alpha, \beta) = (\sqrt{\zeta}, 0)$, together with $a = 0$, or equivalently $A = 1/2 A_0$, is obviously a solution to the 5d Seiberg-Witten equations.

The remaining BPS equation is

$$\iota_R F_{A_0/2} = 0 \tag{4.2.43}$$

If A_0 is chosen to be induced from 6d Chern connection, this may be not true on a general K-contact background; however, if the K-contact structure is Sasakian, then (4.2.43) indeed holds

[42][40]. Therefore on a Sasakian structure, one always has at least one most simple solution, which we will call a *Higgs vacuum*.

Properties of general solutions

Let us now focus on the 5d Seiberg-Witten equations on a K-contact structure (with emphasis on Sasakian structures). First of all, the Dirac equations imply

$$\begin{aligned} &\bar{\partial}_a \bar{\partial}_a \alpha + \bar{\partial}_a \bar{\partial}_a^* \beta = 0 \Rightarrow -i F_a^{0,2} \alpha - N\left(\partial_a \alpha\right) + \bar{\partial}_a \bar{\partial}_a^* \beta = 0 \\ &\Rightarrow 2 \int_M |\alpha|^2 |\beta|^2 - \int_M \beta \wedge *_{\mathbb{C}} N\left(\partial_a \alpha\right) + \int_M \left|\bar{\partial}_a^* \beta\right|^2 = 0. \end{aligned} \tag{4.2.44}$$

where N is the Nijenhuis tensor $N : T^{1,0}M_H^* \to T^{0,2}M_H^*$, which vanishes for any Sasakian structure. Therefore, when (κ, R, g, Φ) is Sasakian, one has

$$\bar{\partial}_a^* \beta = \bar{\partial}_a \alpha = |\alpha|\,|\beta| = 0. \tag{4.2.45}$$

Namely, either α or β must vanish, and the two types of solutions are

$$\text{Sasakian:} \quad \begin{cases} \beta = 0 \\ \bar{\partial}_a \alpha = 0 \end{cases} \quad \text{or} \quad \begin{cases} \alpha = 0 \\ \bar{\partial}_a^* \beta = 0 \end{cases}. \tag{4.2.46}$$

However, unlike the case of 4-dimensional Kahler manifold, at the moment we do not have a topological characterization of the two types of solutions. Let us consider the curvature equation integrated over M

$$\int_M F_a^{d\kappa} \wedge * d\kappa = \int_M F_a^{d\kappa} \wedge \kappa \wedge d\kappa = \frac{1}{2} \int_M \left(\zeta - |\alpha|^2 + |\beta|^2\right) d\kappa \wedge * d\kappa. \tag{4.2.47}$$

In the case of a 4-dimensional Kahler manifold, the left hand side would be replaced by the intersection number $c_1(E) \cdot [\omega]$, a topological number independent on ζ. Therefore, when $\zeta = 0$, the sign of $c_1(E) \cdot [\omega]$ will determine whether α or β will survive; in particular, in the limit $\zeta \gg +1$, only the solutions with $\beta = 0$ survive. On a 5-dimensional Sasakian manifold, however, the left hand side is not a topological number, and therefore at the moment we do not have a topological criteria to determine which of the (4.2.46) will survive.

For non Sasakian K-contact structure, one needs to take the Nijenhuis tensor into account. Combining the Weitzenbock formula, Kahler identities and triangle inequalities, we obtain several estimates (where we rescaled $(\alpha, \beta) \to (\sqrt{\zeta}\alpha, \sqrt{\zeta}\beta)$, z is some constant, and $\lambda > 1$ is a real constant)

$$\begin{aligned} 2 \int_M F_a^{d\kappa} \wedge * d\kappa \geqslant & \left(1 - \frac{2z}{\zeta}\right) \int_M \left|d_a^J \alpha\right|^2 \\ & + 2\zeta \int_M \left(1 - |\alpha|^2\right)^2 + 2\zeta \int_M |\alpha|^2 |\beta|^2 + 2\zeta \left(1 - \frac{1}{\lambda}\right) \int_M |\beta|^2 \end{aligned}, \tag{4.2.48}$$

and

$$\int_M \rho_{A_0} |\beta|^2 + \frac{1}{2} \int_M \left|\nabla_{A_0 + a} \beta\right|^2 + \zeta \int |\beta|^4 + \frac{\zeta}{2} \int |\beta|^2 < \frac{z}{\zeta} \int \left|d_a^J \alpha\right|^2, \tag{4.2.49}$$

In the inequalities, ∇_{A_0+a} is the connection on $K_M \otimes E$, ρ_{A_0} is some function depending on A_0 but not on ζ. Again, *if* the integral on the left in the first estimate is bounded from above, or it scales at most of order $\zeta^{\epsilon<1}$ ($\epsilon = 0$ in 4-dimension, since it is topological and independent on ζ), then the above estimate tells us as $\zeta \to +\infty$, almost everywhere on M

$$|\beta| \to 0, \quad |\alpha| \to 1, \tag{4.2.50}$$

and $|d^a_J\alpha|$ does not grow faster than ζ. The second estimate then implies the overall derivative $\nabla_{A_0+a}\beta \to 0$ faster than $\zeta^{\epsilon-1}$, and therefore $\left|\bar{\partial}^*_a\beta\right| = \left|\bar{\partial}_a\alpha\right| \to 0$ as well.

Therefore, let us make a bold conjecture that we have a similar situation as in 4-dimension. Namely for a general K-contact manifold, as $\zeta \to +\infty$, β is highly suppressed, and we are left with α satisfying $\bar{\partial}_a\alpha = 0$, which approaches $\alpha \to 1$ rapidly once away from any zeros $\alpha^{-1}(0) \in M$. In the case of Sasakian manifold, the type of solutions with non-zero β are less and less likely to survive when $\zeta \to +\infty$. With this conjecture in mind, we study the local behavior of 5d Seiberg-Witten equations with large positive ζ near any closed Reeb orbit.

4.2.5 The Local Model Near Closed Reeb Orbits

On a generic contact manifold, the integral curve of the Reeb vector field may have uncontrollable behavior, as we mentioned early on. However, if the structure is K-contact, then the contact flow, viewed as a subgroup of the group $\mathrm{Isom}(M, g)$ of isometries, has a closure of $T^k \subset \mathrm{Isom}(M, g)$.

In other words, the integral curve of the Reeb vector field going through a point $p \in M$ forms a torus of dimension less than or equal to k. One can think of the curves as similar to irrational flows on a torus. The integer $k \leq 3$ for a K-contact five-manifold, and is called the *rank* of the structure. So, a rank-1 K-contact structure is a quasi-regular or regular contact structure, and $k \geq 2$ are all irregular.

The isometric T^k-action highly degenerates at the closed Reeb orbits, namely $k-1$ of the generators do nothing to the points on closed Reeb orbits. Therefore, at a small neighborhood $\mathcal{C} \times \mathbb{C}^2$ of a closed Reeb orbits $\mathcal{C}$, the $k-1$ generators rotates the $\mathbb{C}^2$ (leaving $\mathcal{C}$ fixed), while the remaining 1 generator, corresponding to the Reeb field R, translates along $\mathcal{C}$.

Bearing this picture in mind, one can write down an adapted coordinate (θ, z_1, z_2) on a small neighborhood $\mathcal{C}\times\mathbb{C}^2$ of any closed orbit $\mathcal{C}$, such that $T^k = \{t_0, ..., t_{k-1}\}$ acts on it in an intuitive way. Such a coordinate system is characterized by the numbers $(\lambda_0; \lambda_j, m_{1j}, m_{2j})$, $j = 1, ..., k-1$, where $\lambda_0, ..., \lambda_j$ are rationally independent positive real numbers, m_{1j} and m_{2j} are two lists of integers. In such a coordinate, the Reeb vector R and contact 1-form κ can be written as

$$\begin{cases} R = \lambda_0 \dfrac{\partial}{\partial\theta} + i \displaystyle\sum_{i=1,2}\sum_{j=1}^{k-1} \lambda_j m_{ij} \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i}\right) \\ \kappa = \dfrac{1}{\lambda_0}\left(1 - \displaystyle\sum_{i=1,2}\sum_{j=1}^{k-1} \lambda_j m_{ij} |z_i|^2\right) d\theta + \dfrac{i}{2}\displaystyle\sum_{i=1,2} z_i d\bar{z}_i - \bar{z}_i dz_i \end{cases} \tag{4.2.51}$$

The isometric subgroup T^k acts on the patch by

$$(t_0, t_1, ..., t_{k-1}) \cdot \left(e^{i\theta}, z_1, z_2\right) = \left(t_0 e^{i\theta}, \prod_{j=1}^{k-1} t_j^{m_{1j}} z_1, \prod_{j=1}^{k-1} t_j^{m_{2j}} z_2\right) \tag{4.2.52}$$

Let us pick a basis for horizontal 1-forms in region $\mathcal{C} \times \mathbb{C}^2$

$$e^5 = \kappa, \quad e^{z_i} \equiv dz_i - i\frac{\Lambda_i}{\lambda_0} z_i d\theta, \quad e^{\bar{z}_i} \equiv d\bar{z}_i + i\frac{\Lambda_i}{\lambda_0}\bar{z}_i d\theta, \tag{4.2.53}$$

where $\Lambda_i \equiv \sum_{j=1}^k \lambda_j m_{ij}$. It is straight-forward to show that $\mathcal{L}_R e^{z_i} = i\Lambda_i e^{z_i}$, $\mathcal{L}_R e^{\bar{z}_i} = -i\Lambda_i e^{\bar{z}_i}$. One can also easily verify that $d\kappa = ie^{z_1} \wedge e^{\bar{z}_1} + ie^{z_2} \wedge e^{\bar{z}_2}$. This suggests that one can view e^{z_i}, $e^{\bar{z}_i}$ as spanning $T^{1,0}M^*$ and $T^{0,1}M^*$. Under such assumption, one can show $\forall \alpha \in \Omega^{0,0}$,

$$\begin{cases} \partial\alpha = \left(\partial_{z_i}\alpha + \dfrac{i}{2}\bar{z}_i \mathcal{L}_R \alpha\right) e^{z_i}, \quad \bar{\partial}\alpha = \left(\partial_{\bar{z}_i}\alpha - \dfrac{i}{2} z_i \mathcal{L}_R \alpha\right) e^{\bar{z}_i} \\ \partial e^{z_i} = \dfrac{\Lambda_i}{2} e^{z_i} \wedge \left(\bar{z}_1 e^{z_1} + \bar{z}_2 e^{z_2}\right), \quad \bar{\partial} e^{z_i} = -\dfrac{\Lambda_i}{2} e^{z_i} \wedge \left(z_1 e^{\bar{z}_1} + z_2 e^{\bar{z}_2}\right) \end{cases} \tag{4.2.54}$$

Examples

Let us look at the example of squashed $S^5 \subset \mathbb{C}^3$

$$S^5_\omega \equiv \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 | \sum_{i=1,2,3} \omega_i^2 |z_i|^2 = 1 \right\} \tag{4.2.55}$$

One can define the Reeb vector field R and contact 1-form κ by restriction of

$$R \equiv i \sum_{i=1,2,3} \omega_i \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right), \quad \kappa \equiv \frac{i}{2} \sum_{i=1,2,3} \left(z_i d\bar{z}_i - \bar{z}_i dz_i \right) \tag{4.2.56}$$

Then it is easy to show that near the orbit $\mathcal{C}_3 \equiv \left\{\theta \in [0, 2\pi] \,|\, \left(0, 0, e^{i\theta}\omega_3^{-1}\right) \in S^5_\omega\right\}$, one can rewrite R and approximate κ in the new coordinate $\theta = (2i)^{-1} \log\left(z_3/\bar{z}_3\right)$, $w_i \equiv \omega_3^{-1}\sqrt{\omega_i} z_i z_3^{-1}$.

$$\begin{cases} R = \omega_3 \dfrac{\partial}{\partial\theta} + i \displaystyle\sum_{i=1,2} \left(\omega_i - \omega_3\right) \left(w_i \frac{\partial}{\partial w_i} - \bar{w}_i \frac{\partial}{\partial \bar{w}_i} \right) \\ \kappa = \dfrac{1}{\omega_3} \left[1 - \displaystyle\sum_{i=1,2} \left(\omega_i - \omega_3\right) |w_i|^2 \right] d\theta + \dfrac{i}{2} \displaystyle\sum_{i=1,2} w_i d\bar{w}_i - \bar{w}_i dw_i \end{cases}. \tag{4.2.57}$$

The natural T^3 action can be rearranged as

$$\left(e^{i\varphi}, e^{i\varphi_1}, e^{i\varphi_2}\right) \cdot (z_1, z_2, z_3) = \left(e^{i\varphi_1} e^{i\varphi} z_1, e^{i\varphi_2} e^{i\varphi} z_2, e^{i\varphi} z_3\right), \tag{4.2.58}$$

so that its action on the local coordinate is $\left(e^{i\varphi}e^{i\theta}, e^{i\varphi_1} w_1, e^{i\varphi_2} w_3\right)$, implying $m_{11} = m_{22} = 1$, and $\lambda_{1,2} = \omega_{1,2} - \omega_3$.

Similar steps can be done on Y^{pq} manifolds, which has K-contact rank $k = 2$. Let us recall how Y^{pq} manifolds are defined [46, 47]. Y^{pq} manifolds are Sasaki-Eintstein manifolds with topology $S^2 \times S^3$. They can be obtained by first looking at $S^3_{z_1,z_2} \times S^3_{z_3,z_4} \subset \mathbb{C}^4$ defined by equations

$$(p+q)\,|z_1|^2 + (p-q)\,|z_2|^2 = 1/2, \; p|z_3|^2 + p|z_4|^2 = 1/2 \tag{4.2.59}$$

Then one can define a nowhere-vanishing $U(1)$-vector field T which rotates the phases of z_i according to the charges $[p+q, p-q, -p, -p]$. The Y^{pq} manifolds is then the quotient $(S^3 \times S^3)/U(1)_T$. The Saaski-Einstein Reeb vector field is defined to be rotations of z_i with irrational charges $[\omega_1, \omega_2, \omega_3, \omega_4]$

$$\omega_1 = 0, \quad \omega_2 = \frac{1}{(p+q)\,l}, \quad \omega_3 = \omega_4 = \frac{3}{2} - \frac{1}{2\,(p+q)\,l}. \tag{4.2.60}$$

It is easy to show that near the closed Reeb orbit $\mathcal{C} \equiv \{(z_i) \in Y^{pq} | z_2 = z_4 = 0\}$, one has

$$\lambda_0 = p\omega_1 + (p+q)\,\omega_3, \quad \lambda_1 = 3, \quad m_{11} = 1, \quad m_{21} = 0. \tag{4.2.61}$$

The 5d Seiberg-Witten equation near $\mathcal{C}$

We study the equations near a closed orbit $\mathcal{C}$. Again, we rescale $(\alpha, \beta) \to (\sqrt{\zeta}\alpha, \sqrt{\zeta}\beta)$ for a better looking equation:

$$F_a^+ = \frac{\zeta}{2}\left(1 - |\alpha|^2 + |\beta|^2\right) d\kappa, \quad F_a^{0,2} = 2i\zeta\bar{\alpha}\beta, \quad \mathcal{L}_R^a\alpha = \mathcal{L}_R^a\beta = 0, \quad \bar{\partial}_a\alpha + \bar{\partial}_a^*\beta = 0 \tag{4.2.62}$$

Using (4.2.54) and its underlying assumption, the last equation in (4.2.62) can be reduced to usual equation on $\mathbb{C}^2$, since $\mathcal{L}_R^a\alpha = \mathcal{L}_R^a\beta = 0$,

$$\bar{\partial}_a\alpha + \bar{\partial}_a^*\beta = 0 \ \text{ on } \mathbb{C}^2. \tag{4.2.63}$$

However, as we discussed early on, we conjecture that when $\zeta \to +\infty$, $\beta, \nabla\beta \to 0$ and therefore the differential equations of α and β reduce to the holomorphic equation on $\mathbb{C}^2$

$$\bar{\partial}_a\alpha = 0, \quad \zeta \to +\infty. \tag{4.2.64}$$

In this sense, the zero set of large-ζ 5d Seiberg-Witten solutions corresponds to pseudo-holomorphic objects in K-contact manifold M. Namely near orbit $\mathcal{C}$, $\alpha^{-1}(0)$ takes the form of $\mathcal{C} \times \Sigma$ where Σ is "pseudo-holomorphically" mapped into M. Of course this is just a naive description and far from rigorous; more careful treatment is needed.

There are known smooth solutions to the 4-dimensional Seiberg-Witten equations, which are lifts of 2-dimensional vortex solutions; however, there are more solutions that we do not yet know how to describe. Nevertheless, let us assume that α has the usual asymptotic behavior $\alpha \to e^{in_0\theta}e^{in_1\varphi_1 + in_2\varphi_2}$, where $n_0 \in \mathbb{Z}$, $n_{1,2} \in \mathbb{Z}_{\geq 0}$ is required by holomorphicity and smoothness at the origin[6]: near the origin, $\alpha \sim e^{in_0\theta}z_1^{n_1}z_2^{n_2}$. Therefore,

$$\mathcal{L}_R^a\alpha = \mathcal{L}_R\alpha - ia\,(R)\,\alpha = 0 \Leftrightarrow \lambda_0 n_0 + n_1\sum_{j=1}^{k-1}\lambda_j m_{1j} + n_2\sum_{j=1}^{k-1}\lambda_j m_{2j} = a\,(R) \tag{4.2.65}$$

Note that the winding number $n_{0,1,2}$ should be bounded by ζ, similar to the situation in [48]. We demonstrate this on a Sasakian structure in the limit $\zeta \gg 1$. We consider the integral

$$\int_M F_a^{d\kappa} \wedge *d\kappa = \frac{\zeta}{2}\int\left(1 - |\alpha|^2\right) d\kappa \wedge *d\kappa \leqslant \frac{\zeta}{2}\,\mathrm{Vol}\,(\kappa)\,, \tag{4.2.66}$$

[6] Not all modes above are possible. The precise range of these integers requires global analysis of the solution, which we will discuss in later examples.

where $\mathrm{Vol}(\kappa) \equiv \int d\kappa \wedge *d\kappa$. On the other hand, if E is a trivial line bundle and thus a can be viewed as a global 1-form,

$$\begin{aligned}\int_M F_a^{d\kappa} \wedge *d\kappa = \int_M da \wedge *d\kappa = \int_M da \wedge \kappa \wedge d\kappa = \int_M a \wedge d\kappa \wedge d\kappa \\ = \int (\iota_R a)\, \kappa \wedge d\kappa \wedge d\kappa\end{aligned} \tag{4.2.67}$$

Notice that if we assume the connections a invariant under $\mathcal{L}_R$, then

$$\iota_R F_a = 0 \Rightarrow \mathcal{L}_R a = d\iota_R a = 0, \tag{4.2.68}$$

which leads to a bound on the winding numbers

$$\lambda_0 n_0 + n_1 \sum_{j=1}^{k-1} m_{1j}\lambda_j + n_2 \sum_{j=1}^{k-1} m_{2j}\lambda_j = \iota_R a \leqslant \frac{\zeta}{2} \tag{4.2.69}$$

Later we will see that this bound corresponds to poles in the perturbative Coulomb branch matrix model. More general situation needs more careful treatment, and we leave it for future study.

4.3 Partition Function: Suppression and Pole Matching

Suppose one obtains a BPS solution to the localization locus (4.2.30), then the contribution to the partition function from this particular solution is the product

$$e^{-S_{\mathrm{cl}}} Z^{\mathrm{vect}}_{\text{1-loop}} Z^{\mathrm{hyper}}_{\text{1-loop}}, \tag{4.3.1}$$

where $\exp[-S_{\mathrm{cl}}]$ is the exponentiated action evaluated on the BPS solution. The 1-loop determinants are

$$Z^{\mathrm{vect}}_{\text{1-loop}} Z^{\mathrm{hyper}}_{\text{1-loop}} = \frac{\mathrm{sdet}_{\mathrm{vect}}\left(-i\mathcal{L}_R + i\left(\sigma + i\iota_R A_{\mathrm{cl}}\right)\right)}{\mathrm{sdet}_{\mathrm{Hyper}}\left(-i\nabla_R^{\mathrm{TW}} + i\left(\sigma + i\iota_R A_{\mathrm{cl}}\right)\right)} \tag{4.3.2}$$

where we have shifted $\sigma \to -i\sigma$, and A_{cl} denotes the value of A as a solution to (4.2.30). Let us denote for a moment $\mathcal{H}_A \equiv \nabla_R^{\mathrm{TW}} - iA(R)$, which we recall is part of the Dirac operator $\not{D}_A^{\mathrm{TW}}$.

In the Coulomb branch, where one does not include the deformation QV_{mixed}, one encounters the BPS equations as a "decoupled" system of differential equations

$$\begin{cases} F_A^+ = 0, \quad d_A\sigma = 0, \quad \iota_R F_A = 0 \\ \not{D}_A \phi_+ + \dfrac{1}{8} d\kappa_{mn}\Gamma^{mn}\phi_+ = 0, \quad \sigma(\phi_+) = 0, \quad F_-^{A=1,2} = 0 \end{cases} \tag{4.3.3}$$

In [25], it is shown that on a Sasaki-Einstein geometry (or other geometry with a large scalar curvature), a solution A to the first line will imply the second line has only trivial solution $\phi_+ = 0$; namely the operator $\not{D}_A^{\mathrm{TW}}$, and in particular $\mathcal{H}_A$ does not have *zero* as one of its eigenvalues. Let $i\lambda_{\mathfrak{m}} \neq 0$ be an eigenvalue of $\mathcal{H}_A$ labeled by some quantum numbers $\mathfrak{m}$, with the corresponding eigenstate $\phi_{\mathfrak{m}}$. Then

$$\mathcal{H}_A \phi_{\mathfrak{m}} = i\lambda_{\mathfrak{m}} \phi_{\mathfrak{m}} \tag{4.3.4}$$

This is equivalent to the statement $\mathcal{H}_{A+\Delta A_{\mathfrak{m}}}\phi_{\mathfrak{m}} = 0$, where the $\Delta A_{\mathfrak{m}}(R) = \lambda_{\mathfrak{m}}$. Namely, there exists certain new gauge field $A + \Delta A_m$ with $\Delta A_{\mathfrak{m}}(R) = \lambda_{\mathfrak{m}}$, such that $\mathcal{H}_{A+\Delta A_{\mathfrak{m}}}$ has zero eigenvalue. Of course, $A + \Delta A$ cannot be a solution to the original Coulomb branch BPS equations, but it could be a solution to some deformed BPS equations. In our case, they are precisely the Higgs branch BPS equations, where the QV_{mixed} is taken into account. Therefore, solutions to the Higgs branch equations are expected to correspond to poles in the Coulomb branch matrix model, which are factors of the form $(i\sigma - i\lambda_{\mathfrak{m}})^{-1}$ coming from the hypermultiplet determinant. We will see this more precisely later in this section.

4.3.1 Suppression of the Deformed Coulomb Branch

In this subsection, we will review the supersymmetric actions for vector and hypermultiplet, and show that it is possible to achieve suppression of perturbative deformed Coulomb branch as $\zeta \to +\infty$ when certain bounds on the Chern-Simons level and the hypermultiplet mass are satisfied. This allows two things for theories containing hypermultiplets and appropriate Chern-Simons level,:

1) one can take a large ζ limit, and only focus on the contributions from 5d Seiberg-Witten solutions to the partition function.

2) One can take the Coulomb branch matrix model, close the integration contour of σ, and identify each pole of the integrand with a 5d Seiberg-Witten solution. Note that this is possible when the integrand is suppressed when $\zeta \to \infty$, and this requires the presence of hypermultiplets.

3) For theories that do not satisfy the bounds, the above two statement are not valid in general. For instance, for pure super-Yang-Mills theory, one cannot close the contour and rewrite the matrix integral into sum of residues, and the deformed Coulomb branch will persist in large ζ limit.

The supersymmetric actions

The Super-Yang-Mills and hypermultiplet action can be obtained by taking rigid limit of supergravity action. The bosonic parts read

$$\begin{aligned}\mathcal{L}_{\text{YM}} = \operatorname{tr}\big[&F \wedge *F - \mathcal{A} \wedge F \wedge F - d_A\sigma \wedge *d_A\sigma - 1/2 D_{IJ}D^{IJ} \\ &-4u\sigma^2 + \sigma\mathcal{F}^{mn}F_{mn} + 2\sigma\left(t^{IJ}D_{IJ}\right) + \sigma F_{mn}\mathcal{P}^{mn}\big]\end{aligned} \tag{4.3.5}$$

$$\mathcal{L}_{\text{Hyper}} = \epsilon^{IJ}\Omega_{AB}\nabla_m\phi_I^A\nabla^m\phi_J^B - \epsilon^{I'J'}\Omega_{AB}\Xi_{I'}^A\Xi_{J'}^B + \epsilon^{IJ}\Omega_{AB}\left(\frac{\mathcal{R}}{4} + h - \frac{1}{4}\mathcal{P}_{mn}\mathcal{P}^{mn}\right)\phi_I^A\phi_J^B \tag{4.3.6}$$

Note that we use the original field variables to write the action, and it is straight forward to use the invertible twisting to convert to new field variables.

One can also add in Q-invariant Chern-Simons terms for the vector multiplet [13], and we have made the shift $\sigma \to i\sigma$ stated earlier

$$\mathcal{L}_{\text{SCS}_5} = \mathcal{L}_{\text{CS}_5}\left(A - i\sigma\kappa\right) - \frac{ik}{8\pi^2}\operatorname{tr}\left(\Psi \wedge \Psi \wedge \kappa \wedge F_{A-i\sigma\kappa}\right), \tag{4.3.7}$$

$$\mathcal{L}_{\text{SCS}_{3,2}} = \mathcal{L}_{\text{CS}_{3,2}}\left(A - i\sigma\kappa\right) - i\operatorname{tr}\left(d\kappa \wedge \kappa \wedge \Psi \wedge \Psi\right), \tag{4.3.8}$$

where the pure Chern-Simons terms are

$$\begin{cases} \mathcal{L}_{\mathrm{CS}_5}(A) = \dfrac{ik}{24\pi^2}\mathrm{tr}\left(A\wedge dA\wedge dA + \dfrac{3}{2}A\wedge A\wedge A\wedge dA + \dfrac{3}{5}A\wedge A\wedge A\wedge A\wedge A\right) \\ \mathcal{L}_{\mathrm{CS}_{3,2}}(A) = i\mathrm{tr}\left(d\kappa\wedge\left(A\wedge dA + \dfrac{2}{3}A\wedge A\wedge A\right)\right) \end{cases} \tag{4.3.9}$$

The 5d Chern-Simons level k is an integer. As noted in [13], $\mathcal{L}_{\mathrm{SCS}_{3,2}}$ is not invariant under rescaling of κ, while $\mathcal{L}_{\mathrm{SCS}_5}$ is invariant.

The classical contributions

The deformed Coulomb branch equations are

$$d_A\sigma = 0, \quad F_A^+ - F_{A_0/2}^+ = \frac{\zeta}{2}d\kappa^+, \quad \iota_R F_A = 0 \tag{4.3.10}$$

On a Sasakian background, $\iota_R F_{A_0/2} = 0$, the perturbative solutions are

$$A = \frac{1}{2}A_0 + \frac{\zeta}{2}\kappa, \quad \sigma = \text{constant} \in \mathfrak{u}(N_c) \tag{4.3.11}$$

Evaluated on (4.3.11), the actions discussed above give the classical perturbative contribution to the partition function. We are interested the asymptotic behavior of these contributions as $\zeta \to +\infty$.

1) The two Chern-Simons terms contribute up to factors of order $\exp O(\zeta)$

$$\exp\left(iS_{\mathrm{SCS}_5} + i\mu S_{\mathrm{SCS}_{3,2}}\right) \to \exp\left[-\mathrm{tr}\left(\frac{k}{24\pi^2}\left(\sigma + \frac{i}{2}\zeta\right)^3 + i\mu\left(\sigma + \frac{i}{2}\zeta\right)^2\right)\mathrm{vol}\,(\kappa)\right] \tag{4.3.12}$$

where we denote the contact volume $\mathrm{Vol}\,(\kappa) = \int_M \kappa\wedge d\kappa^+\wedge d\kappa^+ = \int_M d\kappa^+\wedge * d\kappa^+$, and μ is a real coupling constant.

2) There is no classical contribution from $\mathcal{L}_{\mathrm{Hyper}}$ since all fields in the hypermultiplet vanish.

3) Finally, there is classical contribution from $\mathcal{L}_{\mathrm{YM}}$. To evaluate it, one needs to consider the field redefinition $H_{mn} = 2F_{mn}^+ + \left(2\sigma t^{IJ} + D^{IJ}\right)(\Theta_{IJ})_{mn} - 4\mathcal{F}_{mn}^+$, the equation of motion of H and BPS equation to solve D_{IJ} in terms of σ

$$H_{mn} = F_{mn}^+ + h(\phi)_{mn}, \qquad F_{mn}^+ + h(\phi)_{mn} = 0. \tag{4.3.13}$$

Using some Fierz-identities, the field redefinition implies

$$D_{IJ} = \left(h_{mn} + 2\mathcal{F}_{mn}^+\right)(\Theta_{IJ})^{mn} - 2\sigma t_{IJ}. \tag{4.3.14}$$

With this one can evaluate the classical contribution of super-Yang-Mills action. In the simplest case with $\mathcal{F} = \mathcal{P} = 0$ (namely on a Sasaki-Einstein background), we have

$$\exp\left[-S_{\mathrm{YM}}\right] = \exp\left[-\frac{1}{2}\mathrm{tr}\left(\sigma + \frac{i}{2}\zeta\right)^2\mathrm{Vol}\,(\kappa) + ...\right] \tag{4.3.15}$$

where ... denotes $O(\zeta)$ terms involving $F_{A_0/2}$. So we see there are competing ζ^2-dependent terms in the norm of the classical contribution when $\zeta \to +\infty$[7]

$$\left| e^{-S_{\text{YM}} + iS_{\text{SCS}_5} + i\mu S_{\text{SCS}_{3,2}}} \right| \sim \exp\left[\frac{1}{8}\text{tr}\left(1 + \frac{k}{4\pi^2}\sigma\right)\text{Vol}(\kappa)\zeta^2\right] \tag{4.3.16}$$

On more general background with non-vanishing $\mathcal{F}$ and $\mathcal{P}$, the classical contribution from $\exp\{-S_{\text{YM}}\}$ has the same leading behavior of ζ^2 as above, although the precise value will depend on the geometric background. The 1-loop determinant will be more complicated products of triple-sine function,

The perturbative 1-loop contributions

The perturbative 1-loop determinant from Coulomb branch was studied in [12, 49, 25]. It was shown that the 1-loop determinant can be expressed in terms of triple sine functions $S_3(z|\omega)$, or their particular products.

The triple sine function $S_3(z|\omega)$ with $\omega = (\omega_1, \omega_2, \omega_3)$ is defined as the regularized infinite product

$$S_3\left(z|\omega\right) \equiv \prod_{n_1,n_2,n_3=0}^{+\infty}\left(\sum_{i=1,2,3}\left(n_i+1\right)\omega_i - z\right)\left(\sum_{i=1,2,3} n_i\omega_i + z\right) \tag{4.3.17}$$

or in terms of generalized Γ-function $\Gamma_3(z|\omega_1, \omega_2, \omega_3)$:

$$S_3\left(z|\omega\right) \equiv \frac{1}{\Gamma_3\left(z|\omega_1,\omega_2,\omega_3\right)\Gamma_3\left(\omega_1+\omega_2+\omega_3 - z|\omega_1,\omega_2,\omega_3\right)} \tag{4.3.18}$$

What is most important to us is the asymptotic behavior of the triple-sine function: when $\omega_i > 0$, we have when $z \to \infty$ ($B_{3,3}$ are multiple Bernoulli functions, see [50, 51])

$$\begin{aligned}\log S_3\left(z|\omega\right) \equiv & -\frac{1}{3!}B_{3,3}\left(z\right)\left(\log z + C\right) - \frac{1}{3!}B_{3,3}\left(|\omega| - z\right)\left(\log\left(|\omega| - z\right) + C\right) \\ & -\gamma\zeta_3\left(0,z\right) - \gamma\zeta_3\left(0,|\omega|-z\right) + O\left(z^{-1}\right) + O\left(\left(|\omega|-z\right)^{-1}\right)\end{aligned} \tag{4.3.19}$$

which implies

$$S_3\left(z|\omega\right) \to \begin{cases} \exp\left[-\dfrac{i\pi}{3!}\dfrac{z^3}{\omega_1\omega_2\omega_3} + O\left(z^2\right)\right], & \text{Im}\, z > 0 \\ \exp\left[\dfrac{i\pi}{3!}\dfrac{z^3}{\omega_1\omega_2\omega_3} + O\left(z^2\right)\right], & \text{Im}\, z < 0 \end{cases} \tag{4.3.20}$$

The 1-loop determinant from perturbative Coulomb branch computed in literatures are products (over weights $\mu \in \mathfrak{R}$ to which the hypermultiplet belong) of triple sine functions, with argument of the form

$$z = i\left\langle \mu, \sigma \right\rangle + im + N\left(\omega\right). \tag{4.3.21}$$

[7]Although we are focusing our discussion on ζ-dependent terms, the ζ-independent terms including $\text{tr}\sigma^2$ are still present in the matrix model integral as $\zeta \to \infty$ as a convergence factor when integrating σ.

Here $N(\omega)$ is a real constant determined by equivariant parameters[8]. For us, $\mathfrak{R}$ is the fundamental or anti-fundamental representation of $U(N_c)$ gauge group.

If we consider the deformed Coulomb branch, then what we need is to compute the super-determinant of

$$iQ^2 = \nabla_R^{\mathrm{TW}} - iA\left(R\right) - \sigma = \nabla_R^{\mathrm{TW}} - \left(\sigma + \frac{i}{2}\zeta + \mathrm{const}\right) \tag{4.3.22}$$

from hypermultiplet[9], which effectively shifts $\sigma \to \sigma + i\zeta/2 + \mathrm{const}$ in the Coulomb branch 1-loop determinant. In the limit of large ζ, each S_3 factor of the 1-loop determinant of hypermultiplet tends to

$$\begin{aligned} \left|S_3\left(z|\omega\right)\right| &\xrightarrow{\langle\mu,\sigma\rangle+m>0,|\zeta|\to\infty} \left|\exp\left[-\frac{i\pi}{6\omega_1\omega_2\omega_3}\left(i\left\langle\mu,\sigma+\frac{i\zeta 1_{N_c\times N_c}}{2}\right\rangle + im + \mathrm{constant}\right)^3\right]\right| \\ &\xrightarrow{\text{leading terms}} \exp\left[\frac{\pi}{8\omega_1\omega_2\omega_3}\left(\langle\mu,\sigma\rangle + m\right)\zeta^2\right] \end{aligned} \tag{4.3.23}$$

Similarly,

$$\begin{aligned} \left|S_3\left(z|\omega\right)\right| &\xrightarrow{\langle\mu,\sigma\rangle+m<0,|\zeta|\to\infty} \left|\exp\left[-\frac{i\pi}{6\omega_1\omega_2\omega_3}\left(i\left\langle\mu,\sigma+\frac{i\zeta 1_{N_c\times N_c}}{2}\right\rangle + im + \mathrm{constant}\right)^3\right]\right| \\ &\xrightarrow{\text{leading terms}} \exp\left[-\frac{\pi}{8\omega_1\omega_2\omega_3}\left(\langle\mu,\sigma\rangle + m\right)\zeta^2\right] \end{aligned} \tag{4.3.24}$$

Note that this asymptotic result is different from that in 3d. In 3d, there is an overall ± 1 factor in the exponent, corresponding to how the $\mathfrak{u}(1)$ parts act on the specific weight, while here such factor is squared to 1. This reflects the symmetry in the matter content, where the fundamental and anti-fundamental (or $\mathfrak{R}$ and $\bar{\mathfrak{R}}$ in general) appear in a symmetric way in the hypermultiplet.

As a simplest example, consider N_f massless hypermultiplets on S^5 charged under gauge group $G = U(1)$. They contribute 1-loop determinant at large ζ

$$\sim \exp\left[-\frac{\pi}{8}N_f\left|\sigma + m\right|\zeta^2\right], \tag{4.3.25}$$

so the overall ζ^2-terms in the norm of the matrix model integrand is

$$\exp\left[\frac{1}{8}\left(1 + \frac{k}{4\pi^2}\sigma\right)4\pi^3\zeta^2 - \frac{\pi}{8}N_f\left|\sigma + m\right|\zeta^2\right]. \tag{4.3.26}$$

Therefore there is a window of suppression as $\zeta \to +\infty$

$$-N_f < k < N_f, \quad \frac{4\pi^2}{g_{\mathrm{YM}}^2 N_f} \leq |m|, \tag{4.3.27}$$

where we have reinstated the g_{YM} which was omitted in front of the Yang-Mills action. In the above, the bound on k comes from the competing σ and $|\sigma|$ as one integrates σ from $-\infty \to +\infty$, while

[8] For the individual triple sine function to converge, $N(\omega)$ is required to have imaginary part, but as discussed in [52], after all ingredients are multiplied together, one can take the real limit.

[9] 1-loop determinant of vector multiplet is not affected by ζ

the bound on m comes from negating the positive ζ^2-term from the Yang-Mills action. Within the suppression window, when performing the full matrix integral, because the integrand as a meromorphic function of σ falls of exponentially fast far way from the real line, one can close the contour in the upper half plane, picking up residues from the poles; or alternatively, one can deform the integration contour from $\mathbb{R}$ to $\mathbb{R} + i\zeta$, and collecting a residue each time the contour passes a pole.

Similar result can be obtained for squashed S^5, where the volume $\mathrm{Vol}\,(\kappa) \propto (\omega_1\omega_2\omega_3)^{-1}$, which only contributes an overall factor of the partition function as $\zeta \to +\infty$. On Y^{pq} manifolds, one needs to replace the 1-loop determinant with generalized triple-sine functions, which are products of original triple-sine functions, and we expect one will have a similar suppression window where the Chern-Simons level and the hypermultiplet mass are constrained as $\zeta \to +\infty$.

One can generalize the above result to other gauge groups with $U(1)$ factors. For instance, consider on squashed S^5 the gauge group G having $U(1)$-generators h_a. Define $\zeta = \zeta^a h_a$. Let the hypermultiplets belong to representations $\mathfrak{R}_{f=1,\ldots,N_f}$, and μ will denote weights in $\mathfrak{R}_f$. The eigen-value of h_a on μ, namely the $U(1)$ charge, is denoted by $q_a^f \equiv \langle \mu, h_a \rangle$. The large-$\zeta$ behavior of the exponent of the integrand is

$$\sim \frac{\pi}{8\omega_1\omega_2\omega_3}\sum_{a,b}\zeta^a\zeta^b\left[4\pi^2\mathrm{tr}\,(h_a h_b) + k\mathrm{tr}\,(\sigma h_a h_b) - \sum_{f=1}^{N_f}\sum_{\mu\in\mathfrak{R}_f} q_a^f q_b^f \left|\langle\mu,\sigma\rangle + m_f\right|\right] \tag{4.3.28}$$

The suppression can be achieved if the representations and the masses are such that the above expression tends to $\exp[-\infty]$ as $\zeta_a \to \pm\infty$ (with some choice of sign). For instance, when $G = U(N_c)$, and N_f hypermultiplets in the fundamental $\underline{N_c}$, the above reduces to

$$\sim \frac{\pi\zeta^2}{8\omega_1\omega_2\omega_3}\left[4\pi^2 N_c + k\mathrm{tr}\,(\sigma) - \sum_{f=1}^{N_f}\sum_{\mu\in\underline{N_c}}\left|\langle\mu,\sigma\rangle + m_f\right|\right], \tag{4.3.29}$$

and therefore suppression can be achieved if

$$|k| < N_f, \quad \sum_{f=1}^{N_f} m_f > \frac{4\pi^2}{g_{YM}^2}. \tag{4.3.30}$$

Finally, we remark that the bound above is a sufficient bound, obtained by only looking at the norm of the integrand. To fully understand when suppression can actually be achieved and whether or not the bound can be relaxed, more careful analyses are required. Also, the meaning of the mass bound is not clear to the authors at the moment, and we hope to get a better understanding in the future.

4.3.2 Matching The Poles And The Shift

Similar to 3-dimensional Higgs branch localization [48], if one performs the integral of the Coulomb branch matrix model by closing the contour appropriately, one picks up residues from the enclosed poles. Before checking the matching between poles and 5d Seiberg-Witten equation, let us first understand the operator $\nabla_R^{\mathrm{TW}} - i\iota_R A$ properly.

The operator $\nabla_{R,A}^{TW}$ and $\mathcal{L}_R$

Let $\phi_+ = \xi \otimes \sigma_E$ be a section of $S_+ \otimes E$, where E is equipped with A as a $U(1)$ connection[10].

Equivalently, noting that $S_+ = K_M^{-1/2} \otimes K_M^{1/2}$, one can choose an appropriate section $\hat{\sigma}$ of $K_M^{1/2}$, and rewrite $\phi_+ = (\xi \otimes \hat{\sigma}) \otimes (\hat{\sigma}^{-1} \otimes \sigma_E)$, where we have factored out a piece $\xi \otimes \hat{\sigma} \in \Gamma\left(W_+^0\right)$. $\hat{\sigma}$ then provides the explicit connection 1-form for the abstract canonical connection "A_0" on K_M:

$$\nabla_{A_0/2}\hat{\sigma} = -i\frac{A_0}{2}\hat{\sigma}, \tag{4.3.31}$$

and hence

$$\nabla_{R,A}^{\mathrm{TW}}\phi_+ = \mathcal{L}_R\left(\xi \otimes \hat{\sigma}\right) \otimes \left(\hat{\sigma}^{-1} \otimes \sigma_E\right) - i\left(\iota_R a\right)\phi_+, \tag{4.3.32}$$

where we have used $\nabla_{R,A_0/2}^{\mathrm{TW}} = \mathcal{L}_R$ on W_+^0, $a = A - A_0/2$ as a connection on $E \otimes K_M^{-1/2}$.

In the case where $A = 0$, namely the perturbative Coulomb branch solution, one has $\iota_R a = -\iota_R A_0/2$ and therefore the shift in eigenvalues of ∇_R^{TW} and $\mathcal{L}_R$

$$\Delta\left(\nabla_R^{\mathrm{TW}}, \mathcal{L}_R\right) = \frac{i}{2}\iota_R A_0. \tag{4.3.33}$$

On the other hand, one of the BPS equation reads

$$\nabla_{R,A}^{\mathrm{TW}}\phi_+ = 0 \Leftrightarrow \mathcal{L}_R\left(\xi \otimes \hat{\sigma}\right) \otimes \left(\hat{\sigma}^{-1} \otimes \sigma_E\right) = i\left(\iota_R a\right)\phi_+ \tag{4.3.34}$$

As a section of $T^{0,0}M^* \oplus T^{0,2}M^*$, $\xi \otimes \hat{\sigma}$ contributes eigenvalues of $\mathcal{L}_R$ of the form

$$\lambda_0 n_0 + n_1\sum_{j=1}^{k-1}\lambda_j m_{1j} + n_2\sum_{j=1}^{k-1}\lambda_j m_{2j}, \qquad n_0 \in \mathbb{Z}, n_{1,2} \in \mathbb{Z}_{\geq 0}. \tag{4.3.35}$$

corresponding to modes with asymptotic behavior $\sim e^{in_0} z_1^{n_1} z_2^{n_2}$ near each closed Reeb orbit. Now the remaining puzzle is to determine the value of $\iota_R A_0$.

Squashed S^5 and $\iota_R A_0$

As an example, let us consider matching the poles of 1-loop determinant on squashed S^5 with the local solutions to the 5d Seiberg-Witten equation. We will focus on the orbit $\mathcal{C}_3$ discussed before, and recall the formula (4.2.57).

Note that one can define local orthonormal vielbein e^A by first defining an orthonormal frame at $\theta = 0$, then use R to translate them to *almost* the whole $\mathcal{C}_3$. In particular, one can define e^A in such a way that it is *adapted* to and *invariant* under the K-contact structure, namely $\mathcal{L}_R e^A = 0$. However, translating e^A back to $\theta = 2\pi$ will in general disagree with the starting value. To obtain

[10] Namely, $\nabla_A\sigma_E = -iA\sigma_E \Rightarrow \nabla_A^{\mathrm{TW}}\left(\xi \otimes \sigma_E\right) = \left(\nabla^{\mathrm{TW}} - iA\right)\xi \otimes \sigma_E$

a vielbein well-defined on $\mathcal{C}_3$, one can rotate the original e^A along the way. For instance, in terms of the complex basis

$$e^{z_i} \to \exp\left(i\frac{\omega_i - \omega_3}{\omega_3}\theta\right) e^{z_i}, \quad e^{\bar{z}_i} \to \exp\left(-i\frac{\omega_i - \omega_3}{\omega_3}\theta\right) e^{\bar{z}_i} \tag{4.3.36}$$

Then we have

$$\mathcal{L}_R e^{\bar{z}_i} = -i\left(\omega_i - \omega_3\right) e^{\bar{z}_i} \Leftrightarrow \begin{cases} \mathcal{L}_R e^{2i-1} = -\left(\omega_i - \omega_3\right) e^{2i} \\ \mathcal{L}_R e^{2i} = \left(\omega_i - \omega_3\right) e^{2i-1} \end{cases} \tag{4.3.37}$$

In this basis, one can compute the derivative along R

$$\nabla^{\text{LC}}_R \psi = R^m \partial_m \psi + \frac{1}{2}\sum_{i=1,2}\left(\omega_i - \omega_2\right)\Gamma^{2i-1}\Gamma^{2i}\psi - \frac{1}{4}d\kappa\cdot\psi \tag{4.3.38}$$

Let $\psi_+ = (a,b)^T \in S_+$. Using the explicit representation (A.1.10) the derivative ∇^{LC}_R reduces to

$$\nabla^{\text{LC}}_R \psi_+ = R^m \partial_m \psi_+ + \frac{1}{2}\sum_{i=1,2}(\omega_i - \omega_3) i\sigma_3\psi_+ - i\sigma_3\psi_+, \tag{4.3.39}$$

where we used $\Gamma^{12}\psi_+ = \Gamma^{34}\psi_+ = i\sigma_3\psi_+$ and $d\kappa\cdot\psi_+ = 4i\sigma_3\psi_+$.

When $\omega_{1,2,3} = 1$, one can define Killing spinor by

$$\nabla^{\text{LC}}_m \xi = -\frac{i}{2}\Gamma_m \xi. \tag{4.3.40}$$

Suppose $\xi_{-1/2} \in K_M^{-1/2}$ is a solution to the above Killing spinor equation, then using the above local expression of ∇^{LC}, one can show that ξ behaves like $\sim \exp\left(\frac{3i}{2}\theta\right)$ along $\mathcal{C}_3$. Finally, if we require $\hat{\sigma}$ to satisfy

$$\xi_{-1/2}\otimes\hat{\sigma} = \text{Const} \in \Gamma(T^{0,0}M^*), \tag{4.3.41}$$

one deduces that along $\mathcal{C}_3$

$$\nabla_{R,A_0/2}\hat{\sigma} = -\frac{3i}{2}\hat{\sigma} = -\frac{i}{2}\left(\iota_R A_0\right)\hat{\sigma}, \tag{4.3.42}$$

namely, along $\mathcal{C}_3$, $\hat{\sigma}$ has periodic behavior $\exp(-\frac{3i}{2}\theta)$ to cancel that of $\xi_{-1/2}$. This implies the shift

$$\Delta\left(\nabla^{\text{TW}}_R, \mathcal{L}_R\right) = \frac{i}{2}\iota_R A_0 = \frac{3i}{2}. \tag{4.3.43}$$

On a general squashed S^5_ω, we continue to choose $\hat{\sigma}$ such that it has $\exp\left(-\frac{3i}{2}\theta\right)$ periodic behavior along all three closed Reeb orbits. Then near any of three orbits, we recover the shift of eigenvalues as in [52][49]

$$\Delta\left(\nabla^{\text{TW}}_R, \mathcal{L}_R\right) = \frac{i}{2}\iota_R A_0 = \frac{i\left(\omega_1+\omega_2+\omega_3\right)}{2} \tag{4.3.44}$$

Finally, the bound (4.2.69) on the winding numbers can now be written as

$$\sum_{i=1,2,3}\left(n_i + \frac{1}{2}\right)\omega_i \leqslant \frac{\zeta}{2} + \frac{\iota_R A_0}{2}. \tag{4.3.45}$$

where we defined $n_3 = n_0 - n_1 - n_2$, which is non-negative if one consider all three closed Reeb orbits $\mathcal{C}_{1,2,3}$. Recall that the 1-loop determinant in deformed Coulomb branch is obtained by a shift in that of Coulomb branch

$$\sigma \to \sigma + i\left(\frac{\zeta}{2} + \frac{\iota_R A_0}{2}\right) \Leftrightarrow \operatorname{Im}\sigma = \frac{\zeta}{2} + \frac{\iota_R A_0}{2}. \tag{4.3.46}$$

Combining with the (4.3.45), bound saturation then means

$$\operatorname{Im}\sigma = \sum_{i=1,2,3}\left(n_i + \frac{1}{2}\right)\omega_i, \quad n_i \geq 0, \tag{4.3.47}$$

Poles of the S^5_ω perturbative 1-loop determinant

Recall that the perturbative 1-loop determinant of a hypermultiplet coupled to a $U(1)$ vector multiplet on S^5_ω is

$$Z^{\text{Hyper}}_{\text{1-loop}}\left(S^5_\omega\right) = \left[S_3\left(i\sigma + im + \frac{\omega_1+\omega_2+\omega_3}{2}|\omega\right)\right]^{-1} \tag{4.3.48}$$

The poles are the zeros of the infinite products

$$\prod_{n\geqslant 0}\left(\sum_{i=1,2,3}\left(n_i + \frac{1}{2}\right)\omega_i - i\left(\sigma+m\right)\right)\prod_{n\geqslant 0}\left(\sum_{i=1,2,3}\left(n_i + \frac{1}{2}\right)\omega_i + i\left(\sigma+m\right)\right), \tag{4.3.49}$$

where we have reinstated the mass induced from a background $U(1)$ vector multiplet. All the possible poles are

$$-m \pm i\sum_{i=1,2,3}\left(n_i + \frac{1}{2}\right)\omega_i = \sigma \Leftrightarrow \operatorname{Re}\sigma = -m,\ \operatorname{Im}\sigma = \pm\sum_{i=1,2,3}\left(n_i + \frac{1}{2}\right)\omega_i, \tag{4.3.50}$$

The first equation above is just the equation $(\sigma + m)\,\phi = 0$ in the Higgs branch, and the second is just the bound we obtained above, if one takes the poles with $+$ sign. These are the poles that will be picked up when one close the contour in the upper half plane of the σ-plane. Note that this is allowed thanks to the suppression of deformed Coulomb branch as $\zeta \sim \operatorname{Im}\sigma \to +\infty$.

The case of Y^{pq} manifolds

Recall (4.2.61) that near the orbit $z_2 = z_4 = 0$, the Sasaki-Einstin Reeb vector field can be written as

$$\begin{aligned} R = {} & \left[p\omega_1 + (p+q)\,\omega_3\right]\frac{\partial}{\partial\theta} \\ & + i\left(\omega_2 + \omega_1 + 2\omega_3\right)\left(u_1\frac{\partial}{\partial u_1} - \bar{u}_1\frac{\partial}{\partial \bar{u}_1}\right) + i\left(\omega_4 - \omega_3\right)\left(u_2\frac{\partial}{\partial u_2} - \bar{u}_2\frac{\partial}{\partial \bar{u}_2}\right) \end{aligned} \tag{4.3.51}$$

where

$$\omega_1 = 0,\ \ \omega_2 = \frac{1}{(p+q)\,l}, \quad \omega_3 = \omega_4 = \frac{1}{2}\left(3 - \frac{1}{(p+q)\,l}\right). \tag{4.3.52}$$

One can then read off again $\iota_R A_0 = 3$ by choosing the section $\hat{\sigma}$ with the same criteria as S^5, and the bound on local winding number is also determined

$$n_0 \left(\frac{3}{2}(p+q) - \frac{1}{2l} \right) + 3n_1 + \frac{3}{2} \leq \frac{\zeta}{2} + \frac{1}{2}\iota_R A_0, \quad n_0 \in \mathbb{Z}, n_1 \in \mathbb{Z}_{\geq 0}. \tag{4.3.53}$$

After redefinition $n_{e_1} \equiv n_1 + n_0 p$, $n_\alpha \equiv n_0$, the bound saturation corresponds to the poles[11]

$$\operatorname{Im}\sigma = 3n_{e_1} + n_\alpha \left(\frac{3}{2}(q-p) - \frac{1}{2l} \right) + \frac{3}{2}. \tag{4.3.56}$$

We remark that the redefinition seems to implies $n_{e_1} \in \mathbb{Z}$, but global analysis, namely, the equation (71) in [52] implies $n_{e_1} + n_\alpha p = n_{e_2} \geq 0$ for the poles in the upper-half σ plane.

4.4 Summary

In this work, we apply the idea of Higgs branch localization to supersymmetric theories of $\mathcal{N} = 1$ vector and hypermultiplet on general K-contact background. We show that with this generality the localization locus are described by perturbed contact instanton equations in the deformed Coulomb branch, and 5d Seiberg-Witten equations in the Higgs branch. Neither of these two types of equations is well understood. We focused on the latter, and some study basic properties of its solutions, including their local behavior near closed Reeb orbits, which is shown to reduce to 4-dimension Seiberg-Witten equations. This seems to implies that these BPS solutions corresponds to "pseudo-holomorphic" objects in K-contact manifolds, if the 4-dimensional story can some how be lifted. Finally, we study the suppression of deformed Coulomb branch as the parameter $\zeta \to +\infty$, and manage to match the poles of perturbative Coulomb branch matrix model with the bound on local winding numbers.

From this point on, it is straight-forward to use the factorization property of perturbative partition function on S^5 and Y^{pq} manifolds to perform the contour integral of σ. The result should produce classical and 1-loop contributions of each local Seiberg-Witten solutions, in a form of products of contributions from each closed Reeb orbit.

Another question that we did not address is that whether the partition function is invariants of certain geometric structure. In [20], it is shown that the generalized Killing spinor equation (4.1.5) has huge degrees of *local* freedom, including the background metric g, κ and R, which are reflected as Q-exact deformations in the partition function. Therefore it would be interesting to

[11] The involved generalized triple sine function is [52]

$$\prod_{\Lambda_n^+} \left[\sum_{i=1}^4 \left(n_{e_i} + \frac{1}{2} \right) \omega_i + i(\sigma + m) \right] \prod_{\Lambda_n^-} \left[\sum_{i=1}^4 \left(n_{e_i} + \frac{1}{2} \right) \omega_i + i(\sigma + m) \right], \tag{4.3.54}$$

where $\Lambda_n^\pm$ denotes restrictions on n_{e_i}

$$\begin{cases} n_{e_1} + n_{e_2} - n_{e_3} - n_{e_4} = n_\alpha q \\ n_{e_1} - n_{e_2} = -n_\alpha p \end{cases}, \quad \begin{cases} n_{e_i} \geqslant 0, & n \in \Lambda_n^+ \\ n_{e_i} < 0, & n \in \Lambda_n^- \end{cases} \tag{4.3.55}$$

explore the geometric or topological meaning of $\mathcal{N} = 1$ partition functions and expectation values of BPS operators. We believe that one needs to look closely the constraint (4.1.6) and understand its geometric meaning. Also, one can further study the 5d Seiberg-Witten equations (4.2.41). For instance, it would be interesting to understand its moduli spaces, which we did not take into account when matching the poles. But it is likely that on generic K-contact structures, the moduli spaces are zero-dimensional, considering the matching of perturbative poles and local solutions. Another interesting question is whether the solutions to (4.2.41) correspond to certain "pseudo-holomorphic" objects, similar to the 4-dimensional story. If so, the partition functions will have more explicit geometrical meaning in terms of a "counting" of these objects.

Finally we have the issue of A_0. In several discussions, including obtaining the bound on winding number, we relied on the assumption that the K-contact structure is Sasakian, in order to have a simplification $\iota_R F_{A_0/2} = 0$. It is not clear if this can always be achieved on general K-contact structures, or if there are other wiser choice of A_0 with the horizontal property, while simultaneously enables the identification $\not{D}^{\mathrm{TW}}_{A_0/2} \leftrightarrow \mathcal{L}_R + \left(\bar{\partial} + \bar{\partial}^*\right)$.

Acknowledgment

We thank Francesco Benini and Wolfger Peelaers for explaining their work in great detail. We thank Francesco Benini, Dario Martelli, Wolfger Peelaers, Martin Roček and Maxim Zabzine for reading the manuscript and their helpful comments. We thank Sean Fitzpatrick for discussions on related mathematics.

Appendices

Appendix A

Gamma Matrices and Spinors

A.1 Gamma Matrices

We denote the 5d Gamma matrices as Γ^m, which is defined by the anti-commutation relation

$$\{\Gamma^m, \Gamma^n\} = 2g^{mn}, \quad m, n = 1, 2, 3, 4, 5. \tag{A.1.1}$$

We require them to be Hermitian

$$(\Gamma^m)^\dagger = \Gamma^m \tag{A.1.2}$$

Also we have charge conjugation matrix $C = C_+$

$$C\Gamma^m C^{-1} = (\Gamma^m)^T = \overline{\Gamma^m} \tag{A.1.3}$$

These matrices have the following symmetry properties:

$$C_{\alpha\beta} = -C_{\beta\alpha}, \;\; (C\Gamma_m)_{\alpha\beta} = -(C\Gamma_m)_{\beta\alpha} \tag{A.1.4}$$

$$(C\Gamma_{mn})_{\alpha\beta} = (C\Gamma_{mn})_{\beta\alpha}, \;\; (C\Gamma_{lmn})_{\alpha\beta} = (C\Gamma_{lmn})_{\beta\alpha} \tag{A.1.5}$$

and complex conjugation properties

$$\sum_\beta \overline{C_{\alpha\beta}} C_{\beta\gamma} = -\delta^\alpha{}_\gamma, \; \overline{(\Gamma^m)^\alpha{}_\beta} = (\Gamma^m)^\beta{}_\alpha \tag{A.1.6}$$

The symmetry properties of $C\Gamma$ results in symmetry properties of bilinears of spinors:

$$\begin{aligned} (\xi_1\xi_2) = -(\xi_2\xi_1), \;\; (\xi_1\Gamma_m\xi_2) = -(\xi_2\Gamma_m\xi_1) \\ (\xi_1\Gamma_{mn}\xi_2) = (\xi_2\Gamma_{mn}\xi_1), \;\; (\xi_1\Gamma_{lmn}\xi_2) = (\xi_2\Gamma_{lmn}\xi_1) \end{aligned} \tag{A.1.7}$$

We define

$$\Gamma_{mn} \equiv \frac{1}{2}(\Gamma_m\Gamma_n - \Gamma_n\Gamma_m) \tag{A.1.8}$$

and similarly for Γ_{mnk}, Γ_{mnkl}. These products of Gamma matrices satisfy

$$\Gamma_{mnk} = -\frac{\sqrt{g}}{2}\epsilon_{mnkpq}\Gamma^{pq}, \quad \Gamma_{mnkl} = \sqrt{g}\epsilon_{mnklp}\Gamma^p \tag{A.1.9}$$

One can define a chiral and anti-chiral decomposition using any unit-normed vector field. In our case, we use the Reeb vector field R and define a chiral operator $\Gamma_C \equiv -R^m\Gamma_m$, and decompose $S = S_+ \oplus S_-$.

An explicit representation of Gamma matrices we will use is

$$\Gamma^1 = \begin{pmatrix} 0 & -i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & -i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix} \tag{A.1.10}$$

Another representation we will use is

$$\Gamma^1 = -\sigma_1 \otimes \mathbb{I}, \quad \Gamma^2 = \sigma_2 \otimes \sigma_1, \quad \Gamma^3 = \sigma_2 \otimes \sigma_2, \quad \Gamma_4 = \sigma_2 \otimes \sigma_3, \quad \Gamma_5 = \sigma^3 \otimes \mathbb{I} \tag{A.1.11}$$

A.2 Spinors and Symplectic-Majorana Spinors

A.2.1 Spinor Products

As opposed to that in 4-dimension, one cannot impose simple Majorana condition on a 5d spinor ξ. But one can define a *symplectic-Majorana spinor*, as a pair of spinors $\xi_I, I = 1, 2$, such that

$$\overline{\xi_I^\alpha} = C_{\alpha\beta}\epsilon^{IJ}\xi_J^\beta. \tag{A.2.1}$$

Note that given any usual spinor ξ, one can upgrade it to the symplectic-Majorana version by setting $\xi_{I=1} = \xi,\ \xi_{I=2} = C^{-1}\bar{\xi}$.

Using C, one can define a $\mathbb{C}$-valued anti-symmetric product of any two arbitrary spinors ξ and χ

$$(\xi\chi) \equiv \sum_{\alpha,\beta=1,2} \xi^\alpha C_{\alpha\beta}\chi^\beta \in \mathbb{C}. \tag{A.2.2}$$

The product satisfies (here we consider Grassmann even spinors)

$$(\xi\chi) = -(\chi\xi), \quad (\xi\Gamma_m\chi) = -(\chi\Gamma_m\xi), \quad (\xi\Gamma_{mn}\chi) = (\chi\Gamma_{mn}\xi) \tag{A.2.3}$$

One can also define an $\mathbb{R}$-valued symmetric inner product on S. Let ξ and χ be any two spinors, and we upgrade them to symplectic-Majorana spinor ξ_I and χ_I. Then the inner product $\langle , \rangle$ is defined as

$$\langle \xi, \chi \rangle \equiv \epsilon^{IJ}(\xi_I\chi_J) = \sum_\alpha \xi_1^\alpha\overline{\chi_1^\alpha} + \overline{\xi_1^\alpha}\chi_1^\alpha = \sum_\alpha \xi^\alpha\overline{\chi^\alpha} + \overline{\xi^\alpha}\chi^\alpha \in \mathbb{R} \tag{A.2.4}$$

In particular, if $\xi \neq 0$ then $\langle \xi, \xi \rangle = 2\sum_\alpha \overline{\xi^\alpha}\xi^\alpha > 0$.

A.2.2 Fierz identities

For arbitrary Grassmann even spinors $\xi_{1,2,3}$, we have the basic Fierz identity

$$\xi_1\left(\xi_2\xi_3\right)=\frac{1}{4}\xi_3\left(\xi_2\xi_1\right)+\frac{1}{4}\Gamma^m\xi_3\left(\xi_2\Gamma_m\xi_1\right)-\frac{1}{8}\Gamma^{mn}\xi_3\left(\xi_2\Gamma_{mn}\xi_1\right) \tag{A.2.5}$$

It follows immediately two useful formula

$$\begin{cases}\xi_1\left(\xi_2\xi_3\right)+\xi_2\left(\xi_1\xi_3\right)=-\dfrac{1}{4}\Gamma^{mn}\xi_3\left(\xi_2\Gamma_{mn}\xi_1\right)\\[2ex] 2\xi_1\left(\xi_2\xi_3\right)-2\xi_2\left(\xi_1\xi_3\right)=\xi_3\left(\xi_2\xi_1\right)+\Gamma^m\xi_3\left(\xi_2\Gamma_m\xi_1\right)\end{cases} \tag{A.2.6}$$

A.2.3 $SU(2)_{\mathcal{R}}$ Indices

In this subsection we review our convention for the $SU(2)_{\mathcal{R}}$ indices $I, J, K, ...$

The $SU(2)$-invariant tensor ϵ is defined as $\epsilon^{12}=\epsilon_{21}=1$, with contraction

$$\epsilon^I{}_K=\epsilon^{IJ}\epsilon_{JK}=-\epsilon^{IJ}\epsilon_{KJ}=-\epsilon_K{}^I=\delta^I{}_K\Rightarrow\epsilon^{IJ}\epsilon_{IJ}=-2 \tag{A.2.7}$$

and raising/lowering rules

$$X^I=\epsilon^{IJ}X_J\Leftrightarrow X_I=\epsilon_{IJ}X^J \tag{A.2.8}$$

With this "metric", we define for any 2 triplets of functions X^{IJ} and Y^{IJ} a product in a natural way:

$$(XY)^{IJ}\equiv\epsilon_{LK}X^{IK}Y^{LJ}=X^I{}_KY^{KJ} \tag{A.2.9}$$

Note that this product has the following symmetry:

$$(XY)^{IJ}=-(YX)^{JI} \tag{A.2.10}$$

and in particular

$$\left(X^2\right)^{IJ}=-\left(X^2\right)^{JI}=\frac{1}{2}\mathrm{tr}(X^2)\epsilon^{IJ} \tag{A.2.11}$$

where we define the trace for triplet products:

$$\mathrm{tr}\left(XY\right)\equiv X_I{}^JY_J{}^I=-X_{IJ}Y^{IJ} \tag{A.2.12}$$

with cyclic symmetry

$$\mathrm{tr}\left(XY\right)=\mathrm{tr}\left(YX\right) \tag{A.2.13}$$

As an example, when $X_I{}^J=\frac{i}{2}(\sigma_3)_I{}^J$

$$\mathrm{tr}X^2=-\frac{1}{2} \tag{A.2.14}$$

Note that if non-zero quantity X_{IJ} satisfies reality condition

$$\overline{X_{IJ}}=\epsilon^{II'}\epsilon^{JJ'}X_{I'J'} \tag{A.2.15}$$

then the trace is negative definite

$$\operatorname{tr}\left(X^2\right) < 0 \tag{A.2.16}$$

For objects of direct sum of triplet and singlet,

$$X_{IJ} \equiv \hat{X}_{IJ} - \frac{1}{2}\epsilon_{IJ}X,\ X^{IJ} \equiv \hat{X}^{IJ} + \frac{1}{2}\epsilon^{IJ}X \tag{A.2.17}$$

with

$$X = \epsilon^{IJ}X_{IJ} = -\epsilon_{IJ}X^{IJ} \tag{A.2.18}$$

A.2.4 Useful identities

The Fierz-identities implies several useful formula. Let ξ_I be a symplectic-Majorana spinor and (s, κ, R, Θ) be the associated quantities described in the main text:

$$s \equiv \varepsilon^{IJ}\left(\xi_I\xi_J\right), \quad R^m \equiv \left(\xi^I\Gamma_m\xi_I\right) = g^{mn}\kappa_n, \quad \left(\Theta_{IJ}\right)_{mn} \equiv \left(\xi_I\Gamma_{mn}\xi_J\right) \tag{A.2.19}$$

Then

$$\begin{cases} R^m\Gamma_m\xi_I = -\xi_I \\ \Omega^-_{mn}\Gamma^{mn}\xi_I = 0 \end{cases}, \quad t^{IJ}(\Theta_{IJ})^m{}_k t^{KL}(\Theta_{KL})^k{}_n = \frac{s^2}{2}\left(t_I{}^J t_J{}^I\right)\left(-\delta^m_n + R^m\kappa_n\right), \tag{A.2.20}$$

for any symmetric tensor t_{IJ} and anti-self-dual (w.r.t to R^m) 2-form Ω^+. In particular, if $t_{IJ} \neq 0$ everywhere and satisfies $\overline{t_{IJ}} = \epsilon^{II'}\epsilon^{JJ'}t_{I'J'}$, then the 2-form $t^{IJ}\Theta_{IJ}$ is nowhere-vanishing, since it squares to

$$\left(t^{IJ}\Theta_{IJ}\right)_{mn}\left(t^{IJ}\Theta_{IJ}\right)^{mn} = -2s^2\left(t^{IJ}t_{IJ}\right) > 0. \tag{A.2.21}$$

$$|R|^2 = R^m R_m = \iota_R\kappa = s^2 \tag{A.2.22}$$

$$\iota_R\Theta^{IJ} = 0 \tag{A.2.23}$$

$$\iota_R * \Theta^{IJ} = -s\Theta^{IJ} \Leftrightarrow R^k\left(\xi^I\Gamma_{mnk}\xi^J\right) = +s\left(\xi^I\Gamma_{mn}\xi^J\right) \tag{A.2.24}$$

$$\kappa \wedge \Theta \wedge \Theta \neq 0 \tag{A.2.25}$$

$$(\lambda_1\Theta)_m{}^p(\lambda_2\Theta)_p{}^n = s(\lambda_1)_I{}^K(\lambda_2)_{KJ}\left(\Theta^{IJ}\right)_m{}^n + \frac{s^2}{2}\operatorname{tr}\left(\lambda_1\lambda_2\right)\delta_m{}^n - \frac{1}{2}\operatorname{tr}\left(\lambda_1\lambda_2\right)\kappa_m R^n \tag{A.2.26}$$

$$(\lambda\Theta)^{mn}(\lambda\Theta)_{mn} = -2s^2\operatorname{tr}(\lambda^2) \tag{A.2.27}$$

$$*\,(\lambda\Theta)_{nkl}(\lambda\Theta)_m{}^l = \frac{s}{2}\operatorname{tr}\left(\lambda^2\right)\left[g_{mk}R_n - g_{mn}R_k\right] \tag{A.2.28}$$

$$(*\lambda\Theta)^{mnk}(\lambda\Theta)_{mn} = 2\operatorname{tr}\left(\lambda^2\right)sR^k \tag{A.2.29}$$

Also there are several useful spinor identities

$$R^m\Gamma_m\xi_I = s\xi_I \tag{A.2.30}$$

$$R^m\Gamma_{nm}\xi_I = \left(s\Gamma_n - R_n\right)\xi_I \tag{A.2.31}$$

$$(\lambda\Theta)_{nm}\Gamma^n\xi_I = \left(R_m - s\Gamma_m\right)\lambda_I{}^J\xi_J \tag{A.2.32}$$

$$(\lambda\Theta)_{nm}\Gamma^{kn}\xi_I = \Gamma^k\left(R_m - s\Gamma_m\right)\lambda^J{}_I\xi_J - (\lambda\Theta)^k{}_m\xi_I \Rightarrow (\lambda\Theta)_{mn}\Gamma^{mn}\xi_I = 4s\lambda^J{}_I\xi_J \tag{A.2.33}$$

Appendix B

Conventions in Differential Geometry

In this section we review our convention in differential forms, spin connection and more tensor analysis.

B.1 Differential forms

For any differential p-form ω, the components $\omega_{m_1...m_p}$ and $\omega_{A_1...A_p}$ are defined as

$$\omega = \frac{1}{p!}\omega_{m_1...m_p} dx^{m_1} \wedge ... \wedge dx^{m_p} = \frac{1}{p!}\omega_{A_1...A_p} e^{A_1} \wedge ... \wedge e^{A_p} \tag{B.1.1}$$

for coordinate $\{x^m\}$ and vielbein $\{e^A\}$. The wedge product is defined such that

$$dx^m \wedge dx^n \,(X, Y) = X^m Y^n - X^n Y^m \tag{B.1.2}$$

The exterior derivative d acting on ω is then

$$d\omega = \frac{1}{p!}\partial_k \omega_{m_1...m_p} dx^k \wedge dx^{m_1} \wedge ... \wedge dx^{m_p} \tag{B.1.3}$$

and therefore $(d\omega)_{km_1...m_p} = (p+1)\,\partial_{[k}\omega_{m_1...m_p]}$. In particular,

$$(d\kappa)_{mn} = \partial_m \kappa_n - \partial_n \kappa_m = \nabla^{\mathrm{LC}}_m \kappa_n - \nabla^{\mathrm{LC}}_n \kappa_m \tag{B.1.4}$$

B.2 Covariant Derivatives

Let ∇ be an arbitrary connection on TM, then for any vector $X = X^m \partial_m$, one defines the connection coefficients $\Gamma^k{}_{mn}$ as $\nabla_m X^k = \partial_m X^k + \Gamma^k{}_{mn} X^n$. The torsion tensor of such a connection is defined as $T^k{}_{mn} \equiv \Gamma^k{}_{mn} - \Gamma^k{}_{nm}$.

B.2.1 Levi-civita Connection

In the main text, we denote Levi-civita connection on M as ∇:

$$\nabla g = 0 \tag{B.2.1}$$

with connection coefficients

$$\Gamma^k{}_{mn} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{ml}}{\partial x^n} + \frac{\partial g_{nl}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^l}\right) \tag{B.2.2}$$

and curvature tensor

$$[\nabla_m, \nabla_n]\, X^k = R_{mnl}{}^k X^l \tag{B.2.3}$$

Ricci tensor is defined as

$$Ric_{mn} = R_{mkn}{}^k \tag{B.2.4}$$

B.2.2 Lie derivative

Lie derivative for (1,1), (0,2) tensor are defined as

$$\mathcal{L}_X T^m{}_n = X^k \nabla_k T^m{}_n - (\nabla_k X^m)\, T^k{}_n + \left(\nabla_n X^k\right) T^m{}_k \tag{B.2.5}$$

$$\mathcal{L}_X T_{mn} = X^k \nabla_k T_{mn} + \left(\nabla_m X^k\right) T_{kn} + \left(\nabla_n X^k\right) T_{mk} \tag{B.2.6}$$

with the obvious relation

$$\mathcal{L}_X T_{mn} = (\nabla_m X_l + \nabla_l X_m)\, T^l{}_n + g_{ml}\mathcal{L}_X T^l{}_n \tag{B.2.7}$$

In particular, when acting on a differential forms ω, one has the Cartan formula

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega. \tag{B.2.8}$$

B.2.3 Vielbein and Spin connection

Let $\{e^A\}$ be an orthonormal basis with respect to metric g. Then given any connection ∇ preserving g, one can write down Cartan structure equation and so define connection 1-form (also called spin connection) $\omega^A{}_B$

$$de^A + \omega^A{}_B \wedge e^B = T^A \Leftrightarrow \nabla_m e_B = \omega_m{}^A{}_B e_A \tag{B.2.9}$$

Preserving the metric g implies anti-symmetric property $\omega^A{}_B + \omega^B{}_A = 0$. $\omega^A{}_B$ can be solved from the structure equation, and expressed in terms of $\Gamma^k{}_{mn}$

$$\omega_m{}^A{}_B = e^A_k e^n_B \Gamma^k{}_{mn} - e^n_B \partial_m e^A_n \tag{B.2.10}$$

It is easy to solve the spin connection for the Levi-Civita connection ∇^{LC} of g. Suppose $de^A = C^A{}_{BC} e^B \wedge e^C$ with $C^A{}_{BC} + C^A{}_{CB} = 0$, and $\omega^A{}_B = \omega_C{}^A{}_B e^C$, then

$$\omega_C{}^A{}_B = -C^A{}_{CB} - C^B{}_{AC} + C^C{}_{BA} \tag{B.2.11}$$

One can use this to obtain $\omega_m{}^A{}_B \Gamma^{AB}$, or one can exploit the identification

$$\sum_{A,B} \omega_m{}^A{}_B \Gamma^{AB} \leftrightarrow \sum_{A,B} \omega_m{}^A{}_B e^A \wedge e^B \tag{B.2.12}$$

to simplify computation:

$$
\begin{aligned}
de^A + \sum_B \omega^A{}_B e^B = 0 &\Leftrightarrow \iota_{\partial_m} de^A + \sum_B \omega_m{}^A{}_B e^B - e^B_m \omega^A{}_B = 0 \\
\Rightarrow \sum_{A,B} \omega_m{}^A{}_B e^A \wedge e^B &= -\sum_A \left(e^A \wedge \iota_{\partial_m} de^A + e^A_m de^A \right)
\end{aligned}
\tag{B.2.13}
$$

Given any connection ∇ that preserves metric g, maybe with torsion, one can induce a connection on the spin bundle S

$$
\nabla_m \psi = \partial_m \psi + \frac{1}{4} \omega_m{}^A{}_B \Gamma^{AB} \psi \tag{B.2.14}
$$

We will sometimes use $\cdot$ to denote Clifford action of any differential p-form ω on spinors:

$$
\omega \cdot \psi = \frac{1}{p!} \omega_{A_1 \ldots A_p} \Gamma^{A_1 \ldots A_p} \psi. \tag{B.2.15}
$$

So in particular, $d\kappa \cdot \psi = \dfrac{1}{2} d\kappa_{mn} \Gamma^{mn} \psi$.

Appendix C

Contact Geometry

In this appendix we review some basics aspects about contact geometry and K-contact structures. Interested readers may refer to [2] for more detail [1].

Symplectic geometry is a well-known type of geometry in even dimensions. There, a symplectic structure is defined to be a closed and non-degenerate 2-form ω. In odd dimensions, there is a similar type of structures, called contact structures, which have many similar and interesting behaviors as symplectic structures.

C.1 Hyperplane Field

A hyperplane field E on a manifold M is a codimension one sub-bundle of the tangent bundle TM. Locally, E can always be defined as the kernel of certain 1-form κ. The Euler number $\chi(M) = 0$ implies that generic vector fields or 1-forms on M have no zeros. So l. In particular, any nowhere-vanishing 1-form κ defines a global hyperplane field $E = \ker(\kappa)$. Note that rescaling $\kappa \to e^f \kappa$ does not change the corresponding hyperplane field. If M is further equipped with a Riemannian metric g, one can define a vector field R associated to κ

$$g\,(R, \cdot) \equiv \kappa\,(\cdot)\,. \tag{C.1.1}$$

C.2 Almost contact structure

Let M be a $2n + 1$ oriented dimensional smooth manifold. An almost contact structure[2] on M consists of a nowhere-vanishing 1-form κ, a nowhere vanishing vector field R and a $(1, 1)$-type tensor $\Phi^m{}_n$ viewed as a map $\Phi : \Gamma(TM) \to \Gamma(TM)$, such that

$$\kappa\,(R) = 1, \quad \Phi^2 = -1 + R \otimes \kappa. \tag{C.2.1}$$

[1] However we point out that the convention of exterior derivative d in [2] is such that, for instance,

$$d\kappa = \frac{1}{2}\,(\partial_m \kappa_n)\,dx^m \wedge dx^n \tag{C.0.1}$$

[2] An almost contact structure can also be defined as a reduction of structure group from $SO(2n + 1)$ to $U(n)$.

Note that the condition $\Phi(R) = \kappa \circ \Phi = 0$ can be derived from the above conditions.

Given an almost contact structure, one can always find a (actually infinitely many) compatible metric g such that

$$g(R, \cdot) = \kappa(\cdot). \tag{C.2.2}$$

Together with the metric, (κ, R, Φ, g) is called an almost contact metric structure.

C.3 Almost CR structure

A almost contact structure is equivalent to the notion of almost CR structure, which emphasizes the decomposition of $TM_H = T^{1,0} \oplus T^{0,1}$, induced by Φ, such that

$$\Phi|_{T^{1,0}} = i, \quad \Phi|_{T^{0,1}} = -i \tag{C.3.1}$$

C.4 Contact Structure

Let M be a $2n+1$-dimensional compact smooth manifold. Let κ be a nowhere-vanishing 1-form. Then it defines the horizontal vector bundle $TM_H \subset TM$, as we mentioned in the section 2.2.1.

In particular, κ defines a *contact structure*, or *contact distribution* TM_H, if it satisfies

$$\kappa \wedge (d\kappa)^n \neq 0, \quad \text{Everywhere on } M. \tag{C.4.1}$$

κ itself is called a *contact* 1-form of the structure. So in odd dimensions, $d\kappa$ plays the role of symplectic form in even dimensions; indeed, it renders the horizontal bundle TM_H as a symplectic vector bundle of rank $2n$.

Once a contact 1-form is given, there is unique vector field R such that

$$\kappa_m R^m = 1, \quad R^m (d\kappa)_{mn} = 0. \tag{C.4.2}$$

and we call it the *Reeb vector field* associated to contact the 1-form κ. The Reeb vector field on a compact contact manifold generates 1-parameter family of diffeomorphisms (an effective smooth $\mathbb{R}$-action on M), which is usually called the *Reeb flow* $\varphi_R(t)$, or the contact flow; the flow translates points along the integral curves of the R. It follows from the definition that the flow preserves the contact structure, since $\mathcal{L}_R = \iota_R d\kappa + d\iota_R$ and $\mathcal{L}_R \kappa = 0$, $\mathcal{L}_R d\kappa = 0$.

It is important to note that the integral curves (or equivalently, the Reeb flow) of R have three types of behaviors:

1) The *regular* type is that all the curves are closed and the Reeb flow generates *free* $U(1)$-action on M, rendering M a principal $U(1)$-bundle over some $2n$-dimensional symplectic manifold.

2) A *quasi-regular* type is that although the curves are all closed, the Reeb flow only generates *locally-free* $U(1)$-action.

3) The *irregular* type captures the generic situations, where not all the integral curves are closed. Irregular Reeb flows can have very bad behaviors, but if the Reeb vector field preserves some metric on M, then the behavior could still be tractable. In other context, irregular Reeb flows are better

than the other two types, in the sense that they are non-degenerate and may provide isolated closed Reeb orbits.

C.5 Contact metric structure

Just as in symplectic geometry, one would like to have some metric and almost conplex structure into the play, so that the contact structure has more "visible" properties.

Given a contact 1-form κ, one can define a set of quantities (κ, R, g, Φ) where g is a metric and Φ is a $(1,1)$-type tensor, such that

$$g_{mn}R^n = \kappa_m, \quad 2g_{mk}\Phi^k{}_n = (d\kappa)_{mn} = \nabla^{\mathrm{LC}}_m \kappa_n - \nabla^{\mathrm{LC}}_n \kappa_m, \quad \Phi^2 = -1 + R \otimes \kappa. \tag{C.5.1}$$

where ∇^{LC} denotes the Levi-Civita connection of g. We call such set of quantities a *contact metric structure.*

There are a few useful algebraic and differential relations between quantities. First we have

$$\Phi^n{}_m R^m = \kappa_n \Phi^n{}_m = 0, \quad \frac{(-1)^n}{2^n n!}\kappa \wedge (d\kappa)^n = \Omega_g. \tag{C.5.2}$$

where Ω_g is the volume form associated to metric g. From this one can show that $d\kappa$ satisfies

$$\iota_R * d\kappa = d\kappa. \tag{C.5.3}$$

And in fact, if one takes an adapted vielbein, for instance in 5-dimension, satisfying $e_5 = R, \ \ \Phi\left(e_{2i-1}\right) = e_{2i}, \ \ \kappa\left(e_{1,2,3,4}\right) = 0, \quad i = 1, 2$, one has

$$d\kappa = 2\left(e^1 \wedge e^2 + e^3 \wedge e^4\right). \tag{C.5.4}$$

Moreover, using $\iota_R d\kappa = 0$ and $\kappa(R) = 1$, it can shown that

$$R^n \nabla_m \kappa_n = \kappa_n \nabla_m R^n = R^m \nabla_m R^n = 0, \tag{C.5.5}$$

namely R is geodesic.

There are useful relations between R and Φ: for any contact metric structure,

$$R^m \nabla^{\mathrm{LC}}_m \Phi^n{}_k = 0. \tag{C.5.6}$$

and also

$$\nabla^{\mathrm{LC}}_m R^n = -\Phi^n{}_m - \frac{1}{2}(\Phi \circ \mathcal{L}_R \Phi)^n{}_m. \tag{C.5.7}$$

C.6 K-contact structure

As we have mentioned earlier, irregular Reeb flows can be more tractable if certain metric is invariant under the flow. This leads to the notion of K-contact structure, where the Reeb vector field is Killing with respect to the metric in a contact metric structure:

It is called a *K-contact structure*, if a contact metric structure satisfies an additional condition

$$\mathcal{L}_R g = 0. \tag{C.6.1}$$

Note that this is equivalent to, since Φ and $d\kappa$ are related by metric g, it is easy to see that $\mathcal{L}_R \Phi = 0$, and consequently, $\nabla_m R^n = -\Phi^n{}_m$.

C.7 Sasakian Structure

A Sasakian structure is a K-contact structure (κ, R, g, Φ) with additional constraint

$$(\nabla_X \Phi)\, Y = g\,(X, Y)\, R - \kappa\,(Y)\, X \tag{C.7.1}$$

Sasakian structures are Kähler structures in the odd dimensional world. Therefore, it enjoys many simple properties that allow simplification in computations.

C.8 Generalized Tanaka-Webster connection

There have been several special connections on contact metric structures introduced in various literatures. For us, the most important one is the generalized Tanaka-Webster connection. There are actually two special connections, both of which are called generalized Tanaka-Webster connection, one introduced by Tanno [53] and the other introduced in [41]. Their names comes from the property that when restricted on a integrable CR structure, the two connections reduces to the usual Tanaka-Webster connection. The former connection satisfies

$$\nabla \kappa = \nabla R = \nabla g = 0. \tag{C.8.1}$$

On a general contact metric structure, the two connections are different. However, when the structure is K-contact, the two connections induces the same Dirac operator on the spin bundle S via the standard formula

$$\nabla\!\!\!/^{\,\mathrm{TW}} \equiv \Gamma^m \nabla^{\mathrm{TW}}_m = \Gamma^m \left(\partial_m + \frac{1}{4} \left(\omega^{\mathrm{TW}}_m\right)^A{}_B \Gamma^{AB} \right). \tag{C.8.2}$$

In terms of the Levi-Civita connection ∇^{LC}, this Dirac operator reads

$$\nabla\!\!\!/^{\,\mathrm{TW}} \psi = \nabla\!\!\!/^{\,\mathrm{LC}} \psi + \frac{1}{4} d\kappa \cdot \psi, \tag{C.8.3}$$

which is the operator that appears in the localization locus (4.2.30). Using the projection $P_\pm$ to chiral and anti-chiral spinors, one has for chiral spinor $\forall \phi_+ \in \Gamma(S_+)$

$$P_- \nabla\!\!\!/^{\,\mathrm{TW}} \phi_+ = P_- \nabla\!\!\!/^{\,\mathrm{LC}} \phi_+, \quad P_+ \nabla\!\!\!/^{\,\mathrm{TW}} \phi_+ = -\left(\nabla^{\mathrm{LC}}_R + \frac{1}{4} d\kappa \cdot \right) \phi_+ = -\nabla^{\mathrm{TW}}_R \phi_+. \tag{C.8.4}$$

C.9 Compatible Connection

Suppose ∇ is any affine connection on TM, then one can define new connection $\hat{\nabla}$ that preserves φ:

$$\hat{\nabla}_m X^n \equiv \nabla_m X^n + K^n{}_{mk} X^k \tag{C.9.1}$$

where

$$K^n{}_{ml} \equiv -\frac{1}{2}\left(\nabla_m \varphi_l{}^k\right)\varphi_k{}^n - \frac{1}{2}\kappa_l \nabla_m R^n + R^n \nabla_m \kappa_l \tag{C.9.2}$$

If the affine connection ∇ is chosen to be the Levi-civita connection associated to the ACMS structure, then one has

$$K_{nml} = -K_{lmn} \tag{C.9.3}$$

As mentioned, we have

$$\hat{\nabla}_m \varphi_n{}^k = 0 \tag{C.9.4}$$

Moreover, for any $X, Y \in \Gamma(TM_H)$, one has

$$g(\hat{\nabla}_X Y, R) = 0 \tag{C.9.5}$$

which means $\hat{\nabla}_X Y \in \Gamma(TM_H)$, the restriction of $\hat{\nabla}$ on TM_H gives directly a connection $\hat{\nabla}|_{TM_H} \equiv \nabla^H$ on TM_H.

The connection coefficients are now

$$\hat{\Gamma}^n{}_{ml} = \Gamma^n{}_{ml} + K^n{}_{ml} \tag{C.9.6}$$

and the corresponding change of spin connection

$$\Delta\omega_{ma}{}^b = \omega_{ma}{}^b + K^b{}_{ma} \tag{C.9.7}$$

where we define the spin connection[3]

$$\omega_{ma}{}^b \equiv e^b\left(\nabla_m e_a\right) = e^b{}_n \nabla_m e_a{}^n = e^b{}_n \partial_m e^n_a + \Gamma^b{}_{ma} \tag{C.9.10}$$

In three dimension, where one has relation

$$\varphi_{mn} = \epsilon_{mnk} R^k, \quad R_m = \kappa_m \tag{C.9.11}$$

K can be simplified as

$$K^n{}_{ml} = R^n \nabla_m R_l - R_l \nabla_m R^n \tag{C.9.12}$$

[3]Note that the position of the flat indices a and b indicates that

$$\nabla_m \psi = \partial_m \psi - \frac{1}{4}\omega_{mab}\Gamma^{ab}\psi \tag{C.9.8}$$

as opposed to the frequently used notation $\omega_m{}^b{}_a$ which indicates

$$\nabla_m \psi = \partial_m \psi + \frac{1}{4}\omega_{mab}\Gamma^{ab}\psi \tag{C.9.9}$$

The covariant derivative on spinor with new connection is now

$$\hat{\nabla}_m \xi_I = \nabla_m \xi_I - \frac{1}{4} K_{lmn} \Gamma^{nl} \xi_I \tag{C.9.13}$$

Now, let's consider the ACMS data coming from $\left(s^{-1}R, s^{-1}\kappa, r\left(t\right) t\Theta, g\right)$, where

$$r\left(t\right) = \frac{1}{s}\sqrt{\frac{-2}{\operatorname{tr}\left(t^2\right)}} \tag{C.9.14}$$

such that

$$r(t)^2 (t\Theta)^2 = -1 + \left(s^{-2}R\right) \otimes \kappa \tag{C.9.15}$$

Substituting all these into definition of K, one has (with the assumption that $\Omega^- = 0$)

$$\begin{aligned} K_{nml} = {} & \frac{1}{s^2}\left(R_n \nabla_m R_l - R_l \nabla_m R_n\right) + \frac{1}{s}\frac{1}{\operatorname{tr}\left(t^2\right)} (T_m)_{IJ} \left(\Theta^{IJ}\right)_{ln} \\ & - r(t)^2 \left[(*V_V)_{kmr} (t\Theta)_l{}^r - (*V_V)_{lmr} (t\Theta)_k{}^r \right] (t\Theta)^k{}_n \end{aligned} \tag{C.9.16}$$

where

$$(T_m)_{IJ} \equiv \left(\nabla^A_m t_I{}^K\right) t_{KJ} \tag{C.9.17}$$

Note that when $s = 1$, $K_{nml} = -K_{lmn}$.

To calculate the spin connection, one needs several convenient formula

$$(t\Theta)_{nm} \Gamma^n \xi_I = \left(s\Gamma_m - R_m\right) t_I{}^J \xi_J \tag{C.9.18}$$

$$(t\Theta)_{nm} \Gamma^{kn} \xi_I = \Gamma^k \left(\Gamma_m s - R_m\right) t^J{}_I \xi_J - (t\Theta)^k{}_m \xi_I \Rightarrow (t\Theta)_{mn} \Gamma^{mn} \xi_I = -4 s t^J{}_I \xi_J \tag{C.9.19}$$

$$R^m \Gamma_{nm} \xi_I = \left(s\Gamma_n - R_n\right) \xi_I \tag{C.9.20}$$

Finally, one has

$$\begin{aligned} \hat{\nabla}_m \xi_I = {} & \nabla_m \xi_I + \frac{1}{\operatorname{tr}\left(t^2\right)} (T_m)^J{}_I \xi_J - \frac{1}{2s} \nabla_m R_n \Gamma^n \xi_I + \frac{1}{2}\left(\nabla_m \log s\right) \xi_I \\ & - \frac{1}{\operatorname{tr}\left(t^2\right)} \eta_q (t\Theta)^q{}_m t_I{}^J \xi_J + \frac{1}{2}\left(*V^V\right)_{mpq} \Gamma^{pq} \xi_I \end{aligned} \tag{C.9.21}$$

Some remark. We used almost contact data φ defined as $\sim t\Theta$, but in fact one could use any $SU(2)$ triplet function λ to define $\varphi_\lambda \sim \lambda\Theta$, and in particular, one could choose $\lambda = \lambda_a \sigma^a$. It also has corresponding compatible connection $\hat{\nabla}_\lambda$, such that

$$\hat{\nabla}_\lambda \varphi_\lambda = 0 \tag{C.9.22}$$

However, the tensor K_{lmn} would not have the above simple form.

Appendix D

Spin$^{\mathbb{C}}$ bundle and the Dolbeault-Dirac operator

In this appendix we review the Spin$^{\mathbb{C}}$ bundles on a contact metric manifold and a canonical Dirac operator on any K-contact structure.

Consider a contact metric structure (κ, R, g, Φ). Then on the horizontal tangent bundle TM_H, Φ defines a complex structure and thus induces a (p,q)-decomposition of the complexification

$$TM_H \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M, \qquad \wedge^{\bullet} TM_H^* \otimes \mathbb{C} = \oplus T^{p,q}M^* \tag{D.0.1}$$

Let us focus on a 5-dimensional contact metric structure $(M; \kappa, R, g, \Phi)$. One can start from an adapted vielbein e^A as discussed before, and consider the complexification

$$e^{z_1} \equiv e^1 + ie^2, \quad e^{z_2} \equiv e^3 + ie^4. \tag{D.0.2}$$

With this complex basis, one sees that $d\kappa$ is of type-$(1,1)$ as expected

$$d\kappa = i\left(e^{z_1} \wedge e^{\bar{z}_1} + e^{z_2} \wedge e^{\bar{z}_2}\right). \tag{D.0.3}$$

The bundle $W^0 \equiv T^{0,\bullet}M^*$ is also a Spin$^{\mathbb{C}}$ bundle in the sense that TM^* acts on it in a Clifford manner

$$\begin{cases} \omega \cdot \psi = \sqrt{2}i\left(\omega_{\bar{i}} \bar{e}^{\bar{i}} \wedge \psi - g^{\bar{i}j}\omega_j \iota_{e_{\bar{i}}}\psi\right), \quad \omega = \omega_i e^i + \omega_{\bar{i}}\bar{e}^{\bar{i}} \in \Gamma\left(TM_H^*\right) \\ \kappa \cdot \psi = e^1 \cdot e^2 \cdot e^3 \cdot e^4 \cdot \psi \end{cases}. \tag{D.0.4}$$

which satisfies the Clifford algebra $\{\omega\cdot, \mu\cdot\} = 2g(\omega, \mu)$. In particular, W^0 decomposes into chiral and anti-chiral spinor bundle according to the eigenvalue ± 1 of $\Gamma_C \equiv -\kappa\,\cdot$

$$W^0 = W^0_+ \oplus W^0_-, \quad W^0_+ \equiv T^{0,0}M^* \oplus T^{0,2}M^*, \quad W^0_- \equiv T^{0,1}M^*. \tag{D.0.5}$$

Using the complex basis $e^{\bar{z}_i}$, one can define an orthonormal basis of W^0:

$$W^0_+ = \operatorname{span}\left\{1, \frac{1}{2}e^{\bar{z}_1} \wedge e^{\bar{z}_2}\right\}, \quad W^0_- = \operatorname{span}\left\{\frac{1}{\sqrt{2}}e^{\bar{z}_1}, \frac{1}{\sqrt{2}}e^{\bar{z}_2}\right\} \tag{D.0.6}$$

If one represents

$$\phi = a_1 + \frac{a_2}{2}e^{\bar{z}_1} \wedge e^{\bar{z}_2} + \frac{a_3}{\sqrt{2}}e^{\bar{z}_1} + \frac{a_4}{\sqrt{2}}e^{\bar{z}_2} \leftrightarrow (a_1, a_2, a_3, a_4)^T, \tag{D.0.7}$$

then the above Clifford action is represented as (A.1.10).

On a contact metric structure, there may be other $\mathrm{Spin}^{\mathbb{C}}$ bundles. They can be obtained by tensoring an arbitrary complex line bundle E:

$$W = W^0 \otimes E, \quad W_\pm = W^0_\pm \otimes E \tag{D.0.8}$$

In particular, when the manifold is spin, the spin bundle S can be obtained by

$$S = W^0 \otimes K_M^{-1/2} \Leftrightarrow W^0 = S \otimes K_M^{1/2} \tag{D.0.9}$$

where $K_M \equiv T^{0,2}M^*$. More generally, any $\mathrm{Spin}^{\mathbb{C}}$ bundle W can be written as $W = S \otimes L^{1/2}$ for some complex line bundle $L^{1/2}$ (and its square L is called the determinant line bundle of W). For instance, $W^0 = S \otimes K_M^{1/2}$ and therefore the determinant line bundle L^0 of W^0 is $L^0 = K_M$. Generally, the determinant line bundle L of $W = W^0 \otimes E$ is $L = K_M \otimes E^2$.

This implies that given a connection on S (which can be induced from a metric connection $\omega^A{}_B$) and a $U(1)$-connection[1] A on $L^{1/2}$, we have a connection on $W = S \otimes L^{1/2}$

$$\nabla_A \psi = \nabla\psi - iA\psi, \quad \forall \psi \in \Gamma(W) \tag{D.0.10}$$

The situation of W^0 is a bit special, since one can induce a canonical $U(1)$-connection A_0 on K_M using the Chern connection ∇^{C} on the almost-hermitian cone $C(M)$. Therefore, taking A_0 as a reference connection, any connection A on a $\mathrm{Spin}^{\mathbb{C}}$ bundle W can be written in terms of a $U(1)$-connection a on E as $A = \frac{1}{2}A_0 + a$.

The above construction is good for any contact metric structure. Now let us focus on a K-contact structure, and use the generalized Tanaka-Webster connection to induce a connection ∇^{TW} on S. Combining with a $U(1)$-connection A on $L^{1/2}$, one can define a Dirac operator $\not{D}_A^{\mathrm{TW}}$ [41, 42, 53]

$$\not{D}_A^{\mathrm{TW}} \equiv \Gamma \cdot \nabla_A^{\mathrm{TW}} \tag{D.0.11}$$

In [41], it is shown that when E is trivial and $a = 0$, namely $A = 1/2A_0$,

$$\not{D}^{\mathrm{TW}}_{A_0/2}(\alpha + \beta) = \mathcal{L}_R(\alpha + \beta) + \bar{\partial}\alpha + \bar{\partial}^*\beta, \quad \alpha + \beta \in \Omega^{0,0} \oplus \Omega^{0,2} = \Gamma(W^0_+) \tag{D.0.12}$$

where the Dolbeault operator ∂ and $\bar{\partial}$ are defined in the usual way[2]

$$\partial \equiv \pi^{p+1,q} \circ d : T^{p,q}M^* \to T^{p+1,q}M^*, \quad \bar{\partial} \equiv \pi^{p,q+1} \circ d : T^{p,q}M^* \to T^{p,q+1}M^* \tag{D.0.14}$$

[1] A local basis σ on $L^{1/2}$ is assumed, such that $\nabla_A(f\sigma) = df \otimes \sigma - iA \otimes (f\sigma)$

[2] On a K-contact structure, on has in general (recall that $\mathcal{L}_R$ preserves Φ and therefore the (p,q)-decomposition)

$$d : T^{p,q}M^* \to \kappa \wedge T^{p,q}M^* \oplus \left(T^{p+1,q}M^* \oplus T^{p,q+1}M^* \oplus T^{p+2,q-1}M^* \oplus T^{p-1,q+1}M^*\right) \tag{D.0.13}$$

Note that the two operators do not square to zero in general; define $N\left(\omega^{p,q}\right) \equiv \pi^{p-1,q+2}\left(d\omega^{p,q}\right)$ and $\bar{N}\left(\omega^{p,q}\right) \equiv \pi^{p+2,q-1}\left(d\omega^{p,q}\right)$, then one has

$$\bar{\partial}^2\alpha^{p,q} = -N\left(\partial\alpha^{p,q}\right) - \partial N\left(\alpha^{p,q}\right), \quad \partial^2\alpha^{p,q} = -\bar{N}\left(\bar{\partial}\alpha^{p,q}\right) - \bar{\partial}\bar{N}\left(\alpha^{p,q}\right), \tag{D.0.15}$$

$$\left\{\partial, \bar{\partial}\right\}\omega^{p,q} = -d\kappa \wedge \mathcal{L}_R\omega^{p,q} - \left\{N, \bar{N}\right\}\left(\omega^{p,q}\right), \tag{D.0.16}$$

which are almost identical to those on symplectic 4-manifolds, except for the term $d\kappa \wedge \mathcal{L}_R$. On a Sasakian structure, the Nijenhuis map N and $\bar{N}$ vanish and $\partial^2 = \bar{\partial}^2 = 0$, similar to Kähler structure.

Weitzenböck Formula

We review several useful formula for studying 5d Seiberg-Witten equations, which are direct generalization from those on symplectic 4-manifolds.

Consider $W = W^0 \otimes E$ with $U(1)$-connection a on E, with curvature $F_a = da$. Then for $\alpha \in \Omega^{0,0}(E)$, $\beta \in \Omega^{0,2}(E)$, one has Weitzenböck formula

$$2\bar{\partial}_a^*\bar{\partial}_a\alpha = d_a^{J*}d_a^J\alpha - \Lambda F_a^{1,1}\alpha + 2i\mathcal{L}_R^a\alpha, \quad 2\bar{\partial}_a\bar{\partial}_a^*\beta = \nabla_{A_0+a}^*\nabla_{A_0+a}\beta - \Lambda F_{A_0+a} + 2i\mathcal{L}_R^a\beta. \tag{D.0.17}$$

where we define operator $d_a^J \equiv \partial_a + \bar{\partial}_a$, ∇_{A_0+a} is the connection induced by A_0 and a on $K_M \otimes E$, Λ as the adjoint of wedging $d\kappa$:

$$\left\langle\alpha^{p-1,q-1}, \Lambda\beta^{p,q}\right\rangle = \frac{1}{2}\left\langle d\kappa \wedge \alpha^{p-1,q-1}, \beta^{p,q}\right\rangle, \quad \langle\alpha, \beta\rangle \equiv \int_M \alpha \wedge *_{\mathbb{C}}\beta \tag{D.0.18}$$

The Weitzenböck formula can be shown using Kähler identities

$$i\bar{\partial}_a^*\omega^{p,q} = \left[\Lambda, \partial_a\right]\omega^{p,q}, \quad -i\partial_a^*\omega^{p,q} = \left[\Lambda, \bar{\partial}_a\right]\omega^{p,q}, \quad \forall\omega^{p,q} \in \Omega^{p,q}(E). \tag{D.0.19}$$

and the fact that the Dolbeault operators can be expressed in terms of ∇^{TW}

$$\bar{\partial} = e^{\bar{z}_i} \wedge \nabla_{e_{z_i}}^{\mathrm{TW}}, \quad \bar{\partial}^* = -2\iota\left(e_{\bar{z}_i}\right)\nabla_{e_{z_i}}^{\mathrm{TW}}. \tag{D.0.20}$$

for an adapted complex vielbein.

Appendix E

The integrability of Transversal Holomorphic Foliation

In this appendix we discuss the spinorial holomorphy condition and the integrability of the canonical almost transversal holomorphic foliation $\Phi = \Phi[t]$ in terms of its Niejenhuis tensor.

E.1 The spinorial holomorphy condition

We prove the spinorial characterization of $T^{1,0}$ and $T^{1,0} \oplus \mathbb{R}R$ in equations (3.2.11) and (3.2.13). Assume $X \in T^{1,0}$, namely $\Phi^m{}_n X^n = iX^m$. Then

$$-\sqrt{\frac{1}{\det \mathfrak{m}}}\mathfrak{m}_I{}^J \left(\xi^I \Gamma^m \Gamma_n \xi_J\right) X^n + i\delta_I^J \left(\xi^I \Gamma^m \Gamma_n \xi_J\right) X^n = 0, \tag{E.1.1}$$

which simplifies to $H_I{}^J \left(\xi^I \Gamma^m \Gamma_n \xi_J\right) X^n = 0$, where $H_I{}^J \equiv (\det \mathfrak{m})^{-1/2} \mathfrak{m}_I{}^J - i\delta_I^J$.

Due to the reality properties of $\mathfrak{m}_{IJ}$ we have the identity $\sum H_I{}^K \overline{H_J{}^K} = 2iH_I{}^J$. Contracting the above with $\overline{X^m}$ and inserting the identity, one has

$$\begin{aligned}
&\overline{X_m} H_I{}^K \overline{H_J{}^K} \varepsilon^{II'} \xi_{I'}^\alpha C_{\alpha\beta} (\Gamma^m)^\beta{}_\gamma (\Gamma_n)^\gamma{}_\delta \xi_J^\delta X^n = 0 \\
\Leftrightarrow\ &\overline{X^m H_K{}^I (\Gamma_m)^\gamma{}_\beta \xi_I^\beta} X^n H_K{}^J (\Gamma_n)^\gamma{}_\delta \xi_J^\delta = 0 \\
\Leftrightarrow\ &\sum_{K,\alpha} \Delta_K^\alpha \overline{\Delta_K^\alpha} = 0
\end{aligned} \tag{E.1.2}$$

This implies $\Delta_K^\alpha = 0$, namely $H_I{}^J X^m \Gamma_m \xi_J = 0$.

It is obvious how to extend to $X \in T^{1,0} \oplus \mathbb{R}R$, one just need to project out the vertical components of X, and the horizontal components should satisfy $H_I{}^J X^m \Gamma_m \xi_J = 0$. Namely,

$$H_I{}^J \Pi^m{}_n X^n \Gamma_m \xi_J = 0, \quad \Pi^m{}_n = \delta_m^n - R^m \kappa_n. \tag{E.1.3}$$

E.2 The Nijenhuis tensor and $[T^{1,0}, T^{1,0}]$

Given an almost CR structure (κ, R, Φ), one can define its Nijenhuis tensor as

$$N_\Phi (X, Y) \equiv -[X, Y] + \kappa ([X, Y]) R + [\Phi X, \Phi Y] - \Phi [\Phi X, Y] - \Phi [X, \Phi Y], \tag{E.2.1}$$

which can be expressed in components

$$N^k{}_{mn} \equiv \Phi^l{}_m \nabla_l \Phi^k{}_n - \Phi^l{}_n \nabla_l \Phi^k{}_m + \Phi^k{}_l \nabla_n \Phi^l{}_m - \Phi^k{}_l \nabla_m \Phi^l{}_n. \tag{E.2.2}$$

For simplicity, we restrict our analysis to the canonical almost CR structure determined by t_{IJ}, namely

$$\Phi^m{}_n \equiv \sqrt{\frac{1}{\det t}} t^{IJ} (\xi_I \Gamma^m{}_n \xi_J). \tag{E.2.3}$$

By explicitly inserting the Killing spinor equation and the dilatino equation into (E.2.2), one finds that

$$N_\Phi (X, Y) + d\kappa (\Phi X, \Phi Y) R = 0, \qquad \forall X, Y \in \Gamma(TM_H), \tag{E.2.4}$$

provided that

$$X^m D_m \left(\frac{t_{IJ}}{\sqrt{\det t}} \right) = 0, \qquad \forall X \in \Gamma(TM_H), \tag{E.2.5}$$

where TM_H is the horizontal part of the tangent bundle. Of course this condition is the same as in (3.3.18).

We will now show that the above condition (E.2.4) is equivalent to the statement that

$$\left[T^{1,0}, T^{1,0}\right] \subset T^{1,0} \oplus \mathbb{C}R. \tag{E.2.6}$$

To do so, consider $X, Y \in T^{1,0}$. Using $\Phi(X) = \imath X$ and $\kappa([X, Y]) = -d\kappa(X, Y)$, one can evaluate (E.2.1):

$$N_\Phi(X, Y) + d\kappa(X, Y) = -2(1 + \imath\Phi)[X, Y] = -2[X, Y]^{0,1}. \tag{E.2.7}$$

It is clear that (E.2.4) implies that $[X, Y] \in T^{1,0} \oplus \mathbb{C}R$ and vice versa.

E.3 $\pounds_{sR}\Phi$ and $[T^{1,0}, R]$

In section 3.4 we showed gravitino and dilatino equations imply that for the canonical almost CR structure $\pounds_{sR}\Phi = 0$. For any $X \in TM$ it follows that

$$\pounds_{sR}(\Phi X) = \Phi(s[R, X] - X(s)R) = s\Phi([R, X]). \tag{E.3.1}$$

On the other hand

$$\pounds_{sR}(\Phi X) = [sR, \Phi X] = s[R, \Phi X] - (\Phi X)(s)R \tag{E.3.2}$$

and thus

$$s\Phi([R, X]) = s[R, \Phi X] - (\Phi X)(s)R, \tag{E.3.3}$$

which we rewrite as

$$[R, \Phi X] = \Phi([R, X]) + (\Phi X)(\log s)R. \tag{E.3.4}$$

Now, consider that any $X^{1,0} \in T^{1,0}$ can be written as $X^{1,0} = X - \imath\Phi X$ for some $X \in TM_H$. Then

$$[X^{1,0}, R] = (1 - \imath\Phi)[X, R] + \imath(\Phi X)(\log s)R = [X, R]^{1,0} + (\kappa([X, R]) + \imath(\Phi X)(\log s))R \in T^{1,0} \oplus \mathbb{C}R. \tag{E.3.5}$$

In other words, we have confirmed that the canonical almost CR structure defines a THF as long as the triplet ${t_I}^J$ is covariantly constant; i.e. equation (3.3.18).

Bibliography

[1] M. Zucker, *Minimal off-shell supergravity in five dimensions*, *Nucl.Phys. B570 (2000) 267-283* **B570** (2000) 267–283, [hep-th/9907082].

[2] D. Blair, *Riemannian Geometry of Contact and Sympletic Manifolds*, vol. 203 of *Progress in Mathematics*. Birkhauser, 2nd edition ed., 2010.

[3] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, *Exploring Curved Superspace*, *JHEP* **1208** (2012) 141, [arXiv:1205.1115].

[4] T. T. Dumitrescu and G. Festuccia, *Exploring Curved Superspace (II)*, *JHEP* **1301, 072** (2013) [arXiv:1209.5408].

[5] H. Baum, *Odd-dimensional riemannian manifolds with imaginary killing spinors*, *Annals of Global Analysis and Geometry* **7** (1989), no. 2 141–153.

[6] H. Baum, *Complete riemannian manifolds with imaginary killing spinors*, *Annals of Global Analysis and Geometry* **7** (1989), no. 3 205–226.

[7] C. Bar, *Real killing spinors and holonomy*, *Communications in Mathematical Physics* **154** (1993), no. 3 509–521.

[8] T. Friedrich and I. Kath, *Einstein manifolds of dimension five with small first eigenvalue of the dirac operator*, *Journal of Differential Geometry* **29** (1989), no. 2 263–279.

[9] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, *Supersymmetric field theories on three-manifolds*, *JHEP* **1305, 017** (12, 2013) [arXiv:1212.3388].

[10] J. Kallen, *Cohomological localization of Chern-Simons theory*, *JHEP* **1108, 008** (2011) [arXiv:1104.5353].

[11] A. Kapustin, B. Willett, and I. Yaakov, *Exact results for wilson loops in superconformal chern-simons theories with matter*, *JHEP* **1003, 089** (09, 2010) [arXiv:0909.4559].

[12] J. Kallen, J. Qiu, and M. Zabzine, *The perturbative partition function of supersymmetric 5d yang-mills theory with matter on the five-sphere*, *JHEP* **1208, 157** (2012) [arXiv:1206.6008].

[13] J. Kallen and M. Zabzine, *Twisted supersymmetric 5d yang-mills theory and contact geometry*, *JHEP* **1205, 125** (2012) [arXiv:1202.1956].

[14] K. Hosomichi, R.-K. Seong, and S. Terashima, *Supersymmetric gauge theories on the five-sphere*, *Nucl.Phys.B865:376-396,2012* (03, 2012) [arXiv:1203.0371].

[15] T. Kugo and K. Ohashi, *Off-shell $d = 5$ Supergravity coupled to Matter-Yang-Mills System*, *Prog.Theor.Phys. 105 (2001) 323-353* [hep-ph/0010288].

[16] K. O. T. Kugo, *Supergravity tensor calculus in 5d from 6d*, *Prog.Theor.Phys.* **104** (2000) 835–865.

[17] D. Cassani, C. Klare, D. Martelli, A. Tomasiello, and A. Zaffaroni, *Supersymmetry in Lorentzian Curved Spaces and Holography*, *Commun.Math.Phys.* **327** (2014) 577–602, [arXiv:1207.2181].

[18] I. Biswas *Conform. Geom. Dyn, 5, 74* (2001).

[19] H. Jacobowitz, *Transversely holomorphic foliations and cr structures*, .

[20] Y. Imamura and H. Matsuno, *Supersymmetric backgrounds from 5d $\mathcal{N} = 1$ supergravity*, arXiv:1404.0210.

[21] Y. Pan, *Rigid Supersymmetry on 5-dimensional Riemannian Manifolds and Contact Geometry*, *JHEP* **1405** (2014) 041, [arXiv:1308.1567].

[22] J. Schmude, *Localisation on Sasaki-Einstein manifolds from holomophic functions on the cone*, arXiv:1401.3266.

[23] L. F. Alday, P. B. Genolini, M. Fluder, P. Richmond, and J. Sparks, *Supersymmetric gauge theories on five-manifolds*, arXiv:1503.0909.

[24] D. Rodriguez-Gomez and J. Schmude, *Partition functions for equivariantly twisted $\mathcal{N} = 2$ gauge theories on toric kähler manifolds*, arXiv:1412.4407.

[25] J. Qiu and M. Zabzine, *5D Super Yang-Mills on Y^{pq} Sasaki-Einstein manifolds*, arXiv:1307.3149.

[26] Y. Pan, *5d Higgs Branch Localization, Seiberg-Witten Equations and Contact Geometry*, arXiv:1406.5236.

[27] E. Witten, *Topological quantum field theory*, *Communications in Mathematical Physics* **117** (1988) 353–386.

[28] A. Karlhede and M. Rocek, *Topological quantum field theory and n=2 conformal supergravity*, *Physics Letters B* **212** (1988) 51–55.

[29] D. Rodriguez-Gomez and J. Schmude, *Supersymmetrizing 5d instanton operators*, *JHEP 1503 (2015) 114*.

[30] L. F. Alday, M. Fluder, C. M. Gregory, P. Richmond, and J. Sparks, *Supersymmetric gauge theories on squashed five-spheres and their gravity duals*, *JHEP* **1409** (2014) 067, [arXiv:1405.7194].

[31] M. Zucker, *Gauged N=2 off-shell supergravity in five-dimensions*, *JHEP* **0008** (2000) 016, [hep-th/9909144].

[32] S. M. Kuzenko, J. Novak, and G. Tartaglino-Mazzucchelli, *Symmetries of curved superspace in five dimensions*, arXiv:1406.0727.

[33] S. M. Kuzenko and J. Novak, *On supersymmetric Chern-Simons-type theories in five dimensions*, *JHEP* **1402** (2014) 096, [arXiv:1309.6803].

[34] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **1106, 114** (05, 2011) [arXiv:1105.0689].

[35] C. Klare, A. Tomasiello, and A. Zaffaroni, *Supersymmetry on Curved Spaces and Holography*, *JHEP* **1208** (2012) 061, [arXiv:1205.1062].

[36] Y. Pan, *Note on a Cohomological Theory of Contact-Instanton and Invariants of Contact Structures*, arXiv:1401.5733.

[37] D. Harland and C. Nölle, *Instantons and Killing Spinors*, *JHEP* **1203, 082** (2012) [arXiv:1109.3552].

[38] M. Wolf, *Contact Manifolds, Contact Instantons, and Twistor Geometry*, *JHEP* **1207** (2012) 074, [arXiv:1203.3423].

[39] D. Baraglia and P. Hekmati, *Moduli spaces of contact instantons*, arXiv:1401.5140.

[40] R. Petit, $\mathrm{Spin}^{\mathbb{C}}$*-Structures and Dirac Operators on Contact Manifolds*, *Differential Geometry and its Applications* **22** (2005), no. 2 229 – 252.

[41] L. I. Nicolaescu, *Geometric Connections and Geometric Dirac Operators on Contact Manifolds*, math/0101155.

[42] N. Degirmenci and S. Bulut, *Seiberg-Witten Like Equations on 5-Dimensional Contact Metric Manifolds*, arXiv:1306.1008.

[43] C. H. Taubes, *SW* $\Rightarrow$ *Gr: From The Seiberg-Witten Equations To Pseudo-Holomorphic Curves*, *Journal of the American Mathematical Society* **9** (1996), no. 3.

[44] E. Witten, *Monopoles and four manifolds*, *Math.Res.Lett.* **1** (1994) 769–796, [hep-th/9411102].

[45] M. Hutchings and C. H. Taubes, *An introduction to the Seiberg-Witten equations on symplectic manifolds.* Park City-IAS Summer Institute, 1997.

[46] D. Martelli and J. Sparks, *Toric Geometry, Sasaki-Einstein Manifolds and a New Infinite Class of AdS/CFT Duals*, *Commun.Math.Phys.* **262** (2006) 51–89, [hep-th/0411238].

[47] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, *Sasaki-Einstein Mmetrics on* $S^2 \times S^3$, *Adv.Theor.Math.Phys.* **8** (2004) 711–734, [hep-th/0403002].

[48] F. Benini and W. Peelaers, *Higgs Branch Localization in Three Dimensions*, *JHEP* **1405** (2014) 030, [arXiv:1312.6078].

[49] Y. Imamura, *Perturbative Partition Function For A Squashed* S^5, arXiv:1210.6308.

[50] A. Narukawa, *The modular properties and the integral representations of the multiple elliptic gamma functions*, *ArXiv Mathematics e-prints* (June, 2003) [math/0306164].

[51] M. Jimbo and T. Miwa, *Quantum KZ equation with* $|q| = 1$ *and correlation functions of the XXZ model in the gapless regime*, *J.Phys.* **A29** (1996) 2923–2958, [hep-th/9601135].

[52] J. Qiu and M. Zabzine, *Factorization of 5D super Yang-Mills on* Y^{pq} *spaces*, *Phys. Rev. D 89, 065040* (12, 2014) [arXiv:1312.3475].

[53] S. Tanno, *Variational problems on contact riemannian manifolds*, *Trans. Amer. Math. Soc.* (1989), no. 341 349–379.