Exact Results in Supersymmetric and Superconformal Quantum Field Theories

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Abstract of the Dissertation

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In this dissertation we perform exact, non-perturbative computations in supersymmetric and superconformal quantum field theories.

In the first part, we show that the conformal bootstrap equations, which implement the requirement that in a conformal quantum field theory any four-point function should be crossing symmetric, have an exactly solvable truncation in theories with extended supersymmetry. As a result, we introduce a new 4d/2d correspondence that assigns to any four-dimensional $\mathcal{N} = 2$ superconformal quantum field theory, Lagrangian or non-Lagrangian, a two-dimensional chiral algebra, and subsequently explore its structure in the context of theories of class $\mathcal{S}$.

In the second part, we extend the application of the so-called Higgs branch localization technique to evaluate exactly the path integral of Lagrangian supersymmetric quantum field theories, placed on compact three- and four-dimensional Euclidean manifolds.

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List of Publications

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[6] **"The superconformal index of class S theories of type D"**[1]
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[1] The contents of this paper will not be dicussed in this dissertation.

Chapter 1

Introduction

Quantum field theory has proved itself to be a versatile and powerful framework to describe nature at the shortest length scales. If a Lagrangian description is available, a quantum field theory can be defined in terms of the path integral over the infinite-dimensional space of fields. The phenomenologically most successful instance of this kind, the Standard Model, adequately captures all elementary particles and their non-gravitational interactions discovered thus far. However, various experimental observations and theoretical considerations indicate that it is only an effective theory with new physics entering at higher energies. One of the theoretically most appealing extensions of the Standard Model arises by enlarging its Poincaré symmetry to (Poincaré) supersymmetry, which is often lauded for providing an elegant solution to the hierarchy problem.[1]

Computations in the Lagrangian subspace of quantum field theories are typically performed in a weakly coupled regime of the theory where standard perturbation theory can be applied. However, our aim should be to obtain exact, non-perturbative results. It is clear then that perturbative expansions around a (possibly non-perturbative) saddle-point are not the correct tool, since they are asymptotic series with zero radius of convergence and, by their very nature, do not include any non-perturbative corrections. The situation can be exemplified as follows: perturbative predictions made within the Standard Model are in remarkable agreement with experimental data, but we stand empty-handed to address problems such as the generation of

[1]The simple observation that the pairing of bosonic degrees of freedom to fermionic ones leads to dramatic cancellations, in this case in the radiative corrections to the Higgs boson mass, will be a recurring feature in Part II of this dissertation.

a mass gap in pure Yang-Mills theories, quark confinement and spontaneous chiral symmetry breaking in quantum chromodynamics,. . . , since they are well outside the regime of validity of perturbation theory.

In combination with the aforementioned supersymmetry, the situation improves dramatically. Indeed, apart from potential phenomenological applications, supersymmetric quantum field theories have long been recognized as appealing theoretical laboratories to test more general ideas in quantum field theories — perhaps their most attractive feature being our ability to obtain exact, non-perturbative results. For example, the exact low energy effective action of four-dimensional $\mathcal{N}=2$ super Yang-Mills theory, determined in [7] by leveraging holomorphy and electric-magnetic duality, when further perturbed by a superpotential term that breaks $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$, exhibits quark confinement via condensation of magnetic monopoles in a concrete way.

Another theoretically interesting class of quantum field theories are those that feature conformal symmetry. In fact, their distinct character among quantum field theories becomes clear when adapting the alternative way of thinking about quantum field theory as the study of renormalization group flows, whose fixed points are precisely conformal field theories.

The purpose of this dissertation now is to develop and apply general tools that enable us to study in detail and in an exact fashion certain (classes of) observables in theories endowed with additional symmetries – the aforementioned supersymmetry, conformal symmetry or a combination thereof – that constrain their dynamics and thereby make them more tractable.

Infinite Chiral Symmetry in Four Dimensions

In recent years, the importance and abundance of superconformal quantum field theories has become increasingly clear. Generically, they do not admit a Lagrangian description and are strongly interacting. Techniques to access the strongly coupled phases of these theories directly are scarce: for example, the localization techniques of the next paragraph crucially rely on a Lagrangian formulation. On the other hand, as conformal field theories, they possess a convergent operator product expansion and are completely specified by their conformal data, *i.e.*, their spectrum of operators and the collection of three-point couplings. The latter must satisfy the stringent consistency conditions arising from the requirement that any four-point function should be crossing symmetric or equivalently, that the operator algebra is associative, commonly referred to as the conformal bootstrap equations. The conformal bootstrap program [8, 9] then embodies the hope that given a few spectral assumptions, crossing symmetry (and unitarity) will completely fix the conformal data of the theory irrespective of the existence of a Lagrangian. Such an approach was successfully carried out for certain classes of two-dimensional conformal field theories, *e.g.*, rational conformal field theories, but seemed to fall short in more general (higher dimensional) CFTs. In combination with extended supersymmetry, however, we will show in Part I that the crossing symmetry equations admit an exactly solvable truncation.

More in detail, in chapter 2 we will show that in any four-dimensional $\mathcal{N} = 2$ superconformal field theory one can identify a subsector of protected operators, restricted to be coplanar and treated at the level of cohomology with respect to a particular nilpotent supercharge, whose operator product expansions and correlation functions are meromorphic functions of their position on the plane and thus define a chiral algebra. The resulting chiral algebra has some remarkable properties: it features an enhancement of global $\mathfrak{sl}(2)$ conformal symmetry on the chiral algebra plane to full-fledged Virasoro symmetry, and of possible flavor symmetries present in the four-dimensional theory to affine symmetries. The two-dimensional translation of exactly marginal gauging in the four-dimensional theory is elegantly given by a BRST procedure. Interestingly, the spectrum and structure constants of the chiral algebra are subject to a non-renormalization theorem and are therefore independent of marginal couplings.

The chiral algebras associated to free theories are easily constructed and are in principle sufficient ingredients to construct the chiral algebra for any interacting Lagrangian theory via the gauging procedure. In practice, the BRST cohomology problem is hard to solve, but we will formulate conjectures and present various checks for the full chiral algebra associated to a variety of Lagrangian theories. However, the 4d/2d correspondence does not rely on the existence of a Lagrangian description in any way. In chapter 3, we initiate the study of chiral algebras associated to $\mathcal{N} = 2$ superconformal theories of class $\mathcal{S}$, which are obtained by a partially twisted compactification of the six-dimensional $(2, 0)$ theory on a punctured Riemann surface [10, 11]. Thanks to their independence of exactly marginal couplings, chiral algebras of class $\mathcal{S}$ are described by a generalized topological quantum field theory (TQFT) on the Riemann surface. We conjecture the full set of generators of the chiral algebras corresponding to three-punctured spheres, *i.e.*, associated to the trinion theories T_n, which are the natural basic building blocks of said TQFT. Subsequently imposing associativity on the operator algebra of these generators is expected to completely and uniquely fix the chiral algebra. For the T_3 theory we verify this expectation and checked the conjecture in a variety of ways in chapter 3, and for the T_4 theory in chapter 4. It should be emphasized that such complete characterization implicitly determines an infinite amount of protected conformal field theory data (spectral data and three-point couplings) of the parent four-dimensional theory. Through gluing, an arbitrary (maximal) punctured Riemann surface can be obtained. The action of S-duality in the four-dimensional theory implies that the TQFT should be associative. The final salient feature of class $\mathcal{S}$ theories is the reduction of flavor symmetry by partial Higgsing: we find convincing evidence that the chiral algebra analogue is implemented by quantum Drinfeld-Sokolov reduction.

Higgs Branch Localization

Recently, the exploration of exactly calculable quantities in supersymmetric quantum field theories has focused on observables in the rich variety of Lagrangian supersymmetric theories on compact Euclidean manifolds: the application of supersymmetric localization techniques [12, 13] allows one to compute exactly their partition function and to evaluate vacuum expectation values of operators — local or non-local, order or disorder — preserving some supersymmetry. In fact, the localization procedure supports (at least) two alternative methods to perform the infinite-dimensional integrals over field space, resulting in very different looking, but ultimately equal, expressions for the partition function. In Part II of this dissertation, we will focus on developing the second alternative method.

Technically, supersymmetric localization techniques guarantee that a one-loop computation around the zeros of a positive definite deformation term satisfying certain properties, gives the exact result for the path integral and thus allow one to dramatically reduce it, in many examples to a finite-dimensional problem. Since the seminal work in [14], a large number of exact results have been obtained using what one may call *Coulomb branch localization*, in which the path integral is reduced to an ordinary integral over a classical Coulomb branch, *i.e.*, the zeros of the canonically chosen deformation term are arbitrary constant values for vector multiplet scalars or holonomies around circles. It was first shown in the localization computation of $\mathcal{N} = (2, 2)$ theories on the two-dimensional sphere [15, 16] that upon choosing a particular additional deformation term (or equivalently, by changing the integration contour of the auxiliary fields in complexified field space) and if certain conditions on the parameters of the theory hold, the localization locus instead consists of a finite number of discrete Higgs vacua, where chiral multiplet scalars can acquire a vacuum expectation value solving the D-term equations, accompanied by an infinite tower of point-like vortex and anti-vortex solutions located at special points in the geometry. This localization method is called *Higgs branch localization*. By closing the contours of the integrals of the Coulomb branch localized result and summing over the residues of the encircled poles, one can find a precise match with the Higgs branch localized result. In chapter 6, we will show that Higgs branch localization can be applied to three-dimensional $\mathcal{N}=2$ supersymmetric theories on the squashed three-sphere S^3_b and on $S^2 \times S^1$, which computes the three-dimensional supersymmetric index, thus providing a concrete derivation (alternative to the holomorphic blocks point of view in [17] and the deformation argument in [18]) of the vortex anti-vortex factorized results obtained by manipulating the Coulomb branch matrix integrals initiated in [19]. In chapter 7, we extend the Higgs branch localization results further to the four-dimensional $\mathcal{N}=1$ superconformal index. In chapters 6,7, we also show in detail that Coulomb branch localization and Higgs branch localization can be viewed as limiting cases of an intermediate situation involving a deformed Coulomb branch and a finite number of finite size vortices supported on the compact geometry.

The contents of part I of this dissertation have appeared in the papers [5, 2, 1], while those of part II have been published in [4, 3].

Part I

Infinite Chiral Symmetry in Four Dimensions

Chapter 2

Infinite Chiral Symmetry in Four Dimensions

2.1 Introduction

It has long been recognized that supersymmetric quantum field theories enjoy many special properties that make them particularly useful testing grounds for more general ideas about quantum field theory. This is largely a consequence of the fact that many observables in such theories are "protected", in the sense of being determined by a semiclassical calculation with a finite number of corrections taken into account, or alternatively by some related "finite-dimensional" problem that admits the type of closed-form solution that is uncharacteristic of interacting quantum field theories. In most circumstances, these techniques have a semiclassical flavor to them. For example, in cases where supersymmetric partition functions can be computed by localization, the calculation is generally performed starting from a weakly coupled Lagrangian description of the theory.

A notable omission from the currently available techniques is a way to directly access the interacting superconformal phases of theories that do not admit a Lagrangian formulation. By now, there exists a veritable menagerie of models in various dimensions that exhibit conformal phases with varying amounts of supersymmetry, but only in the nicest cases do such models belong to families that include free theories as special points, allowing for properties of the interacting theory to be studied semiclassically. Even for those Lagrangian models, the standard supersymmetric toolkit does not seem to exploit some of the most powerful structures of conformal field theory, such as the existence of a state/operator map and of a well-controlled and convergent operator product expansion.

Meanwhile, recent years have witnessed a surprising resurgence of progress centering around precisely these aspects of conformal field theory in the form of the conformal bootstrap [8, 9]. In large part, this progress has been inspired by the development of numerical techniques for extracting constraints on the defining data of a CFT using unitarity and crossing symmetry [20, 21]. Generally speaking, these techniques are equally applicable to theories with and without supersymmetry, and despite promising early results [22, 23, 24, 25], it has not been entirely clear the extent to which supersymmetry improves the situation. Nevertheless, the possibility that supersymmetry may act as a crucible in which exact results can be forged even for strongly interacting CFTs is irresistible, and we are led to ask the question:

> *Do the conformal bootstrap equations in dimension $d > 2$ admit a solvable truncation in the case of superconformal field theories?*

Having formulated the question, it is worth pausing to consider in what sense the answer could be "yes". The most natural possibilities correspond to known situations in which bootstrap-type equations are rendered solvable. There are two primary scenarios in which the constraints of crossing symmetry are nontrivial, yet solvable:

- **(I)** Meromorphic (and rational) conformal field theories in two dimensions.
- **(II)** Topological quantum field theories.

The subject of this chapter is the realization of the first option in the context of $\mathcal{N} \geqslant 2$ superconformal field theories in four dimensions. The same option is in fact viable for $(2,0)$ superconformal theories in six dimensions and was worked out in [26]. Although we will not discuss the subject at any length in the present chapter, the second option can also be realized using similar techniques to those discussed herein.

The primary hint that such an embedding should be possible was already observed in [27, 24], building upon the work of [28, 29, 30, 31, 32, 33]. In a remarkable series of papers [28, 29, 30, 31, 32, 33, 27], the constraints of superconformal symmetry on four-point functions of half-BPS operators in

$\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal field theories were studied in detail. This analysis revealed that the superconformal Ward identities obeyed by these correlators can be conveniently solved in terms of a set of arbitrary real-analytic functions of the two conformal cross ratios $(z,\bar{z})$, along with a set of *meromorphic* functions of z alone. In a decomposition of the four-point function as an infinite sum of conformal blocks, these meromorphic functions capture the contribution to the double operator product expansion of intermediate "protected" operators belonging to shortened representations. The real surprise arises when these results are combined with the constraints of crossing symmetry. One then finds [24, 27] that the meromorphic functions obey a *decoupled* set of crossing equations, whose general solution can be parametrized in terms of a finite number of coefficients. For example, in the important case of the four-point function of stress-tensor multiplets in an $\mathcal{N}=4$ theory, there is a one-parameter family of solutions, where the parameter has a direct physical interpretation as the central charge (conformal anomaly) of the theory. The upshot is that the protected part of this correlator is entirely determined by abstract symmetry considerations, with no reference to a free-field description of the theory.

In this chapter we establish a conceptual framework that explains and vastly generalizes this observation. For a general $\mathcal{N}=2$ superconformal field theory, we define a protected subsector by passing to the cohomology of a certain nilpotent supercharge $\mathbb{Q}$. This is a familiar strategy – for example, the definition of the chiral ring in an $\mathcal{N}=1$ theory follows the same pattern – but our version of this maneuver will be slightly unconventional, in that we take $\mathbb{Q}=\mathcal{Q}+\mathcal{S}$ to be a linear combination of a Poincaré and a conformal supercharge. In order to be in the cohomology of $\mathbb{Q}$, local operators must lie in a fixed plane $\mathbb{R}^2\subset\mathbb{R}^4$. Crucially, their correlators can be shown to be non-trivial *meromorphic* functions of their positions. This is in contrast to correlators of $\mathcal{N}=1$ chiral operators, which are purely topological in a general $\mathcal{N}=1$ model, and strictly vanish in an $\mathcal{N}=1$ *conformal* theory due to R-charge conservation.

The meromorphic correlators identified by this cohomological construction are precisely the ingredients that define a two-dimensional *chiral algebra*.[1] Our main result is thus the definition of a map χ from the space of

[1]We have settled on the expression "chiral algebra" as it is the most common in the physics literature. We consider it to be synonymous with "vertex operator algebra", though in the mathematical literature some authors make a distinction between the two

four-dimensional $\mathcal{N}=2$ superconformal field theories to the space of two-dimensional chiral algebras,

$$\chi \; : \; \text{4d } \mathcal{N}=2 \text{ SCFT} \;\longrightarrow\; \text{2d Chiral Algebra.}$$

In concrete terms, the chiral algebra computes correlation functions of certain operators in the four-dimensional theory, which are restricted to be coplanar and further given an *explicit* space-time dependence correlating their $SU(2)_R$ orientation with their positions, see (2.2.27). For the case of four-point functions of half-BPS operators, assigning the external operators this "twisted" space-time dependence accomplishes precisely the task of projecting the full correlator onto the meromorphic functions appearing in the solution to the superconformal Ward identities. To recapitulate, those mysterious meromorphic functions are given a direct interpretation as correlators in the associated chiral algebra, and turn out to be special instances of a much more general structure.

The explicit space-time dependence of the four-dimensional operators is instrumental in making sure that they are annihilated by a common supercharge $\mathbb{Q}$ for any insertion point on the plane. From this viewpoint, our construction is in the same general spirit as [34] (see also [35]). These authors considered particular examples of correlators in $\mathcal{N}=4$ super Yang-Mills theory that are invariant under supercharges of the same schematic form $\mathcal{Q}+\mathcal{S}$. Their choices of supercharges are inequivalent to ours, and do not lead to meromorphic correlators.

The operators captured by the chiral algebra are precisely the operators that contribute to the Schur limit of the superconformal index [36, 37, 38], and we will refer to them as *Schur operators*. Important examples are the half-BPS operators that are charged under $SU(2)_R$ but neutral under $U(1)_r$, whose vacuum expectation values parameterize the Higgs branch of the theory, and the $SU(2)_R$ Noether current. The class of Schur operators is much larger, though, and encompasses a variety of supermultiplets obeying less familiar semi-shortening conditions. Operators associated to the Coulomb branch of the theory (such as the half-BPS operators charged under $U(1)_r$ but neutral under $SU(2)_R$) are *not* of Schur type. In a pithy summary, the cohomology of $\mathbb{Q}$ provides a "categorification" of the Schur index. It is

notions. We trust no confusion will arise with the overloading of the word "chiral" due to its unavoidable use in the four-dimensional context, *e.g.*, "chiral and anti-chiral $4d$ supercharges", "the $\mathcal{N}=1$ chiral ring", etc.

a surprising and useful fact that this vector space naturally possesses the additional structure of a chiral algebra.

Chiral algebras are rigid structures. Associativity of their operator algebra translates into strong constraints on the spectrum and OPE coefficients of Schur operators in the parent four-dimensional theory. We have already pointed out that this leads to a unique determination of the protected part of four-point function of stress-tensor multiplets in the $\mathcal{N} = 4$ context [24]. Another canonical example is the four-point function of "moment map" operators in a general $\mathcal{N} = 2$ superconformal field theory. The moment map M is the lowest component of the supermultiplet that contains the conserved flavor current of the theory, and as such it transforms in the adjoint representation of the flavor group G. We find that the associated two-dimensional meromorphic operator $J(z) := \chi[M]$ is the dimension-one generating current of an affine Lie algebra $\hat{\mathfrak{g}}_{k_{2d}}$, with level k_{2d} fixed in terms of the four-dimensional flavor central charge. As the four-point function of affine currents is uniquely fixed, this relation completely determines the protected part of the moment map four-point function. In turn, this information serves as essential input to the full-fledged bootstrap equations that govern the contributions from generic long multiplets in the conformal block decomposition of these four-point functions. These equations can be studied numerically to derive interesting bounds on non-protected quantities, following the approach of [24]. Numerical bounds that arise for various choices of G were indeed obtained in [39]. It is worth emphasizing that the protected part of the four-point function receives contributions from an *infinite* tower of intermediate shortened multiplets, and without knowledge of its precise form the numerical bootstrap program would never get off the ground. In theories that admit a Lagrangian description, one could appeal to non-renormalization theorems and derive the same protected information in the free field limit; the chiral algebra then just serves as a powerful organizing principle to help obtain the same result. However, the abstract chiral algebra approach seems indispensable for the analysis of non-Lagrangian theories – for example, when G is an exceptional group.

As a byproduct of a detailed study of the moment map four-point function, we are able to derive new unitarity bounds that must be obeyed by the central charges of any interacting $\mathcal{N} = 2$ superconformal field theory. By exploiting the relation between the two- and four-dimensional perspectives, we are able to express certain coefficients of the four-dimensional conformal block decomposition of the four-point function in terms of central charges;

the new bounds arise because those coefficients must be non-negative in a unitary theory. Saturation of the bounds signals special properties of the Higgs branch chiral ring. This is a particular instance of a more general encoding of four-dimensional physics in the chiral algebra, the surface of which we have only barely scratched. One notable aspect of this correspondence is the interplay between the geometry of the Higgs branch and the representation theory of the chiral algebra; for example, null vectors that appear at special values of the affine level imply Higgs branch relations.

We describe several structural properties of the map χ. Two universal features are the affine enhancement of the global flavor symmetry, and the Virasoro enhancement of the global conformal symmetry. The affine level in the chiral algebra is related to the flavor central charge in four dimensions as $k_{2d} = -\frac{1}{2}k_{4d}$, while the Virasoro central charge is proportional to the four-dimensional conformal anomaly coefficient,[2] $c_{2d} = -12c_{4d}$. A perhaps surprising feature of these relations is that the two-dimensional central charges and affine levels must be negative. Another universal aspect of the correspondence is a general prescription to derive the chiral algebra associated to a gauge theory whenever the chiral algebra of the original theory whose global symmetry is being gauged is known.

Turning to concrete examples, we start with the SCFTs of free hypermultiplets and free vector multiplets, which are associated to free chiral algebras. With the help of the general gauging prescription, we can combine these ingredients to find the chiral algebra associated to an arbitrary Lagrangian SCFT. We also sketch the structure of the chiral algebras associated to SCFTs of class $\mathcal{S}$, which are generally non-Lagrangian. In several concrete examples, we present evidence that the chiral algebra has an economical presentation as a $\mathcal{W}$-algebra, *i.e.*, as a chiral algebra with a finite set of generators [42]. We do not know whether all chiral algebras associated to SCFTs are finitely generated, or how to identify the complete set of generators in the general case. Indeed, an important open problem is to give a more precise characterization of the class of chiral algebras that can arise from physical four-dimensional theories. Ideally the distinguishing features of this class could be codified in a set of additional axioms. Since chiral algebras are on sounder mathematical footing than four-dimensional quantum

[2] There are two tensorial structures in the four-dimensional trace anomaly, whose coefficients are conventionally denoted a and c. It is the c anomaly that is relevant for us, in contrast to the better studied a anomaly, which decreases monotonically under RG flow [40, 41].

field theories, it is imaginable that this could lead to a well-defined algebraic classification problem. If successful, this approach would represent concrete progress towards the loftier goal of classifying all possible $\mathcal{N} = 2$ SCFTs.

On a more formal note, four-dimensional intuition leads us to formulate a number of new conjectures about chiral algebras that may be of interest in their own right. The conjectures generally take the form of an ansatz for the cohomology of a BRST complex, and include new free-field realizations of affine Lie algebras at special values of the level and new examples of quantum Drinfeld-Sokolov reduction for nontrivial modules. We present evidence for our conjectures obtained from a low-brow, level-by-level analysis, but we suspect that more powerful algebraic tools may lead to rigorous proofs.

The organization of this chapter is as follows. In §2.2 we review the arguments behind the appearance of infinite-dimensional chiral symmetry algebras in the context of two-dimensional conformal field theories. We explain how the same structure can be recovered in the context of $\mathcal{N} = 2$ superconformal field theories in four dimensions by studying observables that are well-defined after passing to the cohomology of a particular nilpotent supercharge in the superconformal algebra. This leads to the immediate conclusion that chiral symmetry algebras will control the structure of this subclass of observables. In §2.3, we describe in greater detail the resulting correspondence between $\mathcal{N} = 2$ superconformal models in four dimensions and their associated two-dimensional chiral algebras. We outline some of the universal features of the correspondence. We further describe an algorithm that defines the chiral algebra for any four-dimensional SCFT with a Lagrangian description in terms of a BRST complex. In §2.4, we describe the immediate consequences of this structure for more conventional observables of the original theory. It turns out that superconformal Ward identities that have previously derived for four-point functions of BPS operators are a natural outcome from our point of view. We further derive new unitarity bounds for the anomaly coefficients of conformal and global symmetries, many of which are saturated by interesting superconformal models. We point out that the state space of the chiral algebra provides a categorification of the Schur limit of the superconformal index. In §2.5, we detail the construction and analysis of the chiral algebras associated to some simple Lagrangian SCFTs. We also make a number of conjectures as to how to describe these chiral algebras as $\mathcal{W}$-algebras. In §2.6 we provide a sketch of the class of chiral algebras that are associated to four-dimensional theories of class $\mathcal{S}$. Appendix A reviews relevant material concerning the superconformal algebras and representation theory used in our constructions.

2.2 Chiral symmetry algebras in four dimensions

The purpose of this section is to establish the existence of infinite chiral symmetry algebras acting on a restricted class of observables in any $\mathcal{N}=2$ superconformal field theory in four dimensions. This is accomplished in two steps. First, working purely in terms of the relevant spacetime symmetry algebras, we identify a particular two-dimensional conformal subalgebra of the four-dimensional superconformal algebra,[3]

$$\mathfrak{sl}(2) \times \widehat{\mathfrak{sl}(2)} \subset \mathfrak{sl}(4\,|\,2)\ ,$$

with the property that the holomorphic factor $\mathfrak{sl}(2)$ commutes with a nilpotent supercharge, $\mathbb{Q}$, while the antiholomorphic factor $\widehat{\mathfrak{sl}(2)}$ is exact with respect to the same supercharge. We then characterize the local operators that represent nontrivial $\mathbb{Q}$-cohomology classes. The only local operators for which this is the case are restricted to lie in a plane $\mathbb{R}^2 \subset \mathbb{R}^4$ that is singled out by the choice of conformal subalgebra. The correlation functions of these operators are meromorphic functions of the insertion points, and thereby define a chiral algebra. As a preliminary aside, we first recall the basic story of infinite chiral symmetry in two dimensions in order to distill the essential ingredients that need to be reproduced in four dimensions. The reader who is familiar with chiral algebras in two-dimensional conformal field theory may safely proceed directly to §2.2.2.

[3] In this section, we adopt the convention of specifying the complexified versions of symmetry algebras. This will turn out to be particularly natural in the discussion of §2.2.2. We generally attempt to select bases for the complexified algebras that are appropriate for a convenient real form. Our basic constructions are insensitive to the signature of spacetime, though in places we explicitly impose constraints that follow from unitarity in Lorentzian signature.

2.2.1 A brief review of chiral symmetry in two dimensions

Let us take as our starting point a two-dimensional quantum field theory that is invariant under the global conformal group $SL(2,\mathbb{C})$. The complexification of the Lie algebra of infinitesimal transformations factorizes into holomorphic and anti-holomorphic generators,

$$\begin{aligned} L_{-1} &= -\partial_z \ , \qquad & L_0 &= -z\partial_z \ , \qquad & L_{+1} &= -z^2\partial_z \ , \\ \bar{L}_{-1} &= -\partial_{\bar{z}} \ , \qquad & \bar{L}_0 &= -\bar{z}\partial_{\bar{z}} \ , \qquad & \bar{L}_{+1} &= -\bar{z}^2\partial_{\bar{z}} \ , \end{aligned} \tag{2.2.1}$$

which obey the usual $\mathfrak{sl}(2)\times\mathfrak{sl}(2)$ commutation relations,

$$\begin{aligned} [L_{+1},L_{-1}] &= 2L_0 \ , \qquad & [L_0,L_{\pm1}] &= \mp L_{\pm1} \ , \\ [\bar{L}_{+1},\bar{L}_{-1}] &= 2L_0 \ , \qquad & [\bar{L}_0,\bar{L}_{\pm1}] &= \mp \bar{L}_{\pm1} \ . \end{aligned} \tag{2.2.2}$$

We need not assume that the theory is unitary, but for simplicity we will assume that the space of local operators decomposes into a direct sum of irreducible highest weight representations of the global conformal group. Such representations are labelled by holomorphic and anti-holomorphic scaling dimensions h and $\bar{h}$ of the highest weight state,

$$L_0|\psi\rangle_{h.w.} = h|\psi\rangle_{h.w.} \ , \qquad \bar{L}_0|\psi\rangle_{h.w.} = \bar{h}|\psi\rangle_{h.w.} \ , \tag{2.2.3}$$

and we further assume that h and $\bar{h}$ are not equal to negative half-integers (in which case one would find finite-dimensional representations of $\mathfrak{sl}(2)$).

Chiral symmetry arises as a consequence of the existence of any local operator $\mathcal{O}(z,\bar{z})$ which obeys a meromorphicity condition of the form

$$\partial_{\bar{z}}\mathcal{O}(z,\bar{z}) = 0 \implies \mathcal{O}(z,\bar{z}) := \mathcal{O}(z) \ . \tag{2.2.4}$$

Under the present assumptions, such an operator will transform in the trivial representation of the anti-holomorphic part of the symmetry algebra and by locality will have h equal to an integer or half-integer. Meromorphicity implies the existence of infinitely many conserved charges (and their associated Ward identities) defined by integrating the meromorphic operator against an arbitrary monomial in z,

$$\mathcal{O}_n := \oint \frac{dz}{2\pi i}\, z^{n+h-1}\, \mathcal{O}(z) \ . \tag{2.2.5}$$

The operator product expansion of two meromorphic operators contains only meromorphic operators, and the singular terms determine the commutation relations among the associated charges, *cf.* [42]. This is the power of meromorphy in two dimensions: an infinite dimensional symmetry algebra organizes the space of local operators into much larger representations, and the associated Ward identities strongly constrain the correlation functions of the theory.

Some examples of this structure are ubiquitous in two-dimensional conformal field theory. The energy-momentum tensor in a two-dimensional CFT is conserved and traceless in flat space, $\partial^\mu T_{\mu\nu} = T_\mu{}^\mu = 0$, leading to two independent conservation equations

$$\begin{aligned} \partial_{\bar z} T_{zz}(z,\bar z) = 0 &\implies T_{zz}(z,\bar z) := T(z) \ , \\ \partial_z T_{\bar z\bar z}(z,\bar z) = 0 &\implies T_{\bar z\bar z}(z,\bar z) := \overline{T}(\bar z) \ . \end{aligned} \tag{2.2.6}$$

The holomorphic stress tensor $T(z)$ is a meromorphic operator with $(h,\bar h) = (2,0)$, and its self-OPE is fixed by conformal symmetry to take the form

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \ , \tag{2.2.7}$$

which implies that the associated conserved charges obey the commutation relations of the Virasoro algebra with central charge c,

$$L_n := \oint \frac{dz}{2\pi i} z^{n+1} T(z) \ , \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ . \tag{2.2.8}$$

Similarly, global symmetries can give rise to conserved holomorphic currents $J^A_z(z,\bar z) =: J^A(z)$ with $(h,\bar h) = (1,0)$. The self-OPEs of such currents are fixed to take the form

$$J^A(z)J^B(w) \sim \frac{k\,\delta^{AB}}{(z-w)^2} + \sum_C i f^{ABC} \frac{J^C(w)}{(z-w)} \ , \tag{2.2.9}$$

with the structure constants f^{ABC} those of the Lie algebra of the global symmetry. The conserved charges in this case obey the commutation relations of an affine Lie algebra at level k,

$$J^A_n := \oint \frac{dz}{2\pi i}\, z^n\, J^A(z) \ , \qquad [J^A_m, J^B_n] = \sum_c i f^{ABC} J^C_{m+n} + mk\,\delta^{AB}\delta_{m+n,0} \ . \tag{2.2.10}$$

The algebra of all meromorphic operators, or alternatively the algebra of their corresponding charges, constitutes the *chiral algebra* of a two-dimensional conformal field theory.

In most physics applications, the spectrum of a CFT will include non-meromorphic operators that reside in modules of the chiral algebra of the theory. In the generic case in which the chiral algebra is the Virasoro algebra, this just means that there are Virasoro primary operators with $\bar{h} \neq 0$. Nevertheless, the correlation functions of the meromorphic operators can be taken in and of themselves to define a certain meromorphic theory. Such theories are referred to by various authors as chiral algebras, vertex operator algebras, or meromorphic conformal field theories. Though some of these names are occasionally assigned to structures that possess some extra nice properties, such as modular invariant partition functions, we will be discussing the most basic version. Henceforth, by chiral algebra we will mean the operator product algebra of a set of meromorphic operators in the plane.[4] So defined, a chiral algebra is strongly constrained by the requirements of crossing symmetry. In what follows, we show that any $\mathcal{N} = 2$ superconformal field theory in four dimensions possesses a class of observables that define a chiral algebra in this sense.

2.2.2 Twisted conformal subalgebras

Chiral algebras are ordinarily thought to be a special feature of conformal-invariant models in two dimensions. Indeed, the appearance of an infinite number of conserved charges as defined in (2.2.5) follows from the interaction of two different ingredients that are special to two dimensions. Firstly, the operators that give rise to the chiral symmetry charges are invariant under (say) the anti-holomorphic factor of the two-dimensional conformal algebra, while transforming in a nontrivial representation of the holomorphic factor, so they are nontrivial holomorphic operators on the plane. The powerful machinery of complex analysis in a single variable then produces the infinity of conserved charges in (2.2.5).[5]

[4] In a preview of later discussions, we mention that by $\mathcal{W}$-algebra we will mean a chiral algebra for which the space of local operators is generated by a finite number of operators via the operations of taking derivatives and normal-ordered multiplication.

[5] From another point of view, one can hardly hope to find a meromorphic sector in a higher dimensional CFT due to Hartogs' theorem, which implies the absence of singularities of codimension greater than one in a meromorphic function of several variables. This

In dimension $d > 2$, it is the first of these conditions that fails the most dramatically, while the latter seems more superficial. Indeed, correlation functions in a conformal field theory in higher dimensions can be restricted so that all operators lie on a plane $\mathbb{R}^2 \subset \mathbb{R}^d$, and the resulting observables will transform covariantly under the subalgebra of the d-dimensional conformal algebra that leaves the $\mathbb{R}^2$ in question fixed,

$$\mathfrak{sl}(2) \times \overline{\mathfrak{sl}(2)} \subset \mathfrak{so}(d+2) \ . \tag{2.2.11}$$

These correlation functions will be largely indistinguishable from those of an authentic two-dimensional CFT, and if one could locate operators that were chiral with respect to this subalgebra, then the arguments of §2.2.1 would go through unhindered and a chiral symmetry algebra could be constructed that would act on $\mathbb{R}^2$-restricted correlation functions. However, a local operator that transforms in the trivial representation of either copy of $\mathfrak{sl}(2)$ in (2.2.11) will necessarily be trivial with respect to *all* of $\mathfrak{so}(d+2)$. As such, the only "meromorphic" operator on the plane in a higher dimensional theory is the identity operator, and no chiral symmetry algebra can be constructed. This is ultimately a consequence of the simple fact that the higher dimensional conformal algebras do not factorize into a holomorphic and anti-holomorphic part: any two $\mathfrak{sl}(2)$ subalgebras will be related by conjugation.

The brief arguments given above are common knowledge, and essentially spell the end to any hopes of recovering chiral symmetry algebras in a general higher-dimensional conformal field theory. We have reproduced them here to clarify the mechanism by which they will be evaded in the coming discussion. In particular, we will see that the additional tools at our disposal in the case of *super*conformal field theories are sufficient to give life to chiral algebras in four dimensions. Before describing the construction, let us recall a simple example which illustrates the mechanism that will be used.

Intermezzo: translation invariance from cohomology

In a quantum field theory with $\mathcal{N} = 1$ supersymmetry in four dimensions, there exists a special class of operators known as *chiral operators* (not to be confused with the meromorphic operators of §2.2.1, which are chiral in a

has been overcome in, *e.g.*, [43, 44] by considering extended operators that intersect in codimension one. The problem, then, is that the meromorphic structure does not impose constraints on the natural objects in the original theory – the local operators.

different sense) that lie in short representations of the supersymmetry algebra and satisfy a shortening condition in terms of a chiral half of the supercharges,

$$\{Q_\alpha, \mathcal{O}(x)] = 0 \ , \qquad \alpha = \pm \, . \tag{2.2.12}$$

The translation generators in $\mathbb{R}^4$ are exact with respect to the chiral supercharges,

$$P_{\alpha\dot\alpha} = \{Q_\alpha, \widetilde{Q}_{\dot\alpha}\} \ , \tag{2.2.13}$$

and consequently, via the Jacobi identity, the derivative of a chiral operator is also exact,

$$[P_{\alpha\dot\alpha}, \mathcal{O}(x)] = \{Q_\alpha, \mathcal{O}'(x)] \ . \tag{2.2.14}$$

Because the chiral supercharges are nilpotent and anti-commute, the cohomology classes of chiral operators with respect to the supercharges Q_α are well-defined and independent of the insertion point of the operator. Schematically, one can write

$$[\mathcal{O}_i(x)]_{Q_\alpha} := \mathcal{O}_i \ . \tag{2.2.15}$$

Products of chiral operators are then free of short distance singularities and form a ring at the level of cohomology. Correlation functions of chiral operators have the excellent property of being independent of the positions of the operators,

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle &= \langle [\mathcal{O}_1(x_1)][\mathcal{O}_2(x_2)]\ldots[\mathcal{O}_n(x_n)]\rangle \\ &= \langle \mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n\rangle \ . \end{aligned} \tag{2.2.16}$$

A suggestive way of phrasing this well-known feature of the chiral ring is that although chiral operators transform in a nontrivial representation of the four-dimensional translation group, their cohomology classes with respect to the chiral supercharges transform in the trivial representation. The passage from local operators to their cohomology classes modifies the transformation properties of these local operators under the spacetime symmetry algebra, in this case rendering them trivial.

Holomorphy from cohomology

To recover chiral algebras in four dimensions, we adopt the same philosophy just illustrated in the example of the chiral ring. We will find a nilpotent supercharge with the property that *cohomology classes* of local operators with respect to said supercharge transform in a chiral representation of an $\mathfrak{sl}(2) \times$

$\widehat{\mathfrak{sl}(2)}$ subalgebra of the full superconformal algebra, and as such behave as meromorphic operators. Such local operators will then necessarily constitute a chiral algebra as described in §2.2.1.

The first task that presents itself is an algebraic one. To realize chiral symmetry at the level of cohomology classes, we identify a two-dimensional conformal subalgebra of the four-dimensional superconformal algebra,

$$\mathfrak{sl}(2) \times \widehat{\mathfrak{sl}(2)} \subset \mathfrak{sl}(4\,|\,2)~,$$

along with a privileged supercharge $\mathbb{Q}$ for which the following criteria are satisfied:

- The supercharge is nilpotent: $\mathbb{Q}^2 = 0$.
- $\mathfrak{sl}(2)$ and $\widehat{\mathfrak{sl}(2)}$ act as the generators of holomorphic and anti-holomorphic Möbius transformations on a complex plane $\mathbb{C} \subset \mathbb{R}^4$.
- The holomorphic generators spanning $\mathfrak{sl}(2)$ commute with $\mathbb{Q}$.
- The anti-holomorphic generators spanning $\widehat{\mathfrak{sl}(2)}$ are $\mathbb{Q}$ commutators.

In searching for such a subalgebra, we can first restrict our attention to subalgebras of $\mathfrak{sl}(4|2)$ that keep the plane fixed set-wise. There are two inequivalent maximal subalgebras of this kind: $\mathfrak{sl}(2|1) \times \mathfrak{sl}(2|1)$, which is the symmetry algebra of an $\mathcal{N} = (2,2)$ SCFT in two dimensions, and $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$, which is the symmetry algebra of an $\mathcal{N} = (0,4)$ SCFT in two dimensions. One easily determines that the first subalgebra cannot produce the desired structure; we proceed directly to consider the second case.

The four-dimensional $\mathcal{N} = 2$ superconformal algebra and the two-dimensional $\mathcal{N} = (0,4)$ superconformal algebra are summarized in Appendix A.1. In embedding the latter into the former, we take the fixed two-dimensional subspace to be the one that is fixed pointwise by the rotation generator

$$\mathcal{M}^{\perp} := \mathcal{M}_{+}{}^{+} - \mathcal{M}^{\dot{+}}{}_{\dot{+}}~. \tag{2.2.17}$$

The generator of rotations acting within the fixed plane is the orthogonal combination,

$$\mathcal{M} := \mathcal{M}_{+}{}^{+} + \mathcal{M}^{\dot{+}}{}_{\dot{+}}~. \tag{2.2.18}$$

In more conventional terms, we are picking out the plane with $x_1 = x_2 = 0$. Introducing complex coordinates $z := x_3 + ix_4$, $\bar{z} := x_3 - ix_4$, the two-dimensional conformal symmetry generators in $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$ can be expressed in terms of the four-dimensional ones as

$$
\begin{aligned}
L_{-1} &= \mathcal{P}_{+\dot{+}} \ , &\quad L_{+1} &= \mathcal{K}^{\dot{+}+} \ , &\quad 2L_0 &= \mathcal{H} + \mathcal{M} \ , \\
\bar{L}_{-1} &= \mathcal{P}_{-\dot{-}} \ , &\quad \bar{L}_{+1} &= \mathcal{K}^{\dot{-}-} \ , &\quad 2\bar{L}_0 &= \mathcal{H} - \mathcal{M} \ .
\end{aligned}
\tag{2.2.19}
$$

The fermionic generators of $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$ are obviously all anti-holomorphic, and upon embedding are identified with four-dimensional supercharges according to

$$
\mathcal{Q}^{\mathcal{I}} = \mathcal{Q}^{\mathcal{I}}_{-} \ , \qquad \widetilde{\mathcal{Q}}_{\mathcal{I}} = \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{-}} \ , \qquad \mathcal{S}_{\mathcal{I}} = \mathcal{S}^{-}_{\mathcal{I}} \ , \qquad \widetilde{\mathcal{S}}^{\mathcal{I}} = \widetilde{\mathcal{S}}^{\mathcal{I}\dot{-}} \ , \tag{2.2.20}
$$

where $\mathcal{I} = 1, 2$ is an $\mathfrak{sl}(2)_R$ index. Finally, the $\mathfrak{sl}(2|2)$ superalgebra has a central element $\mathcal{Z}$, which upon embedding is given in terms of four-dimensional symmetry generators as

$$
\mathcal{Z} = r + \mathcal{M}^{\perp} \ , \tag{2.2.21}
$$

where r is the generator of $U(1)_r$.

Amongst the supercharges listed in (2.2.20), one finds a variety of nilpotent operators. Any such operator will necessarily commute with the generators $L_{\pm 1}$ and L_0 in (2.2.19) since all of the supercharges do so. The requirement of $\mathbb{Q}$-exact anti-holomorphic Möbius transformations is harder to satisfy. In fact, up to similarity transformation using generators of the bosonic symmetry algebra, there are only two possible choices:

$$
\begin{aligned}
\mathbb{Q}_1 &:= \mathcal{Q}^1 + \tilde{\mathcal{S}}^2 \ , &\quad \mathbb{Q}_2 &:= \mathcal{S}_1 - \tilde{\mathcal{Q}}_2 \ , \\
\mathbb{Q}_1^{\dagger} &:= \mathcal{S}_1 + \tilde{\mathcal{Q}}_2 \ , &\quad \mathbb{Q}_2^{\dagger} &:= \mathcal{Q}^1 - \tilde{\mathcal{S}}^2 \ .
\end{aligned}
\tag{2.2.22}
$$

Interestingly, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ give rise to *the same* $\mathbb{Q}$-exact generators of an anti-holomorphic $\widehat{\mathfrak{sl}(2)}$ algebra,

$$
\begin{aligned}
\{\mathbb{Q}_1 \, , \tilde{\mathcal{Q}}_1\} &= \{\mathbb{Q}_2 \, , -\mathcal{Q}^2\} &= \bar{L}_{-1} + \mathcal{R}^- &=: \widehat{L}_{-1} \ , \\
\{\mathbb{Q}_1 \, , \mathcal{S}_2\} &= \{\mathbb{Q}_2 \, , \tilde{\mathcal{S}}^1\} &= \bar{L}_{+1} - \mathcal{R}^+ &=: \widehat{L}_{+1} \ , \\
\{\mathbb{Q}_1 \, , \mathbb{Q}_1^{\dagger}\} &= \{\mathbb{Q}_2 \, , \mathbb{Q}_2^{\dagger}\} &= 2(\bar{L}_0 - \mathcal{R}) &=: 2\widehat{L}_0 \ .
\end{aligned}
\tag{2.2.23}
$$

In addition, the central element of $\mathfrak{sl}(2|2)$ is exact with respect to both supercharges,

$$
\{\mathbb{Q}_1 \, , \mathbb{Q}_2\} = -\mathcal{Z} \ . \tag{2.2.24}
$$

Note that while $\widehat{\mathfrak{sl}(2)}$ *does* act on the plane by anti-holomorphic conformal transformations, it is not simply a subalgebra of the original global conformal algebra. Rather, it is an $\mathfrak{sl}(2)_R$ twist of $\overline{\mathfrak{sl}(2)}$.[6] Because the relevant real forms of the $\overline{\mathfrak{sl}(2)}$ conformal algebra and $\mathfrak{sl}(2)_R$ are different, the generators of $\widehat{\mathfrak{sl}}(2)$ do not enjoy any reasonable hermiticity properties when acting on the Hilbert space of the four-dimensional theory. In particular, we can immediately see that $\widehat{L}^{\dagger}_{\pm 1} \neq \widehat{L}_{\mp 1}$. This would complicate matters considerably if our intention was to study operators that transform in nontrivial representations of this twisted algebra. Fortunately, our plan is precisely the opposite: chiral algebras can appear after passing to $\mathbb{Q}$-cohomology, at which point all of the objects of interest will effectively be invariant under the action of $\widehat{\mathfrak{sl}(2)}$. Consequently, reality/hermiticity conditions will play no role in the structure of the "physical" operators/observables defined at the level of cohomology.

2.2.3 The cohomology classes of local operators

Our next task is to study the properties of operators that define non-trivial $\mathbb{Q}_i$-cohomology classes. For the purposes of the present chapter, we are restricting our attention to *local operators* in four dimensions; the inclusion of non-local operators, such as line or surface operators, is an interesting extension that will be addressed in future work.

We begin by identifying the requirements for an operator inserted at the origin to define a nontrivial $\mathbb{Q}_i$-cohomology class. In particular, we will derive the conditions under which an operator $\mathcal{O}(x)$ obeys

$$\{\mathbb{Q}_i, \mathcal{O}(0)] = 0 \ , \qquad \mathcal{O}(0) \neq \{\mathbb{Q}_i, \mathcal{O}'(0)] \ , \tag{2.2.25}$$

for $i = 1$ or $i = 2$. Because both $\mathbb{Q}_i$ commute with $\widehat{L}_0$ and $\mathcal{Z}$, we lose no generality in restricting to definite eigenspaces of these charges. A standard cohomological argument then implies that since $\widehat{L}_0$ and $\mathcal{Z}$ are actually $\mathbb{Q}_i$-*exact*, an operator satisfying (2.2.25) must lie in the zero eigenspace of both charges. In terms of four-dimensional quantum numbers, this means that

[6] In light of this, we may understand the absence of a similar construction using the $\mathfrak{sl}(2|1) \times \mathfrak{sl}(2|1)$ algebra as a consequence of there being no $\mathfrak{sl}(2)_R$ with which to twist. Similarly, our construction does not extend to $\mathcal{N} = 1$ superconformal theories since they only have an abelian R-symmetry.

such an operator must obey[7]

$$\tfrac{1}{2}(E-(j_1+j_2))-R=0~,\quad r+(j_1-j_2)=0~, \tag{2.2.26}$$

where E is the conformal dimension/eigenvalue of $\mathcal{H}$, j_1 and j_2 are $\mathfrak{sl}(2)_1$ and $\mathfrak{sl}(2)_2$ Lorentz quantum numbers/eigenvalues of $\mathcal{M}_+{}^+$ and $\mathcal{M}^{\dot{+}}{}_{\dot{+}}$, and R is the $\mathfrak{sl}(2)_R$ charge/eigenvalue of $\mathcal{R}$. As long as the four-dimensional SCFT is unitary, the last line of (2.2.23) implies that any operator with zero eigenvalue under $\widehat{L}_0$ must be annihilated by $\mathbb{Q}_i$ and $\mathbb{Q}_i^\dagger$ for *both* $i=1$ and $i=2$. The relations in (2.2.26) therefore characterize the harmonic representatives of $\mathbb{Q}_i$-cohomology classes of operators at the origin, and we see that the two supercharges actually define the *same* cohomology. Notably, these relations are known to characterize the operators that contribute to the Schur (and Macdonald) limits of the superconformal index in four dimensions [38], suggesting that the cohomology will be non-empty in any nontrivial $\mathcal{N}=2$ SCFT. We will refer to the class of local operators obeying (2.2.26) as the *Schur operators* of the SCFT. We will have more to say about the features of these operators in §2.3.

Note that in contrast to the case of ordinary chiral operators in a supersymmetric theory, which are annihilated by a given Poincaré supercharge regardless of the insertion point, for operators to be annihilated by the $\mathbb{Q}_i$ when inserted away from the origin requires that they acquire a more intricate dependence on their position in $\mathbb{R}^4$. This is a consequence of the fact that the translation generators do not commute with the superconformal charges $\mathcal{S}_1^-$ and $\tilde{\mathcal{S}}^{2\dot{-}}$ appearing in the definitions of the $\mathbb{Q}_i$. Indeed, there is no way to define the translation of a Schur operator from the origin to a point outside of the $(z,\bar{z})$ plane so that it continues to represent a $\mathbb{Q}_i$-cohomology class. Within the plane, though, we can accomplish this task using the $\mathbb{Q}_i$-exact, twisted $\widehat{\mathfrak{sl}(2)}$ of the previous subsection. In particular, because the twisted anti-holomorphic translation generator $\widehat{L}_{-1}$ is a $\mathbb{Q}_i$ anti-commutator and the holomorphic translation generator L_{-1} is $\mathbb{Q}_i$-closed, we can define the *twisted-translated* operators

$$\mathcal{O}(z,\bar{z})=e^{zL_{-1}+\bar{z}\widehat{L}_{-1}}\,\mathcal{O}(0)\,e^{-zL_{-1}-\bar{z}\widehat{L}_{-1}}~, \tag{2.2.27}$$

[7]In fact, the second relation in (2.2.26) follows from the first as a consequence of unitarity and the four-dimensional superconformal algebra (see §2.3.1). We list it separately here since it is an algebraically independent constraint at the level of the quantum numbers.

where $\mathcal{O}(0)$ is a Schur operator. One way of thinking about this prescription for the translation of local operators is as the consequence of introducing a constant, complex background gauge field for the $\mathfrak{sl}(2)_R$ symmetry that is proportional to the $\mathfrak{sl}(2)$ raising operator. By construction, the twisted-translated operator is $\mathbb{Q}_i$ closed for both $i = 1, 2$, and the cohomology class of this operator is well-defined and depends on the insertion point holomorphically,

$$[\mathcal{O}(z, \bar{z})]_{\mathbb{Q}} \quad \Longrightarrow \quad \mathcal{O}(z) \; . \tag{2.2.28}$$

What does such an operator look like in terms of a more standard basis of local operators at the point $(z, \bar{z})$? To answer this, we must first note that Schur operators at the origin occupy the highest-weight states of their respective $\mathfrak{sl}(2)_R$ representation (this fact will be explained in greater detail in §2.3). If we denote the whole spin k representation of $\mathfrak{sl}(2)_R$ as $\mathcal{O}^{\mathcal{I}_1\mathcal{I}_2\cdots\mathcal{I}_{2k}}$ with $\mathcal{I}_i = 1, 2$, then the Schur operator at the origin is $\mathcal{O}^{11\cdots1}(0)$, and the twisted-translated operator at any other point is defined as

$$\mathcal{O}(z, \bar{z}) := u_{\mathcal{I}_1}(\bar{z}) \cdots u_{\mathcal{I}_{2k}}(\bar{z})\, \mathcal{O}^{\mathcal{I}_1\ldots\mathcal{I}_{2k}}(z, \bar{z})\,, \qquad u_{\mathcal{I}}(\bar{z}) := (1, \bar{z})\,. \tag{2.2.29}$$

At any given point $(z, \bar{z})$, this is a particular complex-linear combination of the different elements of the $\mathfrak{sl}(2)_R$ representation of the corresponding Schur operator. The precise combination depends on the insertion point as indicated. What we have discovered is that the correlation functions of these operators are determined at the level of their $\mathbb{Q}_i$-cohomology classes, and are therefore meromorphic functions of the insertion points.[8]

2.2.4 A chiral operator product expansion

The most efficient language for describing chiral algebras is that of the operator product expansion. Let us therefore study the structure of the operator product expansion of the twisted-translated Schur operators in order to see the emergence of meromorphic OPEs befitting a chiral algebra.

Consider two operators: $\mathcal{O}_1(z, \bar{z})$ is the twisted translation of a Schur operator from the origin to $(z, \bar{z})$, and $\mathcal{O}_2(0, 0)$ is a Schur operator inserted at the origin. Given the general expression for the twisted-translated operator

[8] For $\mathcal{N} = 4$ SYM, a similar contraction of the $SU(4)_R$ indices with position-dependent vectors was studied in [34]. The twists considered in that paper are different, and do not give rise to meromorphic operators and chiral algebras.

given in (2.2.29), the OPE of these two operators should take the form

$$\mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0) = \sum_k \lambda_{12k} \frac{\bar{z}^{R_1+R_2-R_k}}{z^{h_1+h_2-h_k}\,\bar{z}^{\bar{h}_1+\bar{h}_2-\bar{h}_k}} \mathcal{O}_k(0)\ , \tag{2.2.30}$$

where the $\bar{z}^{R_1+R_2-R_k}$ in the numerator comes from the explicit factors of $\bar{z}$ appearing in (2.2.29), and R_k is the R-charge of the operator $\mathcal{O}_k$. This form of the OPE is so far a consequence of two-dimensional conformal invariance and conservation of R-charge under multiplication. We have introduced the two-dimensional quantum numbers h and $\bar{h}$, which are expressible in terms of four-dimensional quantum numbers as

$$h = \frac{E+(j_1+j_2)}{2}\ , \qquad \bar{h} = \frac{E-(j_1+j_2)}{2}\ . \tag{2.2.31}$$

Though the OPE does not look meromorphic yet, we are already well on our way. The left hand side of (2.2.30) is $\mathbb{Q}_i$-closed for any $(z,\bar{z})$, with the $\bar{z}$ dependence being $\mathbb{Q}_i$-exact. As a result, each individual term on the right hand side must be $\mathbb{Q}_i$-closed, and the sum should be reorganized into two groups. The first group will consist of the terms in which the operator $\mathcal{O}_k(0)$ is a Schur operator, while the second will consist of the remaining terms, for which the operator $\mathcal{O}_k(0)$ is $\mathbb{Q}_i$-exact. Recalling that the quantum numbers of Schur operators obey $\bar{h}=R$, we immediately see that for those terms in the OPE the $\bar{z}$ dependence cancels between the denominator and the numerator, thus providing the desired meromorphicity result:

$$\mathcal{O}_1(z,\bar{z})\,\mathcal{O}_2(0,0)\ = \sum_{k_{\text{Schur}}} \frac{\lambda_{12k}}{z^{h_1+h_2-h_k}}\,\mathcal{O}_k(0)\ + \{\mathbb{Q},\dots]\ . \tag{2.2.32}$$

From the four-dimensional construction, we expect this OPE to be single-valued, which implies that $h_1+h_2-h_k$ should be an integer. Indeed, this integrality follows from the fact that h is a sum of $SU(2)$ Cartans after applying $SU(2)$ selection rules. Clearly, in passing to $\mathbb{Q}_i$-cohomology classes the OPE stays well-defined and the $\mathbb{Q}_i$-exact piece can be set to zero. Thus at the level of cohomology, the twisted-translated operators can be reinterpreted as two-dimensional meromorphic operators with interesting singular OPEs.

It may be instructive to see how this meromorphic OPE plays out in a simple example. An extremely simple case, to which we shall return in §2.3, is that of free hypermultiplets in four dimensions. The scalar squarks

Q and $\tilde{Q}$ of the hypermultiplet are Schur operators, and the corresponding twisted-translated operators take the form

$$q(z) := [Q(z,\bar{z}) + \bar{z}\tilde{Q}^*(z,\bar{z})]_{\mathbb{Q}} \,, \qquad \tilde{q}(z) := [\tilde{Q}(z,\bar{z}) - \bar{z}Q^*(z,\bar{z})]_{\mathbb{Q}} \,. \quad (2.2.33)$$

The singular OPE of these twisted operators can be easily worked out using the free OPE in four dimensions; we have

$$\begin{aligned} q(z)q(w) &\sim \text{regular} \,, & \tilde{q}(z)\tilde{q}(w) &\sim \text{regular} \,, \\ q(z)\tilde{q}(w) &\sim \frac{1}{z-w} \,, & \tilde{q}(z)q(w) &\sim -\frac{1}{z-w} \,. \end{aligned} \quad (2.2.34)$$

This is example is in some respects deceptively simple, in that the terms appearing in the singular part of the OPE are meromorphic on the nose. In more complicated theories, there will be cohomologically trivial terms appearing in the singular part of the OPE, and meromorphicity will depend on a more detailed knowledge of the action of the nilpotent supercharges.

Let us briefly point out one difference between the structure observed here and that of a more conventional cohomological subalgebra. The chiral ring in the free hypermultiplet theory is generated by the operators $q(x)$ and $\tilde{q}(x)$. Because these operators both have $R = 1/2$, there can be no nonzero correlation functions in the chiral ring. The existence of nontrivial correlation functions in the chiral algebra described here follows precisely from the presence of subleading terms in the $\bar{z}$ expansion (2.2.33) with $SU(2)_R$ quantum numbers of opposite sign relative to the leading term.

Having established existence of nontrivial $\mathbb{Q}$-cohomology classes with meromorphic OPEs and correlators, we now take some time to develop the dictionary between four-dimensional SCFT structures and their two-dimensional counterparts.

2.3 The SCFT/chiral algebra correspondence

For any four-dimensional $\mathcal{N} = 2$ superconformal field theory, we have identified a subsector of operators whose correlation functions are meromorphic when they are restricted to be coplanar. This sector thus defines a map from four-dimensional SCFTs to two-dimensional chiral algebras:

$$\chi \ : \ \text{4d SCFT} \ \longrightarrow \ \text{2d Chiral Algebra.}$$

The aim of this section is to elaborate on the structure of this correspondence, focusing primarily on its more universal aspects. We begin with a short preview of some of the more prominent features of the correspondence.

Our first main result is the generic enhancement of the global $\mathfrak{sl}(2)$ conformal symmetry algebra to a full fledged Virasoro algebra. In other words, for any SCFT $\mathcal{T}$, we find that $\chi[\mathcal{T}]$ contains a meromorphic stress tensor. The two-dimensional central charge turns out to have a simple relationship to the four-dimensional conformal anomaly coefficient,

$$c_{2d} = -12 c_{4d} \ .$$

In particular, this implies that when $\mathcal{T}$ is unitary (which we always take to be the case), $\chi[\mathcal{T}]$ is necessarily non-unitary. In a similar vein, we find that global symmetries of $\mathcal{T}$ are always enhanced into affine symmetries of $\chi[\mathcal{T}]$, and the respective central charges of these flavor symmetries enjoy another simple relationship,

$$k_{2d} = -\frac{1}{2} k_{4d} \ .$$

It is often helpful to think of a chiral algebra in terms of its generators. In the chiral algebra sense of the word, generators are those operators that cannot be expressed as the conformally normal-ordered products of derivatives of other operators. While we do not find a complete characterization of the generators of our chiral algebras, we do identify certain operators in four dimensions whose corresponding chiral operator will necessarily be generators. In particular, operators that are $\mathcal{N}=1$ chiral *and* satisfy the Schur shortening condition form a ring which is a consistent truncation of the $\mathcal{N}=1$ chiral ring, to which we refer as the Hall-Littlewood (HL) chiral ring. We find that every generator of the HL chiral ring necessarily leads to a generator of the associated chiral algebra. There may be additional generators of the chiral algebra beyond the stress tensor and the operators associated to generators of the HL chiral ring. We will find such additional generators in the example of §2.5.4.

For the special case of free SCFTs we completely characterize the associated chiral algebras. Unsurprisingly, free SCFTs give rise to free chiral algebras. In particular, free hypermultiplets correspond to the chiral algebra of dimension $1/2$ symplectic bosons, while free vector multiplets correspond to the small algebra of a (b, c) ghost system of dimension $(1, 0)$.

Finally, we describe the two-dimensional counterpart of gauging a flavor symmetry G in some general SCFT $\mathcal{T}_G$. Assuming that the chiral algebra associated to the ungauged SCFT is known, the prescription to find the chiral algebra of the new theory is as follows. The direct product of the original chiral algebra $\chi[\mathcal{T}_G]$ with a (b,c) system in the adjoint representation of G admits a nilpotent BRST operator precisely when the beta function for the four-dimensional gauge coupling vanishes. The chiral algebra of the gauged theory is then obtained by restricting to the BRST coholomogy. We find that this BRST operator precisely captures the one-loop correction to a certain four-dimensional supercharge, so that restricting to its cohomology is equivalent to the requirement that one should only retain those states that remain in their original short representations once one-loop corrections are taken into account.

2.3.1 Schur operators

As a first order of business, we pursue a more concrete characterization of the four-dimensional operators whose correlation functions are captured by the chiral algebra. Let us first reiterate the basic facts about these operators that were derived in §2.2. The chiral algebra computes correlation functions of operators that define nontrivial cohomology classes of the nilpotent supercharges $\mathbb{Q}_i$. Such operators are obtained by *twisted translations* (2.2.29) of Schur operators from the origin to an arbitrary point $(z,\bar{z})$ on the plane. A Schur operator is any operator that satisfies

$$[\widehat{L}_0,\mathcal{O}]=0 \qquad \Longleftrightarrow \qquad \tfrac{1}{2}\left(E-(j_1+j_2)\right)-R=0\ , \tag{2.3.1}$$
$$[\mathcal{Z},\mathcal{O}]=0 \qquad \Longleftrightarrow \qquad r+j_1-j_2=0\ . \tag{2.3.2}$$

If $\mathcal{T}$ is unitary, then these conditions can be equivalently formulated as the requirement that when inserted *at the origin*, an operator is annihilated by the two Poincaré and the two conformal supercharges that enter in the definition of the $\mathbb{Q}_i$, *i.e.*,

$$[\mathcal{Q}^{\,1}_{\,-},\mathcal{O}(0)]=[\widetilde{\mathcal{Q}}_{\,2\dot{-}},\mathcal{O}(0)]=[\mathcal{S}_1^{-},\mathcal{O}(0)]=[\widetilde{\mathcal{S}}^{2\dot{-}},\mathcal{O}(0)]=0\ . \tag{2.3.3}$$

This follows from the hermiticity conditions $\mathcal{Q}^{1\dagger}_{-}:=\mathcal{S}_1^{-}$ and $\mathcal{Q}^{\dagger}_{2\dot{-}}:=\widetilde{\mathcal{S}}^{2\dot{-}}$ in conjunction with the relevant anticommutators from Appendix A.1,

$$\{\mathcal{Q}^1_{-}\,,\mathcal{Q}^{1\dagger}_{-}\}=\widehat{L}_0-\frac{1}{2}\mathcal{Z}\,,\quad \{\widetilde{\mathcal{Q}}_{2\dot{-}}\,,\widetilde{\mathcal{Q}}^{\dagger}_{2\dot{-}}\}=\widehat{L}_0+\frac{1}{2}\mathcal{Z}\,. \tag{2.3.4}$$

It follows immediately that the state $\mathcal{O}(0)|0\rangle$ is annihilated by all four supercharges if and only if its quantum numbers obey (2.3.1) and (2.3.2). Actually, (2.3.4) implies the additional inequality

$$\widehat{L}_0 \geqslant \frac{|\mathcal{Z}|}{2} , \tag{2.3.5}$$

from which we may conclude that imposing only (2.3.1) is a necessary and sufficient condition to define a Schur operator. We further note that Schur operators are necessarily the highest-weight states of their respective $SU(2)_R$ representations, and so carry the maximum eigenvalue R of the Cartan generator. If this were not the case, states with greater R would have negative $\widehat{L}_0$ eigenvalues, in contradiction with unitarity. Similarly, Schur operators are necessarily the highest weight states of their $SU(2)_1 \times SU(2)_2$ Lorentz symmetry representation, carrying the largest eigenvalues for j_1 and j_2. The index structure of a Schur operator is therefore of the form $\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}$.

From the definition of L_0 in (2.2.19) and (2.3.1) we find that the holomorphic dimension h of a Schur operator is non-zero and fixed in terms of its quantum numbers,

$$h = \tfrac{1}{2}\left(E + j_1 + j_2\right) = R + j_1 + j_2 \ . \tag{2.3.6}$$

This is always a half integer, since R, j_1 and j_2 are all $SU(2)$ Cartans. It follows from (2.3.2) and (2.3.6), in conjunction with the non-negativity of j_1 and j_2, that the holomorphic dimension of a Schur operator is bounded from below in terms of its four-dimensional R-charges,

$$h = R + j_1 + j_2 \geqslant R + |j_1 - j_2| = R + |r| \ . \tag{2.3.7}$$

The inequality is saturated if and only if j_1 or j_2 is zero.

Supermultiplets of Schur type

Schur operators belong to shortened representations of the $\mathcal{N} = 2$ superconformal algebra. The complete list of possible shortening conditions is reviewed in Appendix A.2. In the notations of [45], the superconformal multiplets that contain Schur operators are the following,

$$\hat{\mathcal{B}}_R \ , \quad \mathcal{D}_{R(0,j_2)} \ , \quad \bar{\mathcal{D}}_{R(j_1,0)} \ , \quad \hat{\mathcal{C}}_{R(j_1,j_2)} \ . \tag{2.3.8}$$

Multiplet	$\mathcal{O}_{\text{Schur}}$	h	r	Lagrangian "letters"
$\hat{\mathcal{B}}_R$	$\Psi^{11\dots1}$	R	0	$Q,\, \tilde{Q}$
$\mathcal{D}_{R(0,j_2)}$	$\widetilde{\mathcal{Q}}^1_{\dot{+}}\, \Psi^{11\dots1}_{\dot{+}\dots\dot{+}}$	$R+j_2+1$	$j_2+\frac{1}{2}$	$Q,\, \tilde{Q},\, \tilde{\lambda}^1_{\dot{+}}$
$\bar{\mathcal{D}}_{R(j_1,0)}$	$\mathcal{Q}^1_+\, \Psi^{11\dots1}_{+\cdots+}$	$R+j_1+1$	$-j_1-\frac{1}{2}$	$Q,\, \tilde{Q},\, \lambda^1_+$
$\hat{\mathcal{C}}_{R(j_1,j_2)}$	$\mathcal{Q}^1_+\, \widetilde{\mathcal{Q}}^1_{\dot{+}}\, \Psi^{11\dots1}_{+\cdots+\,\dot{+}\dots\dot{+}}$	$R+j_1+j_2+2$	j_2-j_1	$D^n_{+\dot{+}}Q,\, D^n_{+\dot{+}}\tilde{Q},$ $D^n_{+\dot{+}}\lambda^1_+,\, D^n_{+\dot{+}}\tilde{\lambda}^1_{\dot{+}}$

Table 2.1: This table summarizes the manner in which Schur operators fit into short multiplets of the $\mathcal{N}=2$ superconformal algebra. For each supermultiplet, we denote by Ψ the superconformal primary. There is then a single conformal primary Schur operator $\mathcal{O}_{\text{Schur}}$, which in general is obtained by the action of some Poincaré supercharges on Ψ. We list the holomorphic dimension h and $U(1)_r$ charge r of $\mathcal{O}_{\text{Schur}}$ in terms of the quantum numbers (R, j_1, j_2) that label the shortened multiplet (left-most column). We also indicate the schematic form that $\mathcal{O}_{\text{Schur}}$ can take in a Lagrangian theory by enumerating the elementary "letters" from which the operator may be built. We denote by Q and $\tilde{Q}$ the complex scalar fields of a hypermultiplet, by $\lambda^{\mathcal{I}}_\alpha$ and $\tilde{\lambda}^{\mathcal{I}}_{\dot{\alpha}}$ the left- and right-moving fermions of a vector multiplet, and by $D_{\alpha\dot{\alpha}}$ the gauge-covariant derivatives.

For the purpose of enumeration, it is sufficient to focus on those Schur operators that are conformal primaries. Given such a *primary Schur operator*, there is a tower of descendant Schur operators that are obtained by the action $L_{-1} = P_{+\dot{+}} = -\partial_{+\dot{+}}$. It turns out that each of the supermultiplets listed in (2.3.8) contains exactly one conformal primary Schur operator. In the case of $\hat{\mathcal{B}}_R$, this is also the *super*conformal primary of the multiplet, whereas in the other cases it is a superconformal descendant. This representation-theoretic information is summarized in Table 2.1, where we also provide the schematic form taken by each type of operator in a Lagrangian theory.

The shortening conditions obeyed by the Schur operators make crucial use of the extended $\mathcal{N} = 2$ supersymmetry. Indeed, the hallmark of a Schur operator is that it is annihilated by two Poincaré supercharges of *opposite* chiralities ($\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$ in our conventions). This defines a consistent shortening condition because the supercharges have the same $SU(2)_R$ weight, and thus anticommute with each other. No analogous shortening condition exists in an $\mathcal{N} = 1$ supersymmetric theory, because the anticommutator of opposite-chirality supercharges necessarily yields a momentum operator, which annihilates only the identity.

Although the most general Schur operators, which are those belonging to $\hat{\mathcal{C}}_{R(j_1,j_2)}$ multiplets, may seem somewhat exotic, the Schur operators of type $\hat{\mathcal{B}}_R$, $\mathcal{D}_{R(0,j_2)}$ and $\bar{\mathcal{D}}_{R(j_1,0)}$ are relatively familiar. Indeed, they can be understood as special cases of conventional $\mathcal{N} = 1$ chiral or anti-chiral operators. Let us focus for the moment on the $\mathcal{N} = 1$ Poincaré subalgebra that contains the supercharges

$$\mathcal{Q}^2_\alpha \ , \quad \widetilde{\mathcal{Q}}_{2\dot{\alpha}} \ . \tag{2.3.9}$$

We then ask what subset of Schur operators are also elements of the chiral ring for this $\mathcal{N} = 1$ subalgebra. In particular, such operators will be annihilated by *both* spinorial components of the anti-chiral supercharge $\widetilde{\mathcal{Q}}_{2\dot{\alpha}}$, $\dot{\alpha} = \pm$. These operators have $j_2 = 0$, and a quick glance at Table 2.1 tells us that they are Schur operators of types $\hat{\mathcal{B}}_R$ and $\bar{\mathcal{D}}_{R(j_1,0)}$. These operators saturate the inequality (2.3.7), with $r = -j_1 < 0$ for $\bar{\mathcal{D}}_{R(j_1,0)}$ and $r = 0$ for the $\hat{\mathcal{B}}_R$. As these are precisely the operators that contribute to the Hall-Littlewood (HL) limit of the superconformal index, we refer to them as *Hall-Littlewood operators*. They form a ring, the *Hall-Littlewood chiral ring*, which is a consistent truncation of the full $\mathcal{N} = 1$ chiral ring.

In a Lagrangian theory, the $\hat{\mathcal{B}}_R$ type Schur operators are gauge-invariant combinations of Q and $\tilde{Q}$, the complex hypermultiplet scalars that are bottom components of $\mathcal{N}=1$ chiral superfields (we are suppressing color and flavor indices). Schur operators of type $\bar{\mathcal{D}}_{R(j_1,0)}$ are obtained by further allowing as possible letters the gauginos λ^1_+, which are the bottom components of the field strength chiral superfield W_+. In the full $\mathcal{N}=1$ chiral ring, one also has the other Lorentz component W_- of the field strength, as well as the $\mathcal{N}=1$ chiral superfield belonging to the $\mathcal{N}=2$ vector multiplet. Operators that contain those letters are, however, *not* a part of the HL chiral ring.

In complete analogy, we may also define a Hall-Littlewood anti-chiral ring, which contains the Schur operators of type $\hat{\mathcal{B}}_R$ and $\mathcal{D}_{R(0,j_2)}$. These operators are annihilated by chiral supercharges $\mathcal{Q}^1_\alpha$, $\alpha=\pm$, and are thus $\mathcal{N}=1$ anti-chiral with respect to the $\mathcal{N}=1$ subalgebra that is orthogonal to (2.3.9). Schur operators of type $\hat{\mathcal{B}}_R$ belong to both HL rings – these are half-BPS operators that are annihilated by both $\mathcal{Q}^1_\alpha$ and $\widetilde{\mathcal{Q}}_{2\dot{\alpha}}$. They form a further truncation of the $\mathcal{N}=1$ chiral ring to the *Higgs chiral ring*, and their vacuum expectation values parametrize the Higgs branch of the theory. We note that in Lagrangian theories that are represented by acyclic quiver diagrams, all $\mathcal{D}$-type multiplets recombine and are lifted from the $\mathcal{N}=1$ chiral ring at one-loop order [38]. In such cases, the HL chiral ring will coincide with the more restricted Higgs branch chiral ring.

Let us now look in greater detail at some Schur-type shortened multiplets of particular physical interest:

- $\hat{\mathcal{C}}_{0(0,0)}$: Stress-tensor multiplet. The superconformal primary is a scalar operator of dimension two that is a singlet under the $SU(2)_R \times U(1)_r$. The $SU(2)_R$ and $U(1)_r$ conserved currents, the supercurrents, and the stress tensor all lie in this multiplet. The Schur operator is the highest weight component of the $SU(2)_R$ current: $J^{11}_{+\dot{+}}$ of the $SU(2)_R$.

- $\hat{\mathcal{C}}_{0(j_1,j_2)}$: Higher-spin currents multiplets. These generalize the stress-tensor multiplet and contain conserved currents of spin higher than two. If any such multiplets are present, the SCFT must contain a decoupled free sector [46]. Requiring the absence of these higher spin multiplets will lead to interesting unitarity bounds for the central charge of interacting SCFTs in §2.4.

- $\hat{\mathcal{B}}_{\frac{1}{2}}$: This is the superconformal multiplet of free hypermultiplets.

- $\hat{\mathcal{B}}_1$: Flavor-current multiplet. The superconformal primary is the "moment map" operator $M^{\mathcal{I}\mathcal{J}}$, which is a scalar operator of dimension two that is an $SU(2)_R$ triplet, is $U(1)_r$ neutral, and transforms in the adjoint representation of the flavor group G_F. The highest weight state of the moment map – M^{11} – is the Schur operator. The claim to fame of $\hat{\mathcal{B}}_1$ multiplets is that they harbor the conserved currents $J^F_{\alpha\dot{\alpha}}$ that generate the continuous "flavor" symmetry group G_F of the SCFT, that is, the symmetry group that commutes with the superconformal group. Because $\hat{\mathcal{B}}_1$ multiplets do not appear in any of the recombination rules for short multiplets listed in Appendix A.2, it is absolutely protected: $J^F_{\alpha\dot{\alpha}}$ remains conserved on the entire conformal manifold of the SCFT.[9]

- $\mathcal{D}_{0(0,0)} \oplus \bar{\mathcal{D}}_{0(0,0)}$: This is the superconformal multiplet of free $\mathcal{N} = 2$ vector multiplets.

- $\mathcal{D}_{\frac{1}{2}(0,0)} \oplus \bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$: "Extra" supercurrent multiplets. The top components of these multiplets are spin 3/2 conserved currents of dimension $\Delta = 7/2$ ($J_{\alpha\dot{\alpha}\dot{\beta}}$ and $J_{\alpha\beta\dot{\alpha}}$). They generate *additional* supersymmetry transformations beyond the $\mathcal{N} = 2$ superalgebra in question. In particular, in the $\mathcal{N} = 2$ description of an $\mathcal{N} = 4$ SCFT, one finds two copies of each of these multiplets transforming as a doublet of the "flavor" $SU(2)_F \subset SU(4)_R$ that commutes with $SU(2)_R \times U(1)_r \subset SU(4)_R$. The Schur operators have $\Delta = 5/2$, and have index structure $\mathcal{O}^{11}_{\dot{+}}$ and $\mathcal{O}^{11}_{+}$. In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, these are the operators $\mathrm{Tr}\, q^1_i \tilde{\lambda}^1_{\dot{+}}$ and $\mathrm{Tr}\, q^1_i \lambda^1_{+}$, where $i = 1, 2$ is the $SU(2)_F$ index.

2.3.2 Notable elements of the chiral algebra

Armed with a working knowledge of the relevant four-dimensional operators, we now proceed to derive some universal entries in the $4d/2d$ dictionary. We first recall from §2.2.3 the process by which a meromorphic operator in two dimensions is obtained from an appropriate protected operator in four dimensions. Starting with a Schur operator in four dimensions, we obtain a

[9] The only other supermultiplet that contains a global flavor symmetry current is $\hat{\mathcal{C}}_{0(\frac{1}{2},\frac{1}{2})}$. However, that multiplet also contains higher-spin currents, thus showing that the only points on a conformal manifold at which the flavor symmetry enhances are the points where the SCFT develops a free decoupled subsector.

two-dimensional chiral operator via the following series of specializations:

$$\begin{array}{cc}
\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}(x) & \text{Schur operator} \\
\downarrow & \\
\mathcal{O}(z,\bar{z}) \cong u_{\mathcal{I}_1}(\bar{z})\cdots u_{\mathcal{I}_{2R}}(\bar{z})\mathcal{O}^{(\mathcal{I}_1\cdots\mathcal{I}_{2R})}(z,\bar{z}) & \text{Twisted-translated Schur operator} \\
\downarrow & \\
[\mathcal{O}(z,\bar{z})]_{\mathtt{Q}} & \text{Chiral cohomology class} \\
\downarrow & \\
\mathcal{O}(z) & \text{Two-dimensional chiral operator}
\end{array}$$

In general we will refer to this associated chiral operator via the following notation:

$$\mathcal{O}(z) = \chi[\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}] \ ,$$

where sometimes we will be lax about the argument of the χ map and allow $\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}$ to be replaced by the more generic form of the operator $\mathcal{O}^{\mathcal{I}_1\cdots\mathcal{I}_{2R}}_{\alpha_1\cdots\alpha_{2j_1}\dot{\alpha}_1\cdots\dot{\alpha}_{2j_2}}$. Our first task will be to understand the chiral operators that are related to certain characteristic Schur operators of a four-dimensional theory. In doing so we will discover some interesting and generic features of this correspondence.

Virasoro enhancement of the $\mathfrak{sl}(2)$ symmetry

The holomorphic $\mathfrak{sl}(2)$ algebra generated by $\{L_{-1}, L_0, L_1\}$ is a manifest symmetry of the chiral algebra. Remarkably, this global conformal symmetry is enhanced to the full Virasoro algebra. The Virasoro algebra is generated by the modes L_n, $n \in \mathbb{Z}$, of a holomorphic stress tensor of dimension two $T(z)$. Surveying Table 2.1, we find a suitable candidate that is present in any theory $\mathcal{T}$: the Schur operator belonging to stress tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$. One should note that the Schur operator in this multiplet is *not* the four-dimensional stress tensor, but rather the component $J^{11}_{+\dot{+}}$ of the $SU(2)_R$ current $J^{\mathcal{I}\mathcal{J}}_{\alpha\dot{\alpha}}$.

The corresponding twisted-translated operator is defined as follows,

$$\mathcal{J}_R(z,\bar{z}) := u_{\mathcal{I}}(\bar{z})\ u_{\mathcal{J}}(\bar{z})\ J^{\mathcal{I}\mathcal{J}}_{+\dot{+}}(z,\bar{z}) \ . \tag{2.3.10}$$

Per the discussion of §2.2, we identify the cohomology class $[\mathcal{J}_R(z,\bar{z})]_{\mathtt{Q}_i}$ with a dimension two meromorphic operator in the chiral algebra $\chi[\mathcal{T}]$,

$$T_{\mathcal{J}}(z) := \kappa\ [\mathcal{J}_R(z,\bar{z})]_{\mathtt{Q}_i} \ . \tag{2.3.11}$$

We provisionally include the subscript $\mathcal{J}$ as a reminder of the definition (2.3.11); we still need to establish that the OPEs of $T_{\mathcal{J}}(z)$ with itself and with other operators in the chiral algebra take the standard forms appropriate to a two-dimensional stress tensor. With this in mind, we have also included a normalization factor κ, to be fixed momentarily in order to recover the canonical TT OPE.

The two- and three-point functions of the R-symmetry current with itself are fixed by $\mathcal{N}=2$ superconformal invariance in terms of a single parameter c_{4d}, which is one of the two conformal anomaly coefficients (the other being a_{4d}). Starting from the OPE of two $SU(2)_R$ currents [47],

$$J_\mu^{\mathcal{I}\mathcal{J}}(x)J_\nu^{\mathcal{K}\mathcal{L}}(0) \sim \frac{3c_{4d}}{4\pi^4}\epsilon^{\mathcal{K}(\mathcal{I}}\epsilon^{\mathcal{J})\mathcal{L}}\frac{x^2 g_{\mu\nu}-2x_\mu x_\nu}{x^8}+\frac{2i}{\pi^2}\frac{x_\mu x_\nu x\cdot J^{(\mathcal{K}(\mathcal{I}}\epsilon^{\mathcal{J})\mathcal{L})}}{x^6}+\cdots\ , \tag{2.3.12}$$

we find the following OPE of twisted-translated Schur operators,

$$\begin{aligned}\mathcal{J}_R(z,\bar{z})\mathcal{J}_R(0,0) \ \sim\ & -\frac{3c_{4d}}{2\pi^4 z^4}-\frac{1}{\pi^2}\frac{\mathcal{J}_R(0,0)}{z^2}\\ & -\frac{1}{\pi^2}\bar{z}\frac{u_{\mathcal{I}}u_{\mathcal{J}}J_{--}^{\mathcal{I}\mathcal{J}}(0)}{z^3}+\frac{i}{\pi^2}\bar{z}\frac{J_{++}^{21}(0)}{z^2}\\ & +\frac{i}{\pi^2}\bar{z}^2\frac{J_{--}^{21}(0)}{z^3}+\cdots\ .\end{aligned} \tag{2.3.13}$$

Because the last three terms have non-zero $\widehat{L}_0$ eigenvalue, they are guaranteed to be $\mathbb{Q}_i$-exact. Upon setting $\kappa=-2\pi^2$, we find the following meromorphic OPE for $T_{\mathcal{J}}$,[10]

$$T_{\mathcal{J}}(z)\ T_{\mathcal{J}}(0) \sim \frac{-6\,c_{4d}}{z^4}+\frac{2\ T_{\mathcal{J}}(0)}{z^2}+\frac{\partial T_{\mathcal{J}}(0)}{z}\ . \tag{2.3.14}$$

Happily, we recognize in (2.3.14) the familiar two-dimensional TT OPE with central charge c_{2d} given by

$$c_{2d}=-12\,c_{4d}\ . \tag{2.3.15}$$

[10]The term corresponding to the simple pole does not immediately follow from the OPE given in (2.3.13). In particular, though the presence of $\partial T_{\mathcal{J}}(0)$ is guaranteed as a consequence of the double pole, we may worry that an additional quasiprimary (in the two-dimensional sense) may also appear. Such a quasiprimary $\mathcal{O}$ would have to be a boson of holomorphic dimension $h=3$ and have nonzero three point function $\langle T_{\mathcal{J}}T_{\mathcal{J}}\mathcal{O}\rangle$. This is forbidden by Bose symmetry.

This is the first major entry in our dictionary. Note that unitarity of the four-dimensional theory requires $c_{4d} > 0$, so the chiral algebra will have negative central charge and will therefore necessarily be non-unitary.

It is not immediately clear from the arguments presented thus far that $T_{\mathcal{J}}(z)$ will have the correct OPE with operators of the chiral algebra. In other words, the assertion that $T_{\mathcal{J}}$ acts as the stress tensor of the chiral algebra means that the "geometric" $\mathfrak{sl}(2)$ generators $\{L_{-1}, L_0, L_{+1}\}$ defined by the embedding (2.2.19) of the two-dimensional conformal algebra into the four-dimensional one should coincide in cohomology with the generators $\{L_{-1}^{\mathcal{J}}, L_0^{\mathcal{J}}, L_{+1}^{\mathcal{J}}\}$ defined by the mode expansion of $T_{\mathcal{J}}(z)$. It would be sufficient to verify that this is the case for quasiprimary operators, by which we mean operators $\mathcal{O}(z)$ that, when inserted at the origin, are annihilated by the holomorphic special conformal generator

$$[L_{+1}, \mathcal{O}(0)] = 0 \ . \tag{2.3.16}$$

In our construction, such an $\mathcal{O}(z)$ arises as the cohomology class of a twisted-translated primary Schur operator. The assertion is then that in the chiral algebra (*i.e.*, up to $\mathbb{Q}_i$-exact terms), the $T_{\mathcal{J}}$ OPEs take the form

$$T_{\mathcal{J}}(z)\mathcal{O}(0) \sim \cdots + \frac{0}{z^3} + \frac{h\ \mathcal{O}(0)}{z^2} + \frac{\partial\mathcal{O}(0)}{z} \ , \tag{2.3.17}$$

where h is the holomorphic dimension of $\mathcal{O}$ and the dots indicate possible poles of order four or higher. Though we have not been able to find a general proof, we believe (2.3.17) to be a universal consequence of superconformal Ward identities. It is thanks to the relation for the conformal dimension $h = R + j_1 + j_2$ that the $SU(2)_R$ current can reproduce the appropriate scaling dimension, and the absence of additional operators should be excluded by selection rules for three-point functions of Schur-type superconformal multiplets. In practice, we have been able to give an abstract argument that this OPE holds only for the case where $\mathcal{O}$ is a scalar operator. For non-scalar operators in the abstract setting, we leave the structure of these OPEs as a conjecture. Later in this section, the OPE (2.3.17) will be shown to hold in full generality in the theories of free hypermultiplets and free vector multiplets. The abstract claim would follow if the most general solution of the requisite Ward identity is expressible as a linear combination of structures corresponding to free field models, which is empirically the case in all analogous situations with which the authors are familiar.

Affine enhancement of the flavor symmetry

We next turn to the role played by the flavor symmetries of $\mathcal{T}$ in the associated chiral algebra. When $\mathcal{T}$ enjoys a flavor symmetry G_F, the corresponding conserved current $J_{\alpha\dot{\alpha}}$ is an element of a $\hat{\mathcal{B}}_1$ supermultiplet, which additionally contains as its Schur primary the moment map operator M^{11} described in the list at the end of §2.3.1. We expect the presence of G_F symmetry to make itself known via the chiral operator associated to the moment map. Following the now-familiar procedure, we define a $\mathbb{Q}_i$-closed operator $M(z,\bar{z})$ via twisted translations of the Schur moment-map operator from the origin, and identify the corresponding cohomology class as a meromorphic operator in the chiral algebra,

$$M(z,\bar{z}) := u_{\mathcal{I}}(\bar{z})u_{\mathcal{J}}(\bar{z})\, M^{\mathcal{I}\mathcal{J}}(z,\bar{z}) \ , \qquad J(z) := \kappa[M(z,\bar{z})]_{\mathbb{Q}_i} \ . \tag{2.3.18}$$

The normalization constant κ will be determined momentarily. The meromorphic operator $J(z)$ has holomorphic dimension $h=1$. We have suppressed flavor indices up to this point, but these operators all transform in the adjoint representation of the flavor symmetry group, and so we actually find $\dim G_F$ dimension one currents $J^A(z)$ in the chiral algebra. It is natural to suspect that these operators will behave as affine currents for the flavor symmetry. Indeed, a little calculation bears out this expectation. First, recall that the central charge k_{4d} of the flavor symmetry is defined in terms of the self-OPE of the conserved flavor symmetry current as follows,

$$J^A_\mu(x)\ J^B_\nu(0) \sim \frac{3k_{4d}}{4\pi^4}\delta^{AB}\frac{x^2 g_{\mu\nu}-2x_\mu x_\nu}{x^8} + \frac{2}{\pi^2}\frac{x_\mu x_\nu f^{ABC} x\cdot J^C(0)}{x^6} + \cdots \ . \tag{2.3.19}$$

Here $A,B,C=1,\ldots,\dim G_F$ are adjoint flavor indices, and we are using normalizations such that long roots of a Lie algebra have length $\sqrt{2}$ as in [47]. In the same conventions, the OPE of two moment maps reads

$$M^{A\mathcal{I}\mathcal{J}}(x)M^{B\mathcal{K}\mathcal{L}}(0) \sim -\frac{3k_{4d}}{48\pi^4}\frac{\epsilon^{\mathcal{K}(\mathcal{I}}\epsilon^{\mathcal{J})\mathcal{L}}\delta^{AB}}{x^4} - \frac{\sqrt{2}}{4\pi^2}\ \frac{f^{ABC}M^{C\,(\mathcal{I}(\mathcal{K}}\epsilon^{\mathcal{L})\mathcal{J})}}{x^2} + \cdots \ . \tag{2.3.20}$$

The OPE for the corresponding twisted-translated operators follows directly,

$$\begin{aligned} M^A(z,\bar{z})M^B(0,0) \sim &-\frac{3k_{4d}}{48\pi^4}\frac{\delta^{AB}}{z^2} + \frac{\sqrt{2}}{4\pi^2}\ i\ \frac{f^{ABC}M^C(0,0)}{z} \\ &+ \frac{\sqrt{2}}{4\pi^2}\ f^{ABC}M^{C\,21}(0)\frac{\bar{z}}{z} + \cdots \ , \end{aligned} \tag{2.3.21}$$

where the last term is $\mathbb{Q}_i$-exact. Setting $\kappa = 2\sqrt{2}\pi^2$, we recognize the canonical current algebra OPE,[11]

$$J^A(z)J^B(w) \sim k_{2d}\frac{\delta^{AB}}{(z-w)^2} + \sum_C if^{ABC}\frac{J^C(w)}{z-w} \ , \tag{2.3.22}$$

where the two-dimensional affine level k_{2d} is related to the four-dimensional flavor central charge k_{4d} by

$$k_{2d} = -\frac{k_{4d}}{2} \ . \tag{2.3.23}$$

This is the second important entry in the dictionary.

The Hall-Littlewood chiral ring and chiral algebra generators

An interesting problem that will be of particular concern in §2.5 is that of giving a simple description of the chiral algebra $\chi[\mathcal{T}]$ associated to a given $\mathcal{T}$ in terms of a set of generating currents. Generators of a chiral algebra are by definition those $\mathfrak{sl}(2)$ primary operators $\{\mathcal{O}_j\}$ for which the normal ordered products of their descendants, *i.e.*, operators of the form $\partial^{n_1}\mathcal{O}_1\partial^{n_2}\mathcal{O}_2\ldots\partial^{n_k}\mathcal{O}_k$, span the whole algebra.[12] When the chiral algebra has only a finite number of generators, it is customary to refer to it as a $\mathcal{W}$-algebra.

While we have given a clear set of rules that identifies the spectrum of the chiral algebra given the spectrum of the four-dimensional theory $\mathcal{T}$, these rules have little to say about the question of what operators are *generators* of $\chi[\mathcal{T}]$. There turns out to be a subset of generators that is always relatively easy to identify. Recall from §2.3.1 that the HL chiral and anti-chiral rings are consistent truncations of the $\mathcal{N} = 1$ chiral and anti-chiral rings of $\mathcal{T}$, respectively. As such, they are commutative rings, and it is often possible

[11] In two dimensions it is standard to define a convention-independent affine level k_{2d} as $k_{2d} := \frac{2\tilde{k}_{2d}}{\theta^2}$, where $\tilde{k}_{2d}$ is the level when the length of the long roots are normalized to be θ. In our conventions $\theta^2 = 2$ and so $\tilde{k}_{2d} = k_{2d}$.

[12] We are adopting the normal ordering conventions of [48], in which a sequence of chiral operators represents left-nesting of conformally normal-ordered products:

$$\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_{n-1}\mathcal{O}_n := (\mathcal{O}_1(\mathcal{O}_2(\cdots(\mathcal{O}_{n-1}\mathcal{O}_n)))) \ . \tag{2.3.24}$$

The algebra of operators so-defined is non-commutative and non-associative.

to give them presentations in terms of generators and relations. What we show now is that the meromorphic operators associated to the generators of the HL chiral and antichiral rings are in fact generators of $\chi[\mathcal{T}]$ in the chiral algebra sense.

Given the shortening conditions they obey, one finds that the chiral algebra operators associated to HL operators have holomorphic dimension $h = R + |r|$. In order to establish the claim made above, we will show that an HL operator can never arise as a normal ordered product of other operators that are not themselves of HL type. Let $\mathcal{O}_1(z, \bar{z})$ and $\mathcal{O}_2(z, \bar{z})$ be two generic twisted-translated Schur operators, and let us assume that their OPE contains an HL operator $\mathcal{O}_3^{\rm HL}$,

$$\mathcal{O}_1(z, \bar{z})\, \mathcal{O}_2(0,0) \sim \frac{1}{z^{h_1+h_2-h_3}} \mathcal{O}_3^{\rm HL}(0,0) + \ldots \tag{2.3.25}$$

By assumption, $h_3 = R_3 + |r_3|$, while (2.3.7) implies that $h_1 \geqslant R_1 + |r_1|$, $h_2 \geqslant R_2 + |r_2|$. The $U(1)_r$ charge is conserved, so $r_3 = r_1 + r_2$ and $|r_3| \leqslant |r_1| + |r_2|$. Furthermore, $SU(2)_R$ selection rules imply the triangular inequality $R_3 \leqslant R_1 + R_2$. Combining these (in)equalities, we find that $h_3 \leqslant h_1 + h_2$, which implies that an HL operator may only appear on the right hand side as a singular term (if $h_3 < h_1 + h_2$) or as the leading non-singular term (if $h_3 = h_1 + h_2$). The latter possibility requires that $\mathcal{O}_1$ and $\mathcal{O}_2$ saturate the respective bounds (2.3.7) for h_1 and h_2, which is to say that they themselves must be HL operators. This argument establishes that HL operators cannot be generated as normal ordered products of non-HL operators, and so the generators of the HL chiral and antichiral rings must necessarily be generators of the chiral algebra.

The Hall-Littlewood chiral ring and Virasoro primaries

A further interesting feature of the HL chiral ring operators is that their corresponding meromorphic operators are always Virasoro primaries. For the generators of the HL chiral ring, this is already clear since the generators of any chiral algebra that includes a stress tensor are necessarily primaries of the Virasoro subalgebra. For other HL operators, though, this is a useful result that will help organize our thinking about some of the examples studied in §2.5.

The statement follows from a relatively straightforward analysis of the OPE of the meromorphic stress tensor with an arbitrary HL operator. In particular, let $\mathcal{O}_1(z)$ be the meromorphic operator associated to an HL operator in four dimensions. The quantum numbers of $\mathcal{O}_1$ obey the HL relation

$$h_1 = R_1 + |r_1| \ . \tag{2.3.26}$$

Now the crucial observation from which our result follows is this: from a four-dimensional perspective, the meromorphic stress tensor is a $\bar{z}$-dependent linear combination of operators with $r = 0$ and $R = 0, \pm 1$. Consequently, in the OPE of the meromorphic stress tensor with $\mathcal{O}_1(0)$, the only operators that may appear will have $R = R_1 \pm 1$ or $R = R_1$ and $r = r_1$. With what power of z can such an operator appear in the OPE? A Schur operator $\mathcal{O}_\gamma(0)$ with $R = R_1 + \gamma$ and $\mathcal{M} = |r_1| + 2\mathrm{min}(j_1, j_2)$ will appear in the OPE as

$$T(z)\mathcal{O}_1(0) \supset \frac{\mathcal{O}_\gamma(0)}{z^{2+R_1+|r_1|-R-\mathcal{M}}} = \frac{\mathcal{O}_\gamma(0)}{z^{2-\gamma-2\mathrm{min}(j_1,j_2)}} \ . \tag{2.3.27}$$

This is at most a pole of order three (when $\gamma = -1$ and $j_1 = 0$ or $j_2 = 0$), but such a pole cannot appear because HL operators are always $\mathfrak{sl}(2)$ primaries – thus the most singular term possible is a pole of order two. This is precisely the property that characterizes Virasoro primary operators, and so we have our result.

2.3.3 The chiral algebras of free theories

The simplest $\mathcal{N} = 2$ SCFTs are the theories of a free hypermultiplet and that of a free vector multiplet. For these special cases, we give a complete description of the associated chiral algebras. These chiral algebras are useful as the building blocks of interacting Lagrangian theories, some of which are discussed in §2.4. We describe in turn the cases of hypermultiplets and vector multiplets.

Free hypermultiplets

Let us consider the field theory of a single free hypermultiplet. The hypermultiplet itself lies in the short supermultiplet $\mathcal{B}_{\frac{1}{2}}$, in which the primary Schur operators are the scalars Q and $\tilde{Q}$. These are the highest weight states in a pair of $SU(2)_R$ doublets,

$$Q^{\mathcal{I}} = \begin{pmatrix} Q \\ \tilde{Q}^* \end{pmatrix} \ , \qquad \tilde{Q}^{\mathcal{I}} = \begin{pmatrix} \tilde{Q} \\ -Q^* \end{pmatrix} \ . \tag{2.3.28}$$

The single free hypermultiplet enjoys an $SU(2)_F$ flavor symmetry, under which $Q^{\mathcal{I}}$ and $\tilde{Q}^{\mathcal{I}}$ transform as a doublet. To work covariantly in terms of this $SU(2)_F$, we can introduce the following tensor,

$$Q^{\mathcal{I}}_{\hat{\mathcal{I}}} := \left(\begin{array}{cc} Q & \tilde{Q} \\ \tilde{Q}^* & -Q^* \end{array} \right) , \tag{2.3.29}$$

where $\hat{\mathcal{I}} = 1, 2$ is the newly minted $SU(2)_F$ index.

The Schur operators in this free theory are all the "words" that can be constructed out of the "letters" $\{Q, \tilde{Q}, \partial_{+\dot{+}}\}$. As there are no singularities in the products of ($\partial_{+\dot{+}}$ derivatives of) Q and $\tilde{Q}$, the operator associated to any given word is well-defined and the Schur operators in this theory form a commutative ring. The set of all meromorphic operators in the free hypermultiplet chiral algebra are therefore precisely the $\mathbb{Q}_i$ cohomology classes of the twisted-translated versions of these words. This chiral algebra is itself a free chiral theory in two dimensions. Let us see how this works.

The twisted-translated operators and the associated meromorphic operators for the hypermultiplet scalars themselves are defined as follows,

$$Q_{\hat{\mathcal{I}}}(z,\bar{z}) := u_{\mathcal{I}}(\bar{z})\, Q^{\mathcal{I}}_{\hat{\mathcal{I}}}(z,\bar{z}) \ , \qquad q_{\hat{\mathcal{I}}}(z) := [Q_{\hat{\mathcal{I}}}(z,\bar{z})]_{\mathbb{Q}_i} \ . \tag{2.3.30}$$

The relation to the operators defined in §2.2.4 is $q_{\hat{\mathcal{I}}}(z) = (q(z), \tilde{q}(z))$. This is an $SU(2)_F$ doublet of dimension 1/2 meromorphic fields, the OPE of which can be computed using the free-field OPE in four dimensions and the definition of the twisted translated operators in (2.3.30),

$$q_{\hat{\mathcal{I}}}(z)\, q_{\hat{\mathcal{J}}}(w) \sim \frac{\varepsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}}{z-w} \ . \tag{2.3.31}$$

It is reasonably easy to see that the entire spectrum of the chiral algebra of four-dimensional hypermultiplets is obtained by taking normal ordered products of the $q_{\hat{\mathcal{I}}}(z)$ and their descendants. In particular, one can show that the following diagram commutes,[13]

$$\begin{tikzcd} \{\mathcal{O}_i\ , \mathcal{O}_j\} \arrow[r, "\times_{4d}"] \arrow[d, "\mathbb{Q}_i"'] & \mathcal{O}_i\mathcal{O}_j \arrow[d, "\mathbb{Q}_i"] \\ \{[\mathcal{O}_i]\ , [\mathcal{O}_j]\} \arrow[r, "\times_{::}"] & :[\mathcal{O}_i][\mathcal{O}_j]: \end{tikzcd} \ , \tag{2.3.32}$$

[13]We will see when we come to consider interacting theories in §2.5 that product structures on Schur operators do not always translate so simply into those of the chiral algebra.

where the top row represents multiplication in the ring of Schur operators, the bottom row represents *creation/annihilation* normal ordered products of chiral vertex operators, and the vertical arrows represent the identification of a Schur operator with its meromorphic counterpart in the chiral algebra. It follows that the meromorphic operator associated to any given word in ($\partial_{+\dot{+}}$derivatives of) Q and $\tilde{Q}$ is simply the corresponding creation/annihilation normal ordered product of (holomorphic derivatives of) q and $\tilde{q}$.

The chiral algebra of the free hypermultiplet is thus none other than the free symplectic boson algebra (*cf.* [49]). This simple example serves to illustrate some of the general points made in the previous subsections. The symplectic boson theory has a canonical stress tensor,

$$T(z) = \frac{1}{2}\varepsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}} q_{\hat{\mathcal{I}}} \partial q_{\hat{\mathcal{J}}}(z) \ , \tag{2.3.33}$$

and it is easy to check that the modes $\{L_{+1}, L_0, L_{-1}\}$ appearing in Laurent expansion of (2.3.33) reproduce the action of the holomorphic $\mathfrak{sl}(2)$ symmetry inherited from four dimensions. Thus the holomorphic $\mathfrak{sl}(2)$ is indeed enhanced to Virasoro symmetry. Moreover, we observe that given the form of the $SU(2)_R$ current in four dimensions

$$\mathcal{J}_\mu^{\mathcal{I}\mathcal{J}}(x) \sim \varepsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}} Q_{\hat{\mathcal{I}}}^{(\mathcal{I}} \partial_\mu Q_{\hat{\mathcal{J}}}^{\mathcal{J})}(x) \ , \tag{2.3.34}$$

The corresponding meromorphic operator $T_{\mathcal{J}}(z)$ will be equivalent to the canonical stress tensor,

$$T(z) = T_{\mathcal{J}}(z) \ . \tag{2.3.35}$$

From the TT OPE we read off the central charge $c_{2d} = -1$. Recalling that the conformal anomaly coefficient of a free hypermultiplet is $c_{4d} = 1/12$, this result is in agreement the universal relation $c_{2d} = -12c_{4d}$. The symplectic boson theory is like the theory of a complex free fermion (which of course has $c_{2d} = 1$), but with opposite statistics, hence the opposite value of the central charge.

Finally we mention a minor generalization of the above story for hypermultiplets. Gauge theories with $\mathcal{N} = 2$ supersymmetry are often described in terms of half-hypermultiplets instead of whole hypermultiplets. The generalization of the chiral algebra to the half-hypermultiplet conventions is straightforward. Let us consider half-hypermultiplets transforming in a pseudo-real representation R of some symmetry group G (at the moment we are working at zero coupling, so G is just a global symmetry group). The corresponding chiral algebra will be generated by $\dim R$ meromorphic fields,

$$q_i \ , \quad i = 1, \ldots, \dim R \ , \tag{2.3.36}$$

and the singular OPE of these operators will be given by

$$q_i(z) q_j(w) \sim \frac{\Omega_{ij}}{z - w} \ . \tag{2.3.37}$$

Here Ω_{ij} is the anti-linear involution that maps the representation R to its conjugate and squares to minus one. The description of the single full hypermultiplet in (2.3.31) actually fits into this framework with $G = SU(2)_F$.

Free vector multiplet

The other key ingredient in Lagrangian SCFTs is the theory of free vector multiplets. Free vectors lie in the short supermultiplet $\bar{\mathcal{D}}_{0(0,0)}$ and its conjugate $\mathcal{D}_{0(0,0)}$, whose superconformal primaries are the complex scalar ϕ and its conjugate $\bar{\phi}$, respectively. The primary Schur operators in these multiplets are the fermions λ^1_+ and $\tilde{\lambda}^1_{\dot{+}}$, and as in the case of hypermultiplets, the entire set of Schur operators in this theory is comprised of the words built out of the letters λ^1_+, $\tilde{\lambda}^1_{\dot{+}}$, and $\partial_{+\dot{+}}$.

The twisted-translated operators associated to the vector multiplet fermions are defined as follows,

$$\lambda(z, \bar{z}) := u_{\mathcal{I}}(\bar{z})\, \lambda^{\mathcal{I}}_+(z, \bar{z}) \ , \quad \tilde{\lambda}(z, \bar{z}) := u_{\mathcal{I}}(\bar{z}) \tilde{\lambda}^{\mathcal{I}}_{\dot{+}}(z, \bar{z}) \ , \tag{2.3.38}$$

and the $\mathtt{Q}_i$-cohomology classes of these operators are Grassmann-odd, holomorphic fields of dimension $h = 1$,

$$\lambda(z) := [\lambda(z, \bar{z})]_{\mathtt{Q}_i} \ , \quad \tilde{\lambda}(z) := [\tilde{\lambda}(z, \bar{z})]_{\mathtt{Q}_i} \ . \tag{2.3.39}$$

Using the four-dimensional free field OPEs, it is easy to derive the OPEs of these holomorphic fields. They are again the OPEs of a free chiral algebra:

$$\tilde{\lambda}(z)\lambda(0) \sim \frac{1}{z^2} \ , \quad \lambda(z)\tilde{\lambda}(0) \sim -\frac{1}{z^2} \ . \tag{2.3.40}$$

Indeed, the free-field form of these OPEs leads to an analogous commutative diagram to (2.3.32), which ensures that all the meromorphic operators in this theory are generated by $\lambda(z)$ and $\tilde{\lambda}(z)$ in the chiral algebra sense. We can recognize this chiral algebra as the (b,c) ghost system of weight $(1,0)$,[14]

$$\tilde{\lambda} := b(z) \ , \quad \lambda(z) := \partial c(z) \ . \tag{2.3.41}$$

In making this identification, we have introduced an extra spurious mode – the zero mode c_0 of $c(z)$ – which is of absent in the algebra generated by $\lambda(z)$ and $\tilde{\lambda}(z)$. Thus, the more precise statement is that the chiral algebra associated to the vector multiplet is the so-called "small algebra" of the (b,c) system, which is by definition the algebra generated by $b(z)$ and $\partial c(z)$ (*cf.* [50, 51]). In other words, the Fock space of the small algebra is the subspace of the (b,c) Fock space that does *not* contain c_0, or equivalently, the subspace annihilated by b_0,

$$\mathcal{F}_{\text{small}} := \{\psi \in \mathcal{F}_{bc} \mid b_0 \psi = 0\} \ . \tag{2.3.42}$$

The small algebra enjoys a global $SL(2,\mathbb{R})$ symmetry under which $\lambda(z)$ and $\tilde{\lambda}(z)$ transform as a doublet. We can make this symmetry manifest by introducing the notation ρ^α with $\alpha = \pm$, where $\rho^+ := \tilde{\lambda}$ and $\rho^- := \lambda$. Note that the Cartan generator of this symmetry acts as the $U(1)_r$ charge. In the language of the small algebra, the OPE can be put in a covariant form,

$$\rho^\alpha(z)\, \rho^\beta(0) \sim \frac{\varepsilon^{\alpha\beta}}{z^2} \ . \tag{2.3.43}$$

As in the hypermultiplet case, the action of the $\{L_{+1}, L_0, L_{-1}\}$ modes of the canonical ghost stress tensor can easily be seen to match the action of the geometric $\mathfrak{sl}(2)$ action inherited from the four-dimensional conformal algebra. Furthermore, given the $SU(2)_R$ current of the free vector theory,

$$\mathcal{J}^{\mathcal{I}\mathcal{J}}_{\alpha\dot{\alpha}}(x) \sim \lambda^{(\mathcal{I}}_{\alpha} \tilde{\lambda}^{\mathcal{J})}_{\dot{\alpha}}(x) \ , \tag{2.3.44}$$

we see that the canonical stress tensor coincides precisely with the dimension two current $T_{\mathcal{J}}$ obtained from the R-symmetry current by the usual map,

$$T(z) = -\frac{1}{2}\varepsilon_{\alpha\beta}\rho^\alpha \rho^\beta(z) = T_{\mathcal{J}}(z) \ . \tag{2.3.45}$$

The central charge of the (b,c) ghost system/small algebra is $c_{2d} = -2$, which can be seen to agree with the relation (2.3.15) upon recalling that $c_{4d} = \frac{1}{6}$ for a free vector multiplet.

[14] Recall that the derivative of a dimension zero conformal primary field – $c(z)$ in this case – is again a conformal primary.

2.3.4 Gauging prescription

The natural next step is to consider interacting SCFTs. Lagrangian $\mathcal{N} = 2$ SCFTs can be described using hypermultiplets and vector multiplets as elementary building blocks (see [52] for a recent classification of all possibilities). In particular, such an SCFT consists of vector multiplets transforming in the adjoint representation of a semisimple gauge group $G = G_1 \times G_2 \cdots \times G_k$, along with a collection of (half)hypermultiplets transforming in some representation R of the gauge group such that the one-loop beta functions for all gauge couplings vanish. Supersymmetry ensures that the theory remains conformal at the full quantum level. The building blocks of the corresponding chiral algebra are a collection of symplectic bosons $\{q\,, \tilde{q}\}$ in the representation R, and a collection of $(b\,, c)$ ghost small algebras in the adjoint representation of G. When the gauge couplings are strictly zero, the chiral algebra is simply obtained by imposing the Gauss law constraint, *i.e.*, by restricting to the gauge-invariant operators of the free chiral algebra of symplectic bosons and ghosts. Our next step will be to determine what happens as we turn on the gauge couplings.

In fact, as Lagrangian theories are a small subset of all possible $\mathcal{N} = 2$ SCFTs, it is worthwhile to put the discussion in a more general context. Given a general superconformal field theory $\mathcal{T}$ with G_F flavor symmetry, a new SCFT is obtained by gauging a subgroup $G \subset G_F$ provided the gauge coupling beta function vanishes. We will denote the gauged theory with a nonzero gauge coupling g as $\mathcal{T}_G$.[15] Though $\mathcal{T}$ may be strongly coupled, the gauging procedure can be described in semi-Lagrangian language. By assumption, $\mathcal{T}$ possesses a conserved flavor symmetry current $J^A_{\alpha\dot{\alpha}}$, where $A = 1, \ldots \dim G$, which by $\mathcal{N} = 2$ supersymmetry is the top component of the moment map supermultiplet $\hat{\mathcal{B}}_1$. The gauged theory $\mathcal{T}_G$ is described by minimally coupling an $\mathcal{N} = 2$ vector multiplet to $\hat{\mathcal{B}}_1$. Of particular importance is the addition to the action, in $\mathcal{N} = 1$ notation, of the superpotential coupling

$$g \int d^2\theta\, \Phi^A\, M^{11,A} + \, h.c.\,, \tag{2.3.46}$$

where Φ is the $\mathcal{N} = 1$ chiral superfield in the $\mathcal{N} = 2$ vector multiplet, and M^{11} is the $\mathcal{N} = 1$ chiral superfield whose bottom component is the complex

[15] More precisely, there is one independent gauge coupling for each simple factor of the gauge group. To avoid clutter we focus on the procedure for gauging one simple factor at the time, so G will taken to be a simple group in what follows.

moment map M^{11}; both transform in the adjoint representation of G.

Let us assume that the chiral algebra $\chi[\mathcal{T}]$ is known. It will suffice to work abstractly, in the sense that the only features of $\chi[\mathcal{T}]$ that we will use follow directly from the existence of the global G symmetry. In particular, there will be an affine current $J^A(z)$ at level $k_{2d} = -\frac{1}{2}k_{4d}$ (*cf.* §2.3.2). As we mentioned above, at zero gauge coupling the chiral algebra of the gauged theory is obtained by imposing the Gauss law constraint on the tensor product algebra of $\chi[\mathcal{T}]$ with the G-ghost small algebra (ρ^+, ρ^-). In fact, it will be more useful to introduce the full (b, c) system and restrict to the small algebra by imposing the auxiliary condition $b_0^A \psi = 0$ for any state ψ.

The affine current associated to the G symmetry in the ghost sector is

$$J_{\text{gh}}^A := -i\, f^{ABC}\, (c^B b^C) \ . \tag{2.3.47}$$

The Gauss law, or gauge-invariance, constraint requires that all physical states should have vanishing total gauge charge, which is measured by the zero mode of the total gauge symmetry current,

$$J_{\text{tot}}^A(z) := J^A(z) + J_{\text{gh}}^A(z) \ . \tag{2.3.48}$$

Symbolically, we can therefore define the chiral algebra at zero gauge coupling as follows:

$$\chi[\mathcal{T}_G^{(0)}] = \{\psi \in \chi[\mathcal{T}] \otimes (b^A, c^A) \,|\, b_0^A \psi = J_{\text{tot}\,0}^A \psi = 0\} \, . \tag{2.3.49}$$

We are now ready to address the problem of identifying the chiral algebra for $\mathcal{T}_G$ with $g \neq 0$.

BRST reduction of the chiral algebra

On general grounds, we expect that the chiral algebra of the interacting gauge theory will contain fewer operators than the non-interacting version, because some of the short multiplets containing Schur operators that are present at zero coupling will recombine into long multiplets and acquire anomalous dimensions. Ideally, we would like to describe this phenomenon using only the general algebraic ingredients that we have introduced so far. A crucial hint comes from phrasing the condition of conformal invariance of the gauge theory more abstractly. The vanishing of the one-loop beta function amounts to the requirement that in the ungauged theory, the flavor symmetry central charge is given by

$$k_{4d} = 4h^\vee \ , \tag{2.3.50}$$

where $h^\vee$ is the dual Coxeter number of the gauge group. This means that in two-dimensional language, the corresponding symmetry in $\chi[\mathcal{T}]$ must have its affine level given by

$$k_{2d} = -2h^\vee \ . \tag{2.3.51}$$

The affine level of the ghost-sector flavor currents $J_{\rm gh}$ is easily calculated to be $2h^\vee$, so the requirement of conformal invariance translates into the condition that the level of the total affine current $J^A_{\rm tot}$ be zero. Precisely in this case, it is possible to construct a nilpotent BRST operator in the chiral algebra. Imitating a construction familiar from coset conformal field theory [53], we define

$$Q_{\rm BRST} := \oint \frac{dz}{2\pi i}\, j_{\rm BRST}(z)\,, \quad j_{\rm BRST} := c_A \left[J^A + \frac{1}{2} J^A_{\rm gh} \right] . \tag{2.3.52}$$

Our contention is that the chiral algebra corresponding to the gauged theory at finite coupling is obtained by passing to the cohomology of $Q_{\rm BRST}$ *relative to* the ghost zero modes b_0^A,[16]

$$\chi[\mathcal{T}_G] = \mathcal{H}^*_{\rm BRST}[\psi \in \chi[\mathcal{T}] \otimes (b^A, c^A) \,\big|\, b_0^A \psi = 0] \,. \tag{2.3.53}$$

Apart from its elegance, there are compelling physical arguments behind this claim. We will show that states of the chiral algebra that define nontrivial cohomology classes of $Q_{\rm BRST}$ correspond to the four-dimensional Schur states that survive in the interacting theory. By construction, all states of $\chi[\mathcal{T}_G^{(0)}]$ are annihilated by the four supercharges in (2.3.3). As we turn on the gauge coupling, those supercharges receive quantum corrections, and only a subset of states remains supersymmetric. We will see that $Q_{\rm BRST}$ precisely implements the $O(g)$ correction to one of the Poincaré supercharges, which will justify our conjecture under the assumption that higher order corrections do not remove any additional states.

A preliminary remark is that the Gauss law constraint is imposed automatically. Because

$$\{b_0^A\,, Q_{\rm BRST}\} = J^A_{{\rm tot}\,0} \ , \tag{2.3.54}$$

states in the small algebra that are $Q_{\rm BRST}$-closed are automatically gauge invariant. Consequently, we have the simpler expression,

$$\chi[\mathcal{T}_G] = \mathcal{H}^*_{\rm BRST}[\chi[\mathcal{T}_G^{(0)}]] \ . \tag{2.3.55}$$

[16] In other terms, the BRST cohomology is being defined entirely in the small algebra: two $Q_{\rm BRST}$-closed states belong to the same cohomology class if and only if they differ by an exact state $Q_{\rm BRST}\lambda$, where λ is also in the small algebra.

We can rewrite Q_{BRST} and separate out the ghost zero modes,

$$Q_{\mathrm{BRST}} = c_0^A J^A_{\mathrm{tot}\,0} + b_0^A X^A + Q^- \ , \tag{2.3.56}$$

where we have defined

$$X^A := -\frac{i}{2} f^{ABC} \left(\sum_{n\neq 0} : c^B_{-n} c^C_n : - c^B_0 c^C_0 \right) \ , \tag{2.3.57}$$

while Q^- anticommutes with both c_0^A and b_0^A and can thus be expressed purely in terms of (ρ^{+A}, ρ^{-A}),

$$Q^- := \sum_{n\neq 0} \frac{1}{n} : \rho^{-A}_{-n} J^A_n : + \frac{i}{2} f^{ABC} \sum_{\substack{n\neq 0 \\ m\neq 0 \\ m\neq n}} \frac{1}{nm} : \rho^{-A}_{-n} \rho^{-B}_{m} \rho^{+C}_{n-m} : \ . \tag{2.3.58}$$

The operator Q^- fails to be nilpotent by a term proportional to $J^A_{tot\,0}$, so it *is* nilpotent when acting on gauge-invariant states. It follows that (2.3.54) can be equivalently written as

$$\chi[\mathcal{T}_G] = \mathcal{H}^*_{Q^-} [\psi \in \chi[\mathcal{T}] \otimes (\rho^{+A}, \rho^{-A}) \, , \text{with } J^A_{\mathrm{tot}\,0} \psi = 0] \, . \tag{2.3.59}$$

This is the form of our conjecture that makes more immediate contact with four-dimensional physics. We will show that the action of Q^- precisely matches to the action of $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$, the $O(g)$ term in the expansion of the supercharge $\widetilde{\mathcal{Q}}_{2\dot{-}}$,

$$\widetilde{\mathcal{Q}}_{2\dot{-}} = \widetilde{\mathcal{Q}}^{(0)}_{2\dot{-}} + g\, \widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}} + O(g^2) \, . \tag{2.3.60}$$

In fact, Q^- is the lowest component of an $SL(2,\mathbb{R})$ doublet of operators Q^α, with

$$Q^+ := \sum_{n\neq 0} \frac{1}{n} : \rho^{+A}_{-n} J^A_n : + \frac{i}{2} f^{ABC} \sum_{\substack{n\neq 0 \\ m\neq 0 \\ m\neq n}} \frac{1}{mn} : \rho^{+A}_{-n} \rho^{+B}_{m} \rho^{-C}_{n-m} : \ . \tag{2.3.61}$$

In complete analogy, the action of Q^+ will be shown to be isomorphic to that of $\mathcal{Q}^{1(1)}_{-}$, the $O(g)$ term in the expansion of $\mathcal{Q}^1_-$. The two Poincaré supercharges $\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$ play a completely symmetric role in the definition of Schur operators. The fact that Q_{BRST} contains Q^- rather than Q^+ is a consequence of our choice (2.3.41), which treated λ and $\tilde{\lambda}$ in a slightly asymmetric fashion.

Fortunately, to leading order in the gauge coupling the action of the relevant supercharges takes a universal form in the subspace of operators that obey the tree-level Schur condition. Such operators are obtained by forming gauge-invariant combinations of more elementary building blocks, namely the conformal primaries of the "matter" SCFT $\mathcal{T}$, the gauge-covariant derivative $D_{+\dot{+}}$, and the gauginos $\tilde{\lambda}^1_{\dot{+}}$ and λ^1_{+}. The supersymmetry variation of a gauge-invariant "word" is found by using the Leibniz rule to act on each elementary "letter".[17] It is then sufficient to specify the SUSY variations of the letters:

1. $\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$ (anti)commute with the conformal primary operators in the matter sector $\mathcal{T}$.

2. For the gauge-covariant derivative $D_{+\dot{+}} := \partial_{+\dot{+}} + gA_{+\dot{+}}$,

$$[\mathcal{Q}^1_-, D_{+\dot{+}}] = g\tilde{\lambda}^1_{\dot{+}}\,, \quad [\widetilde{\mathcal{Q}}_{2\dot{-}}, D_{+\dot{+}}] = g\lambda^1_{+}\,, \tag{2.3.62}$$

   where we have just used the tree-level variation of the gauge field, times the explicit factor of g.

3. Finally the variations of the gauginos can be deduced from the non-linear classical equations of motions of the vector multiplet, minimally coupled to the moment map supermultiplet $\hat{\mathcal{B}}_1$,

$$\begin{aligned}\{\widetilde{\mathcal{Q}}_{2\dot{-}}, \tilde{\lambda}^1_{\dot{+}}\} &= \{\mathcal{Q}^1_-, \lambda^1_+\} = F^{11} = gM^{11} \\ \{\widetilde{\mathcal{Q}}_{2\dot{-}}, \lambda^1_{+}\} &= \{\mathcal{Q}^1_-, \tilde{\lambda}^1_{\dot{+}}\} = 0\,,\end{aligned} \tag{2.3.63}$$

   where F^{11} is the highest-weight of the $SU(2)_R$ triplet of auxiliary fields in the $\mathcal{N}=2$ vector multiplet.[18]

If a Schur operator in the free theory is to retain its Schur status at $O(g)$, then when inserted at the origin it must be annihilated by the one-loop corrections to the four relevant supercharges, $\{\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}, (\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}})^\dagger, \mathcal{Q}^{1(1)}_-, (\mathcal{Q}^{1(1)}_-)^\dagger\}$.

[17] For the special case of $\mathcal{N}=2$ superconformal QCD, a very explicit description of the action of $\mathcal{Q}^{1(1)}_-$ in the subsector of tree-level Schur operators can be found in Section 5 of [54].

[18] In an $\mathcal{N}=1$ description of the $\mathcal{N}=2$ vector multiplet, $F^{11}=\bar{F}$, where F is the top component of chiral superfield ϕ, whose superpotential coupling with the moment map is given in (2.3.46).

Equivalently, it must define a nontrivial cohomology class with respect to $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$ and $\mathcal{Q}^{1(1)}_{-}$. Conveniently, the recombination rules for shortened multiplets of Schur type (*cf.* Appendix A.2) are such that in any such recombination, the Schur operators of $\mathcal{T}^{(0)}$ are lifted in quartets that are related by the action of these two supercharges in the manner indicated in the following diagram:

$$\begin{tikzcd}
 & \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \arrow[dr, "\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}"] & \\
\hat{\mathcal{C}}_{R(j_1,j_2)} \arrow[ur, "\mathcal{Q}^{1(1)}_{-}"] \arrow[dr, "\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}"'] & & \hat{\mathcal{C}}_{R+1(j_1-\frac{1}{2},j_2-\frac{1}{2})} \\
 & \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j_2-\frac{1}{2})} \arrow[ur, "\mathcal{Q}^{1(1)}_{-}"'] &
\end{tikzcd} \tag{2.3.64}$$

In the diagram, we are labeling Schur operators by the name of the supermultiplet to which they belong.[19] Consequently, if an operator remains in the cohomology of *either* supercharge, it necessarily remains in the cohomology of both, and so stays a Schur operator at one-loop order. For example, if an operator becomes $\mathcal{Q}^{1(1)}_{-}$ exact then it is either at the right or at the top of the diagram and it follows that it is either $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$ exact or not $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$ closed, respectively. The other cases can be treated analogously.

Under the $4d/2d$ identifications

$$\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}} \to Q^- \,, \quad \mathcal{Q}^{1(1)}_{-} \to Q^+ \,, \quad D_{+\dot{+}} \to \partial \,, \quad \lambda^1_+ \to \rho^- \,, \quad \tilde{\lambda}^1_{\dot{+}} \to \rho^+ \,, \tag{2.3.65}$$

one easily checks that (2.3.58) and (2.3.61) have precisely the right form to reproduce the action of the $O(g)$ correction to the four-dimensional supercharges. Thus, the BRST cohomology specified in (2.3.53) is just the right thing to project out states whose corresponding Schur operators are lifted at one-loop order.

It is of some interest to note that this story of one-loop corrections to the spectrum of Schur operators admits a simple truncation to the case of HL chiral ring operators. The tree-level HL operators will be gauge-invariant

[19] To include all possible recombinations, we must formally allow j_1 and j_2 to take the value $-\frac{1}{2}$ as well, and re-interpret a $\hat{\mathcal{C}}$ multiplet with negative spins as a $\hat{\mathcal{B}}$, $\mathcal{D}$ or $\bar{\mathcal{D}}$ multiplet, according to the rules:
$\hat{\mathcal{C}}_{R(j_1,-\frac{1}{2})} := \bar{\mathcal{D}}_{R+\frac{1}{2}(j_1,0)}$, $\hat{\mathcal{C}}_{R(-\frac{1}{2},j_2)} := \mathcal{D}_{R+\frac{1}{2}(0,j_2)}$, $\hat{\mathcal{C}}_{R(-\frac{1}{2},-\frac{1}{2})} := \hat{\mathcal{B}}_{R+1}$.

combinations of the HL operators of $\mathcal{T}$ and the gaugino λ^1_+. The operators that are lifted from the spectrum at one-loop will be those that are related by the corrected supercharge $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$, whose action in this sector is completely determined by (2.3.63). The problem of finding the HL operators in the spectrum of the interacting theory thus becomes a miniature "HL-cohomology" problem. In examples, it is sometimes useful to solve this problem as a first step in order to determine some important operators that will necessarily make an appearance in the chiral algebra.

Finally, a caveat is in order. We have assumed that the Schur operators that persist at infinitesimal coupling will remain protected at any finite value of the coupling. In some concrete cases, it can be demonstrated that no further recombination of shortened multiplets is possible. Moreover, in the examples of §2.5 we will propose simple economical descriptions for the chiral algebras defined by this cohomological recipe, and demonstrate that they have the symmetries expected at finite coupling from S-duality, giving strong evidence for our proposal, at least in those examples.

Non-renormalization of three-point couplings

So far, we have studied how the spectrum of operators is modified when the coupling is turned on, but we have said nothing about the OPE coefficients of the remaining physical operators in the gauged theory. Our implicit assumption has been that the OPE coefficients of operators that remain protected at finite coupling are actually independent of the coupling. From a two-dimensional perspective, it seems unlikely that the OPE coefficients could change due to the extremely rigid structure of chiral algebras, and we expect a corresponding non-renormalization statement to hold in four dimensions. Indeed, such a non-renormalization theorem directly follows from the methods and results of [55]. Let us consider the four-point function of three Schur-type operators and of the exactly marginal operator $\mathcal{O}_\tau$ responsible for changing the complexified gauge coupling,

$$\langle\, \mathcal{O}_1^{\mathcal{I}_1}(x_1)\, \mathcal{O}_2^{\mathcal{I}_2}(x_2)\, \mathcal{O}_3^{\mathcal{I}_3}(x_3)\, \mathcal{O}_\tau(x_4)\,\rangle\,, \tag{2.3.66}$$

where $\mathcal{I} = (\mathcal{I}^{(1)} \ldots \mathcal{I}^{(k)})$ with $\mathcal{I}^{(i)} = 1, 2$ are $SU(2)_R$ multi-indices and we have suppressed Lorentz indices. Non-renormalization of the appropriate three-point function of Schur-type operators will follow at once if we can argue that the above four-point function vanishes for any x_4 when $x_{1,2,3}$ all lie on the plane. By a conformal transformation, we can always take the fourth operator to lie on the same plane, and then focus on the $SU(1,1|2)$ subalgebra of $SU(2,2|2)$ defined by the embedding (2.2.20). The Schur-type operators are chiral primaries of this subalgebra. The marginal operator $\mathcal{O}_\tau$, being the top component of an $\bar{\mathcal{E}}_2$ multiplet of $SU(2,2|2)$, is of the form $\mathcal{O}_\tau = \{\mathcal{Q}^1, [\mathcal{Q}^2, \dots]\}$ where $\mathcal{Q}^{\mathcal{I}} := \mathcal{Q}^{\mathcal{I}}_-$ are supercharges of $SU(1,1|2)$.[20] All the properties exploited in [55] to show the vanishing of the four-point function (2.3.66) are satisfied. The authors of [55] interpreted this result as a non-renormalization theorem for three-point functions of chiral primaries of two-dimensional $(0,4)$ theories, but exactly the same argument applies to our case as well.

We close this section by pointing out a curious aspect of the gauging prescription given here. Given a chiral CFT $\chi[\mathcal{T}]$ with affine G symmetry, one can introduce a two-dimensional vector field $A_{\bar{z}}$ and gauge G. Following standard arguments (for example, see [56, 53]), a change of variables in the path integral eliminates the gauge field in favor of an extra G current algebra at level $-(2h^\vee + k_{2d})$ and an adjoint-valued (b,c) ghost system. One must also impose invariance under the standard BRST operator associated to the gauge symmetry. In our case, $2h^\vee + k_{2d} = 0$ so the extra current algebra is trivial, and the BRST operator associated to the two-dimensional gauging takes precisely the form of (2.3.52). In some sense, we have found that "$4d$ gauging = $2d$ gauging". We find it plausible that a localization-style argument may shed light on this correspondence.

2.4 Consequences for four-dimensional physics

The chiral symmetry algebras that we have uncovered have extensive consequences for the spectrum and structure constants of any $\mathcal{N}=2$ SCFT. To give a simple example, Virasoro symmetry implies that any Higgs branch half-BPS supermultiplet $\hat{\mathcal{B}}_R$ is accompanied by an entire module of semi-short $\hat{\mathcal{C}}_{R'(j,j)}$ multiplets with $R' = R-1, R, R+1$. In the four-dimensional theory, the descendant operators arise by taking repeated normal ordered

[20] Similarly, the conjugate operator $\bar{\mathcal{O}}_\tau$ is the top component of an $\mathcal{E}_2$ and can be written as $\{\widetilde{\mathcal{Q}}_1, [\widetilde{\mathcal{Q}}_2, \dots]\}$. An entirely analogous argument holds for the four-point function containing $\bar{\mathcal{O}}_\tau$.

products with certain components of the $SU(2)_R$ current, but the chiral algebra perspective makes this structure much more transparent.

In this section we elaborate on the relationship between the observables associated to the chiral algebra (*i.e.*, its correlation functions and torus partition function) and those of the parent four-dimensional theory. We first point out that the superconformal Ward identities for four-point functions of $\hat{\mathcal{B}}_R$ operators [32, 33] are a simple consequence of our cohomological construction. This new perspective makes it clear that analogous Ward identities must hold for four-point functions of general Schur operators. The presence of meromorphic functions in the solution of the Ward identities of [28, 32, 33] was one of the initial clues that led to our work. We now have a neat conceptual interpretation for them: they are nothing but the correlation functions of the associated chiral algebra. By exploiting the relationship between the two-dimensional and four-dimensional perspectives we are able to derive new unitarity bounds that must be satisfied by the conformal and flavor anomalies of a general *interacting* $\mathcal{N} = 2$ SCFT. Finally, we delineate the relationship between the torus partition function of the chiral algebra and the superconformal index of the parent four-dimensional theory.

2.4.1 Conformal twisting and superconformal Ward identities

By construction, for a given SCFT $\mathcal{T}$, the correlation functions of $\chi[\mathcal{T}]$ are equal to certain correlation functions of physical operators in $\mathcal{T}$ restricted to lie on the plane. From the four-dimensional point of view these are somewhat unnatural correlators to study, as they have explicit space-time dependence built into the operators. On the other hand, each correlation function of $\chi[\mathcal{T}]$ is canonically associated to a family of more natural correlation functions of $\mathcal{T}$ that are obtained by replacing the twisted-translated operators with the corresponding untwisted operators at the same points in $\mathbb{R}^2$.

Let us consider such a correlator now. For simplicity, we specialize to a four-point function, in which case there is actually no loss of generality in restricting the operators to be coplanar. We denote the untwisted operators as $\mathcal{O}^{\mathcal{I}}(z, \bar{z})$, with $SU(2)_R$ multi-indices $\mathcal{I} = (\mathcal{I}^{(1)}, \ldots, \mathcal{I}^{(k)})$ where $\mathcal{I}^{(i)} = 1, 2$. The components of the multi-index are symmetrized; the operator transforms in the spin $k/2$ representation of $SU(2)_R$. Recall that in our conventions, the Schur operator in this $SU(2)_R$ multiplet is the highest-weight state

$\mathcal{O}^{1\cdots 1}(z,\bar{z})$. We represent the four-point function of such operators as

$$\mathcal{F}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\mathcal{I}_4}(z_i,\bar{z}_i) = \langle\, \mathcal{O}_1^{\mathcal{I}_1}(z_1,\bar{z}_1)\, \mathcal{O}_2^{\mathcal{I}_2}(z_2,\bar{z}_2)\, \mathcal{O}_3^{\mathcal{I}_3}(z_3,\bar{z}_3)\, \mathcal{O}_4^{\mathcal{I}_4}(z_4,\bar{z}_4)\, \rangle\ . \quad (2.4.1)$$

This is actually a collection of four-point functions labelled by the different possible assignments for the R-symmetry indices. The full collection of four-point functions can be conveniently packaged by introducing two-component $SU(2)_R$ vectors $u(y_i) = (1, y_i)$ and defining contracted operators that depend on the auxiliary variable y as follows [32, 33]

$$\mathcal{O}_i(z_i,\bar{z}_i;y_i) = u_{I_1}(y_i)\cdots u_{\mathcal{I}_{k_i}}(y_i)\mathcal{O}_i^{(\mathcal{I}_1\cdots\mathcal{I}_{k_i})}(z_i,\bar{z}_i)\ . \quad (2.4.2)$$

A single function of x_i and y_i can be defined that encodes the full content of the collection of correlation functions in (2.4.1),

$$\mathcal{F}(z_i,\bar{z}_i;y_i) = \langle\, \mathcal{O}_1(z_1,\bar{z}_1;y_1)\, \mathcal{O}_2(z_2,\bar{z}_2;y_2)\, \mathcal{O}_3(z_3,\bar{z}_3;y_3)\, \mathcal{O}_4(z_4,\bar{z}_4;y_4)\, \rangle\ . \quad (2.4.3)$$

Charge conservation ensures that this function is homogeneous in the auxiliary y_i with weight $\frac{1}{2}\sum k_i$, and the correlation function for a given choice of external R-symmetry indices can be read off by selecting the coefficient of the appropriate monomial in the y_i variables.

This repackaging makes it simple to state the relationship with correlation functions of $\chi[\mathcal{T}]$. The twisted chiral operators defined in §2.2.2 are the specialization of the repackaged operators in (2.4.2) to $y_i = \bar{z}_i$. So if the related four-point function of meromorphic operators $\mathcal{O}_i(z) = \chi[\mathcal{O}_i(z,\bar{z})]$ is defined as

$$f(z_1,z_2,z_3,z_4) = \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle\ , \quad (2.4.4)$$

then the correlation functions are related according to

$$f(z_i) = \mathcal{F}(z_i,\bar{z}_i;y_i)\Big|_{y_i\to\bar{z}_i}\ . \quad (2.4.5)$$

The fact that the left-hand side of this equation is a meromorphic function of the operator insertion points is a consequence of the cohomological arguments of the previous sections, but it is also *precisely* the final form of the superconformal Ward identities for such a correlation function [28, 29, 30, 31, 32, 33].

This is a rather wonderful result: the entirety of the constraints imposed by superconformal Ward identities on the four-point function of half-BPS operators are captured by the existence of the twist of §2.2.2. It is worth noting that while the Ward identities of [32] were derived specifically for half-BPS operators in $\hat{\mathcal{B}}_R$ multiplets, here we see that the same type of Ward identities holds more generally for any Schur-type operators.

2.4.2 Four-dimensional unitarity and central charge bounds

The natural inner product on the Hilbert space of the radially quantized four-dimensional theory $\mathcal{T}$ does not survive the passage to $\mathbb{Q}$ cohomology. This is an immediate consequence of the fact that $\mathbb{Q}$ is not hermitian. Hence, unitarity in four dimensions does not imply unitarity in the chiral algebra. In fact, we have seen that a unitary theory $\mathcal{T}$ always gives rise to a chiral algebra $\chi[\mathcal{T}]$ with negative central charge, which is necessarily non-unitary. Nevertheless, there is an interesting interplay between the structure of the chiral algebra and four-dimensional unitarity. This leads to new unitarity bounds for the anomaly coefficients of any four-dimensional SCFT. In this section, we explore an elementary example that provides us with such bounds. It is possible that more extensive analysis could lead to further constraints; we leave such an analysis for future study.

The origin of nontrivial consistency conditions can be found in the fact that, as summarized in (2.4.5), the meromorphic correlator $f(z_i)$ can be computed in two different ways that must agree. The first computation is the two-dimensional one: once the singular OPEs of the meromorphic operators appearing in the correlator are known, the full correlation function is completely fixed by meromorphy. The meromorphic correlator further admits a unique decomposition into $\mathfrak{sl}2$ conformal blocks,[21] leading to an expression of the form

$$f(z_i) = \left(\frac{z_{24}}{z_{14}}\right)^{h_{12}} \left(\frac{z_{14}}{z_{13}}\right)^{h_{34}} \frac{1}{z_{12}^{h_1+h_2}\, z_{34}^{h_3+h_4}} \sum_{\ell=0}^{\infty} (-1)^{\ell}\, a_\ell\, g_\ell(z) \ , \tag{2.4.6}$$

where

$$g_\ell(z) := (-\tfrac{1}{2}z)^{\ell-1} z\ {}_2F_1(\ell,\ell;2\ell;z) \ , \tag{2.4.7}$$

and we have adopted the standard notation $z_{ij} := z_i - z_j$ and $z := \frac{z_{12}z_{34}}{z_{13}z_{24}}$. Additionally, h_i is the holomorphic scaling dimension of the i'th operator, and we have defined $h_{ij} = h_i - h_j$.

The second computation is the four-dimensional one. The correlator in (2.4.1) admits a decomposition into $\mathfrak{su}(2,2|2)$ superconformal blocks that each represent the contribution of a given superconformal multiplet to the

[21] The result could also be expanded in Virasoro conformal blocks, but this is less natural for comparison to four-dimensional quantities.

four-point function. The contribution of each superconformal block to the meromorphic part of the amplitude defined by (2.4.5) is fixed up to the three-point coefficients. Thus for a given theory $\mathcal{T}$, the spectrum and three-point coefficients of BPS operators appearing in the conformal block expansion of a given correlation function can be determined directly from the correlation functions of $\chi[\mathcal{T}]$. Non-trivial constraints arise when we require that the three-point coefficients determined in this manner be consistent with unitarity.

Let us now turn to a specific example to study in detail. We consider the four-point function of superconformal primary operators in $\hat{\mathcal{B}}_1$ multiplets. As was explained in §2.3, these multiplets contain the spin one conserved currents that generate the global (non-R) symmetry of the theory, and the superconformal primaries are scalar moment map operators M^A. Consequently the results derived from this example will be relevant to any theory with non-trivial flavor symmetry. The moment map operators have dimension two and transform in the adjoint representations of both the flavor group G_F and $SU(2)_R$. The four-point function of such operators can be expanded in channels corresponding to each irreducible representation $\mathcal{R}$ of G_F in which the exchanged operators in the conformal block expansion may transform,

$$\begin{aligned}\langle M^A(z_1,\bar{z}_1;y_1)M^B(z_2,\bar{z}_2;y_2)M^C(z_3,\bar{z}_3;y_3)M^D(z_4,\bar{z}_4;y_4)\rangle \\ = \sum_{\mathcal{R}\in\otimes^2\mathbf{adj}} P_{\mathcal{R}}^{ABCD}\mathcal{F}_{\mathcal{R}}(z_i,\bar{z}_i;y_i)\ ,\end{aligned} \quad (2.4.8)$$

where $P_{\mathcal{R}}^{ABCD}$ is the projector onto the irreducible representation denoted by $\mathcal{R}$. The projectors for the various groups can be obtained following the procedures described in [57].

Per the discussion of §2.3.2, the chiral operators $J^A = \chi[M^A]$ are affine currents, and the mermorphic correlators that emerge in the limit $y_i \to \bar{z}_i$ are equal to the four-point functions in the corresponding chiral algebra,

$$z_{12}^2 z_{34}^2\langle J^A(z_1)J^B(z_2)J^C(z_3)J^D(z_4)\rangle = f^{ABCD}(z) = \sum_{\mathcal{R}} P_{\mathcal{R}}^{ABCD} f_{\mathcal{R}}(z)\,. \quad (2.4.9)$$

Each such function can be examined independently as a potential source of nontrivial consistency conditions. In §2.3 we found that the level of the affine Lie algebra symmetry generated by these currents is $k_{2d} = -\frac{1}{2}k_{4d}$, so this meromorphic four-point function is completely fixed in terms of the structure constants of the associated non-affine Lie algebra and the flavor central charge,[22]

$$
\begin{aligned}
f^{ABCD}(z) =&\delta^{AB}\delta^{CD} + z^2\delta^{AC}\delta^{BD} + \frac{z^2}{(1-z)^2}\delta^{AD}\delta^{CB} - \frac{z}{k_{2d}}f^{ACE}f^{BDE} \\
&- \frac{z}{k_{2d}(z-1)}f^{ADE}f^{BCE}\,.
\end{aligned} \tag{2.4.10}
$$

This correlator can be decomposed into G_F channels, each of which can be expanded in $\mathfrak{sl}(2)$ conformal blocks as in (2.4.6). For example, for the singlet channel $\mathcal{R} = \mathbf{1}$, the above correlator gives

$$
\begin{aligned}
f_{\mathcal{R}=\mathbf{1}} &= \dim G_F + z^2\left(1 + \frac{1}{(1-z)^2}\right) + \frac{4z^2 h^\vee}{k_{2d}(z-1)} \\
&= \dim G_F - \sum_{\ell=0,2,\cdots} \frac{2^\ell(\ell+1)(\ell!)^2\left(2(\ell+1)(\ell+2)k_{2d} - 8\,h^\vee\right)}{k_{2d}(2\ell+1)!} g_{\ell+2}(z)\ ,
\end{aligned} \tag{2.4.11}
$$

where $h^\vee$ is the dual Coxeter number.

This operator product expansion can be compared with that of the full four-point function in four dimensions. The superconformal block decomposition of such a four-point function has been worked out in [30]. In particular, operators that can potentially appear in the intermediate channel must belong to one of the following superconformal multiplets:

- $\mathcal{A}_{\Delta(j,j)}$: Long multiplets that are $SU(2)_R$ singlets with $j_1 = j_2 = j$.
- $\hat{\mathcal{C}}_{0(j,j)}$: Semishort multiplets with $j_1 = j_2 = j$ that contain conserved currents of spin $2j+2$.
- $\hat{\mathcal{C}}_{1(j,j)}$: Semishort multiplets with $j_1 = j_2 = j$.
- $\hat{\mathcal{B}}_1$: Half-BPS multiplets containing Higgs branch moment map operators.
- $\hat{\mathcal{B}}_2$: Half-BPS multiplets containing Higgs branch chiral ring operators of dimension four.

[22]Here we have rescaled the currents in such a way that the identity operator appears with unit normalization in the current-current OPE.

- $\mathcal{I}$: The identity operator.

The contribution of each such multiplet to the full four-point function is fixed up to a single coefficient corresponding to the three-point coupling (squared), and unitarity requires that this coefficient be real and positive. The contribution of each multiplet to the meromorphic functions $f_{\mathcal{R}}(z)$ appearing in the superconformal Ward identities has also been determined in [30]. The results are summarized as follows:

$$\begin{array}{lcl}
\mathcal{A}_{\Delta(\frac{\ell}{2},\frac{\ell}{2})} & : & 0 \ , \\
\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})} & : & \lambda^2_{\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}} g_{\ell+2}(z) \ , \\
\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} & : & -2\lambda^2_{\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}} g_{\ell+3}(z) \ , \\
\hat{\mathcal{B}}_1 & : & \lambda^2_{\hat{\mathcal{B}}_1} g_1(z) \ , \\
\hat{\mathcal{B}}_2 & : & -2\lambda^2_{\hat{\mathcal{B}}_2} g_2(z) \ , \\
\mathrm{Id} & : & \lambda^2_{\mathrm{Id}} \ .
\end{array} \tag{2.4.12}$$

The coefficient $\lambda^2_{\bullet}$ of each contribution is required by unitarity to be non-negative.

Some of the coefficients appearing in (2.4.12) can be completely fixed by symmetry. For example, the identity operator can only appear in the singlet channel $f_{\mathcal{R}=\mathbf{1}}(z)$, where the corresponding coefficient is necessarily given by

$$\lambda^2_{\mathrm{Id}} = \dim G_F \ . \tag{2.4.13}$$

The multiplet $\hat{\mathcal{C}}_{0(0,0)}$ contains a spin two conserved current, *i.e.*, the stress tensor. There can only be one such multiplet, and it contributes to the meromorphic part of the four point function only in the singlet channel. The three-point coupling is fixed in terms of the four-dimensional central charge. In particular, one finds that in $f_{\mathcal{R}=\mathbf{1}}(z)$,

$$\lambda^2_{\hat{\mathcal{C}}_{0(0,0)}} = \frac{\dim G_F}{3c_{4d}} \ . \tag{2.4.14}$$

Finally, multiplets of type $\hat{\mathcal{B}}_1$ can contributes only to the adjoint channel, and the corresponding three-point coupling in $f_{\mathrm{adj}}(z)$ is fixed to be

$$\lambda^2_{\hat{\mathcal{B}}_1} = \frac{4h^\vee}{k_{4d}} \, . \tag{2.4.15}$$

G_F	$h^\vee$	$\dim G_F$	G_F	$h^\vee$	$\dim G_F$
SU(N)	N	N^2-1	E_6	12	78
SO(N)	$N-2$	$\frac{N(N-1)}{2}$	E_7	18	133
USp($2N$)	$N+1$	$N(2N+1)$	E_8	30	248
G_2	4	14	F_4	9	52

Table 2.2: Dual Coxeter number and dimensions for simple Lie groups.

As far as we know, these are the only contributions to this four-point function that are fixed by symmetry in terms of anomaly coefficients. Additionally, the multiplets $\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}$ for $\ell \neq 0$ necessarily contain conserved currents of spin greater than two, and so are expected to be absent in interacting theories [46]. We will take this to be the case in the following analysis.

We can determine the three-point coefficients in, say, the $\mathcal{R}=\mathbf{1}$ channel by comparing with the expansion of the $\chi[\mathcal{T}]$ four-point function in (2.4.11). In particular, we find

$$\begin{aligned}\lambda^2_{\mathrm{Id}} &= \dim G_F \ , \\ \lambda^2_{\hat{\mathcal{C}}_{0(0,0)}} - 2\lambda^2_{\hat{\mathcal{B}}_2} &= \frac{8h^\vee}{k_{4d}} - 4 \ , \\ \lambda^2_{\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}} &= \frac{2^{\ell+1}(\ell+2)((\ell+1)!)^2}{k_{4d}(2\ell+3)!}\left((\ell+2)(\ell+3)k_{4d} - 4h^\vee\right) \ ,\end{aligned} \tag{2.4.16}$$

where in the last line only odd ℓ may appear. The second line of (2.4.16), after substituting the contribution of the stress tensor multiplet from (2.4.14), implies a nontrivial bound that must be satisfied in order for the contribution of the $\hat{\mathcal{B}}_2$ multiplet to be consistent with unitarity,

$$\frac{\dim G_F}{c_{4d}} \geqslant \frac{24h^\vee}{k_{4d}} - 12 \ . \tag{2.4.17}$$

For reference, the dimensions and dual Coxeter numbers of the semi-simple Lie algebras are displayed in Table 2.2. Similarly, the positivity of the last line in (2.4.16) for $\ell = 1$ implies the bound

$$k_{4d} \geqslant \frac{h^\vee}{3} \ . \tag{2.4.18}$$

G_F		Bound	Representation
SU(N)	$N \geqslant 3$	$k_{4d} \geqslant N$	$\mathbf{N^2-1}_{\text{symm}}$
SO(N)	$N = 4, \ldots, 8$	$k_{4d} \geqslant 4$	$\mathbf{\frac{1}{24}N(N-1)(N-2)(N-3)}$
SO(N)	$N \geqslant 8$	$k_{4d} \geqslant N-4$	$\mathbf{\frac{1}{2}(N+2)(N-1)}$
USp($2N$)	$N \geqslant 3$	$k_{4d} \geqslant N+2$	$\mathbf{\frac{1}{2}(2N+1)(2N-2)}$
G_2		$k_{4d} \geqslant \frac{10}{3}$	$\mathbf{27}$
F_4		$k_{4d} \geqslant 5$	$\mathbf{324}$
E_6		$k_{4d} \geqslant 6$	$\mathbf{650}$
E_7		$k_{4d} \geqslant 8$	$\mathbf{1539}$
E_8		$k_{4d} \geqslant 12$	$\mathbf{3875}$

Table 2.3: Unitarity bounds for the anomaly coefficient k_{4d} arising from positivity of the $\hat{\mathcal{B}}_2$ three-point function in non-singlet channels.

The same analysis can be performed for the functions $f_{\mathcal{R}\neq\mathbf{1}}(z_i)$. In these channels there will be no contribution from the stress tensor multiplet, so the resulting bounds make reference only to the anomaly coefficient k_{4d}, as in (2.4.18). *A priori*, an independent bound may be obtained for each representation $\mathcal{R}$ appearing in the tensor product of two copies of the adjoint. For example, in the adjoint channel itself, there can be contributions from $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}$ multiplets with even ℓ. Unitarity then imposes a bound on k_{4d} that turns out to be equivalent to that of (2.4.18). Stronger bounds can be found by considering other choices of $\mathcal{R}$, the possible values of which will depend on the particular choice of simple Lie algebra we consider. In general, we find that for a given choice of G_F, the strongest bound comes from requiring positivity of the contributions of $\hat{\mathcal{B}}_2$ multiplets in a single channel. The bounds from other channels are then automatically satisfied when the strongest bound is imposed. These strongest bounds are displayed in Table 2.3, where we also indicate the representation $\mathcal{R} \in \otimes^2\mathbf{adj}$ that leads to the bound in question. It should be noted that for the special case $G_F = SO(8)$, the same strongest bound is obtained from multiple channels. The representation appearing in the third line of Table 2.3 is in fact decomposable

G_F	A_1	A_2	D_4	E_6	E_7	E_8	F_4	G_2
$h^\vee$	2	3	6	12	18	30	9	4
k_{4d}	$\frac{8}{3}$	3	4	6	8	12	5	$\frac{10}{3}$
c_{4d}	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{6}$	$\frac{13}{6}$	$\frac{19}{6}$	$\frac{31}{6}$	$\frac{5}{3}$	$\frac{5}{6}$

Table 2.4: Central charges for $\mathcal{N} = 2$ SCFTs with Higgs branches given by one-instanton moduli spaces for G_F instantons. Models corresponding to the right-most two columns are not known to exist, but must satisfy these conditions for their central charges if they do.

as $\mathbf{70} = \mathbf{35}_s \oplus \mathbf{35}_c$, and the degeneracy in the bounds can be understood as a consequence of $SO(8)$ triality. For $G_F = SU(2)$ one finds no additional bounds to the ones given in (2.4.17) and in (2.4.18). Finally, we can see that the bound (2.4.18) arising from positivity of the $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$ multiplet in the singlet channel is made obsolete by bounds arising from other channels for all choices of G_F listed in the table.

2.4.3 Saturation of unitarity bounds

Given the existence of these unitarity bounds, it is incumbent upon us to consider the question of whether the bounds are saturated in any known superconformal models. To understand what sort of theory might saturate the bounds, it helps to identify any physical properties that a theory will necessarily possess if it saturates a bound. When the inequalities in (2.4.17) or Table 2.3 are saturated, it means precisely that there is no $\hat{\mathcal{B}}_2$ multiplet in the corresponding representation of G_F contributing to the four-point function in question. The absence of such an operator is intimately connected with a well-known feature of theories with $\mathcal{N} = 2$ supersymmetry in four dimensions. Recalling that the Schur operators in the $\hat{\mathcal{B}}_R$ multiplets are Higgs branch chiral ring operators, the absence of a $\hat{\mathcal{B}}_2$ multiplet contributing to the four-point function of $\hat{\mathcal{B}}_1$ multiplets in the $\mathcal{R}$ channel amounts to a relation in the Higgs branch chiral ring of the form

$$(M \otimes M)\big|_{\mathcal{R}} = 0 \ , \tag{2.4.19}$$

where M is the moment map operator and the tensor product is taken in the chiral ring.

There exists an interesting set of theories for which precisely such relations are known to hold. These are the superconformal field theories that arise on a single $D3$ brane probing a codimension one singularity in F-theory on which the dilaton is constant [58, 59, 60, 61, 62, 63]. There are seven such singularities, labelled $H_0, H_1, H_2, D_4, E_6, E_7, E_8$, for which the corresponding SCFT has global symmetry given by the corresponding group (with $H_i \to A_i$). The Higgs branch of each such theory is isomorphic to the minimal nilpotent orbit of the flavor group G_F. These minimal nilpotent orbits admit a simple description: they are generated by a complex, adjoint-valued moment map M, subject to a set of relations that defined the so-called "Joseph ideal" (see [64] for a nice discussion),

$$(M \otimes M)\big|_{\mathcal{I}_2} = 0 \ , \qquad \mathrm{Sym}^2(\mathbf{adj}) = (2\,\mathbf{adj}) \oplus \mathcal{I}_2 \ , \tag{2.4.20}$$

where $(2\,\mathbf{adj})$ is the representation with Dynkin indices twice those of the adjoint representation.

This leads to an interesting set of conclusions. For one, these theories must saturate some of the $\hat{\mathcal{B}}_2$-type bounds listed above. In particular, this allows us to predict the value of c_{4d} and k_{4d} for these theories as a direct consequence of the Higgs branch relations. These predictions are listed in Table 2.4. Indeed, these anomaly coefficients have been computed by other means and the results agree [65]. On the other hand, an $\mathcal{N}=2$ superconformal theory with G_F symmetry can have as its Higgs branch the one-instanton moduli space of G_F instantons *only* if the $\hat{\mathcal{B}}_2$ bound for all representations in $\mathcal{I}_2$ can be simultaneously saturated. It is not hard to verify that the list of cases for which this can be true includes the cases described above in F-theory, along with $G_F = F_4$ and $G_F = G_2$. Theories with Higgs branches isomorphic to the one-instanton F_4 and G_2 moduli spaces appear to be absent from the literature, and it is tempting to speculate that such theories should nonetheless exist and have as their central charges the values listed in the right-most two columns of Table 2.4.

Finally, it is interesting to rephrase the above discussion purely in the language of the chiral algebra $\chi[\mathcal{T}]$. From this perspective, there is a marked difference between the bound (2.4.17) for the singlet sector and those of Table 2.3 for non-singlets. In a theory saturating the non-singlet bounds, the coefficient of a conformal block is actually set to zero in the OPE of 2.4.6. This should be considered in contrast to a theory that saturates the singlet bound, in which case all of the $\mathfrak{sl}(2)$ conformal blocks are present with nonzero coefficients. It follows that saturation of a non-singlet bound is equivalent to the presence of a null state in the chiral algebra. In particular, because the bounds in question appear in the $\hat{\mathcal{B}}_1$ four-point function, such null states can be understood entirely in terms of the affine Lie subalgebra of the chiral algebra. This interpretation can be verified directly by studying an affine Lie algebra with the level listed in Table 2.3.

The bound (2.4.17), on the other hand, does *not* imply the presence of a null state in the chiral algebra. Instead, a theory $\chi[\mathcal{T}]$ that saturates the singlet bound should have the property that the only $\mathfrak{sl}(2)$ primary of dimension two that appears in the OPE of two affine currents is identically equal to the chiral vertex operator that arises from the $\hat{\mathcal{C}}_{0(0,0)}$ multiplet in four dimensions, *i.e.*, it should be the two-dimensional stress tensor. We thus identify saturation of the singlet bound with the property that the Sugawara construction gives the true stress tensor of the chiral algebra,

$$T_{2d} = \frac{1}{k_{2d} + h^{\vee}} (J^a J^a) \ . \tag{2.4.21}$$

Sure enough, if the bound (2.4.17) is saturated, then we can rewrite the bound as an equation for the central charge

$$c_{2d} = \frac{k_{2d} \dim G_F}{k_{2d} + h^{\vee}} \ . \tag{2.4.22}$$

This is precisely the central charge associated with the Sugawara construction for the stress tensor of an affine Lie algebra.

Finally, we mention a number of additional theories that saturate some of the unitarity bounds derived here. In particular, though the rank one theory corresponding to the H_0 singularity has no flavor symmetry, it will have an extra $SU(2)$ symmetry for rank larger than one (as will all the other rank $\geqslant 1$ theories). In particular, for the case of rank two the flavor central charge corresponding to this extra $SU(2)$ is $\frac{17}{5}$ and the central charge is $c_{4d} = \frac{17}{12}$ [65]. This theory therefore saturates the bound (2.4.17). Additionally, we have found a number of theories that saturate bounds appearing in Table 2.3. In particular, the new rank one SCFTs found in [66] with flavor symmetry $USp(10)_7$ and $USp(6)_5 \times SU(2)_8$, where k_{4d} is indicated as a subscript for each group, saturate the bounds on k_{4d} for the USp factors. However for these theories the central charge bound is not saturated. The following theories described in [67] also saturate bounds on k_{4d}: S_5 with flavor symmetry

$SU(10)_{10}$ (but not the rest of the S_N series), the $R_{0,N}$ series with flavor symmetry $SU(2)_6 \times SU(2N)_{2N}$, and the $R_{2,N}$ series with $SO(2N+4)_{2N} \times U(1)$ flavor symmetry.

2.4.4 Torus partition function and the superconformal index

Just as correlators of the chiral algebra are related to certain supersymmetric correlators of the parent four-dimensional theory, it will not come as a surprise that the torus partition function of the chiral algebra is related to a certain four-dimensional supersymmetric index – indeed, to the Schur limit of the superconformal index, as foreshadowed in our terminology.

We should first identify which quantum numbers can be meaningfully assigned to chiral algebra operators. Of the various Cartan generators of the four-dimensional superconformal algebra, only the holomorphic dimension L_0 and the transverse spin $M^{\perp} = j_1 - j_2$ (which is equal to $-r$ for Schur operators) survive as independent conserved charges of the chiral algebra. The torus partition function therefore takes the form[23]

$$Z(x,q) := \operatorname{Tr} x^{M^{\perp}} q^{L_0} \ . \tag{2.4.23}$$

As usual, the trace is over the Hilbert space in radial quantization, or equivalently over the local operators of the chiral algebra.

Specializing to $x = -1$, and noting that by the four-dimensional spin-statistics connection implies $(-1)^{j_1 - j_2} = (-1)^F$, where F is the fermion number, we find a weighted Witten index,

$$\mathcal{I}(q) := Z(-1,q) = \operatorname{Tr}(-1)^F q^{L_0} = \operatorname{Tr}(-1)^F q^{E-R} \ . \tag{2.4.24}$$

We recognize this as the trace formula that defines the Schur limit of the superconformal index [38], *cf.* Appendix A.2.[24] We should check that in the two-dimensional and four-dimensional interpretations of this formula the

[23] To avoid clutter, we have omitted the obvious refinement by flavor fugacities. If the theory is invariant under some global symmetry group G_F, we may refine the trace formula by $\prod_i a_i^{f_i}$, where the f_i are Cartan generators of G_F and a_i the associated fugacities.

[24] It was observed in [68] that the Schur index has interesting modular properties under the action of $SL(2,\mathbb{Z})$ on the superconformal and flavor fugacities. The identification of the Schur index with a two-dimensional index may serve to shed some light on these observations.

trace can be taken over the same space of states. Strictly speaking, in the four-dimensional interpretation the trace is over the entire Hilbert space of the radially quantized theory. However, the point of the Schur index is that only states obeying the Schur condition can conceivably contribute – the contributions of all other states cancel pairwise. As the states of the chiral algebra are in one-to-one correspondence with Schur states, the chiral algebra index (2.4.24) is indeed equivalent to the Schur index.

The index is a cruder observable than the partition function, but because it is invariant under exactly marginal deformations, it is generally easier to evaluate. In practice, to evaluate the index of a Lagrangian SCFT, one enumerates all gauge-invariant states that can be formed by combining the elementary "letters" that obey the Schur condition, see Table 2.1. This combinatorial exercise is efficiently solved with the help of a matrix integral, where the integration over the gauge group enforces the projection onto gauge singlets. Examples of this prescription will be seen in the following section. By this procedure, one enumerates all gauge-invariant states that obey the tree-level Schur condition; there will be cancellations in the index corresponding to the recombinations of Schur multiplets into long multiplets that are *a priori* allowed by representation theory.

There is an entirely isomorphic computation in the associated chiral algebra. The "letters" obeying the tree-level Schur condition are nothing but the states of the symplectic bosons and the ghost small algebra (in the appropriate representations), and one is again instructed to project onto gauge singlets. To reiterate, to evaluate the index we do not really need to compute the cohomology of Q^{-}, which defines the states of the chiral algebra of the interacting gauge theory, *cf.* (2.3.59). We can simply let the trace run over the redundant set of states of the free theory. By contrast, the trace in the partition function (2.4.23) must be taken over only the states of the chiral algebra for the interacting theory, which are the cohomology classes of Q^{-}.

At the risk of being overly formal, we may point out that the physical state space of the chiral algebra (which for gauge theories is defined by the cohomological problem (2.3.59)), acts as a *categorification* of the Schur index. Once this vector space and the action of the charges are known, we can perform the more refined counting (2.4.23). In physical terms, the categorification contains extra information relative to the Schur index in that it knows about sets of short multiplets that are kinematically allowed to recombine but do not. In addition, there may be multiplets that cannot recombine but nonetheless make accidentally cancelling contributions to the index, and these are also seen in the categorification. Of course, the chiral algebra structure goes well beyond categorification – it is a rich algebraic system that also encodes the OPE coefficients of the Schur operators, and is subject to non-trivial associativity constraints.

It should be noted that as a graded vector space, we also have a categorification of the Macdonald limit of the superconformal index. Recall that the states contributing to the Macdonald index are really the same as the states that contribute to the Schur index, but their counting is refined by an extra fugacity t/q associated to the charge $r + R$ (for $t = q$ we recover the Schur index). Since each state in the vector space defined by the chiral algebra corresponds to a Schur operator, the additional grading by $r + R$ is perfectly well-defined. However, there is no obvious chiral algebra interpretation of the Macdonald limit of the superconformal index, because the additional grading is incompatible with the chiral algebra structure. More precisely, while L_0 and r are conserved charges for the twisted-translated operators (2.2.29), $r + R$ is not, since away from the origin the operators are linear combinations of operators with different R eigenvalues. In particular $r + R$ is *not* preserved by the OPE.

2.5 Examples and conjectures

In this section we consider a number of illustrative examples in which the four-dimensional superconformal field theory $\mathcal{T}$ admits a weakly coupled Lagrangian description. In such cases, the chiral algebra $\chi[\mathcal{T}]$ can be defined via the BRST procedure of §2.3, which at the very least allows for a level-by-level analysis of the physical states/operators in the algebra.

We can also consider the problem of giving an economical description of the chiral algebra in terms of a set of generators and their singular OPEs. A natural question is whether this set is finite, or in other words whether the chiral algebra is a $\mathcal{W}$-algebra. The results of §2.3.2 suggest a very general ansatz for a possible $\mathcal{W}$-algebra structure: the generators should be the operators associated to HL chiral ring generators in four dimensions, and possibly in addition the stress tensor. In each of the first three examples, our results are compatible with this guess, and we formulate concrete conjectures for the precise definition of each chiral algebra as a $\mathcal{W}$-algebra. In the final example, we find a counterexample to this simplistic picture. Namely, we find a theory for which the chiral algebra contains at least one additional generator beyond those included in our basic ansatz.

For the first example, we turn to perhaps the most familiar $\mathcal{N} = 2$ superconformal gauge theory.

2.5.1 $SU(2)$ superconformal QCD

The theory of interest is the $SU(2)$ gauge theory with four fundamental hypermultiplets. Many aspects of this theory that are relevant to the structure of the associated chiral algebra have been analyzed in, *e.g.*, [69]. The field content is an $SU(2)$ vector multiplet and four fundamental hypermultiplets. Because the fundamental representation of $SU(2)$ is pseudo-real, the obvious $U(4)$ global symmetry is enhanced to $SO(8)$, with the four fundamental hypermultiplets being reinterpreted as eight half-hypermultiplets. In $\mathcal{N} = 1$ notation we then have an adjoint-valued $\mathcal{N} = 1$ field strength superfield W_α^A, an adjoint-valued chiral multiplet Φ^B, and fundamental chiral multiplets Q_a^i transforming in the $\mathbf{8}_v$ of $SO(8)$. Here $a, b = 1, 2$ are vector color indices that can be raised and lowered with epsilon tensors, $A, B = 1, 2, 3$ are adjoint color indices, and $i = 1, \ldots, 8$ are $SO(8)$ vector indices. By a common abuse of notation, we use the same symbol for the scalar squarks in the matter chiral multiplets as for the superfields, whereas the gauginos in the vector multiplet are denoted λ_α^A and $\tilde{\lambda}_{A\dot{\alpha}}$. In terms of the $\mathcal{N} = 1$ superfields listed above, the Lagrangian density takes the form

$$\begin{aligned}\mathcal{L} = \mathrm{Im}\Bigg[\tau \int d^2\,\theta d^2\bar{\theta}\ \mathrm{Tr}\left(\Phi^\dagger e^V \Phi + Q_i^\dagger e^V Q^i\right) \\ +\tau \int d^2\theta\ \left(\tfrac{1}{2}\mathrm{Tr}\, W_\alpha W^\alpha + \sqrt{2} Q_a^i \Phi_b^a Q^{ib}\right)\Bigg]\ ,\end{aligned} \quad (2.5.1)$$

Where $\tau = \theta/2\pi + 4\pi i/g_{\mathrm{YM}}^2$ is the complexified gauge coupling. The central charge of the $SU(2)$ color symmetry acting on the hypermultiplets is $k_{4d}^{SU(2)} = 8$, which satisfies condition (2.3.50) for τ to be an exactly marginal coupling. The central charge for the $SO(8)$ flavor symmetry and the conformal anomaly c_{4d} can also be read off directly from the field content,

$$k_{4d}^{SO(8)} = 4\ , \quad c_{4d} = \frac{7}{6}\ . \quad (2.5.2)$$

Although this description is sufficient to set up a BRST cohomology problem that defines the chiral algebra in the manner of §2.3, it is useful to first review some of the features of this theory that we expect to see reflected in the two-dimensional analysis. We have seen that a special role is played in the chiral algebra by the HL chiral ring, the elements of which are the superconformal primary operators in $\hat{\mathcal{B}}$ and $\mathcal{D}$-type multiplets. In this example, these are the lowest components of $\mathcal{N}=1$ chiral superfields that are gauge-invariant polynomials in Q_a^i and W_α^A. As this theory is represented by an acyclic quiver diagram, all $\mathcal{D}$-type multiplets recombine and the HL chiral ring is identically the Higgs chiral ring.

In purely gauge invariant terms, the Higgs branch chiral ring is generated by a single dimension two operator in the adjoint of $SO(8)$,

$$M^{[ij]} = Q_a^i Q^{aj} \ . \tag{2.5.3}$$

This is the moment map for the action of $SO(8)$ on the Higgs branch.[25] There are additional relations that make the structure of the Higgs branch more interesting. Already at tree-level, there are relations that follow automatically from the underlying description in terms of squarks. When organized in representations of $SO(8)$, the of generators of these relations are as follows,

$$M \otimes M\big|_{\mathbf{35}_s} = 0 \ , \qquad M \otimes M\big|_{\mathbf{35}_c} = 0 \ . \tag{2.5.4}$$

On the other hand, there are F-term relations as a consequence of the superpotential in (2.5.1). They are absent in the theory with strictly zero gauge coupling, and encode the fact that certain operators that are present in the chiral ring of the free theory recombine and are lifted from the protected part of the spectrum when the coupling is turned on. The generators of F-term relations, again organized according to $SO(8)$ representation, are as follows,

$$M \otimes M\big|_{\mathbf{35}_v} = 0 \ , \qquad M \otimes M\big|_{\mathbf{1}} = 0 \ . \tag{2.5.5}$$

One immediately recognizes the complete set of relations in (2.5.4) and (2.5.5) as defining the $SO(8)$ Joseph ideal described in §2.4. Indeed, for the particular case of $G_F = SO(8)$ we have $\mathcal{I}_2 = \mathbf{1} \oplus \mathbf{35}_v \oplus \mathbf{35}_s \oplus \mathbf{35}_c$. The Higgs branch of this theory is known to be isomorphic to the $SO(8)$ one-instanton moduli space, and the central charges (2.5.2) do in fact saturate the appropriate unitarity bounds outlined in §2.4.

[25] It is a special feature of this theory (in contrast to, say, the $N_f = 2N_c$ theories with $N_c > 2$ that will be considered next) that the generators of the Higgs branch chiral ring all have dimension two. In general, there will be higher-dimensional baryonic generators that are not directly related to the global symmetry currents of the theory.

As a final comment, let us recall that the gauge coupling appearing in the Lagrangian (2.5.1) is exactly marginal and parameterizes a one-complex-dimensional conformal manifold. S-duality acts by $SL(2,\mathbb{Z})$ transformations on τ, and the conformal manifold is identified with the familiar fundamental domain of $SL(2,\mathbb{Z})$ in the upper half plane. In the various weak-coupling limits the theory can always be described using the same $SU(2)$ gauge theory, but in comparing one such limit to another, the duality transformations act by triality on the $SO(8)$ flavor symmetry. Consequently, though a given Lagrangian description of this theory (and of the chiral algebra in the next subsection) singles out a certain triality frame, the protected spectrum of the theory, and so in particular the chiral algebra, should be triality invariant.

BRST construction of the associated chiral algebra

The chiral algebra can now be constructed using the procedure of §2.3. We first define the chiral algebra $\chi[\,\mathcal{T}_{\text{free}}\,]$ of the free theory. Each half-hypermultiplet gives rise to a pair of commuting, dimension 1/2 currents, whose OPE is that of symplectic bosons

$$q_a^i(z) := \chi[\,Q_a^i\,]\ , \qquad q_a^i(z)\,q_b^j(w) \sim \frac{\delta^{ij}\epsilon_{ab}}{z-w}\ . \tag{2.5.6}$$

Meanwhile, the vector multiplet contributes a set of adjoint-valued (b,c) ghosts of dimension $(1,0)$ with the standard OPE,

$$b^A(z) := \chi[\,\tilde{\lambda}^A]\ , \qquad \partial c^B(z) := \chi[\,\lambda^B]\ , \qquad b^A(z)c^B(w) \sim \frac{\delta^{AB}}{z-w}\ . \tag{2.5.7}$$

The generators of the $SU(2)$ gauge symmetry in the matter sector arise from the moment maps in the free theory, while in the ghost system they take the canonical form described in §2.3,

$$J^A(T^A)_a^{\ b} = q_a^i q^{ib}\ , \qquad J_{\text{gh}}^A = -if^{ABC}(c^B\,b^C)\ . \tag{2.5.8}$$

The chiral algebra of the free theory is then given by the gauge-invariant part of the tensor product of the symplectic boson and small algebra Fock spaces,

$$\chi[\mathcal{T}_{\text{free}}] = \{\psi \in \mathcal{F}(q_a^i,\rho_+^A,\rho_-^A)\,|\,J_{\text{tot},0}^A\psi = 0\}\ . \tag{2.5.9}$$

The current algebra generated by the J_{mat}^A has level $k_{2d}^{SU(2)} = -4 = -2h^\vee$, which ensures the existence of a nilpotent BRST differential. The BRST current and differential are then constructed in terms of these currents,

$$J_{\text{BRST}} = c^A \left(J^A + \frac{1}{2} J^A_{\text{gh}} \right) , \qquad Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} J_{\text{BRST}}(z) . \tag{2.5.10}$$

The chiral algebra of the interacting theory is now the $BRST$ cohomology

$$\chi[\,\mathcal{T}\,] = \mathcal{H}^*_{\text{BRST}} \left[\chi[\mathcal{T}_{\text{free}}] \right] . \tag{2.5.11}$$

We now perform a basic analysis of this cohomology. Already at this rudimentary level, we will find that a substantial amount of four-dimensional physics is packaged elegantly into the chiral algebra framework.

Enumerating physical states

It is a straightforward exercise to enumerate the physical operators up to any given dimension and to compute the singular terms in their OPEs. This is made easier with computer assistance – we have made extensive use of K. Thielemans' `Mathematica` package [48]. We now describe this enumeration in detail for operators of dimension one and two in the chiral algebra. In this example, the material we have reviewed above is already enough to predict the results of this enumeration. We will nevertheless find it instructive to explore in some detail how the inevitable spectrum comes about.

We begin at dimension one. Dimension one currents in the chiral algebra can only originate in $\mathcal{D}_{0(0,0)}$ and $\hat{\mathcal{B}}_1$ multiplets (*cf.* Table 2.1). The former contain free vector multiplets, and so are not gauge invariant. Thus the physical spectrum at dimension one should be isomorphic to the spectrum of $\hat{\mathcal{B}}_1$ multiplets. Sure enough, the complete list dimension-one operators in $\chi[\,\mathcal{T}_{\text{free}}\,]$ is the following,

$$J^{[ij]} = q^i_a q^{ja} , \tag{2.5.12}$$

and these operators are the chiral counterparts of the $SO(8)$ moment maps, *i.e.*,

$$J^{[ij]} = \chi[M^{[ij]}] . \tag{2.5.13}$$

Direct computation further verifies that these operators exhaust the nontrivial BRST cohomology at dimension one. It is also straightforward to determine the singular terms in the OPEs of these currents,

$$J^{[ij]}(z) J^{[kl]}(0) \sim \frac{-2(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})}{z^2} + \frac{i f^{[ij][kl]}_{[mn]} J^{[mn]}(0)}{z} . \tag{2.5.14}$$

This is just an $\mathfrak{so}(8)$ affine Lie algebra at level $k_{2d} = -2$, which confirms the general prediction of §2.3 that flavor symmetries are affinized in the chiral algebra, subject to the relation $k_{2d} = -\frac{1}{2}k_{4d}$.

Moving on, the four-dimensional multiplets that can give rise to two-dimensional quasi-primary currents of dimension two are $\hat{\mathcal{C}}_{0(0,0)}$, $\hat{\mathcal{B}}_2$, $\mathcal{D}_{0(0,1)}$, and $\mathcal{D}_{\frac{1}{2}(0,\frac{1}{2})}$ multiplets (along with the conjugates of the last two). In addition, conformal descendants of dimension two can arise from holomorphic derivatives of the dimension one operators. Since no $\mathcal{D}$-type multiplets appear in this theory, the only quasi-primaries at dimension two will correspond to Higgs branch operators and the two-dimensional stress tensor.

The latter descends from the four-dimensional $SU(2)_R$ current. That current being bilinear in the free fields of the noninteracting theory, the corresponding two-dimensional operator can be obtained by simply replacing the four-dimensional fields with their chiral counterparts and conformally normal ordering,

$$T_{2d} = \tfrac{1}{2}q^i_a \partial q^{ia} - b^A \partial c_A \ . \tag{2.5.15}$$

Alternatively, this is just the canonical stress tensor for the combined system of free symplectic bosons and ghosts. Given the multiplicities of matter and ghost fields, the two-dimensional central charge is easily determined to be $c_{2d} = -14$.

The remaining BRST-invariant currents of dimension two can be constructed as normal ordered products and derivatives of the $\mathfrak{so}(8)$ affine currents,

$$\partial J^{[ij]} \ , \quad (J \otimes J)\big|_{\mathbf{1},\mathbf{35},\mathbf{35},\mathbf{35},\mathbf{300}} \ . \tag{2.5.16}$$

The singlet term in the tensor product above, once appropriately normalized, is the Sugawara stress tensor of the $\mathfrak{so}(8)$ affine Lie algebra,

$$T^{\mathfrak{so}(8)}_{\text{sug}} = \tfrac{1}{8}(J^{[ij]}J^{[ij]}) \ . \tag{2.5.17}$$

The Sugawara central charge is determined by the usual formula,

$$c_{\text{sug}} = \frac{k_{2d} \dim G_F}{k_{2d} + h^\vee} = -14 \ . \tag{2.5.18}$$

This matches the value for the canonical stress-tensor. This comes as no surprise, since the central charges of this theory saturate the unitarity bound (2.4.17), which implies that the canonical stress tensor should be equivalent to the Sugawara stress tensor. Indeed, (2.5.15) and (2.5.16) constitute an overcomplete list, and we in fact have the following relations,

$$
\begin{aligned}
J\otimes J\big|_{\mathbf{1}} &= T_{2d}+\{Q_{\mathrm{BRST}},q^i_a q^{ib} b^a_b\}\ , && \text{(2.5.19a)}\\
J\otimes J\big|_{\mathbf{35}_v} &= \{Q_{\mathrm{BRST}},q^{(i}_a q^{j)b} b^a_b\}\ , && \text{(2.5.19b)}\\
J\otimes J\big|_{\mathbf{35}_c} &= 0\ , && \text{(2.5.19c)}\\
J\otimes J\big|_{\mathbf{35}_s} &= 0\ , && \text{(2.5.19d)}
\end{aligned}
$$

The relations appearing here can be traced back to different aspects of the four-dimensional physics. Relations (2.5.19a) and (2.5.19b) are the two-dimensional avatars of the F-term relations in (2.5.5). Note that the first relation appears differently in this two-dimensional context due to the presence of the two-dimensional stress tensor on the right hand side. This is a remnant of the more complicated structure of normal ordering in the chiral algebra as compared to the chiral ring. Relations (2.5.19c) and (2.5.19d) are the tree-level relations. In the context of the chiral algebra, they can be seen as a simple consequence of Bose symmetry and normal ordering without making any reference to the BRST differential. This perfectly mirrors of the nature of tree-level relations in four dimensions.

A $\mathcal{W}$-algebra conjecture

Although the cohomological description of the chiral algebra is sufficient to compute the physical operators to any given level, it would be ideal to have a characterization entirely in terms of physical operators – for example, we may hope for a description as a $\mathcal{W}$ algebra. We have seen that the physical dimension two currents are all generated by the affine currents of dimension one, *i.e.*, the physical states enumerated so far all lie in the vacuum module of the $\mathfrak{so}(8)$ affine Lie algebra at level $k=-2$. What's more, these operators exhaust the list of operators that are guaranteed to be generators of the chiral algebra according to §2.3. We are thus led to a natural conjecture:

Conjecture 1 *When $\mathcal{T}$ is $\mathcal{N}=2$ $SU(2)$ SQCD with four fundamental flavors, then $\chi[\mathcal{T}]$ is isomorphic to the $\mathfrak{so}(8)$ affine Lie algebra at level $k_{2d}=-2$.*

This is a mathematically well-posed conjecture, since the cohomological characterization of the chiral algebra is entirely concrete. It seems plausible that a more sophisticated approach to the cohomological problem could lead to a proof of the conjecture. We will be satisfied in the present work to test it indirectly.

The superconformal index and affine characters

Conjecture 1 can be tested at the level of the indices of these theories. In particular, we have the following conjectural relationship

$$\mathcal{I}_{\text{Schur}}(q;\vec{a}) = \text{Tr}_{\chi[\mathcal{T}_{\text{free}}]}(-1)^F q^{L_0} \prod_{i=1}^{4} a_i^{\mu_i} = \text{Tr}_{\mathfrak{so}(8)_{-2}}(-1)^F q^{L_0} \prod_{i=1}^{4} a_i^{\mu_i} \ . \quad (2.5.20)$$

The shorthand $\vec{a} = (a_1, a_2, a_3, a_4)$ denotes the $SO(8)$ fugacities. Of course, the affine Lie algebra has only bosonic states, so the factor of $(-1)^F$ is immaterial. In particular this observation implies that if Conjecture 1 is correct, then all possible recombinations of tree-level Schur operators occur already at one loop.

On the one hand, the Schur limit of the superconformal index for this theory can be computed directly to fairly high orders in the q expansion by starting with the defining matrix integral,

$$\mathcal{I}_{\text{Schur}}(q;\vec{a}) = \oint [db] \text{P.E.} \left[\left(\frac{\sqrt{q}}{1-q} \right) \chi^{\mathbf{8}}_{SO(8)}(\vec{a}) \chi^{\mathbf{2}}_{SU(2)}(b) + \left(\frac{-2q}{1-q} \right) \chi^{\mathbf{3}}_{SU(2)}(b) \right], \quad (2.5.21)$$

and expanding the exponential. Here $\oint [db]$ denotes integration over the fugacity for the gauge group with the Haar measure.

On the other hand, the vacuum character of the $\mathfrak{so}(8)$ affine Lie algebra at level $k = -2$ can be computed once the spectrum of null primaries is known. Said spectrum can be determined with the aid of the Kazhdan-Lusztig polynomials, as we review in Appendix A.3. Ultimately, both the character and the index are expanded in the form

$$1 + \sum_{i=1}^{\infty} q^n \left(\sum_{R} d_{\mathcal{R}} \chi^{\mathcal{R}}(\vec{a}) \right) \ ,$$

where the $d_{\mathcal{R}}$ are positive integer multiplicities. At a given power of q, there are only a finite number of non-zero $d_{\mathcal{R}}$. Up to $O(q^5)$, the resulting degeneracies have been computed in both manners and agree. They are displayed in Table 2.5.

level	$SO(8)$ representations and their multiplicities
0	$\mathbf{1}$
1	$\mathbf{28}$
2	$\mathbf{1},\ \mathbf{28},\ \mathbf{300}$
3	$\mathbf{1},\ 2\times\mathbf{28},\ \mathbf{300},\ \mathbf{350},\ \mathbf{1925}$
4	$2\times\mathbf{1},\ 3\times\mathbf{28},\ \mathbf{35}_v,\ \mathbf{35}_s,\ \mathbf{35}_c,\ 3\times\mathbf{300},\ \mathbf{350},\ \mathbf{1925},\ \mathbf{4096},\ \mathbf{8918}$
5	$2\times\mathbf{1},\ 6\times\mathbf{28},\ \mathbf{35}_v,\ \mathbf{35}_s,\ \mathbf{35}_c,\ 4\times\mathbf{300},\ 3\times\mathbf{350},\ \mathbf{567}_v,\ \mathbf{567}_s,$
	$\mathbf{567}_c,\ 3\times\mathbf{1925},\ 2\times\mathbf{4096},\ \mathbf{8918},\ \mathbf{25725},\ \mathbf{32928}'$

Table 2.5: The operator content of the chiral algebra up to level 5.

2.5.2 $SU(N)$ superconformal QCD with $N \geqslant 3$

We next consider the generalization of the previous example to the case of $SU(N)$ superconformal QCD with $N \geqslant 3$. In these theories, the Higgs branch has generators of dimension greater than two, thus guaranteeing the existence of nonlinear $\mathcal{W}$-symmetry generators in the chiral algebra. The cohomological construction of the corresponding chiral algebra is analogous to the $SU(2)$ case, *mutatis mutandi*. We will not repeat the description here in any detail. We first provide a brief outline of the relevant four-dimensional physics of these models, and then perform a systematic analysis of the physical operators of low dimension in the associated chiral algebra.

As in the $SU(2)$ theory, there is a Lagrangian description of these models in terms of the $\mathcal{N}=1$ chiral superfields

$$W_\alpha^A\ ,\quad \Phi^B\ ,\quad Q_a^i\ ,\quad \widetilde{Q}_j^b\ , \tag{2.5.22}$$

where $a,b=1,\dots,N$ are vector color indices, $A,B=1,\dots,N^2-1$ are adjoint color indices, and $i,j=1,\dots,N_f$ with $N_f=2N$ are vector flavor indices. The central charge is fixed by the field content to $c_{4d}=\frac{2N^2-1}{6}$.

For our purposes, the principal difference between the $N \geqslant 3$ theories and the $N=2$ case is in the structure of the Higgs branch chiral ring. In the higher rank theories, the hypermultiplets transform in a complex representation of the gauge group, so the global symmetry is not enhanced and we have

$G_F = SU(N_f) \times U(1)$. The moment map operators for the global symmetry reside in mesonic $\hat{\mathcal{B}}_1$ multiplets, which can be separated into $SU(N_f)$ and $U(1)$ parts,

$$M_j{}^i := \widetilde{Q}_j^a Q_a^i \quad \Longrightarrow \qquad \mu := M_i{}^i \ , \quad \mu_j{}^i := M_j{}^i - \frac{1}{N_f}\mu\,\delta_j{}^i \ . \tag{2.5.23}$$

The level of the non-Abelian part of the global symmetry is $k_{4d}^{SU(N_f)} = 2N$. The baryons are of dimension N and no longer generate any additional global symmetries. Rather, they transform in the N-fold antisymmetric tensor representations of the flavor symmetry:

$$\begin{aligned} B^{i_1 \ldots i_N} &:= Q_{a_1}^{i_1} \cdots Q_{a_N}^{i_N} \epsilon^{a_1 \ldots a_N} \ , \\ \widetilde{B}_{i_1 \ldots i_N} &:= \widetilde{Q}_{i_1}^{a_1} \cdots \widetilde{Q}_{i_N}^{a_N} \epsilon_{a_1 \ldots a_N} \ . \end{aligned} \tag{2.5.24}$$

The mesons and baryons satisfy a set of polynomial relations. Following [69], we introduce notation where "$\cdot$" denotes contraction of an upper and a lower index and "$*$" denotes the contraction of flavor indices with the completely antisymmetric tensor in N_f indices. The relations are then given by

$$\begin{aligned} (*B)\widetilde{B} \ &= \quad *(M^N) \ , \qquad & M \cdot *B \ &= \ M \cdot *\widetilde{B} = 0 \ , \\ M' \cdot B \ &= \ \widetilde{B} \cdot M' \ = \ 0 \ , \qquad & M \cdot M' \ &= \ 0 \ , \end{aligned} \tag{2.5.25}$$

where $(M')_i{}^j := M_i{}^j - \frac{1}{N}\mu\delta_i^j = \mu_i{}^j - \frac{1}{2N}\mu\delta_i^j$. Additionally, all quantities antisymmetrized in more than N flavor indices must vanish.

This completes the description of the Hall-Littlewood chiral ring, since again this theory admits a linear quiver description, so there are no $\mathcal{D}$-type multiplets after turning on interactions. The final representation of canonical interest is the $\hat{\mathcal{C}}_{0(0,0)}$ multiplet, which again contributes an important Schur operator in the form of the $R = 1$ component of the $SU(2)_R$ current:

$$\mathcal{J}_{+\dot{+}}^{R=1} \sim \frac{1}{2}\left(Q_a^i \partial_{+\dot{+}} \widetilde{Q}_i^a - \widetilde{Q}_i^a \partial_{+\dot{+}} Q_a^i\right) + \lambda_+^A \tilde{\lambda}_{\dot{+}A} \ . \tag{2.5.26}$$

Like the $SU(2)$ theory, these models all have one-complex-dimensional conformal manifolds with interesting behaviors at the boundary points, where S-dual descriptions become appropriate. In contrast to the $SU(2)$ theory, these S-dual descriptions are not the same as the original description, and rather involve intrinsically strongly-coupled non-Lagrangian sectors. While such dualities imply interesting structures for the associated chiral algebras, their dependence on non-Lagrangian theories takes us outside the scope of the current examples. This is discussed in much greater detail in chapter 3.

Physical operators of low dimension

The nontrivial BRST cohomology classes of the chiral algebra can be computed by hand for small values of the dimension. The physical operators of dimension one again correspond to the moment map operators of the global symmetry, which in this case includes only the mesonic chiral ring operators,

$$J_i^j \;:=\; q_{ai}\tilde{q}^{aj} - \frac{1}{N_f}\delta_i^j q_{ak}\tilde{q}^{ak} = \chi[\mu_i^j] \;, \tag{2.5.27}$$

$$J \;:=\; q_{ak}\tilde{q}^{ak} = \chi[\mu] \;. \tag{2.5.28}$$

The singular OPEs of these currents are given by

$$\begin{aligned} J_i^j(z)J_k^l(0) \quad &\sim \quad -\frac{N(\delta_i^l\delta_k^j - \text{trace})}{z^2} \;+\; \frac{\delta_i^l J_k^j(z) - \delta_k^j J_i^l(z)}{z} \;, \\ J(z)J(0) \quad &\sim \quad -\frac{2N^2}{z^2} \;. \end{aligned} \tag{2.5.29}$$

This is an $\mathfrak{su}(N_f)\times\mathfrak{u}(1)$ affine Lie algebra at level $k_{2d} = -N$.

At dimension two, we first consider the operators that are invariant under the flavor symmetry. As expected, there is a canonical stress tensor,

$$T := \frac{1}{2}\left(q_{ai}\partial\tilde{q}^{ai} - \tilde{q}^{ai}\partial q_{ai}\right) - b_b^a\partial c_a^b = \chi[\mathcal{J}^1_{++}] \;, \tag{2.5.30}$$

whose self-OPE fixes the two-dimensional central charge,

$$c_{2d} = 2 - 4N^2 \;. \tag{2.5.31}$$

Additionally, the algebra generated by the affine $\mathfrak{su}(N_f)\times\mathfrak{u}(1)$ currents (2.5.27) contains a dimension two singlet that is the Sugawara stress tensor of the current algebra,

$$T_{\text{sug}} := \frac{1}{N_f}\left(J_i^jJ_j^i - \frac{1}{N_f}JJ\right) \;. \tag{2.5.32}$$

The corresponding Sugawara central charge is also equal to $2-4N^2$, which suggests that the two stress tensors T and T^{sug} may be equivalent operators as they were in the $N=2$ theory. Indeed, we expect this to be the case since the central charges in this theory again saturate the unitarity bound (2.4.17). A short computation verifies that their difference is BRST exact,

$$T - T_{\text{sug}} = \frac{1}{N_f}\{Q_{\text{BRST}}, q_{ai}\tilde{q}^{bj}b_b^a\} \;. \tag{2.5.33}$$

A complete basis for the physical flavor singlets of dimension two is given by T, JJ, and ∂J.

The remaining physical operators of dimension two are charged under $U(N_f)$. An overcomplete basis of such operators is given by flavored current bilinears $J_i^j J_k^l$ and $J_i^j J$, in addition to derivatives of the currents ∂J_i^j. These operators are not all independent. For example, the usual rules of conformal normal ordering imply that

$$J_i^j J_k^l - J_k^l J_i^j = \delta_i^l \partial J_k^j - \delta_k^j \partial J_i^l \ , \tag{2.5.34}$$

so the antisymmetric normal ordered product of two $SU(N_f)$ currents is a combination of descendants. For the symmetrized normal ordered product there exists another relation:

$$\frac{1}{2}(J_i^k J_k^j + J_k^j J_i^k) = \delta_i^j \left(\frac{1}{N_f^2} JJ + T \right) - \{Q_{\text{BRST}}, q_{\alpha i} \tilde{q}^{\beta j} b_\beta^\alpha\} \, . \tag{2.5.35}$$

In group-theoretic terms, the relations amount to the statement that the parts of the symmetric product of two currents that transform in the singlet and adjoint representations do not correspond to independent operators.

It is worth jumping ahead to the case of dimension $N/2$, where we find operators that correspond to the baryonic chiral ring generators (2.5.24):

$$\begin{aligned} b_{i_1 i_2 \ldots i_{N_c}} &:= \varepsilon^{\alpha_1 \alpha_2 \ldots \alpha_{N_c}} q_{\alpha_1 i_1} q_{\alpha_2 i_2} \ldots q_{\alpha_{N_c} i_{N_c}} = \chi[B_{i_1 i_2 \cdots i_N}] \ , \\ \tilde{b}^{i_1 i_2 \ldots i_{N_c}} &:= \varepsilon_{\alpha_1 \alpha_2 \ldots \alpha_{N_c}} \tilde{q}^{\alpha_1 i_1} \tilde{q}^{\alpha_2 i_2} \ldots \tilde{q}^{\alpha_{N_c} i_{N_c}} = \chi[\tilde{B}^{i_1 i_2 \cdots i_N}] \ . \end{aligned} \tag{2.5.36}$$

These are Virasoro primaries of dimension $N_f/4$. The only non-trivial OPE that is not entirely fixed by symmetry is the $b \times \tilde{b}$ OPE. For $N_c = 3$, for example, it is given by

$$\begin{aligned} b_{i_1 i_2 i_3}(z) \tilde{b}^{j_1 j_2 j_3}(0) \sim & \frac{36\, \delta_{[i_1}^{[j_1} \delta_{i_2}^{j_2} \delta_{i_3]}^{j_3]}}{z^3} - \frac{36\, \delta_{[i_1}^{[j_1} \delta_{i_2}^{j_2} J_{i_3]}^{j_3]}(0)}{z^2} \\ & + \frac{18\, \delta_{[i_1}^{[j_1} J_{i_2}^{j_2} J_{i_3]}^{j_3]}(0) - 18\, \delta_{[i_1}^{[j_1} \delta_{i_2}^{j_2} \partial J_{i_3]}^{j_3]}(0)}{z} \ , \end{aligned} \tag{2.5.37}$$

where square brackets denote antisymmetrization with weight one.

Relation to the Higgs branch chiral ring

Again, certain features of the Higgs branch chiral ring arise organically from the chiral algebra. According to the general discussion in §2.3.2, the dimension two operators in the chiral algebra should in particular contain the image of the Schur operators in $\hat{\mathcal{B}}_2$ multiplets, which in the theories under consideration simply correspond to the product of two of the mesonic operators μ and μ_i^j subject to the final relation in (2.5.25). Furthermore, these Schur operators necessarily become Virasoro primary operators in the chiral algebra.

From amongst the BRST cohomology classes at level two – spanned by T, JJ, $J_i^j J$, the symmetrized combination $J_i^j J_k^l + J_k^l J_i^j$ modulo relation (2.5.35), and derivatives of level one currents – we find exactly three Virasoro primary operators:

$$\begin{aligned}
\mathcal{X} &:= JJ - \frac{N_f^2}{N_f^2-2}T\,, \\
\mathcal{X}_i^j &:= J_i^j J\,, \\
\mathcal{X}_{ik}^{jl} &:= \frac{1}{2}(J_i^j J_k^l + J_k^l J_i^j) - \frac{N_f}{N_f^2-2}\left(\delta_i^l\delta_k^j - \frac{1}{N_f}\delta_i^j\delta_k^l\right)T\,,
\end{aligned} \tag{2.5.38}$$

which are subject to the additional constraints,

$$\mathcal{X}_{ik}^{jl} = \mathcal{X}_{ki}^{lj}\,, \qquad \mathcal{X}_{ik}^{il} = 0\,, \qquad \mathcal{X}_{ij}^{jk} = \frac{1}{N_f^2}\delta_i^k\mathcal{X} + \{Q_{\rm BRST}, \ldots\}\,. \tag{2.5.39}$$

We see that we should identify $\mathcal{X} = \chi[\,\mu\mu\,]$, $\mathcal{X}_i^j = \chi[\,\mu\mu_i^j\,]$ and $\mathcal{X}_{ik}^{jl} = \chi[\,\mu_i^j\mu_k^l\,]$. The first two relations in (2.5.39) then reflect the natural symmetry properties of the original Schur operator, whilst the last equation precisely reproduces the final equation in (2.5.25).

We note that the definitions (2.5.38) somewhat obscure the relationship to four-dimensional physics because of the conformal normal ordering used to define the products of interacting fields. The same dimension two operators take a completely natural form in terms of creation/annihilation normal ordered products of symplectic bosons,

$$\begin{aligned}
\mathcal{X} &= \quad : q_{\alpha i}\tilde{q}^{\alpha i} q_{\beta j}\tilde{q}^{\beta j} : \;, \\
\mathcal{X}_i^j &= \quad : q_{\alpha i}\tilde{q}^{\alpha j} q_{\beta k}\tilde{q}^{\beta k} : \;, \\
\mathcal{X}_{ik}^{jl} &= \quad : q_{\alpha i}\tilde{q}^{\alpha j} q_{\beta k}\tilde{q}^{\beta l} : \;,
\end{aligned} \tag{2.5.40}$$

and this description also nicely illustrates the commutative diagram of §2.3.3.

Finally, at the level of Virasoro representations, the OPEs of the dimension one currents can now be summarized by the following fusion rules,

$$\begin{aligned} J_i^j \times J_k^l \quad &\rightarrow \quad - N(\delta_i^l \delta_k^j - \text{trace})\mathbb{1} + (\delta_i^l J_k^j - \delta_k^j J_i^l) + \mathcal{X}_{ik}^{jl} + \dots \ , \\ J_i^j \times J \quad &\rightarrow \quad \mathcal{X}_i^j + \dots \ , \\ J \ \times J \quad &\rightarrow \quad - 2N^2 \mathbb{1} + \mathcal{X} + \dots \ , \end{aligned} \tag{2.5.41}$$

where we have omitted operators of dimension higher than two. We see that the product structure of the Higgs branch chiral ring is reproduced precisely by the $O(1)$ terms in these fusion rules.[26]

A $\mathcal{W}$-algebra conjecture

The chiral algebra is not as simple in this case as it was for the $SU(2)$ theory, since the generators b and $\tilde{b}$ are higher-spin $\mathcal{W}$-symmetry generators rather than simple affine currents. Nevertheless, there is a natural guess as to how to describe this more involved theory as a $\mathcal{W}$ algebra. It is useful to think of the operator content of the algebra in terms of representations of the affine $\mathfrak{u}(N_f)$ current algebra. From the analysis of levels one and two, we know that there is the vacuum representation – which in particular contains the affine currents and the stress tensor – and the "baryonic" representations, for which the highest weight state is given by the baryon or anti-baryon of (2.5.36). Other representations of the affine Lie algebra can only come from multi-baryon states or from new generators of dimension greater than two, where we have not performed a detailed analysis of the cohomology.

In four dimensions the mesons and the baryons are the complete set of generators for the Hall-Littlewood chiral ring. The most obvious conjecture is then that the corresponding two-dimensional operators generate the entire $\mathcal{W}$-algebra:

Conjecture 2 *When $\mathcal{T}$ is $\mathcal{N}=2$ $SU(N)$ superconformal QCD for with $2N$ flavors for $N>2$, then $\chi[\mathcal{T}]$ is isomorphic to the $\mathcal{W}$ algebra generated by affine $\mathfrak{u}(N_f)$ currents at level $k_{\mathfrak{su}(N_f)}=-N$ along with baryonic generators b and $\tilde{b}$ with the OPE* (2.5.37) *(or its generalizations to $N \geqslant 4$).*

[26] We may similarly speculate that the Poisson bracket is encoded in the terms of the OPE that correspond to simple poles, but we have not checked this in detail.

Because no additional generators make an appearance in the singular OPEs of the affine currents and baryons, it is guaranteed to be the case that the $\mathcal{W}$ algebra we have just described forms a chiral subalgebra of $\chi[\mathcal{T}]$. Our conjecture is that this is in fact the whole thing. If true, this conjecture would imply that the Schur index for the $N_f = 2N$ theories decomposes into characters of affine $\mathfrak{u}(2N)_{-N}$ with highest weights given by the vacuum or by one or more baryons.

Superconformal Index

We can provide support for this conjecture by comparing with the superconformal index. The Schur index of the theory is given by the following contour integral,

$$\mathcal{I}_{\text{Schur}}(q; c, \vec{a}) = \tag{2.5.42}$$

$$\int [d\vec{b}] P.E. \left[\frac{\sqrt{q}}{1-q} \left(c\, \chi^{\mathbf{N_f}}_{SU(N_f)}(\vec{a}) \chi^{\mathbf{N}}_{SU(N)}(\vec{b}) + c^{-1}\, \chi^{\mathbf{N_f}}_{SU(N_f)}(\vec{a}^{-1}) \chi^{\mathbf{N}}_{SU(N)}(\vec{b}^{-1}) \right) + \left(\frac{-2q}{1-q} \right) \chi^{\mathbf{N^2-1}}_{SU(N)}(\vec{b}) \right], \tag{2.5.43}$$

where c is the $U(1)$ fugacity and $\vec{a} = (a_1, a_2, \ldots, a_{N_f-1})$ denotes $SU(N_f)$ fugacities. For $N = 3$, the first few orders are given by

$$\begin{aligned} \mathcal{I}_{\text{Schur}}(q; c, \vec{a}) =& 1 + \left(1 + \chi^{\mathbf{35}}_{SU(6)}(\vec{a})\right) q + (c^3 + c^{-3}) \chi^{\mathbf{20}}_{SU(6)}(\vec{a}) q^{3/2} \\ &+ \left(\left(\chi^{sym^2(\mathbf{35})}_{SU(6)}(\vec{a}) - \chi^{\mathbf{35}}_{SU(6)}(\vec{a}) \right) + 2\chi^{\mathbf{35}}_{SU(6)}(\vec{a}) + 2 \right) q^2 \\ &+ (c^3 + c^{-3}) \left(2\chi^{\mathbf{20}}_{SU(6)}(\vec{a}) + \left(\chi^{\mathbf{35}\otimes\mathbf{20}}_{SU(6)}(\vec{a}) - \chi^{\mathbf{20}\oplus\mathbf{70}\oplus\overline{\mathbf{70}}}_{SU(6)}(\vec{a}) \right) \right) q^{5/2} \\ &+ \ldots, \end{aligned} \tag{2.5.44}$$

where we have explicitly indicated the presence of relations by listing them with a minus sign. The dimension two relations in the chiral algebra were elaborated upon in the previous subsection. At level 5/2, we can similarly determine the Virasoro primaries

$$Y_{ijk} = J b_{ijk} + \partial b_{ijk}, \qquad \widetilde{Y}^{ijk} = J \widetilde{b}^{ijk} - \partial \widetilde{b}^{ijk} \tag{2.5.45}$$

$$Y^j_{i,klm} = \frac{1}{2} \left(J_i{}^j b_{klm} + b_{klm} J_i{}^j - \frac{1}{6} \delta^j_i \partial b_{klm} + \delta^j_{[k} \partial b_{|i|lm]} \right) \tag{2.5.46}$$

$$\widetilde{Y}^{j,klm}_i = \frac{1}{2} \left(J_i{}^j \widetilde{b}^{klm} + \widetilde{b}^{klm} J_i{}^j + \frac{1}{6} \delta^j_i \partial \widetilde{b}^{klm} - \delta^{[k}_i \partial \widetilde{b}^{|j|lm]} \right), \tag{2.5.47}$$

subject to the constraints

$$\epsilon^{iklmnp}\left(Y^j_{i,mnp}+\frac{1}{6}\delta^j_i Y_{mnp}\right)=0, \qquad Y^j_{i,jlm}-\frac{1}{6}Y_{ilm}=\{Q_{\rm BRST},\ldots\} \tag{2.5.48}$$

$$\epsilon_{jklmnp}\left(\widetilde{Y}_i^{j,mnp}+\frac{1}{6}\delta^j_i \widetilde{Y}^{mnp}\right)=0, \qquad \widetilde{Y}_i^{j,kli}-\frac{1}{6}\widetilde{Y}^{jkl}=\{Q_{\rm BRST},\ldots\}\,, \tag{2.5.49}$$

which again encode precisely the Higgs branch relations.

At level three, we have checked agreement between the Schur index and the cohomology generated by the $SU(6)\times U(1)$ currents and the baryons by explicitly computing the null states.

2.5.3 $\mathcal{N}=4$ supersymmetric Yang-Mills theory

The theories considered in the previous two subsections all shared the special quality of admitting descriptions as linear quiver gauge theories, which meant that $\mathcal{D}$-type multiplets played no role in the analysis. We now turn to a case where this simplification no longer holds, and so there will necessarily be generators outside of the Higgs chiral ring. The theory in question is $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$. For our purposes, this is an $\mathcal{N}=2$ theory with an $SU(N)$ vector multiplet and a single adjoint-valued hypermultiplet. In $\mathcal{N}=1$ notation, we have the following chiral superfields,

$$W^A_\alpha\ , \qquad \Phi^A\ , \qquad Q^A_i\ , \tag{2.5.50}$$

where $A=1,\ldots N^2-1$ an $SU(N)$ adjoint index and $i=1,2$ is an $SU(2)_F$ vector index. The flavor symmetry $SU(2)_F$ is the commutant of $SU(2)_R\times U(1)_r\subset SU(4)_R$, and so is an R-symmetry with respect to the full superalgebra. The central charges of the theory are given by

$$k^{SU(2)}_{4d}=N^2-1\ , \quad c_{4d}=\frac{(N^2-1)}{4}\ . \tag{2.5.51}$$

The Higgs branch chiral ring has $N-1$ generators. In terms of the $N\times N$ matrices $Q_i:=Q^A_i t^A$, these are given by

$$\operatorname{Tr} Q_{(i_1}\cdots Q_{i_k)}\ , \qquad k=1,\ldots,N-1\ , \tag{2.5.52}$$

subject to trace relations. In this theory, the Hall-Littlewood chiral ring contains additional $\mathcal{D}$-type multiplets that are not described by the Higgs chiral ring. More specifically, for $SU(N)$ gauge group there are an additional $N-1$ HL generators given by

$$\mathrm{Tr}\, Q_{(i_1} \cdots Q_{i_k)} \tilde{\lambda}^1_{\dot{+}} \ , \qquad k = 1, \ldots, N-1 \ . \tag{2.5.53}$$

There are corresponding generators of the HL anti-chiral ring that lie in $\overline{\mathcal{D}}$ multiplets and take the same form with $\tilde{\lambda}^1_{\dot{+}}$ replaced by λ^1_{+}. Finally, the Schur component of the $SU(2)_R$ current, which will give rise to the stress tensor in two-dimensions, is given in terms of four-dimensional fields by

$$\mathcal{J}^{R=1}_{+\dot{+}} \sim \frac{1}{2} \mathrm{Tr}\, Q_i \partial_{+\dot{+}} Q_j \varepsilon^{ij} - \mathrm{Tr}\, \tilde{\lambda}_{\dot{+}} \lambda_{+} \ . \tag{2.5.54}$$

Cohomological description of the associated chiral algebra

The free chiral algebra follows the same pattern as the previous examples. The two dimensional counterparts of the hypermultiplet scalars and gauginos can be introduced as usual,

$$q_i^A(z) := \chi[Q_i^A] \ , \qquad b^A(z) := \chi[\tilde{\lambda}^A] \ , \qquad \partial c^A(z) := \chi[\lambda^A] \ . \tag{2.5.55}$$

The free chiral algebra has the free OPEs,

$$q_i^A(z) q_j^B(0) \sim \frac{\varepsilon_{ij} \delta^{AB}}{z} \ , \qquad b^A(z) c^B(0) \sim \frac{\delta^{AB}}{z} \ .$$

The stress tensor is given by the usual canonical expression

$$T = \frac{1}{2} q_i^A \partial q_j^B \ \varepsilon^{ij} - b^A \partial c^A \ , \tag{2.5.56}$$

which has a central charge of $c_{2d} = -3(N^2-1)$. The $SU(2)_F$ currents are given by

$$J_{ij} = -\frac{1}{2} q_i^A q_j^A \ , \tag{2.5.57}$$

and satisfy a current algebra at level $k_{2d} = -\frac{N^2-1}{2}$. The current algebra contains a Sugawara stress tensor of the usual form,

$$T_{\mathrm{Sug}}(z) = \frac{1}{N^2-5} \ J_{ij} J_{kl} \ \varepsilon^{ik} \varepsilon^{jl} \ , \tag{2.5.58}$$

with central charge equal to $\frac{3(N^2-1)}{N^2-5}$. Note that precisely for $N = 2$ and for no other value of N, the Sugawara central charge matches with the true central charge. As we will see, this is again a consequence of the two stress tensors being equivalent in BRST cohomology.

The $SU(N)$ currents for the matter and ghost sectors are given by

$$J^A = \frac{i}{2} f^{ABC} q_i^B q_j^C \, \varepsilon^{ij} \, , \qquad J_{\text{gh}}^A = -i f^{ABC} c^B b^C \, . \tag{2.5.59}$$

The levels for the corresponding current algebras are $-2N$ and $2N$, respectively. The BRST current is constructed as usual,

$$J_{\text{BRST}} = c^A \left(J_{SU(N)}^A + \frac{1}{2} J_{\text{gh}}^A \right) \, , \tag{2.5.60}$$

and its zero mode defines the nilpotent BRST operator Q_{BRST}.

Low-lying physical states

Let us first consider the case of $SU(2)$ gauge group. In this case the difference between the Sugawara stress tensor and the canonical stress tensor is BRST exact,

$$T - T_{\text{Sug}} \quad \sim \quad \{ Q_{\text{BRST}} \, , \, f^{ABC} q_i^A q_j^B b^C \, \varepsilon^{ij} \} \, . \tag{2.5.61}$$

Based on the description of the HL chiral ring generators, we expect that amongst the physical states should be an $SU(2)_F$ triplet of affine currents and an $SU(2)_F$ doublet of dimension $3/2$ fermionic generators. Up to dimension two, the cohomology is generated by precisely these operators,

$$\begin{aligned} J_{ij} = & \quad -\frac{1}{2}(q_i^A q_j^A) = \chi[\operatorname{Tr} Q_i Q_j] \, , \\ G_i := & \quad \sqrt{2}(q_i^A b^A) = \chi[\operatorname{Tr} Q_i \tilde{\lambda}_+] \, , \\ \tilde{G}_i := & -\sqrt{2}(q_i^A \partial c^A) = \chi[\operatorname{Tr} Q_i \lambda_+] \, . \end{aligned} \tag{2.5.62}$$

The OPEs of these generators can be computed directly,

$$J_{ij}(z)J_{kl}(w) \sim -\frac{N^2-1}{2}\frac{\varepsilon_{l(i}\varepsilon_{j)k}}{(z-w)^2} + \frac{2\varepsilon_{(k(i}J_{j)l)}}{z-w}\,, \tag{2.5.63}$$

$$J_{ij}(z)G_k(w) \sim \frac{\frac{1}{2}(\varepsilon_{ki}G_j(w)+\varepsilon_{kj}G_i(w))}{z-w}\,, \tag{2.5.64}$$

$$J_{ij}(z)\tilde{G}_k(w) \sim \frac{\frac{1}{2}(\varepsilon_{ki}\tilde{G}_j(w)+\varepsilon_{kj}\tilde{G}_i(w))}{z-w}\,, \tag{2.5.65}$$

$$G_i(z)G_j(w) \sim 0\,, \tag{2.5.66}$$

$$\tilde{G}_i(z)\tilde{G}_j(w) \sim 0\,, \tag{2.5.67}$$

$$G_i(z)\tilde{G}_j(w) \sim -\frac{2(N^2-1)\varepsilon_{ij}}{(z-w)^3} + \frac{4J_{ij}(w)}{(z-w)^2} + \frac{2\varepsilon_{ij}T(w)+2\partial J_{ij}(w)}{z-w}\,, \tag{2.5.68}$$

where $N=2$ and the symmetrization in the indices i,j and k,l has weight one. The value of N has been left unspecified in (2.5.63) because the OPEs will continue to hold for higher rank gauge groups. For the same reason, $T(z)$ has been included separately, though for $N=2$ it not a distinct generator, but rather is identified with the Sugawara stress tensor.

The operator product algebra in (2.5.63) can be immediately recognized to be the "small" $\mathcal{N}=4$ superconformal algebra with central charge $c_{2d} = -3(N^2-1)$ [70]. It is natural that there should be supersymmetry acting in the chiral algebra, since the holomorphic $\mathfrak{sl}(2)$ that commutes with the supercharges $\mathbb{Q}_i$ is in enhanced to a holomorphic $\mathfrak{sl}(2\,|\,2)$ when the four-dimensional theory is $\mathcal{N}=4$ supersymmetric. However, like the case of the global conformal algebra being generated not by the four-dimensional stress tensor but by the chiral operator associated to the $SU(2)_R$ current, here the enhanced supersymmetry in the chiral algebra is generated *not* by the four-dimensional supercurrents, but by the Schur operators that lie in the same $\mathcal{D}_{\frac{1}{2}(0,0)}$ and $\overline{\mathcal{D}}_{\frac{1}{2}(0,0)}$ multiplets with them. Those are the Schur operators that are transmuted into the two-dimensional supercurrents G_i and $\tilde{G}_i$.

In $SU(3)$ theory there are additional generators arising from the additional HL generators. Sure enough, direct computation produces the following list of new generators of dimension less than or equal to 5/2:

$$\begin{aligned}
B_{ijk} &:= \operatorname{Tr} q_i q_j q_k &&= \chi[\operatorname{Tr} Q_i Q_j Q_k]\ , \\
B_{ij} &:= \operatorname{Tr} q_i q_j b &&= \chi[\operatorname{Tr} Q_i Q_j \tilde{\lambda}_+]\ , \\
\tilde{B}_{ij} &:= \operatorname{Tr} q_i q_j \partial c &&= \chi[\operatorname{Tr} Q_i Q_j \lambda_{\dot{+}}]\ , \\
B_i &:= 3\operatorname{Tr} q_i b \partial c + \operatorname{Tr} \partial q_j q^j q_i &&= \chi[3\operatorname{Tr} Q_i \tilde{\lambda}_+ \lambda_{\dot{+}} + \operatorname{Tr} \partial_{+\dot{+}} Q_j Q^j Q_i]\ .
\end{aligned} \tag{2.5.69}$$

Precisely for the $SU(3)$ case, the operator B_i is in fact equivalent to a composite operator,

$$B_i \sim \varepsilon^{jj'} \varepsilon^{kk'} J_{jk} B_{ij'k'}\ . \tag{2.5.70}$$

This is a consequence of a chiral ring relation for this value of N which sets $\varepsilon^{jj'}\varepsilon^{kk'} \operatorname{Tr} Q_j Q_k \operatorname{Tr} Q_i Q_{j'} Q_{k'}$ to zero. This will not be the case for higher rank gauge groups, and B_i will be an authentic generator of the algebra.

A super $\mathcal{W}$-algebra conjecture

Because the chiral algebras of $\mathcal{N}=4$ SYM theories are supersymmetric, we can introduce a more restrictive notion of generators for these algebras. More precisely, we would like to identify those operators that generate the chiral algebra under the operations of normal ordered products and *super*derivatives, or the action of $\mathfrak{sl}(2\,|\,2)$. In other words, we allow not just L_1 descendants, but also $G_{i,-\frac{1}{2}}$ and $\tilde{G}_{i,-\frac{1}{2}}$ descendants.

The last three generators in (2.5.69) are superdescendants of B_{ijk}, so we have really only found one additional super-generator in the $SU(3)$ theory. In general, HL operators will be grouped by $\mathcal{N}=4$ supersymmetry into multiplets comprised of a single $\hat{\mathcal{B}}$-type operator, an $SU(2)_F$ doublet of $\mathcal{D}$-type operators, and an $SU(2)_F$ doublet worth of $\bar{\mathcal{D}}$-type operators.

For a general simple gauge group, the natural guess is that the chiral algebra is generated by the small $\mathcal{N}=4$ superconformal algebra along with additional chiral primary operators arising from the Higgs chiral ring generators. Our conjecture is then the following:

Conjecture 3 *The chiral algebra for $\mathcal{N}=4$ SYM theory with gauge group G is isomorphic to an $\mathcal{N}=4$ super $\mathcal{W}$-algebra with* $\operatorname{rank}(G)$ *generators given by chiral primaries of dimensions $\frac{d_i}{2}$, where d_i are the degrees of the Casimir invariants of G.*

We now perform some tests of this conjecture at the level of the superconformal index.

The superconformal index

Conjecture 3 can be tested up to any given level by comparing the index of the chiral algebra defined in the conjecture with the superconformal index of $\mathcal{N}=4$ SYM in the Schur limit. For gauge group $SU(N)$, the Schur index is given by a contour integral,

$$\mathcal{I}_{\text{Schur}}(q;a) = \oint [d\vec{b}]\text{P.E.}\left[\left(\frac{\sqrt{q}}{1-q}\right)\chi^{\mathbf{2}}(a)\chi^{\mathbf{N^2-1}}(\vec{b}) + \left(\frac{-2q}{1-q}\right)\chi^{\mathbf{N^2-1}}(\vec{b})\right], \tag{2.5.71}$$

where a is an $SU(2)_F$ flavor fugacity. For $SU(2)$ gauge group, expanding the integrand in powers of q and integrating gives the following result up to $O(q^4)$, where we have collected terms into $SU(2)_F$ characters $\chi^{\mathbf{R}}(a)$,

$$\begin{aligned}\mathcal{I}_{\text{Schur}}(q;a) = {} & 1 + \chi^{\mathbf{3}}(a)q - 2\chi^{\mathbf{2}}(a)q^{3/2} + \left(\chi^{\mathbf{1}}(a) + \chi^{\mathbf{3}}(a) + \chi^{\mathbf{5}}(a)\right)q^2 \\ & - 2\left(\chi^{\mathbf{2}}(a) + \chi^{\mathbf{4}}(a)\right)q^{5/2} + \left(\chi^{\mathbf{1}}(a) + 3\chi^{\mathbf{3}}(a) + \chi^{\mathbf{5}}(a) + \chi^{\mathbf{7}}(a)\right)q^3 \\ & - \left(4\chi^{\mathbf{2}}(a) + 4\chi^{\mathbf{4}}(a) + 2\chi^{\mathbf{6}}(a)\right)q^{7/2} \\ & + \left(3\chi^{\mathbf{1}}(a) + 7\chi^{\mathbf{3}}(a) + 4\chi^{\mathbf{5}}(a) + \chi^{\mathbf{7}}(a) + \chi^{\mathbf{9}}(a)\right)q^4 + \ldots \ .\end{aligned} \tag{2.5.72}$$

We can compare this result with the index of the $\mathcal{W}$-algebra appearing in the conjecture (in this case, just the small superconformal algebra with the appropriate value of the central charge) by enumerating the states of the chiral algebra and then finding and subtracting the null states at each level. We have checked up to level four, and the results match exactly.

The same comparison can be done for the $SU(3)$ case, where the Schur index to $O(q^3)$ is given by

$$\begin{aligned}&\mathcal{I}_{\text{Schur}}(q;a) = \\ &1 + \chi^{\mathbf{3}}(a)q + \left(\chi^{\mathbf{4}}(a) - 2\chi^{\mathbf{2}}(a)\right)q^{3/2} + \left(2\chi^{\mathbf{1}}(a) + \chi^{\mathbf{5}}(a) - \chi^{\mathbf{3}}(a)\right)q^2 \\ &+ \left(\chi^{\mathbf{6}}(a) - 3\chi^{\mathbf{2}}(a)\right)q^{5/2} + \left(5\chi^{\mathbf{1}}(a) + \chi^{\mathbf{3}}(a) + 2\chi^{\mathbf{7}}(a) - 3\chi^{\mathbf{5}}(a)\right)q^3 + \ldots \ .\end{aligned} \tag{2.5.73}$$

Up to level three the nulls were computed and they agree with the index. Note that in this case there are cancellations in the index of the chiral algebra, since there are bosonic and fermionic states appearing at the same level.

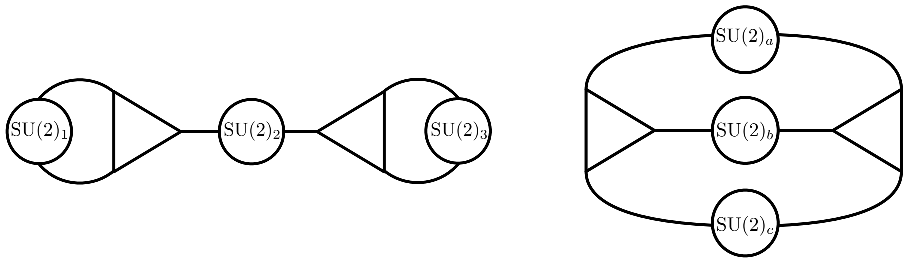


Figure 2.1: Weak coupling limits of the genus two class $\mathcal{S}$ theory.

2.5.4 Class $\mathcal{S}$ at genus two

At this point, the reader may be starting to get the impression that the chiral algebra of any four-dimensional theory be entirely determined by the structure of its various chiral rings. The purpose of this next example is to show that such a simplistic picture is untenable.

Our example is the rank one class $\mathcal{S}$ theory associated to an unpunctured genus two Riemann surface [10, 11]. The theory admits two inequivalent weak-coupling limits, or S-duality frames, corresponding to the two generalized quiver constructions illustrated in Fig. 2.1. We will focus on the first case, which is sometimes called the dumbbell quiver. The gauge groups are denoted $SU(2)_1$ for the left loop, $SU(2)_2$ for central line, and $SU(2)_3$ for the right loop. The fields of the theory are two sets of half-hypermultiplets transforming in the trifundamental representation of $SU(2)^3$ and three $SU(2)$ vector multiplets. In $\mathcal{N}=1$ notation, we denote these by

$$Q_{a_1b_1a_2}\ , \qquad S_{a_3b_3a_2}\ , \qquad W^{(\nu)}_{\alpha\ A_\nu}\ , \qquad \Phi^{(\nu)}_{B_\nu}\ , \tag{2.5.74}$$

where $\nu=1,2,3$ indexes the three $SU(2)$ gauge groups, a_ν, b_ν are fundamental indices of $SU(2)_\nu$, and A_ν, B_ν are adjoint indices of $SU(2)_\nu$. It is convenient to rearrange the fields $Q_{a_1b_1a_2}$ and $S_{a_3b_3a_2}$ in terms of irreducible representations of the gauge groups. In particular, we can define

$$\begin{aligned} Q_{A_1a_2} &:= -iQ_{a_1b_1a_2}(T_{A_1})^{a_1b_1}\ , \quad Q_{a_2} := \frac{1}{\sqrt{2}}\varepsilon^{a_1b_1}Q_{a_1b_1a_2}\ , \\ S_{A_3a_2} &:= -iS_{a_3b_3a_2}(T_{A_3})^{a_3b_3}\ , \quad S_{a_2} := \frac{1}{\sqrt{2}}\varepsilon^{a_3b_3}S_{a_3b_3a_2}\ . \end{aligned} \tag{2.5.75}$$

Finally, we introduce the fields

$$\phi_{a_2} = \frac{1}{\sqrt{2}}(Q_{a_2} + iS_{a_2}) \ , \qquad \bar{\phi}_{a_2} = \frac{1}{\sqrt{2}}(Q_{a_2} - iS_{a_2}) \ . \tag{2.5.76}$$

The theory has a $U(1)_F$ flavor symmetry that is not completely obvious given the usual structure of flavor symmetries in class $\mathcal{S}$ theories. The fields ϕ and $\bar{\phi}$ have charges $+1$ and -1 respectively under the flavor symmetry, and the remaining fields are neutral.

The BRST cohomology problem for this theory can be set up as in the previous sections. In fact, the analysis may be somewhat simplified by leveraging the $\mathcal{N} = 4$ analysis of the previous section. In particular, each loop in the quiver corresponds to a small $\mathcal{N} = 4$ superconformal algebra along with a decoupled $SU(2)$ doublet of symplectic bosons. The genus two theory is obtained by gauging the diagonal subgroup of the $SU(2)$ flavor symmetries for each side. Nevertheless, the resulting cohomology problem is substantially more intricate than those of the previous examples, and we will not describe the level-by-level analysis.

Instead, we will take an indirect approach to understand the spectrum of *generators* of this chiral algebra at low levels. In particular, by analyzing various superconformal indices of this theory and comparing with the structure of the HL chiral ring, we will be able to prove that the full chiral algebra must have generators in addition to those related to HL chiral ring generators and the stress tensor. More precisely, by studying the spectrum up to dimension three, we find that there must be additional generators that arise from $\hat{\mathcal{C}}_{1(0,0)}$ multiplets in four dimensions.

The Higgs branch chiral ring for this theory has been analyzed in [71]. It has three generators: a $U(1)_F$ neutral operator of dimension two, which is actually the moment map for $U(1)_F$,

$$M = -\epsilon^{a_2 a_2'} \phi_{a_2} \bar{\phi}_{a_2'} \ , \tag{2.5.77}$$

and two operators of dimension four,

$$\mathcal{O}_1 = 2\ \phi_{a_2}\phi_{a_2'}\ \epsilon^{a_2 b_2}\epsilon^{a_2' b_2'}\ Q_{A_1 b_2} Q_{B_1 b_2'}\ \delta^{A_1 B_1} \ . \tag{2.5.78}$$

$$\mathcal{O}_2 = 2\ \bar{\phi}_{a_2}\bar{\phi}_{a_2'}\ \epsilon^{a_2 b_2}\epsilon^{a_2' b_2'}\ Q_{A_1 b_2} Q_{B_1 b_2'}\ \delta^{A_1 B_1} \ , \tag{2.5.79}$$

that have charges $+2$ and -2 under the flavor symmetry. These generators satisfy a flavor neutral relation of dimension eight:

$$\mathcal{O}_1 \mathcal{O}_2 = M^4 \ . \tag{2.5.80}$$

It will be helpful for us to write down the Hilbert series [71] for this theory, refined by the $U(1)_F$ flavor symmetry:

$$\begin{aligned} g(\tau,a) &= \frac{1-t^4}{(1-t)(1-a^2t^2)(1-a^{-2}t^2)} \\ &= 1+t+\left(a^2+a^{-2}+1\right)t^2+\left(a^2+a^{-2}+1\right)t^3+\dots\ , \end{aligned} \tag{2.5.81}$$

where a is the $U(1)_F$ fugacity, and t is the fugacity for the dimension of the operator.

The generalized quiver for this theory has closed loops, so there will be additional elements of the HL chiral ring coming from $\mathcal{D}$-type multiplets. The HL index for this theory can be computed by standard methods, and is given by

$$\begin{aligned} \mathcal{I}_{\text{HL}}(t;a) =& 1+t+(a^2+a^{-2}-2a-2a^{-1}+1)t^2 \\ &+(a^2+a^{-2}-2a-2a^{-1}+2)t^3+\dots\ . \end{aligned} \tag{2.5.82}$$

By subtracting off the contributions of the Higgs chiral ring operators (obtained from (2.5.81)), we can find the contributions of just the $\mathcal{D}$-type multiplets. In turn, we can extract the structure of the $\mathcal{D}$-type generators.[27] All told, at dimension two there are two $\mathcal{D}_{1(0,0)}$ multiplets with $U(1)_F$ charge $+1$ and two with charge -1, and at dimension three there is a single $\mathcal{D}_{\frac{3}{2}(0,\frac{1}{2})}$ multiplet that is $U(1)_F$ neutral. The two-dimensional counterparts of these operators can be defined in an explicit calculation of the BRST cohomology.

Up to dimension three, we have now determined all of the generators of the HL chiral ring. The question is whether these operators (along with the conjugates of the $\mathcal{D}$-type operators), in addition to the two-dimensional stress tensor, are sufficient to explain the full spectrum of the chiral algebra (up to dimension three). The generators are listed in the three blocks of Table 2.6, together with their contribution to the Macdonald index and the quantum numbers of the corresponding Schur operators.

The Macdonald limit of the superconformal index of this theory is ob-

[27] We have checked by a computation of the HL cohomology that the HL index captures faithfully the complete spectrum of $\mathcal{D}$-type multiplets up to dimension three.

tained from the following contour integral,

$$
\begin{aligned}
&\mathcal{I}_{\mathrm{MD}}(q,t;a) = \\
&\oint [db_1][db_2][db_3]\, P.E.\left[\frac{\sqrt{t}}{1-q}\left[\left(\chi^{\mathbf{3}}(b_1)\chi^{\mathbf{2}}(b_3)+\chi^{\mathbf{3}}(b_2)\chi^{\mathbf{2}}(b_3)\right)+(a+a^{-1})\chi^{\mathbf{2}}(b_3)\right]\right. \\
&\qquad\qquad\left. +\left(\frac{-t-q}{1-q}\right)\left(\chi^{\mathbf{3}}(b_1)+\chi^{\mathbf{3}}(b_2)+\chi^{\mathbf{3}}(b_3)\right)\right]\,, \qquad (2.5.83)
\end{aligned}
$$

and the expansion including all operators up to dimension three is as follows,

$$
\begin{aligned}
\mathcal{I}_{\mathrm{MD}}(q,t;a) =& 1+t+(a^2+a^{-2}-2a-2a^{-1}+1)t^2+(-2a-2a^{-1}+2)qt+ \\
&+(a^2+a^{-2}-2(a+a^{-1})+2)t^3+(3-2(a+a^{-1}))q^2t+ \\
&+(a^2+a^{-2}-4(a+a^{-1})+5)t^2q+\ldots\ .
\end{aligned}
$$

We find that not all of the terms in this expansion can be accounted for by enumerating normal ordered products of generators and their descendants. In particular, from the list of known generators, the only operators that could contribute as t^2q to the index (with no flavor fugacity) are the normal-ordered product of a $\hat{\mathcal{B}}_1$ and a $\hat{\mathcal{C}}_{0(0,0)}$ and the derivative of the normal-ordered product of two $\hat{\mathcal{B}}_1$ operators. This leaves a contribution of $3t^2q$ remains to be explained. We can thus conclude that there are at least three new operators, and they must all must correspond to $\hat{\mathcal{C}}_{1,(0,0)}$ multiplets that are uncharged under the flavor symmetry. We have included these as the last entry in Table 2.6. The argument presented above shows that at least these three multiplets must be present, however it does not take into account possible cancellations in the index, which could hide even more additional generators.

2.6 Beyond Lagrangian theories

Although the discussion of the previous section focused on theories admitting Lagrangian descriptions, the correspondence between $\mathcal{N}=2$ SCFTs and chiral algebras is of course much more general. In particular, the vast landscape of superconformal theories of class $\mathcal{S}$, most of which are non-Lagrangian in nature, will be mapped to an intricate and interesting class of chiral algebras. A detailed study of the class of chiral algebras defined by this map is the content of chapter 3.

Multiplet	Index contribution	h	$U(1)_R$	$U(1)_F$
$\hat{\mathcal{B}}_1$	$\frac{t}{1-q}$	1	0	0
$\hat{\mathcal{B}}_2$	$\frac{t^2a^2}{1-q}$	2	0	+2
$\hat{\mathcal{B}}_2$	$\frac{t^2/a^2}{1-q}$	2	0	-2
$2\times\mathcal{D}_{1\,(0,0)}$	$-2\frac{t^2a}{1-q}$	2	$\frac{1}{2}$	+1
$2\times\bar{\mathcal{D}}_{1\,(0,0)}$	$-2\frac{tqa}{1-q}$	2	$-\frac{1}{2}$	+1
$2\times\mathcal{D}_{1\,(0,0)}$	$-2\frac{t^2/a}{1-q}$	2	$\frac{1}{2}$	-1
$2\times\bar{\mathcal{D}}_{1\,(0,0)}$	$-2\frac{tq/a}{1-q}$	2	$-\frac{1}{2}$	-1
$\mathcal{D}_{\frac{3}{2}\,(0,\frac{1}{2})}$	$\frac{t^3}{1-q}$	3	1	0
$\bar{\mathcal{D}}_{\frac{3}{2}\,(\frac{1}{2},0)}$	$\frac{tq^2}{1-q}$	3	-1	0
$\hat{\mathcal{C}}_{0(0,0)}$	$\frac{tq}{1-q}$	2	0	0
$3\times\hat{\mathcal{C}}_{1(0,0)}$	$3\frac{t^2q}{1-q}$	3	0	0

Table 2.6: Chiral algebra generators for the genus two theory with $h \leqslant 3$. The first columns lists the name and multiplicity of the four dimensional multiplets giving rise to the generators. The second column lists the contribution of each multiplet to the Macdonald superconformal index, including the flavor fugacity. The last columns list the two-dimensional quantum numbers of the generators. The first block of the table consists of Higgs chiral ring generators, the second the remaining HL chiral and anti-chiral ring generators, the third the two-dimensional stress tensor, and the last block the extra generators deduced from the superconformal index.

Chapter 3

Chiral Algebras of class $\mathcal{S}$

3.1 Introduction

A large and interesting class of interacting quantum field theories are the *theories of class* $\mathcal{S}$ [10, 11]. These are superconformal field theories (SCFTs) with half-maximal (*i.e.*, $\mathcal{N} = 2$) supersymmetry in four dimensions. The most striking feature of this class of theories is that they assemble into vast duality webs that are neatly describable in the language of two-dimensional conformal geometry. This structure follows from the defining property of theories of class $\mathcal{S}$: they can be realized as the low energy limits of (partially twisted) compactifications of six-dimensional CFTs with $(2,0)$ supersymmetry on punctured Riemann surfaces.

Generic theories of class $\mathcal{S}$ are strongly interacting. (In many cases they possess generalized weak-coupling limits wherein the neighborhood of a certain limit point on their conformal manifold can be described by a collection of *isolated* strongly coupled SCFTs with weakly gauged flavor symmetries.) It is remarkable, then, that one can say much of anything about these theories in the general case. One classic and successful approach has been to restrict attention to the weakly coupled phases of these theories by, for example, studying the physics of Coulomb branch vacua at the level of the low energy effective Lagrangian and the spectrum of BPS states. Relatedly, one may utilize brane constructions of these theories to extract some features of the Coulomb branch physics [72, 73].

An alternative – and perhaps more modern – tactic is to try to constrain or solve for various aspects of these theories using consistency conditions that follow from duality. This approach was successfully carried out in [74] (building on the work of [75, 76, 37, 38]) to compute the superconformal index of a very general set of class $\mathcal{S}$ fixed points (see also [6, 77] for extensions to even more general cases). Subsequently, the framework for implementing this approach to study the (maximal) Higgs branch was established in [78]. The general aspiration in this sort of program is that the consistency conditions imposed by generalized S-duality and the (known) behavior of these theories under certain partial Higgsing and weak gauging operations may be sufficient to completely determine certain nice observables. In this sense the approach might be thought of as a sort of "theory space bootstrap". One expects that this approach has the greatest probability of success when applied to observables of class $\mathcal{S}$ theories that are protected against corrections when changing exactly marginal couplings, thus leading to objects that are labelled by topological data and have no dependence on continuous parameters.[1]

A new class of protected observables for four-dimensional $\mathcal{N}=2$ SCFTs was introduced in chapter 2. There it was shown that certain carefully selected local operators, restricted to be coplanar and treated at the level of cohomology with respect to a particular nilpotent supercharge, form a closed subalgebra of the operator algebra. Moreover their operator product expansions and correlation functions are meromorphic functions of the operator insertion points on the plane. This subalgebra consequently acquires the structure of a two-dimensional chiral algebra. The spectrum and structure constants of this chiral algebra are subject to a non-renormalization theorem that renders them independent of marginal couplings. The existence of this sector can formally be summarized by defining a map that associates to any $\mathcal{N}=2$ SCFT in four dimensions the chiral algebra that computes the appropriate protected correlation functions,

$$\chi \; : \; \left\{ {}^{\mathcal{N}=2 \text{ SCFTs}}\big/_{\text{Marginal deformations}} \right\} \longrightarrow \left\{ \text{Chiral algebras} \right\} .$$

Chiral algebras with the potential to appear on the right hand side of this map are not generic – they must possess a number of interesting properties that reflect the physics of their four-dimensional ancestors.

In this chapter we initiate the investigation of chiral algebras that are associated in this manner with four-dimensional theories of class $\mathcal{S}$. For lack

[1] Observables with a manageable dependence on the marginal couplings, such as $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ and $\mathbb{S}^4$ partition functions, also provide natural settings for this type of argument.

of imagination, we refer to the chiral algebras appearing in this fashion as *chiral algebras of class $\mathcal{S}$*. For a general strongly interacting SCFT, there is at present no straightforward method for identifying the associated chiral algebra. Success in this task would implicitly fix an infinite amount of protected CFT data (spectral data and three-point couplings) that is generally difficult to determine. However, given the rigid nature of chiral algebras, one may be optimistic that chiral algebras of class $\mathcal{S}$ can be understood in some detail by leveraging the constraints of generalized S-duality and the wealth of information already available about the protected spectrum of these theories. In the present work, we set up the abstract framework of this bootstrap problem in the language of generalized topological quantum field theory, and put into place as many ingredients as possible to define the problem concretely. We perform some explicit calculations in the case of theories of rank one and rank two, and formulate a number of conjectures for the higher rank case. One of our main results is a general prescription to obtain the chiral algebra of a theory with sub-maximal punctures given that of the related theory with all maximal punctures. We demonstrate that the reduction in the rank of a puncture is accomplished in the chiral algebra by quantum Drinfeld-Sokolov reduction, with the chiral algebra procedure mirroring the corresponding four-dimensional procedure involving a particular Higgsing of flavor symmetries.

Ultimately we believe that the bootstrap problem for chiral algebras of class $\mathcal{S}$ may prove solvable, and we hope that the existence of this remarkable structure will pique the interest of readers with a passion for vertex operator algebras. Characterizing these algebras should prove to be both mathematically and physically rewarding.

The organization of this chapter is as follows. Section 3.2 is a two-part review: first of the protected chiral algebra of $\mathcal{N} = 2$ SCFTs, and then of $\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$. In Section 3.3, we outline the structure of the chiral algebras of class $\mathcal{S}$, using the A_1 and A_2 cases as examples. We also take some steps to formalize the TQFT structure of the chiral algebras of class $\mathcal{S}$ so as to emphasize that the structures outlined here are susceptible to rigorous mathematical analysis. In Section 3.4, we describe the generalization of our story to the case of theories with sub-maximal punctures. In the process, we are led to consider the problem of quantum Drinfeld-Sokolov reduction for modules of affine Lie algebras. In Section 3.5, we offer some comments on unphysical chiral algebras that are expected to exist at a formal level in order to complete the TQFT structure. A number of technical details having to do with rank two theories are included in Appendix B.1. Details having to do with unphysical cylinder and cap chiral algebras appear in Appendix B.2. Finally, in Appendix B.3 we review the methods for computing the cohomology of a double complex using spectral sequences. These methods are instrumental to the analysis of Section 3.4.

3.2 Background

We begin with a review of the two main topics being synthesized in this chapter: the protected chiral algebras of $\mathcal{N}=2$ SCFTs and superconformal theories of class $\mathcal{S}$. Readers who have studied chapter 2 should be fine skipping Section 3.2.1, while those familiar with the class $\mathcal{S}$ literature (for example, [10, 75, 74, 67]) may safely skip Section 3.2.2.

3.2.1 Review of protected chiral algebras

The observables we aim to study for class $\mathcal{S}$ fixed points are those described by the protected chiral algebras introduced in chapter 2 (see also [26] for the extension to six dimensions). The purpose of this section is to provide a short overview of how those chiral algebras come about and the properties that were deduced for them in the original papers. We simply state the facts in this section; the interested reader is encouraged to consult the previous chapter for explanations.

The starting point is the $\mathcal{N}=2$ superconformal algebra $\mathfrak{su}(2,2|2)$. The fermionic generators of the algebra are Poincaré supercharges $\{\mathcal{Q}^{\mathcal{I}}_{\alpha}, \tilde{\mathcal{Q}}_{\dot{\alpha}\mathcal{J}}\}$ and special conformal supercharges $\{\mathcal{S}^{\alpha}_{\mathcal{I}}, \tilde{\mathcal{S}}^{\dot{\alpha}\mathcal{J}}\}$. From these, one can form two interesting nilpotent supercharges that are mixtures of Poincaré and special conformal supercharges,

$$\mathbb{Q}_1 := \mathcal{Q}^1_- + \tilde{\mathcal{S}}^{\dot{-}2} , \qquad \mathbb{Q}_2 := \tilde{\mathcal{Q}}_{\dot{-}2} + \mathcal{S}^-_1 . \tag{3.2.1}$$

These supercharges have the following interesting property. Let us define the subalgebra of the four-dimensional conformal symmetry algebra that acts on a plane $\mathbb{R}^2 \subset \mathbb{R}^4$ as $\mathfrak{sl}(2) \times \overline{\mathfrak{sl}(2)}$. Let us further denote the complexification of the $\mathfrak{su}(2)_R$ R-symmetry as $\mathfrak{sl}(2)_R$. These subalgebras have the following nice relationship to the supercharges $\mathbb{Q}_i$,

$$[\mathbb{Q}_i, \mathfrak{sl}(2)] = 0 , \qquad \{\mathbb{Q}_i, \cdot\} = \mathrm{diag}\left[\overline{\mathfrak{sl}(2)} \times \mathfrak{sl}(2)_R\right] . \tag{3.2.2}$$

It follows from these relations that operators that are $\mathbb{Q}$-closed must behave as meromorphic operators in the plane. They have meromorphic operator product expansions (modulo $\mathbb{Q}$-exact terms) and their correlation functions are meromorphic functions of the positions. Restricting from the full $\mathcal{N}=2$ SCFT to $\mathbb{Q}$-cohomology therefore defines a two-dimensional chiral algebra. For a pedagogical discussion of chiral algebras, see [42].

The conditions for a local operator to define a nontrivial $\mathbb{Q}$-cohomology element were worked out in chapter 2. It turns out that such operators are restricted to lie in the *chiral algebra plane*: $\{x_3 = x_4 = 0\}$. When inserted at the origin, an operator belongs to a well-defined cohomology class if and only if it obeys the conditions

$$\hat{h} := \frac{E-(j_1+j_2)}{2} - R = 0 \ , \qquad \mathcal{Z} := j_1 - j_2 + r = 0 \ . \tag{3.2.3}$$

Unitarity of the superconformal representation requires $\hat{h} \geqslant \frac{|\mathcal{Z}|}{2}$, so the first condition actually implies the second. We refer to operators obeying $\hat{h}=0$ as *Schur operators*. All Schur operators are necessarily $\mathfrak{su}(2)_R$ highest weight states. Indeed, if the $\mathfrak{su}(2)_R$ raising generator did *not* annihilate a Schur operator, it would generate an operator with $\hat{h} < 0$, which would violate unitarity.

As $\overline{\mathfrak{sl}(2)}$ does not commute with $\mathbb{Q}$, ordinary translations of Schur operators in the chiral algebra plane fail to be $\mathbb{Q}$-closed away from the origin. Rather, we translate operators using the twisted translation generator $\widehat{L}_{-1} := \overline{L}_{-1} + \mathcal{R}_{-}$, where $\mathcal{R}_{-}$ is the lowering operator of $\mathfrak{su}(2)_R$. As shown in Eqn. (3.2.2), this is a $\mathbb{Q}$-exact operation. We find that local operators defining nontrivial $\mathbb{Q}$-cohomology classes can be written in the form

$$\mathcal{O}(z,\bar{z}) := u_{\mathcal{I}_1}(\bar{z}) \cdots u_{\mathcal{I}_k}(\bar{z}) \mathcal{O}^{\{\mathcal{I}_1 \cdots \mathcal{I}_k\}}(z,\bar{z}) \ , \qquad \text{where} \qquad u_{\mathcal{I}}(z) := \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} \ . \tag{3.2.4}$$

Here $\mathcal{O}^{1\cdots 1}(0)$ is a Schur operator, and we are suppressing Lorentz indices. It is these *twisted-translated* Schur operators, taken at the level of cohomology, that behave as meromorphic operators in two dimensions,

$$\mathcal{O}(z) := [\mathcal{O}(z,\bar{z})]_{\mathbb{Q}_i} \ . \tag{3.2.5}$$

We now turn to a recap of the various types of four-dimensional operators that may satisfy the Schur condition, and thus participate in the protected chiral algebra.

Multiplet	$\mathcal{O}_{\text{Schur}}$	h	r	Lagrangian "letters"
$\hat{\mathcal{B}}_R$	$\Psi^{11\ldots1}$	R	0	Q, $\tilde{Q}$
$\mathcal{D}_{R(0,j_2)}$	$\widetilde{\mathcal{Q}}^1_{\dot{+}}\,\Psi^{11\ldots1}_{\dot{+}\ldots\dot{+}}$	$R+j_2+1$	$j_2+\frac{1}{2}$	Q, $\tilde{Q}$, $\tilde{\lambda}^1_{\dot{+}}$
$\bar{\mathcal{D}}_{R(j_1,0)}$	$\mathcal{Q}^1_{+}\,\Psi^{11\ldots1}_{+\cdots+}$	$R+j_1+1$	$-j_1-\frac{1}{2}$	Q, $\tilde{Q}$, λ^1_{+}
$\hat{\mathcal{C}}_{R(j_1,j_2)}$	$\mathcal{Q}^1_{+}\,\widetilde{\mathcal{Q}}^1_{\dot{+}}\,\Psi^{11\ldots1}_{+\cdots+\,\dot{+}\ldots\dot{+}}$	$R+j_1+j_2+2$	j_2-j_1	$D^n_{+\dot{+}}Q$, $D^n_{+\dot{+}}\tilde{Q}$, $D^n_{+\dot{+}}\lambda^1_{+}$, $D^n_{+\dot{+}}\tilde{\lambda}^1_{\dot{+}}$

Table 3.1: This table summarizes the manner in which Schur operators fit into short multiplets of the $\mathcal{N}=2$ superconformal algebra. For each supermultiplet, we denote by Ψ the superconformal primary. There is then a single conformal primary Schur operator $\mathcal{O}_{\text{Schur}}$, which in general is obtained by the action of some Poincaré supercharges on Ψ. We list the holomorphic dimension h and $U(1)_r$ charge r of $\mathcal{O}_{\text{Schur}}$ in terms of the quantum numbers (R, j_1, j_2) that label the shortened multiplet (left-most column). We also indicate the schematic form that $\mathcal{O}_{\text{Schur}}$ can take in a Lagrangian theory by enumerating the elementary "letters" from which the operator may be built. We denote by Q and $\tilde{Q}$ the complex scalar fields of a hypermultiplet, by $\lambda^{\mathcal{I}}_{\alpha}$ and $\tilde{\lambda}^{\mathcal{I}}_{\dot{\alpha}}$ the left- and right-moving fermions of a vector multiplet, and by $D_{\alpha\dot{\alpha}}$ the gauge-covariant derivatives.

Taxonomy of Schur operators

A Schur operator is annihilated by two Poincaré supercharges of opposite chiralities ($\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$ in our conventions). A summary of the different classes of Schur operators, organized according to how they fit in shortened multiplets of the superconformal algebra, is given in Table 3.1 (reproduced from chapter 2). Let us briefly discuss each row in turn.

The first row describes half-BPS operators that are a part of the Higgs branch chiral ring. These have $E = 2R$ and $j_1 = j_2 = 0$. In a Lagrangian theory, operators of this type schematically take the form $QQ\cdots\tilde{Q}\tilde{Q}$. A special case is when $R = 1$, in which case a conserved current is amongst the super-descendants of the primary. The half-BPS primary is then the "moment map" operator μ_A which has dimension two and transforms in the adjoint representation of the flavor symmetry. The $\mathfrak{su}(2)_R$ highest weight state of the moment map is a Schur operator.

The operators in the second row are more general $\mathcal{N} = 1$ chiral operators, obeying $E = 2R + |r|$ and $r = -j_1 - \frac{1}{2}$. Together with the Higgs branch chiral ring operators (which can be regarded as the special case with $r = 0$), they make up the so-called *Hall-Littlewood chiral ring*. These are precisely the operators that are counted by the Hall-Littlewood limit of the superconformal index [38]. In a Lagrangian theory, these operators are obtained by constructing gauge-invariant words out of Q, $\tilde{Q}$, and the gaugino field λ^1_+ (the bottom component of the field strength chiral superfield W_α with $\alpha = +$). In complete analogy, the third line describes $\mathcal{N} = 1$ *anti*-chiral operators obeying $E = 2R + |r|$, $r = j_2 + \frac{1}{2}$, which belong to the Hall-Littlewood anti-chiral ring. The second and third lines are CPT conjugate to each other. It is believed that $\mathcal{D}$ and $\overline{\mathcal{D}}$ type operators are absent in any theory arising from a (generalized) quiver description with no loops (*i.e.*, an *acyclic quiver*). These are theories for which the Hall-Littlewood superconformal index matches [38] the "Hilbert series" for the Higgs branch [79]. Equivalently, these are the theories for which the maximal Higgs branch is an honest Higgs branch, with no low-energy abelian gauge field degrees of freedom surviving.

The fourth line describes the most general type of Schur operators, which belong to supermultiplets that obey less familiar semi-shortening conditions. An important operator in this class is the conserved current for $\mathfrak{su}(2)_R$, which belongs to the $\hat{\mathcal{C}}_{0(0,0)}$ supermultiplet which also contains the stress-energy tensor and is therefore universally present in any $\mathcal{N} = 2$ SCFT. This current has one component with $E = 3$, $R = 1$, $j_1 = j_2 = \frac{1}{2}$ which is a Schur operator.

Finally, let us point out the conspicuous absence of half-BPS operators that belong to the *Coulomb* branch chiral ring (these take the form $\mathrm{Tr}\,\phi^k$ in a Lagrangian theory, where ϕ is the complex scalar of the $\mathcal{N} = 2$ vector multiplet). These operators are in many ways more familiar than those appearing above due to their connection with Coulomb branch physics. The protected chiral algebra is thus complementary, rather than overlapping, with a Coulomb branch based analysis of class $\mathcal{S}$ physics.

The $4d/2d$ dictionary

There is a rich dictionary relating properties of a four-dimensional SCFT with properties of its associated chiral algebra. Let us briefly review some of the universal entries in this dictionary that were worked out in chapter 2. Interested readers should consult that chapter for more detailed explanations.

Virasoro symmetry

The stress tensor in a four-dimensional $\mathcal{N} = 2$ SCFT lives in the $\hat{\mathcal{C}}_{0(0,0)}$ supermultiplet, which contains as a Schur operator a component of the $\mathfrak{su}(2)_R$ conserved current $\mathcal{J}_{\alpha\dot{\alpha}}^{(\mathcal{I}\mathcal{J})}$. The corresponding twisted-translated operator gives rise in cohomology to a two-dimensional meromorphic operator of dimension two, which acts as a two-dimensional stress tensor, $T(z) := [\mathcal{J}_{+\dot{+}}(z,\bar{z})]_{\mathbb{Q}}$. As a result, the global $\mathfrak{sl}(2)$ symmetry that is inherited from four dimensions is always enhanced to a local Virasoro symmetry acting on the chiral algebra. From the current-current OPE, which is governed by superconformal Ward identities, one finds a universal expression for the Virasoro central charge,

$$c_{2d} = -12\, c_{4d}\ , \tag{3.2.6}$$

where c_{4d} is the conformal anomaly coefficients of the four-dimensional theory associated to the square of the Weyl tensor. Note that the chiral algebra is necessarily non-unitary due to the negative sign in Eqn. (3.2.6).

Affine symmetry

Similarly, continuous global symmetries of the four-dimensional SCFT (when present) are enhanced to local affine symmetries at the level of the associated chiral algebra. This comes about because the conserved flavor symmetry current sits in the $\hat{\mathcal{B}}_1$ supermultiplet, whose bottom component is the moment-map operator discussed above. The $\mathfrak{su}(2)_R$ highest weight component of the moment map operator then gives rise to an affine current, $J_A(z) := [\mu_A(z,\bar{z})]_{\mathbb{Q}}$. The level of the affine current algebra is related to the four-dimensional flavor central charge by another universal relation,

$$k_{2d} = -\frac{1}{2}k_{4d} \ . \tag{3.2.7}$$

Hall-Littlewood ring generators as chiral algebra generators

Identifying chiral algebra generators is of crucial importance if one is to find an intrinsic characterization of any particular chiral algebra without reference to its four-dimensional parent. A very useful fact is that generators of the Hall-Littlewood chiral ring (and in particular those of the Higgs branch chiral ring) necessarily give rise to generators of the protected chiral algebra after passing to $\mathbb{Q}$-cohomology. This follows from $\mathfrak{su}(2)_R$ and $\mathfrak{u}(1)_r$ selection rules, which forbid such an operator from appearing in any non-singular OPEs. A special case is the aforementioned affine currents, which arise from Higgs branch moment map operators with $E = 2R = 2$. With the exception of theories with free hypermultiplets, these are always generators.

Exactly marginal gauging

Given an SCFT $\mathcal{T}$ with a flavor symmetry G that has flavor central charge $k_{4d} = 4h^\vee$, one may form a new family of SCFTs $\mathcal{T}_G$ by introducing an $\mathcal{N}=2$ vector multiplet in the adjoint representation of G and gauging the symmetry. This specific value of the flavor central charge ensures that the gauge coupling beta function vanishes, so the procedure preserves conformal invariance.

There exists a corresponding procedure at the level of chiral algebras that produces the chiral algebra $\chi[\mathcal{T}_G]$ given that of the original theory $\chi[\mathcal{T}]$. In parallel with the introduction of a G-valued vector multiplet, one introduces a dimension $(1,0)$ ghost system (b_A, c^A) with $A = 1,\ldots,\dim G$. In the tensor product of this ghost system and the chiral algebra $\chi[\mathcal{T}]$, one may form a canonical nilpotent BRST operator given by

$$Q_{\rm BRST} := \oint \frac{dz}{2\pi i}\, j_{\rm BRST}(z) \ , \qquad j_{\rm BRST}(z) := \left(c^A \left[J_A - \frac{1}{2} f_{AB}{}^C\, c^B\, b_C\right]\right)(z) \ , \tag{3.2.8}$$

where the affine currents $J_A(z)$ are those associated with the G symmetry of $\chi[\mathcal{T}]$, and $f_{AB}{}^C$ are the structure constants for G. Nilpotency of this BRST operator depends on the precise value of the affine level $k_{2d} = -2h^\vee$, and so the self-consistency of this procedure is intimately connected with the preservation of conformal invariance in four dimensions. The gauged chiral algebra is then obtained as the cohomology of the BRST operator relative to the b-ghost zero modes,

$$\chi[\mathcal{T}_G] = H^\star_{\text{BRST}}\left[\psi \in \chi[\mathcal{T}] \otimes \chi_{(b,c)} \mid b_0^A \psi = 0\right] \tag{3.2.9}$$

Superconformal index

The superconformal index of a superconformal field theory is the Witten index of the radially quantized theory, refined by a set of fugacities that keep track of the maximal set of charges commuting with each other and with a chosen supercharge. For our purposes, we consider the specialization of the index of an $\mathcal{N} = 2$ SCFT known as the Schur index [37, 38]. The trace formula for the Schur index reads

$$\mathcal{I}^{(\text{Schur})}(q;\mathbf{x}) = \text{Tr}_{\mathcal{H}[\mathbb{S}^3]}(-1)^F q^{E-R} \prod_i x_i{}^{f_i} \ , \tag{3.2.10}$$

where F denotes the fermion number and $\{f_i\}$ the Cartan generators of the flavor group. The Schur index counts (with signs) precisely the operators obeying the condition (3.2.3). Moreover, for Schur operators $E - R$ coincides with the left-moving conformal weight h (the eigenvalue of L_0),

$$E - R \ = \ \frac{E + j_1 + j_2}{2} \ =: \ h \ . \tag{3.2.11}$$

It follows that the graded character of the chiral algebra is identical to the Schur index,

$$\mathcal{I}_\chi(q;\mathbf{x}) \ := \ \text{Tr}_{\mathcal{H}_\chi}(-1)^F q^{L_0} \ = \ \mathcal{I}^{\text{Schur}}(q;\mathbf{x}) \ , \tag{3.2.12}$$

where $\mathcal{H}_\chi$ denotes the state space of the chiral algebra. Note that this object is not interpreted as an index when taken as a partition function of the chiral algebra, because (with the exception of chiral algebras associated to $\mathcal{N} = 4$ theories in four dimensions) the protected chiral algebra itself is not supersymmetric.

3.2.2 Review of theories of class $\mathcal{S}$

Four-dimensional superconformal field theories of class $\mathcal{S}$ may be realized as the low-energy limit of twisted compactifications of an $\mathcal{N} = (2,0)$ superconformal field theory in six dimensions on a Riemann surface, possibly in the presence of half-BPS codimension-two defect operators. The resulting four-dimensional theory is specified by the following data:[2]

- A simply-laced Lie algebra $\mathfrak{g} = \{A_n, D_n, E_6, E_7, E_8\}$. This specifies the choice of six-dimensional $(2,0)$ theory.

- A (punctured) Riemann surface $\mathcal{C}_{g,s}$ known as the *UV curve*, where g indicates the genus and s the number of punctures. In the low energy limit, only the complex structure of $\mathcal{C}_{g,s}$ plays a role. The complex structure moduli of the curve are identified with exactly marginal couplings in the SCFT.

- A choice of embedding $\Lambda_i : \mathfrak{su}(2) \to \mathfrak{g}$ (up to conjugacy) for each puncture $i = 1, \ldots, s$. These choices reflect the choice of codimension-two defect that is present at each puncture in the six-dimensional construction. The centralizer $\mathfrak{h}_{\Lambda_i} \subset \mathfrak{g}$ of the embedding is the global symmetry associated to the defect. The theory enjoys a global flavor symmetry algebra given by at least $\oplus_{i=1}^{s} \mathfrak{h}_{\Lambda_i}$.[3]

When necessary, we will label the corresponding four-dimensional SCFT as $\mathcal{T}[\mathfrak{g}; \mathcal{C}_{g,s}; \{\Lambda_i\}]$. Because we are ultimately only interested in theories modulo their exactly marginal couplings, we will not keep track of a point in the complex structure moduli space of the UV curve.

For the sake of simplicity, we will restrict our attention to theories where $\mathfrak{g}$ is in the A series. The generalization to D and E series theories (at least in the abstract discussion) should be possible to carry out without a great deal of additional difficulty. In the A_{n-1} case – *i.e.*, $\mathfrak{g} = \mathfrak{su}(n)$ – the data at punctures can be reformulated as a partition of n: $[n_1^{\ell_1}\, n_2^{\ell_2}\, \ldots\, n_k^{\ell_k}]$ with

[2]We restrict our attention in this note to *regular* theories. A larger class of theories can be obtained by additionally allowing for *irregular* punctures [80]. Still more possibilities appear when the UV curve is decorated with outer automorphisms twist lines [81, 82].

[3]In some exceptional cases the global symmetry of the theory is enhanced due to the existence of additional symmetry generators that are not naturally associated to an individual puncture.

$\sum_i \ell_i n_i = n$ and $n_i > n_{i+1}$. Such a partition indicates how the fundamental representation $\mathfrak{f}$ of $\mathfrak{su}(n)$ decomposes into irreps of $\Lambda(\mathfrak{su}(2))$,

$$\mathfrak{f} \to \bigoplus_{i=1}^{k} \ell_i V_{\frac{1}{2}(n_i-1)} \ , \tag{3.2.13}$$

where V_j denotes the spin j representation of $\mathfrak{su}(2)$. An equivalent description comes from specifying a nilpotent element e in $\mathfrak{su}(n)$, *i.e.*, an element for which $(\mathrm{ad}_e)^r = 0$ for some positive integer r. The Jordan normal form of such a nilpotent element is given by

$$e = \bigoplus_{i=1}^{k} \overbrace{J_{n_i} \oplus \cdots \oplus J_{n_i}}^{\ell_i \text{ times}} \ , \tag{3.2.14}$$

where J_m is the elementary Jordan block of size m, *i.e.*, a sparse $m \times m$ matrix with only ones along the superdiagonal. Thus every nilpotent element specifies a partition of n and vice versa. The $\mathfrak{su}(2)$ embedding comes from defining $\mathfrak{su}(2)$ generators $t_0, t_\pm$ and demanding that $\Lambda(t_-) = e$.

The trivial embedding is identified with the partition $[1^n]$ and leads to a defect with maximal flavor symmetry $\mathfrak{h} = \mathfrak{su}(n)$. A puncture labelled by this embedding is called *full* or *maximal*. The opposite extreme is the principal embedding, which has partition $[n^1]$. This embedding leads to $\mathfrak{h} = \varnothing$, and the puncture is effectively absent. Another important case is the subregular embedding, with partition $[n-1,1]$, which leads to $\mathfrak{h} = \mathfrak{u}(1)$ (as long as $n > 2$). Punctures labelled by the subregular embedding are called *minimal* or *simple*.

The basic entities of class $\mathcal{S}$ are the theories associated to thrice-punctured spheres, or *trinions*. The designations of these theories are conventionally shortened as

$$T_n^{\Lambda_1\Lambda_2\Lambda_3} := \mathcal{T}[\mathfrak{su}(n); \mathcal{C}_{0,3}; \{\Lambda_1\,\Lambda_2\,\Lambda_3\}] \ . \tag{3.2.15}$$

For the special case of all maximal punctures, the convention is to further define $T_n := T_n^{[1^n][1^n][1^n]}$. All of the trinion theories are isolated SCFTs – they have no marginal couplings. For most of these theories, no Lagrangian description is known. An important class of exceptions are the theories with two maximal punctures and one minimal puncture: $T_n^{[1^n][1^n][n-1,1]}$. These are theories of n^2 free hypermultiplets, which in this context are naturally thought of as transforming in the bifundamental representation of $\mathfrak{su}(n) \times \mathfrak{su}(n)$. In the case $n = 2$, the minimal and maximal punctures are the same and the theory of four free hypermultiplets (equivalently, eight free half-hypermultiplets) is the T_2 theory. In this case the global symmetry associated to the punctures is $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(2)$ which is a subgroup of the full global symmetry $\mathfrak{usp}(8)$.

At the level of two-dimensional topology, an arbitrary surface $\mathcal{C}_{g,s}$ can be assembled by taking $2g-2+s$ copies of the three-punctured sphere, or "pairs of pants", and gluing legs together pairwise $3g-3+s$ times. Each gluing introduces a complex *plumbing parameter* and for a given construction of this type the plumbing parameters form a set of coordinates for a patch of the Teichmuller space of Riemann surfaces of genus g with s punctures. A parallel procedure is used to construct the class $\mathcal{S}$ theory associated to an arbitrary UV curve using the basic trinion theories. Starting with $2g-2+s$ copies of the trinion theory T_n, one glues along maximal punctures by gauging the diagonal subgroup of the $\mathfrak{su}(n) \times \mathfrak{su}(n)$ flavor symmetry associated to the punctures. This introduces an $\mathfrak{su}(n)$ gauge group in the four-dimensional SCFT, and the marginal gauge coupling is related to the plumbing parameter. If one wants, the remaining maximal punctures can then be reduced to sub-maximal punctures using the Higgsing procedure described below.[4] To a given pants decomposition of a UV curve, one associates a "weakly coupled" frame of the corresponding SCFT in which the flavor symmetries of a collection of trinion theories are being weakly gauged. The equivalence of different pants decompositions amounts to S-duality. It is only in very special cases that a weakly coupled duality frame of this type will actually be described by a Lagrangian field theory.

By now, quite a few general facts are known about theories of class $\mathcal{S}$. Here we simply review some relevant ones while providing pointers to the original literature. The list is not meant to be comprehensive in any sense.

[4]In terms of the low energy SCFT, the operations of Higgsing at external punctures and gauging of internal ones commute, so one may equally well think of gluing together trinions some of whose punctures are not maximal. Our presentation here is not meant to convey the full depth of what is possible in class $\mathcal{S}$.

Central charges

The a and c conformal anomalies have been determined for all of the regular A-type theories in [67, 83]. The answer takes the following form,

$$c_{4d} = \frac{2n_v + n_h}{12} \ , \qquad a = \frac{5n_v + n_h}{24} \ , \tag{3.2.16}$$

where

$$\begin{aligned} n_v &= \sum_{i=1}^{s} n_v(\Lambda_i) + (g-1)\left(\tfrac{4}{3}h^\vee \dim \mathfrak{g} + \text{rank } \mathfrak{g}\right) \ , \\ n_h &= \sum_{i=1}^{s} n_h(\Lambda_i) + (g-1)\left(\tfrac{4}{3}h^\vee \dim \mathfrak{g}\right) \ , \end{aligned} \tag{3.2.17}$$

and

$$\begin{aligned} n_v(\Lambda) &= 8\left(\rho\cdot\rho - \rho\cdot\Lambda(t_0)\right) + \tfrac{1}{2}(\text{rank } \mathfrak{g} - \dim \mathfrak{g}_0) \ , \\ n_h(\Lambda) &= 8\left(\rho\cdot\rho - \rho\cdot\Lambda(t_0)\right) + \tfrac{1}{2}\dim \mathfrak{g}_{\frac{1}{2}} \ . \end{aligned} \tag{3.2.18}$$

In these equations, ρ is the Weyl vector of $\mathfrak{su}(n)$ and $h^\vee$ is the dual coxeter number, which is equal to n for $\mathfrak{g} = \mathfrak{su}(n)$. The Freudenthal-de Vries strange formula states that $|\rho|^2 = \frac{h^\vee}{12}\dim \mathfrak{g}$, which is useful in evaluating these expressions. Additionally, the embedded Cartan generator $\Lambda(t_0)$ has been used to define a grading on the Lie-algebra,

$$\mathfrak{g} = \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_m \ , \qquad \mathfrak{g}_m := \left\{t \in \mathfrak{g} \mid \text{ad}_{\Lambda(t_0)} t = mt\right\} \ . \tag{3.2.19}$$

This grading will make another appearance in Sec. 3.4.

The $\mathfrak{su}(n)$ flavor symmetry associated to a full puncture comes with flavor central charge $k_{\mathfrak{su}(n)} = 2n$. This is a specialization of the general formula $k_{\text{ADE}} = 2h^\vee$. For a non-maximal puncture, the flavor central charge for a given simple factor $\mathfrak{h}_{\text{simp}} \subseteq \mathfrak{h}$ is given by [83],

$$k_{\mathfrak{h}_{\text{simp}}}\delta_{AB} = 2\sum_j \text{Tr}_{\mathcal{R}_j^{(\text{adj})}} T_A T_B \ , \tag{3.2.20}$$

where T_A, T_B are generators of $\mathfrak{h}_{\text{simp}}$ satisfying the normalization $\text{Tr}_{\mathfrak{h}_{\text{simp}}} T_A T_B = h^\vee_{\mathfrak{h}_{\text{simp}}}\delta_{AB}$ and we have introduced the decomposition of the adjoint representation of $\mathfrak{su}(n)$ into representations of $\mathfrak{h}_\Lambda \otimes \Lambda(\mathfrak{su}(2))$,

$$\text{adj}_{\mathfrak{g}} = \bigoplus_j \mathcal{R}_j^{(\text{adj})} \otimes V_j \ . \tag{3.2.21}$$

In cases where there are global symmetries that extend the symmetries associated to punctures, the central charge can be deduced in terms of the embedding index.

Higgs branch chiral ring and their relations

Operators in an $\mathcal{N}=2$ SCFT whose conformal dimension is equal to twice their $\mathfrak{su}(2)_R$ spin ($E=2R$) form a ring called the *Higgs branch chiral ring.* This ring is generally believed to be the ring of holomorphic functions (in a particular complex structure) on the Higgs branch of the moduli space of vacua of the theory. It is expected to be finitely generated, with the generators generally obeying nontrivial algebraic relations. For theories of class $\mathcal{S}$ the most general such relations have not been worked out explicitly to the best of our knowledge. However, certain cases of the relations can be understood.

For any puncture there is an associated global symmetry $\mathfrak{h}$, and the conserved currents for that global symmetry will lie in superconformal representations that include *moment map* operators μ^A, $A=1,\ldots,\dim\mathfrak{h}$ that belong to the Higgs branch chiral ring. Of primary interest to us are the relations that involve solely these moment map operators. Let us specialize to the case where all punctures are maximal, so $\mathfrak{h}_i=\mathfrak{g}$ for all $i=1,\ldots,s$. There are then chiral ring relations given by

$$\mathrm{Tr}\mu_1^k=\mathrm{Tr}\mu_2^k=\cdots=\mathrm{Tr}\mu_s^k\ ,\qquad k=1,2\ldots\ . \tag{3.2.22}$$

There are additional Higgs branch chiral ring generators for a general class $\mathcal{S}$ theory of the form

$$Q_{(k)}^{\mathcal{I}_1^{(k)}\cdots\mathcal{I}_s^{(k)}}\ ,\qquad k=1,\ldots,n-1\ , \tag{3.2.23}$$

of dimension $E_k=2R_k=\frac{1}{2}k(n-k)(2g-2+s)$. The multi-indices $\mathcal{I}^{(k)}$ index the k-fold antisymmetric tensor representation of $\mathfrak{su}(n)$. There are generally additional chiral ring relations involving these $Q_{(k)}$ operators, some of which mix them with the moment maps [84]. The complete form of these extra relations has not been worked out – a knowledge of such relations would characterize the Higgs branch of that theory as a complex algebraic variety, and such a characterization is presently lacking for all but a small number of special cases. We will not make explicit use of such additional relations in what follows.

Higgsing and reduction of punctures: generalities

Theories with non-maximal punctures can be obtained by starting with a theory with maximal punctures and going to a particular locus on the Higgs branch [73, 85, 83, 86]. The flavor symmetry associated to a puncture is reflected in the existence of the above-mentioned half-BPS moment map operators, μ_A, that transform in the adjoint representation of the flavor symmetry with corresponding index $A = 1, \ldots, n^2 - 1$. In reducing the flavor symmetry via Higgsing, one aims to give an expectation value to one of the μ_i's, say μ_1, while keeping $\langle \mu_{i\neq 1} \rangle = 0$. Consistency with Eqn. (3.2.22) then requires that $\langle \mathrm{Tr}\mu_1^k \rangle = 0$ for any k, or put differently, $\langle \mu_1 \rangle$ is a nilpotent $\mathfrak{su}(n)$ matrix. Since any nilpotent element can be realized as the image of $t_- \in \mathfrak{su}(2)$ with respect to some embedding $\Lambda : \mathfrak{su}(2) \hookrightarrow \mathfrak{su}(n)$, the relevant loci on the Higgs branch are characterized by such an embedding, where we have

$$\langle \mu_1 \rangle = v\, \Lambda(t_-) \ . \tag{3.2.24}$$

The expectation value breaks the $\mathfrak{su}(n)$ flavor symmetry associated with the puncture to $\mathfrak{h}_\Lambda$, the centralizer of the embedded $\mathfrak{su}(2)$, as well as the $\mathfrak{su}(2)_R$ symmetry (and also conformal symmetry). It will be important in the following that a linear combination of the flavor and $\mathfrak{su}(2)_R$ Cartan generators remains unbroken,[5] namely

$$\tilde{R} := R + J_0 \ , \qquad J_0 := \Lambda(t_0) \ . \tag{3.2.25}$$

In such a vacuum, the low energy limit of the theory is described by the interacting class $\mathcal{S}$ SCFT with the same UV curve as the original theory, but with the first puncture replaced by a puncture of type Λ. Additionally there will be decoupled free fields arising from the Nambu-Goldstone fields associated to the symmetry breaking [86, 85]. We identify $\tilde{R}$ as the Cartan generator of the $\mathfrak{su}(2)_{\tilde{R}}$ symmetry of the infrared fixed point.

It will prove useful to introduce notation to describe the breaking of $\mathfrak{su}(n)$ symmetry in greater detail. The generators of $\mathfrak{su}(n)$ can be relabeled according to the decomposition of Eqn. (3.2.21),

$$T_A \quad \Longrightarrow \quad T_{j,m;\mathcal{W}(\mathcal{R}_j)} \ , \tag{3.2.26}$$

[5] We suspect not only this Cartan generator, but the full diagonal subalgebra of $\mathfrak{su}(2)_R$ and the embedded $\mathfrak{su}(2)$ is preserved on the sublocus of the Higgs branch in question. It should be possible to prove such a thing using the hyperkahler structure on nilpotent cones described in [87]. We thank D. Gaiotto, A. Neitzke, and Y. Tachikawa for helpful conversations on this point.

where $m = -j, -j+1, \ldots, +j$ is the eigenvalue of the generator with respect to $\Lambda(t_0)$, and $\mathcal{W}(\mathcal{R}_j)$ runs over the various weights of the representation $\mathcal{R}_j$ of $\mathfrak{h}_\Lambda$. Expanding μ_1 around its expectation value, we have

$$\mu_1 = v\,\Lambda(t_-) + \sum_j \sum_{m=-j}^{+j} \sum_{\mathcal{W}(\mathcal{R}_j)} (\tilde{\mu}_1)_{j;m,\mathcal{W}(\mathcal{R}_j)} T_{j;m,\mathcal{W}(\mathcal{R}_j)} \ . \tag{3.2.27}$$

The operators $(\tilde{\mu}_1)_{j;m,\mathcal{W}(\mathcal{R}_j)}$ with $m < j$ become the field operators of the Nambu-Goldstone modes. Their number is given by $\dim_{\mathbb{C}} O^{\mathfrak{g}}_{\Lambda(t_-)}$ – the complex dimension of the nilpotent orbit of $\Lambda(t_-)$. They are ultimately organized into $\frac{1}{2}\dim_{\mathbb{C}} O^{\mathfrak{g}}_{\Lambda(t_-)}$ free hypermultiplets.

Superconformal index

The superconformal index of an SCFT is an invariant on its conformal manifold. For theories of class $\mathcal{S}$, this means that the index does not depend on the complex structure moduli of the UV curve. On general grounds, one then expects the class $\mathcal{S}$ index to be computed by a topological quantum field theory living on the UV curve [75]. This expectation is borne out in detail, with a complete characterization of the requisite TQFT achieved in a series of papers [37, 38, 74]. Our interest is in the Schur specialization of the index, which is identical to the graded character of the protected chiral algebra, see Eqn (3.2.12). In [37], the corresponding TQFT was recognized as a q-deformed version of two-dimensional Yang-Mills theory in the zero-area limit. Here we will summarize this result and introduce appropriate notation that will be useful in Sec. 3.4.3.

For the class $\mathcal{S}$ theory $\mathcal{T}[\mathfrak{g}; \mathcal{C}_{g,s}; \{\Lambda_i\}]$, the Schur index takes the form[6]

$$\mathcal{I}^{\text{Schur}}(q; \mathbf{x}) = \sum_{\mathfrak{R}} C_{\mathfrak{R}}(q)^{2g-2+s} \prod_{i=1}^{s} \psi^{\Lambda_i}_{\mathfrak{R}}(\mathbf{x}_{\Lambda_i}; q) \ . \tag{3.2.28}$$

The sum runs over the set of finite-dimensional irreducible representations $\mathfrak{R}$ of the Lie algebra $\mathfrak{g}$. Each puncture contributes a "wavefunction" $\psi^{\Lambda_i}_{\mathfrak{R}}(\mathbf{x}_{\Lambda_i}; q)$,

[6]Not every possible choice of Riemann surface decorated by a choice of $\{\Lambda_i\}$ at the punctures corresponds to a physical SCFT. An indication that a choice of decorated surface may be unphysical is if the sum in (3.2.28) diverges, which happens when the flavor symmetry is "too small". There are subtle borderline cases where the sum diverges, but the theory is perfectly physical. These cases have to be treated with more care [88].

while the Euler character of the UV curve determines the power of the "structure constants" $C_{\mathfrak{R}}(q)$ that appear. Each wavefunction depends on fugacities $\mathbf{x}_\Lambda$ conjugate to the Cartan generators of the flavor group $\mathfrak{h}_\Lambda$ associated to the puncture in question. Note that by definition, the structure constants are related to wave functions for the principal embedding, which corresponds to having no puncture at all, *i.e.*,

$$C_{\mathfrak{R}}(q)^{-1} \equiv \psi^{\rho}_{\mathfrak{R}}(q) \ , \tag{3.2.29}$$

where ρ denotes the principal embedding.[7]

To write down the general wavefunction we need to discuss some group theory preliminaries. Under the embedding $\Lambda : \mathfrak{su}(2) \hookrightarrow \mathfrak{g}$, a generic representation $\mathfrak{R}$ of $\mathfrak{g}$ decomposes into $\mathfrak{h}_\Lambda \otimes \Lambda(\mathfrak{su}(2))$ representations,

$$\mathfrak{R} = \bigoplus_j \mathcal{R}_j^{(\mathfrak{R})} \otimes V_j \ , \tag{3.2.30}$$

where $\mathcal{R}_j^{(\mathfrak{R})}$ is some (generically reducible) representation of $\mathfrak{h}_\Lambda$. We define the fugacity assignment $\mathrm{fug}_\Lambda(\mathbf{x}_\Lambda; q)$ as the solution (for $\mathbf{x}$) of the following character decomposition equation,[8]

$$\chi^{\mathfrak{g}}_{\mathfrak{f}}(\mathbf{x}) = \sum_j \chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}_j^{(\mathfrak{f})}}(\mathbf{x}_\Lambda)\, \chi^{\mathfrak{su}(2)}_{V_j}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}) \ , \tag{3.2.31}$$

where $\chi^{\mathfrak{g}}_{\mathfrak{f}}(\mathbf{x})$ is the character of $\mathfrak{g}$ in the fundamental representation (denoted by $\mathfrak{f}$), and the right hand side is determined by the decomposition of Eqn. (3.2.30) with $\mathfrak{R} \equiv \mathfrak{f}$. Note that $\mathbf{x} = \mathrm{fug}_\Lambda(\mathbf{x}_\Lambda; q)$ also solves the more general character equation

$$\chi^{\mathfrak{g}}_{\mathfrak{R}}(\mathbf{x}) = \sum_j \chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}_j^{(\mathfrak{R})}}(\mathbf{x}_\Lambda)\, \chi^{\mathfrak{su}(2)}_{V_j}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}) \ , \tag{3.2.32}$$

for any other representation $\mathfrak{R}$. A couple of simple examples help to clarify these definitions. Taking $\mathfrak{g} = \mathfrak{su}(2)$ and $\Lambda : \mathfrak{su}(2) \hookrightarrow \mathfrak{su}(2)$ the principal embedding – in this case is just the identity map – the centralizer is trivial and Eqn. (3.2.31) becomes

$$a + a^{-1} = q^{\frac{1}{2}} + q^{-\frac{1}{2}} \ , \tag{3.2.33}$$

[7]The discussion so far applies to a general simply-laced Lie algebra $\mathfrak{g}$. Recall that when $\mathfrak{g} = \mathfrak{su}(n)$, the principal embedding corresponds to the partition $[n^1]$.

[8]For $\mathfrak{g} = \mathfrak{su}(n)$ the solution is unique up to the action of the Weyl group.

which has the two solutions $a = q^{\frac{1}{2}}$ and $a = q^{-\frac{1}{2}}$, which are related to each other by the action of the Weyl group $a \leftrightarrow a^{-1}$. A more complicated example is $\mathfrak{g} = \mathfrak{su}(3)$ and Λ the subregular embedding, which corresponds to the partition $[2^1, 1^1]$. The centralizer is $\mathfrak{h}_\Lambda = \mathfrak{u}(1)$. Given $\mathfrak{su}(3)$ fugacities (a_1, a_2, a_3) with $a_1 a_2 a_3 = 1$, we denote the $\mathfrak{u}(1)$ fugacity by b and then Eqn. (3.2.31) takes the form

$$a_1 + a_2 + a_3 = b^{-2} + b\,(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\ . \tag{3.2.34}$$

Up to the action of the Weyl group, which permutes the a_i, the unique solution is given by $(a_1, a_2, a_3) = (q^{\frac{1}{2}} b, q^{-\frac{1}{2}} b, b^{-2})$.

The wavefunction for a general choice of embedding and representation now takes the following form,

$$\psi^{\Lambda}_{\mathfrak{R}}(\mathbf{x}_\Lambda; q) := K_\Lambda(\mathbf{x}_\Lambda; q)\,\chi^{\mathfrak{g}}_{\mathfrak{R}}(\mathrm{fug}_\Lambda(\mathbf{x}_\Lambda; q))\ . \tag{3.2.35}$$

The *K-factors* admit a compact expression as a plethystic exponential [77],

$$K_\Lambda(\mathbf{x}_\Lambda; q) := \mathrm{PE}\left[\sum_j \frac{q^{j+1}}{1-q}\chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}^{(\mathrm{adj})}_j}(\mathbf{x}_\Lambda)\right]\ , \tag{3.2.36}$$

where the summation is over the terms appearing in the decomposition of Eqn. (3.2.30) applied to the adjoint representation,

$$\mathrm{adj}_{\mathfrak{g}} = \bigoplus_j \mathcal{R}^{(\mathrm{adj})}_j \otimes V_j\ . \tag{3.2.37}$$

Note that $\mathcal{R}^{(\mathrm{adj})}_0 = \mathrm{adj}_{\mathfrak{h}_\Lambda} \oplus$singlets. For the maximal puncture, corresponding to the trivial embedding $\Lambda_{\max} \equiv 0$, the wavefunction reads

$$\psi^{\Lambda_{\max}}_{\mathfrak{R}}(\mathbf{x}; q) = K_{\max}(\mathbf{x}; q)\,\chi^{\mathfrak{g}}_{\mathfrak{R}}(\mathbf{x})\,, \quad K_{\max}(\mathbf{x}; q) := \mathrm{PE}\left[\frac{q}{1-q}\chi^{\mathfrak{g}}_{\mathrm{adj}}(\mathbf{x})\right]\ . \tag{3.2.38}$$

At the other extreme, for the principal embedding $\Lambda = \rho$, the decomposition of Eqn. (3.2.37) reads

$$\mathrm{adj}_{\mathfrak{g}} = \bigoplus_{i=1}^{\mathrm{rank}\,\mathfrak{g}} V_{d_i - 1}\ , \tag{3.2.39}$$

where $\{d_i\}$ are the degrees of invariants of $\mathfrak{g}$, so in particular $d_i = i+1$ for $\mathfrak{su}(n)$. We then find

$$\psi^{\rho}_{\mathfrak{R}}(q) = \mathrm{PE}\left[\sum_{i}^{\mathrm{rank}\,\mathfrak{g}} \frac{q^{d_i}}{1-q}\right] \chi^{\mathfrak{g}}_{\mathfrak{R}}(\mathrm{fug}_{\rho}(q)) \ . \tag{3.2.40}$$

For $\mathfrak{g} = \mathfrak{su}(n)$, the fugacity assignment associated to the principal embedding takes a particularly simple form,

$$\mathrm{fug}_{\rho}(q) = (q^{\frac{n-1}{2}}, q^{\frac{n-3}{2}}, \ldots q^{-\frac{n-1}{2}}) \ . \tag{3.2.41}$$

Together, Eqns. (3.2.29), (3.2.40), and (3.2.41) provide an explicit expression for the structure constants $C_{\mathfrak{R}}(q)$.

Finally, let us recall the procedure for gluing two theories along maximal punctures at the level of the index. Consider two theories $\mathcal{T}_1 = \mathcal{T}[\mathfrak{g}; \mathcal{C}_{g_1,s_1}; \{\Lambda_i\}]$ and $\mathcal{T}_2 = \mathcal{T}[\mathfrak{g}; \mathcal{C}_{g_2,s_2}; \{\Lambda_i\}]$, each of which are assumed to have at least one maximal puncture. We denote their Schur indices as $\mathcal{I}^{\mathrm{Schur}}_{\mathcal{T}_1}(q; \mathbf{a}, \ldots)$ and $\mathcal{I}^{\mathrm{Schur}}_{\mathcal{T}_2}(q; \mathbf{b}, \ldots)$, where we have singled out the dependence on flavor fugacities $\mathbf{a}$ and $\mathbf{b}$ of the two maximal punctures that are going to be glued. As usual, gluing corresponds to gauging the diagonal subgroup of the flavor symmetry $G \times G$ associated to the two maximal punctures. The index of the glued theory is then given by

$$\oint [d\mathbf{a}] \Delta(\mathbf{a}) \ I_V(q; \mathbf{a}) \ \mathcal{I}^{\mathrm{Schur}}_{\mathcal{T}_1}(q; \mathbf{a}, \ldots) \ \mathcal{I}^{\mathrm{Schur}}_{\mathcal{T}_2}(q; \mathbf{a}^{-1}, \ldots) \ , \tag{3.2.42}$$

where $[d\mathbf{a}] := \prod_{j=1}^{r} \frac{da_j}{2\pi i a_j}$, $\Delta(\mathbf{a})$ is the Haar measure, and $I_V(q; \mathbf{a})$ is the index of an $\mathcal{N} = 2$ vector multiplet in the Schur limit,

$$I_V(q; \mathbf{a}) = \mathrm{PE}\left[\frac{-2q}{1-q} \chi_{\mathrm{adj}}(\mathbf{a})\right] = K_{\max}(\mathbf{a}; q)^{-2} \ . \tag{3.2.43}$$

If we write the indices of $\mathcal{T}_1$ and $\mathcal{T}_2$ in the form dictated by Eqn. (3.2.28), then the contour integral is rendered trivial because the K-factors in the wave functions that are being glued cancel against the index of the vector multiplet and the characters $\chi^{\mathfrak{g}}_{\mathfrak{R}}$ are orthonormal with respect to the Haar measure. The result is that we obtain an expression that takes the form of Eqn. (3.2.28), but with $g = g_1 + g_2$ and $s = s_1 + s_2 - 2$.

Higgsing and reduction of punctures: superconformal index

We will now argue that the expression given in Eqn. (3.2.35) for the general wavefunction of type Λ is dictated by the Higgsing procedure if one takes for granted the formula given in Eqn. (3.2.38) for the maximal wavefunction. In fact, the argument we are about to present should be applicable outside of the narrow context under consideration here, so for some parts of the argument we will use a fairly general language.

We are interested in the relationship between the Schur limit of the superconformal index of an $\mathcal{N} = 2$ SCFT and that of the low energy theory at a point on the Higgs branch. It is a familiar feature of supersymmetric indices that in some sense the only difference between the indices of UV and IR fixed points is a possible redefinition of fugacities. In particular, if a renormalization group flow is triggered by a vev that breaks some global symmetry, then the fugacities dual to the broken generators must be set to zero. Furthermore, if the index is to be interpreted as a *superconformal* index of the IR fixed point, then the appropriate R-symmetries that appear in the superconformal algebra of that fixed point must be identified and the fugacities redefined appropriately.

There are two related obstacles to applying this simple reasoning in many cases. One is the appearance of accidental symmetries at the IR fixed point. Fugacities dual to the generators of accidental symmetries cannot be introduced in the UV description of the index, and so in particular if the superconformal R-symmetry in the IR mixes with accidental symmetries, then the superconformal index is inaccessible. The second obstacle is the possible presence of decoupled free fields in addition to the degrees of freedom of interest at low energies. These two issues are related because whenever decoupled free fields emerge at low energies, there will necessarily be an accidental global symmetry that acts just on those fields, and this symmetry will generally contribute to the superconformal R-symmetry.

In nice cases it is possible to overcome these obstacles and write the superconformal index of the IR theory in terms of that of the UV fixed point in a fairly simple way. Sufficient conditions for us to be able to do this are:

- The only accidental symmetries at the IR fixed point are those associated to the decoupled Nambu-Goldstone bosons of spontaneous symmetry breaking.
- The Cartan generator of the $\mathfrak{su}(2)_R$ symmetry of the IR fixed point,

when restricted to act on operators in the interacting sector, can be identified and written as a linear combination of UV symmetries.

- The Higgs branch chiral ring operators that become the field operators for Nambu-Goldstone bosons in the infrared are identifiable, and their quantum numbers with respect to UV symmetries known.

When these conditions are met, the prescription for computing the index of the IR fixed point is simple, and amounts to subtracting out the contributions of the decoupled free fields to the index,

$$\mathcal{I}_{\mathrm{IR}}(q;\mathbf{x}_{\mathrm{IR}}) = \lim_{\mathbf{x}_{\mathrm{UV}}\to\mathbf{x}_{\mathrm{IR}}} \frac{\mathcal{I}_{\mathrm{UV}}(q;\mathbf{x}_{\mathrm{UV}})}{\mathcal{I}_{\mathrm{NGB}}(q;\mathbf{x}_{\mathrm{UV}})}~. \tag{3.2.44}$$

Here $\mathbf{x}_{\mathrm{UV}}$ are the fugacities dual to the UV global symmetries, while $\mathbf{x}_{\mathrm{IR}}$ are those dual to the IR global symmetries. The two sets of fugacities are related to one another by a specialization. The denominator on the right hand side is the index of $\frac{1}{2}N_{\mathrm{NGB}}$ free hypermultiplets, where N_{NGB} is the number of complex Nambu-Goldstone bosons at the chosen locus of the Higgs branch. The only subtlety is that the contributions of these free hypermultiplets are graded according to the charges of the Higgs branch chiral ring operator that becomes the field operator for the Nambu-Goldstone boson in the IR, so we have

$$\mathcal{I}_{\mathrm{NGB}}(\mathbf{x}_{\mathrm{UV}};q) := \mathrm{PE}\left(\sum_{\mathcal{O}_i}\frac{q^{R_{\mathcal{O}_i}}\mathbf{x}^{f_{\mathcal{O}_i}}}{1-q}\right)~. \tag{3.2.45}$$

The reason that Eqn. (3.2.44) involves a limit is that the index will have a pole at the specialized values of the fugacities. It is easy to see that this will be the case because operators that acquire expectation values in the Higgs branch vacuum of interest will always be uncharged under all of the fugacities appearing in the specialized index. This invariably leads to a divergence in the index.

Now let us return to the specific case of interest: the reduction of punctures in class $\mathcal{S}$ theories. All of the conditions listed above are met. The only accidental symmetries are those that act only on the decoupled Nambu-Goldstone bosons arising from the spontaneous breaking of global and scale symmetries. The Cartan generator of the low energy $\mathfrak{su}(2)_R$ (when restricted to act in the interacting sector) was identified in Eqn. (3.2.25). Finally, we know precisely which operators in the UV theory will become the field operators for the Nambu-Goldstone bosons (*cf.* Eqn. (3.2.27)). Consequently we know how these decoupling operators are acted upon by the UV symmetries.

Describing the index of the (interacting part) of the IR theory resulting from the Higgsing associated to an embedding Λ in terms of the theory with maximal punctures is now a simple exercise. The relevant specialization is accomplished by redefining the $\mathfrak{su}(2)_R$ Cartan in the index according to Eqn. (3.2.25), which leads to the replacement rule $\mathbf{x} \mapsto \mathrm{fug}_\Lambda(\mathbf{x}_\Lambda; q)$. The character $\chi^{\mathfrak{g}}_{\mathfrak{R}}$ is regular under this specialization. To check that we obtain the expected wavefunction for the reduced puncture given in Eqns. (3.2.35) and (3.2.36) it only remains to verify that the K-factors behave in the expected manner. The fugacity replacement in the K-factor of the maximal puncture leads to the following rewriting,

$$K_{\max}(\mathbf{a}; q) = \mathrm{PE}\left[\frac{q}{1-q}\chi_{\mathrm{adj}_{\mathfrak{g}}}(\mathbf{a})\right] \;\to\; \mathrm{PE}\left[\frac{q}{1-q}\sum_j \chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}_j^{(\mathrm{adj})}}(\mathbf{a}_\Lambda)\;\chi^{\mathfrak{su}(2)}_{V_j}(q^{\frac{1}{2}}, q^{-\frac{1}{2}})\right], \tag{3.2.46}$$

and upon expanding out the character $\chi^{\mathfrak{su}(2)}_{V_j}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}) = \sum_{m=-j}^{+j} q^m$, we find the expression

$$K_{\max}(\mathbf{a}; q) \to \mathrm{PE}\left[\sum_j \frac{q^{j+1}}{1-q}\chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}_j^{(\mathrm{adj})}}(\mathbf{a}_\Lambda)\right]\mathrm{PE}\left[\frac{q}{1-q}\sum_j \chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}_j^{(\mathrm{adj})}}(\mathbf{a}_\Lambda)\sum_{m=-j}^{+j-1} q^m\right]. \tag{3.2.47}$$

The first factor here reproduces the K-factor of the reduced flavor puncture given in Eqn. (3.2.36). The second factor is strictly divergent because there are constant terms in the plethystic exponent. However it is precisely this second factor that is cancelled by the denominator in Eqn. (3.2.44). We have been a little careless in this treatment by making a formal fugacity replacement and then cancelling an infinite factor. A more rigorous treatment proceeds via the limiting procedure described above, and produces the same result.

3.3 Chiral algebras of class $\mathcal{S}$

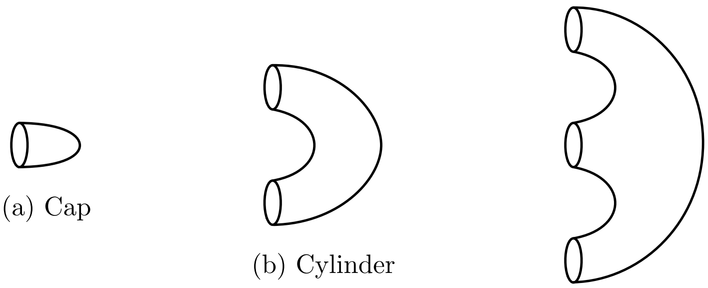
The organization of class $\mathcal{S}$ theories in terms of two-dimensional conformal geometry has important implications for observables of these theories. In particular, any observable that is independent of exactly marginal couplings should give rise to a (generalized) topological quantum field theory upon identifying a given theory with its UV curve. As reviewed above, this insight was originally exploited in the study of the superconformal index [74, 75, 76, 37]. Subsequently the strategy was formalized and extended to the case of the (maximal) Higgs branch in [78]. There it was emphasized that this approach has the additional benefit of providing a way to study the superstructure of class $\mathcal{S}$ with some degree of mathematical rigor, evading problems associated with the definition of interacting quantum field theories. The basic idea is summarized in the following commutative diagram.

$$\begin{tikzcd} & \mathcal{T}\,[\,\mathcal{C}_{g,s};\mathfrak{g};\{\Lambda_i\}\,] \arrow[dr, "\mathbb{P}"] & \\ \{\mathcal{C}_{g,s};\{\Lambda_i\}\} \arrow[ur, "\mathcal{T}_{\mathfrak{g}}"] \arrow[rr, "\mathbb{P}\circ\mathcal{T}_{\mathfrak{g}}"] & & \mathbb{P}\,[\,\mathcal{T}\,[\,\mathcal{C}_{g,s};\mathfrak{g};\{\Lambda_i\}\,]\,] \end{tikzcd} \tag{3.3.1}$$

For some protected observable $\mathbb{P}$ that can be defined for an $\mathcal{N}=2$ SCFT, one defines the composition $\mathbb{P}\circ\mathcal{T}_{\mathfrak{g}}$ that associates the observable in question directly to a UV curve. When the observable is something relatively simple – like the holomorphic symplectic manifolds studied in [78] – one should be able to define this composition in a rigorous fashion without having to define the more complicated $\mathcal{T}_{\mathfrak{g}}$-functor at all.

In the present work we take as our "observable" the protected chiral algebra, which is indeed independent of marginal couplings. The composition $\chi\circ\mathcal{T}_{\mathfrak{g}}$ has as its image the chiral algebras of class $\mathcal{S}$, which are labelled by Riemann surfaces whose punctures are decorated by embeddings $\Lambda:\mathfrak{sl}(2)\hookrightarrow\mathfrak{sl}(n)$. This class of chiral algebras has the form of a generalized topological quantum field theory.

The aim of this section is to develop a basic picture of the structure of this TQFT and to characterize it to the extent possible. In the first subsection, we make some general statements about the implications of the TQFT structure from a physicist's point of view. We also make a modest attempt to formalize the predicted structure in a language closer to that employed in the mathematics literature. In the second subsection, we discuss the basic building blocks of the TQFT for the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ cases. We also make a conjecture about the general case. In the last subsection we make some comments about the constraints of associativity and possible approaches to solving for the class $\mathcal{S}$ chiral algebras at various levels of generality.



(a) Cap

(b) Cylinder

(c) Trinion

Figure 3.1: Elementary building blocks of a two-dimensional TQFT.

3.3.1 A TQFT valued in chiral algebras

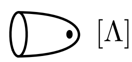
In a physicist's language, the type of generalized TQFT we have in mind is specified by associating a chiral algebra with each of a small number of (topological) Riemann surfaces with boundary, namely the genus zero surface with one, two, or three boundary circles (see Fig. 3.1).[9] We must further give a meaning to the procedure of gluing Riemann surfaces along common boundaries at the level of the chiral algebra. Self-consistency of the generalized TQFT then requires that the resulting structure be associative in that it reflects the equivalence of Fig. 3.3.

The full class $\mathcal{S}$ structure is more complicated than can be captured by this basic version of a generalized TQFT due to the possibility of choosing nontrivial embeddings to decorate the punctures. We can partially introduce this additional structure by allowing the decorated objects illustrated in Fig. 3.2. For our purposes these will be thought of as decorated versions of the cap and cylinder. In choosing this interpretation, we are ignoring the fact that in class $\mathcal{S}$ one can in certain cases glue along a non-maximal puncture. This fact plays an important role already in the basic example of Argyres-Seiberg duality interpreted as a class $\mathcal{S}$ duality. These decorated fixtures will also be required to satisfy certain obvious associativity conditions.

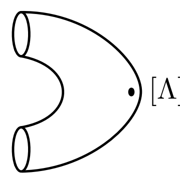
We can define the gluing operation for chiral algebras associated to these elementary surfaces by knowing a few of the general features of these chiral

[9]Strictly speaking, this is a redundant amount of information because composing a trinion with a cap produces a cylinder. In anticipating the fact that the chiral algebra for the cap is somewhat difficult to understand, we are considering them independently.

algebras. Namely, it is guaranteed that the chiral algebras associated to these surfaces will include affine current subalgebras associated to their boundary circles. Indeed, to every full puncture in a class $\mathcal{S}$ theory of type $\mathfrak{su}(n)$ there is associated an $\mathfrak{su}(n)$ global symmetry with central charge $k_{4d} = 2n$. Correspondingly, the associated chiral algebra will have an $\widehat{\mathfrak{su}(n)}_{-n}$ affine current subalgebra. Knowing this, the composition rule for chiral algebras follows more or less immediately from the rules for gauging reviewed in Sec. 3.2.1. Two legs with maximal punctures can be glued by introducing (b, c) ghosts transforming in the adjoint of $\mathfrak{su}(n)$ and passing to the cohomology of a BRST operator formed with the diagonal combination of the two affine current algebras.



(a) Decorated cap



(b) Decorated cylinder

Figure 3.2: Additional building blocks of a class $\mathcal{S}$ TQFT.

Given the fairly involved nature of this gluing operation, associativity for the TQFT as illustrated in Fig. 3.3 is an extremely nontrivial property. Indeed, it is the reflection of generalized S-duality of the four-dimensional SCFTs of class $\mathcal{S}$ at the level of chiral algebras. It is not *a priori* obvious that it should even be possible to find chiral algebras for which this gluing will satisfy the associativity conditions, and the existence of such a family of chiral algebras is an interesting prediction that follows from the existence of the class $\mathcal{S}$ landscape.

For the sake of the mathematically inclined reader, we can now formalize this structure a bit more to bring the definition of this generalized TQFT into line with the standard mathematical description. This type of a formalization has also been presented by Yuji Tachikawa [89]. The structure in question is a strict symmetric monoidal functor between two symmetric monoidal categories that we outline now.

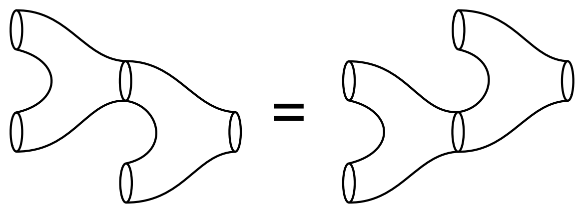


Figure 3.3: Associativity of composition of T_n chiral algebras.

The source category

The source category is a decorated version of the usual bordism category Bo_2. It has previously appeared in [78] for the same purpose. In fact, there is a separate such category for each simply laced Lie algebra $\mathfrak{g}$ (which for us will always be $\mathfrak{su}(n)$ for some n), and we will denote it as $\mathrm{Bo}_2^{(\mathfrak{g})}$. The category has the following structure:

- The objects of $\mathrm{Bo}_2^{(\mathfrak{g})}$ are the same as for the Bo_2 – they are closed oriented one-manifolds (*i.e.*, disjoint unions of circles).
- A morphism in Bo_2 between two objects B_1 and B_2 is a two-dimensional oriented manifold B that is a bordism from B_1 to B_2. A morphism in $\mathrm{Bo}_2^{(\mathfrak{g})}$ is a morphism from Bo_2 that is additionally decorated by an arbitrary finite number of marked points $\{s_i\}$, each of which is labelled by an embedding $\Lambda_i : \mathfrak{su}(2) \hookrightarrow \mathfrak{g}$.
- Composition is the usual composition of bordisms by gluing along boundaries.
- The symmetric monoidal structure is given by taking disjoint unions.
- This category has duality, which follows from the existence of left- and right-facing cylinders for which the S-bordisms of Fig. 3.4 are equivalent to the identity.

The target category

The target category is a certain category of chiral algebras that we will call $\mathbb{CA}_{\mathfrak{g}}$. We define it as follows

- The objects are finite tensor powers of the $\mathfrak{g}$ affine current algebra at the critical level. This includes the case where the power is zero, which corresponds to the trivial chiral algebra for which only the identity operator is present.

$$\mathbf{Obj}(\mathbb{CA}_{\mathfrak{g}}) = \coprod_{n=0}^{\infty} \left(\otimes^n \hat{\mathfrak{g}}_{-h^\vee}\right) \ .$$

- Given two objects $\mathfrak{o}_1 = \otimes^{n_1} \hat{\mathfrak{g}}_{-h^\vee}$ and $\mathfrak{o}_2 = \otimes^{n_2} \hat{\mathfrak{g}}_{-h^\vee}$, the morphisms $\mathbf{Hom}(\mathfrak{o}_1, \mathfrak{o}_2)$ are *conformal* chiral algebras containing $\mathfrak{o}_1 \otimes \mathfrak{o}_2$ as a subalgebra. Note that this precludes a morphism which is just equal to several copies of the critical affine Lie algebra, since there would be no stress tensor.

- For $\chi_1 \in \mathbf{Hom}(\mathfrak{o}_1, \mathfrak{o}_2)$ and $\chi_2 \in \mathbf{Hom}(\mathfrak{o}_2, \mathfrak{o}_3)$, the composition $\chi_2 \circ \chi_1 \in \mathbf{Hom}(\mathfrak{o}_1, \mathfrak{o}_3)$ is obtained by the BRST construction of Sec. 3.2.1. That is, one first introduces $\dim \mathfrak{g}$ copies of the $(1,0)$ ghost system and then passes to the cohomology of the nilpotent BRST operator relative to the b-ghost zero modes,

$$\chi_2 \circ \chi_1 = H^*_{\text{BRST}}(\psi \in \chi_1 \otimes \chi_2 \otimes \chi_{(b,c)_{\mathfrak{g}}} \,\big|\, b_0 \psi = 0) \ .$$

  It is straightforward to show that this composition rule is associative.

- The symmetric monoidal structure is given by taking tensor products of chiral algebras.

- The duality structure in this category is somewhat complicated and involves the precise form of the chiral algebra that is the image of the cylinder in $\mathbf{Hom}(S^1 \sqcup S^1, \varnothing)$. We delay discussion of this chiral algebra until Sec. 3.5.1. For now, we define a weaker version of duality – namely that there exists a certain action of $(\mathbb{Z}_2)^r$ on the collection $\coprod_{p+q=r} \mathbf{Hom}((\hat{\mathfrak{g}}_{-h^\vee})^p, (\hat{\mathfrak{g}}_{-h^\vee})^q)$ that corresponds to the action of changing external legs of a bordism from ingoing to outgoing and vice versa. This action is simple to describe. Note that a chiral algebra belonging to the above collection of **Hom** spaces can be described as r copies of the critical $\hat{\mathfrak{g}}$ current algebra along with (possibly infinitely many) additional generators transforming as modules. The primary states of each such module with respect to the affine current algebras will transform

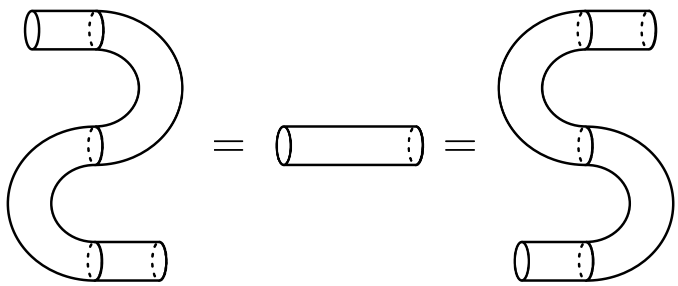


Figure 3.4: Duality and the S-diagram.

in some representation $\mathfrak{R}_1 \otimes \cdots \otimes \mathfrak{R}_r$ of the global $\mathfrak{su}(n)^r$ symmetry. The duality action associated to flipping the i'th leg of a bordism then acts as $\mathfrak{R}_i \mapsto \mathfrak{R}_i^*$, and this action lifts to the full chiral algebra in the obvious way.

The functor

A chiral algebra-valued TQFT of type $\mathfrak{g}$ can now be defined as a functor that realizes the horizontal arrow in diagram (3.3.1),

$$\chi \circ \mathcal{T} : \mathrm{Bo}_2^{(\mathfrak{g})} \to \mathbb{C}\mathbb{A}_{\mathfrak{g}} \ .$$

The image of such a functor in $\mathbb{C}\mathbb{A}_{\mathfrak{g}}$ defines a very interesting set of chiral algebras. The necessary ingredients to define this functor are those outlined in the previous discussion. Namely, we need to specify the images of the basic topological Riemann surfaces in Fig. 3.1 and the decorated versions in Fig. 3.2. In order for this to be a functor, the composition of Riemann surfaces with three boundary components must be associative in the sense of Fig. 3.3. A similar associativity condition is obtained by replacing any of the boundary components in Fig. 3.3 with a general decoration Λ.

The problem of including decorations can be self-consistently ignored in order to focus on the subproblem in which the source category is the more traditional bordism category Bo_2. In the remainder of this section we will address the problem of understanding this more basic version of the TQFT, sometimes with the addition of simple punctures, but not the most general case. The addition of arbitrary decorations will be discussed in Sec. 3.4.

3.3.2 Lagrangian class $\mathcal{S}$ building blocks

The basic building blocks of class $\mathcal{S}$ SCFTs are the theories associated to spheres with three punctures. Of these, the simplest case is the theory with two maximal punctures and one minimal puncture. This is the only regular configuration which gives rise to a Lagrangian theory for arbitrary choice of ADE algebra. For the $\mathfrak{su}(n)$ theory, it is the theory of n^2 free hypermultiplets, so the associated chiral algebra is the theory of n^2 symplectic boson pairs, see chapter 2. Though this chiral algebra has a full $\mathfrak{usp}(2n^2)$ symmetry, it is natural to use a basis which makes manifest the $\mathfrak{su}(n)_1 \times \mathfrak{su}(n)_2 \times \mathfrak{u}(1)$ symmetry associated to the punctures,

$$q_a^i(z)\tilde{q}_j^b(0) \sim \frac{\delta_j^i \delta_a^b}{z} , \qquad i = 1, \ldots, n \quad a = 1, \ldots, n . \tag{3.3.2}$$

The currents generating the puncture symmetries are the chiral algebra relatives of the moment map operators in the free hypermultiplet theory,

$$\begin{aligned}
(J_{\mathfrak{su}(n)_1})_j^i(z) &:= -(q_a^i \tilde{q}_j^a)(z) + \tfrac{1}{n}\delta_i^j (q_a^k \tilde{q}_k^a)(z) , \\
(J_{\mathfrak{su}(n)_2})_a^b(z) &:= -(q_a^i \tilde{q}_i^b)(z) + \tfrac{1}{n}\delta_a^b (q_c^i \tilde{q}_i^c)(z) , \\
J_{\mathfrak{u}(1)}(z) &:= -(q_a^i \tilde{q}_i^a)(z) .
\end{aligned} \tag{3.3.3}$$

The $\mathfrak{su}(n)$ current algebras are each at level $k_{\mathfrak{su}(n)} = -n$. Additionally, the canonical stress tensor for this chiral algebra descends from the $\mathfrak{su}(2)_R$ current of the free hypermultiplet theory

$$T(z) := (q_i^a \partial \tilde{q}_a^i)(z) - (\tilde{q}_a^i \partial q_i^a)(z) . \tag{3.3.4}$$

The central charge of the Virasoro symmetry generated by this operator is given by $c = -n^2$.

These Lagrangian building blocks can be used to build up the chiral algebras associated to any of the Lagrangian class $\mathcal{S}$ theories, *i.e.*, to those theories constructed from linear or circular quivers. For example, the chiral algebras for $\mathcal{N} = 2$ superconformal QCD were studied in chapter 2, and these theories are constructed from a pair of these free field trinions by gauging a single $\mathfrak{su}(n)$ symmetry. The TQFT structure associated to these Lagrangian theories is already quite interesting, but we will not dwell on the subject here since these Lagrangian constructions are only the tip of the iceberg for class $\mathcal{S}$. Indeed, from an abstract point of view there is a different set of theories that are the most natural starting point for an investigation of class $\mathcal{S}$ chiral algebras. These are the chiral algebras associated to spheres with three maximal punctures.

3.3.3 Trinion chiral algebras

Our first order of business should then be to understand the elementary building blocks for class $\mathcal{S}$ chiral algebras of type $\mathfrak{g} = \mathfrak{su}(n)$, which are the trinion chiral algebras $\chi[T_n]$. In this section we will try to outline the general properties of these chiral algebras. It is possible that these properties will actually make it possible to fix the chiral algebras completely. It is a hard problem to characterize these algebras for arbitrary n. Doing so implicitly involves fixing an infinite amount of CFT data (*i.e.*, operator dimensions and OPE coefficients) for the T_n SCFTs, and this data is apparently inaccessible to the usual techniques used to study these theories. Nevertheless, many properties for these chiral algebras can be deduced from the structure of the χ map and from generalized S-duality.

Central charge

From the general results reviewed in Section 3.2.1, we know that the chiral algebra of any T_n theory should include a Virasoro subalgebra, the central charge of which is determined by the c-type Weyl anomaly coefficient of the parent theory according to the relation $c_{2d} = -12c_{4d}$. The central charges of the T_n theories have been computed in [90], and from those results we conclude that the corresponding chiral algebras will have Virasoro central charges given by

$$c_{2d}(\chi[\mathcal{T}_n]) = -2n^3 + 3n^2 + n - 2 \ . \tag{3.3.5}$$

For any value of n the Virasoro central charge predicted by this equation is an even negative integer. These chiral algebras will necessarily be non-unitary, as is always the case for the protected chiral algebras of four-dimensional theories. For reference, we display the Virasoro central charges for $\chi[T_n]$ for low values of n in Table 3.2.

n	2	3	4	5	6	7
c_{2d}	-4	-26	-78	-172	-320	-534

Table 3.2: Central charges of the chiral algebras $\chi[T_n]$ for small values of n.

Affine current subalgebras

Global symmetries of the T_n theories imply the presence of chiral subalgebras that are isomorphic to the affine current algebras for the same symmetry algebra. The levels k_{2d} of these affine current algebras are fixed in terms of the four-dimensional flavor central charges k_{4d} according to $k_{2d} = -\frac{1}{2}k_{4d}$. The T_n theories have $\mathfrak{su}(n)^3$ global symmetry with each $\mathfrak{su}(n)$ factor associated to one of the punctures on the UV curve. The flavor central charge for each $\mathfrak{su}(n)$ is given by $k_{4d} = 2n$. Consequently, the chiral algebras $\chi[T_n]$ will have affine current subalgebras of the form

$$\widehat{\mathfrak{s}u(n)}_{-n} \times \widehat{\mathfrak{s}u(n)}_{-n} \times \widehat{\mathfrak{s}u(n)}_{-n} \subset \chi[T_n]\ . \tag{3.3.6}$$

Note that $k_{2d} = -n$ is the *critical level* for an $\widehat{\mathfrak{su}(n)}$ current algebra, which means that the Sugawara construction of a stress tensor fails to be normalizable. The chiral algebras $\chi[T_n]$ will still have perfectly good stress tensors, but they will not be given by the Sugawara construction. Precisely the critical affine current algebra $\mathfrak{su}(n)_{-n}$ has been argued in [26] to describe the protected chiral algebra that lives on maximal codimension two defects of the six-dimensional $(2,0)$ theory in flat six dimensional space. Its reappearance as a subalgebra of the class $\mathcal{S}$ chiral algebra is then quite natural. It would be interesting to develop a better first-principles understanding of the relationship between BPS local operators supported on codimension two defects in six dimensions and local operators in the class $\mathcal{S}$ theories obtained by compactification in the presence of said defects.

Chiral algebra generators from the Higgs branch

A definitive characterization of the generators of the protected chiral algebra in terms of the operator spectrum of the parent theory is presently lacking. However, as we reviewed in Section 3.2.1, any generator of the Hall-Littlewood chiral ring is guaranteed to a generator of the chiral algebra. For the T_n theories, the Hall-Littlewood chiral ring is actually the same thing as the Higgs branch chiral ring due to the absence of $\mathcal{D}$ and $\bar{\mathcal{D}}$ multiplets in genus zero class $\mathcal{S}$ theories. The list of generators of the Higgs branch chiral ring is known for the T_n theories, so we have a natural first guess for the list of generators of these chiral algebras.

In the interacting theories (all but the T_2 case), the moment map operators for the flavor symmetry acting on the Higgs branch are chiral ring generators. The corresponding chiral algebra generators are the affine currents described above. There are additional generators of the form [84]

$$Q_{(\ell)}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}~, \qquad \ell = 1, \cdots, n-1~. \tag{3.3.7}$$

These operators are scalars of dimension $\Delta = \ell(n-\ell)$ that transform in the $\wedge^\ell$ representation (the ℓ-fold antisymmetric tensor) of each of the $\mathfrak{su}(n)$ flavor symmetries. There must therefore be *at least* this many additional chiral algebra generators. We may denote these chiral algebra generators as

$$W_{(\ell)}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}(z)~, \qquad \mathcal{I} = [i_1 \cdots i_\ell]~, \quad i_* = 1, \ldots, n~. \tag{3.3.8}$$

These operators will have dimension $h_\ell = \frac{1}{2}\ell(n-\ell)$, so for $n > 3$ we are guaranteed to have non-linear chiral algebras.

For $n > 3$ the stress tensor must be an independent generator of the chiral algebra. This is because the stress tensor can only be a composite of other chiral algebra operators with dimension $h \leqslant 1$. For an interacting theory there can be no chiral algebra operators of dimension $h = 1/2$, so the only possibility is that the stress tensor is a Sugawara stress tensor built as a composite of affine currents. This can only happen if the $\mathfrak{su}(n)^3$ symmetry is enhanced, since as we have seen above the affine currents associated to the $\mathfrak{su}(n)$ symmetries are at the critical level and therefore do not admit a normalizable Sugawara stress tensor. Such an enhancement of the flavor symmetry only happens for the $n = 3$ case, as will be discussed in greater detail below.

Let us now consider the two simplest cases of trinion chiral algebras: $n = 2$ and $n = 3$. These are both exceptional in some sense compared to our expectations for generic n, which will ultimately makes them easier to work with in our examples.

The $\chi[T_2]$ chiral algebra

In the rank one case, the trinion SCFT is a theory of free hypermultiplets. This case is exceptional compared to the general free hypermultiplets discussed in Section 3.3.2 because for $\mathfrak{su}(2)$ the maximal puncture and minimal puncture are the same, so the minimal puncture also carries an $\mathfrak{su}(2)$ flavor symmetry, and instead of n^2 hypermultiplets transforming in the bifundamental of $\mathfrak{su}(n) \times \mathfrak{su}(n)$, one instead describes the free fields as $2^3 = 8$ half hypermultiplets transforming in the trifundamental representation of $\mathfrak{su}(2)^3$. Consequently the symplectic bosons describing this theory are organized into a trifundamental field $q_{abc}(z)$ with $a, b, c = 1, 2$, with OPE given by

$$q_{abc}(z) q_{a'b'c'}(w) \sim \frac{\epsilon_{aa'}\epsilon_{bb'}\epsilon_{cc'}}{z-w} \ . \tag{3.3.9}$$

Each of the three $\mathfrak{su}(2)$ subalgebras has a corresponding $\widehat{\mathfrak{su}(2)}_{-2}$ affine current algebra in the chiral algebra. For example, the currents associated to the first puncture are given by

$$\begin{aligned} J_1^+(z) &:= \frac{1}{2}\epsilon^{bb'}\epsilon^{cc'}(q_{1bc}q_{1b'c'})(z) \ , \\ J_1^-(z) &:= \frac{1}{2}\epsilon^{bb'}\epsilon^{cc'}(q_{2bc}q_{2b'c'})(z) \ , \\ J_1^0(z) &:= \frac{1}{4}\epsilon^{bb'}\epsilon^{cc'}\Big[(q_{1bc}q_{2b'c'})(z) + (q_{2bc}q_{1b'c'})(z)\Big] \ . \end{aligned} \tag{3.3.10}$$

The currents associated to the second and third punctures are constructed analogously. The stress tensor is now given by

$$T(z) := \epsilon^{aa'}\epsilon^{bb'}\epsilon^{cc'}(q_{abc}\partial q_{a'b'c'})(z) \ , \tag{3.3.11}$$

with corresponding Virasoro central charge given by $c_{2d} = -4$.

In this simple case it is easy to explicitly compare the Schur superconformal index for the T_2 theory with the vacuum character of the chiral algebra. The Schur index has appeared explicitly in, *e.g.*, [75]. It is given by a single plethystic exponential,

$$\mathcal{I}(q; \mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathrm{PE}\left[\frac{q^{\frac{1}{2}}}{1-q}\chi_{\Box}(\mathbf{a})\chi_{\Box}(\mathbf{b})\chi_{\Box}(\mathbf{c})\right] \ . \tag{3.3.12}$$

This is easily recognized as the vacuum character of the symplectic boson system defined here. The only comment that needs to be made is that there are no null states that have to be removed from the freely generated character of the symplectic boson algebra. In the next example this simplifying characteristic will be absent.

Crossing symmetry, or associativity of gluing, was investigated for this chiral algebra in chapter 2. There it was proposed that the complete chiral algebra obtained when gluing two copies of $\chi[T_2]$ is the $\widehat{\mathfrak{so}(8)}$ affine current algebra at level $k_{\mathfrak{so}(8)} = -2$, and this proposal was checked up to level $h = 5$. If the chiral algebra of the four-punctured sphere is precisely this current algebra, then the crossing symmetry relation is implied immediately. This is because the $\mathfrak{so}(8)$ current algebra has an automorphism as a consequence of triality that exchanges the $\mathfrak{su}(2)$ subalgebras in accordance with Figure 3.3. If one could prove that the solution to the BRST problem for this gluing is the $\widehat{\mathfrak{so}(8)}$ current algebra, one would therefore have a proof of generalized S-duality at the level of the chiral algebra for all rank one theories of class $\mathcal{S}$. We hope that such a proof will turn out to be attainable in the future.

The $\chi[T_3]$ chiral algebra

The T_3 theory is the rank-one $\mathfrak{e}_6$ theory of Minahan and Nemeschanksky [61]. Before describing its chiral algebra, let us list a number of known properties of this theory.

- The a and c_{4d} anomaly coefficients are known to be given by $a = \frac{41}{24}$ and $c_{4d} = \frac{13}{6}$.
- The global symmetry is $\mathfrak{e}_6$, for which the flavor central charge is $k_{\mathfrak{e}_6} = 6$. This is an enhancement of the $\mathfrak{su}(3)^3$ symmetry associated with the punctures. It can be understood as a consequence of the fact that the extra Higgs branch generators have dimension two in this case, which means that they behave as moment maps for additional symmetry generators.
- The Higgs branch of this theory is the $\mathfrak{e}_6$ one-instanton moduli space, which is the same thing as the minimal nilpotent orbit of $\mathfrak{e}_6$. This property follows immediately from the realization of this theory as a single D3 brane probing an $\mathfrak{e}_6$ singularity in F-theory.
- A corollary of this characterization of the Higgs branch is that the Higgs branch chiral ring is finitely generated by the moment map operators μ_A for $A = 1, \ldots, 78$, subject to the *Joseph relations* (see *e.g.* [64]),

$$(\mu \otimes \mu)\big|_{\mathbf{1}\oplus\mathbf{650}} = 0 \ .$$

- The superconformal index of the T_3 theory was computed in [76]. This leads to a formula for the Schur limit of the index given by

$$
\begin{aligned}
\mathcal{I}_{T_3}(q) = 1 \; &+ q\chi_{[\mathbf{0,0,0,0,0,1}]} \\
&+ q^2(\chi_{[\mathbf{0,0,0,0,0,2}]} + \chi_{[\mathbf{0,0,0,0,0,1}]} + 1) \\
&+ q^3(\chi_{[\mathbf{0,0,0,0,0,3}]} + \chi_{[\mathbf{0,0,0,0,0,2}]} + \chi_{[\mathbf{0,0,1,0,0,0}]} + 2\;\chi_{[\mathbf{0,0,0,0,0,1}]} + 1) \\
&+ q^4(\chi_{[\mathbf{0,0,0,0,0,4}]} + \chi_{[\mathbf{0,0,0,0,0,3}]} + \chi_{[\mathbf{0,0,1,0,0,1}]} + 3\;\chi_{[\mathbf{0,0,0,0,0,2}]} \\
&\qquad + \chi_{[\mathbf{0,0,1,0,0,0}]} + \chi_{[\mathbf{1,0,0,0,1,0}]} + 3\;\chi_{[\mathbf{0,0,0,0,0,1}]} + 2) \\
&+ \ldots
\end{aligned}
$$

  where we denoted the $\mathfrak{e}_6$ representations by their Dynkin labels and suppressed the fugacity-dependence.

The only chiral algebra generators that are guaranteed to be present on general grounds are the seventy-eight affine currents that descend from the four-dimensional moment map operators. The level of the affine current algebra generated by these operators will be $k = -3$. Note that this is *not* the critical level for $\mathfrak{e}_6$. The $\mathfrak{su}(3)^3$ symmetry associated to the punctures is enhanced, and criticality of the subalgebras does not imply criticality of the enhanced symmetry algebra. For this reason, it is possible to construct a Sugawara stress tensor for the current algebra that is properly normalized, and indeed the correct value of the central charge is given by

$$
c_{2d} = -26 = \frac{-3\dim(\mathfrak{e}_6)}{-3 + h^\vee_{\mathfrak{e}_6}} = c_{\text{Sugawara}} \; . \tag{3.3.13}
$$

One then suspects that the chiral algebra does not have an independent stress tensor as a generator, but instead the Sugawara construction yields the true stress tensor. Indeed, this was proven in chapter 2 to follow from the saturation of certain unitarity bounds by the central charges of this theory.

This leads to a natural proposal for the $\chi[T_3]$ chiral algebra:

Conjecture 4 *The chiral algebra for the rank one E_6 theory, also known as T_3, is isomorphic to the E_6 affine Lie algebra at level $k_{2d} = -3$.*

The singular OPEs of the seventy-eight affine currents are fixed to the canonical form,[10]

$$
J_A(z)J_B(0) \;\sim\; \frac{-3\,\delta_{AB}}{z^2} + \frac{f_{AB}{}^C\,J_C(0)}{z} \; . \tag{3.3.14}
$$

[10] Our conventions are that the roots of $\mathfrak{e}_6$ have squared length equal to two.

It is natural to consider the subalgebra $\mathfrak{su}(3)^3 \subset \mathfrak{e}_6$ associated to the three punctures on the UV curve and to decompose the currents accordingly. The adjoint representation of $\mathfrak{e}_6$ decomposes as

$$\mathbf{78} \longrightarrow (\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8}) + (\mathbf{3},\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}) \ . \tag{3.3.15}$$

The affine currents are therefore rearranged into three sets of $\mathfrak{su}(3)$ affine currents along with one tri-fundamental and one tri-antifundamental set of dimension one currents,

$$J_A(z) \longrightarrow \left\{(J^1)_a^{a'}(z)\ ,\ (J^2)_b^{b'}(z)\ ,\ (J^3)_c^{c'}(z)\ ,\ W_{abc}(z)\ ,\ \widetilde{W}^{abc}(z)\right\} \ . \tag{3.3.16}$$

The singular OPEs for this basis of generators are listed in Appendix B.1. It is perhaps interesting to note that given this list of generators and the requirement that the $\mathfrak{su}(3)$ current algebras are all at the critical level, the only solution to crossing symmetry for the chiral algebra that includes no additional generators is the $\widehat{\mathfrak{e}}_6$ current algebra with $k = -3$. So the chiral algebra is completely inflexible once the generators and their symmetry properties are specified.

A nice check of the whole story is that the Joseph relations are reproduced automatically by the chiral algebra. For the non-singlet relation, this follows in a simple way from the presence of a set of null states in the chiral algebra.

$$\left\| P^{AB}_{\mathbf{650}}(J_A J_B)(z)\right\|^2 = 0 \quad \Longleftrightarrow \quad (\mu\otimes\mu)\big|_{\mathbf{650}} = 0 \ , \tag{3.3.17}$$

where $P^{AB}_{\mathbf{650}}$ is a projector onto the **650** representation. These states are only null at this particular value of the level, so we see a close relationship between the flavor central charge and the geometry of the Higgs branch. Similarly, the singlet relation follows from the identification of the Sugawara stress tensor with the true stress tensor of the chiral algebra,

$$T(z) = \frac{1}{-3+h^\vee}(J_A J_A)(z) \quad \Longleftrightarrow \quad (\mu\otimes\mu)\big|_{\mathbf{1}} = 0 \ . \tag{3.3.18}$$

So in this relation we see that the geometry of the Higgs branch is further tied in with the value of the c-type central charge in four dimensions.

Note that these successes at the level of reproducing the Higgs branch chiral ring relations follow entirely from the existence of an $\widehat{\mathfrak{e}}_6$ current algebra at level $k = -3$ in the chiral algebra. However what is not necessarily implied is the absence of additional chiral algebra generators transforming as some module of the affine Lie algebra. We can test the claim that there are no additional generators by comparing the partition function of the current algebra to the Schur limit of the superconformal index for T_3 (*cf.* [76]).[11] This comparison is made somewhat difficult by the fact that affine Lie algebras at negative integer dimension have complicated sets of null states in their vacuum module, and these must be subtracted to produce the correct index. The upshot is that up to level four, the vacuum character does indeed match the superconformal index. In order for this match to work, it is crucial that the $\widehat{\mathfrak{e}}_6$ current algebra has certain null states at the special value $k = -3$. In Table 3.3, we show the operator content up to level four of a generic $\widehat{\mathfrak{e}}_6$ current algebra along with the subtractions that occur at this particular value of the level. It is only after making these subtractions that the vacuum character matches the Schur index. Thus we conclude that if there are any additional generators of the $\chi[T_3]$ chiral algebra, they must have dimension greater than or equal to five.

A more refined test of our identification of the $\chi[T_3]$ chiral algebra comes from the requirement of compatibility with Argyres-Seiberg duality [47]. The meaning of Argyres-Seiberg duality at the level of the chiral algebra is as follows. Introduce a pair of symplectic bosons transforming in the fundamental representation of an $\mathfrak{su}(2)$ flavor symmetry,

$$q_\alpha(z)\tilde{q}^\beta(0) \sim \frac{\delta_\alpha{}^\beta}{z} , \qquad \alpha, \beta = 1, 2 . \tag{3.3.19}$$

In this symplectic boson algebra one can construct an $\mathfrak{su}(2)$ current algebra at level $k = -1$. Now take the $\mathfrak{e}_6$ current algebra and consider an $\mathfrak{su}(2) \times \mathfrak{su}(6) \subset \mathfrak{e}_6$ maximal subalgebra. The $\mathfrak{su}(2)$ current algebra coming from this subalgebra has level $k = -3$. Thus the combined level of the symplectic-boson-plus-$\chi[T_3]$ system is $k_{tot} = -4$, and consequently this current algebra can be gauged in the manner described in Section 3.2.1 by introducing a (b, c) ghost system in the adjoint of $\mathfrak{su}(2)$ and passing to the cohomology of the appropriate BRST operator. The resulting chiral algebra should be *identical* to the chiral algebra obtained by taking two copies of the $n = 3$ free

[11] Because the current algebra is entirely bosonic, the $\mathbb{Z}_2$ graded vacuum character is the same as the ungraded vacuum character. Indeed, it is a prediction of our identification of the $\chi[T_3]$ chiral algebra that there are no cancellations in the Schur index between operators that individually contribute.

dimension	$\mathfrak{e}_6$ representations with multiplicities $m_{\text{generic}}/m_{k=-3}$
0	$1 \times [\mathbf{0,0,0,0,0,0}]$
1	$1 \times [\mathbf{0,0,0,0,0,1}]$
2	$1 \times [\mathbf{0,0,0,0,0,2}]$, $1/0 \times [\mathbf{1,0,0,0,1,0}]$, $1 \times [\mathbf{0,0,0,0,0,1}]$,
	$1 \times [\mathbf{0,0,0,0,0,0}]$
3	$1 \times [\mathbf{0,0,0,0,0,3}]$, $1/0 \times [\mathbf{1,0,0,0,1,1}]$, $1 \times [\mathbf{0,0,0,0,0,2}]$,
	$2/0 \times [\mathbf{1,0,0,0,1,0}]$, $3/2 \times [\mathbf{0,0,0,0,0,1}]$, $1 \times [\mathbf{0,0,0,0,0,0}]$
	$2/1 \times [\mathbf{0,0,1,0,0,0}]$,
4	$1 \times [\mathbf{0,0,0,0,0,4}]$, $1/0 \times [\mathbf{1,0,0,0,1,2}]$, $1/0 \times [\mathbf{2,0,0,0,2,0}]$,
	$2/1 \times [\mathbf{0,0,1,0,0,1}]$, $1/0 \times [\mathbf{0,1,0,1,0,0}]$, $3/0 \times [\mathbf{1,0,0,0,1,1}]$,
	$2/0 \times [\mathbf{0,0,0,1,1,0}]$, $5/3 \times [\mathbf{0,0,0,0,0,2}]$, $3/1 \times [\mathbf{0,0,1,0,0,0}]$,
	$6/3 \times [\mathbf{0,0,0,0,0,1}]$, $3/2 \times [\mathbf{0,0,0,0,0,0}]$, $1 \times [\mathbf{0,0,0,0,0,3}]$,
	$2/0 \times [\mathbf{1,1,0,0,0,0}]$, $6/1 \times [\mathbf{1,0,0,0,1,0}]$

Table 3.3: The operator content of the $\mathfrak{e}_6$ current algebra up to dimension four. The first multiplicity is valid for generic values of the level, *i.e.*, any value of k where null states are completely absent. The second multiplicity is valid for $k = -3$, and if no second multiplicity is given then the original multiplicity is also the correct one for $k = -3$. These latter multiplicities precisely reproduce the coefficients appearing in the Schur superconformal index for the T_3 theory.

hypermultiplet chiral algebra of Section 3.3.2 and gauging a diagonal $\mathfrak{su}(3)$ current algebra. This comparison is detailed in Appendix B.1.

Although we have not been able to completely prove the equivalence of these two chiral algebras (the BRST problem for this type of gauging is not easy to solve), we do find the following. On each side of the duality, we are able to determine the generators of dimensions $h = 1$ and $h = 3/2$ which amount to a $\widehat{\mathfrak{u}(6)}_{-6}$ current algebra in addition to a pair of dimension $h = \frac{3}{2}$ generators transforming in the tri-fundamental and tri-antifundamental representations of $\mathfrak{u}(6)$, with singular OPEs given by

$$\begin{aligned} b_{i_1 i_2 i_3}(z)\tilde{b}^{j_1 j_2 j_3}(0) \sim & \frac{36\,\delta^{[j_1}_{[i_1}\delta^{j_2}_{i_2}\delta^{j_3]}_{i_3]}}{z^3} - \frac{36\,\delta^{[j_1}_{[i_1}\delta^{j_2}_{i_2}\hat{J}^{j_3]}_{i_3]}(0)}{z^2} \\ & + \frac{18\,\delta^{[j_1}_{[i_1}\hat{J}^{j_2}_{i_2}\hat{J}^{j_3]}_{i_3]}(0) - 18\,\delta^{[j_1}_{[i_1}\delta^{j_2}_{i_2}\partial\hat{J}^{j_3]}_{i_3]}(0)}{z}\ . \end{aligned} \tag{3.3.20}$$

Thus these operators in addition to the $\mathfrak{u}(6)$ currents form a closed $\mathcal{W}$-algebra which is common to both sides of the duality. We expect that these $\mathcal{W}$-algebras are in fact the entire chiral algebras in question. However, it should be noted that the existence of this $\mathcal{W}$-algebra actually follows from what we have established about the $\chi[T_3]$ chiral algebra without any additional assumptions. That is to say, the possible addition of generators of dimension greater than four could not disrupt the presence of this $\mathcal{W}$-algebra. In this sense, common appearance of this algebra can be taken as a check of Argyres-Seiberg duality that goes well beyond the check of [64] at the level of the Higgs branch chiral ring. It not only implies a match of a much larger set of operators than just those appearing in the chiral ring, but it also amounts to a match of the three-point functions for those operators, which include the Higgs branch chiral ring operators.

Finally, let us mention one last consistency check on the identification of $\chi[T_3]$ to which we will return in Section 3.4.4. When one of the three maximal punctures of the T_3 theory is reduced to a minimal puncture by Higgsing, the resulting theory is simply that of nine free hypermultiplets transforming in the bifundamental representation of the remaining $\mathfrak{su}(3) \times \mathfrak{su}(3)$ flavor symmetry (along with a $\mathfrak{u}(1)$ baryon number symmetry associated to the minimal puncture). Therefore if we have correctly identified the $\chi[T_3]$ chiral algebra, then it should have the property that when the corresponding reduction procedure is carried out, the result is the symplectic boson chiral algebra of Section 3.3.2. The proposal we have given will indeed pass this check, but we postpone the discussion until after we present the reduction procedure in Section 3.4.

A proposal for $\chi[T_n]$

We have seen above that for ranks one and two, the trinion chiral algebras are finitely generated (in the chiral algebra sense) by currents that descend from four-dimensional generators of the Higgs branch chiral ring. We know from the results of chapter 2 that this cannot be a characterization that holds true for the chiral algebra of an *arbitrary* $\mathcal{N} = 2$ SCFT. Moreover, in an interacting theory where the $\mathfrak{su}(n)^3$ symmetry is not enhanced to a larger global symmetry algebra, the chiral algebra stress tensor cannot be the Sugawara stress tensor of the dimension one currents. This follows from the fact that the $\mathfrak{su}(n)$ current algebras are at the critical level, so the Sugawara construction fails to produce an appropriate stress tensor. Therefore there must be at least an additional generator corresponding to the stress tensor. Aside from that, the analysis of the index performed in chapter 4 indicates that there are more higher dimensional singlet generators, leading to the conjecture

Conjecture 5 (T_n chiral algebra) *The T_n chiral algebra $\chi(T_n)$ is generated by*

- *The set of operators, $\mathcal{H}$, arising from the Higgs branch chiral ring:*
  - *Three $\widehat{\mathfrak{su}(n)}$ affine currents J^1, J^2, J^3, at the critical level $k_{2d} = -n$, one for each factor in the flavor symmetry group of the theory,*
  - *Generators $W^{(k)}$, $k = 1, \ldots, n-1$ in the $(\wedge^k, \wedge^k, \wedge^k)$ representation of $\bigotimes_{j=1}^3 \mathfrak{su}(n)_j$, where $\wedge^k$ denotes the $k-$index antisymmetric representation of $\mathfrak{su}(n)$. These generators have dimensions $\frac{k(n-k)}{2}$,*
- *Operators $\mathcal{O}_i$, $i = 1, \ldots n-1$, of dimension $h_i = i + 1$ and singlets under $\bigotimes_{j=1}^3 \mathfrak{su}(n)_j$, with the dimension 2 operator corresponding to the stress tensor T of central charge $c_{2d} = -2n^3 + 3n^2 + n - 2$,*

modulo possible relations which set some of the operators listed above equal to composites of the remaining generators.

At any $n \geqslant 4$, the very existence of such a $\mathcal{W}$ algebra is quite nontrivial, since for a randomly chosen set of generators one doesn't expect to be able to solve the associated Jacobi identities. In fact if the singular OPEs of such a $\mathcal{W}$ algebra can be chosen so that the algebra is associative, it seems likely that the requirements of associativity will *completely fix* the structure constants, rendering the chiral algebra unique. It is worth observing that precisely such uniqueness occurs in the case of the T_3 chiral algebra. The characterization given by the conjecture above for $n = 3$ doesn't explicitly imply $\mathfrak{e}_6$ symmetry enhancement, but the unique chiral algebra satisfying the requirements listed is precisely the $\mathfrak{e}_6$ current algebra at the appropriate level. A similar uniqueness result for the T_4 chiral algebra will be presented in chapter 4.

Before moving on, let us extrapolation a bit from Conjecture 5 to make a further conjecture that, while not extremely well-supported, is consistent with everything we know at this time.

Conjecture 6 (Genus zero chiral algebras) *The protected chiral algebra of any class $\mathcal{S}$ SCFT whose UV curve has genus zero is a $\mathcal{W}$ algebra whose only generators are a stress tensor and the additional currents associated to Higgs branch chiral ring generators of the four-dimensional theory. In the special case when the central charge is equal to its Sugawara value with respect to the affine currents, then the stress tensor is a composite.*

The modest evidence in favor of this proposal is that genus zero theories have honest Higgs branches with no residual $U(1)$ gauge fields in the IR, so they don't have any of the additional $\mathcal{N} = 1$ chiral ring generators discussed in Section 3.2.1. Additionally the examples of chapter 2 for which there were chiral algebra generators unrelated to four-dimensional chiral ring generators was a genus one and two theories. It would be interesting to explore this conjecture further, even in the Lagrangian case.

3.3.4 A theory space bootstrap?

Chiral algebras of general theories with maximal punctures can be constructed from the T_n chiral algebra by means of the BRST procedure reviewed in Sec. 3.2.1. Namely, let us suppose that we are handed the chiral algebra $\mathcal{T}$ associated to some (possibly disconnected) UV curve with at least

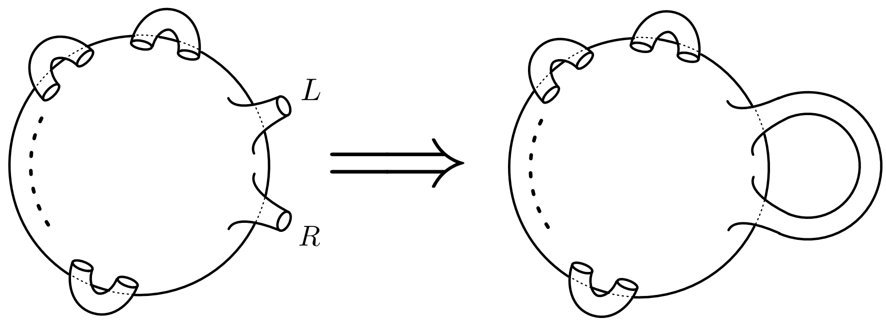


Figure 3.5: Gluing together maximal punctures.

two maximal punctures, that we will label L and R. The chiral algebra associated to the UV curve where these two punctures are glued together, which we will call $\mathcal{T}_{\text{glued}}$ is obtained in two steps. We first introduce a system of n^2-1 (b,c) ghost pairs of dimensions $(1,0)$,

$$b_A(z)c^B(w) \sim \frac{\delta_A{}^B}{z-w}~, \qquad A,B=1,\ldots,n^2-1~. \tag{3.3.21}$$

These are taken to transform in the adjoint representation of $\mathfrak{su}(n)$, and we can construct the $\mathfrak{su}(n)$ affine currents for that symmetry accordingly,

$$J_A^{bc}(z) := -f_{AB}{}^C(c^B b_C)(z)~. \tag{3.3.22}$$

The ghost current algebra has level $k_{2d}^{(b,c)} = 2h^\vee = 2n$. The chiral algebra of the glued configuration is now defined in terms of the ghosts and the chiral algebra of the original system by the BRST procedure of Sec. 3.2.1. In addition to $\mathfrak{su}(n)$ currents coming from the ghost sector, there will be two more $\mathfrak{su}(n)$ currents $J_A^L(z)$ and $J_A^R(z)$ associated to the two punctures being glued. A nilpotent BRST operator is defined using these various $\mathfrak{su}(n)$ currents,

$$Q_{\text{BRST}} := \oint \frac{dz}{2\pi i} j_{\text{BRST}}(z)~, \tag{3.3.23}$$

$$j_{\text{BRST}}(z) := (c^A J_A^L)(z) + (c^A J_A^R)(z) + \frac{1}{2}(c^A J_A^{bc})(z)~. \tag{3.3.24}$$

The nilpotency of Q_{BRST} requires that the sum of the levels of the two matter sector affine currents be given by $k_L + k_R = -2h^\vee$. As usual, this is a reflection of the requirement that the beta function for the newly introduced four-dimensional gauge coupling vanishes. The new chiral algebra is given by

$$\chi[\mathcal{T}_{\text{glued}}] = H^*_{\text{BRST}}\left[\psi \in \chi[\mathcal{T}] \otimes \chi_{(b,c)} \mid b_0\psi = 0\right] \ . \tag{3.3.25}$$

Using this gluing procedure, one may start with a collection of disconnected $\chi[T_n]$ chiral algebras and build up the chiral algebra for an arbitrary generalized quiver diagram with maximal punctures.

The deepest property of the chiral algebras obtained in this manner, which is also the principal condition that must be imposed in order for the map described in the previous section to be a functor, is that they depend only on the topology of the generalized quiver. Of course this is the chiral algebra reflection of generalized S-duality in four dimensions, and follows from the more elementary requirement that the gluing described here is associative (alternatively, crossing-symmetric) in the manner represented pictorially in Fig. 3.3. This is a very strict requirement, and it is conceivable that the $\chi[T_n]$ chiral algebras might be the unique possible choices for the image of the trinion in $\mathbb{CA}_{\mathfrak{su}(n)}$ that satisfy this condition. Indeed, this requirement of theory-space crossing symmetry imposes a strong constraint on any proposal for the $\chi[T_n]$ chiral algebras. For the $\chi[T_3]$ theory, where we have a proposal for the chiral algebra, it would be interesting to investigate this associativity condition. For the general case, it is interesting to ask whether this constraint might help to *determine* the appropriate trinion chiral algebras. At present, we see no obvious strategies that would utilize this direct approach.

Although we will have more to say about reduced punctures in Sec. 3.4, we should point out that the associativity conditions described here apply equally well to the case when not all punctures are maximal. A particularly interesting case that we can consider immediately is when one puncture is minimal. In this case, the requirement of associativity is the one illustrated in Fig. 3.6. This relation is interesting because the theory with two maximal punctures and one minimal puncture is a known quantity – the free hypermultiplet chiral algebra of Sec. 3.3.2 – and so the relation amounts to probing the unknown trinion chiral algebra by coupling it to a known theory. One may hope that this is a sufficient condition in place of the full T_n associativity from which to try to bootstrap the class $\mathcal{S}$ chiral algebras. In fact, as we will see in the next section, this condition does follow directly from the full puncture condition, though the converse is not obvious.

Leaving direct approaches to the theory space bootstrap as an open prob-

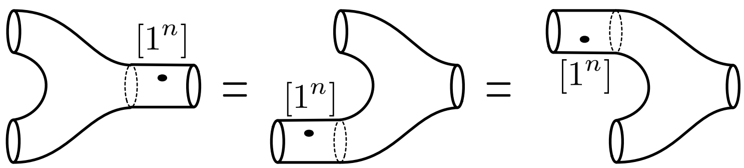


Figure 3.6: Associativity with respect to gluing in free hypermultiplets.

lem, let us note that associativity combined with the conjectures of the previous subsection provide a very constraining framework within which we can attempt to characterize various class $\mathcal{S}$ chiral algebras. Namely, Conjecture 6 suggests a list of generators for an arbitrary genus zero chiral algebra, and the requirement of associativity implies the presence of an automorphism that acts as permutations on the $\mathfrak{su}(n)$ subalgebras associated to the various punctures. This permutation symmetry vastly constrains the possible OPE coefficients of the aforementioned generators, which leads to a straightforward problem of solving the Jacobi identities for such a chiral algebra.

As a simple example of this approach, let us consider the rank one chiral algebra associated to the sphere with five punctures. In this case, the chiral algebra generators associated to Higgs branch chiral ring generators are five sets of $\mathfrak{su}(2)$ affine currents at level $k = -2$ along with a single additional generator of dimension $h = 3/2$ with a fundamental index with respect to each $\mathfrak{su}(2)$ symmetry. Since this is a generic case, the stress tensor will be an independent generator. If Conjecture 6 is correct, then there should be a $\mathcal{W}$-algebra with precisely these generators that, due to associativity, has an S_5 automorphism group that acts as permutations on the five $\mathfrak{su}(2)$ subalgebras. Consequently, the number of independent parameters in the singular OPE of the $\mathcal{W}$-algebra generators is quite small. The only singular OPE not fixed by flavor symmetries and Virasoro symmetry is that of two copies of the quinfundamental field,

$$Q_{abcde}(z)Q_{a'b'c'd'e'}(w) \;\sim\; \frac{\epsilon_{aa'}\epsilon_{bb'}\epsilon_{cc'}\epsilon_{dd'}\epsilon_{ee'}}{(z-w)^3}+ \tag{3.3.26}$$

$$+\,\frac{\alpha(J_{aa'}\epsilon_{bb'}\epsilon_{cc'}\epsilon_{dd'}\epsilon_{ee'}+\text{permutations})}{(z-w)^2} \tag{3.3.27}$$

$$+\,\frac{(\beta\, T+\gamma(\epsilon^{ff'}\epsilon^{gg'}J_{fg}J_{f'g'}+4\text{ more}))\epsilon_{aa'}\epsilon_{bb'}\epsilon_{cc'}\epsilon_{dd'}\epsilon_{ee'}}{(z-w)} \tag{3.3.28}$$

$$+\,\frac{\zeta(\partial J_{aa'}\epsilon_{bb'}\epsilon_{cc'}\epsilon_{dd'}\epsilon_{ee'}+\text{permutations})}{z-w}+\frac{\eta(J_{aa'}J_{bb'}\epsilon_{cc'}\epsilon_{dd'}\epsilon_{ee'}+\text{permutations})}{(z-w)}\;. \tag{3.3.29}$$

The parameters α, β and ζ are constrained in terms of the central charges $c=-24$ and $k=-2$ by comparing with the $\langle QQT\rangle$ and $\langle QQJ\rangle$ three-point functions:

$$-8\,\beta+20\,\gamma=1\;,\qquad \zeta=\frac{1}{4}\;,\qquad \alpha=\frac{1}{2}\;. \tag{3.3.30}$$

This leaves a total of two adjustable parameters, which we may take to be $\{\gamma,\eta\}$. It is a highly nontrivial fact then that the Jacobi identities for this $\mathcal{W}$-algebra can indeed be solved for a unique choice of these parameters,

$$\gamma=-\frac{1}{20}\;,\quad \eta=\frac{1}{4}\;. \tag{3.3.31}$$

Interestingly, this solution of crossing symmetry is special to the $\mathfrak{su}(2)$ level taking the critical value $k=-2$ and the Virasoro central charge taking the expected value $c_{2d}=-24$. Had we not fixed them by hand, we could have derived them from crossing symmetry here.

We consider the existence and uniqueness of this solution as strong evidence in favor of the validity of Conjecture 6 in this instance, seeing as the existence of such a $\mathcal{W}$-algebra would otherwise be somewhat unexpected. Indeed, this characterization of the class $\mathcal{S}$ chiral algebras becomes all the more invaluable for non-Lagrangian theories. See chapter 4 for a discussion of the case of $\chi[T_4]$.

3.4 Reduced punctures

The T_n building blocks outlined in Sec. 3.3.3 only allow us to construct class $\mathcal{S}$ chiral algebras associated to undecorated UV curves, while the inclusion of the free hypermultiplet chiral algebras of Sec. 3.3.2 allow for decoration by minimal punctures only. The purpose of this section is to develop the tools necessary to describe theories that correspond to UV curves with general non-trivial embeddings decorating some of their punctures.

From the TQFT perspective, the most natural way to introduce the necessary additional ingredients is to find a chiral algebra associated to the decorated cap of Fig. 3.2a. This turns out not to be the most obvious approach from a physical perspective since the cap doesn't correspond to any four-dimensional SCFT.[12] Rather, it is more natural to develop a procedure for reducing a maximal puncture to a non-maximal that mimics the Higgsing procedure reviewed in Sec. 3.2.2. Naively, the four-dimensional Higgsing prescription need not lead to a simple recipe for producing the chiral algebra of the Higgsed theory in terms of that of the original theory. This is because the Higgsing spontaneously breaks the superconformal symmetry that is used to argue for the very existence of a chiral algebra, with the theory only recovering superconformal invariance in the low energy limit. Consequently one could imagine that the Higgsing procedure irrecoverably requires that we abandon the chiral algebraic language until reaching the far infrared.

Nevertheless, it turns out that the chiral algebra does admit its own Higgsing procedure that has the desired result. Such a procedure cannot literally amount to Higgsing in the chiral algebra, because quantum mechanically in two dimensions there are no continuous moduli spaces of vacua. The best that we can do is to try to impose a quantum-mechanical *constraint* on the chiral algebra. A natural expectation for the constraint is that it should fix to a non-zero value the chiral algebra operator that corresponds to the Higgs branch chiral ring operator that gets an expectation value. This means imposing the constraint

$$J_{\alpha_-}(z) = A \ , \tag{3.4.1}$$

where $T_{\alpha_-} = \Lambda(t_-)$. Here A is a dimensionful constant that will be irrelevant to the final answer as long as it is nonzero. We might also expect that we should constrain some of the remaining currents to vanish. A motivation for such additional constraints is that when expanded around the new vacuum on the Higgs branch, many of the moment map operators become field operators for the Nambu-Goldstone bosons of spontaneously broken flavor symmetry, and we want to ignore those and focus on the chiral algebra associated to

[12]It does however correspond to a true compactification of the six-dimensional $(2,0)$ theory [91]. We will return to the notion of such a decorated cap in Sec. 3.5.2.

just the interacting part of the reduced theory.

There happens to be a natural conjecture for the full set of constraints that should be imposed. This conjecture is as follows:

Conjecture 7 *The chiral algebra associated to a class $\mathcal{S}$ theory with a puncture of type Λ is obtained by performing quantum Drinfeld-Sokolov (qDS) reduction with respect to the embedding Λ on the chiral algebra for the theory where the same puncture is maximal.*

Quantum Drinfeld-Sokolov in its most basic form is a procedure by which one obtains a new chiral algebra by imposing constraints on an affine Lie algebra $\hat{\mathfrak{g}}$, with the constraints being specified by an embedding $\Lambda : \mathfrak{su}(2) \hookrightarrow \mathfrak{g}$. In the case of interest to us, the chiral algebra on which we will impose these constraints is generally larger than just an affine Lie algebra. Nevertheless, these constraints can still be consistently imposed in the same manner. This conjecture therefore amounts to a choice of the additional constraints beyond (3.4.1) that should be imposed in order to reduce a puncture. It is interesting to note that the right set of constraints will turn out to fix only *half* of the currents that are expected to become Nambu-Goldstone bosons. We will see that the removal of the remaining Nambu-Goldstone bosons occurs in a more subtle manner.

Before delving into the details, we should make the observation that this answer is not unexpected in light of the pre-existing connections between non-maximal defects in the $(2,0)$ theory and qDS reduction [92, 83]. Though a sharp connection between the AGT story and the protected chiral algebra construction is still lacking, we take this as a positive indication that such a connection is there and remains to be clarified. We now turn to a more precise description of qDS reduction for chiral algebras with affine symmetry. We will first develop the general machinery for performing such a reduction in the cases of interest, whereafter we will perform a number of tests of the claim that this is the correct procedure for reducing the ranks of punctures in class $\mathcal{S}$ chiral algebras.

3.4.1 Quantum Drinfeld-Sokolov for modules

Quantum Drinfeld-Sokolov reduction is a procedure for imposing a set of constraints given below in Eqn. (3.4.3) at the quantum level for an affine Lie algebra $\hat{\mathfrak{g}}$ at any level. In the following discussion, we will closely follow the analysis of [93] (see also [94] for a similar discussion for finite dimensional algebras). Although traditionally the starting point for this procedure is a pure affine Lie algebra, our interest is in the case of a more general chiral algebra with an affine Lie subalgebra at the critical level. Said differently, we are interested in performing qDS reduction for nontrivial $\hat{\mathfrak{g}}_{-h^\vee}$ modules. We will utilize essentially the same spectral sequence argument as was used in [93]. Some basic facts about spectral sequences are collected in Appendix B.3 for the convenience of the reader.

The general setup with which we are concerned is the following. We begin with a chiral algebra (for simplicity we take it to be finitely generated) with an $\widehat{\mathfrak{su}(n)}_k$ affine subalgebra. We denote the generating currents of the affine subalgebra as $J_A(z)$, while the additional generators of the chiral algebra will be denoted as $\{\phi^i(z)\}$, each of which transforms in some representation $\mathfrak{R}_i$ of $\mathfrak{su}(n)$.

We now choose some embedding $\Lambda : \mathfrak{su}(2) \hookrightarrow \mathfrak{su}(N)$, for which the images of the $\mathfrak{su}(2)$ generators $\{t_0, t_+, t_-\}$ will be denoted by $\{\Lambda(t_0), \Lambda(t_+), \Lambda(t_-\}$. The embedded Cartan then defines a grading on the Lie-algebra,

$$\mathfrak{g} = \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_m \ , \qquad \mathfrak{g}_m := \left\{T_A \in \mathfrak{g} \mid \mathrm{ad}_{\Lambda(t_0)} T_A = m\, T_A\right\} \ . \tag{3.4.2}$$

When the embedded Cartan is chosen such that some of the currents have half-integral grading, then some of the associated constraints are second-class and cannot be enforced by a straightforward BRST procedure. Fortunately, it has been shown that one may circumvent this problem by selecting an alternative Cartan generator δ which exhibits integer grading and imposing the corresponding first class constraints [95, 94, 93]. We will adopt the convention that an index α ($\bar{\alpha}$) runs over all roots with negative (non-negative) grading with respect to δ, while Latin indices run over all roots. The first-class constraints to be imposed are then as follows,

$$J_\alpha = A\, \delta_{\alpha\alpha_-} \ , \tag{3.4.3}$$

where $\Lambda(t_-) = T_{\alpha_-}$. These constraints are imposed à la BRST by introducing dimension $(1,0)$ ghost pairs (c^α, b_α) in one-to-one correspondence with the generators T_α. These ghosts have the usual singular OPE

$$c^\alpha(z) b_\beta(0) \sim \frac{\delta^\alpha{}_\beta}{z} \ , \tag{3.4.4}$$

and allow us to define a BRST current

$$d(z) = \big(J_\alpha(z) - A\,\delta_{\alpha\alpha_-}\big)\,c^\alpha(z) - \frac{1}{2} f_{\alpha\beta}{}^\gamma (b_\gamma(c^\alpha c^\beta))(z)\ . \tag{3.4.5}$$

The reduced chiral algebra is defined to be the BRST-cohomology of the combined ghost/matter system. Note that this definition is perfectly reasonable for the case where we are reducing not just the affine current algebra, but a module thereof. The presence of the module doesn't modify the system of constraints of the BRST differential, but as we shall see, the operators in the modules will be modified in a nontrivial way in the constrained theory.

This cohomological problem can be solved via a modest generalization of the approach of [96, 93]. We first split the BRST current into a sum of two terms,

$$\begin{aligned} d_0(z) &= \big(-A\,\delta_{\alpha\alpha_-}\big)\,c^\alpha(z)\ , \\ d_1(z) &= J_\alpha(z)c^\alpha(z) - \frac{1}{2} f_{\alpha\beta}{}^\gamma (b_\gamma(c^\alpha c^\beta))(z)\ . \end{aligned} \tag{3.4.6}$$

We now introduce a bi-grading for the currents and ghosts so that the differentials (d_0, d_1) have bi-grades $(1,0)$ and $(0,1)$, respectively,

$$\begin{aligned} \deg\,(J_A(z)) &= (m, -m)\ , & T_A &\in \mathfrak{g}_m\ , \\ \deg\,(c^\alpha(z)) &= (-m, 1+m)\ , & T_\alpha &\in \mathfrak{g}_m\ , \\ \deg\,(b_\alpha(z)) &= (m, -m-1)\ , & T_\alpha &\in \mathfrak{g}_m\ . \end{aligned} \tag{3.4.7}$$

This bi-grading can also be extended to the additional generators ϕ^i. We decompose each such generator into weight vectors of $\mathfrak{su}(n)$ according to

$$\phi^i = \phi^i_I t_I^{(\mathfrak{R}_i)}\ , \qquad I = 1, \ldots, \dim \mathfrak{R}_i\ , \tag{3.4.8}$$

where the $t_I^{(\mathfrak{R}_i)}$ form a weight basis for the representation $\mathfrak{R}_i$ with weights defined according to

$$H_\alpha \cdot t_I^{(\mathfrak{R}_i)} = \lambda_{I,\alpha}^{(\mathfrak{R}_i)}\, t_I^{(\mathfrak{R}_i)}\ , \tag{3.4.9}$$

where H_α is an element of the Cartan subalgebra of $\mathfrak{su}(n)$. Given the element δ in terms of which our grading is defined, the bi-grading of the extra generators can be defined according to

$$\deg\,(\phi^i_I) = (\delta \cdot t_I^{(\mathfrak{R}_i)}, -\delta \cdot t_I^{(\mathfrak{R}_i)})\ . \tag{3.4.10}$$

The differentials (d_0, d_1) are each differentials in their own right, that is, they satisfy

$$d_0^2 = d_1^2 = d_0 d_1 + d_1 d_0 = 0 \ . \tag{3.4.11}$$

Therefore they define a double complex on the Hilbert space of the ghost/ matter chiral algebra, which is the starting point for a spectral sequence computation of the cohomology.

It turns out that a simplification occurs if instead of trying to compute the cohomology of the double complex straight off, we first introduce "hatted currents" [96, 93],

$$\hat{J}_A(z) = J_A(z) + f_{A\beta}{}^{\gamma}(b_\gamma c^\beta)(z) \ . \tag{3.4.12}$$

Let us denote by $\mathbb{A}_1$ the subalgebra generated by $b_\alpha(z)$ and $\hat{J}_\alpha(z)$, and by $\mathbb{A}_2$ the subalgebra produced by the remaining generators $c^\alpha(z)$, $\hat{J}_{\bar{\alpha}}(z)$, and $\phi^i(z)$. One then finds that $d(\mathbb{A}_1) \subseteq \mathbb{A}_1$ and $d(\mathbb{A}_2) \subseteq \mathbb{A}_2$, with the generators of $\mathbb{A}_1$ additionally obeying

$$d(b_\alpha(z)) = \hat{J}_\alpha(z) - A\delta_{\alpha\alpha_-} \ , \qquad d(\hat{J}_\alpha(z)) = 0 \ . \tag{3.4.13}$$

It follows that the BRST cohomology of $\mathbb{A}_1$ is trivial: $H^*(\mathbb{A}_1, d) = \mathbb{C}$. From the Künneth formula (see Appendix B.3), it follows that the BRST cohomology of the chiral algebra is isomorphic to the cohomology of the smaller algebra $\mathbb{A}_2$,

$$H^*(\mathbb{A}, d) \cong H^*(\mathbb{A}_2, d) \ . \tag{3.4.14}$$

Our task then simplifies: we need only compute the cohomology of $\mathbb{A}_2$. We will address this smaller problem by means of a spectral sequence for the double complex $(\mathbb{A}_2, d_0, d_1)$.

The first step in the spectral sequence computation is to compute the cohomology $H^*(\mathbb{A}_2, d_0)$. The only nontrivial part of this computation is the same as in the case without modules. This is because the additional generators $\phi_I^i(z)$ have vanishing singular OPE with the c-ghosts, rendering them d_0-closed. Moreover, they can never be d_0-exact because the b-ghosts are absent from $\mathbb{A}_2$. For the currents and ghosts, one first computes

$$d_0(\hat{J}_{\bar{\alpha}}(z)) = -A f_{\bar{\alpha}\beta}{}^{\gamma}\delta_{\gamma\alpha_-} c^\beta(z) = -\mathrm{Tr}\left(\mathrm{ad}_{\Lambda(t_+)} T_{\bar{\alpha}} \cdot T_\beta\right) c^\beta(z) \ . \tag{3.4.15}$$

It follows that $d_0(\hat{J}_{\bar{\alpha}}(z)) = 0$ if and only if $T_{\bar{\alpha}} \in \ker(\mathrm{ad}_{\Lambda(t_+)})$. The same equation implies that the $c^\alpha(z)$ ghosts are d_0-exact for any α. Because the

d_0-cohomology thus computed is supported entirely at ghost number zero, the spectral sequence terminates at the first step. At the level of vector spaces we find

$$H^*(\mathbb{A}, d) \cong H^*(\mathbb{A}_2, d_0) \ , \tag{3.4.16}$$

with $H^*(\mathbb{A}_2, d_0)$ being generated by the $\phi^i_I(z)$ and by $J_{\bar{\alpha}}(z)$ for $T_{\bar{\alpha}} \in \ker(\mathrm{ad}_{\Lambda(t_+)})$.

In order to improve this result to produce the vertex operator algebra structure on this vector space, we can construct representatives of these with the correct OPEs using the tic-tac-toe procedure. Letting $\psi(z)$ be a generator satisfying $d_0(\psi(z)) = 0$, the corresponding chiral algebra generator $\Psi(z)$ is given by

$$\Psi(z) = \sum_l (-1)^l \psi_l(z) \ , \tag{3.4.17}$$

where $\psi_l(z)$ is fixed by the condition

$$\psi_0(z) := \psi(z) \ , \quad d_1(\psi_l(z)) = d_0(\psi_{l+1}(z)) \ . \tag{3.4.18}$$

At the end, this procedure will give a collection of generators of the qDS reduced theory along with their singular OPEs and it would seem that we are finished. However, it is important to realize that this may not be a minimal set of generators, in that some of the generators may be expressible as composites of lower dimension generators due to null states. The existence of null relations of this type is very sensitive to the detailed structure of the original chiral algebra. For example, the level of the current algebra being reduced plays an important role. In practice, we will find for the class $\mathcal{S}$ chiral algebras, *most* of the generators $\Psi(z)$ produced by the above construction do in fact participate in such null relations.

Some null states of the reduced theory can be deduced from the presence of null states in the starting chiral algebra. This can be an efficient way to generate redundancies amongst the naive generators of the qDS reduced theory like the ones described above. Abstractly, we can understand this phenomenon as follows. Consider a null operator $N^K(z)$ that is present in the original $\mathcal{W}$-algebra, and that transforms in some representation $\mathfrak{R}$ of the symmetry algebra that is being reduced. Given an embedding Λ, the representation $\mathfrak{R}$ decomposes as in (3.2.30) under $\mathfrak{g}_\Lambda \oplus \Lambda(\mathfrak{su}(2))$. We can thus split the index K accordingly and obtain $\{N^{k_j,m_j}(z)\}_{j\geqslant 0}$, where k_j is an index labeling the representation $\mathcal{R}^{(\mathfrak{R})}_j$ and m_j labels the Cartan of the spin j representation V_j. For fixed values of the index m_j we find an operator that will have proper dimension with respect to the new stress tensor (3.4.19). Moreover, since introducing a set of free ghost pairs naturally preserves the null property of the original operator and restricting oneself to the BRST cohomology does not spoil it either, we find that this operator is null in the qDS reduced theory. In practice, for each value of m_j one chooses a representative of the BRST class $N^{k_j,m_j}(z) + d(\ldots)$ that only involves the generators of the qDS reduced theory.

There are a couple of features of the qDS reduced theory that can be deduced without studying the full procedure in specific examples. These features provide us with the most general test of the conjecture that qDS reduction is the correct way to reduce the ranks of punctures in the chiral algebra. The first of these features is the Virasoro central charge of the reduced theory, a subject to which we turn presently.

3.4.2 Virasoro central charge and the reduced stress tensor

A useful feature of qDS reduction is that the stress tensor of a qDS reduced chiral algebra takes a canonical form (up to BRST-exact terms) in which it is written as a shift of the stress tensor of the unreduced theory,

$$T = T_\star - \partial J_0 + \partial b_\alpha c^\alpha - (1+\lambda_\alpha)\partial(b_\alpha c^\alpha)\ . \qquad (3.4.19)$$

Here $T_\star$ is the stress tensor of the unreduced theory, J_0 is the affine current of the $U(1)$ symmetry corresponding to $\Lambda(t_0)$, and λ_α is the weight for T_α with respect to $\Lambda(t_0)$ as defined by Eqn. (3.4.9).[13] The dimensions of the ghosts measured by this new stress tensor are $h_{b_\alpha} = 1+\lambda_\alpha$ and $h_{c^\alpha} = -\lambda_\alpha$. Meanwhile the dimensions of all remaining fields are simply shifted by their J_0 charge.

The central charge of the reduced theory can be read off from the most singular term in the self-OPE of the reduced stress tensor. The result is given

[13] Note that in the case of half-integral gradings, the weights λ_α are defined with respect to $\Lambda(t_0)$ and *not* with respect to the alternate Cartan element δ.

by [95]

$$
\begin{aligned}
c - c_\star &= \left(\dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{\frac{1}{2}} - 12\left|\sqrt{k+h^\vee}\Lambda(t_0) - \frac{\rho}{\sqrt{k+h^\vee}}\right|^2\right) - \left(\frac{k\dim\mathfrak{g}}{k+h^\vee}\right) \ , \\
&= \dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{\frac{1}{2}} - 12(k+h^\vee)\left|\Lambda(t_0)\right|^2 + 24\Lambda(t_0)\cdot\rho - \dim\mathfrak{g} \ .
\end{aligned}
\tag{3.4.20}
$$

Here ρ is the Weyl vector of $\mathfrak{su}(n)$, and in passing to the second line, we have used the Freudenthal-de Vries strange formula $|\rho|^2 = \frac{h^\vee}{12}\dim\mathfrak{g}$. In the cases of interest the level of the current algebra is always given by $k = -h^\vee$ and there is a further simplification,

$$
c = c_\star + \dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{\frac{1}{2}} + 24\Lambda(t_0)\cdot\rho - \dim\mathfrak{g} \ . \tag{3.4.21}
$$

This shift of two-dimensional central charge can be compared to our expectations based on the four-dimensional results in Eqns. (3.2.16)-(3.2.18). The change of the four-dimensional central charge that occurs upon reducing a maximal puncture down to a smaller puncture labelled by the embedding Λ is given by

$$
\begin{aligned}
-12(c_{4d} - c_{4d,\text{orig.}}) &= 2(n_v(\text{max.}) - n_v(\Lambda)) + (n_h(\text{max.}) - n_h(\Lambda)) \ , \\
&= \dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{\frac{1}{2}} + 24\Lambda(t_0)\cdot\rho - \dim\mathfrak{g} \ .
\end{aligned}
\tag{3.4.22}
$$

Thus we see precise agreement with the change in two-dimensional central charge induced by qDS reduction and that of the four-dimensional charge induced by Higgsing after accounting for the relation $c_{2d} = -12c_{4d}$. We take this as a strong indication the the qDS prescription for reducing chiral algebras is indeed the correct one.

3.4.3 Reduction of the superconformal index

We can now check that the qDS reduction procedure has an effect on the (graded) partition function of the chiral algebra that mimics the prescription for reducing the Schur superconformal index described in Sec. 3.2.2. As was reviewed above, the Schur limit of the superconformal index is equivalent to a graded partition function of the corresponding chiral algebra,

$$\mathcal{I}_{\chi}(q;\mathbf{x}) \;:=\; \mathrm{Tr}_{\mathcal{H}_\chi}\,(-1)^F q^{L_0} \;=\; \mathcal{I}^{\mathrm{Schur}}(q;\mathbf{x})\ . \tag{3.4.23}$$

Computing this graded partition function is straightforward for the qDS reduced theory owing to the fact that the BRST differential commutes with all of the fugacities $\mathbf{x}$ that may appear in the index and has odd fermion number. This means that we can ignore the cohomological aspect of the reduction and simply compute the partition function of the larger Hilbert space obtained by tensoring the unreduced chiral algebra with the appropriate ghosts system.[14]

This simpler problem of computing the partition function of the larger Hilbert space parallels the index computation described in Sec. 3.2.2. There are again two steps – the inclusion of the ghosts, and the specialization of fugacities to reflect the symmetries preserved by the BRST differential. Including the ghosts in the partition function before specializing the fugacities requires us to assign them charges with respect to the UV symmetries. This can be done in a canonical fashion so that upon specializing the fugacities the BRST current will be neutral with respect to IR symmetries and have conformal dimension one.

Recall that the ghost sector involves one pair of ghosts (b_α, c^α) for each generator T_α that is negatively graded with respect to δ. The charge assignments are then the obvious ones – namely the charges of b_α are the same as those of T_α (let us call them f_α), while those of c^α are minus those of b_α. With these charge assignments, the graded partition function of the reduced chiral algebra can be obtained as a specialization that mimics that which led to the superconformal index,

$$\mathcal{I}_{\chi_\Lambda}(q;\mathbf{x}_\Lambda) = \lim_{\mathbf{x}\to\mathbf{x}_\Lambda} \mathcal{I}_{\chi}(q;\mathbf{x})\,\mathcal{I}_{(b,c)_\Lambda}(q;\mathbf{x})\ , \tag{3.4.24}$$

$$\mathcal{I}_{(b,c)_\Lambda} := \mathrm{PE}\left[-\sum_{T_\alpha\in\mathfrak{g}_{<0}}\left(\frac{q\,\mathbf{x}^{f_\alpha}}{1-q} + \frac{\mathbf{x}^{-f_\alpha}}{1-q}\right)\right]. \tag{3.4.25}$$

As in the discussion of the index in Sec. 3.4.3, we can formally perform the specialization ignoring divergences that occur in both the numerator and the

[14] There is a caveat to this argument, which is that if there are null states in the reduced theory that do not originate as null states in the parent theory, then their subtraction will not necessarily be accomplished by this procedure. We operate under the assumption that such spurious null states do not appear. This assumption appears to be confirmed by the coherence between this procedure and that discussed in Sec. 3.2.2.

denominator as a consequence of constant terms in the plethystic exponent. In doing this, the flavor fugacities are replaced by fugacities for the Cartan generators of $\mathfrak{h}_\Lambda$, while the q-grading is shifted by the Cartan element of the embedded $\mathfrak{su}(2)$. This leads to the following formal expression for the contribution of the ghosts,[15]

$$\mathcal{I}_{(b,c)_\Lambda} \text{ “ = ” } \mathrm{PE}\left[\frac{-q}{1-q}\sum_j \chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}_j^{(\mathrm{adj})}}(\mathbf{a}_\Lambda)\sum_{i=-j}^{-1} q^i - \frac{1}{1-q}\sum_j \chi^{\mathfrak{h}_\Lambda}_{\mathcal{R}_j^{(\mathrm{adj})}}(\mathbf{a}_\Lambda^{-1})\sum_{i=1}^{j} q^i\right]. \tag{3.4.26}$$

After a small amount of rearrangement and the recognition that the representations $\mathcal{R}_j^{(\mathrm{adj})}$ are pseudoreal, one finds that this exactly reproduces the formal denominator in Eqn. (3.2.47). Again, when the limit in Eqn. (3.4.24) is taken carefully, the divergences in this formal denominator cancel against equivalent terms in the K-factors of the numerator to produce a finite result. It is interesting that in spite of the asymmetry between b and c ghosts in this procedure, they ultimately play the same role from the point of view of four-dimensional physics – each ghost is responsible for cancelling the effect of a single Nambu-Goldstone boson from the index.

Before moving on to examples, we recall that in [26] it was observed that the K-factor for a maximal puncture matches the character of the corresponding affine Lie algebra at the critical level, and it was conjectured that a similar statement would be true for reduced punctures. That is to say, the K-factor associated to the reduction of type Λ should be the character of the qDS reduction of type Λ of the critical affine Lie algebra. Given the analysis to this point, this statement becomes almost a triviality. The qDS reduction of the affine current algebra proceeds by introducing the same collection of ghosts as we have used here, and so the effect on the graded partition function is the introduction of the same ghost term given in Eqn. (3.4.26) and the same specialization of fugacities. Thus, the identification of the K-factors given in Eqn. (3.2.36) with the character of the qDS reduction of the critical affine Lie algebra depends only on our ability to equate the index (*i.e.*, the partition function graded by $(-1)^F$) with the ungraded vacuum character. This is a simple consequence of the fact that the starting current algebra consists of all bosonic operators and the spectral sequence calculation of Sec. 3.4.1 only

[15] For simplicity, we write the expression here for the case where $\Lambda(t_0)$ provides an integral grading so there is no auxiliary δ. The case of half-interal grading can be treated with modest modifications.

found BRST cohomology elements at ghost number zero.

3.4.4 Simple examples

In light of the analysis in Section 3.4.1, the reduction problem admits an algorithmic solution subject to two conditions. (A) the starting chiral algebra should be finitely generated, *i.e.*, it admits a description as a $\mathcal{W}$-algebra. (B) the L_0 operator of the reduced theory should have a positive definite spectrum. The latter condition must hold for any reductions where the end-point corresponds to a physical class $\mathcal{S}$ theory, while the former conditions is conjectured to be true for general class $\mathcal{S}$ theories but is more certainly true in some simple examples. Given these conditions, the procedure is as follows:

- List the (possibly redundant) generators of the qDS reduced chiral algebra at the level of vector spaces. These are given by the hatted currents $\hat{J}_{\bar{\alpha}}$ for which $T_{\bar{\alpha}} \in \ker(\mathrm{ad}_{\Lambda(t_+)})$, along with all of the additional generators $\{\phi_i\}$.

- Apply the tic-tac-toe algorithm to construct genuine generators of the chiral algebra. The OPEs of these reduced chiral algebra generators can be computed directly using the OPEs of the original, unreduced fields.

- Compute the null states at each level up to that of the highest-dimensional generator in order to check for redundancy. Remove any redundant generators. What remains is a description of the reduced chiral algebra as a $\mathcal{W}$-algebra.

This procedure is still morally a correct one when the two conditions listed above fail to be met, but in those cases the algorithm will not necessarily terminate in finite time. In the examples discussed in this subsection, both conditions above will indeed be satisfied, so this algorithm will be sufficient to determine the answer entirely.

We now consider a pair of simple cases in which the reduction can be performed quite explicitly. Our first example will be the complete closure of a single puncture in the rank one theory of a four-punctured sphere, which as we reviewed above has as its chiral algebra the affine Lie algebra $\widehat{\mathfrak{so}(8)}_{-2}$. The result of this closure is expected to be the T_2 theory (see Figure 3.7). The second example will be the partial reduction (corresponding to the semi-regular embedding) of one puncture in the T_3 theory to produce a theory of free bifundamental hypermultiplets, which should correspond to free symplectic bosons at the level of the chiral algebra. Details of the second calculation beyond what is included in this summary can be found in Appendix B.1.2.

Reducing $\widehat{\mathfrak{so}(8)}_{-2}$ to $\chi[T_2]$

The starting point for our first reduction is the affine Lie algebra $\widehat{\mathfrak{so}(8)}_{-2}$. We first introduce a basis for the affine currents that is appropriate for class $\mathcal{S}$ and for the reduction we aim to perform. The adjoint of $\mathfrak{so}(8)$ decomposes into irreps of the $\mathfrak{su}(2)^{(1)}\times\mathfrak{su}(2)^{(2)}\times\mathfrak{su}(2)^{(3)}\times\mathfrak{su}(2)^{(4)}$ symmetries associated to punctures according to

$$\mathbf{28}_{\mathfrak{so}(8)} \;\to\; (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})\oplus(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})\;. \tag{3.4.27}$$

Accordingly, we assemble the twenty-eight affine currents into these irreps,

$$J_A(z) \;\to\; \{J^{(1)}_{(a_1b_1)}(z)\;,\;J^{(2)}_{(a_2b_2)}(z)\;,\;J^{(3)}_{(a_3b_3)}(z)\;,\;J^{(4)}_{(a_4b_4)}(z)\;,\;J_{a_1a_2a_3a_4}(z)\}\;, \tag{3.4.28}$$

where a_I, b_I are fundamental indices of $\mathfrak{su}(2)^{(I)}$. In this basis, the OPEs of the affine Lie algebra are given by

$$\begin{aligned}
J^{(I)}_{ab}(z)J^{(J)}_{cd}(w) &\sim \frac{-k(\epsilon_{ac}\epsilon_{bd}+\epsilon_{ad}\epsilon_{bc})\delta^{IJ}}{2(z-w)^2}+\frac{f_{ab;cd}^{ef}J^{(I)}_{ef}\delta^{IJ}}{z-w}\;,\\
J^{(1)}_{ab}(z)J_{cdef}(w) &\sim \frac{\epsilon_{ac}J_{bdef}+\epsilon_{bc}J_{adef}}{2(z-w)}\;,\\
J_{abcd}(z)J_{efgh}(w) &\sim \frac{k\epsilon_{ae}\epsilon_{bf}\epsilon_{cg}\epsilon_{dh}}{(z-w)^2}\\
&\quad+\frac{J^{(1)}_{ae}\epsilon_{bf}\epsilon_{cg}\epsilon_{dh}+\epsilon_{ae}J^{(2)}_{bf}\epsilon_{cg}\epsilon_{dh}+\epsilon_{ae}\epsilon_{bf}J^{(3)}_{cg}\epsilon_{dh}+\epsilon_{ae}\epsilon_{bf}\epsilon_{cg}J^{(4)}_{dh}}{z-w}\;,
\end{aligned} \tag{3.4.29}$$

and similarly for the other $J^{(I)}$. Here the $\mathfrak{su}(2)$ structure constants are given by $f_{ab;cd}^{ef}=\frac{1}{2}(\epsilon_{ac}\delta_b^e\delta_d^f+\epsilon_{bc}\delta_a^e\delta_d^f+\epsilon_{ad}\delta_b^e\delta_c^f+\epsilon_{bd}\delta_a^e\delta_c^f)$, and for our case of interest level is fixed to $k=-2$.

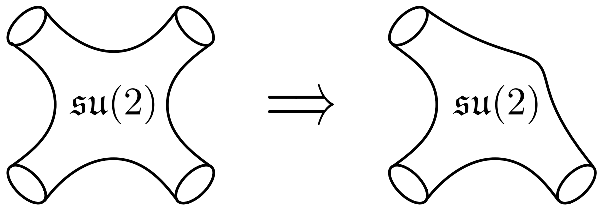


Figure 3.7: Reduction from the $\mathfrak{so}(8)$ theory to T_2.

We will choose the first puncture to close, meaning we will perform qDS reduction on the current algebra generated by $J^{(1)}_{(ab)}$ with respect to the principal embedding,

$$\Lambda(t_+) = -T_{11} \ , \qquad \Lambda(t_-) = T_{22} \ , \qquad \Lambda(t_0) = -T_{(12)} \ . \tag{3.4.30}$$

The grading provided by $\Lambda(t_0)$ is integral, so we can proceed without introducing any auxiliary grading. The only constraint to be imposed in this case is $J^{(1)}_{22}(z) = 1$. This is accomplished with the help of a single ghost pair (c^{22}, b_{22}), in terms of which the BRST operator is given by

$$d(z) = c^{22}(J_{22} - 1)(z) \ . \tag{3.4.31}$$

The remaining three sets of $\mathfrak{su}(2)$ affine currents can be thought of as trivial modules of the reduced currents, while the quadrilinear currents provide a nontrivial module. In the language of the previous subsection we have[16]

$$\{\phi^i\} = \{J^{(2)}_{(a_2b_2)} \ , J^{(3)}_{(a_3b_3)} \ , J^{(4)}_{(a_4b_4)} \ , J_{a_1a_2a_3a_4}\} \ . \tag{3.4.32}$$

The reduced generators of step one are simply the hatted current $\hat{J}^{(1)}_{11} = J^{(1)}_{11}$ along with the additional generators in (3.4.32). Applying the tic-tac-toe

[16]We should note that there is something slightly unconventional about the reduction procedure here. In this example the entire starting chiral algebra is an affine current algebra, so one could in principle perform qDS reduction in the entirely standard manner. This is *not* what our prescription tells us to do. Instead, we treat a single $\mathfrak{su}(2)$ subalgebra as the target of the reduction, and the rest as modules. The two procedures are naively inequivalent, although we have not checked in detail to make sure that the results don't turn out the same.

procedure produces true generators of the reduced chiral algebra,

$$\begin{aligned}
\hat{\mathcal{J}}_{11}^{(1)} &:= \hat{J}_{11}^{(1)} - \hat{J}_{12}^{(1)}\hat{J}_{12}^{(1)} - (k+1)\partial(\hat{J}_{12}^{(1)}) \ , \\
(\mathcal{J}_1)_{a_2a_3a_4} &:= J_{1a_2a_3a_4} - \hat{J}_{12}^{(1)} J_{2a_2a_3a_4} \ , \\
(\mathcal{J}_2)_{a_2a_3a_4} &:= J_{2a_2a_3a_4} \ , \\
\mathcal{J}_{a_Ib_I}^{(I=\{2,3,4\})} &:= J_{a_Ib_I}^{(I=\{2,3,4\})} \ ,
\end{aligned} \tag{3.4.33}$$

where $\hat{J}_{12}^{(1)} := J_{12}^{(1)} + b_{22}c^{22}$.

The stress tensor of the reduced algebra takes the form given in Eqn. (3.4.19), where the original stress tensor was the Sugawara stress tensor of $\widehat{\mathfrak{so}(8)}_{-2}$ and the generator of the embedded Cartan is $J_0 = -J_{12}^{(1)}$. We can then compute the conformal dimensions of the new generators and we find

$$\begin{aligned}
&[\hat{\mathcal{J}}_{11}^{(1)}] = 2 \ , && [\mathcal{J}_{a_Ib_I}^{(I)}] = 1 \ , \\
&[(\mathcal{J}_1)_{a_2a_3a_4}] = 3/2 \ , && [(\mathcal{J}_2)_{a_2a_3a_4}] = 1/2 \ .
\end{aligned} \tag{3.4.34}$$

The currents $\mathcal{J}_{a_Ib_I}^{(I)}$ persist as affine currents of $\mathfrak{su}(2)$ subalgebras, so all of their singular OPEs with other generators are determined by the symmetry properties of the latter. Explicit calculation determines the OPEs that are not fixed by symmetry to take the following form,

$$
\begin{aligned}
\hat{\mathcal{J}}^{(1)}_{11}(z)\hat{\mathcal{J}}^{(1)}_{11}(0) \sim & -\frac{1}{2}\frac{(2+k)(1+2k)(4+3k)}{z^4} - \frac{2(2+k)\hat{\mathcal{J}}^{(1)}_{11}(0)}{z^2} \\
& - \frac{(2+k)\partial\hat{\mathcal{J}}^{(1)}_{11}(0)}{z} \\
\hat{\mathcal{J}}^{(1)}_{11}(z)(\mathcal{J}_1)_{a_2a_3a_4}(0) \sim & -\frac{1}{2}\frac{(2+k)(1+2k)(\mathcal{J}_2)_{a_2a_3a_4}(0)}{z^3} - \frac{1}{4}\frac{(7+2k)(\mathcal{J}_1)_{a_2a_3a_4}(0)}{z^2} \\
& - \frac{(\hat{\mathcal{J}}^{(1)}_{11}(\mathcal{J}_2)_{a_2a_3a_4})(0)}{z} \\
\hat{\mathcal{J}}^{(1)}_{11}(z)(\mathcal{J}_2)_{a_2a_3a_4}(0) \sim & \ \frac{1}{4}\frac{(1+2k)(\mathcal{J}_2)_{a_2a_3a_4}(0)}{z^2} + \frac{(\mathcal{J}_1)_{a_2a_3a_4}(0)}{z} \\
(\mathcal{J}_1)_{a_2a_3a_4}(z)(\mathcal{J}_2)_{b_2b_3b_4}(0) \sim & -\frac{1}{2}\frac{(1+2k)\epsilon_{a_2b_2}\epsilon_{a_3b_3}\epsilon_{a_4b_4}}{z^2} \\
& + \frac{-\frac{1}{2}((\mathcal{J}_2)_{a_2a_3a_4}(\mathcal{J}_2)_{b_2b_3b_4})(0) + \mathfrak{J}_{a_2a_3a_4;b_2b_3b_4}(0)}{z} \\
(\mathcal{J}_2)_{a_2a_3a_4}(z)(\mathcal{J}_2)_{b_2b_3b_4}(0) \sim & \ \frac{\epsilon_{a_2b_2}\epsilon_{a_3b_3}\epsilon_{a_4b_4}}{z} \\
(\mathcal{J}_1)_{a_2a_3a_4}(z)(\mathcal{J}_1)_{b_2b_3b_4}(0) \sim & \ \frac{3}{4}\frac{(1+2k)\epsilon_{a_2b_2}\epsilon_{a_3b_3}\epsilon_{a_4b_4}}{z^3} \\
& + \frac{\frac{1}{4}(3+2k)((\mathcal{J}_2)_{a_2a_3a_4}(\mathcal{J}_2)_{b_2b_3b_4})(0) - \mathfrak{J}_{a_2a_3a_4;b_2b_3b_4}(0)}{z^2} \\
& + \frac{\frac{1}{4}((\mathcal{J}_2)_{a_2a_3a_4}\partial(\mathcal{J}_2)_{b_2b_3b_4})(0)}{z} \\
& + \frac{\frac{1}{2}(1+k)(\partial(\mathcal{J}_2)_{a_2a_3a_4}(\mathcal{J}_2)_{b_2b_3b_4})(0)}{z} \\
& - \frac{1}{2}\frac{\partial\mathfrak{J}_{a_2a_3a_4;b_2b_3b_4}(0)}{z} \ ,
\end{aligned}
\tag{3.4.35}
$$

where

$$
\mathfrak{J}_{a_2a_3a_4;b_2b_3b_4}(z) = \mathcal{J}^{(2)}_{a_2b_2}(z)\epsilon_{a_3b_3}\epsilon_{a_4b_4} + \mathcal{J}^{(3)}_{a_3b_3}(z)\epsilon_{a_2b_2}\epsilon_{a_4b_4} + \mathcal{J}^{(4)}_{a_4b_4}(z)\epsilon_{a_2b_2}\epsilon_{a_3b_3} \ , \tag{3.4.36}
$$

and we have removed d-exact terms.

We expect the result of this reduction procedure to be the trifundamental symplectic boson algebra $\chi[T_2]$, and $(\mathcal{J}_2)_{a_2a_3a_4}(z)$ has the correct dimension and OPE to be identified with the trifundamental generator $q_{a_2a_3a_4}$. In order to complete the argument, we need all of the remaining reduced generators to be expressible as composites of this basic generator. Indeed it turns out to be a straightforward exercise to compute the null states in the reduced algebra at dimensions $h = 1, \frac{3}{2}, 2$ and to verify that null relations allow all the other generators to be written as normal ordered products of (derivatives of) $(\mathcal{J}_2)_{a_2a_3a_4}(z)$. For example, we should expect that the $\mathfrak{su}(2)$ affine currents should be equivalent to the bilinears currents of Eqn. (3.3.10), and indeed there are null relations (only for $k=-2$) that allow us to declare such an equivalence,

$$
\begin{aligned}
\tfrac{1}{2}(\mathcal{J}_2)_{abc}(\mathcal{J}_2)_{a'b'c'}\epsilon^{bb'}\epsilon^{cc'} &= \mathcal{J}^{(2)}_{aa'}\ , \\
\tfrac{1}{2}(\mathcal{J}_2)_{abc}(\mathcal{J}_2)_{a'b'c'}\epsilon^{aa'}\epsilon^{cc'} &= \mathcal{J}^{(3)}_{bb'}\ , \\
\tfrac{1}{2}(\mathcal{J}_2)_{abc}(\mathcal{J}_2)_{a'b'c'}\epsilon^{aa'}\epsilon^{bb'} &= \mathcal{J}^{(4)}_{cc'}\ ,
\end{aligned}
\tag{3.4.37}
$$

At dimensions $h=3/2$ and $h=2$ there are additional null states for our special value of the level,

$$
(\mathcal{J}_1)_{bcd} = -\tfrac{3}{2}\partial(\mathcal{J}_2)_{bcd} + \tfrac{2}{3}(\mathcal{J}_2)_{(b_1(c_1(d_1}(\mathcal{J}_2)_{b_2)c_2)d_2)}(\mathcal{J}_2)_{b_3c_3d_3}\epsilon^{b_2b_3}\epsilon^{c_2c_3}\epsilon^{d_2d_3}\ , \tag{3.4.38}
$$

$$
\begin{aligned}
\hat{\mathcal{J}}^{(1)}_{11} = &-\tfrac{3}{4}(\mathcal{J}_2)_{b_1c_1d_1}\partial(\mathcal{J}_2)_{b_2c_2d_2}\epsilon^{b_1b_2}\epsilon^{c_1c_2}\epsilon^{d_1d_2} \\
&-\tfrac{1}{6}(\mathcal{J}_2)_{b_1c_1d_1}(\mathcal{J}_2)_{(b_2(c_2(d_2}(\mathcal{J}_2)_{b_3)c_3)d_3)}(\mathcal{J}_2)_{b_4c_4d_4}\epsilon^{b_1b_2}\epsilon^{c_1c_2}\epsilon^{d_1d_2}\epsilon^{b_3b_4}\epsilon^{c_3c_4}\epsilon^{d_3d_4}\ .
\end{aligned}
\tag{3.4.39}
$$

Thus all of the additional generators are realized as composites of the basic field $(\mathcal{J}_2)_{abc}(z)$, and we have reproduced the $\chi[T_2]$ chiral algebra from qDS reduction of the $\mathfrak{so}(8)$ affine current algebra at level $k=-2$. We should re-emphasize that the redundancy amongst generators due to null states depends crucially on the precise value of the level. This is another instance of a general lesson that we have learned: the protected chiral algebras of $\mathcal{N}=2$ SCFTs realize very special values of their central charges and levels at which nontrivial cancellations tend to take place. We will see more of this phenomenon in the next example.

Reducing $(\widehat{\mathfrak{e}}_6)_{-3}$ to symplectic bosons

In this case, our starting point is again an affine Lie algebra, this time $(\widehat{\mathfrak{e}}_6)_{-3}$. Also we are again led to decompose the adjoint representation of $\mathfrak{e}_6$ under the maximal $\mathfrak{su}(3)_1\times\mathfrak{su}(3)_2\times\mathfrak{su}(3)_3$ subalgebra associated to the punctures on the UV curve as was done in (3.3.15), leading to a basis of currents given by (3.3.16) subject to singular OPEs given by Eqn. (B.1.1). Our aim is now

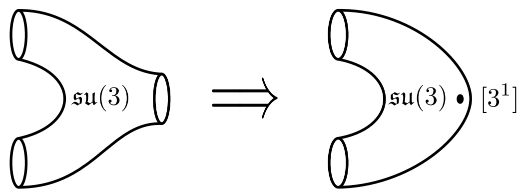


Figure 3.8: Reduction from the $\mathfrak{e}_6$ theory to free hypermultiplets.

to perform a partial reduction of the first puncture. Accordingly, we divide the generating currents as usual,

$$(J^1)_a{}^{a'} \ , \qquad\qquad \{\phi_i\} = \{(J^2)_b{}^{b'} \ , (J^3)_c{}^{c'} \ , W_{abc} \ , \tilde{W}^{abc}\} \ , \tag{3.4.40}$$

where now a, b, c are fundamental indices of $\mathfrak{su}(3)_{1,2,3}$, and the adjoint representation is represented by a fundamental and antifundamental index subject to a tracelessness condition.

The partial closing down to a minimal puncture is accomplished by means of the subregular embedding,

$$\Lambda(t_0) = \frac{1}{2}(T_1{}^1 - T_3{}^3) \ , \qquad \Lambda(t_-) = T_3{}^1 \ , \qquad \Lambda(t_+) = T_1{}^3 \ . \tag{3.4.41}$$

The grading induced by the embedded Cartan turns out to be half-integral in this case and must therefore be supplanted by the integral δ grading. Under this grading the generators $\Lambda(t_-) = T_3{}^1$ and $T_3{}^2$ are negative and of grade minus one. The relevant constraints are thus $(J^1)_3{}^1 = 1$ and $(J^1)_3{}^2 = 0$. The implementation of these constraints via the BRST procedure introduces two ghost pairs $b_3{}^1, c_1{}^3$ and $b_3{}^2, c_2{}^3$.

In the reduction of $\chi[T_3]$, one finds that the currents $(\hat{J}^1)_{\bar{\alpha}}$ such that $T_{\bar{\alpha}} \in \ker(ad(\Lambda(t_+)))$, are given by $(\hat{J}^1)_1{}^2$, $(\hat{J}^1)_1{}^3$, $(\hat{J}^1)_2{}^3$, and the current generating the reduced $\mathfrak{u}(1)$ symmetry

$$\mathcal{J}_{\mathfrak{u}(1)} = (\hat{J}^1)_1{}^1 - 2(\hat{J}^1)_2{}^2 + (\hat{J}^1)_3{}^3 \ . \tag{3.4.42}$$

Together with the additional generators in (3.4.40), these constitute the generators of the cohomology at the level of vector spaces. The tic-tac-toe procedure produces honest chiral algebra generators, which we denote by the calligraphic version of the same letter as the vector space generator. The quantum numbers of these redundant generators are summarized in Table 3.4. Their precise expressions can be found in Appendix B.1.2.

Generator	Dimension	$U(1)$	$SU(3)_2$	$SU(3)_3$
$\mathcal{J}_{\mathfrak{u}(1)}$	1	0	$\mathbf{1}$	$\mathbf{1}$
$(\hat{\mathcal{J}}^1)_1{}^2$	$\frac{3}{2}$	3	$\mathbf{1}$	$\mathbf{1}$
$(\hat{\mathcal{J}}^1)_1{}^3$	2	0	$\mathbf{1}$	$\mathbf{1}$
$(\hat{\mathcal{J}}^1)_2{}^3$	$\frac{3}{2}$	-3	$\mathbf{1}$	$\mathbf{1}$
$\mathcal{W}_{1bc}$	$\frac{3}{2}$	1	$\mathbf{3}$	$\mathbf{3}$
$\mathcal{W}_{2bc}$	1	-2	$\mathbf{3}$	$\mathbf{3}$
$\mathcal{W}_{3bc}$	$\frac{1}{2}$	1	$\mathbf{3}$	$\mathbf{3}$
$\tilde{\mathcal{W}}^{1bc}$	$\frac{1}{2}$	-1	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$
$\tilde{\mathcal{W}}^{2bc}$	1	2	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$
$\tilde{\mathcal{W}}^{3bc}$	$\frac{3}{2}$	-1	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$
$(\mathcal{J}^2)_b{}^{b'}$	1	0	$\mathbf{8}$	$\mathbf{1}$
$(\mathcal{J}^3)_c{}^{c'}$	1	0	$\mathbf{1}$	$\mathbf{8}$

Table 3.4: The quantum numbers of redundant generators of the reduced T_3 chiral algebra.

Again, we see that there are dimension one half generators $(\mathcal{W}_3)_{bc} = W_{3bc}$ and $(\tilde{\mathcal{W}}^1)^{bc} = \tilde{W}^{1bc}$ that one naturally expects should be identified as the symplectic bosons of the reduced theory. Indeed, up to d-exact terms, the OPE for these generators is exactly what we expect from the desired symplectic bosons,

$$(\mathcal{W}_3)_{bc}(z)(\tilde{\mathcal{W}}^1)^{b'c'}(0) \sim \frac{\delta_b{}^{b'}\delta_c{}^{c'}}{z}\ . \tag{3.4.43}$$

These generators thus have the correct dimension, charges and OPE to be identified with the expected hypermultiplet generators. Again, by studying the null relations of the reduced chiral algebra at levels $h = 1, \frac{3}{2}, 2$ one finds that precisely when the level $k = -3$, all of the higher dimensional generators in Table 3.4 are related to composites of $(\mathcal{W}_3)_{bc}$ and $(\tilde{\mathcal{W}}^1)^{bc}$ (see Appendix

B.1.2). In particular, one can verify that the $\mathfrak{u}(1) \oplus \mathfrak{su}(3)_2 \oplus \mathfrak{su}(3)_3$ currents are equal to their usual free field expression modulo null states.

3.5 Cylinders and Caps

The procedure we have introduced for reducing punctures is sufficiently general that there is no obstacle to formally defining chiral algebras associated to unphysical curves such as the cylinder and (decorated) cap. These are unphysical curves from the point of view of class $\mathcal{S}$ SCFTs, although they have a physical interpretation in terms of theories perturbed by irrelevant operators that correspond to assigning a finite area to the UV curve [91]. It would be interesting to interpret the chiral algebras associated with these curves in terms of those constructions, although naively extrapolating away from conformal fixed points seems impossible. (There are other unphysical curves, such as a thrice-punctured sphere with two minimal punctures and one maximal puncture, and the chiral algebras for these can also be defined. We focus on cylinders and caps in this section as they are particularly natural objects in the TQFT.)

The chiral algebra associated to a cylinder is a particularly natural object to consider from the TQFT perspective because it corresponds to the identity morphism (when taken with one ingoing and one outgoing leg). When taken with two ingoing or two outgoing legs, it is the chiral algebra avatar of the evaluation and coevaluation maps, respectively, of an ordinary two-dimensional TQFT. Similarly, the chiral algebra of the undecorated cap is the chiral algebra version of the trace map.

On the whole, we have not been able to systematically solve the BRST problem for these theories in the general case. This is because, as we shall see, the chiral algebras involve dimension zero (or negative dimension) operators, which prevent us from applying the simple algorithm set forth in Sec. 3.4. Nevertheless, we are able to develop a compelling picture of the mechanics of the cylinder chiral algebra. It would be interesting from a purely vertex operator algebra point of view to construct these algebras rigorously.

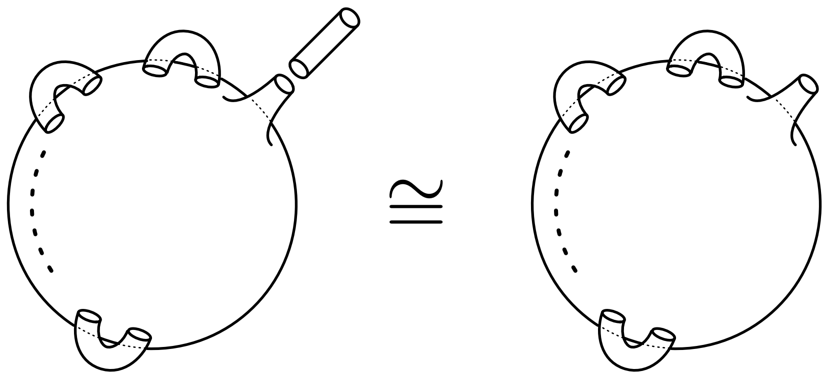


Figure 3.9: Characteristic property of the identity morphism.

3.5.1 The cylinder chiral algebra

The chiral algebra associated to a cylinder should be obtained by performing a complete qDS reduction on one puncture of the trinion chiral algebra $\chi[T_n]$. In the generalized TQFT, the cylinder chiral algebra plays the role of the identity morphism for a single copy of the affine Lie algebra, $\mathrm{Id}: \widehat{\mathfrak{su}(n)}_{-n} \mapsto \widehat{\mathfrak{su}(n)}_{-n}$. The essential property associated with an identity morphism is illustrated in Figure 3.9. As a statement about chiral algebras, the identity property is quite interesting. It means that the chiral algebra should have the property that when tensored with another class $\mathcal{S}$ chiral algebra $\chi[\mathcal{T}]$ along with the usual (b,c) ghosts, restriction to the appropriate BRST cohomology produces a chiral algebra that is isomorphic to the original class $\mathcal{S}$ chiral algebra,

$$H^*_{\mathrm{BRST}}\left(\psi \in \chi_{\mathrm{Id}} \otimes \chi[\mathcal{T}] \otimes \chi_{bc} \,\middle|\, b_0\psi = 0\right) \cong \chi[\mathcal{T}] \ . \tag{3.5.1}$$

As stated above, the qDS reduction problem in this case is substantially complicated by the fact that amongst the list of naive generators of the reduced chiral algebra, there will always be dimension zero currents. Consequently, a systematic solution of the BRST problem that removes redundancies from the list of generators is difficult even in the case of the $\chi[T_2]$ and $\chi[T_3]$ theories, for which the starting point of the reduction is known. A somewhat detailed analysis of the $\mathfrak{su}(3)$ case can be found in Appendix B.2.

Although we don't have a general first principles solution, the general structure of the reduction and our intuition gained from other examples suggests a simple characterization of the cylinder chiral algebra. We state this here as a conjecture.

Conjecture 8 (Cylinder chiral algebra) *The chiral algebra associated to a cylinder of type* $\mathfrak{su}(n)$ *is finitely generated by an* $\widehat{\mathfrak{su}(n)}_{-n}$ *affine current algebra* $\{(\mathcal{J}_L)_A(z),\ A=1,\ldots,n^2-1\}$*, along with dimension zero currents* $\{g_{ab}(z),\ a,b=1,\ldots,n\}$ *that are acted upon on the left by the affine currents. These dimension zero currents further obey a determinant condition* $\det g = 1$*, i.e., they form a matrix that belongs to* $SL(n,\mathbb{C})$*.*

This turns out to be a surprisingly interesting chiral algebra. Let us mention a few of its properties.

The first key property – one which is not completely obvious from the description – is that this chiral algebra actually has two commuting $\widehat{\mathfrak{su}(n)}_{-n}$ current algebras. The second set of affine currents are defined as follows

$$\left(\mathcal{J}_c{}^{c'}\right)_R(z) := \left(\mathcal{J}_{b'}{}^{b}\right)_L \ g_{bc}\ g^{b'c'} + n\left(g_{bc}\ \partial g^{bc'} - \frac{1}{n}\delta_c^{c'} g_{bd}\ \partial g^{bd}\right) \ , \tag{3.5.2}$$

where we have traded the adjoint index for a fundamental and antifundamental index satisfying a tracelessness condition, and we've also introduced the shorthand

$$g^{ab}(z) = \frac{1}{n!}\epsilon^{aa_2\ldots a_n}\epsilon^{bb_2\ldots b_n}\left(g_{a_2b_2}\ldots g_{a_nb_n}\right)(z) \ . \tag{3.5.3}$$

Because of the determinant condition, this can be thought of as the inverse of $g_{ab}(z)$. The currents $(\mathcal{J}_R)_A(z)$ act on the dimension zero currents on the right. The $\mathcal{J}_R$ currents and the $\mathcal{J}_L$ currents have nonsingular OPE with one another, so they generate commuting affine symmetries. These are the symmetries associated with the two full punctures of the cylinder.

The key feature of this chiral algebra should be its behavior as the identity under gluing to other class $\mathcal{S}$ chiral algebras. Let us thus consider a chiral algebra associated to a UV curve $\mathcal{C}_{g,s\geq 1}$ with at least one maximal puncture. Let us consider a general operator in this theory which will take the form $X_{a_1a_2\ldots a_p}^{b_1b_2\ldots b_q}$, with p fundamental indices and q antifundamental indices (possibly subject to (anti)symmetrizations and tracelessness conditions) of the flavor symmetry associated to the maximal puncture and with its transformation properties under other flavor symmetries suppressed. Then our expectations is that after gluing in the cylinder, there will be a new operator of the same dimension of the same form, but where its transformation under the symmetry of the original maximal puncture has been replaced with a transformation under the symmetry at the unglued end of the cylinder.

We can see how this might come about. Gluing a cylinder to the maximal puncture means tensoring the original chiral algebra with the chiral algebra of conjecture 8 in addition to the usual adjoint (b, c) system of dimensions $(1, 0)$. We then restrict ourselves to the BRST cohomology (relative to the b-ghost zero modes) of the nilpotent operator

$$Q_{\mathrm{BRST}} = \oint dz\, c^A((\mathcal{J}_L)_A + J_A^{\mathcal{T}} + \frac{1}{2}J_A^{\mathrm{gh}}) \;, \tag{3.5.4}$$

where $J_A^{\mathcal{T}}$ is the current for the symmetry associated to the puncture on $\mathcal{C}_{g,s\geq 1}$ that is being glued. Our original operator, which was charged under the $\mathfrak{su}(n)$ that is being gauged and therefore does not survive the passage to BRST cohomology, has a related *transferred operator* of the following form

$$\hat{X}^{c_1c_2\ldots c_p}_{d_1d_2\ldots d_q} = X^{b_1b_2\ldots b_q}_{a_1a_2\ldots a_p}\; g^{a_1c_1}\; g^{a_2c_2}\ldots g^{a_pc_p}\; g_{b_1d_1}\; g_{b_2d_2}\ldots g_{b_qd_q} \;. \tag{3.5.5}$$

This operator *is* gauge invariant, since the gauged symmetry acts on g_{ab}, g^{ab} on the left. In this sense the g_{ab} fields effectively transfer and conjugate the symmetry from one end of the cylinder to the other. Notice that the transferred operators have the same dimension as before, because the $g_{ab}(z)$ have dimension zero. What's more, by virtue of the unit determinant condition on g_{ab}, we see that the OPE of the transferred fields ends up being exactly the conjugate of the OPE of the original fields. It therefore seems likely that we recover precisely the same chiral algebra on the other end of the cylinder (up to conjugation of $\mathfrak{su}(n)$ representations). Of course, for this construction to work we have to assume that the spectrum of physical operators will consist only of the transferred operators. It would be interesting to prove this conjecture.

Finally, one can't help but notice the similarities between this description of the cylinder chiral algebra and the discussions of [78] regarding the holomorphic symplectic manifold associated with the cylinder in the Higgs branch TQFT. In that work, the hyperkähler manifold $T^*G_{\mathbb{C}}$ was associated to the cylinder. It is interesting to note that the chiral algebra we have described in Conjecture 8 seems to be precisely what one obtains from studying the half-twisted $(0, 2)$ supersymmetric sigma model on $G_{\mathbb{C}}$ [97, 98]. Alternatively, it describes the global sections of the sheaf of chiral differential operators on $G_{\mathbb{C}}$ as defined in [99, 100, 101, 102, 103]. This connection is exciting, but remains mostly mysterious to the authors at present.

3.5.2 The (decorated) cap chiral algebra

The chiral algebra associated to a decorated cap can be defined by partially reducing one puncture of the cylinder chiral algebra. The resulting chiral algebra should have the interesting property that if you glue it to another class $\mathcal{S}$ chiral algebra using the standard gauging BRST construction, it effectively performs the appropriate qDS reduction on the original chiral algebra.

In trying to characterize this chiral algebra, one immediately encounters the problem that it includes operators of negative dimension. Namely, consider the first steps of the general reduction procedure as applied to the cylinder chiral algebra. The (potentially redundant) generators for the decorated cap labeled by an embedding Λ include the usual currents $J_{\bar{\alpha}}$ for $T_{\bar{\alpha}} \in \ker(ad_{\Lambda(t_+)})$, the dimensions of which are shifted by their $\Lambda(t_0)$ weight. However, there are additional generators coming from the dimension zero bifundamental fields g_{ab} of the cylinder theory. In terms of the reduced symmetry associated with the decoration, these fields are reorganized as follows: for each irrep of $\mathfrak{su}(2)$ in the decomposition (3.2.30) of the fundamental representation there are $2j+1$ generators transforming in representation $\mathfrak{f}\otimes\mathcal{R}_j^{(\mathfrak{f})}$ with dimensions $-j, -j+1, \ldots, j$. The dimension zero null relation corresponding to the determinant condition in the cylinder theory of the cylinder theory is expected to descend to the cap theory. The superconformal index (see App. B.2.1) supports this expectation, and further suggests that there may be no additional redundancies.

The existence of negative dimension operators makes this a rather exotic chiral algebra, and we will not explore it much further. Nevertheless, let us offer a couple of brief comments. In the description of the cap chiral algebra given in the previous paragraph, it is not immediately clear that an affine current algebra associated to the maximal puncture survives. However, one finds that the necessary dimension one currents can be constructed using the above fields in a manner similar to (3.5.2), using only those elements of the left current algebra that survive in the cap chiral algebra. When gluing the cap to another theory $\mathcal{T}$, this current algebra will enter in the BRST current (3.5.4). As in the case of the cylinder, the Gauss law constraint can be solved by constructing transferred fields, which thanks to nonzero conformal dimension of the various components of g_{ab} end up with their dimensions shifted correctly. It remains to verify that restricting to the BRST cohomology removes the transferred versions of the currents $J_A^{\mathcal{T}}$ for

$T_A \notin \ker(ad_{\Lambda(t_+)})$.

Chapter 4

Chiral Algebras for Trinion Theories

4.1 Introduction

In chapter 2 and reference [26] it was shown that even-dimensional extended superconformal field theories (SCFTs)[1] contain a protected subsector that is isomorphic to a two-dimensional chiral algebra. This subsector is obtained by restricting operators to be coplanar and treating them at the level of cohomology with respect to a particular nilpotent supercharge, obtained as a combination of a supercharge and a superconformal charge of the theory. In showing the existence of the chiral algebra one relies only on the symmetries of the theory and there is no need to have a Lagrangian description — a fact that was used to study the chiral algebras associated with the six-dimensional $(2,0)$-theory in [26] and with those obtained from four-dimensional theories of class $\mathcal{S}$ in chapter 3. In this note we will focus on the chiral algebras associated with the so-called trinion or T_n theories of class $\mathcal{S}$.

Chiral algebras of class $\mathcal{S}$, *i.e.*, the collection of chiral algebras associated with four-dimensional theories of class $\mathcal{S}$ [10, 11], were argued to take the form of a generalized topological quantum field theory (TQFT) in chapter 3. Within this TQFT, gluing, the operation associated to four-dimensional exactly marginal gauging, is achieved by solving a BRST cohomology problem, and partially closing a puncture is implemented via a quantum Drinfeld-

[1]More precisely $\mathcal{N}=(2,0)$ in $d=6$, $\mathcal{N}\geq 2$ in $d=4$, and "small" $\mathcal{N}=(0,4)$ and $\mathcal{N}=(4,4)$ in $d=2$.

Sokolov reduction. Furthermore, just as the isolated, strongly interacting T_n theories, *i.e.*, the theories whose UV-curve is a sphere with three punctures of maximal type, are the basic building blocks of class $\mathcal{S}$ theories, so are their associated chiral algebras the basic building blocks of said TQFT. Characterizing the T_n chiral algebras is thus a prerequisite for an in principle complete understanding of chiral algebras of class $\mathcal{S}$.

However, while the existence of a chiral algebra inside a generic $\mathcal{N}=2$ SCFT can be argued in general terms, a complete characterization of its generators is currently lacking.[2] As for a partial characterization, it was argued in chapter 2 that one is guaranteed to have at least generators in one-to-one correspondence with the Higgs branch chiral ring generators.[3] In particular, the T_n Higgs branch chiral ring contains as generators three moment map operators, one for each factor in the T_n flavor symmetry algebra $\bigotimes_{i=1}^{3}\mathfrak{su}(n)_i$, and it was shown in chapter 2 that their corresponding chiral algebra generators are three affine currents with affine levels $k_{2d,i}$ determined in terms of the four-dimensional flavor central charges $k_{4d,i}$ as $k_{2d,i}=-\frac{k_{4d,i}}{2}$. These central charges are equal for the three factors, $k_{4d,i}=2n$, and thus the affine current algebras $\widehat{\mathfrak{su}(n)}$ have critical level $k_{2d}\equiv k_{2d,i}=-n$. The remaining generators of the T_n Higgs branch chiral ring give rise to additional generators of the chiral algebra, which must be primaries of the affine Kac-Moody (AKM) algebras.

It was also shown in chapter 2 that the existence of a four-dimensional stress tensor implies that the chiral algebra must contain a meromorphic stress tensor. Therefore the global $\mathfrak{sl}(2)$ conformal algebra enhances to a Virasoro algebra, with the central charge fixed in terms of the four-dimensional c-anomaly coefficient by $c_{2d}=-12c_{4d}$. However, the stress tensor is not necessarily a new generator of the chiral algebra, as it could be a composite operator (*i.e.*, obtained from normal-ordered products of the generators and of their derivatives). Since the AKM current algebras are at the critical level, they do not admit a normalizable Sugawara stress tensor, and therefore the stress tensor can only be a composite if additional dimension two singlet composites can be constructed. This is only possible (and in fact happens)

[2]For Lagrangian theories, this problem can (in principle) be circumvented by explicitly constructing the full chiral algebra from the basic known chiral algebras associated with the free hypermultiplet and vector multiplet.

[3]More generally, all generators of the so-called Hall-Littlewood chiral ring give rise to generators of the chiral algebra. For class $\mathcal{S}$ theories with acyclic generalized quivers, such as the T_n theories, the Hall-Littlewood chiral ring equals the Higgs branch chiral ring.

for $n = 2$ and 3.

In section 4.2 we perform a detailed study of the graded partition function of the T_n chiral algebra, which can be computed thanks to its equality to the so-called Schur limit of the $\mathcal{N} = 2$ superconformal index [37, 38], and which shows that the collection of generators listed so far is not complete for $n > 4$ (see section 4.2). Motivated by this analysis, we conjectured the complete set of generators to be as in conjecture 5, which we reproduce here for convenience:

Conjecture 9 (T_n chiral algebra) *The T_n chiral algebra $\chi(T_n)$ is generated by*

- *The set of operators, $\mathcal{H}$, arising from the Higgs branch chiral ring:*
  - *Three $\widehat{\mathfrak{su}(n)}$ affine currents J^1, J^2, J^3, at the critical level $k_{2d} = -n$, one for each factor in the flavor symmetry group of the theory,*
  - *Generators $W^{(k)}$, $k = 1, \ldots, n-1$ in the $(\wedge^k, \wedge^k, \wedge^k)$ representation of $\bigotimes_{j=1}^{3} \mathfrak{su}(n)_j$, where $\wedge^k$ denotes the $k-$index antisymmetric representation of $\mathfrak{su}(n)$. These generators have dimensions $\frac{k(n-k)}{2}$,*
- *Operators $\mathcal{O}_i$, $i = 1, \ldots n-1$, of dimension $h_i = i + 1$ and singlets under $\bigotimes_{j=1}^{3} \mathfrak{su}(n)_j$, with the dimension 2 operator corresponding to the stress tensor T of central charge $c_{2d} = -2n^3 + 3n^2 + n - 2$,*

modulo possible relations which set some of the operators listed above equal to composites of the remaining generators.

In other words, if one were to start with all generators of the above conjecture, one would find that some of them could be involved in null relations with composite operators, thereby being redundant. For example, in the chiral algebra associated with T_2, *i.e.*, the theory of eight free half-hypermultiplets, the affine currents and the stress tensor can be written as composites of the dimension $\frac{1}{2}$ generator $W^{(1)}$. For the case of T_3, which corresponds to the E_6 theory of [61], convincing evidence was provided in chapter 3 that its chiral algebra $\chi(T_3)$ is fully generated by operators originating from the Higgs branch chiral ring. The stress tensor can be written as a composite and also, although not explicitly constructed in chapter 3, the dimension three singlet operator is accounted for as a composite. For $n > 3$, as argued above, the stress tensor cannot be a composite of generators in $\mathcal{H}$, but the remaining dimension $3, \ldots, n$ singlet generators could still be. In the case of the T_4 chiral algebra the dimensions three and four singlet generators are redundant, as will be shown in section 4.3.

Our aim in section 4.3 is to verify Conjecture 9 for T_4, in which case the chiral algebra is generated by the operators in $\mathcal{H}$ and the stress tensor, by explicitly constructing an associative algebra with these generators. Our approach to bootstrap this problem is to write down the most general operator product expansions (OPEs) between the generators, and to demand associativity of the operator product algebra by imposing the Jacobi-identities. Since chiral algebras are very rigid, one can hope that these constraints are sufficiently stringent to completely fix the operator algebra, as was famously shown to be the case for the first time for the $\mathcal{W}_3$ algebra in [104] (see for example [42] for a review of other cases). We indeed find that the OPEs are completely and uniquely fixed. The analysis of the Jacobi-identities becomes technically involved in several instances, and as a result we can only claim that the conditions analyzed are necessary for an associative operator product algebra. However we believe that the remaining Jacobi-identities provide redundant constraints. As an interesting by-product of the explicit T_4 chiral algebra, we can compute four-dimensional Higgs branch chiral ring relations, which appear as null relations in the chiral algebra setting. Some of these relations are already known in the literature, (*e.g.*, [105, 86]), and recovering them here provides a further check of the chiral algebra, while others are new.

As mentioned, four-dimensional Higgs branch chiral ring relations can be obtained from null relations in the chiral algebra. The explicit construction of $\chi(T_4)$ we present here thus provides a new, conceptually clear method to obtain all Higgs branch chiral ring relations for the T_4 theory. It seems plausible that once their structure is understood, they can be generalized to arbitrary T_n. In this chapter we obtain all $\chi(T_4)$ null relations of dimension smaller than four, already uncovering new Higgs branch chiral ring relations, but the procedure can be taken further. For example, it would be possible to verify the recently proposed null relation of [106], as well as uncover further unknown ones. Furthermore, as will be elaborated upon in the next sections, our interpretation of the $\chi(T_n)$ chiral algebra partition function also predicts the existence of certain types of null relations, facilitating the task of explicitly computing them in the chiral algebra setting.

Further checks of the $\chi(T_4)$ chiral algebra could be performed by partially closing punctures (via a quantum Drinfeld-Sokolov (qDS) reduction (see chapter 3)) to obtain the free hypermultiplet, the E_7 theory of [62], or more generally the other fixtures of [67]. For example, the chiral algebra associated with the E_7 theory is conjectured to be described by an affine $\widehat{\mathfrak{e}_7}$ current algebra at level $k_{2d} = -4$ and it is easy to convince oneself that the qDS procedure associated with the relevant $\mathfrak{su}(2)$ embedding will indeed result in dimension one currents corresponding to the decomposition of the $\mathfrak{e}_7$ adjoint representation. As shown in chapter 3, to complete the reduction argument, certain null relations need to exist in order to remove redundant generators in the reduced algebra. Such null relations are expected to descend from those of $\chi(T_4)$.

The construction of $\chi(T_4)$ in this chapter makes use of the constraints arising from associativity of the operator algebra. It would also be interesting to study if the theory space bootstrap, as introduced in chapter 3, which imposes instead associativity of the TQFT structure, might result in a complementary route to construct the chiral algebra. In particular with an eye towards a construction of $\chi(T_n)$, for $n > 4$, an alternative (or a combined) approach might prove useful.

The organization of this chapter is as follows. In section 4.2 we analyze the partition function of $\chi(T_n)$ employing its equality to the superconformal index of T_n theories, and show how it motivates Conjecture 9, as well as some other expectations about the chiral algebra. In section 4.3 we present the explicit construction of the T_4 chiral algebra and give explicit expressions for various null relations. We also show how our expectations deduced from the superconformal index are realized for T_4. The readers interested only in the explicit construction of $\chi(T_4)$ can safely skip section 4.2 as section 4.3 is mostly independent from it. Appendix C.1 contains some technical details on the relation between critical affine characters and the superconformal index, and in appendix C.2 we collect all singular OPEs defining the chiral algebra $\chi(T_4)$.

4.2 T_n indexology

In this section we analyze the partition function of the T_n chiral algebra, which gives insights into its generators and relations. By writing the partition function in a suggestive way we can justify Conjecture 9 and infer some properties of the structure of the chiral algebra, such as its null relations.

As shown in chapter 2, the graded partition function of the chiral algebra $\chi(T_n)$ equals the so-called Schur limit of the superconformal index of the T_n theory [37, 38]. We work under the assumption that all generators are bosonic and thus the grading is immaterial. In appendix C.1 we show that the index can be written in a way suggestive of its interpretation as a two-dimensional partition function as

$$\mathcal{Z}_{\chi(T_n)}(q;\mathbf{x}_i) = \sum_{\mathfrak{R}_\lambda} q^{\langle\lambda,\rho\rangle} C_{\mathfrak{R}_\lambda}(q) \prod_{i=1}^{3} \mathrm{ch}_{\mathfrak{R}_\lambda}(q,\mathbf{x}_i)\,. \tag{4.2.1}$$

Here $\mathbf{x}_i$ denote flavor fugacities conjugate to the Cartan generators of the $\mathfrak{su}(n)_i$ flavor symmetry associated with each of the three punctures, and the sum runs over all irreducible $\mathfrak{su}(n)$ representations $\mathfrak{R}_\lambda$ of highest weight λ. The summand contains the product of three copies — one for each puncture — of $\mathrm{ch}_{\mathfrak{R}_\lambda}(q,\mathbf{x})$, the character of the critical irreducible highest weight representation of the affine current algebra $\widehat{\mathfrak{su}(n)}_{-n}$ with highest weight $\hat{\lambda}$, whose restriction to $\mathfrak{su}(n)$ is the highest weight λ [107].[4] Furthermore, ρ is the Weyl vector and $\langle\cdot,\cdot\rangle$ denotes the Killing inner product. Finally, the structure constants $C_{\mathfrak{R}_\lambda}(q)$ can be written as

$$C_{\mathfrak{R}_\lambda}(q) = \text{P.E.}\left[2\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q} + 2\sum_{j=1}^{n-1}(n-j)\,q^j - 2\sum_{j=2}^{n}\sum_{1\leq i<j} q^{\ell_i-\ell_j+j-i}\right], \tag{4.2.2}$$

where $\ell_{i=1,\ldots,n}$ denote the lengths of rows of the Young tableau describing representation $\mathfrak{R}_\lambda$ with $\ell_n=0$, d_j are the degrees of invariants, *i.e.* $d_j = j+1$ for $\mathfrak{su}(n)$, and finally P.E. denotes the standard plethystic exponential

$$\text{P.E.}\,[f(x)] = \exp\left(\sum_{m=1}^{\infty}\frac{f(x^m)}{m}\right)\,. \tag{4.2.3}$$

Let us provide some preliminary interpretative comments:

- We have obtained an expression for the partition function (4.2.1) that is manifestly organized in terms of modules of the direct product of the three critical affine current algebras $\bigotimes_{i=1}^{3}(\widehat{\mathfrak{su}(n)_i})_{-n}$. Indeed, the factor $q^{\langle\lambda,\rho\rangle}\prod_{i=1}^{3}\mathrm{ch}_{\mathfrak{R}_\lambda}(q,\mathbf{x}_i)$ in (4.2.1) captures threefold AKM primaries

[4] Our notation here and in appendix C.1 follows that of [108].

of dimension $\langle \lambda, \rho \rangle$, transforming in representations $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$, including for example all the $W^{(k)}$, and all of their AKM descendants.

- The role of the structure constants is to encode additional operators beyond those captured by the threefold AKM modules. In particular, in the term $\mathfrak{R}_{\lambda=0}$ in the sum over representations, the structure constant $C_{\mathfrak{R}_{\lambda=0}}(q) = \text{P.E.}\left[2\sum_{j=1}^{n-1} \frac{q^{d_j}}{1-q}\right]$ encodes two sets of additional operators of dimensions $d_j = j+1$, for $j = 1, \ldots, n-1$, (and their $\mathfrak{sl}(2)$ descendants) acting on the vacuum module. These operators can either be new generators, or obtained as singlet composites of the generators captured by the AKM modules, which themselves are not present in the modules. Let us now describe these two sets:

  1. The fact that the three AKM current algebras are at the critical level implies that all the Casimir operators $\text{Tr}(J^1)^k$, $\text{Tr}(J^2)^k$, $\text{Tr}(J^3)^k$ with $k = 2, 3, \ldots, n$ are null within their respective AKM algebra, and therefore that their action is not included in the affine modules. However, these operators do not remain null in the full chiral algebra, as it contains a stress tensor as well. In fact, null relations set all Casimirs equal $\text{Tr}(J^1)^k = \text{Tr}(J^2)^k = \text{Tr}(J^3)^k$.[5] These $n-1$ Casimirs correspond to the first set of operators reinstated by the structure constants.
  2. The second set of operators motivates our conjecture that there can be extra generators $\mathcal{O}_i$ with precisely dimensions $h_i = d_i = i+1$.

A more detailed discussion of these statements, and the interpretation of the two remaining factors in (4.2.2) is given in the remainder of this section. Readers not interested in this technical analysis can safely skip the remainder of this section.

The AKM modules

Ignoring for a moment the structure constants, each term in the sum over representations $\mathfrak{R}_\lambda$ of (4.2.1) captures the states in the direct product of

[5]The existence of these null relations follows directly from the existence of relations on the Higgs branch chiral ring setting the Casimir operators formed out of the moment map operators of the three flavor symmetries equal [86]. The corresponding chiral algebra null relations will be recovered in the next section.

three critical affine modules with primary transforming in representation $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$. The dimension of the threefold AKM primary is implemented by the factor $q^{\langle \lambda, \rho \rangle}$, yielding

$$h_{(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)} = \langle \lambda, \rho \rangle = \sum_{i=1}^{n-1} \frac{n-(2i-1)}{2} \ell_i \,. \tag{4.2.4}$$

These pairings of dimension and representations include all the threefold AKM primary generators $W^{(k)}$, $k = 1, \ldots, n-1$ in Conjecture 9. (Note that the currents themselves are AKM descendants of the identity operator and appear in the vacuum module.) We expect that the remaining threefold AKM primaries in the sum over $\mathfrak{R}_\lambda$ all arise from combinations of normal-ordered products of generators in $\mathcal{H}$ (the set of generators originating from the Higgs branch chiral ring generators), and do not give rise to additional generators. It is clear that for each representation $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$ one can write down a composite operator of the $W^{(k)}$, transforming in such representation, and with the appropriate dimension. Then, it seems plausible that such operator can always be made into a threefold AKM primary by — if necessary — adding composites of the remaining operators in $\mathcal{H}$. We have checked this statement in a few low-dimensional examples for T_4 (see equation (4.3.8) for an explicit example). All in all, the AKM modules capture the generators $W^{(k)}$, as well as other threefold AKM primaries obtained as their normal-ordered product, and all of their AKM descendants.

The structure constants

The structure constants (4.2.2) encode additional operators on top of those captured by the AKM modules already described. Let us start by analyzing the factor

$$\text{P.E.}\left[2 \sum_{j=1}^{n-1} \frac{q^{d_j}}{1-q} \right] \,. \tag{4.2.5}$$

When inserted in (4.2.1), it encodes two sets of operators of dimensions d_j and their $\mathfrak{sl}(2)$ descendants (taken into account by the denominator $\frac{1}{1-q}$), normal-ordered with all operators in any given AKM module $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$. As described before, one set adds back the Casimir operators $\text{Tr}(J^1)^k = \text{Tr}(J^2)^k = \text{Tr}(J^3)^k$ of the AKM algebras,[6] and the second set motivates

[6]Both the Casimirs and the Casimirs normal-ordered with threefold AKM primaries are new threefold AKM primaries, since they were null if one were to consider each AKM

the claim that there can be additional generators $\mathcal{O}_{i=1,\ldots,n-1}$ of dimensions $h_i = d_i = i+1$.[7] However, one should bear in mind that in some cases one can construct (non-null) non-AKM-descendant singlet operators as composites of the $W^{(k)}$ of dimensions h equal to one of these dimensions. Since the only singlet operator in the sum over AKM modules, which is not an AKM descendant, corresponds to the identity operator, such operators must be accounted for by (4.2.5). This leaves two possibilities: it is either equal (or set equal by a null relation) to a composite of smaller dimensional $\mathcal{O}_i$ operators and/or of Casimirs, and consequently taken into account by the plethystic exponentiation in (4.2.5). Or it must take the place of the would-be generator $\mathcal{O}$ of dimension h. In other words, if one were to include $\mathcal{O}$, one would find a null relation between this would-be generator and the composite of $W^{(k)}$. As was mentioned before, the simplest example is the stress tensor $T \equiv \mathcal{O}_1$, which for T_2 and T_3 is a composite, but for $T_{n\geq 4}$ must be a new generator. In the next section we will show that for T_4 the generators of dimension three and four are absent, as the type of composites described above exist. However for $n \geq 5$ it is not possible to write such a composite of dimension three, and $\mathcal{O}_2$ must be a generator.

We now turn to the next factor in the structure constants (4.2.2)

$$\text{P.E.}\left[2\sum_{k=1}^{n-1}(n-k)\,q^k\right] . \tag{4.2.6}$$

Recalling that at the critical level the stress tensor is not obtained from the Sugawara construction, the critical modules do not contain derivatives[8] of the threefold AKM primaries, although the full chiral algebra must. Similarly, the action of the modes $(\mathcal{O}_i)_{-1}$, $i = 2, \ldots, n-1$, $(\mathcal{O}_i)_{-2}$, $i = 2, \ldots, n-1$, $(\mathcal{O}_i)_{-3}$, $i = 3, \ldots, n-1$, $\ldots$ of the remaining singlet operators of the chiral algebra are not yet included. The number of modes we have to take into account at grade k is precisely given by $n-k$, and these modes are added by one of the factors in (4.2.6). The other factor adds back similar modes of the Casimir operators of the AKM current algebras, an explicit example of which

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current algebra in isolation.

[7]For readers familiar with the classification of four-dimensional superconformal multiplets of [45], these generators arise from four-dimensional operators in the $\hat{\mathcal{C}}$ multiplets.

[8]*i.e.*, the action of the mode L_{-1}. As is common practice we use the mode expansion $\mathcal{O}(z) = \sum_n \frac{\mathcal{O}_n}{z^{n+h}}$ of an operator $\mathcal{O}$ of dimension h, and L_n denotes the modes of the stress tensor T.

will be given in the next section (see (4.3.7)). It is clear that these modes cannot be added for all representations: for example, they cannot be added when considering the vacuum, since it is killed by all of them. Similarly, (and here we restrict to $n > 2$) the only grade one modes acting on any of the $W^{(k)}$ that do not kill it must be the ones which correspond to either acting on it with a derivative, or normal-ordering it with a current, since these are the only ways one can write a dimension $\frac{k(n-k)}{2} + 1$ composite in representation $(\wedge^k, \wedge^k, \wedge^k)$.[9] These facts are taken into account by the factor

$$\text{P.E.}\left[-2\sum_{j=2}^{n}\sum_{1\leq i<j} q^{\ell_i-\ell_j+j-i}\right], \tag{4.2.7}$$

which must subtract such relations, as well as other possible relations specific of each representation. Indeed, it is for example easy to verify that (4.2.6) and (4.2.7) cancel each other for the vacuum module. For representations $(\wedge^k, \wedge^k, \wedge^k)$ only two q terms survive in the plethystic exponential in the product of (4.2.6) and (4.2.7), which means that we are left with two grade one modes. One might have expected four grade one modes: one corresponding to acting with a derivative and three to normal-ordering with the three currents, but, as we will see in the next section, normal-ordering the three currents with $W^{(k)}$ (making an operator in representation $(\wedge^k, \wedge^k, \wedge^k)$) results in equal operators up to nulls (see equations (4.3.4) and (4.3.7)).

As a final observation we note that the sum in (4.2.1) only runs over flavor symmetry representations of the type $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$, and the structure constants (4.2.2) cannot alter flavor symmetry information. Therefore the partition function predicts that any operator transforming in a representation $(\mathfrak{R}_{\lambda_1}, \mathfrak{R}_{\lambda_2}, \mathfrak{R}_{\lambda_3})$ with not all equal λ_i cannot be a threefold AKM primary. More precisely, if we encounter an operator in unequal representations $(\mathfrak{R}_{\lambda_1}, \mathfrak{R}_{\lambda_2}, \mathfrak{R}_{\lambda_3})$ it must either be an AKM descendant, or obtained from one via the operators taken into account by the structure constants (namely by the action of any operators contributing to (4.2.5) and (4.2.6)). We will get back to this point in the next section (around example (4.3.5)).

[9] Note that the normal-ordered product $(JW^{(k)})$ in representation $(\wedge^k, \wedge^k, \wedge^k)$ is absent in the critical module.

4.3 The T_4 chiral algebra

For the chiral algebra associated with the T_4 theory, Conjecture 9 states that the collection of generators $\mathcal{G}$ contains three $\widehat{\mathfrak{su}(4)}$ affine currents at the critical level $k_{2d} = -4$, which we denote by $(J^1)^{b_1}_{a_1}$, $(J^2)^{b_2}_{a_2}$, $(J^3)^{b_3}_{a_3}$, two dimension $\frac{3}{2}$ generators, $W^{(1)}$ and $W^{(3)}$, in the tri-fundamental and tri-antifundamental representations of the flavor symmetry group respectively, which we rename $W_{a_1a_2a_3}$ and $\widetilde{W}^{b_1b_2b_3}$, and one dimension two generator, $W^{(2)}$, in the $\mathbf{6} \times \mathbf{6} \times \mathbf{6}$ representation which we denote explicitly as $V_{[a_1b_1][a_2b_2][a_3b_3]}$. Here $a_i, b_i, c_i, \ldots = 1, 2, 3, 4$ are (anti)fundamental indices corresponding to the flavor symmetry factor $\mathfrak{su}(4)_{i=1,2,3}$. Moreover, we must add the stress tensor T as an independent generator, with central charge $c_{2d} = -78$, but we claim that the dimension three and four singlets operators can be obtained as composites. As will be shown later the dimension three operator is argued to be a Virasoro primary involving $W\widetilde{W}|_{\text{sing}}$, where $|_{\text{sing}}$ means we take the singlet combination, while the dimension four generator is a Virasoro primary combination involving $VV|_{\text{sing}}$. We summarize the conjectured generators in Table 4.1.

generator $\mathcal{G}$	$h_{\mathcal{G}}$	$\mathcal{R}_{\mathcal{G}}$
$(J^1)^{a_1}_{b_1}$	1	$(\mathbf{15},\mathbf{1},\mathbf{1})$
$(J^2)^{a_2}_{b_2}$	1	$(\mathbf{1},\mathbf{15},\mathbf{1})$
$(J^3)^{a_3}_{b_3}$	1	$(\mathbf{1}, \mathbf{1},\mathbf{15})$
T	2	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$W_{a_1a_2a_3}$	$\frac{3}{2}$	$(\mathbf{4},\mathbf{4},\mathbf{4})$
$\widetilde{W}^{a_1a_2a_3}$	$\frac{3}{2}$	$(\bar{\mathbf{4}},\bar{\mathbf{4}},\bar{\mathbf{4}})$
$V_{[a_1b_1][a_2b_2][a_3b_3]}$	2	$(\mathbf{6},\mathbf{6},\mathbf{6})$

Table 4.1: T_4 generators $\mathcal{G}$, their dimension $h_{\mathcal{G}}$ and their $\mathfrak{su}(4)^3$ representation $\mathfrak{R}_{\mathcal{G}}$.

As mentioned before, our strategy for finding the T_4 chiral algebra is a concrete implementation of the conformal bootstrap program. We start by writing down the most general OPEs for this set of generators consistent with the symmetries of the theory, and in particular we impose that the three different flavor symmetry groups appear on equal footing. This of course implies that the three flavor currents have the same affine level, simply denoted by k_{2d}. The OPEs of all the generators with the stress tensor are naturally fixed to be those of Virasoro primaries with the respective dimensions. Moreover, all generators listed in Table 4.1, with the exception of the stress tensor[10], are affine Kac-Moody primaries of the three current algebras, transforming in the indicated representation. Therefore their OPEs with the currents are also completely fixed. In the self-OPEs of the AKM currents and the stress tensor, we could fix the affine level and the central charge to the values corresponding to the $\chi(T_4)$ chiral algebra, $k_{2d} = -4$ and $c_{2d} = -78$. Instead we leave them as free parameters and try to fix them the same way as any other OPE coefficient. For the remaining OPEs we write all possible operators allowed by the symmetries of the theory with arbitrary coefficients. Our expectation is that this chiral algebra is unique, and that by imposing associativity one can fix all the OPE coefficients, including k_{2d} and c_{2d}. This indeed turns out to be true. Some of the resulting OPEs are quite long so we collect them all in appendix C.2.[11]

The next step is to fix all the arbitrary coefficients by imposing Jacobi-identities, implementing in this way the requirement that the operator algebra is associative. Concretely, we impose on any combination of three generators A, B, C the Jacobi-identities (see, *e.g.*, [109])

$$[A(z)\,[B(w)C(u)]] - [B(w)\,[A(z)C(u)]] - [[A(z)B(w))]\,C(u)] = 0\,, \quad (4.3.1)$$

for $|w-u| < |z-u|$, where $[A(z)B(w)]$ denotes taking the singular part of the OPE of $A(z)$ and $B(w)$, and where we already took into account that our generators are bosonic and no extra signs are needed. It is important to note that the Jacobi-identities do not need to be exactly zero, but they can be proportional to null operators. Since null operators decouple, associativity of the algebra is not spoiled. For analyzing the Jacobi-identities we make use of the Mathematica package described in [48]. Even so, the analysis is quite cumbersome due to the large number of fields appearing in the OPEs

[10] It is clear that the stress tensor cannot be an AKM primary, as the OPE between a dimension one operator (the current) and the stress tensor must have necessarily a $\frac{1}{z^2}$ pole. It is also not an AKM descendant, since at the critical level it cannot be given by the Sugawara construction.

[11] In there and in what follows we adopt the standard conventions for the normal-ordering of operators such that $\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_{\ell-1}\mathcal{O}_\ell = (\mathcal{O}_1\,(\mathcal{O}_2\ldots(\mathcal{O}_{\ell-1}\mathcal{O}_\ell)\,...))$.

and the necessity of removing null relations, especially so for the Jacobi-identities involving the generator V. These null relations are not known *a priori*, therefore part of the task consists of obtaining all null operators at each dimension and in a given representation of the flavor symmetry. Due to these technical limitations we have only found necessary conditions for the Jacobi-identities to be satisfied, not sufficient ones. Nevertheless these conditions turn out to fix completely all the OPE coefficients, including the level and the central charge of the theory, meaning the chiral algebra with this particular set of generators is unique. After all coefficients are fixed, the remaining Jacobi-identities analyzed serve as a test on the consistency of our chiral algebra. We have checked a large enough set of Jacobi-identities to be convinced that the remaining ones will not give any additional constraints. If that is the case we have found an associative operator algebra with the same set of generators and the same central charges as conjectured for the T_4 chiral algebra. A further check that the chiral algebra we constructed corresponds indeed to the T_4 chiral algebra can be performed by comparing the partition function of the former to the one of the latter (which is nothing else than the superconformal index of T_4). Whereas in section 4.2 we have exploited the index to motivate our claim about the full set of generators of the T_4 chiral algebra, in what follows we perform a partial check of the equality of the actual partition function of the constructed chiral algebra with the index by comparing the null states of the chiral algebra to the ones predicted by the superconformal index up to dimension $\frac{7}{2}$. Even if there were generators that we have missed in this analysis, the facts that the generators in our chiral algebra must be present, and that the chiral algebra we constructed is closed (assuming we have solved all constraints from the Jacobi-identities), imply that we have found a closed subalgebra of the full T_4 chiral algebra.

For practical purposes, it is useful to rewrite the partition function of the T_n chiral algebra (4.2.1) alternatively as

$$\mathcal{Z}_{\chi(T_n)}(q;\mathbf{x}_i) = \text{P.E.}\left[\frac{1}{1-q}\sum_{\text{generators }\mathcal{G}} q^{h_{\mathcal{G}}}\chi^{\mathfrak{su}(n)^3}_{\mathcal{R}_{\mathcal{G}}}(\mathbf{x}_i)\right] - \sum_{\text{nulls }\mathcal{N}} q^{h_{\mathcal{N}}}\chi^{\mathfrak{su}(n)^3}_{\mathcal{R}_{\mathcal{N}}}(\mathbf{x}_i)\,, \tag{4.3.2}$$

in terms of a piece that describes the generators $\mathcal{G}$ of dimensions $h_{\mathcal{G}}$ transforming in representations $\mathcal{R}_{\mathcal{G}}$ of the $\mathfrak{su}(n)^3$ flavor symmetry and their $\mathfrak{sl}(2)$ descendants, and a term that subtracts off explicitly the null operators $\mathcal{N}$, of dimension $h_{\mathcal{N}}$ and in representation $\mathcal{R}_{\mathcal{N}}$. By comparing the expansion in

h	mult.	$\mathcal{R}$
2	2	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$\frac{5}{2}$	2	$(\mathbf{4},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{4}},\bar{\mathbf{4}},\bar{\mathbf{4}})$
3	4	$(\mathbf{1},\mathbf{1},\mathbf{1})$
	2	$(\mathbf{6},\mathbf{6},\mathbf{6})$
	1	$(\mathbf{6},\mathbf{6},\mathbf{10})$, and perms.
	3	$(\mathbf{15},\mathbf{1},\mathbf{1})$, and perms.
	1	$(\mathbf{15},\mathbf{15},\mathbf{1})$, and perms.
$\frac{7}{2}$	10	$(\mathbf{4},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{4}},\bar{\mathbf{4}},\bar{\mathbf{4}})$
	2	$(\mathbf{36},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{36}},\bar{\mathbf{4}},\bar{\mathbf{4}})$, and perms.
	1	$(\mathbf{20},\mathbf{20},\mathbf{4})$, $(\bar{\mathbf{20}},\bar{\mathbf{20}},\bar{\mathbf{4}})$, and perms.
	3	$(\mathbf{20},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{20}},\bar{\mathbf{4}},\bar{\mathbf{4}})$, and perms.

Table 4.2: Quantum numbers and multiplicities of T_4 null operators up to dimension $\frac{7}{2}$.

powers of q of (4.2.1) with that of (4.3.2) (and under the assumption that the full list of generators is as in Table 4.1) we can predict how many nulls to expect in each representation and at each dimension. In Table 4.2 we summarize the resulting quantum numbers of the low-lying null operators $\mathcal{N}$. We have explicitly constructed the null operators corresponding to the entries in Table 4.2; the full list is given in Tables 4.3 and 4.4, where we have defined S^i to be the quadratic Casimir $S^i = (J^i)^{b_i}_{a_i}(J^i)^{a_i}_{b_i}$.

$h_{\mathcal{N}}$	$\mathcal{R}_{\mathcal{N}}$	Null relations
2	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(J^1)_{a_1}^{b_1}(J^1)_{b_1}^{a_1} = (J^2)_{a_2}^{b_2}(J^2)_{b_2}^{a_2} = (J^3)_{a_3}^{b_3}(J^3)_{b_3}^{a_3}$
5/2	$(\mathbf{4},\mathbf{4},\mathbf{4})$	$(J^1)_{a_1}^{b_1}W_{b_1a_2a_3} = (J^2)_{a_2}^{b_2}W_{a_1b_2a_3} = (J^3)_{a_3}^{b_3}W_{a_1a_2b_3}$
3	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(J^1)_{a_1}^{b_1}(J^1)_{b_1}^{c_1}(J^1)_{c_1}^{a_1} = (J^2)_{a_2}^{b_2}(J^2)_{b_2}^{c_2}(J^2)_{c_2}^{a_2} = (J^3)_{a_3}^{b_3}(J^3)_{b_3}^{c_3}(J^3)_{c_3}^{a_3}$ $\partial\left((J^1)_{a_1}^{b_1}(J^1)_{b_1}^{a_1}\right) = \partial\left((J^2)_{a_2}^{b_2}(J^2)_{b_2}^{a_2}\right) = \partial\left((J^3)_{a_3}^{b_3}(J^3)_{b_3}^{a_3}\right)$
3	$(\mathbf{6},\mathbf{6},\mathbf{6})$	$(J^1)_{[a_1}^{c_1}V_{[b_1]c_1][a_2b_2][a_3b_3]} = (J^2)_{[a_2}^{c_2}V_{[a_1b_1][b_2]c_2][a_3b_3]} = (J^3)_{[a_3}^{c_3}V_{[a_1b_1][a_2b_2][b_3]c_3]}$
3	$(\mathbf{10},\mathbf{6},\mathbf{6})$	$W_{(a_1[a_2[a_3}W_{b_1)b_2]b_3]} = -\frac{1}{4}J_{(a_1}^{c_1}V_{[\vert c_1\vert b_1)][a_2b_2][a_3b_3]}$
3	$(\mathbf{15},\mathbf{1},\mathbf{1})$	$(J^1)_{a_1}^{b_1}(J^1)_{c_1}^{d_1}(J^1)_{d_1}^{c_1} = (J^1)_{a_1}^{b_1}(J^2)_{a_2}^{b_2}(J^2)_{b_2}^{a_2} = (J^1)_{a_1}^{b_1}(J^3)_{a_3}^{b_3}(J^3)_{b_3}^{a_3}$ $(W_{a_1a_2a_3}\tilde{W}^{b_1a_2a_3} - \text{trace}) = \frac{1}{16}(J^1)_{a_1}^{b_1}(J^3)_{a_3}^{b_3}(J^3)_{b_3}^{a_3} + \frac{1}{16}(J^1)_{a_1}^{b_1}T$ $+\left[((J^1)_{a_1}^{c_1}\partial(J^1)_{c_1}^{b_1} - \text{trace}) + ((J^1)_{c_1}^{b_1}\partial(J^1)_{a_1}^{c_1} - \text{trace})\right]$ $-\frac{1}{4}((J^1)_{a_1}^{c_1}(J^1)_{c_1}^{d_1}(J^1)_{d_1}^{b_1} - \text{trace})$
3	$(\mathbf{15},\mathbf{15},\mathbf{1})$	$(W_{a_1a_2a_3}\tilde{W}^{b_1b_2a_3} - \text{traces}) = \frac{1}{4}\left[(J^1)_{a_1}^{b_1}\partial(J^2)_{a_2}^{b_2} + (J^2)_{a_2}^{b_2}\partial(J^1)_{a_1}^{b_1}\right]$ $-\frac{1}{16}\left[((J^1)_{a_1}^{b_1}(J^2)_{a_2}^{c_2}(J^2)_{c_2}^{b_2} - \text{trace}) + ((J^2)_{a_2}^{b_2}(J^1)_{a_1}^{c_1}(J^1)_{c_1}^{b_1} - \text{trace})\right]$

Table 4.3: Explicit null relations up to dimension three, which can be uplifted to four-dimensional Higgs branch chiral ring relations. Representations which are not real give rise to a similar null in the complex conjugate representation, and representations which are not equal in the three flavor groups give rise to similar null relations with permutations of the flavor group indices. Note that these, together with Table 4.4, are in one-to-one correspondence to the null relations subtracted from the index given in Table 4.2.

Null relations in the two-dimensional chiral algebra can be uplifted to four-dimensional Higgs branch chiral ring relations, a partial list of which is given in [86], by setting to zero all derivatives and generators not coming from the Higgs branch chiral ring (in particular the stress tensor and the other singlet generators, if present as independent generators). The nulls of Tables 4.3 and 4.4 allow one to recover the low-dimensional Higgs branch chiral ring relations in [86], and to find additional ones. Let us give a few illustrative examples.

$h_{\mathcal{N}}$	$\mathcal{R}_{\mathcal{N}}$	Null relations
$7/2$	$(\mathbf{20},\mathbf{4},\mathbf{4})$	$8\widetilde{W}^{f_1b_2b_3}V_{[d_1(e_1][a_2b_2][a_3b_3]}\ \epsilon_{\|f_1\|a_1)b_1c_1}$ $= 9(J^1)^{f_1}_{[a_1}(J^1)^{g_1}_{b_1}W_{(c_1]a_2a_3}\ \epsilon_{d_1)e_1f_1g_1} - 2(J^1)^{f_1}_{[d_1}(J^3)^{b_3}_{a_3}W_{(e_1]a_2b_3}\ \epsilon_{\|f_1\|a_1)b_1c_1}$ $+3\partial(J^1)^{f_1}_{[d_1}W_{(e_1]a_2a_3}\ \epsilon_{\|f_1\|a_1)b_1c_1} + 6\partial\left((J^1)^{f_1}_{[d_1}W_{(e_1]a_2a_3}\right)\ \epsilon_{\|f_1\|a_1)b_1c_1}$ $(J^1)^{f_1}_{[d_1}(J^2)^{b_2}_{a_2}W_{(e_1]b_2a_3}\ \epsilon_{\|f_1\|a_1)b_1c_1} = (J^1)^{f_1}_{[d_1}(J^3)^{b_3}_{a_3}W_{(e_1]a_2b_3}\ \epsilon_{\|f_1\|a_1)b_1c_1}$ $= (J^1)^{f_1}_{[d_1}(J^1)^{g_1}_{(e_1]}W_{\|g_1\|a_2a_3}\ \epsilon_{\|f_1\|a_1)b_1c_1}$
$7/2$	$(\mathbf{20},\mathbf{20},\mathbf{4})$	$\widetilde{W}^{f_1f_2b_3}V_{[d_1(e_1][d_2(e_2][a_3b_3]}\ \epsilon_{\|f_1\|a_1)b_1c_1}\ \epsilon_{\|f_2\|a_2)b_2c_2}$ $= -\frac{1}{2}(J^1)^{f_1}_{[d_1}(J^2)^{f_2}_{[d_2}W_{(e_1]\,(e_2]a_3}\ \epsilon_{\|f_1\|a_1)b_1c_1}\ \epsilon_{\|f_2\|a_2)b_2c_2}$
$7/2$	$(\mathbf{36},\mathbf{4},\mathbf{4})$	$(J^1)^{b_1}_{(a_1}(J^2)^{b_2}_{a_2}W_{c_1b_2a_3}\ \epsilon_{d_1)b_1e_1f_1} = (J^1)^{b_1}_{(a_1}(J^3)^{b_3}_{a_3}W_{c_1a_2b_3}\ \epsilon_{d_1)b_1e_1f_1} =$ $(J^1)^{b_1}_{(a_1}(J^1)^{h_1}_{c_1}W_{\|h_1\|a_2a_3}\ \epsilon_{d_1)b_1e_1f_1}$
$7/2$	$(\mathbf{4},\mathbf{4},\mathbf{4})$	$S^1W_{a_1a_2a_3} = S^2W_{a_1a_2a_3} = S^3W_{a_1a_2a_3}$ $8\widetilde{W}^{b_1b_2b_3}V_{[b_1a_1][b_2a_2][b_3a_3]} = 2(J^1)^{b_1}_{a_1}(J^3)^{b_3}_{a_3}W_{b_1a_2b_3} + 9TW_{a_1a_2a_3}$ $+15\partial\left((J^3)^{b_3}_{a_3}W_{a_1a_2b_3}\right) - \frac{9}{2}\partial^2W_{a_1a_2a_3} - \frac{3}{2}S^1W_{a_1a_2a_3}$ $-9\left((J^1)^{b_1}_{a_1}\partial W_{b_1a_2a_3} + (J^2)^{b_2}_{a_2}\partial W_{a_1b_2a_3} + (J^3)^{b_3}_{a_3}\partial W_{a_1a_2b_3}\right)$ $(J^1)^{b_1}_{a_1}(J^2)^{b_2}_{a_2}W_{b_1b_2a_3} = (J^1)^{b_1}_{a_1}(J^3)^{b_3}_{a_3}W_{b_1a_2b_3} = (J^2)^{b_2}_{a_2}(J^3)^{b_3}_{a_3}W_{a_1b_2b_3}$ $= (J^1)^{b_1}_{a_1}(J^1)^{c_1}_{b_1}W_{c_1a_2a_3} = (J^2)^{b_2}_{a_2}(J^2)^{c_2}_{b_2}W_{a_1c_2a_3} = (J^3)^{b_3}_{a_3}(J^3)^{c_3}_{b_3}W_{a_1a_2c_3}$ $\partial\left[(J^1)^{b_1}_{a_1}W_{b_1a_2a_3}\right] = \partial\left[(J^2)^{b_2}_{a_2}W_{a_1b_2a_3}\right] = \partial\left[(J^3)^{b_3}_{a_3}W_{a_1a_2b_3}\right]$

Table 4.4: Explicit null relations at dimension 7/2, which can be uplifted to four-dimensional Higgs branch chiral ring relations. Similar comments as for Table 4.3 apply.

A simple calculation shows that the null relations

$$\mathrm{Tr}(J^1)^2 = \mathrm{Tr}(J^2)^2 = \mathrm{Tr}(J^3)^2\,, \tag{4.3.3}$$

hold true. Each of these operators separately is null within its respective critical current algebra, but thanks to the presence of the stress tensor T in the full chiral algebra, one finds that only their differences are null. Similarly, we have explicitly recovered the analogous relation for the third order Casimir operators. These null relations are just two instances of the general null relations setting equal the Casimir operators of the three current algebras, which are similarly valid for general T_n. The corresponding Higgs branch chiral ring relations on the moment map operators are well-known (see for example [86]).

Another nice set of null relations is obtained by acting with a current on the generators $W^{(k)}$:

$$\begin{aligned}
&(J^1)_{a_1}^{b_1} W_{b_1 a_2 a_3} = (J^2)_{a_2}^{b_2} W_{a_1 b_2 a_3} = (J^3)_{a_3}^{b_3} W_{a_1 a_2 b_3}\,,\\
&(J^1)_{b_1}^{a_1} \widetilde{W}^{b_1 a_2 a_3} = (J^2)_{b_2}^{a_2} \widetilde{W}^{a_1 b_2 a_3} = (J^3)_{b_3}^{a_3} \widetilde{W}^{a_1 a_2 b_3}\,,\\
&(J^1)_{[a_1}^{c_1} V_{[b_1]c_1][a_2 b_2][a_3 b_3]} = (J^2)_{[a_2}^{c_2} V_{[a_1 b_1][b_2]c_2][a_3 b_3]} = (J^3)_{[a_3}^{c_3} V_{[a_1 b_1][a_2 b_2][b_3]c_3]}\,.
\end{aligned} \tag{4.3.4}$$

Null relations of this type are expected to be valid in general T_n as well, and extend the ones listed in [86] for $W^{(1)}, W^{(n-1)}$. Some of the null relations presented in Tables 4.3 and 4.4 are direct consequences of these nulls, obtained by either acting with derivatives or normal-ordering them with other operators, but others are new. For example, the last two nulls given in Table 4.3 are not obtained from previous nulls, and they give rise to known Higgs branch chiral ring relations (they precisely turn into the relations given in equation (2.7) of [86] after setting all derivatives and the stress tensor to zero, and taking into account the different normalizations of the two- and four-dimensional operators). All null relations involving the generator V in Tables 4.3 and 4.4 give rise to new Higgs branch chiral ring relations.

As mentioned, when computing Jacobi-identities one might find that some of them are not zero on the nose, but end up being proportional to null states. In practice this happens quite often, and we find that consistency of the Jacobi-identities relies precisely on the existence of some of these nulls. For example, when examining the Jacobi-identities involving W, $\widetilde{W}$ and V, one encounters the following null relation:

$$W_{(a_1[a_2[a_3}W_{b_1)b_2]b_3]} = -\frac{1}{4}J^{c_1}_{(a_1}V_{[|c_1|b_1)][a_2b_2][a_3b_3]}\,, \tag{4.3.5}$$

which only exists at $k_{2d} = -4$.

We can now check a prediction made in section 4.2, namely that any operator transforming in a representation $(\mathfrak{R}_{\lambda_1}, \mathfrak{R}_{\lambda_2}, \mathfrak{R}_{\lambda_3})$ for not all equal $\mathfrak{R}_{\lambda_i}$ must be an AKM descendant (or be obtained from an AKM descendant by acting on it with the operators which contribute to the structure constants). The operator $W_{(a_1[a_2[a_3}W_{b_1)b_2]b_3]}$ would seem to contradict this statement, since it transforms in the representation $(\mathbf{10}, \mathbf{6}, \mathbf{6})$, and it clearly cannot be obtained from an AKM descendant. Fortunately, there is no contradiction with the superconformal index as this operator is set equal to an AKM descendant by the null relation (4.3.5). More generally, we have verified in several cases that threefold AKM primaries either appear in representations of the type $(\mathfrak{R}_{\lambda}, \mathfrak{R}_{\lambda}, \mathfrak{R}_{\lambda})$, or are null. Moreover we have checked that all operators in representations which are not of the type $(\mathfrak{R}_{\lambda}, \mathfrak{R}_{\lambda}, \mathfrak{R}_{\lambda})$ are either AKM descendants or obtained from them by acting with the operators which contribute to the structure constants, such as a derivative, or normal-ordering it with the stress tensor. A direct consequence of this interpretation of the partition function is that we can predict the existence of certain types of relations: whenever we can write an operator in a representation not of the type $(\mathfrak{R}_{\lambda}, \mathfrak{R}_{\lambda}, \mathfrak{R}_{\lambda})$ which is neither a descendant nor obtained from one in the manner described above, there has to be a null relation involving it. Since null operators are threefold AKM primary, obtaining AKM primaries in said representation provides a faster way to write down the null combinations than to diagonalize norm matrices.

Finally we must point out that there exist operators that are not AKM descendants and can never take part in an AKM primary combination. We already encountered such an example, namely the stress tensor: since it is not of Sugawara type it cannot be an AKM descendant, and the requirement that the AKM currents are Virasoro primaries implies that it also is not an AKM primary. Since the only other dimension two singlets are given by the quadratic Casimir operators, which have zero OPEs with the currents, one concludes that it is impossible to make an AKM primary combination involving the stress tensor. Another example of an operator which cannot be involved in any AKM primary combination is $(W\widetilde{W})|_{\text{sing.}}$. We expect that the existence of this operator, as well as $(VV)|_{\text{sing.}}$ is precisely the reason why the T_4 chiral algebra does not require (Virasoro primary) singlet generators of dimension three and four to close. Although these operators are not Virasoro primaries on their own, they take part in Virasoro primary combinations, of dimensions three and four respectively, which are not AKM primaries. Note that by being neither AKM primaries nor descendants, their contribution to the partition function is necessarily encrypted in the structure constant. As explained in section 4.2, their contribution is indeed captured by the P.E. $\left[\frac{q^3+q^4}{1-q}\right]$ factor in the T_4 structure constant (see (4.2.2)).

Looking at these operators it is natural to ask if the stress tensor and the Virasoro primary singlet operators obtained from $(W\widetilde{W})|_{\text{sing.}}$ and $(VV)|_{\text{sing.}}$ form a closed subalgebra. If such an algebra closes, it must correspond to the $\mathcal{W}_4$ algebra, which is the unique (up to the choice of central charge) closed algebra with such a set of generators [110, 111]. In principle this could be checked using our explicit construction; however, it is computationally challenging and we have not pursued it. More generally, one could wonder whether the set of operators $\mathcal{O}_i$ in Conjecture 9 could form a closed subalgebra, which then should be a $\mathcal{W}_n = \mathcal{W}(2, 3, \ldots, n)$ algebra with central charge $c_{2d} = -2n^3 + 3n^2 + n - 2$. In particular, one should also be able to test this statement for $\chi(T_3)$ using the explicit construction of chapter 3, in which case one would obtain the $\mathcal{W}_3$ algebra of [104].

In section 4.2 we argued that the structure constant factor of (4.2.6) would add negative modes of the current algebra Casimir operators. Now we can give an explicit example: the operator $(J^1)_{a_1}^{b_1} W_{b_1 a_2 a_3}$, which precisely at the critical level becomes an AKM primary, and thus is not included in the critical module of $W_{a_1 a_2 a_3}$. Taking the OPE of S^1 with $W_{a_1 a_2 a_3}$ we find

$$
\begin{aligned}
S^1(z) W_{a_1 a_2 a_3}(0) &\sim \frac{15}{4}\frac{W_{a_1 a_2 a_3}}{z^2} + 2\frac{(J_1)_{a_1}^{b_1} W_{b_1 a_2 a_3}}{z}\,, \\
\Longleftrightarrow \left[(S^1)_m, (W_{a_1 a_2 a_3})_n\right] &= \frac{15(m+1)}{4}(W_{a_1 a_2 a_3})_{m+n} + 2\left((J_1)_{a_1}^{b_1} W_{b_1 a_2 a_3}\right)_{m+n}\,,
\end{aligned}
\tag{4.3.6}
$$

where $(\mathcal{O})_n$ denote the modes of operator $\mathcal{O}$, which in the case of $\big((J_1)_{a_1}^{b_1} W_{b_1 a_2 a_3}\big)_{m+n}$ correspond to the modes of the normal-ordered product. Acting with the $(S_1)_{-1}$ mode of S_1 on the AKM primary yields

$$
(S^1)_{-1}|W_{a_1 a_2 a_3}\rangle = (S^1)_{-1}(W_{a_1 a_2 a_3})_{-\frac{3}{2}}|0\rangle = 2\left((J_1)_{a_1}^{b_1} W_{b_1 a_2 a_3}\right)_{-\frac{5}{2}}|0\rangle\,, \tag{4.3.7}
$$

which exactly adds $(J^1)_{a_1}^{b_1} W_{b_1 a_2 a_3}$.[12]

When analyzing the superconformal index we also argued that threefold AKM primaries in the sum over AKM modules, that do not correspond to generators $W^{(k)}$ must be obtained by normal-ordered products of generators of Higgs branch chiral ring origin. We can now give explicit examples. Let us start by considering representation $(\mathbf{15}, \mathbf{15}, \mathbf{15})$, for which the corresponding primary must have dimension three. As described in the previous section we can always write down a composite operator with the right quantum numbers, in this case it is $W\widetilde{W}|_{(\mathbf{15},\mathbf{15},\mathbf{15})}$. Even though this operator is not a threefold AKM primary, the following combination is:

$$W\widetilde{W}|_{(\mathbf{15},\mathbf{15},\mathbf{15})} + \frac{1}{64}(J^1)(J^2)(J^3)\,, \tag{4.3.8}$$

and it is precisely this combination that is accounted for by the $\mathfrak{R} = \mathbf{15}$ term in (4.2.1). Other examples at dimension three correspond to $(\mathbf{10}, \mathbf{10}, \mathbf{10})$ (and its conjugate), in which case the threefold AKM primary is simply $WW|_{(\mathbf{10},\mathbf{10},\mathbf{10})}$ (and $\widetilde{W}\widetilde{W}|_{(\overline{\mathbf{10}},\overline{\mathbf{10}},\overline{\mathbf{10}})}$).

[12]Recalling that the first null relation in (4.3.4), sets equal $(J^1)_{a_1}^{b_1} W_{b_1 a_2 a_3} = (J^2)_{a_2}^{b_2} W_{a_1 b_2 a_3} = (J^3)_{a_3}^{b_3} W_{a_1 a_2 b_3}$, this term and $L_{-1}|W_{a_1 a_2 a_3}\rangle$ (which produces $\partial W_{a_1 a_2 a_3}$) account for all the powers of q in the structure constants, since for the fundamental representation only $2q$ survives in the plethystic exponential after combining (4.2.6) and (4.2.7). It can be shown that the OPEs of higher dimensional Casimirs with $W_{a_1 a_2 a_3}$ do not produce anything new.

Chapter 5

Conclusions

We have outlined the main features of a new surprising correspondence between the four-dimensional $\mathcal{N} = 2$ superconformal field theories and chiral algebras. It should be apparent that there is a great deal more to learn about this rich structure. There are many aspects that should be clarified further, and many natural directions in which the construction could be generalized. We will simply provide a concise list of what we consider to be the most salient open questions, some of which are currently under investigation.

- For the Lagrangian examples considered in §2.5, as well as the class $\mathcal{S}$ examples of chapter 3, we have made specific conjectures for the description of the resulting chiral algebras as $\mathcal{W}$-algebras. We hope that some of these conjectures can be proved by more advanced homological-algebraic techniques.

- A detailed analysis of the $\hat{\mathcal{B}}_1$ four-point function that compared $4d$ and $2d$ perspectives led to powerful new unitarity bounds that must be obeyed in any interacting $\mathcal{N} = 2$ SCFT with flavor symmetry. It is likely that applying the same methods to more general correlators will lead to further unitarity constraints.

- A better understanding of the implications of four-dimensional unitarity may help clarify what sort of chiral algebra can be associated to a four-dimensional theory. A sharp characterization of the class of chiral algebras that descend from four-dimensional SCFTs could prove invaluable, both as a source of structural insights and as a possible first step towards a classification program for $\mathcal{N} = 2$ SCFTs.

- We have seen that the four-dimensional operators that play a role in the chiral algebra are closely related to the ones that contribute to the Schur and Macdonald limits of the superconformal index. While the Schur limit has been interpreted in §2.4.4 as an index of the chiral algebra, the additional grading that appears in the Macdonald index is not natural in the framework that we have developed. It would be interesting if the additional refinement of the Macdonald index could be captured by a deformation of the chiral algebra structure, perhaps along the lines of [112].

- It seems inevitable that extended operators will ultimately find a place in our construction. We expect that codimension-two defects orthogonal to the chiral algebra plane will play the role of vertex operators transforming as non-trivial modules of the chiral algebra. One could also apply the tools developed here to study protected operators that live on conformal defects that fill the chiral algebra plane.

- As it was presented here, the definition of a protected chiral algebra appears to use extended superconformal symmetry in an essential way. Nevertheless, one wonders whether some aspects of this structure may survive away from conformality, perhaps after putting the theory on a nontrivial geometry.

- A related question is whether some aspects of our construction for Lagrangian theories may be accessible to the techniques of supersymmetric localization. The chiral algebra itself may emerge after an appropriate localization of the four-dimensional path integral.

- In many examples, the structure of the $4d$ Higgs branch appears to play a dominant role in determining the structure of the associated chiral algebra. It is an interesting question whether there is a sense in which the chiral algebra is an intrinsic property of the Higgs branch, possibly with some additional structure added as decoration.

- The structure that we have utilized in this part does not admit a direct generalization to odd space-time dimensions. However, a philosophically similar approach leads to a correspondence between three-dimensional $\mathcal{N}=4$ superconformal field theories and one-dimensional topological field theories. The topological field theory captures twisted correlators of three-dimensional BPS operators whose positions are constrained to a line.

- The cohomological approach to chiral algebras that was successfully pursued in this part can be repeated in two-dimensional theories with at least $\mathcal{N} = (0, 4)$ superconformal symmetry and six-dimensional theories with $\mathcal{N} = (2, 0)$ superconformal symmetry [26]. As it was in the four-dimensional case, correlation functions of the six-dimensional chiral algebra should provide the jumping off point for a numerical bootstrap analysis of the elusive $(2, 0)$ theories.

- Combining the extension of this story to six dimensions with the inclusion of defect operators has the potential to provide a direct explanation for the AGT relation between conformal field theory in two-dimensions and $\mathcal{N} = 2$ supersymmetric field theories in four dimensions.

Part II

Higgs Branch Localization

Chapter 6

Higgs Branch Localization in Three Dimensions

6.1 Introduction

In the last few years there has been a huge development in the study of supersymmetric quantum field theories on compact manifolds, without topological twist. A stunning feature is that, in many cases, we are able to compute *exactly* the path integral and the expectation values of (local and non-local) operators that preserve some supersymmetry, with localization techniques [12, 13]. The path integral can be reduced to something much simpler, like a matrix integral or a counting problem, and explicitly evaluated. After the seminal work of Pestun on S^4 [14], the techniques have been developed in many different contexts, essentially from two to five dimensions (see [15, 16, 113, 114, 115, 116, 117, 118, 119, 120, 18, 121, 122, 14, 123, 124, 125, 126, 127, 128] for a non-exhaustive list).

Most of the work on supersymmetric theories with no twisting has been within the so-called *Coulomb branch localization*: the path integral is reduced to an ordinary integral over a "classical Coulomb branch",[1] parametrized either by scalars in the vector multiplets, or by holonomies around circles. The integrand can contain non-perturbative contributions (*e.g.* if the geometry contains an S^4 or S^5), or not. For instance, in three dimensions [116, 117, 118, 119, 120, 121, 122] the integrand is simply the one-loop de-

[1]We used quotation marks because that would be the classical Coulomb branch on flat space, while the theories we consider are on compact Euclidean curved manifolds.

terminant of all fields around the Coulomb branch configurations. It was observed in [19] (inspired by [129]), though, that the S^3 partition function can be rewritten as a sum over a finite set of points on the Coulomb branch, of the vortex times the antivortex partition functions [130],[2] which do have a non-perturbative origin. In this chapter we would like to gain a better understanding of this phenomenon, from the point of view of localization.

A mechanism responsible for such a "factorization" was first understood in [15, 16], in the analogous context of 2d $\mathcal{N} = (2, 2)$ theories on S^2. It is possible to perform localization in an alternative way (that can be thought of either as adding a different deformation term, or as choosing a different path integration contour in complexified field space), dubbed *Higgs branch localization*, such that the BPS configurations contributing to the path integral are vortices at the north pole and antivortices at the south pole of S^2. Notice that such 2d factorization for supersymmetric non-twisted theories is tightly related to the more general tt^* setup [132].

In three dimensions quite some work has been done to understand factorization. Building on [129], the authors of [17] gave very general arguments why factorization should take place in terms of "holomorphic blocks". Factorization has been explicitly checked for $U(N)$ theories with (anti)fundamentals on S^3 [133] and $S^2 \times S^1$ [134, 135],[3] manipulating the Coulomb branch integrals. General continuous deformations of the geometry have been studied in [18, 136]. Finally, the more general tt^* setup has been developed in three and four dimensions [137]. Our approach is different.

In this chapter we are after a *Higgs branch localization* mechanism in three-dimensional $\mathcal{N} = 2$ R-symmetric Chern-Simons-matter theories, similar to the two-dimensional one [15]. We focus on the squashed sphere S^3_b and on $S^2 \times S^1$, knowing that more general backgrounds could be analyzed with the tools of [138, 139, 136]. We show that both on S^3_b and $S^2 \times S^1$, as in [15], an alternative localization (based on a different deformation term) is possible which directly yields an expression

$$Z = \sum_{\text{vacua}} Z_{\text{cl}}\, Z'_{\text{1-loop}}\, Z_{\text{v}}\, Z_{\text{av}} \;,$$

whenever the flat-space theory could be completely Higgsed by a Fayet-Iliopoulos term, and with some bounds on the Chern-Simons levels which

[2]The vortex partition function counts vortices in the Ω-background on $\mathbb{R}^2$, in the same way as the instanton partition function of [131] counts instantons on $\mathbb{R}^4$.

[3]We slightly revisit some manipulations in [133, 135].

apparently have been overlooked before. The sum is over a finite set of points on the would-be "Coulomb branch", where some chiral multiplets get a VEV solving the D-term equations and completely Higgsing the gauge group. What is summed is a classical and one-loop contribution, evaluated on the vacua, times a vortex and an antivortex contributions, coming from BPS vortex-strings at the northern and southern circles of S_b^3 or $S^2 \times S^1$. Both can be expressed in terms of the *vortex partition function* (VPF) on the twisted $\mathbb{R}^2_\epsilon \times S^1$ (a version of the VPF on the Ω-deformed $\mathbb{R}^2$ [130] dressed by the KK modes on S^1, much like the 5d instanton partition function of [140] on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1$). The precise identification of parameters depends on the geometry.

We expect the same method to work on other three-manifolds, for instance for the lens space index on $S_b^3/\mathbb{Z}_p$ [141, 142], and also in four dimensions on manifolds like $S^3 \times S^1$ [143, 75], $S_b^3/\mathbb{Z}_p \times S^1$ [141, 144] and $S^2 \times T^2$ [124]. The case of $S^3 \times S^1$ will be studied in chapter 7

The chapter is organized as follows. In section 6.2 we study the case of S_b^3: we analyze the BPS equations and their solutions, we study the effect of the new deformation term responsible for Higgs branch localization, and write the general form of the partition function. We conclude with the example of a $U(N)$ gauge theory with (anti)fundamentals [133]. In section 6.3 we do the same in the case of $S^2 \times S^1$. We also consider the example of $U(N)$ [135], and show that S_b^3 and $S^2 \times S^1$ are controlled by the very same vortex partition function.

6.2 Higgs branch localization on S_b^3

We start by studying the path integral of three-dimensional $\mathcal{N} = 2$ R-symmetric Yang-Mills-Chern-Simons-matter theories on the squashed three-sphere S_b^3, where b is a squashing parameter, and its supersymmetric localization. Such a path integral has been computed, with localization techniques, in [119], building on the works [116, 117, 118] (see also [120]). In their framework the path integral is dominated by BPS configurations that look like a classical Coulomb branch: the only non-vanishing field is an adjoint-valued real scalar in the vector multiplet (together with an auxiliary scalar), which can be diagonalized to the maximal torus. We thus dub this "Coulomb branch localization": the resulting expression in [119] is a matrix-model-like partition function, that we review in section 6.2.4.

Our goal is to perform localization in a different way, by including an extra $\mathcal{Q}$-exact term in the deformation action,[4] so that the path integral is dominated by BPS configurations that look like vortex strings at a northern circle and antivortex strings at a southern circle. Vortices exist on the Higgs branch, therefore we dub this *Higgs branch localization*, as in [15].

We will focus on a special class of backgrounds with three-sphere topology, the squashed three-sphere S^3_b of [119] as we said, because our goal is to spell out how Higgs branch localization works. Much more general backgrounds are possible on S^3 [138, 139], and we expect Higgs branch localization to be extendable to all those backgrounds easily. Moreover it has been shown in [136] that the supersymmetric partition function depends on the background through a single continuous parameter b (there might be multiple connected components, though), therefore the computation on S^3_b produces the full set of possible functions one can obtain in this way from the field theory.

6.2.1 Killing spinors on S^3_b

We consider a squashed three-sphere S^3_b with metric [119]

$$ds^2 = f(\theta)^2 d\theta^2 + \tilde{\ell}^2 \sin^2\theta\, d\chi^2 + \ell^2 \cos^2\theta\, d\varphi^2 \;, \tag{6.2.1}$$

where $f(\theta) = \sqrt{\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta}$ and the squashing parameter b is defined as $b = \sqrt{\tilde{\ell}/\ell}$. The ranges of coordinates are $\theta \in [0, \frac{\pi}{2}]$ and $\chi, \varphi \in [0, 2\pi)$. In fact, as apparent in [119] and remarked in [145] (see also [18]), any function $f(\theta)$ which asymptotes to $\tilde{\ell}, \ell$ at $\theta = 0, \frac{\pi}{2}$ respectively and which gives a smooth metric, would lead to the same results. We choose the vielbein one-forms as

$$e^{\underline{1}} = \ell \cos\theta\, d\varphi \;, \qquad e^{\underline{2}} = -\tilde{\ell} \sin\theta\, d\chi \;, \qquad e^{\underline{3}} = f(\theta) d\theta \;, \tag{6.2.2}$$

yielding the non-zero components of the spin connection $\omega^{\underline{13}} = -\frac{\ell}{f} \sin\theta\, d\varphi$ and $\omega^{\underline{23}} = -\frac{\tilde{\ell}}{f} \cos\theta d\chi$. We underline the flat coordinates in this frame. We also turn on a background gauge field that couples to the $U(1)_R$ R-symmetry current:

$$V = \frac{1}{2}\Big(1 - \frac{\ell}{f}\Big)\, d\varphi + \frac{1}{2}\Big(1 - \frac{\tilde{\ell}}{f}\Big)\, d\chi \;. \tag{6.2.3}$$

[4] $\mathcal{Q}$ is a supercharge, and the path integral is not affected by the insertion of $\mathcal{Q}$-exact terms [12, 13].

The twisted Killing spinor equation[5] $D_\mu \epsilon = \gamma_\mu \hat{\epsilon}$ (where $\gamma_{\underline{a}}$ are Pauli matrices) is then solved by the two spinors [119]

$$\epsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{i}{2}(\varphi+\chi-\theta)} \\ -e^{-\frac{i}{2}(\varphi+\chi+\theta)} \end{pmatrix} , \qquad \bar{\epsilon} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(\varphi+\chi+\theta)} \\ e^{\frac{i}{2}(\varphi+\chi-\theta)} \end{pmatrix} \tag{6.2.4}$$

by assigning R-charges $R[\epsilon] = -1$ and $R[\bar{\epsilon}] = 1$. In fact they satisfy

$$D_\mu \epsilon = \frac{i}{2f} \gamma_\mu \epsilon \,, \qquad D_\mu \bar{\epsilon} = \frac{i}{2f} \gamma_\mu \bar{\epsilon} \,. \tag{6.2.5}$$

We also define the charge conjugate spinor $\tilde{\epsilon} \equiv -\bar{\epsilon}^c = i\epsilon$. For spinor conventions see appendix D.1.

Two bilinears that we will need are:

$$\xi^{\underline{a}} = i\bar{\epsilon}\gamma^{\underline{a}}\epsilon = -\epsilon^\dagger \gamma^{\underline{a}} \epsilon = \big(-i\cos\theta, i\sin\theta, 0 \big) \,, \qquad \bar{\epsilon}\epsilon = i\epsilon^\dagger \epsilon = i \,. \tag{6.2.6}$$

Using the coordinate frame (φ, χ, θ) we have

$$\xi^\mu = i\bar{\epsilon}\gamma^\mu \epsilon = \left(\frac{1}{\ell}, \frac{1}{\tilde{\ell}}, 0 \right) . \tag{6.2.7}$$

There are also two useful scalar bilinears, ρ and α defined in (D.2.9), which take values $\rho = 0$ and $\alpha = -\frac{1}{f} - \xi^\mu V_\mu = -\frac{1}{2}\big(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\big)$. Therefore the commutator of SUSY transformations (D.2.6) is

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \mathcal{L}_\xi^A - \sigma - \frac{i}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}} \right) R \,. \tag{6.2.8}$$

It will be useful to perform a frame rotation such that the Killing vector field $\xi = \xi^\mu \partial_\mu$ becomes one of the frame vectors. We then define the non-underlined frame and its dual basis of vectors:

$$\begin{aligned} e^1 &= -f(\theta) d\theta & e_1 &= -f(\theta)^{-1} \partial_\theta \\ e^2 &= \cos\theta \sin\theta \, (\ell \, d\varphi - \tilde{\ell} \, d\chi) & e_2 &= \ell^{-1} \tan\theta \, \partial_\varphi - \tilde{\ell}^{-1} \cot\theta \, \partial_\chi \\ e^3 &= \ell \cos^2\theta \, d\varphi + \tilde{\ell} \sin^2\theta \, d\chi & e_3 &= \ell^{-1} \partial_\varphi + \tilde{\ell}^{-1} \partial_\chi \,. \end{aligned} \tag{6.2.9}$$

[5] In our conventions $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - iV_\mu$. Charge conjugation is $\epsilon^c = C\epsilon^* = \gamma_{\underline{2}}\epsilon^*$, having chosen $C = \gamma_{\underline{2}}$.

In particular $\xi = e_3$. In this basis the spin connection reads

$$\omega^{ab} = \begin{pmatrix} 0 & -\frac{\ell}{f}\sin^2\theta\, d\varphi - \frac{\tilde\ell}{f}\cos^2\theta\, d\chi & \frac{\sin 2\theta}{2f}(-\ell\, d\varphi + \tilde\ell\, d\chi) \\ \frac{\ell}{f}\sin^2\theta\, d\varphi + \frac{\tilde\ell}{f}\cos^2\theta\, d\chi & 0 & -d\theta \\ \frac{\sin 2\theta}{2f}(\ell\, d\varphi - \tilde\ell\, d\chi) & d\theta & 0 \end{pmatrix} \tag{6.2.10}$$

and the Killing spinors become

$$\epsilon = \begin{pmatrix} 0 \\ -e^{-\frac{i}{2}(\varphi+\chi)} \end{pmatrix} , \qquad \bar\epsilon = \begin{pmatrix} e^{\frac{i}{2}(\varphi+\chi)} \\ 0 \end{pmatrix} \tag{6.2.11}$$

as well as $\tilde\epsilon = -\bar\epsilon^c = i\epsilon$. The relation between the two bases is $e^a = \begin{pmatrix} 0 & 0 & -1 \\ \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}_{a\underline{a}} e^{\underline{a}}$, where the matrix has determinant one. In the rest of this section we will use the non-underlined frame.

To conclude let us describe the metric of the squashed three-sphere using Hopf coordinates $\phi_H = \varphi - \chi$ and $\psi_H = \varphi + \chi$, in which the Killing vector $\xi = \big(\frac{1}{\ell} + \frac{1}{\tilde\ell}\big)\partial_{\psi_H} + \big(\frac{1}{\ell} - \frac{1}{\tilde\ell}\big)\partial_{\phi_H}$. On the round sphere of radius 1, $\xi = 2\partial_{\psi_H}$ generates pure motion around the Hopf fiber, whilst the squashing introduces an additional rotation of the base space S^2 with fixed points at $\theta = 0$ and $\theta = \frac{\pi}{2}$. The metric (6.2.1) reads in these coordinates:

$$\begin{aligned} ds^2 = f(\theta)^2 d\theta^2 &+ \frac{\ell^2\tilde\ell^2 \sin^2 2\theta}{4(\ell^2\cos^2\theta + \tilde\ell^2\sin^2\theta)} d\phi_H^2 \\ &+ \frac{1}{4}(\ell^2\cos^2\theta + \tilde\ell^2\sin^2\theta)\Big(d\psi_H + \frac{\ell^2\cos^2\theta - \tilde\ell^2\sin^2\theta}{\ell^2\cos^2\theta + \tilde\ell^2\sin^2\theta} d\phi_H\Big)^2 . \end{aligned} \tag{6.2.12}$$

In fact one could instead take ∂_{ϕ_H} as the Hopf vector field, and rewrite the metric in the same form as above but with $\psi_H \leftrightarrow \phi_H$.

6.2.2 The BPS equations

We will now consider the BPS equations for vector and chiral multiplets, and how they can be obtained as the zero-locus of the bosonic part of a $\mathcal{Q}$-exact deformation action. See appendix D.2 for the SUSY transformations.

First we define

$$W^r = \frac{1}{2}\varepsilon^{rmn}F_{mn} , \qquad F_{mn} = \varepsilon_{mnr}W^r , \tag{6.2.13}$$

so that $\frac{1}{2}F_{mn}F^{mn} = W_m W^m$. Then, from (D.2.14), the BPS equations for the vector multiplet are

$$\begin{aligned} 0 &= \mathcal{Q}\lambda = i\big(W_\mu + D_\mu\sigma\big)\gamma^\mu\epsilon - \Big(D + \frac{\sigma}{f}\Big)\epsilon \\ 0 &= \mathcal{Q}\lambda^\dagger = -i\tilde{\epsilon}^\dagger\gamma^\mu\big(W_\mu - D_\mu\sigma\big) + \tilde{\epsilon}^\dagger\Big(D + \frac{\sigma}{f}\Big) \,. \end{aligned} \tag{6.2.14}$$

Recall that in Euclidean signature we regard λ and $\lambda^\dagger$ as independent fields. It is convenient to use the non-underlined frame and the Killing spinors in (6.2.11); after taking sums and differences of the components, we get the BPS equations:

$$0 = W_1 - iD_2\sigma \;, \qquad 0 = W_2 + iD_1\sigma \;, \qquad 0 = W_3 - i\Big(D + \frac{\sigma}{f}\Big) \;, \qquad 0 = D_3\sigma \;. \tag{6.2.15}$$

In fact—as it is standard—the equations (6.2.15) can be derived as the zero-locus of the bosonic part of a $\mathcal{Q}$-exact deformation action, whose Lagrangian is

$$\mathcal{L}^{\text{def}}_{\text{YM}} = \mathcal{Q}\,\text{Tr}\Big[\frac{(\mathcal{Q}\lambda)^\ddagger\lambda + \lambda^\dagger(\mathcal{Q}\lambda^\dagger)^\ddagger}{4}\Big] \,. \tag{6.2.16}$$

Here the action of the formal adjoint operator $\ddagger$ on $\mathcal{Q}\lambda$ and $\mathcal{Q}\lambda^\dagger$ is:

$$\begin{aligned} (\mathcal{Q}\lambda)^\ddagger &= \epsilon^\dagger\Big[-i\gamma^\mu(W_\mu + D_\mu\sigma^\dagger) - \Big(D + \frac{\sigma^\dagger}{f}\Big)\Big] \\ (\mathcal{Q}\lambda^\dagger)^\ddagger &= \Big[i(W_\mu - D_\mu\sigma^\dagger)\gamma^\mu + \Big(D + \frac{\sigma^\dagger}{f}\Big)\Big]\tilde{\epsilon} \,, \end{aligned} \tag{6.2.17}$$

where we treat σ as a complex field. The operator $\ddagger$ reduces to $\dagger$ when A_μ and D are taken real. Decomposing $\sigma = \sigma_R + i\sigma_I$ into its real and imaginary parts, we find that the bosonic part of $\mathcal{L}^{\text{def}}_{\text{YM}}$ is a positive sum of squares:

$$\begin{aligned} \frac{1}{4}\text{Tr}\Big[(\mathcal{Q}\lambda)^\ddagger\mathcal{Q}\lambda + \mathcal{Q}\lambda^\dagger(\mathcal{Q}\lambda^\dagger)^\ddagger\Big] &= \text{Tr}\bigg\{\frac{1}{2}\big(W_1 + D_2\sigma_I\big)^2 + \frac{1}{2}\big(W_2 - D_1\sigma_I\big)^2 \\ &\quad + \frac{1}{2}\Big(W_3 + \frac{\sigma_I}{f}\Big)^2 + \frac{1}{2}(D_3\sigma_I)^2 + \frac{1}{2}\sum_{a=1,2,3}(D_a\sigma_R)^2 + \frac{1}{2}\Big(D + \frac{\sigma_R}{f}\Big)^2\bigg\} \,. \end{aligned} \tag{6.2.18}$$

If we restrict to real fields, $\sigma_I = 0$, from the zero locus of this action we recover the localization locus $F_{\mu\nu} = 0$ and $\sigma = -fD = \text{const}$, as in [116]. On a three-sphere, $F_{\mu\nu} = 0$ allows us to set $A_\mu = 0$, then $D_\mu\sigma = \partial_\mu\sigma$ and finally σ can be diagonalized. On the other hand the equations (6.2.15) allow for more general solutions with complex σ.

As in [15], Higgs branch localization can be achieved by adding another $\mathcal{Q}$-exact term to the deformation action. Consider

$$\mathcal{L}_{\text{H}}^{\text{def}} = \mathcal{Q}\text{Tr}\Big[\frac{i(\epsilon^\dagger\lambda - \lambda^\dagger\tilde{\epsilon})\, H(\phi)}{2}\Big] \;, \tag{6.2.19}$$

whose bosonic part is

$$\mathcal{L}_{\text{H}}^{\text{def}}\Big|_{\text{bos}} = \text{Tr}\Big[\Big(W_3 - i\big(D + \tfrac{\sigma}{f}\big)\Big) H(\phi)\Big] \;. \tag{6.2.20}$$

The action $S_{\text{H}}^{\text{def}} = \int \mathcal{L}_{\text{H}}^{\text{def}}$ is both $\mathcal{Q}$-exact and $\mathcal{Q}$-closed.[6] $H(\phi)$ is a generic real function of the complex scalar fields $\phi, \phi^\dagger$ in chiral multiplets,[7] taking values in the adjoint representation. Actually one could even consider more general functions $H(\phi,\sigma)$—and we mention the interesting fact that $H(\phi,\sigma) = H(\phi) + \kappa\sigma_I$ would lead to Yang-Mills-Chern-Simons vortex equations—but we will not do so in this chapter.

The bosonic part of the new deformation term $\mathcal{L}_{\text{H}}^{\text{def}}$ is not positive definite. However if we consider the sum $\mathcal{L}_{\text{YM}}^{\text{def}} + \mathcal{L}_{\text{H}}^{\text{def}}$, the auxiliary field D appears quadratically without derivatives and can be integrated out exactly by performing the Gaussian path integral. This corresponds to imposing

$$D + \frac{\sigma_R}{f} = iH(\phi) \;, \tag{6.2.21}$$

in other words $D + \sigma_R/f$ is formally taken out of the real contour. The bosonic part of what we are left with is a positive sum of squares:

$$\begin{aligned} \mathcal{L}_{\text{YM}}^{\text{def}} + \mathcal{L}_{\text{H}}^{\text{def}}\Big|_{D,\text{ bos}} = \text{Tr}\Bigg[&\frac{1}{2}\big(W_1 + D_2\sigma_I\big)^2 + \frac{1}{2}\big(W_2 - D_1\sigma_I\big)^2 \\ &+ \frac{1}{2}\Big(W_3 + \frac{\sigma_I}{f} + H(\phi)\Big)^2 + \frac{1}{2}(D_3\sigma_I)^2 + \frac{1}{2}\sum_{a=1,2,3}(D_a\sigma_R)^2\Bigg] \;. \end{aligned} \tag{6.2.22}$$

[6]While exactness is manifest in (6.2.19), closeness follows from an argument in [126]. If $\epsilon^\dagger, \tilde{\epsilon}$ were fields, the integral of the trace in (6.2.19) would be invariant under $\mathcal{Q}^2$ because it is a neutral scalar. Therefore if ϵ is invariant under the bosonic operator $\mathcal{Q}^2$, then $S_{\text{H}}^{\text{def}}$ is $\mathcal{Q}$-closed. It is easy to check that $\mathcal{Q}^2\epsilon = \mathcal{L}_\xi\epsilon + \frac{i}{2}(\ell^{-1} + \tilde{\ell}^{-1})\epsilon = 0$.

[7]If we want to be sure that $\mathcal{L}_{\text{H}}^{\text{def}}$ does not change the vacuum structure of the theory, we should limit ourselves to functions H that do not modify the behavior of the action at infinity in field space [146]. This is the case if $H(\phi)$ is quadratic.

The BPS equations describing its zero-locus are then

$$\begin{aligned} &0 = W_1 + D_2\sigma_I \ , \qquad 0 = W_2 - D_1\sigma_I \ , \quad 0 = W_3 + \frac{\sigma_I}{f} + H(\phi) \ , \\ &0 = D_3\sigma_I \ , \qquad\qquad\qquad\qquad\qquad\qquad\quad 0 = D_a\sigma_R \ . \end{aligned} \tag{6.2.23}$$

These equations differ from (6.2.15) only by the fact that the "D-term equation" (6.2.21) has been imposed.

Let us now consider the chiral multiplets, transforming in some (possibly reducible) representation of the gauge and flavor symmetry group. At this point it is useful to introduce some notation. We call $\mathfrak{R}$ the (possibly reducible) representation of the gauge and flavor symmetry group under which all chiral multiplets transform. Accordingly, we consider a vector multiplet for the full gauge and flavor symmetry, the components for the gauge group being dynamical and those for the flavor group being external, and whose real scalar we call $\mathfrak{S}$. On a supersymmetric background, external vector multiplets should satisfy the same BPS equations (6.2.14), but of course they do not have a kinetic action. Real expectation values of the external components of $\mathfrak{S}$ are the so-called real masses, so coupling a chiral multiplet in representation $\mathfrak{R}$ to $\mathfrak{S}$ includes real masses as well.[8] On the other hand, we decompose $\mathfrak{R}$ into irreducible representations of the gauge group: $\mathfrak{R} = \bigoplus_i \mathcal{R}_i$. In this notation, each chiral multiplet in representation $\mathcal{R}_i$ couples to σ and to its real mass term m_i. The projection of $\mathfrak{S}$ on the representation $\mathcal{R}_i$ is $\mathfrak{S}\big|_{\mathcal{R}_i} = \sigma + m_i$.

For each irreducible gauge representation $\mathcal{R}$, the BPS equations $\mathcal{Q}\psi = \mathcal{Q}\psi^\dagger = 0$ give

$$\begin{aligned} &0 = D_3\phi - \Big(\sigma + m + i\frac{q}{f}\Big)\phi \qquad && 0 = e^{-\frac{i}{2}(\chi+\varphi)}(D_1 - iD_2)\phi + ie^{\frac{i}{2}(\chi+\varphi)}F \\ &0 = D_3\phi^\dagger + \phi^\dagger\Big(\sigma + m + i\frac{q}{f}\Big) \qquad && 0 = e^{\frac{i}{2}(\chi+\varphi)}(D_1 + iD_2)\phi^\dagger + ie^{-\frac{i}{2}(\chi+\varphi)}F^\dagger \ , \end{aligned} \tag{6.2.24}$$

[8] In our discussion we are not completely general. In three dimensions, the flavor symmetry group usually includes topological (or magnetic) symmetries which do not act on the microscopic chiral multiplets in the Lagrangian, but rather on monopole operators, and real mass parameters can be included for those symmetries as well. For instance, a $U(1)$ gauge theory has a $U(1)_T$ topological symmetry and a real mass for it is the Fayet-Iliopoulos term. However in our formalism FI terms have to be included by hand, rather than turning on the corresponding component of $\mathfrak{S}$.

where m is the mass and q is the R-charge (all fields in $\mathcal{R}$ must have the same mass and R-charge). Imposing the reality conditions $\phi = (\phi^\dagger)^\dagger$, $F = (F^\dagger)^\dagger$ and decomposing σ into real and imaginary parts as before, the equations simplify to

$$(\sigma_R + m)\phi = 0 \;, \quad D_3\phi - i\Big(\sigma_I + \frac{q}{f}\Big)\phi = 0 \;, \quad (D_1 - iD_2)\phi = 0 \;, \quad F = 0 \;. \tag{6.2.25}$$

In passing we note that, since $\xi = e_3$ and using the first equation $\mathbb{R}\mathrm{e}\,\mathfrak{S}\big|_{\mathcal{R}_i}\phi = 0$, the second one is

$$0 = \xi^\mu\big(\partial_\mu - iA_\mu - iqV_\mu\big)\phi - i\Big(\sigma_I + \frac{q}{f}\Big)\phi = \Big[\mathcal{L}^A_\xi - \frac{iq}{2}\Big(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\Big) - \mathfrak{S}\big|_{\mathcal{R}_i}\Big]\phi = \mathcal{Q}^2\phi \;. \tag{6.2.26}$$

As before, these equations can also be obtained from the canonical deformation action

$$\mathcal{L}^{\text{def}}_{\text{mat}} = \mathcal{Q}\,\frac{(\mathcal{Q}\psi)^\dagger\psi + \psi^\dagger(\mathcal{Q}\psi^\dagger)^\dagger}{4} \;. \tag{6.2.27}$$

Up to total derivatives, its bosonic part reads

$$\mathcal{L}^{\text{def}}_{\text{mat}}\Big|_{\text{bos}} = \frac{1}{2}\Big|D_3\phi - i\big(\sigma_I + \tfrac{q}{f}\big)\phi\Big|^2 + \frac{1}{2}\big|(D_1 - iD_2)\phi\big|^2 + \frac{1}{2}\big|(\sigma_R + m)\phi\big|^2 + \frac{1}{2}|F|^2 \;, \tag{6.2.28}$$

where we recognize once again the BPS equations.

To conclude this section, let us rewrite the BPS equations in components since it will be useful later on. For the vector multiplet we find

$$\begin{aligned} \ell^{-1}\tilde{\ell}^{-1}F_{\varphi\chi} &= \big(-\ell^{-1}\sin^2\theta\, D_\varphi + \tilde{\ell}^{-1}\cos^2\theta\, D_\chi\big)\sigma_I \\ \ell^{-1}F_{\theta\varphi} + \tilde{\ell}^{-1}F_{\theta\chi} &= -D_\theta\sigma_I \\ \ell^{-1}\tan\theta F_{\theta\varphi} - \tilde{\ell}^{-1}\cot\theta\, F_{\theta\chi} &= f(\theta)\, H(\phi) + \sigma_I \\ 0 &= \big(\ell^{-1}D_\varphi + \tilde{\ell}^{-1}D_\chi\big)\sigma_I \\ 0 &= D_\mu\sigma_R \end{aligned} \tag{6.2.29}$$

and for the chiral multiplet we get $0 = (\sigma_R + m)\phi = F$ as well as

$$\begin{aligned} \big(\ell^{-1}D_\varphi + \tilde{\ell}^{-1}D_\chi\big)\phi &= i\Big(\sigma_I + \frac{q}{f}\Big)\phi \\ \big(f(\theta)^{-1}D_\theta + i\ell^{-1}\tan\theta\, D_\varphi - i\tilde{\ell}^{-1}\cot\theta\, D_\chi\big)\phi &= 0 \;. \end{aligned} \tag{6.2.30}$$

6.2.3 BPS solutions: Coulomb, Higgs and vortices

We will now analyze the solutions to (6.2.15), (6.2.23) and (6.2.25). First, let us recall the solutions for the standard choice $H(\phi) = 0$.

Coulomb-like solutions. Consider (6.2.15) and (6.2.25). We solve them along a "real" contour where A_μ, σ, D are real, in particular $\sigma_I = 0$, and $(\phi, \phi^\dagger)$, $(F, F^\dagger)$ are conjugate pairs. Moreover we assume that all chiral multiplets have positive R-charge. As mentioned before, the solutions are [116]

$$A_\mu = 0 \;, \qquad \sigma = -fD = \text{const} \;, \qquad \phi = F = 0 \;. \tag{6.2.31}$$

Let us check that there are no solutions with non-trivial ϕ. We can Fourier expand along the compact directions φ, χ:

$$\phi(\theta,\varphi,\chi) = \sum_{m,n\,\in\,\mathbb{Z}} c_{mn}(\theta)\, e^{in\varphi} e^{im\chi} \;. \tag{6.2.32}$$

The first equation in (6.2.30) imposes the constraint $q = 2(m\ell + n\tilde{\ell})/(\ell+\tilde{\ell})$ for $m, n \in \mathbb{Z}$. In particular for incommensurable values of $\ell, \tilde{\ell}$, either q is one of the special values above and in this case there is only one Fourier mode (m,n), or $\phi = 0$ is the only solution. Assuming that $\ell, \tilde{\ell}$ are incommensurable and that m, n are fixed and solve the constraint, the second equation in (6.2.30) reduces to $\big(\sin 2\theta\, \partial_\theta + q \cos 2\theta + L f(\theta)\big)\phi = 0$ with $L = 2(m-n)/(\ell+\tilde{\ell})$. The solution is

$$\phi(\theta,\varphi,\chi) = \Big(\frac{1-s(\theta)}{1+s(\theta)} \Big)^{\frac{L\ell}{4}} \Big(\frac{1-\tilde{s}(\theta)}{1+\tilde{s}(\theta)} \Big)^{-\frac{L\tilde{\ell}}{4}} (\sin 2\theta)^{-q/2}\, e^{in\varphi} e^{im\chi} \tag{6.2.33}$$

with

$$s(\theta) = \sqrt{ \frac{\ell^2 + \tilde{\ell}^2 - (\ell^2 - \tilde{\ell}^2)\cos 2\theta}{2\ell^2} } \;, \qquad \tilde{s}(\theta) = \sqrt{ \frac{\ell^2 + \tilde{\ell}^2 - (\ell^2 - \tilde{\ell}^2)\cos 2\theta}{2\tilde{\ell}^2} } \;. \tag{6.2.34}$$

The functions $s, \tilde{s}$ are monotonic and positive, with $s(0) = \tilde{s}(\frac{\pi}{2})^{-1} = \tilde{\ell}/\ell$ and $s(\frac{\pi}{2}) = \tilde{s}(0) = 1$. For $q > 0$ there are no smooth solutions. For $q = 0$ (then $m = n = 0$) there is the constant Higgs-like solution $\phi = \phi_0$ that we will re-encounter below (in this case, σ_R is constrained by $(\sigma_R + m)\phi = 0$), but we will not consider it here since we assumed that R-charges are positive.

Now let us study the new solutions with non-trivial $H(\phi)$. We integrate D out first, *i.e.* we solve (6.2.23) and (6.2.25) and impose a "real" contour for all fields but D (in particular $\sigma_I = 0$ again). We also take vanishing R-charges, $q = 0$: arbitrary R-charges can be recovered by analytic continuation of the final result in the real masses, as in [15]. We make the following choice for $H(\phi)$:

$$H(\phi) = \zeta - \sum_{i,a} T^a_{\text{adj}}\, \phi_i^\dagger T^a_{\mathcal{R}_i} \phi_i \tag{6.2.35}$$

where the sum is over the representations $\mathcal{R}_i$ and the gauge symmetry generators T^a in representation $\mathcal{R}_i$. The adjoint-valued parameter ζ is defined as

$$\zeta = \sum_{a:\,U(1)} \zeta_a h_a \;, \tag{6.2.36}$$

i.e. a sum over the Cartan generators h_a of the Abelian factors in the gauge group, in terms of the real parameters ζ_a. We find the following classes of solutions.

Deformed Coulomb branch. It is characterized by $\phi = 0$, therefore from (6.2.23):

$$F = \zeta \sin\theta \cos\theta\, f(\theta)\, d\theta \wedge (\ell\, d\varphi - \tilde{\ell}\, d\chi) \;. \tag{6.2.37}$$

Since S^3_b has trivial second cohomology, any line bundle is trivial and we can find a globally defined and smooth potential:

$$A = \zeta \Big[\big(G(\theta) - G(\pi/2) \big) \ell\, d\varphi + \big(G(0) - G(\theta) \big) \tilde{\ell}\, d\chi \Big] \tag{6.2.38}$$

where $G'(\theta) = \sin\theta\cos\theta\, f(\theta)$. We find

$$G(\theta) = \frac{\big(\ell^2 + \tilde{\ell}^2 - (\ell^2 - \tilde{\ell}^2)\cos 2\theta \big)^{3/2}}{6\sqrt{2}\,(\ell^2 - \tilde{\ell}^2)} + \text{const} \tag{6.2.39}$$

$$G\big(\tfrac{\pi}{2}\big) - G(0) = \frac{\ell^2 + \ell\tilde{\ell} + \tilde{\ell}^2}{3(\ell + \tilde{\ell})} = \frac{\text{vol}(S^3_b)}{4\pi^2 \ell\tilde{\ell}} \;. \tag{6.2.40}$$

The scalar σ is constant and it commutes with F, in particular we can choose a gauge where it is along the Cartan subalgebra.

Higgs-like solutions. They are characterized by $H(\phi) = 0$ (we will relax this condition momentarily). This implies $F_{\mu\nu} = 0$ and, choosing $A_\mu = 0$, also $0 = \partial_\mu \sigma = \partial_\mu \phi$ (one has to exclude non-constant solutions for ϕ with the same argument as above). Therefore σ can be diagonalized, and one is left with the algebraic equations

$$H(\phi) = 0 \,, \qquad\qquad \big(\sigma + m_i\big)\, \phi_i = 0 \qquad \forall\, i \,. \tag{6.2.41}$$

The last equation can be more compactly written as $\mathfrak{S}\phi = 0$. These are the standard D-term equations, and their solutions strongly depend on the gauge group and matter content of the theory.

We will be interested in gauge groups and matter representations for which generic parameters ζ_a and generic masses m_i lead to solutions to (6.2.41) that completely break the gauge group. More specifically, we will be focusing on theories for which the Coulomb branch parameters σ_α, for $\alpha = 1, \ldots, \operatorname{rank} G$, are fixed (depending on the Higgs-like solution) in terms of the masses m_i, and for generic masses they are different breaking the gauge group to $U(1)^{\operatorname{rank} G}$. Each $U(1)$ is then Higgsed by one component of ϕ, along a weight $w \in \mathfrak{R}$, getting VEV. One gets a discrete set of Higgs vacua. If the gauge group is not completely broken (including the case of an unbroken discrete gauge group), or if some continuous Higgs branch is left, the situation is more involved and we will not study it here.

Vortices. Each Higgs-like solution is accompanied by a tower of other solutions with arbitrary numbers of vortices at the north and at the south circles (the Higgs-like solution should be thought of as the one with zero vortex numbers). To see this, expand the BPS equations around $\theta = 0$ at first order in θ. Defining the coordinate $r = \tilde{\ell}\theta$, the metric reads

$$ds^2 \simeq dr^2 + r^2 d\chi^2 + \ell^2 d\varphi^2 \qquad\qquad \text{around } \theta = 0 \tag{6.2.42}$$

which is $\mathbb{R}^2 \times S^1$. The BPS equations (6.2.29) and (6.2.30) reduce to

$$\begin{aligned} r^{-1} F_{r\chi} &= -H(\phi) \qquad & F_{r\varphi} &= -\frac{\ell}{\tilde{\ell}}\, F_{r\chi} \qquad & F_{\varphi\chi} &= 0 \\ 0 &= \Big(D_r - \frac{i}{r} D_\chi\Big)\phi \qquad & D_\varphi \phi &= -\frac{\ell}{\tilde{\ell}} D_\chi \phi \,. \end{aligned} \tag{6.2.43}$$

The two equations on the left are the usual vortex equations[9] on $\mathbb{R}^2$, while the other equations complete the solutions to vortices on $\mathbb{R}^2 \times S^1$ once the solutions on $\mathbb{R}^2$ are found. The equations cannot be solved analytically, therefore let us qualitatively describe the solutions in the $U(1)$ case with a single chiral of charge 1, since—up to a rescaling of the charge—this is the generic situation once the gauge group has been broken to $U(1)^{\operatorname{rank} G}$ by the VEV of σ. We take $\zeta > 0$, in order to have solutions. Far from the core of the vortex, for $r \gg \sqrt{m/\zeta}$ (the integer m will be defined momentarily), we have $0 = H(\phi) = F_{r\chi} = F_{r\varphi}$ therefore

$$\phi \simeq \sqrt{\zeta}\, e^{-in\varphi - im\chi} , \qquad\qquad A \simeq -n\, d\varphi - m\, d\chi . \tag{6.2.44}$$

Stokes' theorem on $\mathbb{R}^2$ implies $\frac{1}{2\pi}\int F = -m$, *i.e.* m is the vortex number at the north circle (while n will be interpreted below). At the core of the vortex ϕ has to vanish in order to be smooth (if $m \neq 0$), therefore close to the core

$$\begin{aligned} &\phi \simeq B(re^{-i\chi})^m e^{-in\varphi} , \qquad\qquad F \simeq \zeta r\, dr \wedge \Big(\frac{\ell}{\tilde\ell}\, d\varphi - d\chi \Big) \\ &A \simeq \Big(-n - \frac{\ell}{\tilde\ell} m + \zeta \frac{\ell}{\tilde\ell} \frac{r^2}{2} \Big) d\varphi - \zeta \frac{r^2}{2}\, d\chi \end{aligned} \qquad \text{for } r \ll \sqrt{m/\zeta} \tag{6.2.45}$$

where B is some constant. In particular, smoothness of ϕ requires $m \in \mathbb{Z}_{\geq 0}$. Note that ϕ vanishes only at $r = 0$, therefore

$$\tilde\ell\, A_\varphi + \ell\, A_\chi = -\tilde\ell\, n - \ell\, m \tag{6.2.46}$$

holds exactly. If we approximate $r^{-1}F_{r\chi}$ by a step function on a disk times $-\zeta$, we get that the size of the vortex is of order $\sqrt{m/\zeta}$ justifying the limits we took. In the limit $\zeta \to \infty$ the vortices squeeze to zero-size, therefore the first-order approximation of the equations around $\theta = 0$ is consistent.

We can similarly study the BPS equations expanded around $\theta = \frac{\pi}{2}$ at first order in $\frac{\pi}{2} - \theta$, defining a coordinate $\tilde r = \ell \left(\frac{\pi}{2} - \theta \right)$. As before, the equations reduce to the 2d antivortex equations (as the orientation induced from S^3_b is opposite) besides some other equations that complete the solutions to 3d. For a $U(1)$ gauge theory with a single chiral, the analysis above goes through *mutatis mutandis*. Far from the core of the vortex, for $\tilde r \gg \sqrt{n/\zeta}$, we have

[9] They are more conventionally antivortex equations, the difference being only the orientation.

the same asymptotic behavior as in (6.2.44). Stokes' theorem on $\mathbb{R}^2$ implies $\frac{1}{2\pi}\int F = -n$, *i.e.* n is the antivortex number at the south circle, and the analysis of the solution for $\tilde{r} \ll \sqrt{n/\zeta}$ reveals that $n \in \mathbb{Z}_{\geq 0}$. The behavior of the fields (6.2.44) in the intermediate region, far from both cores, provides a link of parameters between the two cores and it is indeed a solution of the full BPS equations.

For finite values of ζ, both curvature and finite size effects play a rôle. From the second and third equations on the left in (6.2.29), integrating over the sphere one can obtain

$$-4\pi^2\ell \int F_{\theta\chi}\, d\theta = 4\pi^2\tilde{\ell}\int F_{\theta\varphi}\, d\theta = \int H(\phi)\, d\mathrm{vol}_{S^3_b} \;\leq\; \zeta \mathrm{vol}(S^3_b)\;, \quad (6.2.47)$$

where we used that $H(\phi)$ is bounded by $0 \leq H(\phi) \leq \zeta$ on vortex solutions, and vortex solutions have only θ dependence. Still working in a gauge with smooth and globally defined connection A, we can define the vortex numbers m, n at the north and south circle as the winding numbers of ϕ around χ, φ respectively. The analyses at the cores are still valid, therefore $m, n \in \mathbb{Z}_{\geq}0$ and

$$-\frac{A_\varphi(0)}{\ell} = -\frac{A_\chi\left(\frac{\pi}{2}\right)}{\tilde{\ell}} = \frac{n}{\ell} + \frac{m}{\tilde{\ell}}\;. \quad (6.2.48)$$

Then the bound above implies a bound on the vortex and antivortex numbers:

$$b\,n + b^{-1}m \;\leq\; \zeta\,\frac{\mathrm{vol}(S^3_b)}{4\pi^2\sqrt{\ell\tilde{\ell}}}\;. \quad (6.2.49)$$

We conclude that for finite values of ζ there is a finite number of vortex/antivortex solutions on the squashed three-sphere; when the bound is saturated, the chiral field ϕ actually vanishes and the gauge field is as in the deformed Coulomb branch described before. We thus get a nice picture of the structure of solutions as we continuously increase ζ from 0 to $+\infty$. The Coulomb branch solution is continuously deformed into the deformed Coulomb branch solution; as ζ crosses one of the thresholds, proportional to $bn + b^{-1}m$, a new (anti)vortex solution branches out, in which the value of the matter field is infinitesimal at the threshold and increases further on. This picture will be useful in the next section to understand how localization changes as we change ζ continuously.

For gauge groups of rank larger than one, there can be mixed Coulomb-Higgs branches where part of the gauge group is broken to a diagonal torus (along those components BPS solutions describe a deformed Coulomb branch) and part is completely broken (admitting vortex solutions).

6.2.4 Computation of the partition function

Given the various classes of solutions to the BPS equations found in the previous section, the computation of the partition function requires two more steps: the evaluation of the classical action and of the one-loop determinant of quadratic fluctuations around the BPS configurations, and the sum/integration over the space of BPS configurations.

One-loop determinants from an index theorem

For the computation of the one-loop determinants around non-constant configurations, one most conveniently makes use of an equivariant index theorem for transversally elliptic operators [147], as in [148]. A similar technique has been used on S^4 [14, 149] and S^2 [15]. One can give a cohomological form to the $\mathcal{Q}$-exact localizing action (this point is well explained in [14, 149]), and, with the equivariant index theorem, the one-loop determinants of quadratic fluctuations only get contributions from the fixed points of the equivariant rotations on the worldvolume. Recall that the localizing supercharge squares to

$$\mathcal{Q}^2 = \mathcal{L}_\xi^A - \mathfrak{G} - \frac{i}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\right) R \,. \tag{6.2.50}$$

The vector field $\xi = \frac{1}{\ell}\partial_\varphi + \frac{1}{\tilde{\ell}}\partial_\chi$ does not have fixed points on S_b^3, on the other hand its orbits do not close for generic values of b (ξ generates a non-compact isometry group $\mathbb{R}$) and since the index theorem requires a compact group action, we cannot use it directly.[10] The idea of [148] is to write $\xi = \big(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\big)\partial_{\psi_H} + \big(\frac{1}{\ell} - \frac{1}{\tilde{\ell}}\big)\partial_{\phi_H}$ in Hopf coordinates: it generates a free rotation of the Hopf fiber and a rotation of the base space. We can reduce the operator for quadratic fluctuations (*i.e.* the operator resulting from the quadratic expansion of the localizing action around the background) along the Hopf

[10] For special values of b, *e.g.* the round sphere $b = 1$, the group action is a compact $U(1)$. Still the index theorem determines the index up to torsion, and in fact in those cases the index turns out to be pure torsion. We thank Takuya Okuda for correspondence on this issue.

fiber, obtaining a transversally elliptic operator on the base S^2. We thus reduce the problem to the computation of a one-loop determinant on the base S^2, dressed by the KK modes on the Hopf fiber. The projection of ξ to S^2 gives a rotation with fixed points at $\theta = 0$ (which we call North) and $\theta = \frac{\pi}{2}$ (which we call South). This is exactly the setup in [15]. Identifying the equivariant parameters of the $U(1)_{\partial_{\phi_H}} \times U(1)_R \times G$ action as $\varepsilon = \frac{1}{\ell} - \frac{1}{\tilde{\ell}}$, $\breve{\varepsilon} = \frac{1}{\ell} + \frac{1}{\tilde{\ell}}$ and $a = -i\big(\frac{1}{\ell}A_\varphi + \frac{1}{\tilde{\ell}}A_\chi\big) - \mathfrak{S}$, following [148] we obtain (see appendix D.3) the one-loop determinant for a chiral multiplet of R-charge q in gauge representation $\mathcal{R}$:

$$Z^{\text{chiral}}_{\text{1-loop}} \text{``}=\text{''} \prod_{w\in\mathcal{R}} \prod_{n\in\mathbb{Z}} \prod_{m\geq 0} \frac{(m+1)\ell^{-1} + n\tilde{\ell}^{-1} - \frac{q}{2}\breve{\varepsilon} - i\, w(a_S)}{n\ell^{-1} - m\tilde{\ell}^{-1} - \frac{q}{2}\breve{\varepsilon} - i\, w(a_N)} \,. \tag{6.2.51}$$

In all BPS configurations that we consider in this section, $a_N = a_S \equiv a$ and some further simplifications take place. It is also convenient to introduce the rescaled variable $\hat{a} \equiv \sqrt{\ell\tilde{\ell}}\, a$, as well as $b \equiv \sqrt{\tilde{\ell}/\ell}$ and $Q = b + b^{-1}$. Rescaling numerator and denominator of (6.2.51) by $\sqrt{\ell\tilde{\ell}}$ and neglecting overall signs, we are led to

$$\begin{aligned} Z^{\text{chiral}}_{\text{1-loop}} \text{``}=\text{''} &\prod_{w\in\mathcal{R}} \prod_{m,n\geq 0} \frac{mb + nb^{-1} + \big(1 - \frac{q}{2}\big)Q - iw(\hat{a})}{mb + nb^{-1} + \frac{q}{2}Q + iw(\hat{a})} && (6.2.52) \\ = &\prod_{w\in\mathcal{R}} s_b\bigg(\frac{iQ}{2}(1-q) + w(\hat{a})\bigg) \,. && (6.2.53) \end{aligned}$$

The last one is the regulated expression found in [119], in terms of the double sine function s_b. The one-loop determinant for the vector multiplet is simply

$$Z^{\text{vec}}_{\text{1-loop}} = \prod_{\alpha>0} 2\sinh\big(\pi b\, \alpha(\hat{a})\big)\, 2\sinh\big(\pi b^{-1}\, \alpha(\hat{a})\big) \,, \tag{6.2.54}$$

where the product is over the positive roots α of the gauge group.

Coulomb branch

Let us first quickly review the Coulomb branch localization formula, obtained by choosing $\zeta = 0$ in $H(\phi)$, or taking positive R-charges. The matrix model was derived in [119]. The only $\mathcal{Q}$-closed but not $\mathcal{Q}$-exact pieces of classical action are the CS and FI terms (that we report in appendix D.2.3). Evaluation on the Coulomb branch configurations gives

$$S_{\text{cl}} = i\pi \text{Tr}_{CS}\hat{\sigma}^2 - 2\pi i \text{Tr}_{FI}\hat{\sigma} \ , \tag{6.2.55}$$

in terms of the rescaled adjoint scalar $\hat{\sigma} \equiv \sqrt{\ell\tilde{\ell}}\,\sigma$. The weighted traces Tr_{CS} and Tr_{FI} are spelled out in appendix D.2.3, and for $U(N)$ at level k they reduce to $S_{\text{cl}} = i\pi k \text{Tr}\hat{\sigma}^2 - 2\pi i\xi \text{Tr}\hat{\sigma}$.

Since the equivariant parameters for gauge transformations are equal at the two fixed circles, $\hat{a}_N = \hat{a}_S = -\hat{\sigma}$, the one-loop determinants (6.2.52) and (6.2.54) are

$$Z^{\text{chiral}}_{\text{1-loop}} = \prod_{w\in\mathcal{R}} s_b\left(\frac{iQ}{2}(1-q) - w(\hat{\sigma})\right) \tag{6.2.56}$$

$$Z^{\text{vec}}_{\text{1-loop}} = \prod_{\alpha>0} 2\sinh\big(\pi b\,\alpha(\hat{\sigma})\big)\, 2\sinh\big(\pi b^{-1}\,\alpha(\hat{\sigma})\big) \ . \tag{6.2.57}$$

This leads to the matrix integral of [119]:

$$Z_{S^3_b} = \frac{1}{|\mathcal{W}|}\int\left(\prod_{a=1}^{\text{rank}\,G} d\hat{\sigma}_a\right) e^{-i\pi \text{Tr}_{CS}\hat{\sigma}^2 + 2\pi i \text{Tr}_{FI}\hat{\sigma}}\ Z^{\text{vec}}_{\text{1-loop}}\ Z^{\text{chiral}}_{\text{1-loop}} \ , \tag{6.2.58}$$

where $|\mathcal{W}|$ is the dimension of the Weyl group. Notice that the Vandermonde determinant for integration over the gauge algebra $\mathfrak{g}$ cancels against the one-loop determinant for gauge-fixing ghosts.

Deformed Coulomb branch

Let us now study the contributions for $\zeta \neq 0$. The classical CS and FI actions evaluated on the deformed Coulomb branch configurations give

$$S^{CS}_{\text{cl}} = i\pi \text{Tr}_{CS}\big(\hat{\sigma} - i\zeta\kappa\big)^2 \ , \qquad S^{FI}_{\text{cl}} = -2\pi i \text{Tr}_{FI}\big(\hat{\sigma} - i\zeta\kappa\big) \ , \tag{6.2.59}$$

and we defined the constant

$$\kappa \ \equiv \ \frac{\text{vol}(S^3_b)}{4\pi^2 r} = \frac{r^2}{3}\big(Q - Q^{-1}\big) = \frac{r}{3}\,\frac{\ell^2 + \ell\tilde{\ell} + \tilde{\ell}^2}{\ell + \tilde{\ell}} \ , \tag{6.2.60}$$

where $r \equiv \sqrt{\ell\tilde{\ell}}$. In both cases the effect of the deformation parameter ζ is effectively to shift the integration variable $\hat{\sigma}$ in the imaginary direction. The same shift occurs in the equivariant gauge parameters

$$\hat{a}_N = \hat{a}_S = -\hat{\mathfrak{S}} + i\zeta\kappa$$

defined above (6.2.51), as it follows from (6.2.38), and so also the one-loop determinants simply suffer an effective imaginary shift of $\hat{\sigma}$. Therefore the whole deformed Coulomb branch contribution is simply obtained from the undeformed Coulomb branch expression (6.2.58) by shifting the integration contours in the imaginary directions.

Since the parameter ζ was introduced via a $\mathcal{Q}$-exact term in the action, the partition function should not depend on it. For $\zeta = 0$ we have the original Coulomb branch integral (6.2.58). Upon turning on ζ we effectively deform the contours, shifting them in the imaginary directions, and the integral remains constant until we cross some pole of the chiral one-loop determinant. One can anticipate what happens when crossing a pole based on the bound (6.2.49): the imaginary coordinates of the poles precisely correspond to values of ζ for which new vortices appear on S^3_b as solutions to the vortex equations, and the contribution from the vortices precisely accounts for the jumps in the deformed Coulomb branch integral.

Suppression. Our goal is to derive a localization procedure that reduces the partition function to a pure sum over vortices, with no spurious contributions from deformed Coulomb branches. In order to do that, we can take a suitable limit $\zeta_a \to \pm\infty$: in favorable situations, there exists (for a choice of signs) a limit in which the deformed Coulomb branch contribution vanishes.

Let us define the $U(1)$ charges of a gauge representation $\mathcal{R}_j$: $\mathfrak{q}_j^{(a)} \equiv w(h_a)$, where h_a are the Cartan generators of the Abelian factors in the gauge group, as in (6.2.36), while w is any one weight of $\mathcal{R}_j$.[11] We also decompose $\hat{\sigma} = \hat{\sigma}_R - i\zeta\kappa$ into its real and imaginary parts. Using the asymptotic behavior of the double-sine function (see *e.g.* the appendix of [150]):

$$s_b(z) \;\to\; \begin{cases} e^{+i\frac{\pi}{2}\left(z^2+\frac{1}{12}(b^2+b^{-2})\right)} & |z|\to\infty\,, \quad |\arg z| < \dfrac{\pi}{2} \\ e^{-i\frac{\pi}{2}\left(z^2+\frac{1}{12}(b^2+b^{-2})\right)} & |z|\to\infty\,, \quad |\arg z| > \dfrac{\pi}{2}\,, \end{cases} \tag{6.2.61}$$

one finds that the absolute value of the integrand in the partition function matrix model has the following suppression factor, for $\zeta_a \to \pm\infty$:

$$\Big|\text{integrand}\Big| \;\sim\; \exp\left[-2\pi\kappa\sum_a \zeta_a\bigg(\mathrm{Tr}_{CS}\hat{\sigma}_R h_a - \mathrm{Tr}_{FI} h_a + \frac{1}{2}\sum_{\mathcal{R}_j}\mathfrak{q}_j^{(a)}\sum_{w\in\mathcal{R}_j}\big|w(\hat{\sigma}_R)+m_j\big|\bigg)\right]\,,$$

[11] There is no dependence on the particular weight w chosen, since the $U(1)$ generators commute with all roots of the simple factors.

where the first two terms in parenthesis originate from the classical action while the last term comes from the chiral multiplets in those representations $\mathcal{R}_j$ with $\mathfrak{q}_j^{(a)} \neq 0$. The one-loop determinants of chiral multiplets with $\mathfrak{q}_j^{(a)} = 0$ and that of vector multiplets are unaffected by ζ. One can achieve a suppression of the deformed Coulomb branch contribution if there exists a choice of signs in the limit $\zeta_a \to \pm\infty$ such that the factor above goes to zero for all values of all components of $\hat{\sigma}_R$.

As a concrete example, consider a $U(N)$ theory with N_f fundamentals, N_a antifundamentals and some adjoint chiral multiplets (there is a single Abelian factor in the gauge group, and $\mathfrak{q}$ equals 1, -1 and 0 respectively). Setting the real masses to zero for simplicity, the factor above provides a suppression of the deformed Coulomb branch for

$$\zeta \to +\infty \qquad \text{and} \qquad -\frac{N_f - N_a}{2} < k < \frac{N_f - N_a}{2} \;, \tag{6.2.62}$$

in particular $N_f > N_a$, where the two constraints come from positive and negative $\hat{\sigma}_R$. Similarly, we have suppression for

$$\zeta \to -\infty \qquad \text{and} \qquad \frac{N_f - N_a}{2} < k < -\frac{N_f - N_a}{2} \;, \tag{6.2.63}$$

In particular $N_a > N_f$. These two cases, $|k| < |N_f - N_a|/2$, are the "maximally chiral" theories of [151]. In case one or both bounds are saturated, then the true FI term ξ needs to have the correct sign.

We stress that if the "maximally chiral" condition (including saturations of the bounds) is not met, *i.e.* if $|k| \leq |N_f - N_a|/2$ is not met, the deformed Coulomb branch contribution is *not* suppressed. As we will see in the next section, this translates to the fact that the Coulomb branch integral cannot be closed neither in the upper nor lower half-plane, and reduction to a sum over residues (as in [19]) requires some more clever procedure (if possible at all).

Higgs branch and vortex partition function

For finite values of the deformation parameters ζ_a, among the BPS configurations of section 6.2.3 we find Higgs vacua and vortex solutions, where the (anti)vortex numbers (m, n) are bounded by (6.2.49) (or its multi-dimensional version). These BPS configurations contribute to the path integral, besides the deformed Coulomb branch discussed before. Let us determine their contribution.

The classical actions can be integrated exactly (even though the vortex solutions cannot be written explicitly) using $D = -\sigma/f + iH(\phi)$, the BPS equations (6.2.29) and the knowledge of $A_\varphi(\theta)$ at $\theta = 0, \frac{\pi}{2}$ in a globally defined gauge with $A_\theta = 0$, as discussed around (6.2.45). One finds

$$S_{\rm cl}^{CS} = i\pi \mathrm{Tr}_{CS}\big(\hat{\sigma} - ib^{-1}m - ibn\big)^2 \;, \qquad S_{\rm cl}^{FI} = -2\pi i \mathrm{Tr}_{FI}\big(\hat{\sigma} - ib^{-1}m - ibn\big) \;. \tag{6.2.64}$$

Here the vortex numbers m, n should really be thought of as GNO quantized [152] elements of the gauge algebra, *i.e.* belonging to the coweight lattice.

The evaluation of the one-loop determinants for the off-diagonal W-bosons and all chiral multiplets not getting a VEV is straightforward: one identifies the equivariant gauge transformation parameters in the vortex background from the expression of $\mathcal{Q}^2$ at the poles:

$$\hat{a}_N = \hat{a}_S = -\big(\hat{\mathfrak{S}} - ib^{-1}m - ibn\big) \;. \tag{6.2.65}$$

These values have to be plugged into (6.2.52) and (6.2.54). For the rank G chiral multiplets that get a VEV and, by Higgs mechanism, pair with the vector multiplets along the maximal torus of the gauge group becoming massive, one has to be more careful. As pointed out in [16], the one-loop determinant for the combined system is just the residue of the chiral one-loop.[12] Therefore the total contribution from the chiral multiplets is

$$Z^{\rm chiral}_{\text{1-loop}} = \operatorname*{Res}_{\mathfrak{S} \to \mathfrak{S}_H} \prod_{w \,\in\, \mathfrak{R}} s_b\bigg(\frac{iQ}{2} - w\big(\hat{\mathfrak{S}} - imb^{-1} - inb\big)\bigg) \;, \tag{6.2.66}$$

where $\mathfrak{S}_H$ denotes $\mathfrak{S}$ evaluated on the particular Higgs vacuum, and the R-charges have been set to zero. Finally, since each BPS solution is a smooth configuration with no moduli, we simply sum over them with weight 1.

From (6.2.66) it is clear that the sum of the contributions from the finite number of vortices satisfying the bound (6.2.49) exactly accounts for the jumps in the deformed Coulomb branch contribution every time the integration contour—which is shifted in the imaginary directions by ζ_a—crosses a pole of the chiral one-loop determinant. This of course is expected, since the path integral should not depend on ζ.

[12]The chiral one-loop diverges because it is evaluated at a point on the Coulomb branch where the chiral multiplet, before pairing with the vector multiplet, is massless. Taking the residue corresponds to removing the zero-mode.

Vortex partition function. We obtain a more interesting result if we take a suitable $\zeta_a \to \pm\infty$ limit in which the deformed Coulomb branch contribution vanishes, and there is no bound on the (anti)vortex numbers. Conditions for the existence of such a limit were discussed in section 6.2.4.

In this limit the path integral is completely dominated by (anti)vortex-string configurations wrapping the northern and southern circles, and whose size shrinks to zero. The resummed contribution of all vortex strings is accounted by the K-theoretic vortex partition function, Z_{vortex}, which can be computed on the twisted $\mathbb{R}^2_\epsilon \times S^1$: $\mathbb{R}^2$ is rotated by the equivariant parameter ϵ as we go around S^1, and this effectively compactifies the space. In fact one associates equivariant parameters to flavor symmetries as well. In a suitable scaling limit in which S^1 shrinks (together with the equivariant parameters), one recovers the vortex partition function in Ω-background of [130]. This all is the 2d analog of the 4d and 5d instanton partition functions constructed in [131, 140].

Let us compute the partition function in this limit. First, we have a finite number of Higgs vacua. In each vacuum, $\hat{\sigma}_\alpha$ are fixed to some specific (real) values that are functions of the real masses. The classical actions (6.2.64) provide a weighting factor to Z_{vortex} for the vortex configurations, times an overall classical contribution:

$$S_{\text{cl}} = i\pi \mathrm{Tr}_{CS}\hat{\sigma}^2 - 2\pi i \mathrm{Tr}_{FI}\hat{\sigma} \ . \tag{6.2.67}$$

The weighting factors for (anti)vortices have a term quadratic in the vortex number and a linear term:

$$\begin{aligned} e^{-S_{\text{v}}} &= \exp\big[i\pi b^{-2}\mathrm{Tr}_{CS} m^2 + 2\pi b^{-1}\big(- \mathrm{Tr}_{CS}\hat{\sigma}\,\cdot + \mathrm{Tr}_{FI}\big) m \big] \\ e^{-S_{\text{av}}} &= \exp\big[i\pi b^{2}\mathrm{Tr}_{CS} n^2 + 2\pi b\big(- \mathrm{Tr}_{CS}\hat{\sigma}\,\cdot + \mathrm{Tr}_{FI}\big) n \big] \ . \end{aligned} \tag{6.2.68}$$

The actions (6.2.64) also give rise to a term $e^{2\pi i \mathrm{Tr}_{CS} mn}$: in the absence of parity anomaly in the matter sector, $\mathrm{Tr}_{CS} mn$ is integer and the term equals 1; otherwise Tr_{CS} is semi-integer and such that the term is a sign precisely canceling the parity anomaly.[13]

Second, the one-loop determinants for the vector multiplet and the chiral multiplets not acquiring a VEV are as in (6.2.56). The rank G chiral multiplets acquiring VEV bring a residue factor, which in this case is just 1.

[13] Concretely, for $U(N)_k$ with N_f fundamentals and N_a antifundamentals, cancelation of the parity anomaly requires $2k + N_f - N_a \in \mathbb{Z}$. The general case is discussed in [153].

Finally, the vortex partition function Z_{vortex} depends on equivariant parameters for rotations of $\mathbb{R}^2$ (ε) and flavor rotations (g): they are identified—at $\theta = 0$ (N) and $\theta = \frac{\pi}{2}$ (S)—from the $SU(1|1)$ complex of the supercharge $\mathcal{Q}$ at the poles, *i.e.* from $\mathcal{Q}^2$ in (6.2.50). We find

$$\varepsilon_N = \frac{2\pi}{b^2} , \qquad g_N = -\frac{2\pi}{b}\hat{\mathfrak{S}} , \qquad\qquad \varepsilon_S = 2\pi b^2 , \qquad g_S = -2\pi b \hat{\mathfrak{S}} . \tag{6.2.69}$$

Eventually, Higgs branch localization gives the following expression of the sphere partition function:

$$Z_{S^3_b} = \sum_{\text{Higgs vacua}} e^{-i\pi \text{Tr}_{CS}\hat{\sigma}^2 + 2\pi i \text{Tr}_{FI}\hat{\sigma}} \; Z'_{\text{1-loop}} \; Z_{\text{v}} \; Z_{\text{av}} . \tag{6.2.70}$$

The sum is over solutions to (6.2.41). The one-loop determinant $Z'_{\text{1-loop}}$ does not contain the $\operatorname{rank} G$ chiral multiplets getting VEV in (6.2.41). The (anti)vortex-string contributions are expressed in terms of the 3d vortex partition function:

$$\begin{aligned} Z_{\text{v}} &= Z_{\text{vortex}}\Big(e^{i\pi b^{-2}\text{Tr}_{CS}\cdot} , \, e^{2\pi b^{-1}(-\text{Tr}_{CS}\hat{\sigma}\cdot + \text{Tr}_{FI}\cdot)} , \, \frac{2\pi}{b^2} , \, -\frac{2\pi}{b}\hat{\mathfrak{S}}\Big) \\ Z_{\text{av}} &= Z_{\text{vortex}}\Big(e^{i\pi b^{2}\text{Tr}_{CS}\cdot} , \, e^{2\pi b(-\text{Tr}_{CS}\hat{\sigma}\cdot + \text{Tr}_{FI}\cdot)} , \, 2\pi b^2 , \, -2\pi b\hat{\mathfrak{S}}\Big) . \end{aligned} \tag{6.2.71}$$

The first two arguments in the vortex partition function are exponentiated linear functions on the gauge algebra, corresponding to the quadratic and linear weights for the vortex numbers; the third is the rotational equivariant parameter and the last one includes all flavor equivariant parameters. Notice that the expression (6.2.70) is very much in the spirit of the "holomorphic blocks" of [17].

We shall give a concrete example in the next section.

6.2.5 Matching with the Coulomb branch integral

We would like to briefly show, in the simple example of a $U(N)$ gauge theory with N_f fundamentals and N_a antifundamentals, that Higgs branch and Coulomb branch localization produce in fact the same partition function, written in a completely different way. This computation has already been done in the case of $U(1)$ in [19], and in the case of $U(N)$ in [133], therefore we will just review it in our conventions. We stress, however, that this computation is only valid for

$$|k| \leq \frac{|N_f - N_a|}{2} \tag{6.2.72}$$

where k is the Chern-Simons level; these theories have been dubbed "maximally chiral" in [151].[14]

The theory has $SU(N_f) \times SU(N_a) \times U(1)_A$ flavor symmetry. We will use a "quiver" notation, in which the fundamentals are in the antifundamental representation of the flavor group $SU(N_f)$, and viceversa. Then we can introduce real masses m_α for fundamentals and $\tilde{m}_\beta$ for antifundamentals, defined up to a common shift (which corresponds to a shift of the adjoint scalar σ). Generic positive R-charges are encoded as imaginary parts of the masses.

The matrix integral (6.2.58) is given by (we removed hat from $\hat{\sigma}$):

$$\begin{aligned} Z_{S_b^3}^{U(N),N_f,N_a} &= \\ \frac{1}{N!}\int d^N\sigma\, e^{-i\pi k\sum\sigma_i^2+2\pi i\xi\sum\sigma_i} &\prod_{i<j}^{N} 4\sinh\left(\pi b^{-1}(\sigma_i-\sigma_j)\right)\sinh\left(\pi b(\sigma_i-\sigma_j)\right) \\ &\times \prod_{i=1}^{N} \frac{\prod_{\beta=1}^{N_a} s_b\left(\frac{iQ}{2}+\sigma_i-\tilde{m}_\beta\right)}{\prod_{\alpha=1}^{N_f} s_b\left(-\frac{iQ}{2}+\sigma_i-m_\alpha\right)}\ , \end{aligned} \tag{6.2.73}$$

where we used $s_b(-x) = s_b^{-1}(x)$. Our goal is to rewrite it as a sum over residues, as done in [19, 133]. First, one can employ twice the Cauchy determinant formula that we use in the following form:

$$\prod_{i<j}^{N} 2\sinh(x_i - x_j) = \frac{1}{\prod_{i<j}^{N} 2\sinh(\chi_i-\chi_j)} \sum_{s\in S^N} (-1)^s \prod_{i=1}^{N}\prod_{j\neq s(i)}^{N} 2\cosh(x_i-\chi_j)\ , \tag{6.2.74}$$

[14] The computation in this section leads to an expression for the S^3 partition function which identically vanish for $\max(N_f, N_a) < N$, signaling supersymmetry breaking. The fact that the maximally chiral theories ($|k| \leq \frac{|N_f-N_a|}{2}$) break supersymmetry for $\max(N_f, N_a) < N$ has been noticed in [151]. On the other hand, the minimally chiral theories ($|k| \geq \frac{|N_f-N_a|}{2}$) generically do *not* break supersymmetry for $\max(N_f, N_a) < N$; the simplest example is pure Chern-Simons theory. In fact the manipulations carried out here are not valid in the latter case. A similar reasoning applies to the index $Z_{S^2\times S^1}$. We thank Ofer Aharony for this observation.

where the auxiliary variables χ_i must satisfy $\chi_i \neq \chi_j \pmod{\pi i}$, to separate the interacting matrix-model into a product of simple integrals. The simple integrals will contain two sets of auxiliary variables χ_i, $\tilde{\chi}_i$. Assuming that $|k| < \frac{N_f - N_a}{2}$ (or $|k| \leq \frac{N_f - N_a}{2}$ and $\xi < 0$), these integrals can be computed by closing the contour in the lower-half plane and then picking up the residues. The regime $|k| \leq \frac{N_a - N_f}{2}$ can be studied in a similar way, closing the contours in the upper-half plane. One gets contributions from the simple poles of the one-loop determinants of fundamentals, located at the zeros of s_b in the denominator: $\sigma_j = m_{\gamma_j} - i\mu_j b - i\nu_j b^{-1} \equiv \tau_j(m_{\gamma_j}, \mu_j, \nu_j)$ for $\mu_j, \nu_j \in \mathbb{Z}_{\geq 0}$ and $\gamma_j = 1, \dots, N_f$. Applying the Cauchy determinant formula backwards, to re-absorb the auxiliary variables, one obtains

$$\begin{aligned} Z_{S_b^3} =& \frac{(-2\pi i)^N}{N!} \sum_{\vec{\gamma} \in (\mathbb{Z}_{N_f})^N} \sum_{\vec{\mu}, \vec{\nu} \in \mathbb{Z}^N_{\geq 0}} e^{-i\pi k \sum \tau_i^2 + 2\pi i \xi \sum \tau_i} \\ & \times \prod_{i<j}^{N} 4 \sinh\big(\pi b(\tau_i - \tau_j)\big) \sinh\big(\pi b^{-1}(\tau_i - \tau_j)\big) \\ & \times \prod_{i=1}^{N} \left(\frac{\prod_{\beta=1}^{N_a} s_b\left(\frac{iQ}{2} + \tau_i - \tilde{m}_\beta\right)}{\prod_{\alpha \neq \gamma_i}^{N_f} s_b\left(-\frac{iQ}{2} + \tau_i - m_\alpha\right)} \operatorname*{Res}_{x \to 0} s_b\Big(\frac{iQ}{2} + i\mu_i b + i\nu_i b^{-1} - x\Big) \right) . \end{aligned} \tag{6.2.75}$$

Of course, one could have just collected the residues of the multi-dimensional integral with no need of the Cauchy formula. The residue can be computed with the identity

$$\begin{aligned} & s_b\Big(x + \frac{iQ}{2} + i\mu b + i\nu b^{-1}\Big) = \\ & \qquad \frac{(-1)^{\mu\nu} \, s_b\Big(x + \dfrac{iQ}{2}\Big)}{\prod_{\lambda=1}^{\mu} 2i \sinh \pi b\big(x + i\lambda b\big) \quad \prod_{\kappa=1}^{\nu} 2i \sinh \pi b^{-1}\big(x + i\kappa b^{-1}\big)} \end{aligned} \tag{6.2.76}$$

and $\operatorname{Res}_{x\to 0} s_b\big(x + iQ/2\big) = 1/2\pi i$. At this point one can factorize the summation into a factor independent of $\vec{\mu}$ and $\vec{\nu}$, a summation over $\vec{\mu}$ and a summation over $\vec{\nu}$. To achieve that one uses $2k + N_f - N_a = 0 \pmod 2$, which is the condition for parity anomaly cancelation, so that $(-1)^{(N_f - N_a + 2k)\sum_i \mu_i \nu_i} = 1$. Finally one observes that each of the two summations over $\vec{\mu}$ and $\vec{\nu}$ vanishes if we choose $\gamma_i = \gamma_j$ for some i, j, and on the other hand it is symmetric under permutations of the γ_i's. Therefore we can restrict the sum over unordered combinations $\vec{\gamma} \in C(N, N_f)$ of N out of the N_f flavors, and cancel the $N!$ in the denominator.

We can also use the following identity (see *e.g.* appendix B of [135]), valid when the γ_i's are distinct:

$$\frac{\prod_{j<k}^N \sinh\big(X_{\gamma_k} - X_{\gamma_j} + i(\mu_k - \mu_j)Y\big)}{\prod_{i=1}^N \prod_{\beta=1}^{N_f} \prod_{\lambda=1}^{\mu_i} \sinh\left(X_{\gamma_i} - X_\beta + i\lambda Y\right)} = \tag{6.2.77}$$
$$= \frac{(-1)^{\sum_j \mu_j} \prod_{j<k}^N \sinh\big(X_{\gamma_k} - X_{\gamma_j}\big) \Big[\prod_{\beta \notin \{\gamma_l\}}^{N_f} \sinh\left(X_{\gamma_i} - X_\beta + i\lambda Y\right)\Big]^{-1}}{\prod_{k=1}^N \prod_{\lambda=1}^{\mu_k} \Big[\prod_{j=1}^N \sinh\big(X_{\gamma_k} - X_{\gamma_j} - i(\mu_j - \lambda + 1)Y\big)\Big]}$$

and the observation $\prod_{i<j}^N (-1)^{\mu_i - \mu_j} = (-1)^{(N-1)\sum_i \mu_i}$, to eventually write:

$$Z_{S_b^3} = \sum_{\vec{\gamma} \,\in\, C(N,N_f)} Z_{\text{cl}}^{(\vec{\gamma})}\, Z'^{(\vec{\gamma})}_{\text{1-loop}}\, Z_{\text{v}}^{(\vec{\gamma})}\, Z_{\text{av}}^{(\vec{\gamma})}\ , \tag{6.2.78}$$

which exactly matches with the general result of Higgs branch localization (6.2.70). The summation is over classical Higgs vacua, *i.e.* over solutions to the algebraic D-term equations (6.2.41). Then we have a simple classical piece, the one-loop determinant of all fields except the N chiral multiplets (specified by $\vec{\gamma}$) getting a VEV and Higgsing the gauge group, the vortex and the anti-vortex contributions; all these functions are evaluated at the point $(\vec{\gamma})$ on the Coulomb branch solving the D-term equations. Using a notation in which $\alpha \in \vec{\gamma}$ denotes the flavor indices in the combination $\vec{\gamma}$, we can write the classical and one-loop contributions as

$$\begin{aligned} Z_{\text{cl}}^{(\vec{\gamma})} &= \prod\nolimits_{\alpha\in\vec{\gamma}} e^{-i\pi k m_\alpha^2 + 2\pi i \xi m_\alpha} \\ Z'^{(\vec{\gamma})}_{\text{1-loop}} &= \prod_{i\in\vec{\gamma}} \frac{\prod_{\beta=1}^{N_a} s_b\left(\frac{iQ}{2} + m_i - \tilde{m}_\beta\right)}{\prod_{\alpha\,(\neq i)}^{N_f} s_b\left(-\frac{iQ}{2} + m_i - m_\alpha\right)} \\ &\qquad \times \prod_{\substack{i,j\in\vec{\gamma} \\ i\neq j}} 4\sinh\big(\pi b(m_i - m_j)\big)\sinh\big(\pi b^{-1}(m_i - m_j)\big)\ , \end{aligned} \tag{6.2.79}$$

the (anti)vortex contributions as

$$\begin{aligned} Z_{\rm v}^{(\vec\gamma)} &= Z_{\rm vortex}^{(\vec\gamma)}\Big(e^{i\pi b^{-2}k} \,,\, e^{2\pi b^{-1}(-km_j+\xi)}\big|_{j\in\vec\gamma} \,,\, \frac{2\pi}{b^2} \,,\, -\frac{2\pi}{b} m_\alpha \,,\, -\frac{2\pi}{b}\tilde m_\beta \Big) \\ Z_{\rm av}^{(\vec\gamma)} &= Z_{\rm vortex}^{(\vec\gamma)}\Big(e^{i\pi b^{2}k} \,,\, e^{2\pi b(-km_j+\xi)}\big|_{j\in\vec\gamma} \,,\, 2\pi b^2 \,,\, -2\pi b m_\alpha \,,\, -2\pi b\tilde m_\beta \Big) \,, \end{aligned} \tag{6.2.80}$$

and the vortex-string partition function turns out to be (for $N_f \geq N_a$):

$$Z_{\rm vortex}^{(\vec\gamma)}\Big(Q_j \,,\, L_j \,,\, \varepsilon \,,\, a_\alpha \,,\, b_\beta\Big) = \sum_{\vec\mu\,\in\,\mathbb{Z}^N_{\geq 0}} \prod_{j\,\in\,\vec\gamma} Q_j^{\mu_j^2}\, L_j^{\mu_j}\, (-1)^{(N_f-N_a)\mu_j}$$
$$\times \prod_{\lambda=0}^{\mu_j-1} \frac{\prod_{\beta=1}^{N_a} 2i\sinh\dfrac{a_j - b_\beta + i\varepsilon\lambda}{2}}{\prod_{l\,\in\,\vec\gamma} 2i\sinh\dfrac{a_j-a_l+i\varepsilon(\lambda-\mu_l)}{2}\, \prod_{\alpha\notin\vec\gamma}^{N_f} 2i\sinh\dfrac{a_\alpha - a_j + i\varepsilon(\lambda-\mu_j)}{2}} \,. \tag{6.2.81}$$

The map of parameters in $Z_{\rm v}$ and $Z_{\rm av}$ precisely agrees with our general expression (6.2.71). As we will see in section 6.3.5, precisely the same function $Z_{\rm vortex}$ controls the partition function on $S^2\times S^1$. Such an expression for $Z_{\rm vortex}$ can be compared with [154].[15]

6.2.6 Comparison with the two-dimensional vortex partition function

Let us check that by taking the limit of small equivariant parameter and scaling at the same time all other parameters in the same way, the 3d vortex partition function (6.2.81) reduces to the 2d vortex partition function. After a redefinition $\varepsilon\to-\varepsilon$, we take a limit $\varepsilon\to 0$ in (6.2.81) keeping the ratios a_α/ε and b_β/ε finite; we also send the CS level $k\to 0$, that corresponds to $Q_j\to 1$ and $L_j\to z$. We get

$$Z_{\rm vortex}^{(\vec\gamma)} \to$$
$$\sum_{\vec\mu\,\in\,\mathbb{Z}^N_{\geq 0}} \frac{z^{|\vec\mu|}}{(-\varepsilon)^{(N_f-N_a)|\vec\mu|}} \prod_{j\,\in\,\vec\gamma} \frac{\prod_{\beta=1}^{N_a}\left(\dfrac{ia_j - ib_\beta}{\varepsilon}\right)_{\mu_j}}{\prod_{l\,\in\,\vec\gamma}\left(\dfrac{ia_j-ia_l}{\varepsilon}-\mu_l\right)_{\mu_j} \prod_{\alpha\notin\vec\gamma}^{N_f}\left(\dfrac{ia_\alpha-ia_j}{\varepsilon}-\mu_j\right)_{\mu_j}} \,. \tag{6.2.82}$$

[15]See also [155].

Here $|\vec{\mu}| = \sum_j \mu_j$ and we used the Pochhammer symbol $(a)_n = \prod_{k=0}^{n-1}(a+k)$. This expression is precisely the standard two-dimensional vortex partition function in Ω-background, see *e.g.* [15].

6.3 Higgs branch localization on $S^2 \times S^1$

We will now move to the similar study of Higgs branch localization for $\mathcal{N} = 2$ theories on $S^2 \times S^1$, whose path integral computes the three-dimensional supersymmetric index [156]. Localization on the Coulomb branch for $\mathcal{N} = 6$ Chern-Simons-matter theories was first performed in [121], and later generalized to $\mathcal{N} = 2$ theories in [122] (see also [157] for a further generalization in which magnetic fluxes for global symmetries are introduced). It was later pointed out in [129] (see also [158]) that in the presence of non-trivial magnetic fluxes, the angular momentum of fields can be shifted by half-integer amounts, thus correcting the naive fermion number: such a different weighing of the magnetic sectors helps to verify various expected dualities.

The expression that results from Coulomb branch localization is a matrix integral over the holonomy of the gauge field. As in the previous section, we will perform an alternative Higgs branch localization, in which the relevant BPS configurations are discrete Higgs branches accompanied by towers of vortex strings at the north and south poles of the two-sphere.

6.3.1 Killing spinors on $S^2 \times S^1$, supersymmetric index and deformed background

Supersymmetric theories on three-manifolds, among which $S^2 \times S^1$, have been studied in [138, 139] considering the rigid limit of supergravity. In this approach, the auxiliary fields of the supergravity multiplet are treated as arbitrary background fields and SUSY backgrounds are found by setting to zero the gravitino variations; in the presence of flavor symmetries, one similarly sets to zero the external gaugino variations.

Here we will take a different approach: we will first recall the Killing spinor solutions on $S^2 \times \mathbb{R}$, and then compactify $\mathbb{R}$ to S^1 with some twisted boundary conditions: the supersymmetric index with respect to the supercharges described by the Killing spinors indeed imposes twisted boundary conditions. In a path integral computation, however, the twisted boundary conditions are most conveniently described by turning on background fields for the charges appearing in the index formula, which finally leads to the desired theory on a deformed background.

We take the metric

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + d\tau^2 \ , \tag{6.3.1}$$

with vielbein $e^1 = r\, d\theta$, $e^2 = r \sin\theta\, d\varphi$, $e^3 = d\tau$, and set the background $U(1)_R$ field V_μ to zero. The spin connection is $\omega^{12} = -\cos\theta\, d\varphi$. Consider the Killing spinor equation

$$D_\mu \epsilon = \gamma_\mu \hat{\varepsilon} \tag{6.3.2}$$

where $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$. Following [122] we consider the factorized ansatz $\epsilon_\pm = f(\tau)\, \epsilon_\pm^{S^2}(\theta, \varphi)$, where the 2d spinor satisfies $D_{\hat{\mu}}\epsilon_\pm^{S^2} = \pm\frac{1}{2r}\gamma_{\hat{\mu}}\gamma^3\epsilon_\pm^{S^2}$ with $\hat{\mu} = \theta, \varphi$. Plugging in (6.3.2) gives

$$\epsilon_\pm = e^{\pm\tau/2r}\, \epsilon_\pm^{S^2}(\theta, \varphi) \ , \qquad\qquad D_\mu \epsilon_\pm = \pm\frac{1}{2r}\gamma_\mu\gamma^3\epsilon_\pm \ . \tag{6.3.3}$$

Notice that the spinors are not periodic on S^1 and twisted boundary conditions will be needed. On the sphere S^2 there are four Killing spinors; then we can write the $S^2 \times \mathbb{R}$ spinors in a compact form as

$$\epsilon_\pm = e^{\pm\tau/2r} \exp\Big(\mp \frac{i\theta}{2}\gamma_2\Big) \exp\Big(\frac{i\varphi}{2}\gamma_3\Big)\, \epsilon_0 \tag{6.3.4}$$

where $\epsilon_0 = \left(\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right)$ is constant.

Killing spinors for supersymmetric index. We will choose the spinor ϵ to be "positive" and with $\epsilon_0 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ (so that $\gamma_3\epsilon_0 = \epsilon_0$) and $\bar{\epsilon}$ to be "negative" and with $\bar{\epsilon}_0 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ (so that $\gamma_3\bar{\epsilon}_0 = -\bar{\epsilon}_0$):

$$\epsilon = e^{\tau/2r} e^{i\frac{\varphi}{2}} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \ , \qquad\qquad \bar{\epsilon} = e^{-\tau/2r} e^{-i\frac{\varphi}{2}} \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \ . \tag{6.3.5}$$

Another useful spinor is

$$\tilde{\epsilon} = -\bar{\epsilon}^c = i\, e^{-\tau/2r} e^{i\frac{\varphi}{2}} \begin{pmatrix} \cos\theta/2 \\ -\sin\theta/2 \end{pmatrix} \tag{6.3.6}$$

which is also a "negative" Killing spinor. We choose them of opposite positivity so that bilinears be independent of τ; this also guarantees that there are no dilations in the algebra ($\rho = 0$). With these choices, the Killing vector and the functions appearing in the algebra are

$$v^a = \bar{\epsilon}\gamma^a\epsilon = -\tilde{\epsilon}^\dagger\gamma^a\epsilon = (0, \sin\theta, i) \; , \quad \bar{\epsilon}\epsilon = -\tilde{\epsilon}^\dagger\epsilon = i\cos\theta \; , \quad \alpha = \frac{1}{r}\bar{\epsilon}\gamma^3\epsilon = \frac{i}{r} \; . \tag{6.3.7}$$

We also have

$$\xi = i(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu = \tfrac{i}{r}\partial_\varphi - \partial_\tau \; . \tag{6.3.8}$$

On the other hand $\epsilon^\dagger\epsilon = e^{\tau/r}$ and $\tilde{\epsilon}^\dagger\tilde{\epsilon} = e^{-\tau/r}$, as required by the dimension Δ (see below). The quantum numbers of the spinors are:

Spinor	Δ	j_3	R
ϵ	$-1/2$	$1/2$	-1
$\bar{\epsilon}$	$1/2$	$-1/2$	1

obtained by acting with the operators Δ and j_3 as defined below; the R-charge follows from the supersymmetry variations. We also have

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \frac{1}{r}\Big(\big(\underbrace{-r\mathcal{L}^A_{\partial_\tau}}_{=\Delta}\big) - \big(\underbrace{-i\mathcal{L}^A_{\partial_\varphi} + r\cos\theta\sigma}_{=j_3}\big) - R\Big) \; . \tag{6.3.9}$$

Supersymmetric index and deformed background. The spinors are preserved by the mutually commuting operators

$$\Delta - j_3 - R \; , \qquad\qquad R + 2j_3 \; .$$

The first one is the commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$. We will compute the index

$$I(x, \zeta_i) = \mathrm{Tr}\,(-1)^{2j_3} e^{-\beta(\Delta - j_3 - R)}\, e^{-\xi(R+2j_3)}\, e^{i\sum_j \mathfrak{z}_j F_j} \; . \tag{6.3.10}$$

with $x = e^{-\xi}$, $\zeta_j = e^{i\mathfrak{z}_j}$. Here F_j are the Cartan generators of the flavor symmetries and the circumference of S^1 is βr. To correctly describe the fermion number in the presence of magnetic fluxes, we have used $2j_3$ [129, 158]. Notice that convergence of the trace requires $|x| < 1$. For each Cartan generator of the flavor symmetry, besides the chemical potential ζ_j one could also turn on a fixed background flux on S^2 [157]: the only example we will consider in this chapter is a flux for the topological symmetry $U(1)_J$.

In the path integral formulation on $S^2 \times S^1$, the index is described by the twisted periodicity conditions

$$\Phi(\tau + \beta r) = e^{\beta(-j_3 - R)} \, e^{\xi(R+2j_3)} \, e^{-i\sum_j \mathfrak{z}_j F_j} \, \Phi(\tau) \, . \tag{6.3.11}$$

These are also the boundary conditions satisfied by the spinors (with $F_j = 0$). By the field redefinition $\tilde{\Phi} \equiv e^{-\frac{\tau}{\beta r}\left(\beta(-j_3-R)+\xi(R+2j_3)-i\sum_j \mathfrak{z}_j F_j\right)}\Phi$, one can make the fields periodic again; such a redefinition is in fact a gauge transformation, indeed one can alternatively turn on background flat connections on S^1:

$$V_\mu = \left(0, 0, -\frac{i}{r} + \frac{i\xi}{\beta r}\right) , \qquad \tilde{V}_\mu^{(j)} = \left(0, 0, \frac{\mathfrak{z}_j}{\beta r}\right) \tag{6.3.12}$$

for the R- and flavor symmetries respectively. The twist by the rotational symmetry imposes the identification $(\tau, \varphi) \sim (\tau + \beta r, \varphi - i(\beta - 2\xi))$. Introducing coordinates $\hat{\tau} = \tau$ and $\hat{\varphi} = \varphi + \frac{i(\beta-2\xi)}{\beta r}\tau$, the identification becomes $(\hat{\tau}, \hat{\varphi}) \sim (\hat{\tau} + \beta r, \hat{\varphi})$. In hatted coordinates the metric (6.3.1) is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta \left[d\hat{\varphi} - \frac{i}{r}\left(1 - \frac{2\xi}{\beta}\right) d\hat{\tau}\right]^2 + d\hat{\tau}^2 \, , \tag{6.3.13}$$

which is complex. This metric can also be rewritten as

$$\begin{aligned} ds^2 = r^2 d\theta^2 &+ \frac{r^2 \sin^2\theta}{1 - \left(1 - \frac{2\xi}{\beta}\right)^2 \sin^2\theta} d\hat{\varphi}^2 \\ &+ \left(1 - \left(1 - \tfrac{2\xi}{\beta}\right)^2 \sin^2\theta\right)\left(d\hat{\tau} - \frac{ir\left(1 - \frac{2\xi}{\beta}\right)\sin^2\theta}{1 - \left(1 - \frac{2\xi}{\beta}\right)^2 \sin^2\theta} d\hat{\varphi}\right)^2 , \end{aligned} \tag{6.3.14}$$

which is a circle-fibration over a squashed two-sphere.

The index is thus computed by the partition function on a deformed background. A vielbein for (6.3.13) is $e^1 = r\,d\theta, e^2 = r\sin\theta\left(d\hat{\varphi} - \frac{i}{r}\left(1 - \frac{2\xi}{\beta}\right)d\hat{\tau}\right)$, $e^3 = d\hat{\tau}$, and the frame vectors are $e_1 = \frac{1}{r}\partial_\theta$, $e_2 = \frac{1}{r\sin\theta}\partial_{\hat{\varphi}}$, $e_3 = \partial_{\hat{\tau}} + \frac{i}{r}\left(1 - \frac{2\xi}{\beta}\right)\partial_{\hat{\varphi}}$. The non-vanishing component of the spin connection is $\omega^{12} = -\cos\theta\left(d\hat{\varphi} - \frac{i}{r}\left(1 - \frac{2\xi}{\beta}\right)d\hat{\tau}\right)$. The Killing spinors corresponding to (6.3.5) are

$$\epsilon = e^{i\hat{\varphi}/2}\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix} , \qquad \bar{\epsilon} = e^{-i\hat{\varphi}/2}\begin{pmatrix}\sin\theta/2 \\ \cos\theta/2\end{pmatrix} \, . \tag{6.3.15}$$

They satisfy $D_\mu\varepsilon = \frac{1}{2r}\gamma_\mu\gamma^3\varepsilon$ and $D_\mu\bar{\varepsilon} = -\frac{1}{2r}\gamma_\mu\gamma^3\bar{\varepsilon}$, where $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - iV_\mu - i\sum_j \tilde{V}_\mu^{(j)}$, and $\tilde{\epsilon} = -\bar{\epsilon}^c$. The Killing vector and the functions appearing in the algebra are

$$v^a = \bar{\epsilon}\gamma^a\epsilon = \big(0, \sin\theta, i\big) \Longrightarrow \xi^\mu = \left(0, \frac{2i\xi}{\beta r}, -1\right) , \quad \bar{\epsilon}\epsilon = i\cos\theta , \quad \alpha = \frac{i\xi}{\beta r} . \tag{6.3.16}$$

We thus find

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = -\mathcal{L}^A_{\partial_{\hat{\tau}}} + \frac{2i\xi}{\beta r}\mathcal{L}^A_{\partial_{\hat{\varphi}}} - \cos\theta\,\sigma - \frac{\xi}{\beta r}R + i\sum\nolimits_j \frac{\mathfrak{z}_j}{\beta r}F_j . \tag{6.3.17}$$

From standard arguments, it is known that the index is independent of the parameter β. A significant simplification takes place by setting $\beta = 2\xi$, since the rotational symmetry charge disappears from the trace (6.3.10), and the complex metric (6.3.13) becomes the real metric on the product space $S^2 \times S^1$. Henceforth, we make this choice for the immaterial parameter β and we further omit the hats.

6.3.2 The BPS equations

We will now proceed to derive the BPS equations. We define the quantities

$$Y_a = W_a + \delta_{a3}\,\frac{\sigma}{r} , \tag{6.3.18}$$

where W_a was defined in (6.2.13). Using the explicit expressions for the Killing spinors (6.3.15), the BPS equations from the gaugino variations (D.2.14) can be written as

$$\begin{aligned}
0 &= \big(Y_3 + iD\big)\cos\frac{\theta}{2} + \big(D_1\sigma - iY_2\big)\sin\frac{\theta}{2} \\
0 &= D_3\sigma\cos\frac{\theta}{2} + \big(Y_1 - iD_2\sigma\big)\sin\frac{\theta}{2} \\
0 &= \big(-Y_3 + iD\big)\sin\frac{\theta}{2} + \big(D_1\sigma + iY_2\big)\cos\frac{\theta}{2} \\
0 &= -D_3\sigma\sin\frac{\theta}{2} + \big(Y_1 + iD_2\sigma\big)\cos\frac{\theta}{2} .
\end{aligned} \tag{6.3.19}$$

The localization locus can also be obtained from the positive definite deformation action (6.2.16), where now the action of ‡ is defined to be

$$\begin{aligned}(\mathcal{Q}\lambda)^\ddagger &= \epsilon^\dagger\Big(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D - i\gamma^\mu D_\mu\sigma - \frac{i}{r}\sigma\gamma^3\Big) \\ &= \epsilon^\dagger\big(-i\gamma^r(Y_r + D_r\sigma) - D\big) \\ (\mathcal{Q}\lambda^\dagger)^\ddagger &= \Big(\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + D - i\gamma^\mu D_\mu\sigma + \frac{i}{r}\sigma\gamma^3\Big)\tilde{\epsilon} = \big(i(Y_r - D_r\sigma)\gamma^r + D\big)\tilde{\epsilon}\ .\end{aligned} \tag{6.3.20}$$

One then obtains $\mathcal{L}^{\text{def}}_{\text{YM}} = \frac{1}{2}\text{Tr}\big[(Y_\mu)^2 + (D_\mu\sigma)^2 + D^2\big]$. Imposing the reality conditions, the Coulomb branch localization locus immediately follows:

$$Y_\mu = 0\ , \qquad D_\mu\sigma = 0\ , \qquad D = 0\ . \tag{6.3.21}$$

Note that the string-like vortices are excluded by these equations since they imply $D_\mu F_{12} = 0$.

Higgs branch localization can be achieved by adding another $\mathcal{Q}$-exact term to the deformation action. We use the same term as in (6.2.19):

$$\mathcal{L}^{\text{def}}_{\text{H}} = \mathcal{Q}\text{Tr}\Big[\frac{i(\epsilon^\dagger\lambda - \lambda^\dagger\tilde{\epsilon})H(\phi)}{2}\Big]\ ,$$

whose bosonic piece is

$$\mathcal{L}^{\text{def}}_{\text{H}}\Big|_{\text{bos}} = -\text{Tr}\Big[\big(\sin\theta(D_1\sigma) + \cos\theta Y_3 + iD\big)\ H(\phi)\Big]\ . \tag{6.3.22}$$

The Gaussian path integral over D imposes

$$D = iH(\phi)\ . \tag{6.3.23}$$

Then one is left with

$$\begin{aligned}\mathcal{L}^{\text{def}}_{\text{YM}} + \mathcal{L}^{\text{def}}_{\text{H}}\Big|_{D,\ \text{bos}} = \frac{1}{2}\text{Tr}\Big[&\big(Y_2\big)^2 + \big(D_2\sigma\big)^2 + \big(Y_1\big)^2 + \big(D_3\sigma\big)^2 + \\ &\big(D_1\sigma\cos\theta - Y_3\sin\theta\big)^2 + \big(H(\phi) - D_1\sigma\sin\theta - Y_3\cos\theta\big)^2\Big]\ ,\end{aligned} \tag{6.3.24}$$

which is a sum of squares. The BPS equations are then

$$\begin{aligned}0 &= D_1\sigma\cos\theta - \Big(F_{12} + \frac{\sigma}{r}\Big)\sin\theta\ , & 0 &= D_2\sigma = D_3\sigma \\ 0 &= H(\phi) - D_1\sigma\sin\theta - \Big(F_{12} + \frac{\sigma}{r}\Big)\cos\theta\ , & 0 &= F_{13} = F_{23}\ .\end{aligned} \tag{6.3.25}$$

Consider now the chiral multiplets, transforming in some representation $\mathfrak{R} = \bigoplus_j \mathcal{R}_j$ of the gauge and flavor group, where $\mathcal{R}_j$ are irreducible gauge representations. Imposing the reality conditions $\bar{\phi}^\dagger = \phi$, $\bar{F}^\dagger = F$ and $\sigma^\dagger = \sigma$, one finds the BPS equations

$$\begin{aligned} 0 &= \sin\frac{\theta}{2}\, D_+\phi + \cos\frac{\theta}{2}\left(\frac{D_3 + D_3^\dagger}{2}\phi + \frac{q}{r}\phi + \sigma\phi\right) , & 0 &= (D_3 - D_3^\dagger)\phi \\ 0 &= \cos\frac{\theta}{2}\, D_-\phi - \sin\frac{\theta}{2}\left(\frac{D_3 + D_3^\dagger}{2}\phi + \frac{q}{r}\phi - \sigma\phi\right) , & 0 &= F \;, \end{aligned} \tag{6.3.26}$$

where $D_\pm \equiv D_1 \mp iD_2$ and $D_3\phi = D_\tau\phi = \left(\partial_\tau - i\frac{a}{2\xi r} - \frac{1}{2r}q - i\frac{\mathfrak{z}}{2\xi r}\right)\phi$.

As before, these equations can be obtained from the canonical deformation action $\mathcal{L}^{\text{def}}_{\text{mat}}$. Its bosonic part reads

$$\begin{aligned} \mathcal{L}^{\text{def}}_{\text{mat}}\Big|_{\text{bos}} = \frac{1}{2}|F|^2 &+ \frac{1}{8}\Big|D_3\phi - D_3^\dagger\phi\Big|^2 \\ &+ \frac{1}{2}\left|\sin\frac{\theta}{2}\, D_+\phi + \cos\frac{\theta}{2}\left(\frac{D_3 + D_3^\dagger}{2}\phi + \frac{q}{r}\phi + \sigma\phi\right)\right|^2 \\ &\quad + \frac{1}{2}\left|\cos\frac{\theta}{2}\, D_-\phi - \sin\frac{\theta}{2}\left(\frac{D_3 + D_3^\dagger}{2}\phi + \frac{q}{r}\phi - \sigma\phi\right)\right|^2 . \end{aligned} \tag{6.3.27}$$

6.3.3 BPS solutions: Coulomb, Higgs and vortices

We will now present the BPS solutions to the equations (6.3.21), (6.3.25) and (6.3.26). First, let us recall the solutions for the standard choice $H(\phi) = 0$.

Coulomb-like solutions. Consider (6.3.21) and (6.3.26). They allow for a field strength

$$F = \frac{\mathfrak{m}}{2}\sin\theta\, d\theta \wedge d\varphi \;, \tag{6.3.28}$$

where $\mathfrak{m}$ can be diagonalized to lie in the Cartan subalgebra and it takes values in the coweight lattice of the gauge group G (it is GNO quantized). The gauge field can be written as

$$A = \frac{\mathfrak{m}}{2}\left(\kappa - \cos\theta\right) d\varphi + \frac{a}{2\xi r}\, d\tau \;, \tag{6.3.29}$$

where in this section $\kappa = 1$ ($\kappa = -1$) on the patch excluding the south (north) pole. We have also included a holonomy a, with $[a, \mathfrak{m}] = 0$, around the temporal circle. The BPS equations fix $\sigma = -\mathfrak{m}/2r$ and $D = 0$, which is the localization locus of [122].

Let us now analyze the BPS equations for a chiral multiplet in gauge representation $\mathcal{R}$, assuming that its R-charge q is positive, and show that the only smooth solution is $\phi = 0$. First, we decompose ϕ in Fourier modes recalling that, in the presence of non-trivial flux on S^2, ϕ is a section of a non-trivial bundle and should be expanded in monopole harmonics [159]:

$$\phi(\tau, \theta, \varphi) = \sum_{p,l,m} c_{p,l,m} \ \exp\left(\frac{2\pi i p \tau}{2\xi r}\right) Y_{\frac{\mathfrak{m}}{2},l,m} \tag{6.3.30}$$

where the range of parameters is $p \in \mathbb{Z}$, $l \in \frac{|\mathfrak{m}|}{2} + \mathbb{N}$ and $m = -l, -l+1, \ldots, +l$. The third component of the angular momentum is given by the eigenvalue of[16]

$$j_3 = -i\partial_\varphi - \kappa\frac{\mathfrak{m}}{2}, \tag{6.3.33}$$

and on the monopole harmonics: $j_3\, Y_{\frac{\mathfrak{m}}{2},l,m} = m\, Y_{\frac{\mathfrak{m}}{2},l,m}$. Imposing a Hermiticity condition on the holonomy a, the equation $(D_3 - D_3^\dagger)\phi = 0$ corresponds to

$$\left(\partial_\tau - i\frac{a}{2\xi r} - i\frac{\mathfrak{z}}{2\xi r}\right)\phi = 0\ . \tag{6.3.34}$$

This implies that only those modes for which $\big(a - 2\pi p + \mathfrak{z}\big)\,\phi = 0$ can survive. Since the time dependence is completely fixed, we can reabsorb p by a large gauge transformation and set $p = 0$. From the equations in the first column of (6.3.26), the expressions for σ and the gauge field found above, we find $\big(j_3 + \frac{q}{2}\big)\phi = 0$ and $j_+\phi = 0$. The first one imposes $m = -q/2$, whereas

[16] From [159], the gauge invariant angular momentum operator on $\mathbb{R}^3$ in a monopole background $\mathfrak{m}$ is given by $\vec{L} = \vec{r} \times (-i\vec{D}) - \hat{r}\,\mathfrak{m}/2$, where $\hat{r}$ is a unit vector along the $\vec{r}$ direction. In particular the third component is

$$j_3 = -iD_\varphi - \mathfrak{m}/2\cos\theta = -i\partial_\varphi - \kappa\frac{\mathfrak{m}}{2}\ . \tag{6.3.31}$$

This result is directly applicable to S^2. For later reference, we also write the operators j_+ and j_-:

$$j_\pm = e^{\pm i\varphi}\Big(\pm\partial_\theta + i\cot\theta\, D_\varphi - \frac{\mathfrak{m}}{2}\sin\theta\Big) = e^{\pm i\varphi}\Big(\pm\partial_\theta + i\cot\theta\,\partial_\varphi + \frac{\mathfrak{m}}{2}\kappa\cot\theta - \frac{\mathfrak{m}}{2}\frac{1}{\sin\theta}\Big)\ . \tag{6.3.32}$$

the second one imposes that the angular momentum eigenvalue m take its maximal value $+l$. For positive R-charge $q > 0$, there are no solutions. For zero R-charge (then $l = m = 0$) one finds the constant Higgs-like solution $\phi = \phi_0$, if $\big(a + \mathfrak{z}\big)\,\phi = 0$.

Now let us see the new solutions with non-trivial $H(\phi)$. We integrate D out first, *i.e.* we set $D = iH(\phi)$, solve (6.3.25) and (6.3.26), and take all vanishing R-charges $q = 0$ (arbitrary R-charges can be recovered by analytic continuation of the result by complexifying flavor fugacities). We take exactly the same deformation function $H(\phi)$ as in (6.2.35). We find the following classes of solutions.

Deformed Coulomb branch. It is characterized by $\phi = 0$, and (in complete analogy with [15]) can be completed to

$$F_{13} = F_{23} = 0\;, \qquad F_{12} = 2\zeta\cos\theta + \frac{\mathfrak{m}}{2r^2}\;, \qquad \sigma = -r\zeta\cos\theta - \frac{\mathfrak{m}}{2r}\;. \tag{6.3.35}$$

We thus have $F_{\theta\varphi} = r^2\sin\theta\,(2\zeta\cos\theta + \mathfrak{m}/2r^2)$. The corresponding gauge field can be written as

$$A = \Big(r^2\zeta\sin^2\theta + \frac{\mathfrak{m}}{2}(\kappa - \cos\theta)\Big)\,d\varphi + \frac{a}{2\xi r}d\tau\;. \tag{6.3.36}$$

Higgs-like solutions. They are characterized by $F_{\mu\nu} = 0$, $\sigma = 0$ and a constant profile ϕ for the matter fields that solves the D-term equations

$$H(\phi) = 0\;, \qquad\qquad \big(a + \mathfrak{z}\big)\phi = 0\;. \tag{6.3.37}$$

The solutions to these algebraic equations are analogous to the Higgs-like solutions of section 6.2.3. We will be mainly interested in gauge groups and matter representations such that, for ζ_a in a suitable range, all VEVs ϕ completely break the gauge group.

Vortices. Each Higgs-like solution is accompanied by a tower of vortex-string solutions with arbitrary numbers of vortices at the north and at the south circles. To see this, we expand the BPS equations around $\theta = 0$ and $\theta = \pi$.

The $S^2 \times S^1$ metric (6.3.1) in the $\theta \to 0$ limit becomes $ds^2 = dR^2 + R^2 d\varphi^2 + d\tau^2$, where $R \equiv r\theta$, which is the metric of $\mathbb{R}^2 \times S^1$. The equations (6.3.25) become, to linear order in R:

$$\begin{aligned} 0 &= D_R\sigma - \frac{1}{r}F_{R\varphi} \,, & 0 &= D_\varphi\sigma = D_\tau\sigma \\ 0 &= H(\phi) - \frac{1}{R}F_{R\varphi} - D_R\Big(\frac{\sigma R}{r}\Big) \,, & 0 &= F_{\varphi\tau} = F_{R\tau} \,, \end{aligned} \tag{6.3.38}$$

whereas the equations for the chiral fields (6.3.26) become

$$\begin{aligned} 0 &= \Big(D_R + \frac{i}{R}D_\varphi + \frac{R}{r}\sigma\Big)\,\phi \,, & 0 &= \big(D_3 - D_3^\dagger\big)\phi \\ 0 &= \Big(-\frac{i}{r}D_\varphi + \sigma + \frac{D_3 + D_3^\dagger}{2}\Big)\,\phi \,, & 0 &= F \,. \end{aligned} \tag{6.3.39}$$

Let us qualitatively describe the solutions for a $U(1)$ theory with a single chiral field of charge 1. Working in the gauge $A_\theta = 0$, (6.3.25) implies that $\partial_\theta\big(r\sigma\cos\theta - A_\varphi\big) = 0$ exactly. We write $r\sigma\cos\theta = A_\varphi - n$, for some integration constant n, so it is sufficient to specify the behavior of ϕ and A_φ. Far from the core (the length scale is set by $\sqrt{\zeta^{-1}}$) one finds

$$\phi \simeq \sqrt{\zeta}\, e^{in\varphi} \,, \qquad A_\varphi \simeq n \,, \tag{6.3.40}$$

and Stokes' theorem implies that $\frac{1}{2\pi}\int F = n$, which is the vortex number. Close to the core:

$$\phi \simeq B\,\big(Re^{i\varphi}\big)^n \,, \qquad A_\varphi \simeq 0 + \mathcal{O}\Big(e^{-\frac{R^2}{2r^2}}\Big) \,, \tag{6.3.41}$$

and in particular $n \geq 0$. A similar analysis can be performed around the south pole in the coordinate $\tilde{R} = r(\pi - \theta)$. This time we write $r\sigma\cos\theta = A_\varphi - m$. Then $A_\varphi \to 0$ near the core and $A_\varphi \to m$, which we identify with the vortex number, far from the core. We also find $|\phi| \to B'\tilde{R}^m$ near the core, while it sits in the vacuum far from it: $|\phi|^2 \to \zeta$. The vortex configurations we wrote around the north and south poles are connected by a gauge transformation on the equator: $\phi^N = e^{i(n-m)\varphi}\phi^S$ and $A_\varphi^N - A_\varphi^S = n - m$.

For finite values of ζ, we can derive a bound on the allowed vortex numbers. From (6.3.25) one deduces $H(\phi)\, r\sin\theta = \partial_\theta\sigma$, which results in the inequality

$$r\sigma(\pi) - r\sigma(0) = \frac{1}{2\pi}\int H(\phi) d\mathrm{vol}(S^2) \leq \zeta\frac{\mathrm{vol}(S^2)}{2\pi} \,, \tag{6.3.42}$$

which upon plugging in the values of σ found above leads to the bound

$$m+n \;\leq\; \zeta\,\frac{\mathrm{vol}(S^2)}{2\pi}\;. \tag{6.3.43}$$

As in section 6.2.3, we conclude that for finite values of ζ there is a finite number of vortex/antivortex solutions on S^2. When the bound is saturated, the chiral field ϕ actually vanishes and the gauge field is as in the deformed Coulomb branch described above. We thus get a similar picture of the structure of solutions as in section 6.2.3.

6.3.4 Computation of the index

We will now evaluate the classical action and the one-loop determinants of quadratic fluctuations, and then sum/integrate over the space of BPS configurations.

One-loop determinants from the index theorem

As in section 6.2.4, we compute the one-loop determinants on non-trivial backgrounds with the equivariant index theorem, following [148]. The localizing supercharge squares to

$$\mathcal{Q}^2 = -\mathcal{L}^A_{\partial_\tau} + \frac{i}{r}\mathcal{L}^A_{\partial_\varphi} - \cos\theta\,\sigma - \frac{1}{2r}R + i\sum_j \frac{\mathfrak{z}_j}{2\xi r}F_j\;. \tag{6.3.44}$$

The action of $\mathcal{Q}^2$ on the worldvolume consists of a free rotation along S^1 generated by $\mathcal{L}_{\partial_\tau}$ and a rotation of S^2 generated by $\mathcal{L}_{\partial_\varphi}$ with fixed points at the north and south poles. The equivariant parameters for the $U(1)_{\partial_\varphi}\times U(1)_R\times U(1)^F_{\text{flavor}}\times G$ are given by $\varepsilon=\frac{i}{r}$, $\hat\varepsilon=-\frac{1}{2r}$, $\check\varepsilon_j = i\frac{\mathfrak{z}_j}{2\xi r}$ and $\hat a = iA_\tau + \frac{1}{r}A_\varphi - \cos\theta\,\sigma$. In appendix D.3 we compute the one-loop determinants in our conventions. For a chiral multiplet in gauge representation $\mathcal{R}$ we have

$$Z^{\text{chiral}}_{\text{1-loop}} \text{ ``}=\text{''} \prod_{w\in\mathcal{R}}\prod_{n\in\mathbb{Z}}\prod_{k\geq 0}\frac{i\pi n-(k+1)\xi+\xi\frac{q}{2}-\xi r\,w(\hat a_S)-\frac{i}{2}\sum\mathfrak{z}_jF_j}{i\pi n+k\xi+\xi\frac{q}{2}-\xi r\,w(\hat a_N)-\frac{i}{2}\sum\mathfrak{z}_jF_j}\;, \tag{6.3.45}$$

which requires regularization. For the gauge multiplet one has

$$Z^{\text{vec}}_{\text{1-loop}} = \prod_{\alpha>0} 2\sinh\big(\xi r\,\alpha(\hat a_N)\big)\,2\sinh\big(-\xi r\,\alpha(\hat a_S)\big)\;. \tag{6.3.46}$$

Coulomb branch

Coulomb branch localization for the 3d index was first performed in [121] for $\mathcal{N} = 6$ Chern-Simons-matter theories, and later generalized to $\mathcal{N} = 2$ theories in [122]. A subtlety involving the fermion number was pointed out in [129] (see also [158]), and was later confirmed in [148] by computing the one-loop determinants with the index theorem. Let us quickly review these results. The Chern-Simons action evaluated on the Coulomb branch configurations gives[17]

$$S_{\rm cl}^{CS} = -\frac{i}{4\pi}\int \mathrm{Tr}_{CS} A\wedge F = -i\mathrm{Tr}_{CS}\, a\,\mathfrak{m}\ . \tag{6.3.47}$$

Due to the modified fermion number, an extra phase $(-1)^{\mathrm{Tr}_{CS}\mathfrak{m}}$ needs to be taken into account [158].

To each Abelian factor (with field strength F) in the gauge group is associated a topological symmetry $U(1)_J$, whose current is $J = *F$. Coupling $U(1)_J$ to an external vector multiplet with bosonic components $(A_{BG}, \sigma_{BG}, D_{BG})$ is equivalent to introducing a mixed supersymmetric Chern-Simons term, whose bosonic part is

$$S_J\Big|_{\rm bos} = \frac{i}{2\pi}\int \mathrm{Tr}\big(A_{BG}\wedge F + \sigma D_{BG} + \sigma_{BG} D\big)\ . \tag{6.3.48}$$

An expectation value for σ_{BG} would correspond to an FI term. In this section, though, we will be interested in turning on a holonomy b and a flux $\mathfrak{n}$. Notice that this is indeed an example of an external flux for a flavor symmetry, in the spirit of [157]. Evaluation on the Coulomb branch BPS configurations yields

$$S_J = i\mathrm{Tr}\big(a\,\mathfrak{n} + b\,\mathfrak{m}\big)\ . \tag{6.3.49}$$

We will introduce the topological fugacity $w = e^{-ib}$. Also in this case extra signs are required: this can be done by taking the index not to be a function of w, but rather of $(-1)^{\mathfrak{n}} w$. Such dependence will always be understood.

The gauge equivariant parameter is $\hat{a} = \dfrac{ia}{2\xi r} + \dfrac{\kappa}{2r}\mathfrak{m}$, where $\kappa = 1$ (-1) on the northern (southern) patch, as in (6.3.29). The chiral one-loop deter-

[17] We recall that in order to correctly evaluate the CS action $\int A\wedge F$, one should construct an extension $\tilde{F}$ of the gauge bundle to $S^2\times D_2$ (where the second factor is a disk) and integrate $\int \tilde{F}\wedge\tilde{F}$.

minant then simplifies and, after regularization, becomes

$$Z^{\text{chiral}}_{\text{1-loop}} = \prod_{w\in\mathcal{R}} \Big(x^{1-q}\, e^{-iw(a)}\, \zeta^{-F}\Big)^{-w(\mathfrak{m})/2} \, \frac{\big(x^{2-q-w(\mathfrak{m})}\, e^{-iw(a)}\, \zeta^{-F}; x^2\big)_\infty}{\big(x^{q-w(\mathfrak{m})}\, e^{iw(a)}\, \zeta^{F}; x^2\big)_\infty} \,, \tag{6.3.50}$$

where $(a;q)_\infty \equiv \prod_{k=0}^\infty (1-aq^k)$ is the q-Pochhammer symbol, we defined $x = e^{-\xi}$ and $\zeta_j = e^{i\mathfrak{z}_j}$, we used the short-hand notation $\zeta^F = \prod_i \zeta_i^{F_i}$, and q is the R-charge. The regularization is similar to [121] (see also [148]). The expression above includes all the correct extra signs. The vector one-loop determinant becomes

$$\begin{aligned} Z^{\text{vec}}_{\text{1-loop}} &= \prod_{\alpha>0} 4\sinh\Big(\frac{1}{2}\alpha\big(ia+\xi\mathfrak{m}\big)\Big) \sinh\Big(-\frac{1}{2}\alpha\big(ia-\xi\mathfrak{m}\big)\Big) & (6.3.51)\\ &= \prod_{\alpha\in G} x^{-\frac{1}{2}|\alpha(\mathfrak{m})|}\Big(1-x^{|\alpha(\mathfrak{m})|}e^{i\alpha(a)}\Big)\,. & (6.3.52)\end{aligned}$$

The index is thus computed by the matrix integral:

$$I\big(x,\zeta_j,(-1)^{\mathfrak{n}}w,\mathfrak{n}\big) = \\ \frac{1}{|\mathcal{W}|}\sum_{\mathfrak{m}\,\in\,\mathbb{Z}^{\operatorname{rank} G}} \int \Bigg(\prod_{j=1}^{\operatorname{rank} G}\frac{dz_j}{2\pi i z_j}\Bigg)\,(-1)^{\operatorname{Tr}_{CS}\mathfrak{m}}\; e^{i\operatorname{Tr}_{CS}a\mathfrak{m} - i\operatorname{Tr}(a\mathfrak{n}+b\mathfrak{m})}\; Z_{\text{1-loop}}\,, \tag{6.3.53}$$

where $|\mathcal{W}|$ is the order of the Weyl group, $z_j = e^{ia_j}$ is the gauge fugacity and the integration contour is counterclockwise along the unit circle.

Deformed Coulomb branch

The full Chern Simons action (D.2.16) and the mixed CS term (6.3.48) evaluated on the deformed Coulomb branch read

$$S^{CS}_{\text{cl}} = -i\operatorname{Tr}_{CS}\big((a-2ir^2\xi\zeta)\,\mathfrak{m}\big)\,, \qquad S_J = \operatorname{Tr}\big((a-2ir^2\xi\zeta)\,\mathfrak{n} + b\,\mathfrak{m}\big)\,, \tag{6.3.54}$$

where, in this subsection, ζ refers to the deformation parameter (6.2.36). We also need to include the phase $(-1)^{\operatorname{Tr}_{CS}\mathfrak{m}}$. The equivariant parameter is given by

$$\hat{a} = i\frac{a-2ir^2\xi\zeta}{2\xi r} + \frac{\kappa}{2r}\mathfrak{m}\,. \tag{6.3.55}$$

As in section 6.2.4, we observe that the net effect of the deformation parameter ζ is an imaginary shift of the integration variable $a \to a - 2ir^2\xi\zeta$, or equivalently $z \equiv e^{ia} \to x^{-2r^2\zeta}z$. Effectively it modifies the radius of the integration contour; since $|x| < 1$, the contour grows for $\zeta > 0$ and shrinks for $\zeta < 0$. The effect on the integral is the same as in section 6.2.4: it remains constant, until the contour crosses some pole and the integral jumps. In view of the bound (6.3.43), this happens precisely when new vortex configuration become allowed, providing the missing residue.

In order to obtain an expression of $Z_{S^2\times S^1}$ purely in terms of vortices, we need to suppress the contribution from the deformed Coulomb branch. Heuristically, this can be achieved if there is no pole at the origin or infinity. As we show in appendix D.4 following [160], for a $U(N)$ theory with N_f fundamentals, N_a antifundamentals and Chern-Simons level k, there is no pole at infinity if $N_f > N_a$ and $|k| \leq \frac{N_f - N_a}{2}$, thus suppression is obtained by sending $\zeta \to +\infty$; for $N_f < N_a$ and $|k| \leq \frac{N_a - N_f}{2}$ there is no pole at the origin, thus suppression is obtained by sending $\zeta \to -\infty$. For $N_f - N_a = k = 0$ there are poles both at the origin and infinity, however the residues vanish.

Higgs branch and vortices

For finite values of the deformation parameters ζ_a, additional BPS configurations are present, namely Higgs vacua and vortex solutions, whose (anti)vortex numbers (m, n) are bounded by (6.3.43) (or its multi-dimensional generalization). We determine here their additional contribution to the path integral, besides the deformed Coulomb branch. The discussion is similar to section 6.2.4, so we will be brief.

The classical actions can be evaluated exactly using $D = iH(\phi)$, the BPS equations (6.3.25), the knowledge of the flux carried by the vortices and of the corresponding values of $A_\varphi(\theta)$ at $\theta = 0, \pi$, in a gauge $A_\theta = 0$. Recall that the equations determine σ exactly in terms of A_φ, see around (6.3.40). One finds

$$S_{\rm cl}^{CS} = -i\mathrm{Tr}_{CS}\Big((n-m)a + i\xi(m^2 - n^2)\Big) \tag{6.3.56}$$

$$S_J = i\mathrm{Tr}\Big[\mathfrak{n}\big(a - i\xi(m+n)\big) + b(n-m)\Big]\,, \tag{6.3.57}$$

where a is evaluated on the Higgs branch, $a = -\mathfrak{z}$. Again we need to include the extra phase $(-1)^{\mathrm{Tr}_{CS}(n-m)}$. The one-loop determinants are evaluated with (6.3.45) and (6.3.46), using the equivariant parameters

$$\hat{a}_N = \frac{ia + 2\xi n}{2\xi r} \,, \qquad\qquad \hat{a}_S = \frac{ia + 2\xi m}{2\xi r} \tag{6.3.58}$$

at the north and south poles, where in both cases a is evaluated on its Higgs branch location a_H. The one-loop determinants for the rank G chiral multiplets Higgsing the gauge group should be computed with a residue prescription. Therefore, after a regularization similar to [148], the one-loop determinant for chiral multiplets is

$$Z^{\text{chiral}}_{\text{1-loop}} = \\ \operatorname*{Res}_{a \to a_H} \left[\prod_{w \in \mathfrak{R}} \Big(x^{1+w(m+n)} \, e^{-iw(a)} \, \zeta^{-F(\phi)} \Big)^{w(m-n)/2} \; \frac{\big(x^{2+2w(m)} \, e^{-iw(a)} \, \zeta^{-F(\phi)}; x^2 \big)_\infty}{\big(x^{-2w(n)} \, e^{iw(a)} \, \zeta^{F(\phi)}; x^2 \big)_\infty} \right] . \tag{6.3.59}$$

Here $F(\phi)$ refers to the chiral multiplets, $\zeta^F = \prod_i \zeta_i^{F_i}$ and we set the R-charges to zero. For the vector one-loop determinant we have

$$\begin{aligned} Z^{\text{gauge}}_{\text{1-loop}} &= \prod_{\alpha>0} 2\sinh\Big(\frac{\alpha(ia+2\xi n)}{2} \Big) \, 2\sinh\Big(-\frac{\alpha(ia+2\xi m)}{2} \Big) \\ &= \prod_{\alpha \in \mathfrak{g}} x^{-\frac{|\alpha(n-m)|}{2}} \Big(1 - x^{|\alpha(n-m)| - \alpha(n+m)} \, e^{i\alpha(a)} \Big) \,, \end{aligned} \tag{6.3.60}$$

evaluated on the Higgs branch location. These expressions, for the vortices that satisfy the bound (6.3.43), precisely reproduce the residues of the integrand in (6.3.53), which are the jumps of the deformed Coulomb branch contribution as the contour crosses the poles.

Vortex partition function. We will now take a suitable limit $\zeta_a \to \pm\infty$, in which the deformed Coulomb branch contribution is suppressed. Then the resummed contribution of all vortex strings is described by the same vortex partition function that we used on S^3_b.

Let us compute the partition function in the limit. First, we have a finite number of Higgs vacua. In each vacuum, the components of the holonomy a_α are fixed to some specific (real) values that are functions of the real masses. The classical actions (6.3.56) provide an overall classical contribution:

$$S_J = i\mathrm{Tr}\big(\mathfrak{n} \, a \big) \,, \tag{6.3.61}$$

as well as the weighting factors for vortices and anti-vortices:

$$\begin{aligned} e^{-S_{\text{v}}} &= \exp\Big[-\xi \text{Tr}_{CS} m^2 + \Big(-i\text{Tr}_{CS} a \cdot \, + \text{Tr}\big(-\xi\mathfrak{n} + ib\big)\cdot \Big) m \Big] \\ e^{-S_{\text{av}}} &= \exp\Big[\xi \text{Tr}_{CS} n^2 + \Big(i\text{Tr}_{CS} a \cdot \, + \text{Tr}\big(-\xi\mathfrak{n} - ib\big)\cdot \Big) n \Big] \,. \end{aligned} \tag{6.3.62}$$

Second, the one-loop determinants for the vector multiplet and the chiral multiplets not acquiring a VEV are as in the Coulomb branch. The rank G chiral multiplets acquiring VEV bring a residue factor, which in this case is some phase. Finally, the vortex partition function Z_{vortex} depends on equivariant parameters for rotations of $\mathbb{R}^2$ (ε) and flavor rotations (g): they are identified—at $\theta = 0$ (N) and $\theta = \pi$ (S)—from the $SU(1|1)$ complex of the supercharge $\mathcal{Q}$ at the poles, *i.e.* from $\mathcal{Q}^2$ in (6.3.44). We find

$$\begin{aligned} \varepsilon_N &= -2i\xi \,, \quad g_N = i\Big(a + \textstyle\sum_j \mathfrak{z}_j F_j\Big) \,, \\ \varepsilon_S &= 2i\xi \,, \quad g_S = -i\Big(a + \textstyle\sum_j \mathfrak{z}_j F_j\Big) \,, \end{aligned} \tag{6.3.63}$$

where the minus sign in the south pole parameters with respect to the north pole ones is due to the opposite orientation.

Eventually, Higgs branch localization gives the following expression for the index:

$$I = \sum_{\text{Higgs vacua}} e^{-i\text{Tr}(\mathfrak{n}\, a)} \, Z'_{\text{1-loop}} \, Z_{\text{v}} \, Z_{\text{av}} \,. \tag{6.3.64}$$

The (anti)vortex-string contributions are expressed in terms of the 3d vortex partition function:

$$\begin{aligned} Z_{\text{v}} &= Z_{\text{vortex}}\Big(e^{-\xi \text{Tr}_{CS}\cdot} \,,\, e^{-i\text{Tr}_{CS} a\cdot + \text{Tr}(-\xi\mathfrak{n} + ib)\cdot} \,,\, -2i\xi \,,\, i\Big(a + \textstyle\sum_j \mathfrak{z}_j F_j\Big)\Big) \\ Z_{\text{av}} &= Z_{\text{vortex}}\Big(e^{\xi \text{Tr}_{CS}\cdot} \,,\, e^{i\text{Tr}_{CS} a\cdot + \text{Tr}(-\xi\mathfrak{n} - ib)\cdot} \,,\, 2i\xi \,,\, -i\Big(a + \textstyle\sum_j \mathfrak{z}_j F_j\Big)\Big) \,. \end{aligned} \tag{6.3.65}$$

As in section 6.2.4, the first two arguments in the vortex partition function are exponentiated linear functions on the gauge algebra, corresponding to the quadratic and linear weights for the vortex numbers. We shall give a concrete example in the next section.

6.3.5 Matching with the Coulomb branch integral

We wish to shortly review, in our conventions, that the superconformal index of a $U(N)$ gauge theory with N_f fundamentals, N_a antifundamentals and CS level

$$|k| \leq \frac{|N_f - N_a|}{2} , \tag{6.3.66}$$

(see footnote 14) can be rewritten in a form that matches with the result of Higgs branch localization, as done in [135],[18] and moreover that the very same Z_{vortex} as in (6.2.81) emerges.

Concretely,

$$\begin{aligned} I^{U(N),N_f,N_a} = &\frac{1}{N!} \sum_{\vec{\mathfrak{m}} \in \mathbb{Z}^N} w^{\sum_j \mathfrak{m}_j} \oint \prod_{j=1}^{N} \left(\frac{dz_j}{2\pi i z_j} (-z_j)^{k\mathfrak{m}_j} z_j^{-\mathfrak{n}} \right) \\ &\times \prod_{\substack{i,j=1 \\ i\neq j}}^{N} x^{-|\mathfrak{m}_i - \mathfrak{m}_j|/2} \Big(1 - z_i z_j^{-1} x^{|\mathfrak{m}_i - \mathfrak{m}_j|} \Big) \\ &\times \prod_{i=1}^{N} \prod_{\alpha=1}^{N_f} \big(x z_i^{-1} \zeta_\alpha \big)^{-\mathfrak{m}_i/2} \frac{\big(z_i^{-1} \zeta_\alpha x^{-\mathfrak{m}_i + 2} \, ; \, x^2 \big)_\infty}{\big(z_i \zeta_\alpha^{-1} x^{-\mathfrak{m}_i} \, ; \, x^2 \big)_\infty} \\ &\quad \prod_{\beta=1}^{N_a} \big(x z_i \tilde{\zeta}_\beta^{-1} \big)^{\mathfrak{m}_i/2} \frac{\big(z_i \tilde{\zeta}_\beta^{-1} x^{\mathfrak{m}_i + 2} \, ; \, x^2 \big)_\infty}{\big(z_i^{-1} \tilde{\zeta}_\beta x^{\mathfrak{m}_i} \, ; \, x^2 \big)_\infty} , \end{aligned} \tag{6.3.67}$$

where $z_j = e^{ia_j}$ and $w = e^{-ib}$. The flavor fugacities $\zeta_\alpha = e^{i\mathfrak{z}_\alpha}$, $\tilde{\zeta}_\beta = e^{i\tilde{\mathfrak{z}}_\beta}$ are defined up to a common rescaling, since the flavor symmetry is $SU(N_f) \times SU(N_a) \times U(1)_A$. The integration contour is along the unit circle for $|\tilde{\zeta}_\beta| < 1 < |\zeta_\alpha|$. We also introduced the extra sign $(-1)^{k\sum \mathfrak{m}_j}$, as explained in section 6.3.4. Note that $k + \frac{N_f + N_a}{2}$ is integer if we impose parity anomaly cancelation: this guarantees that the integrand is a single-valued function of z_j.

For $N_f > N_a$ there is no pole at infinity (see appendix D.4). Moreover, since $|\tilde{\zeta}_\beta| < 1 < |\zeta_\alpha|$ and $|x| < 1$, only the one-loop determinants of fundamentals have poles outside the unit circle. More precisely, the numerator of the one-loop determinants of fundamentals has zeros at $z_j = \zeta_{\alpha_j} x^{-\mathfrak{m}_j + 2r_j}$, for all $r_j \geq 1$ and $j = 1, \ldots, N$, while the denominator has zeros at $z_j =$

[18] See also [134], where the factorized form of the index was first observed in the $U(1)$ case.

$\zeta_{\alpha_j} x^{\mathfrak{m}_j - 2r_j}$ for all $r_j \geq 0$. For $\mathfrak{m}_j \leq 0$ there is no superposition of zeros, while for $\mathfrak{m}_j > 0$ there is superposition and some of them cancel. The net result is that the poles outside the unit circle are located at

$$z_j = \zeta_{\gamma_j}\, x^{-|\mathfrak{m}_j| - 2r_j} \;, \qquad r_j \in \mathbb{Z}_{\geq 0} \;, \qquad \gamma_j = 1, \dots, N_f \;, \qquad j = 1, \dots, N \;. \tag{6.3.68}$$

Summing the residues, one obtains:

$$\begin{aligned} I = \frac{1}{N!} \sum_{\vec{\gamma}\,\in\,(\mathbb{Z}_{N_f})^N} \; \sum_{\vec{\mu},\,\vec{\nu}\,\in\,\mathbb{Z}^N_{\geq 0}} (-1)^{-k\sum_j(\mu_j-\nu_j)}\, w^{\sum_j(\mu_j-\nu_j)} \prod_{i=1}^N \big(\zeta_{\gamma_i}^{-1} x^{\mu_i+\nu_i}\big)^{-k(\mu_i-\nu_i)+\mathfrak{n}} \\ \times \prod_{i\neq j}^N x^{-\frac{1}{2}|(\mu_i-\nu_i)-(\mu_j-\nu_j)|} \bigg(1 - \frac{\zeta_{\gamma_j}^{-1} x^{\mu_j+\nu_j}}{\zeta_{\gamma_i}^{-1} x^{\mu_i+\nu_i}}\, x^{|(\mu_i-\nu_i)-(\mu_j-\nu_j)|}\bigg) \prod_{i=1}^N \frac{\big(x^{\mu_i+\nu_i+1}\big)^{-(\mu_i-\nu_i)/2}}{(x^{-2};x^{-2})_{\mu_i}(x^2;x^2)_{\nu_i}} \\ \times \prod_{i=1}^N \prod_{\alpha\,(\neq\gamma_i)}^{N_f} \big(\zeta_\alpha \zeta_{\gamma_i}^{-1} x^{\mu_i+\nu_i+1}\big)^{-(\mu_i-\nu_i)/2} \frac{\big(\zeta_\alpha\zeta_{\gamma_i}^{-1} x^{2\nu_i+2}\,;\, x^2\big)_\infty}{\big(\zeta_\alpha^{-1}\zeta_{\gamma_i} x^{-2\mu_i}\,;\, x^2\big)_\infty} \\ \times \prod_{i=1}^N \prod_{\beta=1}^{N_a} \big(\tilde{\zeta}_\beta^{-1}\zeta_{\gamma_i} x^{-\mu_i-\nu_i+1}\big)^{(\mu_i-\nu_i)/2} \frac{\big(\tilde{\zeta}_\beta^{-1}\zeta_{\gamma_i} x^{-2\nu_i+2}\,;\, x^2\big)_\infty}{\big(\tilde{\zeta}_\beta\zeta_{\gamma_i}^{-1} x^{2\mu_i}\,;\, x^2\big)_\infty} \;, \end{aligned} \tag{6.3.69}$$

where we decomposed the summation over $\mu_i = r_i + \frac{\mathfrak{m}_i + |\mathfrak{m}_i|}{2}$ and $\nu_i = \mu_i - \mathfrak{m}_i$. The q-Pochhammer symbol is $(a;q)_n = \prod_{k=0}^{n-1}(1-q^k a)$.

At this point one can factorize the summation into a factor independent of $\vec{\mu}$ and $\vec{\nu}$, a summation over $\vec{\mu}$ and a summation over $\vec{\nu}$. One observes that each of the two summations over $\vec{\mu}$ and $\vec{\nu}$ vanishes if we choose $\gamma_i = \gamma_j$ for some i, j, and on the other hand it is symmetric under permutations of the γ_i's. Therefore we can restrict the sum over unordered combinations $\vec{\gamma} \in C(N, N_f)$ of N out of the N_f flavors, and cancel the $N!$ in the denominator. Finally, rewriting the q-Pochhammer symbols in terms of sinh and using the identity (6.2.77) one obtains

$$I = \sum_{\vec{\gamma}\,\in\, C(N,N_f)} Z_{\text{cl}}^{(\vec{\gamma})}\; Z'^{(\vec{\gamma})}_{\text{1-loop}}\; Z_{\text{v}}^{(\vec{\gamma})}\; Z_{\text{av}}^{(\vec{\gamma})} \;. \tag{6.3.70}$$

The classical and one-loop contributions are

$$
\begin{aligned}
Z_{\text{cl}}^{(\vec{\gamma})} &= \prod_{j\,\in\,\vec{\gamma}} \zeta_j^{-\mathfrak{n}} \\
Z'^{(\vec{\gamma})}_{\text{1-loop}} &= \prod_{j\,\in\,\vec{\gamma}} \prod_{\alpha\,(\neq j)}^{N_f} \frac{\big(\zeta_j^{-1}\zeta_\alpha x^2\,;\,x^2\big)_\infty}{\big(\zeta_j\zeta_\alpha^{-1}\,;\,x^2\big)_\infty} \prod_{\beta=1}^{N_a} \frac{\big(\zeta_j\tilde{\zeta}_\beta^{-1}x^2\,;\,x^2\big)_\infty}{\big(\zeta_j^{-1}\tilde{\zeta}_\beta\,;\,x^2\big)_\infty} \\
&\qquad \cdot \prod_{\substack{i,j\,\in\,\vec{\gamma}\\ i\neq j}} 2\sinh\Big(\frac{i\mathfrak{z}_i - i\mathfrak{z}_j}{2}\Big)\;.
\end{aligned}
\tag{6.3.71}
$$

The vortex and antivortex contribution can be written as

$$
\begin{aligned}
Z_{\text{v}}^{(\vec{\gamma})} &= Z_{\text{vortex}}^{(\vec{\gamma})}\Big(e^{-\xi k}\;,\;w_{\text{v}}^{-1}e^{(-i\mathfrak{z}_jk-\xi\mathfrak{n})}\big|_{j\in\vec{\gamma}}\;,\;-2i\xi\;,\;i\mathfrak{z}_\alpha\;,\;i\tilde{\mathfrak{z}}_\beta\Big) \\
Z_{\text{av}}^{(\vec{\gamma})} &= Z_{\text{vortex}}^{(\vec{\gamma})}\Big(e^{\xi k}\;,\;w_{\text{av}}\,e^{(i\mathfrak{z}_ik-\xi\mathfrak{n})}\big|_{i\in\vec{\gamma}}\;,\;2i\xi\;,\;-i\mathfrak{z}_\alpha\;,\;-i\tilde{\mathfrak{z}}_\beta\Big)\;,
\end{aligned}
\tag{6.3.72}
$$

and the vortex-string partition function turns out to be exactly the same (6.2.81) as for the computation on S_b^3, namely:

$$
Z_{\text{vortex}}^{(\vec{\gamma})}\Big(Q_j\;,\;L_j\;,\;\varepsilon\;,\;a_\alpha\;,\;b_\beta\Big) = \sum_{\vec{\mu}\,\in\,\mathbb{Z}^N_{\geq 0}} \prod_{j\,\in\,\vec{\gamma}} Q_j^{\mu_j^2} L_j^{\mu_j}\;(-1)^{(N_f-N_a)\mu_j}
$$

$$
\times \prod_{\lambda=0}^{\mu_j-1} \frac{\prod_{\beta=1}^{N_a} 2i\sinh\dfrac{a_j-b_\beta+i\varepsilon\lambda}{2}}{\prod_{l\,\in\,\vec{\gamma}} 2i\sinh\dfrac{a_j-a_l+i\varepsilon(\lambda-\mu_l)}{2}\;\prod_{\alpha\notin\vec{\gamma}}^{N_f} 2i\sinh\dfrac{a_\alpha-a_j+i\varepsilon(\lambda-\mu_j)}{2}}\;.
$$

The fugacity w for the topological charge is rotated by a phase: $w_{\text{v}} = (-i)^{N_f-N_a}(-1)^{k+N-1}w$, $w_{\text{av}} = i^{N_f-N_a}(-1)^{k+N-1}w$. The parameters that determine Z_{v} and Z_{av} in terms of Z_{vortex} are exactly as prescribed by our general discussion in section 6.3.4.

Chapter 7

Higgs branch localization of $\mathcal{N} = 1$ theories on $S^3 \times S^1$

7.1 Introduction

Given the results in chapter 6 it is now a natural question to ask if Higgs branch localization can be applied to four-dimensional theories as well. This would imply that the partition function can be factorized. In this chapter, we address this question – and answer it positively – for $\mathcal{N} = 1$ supersymmetric gauge theories on $S^3 \times S^1$.[1] At an RG fixed point, the partition function on this geometry is known to describe the superconformal index [36], which encodes information about the protected spectrum of the corresponding flat space theory. A prescription to write down the Coulomb branch expression computing the index of the IR fixed point to which a given Lagrangian UV theory flows, was first given by Römelsberger in [143, 162] and it takes the form of a matrix integral over the holonomy along the temporal circle S^1 of the one-loop determinants, which are typically expressed in terms of the plethystic exponential of the single letter partition functions of the fields in the UV theory, but can also be written in terms of elliptic hypergeometric functions [163]. Our main result shows that the index can alternatively be written as

$$I = \sum_{\text{Higgs vacua}} Z_{\text{cl}}\, Z'_{\text{1-loop}}\, Z_{\text{v}}\, Z_{\text{av}}\ , \tag{7.1.1}$$

[1] Recently the gravity dual of these theories has been studied in [161].

which has the typical form of a Higgs branch localized result. Here Z_{v} and Z_{av} are the contributions from vortex-membranes wrapping a torus at two distinct points in the geometry. As such they are given in terms of the elliptic uplift of the usual vortex partition function in the Ω background [130].

The superconformal index is a powerful tool in checking various dualities, see for example [163, 164, 165]. It would be very interesting to study the effects of such dualities on the vortex-membrane partition function. Moreover, the elliptic vortex partition functions we encounter are naturally expected to have nice modular properties. It would be interesting to study these alongside the modular properties of the index itself [68]. The factorization results obtained in this chapter are expected to be just one instance of a rich structure involving the four-dimensional uplift of the holomorphic blocks of [17]. Unraveling this structure is an outstanding problem. Finally, the $\mathcal{N} = 1$ index can be further decorated with surface operators. Three possible approaches can be used to introduce them, namely to construct them as the IR limit of vortex configurations as in [74], to perform a localization computation as in [148] for vortex-loops, or to consider a coupled 2d-4d system as in [113]. Their connections among each other and with the vortex factorization should be study-worthy.

On the other hand, we expect the techniques used in this chapter to be applicable to $\mathcal{N} = 1$ supersymmetric theories on different geometries as well, most obviously $L(r, 1) \times S^1$[141, 144], but also more generally in theories with more supersymmetry. For example, the $\mathcal{N} = 2$ superconformal index for theories of class $\mathcal{S}$ is computed by a TQFT correlator [75, 38] and it would be very interesting to study its interplay with a possible vortex anti-vortex factorization.

The outline of this chapter is as follows. In section 7.2 we introduce the index we want to compute and construct the deformed background on which the computation of the partition function achieves that goal. Next, we derive the BPS equations in section 7.3 and find various classes of solutions to them in section 7.4. We compute the index on the various solutions in section 7.5. In section 7.6 we match our Higgs branch expression with the Coulomb branch expression in some examples by manipulating the matrix integral. Here we also explain how our results apply in the absence of an abelian factor in the gauge group. Finally, the appendices contain the spinor conventions we use, the $\mathcal{N} = 1$ algebra and some useful identities satisfied by the elliptic gamma function.

7.2 Killing spinors on $S^3 \times S^1$, supersymmetric index and deformed background

$\mathcal{N} = 1$ supersymmetric theories on $S^3 \times \mathbb{R}$ were explicitly constructed in [166] and later also in [143]. A systematic study of supersymmetric theories on Euclidean four-manifolds, among which $S^3 \times S^1$, with four or less supercharges was performed in [167, 168, 169] by considering the rigid limit of supergravity. Their method constructs supersymmetric backgrounds as solutions to the Killing spinor equation, which in turn is obtained by setting to zero the gravitino variations, as well as – in the presence of flavor symmetries – to the equations that set to zero the gaugino variations, while treating the bosonic auxiliary fields as arbitrary background fields. [2]

For the particular case of the index, it is illustrative to construct the sought-after supersymmetric background differently, namely by turning on background gauge fields associated to the charges appearing in the supersymmetric index such that in a path integral formulation they have the effect of precisely implementing the twisted boundary conditions along the temporal circle imposed by the index. As a preliminary step, we first construct the solutions to the conformal Killing spinor equations on $S^3 \times \mathbb{R}$ and select the Killing spinor associated to the supercharge with respect to which we will compute the index. Its lack of periodicity along $\mathbb{R}$ is remedied by the twisted boundary conditions imposed by the associated index.

Killing spinors on $S^3 \times \mathbb{R}$ We would like to solve the Killing spinor equation[3]

$$D_\mu \varepsilon = \left(\partial_\mu + \frac{1}{4} \omega_\mu{}^{mn} \gamma_{mn} - i V_\mu \gamma_5 \right) \varepsilon = \gamma_\mu \tilde{\varepsilon} \tag{7.2.1}$$

on $S^3 \times \mathbb{R}$ with metric

$$ds^2 = ds^2_{S^3} + d\tau^2 = \ell^2 \left(d\theta^2 + \cos^2\theta \; d\varphi^2 + \sin^2\theta \; d\chi^2 \right) + d\tau^2 \; . \tag{7.2.2}$$

Upon choosing the vielbeins $e^1 = \ell \cos\theta \; d\varphi$, $e^2 = \ell \sin\theta \; d\chi$, $e^3 = \ell d\theta$, $e^4 = d\tau$, one finds the non-zero components of the spin connection to be $\omega^{13} = -\sin\theta \; d\varphi$ and $\omega^{23} = \cos\theta d\chi$. At this point we also set the $U(1)_R$ background field V_μ to zero.

[2]The parameter dependence of partition functions on these four-dimensional backgrounds was studied in [136].

[3]Our spinor conventions are summarized in appendix E.1.

Our first step in solving (7.2.1) is to write $\tilde{\varepsilon} = \gamma^4 \hat{\varepsilon}$ and decompose ε and $\hat{\varepsilon}$ into their right and left-handed components, which we denote as $\varepsilon = \binom{\eta}{\zeta}$ and similarly $\hat{\varepsilon} = \begin{pmatrix} \hat{\eta} \\ \hat{\zeta} \end{pmatrix}$. The equation (7.2.1) then splits as

$$\left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{mn}\sigma_{mn}\right)\eta = -i\sigma_\mu \hat{\eta} \tag{7.2.3}$$

$$\left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{mn}\bar{\sigma}_{mn}\right)\zeta = i\bar{\sigma}_\mu \hat{\zeta}\ . \tag{7.2.4}$$

Next, making a factorized Ansatz $\eta = f(\tau)\ \eta_{S^3}$ and $\hat{\eta} = f(\tau)\ \hat{\eta}_{S^3}$, where η_{S^3} and $\hat{\eta}_{S^3}$ only depend on the coordinates on the three-sphere, one immediately recognizes that the spatial part of the Killing spinor equation simplifies to the Killing spinor equation on S^3

$$\left(\partial_{\hat{\mu}} + \frac{1}{4}\omega_{\hat{\mu}}{}^{\hat{m}\hat{n}}\sigma_{\hat{m}\hat{n}}\right)\eta_{S^3}^{(s_1,t_1)} = -i\sigma_{\hat{\mu}}\hat{\eta}_{S^3}^{(s_1,t_1)}\ , \tag{7.2.5}$$

where $\hat{\mu} = \varphi, \chi, \theta$. Its solutions are given by [119]

$$\eta_{S^3}^{(s_1,t_1)} = \begin{pmatrix} e^{\frac{i}{2}(s_1\chi + t_1\varphi - s_1 t_1 \theta)} \\ -s_1\ e^{\frac{i}{2}(s_1\chi + t_1\varphi + s_1 t_1 \theta)} \end{pmatrix}\ , \qquad (s_1, t_1 = \pm) \tag{7.2.6}$$

if $\hat{\eta}_{S^3}^{(s_1,t_1)} = \frac{s_1 t_1}{2\ell}\eta_{S^3}^{(s_1,t_1)}$. The time dependence is then determined by $\partial_\tau f(\tau) = \frac{s_1 t_1}{2\ell} f(\tau)$ which implies that $f(\tau) = e^{\frac{s_1 t_1}{2\ell}\tau}$. In total one thus finds that

$$\eta^{(s_1,t_1)} = e^{\frac{s_1 t_1}{2\ell}\tau}\eta_{S^3}^{(s_1,t_1)}\ , \qquad \hat{\eta}^{(s_1,t_1)} = \frac{s_1 t_1}{2\ell}\eta^{(s_1,t_1)}\ . \tag{7.2.7}$$

Similarly, one finds that

$$\zeta^{(s_2,t_2)} = e^{-\frac{s_2 t_2}{2\ell}\tau}\zeta_{S^3}^{(s_2,t_2)}\ , \qquad \hat{\zeta}^{(s_2,t_2)} = -\frac{s_2 t_2}{2\ell}\zeta^{(s_2,t_2)}\ , \tag{7.2.8}$$

and the most general four-component solution is thus

$$\varepsilon = \sum_{s_1,t_1=\pm} a_{s_1,t_1}\begin{pmatrix} \eta^{(s_1,t_1)} \\ 0 \end{pmatrix} + \sum_{s_2,t_2=\pm} b_{s_2,t_2}\begin{pmatrix} 0 \\ \zeta^{(s_2,t_2)} \end{pmatrix} \tag{7.2.9}$$

We found eight independent supercharges as expected for a superconformal $\mathcal{N} = 1$ theory. Note that the Killing spinors are not periodic along the time circle which signals the need of twisted boundary conditions.

Killing spinors for supersymmetric index We choose the combination of supercharges described by the four-component spinor ε in (7.2.9) with as only non-zero coefficients $a_{++} = 1$ and $b_{--} = 1$,

$$\varepsilon_1 = \begin{pmatrix} \eta^{(+,+)} \\ \zeta^{(-,-)} \end{pmatrix} . \tag{7.2.10}$$

It satisfies $D_\mu \varepsilon_1 = \frac{1}{2\ell}\gamma_\mu\gamma_4\gamma_5\varepsilon_1$. The bilinears appearing in the algebra (see formula (E.2.12) in appendix E.2) are then given by

$$v_1^\mu \partial_\mu = \frac{2}{\ell}\left(-i(\partial_\varphi + \partial_\chi) + \ell\partial_\tau\right) , \qquad \rho_1 = 0 , \qquad \alpha_1 = \frac{3i}{\ell} , \tag{7.2.11}$$

which upon plugging in (E.2.9) result in

$$\delta^2_{\varepsilon_1} = -\frac{2}{\ell}\left(-\ell\mathcal{L}^A_{\partial_\tau} + i\mathcal{L}^A_{\partial_\varphi+\partial_\chi} + \frac{3}{2}R\right) . \tag{7.2.12}$$

Introducing the operators

$$\Delta = -\ell\mathcal{L}^A_{\partial_\tau} , \qquad j_1 = -\frac{i}{2}\mathcal{L}^A_{\partial_\chi+\partial_\varphi} , \qquad j_2 = -\frac{i}{2}\mathcal{L}^A_{\partial_\chi-\partial_\varphi} , \tag{7.2.13}$$

one can also write

$$\delta^2_{\varepsilon_1} = -\frac{2}{\ell}\left(\Delta - 2j_1 + \frac{3}{2}R\right) . \tag{7.2.14}$$

The action of the operators Δ, j_1, j_2, and R on ε_1 is given by

$$\Delta\epsilon_1 = -\frac{1}{2}\gamma_5\varepsilon_1 , \qquad j_1\epsilon_1 = \frac{1}{2}\gamma_5\varepsilon_1 , \qquad j_2\epsilon_1 = 0 , \qquad R\epsilon_1 = \gamma_5\varepsilon_1 . \tag{7.2.15}$$

Note that as expected $\left(\Delta - 2j_1 + \frac{3}{2}R\right)\varepsilon_1 = 0$. We can find two more linearly independent charges that vanish on the Killing spinor, namely $2j_1 - R$ and j_2.

Alternatively, we could choose the combination of supercharges described by the four-component spinor with as only non-zero coefficients $a_{-+} = 1$ and $b_{+-} = 1$

$$\varepsilon_2 = \begin{pmatrix} \eta^{(-,+)} \\ \zeta^{(+,-)} \end{pmatrix} , \tag{7.2.16}$$

which satisfies $D_\mu \varepsilon_2 = -\frac{1}{2\ell}\gamma_\mu\gamma_4\gamma_5\varepsilon_2$. Then we find

$$\delta^2_{\varepsilon_2} = \frac{2}{\ell}\left(-\ell\mathcal{L}^A_{\partial_\tau} + i\mathcal{L}^A_{\partial_\chi-\partial_\varphi} - \frac{3}{2}R\right) = \frac{2}{\ell}\left(\Delta - 2j_2 - \frac{3}{2}R\right) . \qquad (7.2.17)$$

Now one has $\Delta\epsilon_2 = \frac{1}{2}\gamma_5\varepsilon_2$, $j_1\epsilon_2 = 0$, $j_2\epsilon_2 = -\frac{1}{2}\gamma_5\varepsilon_2$, and $R\epsilon_2 = \gamma_5\varepsilon_2$. We find three linearly independent charges that vanish on the Killing spinor, $\Delta - 2j_2 - \frac{3}{2}R$, $2j_2 + R$ and j_1.

Supersymmetric index and deformed background One can introduce two inequivalent superconformal indices in $\mathcal{N} = 1$ theories, a left-handed one and a right-handed one, namely

$$\begin{aligned} I_1(t, y, \zeta_j) &= \mathrm{Tr}(-)^F\ e^{-\beta\left(\Delta-2j_1+\frac{3}{2}R\right)}\ t^{3(2j_1-R)}\ y^{2j_2}\ \prod\nolimits_j \zeta_j^{F_j} \\ I_2(t, y, \zeta_j) &= \mathrm{Tr}(-)^F\ e^{-\beta\left(\Delta-2j_2-\frac{3}{2}R\right)}\ t^{3(2j_2+R)}\ y^{2j_1}\ \prod\nolimits_j \zeta_j^{F_j} \end{aligned} \qquad (7.2.18)$$

where $t = e^{-\xi}$, $y = e^{i\eta}$, and $\zeta_j = e^{i\mathfrak{z}_j}$. Moreover, $\beta\ell$ is the circumference of the temporal circle and F_j are the Cartan generators of the flavor symmetry group. Convergence requires that $|t| < 1$. These indices are precisely computed with respect to the charges described by the Killing spinors ε_1 and ε_2 respectively. It is very important to remark that all charges appearing in the index need to be non-anomalous – we will always assume this to be the case. From here onward, we will focus on the index I_1, knowing that I_2 can be dealt with completely similarly.

In the path integral formulation, the insertion of the chemical potentials in the trace leads to twisted boundary conditions on all fields

$$\Phi(\tau + \beta\ell) = e^{\beta\left(-2j_1+\frac{3}{2}R\right)}\ t^{-3(2j_1-R)}\ y^{-2j_2}\ \prod_j \zeta_j^{-F_j}\ \Phi(\tau) , \qquad (7.2.19)$$

which are indeed also the boundary conditions satisfied by the Killing spinor ε_1. Alternatively, one can turn on flat background gauge connections along the temporal circle

$$V_\mu = \left(0, 0, 0, i\left(\frac{3\beta - 6\xi}{2\beta\ell}\right)\right) , \qquad \tilde{V}^{(j)}_\mu = \left(0, 0, 0, \frac{\mathfrak{z}_j}{\beta\ell}\right) , \qquad (7.2.20)$$

for the R-symmetry and the flavor symmetry respectively. The twists by the rotational charges j_1 and j_2 furthermore impose the identification

$$(\varphi, \chi, \theta, \tau) \sim \left(\varphi + \frac{i}{2}(-2\beta + 6\xi + 2i\eta), \chi + \frac{i}{2}(-2\beta + 6\xi - 2i\eta), \theta, \tau + \beta\ell\right) . \qquad (7.2.21)$$

Introducing the coordinates

$$\hat{\varphi} = \varphi - \frac{i}{2}(-2\beta + 6\xi + 2i\eta)\frac{\tau}{\beta\ell} \ , \ \hat{\chi} = \chi - \frac{i}{2}(-2\beta + 6\xi - 2i\eta)\frac{\tau}{\beta\ell} \ , \ \hat{\theta} = \theta \ , \ \hat{\tau} = \tau \ , \tag{7.2.22}$$

the identification simplifies to $(\hat{\varphi}, \hat{\chi}, \hat{\theta}, \hat{\tau}) \sim (\hat{\varphi}, \hat{\chi}, \hat{\theta}, \hat{\tau} + \beta\ell)$. The metric in these hatted coordinates reads

$$\begin{aligned} ds^2 = \ell^2 \cos^2\hat{\theta} \ &\left(d\hat{\varphi} + \frac{i}{2\beta\ell}(-2\beta + 6\xi + 2i\eta) \ d\hat{\tau} \right)^2 + \\ + \ell^2 \sin^2\hat{\theta} \ &\left(d\hat{\chi} + \frac{i}{2\beta\ell}(-2\beta + 6\xi - 2i\eta) \ d\hat{\tau} \right)^2 + \ell^2 d\hat{\theta}^2 + d\hat{\tau}^2 \ , \end{aligned} \tag{7.2.23}$$

and is complexified. Its vielbeins are

$$e^1 = \ell\cos\hat{\theta} \ \left(d\hat{\varphi} + \frac{i}{2\beta\ell}(-2\beta + 6\xi + 2i\eta) \ d\hat{\tau} \right) \ , \qquad e^3 = \ell d\hat{\theta} \ , \tag{7.2.24}$$

$$e^2 = \ell\sin\hat{\theta} \ \left(d\hat{\chi} + \frac{i}{2\beta\ell}(-2\beta + 6\xi - 2i\eta) \ d\hat{\tau} \right) \ , \qquad e^4 = d\hat{\tau} \ , \tag{7.2.25}$$

while the dual frame vectors are given by

$$e_1 = \left(\ell\cos\hat{\theta}\right)^{-1} \ \partial_{\hat{\varphi}} \ , \qquad e_2 = \left(\ell\sin\hat{\theta}\right)^{-1} \ \partial_{\hat{\chi}} \ , \qquad e_3 = \ell^{-1}\partial_{\hat{\theta}} \ , \tag{7.2.26}$$

$$e_4 = \partial_{\hat{\tau}} - \frac{i}{2\beta\ell}(-2\beta + 6\xi + 2i\eta) \ \partial_{\hat{\varphi}} - \frac{i}{2\beta\ell}(-2\beta + 6\xi - 2i\eta) \ \partial_{\hat{\chi}} \ , \tag{7.2.27}$$

and the non-zero components of the spin connection read

$$\omega^{13} = -\sin\hat{\theta} \ \left(d\hat{\varphi} + \frac{i}{2\beta\ell}(-2\beta + 6\xi + 2i\eta) \ d\hat{\tau} \right) \tag{7.2.28}$$

$$\omega^{23} = \cos\hat{\theta} \ \left(d\hat{\chi} + \frac{i}{2\beta\ell}(-2\beta + 6\xi - 2i\eta) \ d\hat{\tau} \right) \ . \tag{7.2.29}$$

The solution to the Killing spinor equation $D_\mu\varepsilon = \gamma_\mu\tilde{\varepsilon}$ on the deformed background, corresponding to ε_1 in (7.2.10), is given by

$$\varepsilon_1 = \begin{pmatrix} \eta_{S^3}^{(+,+)} \\ \zeta_{S^3}^{(-,-)} \end{pmatrix} \ , \tag{7.2.30}$$

and satisfies $D_\mu\varepsilon_1 = \frac{1}{2\ell}\gamma_\mu\gamma_4\gamma_5\varepsilon_1$. The square of the supersymmetry variation equals

$$\delta^2_{\varepsilon_1} = -\frac{2}{\ell}\left[-\ell\ \mathcal{L}^A_{\partial_{\hat\tau}} + \frac{6i\xi}{2\beta}\ \mathcal{L}^A_{\partial_{\hat\varphi}+\partial_{\hat\chi}} + \frac{\eta}{\beta}\ \mathcal{L}^A_{\partial_{\hat\chi}-\partial_{\hat\varphi}} + \frac{3\xi}{\beta}R + \frac{i}{\beta}\sum_j \mathfrak{z}_j F_j\right] . \tag{7.2.31}$$

Thanks to pairwise cancellation, the index only receives contributions from states satisfying $\delta^2_{\varepsilon_1} = 0$. It is thus independent of the parameter β, and it will be convenient to choose it such that the metric (7.2.23) is real, namely $\beta = 3\xi$. From now on, we make this choice for β and further omit the hats.

Fayet-Iliopoulos term It is well known that both the gauge and the matter Lagrangian are $\mathcal{Q}$-exact.[4] However, if the gauge group contains an abelian factor[5], we can write down a Fayet-Iliopoulos term [143]. Indeed, if the Killing spinor satisfies $D_\mu\varepsilon_1 = \frac{1}{2\ell}\gamma_\mu\gamma_4\gamma_5\varepsilon_1$, then it is easy to convince oneself that $\delta_{\varepsilon_1}(D + \frac{2}{\ell}A_4) = D_\mu(\bar\varepsilon_1\gamma_5\gamma^\mu\lambda)$. When integrated over the compact space, the variation of $D + \frac{2}{\ell}A_4$ vanishes and thus results in an invariant action. Note however, that in order for the action to be invariant under large gauge transformations along the 4-direction the properly normalized FI parameter needs to be an integer. Due to its discrete nature it avoids the common lore that the index does not depend on continuous parameters.

7.3 The BPS equations

The BPS equations for the vectormultiplet of gauge group G are obtained by setting to zero the gaugino variation

$$0 = \delta_{\varepsilon_1}\lambda = -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\ \varepsilon_1 - \gamma_5\ D\ \varepsilon_1\ . \tag{7.3.1}$$

Upon solving the resulting four equations for $F_{14}, F_{24}, F_{34}, D$ one obtains

$$F_{14} = i\sin\theta\ F_{12}\ , \qquad F_{34} = -i\left(\cos\theta\ F_{13} + \sin\theta\ F_{23}\right)\ , \tag{7.3.2}$$

$$F_{24} = -i\cos\theta\ F_{12}\ , \qquad D = i\cos\theta\ F_{23} - i\sin\theta\ F_{13}\ . \tag{7.3.3}$$

[4] We use δ_{ϵ_1} and $\mathcal{Q}$ interchangeably.

[5] In the presence of an abelian factor, the theory develops a Landau pole. However, as was also argued in [74], one can exploit the independence of the index on the gauge coupling to suppress the Landau pole arbitrarily by making the gauge coupling smaller and smaller.

Declaring that all fields are real, immediately leads to the localization locus $F_{\mu\nu} = D = 0$. Flat connections on $S^3 \times S^1$ are given by $A = \frac{a}{3\xi\ell} d\tau$, for arbitrary holonomy a. Alternatively, we can obtain the localization equations as the zero-locus of the bosonic part of the deformation action

$$\mathcal{L}^{\text{def}}_{\text{YM}} = \frac{1}{4}\mathcal{Q}\ \text{Tr}\ (\mathcal{Q}\lambda)^{\ddagger}\lambda, \tag{7.3.4}$$

where the action of the formal hermitian conjugate $\ddagger$ operator on $\mathcal{Q}\lambda$ is

$$(\mathcal{Q}\lambda)^{\ddagger} = \frac{1}{2}\varepsilon_1^{\dagger}\ \gamma^{\mu\nu}F_{\mu\nu} - \varepsilon_1^{\dagger}\ \gamma_5\ D\ . \tag{7.3.5}$$

One then obtains for the bosonic piece

$$\frac{1}{4}\text{Tr}\ (\mathcal{Q}\lambda)^{\ddagger}\ \mathcal{Q}\lambda = \frac{1}{2}\text{Tr}\ \left(D^2 + \frac{1}{2}\sum_{m,n}(F_{mn})^2\right), \tag{7.3.6}$$

whose zero-locus is indeed $D = F_{mn} = 0$.

Higgs branch localization requires the addition of an extra $\mathcal{Q}-$exact deformation term

$$\mathcal{L}^{\text{def}}_{\text{H}} = \frac{i}{2}\mathcal{Q}\ \text{Tr}\ \varepsilon_1^{\dagger}\gamma_5\lambda\ H(\phi)\ , \tag{7.3.7}$$

whose bosonic part is

$$\mathcal{L}^{\text{def}}_{\text{H}}\Big|_{\text{bos}} = -\text{Tr}\ \left(iD + \cos\theta\ F_{23} - \sin\theta\ F_{13}\right)\ H(\phi)\ . \tag{7.3.8}$$

Upon adding $\mathcal{L}^{\text{def}}_{\text{YM}}$ and $\mathcal{L}^{\text{def}}_{\text{H}}$, the auxiliary field D can be integrated out exactly by performing the Gaussian path integral. Correspondingly, one imposes its field equation $D = iH(\phi)$. The auxiliary field D is thus taken out of its real contour. The bosonic part of the total deformation Lagrangian can then be written as a sum of squares once again:

$$\begin{aligned}\mathcal{L}^{\text{def}}_{\text{YM}}\Big|_{\text{bos}} + \mathcal{L}^{\text{def}}_{\text{H}}\Big|_{\text{bos}} = \frac{1}{2}\text{Tr}\ \big(&(F_{12})^2 + (F_{14})^2 + (F_{24})^2 + (F_{34})^2 + \\ &+(-H(\phi) - \sin\theta\ F_{13} + \cos\theta\ F_{23})^2 + (\cos\theta\ F_{13} + \sin\theta\ F_{23})^2\big)\ ,\end{aligned}$$

from which we read off the BPS equations

$$0 = F_{12} = F_{14} = F_{24} = F_{34} = -H(\phi) - \sin\theta\ F_{13} + \cos\theta\ F_{23} \tag{7.3.9}$$
$$= \cos\theta\ F_{13} + \sin\theta\ F_{23}\ . \tag{7.3.10}$$

Note that these equations could have been obtained equivalently from (7.3.2)-(7.3.3) by imposing the D-term equation. More explicitly, these equations read in the coordinate frame

$$
\begin{aligned}
0 &= F_{\varphi\chi} = F_{\varphi\tau} = F_{\chi\tau} \\
F_{\theta\tau} &= \frac{2\eta}{3\xi\ell} F_{\varphi\theta} = -\frac{2\eta}{3\xi\ell} F_{\chi\theta} \\
-\ell^2 H(\phi) &= \frac{F_{\varphi\theta}}{\sin\theta\ \cos\theta} = -\frac{F_{\chi\theta}}{\sin\theta\ \cos\theta} \ .
\end{aligned}
\tag{7.3.11}
$$

Let us next turn our attention to chiral multiplets. We take them to transform under some generic representation $\mathfrak{R}$ of the flavor and gauge group. Let us denote its decomposition in irreducible gauge representations as $\mathfrak{R} = \sum_i \mathcal{R}_i$. The BPS equations for a single chiral multiplet transforming in representation $\mathcal{R}$ are found by setting to zero each component of the variation of the fermion χ under the supersymmetry transformation by ε_1. Subsequently imposing the reality property $\phi^\dagger = \bar{\phi}$ and $\mathcal{F}^\dagger = \bar{\mathcal{F}}$ and taking appropriate linear combinations, one obtains

$$
\begin{aligned}
&0 = (D_4 - D_4^\dagger)\phi \qquad && 0 = \cos\theta\ D_2\phi - \sin\theta\ D_1\phi + iD_3\phi \\
&0 = \mathcal{F} && 0 = 3r\phi + \ell\left(2i(\cos\theta\ D_1\phi + \sin\theta\ D_2\phi) - (D_4 + D_4^\dagger)\phi\right)
\end{aligned}
\tag{7.3.12}
$$

Using that

$$
D_4\phi = \left(D_\tau - \frac{\eta}{3\xi\ell}(D_\chi - D_\varphi) + \frac{r}{2\ell} - \frac{i}{3\xi\ell}\mathfrak{z}\right)\phi \ , \tag{7.3.13}
$$

these equations can be written explicitly as (assuming the gauge field is real)

$$
0 = \left(D_\tau - \frac{\eta}{3\xi\ell}(D_\chi - D_\varphi) - \frac{i}{3\xi\ell}\mathfrak{z}\right)\phi \ , \tag{7.3.14}
$$

$$
0 = \mathcal{F} \ , \tag{7.3.15}
$$

$$
0 = r\phi + i(D_\varphi\phi + D_\chi\phi) \ , \tag{7.3.16}
$$

$$
0 = \cot\theta\ D_\chi\phi - \tan\theta\ D_\varphi\phi + iD_\theta\phi \ . \tag{7.3.17}
$$

7.4 BPS solutions: Coulomb, Higgs and vortices

In this section we set out to solve the BPS equations. Let us first recall the Coulomb branch solutions.

Coulomb branch The Coulomb branch solution was already mentioned above:

$$D = 0 \, , \qquad A = \frac{a}{3\xi\ell} d\tau \, . \tag{7.4.1}$$

As usual a can be taken to lie in the Cartan algebra. Let us verify that there are no solutions to the chiral multiplet equations for positive R-charges. Fourier expanding the chiral field as

$$\phi = \sum_{p,m,n} e^{2\pi i p\tau/3\xi\ell} \; e^{in\varphi} \; e^{im\chi} \; c_{pmn}(\theta) \, , \tag{7.4.2}$$

one finds from (7.3.14) that only modes for which

$$(a + \mathfrak{z})\phi = 2\pi p - \eta(m - n) \tag{7.4.3}$$

can exist. Via a large gauge transformation, we can set $p = 0$. Next, equation (7.3.16) further imposes that $r = n + m$. Finally, equation (7.3.17) reduces to the differential equation

$$\partial_\theta c_{pmn} = -(m \cot\theta - n \tan\theta) c_{pmn} \, , \tag{7.4.4}$$

which solves to $c_{pmn}(\theta) = \phi_0 (\cos\theta)^{-n} (\sin\theta)^{-m}$, for some constant ϕ_0 . Smoothness at $\theta = 0$ and $\theta = \frac{\pi}{2}$ demands that $m \leq 0$ and $n \leq 0$ respectively. Therefore, for positive R-charges, no solutions exist. For zero R-charge (then $m = n = 0$), we find the constant Higgs like solution $\phi = \phi_0$, if $(a + \mathfrak{z})\phi = 0$.

Next, we study the new solutions which become available upon choosing a non-trivial $H(\phi)$, *i.e.* we want to solve (7.3.9) and (7.3.12). We set the R-charges to zero, $r = 0$: the exact non-anomalous R-charge should be restored by giving an imaginary part to the flavor fugacities. We make the standard choice for $H(\phi)$:

$$H(\phi) = \zeta - \sum_{i,a} T^a_{\text{adj}} \; \phi_i^\dagger T^a_{\mathcal{R}_i} \phi_i \, , \tag{7.4.5}$$

where the sum runs over the matter representations $\mathcal{R}_i$ and its generators $T^a_{\mathcal{R}_i}$. Here, ζ is adjoint-valued and defined as a real linear combination of the Cartan generators h_a of the Abelian factors in the gauge group

$$\zeta = \sum_{a:\ U(1)} \zeta_a h_a \ . \tag{7.4.6}$$

We find the following classes of solutions.

Deformed Coulomb branch The deformed Coulomb branch is characterized by $\phi = 0$. A solution to the vector multiplet BPS equations (7.3.9) is then given by

$$F = \zeta \ell^2 \sin\theta \ \cos\theta \ d\theta \wedge \left(d\varphi - d\chi - \frac{2\eta}{3\xi\ell} d\tau \right) \ , \tag{7.4.7}$$

which can be integrated to

$$A = -\zeta\ell^2 \left(\frac{1}{2}\cos^2\theta \left(d\varphi - \eta\frac{d\tau}{3\xi\ell} \right) + \frac{1}{2}\sin^2\theta \left(d\chi + \eta\frac{d\tau}{3\xi\ell} \right) \right) + \frac{a}{3\xi\ell} d\tau \ . \tag{7.4.8}$$

Higgs-like solutions Higgs-like solutions are defined by setting $H(\phi) = 0$. Then it follows that also $F_{\mu\nu} = 0$. From above, we know that $\phi = \phi_0$ is a constant constrained by the condition $(a + \mathfrak{z})\phi_0 = 0$.

Solutions to the D-term equations

$$H(\phi) = 0 \ , \qquad (a + \mathfrak{z})\phi = 0 \ , \tag{7.4.9}$$

depend both on the gauge group and on the matter representations. Here we will restrict ourselves to cases where the vacuum expectation values of ϕ completely break the gauge group.

Vortices Each Higgs-like solution is the root of a tower of vortex solutions at the north and south torus. Indeed, using the other BPS equations, the BPS equations (7.3.17) and the last equation in (7.3.11) become for $\theta \to 0$, and introducing $R \equiv \ell\theta$,

$$0 = \left(D_R - \frac{i}{R} D_\chi \right) \phi \ , \qquad H(\phi) = -\frac{1}{R} F_{R\chi} \ , \tag{7.4.10}$$

which we recognize as the standard (anti)vortex equations on $\mathbb{R}^2$. Once the solutions to these equations are found, the other BPS equations will complete it to solutions on $\mathbb{R}^2 \times T^2$. The vortex equations cannot be solved analytically, so we shall content ourselves with qualitatively analyzing the behavior of the solutions. We consider the case of a $U(1)$ theory with a single chiral multiplet of (gauge) charge $+1$. Up to rescalings of the latter, this is the generic case once the gauge group is broken to its maximal torus. Let us start by making the Ansatz

$$\phi = e^{-in\varphi} e^{-im\chi} \phi_0(R) \ , \qquad A = A_\tau(R) d\tau + A_\varphi(R) d\varphi + A_\chi(R) d\chi \ , \quad (7.4.11)$$

where we didn't include a time dependence since it can be removed by the same large gauge transformation we employed earlier. When $\phi_0 \neq 0$ one finds from (7.3.14) and (7.3.17) the exact relations

$$A_\tau = \frac{1}{3\xi\ell}\Big(\eta\left((A_\chi + m) - (A_\varphi + n)\right) - \mathfrak{z}\Big) \ , \qquad A_\varphi + A_\chi = -(n+m) \ . \quad (7.4.12)$$

Given these exact relations, all BPS equations are satisfied except for the vortex equations (7.4.10) themselves:

$$\partial_R \phi_0 - \frac{1}{R}(m + A_\chi)\phi_0 = 0 \ , \qquad \zeta - \phi_0^2 = -\frac{1}{R}\partial_R A_\chi \ , \quad (7.4.13)$$

and moreover it is sufficient to outline the behavior of A_χ and ϕ. When $R \to 0$ (more precisely, for $R \ll \sqrt{\frac{m}{\zeta}}$), in order to have a smooth connection, one necessarily has $A_\chi \to 0$. The first equation then further implies that $\phi_0 = BR^m$. In particular we deduce that $m > 0$. From the second equation to leading order in R we deduce that $\partial_R A_\chi = -R\zeta$ and thus $A_\chi = -\frac{\zeta R^2}{2}$. For $R \to \infty$ ($R \gg \sqrt{\frac{m}{\zeta}}$), ϕ sits in its Higgs vacuum $\phi_0 \to \zeta$. Then one finds that $A_\chi \to -m$. Integrating the field strength over $\mathbb{R}^2$, one finds that m can be interpreted as the vortex number $\frac{1}{2\pi}\int F = -m$. When approximating $R^{-1}F_{R\chi}$ by a step function of height $-\zeta$, we immediately find a measure for the size of the vortex to be $\sqrt{\frac{m}{\zeta}}$. For sufficiently large values of ζ the vortex shrinks to zero size and the first order approximations we took are justified. Momentarily, we will give an interpretation to n as well.

It is noteworthy that A_τ only asymptotically sits in its Higgs vacuum: for $R \to 0$ one finds $A_\tau = \frac{1}{3\xi\ell}(2m\eta - \mathfrak{z}) - \frac{\eta}{3\xi\ell}\zeta R^2$.

One can similarly analyze the behavior for $\theta \to \frac{\pi}{2}$. Introducing $\rho = \ell\left(\frac{\pi}{2} - \theta\right)$, one again finds the vortex equations among the BPS equations

$$0 = \left(D_\rho - \frac{i}{\rho} D_\varphi\right)\phi \ , \qquad H(\phi) = -\frac{1}{\rho} F_{\rho\varphi} \ . \tag{7.4.14}$$

Let us also here analyze the qualitative behavior for the case of a $U(1)$ theory with a single chiral of charge $+1$. Starting by making the Ansatz

$$\phi = e^{-in\varphi} e^{-im\chi} \phi_0(\rho) \ , \qquad A = A_\tau(\rho) d\tau + A_\varphi(\rho) d\varphi + A_\chi(\rho) d\chi \ . \tag{7.4.15}$$

we rediscover the exact relations (7.4.12) which solve all BPS equations but

$$\partial_\rho \phi_0 - \frac{1}{\rho}(n + A_\varphi)\phi_0 = 0 \ , \qquad \zeta - \phi_0^2 = -\frac{1}{\rho} \partial_\rho A_\varphi \ . \tag{7.4.16}$$

For $\rho \ll \sqrt{\frac{n}{\zeta}}$, smoothness demands that $A_\varphi \to 0$. In this region we then find from the first equation that $\phi_0 = B' \rho^n$, implying that $n > 0$. To leading order in ρ the second equations teaches that $\partial_\rho A_\varphi = -\rho\zeta$ and thus $A_\varphi = -\frac{\zeta\rho^2}{2}$. For $\rho \gg \sqrt{\frac{n}{\zeta}}$, we have $\phi_0 \to \zeta$ and $A_\varphi \to -n$. Since integrating over $\mathbb{R}^2$ gives $\frac{1}{2\pi}\int F = -n$, we interpret n as the vortex number at the south torus.

Also here A_τ sits only asymptotically in its Higgs vacuum. Note also that in the intermediate region both solutions glue together appropriately.

For smaller values of ζ both the presence of curvature in and the finite volume of space will start affecting the solutions. However, we can derive an exact bound by integrating $H(\phi)$ over spacetime and using the last BPS equation in (7.3.11)

$$\begin{aligned} \zeta \mathrm{vol}(S^3 \times S^1) &\geq \int_{S^3 \times S^1} H(\phi) \ d\mathrm{vol}(S^3 \times S^1) \\ &= 4\pi^2 \ell \mathrm{vol}(S^1) \int_0^{\frac{\pi}{2}} d\theta \partial_\theta A_\varphi = -4\pi^2 \ell \mathrm{vol}(S^1) \int_0^{\frac{\pi}{2}} d\theta \partial_\theta A_\chi \end{aligned} \tag{7.4.17}$$

Here we used that on vortex solutions $0 \leq H(\phi) \leq \zeta$ and that vortex solutions don't have θ dependence. Defining the vorticities as the winding numbers of ϕ around χ, φ respectively and employing the analysis at the core of the vortex, we then find that

$$4\pi^2 \ell (n + m) \leq \zeta \mathrm{vol}(S^3) \Rightarrow n + m \leq \zeta \frac{\ell^2}{2} \ . \tag{7.4.18}$$

One observes that for finite values of ζ only a finite number of vortices are supported on $S^3 \times S^1$. The bound is saturated precisely when ϕ vanishes; the solution is then described by the deformed Coulomb branch solution.

We thus find essentially the same interpretation as in chapter 6. Upon increasing ζ from 0 to $+\infty$ the original Coulomb branch solution is deformed into the deformed Coulomb branch and each time ζ crosses a bound (7.4.18), a collection of new vortices branches out.

7.5 Computation of the index

In the previous section, we found various classes of BPS solutions. The final steps in the computation of the index using localization, are then to first evaluate the classical action on and the one-loop determinants of quadratic fluctuations around these solutions, and next integrate and/or sum over the space of BPS configurations.

7.5.1 One-loop determinants from an index theorem

Although the computation of the one-loop determinants can be straightforwardly performed on the Coulomb branch (in a Lagrangian theory like the ones at hand) by enumerating letters, constructing the single letter partition function, subsequently plethystically exponentiating these and finally imposing the Gauss law constraint by projecting onto gauge singlets, the computation on non-constant configurations is most easily performed using an equivariant index theorem for transversally elliptic operators [147]. The idea is to bring the problem in cohomological form, and make use of the fact that, via the equivariant index theorem, only the fixed points of the equivariant spatial rotations contribute to the one-loop determinants. A detailed discussion can be found in [14].

Recall from (7.2.31) that the supercharge squares to

$$\delta_{\varepsilon_1}^2 = -\frac{2}{3\xi\ell}\left[-3\xi\ell\ \mathcal{L}^A_{\partial_{\hat\tau}} + 6\xi i\ \mathcal{L}^A_{\frac{1}{2}(\partial_{\hat\varphi}+\partial_{\hat\chi})} + 2\eta\ \mathcal{L}^A_{\frac{1}{2}(\partial_{\hat\chi}-\partial_{\hat\varphi})} + 3\xi R + i\sum_j \mathfrak{z}_j F_j\right] \tag{7.5.1}$$

$$= -\frac{2}{3\xi\ell}\left[-3\xi\ell\ \mathcal{L}^A_{\partial_{\hat\tau}} + (3\xi i-\eta)\ \mathcal{L}^A_{\partial_{\hat\varphi}} + (3\xi i+\eta)\ \mathcal{L}^A_{\partial_{\hat\chi}} + 3\xi R + i\sum_j \mathfrak{z}_j F_j\right]. \tag{7.5.2}$$

where we used the value $\beta = 3\xi$. An important observation is that $\delta_{\varepsilon_1}^2$ precisely equals (upon properly identifying the equivariant parameters[6]) the square of the supercharge used in the localization on S_b^3 in [148] (see also 6) with an additional free motion along the temporal circle generated by $-3\xi\ell\ \mathcal{L}^A_{\partial_{\hat\tau}}$. Thus, taking into account the Kaluza-Klein modes along the temporal circle, the computation of the equivariant index on $S^3\times S^1$ can be effectively reduced to that on a squashed three-sphere. This latter computation was performed in [148] (see also appendix D) and involves a further reduction along the Hopf fiber. The base space of the double reduction, which is topologically a two-sphere, has two fixed points (one at $\theta=0$ which we call North and one at $\theta=\frac{\pi}{2}$ (South)) under the reduction of the spatial rotations appearing in $\delta_{\varepsilon_1}^2$. The equivariant index only receives contributions from these two points.

Introducing the equivariant parameter for gauge transformations

$$i\hat a = -3\xi\ell(-iA_\tau) + 3\xi i(-i(A_\varphi + A_\chi)) + \eta(-i(A_\chi - A_\varphi))\ , \tag{7.5.3}$$

we can now immediately write the one-loop determinant for the vector multiplet

$$Z^{\text{vector}}_{\text{1-loop}}\ \text{“}=\text{”}$$
$$\prod_{\substack{n,m\in\mathbb{Z}\\ \alpha\in\mathfrak{g}}}\left(\pi i n - \frac{i}{2}(3\xi i-\eta)m - \frac{i}{2}\alpha(\hat a_N)\right)^{1/2}\left(\pi i n - \frac{i}{2}(3\xi i+\eta)m - \frac{i}{2}\alpha(\hat a_S)\right)^{1/2}\ , \tag{7.5.4}$$

where $\alpha\in\mathfrak{g}$ denotes the roots of the gauge algebra $\mathfrak{g}$. Compared to the unregularized vector multiplet one-loop determinant on the squashed three-sphere an extra product over the integer n appears, which precisely captures

[6] The precise identifications between the equivariant parameters here and those on the squashed three-sphere (see for example expression (D.3.3) in 6) are $b = 3\xi i-\eta$, $b^{-1} = 3\xi i+\eta$ up to a constant rescaling.

the contribution of the Kaluza-Klein modes along the temporal circle. Regularizing the infinite products results in

$$
\begin{aligned}
&Z^{\text{vector}}_{\text{1-loop}} = \\
&\left[\left(t^3y^{-1}\ ;\ t^3y^{-1}\right)_\infty \left(t^3y\ ;\ t^3y\right)_\infty\right]^{\operatorname{rank}\mathfrak{g}} \prod_{\alpha\neq 0}\left(1-e^{i\alpha(\hat{a}_N)}\right)^{1/2}\left(1-e^{i\alpha(\hat{a}_S)}\right)^{1/2} \\
&\times \prod_{\alpha\neq 0}\left(t^3y^{-1}\ e^{i\alpha(\hat{a}_N)}\ ;\ t^3y^{-1}\right)_\infty \left(t^3y\ e^{i\alpha(\hat{a}_S)}\ ;\ t^3y\right)_\infty\ ,
\end{aligned}
\tag{7.5.5}
$$

in terms of the infinite q-Pochhammer symbol $(z,q)_\infty = \prod_{j=0}^\infty (1-zq^j)$. Using the standard plethystic exponential, it can be written as

$$
\begin{aligned}
Z^{\text{vector}}_{\text{1-loop}} &= \prod_{\alpha\neq 0}\left(1-e^{i\alpha(\hat{a}_N)}\right)^{1/2}\left(1-e^{i\alpha(\hat{a}_S)}\right)^{1/2} \\
&\times \text{P.E.}\left[-\frac{t^3y^{-1}}{1-t^3y^{-1}}\left(\operatorname{rank}\mathfrak{g}+\sum_{\alpha\neq 0}e^{i\alpha(\hat{a}_N)}\right)-\frac{t^3y}{1-t^3y}\left(\operatorname{rank}\mathfrak{g}+\sum_{\alpha\neq 0}e^{i\alpha(\hat{a}_S)}\right)\right]\ ,
\end{aligned}
\tag{7.5.6}
$$

For all BPS configurations we will consider $\hat{a}_N = \hat{a}_S = \hat{a}$. The vector multiplet one-loop determinant simplifies then further to

$$
\begin{aligned}
&Z^{\text{vector}}_{\text{1-loop}} = \\
&\prod_{\alpha\neq 0}\left(1-e^{i\alpha(\hat{a})}\right)\ \text{P.E.}\left[-\left(\frac{t^3y^{-1}}{1-t^3y^{-1}}+\frac{t^3y}{1-t^3y}\right)\left(\operatorname{rank}\mathfrak{g}+\sum_{\alpha\neq 0}e^{i\alpha(\hat{a})}\right)\right]\ .
\end{aligned}
\tag{7.5.7}
$$

Observing that $-\left(\frac{t^3y^{-1}}{1-t^3y^{-1}}+\frac{t^3y}{1-t^3y}\right) = \frac{2t^6-t^3(y+y^{-1})}{(1-t^3y^{-1})(1-t^3y)}$, one recognizes the single letter partition function of the vector multiplet [162]. Using that

$$
-\left(\frac{t^3y^{-1}}{1-t^3y^{-1}}+\frac{t^3y}{1-t^3y}\right) = 1-\frac{1-t^6}{(1-t^3y^{-1})(1-t^3y)}
$$

it can be written alternatively as [163]

$$
\begin{aligned}
&Z^{\text{vector}}_{\text{1-loop}} = \\
&= \left((t^3y\ ;\ t^3y)_\infty (t^3y^{-1}\ ;\ t^3y^{-1})_\infty\right)^{\operatorname{rank}\mathfrak{g}} \prod_{\substack{n,m\geq 0\\ \alpha\neq 0}} \frac{1-e^{i\alpha(\hat{a})}(t^3y)^n(t^3y^{-1})^m}{(1-e^{i\alpha(\hat{a})}(t^3y)^{n+1}(t^3y^{-1})^{m+1})} \\
&= \left((t^3y\ ;\ t^3y)_\infty (t^3y^{-1}\ ;\ t^3y^{-1})_\infty\right)^{\operatorname{rank}\mathfrak{g}} \prod_{\alpha\neq 0} \frac{1}{\Gamma(e^{i\alpha(\hat{a})}, t^3y, t^3y^{-1})}\ , \qquad (7.5.8)
\end{aligned}
$$

in terms of the standard elliptic gamma function

$$
\Gamma(z,p,q) = \prod_{j,k\geq 0} \frac{1-p^{j+1}q^{k+1}/z}{1-p^jq^kz}\ . \qquad (7.5.9)
$$

For the one-loop determinant of a chiral multiplet of R-charge r transforming in gauge representation $\mathcal{R}$ we find the unregularized expression

$$
\begin{aligned}
&Z^{\text{chiral}}_{\text{1-loop}}\ \text{``}=\text{''} \\
&\prod_{w\in\mathcal{R}} \prod_{\substack{n,m\in\mathbb{Z}\\ p\geq 0}} \frac{-\pi i n + \frac{i}{2}(3\xi i+\eta)m + \frac{i}{2}(3\xi i-\eta)(p+1) + \frac{3}{2}\xi r + \frac{i}{2}w(\hat{a}_S) + \frac{i}{2}\mathfrak{z}}{-\pi i n + \frac{i}{2}(3\xi i-\eta)m - \frac{i}{2}(3\xi i+\eta)p + \frac{3}{2}\xi r + \frac{i}{2}w(\hat{a}_N) + \frac{i}{2}\mathfrak{z}}\ , \\
&\qquad (7.5.10)
\end{aligned}
$$

where $w\in\mathcal{R}$ denotes the weights of the representation $\mathcal{R}$. Also here the extra contribution of the Kaluza-Klein modes along the temporal circle is given by the infinite product over the integer n.

When $\hat{a}_N = \hat{a}_S = \hat{a}$, it can be regularized to

$$
\begin{aligned}
Z^{\text{chiral}}_{\text{1-loop}} &= \prod_{w\in\mathcal{R}} \Gamma\left(t^{3r}e^{-iw(\hat{a})-i\mathfrak{z}}\ ,\ t^3y\ ,\ t^3y^{-1}\right) \qquad (7.5.11)\\
&= \prod_{w\in\mathcal{R}} \text{P.E.}\left[\frac{t^{3r}e^{-i\mathfrak{z}}\ e^{-iw(\hat{a})} - t^{3(2-r)}e^{i\mathfrak{z}}\ e^{iw(\hat{a})}}{(1-t^3y^{-1})(1-t^3y)}\right]\ , \qquad (7.5.12)
\end{aligned}
$$

where again one recognizes the correct single letter partition function [162].

7.5.2 Coulomb branch

Let us first briefly recall the Coulomb branch expression [162, 163]. As was mentioned before, both the gauge and matter Lagrangians are $\mathcal{Q}$-exact, and we only have to evaluate the Fayet-Iliopoulos term:

$$S_{FI} = \frac{-i\ell}{2\mathrm{vol}(S^3)}\mathrm{Tr}_{FI}\int_{S^3\times S^1}\left(D + \frac{2}{\ell}A_\tau\right)d\mathrm{vol}(S^3\times S^1) = -i\mathrm{Tr}_{FI}a\ . \quad (7.5.13)$$

The equivariant parameter for the gauge transformation $i\hat{a} = 3\xi i\ell A_\tau + 3\xi(A_\varphi + A_\chi) - i\eta(A_\chi - A_\varphi)$ simply gives $\hat{a}_N = \hat{a}_S = a$. The one-loop determinants (7.5.8) and (7.5.11) are thus

$$Z^{\text{vector}}_{\text{1-loop}} = \frac{\left((t^3y\ ;\ t^3y)_\infty(t^3y^{-1}\ ;\ t^3y^{-1})_\infty\right)^{\text{rank}\,\mathfrak{g}}}{\prod_{\alpha\neq 0}\Gamma(e^{i\alpha(a)}, t^3y, t^3y^{-1})}\ , \quad (7.5.14)$$

$$Z^{\text{chiral}}_{\text{1-loop}} = \prod_{w\in\mathcal{R}}\Gamma\left(t^{3r}e^{-iw(a)-i\mathfrak{z}}\ ,\ t^3y\ ,\ t^3y^{-1}\right)\ , \quad (7.5.15)$$

and the index can be computed by

$$I = \frac{1}{|\mathcal{W}|}\oint\left(\prod_{j=1}^{\text{rank}\,G}\frac{dz_j}{2\pi i z_j}\right)\ e^{i\mathrm{Tr}_{FI}a}\ Z_{\text{1-loop}}\ , \quad (7.5.16)$$

where $|\mathcal{W}|$ denotes the dimension of the Weyl group of the gauge group G, $z_j = e^{ia_j}$ and the integration contour is along the unit circle. Note that the quantized nature of the FI parameter can now be seen to ensure that the integrand remains meromorphic. We should also mention that the usual Vandermonde determinant cancels against the contribution of the gauge-fixing ghosts.

7.5.3 Deformed Coulomb branch

Next, we study the situation on the deformed Coulomb branch. Using $D = iH(\phi)$ which equals $i\zeta$ on this solution, we obtain for the classical action

$$S_{FI} = -i\mathrm{Tr}_{FI}\left(a + i\frac{3}{2}\xi\ell^2\zeta\right)\ , \quad (7.5.17)$$

where we also used that $A_\tau = \frac{1}{3\xi\ell}\left(a + \frac{\eta\ell^2}{2}\zeta\cos 2\theta\right)$. For the equivariant parameter $\hat{a}$ we find

$$\hat{a} = \hat{a}_N = \hat{a}_S = a + i\frac{3}{2}\xi\ell^2\zeta\ . \quad (7.5.18)$$

Both in the classical action and the one-loop determinants, the effect of the deformation is seen to be given by an imaginary shift of the holonomy variable $a \to a + i\frac{3}{2}\xi\ell^2\zeta$ or thus $z = e^{ia} \to z\, t^{\frac{3}{2}\ell^2\zeta}$. When used in the matrix integral (7.5.16), one effectively changes the radius of the integration contour. Indeed, since $t < 1$, one finds that the contour shrinks (grows) for $\zeta \to +\infty$ ($\zeta \to -\infty$). When turning on the deformation parameter ζ, the integral remains constant as long as no poles of the integrand are crossed. Moreover, one can understand by looking at the bound (7.4.18), that the jumps in the integral, which are equal to the residues of the crossed poles, are precisely the contributions of the newly available vortex configurations. We thus recover the same picture as was found in chapter 6 in three dimensions.

Of particular interest is the situation where the index is expressed only in terms of vortices. This can be achieved if there exists a certain limit for the parameters $\zeta^a \to \pm\infty$ such that the deformed Coulomb branch is suppressed. In view of the shrinking/growing contour, such suppression can be obtained heuristically if the residue at the origin or infinity vanishes.

7.5.4 Higgs branch and vortices

For finite values of the deformation parameters ζ^a, the deformed Coulomb branch contribution of the previous subsection is complemented by finite size vortex configurations satisfying the bound (7.4.18). Evaluation of their classical action can be done exactly in a gauge $A_\theta = 0$, using the BPS equation (7.3.11), the behavior of the vortices in their core and the exact relations (7.4.12). We then find

$$S_{FI} = -i\mathrm{Tr}_{FI}\left(3\xi i(n+m) - \mathfrak{z} + \eta(m-n)\right)\ , \tag{7.5.19}$$

where the vortex numbers m, n are GNO quantized elements of the coweight lattice. For the evaluation of the one-loop determinants, we first consider the contribution from the off-diagonal W-bosons and those chiral multiplets that do not acquire a vacuum expectation value. Their one-loop determinant is simply found by inserting the equivariant parameter evaluated on the vortex background

$$\hat{a} = \hat{a}_N = \hat{a}_S = -\mathfrak{z} + 3\xi i(m+n) + \eta(m-n)\ , \tag{7.5.20}$$

in the expressions for the one-loop determinants (7.5.8), with the contribution of the diagonal vector multiplets, *i.e.* $\left((t^3y\ ;\ t^3y)_\infty(t^3y^{-1}\ ;\ t^3y^{-1})_\infty\right)^{\mathrm{rank}\,\mathfrak{g}}$, removed, and (7.5.11). The rank $\mathfrak{g}$ chiral multiplets that do get a VEV are eaten by the diagonal vector multiplets, which in turn become massive, via the Higgs mechanism. As was explained in [16], the one-loop determinant of this paired system is found as the residue of the product of their one-loop determinants. In total one thus finds

$$Z^{\text{vector}}_{\text{1-loop}} = \frac{1}{\prod_{\alpha\neq 0} \Gamma\left(e^{i\alpha(a_H)} \ (t^3 y)^{\alpha(m)} \ (t^3 y^{-1})^{\alpha(n)}, t^3 y, t^3 y^{-1}\right)} , \tag{7.5.21}$$

and

$$\begin{aligned} Z^{\text{chiral}}_{\text{1-loop}} &= \left((t^3 y \ ; \ t^3 y)_\infty (t^3 y^{-1} \ ; \ t^3 y^{-1})_\infty\right)^{\operatorname{rank}\mathfrak{g}} \\ &\times \operatorname*{Res}_{a\to a_H} \prod_{w\in\mathcal{R}} \Gamma\left(t^{3r} e^{-iw(a)-i\mathfrak{z}} \ \left(t^3 y\right)^{-w(m)} \ \left(t^3 y^{-1}\right)^{-w(n)} \ , \ t^3 y \ , \ t^3 y^{-1}\right) , \end{aligned} \tag{7.5.22}$$

where a_H is the holonomy evaluated in its Higgs vacuum.

It is clear from (7.5.19), (7.5.21) and in particular (7.5.22), that when adding the contribution of the vortices satisfying the bound (7.4.18) to the deformed Coulomb branch integral, we precisely recover the original Coulomb branch expression, since they precisely contribute the residues of the crossed poles. Since the deformation parameters enter our analysis via a $\mathcal{Q}$-exact piece, such picture was expected.

Elliptic vortex partition function Let us now send the deformation parameters ζ^a to infinity in such a way that the contribution of the deformed Coulomb branch vanishes. The index is then described purely in terms of point-like vortices which wrap the torus and have arbitrary vortex numbers. The elliptic uplift of the standard vortex partition function [130] describes their total contribution and can be independently computed by considering the theory on $\mathbb{R}^2_\epsilon \times T^2_\tau$ in the Ω-background. The plane $\mathbb{R}^2$ is effectively compactified, since it is rotated as we go around either cycle of the torus. The resulting elliptic vortex partition function Z_{vortex} can depend on the rotational parameter ϵ, the complex structure of the torus τ, flavor fugacities g and a fugacity coupling to leftmoving fermion number. This is all the two dimensional analog of the elliptic instanton partition function obtained by studying the theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times T^2_\tau$, see for example [170].

In the computation of the partition function in this limit, there are three contributions to be considered. First, there is the classical action evaluated on the vortex configuration (7.5.19) which splits into an overall classical action

$$S_{FI} = -i\mathrm{Tr}_{FI}\left(a_H\right) \ , \tag{7.5.23}$$

and a weighting factor for the vortex partition functions

$$e^{-S_{\rm v}} = (t^3y)^{\mathrm{Tr}_{FI}m}, \qquad e^{-S_{\rm av}} = (t^3y^{-1})^{\mathrm{Tr}_{FI}n} \ . \tag{7.5.24}$$

Second, the contribution of the off-diagonal vectormultiplets and the chiral multiplets not taking on a vacuum expectation value is as in the Coulomb branch (7.5.14), but evaluated on the Higgs branch location, *i.e.* $a \to a_H$, and with the contribution of the diagonal vector multiplets removed. The contributions of the rank $\mathfrak{g}$ chiral multiplets acquiring a vacuum expectation value and the diagonal vector multiplets cancel each other. Third, there is the vortex partition function itself. Its parameters can be read off from (7.5.1):

$$\epsilon_N = 3\xi i + \eta \ , \ \tau_N = \frac{3\xi i - \eta}{2\pi} + i(-i) \ , \qquad g_N = a_H + \sum_j \mathfrak{z}_j F_j \ , \tag{7.5.25}$$

$$\epsilon_S = 3\xi i - \eta \ , \ \tau_S = \frac{3\xi i + \eta}{2\pi} + i(-i) \ , \qquad g_S = a_H + \sum_j \mathfrak{z}_j F_j \ . \tag{7.5.26}$$

The extra factor of i in the modular parameter is explained by the fact that in our setup $\Delta \sim \partial_\tau$ while the momenta are $P_\varphi, P_\chi \sim i\partial_\varphi, i\partial_\chi$. The final expression for the index as obtained by Higgs branch localization is thus

$$I = \sum_{\text{Higgs vacua}} e^{i\mathrm{Tr}_{FI}(a_H)} \ Z'_{\text{1-loop}} \ Z_{\rm v} \ Z_{\rm av} \ , \tag{7.5.27}$$

where the sum runs over solutions to the D-term equations (7.4.9) and the one-loop determinant excludes the chiral multiplets acquiring a VEV and the diagonal vector multiplets. Finally,

$$Z_{\rm v} = Z_{\rm vortex}\left((t^3y)^{\mathrm{Tr}_{FI}\cdot} \ ; \ t^3y, t^3y^{-1}, e^{i(a_H+\sum_j \mathfrak{z}_j F_j)}\right) \tag{7.5.28}$$

$$Z_{\rm av} = Z_{\rm vortex}\left((t^3y^{-1})^{\mathrm{Tr}_{FI}\cdot} \ ; \ t^3y^{-1}, t^3y, e^{i(a_H+\sum_j \mathfrak{z}_j F_j)}\right) \ . \tag{7.5.29}$$

Here the first argument encodes the weight of the vortex sum and is given as an exponentiated linear function on the gauge algebra, the second and third argument are the exponentiated rotational parameter, $e^{i\epsilon}$, and complex structure, $q = e^{2\pi i\tau}$, respectively, and the last argument is the exponentiated flavor equivariant parameter.

7.6 Matching the Coulomb branch expression

In this section we give some examples of how manipulating the Coulomb branch integral gives rise to our Higgs branch result (7.5.27).

Free chiral multiplet For completeness, let us first mention the factorization of the simplest theory, namely the free chiral. Its index was given in (7.5.11) and can be factorized as [144]

$$I = \Gamma\left(t^{3r}\zeta\ ,\ t^3 y\ ,\ t^3 y^{-1}\right) = \Gamma\left(t^{3r}\zeta, t^3 y^{-1}, t^6\right)\Gamma\left(t^{3r+3} y\zeta, t^3 y, t^6\right) \tag{7.6.1}$$

$$= \Gamma\left(t^{3r+3} y^{-1}\zeta, t^3 y^{-1}, t^6\right)\Gamma\left(t^{3r}\zeta, t^3 y, t^6\right)\ . \tag{7.6.2}$$

U(1) gauge theory Next, we consider the example of a $U(1)$ gauge theory with an equal number N of fundamental and antifundamental chiral multiplets, which is necessary to cancel the $U(1)_{\text{gauge}}\, U(1)_{\text{gauge}}\, U(1)_{\text{gauge}}$ anomaly. The $U(1)_R\; U(1)_R\; U(1)_{\text{gauge}}$ anomaly then also cancels. The non-anomalous R-charge assignment is determined by requiring the $U(1)_R\, U(1)_{\text{gauge}}\, U(1)_{\text{gauge}}$ anomaly to vanish. This anomaly is obviously proportional to the R-charge of the chiral fermion, namely $r-1$, which implies that one should take $r=1$. Note that these are not the superconformal R-charges of the free IR theory, which equal $r=\frac{2}{3}$.

The matrix integral (7.5.16) reads explicitly

$$I = (p,p)_\infty\ (q,q)_\infty \times$$
$$\oint \frac{dz}{2\pi i z} z^{\xi_{FI}} \prod_{\alpha=1}^{N} \Gamma(z^{-1}\zeta_\alpha (pq)^{r/2}, p, q)\ \prod_{\beta=1}^{N} \Gamma(z\tilde{\zeta}_\beta^{-1}(pq)^{r/2}, p, q)\ , \tag{7.6.3}$$

where we introduced the notation that $p = t^3 y$ and $q = t^3 y^{-1}$. We introduced fugacities ζ_α and $\tilde{\zeta}_\beta$ for the $SU(N)\times SU(N)$ flavor symmetry. For notational simplicity, let us absorb the R-charges in the flavor fugacities as $Z_\alpha = \zeta_\alpha (pq)^{r/2}$ and $\tilde{Z}_\beta^{-1} = \tilde{\zeta}_\beta^{-1}(pq)^{r/2}$.

The fundamentals contribute zeros at $z = p^{-\kappa-1} q^{-\lambda-1} Z_\gamma$ and poles at $z = p^\kappa q^\lambda Z_\gamma$. The antifundamentals have zeros at $z = p^{\kappa+1} q^{\lambda+1}\tilde{Z}_\delta$ and poles at $z = p^{-\kappa} q^{-\lambda}\tilde{Z}_\delta$. Picking up the poles inside the unit circle[7], *i.e.* the poles

[7] Here and in the next examples we are not careful about the pole at the origin. If it has a non-zero residue, it would give rise to a not completely suppressed deformed Coulomb branch contribution.

arising from the fundamentals, we obtain using the formulas in appendix E.3

$$I = \sum_{\gamma=1}^{N} Z_\gamma^{\xi_{FI}} \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{N} \Gamma(Z_\gamma^{-1} Z_\alpha, p, q) \prod_{\beta=1}^{N} \Gamma(Z_\gamma \tilde{Z}_\beta^{-1}, p, q) \sum_{\kappa,\lambda\geq 0} (p^\kappa q^\lambda)^{\xi_{FI}} (pq)^{\kappa\lambda N}$$
$$\times \prod_{\alpha=1}^{N} (\tilde{Z}^{-1} Z_\alpha)^{-\kappa\lambda} \frac{\prod_{\beta=1}^{N}\prod_{j=0}^{\lambda-1} \theta(q^j Z_\gamma \tilde{Z}_\beta^{-1}, p)\ \prod_{i=0}^{\kappa-1} \theta(p^i Z_\gamma \tilde{Z}_\beta^{-1}, q)}{\prod_{\alpha=1}^{N}\prod_{j=1}^{\lambda} \theta(q^{-j} Z_\gamma^{-1} Z_\alpha, p)\ \prod_{i=1}^{\kappa} \theta(p^{-i} Z_\gamma^{-1} Z_\alpha, q)}\ . \quad (7.6.4)$$

The intertwining factor vanishes as expected when reinstating the non-anomalous R-charges,

$$(pq)^N \prod_{\alpha=1}^{N} (\tilde{Z}^{-1} Z_\alpha)^{-1} = \left((pq)^{1-r}\right)^N = 1, \quad (7.6.5)$$

where we used that $\prod_\alpha \zeta_\alpha = \prod_\alpha \tilde{\zeta}_\alpha = 1$. We then find

$$I = \sum_\gamma Z_{\text{cl}}^{(\gamma)}\ Z'^{(\gamma)}_{\text{1-loop}}\ Z_{\text{v}}^{(\gamma)}\ Z_{\text{av}}^{(\gamma)}, \quad (7.6.6)$$

where the classical and one-loop contribution are given by

$$Z_{\text{cl}}^{(\gamma)} = Z_\gamma^{\xi_{FI}} \quad (7.6.7)$$
$$Z'^{(\gamma)}_{\text{1-loop}} = \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{N} \Gamma(Z_\gamma^{-1} Z_\alpha, p, q) \prod_{\beta=1}^{N} \Gamma(Z_\gamma \tilde{Z}_\beta^{-1}, p, q)\ . \quad (7.6.8)$$

The vortex contributions can be written as

$$Z_{\text{v}}^{(\gamma)} = Z_{\text{vortex}}^{(\gamma)}\left(p^{\xi_{FI}}\ ;\ p, q, \zeta_\alpha, \tilde{\zeta}_\beta\right)\ , \qquad Z_{\text{av}}^{(\gamma)} = Z_{\text{vortex}}^{(\gamma)}\left(q^{\xi_{FI}}\ ;\ q, p, \zeta_\alpha, \tilde{\zeta}_\beta\right)\ , \quad (7.6.9)$$

in terms of the vortex membrane partition function

$$Z_{\text{vortex}}^{(\gamma)}\left(L\ ;\ e^{i\epsilon}, q = e^{2\pi i\tau}, a_\alpha, b_\beta\right) =$$
$$\sum_{\kappa\geq 0} L^\kappa \frac{\prod_{j=0}^{\kappa-1}\prod_{\beta=1}^{N} \theta((e^{i\epsilon})^j\ A_\gamma B_\beta^{-1}, q)}{\prod_{j=1}^{\kappa} \theta((e^{i\epsilon})^{-j}, q) \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{N} \theta((e^{i\epsilon})^{-j}\ A_\gamma^{-1}\ A_\alpha, q)}\ , \quad (7.6.10)$$

where $A_\alpha = a_\alpha\,(e^{i\epsilon} q)^{\frac{1}{2}}$ and $B_\beta = b_\beta\,(e^{i\epsilon} q)^{-\frac{1}{2}}$.

U(N) gauge theory For a $U(N_c) = U(1) \times SU(N_c)$ gauge theory with $N_f = N_a = N$ fundamentals and antifundamentals, we should check cancellation of two potential anomalies, namely the $U(1)_{\text{gauge}}\, U(1)_{\text{gauge}}\, U(1)_R$ anomaly and the $SU(N_c)\, SU(N_c)\, U(1)_R$ anomaly. The $U(1)_R\, U(1)_R\, U(1)_{\text{gauge}}$ anomaly cancels thanks to $N_f = N_a$. While the first anomaly is again proportional to $r - 1$, and thus imposes that $r = 1$, the second one leads to the usual R-charge assignment $r = \frac{N_f - N_c}{N_f}$. These are not compatible for $N_c \neq 0$. One should thus not hope to achieve factorization in a $U(N_c)$ theory with only fundamentals and antifundamentals. One resolution, also used in two dimensions [113, 115], might be to add extra matter to cancel the anomaly. We will not pursue this resolution here.

Associated Cartan theory At first sight, Higgs branch localization breaks down in the absence of an abelian factor in the gauge group since one cannot introduce the Fayet-Iliopoulos parameter ζ of (7.4.6), which played such an essential role. However, we will now argue that one can associate to any theory with gauge group G a theory with gauge group $U(1)^{\text{rank}\,\mathfrak{g}}$ with equal index up to numerical and other holonomy independent factors. A similar observation was made in [171] for the two-sphere partition function. This associated Cartan theory can be subjected to Higgs branch localization.

First, one remarks that the integration measure of the matrix integral (7.5.16) for gauge group G is naturally equal to that of $U(1)^{\text{rank}\,\mathfrak{g}}$ up to the numerical prefactor $|\mathcal{W}|^{-1}$. Next, the one-loop determinant of a chiral field in gauge representation $\mathcal{R}$ of G can be equivalently thought of as the product of one-loop determinants of chiral fields with $U(1)^{\text{rank}\,\mathfrak{g}}$ charges determined by the weights $w \in \mathcal{R}$. Finally, using the simple observation that the one-loop determinant of the vector multiplet can be rewritten as

$$Z^{\text{vector}}_{\text{1-loop}} = \frac{\left((t^3y\ ;\ t^3y)_\infty (t^3y^{-1}\ ;\ t^3y^{-1})_\infty\right)^{\text{rank}\,\mathfrak{g}}}{\prod_{\alpha\neq 0} \Gamma(e^{i\alpha(\hat{a})}, t^3y, t^3y^{-1})} \tag{7.6.11}$$

$$= \left((t^3y\ ;\ t^3y)_\infty (t^3y^{-1}\ ;\ t^3y^{-1})_\infty\right)^{\text{rank}\,\mathfrak{g}} \prod_{\alpha\neq 0} \Gamma(t^6 e^{-i\alpha(\hat{a})}, t^3y, t^3y^{-1})\ , \tag{7.6.12}$$

where we used the elliptic gamma function identity $\Gamma(z,p,q)\,\Gamma(pq/z,p,q) = 1$, one can equivalently think of the vector one-loop determinant (up to a holonomy independent prefactor) as the product of one-loop determinants of chiral fields with $U(1)_R$ charge equal to two and with $U(1)^{\text{rank}\,\mathfrak{g}}$ charges determined by the non-zero roots $\alpha \neq 0$.

SU(2) gauge theory Let us finally then consider the simplest physically relevant example, namely an $SU(2)$ gauge theory with $N_f = N_a = N$ fundamental and antifundamental chiral multiplets. The argument presented above, indicates that factorization can be achieved provided that the R-symmetry is not anomalous, *i.e.* if we use the well-known non-anomalous R-charge assignment $r = \frac{N_f - N_c}{N_f} = \frac{N-2}{N}$.

The index is computed by

$$
\begin{aligned}
I = \frac{1}{2}(p,p)_\infty\ (q,q)_\infty \oint \frac{dz}{2\pi i z} & \frac{1}{\Gamma(z^2,p,q)\Gamma(z^{-2},p,q)} \\
& \times \prod_{\alpha=1}^{N} \Gamma(z^{-1}\zeta_\alpha (pq)^{r/2},p,q)\ \Gamma(z\zeta_\alpha (pq)^{r/2},p,q) \\
& \times \prod_{\beta=1}^{N} \Gamma(z\tilde{\zeta}_\beta^{-1} (pq)^{r/2},p,q)\ \Gamma(z^{-1}\tilde{\zeta}_\beta^{-1} (pq)^{r/2},p,q) \\
= \frac{1}{2}(p,p)_\infty\ (q,q)_\infty \oint \frac{dz}{2\pi i z} & \frac{1}{\Gamma(z^2,p,q)\Gamma(z^{-2},p,q)} \\
& \times \prod_{A=1}^{2N} \Gamma(z^{-1}Y_A,p,q)\ \Gamma(zY_A,p,q)\ ,
\end{aligned}
$$

where we introduced fugacities $\zeta_\alpha, \tilde{\zeta}_\beta$ for the $SU(N) \times SU(N)$ flavor symmetry. Since the fundamental representation of $SU(2)$ is pseudoreal, we get an enhanced flavor symmetry, with fugacities $Z_A = (\zeta_\alpha, \tilde{\zeta}_\beta^{-1})$. Finally, we introduced $Y_A = Z_A (pq)^{r/2}$.

The poles from the one factor of the vectormultiplet cancel against the zeros of the other factor and vice versa. The integrand further has zeros at $z = p^{-\kappa-1} q^{-\lambda-1} Y_B$ and $z = p^{\kappa+1} q^{\lambda+1} Y_C^{-1}$ and poles at $z = p^\kappa q^\lambda Y_B$ and $z = p^{-\kappa} q^{-\lambda} Y_C^{-1}$. Picking up the poles inside the unit circle, *i.e.* the poles at

$z = p^\kappa q^\lambda Y_B$, we obtain using the formulas in appendix E.3

$$
\begin{aligned}
I = \frac{1}{2} \sum_{B=1}^{2N} & \frac{\prod_{A=1}^{2N} \Gamma(Y_B Y_A;\ p,q) \prod_{\substack{A=1 \\ A\neq B}}^{2N} \Gamma(Y_B^{-1} Y_A\ ;\ p,q)}{\Gamma(Y_B^2\ ;\ p,q)\Gamma(Y_B^{-2}\ ;\ p,q)} \\
& \times \sum_{\kappa,\lambda \geq 0} (pq)^{-2\kappa\lambda(2-N)} \prod_{A=1}^{2N} (Y_A)^{-2\kappa\lambda} \\
& \times \frac{\prod_{j=1}^{2\lambda} \theta(q^{-j} Y_B^{-2}, p) \prod_{i=1}^{2\kappa} \theta(p^{-i} Y_B^{-2}, q)}{\prod_{j=0}^{2\lambda-1} \theta(q^j Y_B^2, p) \prod_{i=0}^{2\kappa-1} \theta(p^i Y_B^2, q)} \\
& \times \prod_{A=1}^{2N} \frac{\prod_{j=0}^{\lambda-1} \theta(q^j Y_A Y_B, p) \prod_{i=0}^{\kappa-1} \theta(p^i Y_A Y_B, q)}{\prod_{j=1}^{\lambda} \theta(q^{-j} Y_B^{-1} Y_A, p) \prod_{i=1}^{\kappa} \theta(p^{-i} Y_B^{-1} Y_A, q)} \ .
\end{aligned}
$$

Note now that the intertwining factor as expected disappears for the correct non-anomalous R-charges: $(pq)^{-2(2-N)} \prod_A (Y_A)^{-2} = (pq)^{-2(2-N+RN)} = 1$ where we used that $\prod_A Z_A = 1$. We thus find complete factorization

$$
I = \frac{1}{2} \sum_{B=1}^{2N} Z'^{(B)}_{\text{1-loop}}\ Z^{(B)}_{\text{v}}\ Z^{(B)}_{\text{av}}, \tag{7.6.13}
$$

where the one-loop contribution is

$$
Z'^{(B)}_{\text{1-loop}} = \frac{\prod_{\substack{A=1 \\ A\neq B}}^{2N} \Gamma(Y_B Y_A, p, q) \Gamma(Y_B^{-1} Y_A, p, q)}{\Gamma(Y_B^{-2}, p, q)} \tag{7.6.14}
$$

and the vortex partition functions are given by

$$
Z^{(B)}_{\text{v}} = Z^{(B)}_{\text{vortex}}(p, q, Z_A)\ , \qquad Z^{(B)}_{\text{av}} = Z^{(B)}_{\text{vortex}}(q, p, Z_A)\ . \tag{7.6.15}
$$

Here the vortex membrane partition function is given by

$$
\begin{aligned}
& Z^{(B)}_{\text{vortex}}(p, q, Z_A) = \\
& \sum_{\kappa \geq 0} \frac{\prod_{i=1}^{2\kappa} \theta(p^{-i} Y_B^{-2}, q)}{\prod_{i=\kappa}^{2\kappa-1} \theta(p^i Y_B^2, q)} \frac{1}{\prod_{i=1}^{\kappa} \theta(p^{-i}, q)} \prod_{\substack{A=1 \\ A\neq B}}^{2N} \frac{\prod_{i=0}^{\kappa-1} \theta(p^i Y_A Y_B, q)}{\prod_{i=1}^{\kappa} \theta(p^{-i} Y_B^{-1} Y_A, q)}\ ,
\end{aligned} \tag{7.6.16}
$$

where $Y_A = Z_A (pq)^{\frac{N-2}{2N}}$.

The generalization of this result to $SU(N)$ gauge group is technically more involved, but is expected to take on a factorized form as well.

Chapter 8

Conclusions

In this part we have extended the Higgs branch localization framework of [15] to three-dimensional $\mathcal{N} = 2$ R-symmetric theories on S^3_b and $S^2 \times S^1$, and to four-dimensional $\mathcal{N} = 1$ theories on $S^3 \times S^1$. We expect the method to work on much more general 3d backgrounds. We also expect to be able to perform Higgs branch localization on four-dimensional $\mathcal{N} = 1$ theories on manifolds like $S^2 \times T^2$. Even more generally, the method should work for theories with 8 supercharges, for instance in 4 and 5 dimensions.

Higgs branch localization expresses the partition function in terms of the K-theoretic/elliptic vortex partition function (VPF), which could also be computed in the Ω-background [131, 140, 130]. In fact, the partition function on different three-dimensional geometries—like S^3_b and $S^2 \times S^1$—is controlled by the very same VPF, with different identifications of the parameters. This has been extensively elaborated upon in [17]. A similar feature is expected to hold for four-dimensional $\mathcal{N} = 1$ theories.

In the special case of three-dimensional QCD-like theories, *i.e.* $U(N)_k$ gauge theories with N_f fundamentals and N_a antifundamentals, we have noticed that the S^3 and $S^2 \times S^1$ partition functions factorize into VPFs only for $|k| \leq \frac{|N_f - N_a|}{2}$, a fact that apparently has been overlooked before. It is a natural question to understand factorization beyond such bound.

It might be worth studying more in detail aspects of the 3d VPF. For instance, 3d mirror symmetry maps particles to vortices [153] and it would be interesting to understand its action on the VPF. Through the mirror map [172] between star-shaped quivers and the 3d reduction of class-S theories [10, 11], this might shed more light on the latter. Similarly, the study of the modular properties of the 4d VPF is of natural interest.

The 3d VPF encodes (equivariant) geometrical information about the Higgs branch of the theory. It might be interesting to investigate how the VPF captures the quantum moduli space [173, 174, 175, 176] of Chern-Simons-matter quiver theories arising from M2-branes at Calabi-Yau fourfold singularities.

Part III

Appendices

Appendix A

Infinite Chiral Symmetry in Four Dimensions

A.1 Superconformal algebras

This appendix lists useful superconformal algebras that are used in chapter 2. We adopt the convention of working in terms of the complexified version of symmetry algebras. We adopt bases for the complexified algebras such that the restriction to the real form that is relevant for physics in Lorentzian signature is the most natural. In general, the structures described in chapter 2 are insensitive to the spacetime signature of the four-dimensional theory, with the caveat that we will assume that the theories in question, when Wick rotated to Lorentzian signature, are unitary.

A.1.1 The four-dimensional superconformal algebra

The spacetime symmetry algebra for $\mathcal{N}=2$ superconformal field theories in four dimensions is the superalgebra $\mathfrak{sl}(4\,|\,2)$. The maximal bosonic subalgebra is $\mathfrak{so}(6,\mathbb{C})\times\mathfrak{sl}(2)_R\times\mathbb{C}^*$. The $\mathfrak{so}(6,\mathbb{C})$ conformal algebra, in a spinorial basis with $\alpha, \dot\alpha = 1, 2$, is given by

$$\begin{aligned}
[\mathcal{M}_\alpha{}^\beta, \mathcal{M}_\gamma{}^\delta] &= \delta_\gamma{}^\beta \mathcal{M}_\alpha{}^\delta - \delta_\alpha{}^\delta \mathcal{M}_\gamma{}^\beta \ , \\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \mathcal{M}^{\dot\gamma}{}_{\dot\delta}] &= \delta^{\dot\alpha}{}_{\dot\delta} \mathcal{M}^{\dot\gamma}{}_{\dot\beta} - \delta^{\dot\gamma}{}_{\dot\beta} \mathcal{M}^{\dot\alpha}{}_{\dot\delta} \ , \\
[\mathcal{M}_\alpha{}^\beta, \mathcal{P}_{\gamma\dot\gamma}] &= \delta_\gamma{}^\beta \mathcal{P}_{\alpha\dot\gamma} - \tfrac{1}{2}\delta_\alpha{}^\beta \mathcal{P}_{\gamma\dot\gamma} \ , \\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \mathcal{P}_{\gamma\dot\gamma}] &= \delta^{\dot\alpha}{}_{\dot\gamma} \mathcal{P}_{\gamma\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} \mathcal{P}_{\gamma\dot\gamma} \ , \\
[\mathcal{M}_\alpha{}^\beta, \mathcal{K}^{\dot\gamma\gamma}] &= -\,\delta_\alpha{}^\gamma \mathcal{K}^{\dot\gamma\beta} + \tfrac{1}{2}\delta_\alpha{}^\beta \mathcal{K}^{\dot\gamma\gamma} \ , \\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \mathcal{K}^{\dot\gamma\gamma}] &= -\,\delta^{\dot\gamma}{}_{\dot\beta} \mathcal{K}^{\dot\alpha\gamma} + \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} \mathcal{K}^{\dot\gamma\gamma} \ , \\
[\mathcal{H}, \mathcal{P}_{\alpha\dot\alpha}] &= \mathcal{P}_{\alpha\dot\alpha} \ , \\
[\mathcal{H}, \mathcal{K}^{\dot\alpha\alpha}] &= -\,\mathcal{K}^{\dot\alpha\alpha} \ , \\
[\mathcal{K}^{\dot\alpha\alpha}, \mathcal{P}_{\beta\dot\beta}] &= \delta_\beta{}^\alpha \delta^{\dot\alpha}{}_{\dot\beta} \mathcal{H} + \delta_\beta{}^\alpha \mathcal{M}^{\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta} \mathcal{M}_\beta{}^\alpha \ .
\end{aligned} \tag{A.1.1}$$

The $\mathfrak{sl}(2)_R$ algebra has a Chevalley basis of generators $\mathcal{R}^\pm$ and $\mathcal{R}$, where

$$[\mathcal{R}^+, \mathcal{R}^-] = 2\mathcal{R} \ , \qquad [\mathcal{R}, \mathcal{R}^\pm] = \pm\mathcal{R}^\pm \ . \tag{A.1.2}$$

In Lorentz signature where the appropriate real form of this algebra is $\mathfrak{su}(2)_R$, these generators will obey hermiticity conditions $(\mathcal{R}^+)^\dagger = \mathcal{R}^-$, $\mathcal{R}^\dagger = \mathcal{R}$. The generator of the Abelian factor $\mathbb{C}^*$ is denoted by r and is central in the bosonic part of the algebra. It is also convenient to introduce the basis $\mathcal{R}^\mathcal{I}{}_\mathcal{J}$, with

$$\mathcal{R}^1{}_2 = \mathcal{R}^+ \ , \qquad \mathcal{R}^2{}_1 = \mathcal{R}^- \ , \qquad \mathcal{R}^1{}_1 = \frac{1}{2}r + \mathcal{R} \ , \qquad \mathcal{R}^2{}_2 = \frac{1}{2}r - \mathcal{R} \ , \tag{A.1.3}$$

where we follow the conventions of [45] for r, and which obey the commutation relations

$$[\mathcal{R}^\mathcal{I}{}_\mathcal{J}, \mathcal{R}^\mathcal{K}{}_\mathcal{L}] = \delta^\mathcal{K}{}_\mathcal{J} \mathcal{R}^\mathcal{I}{}_\mathcal{L} - \delta^\mathcal{I}{}_\mathcal{L} \mathcal{R}^\mathcal{K}{}_\mathcal{J} \ . \tag{A.1.4}$$

There are sixteen fermionic generators in this superconformal algebra – eight Poincaré supercharges and eight conformal supercharges – denoted $\{\mathcal{Q}^\mathcal{I}_\alpha,\ \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha},\ \mathcal{S}^\alpha_\mathcal{J},\ \widetilde{\mathcal{S}}^{\mathcal{J}\dot\alpha}\}$. The nonvanishing commutators amongst them are as follows,

$$\begin{aligned}
\{\mathcal{Q}^\mathcal{I}_\alpha,\ \widetilde{\mathcal{Q}}_{\mathcal{J}\dot\alpha}\} &= \delta^\mathcal{I}{}_\mathcal{J} \mathcal{P}_{\alpha\dot\alpha} \ , \\
\{\widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha},\ \mathcal{S}_\mathcal{J}{}^\alpha\} &= \delta^\mathcal{I}{}_\mathcal{J} \mathcal{K}^{\dot\alpha\alpha} \ , \\
\{\mathcal{Q}^\mathcal{I}_\alpha,\ \mathcal{S}_\mathcal{J}{}^\beta\} &= \tfrac{1}{2}\delta^\mathcal{I}{}_\mathcal{J} \delta_\alpha{}^\beta \mathcal{H} + \delta^\mathcal{I}{}_\mathcal{J} \mathcal{M}_\alpha{}^\beta - \delta_\alpha{}^\beta \mathcal{R}^\mathcal{I}{}_\mathcal{J} \ , \\
\{\tilde{\mathcal{S}}^{\mathcal{I}\dot\alpha},\ \tilde{\mathcal{Q}}_{\mathcal{J}\dot\beta}\} &= \tfrac{1}{2}\delta^\mathcal{I}{}_\mathcal{J} \delta^{\dot\alpha}{}_{\dot\beta} \mathcal{H} + \delta^\mathcal{I}{}_\mathcal{J} \mathcal{M}^{\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta} \mathcal{R}^\mathcal{I}{}_\mathcal{J} \ .
\end{aligned} \tag{A.1.5}$$

Finally, the commutators of the supercharges with the bosonic symmetry generators are the following:

$$
\begin{aligned}
[\mathcal{M}_\alpha{}^\beta, \mathcal{Q}^{\mathcal{I}}_\gamma] &= \delta_\gamma{}^\beta \mathcal{Q}^{\mathcal{I}}_\alpha - \tfrac{1}{2}\delta_\alpha{}^\beta \mathcal{Q}^{\mathcal{I}}_\gamma \ , \\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\delta}] &= \delta^{\dot\alpha}{}_{\dot\delta} \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\delta} \ , \\
[\mathcal{M}_\alpha{}^\beta, \mathcal{S}_{\mathcal{I}}{}^\gamma] &= -\,\delta_\alpha{}^\gamma \mathcal{S}_{\mathcal{I}}{}^\beta + \tfrac{1}{2}\delta_\alpha{}^\beta \mathcal{S}_{\mathcal{I}}{}^\gamma \ , \\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \widetilde{\mathcal{S}}^{\mathcal{I}\dot\gamma}] &= -\,\delta^{\dot\gamma}{}_{\dot\beta} \widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha} + \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} \widetilde{\mathcal{S}}^{\mathcal{I}\dot\gamma} \ , \\
[\mathcal{H}, \mathcal{Q}^{\mathcal{I}}_\alpha] &= \tfrac{1}{2}\mathcal{Q}^{\mathcal{I}}_\alpha \ , \\
[\mathcal{H}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha}] &= \tfrac{1}{2}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha} \ , \\
[\mathcal{H}, \mathcal{S}_{\mathcal{I}}{}^\alpha] &= -\,\tfrac{1}{2}\mathcal{S}_{\mathcal{I}}{}^\alpha \ , \\
[\mathcal{H}, \tilde{\mathcal{S}}^{\mathcal{I}\dot\alpha}] &= -\,\tfrac{1}{2}\tilde{\mathcal{S}}^{\mathcal{I}\dot\alpha} \ , \\
[\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}, \mathcal{Q}^{\mathcal{K}}_\alpha] &= \delta_{\mathcal{J}}^{\ \mathcal{K}} \mathcal{Q}^{\mathcal{I}}_\alpha - \frac{1}{4}\delta^{\mathcal{I}}_{\mathcal{J}} \mathcal{Q}^{\mathcal{K}}_\alpha \ , \\
[\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}, \widetilde{\mathcal{Q}}_{\mathcal{K}\dot\alpha}] &= -\,\delta_{\mathcal{K}}^{\ \mathcal{I}} \widetilde{\mathcal{Q}}_{\mathcal{J}\dot\alpha} + \frac{1}{4}\delta^{\mathcal{I}}_{\mathcal{J}} \widetilde{\mathcal{Q}}_{\mathcal{K}\dot\alpha} \ , \\
[\mathcal{K}^{\dot\alpha\alpha}, \mathcal{Q}^{\mathcal{I}}_\beta] &= \delta_\beta{}^\alpha \widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha} \ , \\
[\mathcal{K}^{\dot\alpha\alpha}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\beta}] &= \delta_{\dot\beta}{}^{\dot\alpha} \mathcal{S}_{\mathcal{I}}{}^\alpha \ , \\
[\mathcal{P}_{\alpha\dot\alpha}, \mathcal{S}_{\mathcal{I}}{}^\beta] &= -\,\delta_\alpha{}^\beta \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha} \ , \\
[\mathcal{P}_{\alpha\dot\alpha}, \tilde{\mathcal{S}}^{\mathcal{I}\dot\beta}] &= -\,\delta_{\dot\alpha}{}^{\dot\beta} \mathcal{Q}^{\mathcal{I}}_\alpha \ .
\end{aligned}
\tag{A.1.6}
$$

A.1.2 The two-dimensional superconformal algebra

The second superalgebra of interest is $\mathfrak{sl}(2|2)$, which corresponds to the right-moving part of the global superconformal algebra in $\mathcal{N} = (0,4)$ SCFTs in two dimensions. The maximal bosonic subgroup is $\mathfrak{sl}(2)\times\mathfrak{sl}(2)_R$, with generators $\{L_0, L_{\pm1}\}$ for $\mathfrak{sl}(2)$ and $\{\mathcal{R}^\pm, \mathcal{R}\}$ for $\mathfrak{sl}(2)_R$. The non-vanishing bosonic commutation relations are given by

$$
\begin{aligned}
[\mathcal{R}, \mathcal{R}^\pm] &= \pm\mathcal{R}^\pm \ , \qquad & [\mathcal{R}^+, \mathcal{R}^-] &= 2\mathcal{R} \ , \\
[\tilde{L}_0, \tilde{L}_{\pm1}] &= \mp\tilde{L}_{\pm1} \ , \qquad & [\tilde{L}_1, \tilde{L}_{-1}] &= 2\tilde{L}_0 \ .
\end{aligned}
$$

There are additionally right-moving Poincaré supercharges $\mathcal{Q}^{\mathcal{I}}$, $\tilde{\mathcal{Q}}_{\mathcal{J}}$ and right-moving superconformal charges $\mathcal{S}_{\mathcal{J}}$, $\tilde{\mathcal{S}}^{\mathcal{I}}$. The commutation relations amongst the fermionic generators are given by

$$\begin{aligned}
\{\mathcal{Q}^{\mathcal{I}}, \tilde{\mathcal{Q}}_{\mathcal{J}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \tilde{L}_{-1} \ , \\
\{\tilde{\mathcal{S}}^{\mathcal{I}}, \mathcal{S}_{\mathcal{J}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \tilde{L}_{+1} \ , \\
\{\mathcal{Q}^{\mathcal{I}}, \mathcal{S}_{\mathcal{J}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \tilde{L}_0 - \mathcal{R}^{\mathcal{I}}_{\mathcal{J}} - \frac{1}{2}\delta^{\mathcal{I}}_{\mathcal{J}} \mathcal{Z} \ , \\
\{\tilde{\mathcal{Q}}_{\mathcal{J}}, \tilde{\mathcal{S}}^{\mathcal{I}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \tilde{L}_0 + \mathcal{R}^{\mathcal{I}}_{\mathcal{J}} + \frac{1}{2}\delta^{\mathcal{I}}_{\mathcal{J}} \mathcal{Z} \ ,
\end{aligned}$$

where ${\mathcal{R}^{\mathcal{I}}}_{\mathcal{J}}$ are defined as in (A.1.3), but with r set to zero. Here $\mathcal{Z}$ is a central element, the removal of which gives the algebra $\mathfrak{psl}(2|2)$. The additional commutators of bosonic symmetry generators with the supercharges are given by

$$\begin{aligned}
[\tilde{L}_{-1}\,, \tilde{\mathcal{S}}^{\mathcal{I}}] &= -\mathcal{Q}^{\mathcal{I}} \ , \\
[\tilde{L}_{-1}\,, \mathcal{S}_{\mathcal{I}}] &= -\tilde{\mathcal{Q}}_{\mathcal{I}} \ , \\
[\tilde{L}_{+1}\,, \tilde{\mathcal{Q}}_{\mathcal{I}}] &= \mathcal{S}_{\mathcal{I}} \ , \\
[\tilde{L}_{+1}\,, \mathcal{Q}^{\mathcal{I}}] &= \tilde{\mathcal{S}}^{\mathcal{I}} \ , \\
[\tilde{L}_0 \quad, \tilde{\mathcal{S}}^{\mathcal{I}}] &= -\tfrac{1}{2}\tilde{\mathcal{S}}^{\mathcal{I}} \ , \\
[\tilde{L}_0 \quad, \mathcal{S}_{\mathcal{I}}] &= -\tfrac{1}{2}\mathcal{S}_{\mathcal{I}} \ , \\
[\tilde{L}_0 \quad, \tilde{\mathcal{Q}}_{\mathcal{I}}] &= \tfrac{1}{2}\tilde{\mathcal{Q}}_{\mathcal{I}} \ , \\
[\tilde{L}_0 \quad, \mathcal{Q}^{\mathcal{I}}] &= \tfrac{1}{2}\mathcal{Q}^{\mathcal{I}} \ .
\end{aligned} \tag{A.1.7}$$

A.2 Shortening conditions and indices of $\mathfrak{su}(2,2\,|\,2)$

The classification of short representations of the four-dimensional $\mathcal{N}=2$ superconformal algebra [177, 45, 36] plays a major role in the structure of the chiral algebras described in chapter 2. This appendix provides a review of the classification, as well as of the various indices that can be defined on any representation of the algebra that are insensitive to the recombination of collections of short multiplets into generic long multiplets.

Short representations occur when the norm of a superconformal descendant state in what would otherwise be a long representation is rendered null by a conspiracy of quantum numbers. The unitarity bounds for a superconformal primary operator are given by

$$\begin{aligned} &E \geqslant E_i \ , && j_i \neq 0 \ , \\ &E = E_i - 2 \quad \text{or} \quad E \geqslant E_i \ , && j_i = 0 \ , \end{aligned} \tag{A.2.1}$$

where we have defined

$$E_1 = 2 + 2j_1 + 2R + r \ , \qquad E_2 = 2 + 2j_2 + 2R - r \ , \tag{A.2.2}$$

and short representations occur when one or more of these bounds are saturated. The different ways in which this can happen correspond to different combinations of Poincaré supercharges that will annihilate the superconformal primary state in the representation. There are two types of shortening conditions, each of which has four incarnations corresponding to an $SU(2)_R$ doublet's worth of conditions for each supercharge chirality:

$$\mathcal{B}^{\mathcal{I}} \quad : \qquad \mathcal{Q}^{\mathcal{I}}_{\alpha}|\psi\rangle = 0 \ , \quad \alpha = 1, 2 \tag{A.2.3}$$

$$\bar{\mathcal{B}}_{\mathcal{I}} \quad : \qquad \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}|\psi\rangle = 0 \ , \quad \dot{\alpha} = 1, 2 \tag{A.2.4}$$

$$\mathcal{C}^{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha} \quad |\psi\rangle_{\beta} = 0 \ , \quad j_1 \neq 0 \\ \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha}\mathcal{Q}^{\mathcal{I}}_{\beta} \quad |\psi\rangle = 0 \ , \quad j_1 = 0 \end{cases} , \tag{A.2.5}$$

$$\bar{\mathcal{C}}_{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}} \quad |\psi\rangle_{\dot{\beta}} = 0 \ , \quad j_2 \neq 0 \\ \epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\beta}} \quad |\psi\rangle = 0 \ , \quad j_2 = 0 \end{cases} , \tag{A.2.6}$$

The different admissible combinations of shortening conditions that can be simultaneously realized by a single unitary representation are summarized in Table A.1, where the reader can also find the precise relations that must be satisfied by the quantum numbers (E, j_1, j_2, r, R) of the superconformal primary operator, as well as the notations used to designate the different representations in [45] (DO) and [36] (KMMR).[1]

[1] We follow the R-charge conventions of DO.

Shortening	Quantum Number Relations	DO	KMMR
$\varnothing$	$E \geqslant 2R + r$	$\mathcal{A}^{\Delta}_{R,r(j_1,j_2)}$	$\mathbf{aa}_{\Delta,j_1,j_2,r,R}$
$\mathcal{B}^1$	$E = 2R + r$ $\quad j_1 = 0$	$\mathcal{B}_{R,r(0,j_2)}$	$\mathbf{ba}_{0,j_2,r,R}$
$\bar{\mathcal{B}}_2$	$E = 2R - r$ $\quad j_2 = 0$	$\bar{\mathcal{B}}_{R,r(j_1,0)}$	$\mathbf{ab}_{j_1,0,r,R}$
$\mathcal{B}^1 \cap \mathcal{B}^2$	$E = r$ $\quad R = 0$	$\mathcal{E}_{r(0,j_2)}$	$\mathbf{ba}_{0,j_2,r,0}$
$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$E = -r$ $\quad R = 0$	$\bar{\mathcal{E}}_{r(j_1,0)}$	$\mathbf{ab}_{j_1,0,r,0}$
$\mathcal{B}^1 \cap \bar{\mathcal{B}}_2$	$E = 2R$ $\quad j_1 = j_2 = r = 0$	$\hat{\mathcal{B}}_R$	$\mathbf{bb}_{0,0,0,R}$
$\mathcal{C}^1$	$E = 2 + 2j_1 + 2R + r$	$\mathcal{C}_{R,r(j_1,j_2)}$	$\mathbf{ca}_{j_1,j_2,r,R}$
$\bar{\mathcal{C}}_2$	$E = 2 + 2j_2 + 2R - r$	$\bar{\mathcal{C}}_{R,r(j_1,j_2)}$	$\mathbf{ac}_{j_1,j_2,r,R}$
$\mathcal{C}^1 \cap \mathcal{C}^2$	$E = 2 + 2j_1 + r$ $\quad R = 0$	$\mathcal{C}_{0,r(j_1,j_2)}$	$\mathbf{ca}_{j_1,j_2,r,0}$
$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$E = 2 + 2j_2 - r$ $\quad R = 0$	$\bar{\mathcal{C}}_{0,r(j_1,j_2)}$	$\mathbf{ac}_{j_1,j_2,r,0}$
$\mathcal{C}^1 \cap \bar{\mathcal{C}}_2$	$E = 2 + 2R + j_1 + j_2$ $\quad r = j_2 - j_1$	$\hat{\mathcal{C}}_{R(j_1,j_2)}$	$\mathbf{cc}_{j_1,j_2,j_2-j_1,R}$
$\mathcal{B}^1 \cap \bar{\mathcal{C}}_2$	$E = 1 + 2R + j_2$ $\quad r = j_2 + 1$	$\mathcal{D}_{R(0,j_2)}$	$\mathbf{bc}_{0,j_2,j_2+1,R}$
$\bar{\mathcal{B}}_2 \cap \mathcal{C}^1$	$E = 1 + 2R + j_1$ $\quad -r = j_1 + 1$	$\bar{\mathcal{D}}_{R(j_1,0)}$	$\mathbf{cb}_{j_1,0,-j_1-1,R}$
$\mathcal{B}^1 \cap \mathcal{B}^2 \cap \bar{\mathcal{C}}_2$	$E = r = 1 + j_2$ $\quad r = j_2 + 1$ $\quad R = 0$	$\mathcal{D}_{0(0,j_2)}$	$\mathbf{bc}_{0,j_2,j_2+1,0}$
$\mathcal{C}^1 \cap \bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$E = -r = 1 + j_1$ $\quad -r = j_1 + 1$ $\quad R = 0$	$\bar{\mathcal{D}}_{0(j_1,0)}$	$\mathbf{cb}_{j_1,0,-j_1-1,0}$

Table A.1: Unitary irreducible representations of the $\mathcal{N} = 2$ superconformal algebra.

At the level of group theory, it is possible for a collection of short representations to recombine into a generic long representation whose dimension is equal to one of the unitarity bounds of (A.2.1). In the DO notation, the generic recombinations are as follows:

$$\begin{aligned}
\mathcal{A}^{2R+r+2+2j_1}_{R,r(j_1,j_2)} &\simeq \mathcal{C}_{R,r(j_1,j_2)} \oplus \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \ , \\
\mathcal{A}^{2R-r+2+2j_2}_{R,r(j_1,j_2)} &\simeq \bar{\mathcal{C}}_{R,r(j_1,j_2)} \oplus \bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}(j_1,j_2-\frac{1}{2})} \ , \\
\mathcal{A}^{2R+j_1+j_2+2}_{R,j_1-j_2(j_1,j_2)} &\simeq \hat{\mathcal{C}}_{R(j_1,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j_2-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{R+1(j_1-\frac{1}{2},j_2-\frac{1}{2})} \ .
\end{aligned} \tag{A.2.7}$$

There are special cases when the quantum numbers of the long multiplet at threshold are such that some Lorentz quantum numbers in (A.2.7) would be negative and unphysical:

$$\begin{aligned}
\mathcal{A}^{2R+r+2}_{R,r(0,j_2)} &\simeq \mathcal{C}_{R,r(0,j_2)} \oplus \mathcal{B}_{R+1,r+\frac{1}{2}(0,j_2)} \ , \\
\mathcal{A}^{2R-r+2}_{R,r(j_1,0)} &\simeq \bar{\mathcal{C}}_{R,r(j_1,0)} \oplus \bar{\mathcal{B}}_{R+1,r-\frac{1}{2}(j_1,0)} \ , \\
\mathcal{A}^{2R+j_2+2}_{R,-j_2(0,j_2)} &\simeq \hat{\mathcal{C}}_{R(0,j_2)} \oplus \mathcal{D}_{R+1(0,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(0,j_2-\frac{1}{2})} \oplus \mathcal{D}_{R+\frac{3}{2}(0,j_2-\frac{1}{2})} \ , \\
\mathcal{A}^{2R+j_1+2}_{R,j_1(j_1,0)} &\simeq \hat{\mathcal{C}}_{R(j_1,0)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},0)} \oplus \bar{\mathcal{D}}_{R+1(j_1,0)} \oplus \bar{\mathcal{D}}_{R+\frac{3}{2}(j_1-\frac{1}{2},0)} \ , \\
\mathcal{A}^{2R+2}_{R,0(0,0)} &\simeq \hat{\mathcal{C}}_{R(0,0)} \oplus \mathcal{D}_{R+1(0,0)} \oplus \bar{\mathcal{D}}_{R+1(0,0)} \oplus \hat{\mathcal{B}}_{R+2} \ .
\end{aligned} \tag{A.2.8}$$

The last three recombinations involve multiplets that make an appearance in the associated chiral algebra described in this work. Note that the $\mathcal{E}$, $\bar{\mathcal{E}}$, $\hat{\mathcal{B}}_{\frac{1}{2}}$, $\hat{\mathcal{B}}_1$, $\hat{\mathcal{B}}_{\frac{3}{2}}$, $\mathcal{D}_0$, $\bar{\mathcal{D}}_0$, $\mathcal{D}_{\frac{1}{2}}$ and $\bar{\mathcal{D}}_{\frac{1}{2}}$ multiplets can never recombine, along with $\mathcal{B}_{\frac{1}{2},r(0,j_2)}$ and $\bar{\mathcal{B}}_{\frac{1}{2},r(j_1,0)}$.

There exist a variety of trace formulas [36, 38] that can be defined on the Hilbert space of an $\mathcal{N}=2$ SCFT such that the result receives contributions only from states that lie in short representations of the superconformal algebra, with the contributions being such that the indices are insensitive to recombinations. The indices are defined and named as follows:

$$\begin{aligned}
\text{Superconformal Index} &: \quad \mathrm{Tr}_{\mathcal{H}}(-1)^F p^{\frac{1}{2}(E+2j_1-2R-r)} q^{\frac{1}{2}(E-2j_1-2R-r)} t^{R+r}, \\
\text{Macdonald} &: \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{M}}}(-1)^F q^{\frac{1}{2}(E-2j_1-2R-r)} t^{R+r} \ , \\
\text{Schur} &: \quad \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{E-R} \ , \\
\text{Hall-Littlewood} &: \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{HL}}}(-1)^F \tau^{2E-2R} \ , \\
\text{Coulomb} &: \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{C}}}(-1)^F \sigma^{\frac{1}{2}(E+2j_1-2R-r)} \rho^{\frac{1}{2}(E-2j_1-2R-r)} \ .
\end{aligned}$$

The specialized Hilbert spaces appearing in the trace formulas above are defined as follows,

$$\mathcal{H}_{\mathrm{M}} := \{\psi \in \mathcal{H} \mid E + 2j_1 - 2R - r = 0\} , \tag{A.2.9}$$

$$\mathcal{H}_{\mathrm{HL}} := \{\psi \in \mathcal{H} \mid E - 2R - r = 0 , j_1 = 0\} , \tag{A.2.10}$$

$$\mathcal{H}_{\mathrm{C}} := \{\psi \in \mathcal{H} \mid E + 2j_1 + r = 0\} . \tag{A.2.11}$$

The different indices are sensitive to different superconformal multiplets. In particular, the Coulomb index counts only $\mathcal{E}$ and $\mathcal{D}_0$ type multiplets. These can be thought of as $\mathcal{N} = 1$ chiral ring operators that are $SU(2)_R$ singlets. Similarly, the Hall-Littlewood index counts only $\hat{\mathcal{B}}_R$ and $\mathcal{D}_R$ multiplets, which can be thought of as the consistent truncation of the $\mathcal{N} = 1$ chiral ring to operators that are neutral under $U(1)_r$. The Schur and Macdonald indices count only the operators that are involved in the chiral algebras of chapter 2: $\hat{\mathcal{B}}_R$, $\hat{\mathcal{C}}_R$, $\mathcal{D}$, and $\bar{\mathcal{D}}$ multiplets. The full index receives contributions from all of the multiplets appearing in Table A.1.

A.3 Kazhdan-Lusztig polynomials and affine characters

Computing the characters of irreducible modules of an affine Lie algebra at a negative integer level is a nontrivial task. For low levels, the multiplicity and norms of states can be found by hand using the mode expansion of the affine currents $J^A(z)$, but this computation quickly becomes rather involved. Fortunately there exists another method to compute these characters, based on the work of Kazhdan and Lusztig [178], which (with the aid of a computer) can produce results to very high order. In this appendix we give a brief introduction to this method. The interested reader is referred to, *e.g.*, [179, 180] for more details.

A generic method to obtain an irreducible representation of any (affine) Lie algebra is to start with the Verma module M built on a certain highest weight state $\psi_{h.w.}$, and then to subtract away all the null states in this module with the correct multiplicities. Let us recall that according to the Poincaré-Birkhoff-Witt theorem, the Verma module is spanned by all the states of the form

$$(E^{-\alpha_1,1})^{n_{1,1}}(E^{-\alpha_1,2})^{n_{1,2}}\ldots(E^{-\alpha_1,m_1})^{n_{1,m_1}}\ldots(E^{-\alpha_2,1})^{n_{2,1}}\ldots(E^{-\alpha_N,m_N})^{n_{N,m_N}}\psi_{h.w.} , \tag{A.3.1}$$

with nonnegative integer coefficients $n_{i,j}$. Here the $E^{-\alpha,k_\alpha}$ are the negative roots with weight $-\alpha$, and the auxiliary index $k_\alpha \in \{1,\ldots,m_\alpha\}$ is only necessary when the multiplicity m_α of the given weight is greater than one. The ordering of the roots in the above equation is arbitrary but fixed. If the highest weight state $\psi_{h.w.}$ has weight μ then the state defined as above has weight

$$\mu - \alpha_1(n_{1,1}+n_{1,2}+\ldots+n_{1,m_1}) - \alpha_2(n_{2,1}+\ldots) - \ldots - \alpha_N(\ldots+n_{N,m_N}) \ , \quad \text{(A.3.2)}$$

and with a moment's thought one sees that the character M_μ of the Verma module is given by

$$\mathrm{char} M_\mu = e^\mu \prod_{\alpha>0} (1-e^{-\alpha})^{-\mathrm{mult}(\alpha)} \ . \quad \text{(A.3.3)}$$

This is the *Kostant partition function*. The product is taken over the set of all the positive roots, which is infinite for an affine Lie algebra.

For a given affine Lie algebra there are special values of the highest weights for which the Verma module becomes reducible due to the existence of null states. We need to subtract all these null states to recover the irreducible module. Since any descendant of a null state is also null, the null states are themselves organized into Verma modules and we can subtract away entire modules at a time. This procedure is further complicated by the existence of "nulls of nulls", *i.e.*, null states inside the Verma module that we are subtracting. In general, this leads to a rather intricate pattern of subtractions. It follows that the character of the irreducible module with highest weight λ, which we denote as L_λ, can be obtained from a possibly infinite sum of the form

$$\mathrm{char} L_\lambda = \sum_{\mu \leqslant \lambda} m_{\lambda,\mu} \mathrm{char} M_\mu \ , \quad \text{(A.3.4)}$$

where the integers $m_{\lambda,\mu}$ are not of definite sign and reflect the aforementioned pattern of null states. Of course $m_{\lambda,\lambda} = 1$. The vectors labeled by μ in the above sum are called the *primitive null vectors* of the Verma module M_λ.

This leaves us with the task of determining the weights μ that appear in (A.3.4) along with their associated multiplicities $m_{\lambda,\mu}$. The first task is accomplished by noting that these weights are necessarily annihilated by all raising operators, and therefore must be highest weight states in themselves. The quadratic Casimir operator of an affine Lie algebra acts simply on highest weight states with weight μ as multiplication by $|\mu+\rho|^2$, where ρ is the Weyl vector with unit Dynkin labels. On the other hand, the eigenvalue should be an invariant of the full representation, which means that the only states μ that can appear in (A.3.4) have to satisfy

$$|\mu + \rho|^2 = |\lambda + \rho|^2 \ . \tag{A.3.5}$$

Notice that so far we have made no distinction between unitary representations, where the highest weight λ is dominant integral (*i.e.*, its Dynkin labels are nonnegative integers), and non-unitary representations like the ones in which we are interested. This distinction becomes crucial in the computation of the multiplicities $m_{\lambda,\mu}$.

For the irreducible representations associated to dominant integral weights, the weight multiplicities are invariant under the action of the Weyl group, and correspondingly $\mathrm{char}L_\lambda$ is invariant under the action of the Weyl group on the fugacities. On the other hand, the Kostant partition function is essentially *odd* under this action (*cf.* [179]),

$$w(e^{-\rho-\mu}\mathrm{char}M_\mu) = \mathrm{sign}(w)e^{-\rho-\mu}\mathrm{char}M_\mu \ , \tag{A.3.6}$$

where the sign of an element w in the Weyl group is simply given by -1 raised to the power of the number of generators used to express w. One can easily convince oneself that the multiplicities $m_{\lambda,\mu}$ therefore necessarily satisfy

$$m_{\lambda,\mu} = \mathrm{sign}(w)m_{\lambda,w\cdot\mu} \ , \tag{A.3.7}$$

where $w \cdot \mu := w(\mu + \rho) - \rho$ is the shifted action of the Weyl group on the weight μ. All the multiplicities $m_{\lambda,\mu}$ for weights μ on the same shifted Weyl orbit are therefore related by factors of $\mathrm{sign}(w)$, and it suffices to know only one multiplicity on each orbit. Happily, if the highest weight λ is dominant integral, then it lies on the shifted Weyl orbit of *any* primitive null vector. This essentially follows from the fact that there is a unique dominant integral weight on every shifted Weyl orbit, and from (A.3.5) it can be shown that this has to be λ. So, using that $m_{\lambda,\lambda} = 1$, we find that all the weights appearing in (A.3.4) are given by the shifted Weyl orbit of λ and have multiplicities equal to $\mathrm{sign}(w)$. In summary, then,

$$\mathrm{char}L_\lambda = \frac{\sum_{w\in W}\mathrm{sign}(w)e^{w(\rho+\lambda)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} \ , \tag{A.3.8}$$

which is the famous Weyl-Kac character formula.

Let us return to the case where the λ is not dominant integral. This is the case that interests us: indeed, for $\mathfrak{so}(8)_{-2}$ the vacuum representation has Dynkin labels $[-2\,0\,0\,0\,0]$ and the zeroth Dynkin label is not positive.[2] For non-dominant integral weights the above derivation already fails at the very first step: the weight multiplicities in the irreducible representation are not invariant under the action of the Weyl group. This is most easily seen by taking the infinite irreducible representation of $\mathfrak{su}(2)$ whose highest weight is negative. In this case the single Weyl reflection maps the highest weight, which of course has multiplicity one, to a positive weight, which has multiplicity zero. The derivation of the coefficients $m_{\lambda,\mu}$ now becomes considerably more involved. Since we find qualitative differences depending on the sign of $k + h^\vee$, we will in the remainder of this appendix focus on the relevant case $k + h^\vee > 0$.

For the non-unitary representations considered here it is still true that all the primitive null vectors lie on the shifted Weyl orbit of the highest weight λ, and for $k + h^\vee > 0$ there is still a unique *dominant* weight Λ on the same orbit such that $\Lambda + \rho$ has nonnegative Dynkin labels. For example, for the vacuum module of $\mathfrak{so}(8)_{-2}$ the dominant weight has Dynkin labels $[0\,0\,{-1}\,0\,0]$ which happens to be related to $[-2\,0\,0\,0\,0]$ by a single elementary reflection. All the weights in (A.3.4), including λ itself, can thus be written as $\mu = w \cdot \Lambda$ for some Weyl element w. We can therefore alternatively try to label these weights with the corresponding element of the Weyl group w instead of μ. We will see that such a relabeling has great benefits, but first we need to mention two important subtleties.

The first subtlety concerns the fact that we may restrict ourselves to elementary reflections of the Weyl group for which the corresponding Dynkin label in Λ is integral, since it is only in those cases that null states can possibly appear. These reflections generate a subgroup of the Weyl group that we will denote as W_Λ. In the case of $\mathfrak{so}(8)_{-2}$ the weights are all integral and $W_\Lambda = W$. The second subtlety is the possibility of the existence of a subgroup W_Λ^0 of W_Λ that leaves Λ invariant. This happens precisely when some of the Dynkin labels of $\Lambda + \rho$ are zero - in our case there is a single such zero. It is clear that the weights μ can then at best be uniquely labeled by elements of the

[2]Recall that the zeroth Dynkin label for a weight vector in an affine Lie algebra $\hat{\mathfrak{g}}$ is given by $k - (\lambda, \theta)$ with λ the part of the weight vector corresponding to the original Lie algebra $\mathfrak{g}$ and θ the highest root of $\mathfrak{g}$.

coset W_Λ/W_Λ^0.

It is now a deep result that the multiplicities $m_{\lambda,\mu}$ depend on the dominant integral weight Λ *only* through the corresponding elements w and w' of the coset W_Λ/W_Λ^0. We may therefore replace

$$m_{\lambda,\mu} \to m_{w,w'} \ , \tag{A.3.9}$$

where $\lambda = w \cdot \Lambda$, $\mu = w' \cdot \Lambda$ and w and w' are elements of the coset. The celebrated *Kazhdan-Lusztig conjecture* tells us that the precise form of these multiplicities is given by

$$m_{w,w'} = \tilde{Q}_{w,w'}(1) \ . \tag{A.3.10}$$

where the Kazhdan-Lusztig polynomial $\tilde{Q}_{w,w'}(q)$ is a single-variable polynomial depending on two elements w and w' of the coset W_Λ/W_Λ^0. These polynomials are determined via rather intricate recursion relations that are explained in detail in [180]. For $k + h^\vee > 0$ and integral weights, which is the case that interests us here, the Kazhdan-Lusztig conjecture was proven in [181, 182].

For the computations mentioned in the main text, we have implemented the recursive definitions of the Kazhdan-Lusztig polynomials on cosets given in [180] in `Mathematica`. Equations (A.3.3), (A.3.4), and (A.3.10) then allow us to compute all the states in the irreducible vacuum character of $\mathfrak{so}(8)_{-2}$ up to level five. The results are shown in Table 2.5.

Appendix B

Chiral algebras of class $\mathcal{S}$

B.1 Details for rank two theories

This appendix includes details regarding a number of calculations having to do with operations on the $\chi[T_3]$ chiral algebra described in Sec. 3.3.3. For all of these calculations, it is useful to have the realization of the $\chi[T_3]$ chiral algebra, which coincides with the affine $\mathfrak{e}_6$ current algebra at level $k=-3$, in the basis relevant for class $\mathcal{S}$ given in Eqn. (3.3.16). In this basis, the singular OPEs are as follows,

$$\begin{aligned}
(J^1)_a^{a'}(z)(J^1)_{\tilde{a}}^{\tilde{a}'}(0) &\sim \frac{k(\delta_a^{\tilde{a}'}\delta_{\tilde{a}}^{a'}-\frac{1}{3}\delta_a^{a'}\delta_{\tilde{a}}^{\tilde{a}'})}{z^2}+\frac{\delta_{\tilde{a}}^{a'}\ (J^1)_a^{\tilde{a}'}-\delta_a^{\tilde{a}'}\ (J^1)_{\tilde{a}}^{a'}}{z}\ , \\
(J^1)_a^{a'}(z)W_{a''bc}(0) &\sim \frac{1}{z}\left(\delta_{a''}^{a'}W_{abc}-\frac{1}{3}\delta_a^{a'}W_{a''bc}\right)\ , \\
(J^1)_a^{a'}(z)\widetilde{W}^{a''bc}(0) &\sim -\frac{1}{z}\left(\delta_a^{a''}\widetilde{W}^{a'bc}-\frac{1}{3}\delta_a^{a'}\widetilde{W}^{a''bc}\right)\ , \\
W_{abc}(z)W_{a'b'c'}(0) &\sim -\frac{1}{z}\epsilon_{aa'a''}\epsilon_{bb'b''}\epsilon_{cc'c''}\widetilde{W}^{a''b''c''}\ , \\
\widetilde{W}^{abc}(z)\widetilde{W}^{a'b'c'}(0) &\sim \frac{1}{z}\epsilon^{aa'a''}\epsilon^{bb'b''}\epsilon^{cc'c''}W_{a''b''c''}\ , \\
W_{abc}(z)\widetilde{W}^{a'b'c'}(0) &\sim \frac{k}{z^2}\delta_a^{a'}\delta_b^{b'}\delta_c^{c'} \\
&\quad+\frac{1}{z}\left((J^1)_a^{a'}\delta_b^{b'}\delta_c^{c'}+\delta_a^{a'}(J^2)_b^{b'}\delta_c^{c'}+\delta_a^{a'}\delta_b^{b'}(J^3)_c^{c'}\right)\ ,
\end{aligned} \tag{B.1.1}$$

and similarly for (J^2) and (J^3).

We saw illustrated in Table 3.3 that there exists for $k = -3$ a null relation in the **650**-dimensional representation of $\mathfrak{e}_6$. This is in agreement with the Higgs branch relation of Eqn. (3.3.17). It will prove useful to have the explicit expression for the components of this null vector upon decomposition in terms of $\oplus_{I=1}^{3}\mathfrak{su}(3)_I$ representations. Group theoretically, the decomposition in question is given by

$$\begin{aligned}\mathbf{650} \;\rightarrow\;\; & 2\times(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{8},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{8})\\ & +2\times(\mathbf{3},\mathbf{3},\mathbf{3})+2\times(\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}})+(\mathbf{8},\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{8},\mathbf{8})+(\mathbf{8},\mathbf{1},\mathbf{8})\\ & +(\mathbf{6},\bar{\mathbf{3}},\bar{\mathbf{3}})+(\bar{\mathbf{6}},\mathbf{3},\mathbf{3})+(\bar{\mathbf{3}},\mathbf{6},\bar{\mathbf{3}})+(\mathbf{3},\bar{\mathbf{6}},\mathbf{3})+(\bar{\mathbf{3}},\bar{\mathbf{3}},\mathbf{6})+(\mathbf{3},\mathbf{3},\bar{\mathbf{6}})\;.\end{aligned}$$

The corresponding null vectors arise from the relations summarized in Table B.1.

Representation	Null relation
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(J^1)_{a_1}{}^{a_2}(J^1)_{a_2}{}^{a_1} = (J^2)_{b_1}{}^{b_2}(J^2)_{b_2}{}^{b_1} = (J^3)_{c_1}{}^{c_2}(J^3)_{c_2}{}^{c_1}$
$(\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8})$	$\frac{1}{3}\left(W_{a_1bc}\tilde{W}^{a_2bc} - \frac{1}{3}\delta^{a_2}_{a_1}W_{abc}\tilde{W}^{abc}\right) + \left((J^1)_{a_1}{}^{a}(J^1)_{a}{}^{a_2} - \frac{1}{3}\delta^{a_2}_{a_1}(J^1)_{\alpha}{}^{a}(J^1)_{a}{}^{\alpha}\right) - 3\partial(J^1)_{a_1}{}^{a_2} = 0$
$(\mathbf{8},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{8}) + (\mathbf{8},\mathbf{1},\mathbf{8})$	$W_{a_1b_1c}\tilde{W}^{a_2b_2c} - \frac{1}{3}\delta^{a_2}_{a_1}W_{ab_1c}\tilde{W}^{ab_2c} - \frac{1}{3}\delta^{b_2}_{b_1}\left(W_{a_1bc}\tilde{W}^{a_2bc}\right) + \frac{1}{9}\delta^{a_2}_{a_1}\delta^{b_2}_{b_1}\left(W_{abc}\tilde{W}^{abc}\right) + (J^1)_{a_1}{}^{a_2}(J^2)_{b_1}{}^{b_2} = 0$
$(\mathbf{3},\mathbf{3},\mathbf{3})$	$(J^1)_a{}^{\alpha}W_{\alpha bc} = (J^2)_b{}^{\beta}W_{a\beta c} = (J^3)_c{}^{\gamma}W_{ab\gamma}$
$(\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}})$	$(J^1)_\alpha{}^a\tilde{W}^{\alpha bc} = (J^2)_\beta{}^b\tilde{W}^{a\beta c} = (J^3)_\gamma{}^c\tilde{W}^{abc}$
$(\bar{\mathbf{6}},\mathbf{3},\mathbf{3}) + (\mathbf{3},\bar{\mathbf{6}},\mathbf{3}) + (\mathbf{3},\mathbf{3},\bar{\mathbf{6}})$	$2(J^1)_{\alpha_1}{}^{(a_1\vert}W_{\alpha_2bc}\epsilon^{\alpha_1\alpha_2\vert a_2)} + \tilde{W}^{(a_1b_1c_1}\tilde{W}^{a_2)b_2c_2}\epsilon_{bb_1b_2}\epsilon_{cc_1c_2} = 0$
$(\mathbf{6},\bar{\mathbf{3}},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\mathbf{6},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\bar{\mathbf{3}},\mathbf{6})$	$2(J^1)_{(a_1\vert}{}^{\alpha_1}\tilde{W}^{\alpha_2bc}\epsilon_{\alpha_1\alpha_2\vert a_2)} + W_{(a_1b_1c_1}W_{a_2)b_2c_2}\epsilon^{bb_1b_2}\epsilon^{cc_1c_2} = 0$

Table B.1: Null state relations at level two in the $\chi[T_3]$ chiral algebra.

B.1.1 Argyres-Seiberg duality

First we describe in detail the check of Argyres-Seiberg duality at the level of chiral algebras described in Section 3.3.3. The first duality frame is that of SQCD, the chiral algebra of which was described in chapter 2. There the generators of the chiral algebra were found to include a $\widehat{\mathfrak{su}(6)}_{-3} \times \widehat{\mathfrak{u}(1)}$ affine current algebra with currents J_i^j and J, along with baryonic and anti-baryonic operators $\{b_{ijk}, \tilde{b}^{ijk}\}$ of dimension $\Delta = \frac{3}{2}$. The singular OPEs for these generators are as follows,

$$
\begin{aligned}
J_i^j(z)J_k^l(0) &\sim -\frac{3(\delta_i^l\delta_k^j - \text{trace})}{z^2} + \frac{\delta_k^j J_i^l(z) - \delta_i^l J_k^j(z)}{z} , \\
J(z)J(0) &\sim -\frac{18}{z^2} , \\
J_i^j(z)b_{k_1k_2k_3}(0) &\sim \frac{3\delta^j_{[k_1|}b_{i|k_2k_3]}(0) - \frac{1}{2}\delta_i^j b_{k_1k_2k_3}(0)}{z} , \\
J(z)b_{k_1k_2k_3}(0) &\sim \frac{3b_{k_1k_2k_3}(0)}{z} , \\
J(z)b^{k_1k_2k_3}(0) &\sim -\frac{3b^{k_1k_2k_3}(0)}{z} , \\
b_{i_1i_2i_3}(z)\tilde{b}^{j_1j_2j_3}(0) &\sim \frac{36\,\delta^{[j_1}_{[i_1}\delta^{j_2}_{i_2}\delta^{j_3]}_{i_3]}}{z^3} - \frac{36\,\delta^{[j_1}_{[i_1}\delta^{j_2}_{i_2}\hat{J}^{j_3]}_{i_3]}(0)}{z^2} \\
&\qquad + \frac{18\,\delta^{[j_1}_{[i_1}\hat{J}^{j_2}_{i_2}\hat{J}^{j_3]}_{i_3]}(0) - 18\,\delta^{[j_1}_{[i_1}\delta^{j_2}_{i_2}\partial\hat{J}^{j_3]}_{i_3]}(0)}{z} .
\end{aligned}
\tag{B.1.2}
$$

Antisymmetrizations are performed with weight one, and lower (upper) indices $i, j, \ldots$ transform in the fundamental (antifundamental) representation of $\mathfrak{su}(6)$. In the last line we have introduced the $\mathfrak{u}(6)$ current $\hat{J}_j^i := J_j^i + \frac{1}{6}\delta_j^i J$. It was conjectured in chapter 2 that the SQCD chiral algebra is a $\mathcal{W}$ algebra with just these generators. This proposal passed a few simple checks. All the generators of the Hall-Littlewood chiral ring have been accounted for and the OPE closes. There is no additional stress tensor as a generator because the Sugawara stress tensor of the $\mathfrak{u}(6)$ current algebra turns out to do the job (this again implies a relation in the Higgs branch chiral ring of SQCD). The spectrum of the chiral algebra generated by these operators also correctly reproduces the low-order expansion of the superconformal index.

Our aim in the remainder of this appendix is to reproduce this chiral algebra from the 'exceptional side' of the duality using our proposal that the chiral algebra $\chi[T_3]$ is the current algebra $(\widehat{\mathfrak{e}}_6)_{-3}$. The two free hypermultiplets contribute symplectic bosons q_α and $\tilde{q}^\alpha$ with $\alpha = 1, 2$ with singular OPE given by

$$q_\alpha(z)\tilde{q}^\beta(0) \sim \frac{\delta_\alpha^\beta}{z} \ . \tag{B.1.3}$$

The $\chi[T_3]$ chiral algebra should be re-expressed in terms of an $\mathfrak{su}(6) \times \mathfrak{su}(2)$ maximal subalgebra, in terms of which the affine currents split as

$$\{J_{A=1,\ldots,78}\} \quad \Longrightarrow \quad \{X_j^i,\ Y_\alpha^{[ijk]},\ Z_\alpha^\beta\} \ . \tag{B.1.4}$$

The operators X_j^i and Z_α^β are the affine currents of $\mathfrak{su}(6)$ and $\mathfrak{su}(2)$, respectively, with $X_i^i = Z_\alpha^\alpha = 0$. The additional operators Y_α^{ijk} transform in the $(\mathbf{20}, \mathbf{2})$ of $\mathfrak{su}(6) \times \mathfrak{su}(2)$. The nonvanishing OPEs amongst these operators are simply a rewriting of the $\widehat{\mathfrak{e}}_6$ current algebra,

$$\begin{aligned}
X_i^j(z)X_k^l(0) \quad &\sim \quad -\frac{3(\delta_i^l\delta_k^j - \text{trace})}{z^2} + \frac{\delta_k^j X_i^l(0) - \delta_i^l X_k^j(0)}{z} \\
Z_\alpha^\beta(z)Z_\gamma^\delta(0) \quad &\sim \quad -\frac{3(\delta_\alpha^\delta\delta_\gamma^\beta - \text{trace})}{z^2} + \frac{\delta_\gamma^\beta Z_\alpha^\partial(0) - \delta_\alpha^\delta Z_\gamma^\beta(0)}{z} \\
X_i^j(z)Y_\alpha^{klm}(0) \quad &\sim \quad -\frac{3\delta_i^{[k}Y_\alpha^{lm]j}(0)}{z} - \text{trace} \\
Z_\alpha^\beta(z)Y_\gamma^{ijk}(0) \quad &\sim \quad \frac{\delta_\gamma^\beta Y_\alpha^{ijk}(0)}{z} - \text{trace} \\
Y_\alpha^{ijk}(z)Y_\beta^{lmn}(0) \quad &\sim \quad \frac{\epsilon_{\alpha\beta}\epsilon^{ijklmn}}{z^2} + \frac{\epsilon^{ijklmn}\epsilon_{\alpha\gamma}Z_\beta^\gamma(0) - 3\epsilon_{\alpha\beta}\epsilon^{[ijklm|p}X_p^{|n]}(0)}{z} \ .
\end{aligned} \tag{B.1.5}$$

Gluing introduces a dimension $(1, 0)$ ghost system in the adjoint of $\mathfrak{su}(2)$ and restricting to the appropriate cohomology of the following BRST operator,

$$J_{\mathrm{BRST}} = \ c_\beta^\alpha(Z_\alpha^\beta - q_\alpha\tilde{q}^\beta) - \frac{1}{2}(\delta_{\alpha_1}^{\alpha_6}\delta_{\alpha_3}^{\alpha_2}\delta_{\alpha_5}^{\alpha_4} - \delta_{\alpha_1}^{\alpha_4}\delta_{\alpha_3}^{\alpha_6}\delta_{\alpha_5}^{\alpha_2})c_{\alpha_2}^{\alpha_1}b_{\alpha_4}^{\alpha_3}c_{\alpha_6}^{\alpha_5}. \tag{B.1.6}$$

The cohomology can be constructed level by level using the `OPEdefs` package for `Mathematica` [48]. Up to dimension $h = \frac{3}{2}$, we find the following operators,

$$X_j^i \ , \qquad q_\alpha\tilde{q}^\alpha \ , \qquad \epsilon_{ijklmn}\tilde{q}^\alpha Y_\alpha^{lmn} \ , \qquad \epsilon^{\alpha\beta}q_\alpha Y_\beta^{ijk} \ . \tag{B.1.7}$$

Up to normalizations, these can naturally be identified with the generators of the SQCD chiral algebra,

$$X_i^j \simeq J_i^j \ , \qquad 3q_\alpha\tilde{q}^\alpha \simeq J \ , \qquad \frac{1}{6}\epsilon_{ijklmn}\tilde{q}^\alpha Y_\alpha^{lmn} \simeq b_{ijk} \ , \qquad \epsilon^{\alpha\beta}q_\alpha Y_\beta^{ijk} \simeq \tilde{b}^{ijk} \ . \tag{B.1.8}$$

The equations relating chiral algebra generators in the two duality frames are the same as the ones obtained in [64], with the operators being viewed as generators of the Higgs branch chiral ring. In that work, establishing them at the level of the Higgs branch required a detailed understanding of the chiral ring relations on both sides. By contrast, to establish equivalence of the chiral algebras we need to check that the above operators have the same singular OPEs. Relations in the chiral ring will then show up automatically as null states.

With the `OPEdefs` package we have also computed the OPEs of the composite operators in (B.1.7) and found perfect agreement with (B.1.2). Most of the OPEs are reproduced in a fairly trivial fashion. However, the simple pole in the baryon-antibaryon OPE can only be matched by realizing that there is a null state at level two in the $(\widehat{\mathfrak{e}}_6)_{-3}$ algebra given by

$$Y_\alpha^{ijk}Y_\beta^{abc}\epsilon^{\alpha\beta}\epsilon_{abclmn}+108\partial X_{[l}^{[i}\delta_m^j\delta_{n]}^{k]}+108X_{[l}^{[i}X_m^j\delta_{n]}^{k]}+\frac{1}{72}Z_\alpha^\beta Z_\beta^\alpha\delta_{[l}^{[i}\delta_m^j\delta_{n]}^{k]} \ . \tag{B.1.9}$$

Thus we have shown that using our proposal for the $\chi[T_3]$ chiral algebra in the Argyres-Seiberg duality problem, one at least produces a self-contained $\mathcal{W}$-algebra that matches between the two sides of the duality. It would be nice to prove that this $\mathcal{W}$ algebra is the full chiral algebra. Indeed, if one could demonstrate this fact for the SQCD side of the duality, it seems likely that it wouldn't be too hard to prove that there can be no additional generators in the $\chi[T_3]$ chiral algebra beyond the affine currents.

B.1.2 Reduction of T_3 to free hypermultiplets

In this appendix we provide some details about the reduction of the $\chi[T_3]$ chiral algebra to free symplectic bosons. This corresponds to the subregular embedding $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(3)$, which is given by

$$\Lambda(t_0) = \frac{1}{2}(T_1{}^1 - T_3{}^3) \ , \qquad \Lambda(t_-) = T_3{}^1 \ , \qquad \Lambda(t_+) = T_1{}^3 \ . \tag{B.1.10}$$

The grading on the Lie algebra by the Cartan element $\Lambda(t_0)$ is half-integral. In order to arrive at first-class constraints, we introduce a different Cartan element δ that gives an integral grading. More specifically, we have $\delta = \frac{1}{3}({T_1}^1 + {T_2}^2 - 2{T_3}^3)$. With respect to the δ-grading there are two positively graded currents and we consequently impose the constraints ${(J^1)_3}^1 = 1$ and ${(J^1)_3}^2 = 0$. These are implemented via a BRST procedure with differential given by

$$d(z) = \left(\left({(J^1)_3}^1 - 1\right) {c_1}^3 + {(J^1)_3}^2 {c_2}^3\right)(z) \;, \tag{B.1.11}$$

where the ghost pairs ${b_3}^1, {c_1}^3$ and ${b_3}^2, {c_2}^3$ have the usual singular OPEs.

Implementing the first step of the qDS procedure, one obtains the (redundant) generators of the chiral algebra at the level of vector spaces. Applying the tic-tac-toe procedure to produce genuine chiral algebra generators, we obtain the set of generators that were listed in Table 3.4. The explicit forms of these generators are given as follows,

$$\begin{aligned}
\mathcal{J}_{\mathrm{u}(1)} &:= {(\hat{J}^1)_1}^1 - 2{(\hat{J}^1)_2}^2 + {(\hat{J}^1)_3}^3 \\
{(\hat{\mathcal{J}}^1)_1}^2 &:= {(\hat{J}^1)_1}^2 \\
{(\hat{\mathcal{J}}^1)_1}^3 &:= {(\hat{J}^1)_1}^3 - \left(-(k+1)\partial{(\hat{J}^1)_3}^3 + {(\hat{J}^1)_1}^1{(\hat{J}^1)_3}^3 - {(\hat{J}^1)_2}^1{(\hat{J}^1)_1}^2\right) \\
{(\hat{\mathcal{J}}^1)_2}^3 &:= {(\hat{J}^1)_2}^3 - \left((k+2)\partial{(\hat{J}^1)_2}^1 + {(\hat{J}^1)_3}^3{(\hat{J}^1)_2}^1 - {(\hat{J}^1)_2}^2{(\hat{J}^1)_2}^1\right) \\
\mathcal{W}_{1bc} &:= W_{1bc} - W_{3bc}{(\hat{J}^1)_1}^1 \\
\mathcal{W}_{2bc} &:= W_{2bc} - W_{3bc}{(\hat{J}^1)_2}^1 \\
\mathcal{W}_{3bc} &:= W_{3bc} \\
\tilde{\mathcal{W}}^{1bc} &:= \tilde{W}^{1bc} \\
\tilde{\mathcal{W}}^{2bc} &:= \tilde{W}^{2bc} \\
\tilde{\mathcal{W}}^{3bc} &:= \tilde{W}^{3bc} - \left(-\tilde{W}^{1bc}{(\hat{J}^1)_1}^1 - \tilde{W}^{2bc}{(\hat{J}^1)_2}^1\right) \\
(\mathcal{J}^2)_{b_1}^{\;b_2} &:= (J^2)_{b_1}^{\;b_2} \\
(\mathcal{J}^3)_{c_1}^{\;c_2} &:= (J^3)_{c_1}^{\;c_2} \;.
\end{aligned} \tag{B.1.12}$$

The generators $\mathcal{W}_{3bc}$ and $\tilde{\mathcal{W}}^{1bc}$ have the correct charges and mutual OPE to be identified as the expected symplectic bosons. It follows that the reduction argument will be complete if we can show that at the specific value of the level of interest $k = -3$, all the other generators listed in Eqn. (B.1.12) participate in a null state condition that allows them to be equated with composites of $\mathcal{W}_{3bc}$ and $\tilde{\mathcal{W}}^{1bc}$.

Indeed, we do find such relations to account for all additional generators. At level $h=1$, we find

$$\mathcal{J}_{\mathfrak{u}(1)} = -\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc} , \tag{B.1.13}$$

$$(\mathcal{J}^2)_{b_1}{}^{b_2} = -\left(\mathcal{W}_{3b_1c}\tilde{\mathcal{W}}^{1b_2c} - \frac{1}{3}\delta_{b_1}^{b_2}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc}\right) , \tag{B.1.14}$$

$$(\mathcal{J}^3)_{c_1}{}^{c_2} = -\left(\mathcal{W}_{3bc_1}\tilde{\mathcal{W}}^{1bc_2} - \frac{1}{3}\delta_{c_1}^{c_2}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc}\right) , \tag{B.1.15}$$

$$\tilde{\mathcal{W}}_{2bc} = \frac{1}{2}\epsilon_{bb_1b_2}\epsilon_{cc_1c_2}\tilde{\mathcal{W}}^{1b_1c_1}\tilde{\mathcal{W}}^{1b_2c_2} , \tag{B.1.16}$$

$$\tilde{\mathcal{W}}^{2bc} = -\frac{1}{2}\epsilon^{bb_1b_2}\epsilon^{cc_1c_2}\mathcal{W}_{3b_1c_1}\mathcal{W}_{3b_2c_2} . \tag{B.1.17}$$

At dimension $h=3/2$, one can find the null relations

$$(\hat{\mathcal{J}}^1)_1{}^2 = \frac{1}{6}\mathcal{W}_{3b_1c_1}\mathcal{W}_{3b_2c_2}\mathcal{W}_{3b_3c_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3} , \tag{B.1.18}$$

$$(\hat{\mathcal{J}}^1)_2{}^3 = -\frac{1}{6}\tilde{\mathcal{W}}^{1b_1c_1}\tilde{\mathcal{W}}^{1b_2c_2}\tilde{\mathcal{W}}^{1b_3c_3}\epsilon_{b_1b_2b_3}\epsilon_{c_1c_2c_3} , \tag{B.1.19}$$

$$\mathcal{W}_{1bc} = 2\partial\mathcal{W}_{3bc} + \frac{5}{12}\mathcal{W}_{3b_1c_1}\mathcal{W}_{3b_2c_2}\tilde{\mathcal{W}}^{1b_3c_3}\epsilon^{\beta b_1b_2}\epsilon^{\gamma c_1c_2}\epsilon_{\beta bb_3}\epsilon_{\gamma cc_3} \tag{B.1.20}$$

$$-\frac{1}{3}\mathcal{W}_{3(b(c}\mathcal{W}_{3b_1)c_1)}\tilde{\mathcal{W}}^{1b_1c_1} , \tag{B.1.21}$$

$$\mathcal{W}^{3bc} = -\partial\tilde{\mathcal{W}}^{1bc} + \frac{1}{3}\tilde{\mathcal{W}}^{1b_1c_1}\tilde{\mathcal{W}}^{1b_2c_2}\mathcal{W}_{3b_3c_3}\epsilon_{\beta b_1b_2}\epsilon_{\gamma c_1c_2}\epsilon^{\beta bb_3}\epsilon^{\gamma cc_3} \tag{B.1.22}$$

$$-\frac{2}{3}\tilde{\mathcal{W}}^{1(b(c}\tilde{\mathcal{W}}^{1b_1)c_1)}\mathcal{W}_{3b_1c_1} . \tag{B.1.23}$$

Finally, at dimension $h=2$, we find

$$(\hat{\mathcal{J}}^1)_1{}^3 = \frac{14}{9}\mathcal{W}_{3bc}\partial\tilde{\mathcal{W}}^{1bc} - \frac{8}{9}\partial\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc} + \frac{2}{9}\mathcal{W}_{3(b_1(c_1}\mathcal{W}_{3b_2)c_2)}\tilde{\mathcal{W}}^{1(b_1(c_1}\tilde{\mathcal{W}}^{1b_2)c_2)}$$
$$- \frac{7}{36}\mathcal{W}_{3b_1c_1}\mathcal{W}_{3b_2c_2}\tilde{\mathcal{W}}^{1b_3c_3}\tilde{\mathcal{W}}^{1b_4c_4}\epsilon^{b_1b_2b}\epsilon^{c_1c_2c}\epsilon_{b_3b_4b}\epsilon_{c_3c_4c} . \tag{B.1.24}$$

It is interesting to see these null relations as a consequence of the nulls in the original chiral algebra. To that effect, let us re-derive the dimension one nulls in this manner. Starting with the $(\mathbf{8},\mathbf{1},\mathbf{1})$ null states in Table B.1 and specializing the indices to $(a_1,a_2)=(3,1)$, we find the null relation

$$0 = \tfrac{1}{3}W_{3bc}\tilde{W}^{1bc} + (J^{(1)})_3{}^a(J^{(1)})_a{}^1 + 3\partial(J^{(1)})_3{}^1 = \tfrac{1}{3}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc} + \tfrac{1}{3}\mathcal{J}_{\mathfrak{u}(1)} + d(\ldots) , \tag{B.1.25}$$

thus reproducing Eqn. (B.1.13). Alternatively, starting with the null states in the $(\mathbf{8},\mathbf{8},\mathbf{1})$ and specializing the indices to $(a_1,a_2)=(3,1)$, we obtain the null relation

$$\begin{aligned}
0 &= \left(W_{3b_1c}\tilde{W}^{1b_2c} - \frac{1}{3}\delta^{b_2}_{b_1}W_{3bc}\tilde{W}^{1bc}\right) - \frac{1}{3}\beta_1 (J^{(1)})_3{}^1 (J^{(2)})_{b_1}{}^{b_2} \\
&= \left(\mathcal{W}_{3b_1c}\tilde{\mathcal{W}}^{1b_2c} - \frac{1}{3}\delta^{b_2}_{b_1}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc}\right) + \left(\mathcal{J}^{(2)}\right)_{b_1}{}^{b_2} + d(\ldots)\ , \qquad \text{(B.1.26)}
\end{aligned}$$

which precisely matches the null relation of Eqn. (B.1.14). Similarly, one can reproduce (B.1.15). It is straightforward to check that the null relations in Eqns. (B.1.16)-(B.1.17) can be obtained from the relations in the $(\bar{\mathbf{6}},\mathbf{3},\mathbf{3})$ and $(\mathbf{6},\bar{\mathbf{3}},\bar{\mathbf{3}})$ and specializing the indices appropriately.

B.2 Cylinder and cap details

This appendix describes the quantum Drinfeld-Sokolov reduction that produces the chiral algebra for cylinder and cap geometries when $\mathfrak{g}=\mathfrak{su}(3)$. We first introduce some general formulas for the Schur superconformal index associated to these geometries. These formulas prove useful for getting a basic intuition for how these chiral algebras may be described.

B.2.1 Schur indices

Although they are only formally defined (there is no true four-dimensional SCFT associated to the cylinder and cap geometries), the reduction rules for the Schur index allow us to define an index for these geometries that must behave appropriately under gluing. Let us determine these indices.

Cylinder Using the general results given in Eqns. (3.2.28) and (3.2.38), the index of the two-punctured sphere theory can be determined immediately

$$\begin{aligned}
\mathcal{I}_{\text{cylinder}}\left(q;\mathbf{a},\mathbf{b}\right) &= K_{\text{max.}}(\mathbf{a};q)K_{\text{max.}}(\mathbf{b};q)\sum_{\mathfrak{R}}\chi_{\mathfrak{R}}(\mathbf{a})\chi_{\mathfrak{R}}(\mathbf{b}) \\
&= \text{PE}\left[\frac{q}{1-q}\left(\chi_{\text{adj}}(\mathbf{a})+\chi_{\text{adj}}(\mathbf{b})\right)\right]\sum_{\mathfrak{R}}\chi_{\mathfrak{R}}(\mathbf{a})\chi_{\mathfrak{R}}(\mathbf{b})\ .
\end{aligned} \qquad \text{(B.2.1)}$$

Upon using the relation $\sum_{\mathfrak{R}} \chi_{\mathfrak{R}}(\mathbf{a})\chi_{\mathfrak{R}}(\mathbf{b}) = \delta(\mathbf{a}, \mathbf{b}^{-1})$, where the delta function is defined with respect to the Haar measure, we can rewrite this index as

$$\mathcal{I}_{\text{cylinder}}(q; \mathbf{a}, \mathbf{b}) = \text{PE}\left[\frac{2q}{1-q}\chi_{\text{adj}}(\mathbf{a})\right]\delta(\mathbf{a}, \mathbf{b}^{-1}) = I_V^{-1}(\mathbf{a}; q)\ \delta(\mathbf{a}, \mathbf{b}^{-1})\ , \tag{B.2.2}$$

where I_V is the vector multiplet index (3.2.43). This makes it clear that when the gluing prescription for the index given in Eqn. (3.2.42) is applied, the index $\mathcal{I}_{\mathcal{T}}(q; \mathbf{a}, \ldots)$ of a generic theory $\mathcal{T}$ containing a maximal puncture with fugacities $\mathbf{a}$ remains the same after gluing a cylinder to that maximal puncture

$$\int [d\mathbf{a}]\Delta(\mathbf{a})I_V(q; \mathbf{a})\ \mathcal{I}_{\mathcal{T}}(q; \mathbf{a}, \ldots)\ \mathcal{I}_{\text{cylinder}}\left(q; \mathbf{a}^{-1}, \mathbf{b}\right) = \mathcal{I}_{\mathcal{T}}(q; \mathbf{b}, \ldots)\ . \tag{B.2.3}$$

Here $[d\mathbf{a}] = \prod_{j=1}^{\text{rank}\mathfrak{g}} \frac{da_j}{2\pi i a_j}$ and $\Delta(\mathbf{a})$ is the Haar measure.

Returning to expression (B.2.1), we wish to rewrite the sum over representations. Let us therefore consider the regularized sum

$$\sum_{\mathfrak{R}} u^{|\mathfrak{R}|}\chi_{\mathfrak{R}}(\mathbf{a})\chi_{\mathfrak{R}}(\mathbf{b}) = \text{PE}\left[u\ \chi_{\mathfrak{f}}(\mathbf{a})\chi_{\mathfrak{f}}(\mathbf{b}) - u^n\right]\ , \tag{B.2.4}$$

where $|\mathfrak{R}|$ denotes the number of boxes in the Young diagram corresponding to the representation $\mathfrak{R}$ of $\mathfrak{g} = \mathfrak{su}(n)$. For $\mathfrak{g} = \mathfrak{su}(2)$ we have checked this equality exactly by performing the geometric sums and for $\mathfrak{su}(3)$, $\mathfrak{su}(4)$ and $\mathfrak{su}(5)$ in a series expansion in u. In the limit $u \to 1$ one can verify that the right hand side behaves as a δ-function with respect to the Haar measure, as expected. Consequently, the cylinder index can then be rewritten in a particularly useful form,

$$\mathcal{I}_{\text{cylinder}}(q; \mathbf{a}, \mathbf{b}) = \text{PE}\left[\frac{q}{1-q}\left(\chi_{\text{adj}}(\mathbf{a}) + \chi_{\text{adj}}(\mathbf{b})\right) + \chi_{\mathfrak{f}}(\mathbf{a})\chi_{\mathfrak{f}}(\mathbf{b}) - 1\right]\ . \tag{B.2.5}$$

By using $\chi_{\text{adj}}(\mathbf{a}) = \chi_{\mathfrak{f}}(\mathbf{a})\chi_{\mathfrak{f}}(\mathbf{a}^{-1}) - 1$ and the δ-function constraint, one can finally rewrite the index as

$$\begin{aligned}\mathcal{I}_{\text{cylinder}}(q; \mathbf{a}, \mathbf{b}) &= \text{PE}\left[\frac{q}{1-q}\left(\chi_{\text{adj}}(\mathbf{b}) + (\chi_{\mathfrak{f}}(\mathbf{a})\chi_{\mathfrak{f}}(\mathbf{b}) - 1)\right) + \chi_{\mathfrak{f}}(\mathbf{a})\chi_{\mathfrak{f}}(\mathbf{b}) - 1\right] \\ &= \text{PE}\left[\frac{q}{1-q}\chi_{\text{adj}}(\mathbf{b}) + \frac{1}{1-q}\left(\chi_{\mathfrak{f}}(\mathbf{a})\chi_{\mathfrak{f}}(\mathbf{b}) - 1\right)\right]\ .\end{aligned} \tag{B.2.6}$$

Note that this looks like the partition function of a finitely generated chiral algebra satisfying a single relation. Namely, it appears that the chiral algebra has one set of dimension one currents transforming in the adjoint of $\mathfrak{su}(n)$, in addition to a bifundamental field g_{ab} of dimension zero subject to a dimension zero constraint in the singlet representation. Going further, using this interpretation of the index and reintroducing the fugacity u as in (B.2.4), we see that u counts the power of the bifundamental generators in an operator, and the constraint should then involve n bifundamental fields. A natural form for such a relation (after proper rescaling of the generators) is the following,

$$\frac{1}{n!}\epsilon^{a_1 a_2 \dots a_n}\epsilon^{b_1 b_2 \dots b_n} g_{a_1 b_1} g_{a_2 b_2} \dots g_{a_n b_n} = 1. \tag{B.2.7}$$

Interpreting g_{ab} as a matrix, this is a unit determinant condition. This picture, guessed on the basis of the superconformal index, will be borne out in the qDS analysis below.

Cap A similarly heuristic analysis is possible for the theory associated to a decorated cap, which is obtained by further partially closing a puncture in the cylinder theory. The index of the decorated cap theory takes the form

$$\begin{aligned}
\mathcal{I}_{\mathrm{cap}(\Lambda)}\,(q;\mathbf{a},\mathbf{b}_\Lambda) &= K_{\mathrm{max.}}(\mathbf{a};q)K_\Lambda(\mathbf{b}_\Lambda,q)\sum_{\mathfrak{R}}\chi_{\mathfrak{R}}(\mathbf{a})\chi_{\mathfrak{R}}(\mathrm{fug}_\Lambda(\mathbf{b}_\Lambda;q)) \\
&= \mathrm{PE}\left[\frac{q}{1-q}\chi_{\mathrm{adj}}(\mathbf{a})+\sum_j\frac{q^{j+1}}{1-q}\mathrm{Tr}_{\mathcal{R}_j^{(\mathrm{adj})}}(\mathbf{b}_\Lambda)\right]\sum_{\mathfrak{R}}\chi_{\mathfrak{R}}(\mathbf{a})\chi_{\mathfrak{R}}(\mathrm{fug}_\Lambda(\mathbf{b}_\Lambda;q)) \\
&= I_V^{-1/2}(\mathbf{a};q)\;K_\Lambda(\mathbf{b}_\Lambda,q)\;\delta(\mathbf{a}^{-1},\mathrm{fug}_\Lambda(\mathbf{b}_\Lambda;q))\;.
\end{aligned} \tag{B.2.8}$$

Again it is clear how gluing this index reduces the flavor symmetry of the puncture. Using (3.2.42) and the general expression for a class $\mathcal{S}$ index (3.2.28) for some theory $\mathcal{T}$ of genus g and containing s punctures, of which the first is maximal with corresponding flavor fugacities $\mathbf{a}$, one obtains by gluing the cap to this maximal puncture

$$\int [d\mathbf{a}]\Delta(\mathbf{a}) I_V(\mathbf{a};q)\ \mathcal{I}_{\text{cap}(\Lambda)}\left(q;\mathbf{a}^{-1},\mathbf{b}_\Lambda\right)$$
$$\times \sum_{\mathfrak{R}} C_{\mathfrak{R}}(q)^{2g-2+s} K_{\text{max.}}(\mathbf{a};q)\chi_{\mathfrak{R}}(\mathbf{a}) \prod_{i=2}^{s} \psi_{\mathfrak{R}}^{\Lambda_i}(\mathbf{x}_{\Lambda_i};q) \tag{B.2.9}$$
$$= \sum_{\mathfrak{R}} C_{\mathfrak{R}}(q)^{2g-2+s} K_{\Lambda}(\mathbf{b}_\Lambda,q)\chi_{\mathfrak{R}}(\text{fug}_\Lambda(\mathbf{b}_\Lambda;q)) \prod_{i=2}^{s} \psi_{\mathfrak{R}}^{\Lambda_i}(\mathbf{x}_{\Lambda_i};q)\ , \tag{B.2.10}$$

where we have again used that $K_{\text{max.}}(\mathbf{a};q) = I_V^{-1/2}(\mathbf{a};q)$.

As in the previous paragraph we can rewrite the index in a suggestive fashion,

$$\mathcal{I}_{\text{cap}(\Lambda)}\left(q;\mathbf{a},\mathbf{b}_\Lambda\right) \ =$$
$$\text{PE}\left[\sum_j \frac{q^{j+1}}{1-q}\text{Tr}_{\mathcal{R}_j^{(\text{adj})}}(\mathbf{b}_\Lambda) + \frac{1}{1-q}\left(\chi_{\mathfrak{f}}(\mathbf{a})\chi_{\mathfrak{f}}(\text{fug}_\Lambda(\mathbf{b}_\Lambda;q)) - 1\right)\right]\ . \tag{B.2.11}$$

A natural interpretation of this index is as that of a chiral algebra with generators given by currents $J_{\bar{\alpha}}$ for $T_{\bar{\alpha}} \in \ker(ad_{\Lambda(t_+)})$ with dimensions shifted by their $\Lambda(t_0)$ weight. Moreover, for each $\mathfrak{su}(2)$ irrep in the decomposition (3.2.30) of the fundamental representation $\mathfrak{f}$ there are an additional $2j+1$ generators transforming in representation $\mathfrak{f}\otimes\mathcal{R}_j^{(\mathfrak{f})}$ with dimensions $-j,-j+1,\ldots,j$. The latter generators satisfy a singlet relation of dimension zero.

B.2.2 QDS argument

Now that we have some intuition for the kinds of chiral algebras to expect, let us study the cylinder theory for $\mathfrak{g} = \mathfrak{su}(2)$ by fully closing a puncture in the $\chi[T_3]$ theory. Full closure is achieved via the principal embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, which is can be specified explicitly in components as

$$\rho(t_-) = 2({T_2}^1 + {T_3}^2)\ ,\qquad \rho(t_0) = {T_1}^1 - {T_3}^3\ ,\qquad \rho(t_+) = {T_1}^2 + {T_2}^3\ . \tag{B.2.12}$$

The grading by $\rho(t_0)$ is integral, with the negatively graded generators being ${T_3}^1$ with grade minus two and ${T_2}^1, {T_3}^2$ with grade minus one. We should

dimension	generators
0	$\mathcal{W}_{3bc}, \tilde{\mathcal{W}}^{1bc}$
1	$\mathcal{W}_{2bc}, \tilde{\mathcal{W}}^{2bc}, (\mathcal{J}^2)_{b_1}{}^{b_2}, (\mathcal{J}^3)_{c_1}{}^{c_2}$
2	$\hat{\mathcal{J}}_{\rm sum}, \mathcal{W}_{1bc}, \tilde{\mathcal{W}}^{3bc}$
3	$(\hat{\mathcal{J}}^1)_1{}^3$

Table B.2: (Redundant) generators of the cylinder theory for $\mathfrak{g} = \mathfrak{su}(3)$.

then impose the constraints

$$\big(J^{(1)}\big)_2{}^1 + \big(J^{(1)}\big)_3{}^2 = 1 \ , \qquad \big(J^{(1)}\big)_2{}^1 - \big(J^{(1)}\big)_3{}^2 = 0 \ , \qquad \big(J^{(1)}\big)_3{}^1 = 0 \ . \tag{B.2.13}$$

Upon introducing three (b,c)-ghost systems – $(b_2{}^1, c_1{}^2)$, $(b_3{}^2, c_2{}^3)$, and $(b_3{}^1, c_1{}^3)$ – these first-class constraints are implemented by a BRST procedure via the current

$$d(z) = \\ (J^{(1)})_2{}^1 c_1{}^2(z) + (J^{(1)})_3{}^2 c_2{}^3(z) + (J^{(1)})_3{}^1 c_1{}^3(z) - \frac{1}{2}(c_1{}^2 + c_2{}^3)(z) - b_3{}^1 c_1{}^2 c_2{}^3(z) \ . \tag{B.2.14}$$

This cohomological problem is partly solved by following the same approach as that advocated in Subsec. 3.4.1. The redundant generators of the reduced algebra are the tic-tac-toed versions of the currents $(\hat{J}^1)_1{}^3$ and $\hat{J}_{\rm sum} \equiv (\hat{J}^1)_1{}^2 + (\hat{J}^1)_2{}^3$, as well as of the generators $\{(J^2)_{b_1}{}^{b_2}, (J^3)_{c_1}{}^{c_2}, W_{abc}, \tilde{W}^{abc}\}$. These currents can be seen arranged according to their dimensions in Table B.2.

The explicit form of the tic-tac-toed generators if dimensions zero and one are fairly simple,

$$\mathcal{W}_{3bc} \ := \ W_{3bc} \ , \tag{B.2.15}$$

$$\tilde{\mathcal{W}}^{1bc} \ := \ \tilde{W}^{1bc} \ , \tag{B.2.16}$$

$$\mathcal{W}_{2bc} \ := \ W_{2bc} + 2W_{3bc}(\hat{J}^{(1)})_3{}^3 \ , \tag{B.2.17}$$

$$\tilde{\mathcal{W}}^{2bc} \ := \ \tilde{W}^{2bc} + 2\tilde{W}^{1bc}(\hat{J}^{(1)})_1{}^1 \ , \tag{B.2.18}$$

$$(\mathcal{J}^2)_{b_1}^{\ b_2} \ := \ (J^2)_{b_1}^{\ b_2} \ , \tag{B.2.19}$$

$$(\mathcal{J}^3)_{c_1}^{\ c_2} \ := \ (J^3)_{c_1}^{\ c_2} \ . \tag{B.2.20}$$

On the other hand, the higher dimensional generators are quite complicated,

$$\begin{aligned}\hat{\mathcal{J}}_{\text{sum}} \ := \ & (\hat{J}^1)_1{}^2 + (\hat{J}^1)_2{}^3 \\ & - \left(-2(2+k)\partial(\hat{J}^1)_2{}^2 - 4(2+k)\partial(\hat{J}^1)_3{}^3\right) \\ & - \left(2(\hat{J}^1)_1{}^1(\hat{J}^1)_2{}^2 + 2(\hat{J}^1)_1{}^1(\hat{J}^1)_3{}^3 + 2(\hat{J}^1)_2{}^2(\hat{J}^1)_3{}^3\right) \ ,\end{aligned} \tag{B.2.21}$$

$$\begin{aligned}\mathcal{W}_{1bc} \ := \ & W_{1bc} - \left(2W_{2bc}(\hat{J}^1)_1{}^1 - 2W_{3bc}(\hat{J}^1)_2{}^3\right) \\ & + \left(-4\left((\hat{J}^1)_1{}^1 + (\hat{J}^1)_2{}^2\right)W_{3bc}(\hat{J}^1)_3{}^3 - \frac{1}{3}(-20-12k)W_{3bc}\partial(\hat{J}^1)_3{}^3\right) \\ & - \frac{8}{3}\partial W_{3bc}(\hat{J}^1)_3{}^3 \ ,\end{aligned} \tag{B.2.22}$$

$$\begin{aligned}\tilde{\mathcal{W}}^{3bc} \ := \ & \tilde{W}^{3bc} - \left(2\tilde{W}^{2bc}(\hat{J}^1)_3{}^3 + 2\tilde{W}^{1bc}(\hat{J}^1)_2{}^3\right) + 4(\hat{J}^1)_3{}^3\tilde{W}^{1bc}(\hat{J}^1)_2{}^2 \\ & - \tilde{W}^{1bc}\partial\left(-\frac{4}{3}(\hat{J}^1)_1{}^1 + (8+4k)(\hat{J}^1)_3{}^3\right) \\ & - \partial\tilde{W}^{1bc}\left(-\frac{4}{3}(\hat{J}^1)_1{}^1 - \frac{4}{3}(\hat{J}^1)_3{}^3\right) \ ,\end{aligned} \tag{B.2.23}$$

$$\begin{aligned}(\hat{\mathcal{J}}^1)_1{}^3 \ := \ & (\hat{J}^1)_1{}^3 - \left(2(k+2)\partial(\hat{J}^1)_1{}^2 - 2(\hat{J}^1)_2{}^3\left((\hat{J}^1)_2{}^2 + (\hat{J}^1)_3{}^3\right)\right) \\ & + 2(\hat{J}^1)_1{}^2\left((\hat{J}^1)_1{}^1 + (\hat{J}^1)_2{}^2\right) + 4(4+4k+k^2)\partial^2(\hat{J}^1)_1{}^1 \\ & - 4(2+k)(\hat{J}^1)_1{}^1\partial(\hat{J}^1)_1{}^1 + 4(2+k)(\hat{J}^1)_1{}^1\partial(\hat{J}^1)_2{}^2 \\ & - 4\left((\hat{J}^1)_1{}^1 + (\hat{J}^1)_3{}^3\right)\left((\hat{J}^1)_2{}^2 + (\hat{J}^1)_3{}^3\right)\left((\hat{J}^1)_1{}^1 + (\hat{J}^1)_2{}^2\right) \ .\end{aligned} \tag{B.2.24}$$

Our next task should be to check for redundancies by computing null relations. This analysis is substantially complicated by the presence of dimension zero fields in the cohomology. This means that we don't have an algorithm for finding such redundancies that must terminate in principle. Instead, we use the nulls of T_3 to predict some of the nulls in the cylinder theory.

Dimension zero nulls Starting with the $(\mathbf{8},\mathbf{1},\mathbf{1})$ nulls and specializing the indices to $(a_1,a_2)=(3,1)$ we obtain the null relation

$$0=\frac{1}{3}W_{3bc}\tilde{W}^{1bc}+(J^1)_3{}^a(J^1)_a{}^1-3\partial(J^1)_3{}^1=\frac{1}{4}+\frac{1}{3}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc}+d(\ldots)\,. \quad \text{(B.2.25)}$$

Similarly, starting with the $(\mathbf{8},\mathbf{8},\mathbf{1})$ nulls and specializing the indices to $(a_1,a_2)=(3,1)$ we obtain the null relation

$$\begin{aligned} 0 &= \left(W_{3b_1c}\tilde{W}^{1b_2c}-\frac{1}{3}\delta_{b_1}^{b_2}W_{3bc}\tilde{W}^{1bc}\right)+(J^{(1)})_3{}^1(J^{(2)})_{b_1}{}^{b_2}\,, \\ &= \left(\mathcal{W}_{3b_1c}\tilde{\mathcal{W}}^{1b_2c}-\frac{1}{3}\delta_{b_1}^{b_2}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc}\right)+d(\ldots)\,. \end{aligned} \quad \text{(B.2.26)}$$

Similar nulls can be found by interchanging the second and third puncture. In summary, we have the relations

$$\mathcal{W}_{3b_1c}\tilde{\mathcal{W}}^{1b_2c}=-\frac{1}{4}\delta_{b_1}^{b_2}\,,\qquad \mathcal{W}_{3bc_1}\tilde{\mathcal{W}}^{1bc_2}=-\frac{1}{4}\delta_{c_1}^{c_2}\,. \quad \text{(B.2.27)}$$

This shows that, up to a rescaling, $\mathcal{W}_{3bc}(z)$ and $\tilde{\mathcal{W}}^{1bc}(z)$ can be thought of as inverses of one another.

Next, we look at the $(\bar{\mathbf{6}},\mathbf{3},\mathbf{3})$ nulls and specialize $a_1=a_2=1$, which gives us

$$\begin{aligned} 0 &= 2(J^1)_{\alpha_1}^{\ 1}W_{\alpha_2bc}\epsilon^{\alpha_1\alpha_21}+\tilde{W}^{1b_1c_1}\tilde{W}^{1b_2c_2}\epsilon_{bb_1b_2}\epsilon_{cc_1c_2}\,, \\ &= \mathcal{W}_{3bc}+\tilde{\mathcal{W}}^{1b_1c_1}\tilde{\mathcal{W}}^{1b_2c_2}\epsilon_{bb_1b_2}\epsilon_{cc_1c_2}+d(\ldots)\,. \end{aligned} \quad \text{(B.2.28)}$$

Similarly from the nulls in the $(\mathbf{6},\bar{\mathbf{3}},\bar{\mathbf{3}})$ we find

$$\begin{aligned} 0 &= (J^1)_3{}^{\alpha_1}\tilde{W}^{\alpha_2bc}\epsilon_{\alpha_1\alpha_23}+\frac{1}{2}W_{3b_1c_1}W_{3b_2c_2}\epsilon^{bb_1b_2}\epsilon^{cc_1c_2}\,, \\ &= -\frac{1}{2}\tilde{\mathcal{W}}^{1bc}+\frac{1}{2}\mathcal{W}_{3b_1c_1}\mathcal{W}_{3b_2c_2}\epsilon^{bb_1b_2}\epsilon^{cc_1c_2}+d(\ldots)\,. \end{aligned} \quad \text{(B.2.29)}$$

Combining these with the previous relations, we find that

$$\frac{1}{3!}\tilde{\mathcal{W}}^{1bc}\tilde{\mathcal{W}}^{1b_1c_1}\tilde{\mathcal{W}}^{1b_2c_2}\epsilon_{bb_1b_2}\epsilon_{cc_1c_2}=-\frac{1}{3!}\tilde{\mathcal{W}}^{1bc}\mathcal{W}_{3bc}=-\frac{1}{3!}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc}=\frac{1}{8}\,, \quad \text{(B.2.30)}$$

and

$$\frac{1}{3!}\mathcal{W}_{3bc}\mathcal{W}_{3b_1c_1}\mathcal{W}_{3b_2c_2}\epsilon^{bb_1b_2}\epsilon^{cc_1c_2} = \frac{1}{3!}\mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc} = -\frac{1}{8} \ . \tag{B.2.31}$$

These are conditions on the determinants of $\mathcal{W}_{3bc}$ and $\tilde{\mathcal{W}}^{1bc}$ thought of as three-by-three matrices. Note that we used the relation $\tilde{\mathcal{W}}^{1bc}\mathcal{W}_{3bc} = \mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc}$, which is true in cohomology:

$$\tilde{\mathcal{W}}^{1bc}\mathcal{W}_{3bc} = \mathcal{W}_{3bc}\tilde{\mathcal{W}}^{1bc} - d(9\partial b_3{}^1) \ . \tag{B.2.32}$$

If we now introduce rescaled operators $g_{bc} := -2\mathcal{W}_{3bc}$ and $\tilde{g}^{bc} := 2\tilde{\mathcal{W}}^{1bc}$, then g and $\tilde{g}$ have unit determinant and are inverses of one another. Because of the determinant condition, this also means that we can rewrite $\tilde{g}$ in terms of positive powers of g, so only one needs to be considered as an honest generator of the chiral algebra.

Dimension one nulls We can continue the same analysis at dimension one. The second relation in the $(\mathbf{3},\mathbf{3},\mathbf{3})$ representation gives us

$$(\mathcal{J}^2)_b{}^\beta\mathcal{W}_{3\beta c_1} = (\mathcal{J}^3)_{c_1}{}^\gamma\mathcal{W}_{3b\gamma} \ . \tag{B.2.33}$$

By taking the normal ordered product of both sides with $\tilde{\mathcal{W}}^{1bc_2}$ and reordering (ignoring BRST exact terms), we can make a sequence of replacements using the dimension zero relations of the previous paragraph and end up with the following implication,

$$\begin{aligned} & \tilde{\mathcal{W}}^{1bc_2}(\mathcal{J}^{(2)})_b{}^\beta\mathcal{W}_{3\beta c_1} &=& \ \tilde{\mathcal{W}}^{1bc_2}(\mathcal{J}^{(3)})_{c_1}{}^\gamma\mathcal{W}_{3b\gamma} \\ \Longrightarrow \quad & (\mathcal{J}^{(2)})_b{}^\beta g_{\beta c_1}\tilde{g}^{bc_2} &=& \ (\mathcal{J}^{(3)})_{c_1}{}^{c_2} - 3\left(g_{\beta c_1}\partial\tilde{g}^{\beta c_2} - \tfrac{1}{3}\delta_{c_1}^{c_2}g_{\beta\gamma}\partial\tilde{g}^{\beta\gamma}\right) \ . \end{aligned} \tag{B.2.34}$$

At last, we see that the current $\mathcal{J}^{(3)}$ is not an independent generator.

Other dimension one nulls can be obtained from the first equality in the $(\mathbf{3},\mathbf{3},\mathbf{3})$. Here we find

$$(J^1)_3{}^\alpha W_{\alpha bc} = (J^2)_b{}^\beta W_{3\beta c} \implies \frac{1}{2}\mathcal{W}_{2bc} + \frac{2}{3}\partial\mathcal{W}_{3bc} = (\mathcal{J}^2)_b{}^\beta\mathcal{W}_{3\beta c} \ , \tag{B.2.35}$$

which implies that the generator $\mathcal{W}_{2bc}$ is not independent. Similarly, from the $(\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}})$ relations one finds

$$(J^1)_\alpha{}^1\tilde{W}^{\alpha bc} = (J^2)_\beta{}^b\tilde{W}^{1\beta c} \implies \frac{1}{2}\tilde{\mathcal{W}}^{2bc} - \frac{2}{3}\partial\tilde{\mathcal{W}}^{1bc} = (\mathcal{J}^2)_\beta{}^b\tilde{\mathcal{W}}^{1\beta c}, \tag{B.2.36}$$

which implies that $\tilde{\mathcal{W}}^{2bc}$ is not an independent generator.

Based on the analysis of the index in B.2.1, we expect that all higher dimensional generators can be similarly related via null relations to composites of $\mathcal{J}^2$ and $g_{bc} = -2\mathcal{W}_{3bc}$. It would be interesting if this could be proven as a consequence of only the null states that are guaranteed to exist based on nulls of the unreduced theory, although such a simplification is not a necessary condition for the existence of the desired nulls.

B.3 Spectral sequences for double complexes

In this appendix we review some of the basics of spectral sequences. Standard references are [183, 184]. One can also consult Section 5.3 of [94] for a concise summary of some of the most useful statements.

One of the simplest spectral sequences makes an appearance when one considers a single cochain complex (M^*, d), where $d : M^p \to M^{p+1}$ is a differential of degree one satisfying $d \circ d = 0$. A *decreasing filtration* of M^* is a family of subspaces $\{F^pM;\ p \in \mathbb{Z}\}$ such that $F^{p+1}M \subseteq F^pM$ and $\cup_p F^pM = M$. We restrict our attention to *bounded differential filtrations*, which satisfy two additional properties:

- There exist $s, t \in \mathbb{Z}$ such that $F^pM = M$ for $p \leqslant t$ and $F^pM = 0$ for $p \geqslant s$.
- The filtration is compatible with the differential, *i.e.*, $d(F^pM) \subseteq F^pM$.

We further introduce the spaces $F^pM^r := F^pM \cap M^r$. One then says that the filtration is bounded in each dimension if it is bounded for each r. The associated graded vector space is defined as

$$E_0^{p,q}(M^*, F) := F^pM^{p+q}/F^{p+1}M^{p+q} \ . \tag{B.3.1}$$

Note that at the level of vector spaces one has $M^r \cong \oplus_{p+q=r} E_0^{p,q}$.

If F is a bounded differential filtration of (M^*, d), then also (F^pM, d) is a complex. The inclusion map $F^pM \hookrightarrow M$ descends to a map in cohomology $H(F^pM, d) \to H(M, d)$ which is however not necessarily injective. We denote the image of $H(F^pM, d)$ under this map as $F^pH(M, d)$. This defines a bounded filtration on $H(M, d)$.

A spectral sequence is defined as a collection of bigraded spaces $(E_r^{*,*}, d_r)$ where $r = 1, 2, \ldots$, the differentials d_r have degrees $(r, 1-r)$, and for all p, q, r one has $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$. A spectral sequence is said to converge to N^* if there exists a filtration F on N^* such that $E_\infty^{p,q} \cong E_0^{p,q}(N^*, F)$. The main theorem of concern is then for any complex (M, d) with a differential filtration F bounded in each dimension, one can find a spectral sequence with $E_1^{p,q} = H^{p+q}(F^pM/F^{p+1}M)$ that converges to $H^*(M, d)$. In favorable situations, one may have $d_r = 0$ for $r \geqslant r_0$ in which case the spectral sequence terminates: $E_{r_0}^{p,q} = E_\infty^{p,q}$.

Let us consider the case of a double complex $(M^{*,*}; d_0, d_1)$, where M is bigraded and d_0, d_1 are maps of degree $(1, 0)$ and $(0, 1)$ respectively, satisfying $d_0 \circ d_0 = d_1 \circ d_1 = d_0 \circ d_1 + d_1 \circ d_0$. Diagrammatically, a double complex is represented as:

$$\begin{tikzcd}
 & {} & {} & \\
{} \arrow[r, "d_0"] & M^{p,q+1} \arrow[u, "d_1"] \arrow[r, "d_0"] & M^{p+1,q+1} \arrow[u, "d_1"] \arrow[r, "d_0"] & {} \\
{} \arrow[r, "d_0"] & M^{p,q} \arrow[u, "d_1"] \arrow[r, "d_0"] & M^{p+1,q} \arrow[u, "d_1"] \arrow[r, "d_0"] & {} \\
 & {} \arrow[u, "d_1"] & {} \arrow[u, "d_1"] &
\end{tikzcd}$$

The associated *total complex* is defined as $\mathrm{Tot}^n M := \oplus_{p+q=n} M^{p,q}$, with total differential $d := d_0 + d_1$. A double complex allows for two filtrations, namely,

$$F_{\mathrm{I}}^p(\mathrm{Tot}^n\ M) = \oplus_{r \geqslant p} M^{r,n-r}\ , \qquad F_{\mathrm{II}}^p(\mathrm{Tot}^n\ M) = \oplus_{r \geqslant p} M^{n-r,r}\ . \qquad \text{(B.3.2)}$$

These filtrations are bounded in each dimension if for each n only a finite number of $M^{p,q}$ with $n = p + q$ are non-zero.

Correspondingly, we can consider two spectral sequences converging to $H^*(\mathrm{Tot}\ M, d)$ with as first terms

$$\begin{aligned}
{}_{\mathrm{I}}E_1^{p,q} &\cong H^{p,q}(M, d_1)\ , & {}_{\mathrm{I}}E_2^{p,q} &\cong H^{p,q}(H^{*,*}(M, d_1), d_0) & \text{(B.3.3)}\\
{}_{\mathrm{II}}E_1^{p,q} &\cong H^{p,q}(M, d_0)\ , & {}_{\mathrm{II}}E_2^{p,q} &\cong H^{p,q}(H^{*,*}(M, d_0), d_1)\ . & \text{(B.3.4)}
\end{aligned}$$

Note that here one can show that the first term of the spectral sequence is equal to the one mentioned in the more general case above. Higher differentials d_{r+1} for $r \geqslant 1$ are defined by $d_{r+1}x = d_1 y$ where y is defined by

$d_0 y = d_r x$. Such a y can be proven to always exist, so that the higher differentials are always well-defined.

Example As a simple example of the utility of spectral sequences, let us reproduce a proof of the Künneth formula [94]. Consider a differential graded algebra $(\mathbb{A}, d)$, *i.e.*, a graded algebra endowed with a differential d of degree one satisfying the Leibniz rule. Let it have two graded subalgebras $\mathbb{A}_1$ and $\mathbb{A}_2$ which are respected by the differential, *i.e.*, $d\mathbb{A}_i \subseteq \mathbb{A}_i$. Let us assume the multiplication map $m : \mathbb{A}_1 \otimes \mathbb{A}_2 \to \mathbb{A}$ is an isomorphism of vector spaces. Then one can define the double complex $(M^{p,q}; d_0, d_1)$ by

$$M^{p,q} := m(\mathbb{A}_1^p \otimes \mathbb{A}_2^q) \ , \quad d_0(a_1 a_2) = d(a_1) a_2 \ , \quad d_1(a_1 a_2) = (-1)^{\deg(a_1)} a_1 d(a_2) \ . \tag{B.3.5}$$

Assume that this double complex is bounded in each dimension; then one can make use of the spectral sequence for the double complex as described above. One finds for the first couple of levels

$$E_1^{p,q} \cong m(\mathbb{A}_1^p \otimes H^q(\mathbb{A}_2, d)) \ , \qquad E_2^{p,q} \cong m(H^p(\mathbb{A}_1, d) \otimes H^q(\mathbb{A}_2, d)) \ . \tag{B.3.6}$$

Higher differentials all manifestly vanish, so the spectral sequence terminates. At the level of vector spaces, the above-stated theorem implies that $H^*(\mathbb{A}, d) \cong m(H^*(\mathbb{A}_1, d) \otimes H^*(\mathbb{A}_2, d))$. This statement can be extended to an isomorphism of algebras because $a_1 a_2$ is a representative of an element in $H^*(\mathbb{A}, d)$.

Appendix C

Chiral Algebras for Trinion Theories

C.1 Affine critical characters and the Schur index

We show how to re-write the superconformal index [36] in the so-called Schur limit [37, 38] in terms of characters of affine Kac-Moody modules at the critical level. The superconformal index of class $\mathcal{S}$ theories was computed in [75, 37, 38, 74], and the characters of affine Kac-Moody algebras at the critical level in [107]. Here we just collect the final expressions and refer the readers to the original work for details.

Our conventions for affine Lie algebras follow those of [108], and here we simply review some notation needed to write the characters. We denote the affine Lie algebra obtained by adding an imaginary root δ to a finite Lie algebra $\mathfrak{g}$ (of rank r) by $\hat{\mathfrak{g}}$. The Cartan subalgebra of $\hat{\mathfrak{g}}$ ($\mathfrak{g}$) is denoted by $\hat{\mathfrak{h}}$ ($\mathfrak{h}$), and the positive roots of $\hat{\mathfrak{g}}$ ($\mathfrak{g}$) by $\hat{\Delta}_+$ (Δ_+). We also denote the real positive roots of the affine Lie algebra, that is positive roots not of the form $n\delta$, by $\hat{\Delta}^{\mathrm{re}}_+$. The character of a critical irreducible highest weight representation $\mathfrak{R}_\lambda$ with highest weight $\hat{\lambda}$, whose restriction to the finite Lie algebra λ is by definition an integral dominant weight is given in [107]. It reads [1]

$$\mathrm{ch}_{\mathfrak{R}_\lambda} = \frac{\sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta_+} (1-q^{\langle\lambda+\rho,\alpha^\vee\rangle}) \prod_{\hat{\alpha}\in\hat{\Delta}_+^{\mathrm{re}}} (1-e^{-\hat{\alpha}})}\,, \tag{C.1.1}$$

where W is the Weyl group of $\mathfrak{g}$, $\epsilon(w)$ is the signature of w, $q=e^{-\delta}$, ρ denotes the Weyl vector, $\langle\cdot,\cdot\rangle$ denotes the Killing inner product and $\alpha^\vee$ is the coroot associated to α.

The Schur limit of the superconformal index of a T_n theory is given by [37, 38]

$$\mathcal{I}_{T_n}(q;\mathbf{x}_i) = \sum_{\mathfrak{R}_\lambda} \frac{\prod_{i=1}^3 \mathcal{K}_\Lambda(q;\mathbf{x}_i)\ \chi_{\mathfrak{R}_\lambda}(\mathbf{x}_i)}{\mathcal{K}_{\Lambda^t}(q)\ \dim_q\mathfrak{R}_\lambda}\,, \tag{C.1.2}$$

with

$$\mathcal{K}_{\Lambda^t}(q) = \mathrm{P.E.}\left[\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q}\right]\,, \qquad \mathcal{K}_\Lambda(q;\mathbf{x}) = \mathrm{P.E.}\left[\frac{q\,\chi_{\mathrm{adj.}}(\mathbf{x})}{1-q}\right]\,. \tag{C.1.3}$$

Here $\mathbf{x}_i$ denotes flavor fugacities conjugate to the Cartan generators of the $\mathfrak{su}(n)_i$ flavor group associated with each of the three punctures, Λ and Λ^t are respectively the trivial and principal embeddings of $\mathfrak{su}(2)\hookrightarrow\mathfrak{su}(n)$, and d_j are the degrees of invariants. Furthermore, $\dim_q\mathfrak{R}_\lambda$ is the q-deformed dimension of the representation $\mathfrak{R}_\lambda$, *i.e.*,

$$\dim_q\mathfrak{R}_\lambda = \prod_{\alpha\in\Delta_+}\frac{[\langle\lambda+\rho,\alpha\rangle]_q}{[\langle\rho,\alpha\rangle]_q}\,, \qquad \text{where} \qquad [x]_q = \frac{q^{-\frac{x}{2}}-q^{\frac{x}{2}}}{q^{-\frac{1}{2}}-q^{\frac{1}{2}}}\,. \tag{C.1.4}$$

As shown in [26], if $\lambda=0$ is the highest weight of the vacuum module

$$\mathrm{ch}_{\mathfrak{R}_{\lambda=0}} = \frac{\mathcal{K}_\Lambda(q;\mathbf{x}_i)}{\mathcal{K}_{\Lambda^t}(q)}\,. \tag{C.1.5}$$

The expectation is that the full index for T_n can be re-written as a sum of characters of critical modules[2]. Re-writing (C.1.1) to make manifest the

[1]Here we used that for a critical highest weight $\hat{\lambda}+\hat{\rho}=\lambda+\rho$, and normalized the character to match the standard conventions for a partition function.

[2]Along similar lines, one can rewrite the Schur limit of the superconformal index of the $T_{SO(2n)}$ theory [77, 6] in terms of critical affine $\widehat{\mathfrak{so}(2n)}$ characters.

vacuum module we find

$$\mathrm{ch}_{\mathfrak{R}_\lambda} = \mathrm{ch}_{\mathfrak{R}_{\lambda=0}} \prod_{\alpha\in\Delta_+} \left(\frac{1-q^{\langle\rho,\alpha^\vee\rangle}}{1-q^{\langle\lambda+\rho,\alpha^\vee\rangle}} \right) \frac{\sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W} \epsilon(w) e^{w(\rho)-\rho}} \,, \tag{C.1.6}$$

where we recognize the last term as the character of the representation with highest weight λ of $\mathfrak{g}$,

$$\chi_\lambda(\mathbf{x}) = \frac{\sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w) e^{w(\rho)}} \,. \tag{C.1.7}$$

After factoring out a $q^{-\langle\lambda,\rho\rangle}$, the middle factor can be written in terms of the q-deformed dimension (C.1.4) of the same representation:

$$\prod_{\alpha\in\Delta_+} \left(\frac{1-q^{\langle\rho,\alpha^\vee\rangle}}{1-q^{\langle\lambda+\rho,\alpha^\vee\rangle}} \right) = \prod_{\alpha\in\Delta_+} q^{-\langle\lambda,\alpha^\vee\rangle/2} \prod_{\alpha\in\Delta_+} \left(\frac{q^{-\langle\rho,\alpha^\vee\rangle/2} - q^{\langle\rho,\alpha^\vee\rangle/2}}{q^{-\langle\lambda+\rho,\alpha^\vee\rangle/2} - q^{\langle\lambda+\rho,\alpha^\vee\rangle/2}} \right) \tag{C.1.8}$$

$$= q^{-\langle\lambda,\rho\rangle} \frac{1}{\dim_q \mathfrak{R}_\lambda} \,, \tag{C.1.9}$$

where we used that $\alpha^\vee = \alpha$, for $\mathfrak{su}(n)$, to identify ρ in the last step. In total we thus find

$$\mathrm{ch}_{\mathfrak{R}_\lambda} = \frac{\mathrm{P.E.}\left[\frac{q\,\chi_{\mathrm{adj.}}(\mathbf{x})}{1-q}\right] \chi_\lambda(\mathbf{x})}{q^{\langle\lambda,\rho\rangle}\mathrm{P.E.}\left[\sum_{j=1}^{n-1} \frac{q^{d_j}}{1-q}\right] \dim_q \mathfrak{R}_\lambda} \,. \tag{C.1.10}$$

Using this result in the expression for the superconformal index (C.1.2) we obtain (4.2.1). To obtain (4.2.2) we also note that the denominator of (C.1.10) can be rewritten as

$$q^{\langle\lambda,\rho\rangle}\ \mathrm{P.E.}\left[\sum_{j=1}^{n-1} \frac{q^{d_j}}{1-q}\right]\ \dim_q \mathfrak{R}_\lambda = \tag{C.1.11}$$

$$= \mathrm{P.E.}\left[\sum_{j=1}^{n-1} \frac{q^{d_j}}{1-q} + \sum_{\alpha\in\Delta_+} q^{\langle\rho,\alpha\rangle} - \sum_{\alpha\in\Delta_+} q^{\langle\lambda+\rho,\alpha\rangle}\right]$$

$$= \mathrm{P.E.}\left[\sum_{j=1}^{n-1} \frac{q^{d_j}}{1-q} + \sum_{j=1}^{n-1} (n-j)\, q^j - \sum_{j=2}^{n} \sum_{1\le i<j} q^{\ell_i - \ell_j + j - i}\right] . \tag{C.1.12}$$

C.2 The OPEs

In this appendix we give all the OPEs between the generators of the T_4 chiral algebra. Here all OPE coefficients (including the central charges) are already set to the values required by the Jacobi-identities, as described in Section 4.3. Since all generators are both Virasoro and AKM primaries, with the exception of the stress tensor which is neither and the AKM currents which are not AKM primaries, all singular OPEs involving the affine currents and the stress tensor are completely fixed by flavor symmetries and Virasoro symmetry, up to the flavor central charges $(k_{2d})_{i=1,2,3} = -4$ and the Virasoro central charge $c_{2d} = -78$ appearing in the most singular term in their respective self-OPEs. Different AKM currents are taken to have zero singular OPE. As discussed in Section 4.3 we consistently treat the three flavor symmetries on equal footing, in particular we require $k_{2d} \equiv (k_{2d})_1 = (k_{2d})_2 = (k_{2d})_3$. We recall that also the precise values of c_{2d} and k_{2d} central charges are a result of imposing the Jacobi-identities.

The singular OPEs of the $W, \widetilde{W}$ generators among themselves were found to be

$$\begin{aligned}
W_{a_1a_2a_3}(z)\ W_{b_1b_2b_3}(0) &\sim \frac{1}{2\,z} V_{[a_1b_1][a_2b_2][a_3b_3]}\ ,\\
\widetilde{W}^{a_1a_2a_3}(z)\ \widetilde{W}^{b_1b_2b_3}(0) &\sim \frac{1}{2\,z}\frac{1}{8}\epsilon^{a_1b_1c_1d_1}\epsilon^{a_2b_2c_2d_2}\epsilon^{a_3b_3c_3d_3}V_{[c_1d_1][c_2d_2][c_3d_3]}\ ,
\end{aligned}$$

and

$$\begin{aligned}
W_{a_1a_2a_3}(z)\ \widetilde{W}^{b_1b_2b_3}(0) \sim&\\
&\frac{1}{z^3}\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\delta_{a_3}^{b_3} - \frac{1}{4\,z^2}\left(\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}(J^3)_{a_3}^{b_3} + \text{perms.}\right)\\
&-\frac{1}{4\,z}\left(\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\partial(J^3)_{a_3}^{b_3} + \text{perms.}\right) + \frac{1}{16\,z}\left(\delta_{a_1}^{b_1}(J^2)_{a_2}^{b_2}(J^3)_{a_3}^{b_3} + \text{perms.}\right)\\
&+\frac{1}{z}\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\delta_{a_3}^{b_3}\left(-\frac{1}{16}T - \frac{1}{96}\left((J^1)_{\beta_1}^{\alpha_1}(J^1)_{\alpha_1}^{\beta_1} + 2\text{ more}\right)\right)\\
&+\frac{1}{16\,z}\left(\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}(J^3)_{a_3}^{\alpha_3}(J^3)_{\alpha_3}^{b_3} + \text{perms.}\right)\ ,
\end{aligned}$$

where we have fixed the normalization of W and $\widetilde{W}$ to convenient values. In all these OPEs "+2 more" means we must add the same term for the remaining two currents, and "+perms." that all independent permutations of the previous term must be added. We also found the OPEs between the $W, \widetilde{W}$ and V generators to be

$$
\begin{aligned}
&W_{a_1a_2a_3}(z)\ V_{[b_1c_1][b_2c_2][b_3c_3]}(0)\ \sim \\
&\qquad \frac{1}{8}\epsilon_{a_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}\left(-\frac{3}{z^2}\widetilde{W}^{d_1d_2d_3}-\frac{1}{z}\partial\widetilde{W}^{d_1d_2d_3}\right)\\
&\qquad -\frac{1}{8}\epsilon_{a_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}\frac{1}{3\,z}\left((J^1)^{d_1}_{\alpha_1}\widetilde{W}^{\alpha_1d_2d_3}+\text{perms.}\right)\\
&\qquad -\frac{1}{8\,z}\left(\epsilon_{\alpha_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}(J^1)^{d_1}_{a_1}\widetilde{W}^{\alpha_1d_2d_3}+\text{perms.}\right)\,,\\
&\widetilde{W}^{a_1a_2a_3}(z)\ V_{[b_1c_1][b_2c_2][b_3c_3]}(0)\ \sim \\
&\qquad \delta^{a_1}_{[b_1}\delta^{a_2}_{[b_2}\delta^{a_3}_{[b_3}\left(\frac{3}{z^2}W_{c_1]c_2]c_3]}+\frac{1}{z}\partial W_{c_1]c_2]c_3]}\right)\\
&\qquad +\frac{1}{3\,z}\delta^{a_1}_{[b_1}\delta^{a_2}_{[b_2}\delta^{a_3}_{[b_3}\left((J^1)^{\alpha_1}_{c_1]}W_{\alpha_1c_2]c_3]}+\text{perms.}\right)\\
&\qquad +\frac{1}{z}\left(\delta^{a_2}_{[b_2}\delta^{a_3}_{[b_3}(J^1)^{a_1}_{[b_1}W_{c_1]c_2]c_3]}+\text{perms.}\right)\,.
\end{aligned}
$$

Finally, the singular VV OPE reads

$$
\begin{aligned}
&V_{[a_1b_1][a_2b_2][a_3b_3]}(z)V^{[c_1d_1][c_2d_2][c_3d_3]}(0)\\
&\sim\ \delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\Bigg(\frac{6}{z^4}-\frac{1}{2\,z^2}T-\frac{1}{4\,z}\partial T-\frac{1}{24\,z^2}\left((J^1)^{\alpha_1}_{\beta_1}(J^1)^{\beta_1}_{\alpha_1}+2\text{ more}\right)\\
&\qquad -\frac{19}{480\,z}\partial\left((J^1)^{\alpha_1}_{\beta_1}(J^1)^{\beta_1}_{\alpha_1}+2\text{ more}\right)-\frac{37}{20\,z}W_{\alpha_1\beta_1\gamma_1}\widetilde{W}^{\alpha_1\beta_1\gamma_1}\\
&\qquad +\frac{1}{40\,z}\left((J^1)^{\beta_1}_{\alpha_1}(J^1)^{\gamma_1}_{\beta_1}(J^1)^{\alpha_1}_{\gamma_1}+2\text{ more}\right)\Bigg)\\
&+\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\Bigg(-\frac{3}{16\,z}\delta^{[d_1}_{[b_1}(J^1)^{|\gamma_1}_{a_1]}(J^1)^{\beta_1|}_{\gamma_1}(J^1)^{c_1]}_{\beta_1}+\frac{33}{80\,z}(J^1)^{\beta_1}_{[a_1}(J^1)^{[d_1}_{|\beta_1|}(J^1)^{c_1]}_{b_1]}\\
&\qquad -\frac{43}{80\,z^2}\partial\left((J^1)^{[d_1}_{[a_1}(J^1)^{c_1]}_{b_1]}\right)-\frac{3}{2\,z^3}(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}+\frac{1}{40\,z^3}T(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}\\
&\qquad -\frac{11}{120\,z}\left((J^1)^{\alpha_1}_{\beta_1}(J^1)^{\beta_1}_{\alpha_1}+2\text{ more}\right)(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}-\frac{5}{4\,z^2}\partial(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}\\
&\qquad +\frac{1}{4\,z^2}(J^1)^{\alpha_1}_{[a_1}(J^1)^{[c_1}_{|\alpha_1|}\delta^{d_1]}_{b_1]}+\frac{17}{40\,z}\partial(J^1)^{\alpha_1}_{[a_1}(J^1)^{[c_1}_{|\alpha_1|}\delta^{d_1]}_{b_1]}\\
&\qquad +\frac{7}{40\,z}(J^1)^{\alpha_1}_{[a_1}\partial(J^1)^{[c_1}_{|\alpha_1|}\delta^{d_1]}_{b_1]}-\frac{23}{80\,z}\partial^2(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}-\frac{1}{4\,z}(J^1)^{[d_1}_{[a_1}(J^1)^{c_1]}_{b_1]}\Bigg)\\
&\qquad +\text{permutations}[1,2,3]
\end{aligned}
$$

$$
\begin{aligned}
&+\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\left(\frac{1}{4\,z}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}(J^1)^{[d_1}_{[a_1}(J^1)^{c_1]}_{b_1]}+\frac{3}{4\,z^2}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}+\right.\\
&\qquad\qquad\left.+\frac{13}{4\,z}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}\partial(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}-\frac{3}{4\,z}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}(J^1)^{\alpha_1}_{[a_1}(J^1)^{c_1]}_{|\alpha_1|}\delta^{d_1}_{b_1]}\right)\\
&\qquad+\text{permutations}[1,2,3]\\
&+\frac{19}{5\,z}\left(\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\delta^{[d_1}_{[b_1}W_{a_1]\beta_2\gamma_3}\widetilde{W}^{c_1]\beta_2\gamma_3}+\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[d_2}_{[b_2}W_{\beta_1 a_2]\gamma_3}\widetilde{W}^{\beta_1 c_2]\gamma_3}\right.\\
&\qquad\left.+\delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[d_3}_{[b_3}W_{\beta_1\gamma_2 a_3]}\widetilde{W}^{\beta_1\gamma_2 c_3]}\right)\\
&-\frac{4}{z}\left(\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\delta^{[d_1}_{[b_1}\delta^{[d_2}_{[b_2}W_{a_1]a_2]\gamma_3}\widetilde{W}^{c_1]c_2]\gamma_3}+\delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[d_2}_{[b_2}\delta^{[d_3}_{[b_3}W_{\gamma_1 a_2]a_3]}\widetilde{W}^{\gamma_1 c_2]c_3]}\right.\\
&\qquad\left.+\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[d_3}_{[b_3}\delta^{[d_1}_{[b_1}W_{a_1]\gamma_2 a_3]}\widetilde{W}^{c_1]\gamma_2 c_3]}\right)\\
&-\frac{1}{z}\delta^{[d_1}_{[b_1}\delta^{[d_2}_{[b_2}\delta^{[d_3}_{[b_3}(J^1)^{c_1]}_{a_1]}(J^2)^{c_2]}_{a_2]}(J^3)^{c_3]}_{a_3]}-\frac{16}{z}\delta^{[d_1}_{[b_1}\delta^{[d_2}_{[b_2}\delta^{[d_3}_{[b_3}W_{a_1]a_2]a_3]}\widetilde{W}^{c_1]c_2]c_3]}\,,
\end{aligned}
$$

where the norm of V was also fixed, and for convenience we defined $V^{[c_1d_1][c_2d_2][c_3d_3]}$ through $V_{[a_1b_1][a_2b_2][a_3b_3]}=\frac{1}{8}\epsilon_{a_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}V^{[c_1d_1][c_2d_2][c_3d_3]}$. Here "permutations[1,2,3]" means we must repeat the previous term with all possible permutations of the flavor groups indices.

Appendix D

Higgs Branch Localization in Three Dimensions

D.1 Spinor conventions

We use essentially the same conventions as in [118, 119, 15]. In vielbein space we take the gamma matrices $\gamma^a = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ which do not have definite symmetry: $[\gamma^1, \gamma^2, \gamma^3]^\mathsf{T} = [\gamma^1, -\gamma^2, \gamma^3]$. We take the charge conjugation matrix C, defined by $C\gamma^\mu C^{-1} = -\gamma^{\mu\mathsf{T}}$, as $C = -i\varepsilon_{\alpha\beta} = \gamma_2$ (where $\varepsilon_{12} = \varepsilon^{12} = 1$) so that

$$C\gamma^\mu C = -\gamma^{\mu\mathsf{T}} , \qquad C^2 = \mathbb{1} . \tag{D.1.1}$$

Indeed $C = C^{-1} = C^\dagger = -C^\mathsf{T} = -C^*$. Since Dirac spinors are in the **2** of $SU(2)$, there are two products we can consider: $\eta^\mathsf{T} C\epsilon \equiv -i\,\eta^\alpha \varepsilon_{\alpha\beta}\epsilon^\beta$ and $\eta^\dagger\epsilon \equiv \eta^*_\alpha \epsilon^\alpha$. When we use the first product, we omit ${}^\mathsf{T}C$ (that is we write $\eta\epsilon \equiv \eta^\mathsf{T} C\epsilon$). The two products are related by charge conjugation: $\epsilon^c \equiv C\epsilon^*$ and $\epsilon^{c\dagger} = \epsilon^\mathsf{T} C$, so that $\eta^\mathsf{T} C\epsilon = \eta^{c\dagger}\epsilon$. Notice that $(\epsilon^c)^c = -\epsilon$ and there are no Majorana spinors.

Barred spinors will simply be independent spinors. Products are constructed as spelled out before: $\bar\epsilon\lambda \equiv \bar\epsilon^\alpha C_{\alpha\beta}\lambda^\beta$, $\bar\epsilon\gamma^\mu\lambda \equiv \bar\epsilon^\alpha (C\gamma^\mu)_{\alpha\beta}\lambda^\beta$, etc... The charge conjugation matrix C is antisymmetric, while $C\gamma^a$ are symmetric and so $C\gamma^\mu$. Since $\gamma^{\mu\nu}$ equals a single gamma matrix or zero, also $C\gamma^{\mu\nu}$ are symmetric. For anticommuting fermions we get:

$$\bar\epsilon\lambda = \lambda\bar\epsilon , \qquad \bar\epsilon\gamma^\mu\lambda = -\lambda\gamma^\mu\bar\epsilon , \qquad \bar\epsilon\gamma^{\mu\nu}\lambda = -\lambda\gamma^{\mu\nu}\bar\epsilon . \tag{D.1.2}$$

Some useful relations among gamma matrices are:

$$
\begin{aligned}
[\gamma_\mu, \gamma_\nu] &= 2g_{\mu\nu} \;, \qquad \gamma_\mu\gamma_\nu = g_{\mu\nu} + \gamma_{\mu\nu} \;, \\
\gamma^{\mu\nu} &= i\varepsilon^{\mu\nu\rho}\gamma_\rho \;, \qquad \gamma^{\mu\nu}\varepsilon_{\mu\nu\rho} = 2i\gamma_\rho \\
\gamma_\mu\gamma^{\nu\rho} &= i\varepsilon^{\nu\rho\sigma}g_{\mu\sigma} + (\delta^\nu_\mu\delta^\rho_\alpha - \delta^\nu_\alpha\delta^\rho_\mu)\gamma^\alpha \;, \qquad \gamma_{\mu\nu}\gamma^\nu = -\gamma^\nu\gamma_{\mu\nu} = 2\gamma_\mu \\
\gamma_\mu\gamma^{\nu\rho}\gamma^\mu &= -\gamma^{\nu\rho} \;, \qquad \gamma_\mu\gamma^\nu\gamma^\mu = -\gamma^\nu \;, \\
\gamma^\mu\gamma_\mu &= 3 \;, \qquad \gamma_{\mu\nu}\gamma^{\rho\nu} = -2\delta^\rho_\mu - \gamma_\mu{}^\rho \\
\gamma^{\mu\nu}\gamma_\rho\gamma_\nu &= -2\delta^\mu_\rho \;, \qquad \gamma^{\mu\nu}\gamma_\rho\gamma_{\mu\nu} = 2\gamma_\rho \;.
\end{aligned}
\tag{D.1.3}
$$

The antisymmetric tensor with flat indices is $\varepsilon^{\hat{1}\hat{2}\hat{3}} = \varepsilon_{\hat{1}\hat{2}\hat{3}} = 1$, and the covariant forms with curved indices are $\varepsilon_{\mu\nu\rho} = \sqrt{g}\,\varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}}$ and $\varepsilon^{\mu\nu\rho} = \frac{1}{\sqrt{g}}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}}$.

The Fierz identity for anticommuting 3d Dirac fermions is

$$
(\bar{\lambda}_1\lambda_2)\lambda_3 = -\frac{1}{2}(\bar{\lambda}_1\lambda_3)\lambda_2 - \frac{1}{2}(\bar{\lambda}_1\gamma^\rho\lambda_3)\,\gamma_\rho\lambda_2 \;. \tag{D.1.4}
$$

Since γ^α and $\gamma^{\mu\nu}$ are dual, one finds

$$
(\gamma_{\mu\rho})_{**}(\gamma^\rho)_{**} = (\gamma^\rho)_{**}(\gamma_{\rho\mu})_{**} \;, \qquad -2\,(\gamma_\mu)_{**}(\gamma^\mu)_{**} = (\gamma_{\nu\rho})_{**}(\gamma^{\nu\rho})_{**} \tag{D.1.5}
$$

where indices are not contracted. It might also be useful:

$$
-\frac{i}{4}\bar{\epsilon}\gamma^\rho\gamma^{\mu\nu}\epsilon\,\gamma_\rho\gamma_\nu\mathcal{O}_\mu\lambda = -\frac{i}{2}\bar{\epsilon}\epsilon\,\gamma^\mu\mathcal{O}_\mu\lambda + \frac{i}{2}\bar{\epsilon}\gamma^\mu\epsilon\,\mathcal{O}_\mu\lambda + \frac{i}{4}\bar{\epsilon}\gamma^{\alpha\rho}\epsilon\,\gamma_\rho\mathcal{O}_\alpha\lambda \;, \tag{D.1.6}
$$

where $\mathcal{O}_\mu$ is any operator, acting on any field.

D.2 Supersymmetric theories on three-manifolds

Following [118, 119], we write the superconformal transformation rules on the gauge and matter multiplets on a three-dimensional manifold. The manifold is restricted by the requirement that it admits solutions to the usual Killing spinor equations, and that the superalgebra closes. After presenting the supersymmetry variations, in section D.2.2 we present the anticommuting supercharges by replacing the anticommuting Killing spinors in δ_ϵ and $\delta_{\bar{\epsilon}}$ with their commuting counterparts. Lagrangians invariant under the supersymmetry transformations were studied in [118, 119]. Most of them are exact and therefore will not contribute in a localization computation. Notable exceptions are the Chern-Simons and Fayet-Iliopoulos actions. A more systematic analysis of SUSY on three-manifolds has been done in [138, 139].

D.2.1 The superconformal algebra

We define the field strength as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, and the gauge and metric covariant derivative as $D_\mu = \nabla_\mu - iA_\mu$, where ∇_μ is the metric-covariant derivative. It follows, for instance, that for an adjoint scalar σ: $[D_\mu, D_\nu]\sigma = -i[F_{\mu\nu}, \sigma]$. We will also turn on a background gauge field V_μ for $U(1)_R$, therefore

$$D_\mu = \nabla_\mu - iA_\mu - iV_\mu \ . \tag{D.2.1}$$

The superconformal transformations of the vector multiplet are

$$\begin{aligned}
\delta A_\mu &= -\frac{i}{2}(\bar\epsilon\gamma_\mu\lambda - \bar\lambda\gamma_\mu\epsilon) \qquad\qquad \delta\sigma = \frac{1}{2}(\bar\epsilon\lambda - \bar\lambda\epsilon) \\
\delta\lambda &= \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon \\
\delta\bar\lambda &= \frac{1}{2}\gamma^{\mu\nu}\bar\epsilon F_{\mu\nu} + D\bar\epsilon - i\gamma^\mu\bar\epsilon D_\mu\sigma - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar\epsilon \\
\delta D &= -\frac{i}{2}\bar\epsilon\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar\lambda\gamma^\mu\epsilon + \frac{i}{2}[\bar\epsilon\lambda, \sigma] + \frac{i}{2}[\bar\lambda\epsilon, \sigma] \\
&\quad - \frac{i}{6}(D_\mu\bar\epsilon\gamma^\mu\lambda + \bar\lambda\gamma^\mu D_\mu\epsilon) \ ,
\end{aligned} \tag{D.2.2}$$

and those of the chiral multiplet are

$$\begin{aligned}
\delta\phi &= \bar\epsilon\psi & \delta\psi &= i\gamma^\mu\epsilon\, D_\mu\phi + i\epsilon\sigma\phi + \frac{2iq}{3}\gamma^\mu D_\mu\epsilon\,\phi + \bar\epsilon F \\
\delta\bar\phi &= \bar\psi\epsilon & \delta\bar\psi &= i\gamma^\mu\bar\epsilon\, D_\mu\bar\phi + i\bar\epsilon\bar\phi\sigma + \frac{2iq}{3}\gamma^\mu D_\mu\bar\epsilon\,\bar\phi + \epsilon\bar F \\
&& \delta F &= \epsilon\big(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi\big) + \frac{i}{3}(2q-1)D_\mu\epsilon\,\gamma^\mu\psi \\
&& \delta\bar F &= \bar\epsilon\big(i\gamma^\mu D_\mu\bar\psi - i\bar\psi\sigma + i\bar\phi\bar\lambda\big) + \frac{i}{3}(2q-1)D_\mu\bar\epsilon\,\gamma^\mu\bar\psi \ .
\end{aligned} \tag{D.2.3}$$

Here ϵ and $\bar\epsilon$ are independent spinors satisfying the Killing spinor equations

$$D_\mu\epsilon = \gamma_\mu\hat\epsilon \ , \qquad\qquad D_\mu\bar\epsilon = \gamma_\mu\hat{\bar\epsilon} \ , \tag{D.2.4}$$

in terms of some other spinors $\hat\epsilon, \hat{\bar\epsilon}$. Closure of the algebra requires the additional constraints:

$$\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon = -\frac{3}{8}\big(R - 2iV_{\mu\nu}\gamma^{\mu\nu}\big)\,\epsilon \ , \qquad \gamma^\mu\gamma^\nu D_\mu D_\nu\bar\epsilon = -\frac{3}{8}\big(R + 2iV_{\mu\nu}\gamma^{\mu\nu}\big)\,\bar\epsilon \tag{D.2.5}$$

with the *same* functions R and $V_{\mu\nu}$ [118, 119]. Consistency implies that R is the scalar curvature of the three-manifold and $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the background gauge field strength. Then the algebra reads

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \mathcal{L}^A_\xi + i\Lambda + \rho\Delta + i\alpha R, \qquad [\delta_\epsilon, \delta_\epsilon] = 0, \qquad [\delta_{\bar{\epsilon}}, \delta_{\bar{\epsilon}}] = 0 \,, \tag{D.2.6}$$

where $\mathcal{L}^A_\xi$ is the gauge-covariant Lie derivative (independent of the metric, see below) along the vector field ξ, $i\Lambda$ denotes a gauge transformation with parameter $i\Lambda$, R is the R-symmetry charge,[1] and Δ the scaling weight.[2] The parameters themselves are given by

$$\begin{aligned} \xi^\mu &= i\bar{\epsilon}\gamma^\mu\epsilon & \rho &= \frac{i}{3}(D_\mu\bar{\epsilon}\gamma^\mu\epsilon + \bar{\epsilon}\gamma^\mu D_\mu\epsilon) = \frac{1}{3}D_\mu\xi^\mu \\ \Lambda &= \bar{\epsilon}\epsilon\sigma & \alpha &= -\frac{1}{3}(D_\mu\bar{\epsilon}\gamma^\mu\epsilon - \bar{\epsilon}\gamma^\mu D_\mu\epsilon) - \xi^\mu V_\mu \,. \end{aligned} \tag{D.2.9}$$

The Lie derivative $\mathcal{L}_X$ with respect to a vector field X is a derivation independent of the metric. On forms it is easily defined as $\mathcal{L}_X = \{d, \iota_X\}$ in terms of the contraction ι_X; using the normalization $\alpha = \frac{1}{n!}\alpha_{\mu_1\cdots\mu_n}dx^{\mu_1\cdots\mu_n}$, in components we have

$$[\mathcal{L}_X\alpha]_{\mu_1\cdots\mu_n} = X^\mu\partial_\mu\alpha_{\mu_1\cdots\mu_n} + n\,(\partial_{[\mu_1}X^\mu)\,\alpha_{\mu|\mu_2\cdots\mu_n]} \,. \tag{D.2.10}$$

The Lie derivative of spinors [185] (see [186] for explanations) is

$$\mathcal{L}_X\psi = X^\mu\nabla_\mu\psi + \frac{1}{4}\nabla_\mu X_\nu\,\gamma^{\mu\nu}\psi \,, \tag{D.2.11}$$

---

[1] The R-charges are:

$$\begin{aligned} &R(A_\mu, \sigma, \lambda, \bar{\lambda}, D) = (0, 0, -1, 1, 0) \,, \quad R(\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F}) = (q, -q, q-1, 1-q, q-2, 2-q), \\ &R(\epsilon, \bar{\epsilon}) = (-1, +1) \,. \,. \end{aligned} \tag{D.2.7}$$

[2] The dilation weights are:

$$\begin{aligned} &\Delta(A_\mu, \sigma, \lambda, \bar{\lambda}, D) = (1, 1, \tfrac{3}{2}, \tfrac{3}{2}, 2) \,, \quad \Delta(\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F}) = (q, q, q+\tfrac{1}{2}, q+\tfrac{1}{2}, q+1, q+1) \,, \\ &\Delta(\epsilon, \bar{\epsilon}) = (\tfrac{1}{2}, \tfrac{1}{2}) \,. \end{aligned} \tag{D.2.8}$$

Note that whereas the 1-form A has weight zero, its components have weight 1. The commutator on A_μ gives the μ-component of the Lie derivative on the 1-form A, without further action of the dilation group.

where the covariant derivative is $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$. Although this definition seems to depend on the metric (through the spin connection and the vielbein), the dependence in fact cancels out. Finally, we can define a "gauge-covariant" Lie derivative that acts on sections of some (gauge) vector bundle. On tensors it is simply obtained by substituting the flat derivative ∂_μ with the covariant derivative, $\partial_\mu \to \partial_\mu^A = \partial_\mu - iA_\mu$, while on spinors it is obtained by substituting $\nabla_\mu \to \nabla_\mu^A$ in the first term. The gauge-covariant Lie derivative of the connection (which does not transform as a section of the adjoint bundle) is defined as

$$\mathcal{L}_X^A A = \mathcal{L}_X A - d^A(\iota_X A) \;, \qquad (\mathcal{L}_X^A A)_\mu = X^\rho F_{\rho\mu} = X^\rho\big(2\partial_{[\rho}A_{\mu]} - i[A_\rho, A_\mu]\big) \;. \tag{D.2.12}$$

D.2.2 Commuting Killing spinors

For given anticommuting spinors $\epsilon, \bar\epsilon$, let us construct the corresponding supercharges $Q, \tilde{Q}$ in terms of *commuting* spinors ϵ and $\tilde\epsilon = -C\bar\epsilon^*$ (so that $\bar\epsilon = \tilde\epsilon^c$). They are constructed as follows:

$$\delta = \delta_\epsilon + \delta_{\bar\epsilon} = \epsilon^\alpha Q_\alpha + \bar\epsilon^\alpha \tilde{Q}_\alpha \;, \qquad Q = \epsilon^\alpha Q_\alpha \;, \qquad \tilde{Q} = \tilde\epsilon^{c\,\alpha}\tilde{Q}_\alpha = -(\tilde\epsilon^\dagger C)^\alpha \tilde{Q}_\alpha \;. \tag{D.2.13}$$

We also need the charge conjugate $\bar\lambda = C(\lambda^\dagger)^\mathsf{T}$. On the vector multiplet we get:

$$\begin{aligned}
QA_\mu &= \frac{i}{2}\lambda^\dagger\gamma_\mu\epsilon \;, \qquad Q\lambda = \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon\, D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon \;, \\
\tilde{Q}A_\mu &= \frac{i}{2}\tilde\epsilon^\dagger\gamma_\mu\lambda \;, \qquad \tilde{Q}\lambda^\dagger = -\frac{1}{2}\tilde\epsilon^\dagger\gamma^{\mu\nu}F_{\mu\nu} + \tilde\epsilon^\dagger D + i\tilde\epsilon^\dagger\gamma^\mu D_\mu\sigma + \frac{2i}{3}D_\mu\tilde\epsilon^\dagger\gamma^\mu\sigma \;, \\
QD &= -\frac{i}{2}D_\mu\lambda^\dagger\gamma^\mu\epsilon + \frac{i}{2}[\lambda^\dagger\epsilon, \sigma] - \frac{i}{6}\lambda^\dagger\gamma^\mu D_\mu\epsilon \;, \\
\tilde{Q}D &= \frac{i}{2}\tilde\epsilon^\dagger\gamma^\mu D_\mu\lambda + \frac{i}{2}[\sigma, \tilde\epsilon^\dagger\lambda] + \frac{i}{6}D_\mu\tilde\epsilon^\dagger\gamma^\mu\lambda \;, \\
\tilde{Q}\lambda &= 0 \;, \qquad Q\sigma = -\frac{1}{2}\lambda^\dagger\epsilon \;, \\
Q\lambda^\dagger &= 0 \;, \qquad \tilde{Q}\sigma = -\frac{1}{2}\tilde\epsilon^\dagger\lambda \;,
\end{aligned} \tag{D.2.14}$$

On the chiral multiplet we get:

$$\begin{aligned}
&Q\phi = 0\ , \qquad \tilde{Q}\phi = -\tilde{\epsilon}^\dagger\psi \\
&Q\phi^\dagger = \psi^\dagger\epsilon\ , \qquad \tilde{Q}\phi^\dagger = 0 \\
&Q\psi = \big(i\gamma^\mu D_\mu\phi + i\sigma\phi\big)\epsilon + \frac{2iq}{3}\phi\,\gamma^\mu D_\mu\epsilon\ , \qquad \tilde{Q}\psi = C\tilde{\epsilon}^* F \\
&\tilde{Q}\psi^\dagger = \tilde{\epsilon}^\dagger\big(- i\gamma^\mu D_\mu\phi^\dagger + i\phi^\dagger\sigma\big) - \frac{2iq}{3}D_\mu\tilde{\epsilon}^\dagger\gamma^\mu\phi^\dagger\ , \qquad Q\psi^\dagger = -\epsilon^\mathsf{T} C F^\dagger \\
&QF = \epsilon^\mathsf{T} C\big(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi\big) + \frac{i(2q-1)}{3}D_\mu\epsilon^\mathsf{T} C\gamma^\mu\psi\ , \\
&\tilde{Q}F^\dagger = \big(- iD_\mu\psi^\dagger\gamma^\mu - i\psi^\dagger\sigma + i\phi^\dagger\lambda^\dagger\big)C\tilde{\epsilon}^* - \frac{i(2q-1)}{3}\psi^\dagger\gamma^\mu C D_\mu\tilde{\epsilon}^*\ , \\
&\tilde{Q}F = 0 \qquad QF^\dagger = 0\ .
\end{aligned} \tag{D.2.15}$$

Finally we define $\mathcal{Q} \equiv Q + \tilde{Q}$.

D.2.3 Supersymmetric actions

Let us write down the $\mathcal{Q}$-closed but not $\mathcal{Q}$-exact actions we consider in chapter 6: they are the Chern-Simons (CS) action and the Fayet-Iliopoulos (FI) action. Since they are non-trivial in $\mathcal{Q}$-cohomology, their evaluation on the BPS configurations is non-trivial. The CS action is

$$S_\mathrm{CS} = -\frac{i}{4\pi}\int \mathrm{Tr}_{CS}\left[A\wedge F - \frac{2i}{3}A\wedge A\wedge A + \Big(2D\sigma - \bar{\lambda}\lambda\Big)d\mathrm{vol}\right]\ , \tag{D.2.16}$$

both on S^3_b and $S^2\times S^1$. The symbol Tr_{CS} (as in [116]) means a trace where each Abelian and simple factor in the gauge group is weighed by its own (quantized) CS level k. For instance, for $SU(N)$ this would just be $\mathrm{Tr}_{CS} = k\mathrm{Tr}$.

The FI action on S^3_b is

$$S_\mathrm{FI} = \frac{i}{2\pi\sqrt{\ell\tilde{\ell}}}\int \mathrm{Tr}_{FI}\Big(D - \frac{\sigma}{f}\Big)\, d\mathrm{vol}(S^3_b)\ , \tag{D.2.17}$$

where again Tr_{FI} is a trace where each Abelian factor is weighed by its own FI term ξ. For $U(N)$, this would just be $\mathrm{Tr}_{FI} = \xi\mathrm{Tr}$.

D.3 One-loop determinants from an index theorem

The one-loop determinants of quadratic fluctuations around a non-trivial background, in particular around our general vortex backgrounds, are most easily evaluated with the help of an equivariant index theorem for transversally elliptic operators [147]. Such a technique was used on S^4 [14, 149] and S^2 [15], while the computations on S_b^3 and $S^2 \times S^1$ have been done in [148]. We will summarize the latter computation here, adapted to our conventions, referring to [14, 149, 15, 148] for details.

After the cancelations between bosons and fermions, the one-loop determinant equals the ratio $\det_{\text{coker } D_{oe}} \mathcal{Q}^2 / \det_{\ker D_{oe}} \mathcal{Q}^2$, where D_{oe} is the projection, from a subset $\{\varphi_e\}$ to a subset $\{\varphi_o\}$ of fields, of the expansion of $\mathcal{Q}$ at linear order around the background. The ratio of weights of the group action of $\mathcal{Q}^2$ on respective spaces can be computed by first evaluating the index

$$\operatorname{ind} D_{oe}(\epsilon) = \operatorname{tr}_{\ker D_{oe}} e^{\mathcal{Q}^2(\epsilon)} - \operatorname{tr}_{\text{coker } D_{oe}} e^{\mathcal{Q}^2(\epsilon)} \,, \tag{D.3.1}$$

where ϵ summarizes the equivariant parameters, and then extracting the determinant with the map

$$\sum\nolimits_\alpha c_\alpha e^{w_\alpha(\epsilon)} \quad \to \quad \prod\nolimits_\alpha w_\alpha(\epsilon)^{c_\alpha} \,. \tag{D.3.2}$$

As explained in the main text, $\operatorname{ind} D_{oe}(\epsilon)$ is computed with the help of the index theorem, and it only gets contributions from the fixed points on the worldvolume of the action of $\mathcal{Q}^2$. However the theorem can be applied if the action is compact, which is not the case on S_b^3 and $S^2 \times S^1$ in general. Then [148] propose to reduce along an S^1 fiber, and be left with the computation on S^2, as in [15]. It turns out that for the chiral multiplet the operator D_{oe} is the Dolbeault operator $D_{\bar{z}}$ with inverted grading acting on $\Omega^{(0,0)}$, whose index is $-\frac{1}{1-z}$, while for the vector multiplet it is the real operator $d^* \oplus d$ acting on Ω^1, whose index is $\frac{1}{2}$.

The sphere S_b^3. We write the metric in Hopf coordinates as in (6.2.12), in terms of $\phi_H = \varphi - \chi$ and $\psi_H = \varphi + \chi$. The square of the supercharge is

$$\begin{aligned}\mathcal{Q}^2 &= \mathcal{L}_\xi^A - \sigma - \frac{i}{2}\Big(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\Big)R = \frac{b}{r}\mathcal{L}^A_{\partial_\varphi} + \frac{b^{-1}}{r}\mathcal{L}^A_{\partial_\chi} - \frac{r\sigma}{r} - \frac{i}{2r}(b+b^{-1})R \\ &= \frac{b+b^{-1}}{r}\mathcal{L}^A_{\psi_H} + \frac{b-b^{-1}}{r}\mathcal{L}^A_{\phi_H} - \frac{r\sigma}{r} - \frac{i}{2r}(b+b^{-1})R\end{aligned} \tag{D.3.3}$$

where we used $r = \sqrt{\ell\tilde{\ell}}$ and $b = \sqrt{\tilde{\ell}/\ell}$.

At the northern circle, $\theta = 0$, the Hopf fiber is parametrized by φ (see (6.2.1)) and $\mathcal{Q}^2$ acts freely on it with equivariant parameter b; the KK modes thus contribute $\sum_{n\in\mathbb{Z}} e^{ibn}$ to the index. On the S^2, parametrized by θ and ϕ_H, resulting from the reduction along the Hopf fiber, $\mathcal{Q}^2$ has a fixed point at $\theta = 0$. There the SUSY variation of a chiral multiplet (see (6.2.30)) is schematically $D_\theta + \frac{i}{\theta}D_{\phi_H} \sim D_{\bar{z}}$ if we identify $z = \theta e^{i\phi_H}$. In fact the one-loop determinant of the chiral multiplet is the index of the Dolbeault operator with inverted grading (as noticed in [14, 149, 15]), which is $-\frac{1}{1-z}$. Now we expand in $t = e^{i\phi_H}$ and use the equivariant parameter $(b - b^{-1})$, getting $-\sum_{m\geq 0} e^{i(b-b^{-1})m}$. Putting everything together, and recalling that the multiplet transforms in a gauge representation $\mathcal{R}$, the contribution to the index of a chiral multiplet from the northern circle is:

$$\text{ind chiral}_N = -\sum_{w\in\mathcal{R}}\sum_{n\in\mathbb{Z}} e^{ibn}\sum_{m\geq 0} e^{i(b-b^{-1})m}e^{-\frac{i}{2}QR}e^{w(\hat{a}_N)} \tag{D.3.4}$$

where $Q \equiv b + b^{-1}$ and $\hat{a} = -i\big(bA_\varphi + b^{-1}A\chi\big) - r\mathfrak{S}$.

At the southern circle, $\theta = \frac{\pi}{2}$, the Hopf fiber is parametrized by χ and $\mathcal{Q}^2$ acts freely on it with equivariant parameter b^{-1}, therefore the KK modes yield $\sum_{n\in\mathbb{Z}} e^{ib^{-1}n}$. The SUSY variation around $\theta = \frac{\pi}{2}$ is schematically $-D_{\tilde{\theta}} + \frac{i}{\tilde{\theta}}D_{\phi_H} \sim D_{\bar{z}}$ (where $\tilde{\theta} = \frac{\pi}{2} - \theta$) if we identify $z = \tilde{\theta}e^{-i\phi_H}$. Again we expand in t and use equivariant parameter $(b - b^{-1})$, getting $\sum_{m\geq 1} e^{i(b-b^{-1})m}$. Putting together:

$$\text{ind chiral}_S = \sum_{w\in\mathcal{R}}\sum_{n\in\mathbb{Z}} e^{ib^{-1}n}\sum_{m\geq 1} e^{i(b-b^{-1})m}e^{-\frac{i}{2}QR}e^{w(\hat{a}_S)}\,. \tag{D.3.5}$$

The one-loop determinant is extracted with (D.3.2). We get the non-regulated expression

$$Z_{\text{1-loop}}^{\text{chiral}} \text{ ``}=\text{''} \prod_{w\in\mathcal{R}} \prod_{n\in\mathbb{Z}} \prod_{m\geq 0} \frac{(m+1)b + nb^{-1} - \frac{Q}{2}R - iw(\hat{a}_S)}{nb - mb^{-1} - \frac{Q}{2}R - iw(\hat{a}_N)} \; . \tag{D.3.6}$$

This is the expression in (6.2.51), after a rescaling by $\sqrt{\ell\tilde{\ell}}$ of both numerator and denominator. If $\hat{a}_N \neq \hat{a}_S$, this expression cannot be further simplified; the regulated expression could be written in terms of infinite q-Pochhammer factors. In our case $\hat{a}_N = \hat{a}_S \equiv \hat{a}$, thus we can simplify coincident factors and, neglecting overall signs, we get

$$\begin{aligned} Z_{\text{1-loop}}^{\text{chiral}} \text{ ``}=\text{''} & \prod_{w\in\mathcal{R}} \prod_{m,n\geq 0} \frac{mb + nb^{-1} + \big(1-\frac{R}{2}\big)Q - iw(\hat{a})}{mb + nb^{-1} + \frac{R}{2}Q + iw(\hat{a})} & \text{(D.3.7)} \\ = & \prod_{w\in\mathcal{R}} s_b\bigg(\frac{iQ}{2}(1-R) + w(\hat{a})\bigg) \; . & \text{(D.3.8)} \end{aligned}$$

This is the expression in (6.2.52), and the last regulated expression was found in [119] in terms of the double sine function.

The one-loop determinant of the vector multiplet is computed in a similar way, observing that the relevant complex is the de Rham complex: the index of its complexification is just 1, therefore we get $\frac{1}{2}$. At the northern and southern circles the indices are $\frac{1}{2}\sum_{n\in\mathbb{Z}} e^{ibn+\alpha(\hat{a}_N)}$ and $\frac{1}{2}\sum_{n\in\mathbb{Z}} e^{ib^{-1}n+\alpha(\hat{a}_S)}$ respectively, summed over the roots α of the gauge group. Extracting the eigenvalues and regularizing, we get

$$Z_{\text{1-loop}}^{\text{vec}} = \prod_{\alpha>0} 2\sinh\big(\pi b^{-1}\,\alpha(\hat{a}_N)\big)\, 2\sinh\big(\pi b\,\alpha(\hat{a}_S)\big) \; , \tag{D.3.9}$$

where the product is over the positive roots and the normalization is somewhat arbitrary.

The space $S^2 \times S^1$. The square of the supercharge reads in this case

$$\mathcal{Q}^2 = -\mathcal{L}^A_{\partial_\tau} + \frac{i}{r}\mathcal{L}^A_{\partial_\varphi} - \cos\theta\,\sigma - \frac{1}{2r}R + i\frac{\mathfrak{z}_j}{2\xi r}F_j \; . \tag{D.3.10}$$

It generates a free rotation along S^1 (of radius $2\xi r$) with equivariant parameter -1, thus resulting in the KK contribution $\sum_{n\in\mathbb{Z}} e^{-i\pi n/\xi r}$, and a rotation of the base S^2 with fixed points at $\theta = 0$ and $\theta = \pi$.

At $\theta = 0$ the SUSY variation of a chiral multiplet is of the form $D_\theta + \frac{i}{\theta} D_\varphi \sim D_{\bar z}$ if we identify $z = \theta e^{i\varphi}$. As above, the one-loop determinant of the chiral multiplet is then obtained from the index of the Dolbeault operator with inverted grading, which is $-\frac{1}{1-z}$. We expand in $t = e^{i\varphi}$ and use the equivariant parameter $\frac{i}{r}$, getting $-\sum_{k\geq 0} e^{-k/r}$. The total index at the north pole is thus:

$$\text{ind chiral}_N = -\sum_{w\in\mathcal{R}} \sum_{n\in\mathbb{Z}} e^{-\pi i n/\xi r} \sum_{k\geq 0} e^{-k/r} e^{-\frac{1}{2r}R} e^{i\frac{\mathfrak{z}_j}{2\xi r}F_j} e^{w(\hat a_N)} \tag{D.3.11}$$

where $\hat a = iA_\tau + \frac{1}{r}A_\varphi - \cos\theta\,\sigma$. Similarly, at $\theta = \pi$ the SUSY variation is of the form $D_{\tilde\theta} + \frac{i}{\tilde\theta} D_\varphi \sim D_{\bar z}$ (where $\tilde\theta = \pi - \theta$) if we identify $z = \tilde\theta e^{i\varphi}$. Now we expand the index of the Dolbeault operator in t^{-1} (since the orientation is opposite) and use the equivariant parameter $\frac{i}{r}$, getting $\sum_{k\geq 1} e^{k/r}$. The total index at the south pole is thus:

$$\text{ind chiral}_S = \sum_{w\in\mathcal{R}} \sum_{n\in\mathbb{Z}} e^{-\pi i n/\xi r} \sum_{k\geq 1} e^{k/r} e^{-\frac{1}{2r}R} e^{i\frac{\mathfrak{z}_j}{2\xi r}F_j} e^{w(\hat a_S)}\ . \tag{D.3.12}$$

The one-loop determinant is extracted with (D.3.2), obtaining the non-regulated expression:

$$Z^{\text{chiral}}_{\text{1-loop}} \text{ “ = ” } \prod_{w\in\mathcal{R}} \prod_{n\in\mathbb{Z}} \prod_{k\geq 0} \frac{-\pi i n + (k+1)\xi - \frac{\xi}{2}R + \frac{i}{2}\sum_j \mathfrak{z}_j F_j + \xi r\, w(\hat a_S)}{-\pi i n - k\xi - \frac{\xi}{2}R + \frac{i}{2}\sum_j \mathfrak{z}_j F_j + \xi r\, w(\hat a_N)}\ . \tag{D.3.13}$$

For the vector multiplet, a computation exactly parallel to the one for S^3_b gives

$$\begin{aligned} Z^{\text{vec}}_{\text{1-loop}} \text{ “ = ” } &\prod_{\alpha\in\mathfrak{g}} \prod_{n\in\mathbb{Z}} \Big(\alpha(\hat a_N) + \frac{2\pi i n}{2\xi r}\Big)^{1/2} \Big(\alpha(\hat a_S) + \frac{2\pi i n}{2\xi r}\Big)^{1/2} \\ &= \prod_{\alpha>0} 2\sinh\big(\xi r\,\alpha(\hat a_N)\big)\, 2\sinh\big(-\xi r\,\alpha(\hat a_S)\big)\ , \end{aligned} \tag{D.3.14}$$

where the product runs over the positive roots.

D.4 One-loop deteminants on $S^2 \times S^1$: poles at zero or infinity

In this appendix we study under what conditions the one-loop determinants on $S^2 \times S^1$ do not have poles at zero or infinity, and therefore the deformed

Coulomb branch contribution can be suppressed in a suitable $\zeta \to \pm\infty$ limit—ζ being the coefficient in (6.2.35) and (6.2.36)—or equivalently the Coulomb branch contribution can be reduced to a sum of residues as in section 6.3.5. For simplicity we consider the case of a $U(1)$ gauge theory with N_f fundamentals, N_a antifundamentals and CS level k; the case of $U(N)$ gauge group is a straightforward generalization. We follow an argument in [160], correcting a small imprecision.

First, we remind that the chiral one-loop determinant on the Coulomb branch can be written in two ways:

$$\begin{aligned} Z^{\text{chiral}}_{\text{1-loop}} &= \prod_{w\in\mathcal{R}} \Big(x^{1-q}\, e^{-iw(a)}\, \zeta^{-F}\Big)^{-w(\mathfrak{m})/2} \, \frac{\big(x^{2-q-w(\mathfrak{m})}\, e^{-iw(a)}\, \zeta^{-F}; x^2\big)_\infty}{\big(x^{q-w(\mathfrak{m})}\, e^{iw(a)}\, \zeta^{F}; x^2\big)_\infty} \\ &= \prod_{w\in\mathcal{R}} (-1)^{\frac{w(\mathfrak{m})+|w(\mathfrak{m})|}{2}} \Big(x^{1-q}\, e^{-iw(a)}\, \zeta^{-F}\Big)^{|w(\mathfrak{m})|/2} \, \frac{\big(x^{2-q+|w(\mathfrak{m})|}\, e^{-iw(a)}\, \zeta^{-F}; x^2\big)_\infty}{\big(x^{q+|w(\mathfrak{m})|}\, e^{iw(a)}\, \zeta^{F}; x^2\big)_\infty} \,. \end{aligned} \tag{D.4.1}$$

The first line is as in (6.3.50), where $\zeta^F \equiv \prod_j \zeta_j^{F_j} = \prod_j e^{i\mathfrak{z}_j F_j}$ are the flavor fugacities; the equality with the second line can be proven easily.

The index of the $U(1)$ theory is then computed by

$$\begin{aligned} I_\infty = \sum_{\mathfrak{m}\in\mathbb{Z}} (-1)^{k\mathfrak{m} + N_f \frac{|\mathfrak{m}|+\mathfrak{m}}{2} + N_a \frac{|\mathfrak{m}|-\mathfrak{m}}{2}}\; w^{\mathfrak{m}}\; x^{\frac{N_f+N_a}{2}|\mathfrak{m}|} \prod_{\alpha=1}^{N_f} \big(\zeta_\alpha\big)^{|\mathfrak{m}|/2} \prod_{\beta=1}^{N_a} \big(\tilde\zeta_\beta^{-1}\big)^{|\mathfrak{m}|/2} \\ \oint \frac{dz}{2\pi i z}\; z^{k\mathfrak{m} - \mathfrak{n} - \frac{1}{2}(N_f-N_a)|\mathfrak{m}|}\; A_\infty(N_f, N_a, x, \zeta, \tilde\zeta, z; \mathfrak{m}) \,, \end{aligned} \tag{D.4.2}$$

where we introduced

$$A_\infty(N_f, N_a, x, \zeta, \tilde\zeta, z; \mathfrak{m}) = \prod_{\alpha=1}^{N_f} \frac{\big(z^{-1}\zeta_\alpha x^{|\mathfrak{m}|+2} \,;\, x^2\big)_\infty}{\big(z\zeta_\alpha^{-1} x^{|\mathfrak{m}|} \,;\, x^2\big)_\infty} \; \prod_{\beta=1}^{N_a} \frac{\big(z\tilde\zeta_\beta^{-1} x^{|\mathfrak{m}|+2} \,;\, x^2\big)_\infty}{\big(z^{-1}\tilde\zeta_\beta x^{|\mathfrak{m}|} \,;\, x^2\big)_\infty} \,. \tag{D.4.3}$$

This is exactly the same integral as in (6.3.67), in the special case $N = 1$. In particular $z = e^{ia}$, the integration contour is along the unit circle $|z| = 1$ for $|\tilde\zeta_\beta| < 1 < |\zeta_\alpha|$, and convergence of the Pochhammer symbols requires $|x| < 1$. For fixed ζ_α, $\tilde\zeta_\beta$, x, the product A_∞ is uniformly convergent on the unit circle $|z| = 1$ and the convergence is faster the larger is $|\mathfrak{m}|$, therefore one can argue as in [160] that for every ε there is an n such that

$$\big| A_\infty(N_f, N_a, x, \zeta, \tilde\zeta, z; \mathfrak{m}) - A_n(N_f, N_a, x, \zeta, \tilde\zeta, z; \mathfrak{m}) \big| < \varepsilon \qquad \forall\, |z| = 1 \,,\ \forall\, \mathfrak{m} \,, \tag{D.4.4}$$

where A_n is the same quantity as in (D.4.3) with ∞ replaced by n in the Pochhammer symbols. Then one can argue that

$$\big|I_\infty - I_n\big| \;\leq\; \varepsilon \sum_{m\,\in\,\mathbb{Z}} \bigg| x^{N_f+N_a} \prod_{\alpha=1}^{N_f} \zeta_\alpha \prod_{\beta=1}^{N_a} \tilde{\zeta}_\beta^{-1} \bigg|^{|\mathfrak{m}|/2} |w|^{\mathfrak{m}} \;, \tag{D.4.5}$$

where the right-hand-side is finite and $\mathcal{O}(\varepsilon)$ for small enough x. We can thus approximate I_∞ arbitrarily well by I_n by choosing a large enough n.

To compute I_n, we can deform its integration contour either towards infinity or zero and pick up residues. We can rewrite

$$A_n = z^{-n(N_f-N_a)} \prod_{j=0}^{n-1} \prod_{\alpha=1}^{N_f} \frac{z - \zeta_\alpha x^{|\mathfrak{m}|+2j+2}}{1 - z\zeta_\alpha^{-1} x^{|\mathfrak{m}|+2j}} \prod_{\beta=1}^{N_a} \frac{1 - z\tilde{\zeta}_\beta^{-1} x^{|\mathfrak{m}|+2j+2}}{z - \tilde{\zeta}_\beta x^{|\mathfrak{m}|+2j}} \;. \tag{D.4.6}$$

The only factor that can contribute poles either at zero or infinity is $z^{-n(N_f-N_a)}$.

For $N_f > N_a$, $A_n(z)$ does not provide poles at infinity for any arbitrarily large n and we can deform the integration contour towards infinity. However the integrand also contains $z^{k\mathfrak{m} - \frac{N_f-N_a}{2}|\mathfrak{m}| - \mathfrak{n}}$; we have absence of poles at infinity for all $\mathfrak{m} \in \mathbb{Z}$ if

$$|k| \leq \frac{N_f - N_a}{2} \;. \tag{D.4.7}$$

Here it is important to note that, when evaluating I_n, n is held fixed while $\mathfrak{m}$ is summed over $\mathbb{Z}$. Theories with $|k|$ within the bound have been dubbed "maximally chiral" in [151]. If $|k|$ is larger than the bound, I_n receives contributions from poles at infinity for infinitely many values of $\mathfrak{m}$, and such contributions do not disappear in the $n \to \infty$ limit; therefore the mere sum of the residues not at infinity does not reproduce the correct result. We will not attempt to perform the complete computation here.

For $N_f < N_a$, $A_n(z)$ does not have poles at zero and we can deform the integration contour towards zero. Because of the extra factor in the integrand, there are no poles at $z = 0$ for all $\mathfrak{m} \in \mathbb{Z}$ if

$$|k| \leq \frac{N_a - N_f}{2} \;. \tag{D.4.8}$$

The case $N_f = N_a$ is a bit special, because $A_n(z)$ has a finite non-zero value both at $z = 0$ and $z = \infty$. Consider $k = 0$. For $\mathfrak{n} \geq 1$ there are no poles at infinity, while for $\mathfrak{n} \leq -1$ there are no poles at $z = 0$. For $\mathfrak{n} = 0$

there are poles both at $z = 0$ and $z = \infty$, controlled by $A_n(z = 0; \mathfrak{m})$ and $A_n(z = \infty; \mathfrak{m})$. However the series $\sum_{\mathfrak{m}\in\mathbb{Z}}$ of such residues is convergent and can be resummed; moreover $\lim_{n\to\infty} A_n(z = 0, \infty; \mathfrak{m}) = 0$. Therefore the contribution of the poles at $z = 0, \infty$ to I_n is smaller and smaller as n is taken larger and larger, and can be neglected. In this case the integration contour can be deformed both towards zero or infinity.

Summarizing, we have shown that for

$$|k| \leq \frac{|N_f - N_a|}{2} \tag{D.4.9}$$

I_∞ can be computed by deforming the integration contour towards $z = 0$ and/or $z = \infty$ and picking up the residues outside $z = 0, \infty$, since the essential singularity at $z = 0$ and/or $z = \infty$ does not contribute to the integral. For $|k|$ larger than the bound, the contributions from the essential singularities should be taken into account, although we will not try to do that here. Notice that exactly the same bound appeared in section 6.2.5 when computing the S^3 partition function.

Appendix E

Higgs branch localization of $\mathcal{N} = 1$ theories on $S^3 \times S^1$

E.1 Spinor conventions

We choose to use four-component spinors. Bars on spinors are taken to be the Majorana conjugate, *i.e.* $\bar{\psi} = \psi^t C$ where C is the antisymmetric charge conjugation matrix satisfying $(\gamma_\mu)^t C = -C\gamma_\mu$. Since we are in Euclidean signature, it is impossible to impose the Majorana conjugate to be equal to the Dirac conjugate, but rather we work 'holomorphically', *i.e.* the hermitian conjugate spinor does not make an appearance.

We take the Euclidean gamma matrices to be

$$\gamma^m = \begin{pmatrix} 0 & -i\sigma^m \\ i\bar{\sigma}^m & 0 \end{pmatrix}, \tag{E.1.1}$$

where $\sigma^m = (\vec{\sigma}, i\mathbb{1}_2)$ and $\bar{\sigma}^m = (\vec{\sigma}, -i\mathbb{1}_2)$, where $\vec{\sigma}$ are the three Pauli matrices. We also introduce $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \left(\begin{smallmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{smallmatrix}\right)$ which squares to one. The charge conjugation matrix is given explicitly by $C = \gamma^4\gamma^2 = \left(\begin{smallmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{smallmatrix}\right)$.

We also introduce $\sigma^{mn} = \frac{1}{2}\left(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m\right)$ and $\bar{\sigma}^{mn} = \frac{1}{2}\left(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m\right)$, in terms of which one can write

$$\gamma_{mn} = \frac{1}{2}\left(\gamma_m\gamma_n - \gamma_n\gamma_m\right) = \begin{pmatrix} \sigma^{mn} & 0 \\ 0 & \bar{\sigma}^{mn} \end{pmatrix} . \tag{E.1.2}$$

Finally, for any four-component spinor ψ, we denote its right and left-handed piece as $\psi_R = \frac{\mathbb{1}_4+\gamma_5}{2}\psi$ and $\psi_L = \frac{\mathbb{1}_4-\gamma_5}{2}\psi$ respectively.

E.2 $\mathcal{N} = 1$ supersymmetry algebra on Euclidean four-manifolds

In this section we present the $\mathcal{N} = 1$ supersymmetry transformation rules on any four-dimensional Euclidean manifold allowing for a solution to the conformal Killing spinor equation $D_\mu\varepsilon = \gamma_\mu\tilde{\varepsilon}$. A more general and systematic analysis of supersymmetry on four-dimensional Euclidean backgrounds has been performed in [167, 168, 169].

The transformation rules on the vectormultiplet are

$$\begin{aligned} \delta A_\mu &= \bar{\varepsilon}\gamma_\mu\lambda && \text{(E.2.1)} \\ \delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\,\varepsilon - \gamma_5\ D\ \varepsilon && \text{(E.2.2)} \\ \delta D &= \bar{\varepsilon}\ \gamma_5 \not{D}\lambda\ , && \text{(E.2.3)} \end{aligned}$$

and those on the chiral multiplet are

$$\delta A = \bar{\varepsilon}\chi \tag{E.2.4}$$
$$\delta B = \bar{\varepsilon}i\gamma_5\chi \tag{E.2.5}$$
$$\delta\chi = (\gamma^\mu D_\mu(A + i\gamma_5 B))\,\varepsilon - i(F + i\gamma_5 G)\varepsilon + \frac{3r}{4}(A - i\gamma_5 B)\not{D}\varepsilon \tag{E.2.6}$$
$$\delta F = i\bar{\varepsilon}\not{D}\chi + i\left(\frac{3r}{4} - \frac{1}{2}\right)\bar{\chi}\not{D}\varepsilon + \bar{\varepsilon}(A + i\gamma_5 B)\lambda \tag{E.2.7}$$
$$\delta G = -\bar{\varepsilon}\gamma_5\not{D}\chi + \left(\frac{3r}{4} - \frac{1}{2}\right)\bar{\chi}\gamma_5\not{D}\varepsilon + \bar{\varepsilon}i\gamma_5(A + i\gamma_5 B)\lambda\ , \tag{E.2.8}$$

for commuting ε. Here D_μ is the covariant derivative $D_\mu = \partial_\mu - iA_\mu - iV_\mu$, where A_μ is the gauge connection and V_μ is a background field for the R-symmetry. In the chiral multiplet we decomposed $\phi = \frac{A-iB}{2}$, $\bar{\phi} = \frac{A+iB}{2}$ and $\mathcal{F} = \frac{F+iG}{2}$, $\bar{\mathcal{F}} = \frac{F-iG}{2}$. The spinor ε needs to satisfy the Killing spinor equation $D_\mu\varepsilon = \gamma_\mu\tilde{\varepsilon}$. One can check that the supersymmetry variations then square to

$$\delta^2 = \mathcal{L}_v^{A+V} + \rho\Delta + i\alpha R\ , \tag{E.2.9}$$

where $\mathcal{L}_v^{A+V}$ is the gauge and background R-symmetry covariant Lie derivative along the vector field v, Δ is the scaling weight[1] and R is the $U(1)_R$ generator[2]. The parameters themselves are given by

$$v_\mu = \bar{\varepsilon}\gamma_\mu\varepsilon\,, \qquad \rho = \frac{1}{4}D_\mu v^\mu\,, \qquad \alpha = 3i\bar{\tilde{\varepsilon}}\gamma_5\varepsilon\,. \tag{E.2.12}$$

E.3 Elliptic gamma function

The elliptic gamma function is defined as

$$\Gamma(z,p,q) = \prod_{j,k\geq 0} \frac{1-p^{j+1}q^{k+1}/z}{1-p^j q^k z}\,. \tag{E.3.1}$$

It satisfies the shift formulas

$$\Gamma(pz,p,q) = \theta(z,q)\Gamma(z,p,q)\,, \qquad \Gamma(qz,p,q) = \theta(z,p)\Gamma(z,p,q)\,, \tag{E.3.2}$$

where $\theta(z,q) = (z,q)_\infty(q/z,q)_\infty$ in terms of the infinite q-Pochhammer symbol $(z,q)_\infty = \prod_{j\geq 0}(1-zq^j)$. Furthermore, one has

$$\Gamma(z,p,q)\ \Gamma(pq/z,p,q) = 1\,. \tag{E.3.3}$$

The θ-function satisfies

$$\theta(z,q) = \theta(q/z,q) = -z\ \theta(z^{-1},q)\,, \tag{E.3.4}$$

[1]The scaling weights are

$$\begin{aligned}\Delta\left(A_\mu,\lambda_R,\lambda_L,D\right) &= \left(1,\frac{3}{2},\frac{3}{2},2\right)\,, \qquad \Delta(\varepsilon_R,\varepsilon_L) = \left(\frac{1}{2},\frac{1}{2}\right)\,,\\ \Delta(\phi,\bar{\phi},\chi_R,\chi_L,\mathcal{F},\bar{\mathcal{F}}) &= \left(\frac{3r}{2},\frac{3r}{2},\frac{3r+1}{2},\frac{3r+1}{2},\frac{3r+2}{2},\frac{3r+2}{2}\right)\,.\end{aligned} \tag{E.2.10}$$

[2]The R-charge assignments are

$$\begin{aligned}R\left(A_\mu,\lambda_R,\lambda_L,D\right) &= (0,1,-1,0)\,, \qquad R(\varepsilon_R,\varepsilon_L) = (1,-1)\,,\\ R(\phi,\bar{\phi},\chi_R,\chi_L,\mathcal{F},\bar{\mathcal{F}}) &= (r,-r,r-1,1-r,r-2,2-r)\,.\end{aligned} \tag{E.2.11}$$

which when iterated gives for positive κ

$$\theta(q^{\kappa} z,q)=\theta\,(z,q)\;\;(-zq^{(\kappa-1)/2})^{-\kappa}, \qquad \theta(q^{-\kappa} z,q)=\theta\,(z,q)\;\;(-z^{-1}q^{(\kappa+1)/2})^{-\kappa} \tag{E.3.5}$$

Given the above formulae, we can derive for positive κ,λ

$$\Gamma(p^{\kappa}q^{\lambda}z,p,q)=\left(-zq^{(\lambda-1)/2}p^{(\kappa-1)/2}\right)^{-\kappa\lambda}\;\Gamma(z,p,q)\;\prod_{j=0}^{\lambda-1}\theta(q^{j}z,p)\;\prod_{i=0}^{\kappa-1}\theta(p^{i}z,q)\;, \tag{E.3.6}$$

and

$$\Gamma(p^{-\kappa}q^{-\lambda}z,p,q)= \\ \frac{\Gamma(z,p,q)}{\left(-z^{-1}q^{(\lambda+1)/2}p^{(\kappa+1)/2}\right)^{-\kappa\lambda}\;\prod_{j=1}^{\lambda}\theta(q^{-j}z,p)\;\prod_{i=1}^{\kappa}\theta(p^{-i}z,q)}\;. \tag{E.3.7}$$

Finally, in order to compute residues, we have the following limit

$$\lim_{z\to 1}(1-z)\Gamma(z,p,q)=\frac{1}{(p,p)_{\infty}\;(q,q)_{\infty}}\;. \tag{E.3.8}$$

Bibliography

[1] M. Lemos and W. Peelaers, *Chiral Algebras for Trinion Theories*, *JHEP* **1502** (2015) 113, [arXiv:1411.3252].

[2] C. Beem, W. Peelaers, L. Rastelli, and B. C. van Rees, *Chiral algebras of class S*, *JHEP* **1505** (2015) 020, [arXiv:1408.6522].

[3] W. Peelaers, *Higgs branch localization of $\mathcal{N} = 1$ theories on S^3 x S^1*, *JHEP* **1408** (2014) 060, [arXiv:1403.2711].

[4] F. Benini and W. Peelaers, *Higgs branch localization in three dimensions*, *JHEP* **1405** (2014) 030, [arXiv:1312.6078].

[5] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, et al., *Infinite Chiral Symmetry in Four Dimensions*, *Commun.Math.Phys.* **336** (2015), no. 3 1359–1433, [arXiv:1312.5344].

[6] M. Lemos, W. Peelaers, and L. Rastelli, *The superconformal index of class S theories of type D*, *JHEP* **1405** (2014) 120, [arXiv:1212.1271].

[7] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, *Nucl.Phys.* **B426** (1994) 19–52, [hep-th/9407087].

[8] A. Polyakov, *Nonhamiltonian approach to conformal quantum field theory*, *Zh.Eksp.Teor.Fiz.* **66** (1974) 23–42.

[9] S. Ferrara, A. Grillo, and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, *Annals Phys.* **76** (1973) 161–188.

[10] D. Gaiotto, *N=2 dualities*, *JHEP* **1208** (2012) 034, [arXiv:0904.2715].

[11] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin Systems, and the WKB Approximation*, arXiv:0907.3987.

[12] E. Witten, *Topological Quantum Field Theory*, *Commun.Math.Phys.* **117** (1988) 353.

[13] E. Witten, *Mirror manifolds and topological field theory*, in *Mirror Symmetry*, pp. 121–160. S. T. Yau, 1991. hep-th/9112056.

[14] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun.Math.Phys.* **313** (2012) 71–129, [arXiv:0712.2824].

[15] F. Benini and S. Cremonesi, *Partition functions of $\mathcal{N} = (2,2)$ gauge theories on S^2 and vortices*, arXiv:1206.2356.

[16] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, *Exact Results in D=2 Supersymmetric Gauge Theories*, *JHEP* **1305** (2013) 093, [arXiv:1206.2606].

[17] C. Beem, T. Dimofte, and S. Pasquetti, *Holomorphic Blocks in Three Dimensions*, arXiv:1211.1986.

[18] L. F. Alday, D. Martelli, P. Richmond, and J. Sparks, *Localization on Three-Manifolds*, arXiv:1307.6848.

[19] S. Pasquetti, *Factorisation of $\mathcal{N} = 2$ Theories on the Squashed 3-Sphere*, *JHEP* **1204** (2012) 120, [arXiv:1111.6905].

[20] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **0812** (2008) 031, [arXiv:0807.0004].

[21] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, et al., *Solving the 3D Ising Model with the Conformal Bootstrap*, *Phys.Rev.* **D86** (2012) 025022, [arXiv:1203.6064].

[22] D. Poland and D. Simmons-Duffin, *Bounds on 4D Conformal and Superconformal Field Theories*, *JHEP* **1105** (2011) 017, [`arXiv:1009.2087`].

[23] D. Poland, D. Simmons-Duffin, and A. Vichi, *Carving Out the Space of 4D CFTs*, *JHEP* **1205** (2012) 110, [`arXiv:1109.5176`].

[24] C. Beem, L. Rastelli, and B. C. van Rees, *The N=4 Superconformal Bootstrap*, *Phys.Rev.Lett.* **111** (2013) 071601, [`arXiv:1304.1803`].

[25] L. F. Alday and A. Bissi, *The superconformal bootstrap for structure constants*, *JHEP* **1409** (2014) 144, [`arXiv:1310.3757`].

[26] C. Beem, L. Rastelli, and B. C. van Rees, *W Symmetry in six dimensions*, `arXiv:1404.1079`.

[27] F. Dolan, M. Nirschl, and H. Osborn, *Conjectures for large N superconformal N=4 chiral primary four point functions*, *Nucl.Phys.* **B749** (2006) 109–152, [`hep-th/0601148`].

[28] B. Eden, A. C. Petkou, C. Schubert, and E. Sokatchev, *Partial nonrenormalization of the stress tensor four point function in N=4 SYM and AdS / CFT*, *Nucl.Phys.* **B607** (2001) 191–212, [`hep-th/0009106`].

[29] B. Eden and E. Sokatchev, *On the OPE of 1/2 BPS short operators in N=4 SCFT(4)*, *Nucl.Phys.* **B618** (2001) 259–276, [`hep-th/0106249`].

[30] F. Dolan and H. Osborn, *Superconformal symmetry, correlation functions and the operator product expansion*, *Nucl.Phys.* **B629** (2002) 3–73, [`hep-th/0112251`].

[31] P. Heslop and P. Howe, *Four point functions in N=4 SYM*, *JHEP* **0301** (2003) 043, [`hep-th/0211252`].

[32] F. A. Dolan, L. Gallot, and E. Sokatchev, *On four-point functions of 1/2-BPS operators in general dimensions*, *JHEP* **0409** (2004) 056, [`hep-th/0405180`].

[33] M. Nirschl and H. Osborn, *Superconformal Ward identities and their solution*, *Nucl.Phys.* **B711** (2005) 409–479, [`hep-th/0407060`].

[34] N. Drukker and J. Plefka, *Superprotected n-point correlation functions of local operators in N=4 super Yang-Mills*, *JHEP* **0904** (2009) 052, [arXiv:0901.3653].

[35] P. de Medeiros, C. M. Hull, B. J. Spence, and J. M. Figueroa-O'Farrill, *Conformal topological Yang-Mills theory and de Sitter holography*, *JHEP* **0208** (2002) 055, [hep-th/0111190].

[36] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun.Math.Phys.* **275** (2007) 209–254, [hep-th/0510251].

[37] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *The 4d Superconformal Index from q-deformed 2d Yang-Mills*, *Phys.Rev.Lett.* **106** (2011) 241602, [arXiv:1104.3850].

[38] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *Gauge Theories and Macdonald Polynomials*, *Commun.Math.Phys.* **319** (2013) 147–193, [arXiv:1110.3740].

[39] C. Beem, M. Lemos, P. Liendo, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 2$ superconformal bootstrap*, arXiv:1412.7541.

[40] J. L. Cardy, *Is There a c Theorem in Four-Dimensions?*, *Phys.Lett.* **B215** (1988) 749–752.

[41] Z. Komargodski and A. Schwimmer, *On Renormalization Group Flows in Four Dimensions*, *JHEP* **1112** (2011) 099, [arXiv:1107.3987].

[42] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys.Rept.* **223** (1993) 183–276, [hep-th/9210010].

[43] A. Johansen, *Infinite conformal algebras in supersymmetric theories on four manifolds*, *Nucl.Phys.* **B436** (1995) 291–341, [hep-th/9407109].

[44] A. Kapustin, *Holomorphic reduction of N=2 gauge theories, Wilson-'t Hooft operators, and S-duality*, hep-th/0612119.

[45] F. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41–89, [hep-th/0209056].

[46] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry*, *J.Phys.* **A46** (2013) 214011, [arXiv:1112.1016].

[47] P. C. Argyres and N. Seiberg, *S-duality in N=2 supersymmetric gauge theories*, *JHEP* **0712** (2007) 088, [arXiv:0711.0054].

[48] K. Thielemans, *A Mathematica package for computing operator product expansions*, *Int.J.Mod.Phys.* **C2** (1991) 787–798.

[49] P. Goddard, D. I. Olive, and G. Waterson, *Superalgebras, Symplectic Bosons and the Sugawara Construction*, *Commun.Math.Phys.* **112** (1987) 591.

[50] D. Friedan, E. J. Martinec, and S. H. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, *Nucl.Phys.* **B271** (1986) 93.

[51] H. G. Kausch, *Curiosities at c = -2*, hep-th/9510149.

[52] L. Bhardwaj and Y. Tachikawa, *Classification of 4d N=2 gauge theories*, *JHEP* **1312** (2013) 100, [arXiv:1309.5160].

[53] D. Karabali and H. J. Schnitzer, *BRST Quantization of the Gauged WZW Action and Coset Conformal Field Theories*, *Nucl.Phys.* **B329** (1990) 649.

[54] P. Liendo, E. Pomoni, and L. Rastelli, *The Complete One-Loop Dilation Operator of N=2 SuperConformal QCD*, *JHEP* **1207** (2012) 003, [arXiv:1105.3972].

[55] M. Baggio, J. de Boer, and K. Papadodimas, *A non-renormalization theorem for chiral primary 3-point functions*, *JHEP* **1207** (2012) 137, [arXiv:1203.1036].

[56] K. Gawedzki and A. Kupiainen, *Coset Construction from Functional Integrals*, *Nucl.Phys.* **B320** (1989) 625.

[57] P. Cvitanovic, *Group theory: Birdtracks, Lie's and exceptional groups*, .

[58] A. Sen, *F theory and orientifolds*, *Nucl.Phys.* **B475** (1996) 562–578, [hep-th/9605150].

[59] T. Banks, M. R. Douglas, and N. Seiberg, *Probing F theory with branes*, *Phys.Lett.* **B387** (1996) 278–281, [hep-th/9605199].

[60] K. Dasgupta and S. Mukhi, *F theory at constant coupling*, *Phys.Lett.* **B385** (1996) 125–131, [hep-th/9606044].

[61] J. A. Minahan and D. Nemeschansky, *An N=2 superconformal fixed point with E(6) global symmetry*, *Nucl.Phys.* **B482** (1996) 142–152, [hep-th/9608047].

[62] J. A. Minahan and D. Nemeschansky, *Superconformal fixed points with E(n) global symmetry*, *Nucl.Phys.* **B489** (1997) 24–46, [hep-th/9610076].

[63] O. Aharony, A. Fayyazuddin, and J. M. Maldacena, *The Large N limit of N=2, N=1 field theories from three-branes in F theory*, *JHEP* **9807** (1998) 013, [hep-th/9806159].

[64] D. Gaiotto, A. Neitzke, and Y. Tachikawa, *Argyres-Seiberg duality and the Higgs branch*, *Commun.Math.Phys.* **294** (2010) 389–410, [arXiv:0810.4541].

[65] O. Aharony and Y. Tachikawa, *A Holographic computation of the central charges of d=4, N=2 SCFTs*, *JHEP* **0801** (2008) 037, [arXiv:0711.4532].

[66] P. C. Argyres and J. R. Wittig, *Infinite coupling duals of N=2 gauge theories and new rank 1 superconformal field theories*, *JHEP* **0801** (2008) 074, [arXiv:0712.2028].

[67] O. Chacaltana and J. Distler, *Tinkertoys for Gaiotto Duality*, *JHEP* **1011** (2010) 099, [arXiv:1008.5203].

[68] S. S. Razamat, *On a modular property of N=2 superconformal theories in four dimensions*, *JHEP* **1210** (2012) 191, [arXiv:1208.5056].

[69] P. C. Argyres, M. R. Plesser, and N. Seiberg, *The Moduli space of vacua of N=2 SUSY QCD and duality in N=1 SUSY QCD*, *Nucl.Phys.* **B471** (1996) 159–194, [hep-th/9603042].

[70] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, et al., *Dual String with U(1) Color Symmetry*, *Nucl.Phys.* **B111** (1976) 77–110.

[71] A. Hanany and N. Mekareeya, *Tri-vertices and SU(2)'s*, *JHEP* **1102** (2011) 069, [arXiv:1012.2119].

[72] E. Witten, *Solutions of four-dimensional field theories via M theory*, *Nucl.Phys.* **B500** (1997) 3–42, [hep-th/9703166].

[73] F. Benini, S. Benvenuti, and Y. Tachikawa, *Webs of five-branes and N=2 superconformal field theories*, *JHEP* **0909** (2009) 052, [arXiv:0906.0359].

[74] D. Gaiotto, L. Rastelli, and S. S. Razamat, *Bootstrapping the superconformal index with surface defects*, *JHEP* **1301** (2013) 022, [arXiv:1207.3577].

[75] A. Gadde, E. Pomoni, L. Rastelli, and S. S. Razamat, *S-duality and 2d Topological QFT*, *JHEP* **1003** (2010) 032, [arXiv:0910.2225].

[76] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *The Superconformal Index of the E_6 SCFT*, *JHEP* **1008** (2010) 107, [arXiv:1003.4244].

[77] N. Mekareeya, J. Song, and Y. Tachikawa, *2d TQFT structure of the superconformal indices with outer-automorphism twists*, *JHEP* **1303** (2013) 171, [arXiv:1212.0545].

[78] G. W. Moore and Y. Tachikawa, *On 2d TQFTs whose values are holomorphic symplectic varieties*, arXiv:1106.5698.

[79] S. Benvenuti, A. Hanany, and N. Mekareeya, *The Hilbert Series of the One Instanton Moduli Space*, *JHEP* **1006** (2010) 100, [arXiv:1005.3026].

[80] E. Witten, *Gauge theory and wild ramification*, arXiv:0710.0631.

[81] Y. Tachikawa, *N=2 S-duality via Outer-automorphism Twists*, *J.Phys.* **A44** (2011) 182001, [arXiv:1009.0339].

[82] O. Chacaltana, J. Distler, and Y. Tachikawa, *Gaiotto Duality for the Twisted A_{2N-1} Series*, arXiv:1212.3952.

[83] O. Chacaltana, J. Distler, and Y. Tachikawa, *Nilpotent orbits and codimension-two defects of 6d N=(2,0) theories*, *Int.J.Mod.Phys.* **A28** (2013) 1340006, [arXiv:1203.2930].

[84] A. Gadde, K. Maruyoshi, Y. Tachikawa, and W. Yan, *New N=1 Dualities*, *JHEP* **1306** (2013) 056, [arXiv:1303.0836].

[85] Y. Tachikawa, *N=2 supersymmetric dynamics for pedestrians*, *Lect.Notes Phys.* **890** (2013) 2014, [arXiv:1312.2684].

[86] K. Maruyoshi, Y. Tachikawa, W. Yan, and K. Yonekura, *N=1 dynamics with T_N theory*, *JHEP* **1310** (2013) 010, [arXiv:1305.5250].

[87] A. Dancer, F. Kirwan, and A. Swann, *Twistor spaces for hyperkaehler implosions*, arXiv:1308.3196.

[88] D. Gaiotto and S. S. Razamat, *Exceptional Indices*, *JHEP* **1205** (2012) 145, [arXiv:1203.5517].

[89] Y. Tachikawa, "Some conjectures on VOAs." http://member.ipmu.jp/yuji.tachikawa/transp/voaconj2.pdf.

[90] D. Gaiotto and J. Maldacena, *The Gravity duals of N=2 superconformal field theories*, *JHEP* **1210** (2012) 189, [arXiv:0904.4466].

[91] D. Gaiotto, G. W. Moore, and Y. Tachikawa, *On 6d N=(2,0) theory compactified on a Riemann surface with finite area*, *Prog. Theor. Exp. Phys.* **2013** (2013) 013B03, [arXiv:1110.2657].

[92] L. F. Alday and Y. Tachikawa, *Affine SL(2) conformal blocks from 4d gauge theories*, *Lett.Math.Phys.* **94** (2010) 87–114, [arXiv:1005.4469].

[93] J. de Boer and T. Tjin, *The Relation between quantum W algebras and Lie algebras*, *Commun.Math.Phys.* **160** (1994) 317–332, [hep-th/9302006].

[94] J. de Boer and T. Tjin, *Quantization and representation theory of finite W algebras*, *Commun.Math.Phys.* **158** (1993) 485–516, [hep-th/9211109].

[95] L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, *On the general structure of Hamiltonian reductions of the WZNW theory*, hep-th/9112068.

[96] B. Feigin and E. Frenkel, *Quantization of the Drinfeld-Sokolov reduction*, *Phys.Lett.* **B246** (1990) 75–81.

[97] E. Witten, *Two-dimensional models with (0,2) supersymmetry: Perturbative aspects*, *Adv.Theor.Math.Phys.* **11** (2007) [hep-th/0504078].

[98] A. Kapustin, *Chiral de Rham complex and the half-twisted sigma-model*, hep-th/0504074.

[99] F. Malikov, V. Schechtman, and A. Vaintrob, *Chiral de Rham complex*, *Comm. Math. Phys.* **204** (1999), no. 2 439–473.

[100] F. Malikov and V. Schechtman, *Chiral de Rham complex. II*, in *Differential topology, infinite-dimensional Lie algebras, and applications*, vol. 194 of *Amer. Math. Soc. Transl. Ser. 2*, pp. 149–188. Amer. Math. Soc., Providence, RI, 1999.

[101] V. Gorbounov, F. Malikov, and V. Schechtman, *Gerbes of chiral differential operators*, *Math. Res. Lett.* **7** (2000), no. 1 55–66.

[102] V. Gorbounov, F. Malikov, and V. Schechtman, *Gerbes of chiral differential operators. II. Vertex algebroids*, *Invent. Math.* **155** (2004), no. 3 605–680.

[103] V. Gorbounov, F. Malikov, and V. Schechtman, *Gerbes of chiral differential operators. III*, in *The orbit method in geometry and physics (Marseille, 2000)*, vol. 213 of *Progr. Math.*, pp. 73–100. Birkhäuser Boston, Boston, MA, 2003.

[104] A. Zamolodchikov, *Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory*, *Theor.Math.Phys.* **65** (1985) 1205–1213.

[105] F. Benini, Y. Tachikawa, and B. Wecht, *Sicilian gauge theories and N=1 dualities*, *JHEP* **1001** (2010) 088, [`arXiv:0909.1327`].

[106] H. Hayashi, Y. Tachikawa, and K. Yonekura, *Mass-deformed T_N as a linear quiver*, `arXiv:1410.6868`.

[107] T. Arakawa, *Characters of representations of affine Kac-Moody Lie algebras at the critical level*, *ArXiv e-prints* (June, 2007) [`arXiv:0706.1817`].

[108] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*, .

[109] K. Thielemans, *An Algorithmic approach to operator product expansions, W algebras and W strings*, `hep-th/9506159`.

[110] H. Kausch and G. Watts, *A Study of W algebras using Jacobi identities*, *Nucl.Phys.* **B354** (1991) 740–768.

[111] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel, et al., *W algebras with two and three generators*, *Nucl.Phys.* **B361** (1991) 255–289.

[112] E. Frenkel and N. Reshetikhin, *Towards Deformed Chiral Algebras*, in *eprint arXiv:q-alg/9706023*, p. 6023, 1997.

[113] A. Gadde and S. Gukov, *2d Index and Surface operators*, `arXiv:1305.0266`.

[114] F. Benini, R. Eager, K. Hori, and Y. Tachikawa, *Elliptic genera of two-dimensional N=2 gauge theories with rank-one gauge groups*, `arXiv:1305.0533`.

[115] F. Benini, R. Eager, K. Hori, and Y. Tachikawa, *Elliptic genera of 2d N=2 gauge theories*, `arXiv:1308.4896`.

[116] A. Kapustin, B. Willett, and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **1003** (2010) 089, [`arXiv:0909.4559`].

[117] D. L. Jafferis, *The Exact Superconformal R-Symmetry Extremizes Z*, *JHEP* **1205** (2012) 159, [`arXiv:1012.3210`].

[118] N. Hama, K. Hosomichi, and S. Lee, *Notes on SUSY Gauge Theories on Three-Sphere*, *JHEP* **1103** (2011) 127, [`arXiv:1012.3512`].

[119] N. Hama, K. Hosomichi, and S. Lee, *SUSY Gauge Theories on Squashed Three-Spheres*, *JHEP* **1105** (2011) 014, [`arXiv:1102.4716`].

[120] Y. Imamura and D. Yokoyama, *N=2 supersymmetric theories on squashed three-sphere*, *Phys.Rev.* **D85** (2012) 025015, [`arXiv:1109.4734`].

[121] S. Kim, *The Complete superconformal index for N=6 Chern-Simons theory*, *Nucl.Phys.* **B821** (2009) 241–284, [`arXiv:0903.4172`].

[122] Y. Imamura and S. Yokoyama, *Index for three dimensional superconformal field theories with general R-charge assignments*, *JHEP* **1104** (2011) 007, [`arXiv:1101.0557`].

[123] N. Hama and K. Hosomichi, *Seiberg-Witten Theories on Ellipsoids*, *JHEP* **1209** (2012) 033, [`arXiv:1206.6359`].

[124] C. Closset and I. Shamir, *The $\mathcal{N}=1$ Chiral Multiplet on $T^2 \times S^2$ and Supersymmetric Localization*, `arXiv:1311.2430`.

[125] J. Källén and M. Zabzine, *Twisted supersymmetric 5D Yang-Mills theory and contact geometry*, *JHEP* **1205** (2012) 125, [`arXiv:1202.1956`].

[126] K. Hosomichi, R.-K. Seong, and S. Terashima, *Supersymmetric Gauge Theories on the Five-Sphere*, *Nucl.Phys.* **B865** (2012) 376–396, [`arXiv:1203.0371`].

[127] H.-C. Kim and S. Kim, *M5-branes from gauge theories on the 5-sphere*, *JHEP* **1305** (2013) 144, [`arXiv:1206.6339`].

[128] Y. Imamura, *Supersymmetric theories on squashed five-sphere*, *PTEP* **2013** (2013) 013B04, [arXiv:1209.0561].

[129] T. Dimofte, D. Gaiotto, and S. Gukov, *Gauge Theories Labelled by Three-Manifolds*, arXiv:1108.4389.

[130] S. Shadchin, *On F-term contribution to effective action*, *JHEP* **0708** (2007) 052, [hep-th/0611278].

[131] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv.Theor.Math.Phys.* **7** (2004) 831–864, [hep-th/0206161].

[132] S. Cecotti and C. Vafa, *Topological antitopological fusion*, *Nucl.Phys.* **B367** (1991) 359–461.

[133] M. Taki, *Holomorphic Blocks for 3d Non-abelian Partition Functions*, arXiv:1303.5915.

[134] C. Krattenthaler, V. Spiridonov, and G. Vartanov, *Superconformal indices of three-dimensional theories related by mirror symmetry*, *JHEP* **1106** (2011) 008, [arXiv:1103.4075].

[135] C. Hwang, H.-C. Kim, and J. Park, *Factorization of the 3d superconformal index*, arXiv:1211.6023.

[136] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, *The Geometry of Supersymmetric Partition Functions*, *JHEP* **1401** (2014) 124, [arXiv:1309.5876].

[137] S. Cecotti, D. Gaiotto, and C. Vafa, *tt^* Geometry in 3 and 4 Dimensions*, arXiv:1312.1008.

[138] C. Klare, A. Tomasiello, and A. Zaffaroni, *Supersymmetry on Curved Spaces and Holography*, *JHEP* **1208** (2012) 061, [arXiv:1205.1062].

[139] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, *Supersymmetric Field Theories on Three-Manifolds*, *JHEP* **1305** (2013) 017, [arXiv:1212.3388].

[140] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238.

[141] F. Benini, T. Nishioka, and M. Yamazaki, *4d Index to 3d Index and 2d TQFT*, *Phys.Rev.* **D86** (2012) 065015, [arXiv:1109.0283].

[142] L. F. Alday, M. Fluder, and J. Sparks, *The Large N limit of M2-branes on Lens spaces*, *JHEP* **1210** (2012) 057, [arXiv:1204.1280].

[143] C. Romelsberger, *Counting chiral primaries in N = 1, d=4 superconformal field theories*, *Nucl.Phys.* **B747** (2006) 329–353, [hep-th/0510060].

[144] S. S. Razamat and B. Willett, *Global Properties of Supersymmetric Theories and the Lens Space*, arXiv:1307.4381.

[145] D. Martelli, A. Passias, and J. Sparks, *The gravity dual of supersymmetric gauge theories on a squashed three-sphere*, *Nucl.Phys.* **B864** (2012) 840–868, [arXiv:1110.6400].

[146] E. Witten, *Two-dimensional gauge theories revisited*, *J.Geom.Phys.* **9** (1992) 303–368, [hep-th/9204083].

[147] M. F. Atiyah, *Elliptic operators and compact groups*, in *Lecture Notes in Mathematics*, vol. 401. Springer-Verlag, Berlin, 1974.

[148] N. Drukker, T. Okuda, and F. Passerini, *Exact results for vortex loop operators in 3d supersymmetric theories*, arXiv:1211.3409.

[149] J. Gomis, T. Okuda, and V. Pestun, *Exact Results for 't Hooft Loops in Gauge Theories on S4*, *JHEP* **1205** (2012) 141, [arXiv:1105.2568].

[150] A. G. Bytsko and J. Teschner, *Quantization of models with non-compact quantum group symmetry: Modular XXZ magnet and lattice sinh-Gordon model*, *J.Phys.* **A39** (2006) 12927–12981, [hep-th/0602093].

[151] F. Benini, C. Closset, and S. Cremonesi, *Comments on 3d Seiberg-like dualities*, *JHEP* **1110** (2011) 075, [arXiv:1108.5373].

[152] P. Goddard, J. Nuyts, and D. I. Olive, *Gauge Theories and Magnetic Charge*, *Nucl.Phys.* **B125** (1977) 1.

[153] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg, and M. Strassler, *Aspects of N=2 supersymmetric gauge theories in three-dimensions*, *Nucl.Phys.* **B499** (1997) 67–99, [hep-th/9703110].

[154] H.-C. Kim, J. Kim, S. Kim, and K. Lee, *Vortices and 3 dimensional dualities*, arXiv:1204.3895.

[155] H.-Y. Chen, H.-Y. Chen, and J.-K. Ho, *Connecting Mirror Symmetry in 3d and 2d via Localization*, arXiv:1312.2361.

[156] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and S. Raju, *Indices for Superconformal Field Theories in 3,5 and 6 Dimensions*, *JHEP* **0802** (2008) 064, [arXiv:0801.1435].

[157] A. Kapustin and B. Willett, *Generalized Superconformal Index for Three Dimensional Field Theories*, arXiv:1106.2484.

[158] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, *3d dualities from 4d dualities*, *JHEP* **1307** (2013) 149, [arXiv:1305.3924].

[159] T. T. Wu and C. N. Yang, *Dirac Monopole Without Strings: Monopole Harmonics*, *Nucl.Phys.* **B107** (1976) 365.

[160] C. Hwang, H.-C. Kim, and J. Park, *Factorization of the 3d superconformal index*, *JHEP* **1408** (2014) 018.

[161] D. Cassani and D. Martelli, *The gravity dual of supersymmetric gauge theories on a squashed $S^1 \times S^3$*, arXiv:1402.2278.

[162] C. Romelsberger, *Calculating the Superconformal Index and Seiberg Duality*, arXiv:0707.3702.

[163] F. Dolan and H. Osborn, *Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to N=1 Dual Theories*, *Nucl.Phys.* **B818** (2009) 137–178, [arXiv:0801.4947].

[164] V. Spiridonov and G. Vartanov, *Elliptic Hypergeometry of Supersymmetric Dualities*, *Commun.Math.Phys.* **304** (2011) 797–874, [arXiv:0910.5944].

[165] V. Spiridonov and G. Vartanov, *Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots, and vortices*, *Commun.Math.Phys.* **325** (2014) 421–486, [arXiv:1107.5788].

[166] D. Sen, *Supersymmetry in the Space-time R X S**3*, *Nucl.Phys.* **B284** (1987) 201.

[167] G. Festuccia and N. Seiberg, *Rigid Supersymmetric Theories in Curved Superspace*, *JHEP* **1106** (2011) 114, [arXiv:1105.0689].

[168] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, *Exploring Curved Superspace*, *JHEP* **1208** (2012) 141, [arXiv:1205.1115].

[169] T. T. Dumitrescu and G. Festuccia, *Exploring Curved Superspace (II)*, *JHEP* **1301** (2013) 072, [arXiv:1209.5408].

[170] T. J. Hollowood, A. Iqbal, and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, *JHEP* **0803** (2008) 069, [hep-th/0310272].

[171] J. Halverson, V. Kumar, and D. R. Morrison, *New Methods for Characterizing Phases of 2D Supersymmetric Gauge Theories*, *JHEP* **1309** (2013) 143, [arXiv:1305.3278].

[172] F. Benini, Y. Tachikawa, and D. Xie, *Mirrors of 3d Sicilian theories*, *JHEP* **1009** (2010) 063, [arXiv:1007.0992].

[173] D. Gaiotto and D. L. Jafferis, *Notes on adding D6 branes wrapping $\mathbb{RP}^3$ in $AdS_4 \times \mathbb{CP}^3$*, *JHEP* **1211** (2012) 015, [arXiv:0903.2175].

[174] F. Benini, C. Closset, and S. Cremonesi, *Chiral flavors and M2-branes at toric CY4 singularities*, *JHEP* **1002** (2010) 036, [arXiv:0911.4127].

[175] D. L. Jafferis, *Quantum corrections to $\mathcal{N} = 2$ Chern-Simons theories with flavor and their AdS_4 duals*, *JHEP* **1308** (2013) 046, [arXiv:0911.4324].

[176] F. Benini, C. Closset, and S. Cremonesi, *Quantum moduli space of Chern-Simons quivers, wrapped D6-branes and AdS4/CFT3*, *JHEP* **1109** (2011) 005, [arXiv:1105.2299].

[177] V. Dobrev and V. Petkova, *All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry*, *Phys.Lett.* **B162** (1985) 127–132.

[178] D. Kazhdan and G. Lusztig, *Representations of coxeter groups and hecke algebras*, *Inventiones mathematicae* **53** (1979), no. 2 165–184.

[179] J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations: a graduate course for physicists*. Cambridge University Press, 1997.

[180] K. De Vos and P. Van Driel, *The Kazhdan-Lusztig conjecture for W algebras*, `hep-th/9508020`.

[181] Kashiwara, Masaki and Tanisaki, Toshiyuki, *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras II*, *Operator algebras, unitary representations, enveloping algebras, and invariant theory* **2** (1990) 159–195.

[182] Casian, Luis, *Kazhdan-Lusztig multiplicity formulas for Kac-Moody algebras*, *Comptes Rendus de l'Academie des Sciences Serie I-Mathematique* **310** (1990), no. 6 333–337.

[183] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*. Graduate texts in Mathematics 82. Springer Verlag, New York, 1982.

[184] J. McCleary, *A User's Guide to Spectral Sequences*. Cambridge Studies in Advanced Mathematics 58. Cambridge University Press, 2001.

[185] Y. Kosmann, *Dérivées de Lie des spineurs*, *Ann.Mat.Pura Appl.* **91** (1971) 317.

[186] M. Godina and P. Matteucci, *Reductive G structures and Lie derivatives*, *J.Geom.Phys.* **47** (2003) 66–86, [`math/0201235`].