Housing Market Dynamics: Financial Risk and Housing Price Bubbles

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Abstract of the Dissertation

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The dissertation investigates the inter-dependence between financial and housing markets and analyzes how uncorrelated financial risk may promote an increase in housing demand and induce bubble-like behavior of residential real estate prices. We show that endogenous relative wealth concerns may play an important role in explaining the emergence and dynamics of housing price bubbles in times of technological innovation that has high level of uncertainty. We present a general equilibrium model in which house-buyers' exposure to financial risk together with concerns about relative wealth translates into housing price volatility. Unlike other models with endogenous relative wealth concerns, our model suggests a non-monotonic relation between technological risk and housing price risk. Our main result is that housing price bubbles are most likely to emerge as a result of house-buyer's financial risk exposure when this exposure is low.

Additionally, the dissertation explores the recent growth of the subprime mortgage market in the United States and its effect on the housing market dynamics. It examines patterns in the borrowing/lending market in the presence of relative wealth concerns and analyzes the effects of mortgage market on the housing price dynamics. A non-monotonic relation between technological risk and housing price risk in our model suggests that high borrower's debt-to-income ratio resulting from high housing price volatility is more likely to be high when financial exposure of lenders is low.

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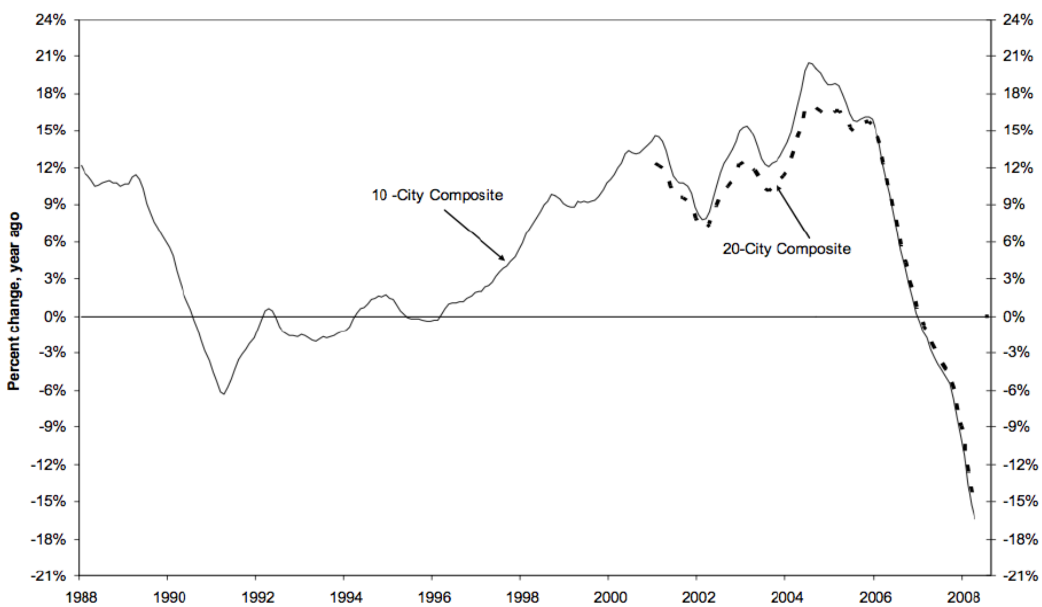
Chapter 1. Investment in Risky Technology Stocks and Housing Price Bubbles

1.1. Introduction

The recent expansion of the mortgage and housing markets, accompanied by significant up and downs in housing prices and levels of borrower default, stimulated much scientific research on speculative price bubbles (see McCarthy, Peach (2004), Himmelberg, Mayer, Sinai (2005), Hong, Scheinkman, Xiong (2008)). Most of the literature studies the boom in residential real estate prices as a consequence of structural changes in the market for mortgage financing. Rather than looking at the housing market in isolation, in this chapter we look at the interrelation between the technology stocks bubble and the housing bubble.

Figure 1: S&P Case-Shiller Home Price Indices from 1988 to 2008



Source: Standard & Poor's and Fiserv

Figure 1 shows the change in the S&P Case-Shiller Home Price Indices over the period from 1988 to 2008. The boom in real estate prices started in the late 1990s and continued until 2005. However, the growth in the Case-Shiller index slowed down between 2000 and 2002. Significant growth that followed this decline was primarily caused by the development of private securitization and the expansion of the subprime mortgage market. As reported by the Census

Bureau, the housing market has been rapidly expanding over the period from 2002 to 2005, with the peak of home sales in 2005. The total home sales of both new and existing houses grew from 6 million in 2002 to about 9 million in 2005 (Fratantoni, Duncan, Brinkmann, and Velz, (2007)).

The goal of this chapter is to examine a potential relation between investment in risky technology stocks and a bubble-like behavior of housing prices. Chapter 1 builds on DeMarzo, Kaniel, and Kremer (2007), henceforth DKK. The authors develop a general equilibrium model showing that rational, risk-averse agents may overinvest in risky technologies, even when the investment is predictably unprofitable. The model is based on the idea that, if future scarce resources are not available in the market today, then competition over future consumption leads to relative wealth concerns that arise endogenously in equilibrium. The authors show that in the presence of relative wealth concerns, when the financial asset in the house-buyer's portfolio is risky enough and the expected return of the asset is fixed, an increase in the risk of the asset encourages more investment in this asset.

We extend the DKK model by adding a housing market. In the real estate market, the main source of risk is induced by the changes in the housing prices. In our model the investment in uncertain technology reverberates on housing prices, so that the price of housing over time is influenced by the realization of the risky investment. Thus, agents end up being exposed to both types of risk, the technological risk and the housing price risk. We show that the presence of aggregate risk can induce a price bubble, even when agents are fully rational.

The housing price movements generated by relative wealth concerns show that house-buyers' exposure to uncorrelated financial risk leads to substantial increase in housing demand and housing prices. To understand this increase, we need to consider the impact of house-buyer's portfolio choice on the future wealth. When house-buyers allocate a significant part of their financial portfolio to the risky assets, uncertainty in realization of these investments generates volatility in house-buyer's future income. Successful realization of risky investments induces significant increase in the wealth of the community and drives housing demand and housing prices up.

Importantly, compared to the benchmark case with complete markets, the housing price volatility in the model is greater for all levels of risk. This finding supports our claim that in the presence of the relative wealth concerns, emergence of uncorrelated financial risks results in greater housing price volatility that can lead to the housing price bubbles.

Several authors have examined the role of relative wealth considerations on consumption and investment decisions. Most of these papers, however, introduce relative wealth concerns exogenously (see Abel (1990), Gali (1994), Dupor and Liu (2003), Cole, Mailath, and Postelwaite (2001), Yeung and Kogan (2002)).

Recent papers that look at general equilibrium models with housing include Davis and Heathcote (2005), who investigate the implications of a real business cycle model with a manufacturing sector, and Ortalo-Magne and Rady (2006), who propose a life-cycle model to study prices in the housing market. However, these papers do not consider financial assets and portfolio choice. Flavin and Yamashita (2002), Cocco (2005), and Flavin and Nakagawa (2005) focus on household's optimal holdings of financial assets in the presence of housing. However, these models do not feature a general equilibrium setup.

There is also much literature on the correlation between technological innovations and stock market dynamics. Quan and Titman (1999) consider the interrelation between real estate and stock prices. Hobijn and Jovanovic (2001) investigate how the IT revolution affects the stock market. Laitner and Stolyarov (2003) analyze how technological innovations may cause stock market fluctuations. Pastor and Veronesi (2006) develop a general equilibrium model in which technological revolutions promote a bubble-like behavior of technology stock prices.

The rest of the chapter is organized as follows. Section 1.2 presents a two-period stochastic general equilibrium model with two types of agents, three goods, and three investments opportunities via real estate and financial markets. Section 1.2 also computes the equilibrium for the incomplete markets case. Section 1.3 further analyzes the model and computes the equilibrium for the complete markets case. Section 1.4 compares the two cases (complete and incomplete markets) and shows some properties of equilibrium returns obtained from MATLAB simulations. Section 1.5 concludes.

1.2. The Model

1.2.1. Setup

We consider a two-period stochastic production economy. There are two types of rational, risk-averse agents in the market: home-buyers and service providers. There is a continuum with mass one of each type of agent. There are three types of goods available in the market: capital goods, services, and housing. House-buyers consume all three types of goods, while service providers only consume capital goods and services. House-buyers consume housing only in period 1, while in period 2 they consume housing, the capital good and services. The utility function is CRRA, so we have ,

$$U_1(h_1) = \frac{1}{1-\gamma} w h_1^{1-\gamma}$$

and

$$U_2(c_{cb}, c_{sb}, h_2) = \frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma} + c_{sb}^{1-\gamma} + w h_2^{1-\gamma}\right),$$

where c_{cb} and c_{sb} denote house-buyer's consumption of capital and consumption of services respectively, while h_1 and h_2 denote the consumption of housing in the first and second periods respectively.

Service providers only appear in period 2, and their utility function is

$$U\left(c_{cp}, c_{sp}\right) = \frac{1}{1-\gamma}\left(c_{cp}^{1-\gamma} + c_{sp}^{1-\gamma}\right),$$

where c_{cp} and c_{sp} denote service provider's consumption of capital and consumption of services respectively.

House-buyers are endowed with one unit of capital and $\bar{h}$ units of housing at date 1. Thus, the first period endowment of an individual house-buyer is given by $\left(1 + p_1\bar{h}\right)$, where p_1 is the housing price at date 1. Service providers are endowed with one unit of services at date 2. The aggregate endowment is therefore also equal to 1.

In the first period, capital can be invested in two types of assets – a risky asset and a non-risky asset. The risky asset returns $\bar{r}$ units of capital for each unit of capital invested, where $\bar{r}$ is a random variable. We assume a simple binary distribution: with probability σ the investment in risky asset is successful and gives return $r > 1$, and with probability $(1-\sigma)$ the risky asset returns zero. The non-risky asset returns one unit of capital for each unit of capital invested. Thus, by investing x_n units of capital in the non-risky asset, and x_r in the risky asset, the house-buyer obtains

$$y(\bar{r}, x_r, x_n) = \bar{r}x_r + x_n,$$

units of capital in period 2. If, in addition, h_1 units of housing are bought then the total wealth available to house-buyers in the second period is

$$y(\bar{r}, x_r, x_n) + p_2(\bar{r})h_1,$$

where we have highlighted the fact that the price of housing in period 2 may depend on the realization of $\bar{r}$. When no confusion arises we will simply write y, ignoring the arguments.

In period 1, house-buyers invest their capital in the risky and non-risky asset and they buy or sell housing. In period 2, house-buyers and service providers trade capital and services competitively in the market and consume. Both types of agents make their investment, consumption, and trading decisions so as to maximize their expected utility, and markets clear.

1.2.2. Equilibrium

The equilibrium of the model is a set of allocations $\{c_{cb}(\bar{r}), c_{sb}(\bar{r}), c_{cp}(\bar{r}), c_{sp}(\bar{r}), h_1, h_2(\bar{r}), x_r, x_n\}$ and a set of prices $\{p_s(\bar{r}), p_1, p_2(\bar{r})\}$, for $\bar{r} \in \{r, 0\}$, such that

- At period one, house-buyers maximize their expected life-time utility

$$\max \frac{1}{1-\gamma} w h_1^{1-\gamma} + E\left\{\frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma}(\bar{r}) + c_{sb}^{1-\gamma}(\bar{r}) + w h_2^{1-\gamma}(\bar{r})\right)\right\}$$

$$\text{s. t. } x_r + x_n + p_1 h_1 = 1 + p_1 \bar{h},$$

$$c_{cb}(\bar{r}) + p_s(\bar{r})c_{sb}(\bar{r}) + p_2(\bar{r})h_2(\bar{r}) = \bar{r}x_r + x_n + p_2(\bar{r})h_1, \text{ each } \bar{r} \in \{r, 0\},$$

where $c_{cb}(\bar{r})$ denotes house-buyer's personal consumption, $c_{sb}(\bar{r})$ denotes house-buyer's consumption of services, and $p_s(\bar{r})$ is the market price for services as a function of the realization of the return on the risky asset.

- For each realization of $\bar{r} \in \{r, 0\}$, service providers maximize

$$\max \frac{1}{1-\gamma}\left(c_{cp}^{1-\gamma}(\bar{r}) + c_{sp}^{1-\gamma}(\bar{r})\right)$$

$$\text{s. t.} \quad c_{cp}(\bar{r}) + p_s(\bar{r})c_{sp}(\bar{r}) = p_s(\bar{r}),$$

where $c_{cp}(\bar{r})$ denotes provider's personal consumption and $c_{sp}(\bar{r})$ denotes provider's consumption of services.

- For each realization of $\bar{r} \in \{r, 0\}$, markets clear:

$$\bar{H} = H_1 = H_2(\bar{r}),$$

$$X_r + X_n = 1,$$

$$C_{cb}(\bar{r}) + C_{cp}(\bar{r}) = Y(\bar{r}),$$

$$C_{sb}(\bar{r}) + C_{sp}(\bar{r}) = 1,$$

where $\mathrm{Y}(\bar{r}) = \bar{r}X_r + X_n$ is the aggregate capital return from both assets. Capital letters denote aggregate demand and supply (e.g. $X_r = \int_0^1 x_r(i)di$ and $H_2(\bar{r}) = \int_0^1 h_2(\bar{r})(i)di$). We consider a symmetric equilibrium where individual decisions are equal to aggregate decisions of agents' population.

1.3. Analysis

1.3.1. Period 2

We begin our analysis by assuming that house-buyers and service providers are unable to trade in period one. In the second period there are two types of agents: house-buyers and service providers. Let y be the realization of $\bar{r}x_r + x_n$ in period 2.

House-buyer's maximization problem is

$$\max \frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma} + c_{sb}^{1-\gamma} + wh_2^{1-\gamma}\right)$$

$$\text{s. t. } c_{cb} + p_s c_{sb} + p_2 h_2 = y + p_2 h_1.$$

Service providers solve

$$\max \frac{1}{1-\gamma}\left(c_{cp}^{1-\gamma} + c_{sp}^{1-\gamma}\right)$$

$$\text{s. t.} \quad c_{cp} + p_s c_{sp} = p_s,$$

Computing the demand of each type we obtain:

$$h_2 = \frac{y + p_2 h_1}{1 + p_s^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2^{1-\frac{1}{\gamma}}} p_2^{\left(-\frac{1}{\gamma}\right)} w^{\frac{1}{\gamma}},$$

$$c_{cb} = \frac{y + p_2 h_1}{1 + p_s^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2^{1-\frac{1}{\gamma}}},$$

$$c_{sb} = \frac{y + p_2 h_1}{1 + p_s^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2^{1-\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)},$$

$$c_{cp} = \frac{1}{1 + p_s^{1-\frac{1}{\gamma}}},$$

$$c_{sp} = \frac{p_s^{1-\frac{1}{\gamma}}}{1 + p_s^{1-\frac{1}{\gamma}}}.$$

Imposing the equilibrium conditions we obtain:

$$p_s(\bar{r}) = \left(Y(\bar{r})\right)^{\gamma} = (\bar{r} X_r + X_n)^{\gamma}.$$

The market price for housing in period 2 is given by

$$p_2(\bar{r}) = \frac{w}{\bar{h}^{\gamma}} \left[\left(\frac{Y(\bar{r})}{1 + \left(Y(\bar{r})\right)^{\gamma-1}}\right)\right]^{\gamma},$$

where we use the fact that in equilibrium it must be

$$h_1 = h_2 = \bar{h}.$$

In the two-state economy, there will be high and low price realizations. With probability σ, successful realizations of the investment in the risky asset will result in the high prices given by $\left(p_s(r), p_2(r)\right)$, while with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the low prices $\left(p_s(0), p_2(0)\right)$.

At the equilibrium prices, a consumption profile of the house-buyer implies that the indirect utility depends on both capital return from financial assets y and on aggregate capital return Y. Let (x_r, x_n, h_1) be the investment strategy chosen by the house-buyer in period 1. For a given investment strategy, with probability σ successful realization of high return on the investment in the risky asset will result in the indirect utility of the house-buyer given by

$$V(r; x_r, x_n, h_1) = \frac{1}{1-\gamma}(rx_r + x_n + p_2(r)h_1)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

While with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the indirect utility of the house-buyer given by

$$V(0; x_r, x_n, h_1) = \frac{1}{1-\gamma}(x_n + p_2(0)h_1)^{1-\gamma}\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

Thus, the expected indirect utility of the house-buyer is given by

$$W(x_r, x_n, h_1) = \sigma V(r; x_r, x_n, h_1) + (1-\sigma)V(0; x_r, x_n, h_1)$$

1.3.2. Period 1

We now solve the house-buyer's maximization problem at period 1. At this time, the house-buyer chooses the investment strategy to maximize his expected life-time utility subject to the first period budget constraint. Therefore, the house-buyer's maximization problem at date 1 is

$$\max \frac{1}{1-\gamma} w h_1^{1-\gamma} + W(x_r, x_n, h_1)$$

$$\text{s. t. } x_r + x_n + p_1 h_1 = 1 + p_1 \bar{h},$$

$$x_r \geq 0,$$

$$x_n \geq 0.$$

We obtain an equation for the investment decision of the house-buyer:

$$\sigma(r-1)\left(\frac{1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{rx_r + x_n + p_2(r)h_1}\right)^{\gamma} - (1-\sigma)\left(\frac{1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{x_n + p_2(0)h_1}\right)^{\gamma} = 0$$

Together with the budget constraint this equation defines the demand functions for the investment decisions in implicit form.

In equilibrium the decision of an individual house-buyer matches the aggregate decision of the house-buyer population, that is $h_1 = H_1,\ h_2 = H_2, x_r = X_r,$ and $x_n = X_n$. And markets clear:

$$\bar{h} = h_1,$$

$$x_r + x_n = 1.$$

An equation for the investment decision of the house-buyer can be rewritten as

$$\sigma(r-1)\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}}}{rx_r + x_n}\right)^{\gamma}\left[\frac{1+\dfrac{w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{1+p_s(r)^{1-\frac{1}{\gamma}}}}{1+\dfrac{p_2(r)h_1}{rx_r+x_n}}\right]^{\gamma} -$$

$$-(1-\sigma)\left(\frac{1+p_s(0)^{1-\frac{1}{\gamma}}}{x_n}\right)^{\gamma}\left[\frac{1+\dfrac{+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{1+p_s(0)^{1-\frac{1}{\gamma}}}}{1+\dfrac{p_2(0)h_1}{x_n}}\right]^{\gamma} = 0$$

If we substitute the expressions for equilibrium prices, the term in square brackets in equilibrium is identical to 1:

$$\left[\frac{1+\dfrac{w^{\frac{1}{\gamma}}p_2(\bar{r})^{1-\frac{1}{\gamma}}}{1+p_s(\bar{r})^{1-\frac{1}{\gamma}}}}{1+\dfrac{p_2(\bar{r})\bar{h}}{\bar{r}X_r+X_n}}\right]^{\gamma} = \left[\frac{1+\dfrac{w^{\frac{1}{\gamma}}}{\bar{h}}\dfrac{Y(\bar{r})}{1+\left(Y(\bar{r})\right)^{\gamma-1}}}{1+\dfrac{w^{\frac{1}{\gamma}}}{\bar{h}}\dfrac{Y(\bar{r})}{1+\left(Y(\bar{r})\right)^{\gamma-1}}}\right]^{\gamma} \equiv 1.$$

Thus, an equation for the investment decision of the house-buyer in equilibrium becomes

$$\sigma(r-1)\left(\frac{1+\left(Y(r)\right)^{\gamma-1}}{Y(r)}\right)^{\gamma} - (1-\sigma)\left(\frac{1+\left(Y(0)\right)^{\gamma-1}}{Y(0)}\right)^{\gamma} = 0$$

or

$$\sigma(r-1)\left(\frac{1+(rX_r+X_n)^{\gamma-1}}{rX+X}\right)^{\gamma}-(1-\sigma)\left(\frac{1+(X_n)^{\gamma-1}}{X_n}\right)^{\gamma}=0.$$

Market price for housing at period 1:

$$p_1=\left[\frac{w}{\bar{h}^{\gamma}}+\sigma\left(\frac{rX_r+X_n}{1+p_s(r)^{1-\frac{1}{\gamma}}}\right)^{\gamma}\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{rX_r+X_n+p_2(r)\bar{h}}\right)^{\gamma}+\right.$$

$$\left.+(1-\sigma)\left(\frac{X_n}{1+p_s(0)^{1-\frac{1}{\gamma}}}\right)^{\gamma}\left(\frac{1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{X_n+p_2(0)\bar{h}}\right)^{\gamma}\right]x$$

$$x\left[\sigma\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{rX_r+X_n+p_2(r)\bar{h}}\right)^{\gamma}+(1-\sigma)\left(\frac{1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{X_n+p_2(0)\bar{h}}\right)^{\gamma}\right]^{-1}$$

This expression can be rewritten as

$$p_1=\frac{1+\sigma\left[\frac{1+\frac{w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{1+p_s(r)^{1-\frac{1}{\gamma}}}}{1+\frac{p_2(r)\bar{h}}{rX+X_n}}\right]^{\gamma}+(1-\sigma)\left[\frac{1+\frac{+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{1+p_s(0)^{1-\frac{1}{\gamma}}}}{1+\frac{p_2(0)\bar{h}}{X_n}}\right]^{\gamma}}{\sigma\left[\frac{1+\frac{w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{1+p_s(r)^{1-\frac{1}{\gamma}}}}{1+\frac{p_2(r)\bar{h}}{rX_r+X}}\right]^{\gamma}p_2(r)^{-1}+(1-\sigma)\left[\frac{1+\frac{+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{1+p_s(0)^{1-\frac{1}{\gamma}}}}{1+\frac{p_2(0)\bar{h}}{X_n}}\right]^{\gamma}p_2(0)^{-1}}$$

Considering the identity above, this expression simplifies to

$$p_1=2\left[\frac{\sigma}{p_2(r)}+\frac{(1-\sigma)}{p_2(0)}\right]^{-1}=$$

$$=2\left[\sigma\left(\frac{1+(rX_r+X_n)^{\gamma-1}}{rX_r+X_n}\right)^{\gamma}+(1-\sigma)\left(\frac{1+(X_n)^{\gamma-1}}{X_n}\right)^{\gamma}\right]^{-1}.$$

1.3.3. Benchmark: Complete Markets

We now consider a benchmark case in which complete contingent contracts exist in period 1. Hence, service providers exist at date 1 and can sell their services for period 2 to house-buyers; and full contingent markets for housing and capital at period 2 exist in period 1.

The equilibrium of the benchmark case is then a set of allocations $\{c_{cb}(\bar{r}), c_{sb}(\bar{r}), c_{cp}(\bar{r}), c_{sp}(\bar{r}), h_1, h_2(\bar{r}), x_{rb}, x_{nb}\}$ and a set of market prices $\{p_s(\bar{r}), p_c(\bar{r}), p_{1b}, p_{2b}(\bar{r})\}$, for $\bar{r} \in \{r, 0\}$, such that

- At period one, house-buyers maximize their expected life-time utility

$$\max \frac{1}{1-\gamma} w h_1^{1-\gamma} + E\left\{\frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma}(\bar{r}) + c_{sb}^{1-\gamma}(\bar{r}) + w h_2^{1-\gamma}(\bar{r})\right)\right\}$$

$$\text{s. t. } p_{1b}h_1 + p_c(r)c_{cb}(r) + p_s(r)c_{sb}(r) + p_{2b}(r)h_2(r) + p_c(0)c_{cb}(0) + p_s(0)c_{sb}(0) + p_{2b}(0)h_2(0) = p_c(r)rx_r + p_c(r)x_n + p_c(0)x_n + p_{2b}(r)h_1 + p_{2b}(0)h_1 + p_{1b}\bar{h},$$

$$x_{rb} + x_{nb} = 1,$$

$$x_r \geq 0,$$

$$x_n \geq 0.$$

- Service providers maximize

$$\max \frac{1}{1-\gamma} E\left\{c_{cp}^{1-\gamma}(\bar{r}) + c_{sp}^{1-\gamma}(\bar{r})\right\}$$

$$\text{s. t. } p_c(r)\ c_{cp}(r) + p_s(r)c_{sp}(r) + p_c(0)\ c_{cp}(0) + p_s(0)c_{sp}(0) = p_s(r) + p_s(0).$$

- For each realization of $\bar{r} \in \{r, 0\}$, markets clear:

$$\bar{H} = H_1 = H_2(\bar{r}),$$

$$X_{rb} + X_{nb} = 1,$$

$$C_{cb}(\bar{r}) + C_{cp}(\bar{r}) = Y(\bar{r}),$$

$$C_{sb}(\bar{r}) + C_{sp}(\bar{r}) = 1.$$

Solving for the benchmark case, we obtain the following set of equations for investment decisions of the house-buyers:

$$x_{rb} = \frac{1-a}{1+a(r-1)}$$

where a is given by the expression

$$a = \left(\frac{1}{r-1}\frac{1-\sigma}{\sigma}\right)^{\frac{1}{\gamma}}$$

$$x_{nb} = 1 - x_{rb} = \frac{ar}{1+a(r-1)}$$

Let $\beta = \sigma r$ be the expected return of the risky technology. Then $a = \left(\frac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}} < 1$ for $\beta > 1$.

For housing prices in the benchmark case we obtain:

$$p_{2b}(r) = \left(\frac{w}{\overline{h}}\right)^{\gamma} \sigma \left\{\frac{p_c(r)(rX_{rb} + X_{nb}) + p_c(0)X_{nb}}{(1 + rX_{rb} + X_{nb})^{\gamma}(\sigma p_c(r)^{\gamma-1})^{\frac{1}{\gamma}} + (1 + X_{nb})^{\gamma}\left((1-\sigma)p_c(0)^{\gamma-1}\right)^{\frac{1}{\gamma}}}\right\}^{\gamma}$$

$p_{2b}(0) =$

$$\left(\frac{w}{\overline{h}}\right)^{\gamma} (1-\sigma) \left\{\frac{p_c(r)(rX_{rb} + X_{nb}) + p_c(0)X_{nb}}{(1 + rX_{rb} + X_{nb})^{\gamma}(\sigma p_c(r)^{\gamma-1})^{\frac{1}{\gamma}} + (1 + X_{nb})^{\gamma}\left((1-\sigma)p_c(0)^{\gamma-1}\right)^{\frac{1}{\gamma}}}\right\}^{\gamma}$$

$$p_{1b} = 2\left(p_{2b}(r) + p_{2b}(0)\right)$$

1.4. Results

1.4.1. Complete Markets

In the complete markets case relative wealth concerns are not relevant. In our setting (adopted from DKK model), with probability $\sigma = \frac{\beta}{r}$ the investment in risky asset is successful and gives return $r > 1$, and with probability $(1-\sigma)$ the risky asset returns zero. Holding the expected return β constant, we consider an increase in the probability of the asset's success σ. This is equivalent to a decrease in risk of the asset $r = \frac{\beta}{\sigma}$. We have the following result.

Proposition 1. Suppose $\beta > 1$. When we keep β constant, the investment x_{rb} in the risky asset is increasing in probability of the asset's success.

The proofs of this and the following propositions are in the appendix. The result is standard. Rational risk-averse investors are expected to invest in a risky asset only when they are compensated for the risk taken. If we keep the expected return fixed but decrease the risk, then investment in the risky asset goes up. Later we will compare these findings with house-buyer's investment decisions in the presence of relative wealth concerns.

In the complete markets case, we also have the following result.

Proposition 2. Housing prices p_{1b} and $p_{2b}(r)$ at $\sigma \to 0$ are lower than at $\sigma \to 1$. The housing price $p_{2b}(0)$ at $\sigma \to 0$ is higher than at $\sigma \to 1$.

Since the supply of housing is fixed, a decrease in housing prices indicates a decline in housing demand in the economy with no trading frictions. To analyze the volatility of investments in housing and in the risky asset, we look at the rates of return on both investments. We compute their variance ratio, i.e. the ratio of variance of housing rate of return to variance of risky asset rate of return.

The rate of return on the risky asset is

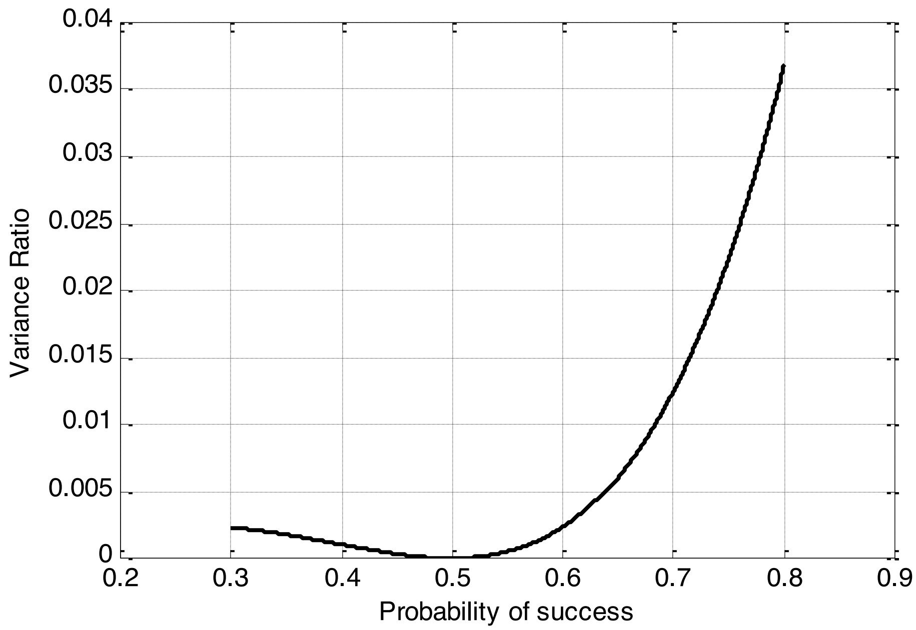
$$\widetilde{R_r} = \begin{cases} r - 1, & \text{with probability } \sigma, \\ -1, & \text{with probability } (1 - \sigma), \end{cases}$$

while the rate of return on the housing is

$$\widetilde{R_{hb}} = \begin{cases} \dfrac{p_{2b}(r) - p_{1b}}{p_{1b}}, & \text{with probability } \sigma, \\ \dfrac{p_{2b}(0) - p_{1b}}{p_{1b}}, & \text{with probability } (1 - \sigma). \end{cases}$$

Figure 2 illustrates the relative risk for the house-buyer's investments. In the economy with rational risk-averse agents and complete markets, investment in the risky asset increases in the probability of this asset's success. This investment in turn affects housing prices specified in the contingent contracts in period 1. These contracts insure positive correlation between house-buyer's investment in the risky asset and housing price $p_{2b}(r)$, while imposing negative correlation with housing price $p_{2b}(0)$. When risk is high, volatility of the housing prices is monotonic in the volatility of the risky asset (i.e., the relative risk function is decreasing). For low levels of risk, the volatility of the housing prices is not monotonically related to the volatility of the risky asset (i.e., relative risk function is increasing). We will compare this result later with the incomplete markets case to see how the presence of relative wealth concerns increases housing price volatility.

Figure 2: Variance ratio in the benchmark case with an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership ω=0.9, and housing market size $\bar{h} = \mathbf{0.5}$



1.4.2. Relative Wealth Concerns and Overinvestment in the Risky Asset

Let us now compare the complete market case with the incomplete markets case. When house-buyers and service providers are unable to trade in period 1, relative wealth concerns arise endogenously in equilibrium, resulting in extensive investment in the risky asset.

If agents are sufficiently risk-averse ($\gamma > 1$), then competition over consumption of services, which arises among house-buyers in the period 2, results in relative wealth concerns. This comes from the fact that for $\gamma > 1$, the marginal utility of income $\partial V(y, Y)/\partial y$ is increasing in aggregate capital income Y. House-buyers are afraid to be relatively poor in comparison to others when the risky asset is successful. An investment in the risky asset reduces the risk of their relative wealth, so house-buyers invest more in the risky asset than they would in the situation with no competition over the scarce future services.

The analysis of market equilibrium shows that relative wealth concerns lead to a situation where investment x_r in the risky asset exceeds the benchmark level of investment x_{rb}(we adopt DKK terminology and call "overinvestment" this investment behavior).

Our model with trading frictions predicts that

- There is a unique equilibrium with positive investment x_r in the risky asset.
- If the asset is risky enough, holding the expected return σr fixed, investment x_r decreases with the probability of the asset's success σ (i.e., investment x_r increases in the risk of the asset r, yielding a larger deviation from the first-best result).
- If the asset has low risk, holding the expected return σr fixed, investment x_r increases with the probability of the asset's success σ (i.e., investment x_r decreases in the risk of the asset r).
- In the presence of relative wealth concerns, investors tend to invest more in the risky asset than they would otherwise ($x_r > x_{rb}$).

Proposition 3. Suppose $\gamma > 2$ and $\beta > 1$. Then for low levels of σ, x_r decreases with the probability of the asset's success. For high levels of σ, x_r increases with the probability of the asset's success.

This result comes from the fact that when agents are sufficiently risk-averse and the expected return on the risky asset is kept constant, an increase in the risk of an asset induces stronger concerns about relative wealth, which in turn lead to house-buyer's extensive investment in the risky asset. For low levels of risk, however, relative wealth concerns are not sufficient, thus marginal utility of the house-buyer is decreasing in aggregate capital income. As a result, for low levels of risk, house-buyer's investment in the risky asset decreases in its risk.

Proposition 4. Suppose $\gamma > 1$. If agents invest in the risky asset $(x > 0)$, then in the presence of relative wealth concerns investors tend to invest more in the risky asset than they would otherwise ($x_r > x_{rb}$).

Thus, if service providers do not trade services in the period 1, investments by different house-buyers are complements. This increases the equilibrium level of investment in the risky asset.

Table 1: Results for equilibrium investment in the risky asset with an expected return of 25%, risk aversion γ = 3, the coefficient of utility benefit of homeownership w=0.9, and different probabilities of asset's success

Probability of success, $\boldsymbol{\sigma}$	$\boldsymbol{x_{rb}}$	$\boldsymbol{x_r}$	Overinvestment
0.2	0.0150	0.8317	0.8167
0.3	0.0252	0.7339	0.7087
0.4	0.0380	0.6387	0.6007
0.5	0.0549	0.5663	0.5114
0.6	0.0780	0.5309	0.4529
0.7	0.1117	0.5309	0.4192
0.8	0.1660	0.5626	0.3966
0.9	0.2722	0.6368	0.3646

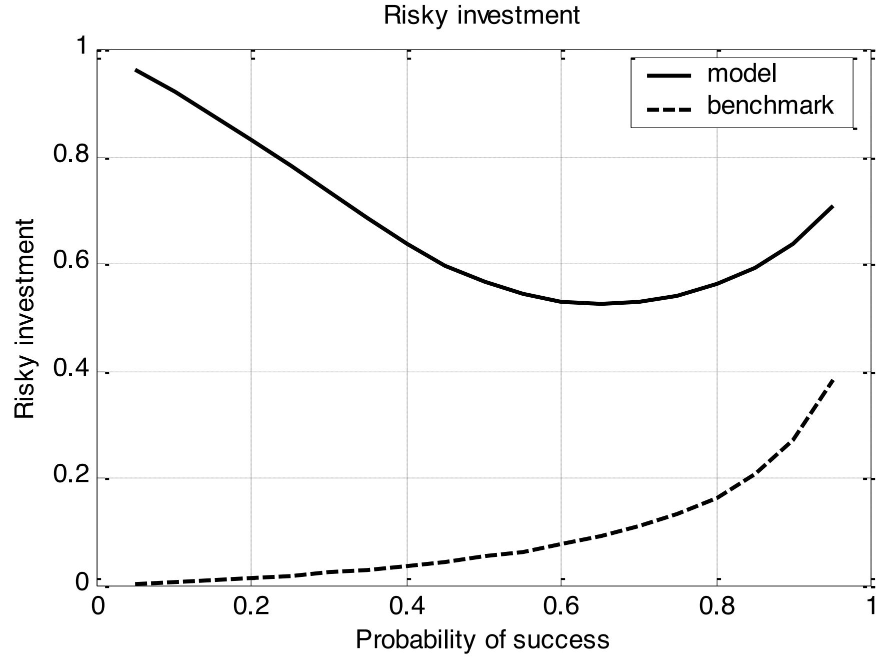
Results for the equilibrium investment decisions of house-buyers are presented in the Table 1. These MATLAB simulation results have been obtained with risk aversion coefficient $\gamma = 3$, an expected return of 25% ($\beta = \sigma r = 1.25$), and the coefficient of utility benefit of homeownership $w = 0.9$. We considered eight possible cases with different levels of risk for the risky asset in the portfolio of the house-buyers. Risk was modeled through the probability of this asset's success. With the expected return of the risky asset fixed, its variance is decreasing in the probability of the asset's success. Thus, holding the expected return fixed an increase in the probability of asset's success decreases the risk associated with this asset. Results in second column show that as we increase the probability of the asset's success σ (i.e., decrease the risk of the asset), investment x_{rb} in this asset increases in the benchmark case.

Table 1 also provides the results for the equilibrium investment decisions of house-buyers in the incomplete markets case. The results presented in the third column of Table 1 illustrate that, for high levels of risk, investment x_r decreases with the probability of the asset's success σ (i.e., investment x_r increases in the risk of the asset r). This finding is an important illustration of the role that relative wealth concerns play in the portfolio decisions of house-buyers. Rational risk-averse house-buyers fear to be relatively poor when others are wealthy. As a result, agents correlate their income with high risk/high return financial assets. As we may see from Table 1, in presence of relative wealth concerns house-buyers allocate more than half of their investment portfolio to the risky asset.

On the other hand, for low levels of risk, investment x_r increases with the probability of the asset's success σ (i.e., investment x_r decreases in the risk of the asset r). When risk is low, volatility of the financial asset is low. Relative wealth concerns are not strong enough to induce positive correlation between investment and its risk. Thus, at low levels of risk, house-buyers investment in the risky asset decreases with the risk of an asset.

The last column of Table 1 presents a quantitative illustration of overinvestment resulting from the introduction of trading frictions in the model. Here we compare investment decisions of the house-buyers in the benchmark case to their investment decisions in our model. The numbers in the fourth column correspond to the quantitative difference between investment in the risky asset in our model and investment in the benchmark case with the same parameterization. As we increase the risk of an asset, overinvestment increases significantly. When risk associated with the risky asset is relatively low and investment in the risky asset is successful with probability $\sigma = 0.9$, in our model house-buyers invest approximately 60 percent more than they do in the benchmark case. That is in the model they allocate roughly two thirds of their capital to the risky asset as opposed to less than 3 percent in the benchmark case. While for the case of high risk associated with the risky asset, when probability of asset's success is $\sigma = 0.2$, effect of relative wealth concerns increases significantly. In this case, in the presence of relative wealth concerns house-buyers invest in the risky asset approximately 82 percent more than they would otherwise (benchmark case). That is house-buyers invest more than 83 percent of their capital in the risky asset when they are worried about future scarce resources; with only 1 percent of the capital invested in the risky asset in absence of these concerns. Equilibrium results for house-buyers' investment in the risky asset are also illustrated in Figure 3.

Figure 3: Benchmark and model investment in the risky asset with an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership ω=0.9



Proposition 5. Suppose $\gamma > 2$ and $\beta = 1$. Then investment x_r decreases with the probability of the asset's success.

Table 2 offers results for the equilibrium investment decisions of house-buyers when the risky asset has zero expected return (i.e. $\beta = \sigma r = 1$,), the risk aversion coefficient is $\gamma = 3$, and the coefficient of utility benefit of homeownership $w = 0.9$. The second column illustrates that when the expected return is zero, there is no investment in the risky asset in the benchmark case. The third column of Table 2 provides the results for the equilibrium investment decisions of house-buyers in the model with trading frictions. Results presented in this column illustrate that in presence of relative wealth concerns there is positive investment in the risky asset even when the asset has zero expected return.

It is clear that when the risky technology has zero expected return, any investment in the risky asset lowers house-buyer's utility. Therefore, levels of house-buyer's investment in the risky asset are dictated solely by the relative wealth concerns mechanism for all levels of asset's risk. This is the reason why an increase in the risk of the asset actually encourages more investment in it, despite the fact that agents are risk averse.

Table 2: Results for equilibrium investment in the risky asset with zero expected return, risk aversion γ = 3, the coefficient of utility benefit of homeownership w=0.9, and different probabilities of asset's success

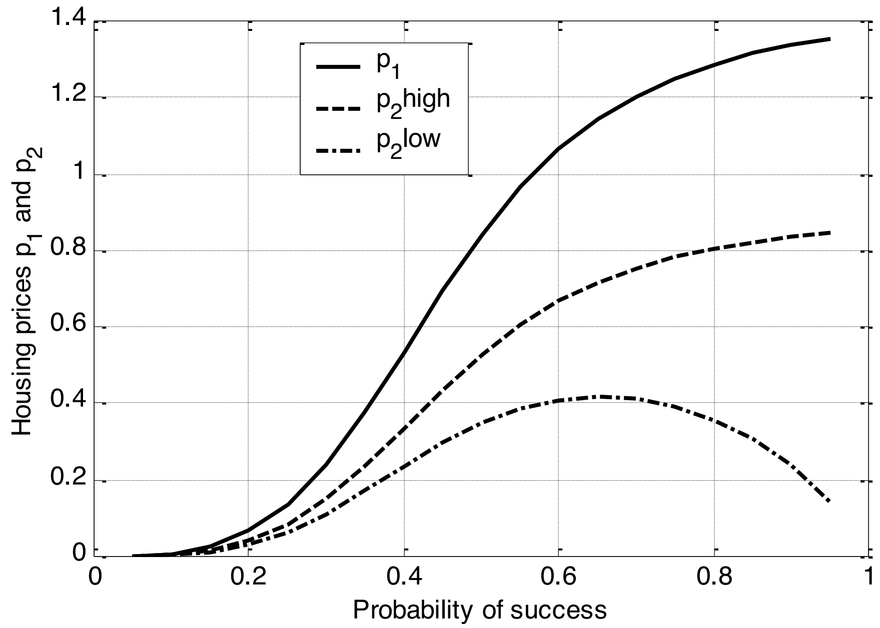
Probability of success, $\boldsymbol{\sigma}$	$\boldsymbol{x_{rb}}$	$\boldsymbol{x_r}$	Overinvestment
0.2	0.0000	0.7511	0.7511
0.3	0.0000	0.5728	0.5728
0.4	0.0000	0.3360	0.3360
0.5	0.0000	0.0697	0.0697
0.6	0.0000	0.0334	0.0334
0.7	0.0000	0.0281	0.0281
0.8	0.0000	0.0263	0.0263
0.9	0.0000	0.0257	0.0257

1.4.3. Relative Wealth Concerns and Housing Market

Let us now examine how relative wealth concerns and house-buyers' portfolio composition affect the housing market. We have already seen that when house-buyers are able to trade future endowments with the service providers, the spot price for housing in period 1 is higher. We will now show that imperfect tradability of future endowments together with the presence of high risk in the financial markets lead to a bubble-like behavior of the housing prices.

We analyze the behavior of housing prices with respect to financial risk and a composition of the house-buyer's portfolio. Housing price dynamics in the model with trading frictions can be inferred from Figure 4. It plots spot housing price p_1 and contingent housing prices $p_2(r)$ and $p_2(0)$ as a function of the probability of risky asset's success when the expected return on the risky asset is kept constant. Figure 4 shows housing prices for different levels of σ when the expected return is 25%, risk aversion is γ=3, the coefficient of utility benefit of homeownership is w=0.9, and housing market size is $\bar{h} = 0.5$.

Figure 4: Housing prices in the model with an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, and housing market size $\bar{h} = 0.5$



The first observation we make is that a success of the risky investment in the financial market results in the housing price realization $p_2(r)$ that is higher than housing price realization $p_2(0)$. The initial housing price p_1 is higher than the housing prices in the second period.

Proposition 6. In our model $p_1 > p_2(r) > p_2(0)$ for any level of γ.

When the asset is successful, the amount of capital in the economy is larger. Given the fixed amount of housing in the economy, this translates into lower relative price for capital or, equivalently, in higher relative housing prices.

Compared to the benchmark case, higher average level of housing prices in the model illustrates that house-buyers' concerns about relative wealth may lead to an increase in housing demand. Relative wealth concerns which arise endogenously in the equilibrium lead to house-buyer's portfolio diversification and extensive investment in high-risk financial instruments. This portfolio composition affects the wealth of house-buyer's population in period 2 and increases demand for housing which in turn results in housing price bubble.

To understand the above statement better, let us now examine in greater details the dynamics of housing prices with respect to risk and analyze the housing price formation mechanism.

Consider the expression for housing price $p_2(r)$:

$$p_2(r) = \frac{w}{\bar{h}^\gamma}\left[\left(\frac{Y(r)}{1 + \left(Y(r)\right)^{\gamma-1}}\right)\right]^\gamma,$$

where $Y(r) = 1 + (r-1)X_r = 1 + \frac{\beta-\sigma}{\sigma}X_r$ is the return function of the house-buyer for high realization of the risky investment. We have already analyzed the behavior of house-buyers' investment function x_r (in equilibrium $x_r = X_r$) with respect to probability of the risky asset's success in Proposition 5. Let us now employ the findings of that analysis.

For small σ (i.e., high risk), investment x_r decreases with σ and for large σ (i.e., low risk), investment x_r increases with σ. For small σ, both term $\frac{\beta-\sigma}{\sigma}$ and investment x_r are decreasing in σ. Therefore, $Y(r)$ is decreasing in σ. For large σ, term $\frac{\beta-\sigma}{\sigma}$ is decreasing and investment x_r is increasing in σ. To find whether $Y(r)$ is decreasing or increasing in σ, we look at the asymptotic behavior of x_r. When $\sigma \to 1$, $a = \left(\frac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}} \to 0$. Then investment in the risky asset is

$$x_r \cong 1 - \left(\frac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}} \frac{\beta}{1+\beta^{\gamma-1}} \to 1.$$

Therefore, for large σ, $Y(r)$ is decreasing in σ as well.

For $\gamma = 3$ and $x_r > 0$, return function $Y(r) > 1$. Then function $p_2(r)$ is decreasing in $Y(r)$, while $Y(r)$ is decreasing in σ for all levels of risk. So, when risky asset is successful, housing price $p_2(r)$ is increasing in the probability of the asset's success.

To understand this increase, we need to consider the market prices of scarce services in our model. When services are not available in today's market, house-buyers correlate their future wealth with the realization of the risky investment. In the booming economy, the price of services is a rapidly increasing function of house-buyer population wealth. Since in the successful economy, when the expected return on the risky asset is kept constant, house-buyer population wealth decreases with the probability of the risky asset's success, service prices decrease as well. Note that house-buyer's wealth decreases much slower than service prices. Hence, given fixed market size, lower service prices result in the higher relative housing prices. This implies that in a successful economy, housing prices $p_2(r)$ increase with the probability of the asset's success.

Now consider the expression for housing price $p_2(0)$:

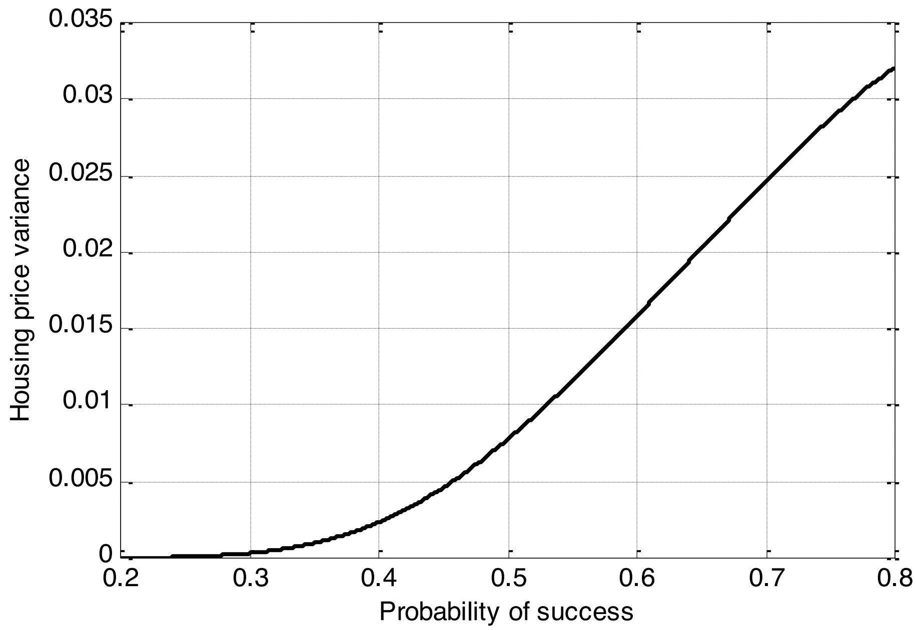
$$p_2(0) = \frac{w}{\bar{h}^{\gamma}}\left[\left(\frac{Y(0)}{1 + \left(Y(0)\right)^{\gamma-1}}\right)\right]^{\gamma},$$

where $Y(0) = 1 - X_r$ is the return function of the house-buyer for low realization of the risky investment. Function $Y(0)$ mirrors function x_r. The more house-buyers invest in the risky asset, the poorer they are when this asset gives low return. For $\gamma = 3$ and $x_r > 0$, we have $Y(0) < 1$. Then the function $p_2(0)$ behaves as $Y(0)$ for all levels of risk, so when the risky asset is unsuccessful, higher levels of wealth in the economy result in higher level of housing price $p_2(0)$. Thus, for small σ, price $p_2(0)$ increases in probability of the asset's success, while for large σ, price $p_2(0)$ decreases in probability of the asset's success.

Figure 4 serves as a good illustration to the fact that the emergence of uncorrelated risk in the financial market, in presence of relative wealth concerns, leads to price volatility in the housing market. Figure 5 depicts the variance of the housing price $p_2(\bar{r})$ showing that price volatility increases in the probability of risky asset's success when the expected return is kept constant. Hence, housing price volatility is the greatest at low levels of financial risk. This result holds for different parameterization. It provides us with an important observation: while an emergence of uncorrelated financial risk itself, in the presence of relative wealth concerns, results in increase of housing demand and bubble-like behavior of housing prices, this impact is strongest at the low levels of house-buyer's risk exposure.

This result comes from the fact that when the asset is successful, the amount of capital in the economy is larger. Given the fixed amount of housing in the economy, this translates into lower relative price for capital or, equivalently, in higher relative housing prices. When the asset is unsuccessful, the amount of capital in the economy is smaller. Given the fixed amount of housing in the economy, this translates into higher relative price for capital or, equivalently, in lower relative housing prices. Since the gap between amount of capital in a successful economy and amount of capital in an unsuccessful economy increases in the probability of the asset's success, so does the housing price volatility.

Figure 5: Housing price variance in the model with an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, and housing market size $\bar{h} = 0.5$



For completeness, let us now examine the relative risk in our model. We look at relative volatility in the rates of returns for housing and risky investment in the house-buyer's portfolio. We have already seen the behavior of housing price volatility in the benchmark case. Let us now compare this to the findings of our model.

The rate of return on the housing in the model is

$$\widetilde{R_h} = \begin{cases} \dfrac{p_2(r) - p_1}{p_1}, & \text{with probability } \sigma, \\ \dfrac{p_2(0) - p_1}{p_1}, & \text{with probability } (1-\sigma), \end{cases}$$

while the rate of return on the risky asset is the same as in the benchmark case is

$$\widetilde{R_r} = \begin{cases} r - 1, & \text{with probability } \sigma, \\ \;\;-1, & \text{with probability } (1-\sigma). \end{cases}$$

Figure 6: Expected return and Variance ratios in the model with an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, and housing market size $\bar{h} = 0.5$

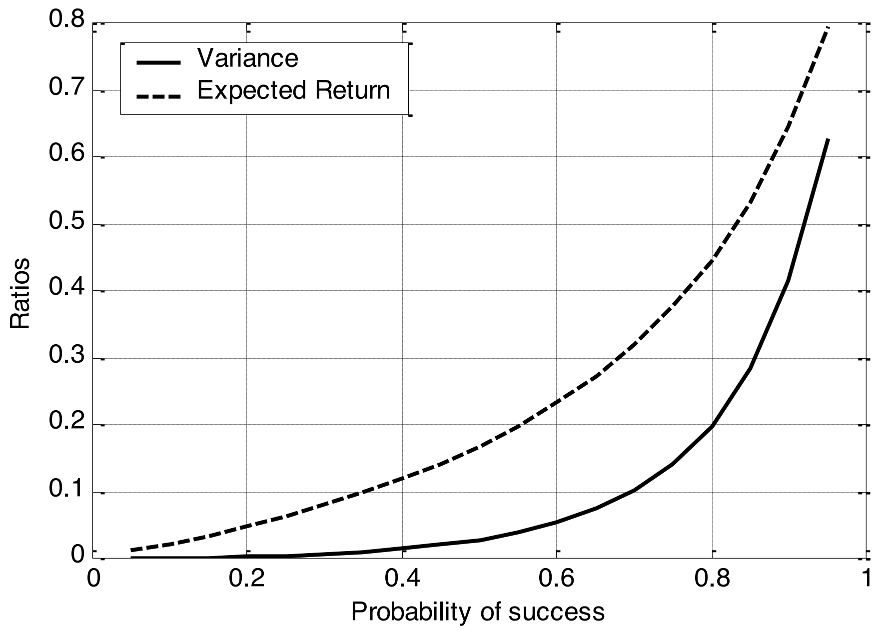


Figure 6 shows expected return and volatility ratios in the model with an expected return of 25% for the risky asset, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, and housing market size $\bar{h} = 0.5$ (the same parameterization we had when analyzing the benchmark case). From Figure 6 we can see that the ratio of expected returns of housing and risky asset investments is below 1, thus expected return on housing is always less than expected return on the risky investment. The ratio is increasing in the probability of the success of the risky asset (i.e. decreases in the risk of an asset).

Variance ratio shows that volatility in the housing prices is less than volatility of the rate of return on the risky asset. However, compared to the benchmark case, the housing price volatility in the model is greater for all levels of risk. This finding supports our claim that in the presence of the relative wealth concerns, emergence of uncorrelated financial risks results in greater housing price volatility that can lead to the housing price bubbles.

Another important finding of our model is that for all levels of risk, housing price volatility in the model is not monotonically related to the volatility of the risky asset. In other words, housing price volatility increases in the probability of the risky asset's success (i.e., decreases in risk).

Thus housing price bubbles are most likely to emerge as a result of house-buyer's risk exposure when this exposure is minimal.

As mentioned above, these finding results from the fact that the gap between amount of capital in a successful market and amount of capital in an unsuccessful market is increasing in the probability of the asset's success. As a result, housing price volatility increases in the probability of the asset's success. This implies that house-buyers face the greatest risk in the housing market (i.e. the greatest housing price volatility), when their financial risk exposure is low.

Chapter 2. Mortgage Market and Housing Price Dynamics

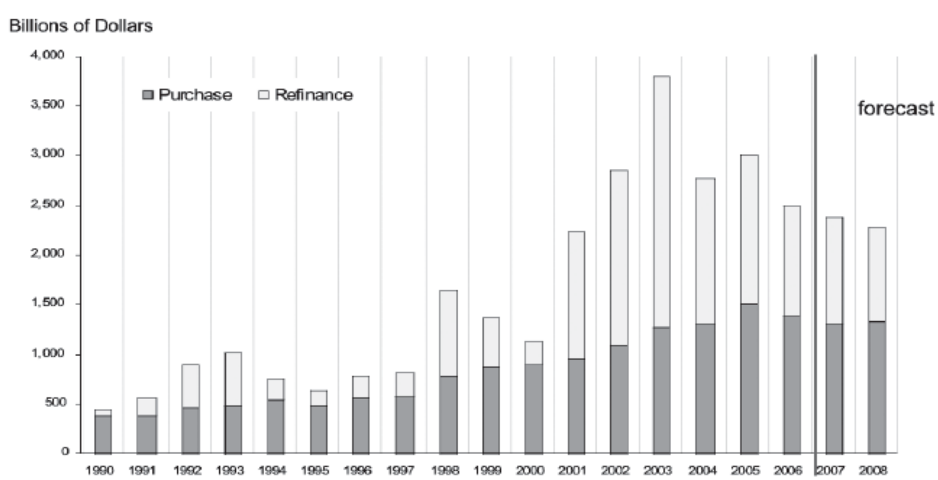
2.1. Introduction

The purchase of a home is a fundamental and life-changing event for most American citizens. However, it is only in the last two decades that many of them, often defined as "subprime" borrowers, gained access to home ownership. The realization of their dream became possible through the expansion of the mortgage market.

Between 1989 and 2006, due to the expansion of mortgage credit, the percentage of American homeowners rose from 64 to 69 percent. This increase corresponds to 12 million additional home owners (Doms and Motika (2006)).

According to 2005 American Housing Survey (AHS) data, 66 percent of American homeowners maintained a mortgage obligation on their houses. The remaining 34 percent owned their residence free of loan obligations.

Figure 7: Annual Mortgage Production



Source: **Mortgage Bankers Association and Department of Housing and Urban Development**

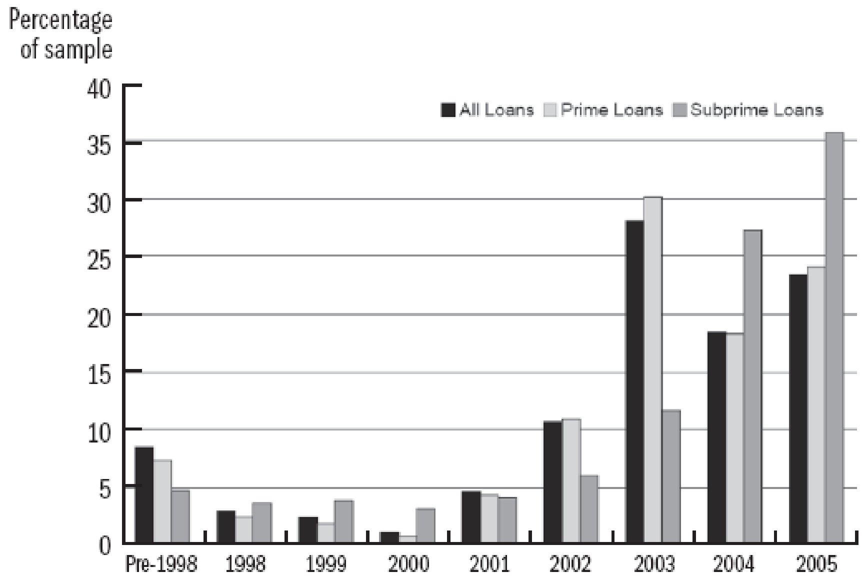
Figure 7 illustrates the rapid growth in the annual mortgage production in the late 1990s and early 2000s. According to the Mortgage Bankers Association and Department of Housing and

Urban Development (2005) the total volume of loan originations has grown from about $400 billion in 1990 to $3.7 trillion in 2003.

When studying the changes in the composition of outstanding mortgages, the trend that comes into view is the growth of the market share for the prime and rapid expansion of the market share for the subprime segment of the market. Only in the late 1990s the fast development of subprime mortgage market attracted national attention. There were many factors that have contributed to the growth of subprime market. First, and obviously, subprime lending became legal. It was only through the Depository Institutions Deregulations and Monetary Control Act (DIDMCA) that charging high rates to borrowers with a limited or poor credit history became possible. DIDMCA was passed in 1980, and following it in 1982 the Alternative Mortgage Transaction Parity Act (AMTPA) allowed lenders to use the adjusted interest rates. The last reform that indeed opened the door for the growth of the subprime market was the Tax Reform Act (TRA) of 1986. It raised the demand for mortgage loans because it disallowed the deduction of interest on consumer loans, yet permitted interest deductions on mortgages for primary and secondary residences. Besides favorable law changes, changes in the market also added a stimulus for the growth in subprime lending. An increase in the interest rates in 1994 reduced the volume of prime loan originations. It made mortgage lending companies to turn to the subprime borrowers in order to maintain the volume of mortgage production. Between 1994 and 2006, the total value of subprime mortgage loans rose from $35 billion to $640 billion (Inside B&C Lending (2007)).

Lenders financed their originations through the securitization of the loans. The securitization rate of subprime loans grew from 28.4 percent in 1995 to 58.7 percent in 2003. At this time subprime mortgages were fairly new and apparently profitable, but the performance of the loans in the long run was not known yet. Now it is a well-known fact that subprime loans are likely to hit their peak delinquency rates three to five years after origination. Analysis of the data shows that more than 86 percent of all outstanding loans have been originated since 2000, and 80 percent - since 2002. This is illustrated in Figure 8, where outstanding loans, outstanding prime loans and outstanding subprime loans originated in a given year are computed as a percentage of total number of loans in each of these categories. That together with a fall in housing prices, that made it impossible for subprime borrowers to refinance, explains the dramatic rise in levels of borrower default and foreclosure in recent years.

Figure 8: Outstanding Loans by Origination Year



Source: **National Delinquency Survey. Mortgage Bankers Association**

The recent expansion of the mortgage and housing markets, accompanied by significant up and downs in housing prices and levels of borrower default, stimulated much scientific research (see Calhoun (2004), Capozza, and Thomson (2005), Chinloy, Macdonald (2005), Cutts, Order, 2005), Staten, Yezer (2005)). Most of the literature examines housing price volatility as a result of structural changes in the mortgage market. In chapter 1 we discussed how exposure to financial risk together with concerns about relative wealth translates into housing price volatility. We now examine a relation between extension of the mortgage credit to the low-income borrowers and high volatility in the real estate prices in the presence of relative wealth concerns.

Chapter 1 is built on DeMarzo, Kaniel, and Kremer (2007). In that chapter we presented a stylized finite-horizon stochastic model that demonstrated that increase in demand for housing may result from financial risk exposure, when house-buyers are subject to relative wealth concerns and competition over future consumption. In this chapter we modify our original model by introducing a mortgage mechanism and investigate how decline in borrower's income level affect the housing price dynamics and borrower's default risk.

The housing price movements generated by relative wealth concerns show that lender's exposure to uncorrelated financial risk together with availability of mortgage credit to the low-income borrowers lead to increase in housing prices and high borrower debt-to-income ratio. To understand this increase, we need to consider the impact of agent's portfolio choice on the future wealth. When lenders allocate a significant part of their financial portfolio to the risky assets, uncertainty in realization of these investments generates volatility in lender's future income. Competition over future consumption leads to relative wealth concerns that arise endogenously in equilibrium and drives housing demand and housing prices up. Additionally, increased demand for borrowing raises mortgage interest rates resulting in high borrower's debt-to-income ratio.

There have been several authors who examined the role of relative wealth concerns on consumption and investment decisions. These models, differently from ours, introduce relative wealth concerns exogenously (see Abel (1990), Gali (1994), Dupor and Liu (2003), Cole, Mailath, and Postelwaite (2001), Nichols, Pennington-Cross, and Yezer (2005), Yeung and Kogan (2002)).

Recent papers that look at general equilibrium models with mortgages include Chambers, Garriga, and Schlagenhauf (2008), who investigate the determinants and implications of mortgage choice, Piskorski and Tchistyi (2008), who look at the optimal mortgage contract in a continuous time setting. However, these papers do not consider financial assets and portfolio choice. Flavin and Yamashita (2002), Cocco (2005), and Flavin and Nakagawa (2005) focus on household's optimal holdings of financial assets in the presence of housing. However, these models do not feature a general equilibrium setup.

The rest of the chapter is organized as follows. Section 2.2 presents a two-period stochastic general equilibrium model with three types of agents, three goods, three investments opportunities via real estate and financial markets, and borrowing/lending opportunity. Section 2.3 further analyzes the model and computes the equilibrium for the no-bankruptcy and contingent bankruptcy cases. Section 2.4 compares the two cases and shows some properties of equilibrium returns obtained from MATLAB simulations. Section 2.5 concludes.

2.2. The Model

2.2.1. Setup

We consider a two-period stochastic production economy. There are three types of rational, risk-averse agents in the market: borrowers, lenders, and service providers. There is a continuum with mass one of each type of agent and three types of goods available in the market: capital goods, services, and housing. Borrowers and lenders consume all three types of goods, while service providers only consume capital goods and services. In period 1, borrowers and lenders consume only housing, while in period 2 they consume housing, the capital good and services. The utility function is CRRA, so the utility function of a borrower is

$$U_{1b}(h_{1b}) = \frac{1}{1-\gamma} w h_{1b}^{1-\gamma}$$

and

$$U_{2b}(c_{cb}, c_{sb}, h_{2b}) = \frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma} + c_{sb}^{1-\gamma} + w h_{2b}^{1-\gamma}\right),$$

where c_{cb} and c_{sb} denote borrower's consumption of capital and consumption of services respectively, while h_{1b} and h_{2b} denote borrower's consumption of housing in the first and second periods respectively. Parameter γ indicates the level of agents' risk-aversion.

The utility function of a lender is

$$U_{1l}(h_{1l}) = \frac{1}{1-\gamma} w h_{1l}^{1-\gamma}$$

and

$$U_{2l}(c_{cl}, c_{sl}, h_{2l}) = \frac{1}{1-\gamma}\left(c_{cl}^{1-\gamma} + c_{sl}^{1-\gamma} + w h_{2l}^{1-\gamma}\right),$$

where c_{cl} and c_{sl} denote lender's consumption of capital and consumption of services respectively, while h_{1l} and h_{2l} denote lender's consumption of housing in the first and second periods respectively.

Service providers only appear in period 2, and their utility function is

$$U(c_{cp}, c_{sp}) = \frac{1}{1-\gamma}\left(c_{cp}^{1-\gamma} + c_{sp}^{1-\gamma}\right),$$

where c_{cp} and c_{sp} denote service provider's consumption of capital and consumption of services respectively.

Lenders are endowed with one unit of capital and $\bar{h}$ units of housing at date 1. Thus, the first period endowment of an individual lender is given by $(1 + p_1\bar{h})$, where p_1 is the housing price at date 1. Borrowers have zero endowment in period 1 and endowment of I units of capital in period 2. Service providers are endowed with one unit of services at date 2. The aggregate endowment is therefore also equal to 1.

In the first period, borrowers take out a mortgage b_b from lenders. With this money, borrowers buy h_{1b} units of housing at market price p_1. At date 1, lenders underwrite a mortgage b_l to borrowers. Lenders also can buy housing at market price p_1 and invest their capital in two types of assets – a risky asset and a non-risky asset. The risky asset returns $\bar{r}$ units of capital for each unit of capital invested, where $\bar{r}$ is a random variable. We assume a simple binary distribution: with probability σ the investment in risky asset is successful and gives return $r > 1$, and with probability $(1-\sigma)$ the risky asset returns zero. The non-risky asset returns one unit of capital for each unit of capital invested. If lenders, after lending b_l to borrowers, invest some of their endowment p_1h_{1l} in housing then the remaining part $(1 + p_1\bar{h} - b_l - p_1h_{1l})$ is invested in risky and non-risky assets. In equilibrium, the housing market clears so $\bar{h} = h_{1b} + h_{1l}$ and the mortgage market clears so $b_b = b_l$.

In the second period, borrowers have endowment of I units of capital, with I being an exogenous parameter of the model. If, in addition, h_{1b} units of housing are bought then the total wealth y_{tb} available to borrowers in the second period is

$$y_{tb}(\bar{r}) = I + p_2(\bar{r})h_{1b},$$

where we have highlighted the fact that the price of housing in period 2 may depend on the realization of $\bar{r}$.

Borrowers have to repay their mortgage at this time, resulting in $\widetilde{b_b}(1 + r_b)$ payment to lenders, where r_b is the interest rate on the mortgage. If the wealth of a borrower is high enough to pay out the mortgage, payment of $b_b(1 + r_b)$ is issued to the lender. If not, the borrower defaults and the lenders is paid the wealth of the borrower. Mortgage payment $\widetilde{b_b}(1+r_b)$ is then given by the following function:

$$\widetilde{b_b}(1+r_b) = \begin{cases} b_b(1+r_b), & if\ b_b(1+r_b) \leq I + p_2(\bar{r})h_{1b}, \\ I + p_2(\bar{r})h_{1b}, & if\ b_b(1+r_b) > I + p_2(\bar{r})h_{1b}. \end{cases}$$

In period 2, borrowers reinvest $p_2 h_{2b}$ in the housing market, by purchasing h_{2b} units of housing at a market price p_2. Then the wealth which house-buyers have available for consumption is

$$y_b(\bar{r}) = y_{tb}(\bar{r}) - \widetilde{b_b}(1+r_b) - p_2(\bar{r})h_{2b} = I + p_2(\bar{r})(h_{1b} - h_{2b}) - \widetilde{b_b}(1+r_b).$$

In the second period, lenders get return from their investments in terms of capital units. Investment in housing results in wealth $p_2 h_{1l}$. Risky asset returns $\bar{r}x_r$ units of capital for an investment of x_r and the x_n units invested in the non-risky asset return x_n with certainty. The total capital return from both assets then is given by $\bar{r}x_r + x_n$, for $x_r + x_n \leq 1 + p_1\bar{h} - b_l - p_1 h_{1l}$. The total return on both investments in assets and housing is then given by $\bar{r}x_r + x_n + p_2(\bar{r})h_{1l}$. The wealth of lenders at date 2 is:

$$y_{tl}(\bar{r}, x_r, x_n) = \bar{r}x_r + x_n + p_2(\bar{r})h_{1l} + \widetilde{b_l}(1+r_b).$$

In period 2, lenders reinvest $p_2(\bar{r})h_{2l}$ in the housing market, by purchasing h_{2l} units of housing at a market price $p_2(\bar{r})$. Then the total income, which lenders have available for a trade with service providers, is

$$y_l(\bar{r}, x_r, x_n) = y_{tl}(\bar{r}, x_r, x_n) - p_2(\bar{r})h_{2l} = \bar{r}x_r + x_n + p_2(\bar{r})(h_{1l} - h_{2l}) + \widetilde{b_l}(1+r_b).$$

There is no housing production in the model, so that $h_{1b} + h_{1l} = h_{2b} + h_{2l} = \bar{h}$ in equilibrium.

Borrowers, lenders, and service providers then trade capital and services competitively in the market and consume. All three types of agents make their investment, consumption, and trading decisions in order to maximize their expected utility, and markets clear.

2.2.2. Equilibrium

Equilibrium of the model is a set of allocations $\{c_{cb}(\bar{r}), c_{sb}(\bar{r}), c_{cl}(\bar{r}), c_{sl}(\bar{r}), c_{cp}(\bar{r}), c_{sp}(\bar{r}), h_{1b}, h_{1l}, h_{2b}(\bar{r}), h_{2l}(\bar{r}), x_r, x_n, b_b, b_l\}$ and a set of prices $\{p_s(\bar{r}), p_1, p_2(\bar{r}), r_b\}$, for $\bar{r} \in \{0, r\}$, such that

- Borrowers choose $c_{cb}(\bar{r}),\ c_{sb}(\bar{r}),\ h_{2b}(\bar{r}), h_{1b}$ and b_b to maximize their life-time utility

$$\frac{1}{1-\gamma} w h_{1b}^{1-\gamma} + E\left\{\frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma}(\bar{r}) + c_{sb}^{1-\gamma}(\bar{r}) + w h_{2b}^{1-\gamma}(\bar{r})\right)\right\}$$

$$\text{s. t. } p_1 h_{1b} = b_b,$$

$$c_{cb}(\bar{r}) + p_s(\bar{r}) c_{sb}(\bar{r}) + \widetilde{b_b}(1 + r_b) + p_2(\bar{r}) h_{2b}(\bar{r}) = I + p_2(\bar{r}) h_{1b},$$

for each realization of $\bar{r} \in \{0, r\}$.

- Lenders choose $c_{cl}(\bar{r}),\ c_{sl}(\bar{r}),\ h_{2l}(\bar{r}), h_{1l}, b_l, x_r$ and x_n to maximize their life-time utility

$$\frac{1}{1-\gamma} w h_{1l}^{1-\gamma} + E\left\{\frac{1}{1-\gamma}\left(c_{cl}^{1-\gamma}(\bar{r}) + c_{sl}^{1-\gamma}(\bar{r}) + w h_{2l}^{1-\gamma}(\bar{r})\right)\right\}$$

$$\text{s. t. } x_r \geq 0,\ x_n \geq 0,$$

$$x_r + x_n + b_l + p_1 h_{1l} = 1 + p_1 \bar{h},$$

$$c_{cl}(\bar{r}) + p_s(\bar{r}) c_{sl}(\bar{r}) + p_2(\bar{r}) h_{2l}(\bar{r}) = \bar{r} x_r + x_n + \widetilde{b_l}(1 + r_b) + p_2(\bar{r}) h_{1l}$$

for each realization of $\bar{r} \in \{0, r\}$.

- Service providers choose $c_{cp}(\bar{r}),\ c_{sp}(\bar{r})$ to maximize

$$\frac{1}{1-\gamma}\left(c_{cp}^{1-\gamma}(\bar{r}) + c_{sp}^{1-\gamma}(\bar{r})\right)$$

$$\text{s. t. } c_{cp}(\bar{r}) + p_s(\bar{r}) c_{sp}(\bar{r}) = p_s(\bar{r}),$$

for each realization of $\bar{r} \in \{0, r\}$.

- At period 1 and for each realization of $\bar{r} \in \{0, r\}$ at period 2, markets clear:

$$H_{1b} + H_{1l} = H_{2b}(\bar{r}) + H_{2l}(\bar{r}) = \bar{H}$$

$$X_r + X_n = 1,$$

$$C_{cb}(\bar{r}) + \; C_{cl}(\bar{r}) + C_{cp}(\bar{r}) = Y,$$

$$C_{sb}(\bar{r}) + \; C_{sl}(\bar{r}) + C_{sp}(\bar{r}) = 1,$$

$$B_b = B_l.$$

where $Y(\bar{r}) = I + \bar{r}X_r + X_n$ is equilibrium aggregate capital income of both borrower and lender population. Capital letters denote aggregate demand and supply (e.g. $X_r = \int_0^1 x_r(i)di$ and $H_{2l}(\bar{r}) = \int_0^1 \; h_{2l}(\bar{r})(i)di$). We consider a symmetric equilibrium where individual decisions are equal to aggregate decisions of agents' population.

2.3. Analysis

2.3.1. Equilibrium with no bankruptcy

2.3.1.1. No bankruptcy occurs in equilibrium. Period 2

We begin our analysis by assuming that in period one there are no markets for goods and services at time 2. In the second period there are three types of agents: borrowers, lenders, and service providers. When bankruptcy does not occur, borrower's maximization problem is given by

$$\max \frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma} + c_{sb}^{1-\gamma} + wh_{2b}^{1-\gamma}\right)$$

$$\text{s. t. } c_{cb} + p_s c_{sb} + p_2 h_{2b} = I + p_2 h_{1b} - b_b(1 + r_b).$$

Lagrangian

$$\frac{c_{cb}^{1-\gamma} + c_{sb}^{1-\gamma} + wh_{2b}^{1-\gamma}}{1-\gamma} - \lambda_{2b}(c_{cb} + p_s c_{sb} + b_b(1 + r_b) + p_2 h_{2b} - I - p_2 h_{1b})$$

FOCs

$$c_{cb} = \lambda_{2b}^{\left(-\frac{1}{\gamma}\right)},$$

$$c_{sb} = \lambda_{2b}^{\left(-\frac{1}{\gamma}\right)} p_s^{\left(-\frac{1}{\gamma}\right)},$$

$$h_{2b} = \lambda_{2b}^{\left(-\frac{1}{\gamma}\right)} p_2^{\left(-\frac{1}{\gamma}\right)} w^{\frac{1}{\gamma}}.$$

From the budget constraint

$$\lambda_{2b}^{\left(-\frac{1}{\gamma}\right)} = \frac{I + p_2 h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}}.$$

If the wealth of a borrower is high enough to pay out the mortgage, lender's maximization problem is given by

$$\max \frac{1}{1-\gamma}\left(c_{cl}^{1-\gamma} + c_{sl}^{1-\gamma} + w h_{2l}^{1-\gamma}\right)$$

$$\text{s. t. } c_{cl} + p_s c_{sl} + p_2 h_{2l} = \bar{r} x_r + \ x_n + b_l(1 + r_b) + p_2 h_{1l}.$$

Lagrangian

$$\frac{c_{cl}^{1-\gamma} + c_{sl}^{1-\gamma} + w h_{2l}^{1-\gamma}}{1-\gamma} - \lambda_{2l}(c_{cl} + p_s c_{sl} + p_2 h_{2l} - \bar{r} x_r - \ x_n - b_l(1 + r_b) - p_2 h_{1l})$$

FOCs

$$c_{cl} = \lambda_{2l}^{\left(-\frac{1}{\gamma}\right)},$$

$$c_{sl} = \lambda_{2l}^{\left(-\frac{1}{\gamma}\right)} p_s^{\left(-\frac{1}{\gamma}\right)},$$

$$h_{2l} = \lambda_{2l}^{\left(-\frac{1}{\gamma}\right)} p_2^{\left(-\frac{1}{\gamma}\right)} w^{\frac{1}{\gamma}}.$$

From the budget constraint

$$\lambda_{2l}^{\left(-\frac{1}{\gamma}\right)} = \frac{\bar{r} x_r + \ x_n + b_l(1 + r_b) + p_2 h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}}.$$

Computing the demand of each type in the situation when the wealth of a borrower is high enough to pay out the mortgage we obtain:

$$h_{2b} = \frac{I + p_2 h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} p_2^{\left(-\frac{1}{\gamma}\right)} w^{\frac{1}{\gamma}},$$

$$c_{cb} = \frac{I + p_2 h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}},$$

$$c_{sb} = \frac{I + p_2 h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)},$$

$$h_{2l} = \frac{\bar{r} x_r + x_n + b_l(1 + r_b) + p_2 h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} p_2^{\left(-\frac{1}{\gamma}\right)} w^{\frac{1}{\gamma}},$$

$$c_{cl} = \frac{\bar{r} x_r + x_n + b_l(1 + r_b) + p_2 h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}},$$

$$c_{sl} = \frac{\bar{r} x_r + x_n + b_l(1 + r_b) + p_2 h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)},$$

$$c_{cp} = \frac{p_s}{1 + p_s^{1-\frac{1}{\gamma}}},$$

$$c_{sp} = \frac{p_s}{1 + p_s^{1-\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)}.$$

Let

$$Y = I + \bar{r} X_r + X_n.$$

The equilibrium condition for the capital good market is

$$\frac{I + p_2 h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} + \frac{\bar{r} X_r + X_n + b_l(1 + r_b) + p_2 h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} + \frac{p_s}{1 + p_s^{1-\frac{1}{\gamma}}} = Y,$$

$$\frac{Y + p_2 \bar{h}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} + \frac{p_s}{1 + p_s^{1-\frac{1}{\gamma}}} = Y.$$

For the service market is

$$p_s^{\left(-\frac{1}{\gamma}\right)} \frac{Y + p_2\bar{h}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} + \frac{p_s}{1 + p_s^{1-\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)} = 1.$$

For the housing market is

$$\frac{Y + p_2\bar{h}}{1 + p_s^{1-\frac{1}{\gamma}} + p_2^{1-\frac{1}{\gamma}} w^{\frac{1}{\gamma}}} p_2^{\left(-\frac{1}{\gamma}\right)} w^{\frac{1}{\gamma}} = \bar{h}.$$

The result of the analysis of the spot market competitive equilibrium at date 2 establishes the price for services. If in equilibrium the total aggregate capital income of borrowers and lenders available for a trade with service providers at date 2 is given by Y and the nominal value of one unit of capital is normalized to 1, then the price of services, in terms of units of capital, is given by

$$p_s(\bar{r}) = \left(Y(\bar{r})\right)^{\gamma} = (I + \bar{r}X_r + X_n)^{\gamma}.$$

Therefore, the price of services is positively correlated with the return on the investment in risky asset in period 1.

The market price for housing in period 2 is given by

$$p_2(\bar{r}) = \frac{w}{\bar{h}^{\gamma}} \left[\left(\frac{Y(\bar{r})}{1 + \left(Y(\bar{r})\right)^{\gamma-1}} \right) \right]^{\gamma},$$

where we use the fact that in equilibrium it must be

$$h_{1b} + h_{1l} = h_{2b} + h_{1l} = \bar{h}.$$

In the two-state economy, with probability σ, successful realization of the investment in the risky asset will result in the high-state service prices given by $\left(p_s(r), p_2(r)\right)$, while with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the low-state prices $\left(p_s(0), p_2(0)\right)$.

At the equilibrium prices, the consumption profiles of the lender and the borrower imply that agents' indirect utilities depend on capital income y of both borrower and lender and on equilibrium aggregate capital income Y of both borrower and lender population.

Let (x_r, x_n, h_{1l}, b_l) be the investment strategy chosen by the lender in period 1. When the wealth of a borrower is high enough to pay out the mortgage, for a given investment strategy, with probability σ successful realization of high return on the investment in the risky asset will result in the indirect utility of the lender given by

$$V_l(r; x_r, x_n, h_{1l}, b_l) = \frac{1}{1-\gamma}\left(\lambda(r)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right) =$$

$$= \frac{1}{1-\gamma}(r x_r + \ x_n + b_l(1+r_b) + p_2(r) h_{1l})^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

While with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the indirect utility of the lender given by

$$V_l(0; x_r, x_n, h_{1l}, b_l) = \frac{1}{1-\gamma}\left(\lambda(0)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right) =$$

$$= \frac{1}{1-\gamma}(\ x_n + b_l(1+r_b) + p_2(0) h_{1l})^{1-\gamma}\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

Thus, the expected indirect utility of the lender is given by

$$W_l(x_r, x_n, h_{1l}, b_l) = \sigma V_l(r; x_r, x_n, h_{1l}, b_l) + (1-\sigma) V_l(0; x_r, x_n, h_{1l}, b_l).$$

Let (h_{1l}, b_b) be the investment strategy chosen by the borrower in period 1. When the wealth of a borrower is high enough to pay out the mortgage, for a given investment strategy, with probability σ successful realization of high return on the investment in the risky asset will result in the indirect utility of the borrower given by

$$V_b(r; h_{1b}, b_b) = \frac{1}{1-\gamma}\left(\lambda(r)_{2b}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right) =$$

$$= \frac{1}{1-\gamma}\left(I + p_2(r) h_{1b} - b_b(1+r_b)\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

While with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the indirect utility of the borrower given by

$$V_b(0;h_{1b},b_b)=\frac{1}{1-\gamma}\left(\lambda(0)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}\right)=$$

$$=\frac{1}{1-\gamma}\left(I+p_2(0)h_{1b}-b_b(1+r_b)\right)^{1-\gamma}\left(1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

Thus, the expected indirect utility of the borrower is given by

$$W_b(h_{1b},b_b)=\sigma V_b(r;h_{1b},b_b)+(1-\sigma)V_b(0;h_{1b},b_b).$$

2.3.1.2. No bankruptcy occurs in equilibrium. Period 1

We now solve the lender's and borrower's maximization problems at period 1 in the case when no bankruptcy occurs in equilibrium. At this time, the agents choose their investment strategies to maximize their expected life-time utilities subject to their first period budget constraints respectively. In this case, the borrower's maximization problem at date 1 is

$$\max\frac{1}{1-\gamma}wh_{1b}^{1-\gamma}+W_b(h_{1b},b_b)$$

$$\text{s. t. }\ p_1h_{1b}=b_b.$$

Lender's maximization problem at date 1 is

$$\max\frac{1}{1-\gamma}wh_{1l}^{1-\gamma}+W_l(x_r,x_n,h_{1l},b_l)$$

$$\text{s. t. }\ x_r+\ x_n+b_l+p_1h_{1l}=1+p_1\bar{h},$$

$$x_r\geq 0,$$

$$x_n\geq 0.$$

Lagrangian for the lender

$$\frac{1}{1-\gamma}wh_{1l}^{1-\gamma}+W_l(x_r,x_n,h_{1l},b_l)-\lambda_{1l}\left(x_r+\ x_n+b_l+p_1h_{1l}-1-p_1\bar{h}\right)$$

Lagrangian for the borrower

$$\frac{1}{1-\gamma}wh_{1b}^{1-\gamma}+W_b(h_{1b},b_b)-\lambda_{1b}(p_1h_{1b}-b_b)$$

Because no bankruptcy occurs in equilibrium, mortgage lending is a risk-free activity. This implies that if there is a strictly positive interest rate on mortgages then the investment in the risk-free asset has to be zero. In equilibrium lenders will put money either in mortgages or the risky asset. Alternatively, we may have $r_b = 0$. In that case, investment in the non-risky asset and investment in mortgages are perfect substitutes and investment in the non-risky technology may be strictly positive.

Let now us examine the case with a strictly positive interest rate on mortgages. In this case the investment in the risk-free asset has to be zero and in equilibrium lenders will put money either in mortgages or the risky asset.

FOCs for the risky investment decision of the lender is

$$\sigma r \left(\frac{1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}}{r x_r + x_n + b_l(1 + r_b) + p_2(r) h_{1l}} \right)^{\gamma} = \lambda_{1l},$$

In equilibrium the decision of an individual lender and borrower match the aggregate decision of the lender population and borrower population respectively, that is $h_{1l} = H_{1l}$, $h_{1b} = H_{1b}$, $b_b = B_b$, $b_l = B_l$, $x_r = X_r$, and $x_n = X_n$. And markets clear:

$$h_{1b} + h_{1l} = \bar{h}$$

$$x_r + x_n = 1,$$

$$b_b = b_l.$$

From FOC for housing in period 1 we obtain two expressions for market price for housing at period 1:

$$p_1 = \left[\frac{w}{{H_{1l}}^{\gamma}} + \sigma \left(\frac{1+r}{1 + p_s(r)^{1-\frac{1}{\gamma}}} \right)^{\gamma} \left(\frac{1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}}{r + B_l(1 + r_b) + p_2(r) H_{1l}} \right)^{\gamma} + \right.$$

$$\left. +(1-\sigma) \left(\frac{1}{1 + p_s(0)^{1-\frac{1}{\gamma}}} \right)^{\gamma} \left(\frac{1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}}{B_l(1 + r_b) + p_2(0) H_{1l}} \right)^{\gamma} \right] x$$

$$x\left[\sigma r\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{r+B_l(1+r_b)+p_2(r)H_{1l}}\right)^{\gamma}\right]^{-1}$$

and

$$p_1=\left[\frac{w}{H_{1b}{}^{\gamma}}+\sigma\left(\frac{1+r}{1+p_s(r)^{1-\frac{1}{\gamma}}}\right)^{\gamma}\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{I-B_b(1+r_b)+p_2(r)H_{1b}}\right)^{\gamma}+\right.$$

$$\left.+(1-\sigma)\left(\frac{1}{1+p_s(0)^{1-\frac{1}{\gamma}}}\right)^{\gamma}\left(\frac{1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{I-B_b(1+r_b)+p_2(0)H_{1b}}\right)^{\gamma}\right]x$$

$$x\left[(1+r_b)\left[\sigma\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{I-B_b(1+r_b)+p_2(r)H_{1b}}\right)^{\gamma}+\right.\right.$$

$$\left.\left.+(1-\sigma)\left(\frac{1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{I-B_b(1+r_b)+p_2(0)H_{1b}}\right)^{\gamma}\right]\right]^{-1}$$

Together with the clearing condition for the housing market these expressions define the housing price at date 1 and the demand functions for the housing investment decisions of the lender and borrower in implicit form.

From the FOCs for the lending and investments decisions of the lender we obtain the expression for lending demand function in implicit form:

$$\sigma(r-1-r_b)\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}}{r+b_l(1+r_b)+p_2(r)h_{1l}}\right)^{\gamma}-(1-\sigma)(1+r_b)\left(\frac{1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}}{b_l(1+r_b)+p_2(0)h_{1l}}\right)^{\gamma}$$

$$=0.$$

Together with the clearing condition for the mortgage market it determines the mortgage rate r_b. There is no closed-form solution for this case. It is solved with MATLAB simulations.

2.3.2. Equilibrium with bankruptcy: contingent bankruptcy

To analyze the case in which the equilibrium involves bankruptcy, first let us prove that there is no equilibrium in which bankruptcy occurs in both states. Let's say that $p_2(0) < p_2(r)$. Without loss of generality, we assume that r_b is such that in equilibrium $b_b(1+r_b) = I + p_2(r)h_{1b}$. If that is the case, the borrower consumes zero at period 2. But then the marginal utility of increasing consumption at period two is infinity, so the borrower would optimally reduce the amount borrowed b_b, reducing housing consumption in period 1 and increasing consumption in period 2. Thus, equilibria with bankruptcy must be such that bankruptcy only occurs when the price of housing is low in period 2. Therefore, to analyze the case in which the wealth of a borrower is not high enough to pay out the mortgage, we will consider the case of contingent bankruptcy.

2.3.2.1. Economy with successful risky asset. No bankruptcy. Period 2

Consider $p_2(0) < p_2(r)$. Then in the economy with successful risky asset there is no bankruptcy and mortgages are paid in full. However, in the state of economy in which risky asset is unsuccessful bankruptcy occurs: borrower defaults and the lender is paid the wealth of the borrower. In this case, the consumption of everything for the borrower must be zero. Let us analyze the model in this case.

When bankruptcy does not occur in the economy with successful risky asset, borrower's maximization problem is given by

$$\max \frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma}(r) + c_{sb}^{1-\gamma}(r) + wh_{2b}^{1-\gamma}(r)\right)$$

$$\text{s. t. } c_{cb}(r) + p_s(r)c_{sb}(r) + p_2(r)h_{2b}(r) = I + p_2(r)h_{1b} - b_b(1+r_b).$$

Lender's maximization problem is given by

$$\max \frac{1}{1-\gamma}\left(c_{cl}^{1-\gamma}(r) + c_{sl}^{1-\gamma}(r) + wh_{2l}^{1-\gamma}(r)\right)$$

$$\text{s. t. } c_{cl}(r) + p_s(r)c_{sl}(r) + p_2(r)h_{2l}(r) = rx_r + x_n + b_l(1+r_b) + p_2(r)h_{1l}.$$

Computing the demand of each type we obtain:

$$h_{2b}(r) = \frac{I + p_2(r)h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}}(r) + p_2^{1-\frac{1}{\gamma}}(r)w^{\frac{1}{\gamma}}} p_2^{\left(-\frac{1}{\gamma}\right)}(r)w^{\frac{1}{\gamma}},$$

$$c_{cb}(r) = \frac{I + p_2(r)h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}}(r) + p_2^{1-\frac{1}{\gamma}}(r)w^{\frac{1}{\gamma}}},$$

$$c_{sb}(r) = \frac{I + p_2(r)h_{1b} - b_b(1 + r_b)}{1 + p_s^{1-\frac{1}{\gamma}}(r) + p_2^{1-\frac{1}{\gamma}}(r)w^{\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)}(r),$$

$$h_{2l}(r) = \frac{rx_r + x_n + b_l(1 + r_b) + p_2(r)h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}}(r) + p_2^{1-\frac{1}{\gamma}}(r)w^{\frac{1}{\gamma}}} p_2^{\left(-\frac{1}{\gamma}\right)}(r)w^{\frac{1}{\gamma}},$$

$$c_{cl}(r) = \frac{rx_r + x_n + b_l(1 + r_b) + p_2(r)h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}}(r) + p_2^{1-\frac{1}{\gamma}}(r)w^{\frac{1}{\gamma}}},$$

$$c_{sl}(r) = \frac{rx_r + x_n + b_l(1 + r_b) + p_2(r)h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}}(r) + p_2^{1-\frac{1}{\gamma}}(r)w^{\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)}(r),$$

$$c_{cp}(r) = \frac{p_s(r)}{1 + p_s^{1-\frac{1}{\gamma}}(r)},$$

$$c_{sp}(r) = \frac{p_s(r)}{1 + p_s^{1-\frac{1}{\gamma}}(r)} p_s^{\left(-\frac{1}{\gamma}\right)}(r).$$

Imposing equilibrium conditions, the analysis of the spot market competitive equilibrium at date 2 establishes the price for services

$$p_s(r) = \left(Y(r)\right)^{\gamma} = (I + rX_r + X_n)^{\gamma}$$

and the market price for housing in period 2

$$p_2(r) = \frac{w}{\bar{h}^{\gamma}} \left[\left(\frac{Y(r)}{1 + \left(Y(r)\right)^{\gamma-1}}\right)\right]^{\gamma}.$$

At the equilibrium prices, a consumption profile of the lender and the borrower imply that agents' indirect utilities depend on capital income y of both borrower and lender and on equilibrium aggregate capital income Y of both borrower and lender population.

Let (x_r, x_n, h_{1l}, b_l) be the investment strategy chosen by the lender in period 1. Then for a given investment strategy, with probability σ successful realization of high return on the investment in the risky asset will result in the indirect utility of the lender given by

$$V_l(r; x_r, x_n, h_{1l}, b_l) = \frac{1}{1-\gamma}\left(\lambda(r)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right) =$$

$$= \frac{1}{1-\gamma}(r x_r + \ x_n + b_l(1+r_b) + p_2(r)h_{1l})^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

Let (h_{1l}, b_b) be the investment strategy chosen by the borrower in period 1. Then the for a given investment strategy, with probability σ successful realization of high return on the investment in the risky asset will result in the indirect utility of the borrower given by

$$V_b(r; h_{1b}, b_b) = \frac{1}{1-\gamma}\left(\lambda(r)_{2b}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right) =$$

$$= \frac{1}{1-\gamma}\left(I + p_2(r)h_{1b} - b_b(1+r_b)\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

2.3.2.2. Economy with unsuccessful risky asset. Bankruptcy occurs. Period 2

Lender's maximization problem in this case becomes

$$\max \frac{1}{1-\gamma}\left(c_{cl}^{1-\gamma}(0) + c_{sl}^{1-\gamma}(0) + w h_{2l}^{1-\gamma}(0)\right)$$

s.t.

$$c_{cl}(0) + p_s(0)c_{sl}(0) + p_2(0)h_{2l}(0) = \ x_n + \frac{I + p_2(0)H_{1b}}{B_l} b_l + p_2(0)h_{1l}.$$

Lagrangian

$$\frac{c_{cl}^{1-\gamma}(0) + c_{sl}^{1-\gamma}(0) + w h_{2l}^{1-\gamma}(0)}{1-\gamma} -$$

$$-\lambda_{2l}\left(c_{cl}(0)+p_s(0)c_{sl}(0)+p_2(0)h_{2l}(0)-\ x_n-\frac{I+p_2(0)H_{1b}}{B_l}b_l-p_2(0)h_{1l}\right)$$

FOCs

$$c_{cl}(0)=\lambda_{2l}^{\left(-\frac{1}{\gamma}\right)},$$

$$c_{sl}(0)=\lambda_{2l}^{\left(-\frac{1}{\gamma}\right)}p_s^{\left(-\frac{1}{\gamma}\right)}(0),$$

$$h_{2l}(0)=\lambda_{2l}^{\left(-\frac{1}{\gamma}\right)}p_2^{\left(-\frac{1}{\gamma}\right)}(0)w^{\frac{1}{\gamma}}.$$

From the budget constraint

$$\lambda_{2l}^{\left(-\frac{1}{\gamma}\right)}=\frac{x_n+\frac{I+p_2(0)H_{1b}}{B_l}b_l+p_2(0)h_{1l}}{1+p_s^{1-\frac{1}{\gamma}}(0)+p_2^{1-\frac{1}{\gamma}}(0)w^{\frac{1}{\gamma}}}.$$

Service providers solve

$$\max\frac{1}{1-\gamma}\left(c_{cp}^{1-\gamma}(0)+c_{sp}^{1-\gamma}(0)\right)$$

$$\text{s. t.}\quad c_{cp}(0)+p_s(0)c_{sp}(0)=p_s(0).$$

Lagrangian

$$\frac{c_{cp}^{1-\gamma}(0)+c_{sp}^{1-\gamma}(0)}{1-\gamma}-\lambda_{2p}\left(c_{cp}(0)+p_s(0)c_{sp}(0)-p_s(0)\right)$$

FOCs

$$c_{cp}(0)=\lambda_{2p}^{\left(-\frac{1}{\gamma}\right)},$$

$$c_{sp}(0)=\lambda_{2p}^{\left(-\frac{1}{\gamma}\right)}p_s^{\left(-\frac{1}{\gamma}\right)}(0).$$

From the budget constraint

$$\lambda_{2p}^{\left(-\frac{1}{\gamma}\right)}=\frac{p_s(0)}{1+p_s^{1-\frac{1}{\gamma}}(0)}.$$

When the wealth of a borrower is not high enough to pay out the mortgage, the borrower consumes zero in period 2:

$$h_{2b}(0) = 0,$$

$$c_{cb}(0) = 0,$$

$$c_{sb}(0) = 0,$$

$$h_{2l}(0) = \frac{x_n + \frac{I + p_2(0)H_{1b}}{B_l} b_l + p_2(0)h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}}(0) + p_2^{1-\frac{1}{\gamma}}(0)w^{\frac{1}{\gamma}}} p_2^{\left(-\frac{1}{\gamma}\right)}(0)w^{\frac{1}{\gamma}},$$

$$c_{cl}(0) = \frac{x_n + \frac{I + p_2(0)H_{1b}}{B_l} b_l + p_2(0)h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}}(0) + p_2^{1-\frac{1}{\gamma}}(0)w^{\frac{1}{\gamma}}},$$

$$c_{sl}(0) = \frac{x_n + \frac{I + p_2(0)H_{1b}}{B_l} b_l + p_2(0)h_{1l}}{1 + p_s^{1-\frac{1}{\gamma}}(0) + p_2^{1-\frac{1}{\gamma}}(0)w^{\frac{1}{\gamma}}} p_s^{\left(-\frac{1}{\gamma}\right)}(0),$$

$$c_{cp}(0) = \frac{p_s(0)}{1 + p_s^{1-\frac{1}{\gamma}}(0)},$$

$$c_{sp}(0) = \frac{p_s(0)}{1 + p_s^{1-\frac{1}{\gamma}}(0)} p_s^{\left(-\frac{1}{\gamma}\right)}(0).$$

Let

$$Y = I + X_n.$$

The equilibrium condition for the capital good market is

$$\frac{Y + p_2\bar{h}}{1 + p_s^{1-\frac{1}{\gamma}}(0) + p_2^{1-\frac{1}{\gamma}}(0)w^{\frac{1}{\gamma}}} + \frac{p_s(0)}{1 + p_s^{1-\frac{1}{\gamma}}(0)} = Y.$$

For the service market is

$$p_s^{\left(-\frac{1}{\gamma}\right)}(0)\frac{Y+p_2(0)\bar{h}}{1+p_s^{1-\frac{1}{\gamma}}(0)+p_2^{1-\frac{1}{\gamma}}(0)w^{\frac{1}{\gamma}}}+\frac{p_s(0)}{1+p_s^{1-\frac{1}{\gamma}}(0)}p_s^{\left(-\frac{1}{\gamma}\right)}(0)=1.$$

For the housing market is

$$\frac{Y+p_2(0)\bar{h}}{1+p_s^{1-\frac{1}{\gamma}}(0)+p_2^{1-\frac{1}{\gamma}}(0)w^{\frac{1}{\gamma}}}p_2^{\left(-\frac{1}{\gamma}\right)}(0)w^{\frac{1}{\gamma}}=\bar{h}.$$

The result of the analysis of the spot market competitive equilibrium at date 2 establishes the price for services. If in equilibrium the total aggregate capital income of borrowers and lenders available for a trade with service providers at date 2 is given by Y and the nominal value of one unit of capital is normalized to 1, then the price of services, in terms of units of capital, is given by

$$p_s(0)=\left(Y(0)\right)^{\gamma}=(I+X_n)^{\gamma}.$$

Therefore, the price of services is positively correlated with the return on the investment in risky asset in period 1.

The market price for housing in period 2 is given by

$$p_2(0)=\frac{w}{\bar{h}^{\gamma}}\left[\left(\frac{Y(0)}{1+\left(Y(0)\right)^{\gamma-1}}\right)\right]^{\gamma},$$

where we use the fact that in equilibrium it must be

$$h_{1b}+h_{1l}=h_{2b}+h_{1l}=\bar{h}.$$

When the wealth of a borrower is not high enough to pay out the mortgage, for a given investment strategy, with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the indirect utility of the lender given by

$$V_l(0;x_r,x_n,h_{1l},b_l,h_{1b})=\frac{1}{1-\gamma}\left(\lambda(0)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}\right)=$$

$$=\frac{1}{1-\gamma}\left(x_n+\frac{I+p_2(0)H_{1b}}{B_l}b_l+p_2(0)h_{1l}\right)^{1-\gamma}\left(1+p_s(0)^{1-\frac{1}{\gamma}}+w^{\frac{1}{\gamma}}p_2(0)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

Thus, the expected indirect utility of the lender in the case with contingent bankruptcy is given by

$$W_l(x_r, x_n, h_{1l}, b_l) = \sigma V_l(r; x_r, x_n, h_{1l}, b_l) + (1-\sigma)V_l(0; x_r, x_n, h_{1l}, b_l, h_{1b}).$$

As was mentioned above, when the wealth of a borrower is not high enough to pay out the mortgage, the consumption of everything for the borrower must be zero. This results in the zero indirect utility function of the borrower in the case of default. Thus, the expected indirect utility of the borrower in the case with contingent bankruptcy is given by

$$W_b(h_{1b}, b_b) = \sigma V_b(r; h_{1b}, b_b).$$

2.3.2.3. Economy with contingent bankruptcy in low state. Period 1

We now solve the lender's and borrower's maximization problems at period 1 in the case with no bankruptcy occurring in equilibrium when the price of housing is high. At this time, the agents choose their investment strategies to maximize their expected life-time utilities subject to their first period budget constraints respectively. In this case, the borrower's maximization problem at date 1 is

$$\max \frac{1}{1-\gamma} w h_{1b}^{1-\gamma} + W_b(h_{1b}, b_b)$$

$$\text{s. t. } p_1 h_{1b} = b_b.$$

Lender's maximization problem at date 1 is

$$\max \frac{1}{1-\gamma} w h_{1l}^{1-\gamma} + W_l(x_r, x_n, h_{1l}, b_l)$$

$$\text{s. t. } x_r + x_n + b_l + p_1 h_{1l} = 1 + p_1 \bar{h},$$

$$x_r \geq 0,$$

$$x_n \geq 0.$$

Lagrangian for the lender

$$\frac{1}{1-\gamma} w h_{1l}^{1-\gamma} + W_l(x_r, x_n, h_{1l}, b_l) - \lambda_{1l}\left(x_r + x_n + b_l + p_1 h_{1l} - 1 - p_1 \bar{h}\right)$$

Lagrangian for the borrower

$$\frac{1}{1-\gamma} w h_{1b}^{1-\gamma} + W_b(h_{1b}, b_b) - \lambda_{1b}(p_1 h_{1b} - b_b)$$

In equilibrium the decision of an individual lender and borrower match the aggregate decision of the lender population and borrower population respectively, that is $h_{1l} = H_{1l}$, $h_{1b} = H_{1b}$, $b_b = B_b$, $b_l = B_l$, $x_r = X_r$, and $x_n = X_n$. And markets clear:

$$h_{1b} + h_{1l} = \bar{h}$$

$$x_r + x_n = 1,$$

$$b_b = b_l.$$

From the FOCs for the lending and risky investment decisions of the lender we obtain the expression for lending demand function in implicit form:

$$\sigma(r-1-r_b)\left(\frac{1+p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}}{r x_r + x_n + b_l(1+r_b) + p_2(r) h_{1l}}\right)^{\gamma}$$

$$-(1-\sigma)\left(\frac{I + p_2(0) H_{1b}}{B_l}\right)\left(\frac{1+p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}}{x_n + \frac{I + p_2(0) H_{1b}}{B_l} b_l + p_2(0) h_{1l}}\right)^{\gamma} = 0.$$

Together with the clearing condition for the mortgage market it determines the mortgage rate r_b.

Computing the demand for housing we obtain

$$h_{1b} = \frac{I}{\left((1+r_b)p_1 - p_2(r) + \left(1+p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}\right)\left[\frac{\sigma\left((1+r_b)p_1 - p_2(r)\right)}{w}\right]^{\frac{1}{\gamma}}\right)},$$

$$h_{1l} = \left[(r-1)\left(I + p_2(0)\bar{h}\right) + r(1+r_b)p_1\bar{h}\right] x$$

$$x\left\{(1+r_b)p_1 - p_2(r) + (\theta + r - 1)\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right) x\right.$$

$$\left. x\left[\frac{(1-\sigma)\left(r p_1 - p_2(r) - (r-1)p_2(0)\right)}{w(r-1)}\right]^{\frac{1}{\gamma}}\right\},$$

where

$$\theta = \left[\frac{\sigma(r-1)}{1-\sigma}\right]^{\frac{1}{\gamma}} \frac{1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}}{1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}}.$$

Together with the clearing condition for the housing market demand functions for housing define the housing price at date 1.

Investment demand is given by

$$x_n = \frac{1}{\theta + r - 1}\left\{r + (1 + r_b)p_1\bar{h} - \left((1 + r_b)p_1 - p_2(r)\right)h_{1l} - \theta\left(I + p_2(0)\bar{h}\right)\right\},$$

$$x_r = \frac{1}{\theta + r - 1}\left\{(\theta - 1) + \theta\left(I + p_2(0)\bar{h}\right) - (1 + r_b)p_1\bar{h} - p_2(r)h_{1l}\right\}.$$

Additionally, the bankruptcy condition in low state

$$b_b(1 + r_b) > I + p_2(0)h_{1b}$$

can be written as

$$p_1 > \frac{I + p_2(0)h_{1b}}{(1 + r_b)h_{1b}}.$$

2.3.2.4. Economy with unsuccessful risky asset. No bankruptcy. Period 2

Consider $p_2(r) < p_2(0)$. Then in the economy with unsuccessful risky asset there is no bankruptcy and mortgages are paid in full. However, in the state of economy in which risky asset is successful bankruptcy occurs: borrower defaults and the lender is paid the wealth of the borrower. In this case, the consumption of everything for the borrower must be zero. Let us analyze the model in this case.

When bankruptcy does not occur in the economy with unsuccessful risky asset, borrower's maximization problem is given by

$$\max \frac{1}{1-\gamma}\left(c_{cb}^{1-\gamma}(0) + c_{sb}^{1-\gamma}(0) + w h_{2b}^{1-\gamma}(0)\right)$$

$$\text{s. t. } c_{cb}(0) + p_s(0)c_{sb}(0) + p_2(0)h_{2b}(0) = I + p_2(0)h_{1b} - b_b(1 + r_b).$$

Lender's maximization problem is given by

$$\max \frac{1}{1-\gamma}\left(c_{cl}^{1-\gamma}(0) + c_{sl}^{1-\gamma}(0) + w h_{2l}^{1-\gamma}(0)\right)$$

$$\text{s. t. } c_{cl}(0) + p_s(0)c_{sl}(0) + p_2(0)h_{2l}(0) = x_n + b_l(1+r_b) + p_2(0)h_{1l}.$$

Computing the demand of each type and imposing equilibrium conditions, the analysis of the spot market competitive equilibrium at date 2 establishes the price for services

$$p_s(0) = \left(Y(0)\right)^{\gamma} = (I + X_n)^{\gamma}$$

and the market price for housing in period 2

$$p_2(0) = \frac{w}{\bar{h}^{\gamma}}\left[\left(\frac{Y(0)}{1+\left(Y(0)\right)^{\gamma-1}}\right)\right]^{\gamma}.$$

At the equilibrium prices, a consumption profiles of the lender and the borrower imply that agents' indirect utilities depend on capital income y of both borrower and lender and on equilibrium aggregate capital income Y of both borrower and lender population.

Let (x_r, x_n, h_{1l}, b_l) be the investment strategy chosen by the lender in period 1. Then for a given investment strategy, with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the indirect utility of the lender given by

$$V_l(0; x_r, x_n, h_{1l}, b_l) = \frac{1}{1-\gamma}\left(\lambda(0)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right) =$$

$$= \frac{1}{1-\gamma}\left(x_n + b_l(1+r_b) + p_2(0)h_{1l}\right)^{1-\gamma}\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

Let (h_{1l}, b_b) be the investment strategy chosen by the borrower in period 1. Then the for a given investment strategy, with probability $(1-\sigma)$, zero return on the investment in the risky asset will result in the indirect utility of the borrower given by

$$V_b(0; h_{1b}, b_b) = \frac{1}{1-\gamma}\left(\lambda(0)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right) =$$

$$= \frac{1}{1-\gamma}\left(I + p_2(0)h_{1b} - b_b(1+r_b)\right)^{1-\gamma}\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

2.3.2.5. Economy with successful risky asset. Bankruptcy occurs. Period 2

Lender's maximization problem in this case becomes

$$\max \frac{1}{1-\gamma}\left(c_{cl}^{1-\gamma}(r) + c_{sl}^{1-\gamma}(r) + wh_{2l}^{1-\gamma}(r)\right)$$

s. t.

$$c_{cl}(r) + p_s(r)c_{sl}(r) + p_2(r)h_{2l}(r) = \bar{r}x_r + \ x_n + \frac{I + p_2(r)H_{1b}}{B_l} b_l + p_2(r)h_{1l}.$$

Service providers solve

$$\max \frac{1}{1-\gamma}\left(c_{cp}^{1-\gamma}(r) + c_{sp}^{1-\gamma}(r)\right)$$

$$\text{s. t.} \quad c_{cp}(r) + p_s(r)c_{sp}(r) = p_s(r).$$

When the wealth of a borrower is not high enough to pay out the mortgage, the borrower consumes zero in period 2. Computing the demand of each type and imposing equilibrium conditions, the analysis of the spot market competitive equilibrium at date 2 establishes the price for services

$$p_s(r) = \left(Y(r)\right)^{\gamma} = (I + \ \bar{r}X_r + X_n)^{\gamma}.$$

The market price for housing in period 2 is given by

$$p_2(r) = \frac{w}{\bar{h}^{\gamma}}\left[\left(\frac{Y(r)}{1 + \left(Y(r)\right)^{\gamma-1}}\right)\right]^{\gamma}.$$

When the wealth of a borrower is not high enough to pay out the mortgage, for a given investment strategy, with probability σ, zero return on the investment in the risky asset will result in the indirect utility of the lender given by

$$V_l(r; x_r, x_n, h_{1l}, b_l, h_{1b}) = \frac{1}{1-\gamma}\left(\lambda(r)_{2l}^{\left(-\frac{1}{\gamma}\right)}\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}\right) =$$

$$\frac{1}{1-\gamma}\left(\bar{r}x_r + \ x_n + \frac{I + p_2(r)H_{1b}}{B_l} b_l + p_2(r)h_{1l}\right)^{1-\gamma}\left(1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}}p_2(r)^{1-\frac{1}{\gamma}}\right)^{\gamma}.$$

Thus, the expected indirect utility of the lender in the case with contingent bankruptcy is given by

$$W_l(x_r, x_n, h_{1l}, b_l) = \sigma V_l(r; x_r, x_n, h_{1l}, b_l) + (1-\sigma)V_l(0; x_r, x_n, h_{1l}, b_l, h_{1b}).$$

As was mentioned above, when the wealth of a borrower is not high enough to pay out the mortgage, the consumption of everything for the borrower must be zero. This results in the zero indirect utility function of the borrower in the case of default. Thus, the expected indirect utility of the borrower in the case with contingent bankruptcy is given by

$$W_b(h_{1b}, b_b) = (1-\sigma)V_b(0; h_{1b}, b_b).$$

2.3.2.6. Economy with contingent bankruptcy in high state. Period 1

We now solve the lender's and borrower's maximization problems at period 1 in the case with no bankruptcy occurring in equilibrium when the price of housing is low. At this time, the agents choose their investment strategies to maximize their expected life-time utilities subject to their first period budget constraints respectively. In this case, the borrower's maximization problem at date 1 is

$$\max \frac{1}{1-\gamma} w h_{1b}^{1-\gamma} + W_b(h_{1b}, b_b)$$

$$\text{s. t. } \; p_1 h_{1b} = b_b.$$

Lender's maximization problem at date 1 is

$$\max \frac{1}{1-\gamma} w h_{1l}^{1-\gamma} + W_l(x_r, x_n, h_{1l}, b_l)$$

$$\text{s. t. } \; x_r + x_n + b_l + p_1 h_{1l} = 1 + p_1 \bar{h},$$

$$x_r \geq 0,$$

$$x_n \geq 0.$$

In equilibrium the decision of an individual lender and borrower match the aggregate decision of the lender population and borrower population respectively, that is $h_{1l} = H_{1l}$, $h_{1b} = H_{1b}$, $b_b = B_b$, $b_l = B_l$, $x_r = X_r$, and $x_n = X_n$. And markets clear:

$$h_{1b} + h_{1l} = \bar{h}$$

$$x_r + x_n = 1,$$

$$b_b = b_l.$$

From the FOCs for the lending and risky investment decisions of the lender we obtain the expression for lending demand function in implicit form:

$$\sigma\left(r - \frac{I + p_2(r)H_{1b}}{B_l}\right)\left(\frac{1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}}{rx_r + x_n + \frac{I + p_2(r)H_{1b}}{B_l} b_l + p_2(r)h_{1l}}\right)^{\gamma}$$

$$- (1-\sigma)(1+r_b)\left(\frac{1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}}{x_n + b_l(1+r_b) + p_2(0)h_{1l}}\right)^{\gamma} = 0.$$

Together with the clearing condition for the mortgage market it determines the mortgage rate r_b.

Computing the demand for housing we obtain

$$h_{1b} = \frac{I}{\left(p_1(1+r_b) - p_2(0) + \left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right)\left[\frac{(1-\sigma)\left(p_1(1+r_b) - p_2(0)\right)}{w}\right]^{\frac{1}{\gamma}}\right)},$$

$$h_{1l} = \left[r + I + \left((r-1)(1+r_b)p_1 + p_2(r)\right)\bar{h}\right]x$$

$$\left\{(r-1)(1+r_b)p_1 - p_2(0) + (\theta + r - 1)\left(1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}\right)x\right.$$

$$\left. x\left[\frac{(1-\sigma)}{w}\left(rp_1 - p_2(0) - \frac{p_2(r)}{(r-1)}\right)\right]^{\frac{1}{\gamma}}\right\}^{-1}$$

where

$$\theta = \left[\frac{\sigma(r-1)}{1-\sigma}\right]^{\frac{1}{\gamma}} \frac{1 + p_s(r)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(r)^{1-\frac{1}{\gamma}}}{1 + p_s(0)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} p_2(0)^{1-\frac{1}{\gamma}}}.$$

Together with the clearing condition for the housing market demand functions for housing define the housing price at date 1.

Investment demand is given by

$$x_n = \frac{1}{\theta + r - 1}\{r + I + p_2(r)\bar{h} - \theta[p_1(1 + r_b)\bar{h} + (p_2(0) - p_1(1 + r_b))h_{1l}]\},$$

$$x_r = \frac{1}{\theta + r - 1}\{\theta[1 + p_1(1 + r_b)\bar{h} + (p_2(0) - p_1(1 + r_b))h_{1l}] - I - p_2(r)\bar{h} - 1\}.$$

Additionally, the bankruptcy condition in high state

$$b_b(1 + r_b) > I + p_2(r)h_{1b}$$

can be written as

$$p_1 > \frac{I + p_2(r)h_{1b}}{(1 + r_b)h_{1b}}.$$

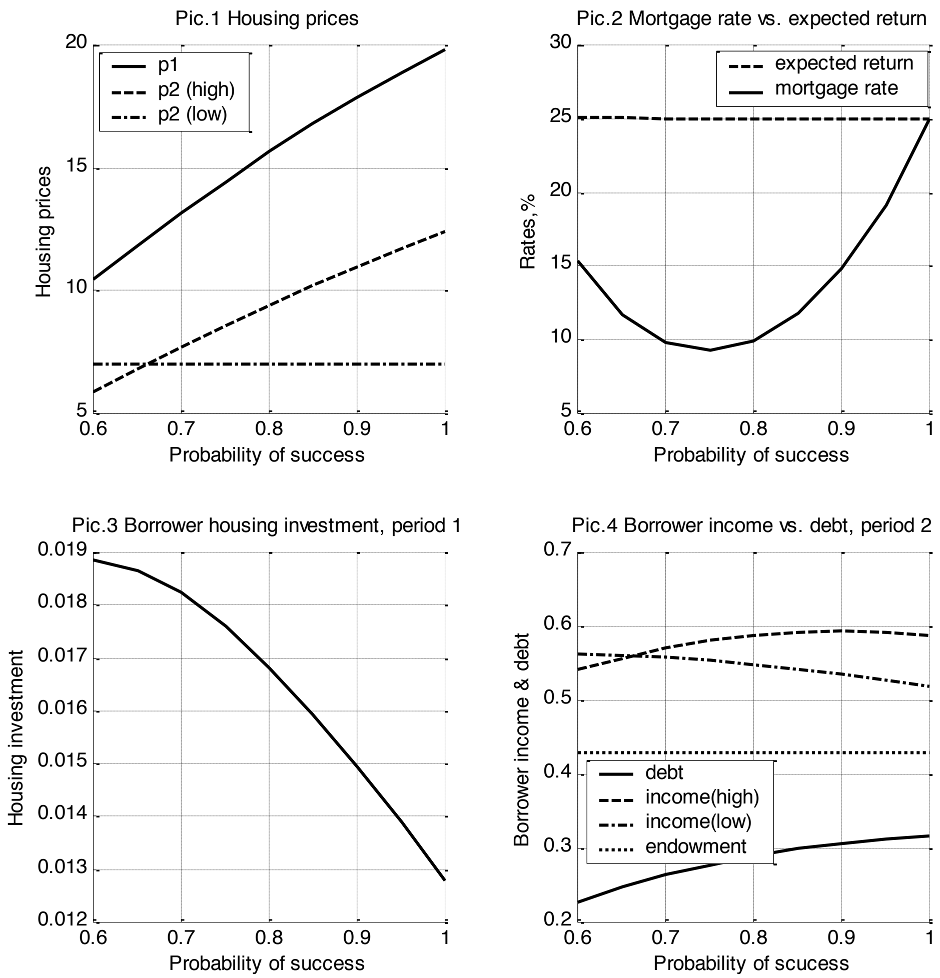
2.4. Results

2.4.1. Equilibrium with no bankruptcy. Results

There is no analytical solution for the case when no bankruptcy occurs. We solve the model with MATLAB simulations to find conditions on the parameters such that equilibrium with no bankruptcy exists. These MATLAB simulation results have been obtained with risk aversion coefficient $\gamma = 3$, an expected return of 25% ($\beta = \sigma r = 1.25$), and the coefficient of utility benefit of homeownership $w = 0.9$. We also parameterize a housing market size to be $\bar{h} =\in [0.16, 0.27]$. Additionally, we consider different levels of borrower's capital endowment/income that has a non-housing investment origin I.

Housing price dynamics, the interest rates behavior, and economics of borrower in the model can be inferred from Figure 9. Figure 9 shows economics of the model for different levels of σ when the expected return is 25%, risk aversion is γ=3, the coefficient of utility benefit of homeownership is w=0.9, housing market size is $\bar{h} = 0.19$, and borrower's endowment parameter is $I = 0.43$.

Figure 9: The model with no bankruptcy, an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, housing market size $\overline{h} = 0.19$, and borrower's endowment parameter l=0.43



Pic.1 plots spot housing price p_1 and contingent housing prices $p_2(r)$ and $p_2(0)$ as a function of the probability of risky asset's success when the expected return on the risky asset is kept constant. In our setting (adopted from DKK model), with probability $\sigma = \frac{\beta}{r}$ the investment in risky asset is successful and gives return $r > 1$, and with probability $(1 - \sigma)$ the risky asset returns zero. Holding the expected return β constant, we consider an increase in the probability of the asset's success σ. This is equivalent to a decrease in risk of the asset $r = \frac{\beta}{\sigma}$. The first observation we make is that price realization $p_2(0)$ is constant. This comes from the fact that in equilibrium with a strictly positive interest rate r_b, investments in the non-risky asset and in mortgages are not perfect substitutes. Lenders invest zero in the non-risky technology, thus correlating their income with the outcome of the risky technology ($x_r = 1$). When the risky asset is unsuccessful, it returns zero units of capital always and the amount of capital in the economy does not depend on the level of financial risk. Thus contingent housing price in unsuccessful economy is constant with respect to financial risk.

Secondly, we observe that housing prices $p_2(r)$ increase with the probability of the asset's success (decreases in risk). To understand this increase, we need to consider the market prices of scarce services in our model. In the booming economy, the price of services is an increasing function of the total aggregate capital income of borrowers and lenders available for a trade with service providers $Y(r)$. Since in the successful economy, when the expected return on the risky asset is kept constant, the total aggregate capital income decreases with the probability of the risky asset's success, service prices decrease as well. Note that the total aggregate capital income decreases much slower than service prices. Hence, given fixed market size, lower service prices result in the higher relative housing prices. This implies that in a successful economy, housing prices $p_2(r)$ increase with the probability of the asset's success.

Pic.2 illustrates the mortgage rate vs. the expected return on the risky asset with respect to financial risk faced by lenders. The expected return is fixed at 25%. For low levels of σ, r_b decreases with the probability of the asset's success. For high levels of σ, r_b increases with the probability of the asset's success. In the economy with high levels of risk (low levels of σ), there is monotonic relation between the mortgage interest rates and level of financial risk. This result comes from the fact that when agents are sufficiently risk-averse and the expected return on the risky asset is kept constant, an increase in the risk of an asset induces stronger concerns about relative wealth, which in turn lead to higher demand for borrowing that drives mortgage rates up. For low levels of risk (high levels of σ), however, relative wealth concerns are not sufficient, demand for borrowing decreases with respect to financial risk driving the mortgage interest rates down.

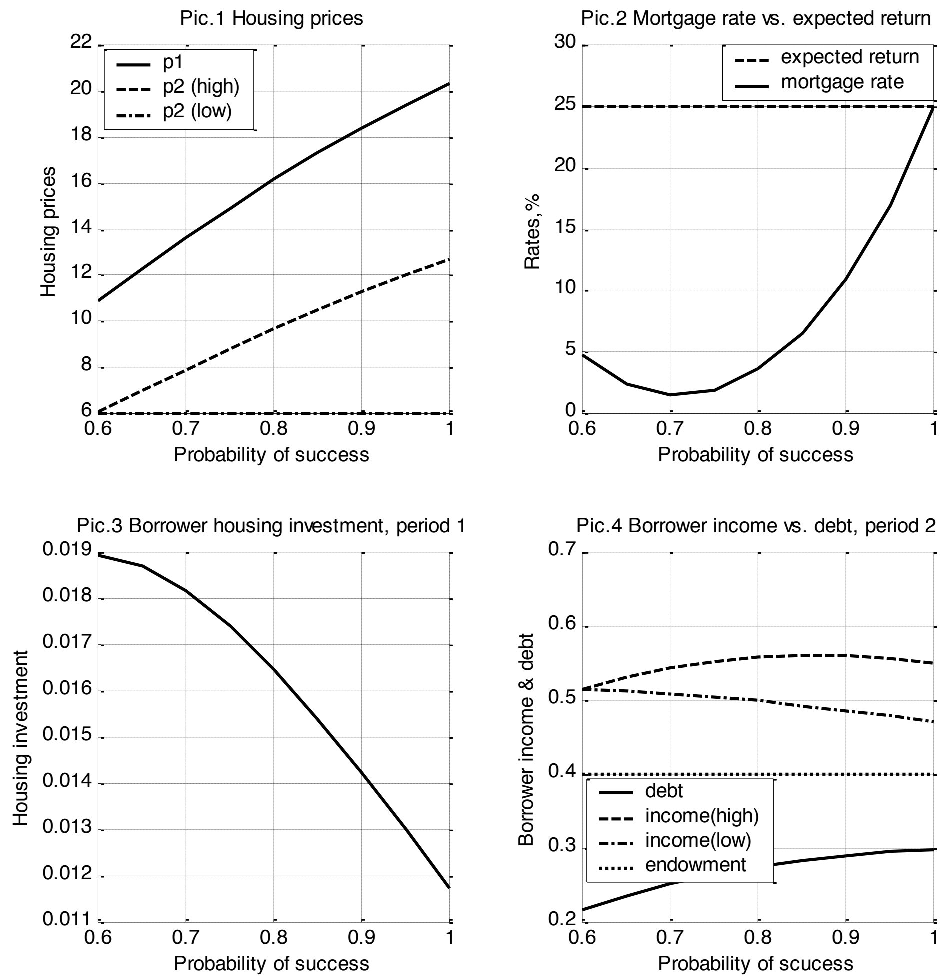
Housing investment decision of borrowers in period 1 mirrors the behavior of the housing prices. This is illustrated in Pic.3. Borrower's housing investment in period 1 increases with financial risk (decreases with the probability of the risky asset's success). Given the competition over the future scarce services, an increase in the risk of an asset induces stronger relative wealth concerns among borrowers. They cannot link their future wealth to the outcome of the risky asset. Instead, they correlate their future wealth with the housing market to insure against bankruptcy.

Borrower's income outcomes and debt in period 2 are shown in Pic.4. Dynamics of borrower's debt are determined by the fact that housing price p_1 is a rapidly increasing function of σ, while borrower's housing investment is a slowly decreasing function of σ. Thus, borrower's debt is an increasing function of σ. Dynamics of borrower's income are determined by dynamics of the spot housing prices in period 2 and borrower's housing investment. Housing price $p_2(r)$ rapidly increases with the probability of the asset's success. Thus, given that borrower's housing investment is slowly decreasing function and non-housing income of the borrower is a constant, borrower's income in the successful economy is an increasing function of σ. Housing prices $p_2(0)$ is constant with respect to the probability of the asset's success. Thus, given that borrower's housing investment is slowly decreasing function and non-housing income of the borrower is a constant, borrower's income in the unsuccessful economy is a slowly decreasing function of σ. We also observe that borrowers have means to repay their mortgage debt for all levels of risk.

Let us now analyze the economics of our model with respect to financial risk and the level of borrower's endowment. As we decrease a non-housing related income of borrowers I, some of the dynamics of the housing prices, mortgage rates, and housing investments change. Figure 10 shows behavior of the variables for different levels of σ when the expected return is 25%, risk aversion is γ=3, the coefficient of utility benefit of homeownership is w=0.9, housing market size is $\bar{h} = 0.19$, and borrower's endowment parameter is $I = 0.4$.

First in Pic.1 we notice that lower levels of borrowers' endowment result in higher housing prices $p_2(r)$ and p_1, but lower housing price $p_2(0)$. To understand the increase in price $p_2(r)$, we need again to consider the market prices of scarce services in our model. In the booming economy, the price of services is a rapidly increasing function of the total aggregate capital income of borrowers and lenders available for a trade with service providers $Y(r)$. Since in the successful economy, when the expected return on the risky asset is kept constant, the total aggregate capital income decreases with the decrease in the borrower's non-housing endowment, service prices decrease as well. Note that the total aggregate capital income decreases much slower than service prices. Hence, given fixed market size, lower service prices result in the higher relative housing prices. This implies that in a successful economy, housing prices $p_2(r)$ increase with the borrower's non-housing endowment.

Figure 10: The model with no bankruptcy, an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, housing market size $\bar{h} = 0.19$, and borrower's endowment parameter l=0.4



To understand the decrease in price $p_2(0)$, we employ a similar analysis of the relative prices in the economy with low realization of the risky asset. In the unsuccessful economy, the price of services is a slowly increasing function of the total aggregate capital income of borrowers and lenders available for a trade with service providers $Y(r)$. Since in the unsuccessful economy, when the expected return on the risky asset is kept constant, the total aggregate capital income decreases with the decrease in the borrower's non-housing endowment, service prices decrease as well. Note that the total aggregate capital income decreases much faster than service prices. Hence, given fixed market size, this results in the lower relative housing prices. This implies that in a successful economy, housing prices $p_2(0)$ decrease with the borrower's non-housing endowment.

Pic.2 illustrates the interest rates on mortgages with respect to financial risk faced by lenders. The dynamics of the mortgage rates analyzed above continue to be the same. However the overall level of the mortgage rates in the economy declines with the decrease in the borrower's level of income.

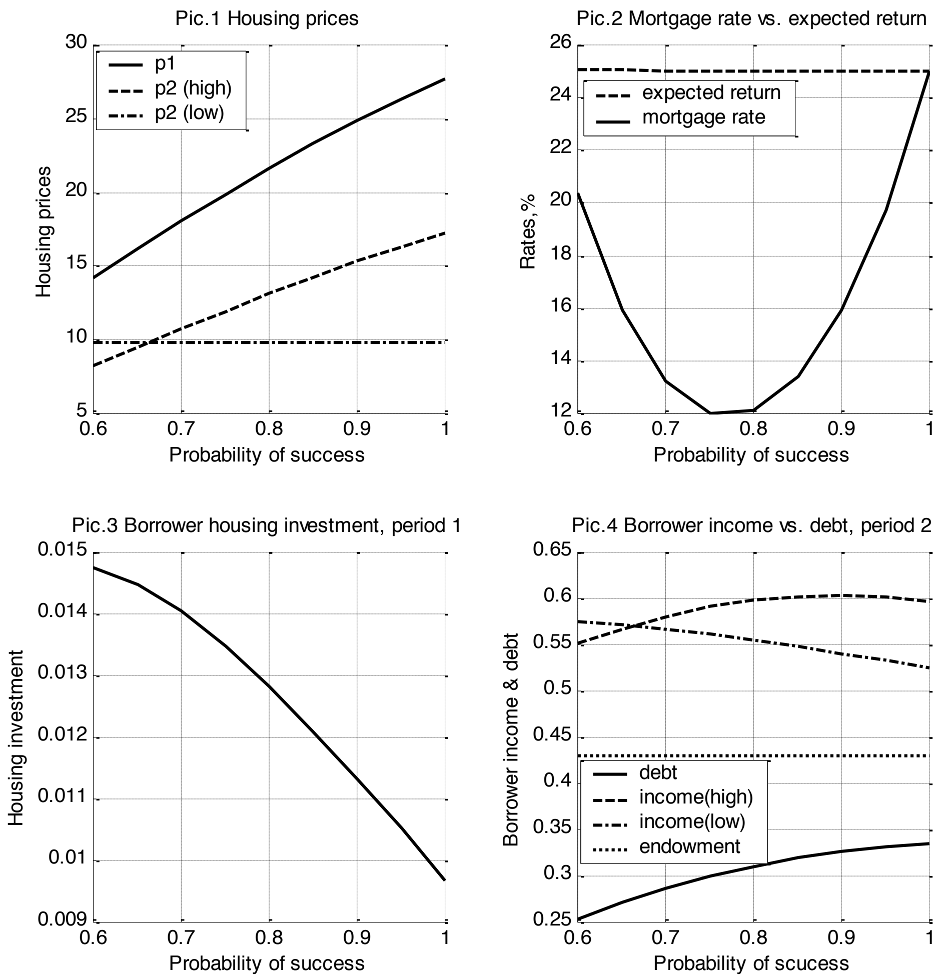
Housing investment decision dynamics of borrowers in period 1 stays the same; as before it mirrors the behavior of the housing prices. Given an increase in the levels of housing price p_1, we observe an overall decline in the levels of the borrower's housing investment. This is illustrated in Pic.3.

Dynamics of borrower's income outcomes and debt in period 2 are unchanged and shown in Pic.4. However the overall levels of borrower's income and debt in the economy declines with the decrease in the borrower's level of endowment. Lower levels of borrower's housing investment and lower mortgage rates decrease borrower's debt, while a drop in the borrower's non-housing related endowment results in lower levels of borrower's income outcomes. In this case, the drop in borrower's levels of income is more significant than the decline in their levels of debt. Thus, we can conclude that decrease in the levels of borrower's non-housing related income results in a smaller gap between borrower's debt and borrower's income.

Let us now consider changes with respect to financial risk and the housing market size. Figure 11 shows the behavior of the variables for different levels of σ when the expected return is 25%, risk aversion is γ=3, the coefficient of utility benefit of homeownership is w=0.9, housing market size is $\bar{h} = 0.17$, and borrower's endowment parameter is $I = 0.43$.

Pic.1 and Pic.2 illustrate that smaller housing market size and thus decrease in supply of housing in the model result in higher housing prices and higher mortgage rates. Consequently, given an increase in the levels of housing price p_1, we observe an overall decline in the levels of the borrower's housing investment. This is illustrated in Pic.3.

Figure 11: The model with no bankruptcy, an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, housing market size $\overline{h} = 0.17$, and borrower's endowment parameter l=0.43



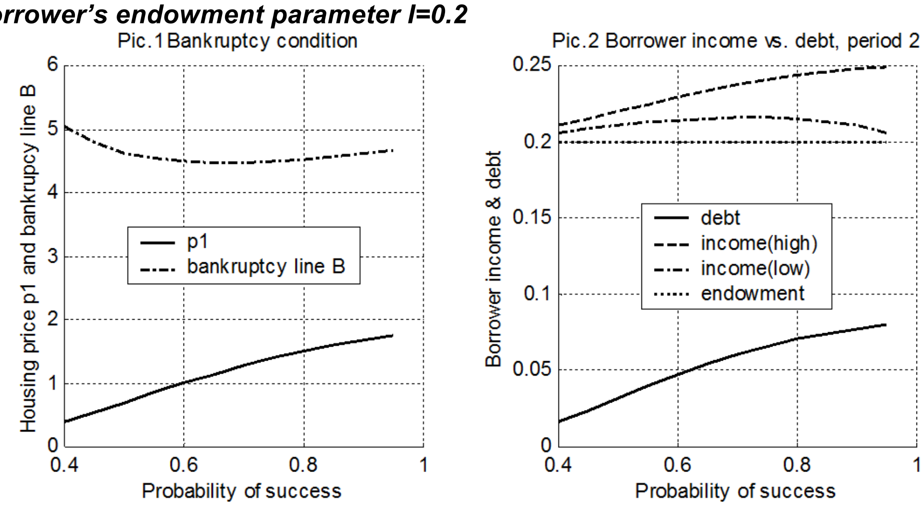
Dynamics of borrower's income outcomes and debt in period 2 are unchanged and shown in Pic.4. However the overall levels of borrower's income and debt in the economy increase with the decrease in the housing market size. This is a result of a significant increase in the levels of the housing prices. However relative increase in borrower's debt is greater than relative increase in their income. Thus we conclude that a decrease in the housing market size results in the higher borrower's debt-to-income ratio.

2.4.2. Equilibrium with contingent bankruptcy. Results

There is no analytical solution for the case when bankruptcy occurs. We attempted to solve the model with MATLAB simulations to find conditions on the parameters such that equilibrium with bankruptcy exists. We considered $p_2(0) < p_2(r)$. Then, according our previous analysis in the economy with successful risky asset there is no bankruptcy and mortgages are paid in full. However, in the state of economy in which risky asset is unsuccessful bankruptcy occurs: borrower defaults and the lender is paid the wealth of the borrower. However, MATLAB simulation showed that the bankruptcy condition is never satisfied in this case and there is no parameterization for which equilibrium with bankruptcy exists.

Figure 12 shows economics of the model for different levels of σ when the expected return is 25%, risk aversion is γ=3, the coefficient of utility benefit of homeownership is w=0.9, housing market size is $\bar{h} = 0.45$, and borrower's endowment parameter is $I = 0.2$.

Figure 12: Bankruptcy condition and borrowers income vs. debt in period 2. The model with no bankruptcy, an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, housing market size $\bar{h} = 0.45$, and borrower's endowment parameter I=0.2



Since the bankruptcy condition is

$$p_1 > \frac{I + p_2(0)h_{1b}}{(1 + r_b)h_{1b}},$$

we denote

$$B = \frac{I + p_2(0)h_{1b}}{(1 + r_b)h_{1b}}.$$

Pic.1 plots housing price p_1 and bankruptcy line B. It illustrates that the bankruptcy condition in this case is not satisfied. Additionally, from Pic.2 we can see that borrowers have means to repay the mortgage debt in this case.

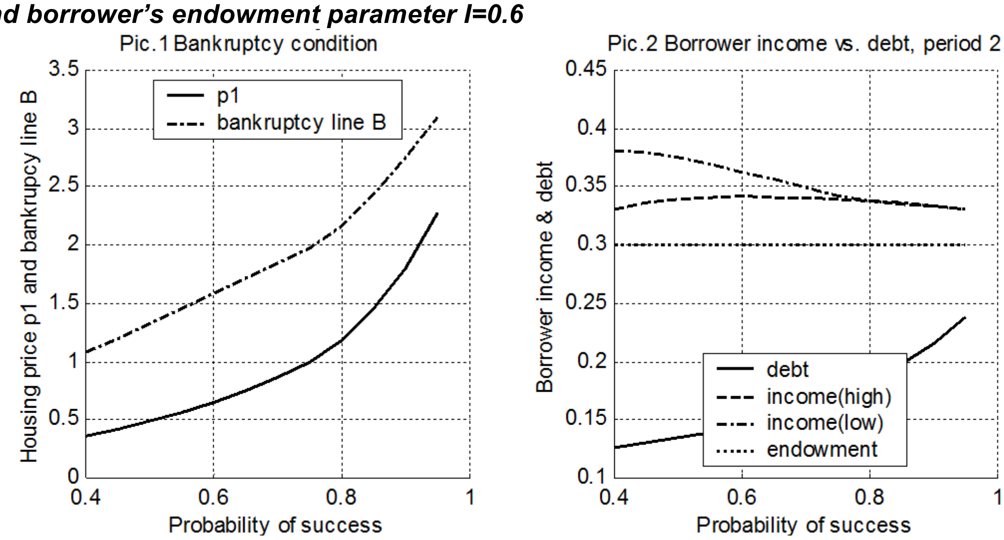
We now consider $p_2(r) < p_2(0)$. Then, according to our previous analysis in the economy with unsuccessful risky asset there is no bankruptcy and mortgages are paid in full. However, in the state of economy in which risky asset is successful bankruptcy occurs: borrower defaults and the lender is paid the wealth of the borrower. However, MATLAB simulation showed that the bankruptcy condition is never satisfied in this case and there is no parameterization for which equilibrium with bankruptcy exists.

Figure 13 shows economics of the model for different levels of σ when the expected return is 25%, risk aversion is γ=3, the coefficient of utility benefit of homeownership is w=0.9, housing market size is $\bar{h} = 0.7$, and borrower's endowment parameter is $I = 0.3$.

Pic.1 plots housing price p_1 and bankruptcy line B.

$$B = \frac{I + p_2(r)h_{1b}}{(1 + r_b)h_{1b}}.$$

Figure 13: Bankruptcy condition and borrowers income vs. debt in period 2. The model with no bankruptcy, an expected return of 25%, risk aversion γ=3, the coefficient of utility benefit of homeownership w=0.9, housing market size $\bar{h} = 0.6$, and borrower's endowment parameter I=0.6



Thus, Pic.1 illustrates that bankruptcy condition in this case is not satisfied. Additionally, from Pic.2 we can see that borrowers have means to repay the mortgage debt in this case.

We were not able to find conditions on the parameters such that equilibrium with bankruptcy exists neither in the case with $p_2(0) < p_2(r)$, nor in the case with $p_2(r) < p_2(0)$.

Conclusions

In this dissertation we develop two stylized computational models that simulate and examine the mechanisms of housing market and explore the possible origins of real estate price bubbles. In chapter 1 we investigate inter-dependence between financial and housing markets and analyze how uncorrelated financial risk may promote housing price bubble. We present a stylized finite-horizon stochastic model that demonstrates that increase in demand for housing may result from financial risk exposure, when house-buyers are subject to relative wealth concerns and competition over future consumption. Our main result shows that housing price bubbles are most likely to emerge as a result of house-buyer's risk exposure when this exposure is minimal.

We show that endogenous relative wealth concerns may play an important role in explaining the emergence and dynamics of housing price bubbles. Our model demonstrates that concerns about relative wealth arise endogenously in equilibrium and affect equilibrium outcomes given that agents endowed with scarce services are unable to sell them in advance. Because future prices of scarce services increase with the agent's wealth, house-buyers' ability to consume will rely on their relative wealth. This leads to house-buyer's portfolio diversification and extensive investment in high-risk financial instruments. Such portfolio composition pattern affects the wealth of house-buyer population and future demand for housing resulting in a residential real estate price bubble.

In chapter 2 we investigate inter-dependence between financial, mortgage, and housing markets and analyze the effects of mortgage market expansion on the housing price dynamics. We present a stylized finite-horizon stochastic model that demonstrates that lender's exposure to uncorrelated financial risk together with availability of mortgage credit to the low-income borrowers lead to increase in housing prices and greater borrower's debt to income ratio. Our main result shows that the high borrower's debt-to-income ratio resulting from high housing price volatility is the most probable when financial exposure of lenders is low. Our model demonstrates that concerns about relative wealth lead to extensive financial investment in high-risk financial instruments for lenders and greater housing investment for borrowers. Such investment behavior affects the wealth of agent population and drives housing demand and housing prices up, while increasing the borrower's debt-to-income ratio.

Bibliography

Abel, Andrew B. 1990. Asset Prices under Habit Formation and Catching Up with the Joneses. A.E.R. Papers and Proceedings 80, 38-42.

Brueckner, Jan K., 1997. Consumption and Investment Motives and the Portfolio Choices of Homeowners. Journal of Real Estate Finance and Economics, 15 (2), 159-180.

Calhoun, Michael D., 2004. Promoting Homeownership by Ensuring Liquidity in the Subprime Mortgage Market. Subcommittee on Financial Institutions and Consumer Credit. Subcommittee on Housing and Community Opportunity.

Capozza, Dennis R. and Thomson, Thomas A., 2005. Optimal Stopping and Losses on Subprime Mortgages. The Journal of Real Estate Finance and Economics, 30:2, 115-131.

Chambers, Matthew, Garriga, Carlos, and Schlagenhauf, Don, 2008. Equilibrium mortgage choice and housing tenure decisions with refinancing. Research Division Federal Reserve Bank of St. Louis.

Chinloy, Peter and Macdonald, Nancy, 2005. Subprime Lenders and Mortgage Market Completion. The Journal of Real Estate Finance and Economics, 30:2, 153-165.

Cocco, J., 2005. Portfolio Choice in the Presence of Housing. Review of Financial Studies 18, 535–567.

Cole, Hal, Mailath, George, and Postelwaite, Andrew, 2001. Investment and Concern for Relative Position. Review of Economic Design 6, 241-261.

Cutts, Amy C. and Robert A. Van Order, 2005. On the Economics of Subprime Lending. The Journal of Real Estate Finance and Economics, 30:2, 167-196.

Davis, M., Heathcote, J., 2005. Housing and the business cycle. International Economic Review 46, 751-784.

DeMarzo, Peter, Kaniel, Ron, and Kremer, Ilan, 2007. Technological Innovation and Real Investment Booms and Busts. Journal of Financial Economics 85, 735-754.

DeMarzo, Peter, Kaniel, Ron, and Kremer, Ilan, 2008. Relative Wealth Concerns and Investment Bubbles. Review of Financial Studies 21(1), 19-50.

Doms, Mark and Motika, Meryl, 2006. The Rise in Homeownership. Federal Reserve Bank of San Francisco Economic Letter 2006-30.

Dupor, William and Liu, Wen-Fang, 2003. Jealousy and Equilibrium Overconsumption. The American Economic Review, 93 (1), 423-428.

Flavin, M., Nakagawa, S., 2005. A model of housing in the presence of adjustment costs: a structural interpretation of habit persistence. Unpublished working paper, university of California, San Diego.

Flavin, M., Yamashita, T., 2002. Owner-occupied housing and the composition of the household portfolio. American Economics Review 92, 345-362.

Fratantoni, Michael, Duncan, Douglas G., Brinkmann, Jay, and Orawin Velz, 2007. The Residential Mortgage Market and Its Economic Context in 2007. Mortgage Bankers Association.

Gali, Jordi, 1994. Keeping up with the Joneses: Consumption Externalities, Portfolio Choice, and Asset Prices. Journal of Money, Credit and Banking 26, 1-8.

Himmelberg, Charles P., Mayer, Christopher J., and Sinai, Todd M., 2005. Assessing High House Prices: Bubbles, Fundamentals, and Misperceptions. Journal of Economic Perspectives 19 (4), 67-92.

Hobijn, Bart, and Jovanovic, Boyan, 2001. The Information-Technology Revolution and the Stock Market: Evidence. American Economic Review 91, 1203–1220.

Hong, Harrison, Scheinkman, Jose, and Xiong, Wei, 2008. Advisors and Asset Prices: A Model of the Origins of Bubbles. Journal of Financial Economics 89, 268-287.

Inside B&C Lending, 2007. Top 25 Lenders in 2006, Feb. 9, 2007; Inside B&C Lending, 2006. Subprime Mortgage Origination Indicators, Nov. 10, 2006.

Laitner, John, and Stolyarov, Dmitriy, 2003. Technological change and the stock market. American Economic Review 93, 1240–1267.

McCarthy, Jonathan, and Richard W. Peach, 2004. Are Home Prices the Next "Bubble"? FRBNY Economic Policy Review, 10 (3), 1-17.

Nichols, Joseph, Pennington-Cross, Anthony, and Yezer, Anthony, 2005. Borrower Self-Selection, Underwriting Costs, and Subprime Mortgage Credit Supply. The Journal of Real Estate Finance and Economics, 30:2, 197-219.

Ortalo-Magne, F., Rady, S., 2006. Housing market dynamics: on the contribution of income shocks and credit constraints. Review of Economics Studies, 73, 459-485.

Pastor, Lubos, and Veronesi, Pietro, 2006. Technological Revolutions and Stock Prices. Unpublished working paper, University of Chicago.

Pastor, Lubos, and Veronesi, Pietro, 2006. Was there a NASDAQ bubble in the late 1990s? Journal of Financial Economics, 81 (1), 61-100.

Piazzesi, Monika, Schneider, Martin, and Tuzel, Selale, 2007. Housing, Consumption, and Asset Pricing. Journal of Financial Economics 83, 531-569.

Piskorski, Tomasz and Tchistyi, Alexei, 2008. Stochastic House Appreciation and Optimal Mortgage Lending. NYU Stern. Working paper series.

Quan, Daniel C., and Titman, Sheridan, 1999. Do Real Estate Prices and Stock Prices Move Together? An International Analysis. Real Estate Economics, 27, 183-207.

Yeung Chan L., and Kogan, Leonid, 2002. Catching up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices. Journal of Political Economy, 110, 1255-1285.

Appendix

Proof of Proposition 1. For β greater than 1 the investment in the risky asset in the benchmark case is:

$$x_{rb} = \frac{1-a}{1+a(r-1)} = \frac{1-a}{1+a\left(\frac{\beta}{\sigma}-1\right)},$$

where a is equal to

$$a = \left(\frac{1}{r-1}\frac{1-\sigma}{\sigma}\right)^{\frac{1}{\gamma}} = \left(\frac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}}.$$

We need to consider the sign of

$$\frac{dx_{rb}}{d\sigma} = \frac{\partial x_{rb}}{\partial \sigma} + \frac{\partial x_{rb}}{\partial a}\frac{da}{d\sigma}.$$

$$\frac{\partial x_{rb}}{\partial \sigma} = (1-a)a\beta\left(\frac{1}{\sigma\left(1+a\left(\frac{\beta}{\sigma}-1\right)\right)}\right)^{-2} > 0,$$

$$\frac{\partial x_{rb}}{\partial a} = -\frac{\frac{\beta}{\sigma}+1-a}{\left(1+a\left(\frac{\beta}{\sigma}-1\right)\right)^2} < 0,$$

$$\frac{da}{d\sigma} = -\frac{1}{\gamma}\left(\frac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}-1}\left(\frac{\beta-1}{(\beta-\sigma)^2}\right) < 0,$$

Thus,

$$\frac{dx_{rb}}{d\sigma} = \frac{\partial x_{rb}}{\partial \sigma} + \frac{\partial x_{rb}}{\partial a}\frac{da}{d\sigma} > 0.$$

Investment x_{rb} is decreasing in a, while a is decreasing in σ. ■

Proof of Proposition 2. Consider the expressions for period 2 housing prices in the benchmark case:

$$p_{2b}(r) = \left(\frac{w}{\bar{\bar{h}}}\right)^{\gamma} \sigma F(\sigma),$$

$$p_{2b}(0) = \left(\frac{w}{\bar{\bar{h}}}\right)^{\gamma} (1-\sigma) F(\sigma),$$

where $F(\sigma)$ is given by

$$F(\sigma) = \left\{ \frac{p_c(r)(rX_{rb} + X_{nb}) + p_c(0)X_{nb}}{(1 + rX_{rb} + X_{nb})^{\gamma} (\sigma p_c(r)^{\gamma-1})^{\frac{1}{\gamma}} + (1 + X_{nb})^{\gamma} \left((1-\sigma) p_c(0)^{\gamma-1}\right)^{\frac{1}{\gamma}}} \right\}^{\gamma}$$

Using analytical solution for the benchmark case, the expression for $F(\sigma)$can be rewritten as

$$F(\sigma) = \left\{ \frac{1 + \frac{1}{r-1}}{\left(2 + x_{rb}(r-1)\right)^{\gamma} \left(\sigma \left(\frac{1}{r-1}\right)^{\gamma-1}\right)^{\frac{1}{\gamma}} + (2 - x_{rb})^{\gamma} (1-\sigma)^{\frac{1}{\gamma}}} \right\}^{\gamma}$$

For $\sigma \to 0$,

$$a = \left(\frac{1-\sigma}{\beta-\sigma}\right)^{\gamma} \to \left(\frac{1}{\beta}\right)^{\gamma},$$

$$x_r = \frac{1-a}{1+a(r-1)} \to 0,$$

$$F(\sigma) \to \left(\frac{1}{2^{\gamma}}\right)^{\gamma},$$

And for $\sigma \to 1$,

$$a = \left(\frac{1-\sigma}{\beta-\sigma}\right)^{\gamma} \to 0,$$

$$x_r = \frac{1-a}{1+a(r-1)} \to 1,$$

$$F(\sigma) \to \left(\frac{1 + \frac{1}{\beta-1}}{\left(2 + (\beta-1)\right)^{\gamma} \left(\frac{1}{\beta-1}\right)^{1-\frac{1}{\gamma}}} \right)^{\gamma}.$$

Thus, housing prices p_{1b} and $p_{2b}(r)$ at $\sigma \to 0$ are lower than at $\sigma \to 1$. Housing price $p_{2b}(0)$ at $\sigma \to 0$ is higher than at $\sigma \to 1$. ■

Moreover, with MATLAB simulations, we can show that $F(\sigma)$ is a smooth and sign conserving (positive) function that is slowly increasing in the whole range of argument. Thus for all σ, $\frac{\partial p_{2b}(r)}{\partial \sigma} > 0$ indicating that price $p_{2b}(r)$ increases in probability of success of the risky asset, while $\frac{\partial p_{2b}(0)}{\partial \sigma} < 0$ indicating that price $p_{2b}(0)$ decreases in probability of success of the risky asset.

Housing price of period 1 summarizes the effect of both housing prices of period 2:

$$p_{1b} = 2\big(p_{2b}(r) + p_{2b}(0)\big) = 2\left(\frac{w}{\bar{h}}\right)^{\gamma} F(\sigma).$$

Therefore, p_{1b} is a smooth increasing function of probability of asset's success.

Proof of Proposition 3. The first order condition for optimality in the model with trading frictions implies that:

$$\sigma(r-1)\left(\frac{1+(1+(r-1)x_r)^{\gamma-1}}{1+(r-1)x_r}\right)^{\gamma} - (1-\sigma)\left(\frac{1+(1-x_r)^{\gamma-1}}{1-x_r}\right)^{\gamma} = 0$$

Derivative of x_r with respect to σ is given by

$$\frac{dx_r}{d\sigma} = \frac{\dfrac{(\gamma-2)\left(\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{\gamma-1}-1\right)\beta x_r}{\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{2}\sigma^2} - \dfrac{1}{\gamma}\left(\dfrac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}-1}\left(\dfrac{\beta-1}{(\beta-\sigma)^2}\right)\left(\dfrac{1+(1-x_r)^{\gamma-1}}{1-x_r}\right)}{\dfrac{(\gamma-2)\left(\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{\gamma-1}-1\right)(\beta-\sigma)}{\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{2}\sigma} + \dfrac{(\gamma-2)((1-x_r)^{\gamma-1}-1)}{(1-x_r)^2}\left(\dfrac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}}}$$

To find the sign of the derivative, we need to consider asymptotic for x_r. For $\sigma \to 1$, let $x_r \neq 1$. Then the second term in the FOC above equals zero. In turn, the numerator of the first term in the FOC must equal zero. However, this cannot be true for $x_r \in [0,1]$. Thus, for $\sigma \to 1$, we have $x_r \to 1$. With this result at hand, for $\sigma \to 1$, the first term in the FOC is a positive constant, while the limiting value for the second term is $\frac{1-\sigma}{(1-x_r)^{\gamma}}$. Consequently, for $\sigma \to 1$, we have $(1 - x_r) \to (1-\sigma)^{\frac{1}{\gamma}}$.

Let us now analyze the derivative $\frac{dx_r}{d\sigma}$. For $\sigma \to 1$, both the first term in the numerator of the derivative and the first term in the denominator of the derivative are small positive constants. On the other hand, for $\sigma \to 1$, considering asymptotic for $(1 - x_r)$, the limiting value for the second term in the denominator is $-(1-\sigma)^{-1}$. The limiting value of the second term in the denominator, in turn, is $-(1-\sigma)^{-\frac{1}{\gamma}}$. Consequently, for $\sigma \to 1$, the limiting value of the derivative is $(1-\sigma)^{\frac{1}{\gamma}-1}$. Thus, as σ approaches 1, we have

$$\frac{dx_r}{d\sigma} > 0.$$

To find the sign of the derivative for small σ, we need again to consider asymptotic for x_r. When $\sigma \to 0$, we have $a \cong \left(\frac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}} = (\beta)^{-\frac{1}{\gamma}}$. Then investment in the risky asset is

$$x_r \cong 1 - \frac{\sigma^{\gamma-2}}{\beta^{\gamma+\frac{1}{\gamma}-2}} \to 1.$$

With this result at hand, for $\sigma \to 0$, the limiting value of the first term in the numerator is $\sigma^{1-\gamma}$, while the second term is a small negative constant. Thus, for $\gamma > 1$, the numerator of the derivative is always positive.

Additionally, for $\sigma \to 0$, the limiting value of the first term in the denominator is $\sigma^{2-\gamma}$, while the limiting value of the second term is $-\sigma^{2(2-\gamma)}$. Therefore, for $\sigma \to 0$ and $\gamma > 2$, the denominator of the derivative is always negative. Thus, as σ approaches 0, we have

$$\frac{dx_r}{d\sigma} < 0.$$

Hence, for low levels of σ, investment x_r decreases with the probability of the asset's success. For high levels of σ, investment x_r increases with the probability of the asset's success. ■

Proof of Proposition 4. Equations for house-buyer's investment in the risky asset in the benchmark and in the model are

$$\frac{1-x_{rb}}{1+x_{rb}(r-1)}=a$$

and

$$\frac{1-x_r}{1+x_r\,(r-1)}\left[\frac{1+\left(1+x_r\,(r-1)\right)^{\gamma-1}}{1+(1-x_r\,)^{\gamma-1}}\right]=a$$

respectively. Let us rewrite the last equation as

$$\frac{1-x_r}{1+x_r\,(r-1)}\left[\frac{z\left(1+x_r\,(r-1)\right)}{z(1-x_r\,)}\right]=a,$$

where $z=1+(arg)^{\gamma-1}$ is an increasing function of the argument, for $\gamma>1$. If $x_r>0$, then $1+x_r\;(r-1)>1-x_r$, which in turn implies that

$$\left[\frac{1+\left(1+x_r\,(r-1)\right)^{\gamma-1}}{1+(1-x_r\,)^{\gamma-1}}\right]>1.$$

So, $x_r>x_{rb}$. ■

Proof of Proposition 5. We have already seen that the derivative of x_r with respect to σ is

$$\frac{dx_r}{d\sigma}=$$

$$\frac{\dfrac{(\gamma-2)\left(\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{\gamma-1}-1\right)\beta x_r}{\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{2}\sigma^2}-\dfrac{1}{\gamma}\left(\dfrac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}-1}\left(\dfrac{\beta-1}{(\beta-\sigma)^2}\right)\left(\dfrac{1+(1-x_r)^{\gamma-1}}{1-x_r}\right)}{\dfrac{(\gamma-2)\left(\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{\gamma-1}-1\right)(\beta-\sigma)}{\left(1+x_r\left(\frac{\beta-\sigma}{\sigma}\right)\right)^{2}\sigma}+\dfrac{(\gamma-2)((1-x_r)^{\square-1}-1)}{(1-x_r)^2}\left(\dfrac{1-\sigma}{\beta-\sigma}\right)^{\frac{1}{\gamma}}}$$

When beta=1 the expression becomes

$$\frac{\frac{\left(\left(1+x_r\left(\frac{1-\sigma}{\sigma}\right)\right)^{\gamma-1}-1\right)x_r}{\left(1+x_r\left(\frac{1-\sigma}{\sigma}\right)\right)^2\sigma^2}}{\frac{\left(\left(1+x_r\left(\frac{1-\sigma}{\sigma}\right)\right)^{\gamma-1}-1\right)(1-\sigma)}{\left(1+x_r\left(\frac{1-\sigma}{\sigma}\right)\right)^2\sigma}+\frac{((1-x_r)^{\gamma-1}-1)}{(1-x_r)^2}}$$

Using the results for asymptotic from the Proof of Proposition 3, it is easy to see that for $x_r > 0$ and $\gamma > 2$, the numerator of the derivate is always positive, while the denominator is negative for both large and small σ. Thus,

$$\frac{dx_r}{d\sigma} < 0. \blacksquare$$

Proof of Proposition 6. Consider the expression for housing prices in period 2:

$$p_2\big(Y(\bar{r})\big) = \frac{w}{\bar{h}^\gamma}\left[\left(\frac{Y(\bar{r})}{1+\big(Y(\bar{r})\big)^{\gamma-1}}\right)\right]^\gamma.$$

The function of housing price $p_2(\bar{r})$ is proportional to

$$p_2\big(Y(\bar{r})\big)\sim\frac{Y(\bar{r})}{1+Y(\bar{r})^{\gamma-1}},$$

therefore, its derivative with respect to $Y(\bar{r})$ is proportional to

$$\frac{\partial p_2\big(Y(\bar{r})\big)}{\partial Y(\bar{r})}\sim 1+(2-\gamma)\big(Y(\bar{r})\big)^{\gamma-1}.$$

This function has a maximum at $Y_m = \left(\frac{1}{\gamma-2}\right)^{\frac{1}{\gamma-1}}$. For $\gamma = 3$, $Y_m = 1$. Expansion of the function $p_2\big(Y(\bar{r})\big)$ in the Taylor series near the point of the maximum gives symmetric parabolic function of the displacement:

$p_2(Y(\bar{r}) - Y_m) =$

$$p_2(Y_m) + \frac{\partial p_2\big(Y(\bar{r})\big)}{\partial Y(\bar{r})}_{Y(\bar{r})=Y_m}(Y(\bar{r}) - Y_m) + \frac{1}{2}\frac{\partial 2 p_2\big(Y(\bar{r})\big)}{\partial Y(\bar{r})^2}_{Y(\bar{r})=Y_m}(Y(\bar{r}) - Y_m)^2 + \cdots$$

Remember that

$$Y(r) = 1 + x_r \ (r-1) = \ Y_m + x_r \ (r-1)$$

and

$$Y(0) = 1 - x_r = \ Y_m - x_r.$$

If $x_r \ > 0$ and $r < 2$, then $|Y(r) - \ Y_m| < |Y(0) - \ Y_m|$. Given the symmetric function of the Taylor series expansion, this implies that $p_2(r) > p_2(0)$.

Now consider the expression for housing price p_1:

$$p_1 \ = 2\left[\frac{\sigma}{p_2(r)} + \frac{(1-\sigma)}{p_2(0)}\right]^{-1}$$

or

$$\frac{1}{p_1} = \frac{\sigma}{2p_2(r)} + \frac{1-\sigma}{2p_2(0)}.$$

It is easy to prove that

$$\frac{1}{p_1} = \frac{\sigma}{2p_2(r)} + \frac{1-\sigma}{2p_2(0)} < \frac{\sigma}{2p_2(r)} + \frac{1-\sigma}{2p_2(r)} = \frac{1}{2\,p_2(r)}.$$

Thus, $p_1 > p_2(r)$. ■