Essays in Game Theory and Its Applications

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Abstract of the Dissertation

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My dissertation comprises the following three essays: the first essay analyzes third-price auctions, the second one considers patent licensing of an innovation with an unknown use; and the third essay studies the existence and uniqueness of Cournot equilibrium.

Since the classic work of Vickrey (1961), the ranking of various auction forms in terms of expected revenue has been the central question of auction theory. When the bidders' information about the value of the object is independently and identically distributed, the so-called revenue equivalence principle provides a complete answer to the revenue ranking question. However, when the assumption of independence is relaxed, the answer is less well-understood. Utilizing the assumption that the bidders' information is affiliated, we analyze the third-price auction where the seller collects the third highest bid from the winner, and show that the third-price auctions

performs better than the first- and second-price auctions.

Although it has not been considered widely (or not at all) in the existing literature on patent licensing, licensing of a new technology could take place in environments where uses for it are, to some extent, unknown. That is, inventor holds the patent of a technology that could potentially reduce the costs of firms operating in a given industry, and some additional effort (or costly test) could discover its use with some positive probability. The inventor thus face the problem: should he first try to discover the use for the technology and then license it, or should he license the technology before a use has been discovered, leaving the discovery task to the licensees? This question has been raised in the second essay, and we show that the answer to this question depends on how discovery by each agent is related to discovery by other agents.

The third essay studies the existence and uniqueness of Cournot equilibrium. Our conditions are weaker than the ones appearing in the literature, at the expense of multiple equilibrium points. However, we show that among these equilibrium points only one has a positive price. If we add the requirement that at least one firm produces at a positive cost whenever the industry aggregate output implies zero market price, then the equilibrium is unique and the equilibrium price is positive.

Keywords: Third-price auction, Affiliation, Private signal, Volterra integral equation; Process innovation, Patent licensing, Cournot competition; Cournot equilibrium, Existence, Uniqueness

JEL classification: C62, D43, D44, D45, D82, L13

Contents

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Chapter 1

Third-price Auctions with Affiliated Signals

Abstract

This paper characterizes the symmetric equilibrium in a third-price sealed-bid auction when players' signals are affiliated, and shows that the expected revenue of the seller from a third-price auction is greater than the expected revenue from a first-price auction and a second-price auction.

Keywords: Third-price auction, Affiliation, Private signal, Volterra integral equation

JEL classification: C62, D44, D82

1.1 Introduction

Since the classic work of Vickrey (1961), the ranking of various auction forms in terms of expected revenue has been the central question of auction theory. When the bidders are risk-neutral and their information about the value of the object is independently and identically distributed, the so-called "revenue equivalence principle" (see, for instance, Myerson (1981)) provides a complete answer to the revenue ranking question. When the assumption of independence is relaxed, the answer is less well-understood. Utilizing the assumption that the bidders' information is affiliated, Milgrom and Weber (1982) develop the most comprehensive set of revenue ranking results. In this paper, we extend their analysis to the third-price auction where the seller collects the third highest bid from the winner, and show that the third-price auctions performs better than the first- and second-price auctions.

The paper is organized as follows. For later purposes, we first consider a functional equation so-called *Volterra integral equation* which later becomes useful to guarantee the existence of monotone equilibrium in a third-price auction. In section 3, we briefly describe the model which is the same as the model in Milgrom and Weber (1982). And section 4 characterizes the monotone equilibrium in a third-price auction, and the performance of the third-price auction is examined in Section 5.

1.2 Volterra Integral Equations

Let X and Z be two *affiliated* random variables that are supported on a bounded interval, say $[0,1]^2$. That is, their joint density function $g : [0,1]^2 \to \mathbb{R}$ satisfies:

$$g(t \vee t')g(t \wedge t') \geq g(t)g(t')$$

where, $t \vee t'$ and $t \wedge t'$ denote the component-wise maximum and the component-wise minimum of t and t'. Assume that g is positive and continuously differentiable. Define a function $k : D \equiv \{(x,z)|0 \leq z \leq x \leq 1\} \to \mathbb{R}$ as:

$$k(x,z) \equiv g_Z(z|X = x, Z \leq x)$$

We are now interested in the following functional equation:

$$\int_0^x k(x,z)b(z)dz = a(x) \tag{1.1}$$

where, the function $a : [0,1] \to \mathbb{R}$ is given, and $b : [0,1] \to \mathbb{R}$ is the unknown function.

Theorem 1. *Suppose that (i) $a(\cdot)$ is twice differentiable and convex (ii) $k(x,x)$ is decreasing in x. Then the equation* (1.1) *has a unique continuous solution. Moreover, the solution $b(\cdot)$ is increasing.*

The functional equations of this form are known as *Volterra integral equations of the first kind*. The function k is called the *kernel* of the equation. By differentiating the equation (1.1) and rearranging it, we derive another equation so-called *Volterra integral equation of the second kind*, which has the form:

$$b(x) = A(x) + \int_0^x K(x,z)b(z)dz \tag{1.2}$$

where, $A(x) = \dfrac{a'(x)}{k(x,x)}$ and $K(x,z) = -\dfrac{k_x(x,z)}{k(x,x)}$.

The existence of a solution to this kind of equation is well-studied and the most fundamental result due to Vito Volterra is stated as below:

Theorem (Volterra, 1896). *If the kernel $\tilde{K}(x,z)$ and the given function $\tilde{A}(x)$ of the integral equation:*

$$b(x) = \tilde{A}(x) + \int_0^x \tilde{K}(x,z)b(z)dz$$

are continuous on their respective domains, then it possesses a unique continuous solution.

Note that Volterra's theorem can not be directly applied to the equation (1.2) of our interest, since the kernel K is discontinuous at $(0,0)$ as:

$$\int_0^x k(x,z)dz = 1 \quad \Rightarrow \quad \int_0^x K(x,z)dz = 1$$

for all $0 \leq x \leq 1$. Having stated that we now prove Theorem 1.

Proof of Theorem 1. First we show the uniqueness and continuity of the solution, and then its monotonicity. Since $k(x,z) = \dfrac{g(x,z)}{\int_0^x g(x,z)dz}$, the equation (1.1) can be written as:

$$\int_0^x g(x,z)b(z)dz = a(x)\int_0^x g(x,z)dz$$

We differentiate the last equation to get the following equation of the second kind:

$$b(x) = \frac{1}{g(x,x)}\frac{d}{dx}\left(a(x)\int_0^x g(x,z)dz\right) + \int_0^x \left(-\frac{g_x(x,z)}{g(x,x)}\right)b(z)dz$$

to which Volterra's theorem can be applied, and we therefore claim that our equation (1.1) possesses a unique continuous solution $b(\cdot)$.

Next, we show the monotonicity of the function $b(\cdot)$. Here we adopt the idea of *Picard iteration* to express the function $b(\cdot)$ as a summation of a sequence of functions, and then show that each term in the summation is non-decreasing function, so that $b(\cdot)$ is non-decreasing. Recall that the function $b(\cdot)$ satisfies the equation (1.2) which is rewritten below:

$$b(x) = A(x) + \int_0^x K(x,z)b(z)dz \tag{1.3}$$

First, define the functions so-called *iterated kernels*, for all $n \geq 1$, as:

$$K_1(x,z) \equiv K(x,z)$$
$$K_{n+1}(x,z) \equiv \int_z^x K(x,t)K_n(t,z)dt$$

and then, substitute the equation (1.3) into itself,

$$\begin{aligned}
b(x) &= A(x) + \int_0^x K(x,z)b(z)dz \\
&= A(x) + \int_0^x K(x,z)\left(A(z) + \int_0^z K(z,t)b(t)dt\right)dz \\
&= A(x) + \int_0^x K(x,z)A(z)dz + \int_0^x \int_0^z K(x,z)K(z,t)b(t)dtdz \\
&= A(x) + \int_0^x K_1(x,z)A(z)dz + \int_0^x \left(\int_t^x K(x,z)K_1(z,t)dz\right)b(t)dt \\
&= A(x) + \int_0^x K_1(x,z)A(z)dz + \int_0^x K_2(x,t)b(t)dt
\end{aligned}$$

By doing this substitution repeatedly, we get following identity:

$$b(x) = A(x) + \sum_{n=1}^{\infty} \underbrace{\int_0^x K_n(x,z)A(z)dz}_{\equiv \varphi_n(x)}$$

$$\Rightarrow \quad b(x) = A(x) + \sum_{n=1}^{\infty} \varphi_n(x) \tag{1.4}$$

Finally, we prove the following claims:

Claim 1. $A(\cdot)$ is increasing.

Claim 2. $\int_0^z K(x,t)dt$ is non-increasing in x.

Claim 3. $\varphi_1(\cdot)$ is non-decreasing.

Claim 4. All $\varphi_n(\cdot)$ are non-decreasing.

Claim 1 immediately follows from the assumptions of Theorem 1, and Claim 2 is a consequence of the affiliation assumption. And, Claim 3 is verified as follows:

$$
\begin{aligned}
\varphi_1(x) &= \int_0^x K(x,z)A(z)dz \\
&= \left(A(z)\int_0^z K(x,t)dt\right)\Bigg|_0^x - \int_0^x \left(\int_0^z K(x,t)dt\right)A'(z)dz \\
&= A(x) - \int_0^x \left(\int_0^z K(x,t)dt\right)A'(z)dz \\
\varphi_1'(x) &= A'(x) - \left(\int_0^x K(x,t)dt\right)A'(x) - \int_0^x \frac{\partial}{\partial x}\left(\int_0^z K(x,t)dt\right)A'(z)dz \\
&= -\int_0^x \frac{\partial}{\partial x}\left(\int_0^z K(x,t)dt\right)A'(z)dz \geq 0
\end{aligned}
$$

The last inequality follows from Claim 1 and 2. Finally, notice that the functions φ_n satisfies the following recurrence relation:

$$
\begin{aligned}
\varphi_{n+1}(x) &= \int_0^x K_{n+1}(x,z)A(z)dz \\
&= \int_0^x \left(\int_z^x K(x,t)K_n(t,z)dt\right)A(z)dz \\
&= \int_0^x K(x,t)\left(\int_0^t K_n(t,z)A(z)dz\right)dt \\
&= \int_0^x K(x,t)\varphi_n(t)dt
\end{aligned}
$$

for all $n \geq 1$. Hence, by induction, Claim 4 is concluded by using the similar arguments in the proof of Claim 3.

Now since each term in the summation (1.4) is non-decreasing and $A(\cdot)$ is increasing, the solution function $b(\cdot)$ must be increasing. This completes the proof of Theorem 1. $\square$

1.3 The Model

We mainly follow the model and notation of Milgrom and Weber (1982). There is a single object to be auctioned and there are n potential buyers (*bidders*) of the object. Each bidder i receives a real-valued private *signal* X_i, prior to the auction that affects the value of the object. And let $S = (S_1, S_2, \ldots, S_n)$ be a vector of additional real-valued variables which influence the value of the object to the bidders but are not observed by any bidder. The actual value of the object to bidder i is then

$$V_i = u(S, X_i, \{X\}_{j \neq i})$$

where $u : \mathbb{R}^{n+m} \to \mathbb{R}$ is non-negative and symmetric in its last $n-1$ variables, and is continuous and non-decreasing in its all variables.

The random variables $S_1, \ldots, S_m, X_1, \ldots, X_n$ are *affiliated*, and assume that the support of the random variables is bounded. Then without loss of generality, we assume that the support is $[0,1]^{n+m}$. Let f be the joint density function of the random elements, and it is assumed to be positive and symmetric in its last n arguments.

Let Y_1 and Y_2 denote the largest and the second largest elements from among the random variables $X_2, X_3, \ldots, X_n$. And let $f_{Y_1}(\cdot|x)$ and $F_{Y_1}(\cdot|x)$ denote the conditional density and the cumulative distribution functions of Y_1 given that $X_1 = x$. Similarly, let $f_{Y_2}(\cdot|x, y)$ and $F_{Y_2}(\cdot|x, y)$ denote the conditional density and the cumulative distribution functions of Y_2 given that $X_1 = x$ and $Y_1 = y$.

Also define $v(x, y) \equiv E[V_1|X_1 = x, Y_1 = y]$. Here we state a few consequences of the affiliation assumption that will be useful later for our analysis. The proof of these facts can be found in Milgrom and Weber (1982).

Fact 1. Random variables X_1, Y_1, and Y_2 are affiliated.

Fact 2. $F_{Y_2}(\cdot|x, y)$ first-order stochastically dominates $F_{Y_2}(\cdot|x', y')$ whenever $(x, y) \geq (x', y')$.

Fact 3. $v(x, y)$ is a non-decreasing function in its arguments.

Finally, we make following additional assumptions to guarantee the existence of a monotone equilibrium in the third-price auctions:

Assumption 1. *$v(x,x)$ is convex in x.*

Assumption 2. *$f_{Y_2}(x|x,x)$ is decreasing in x.*

1.4 Equilibrium in the Third-price Auction

In this section, we characterize the symmetric and monotone equilibrium in a third-price auction where the seller collects the third-highest bid from the winner who submits the highest bid.

Theorem 2. *There exists a unique equilibrium in a third-price auction that is symmetric and increasing. Moreover, the equilibrium strategy $b(\cdot)$ is completely characterized by the following condition:*

$$\int_0^x b(z) f_{Y_2}(z|x,x) = v(x,x) \tag{1.5}$$

Proof. By Theorem 1, the equation (1.5) has a unique solution which is, in fact, increasing. We show that this condition is necessary and sufficient for its solution $b(\cdot)$ to be an equilibrium in a third-price auction.

Necessity: Let an increasing function $b(\cdot)$ be an equilibrium strategy. Suppose that every bidder, except bidder 1, adopts the strategy $b(\cdot)$. If bidder 1 who receives signal $X_1 = x$ bids $b(t)$, then his expected payoff is given by:

$$\begin{aligned}\Pi(t;x) &= \int_0^t \left(v(x,y) - E\Big[b(Y_2)|X_1 = x, Y_1 = y\Big]\right) f_{Y_1}(y|x)dy \\ &= \int_0^t \left(v(x,y) - \int_0^y b(z) f_{Y_2}(z|x,y)dz\right) f_{Y_1}(y|x)dy\end{aligned}$$

which must be maximized at $t = x$. That is,

$$\frac{\partial}{\partial t}\Pi(t;x)\Big|_{t=x} = 0$$

And, the last equation is equivalent to our condition (1.5).

Sufficiency: Let $b(\cdot)$ be the unique solution to the equation (1.5). And suppose that every bidder, except bidder 1, adopts the strategy $b(\cdot)$. If bidder 1 bids $b(t)$ when his signal is $X_1 = x$, then his expected payoff is:

$$\Pi(t;x) = \int_0^t \underbrace{\left[v(x,y) - \int_0^y b(z) f(z|x,y) dz \right]}_{\equiv\, p(x,y)} f(y|x) dy$$

We now prove that:

$$\begin{cases} p(x,y) \geq 0, \text{ for all } \; y < x \\ p(x,y) \leq 0, \text{ for all } \; y > x \end{cases}$$

so that $\Pi(t;x)$ is maximized at $t = x$, and the best response of bidder 1 is then to bid $b(x)$ when he receives signal $X_1 = x$.

Here we provide a proof of above statement for a special case of the private value auction in which $v(x,y) = x$. In this case:

$$\begin{aligned} p(x,y) &= v(x,y) - \int_0^y b(z) f(z|x,y) dz \\ &\geq v(x,y) - \int_0^x b(z) f(z|x,x) dz \\ &= v(x,y) - v(x,x) = 0 \end{aligned}$$

for all $y < x$. Similarly, for all $y > x$:

$$\begin{aligned} p(x,y) &= v(x,y) - \int_0^y b(z) f(z|x,y) dz \\ &\leq v(x,y) - \int_0^x b(z) f(z|x,x) dz \\ &= v(x,y) - v(x,x) = 0 \end{aligned}$$

The above inequalities follow from Fact 2. $\square$

1.5 Revenue Comparison

In this section, we examine the performance of a third-price auction in terms of the expected revenue of the seller compared with a first-price auction and with a second-price auction.

Theorem 3. *The expected revenue of the seller from a third-price auction is greater than or equal to the expected revenue from a first-price auction and a second-price auction.*

Proof. Here we show that the third-price auction performs better than the second-price auction. Consequently, it performs better than the first-price auction since Milgrom and Weber (1982) shows that the second-price auction performs better than the first-price auction.

In a second-price auction, the equilibrium bid of bidder 1 who receives a signal of $X_1 = x$ is $v(x, x)$, and thus his expected payment is:

$$e^{II}(x) = \int_0^x v(y, y) f_{Y_1}(y|x) dy$$

In a third-price auction, the expected payment of the same bidder (bidder 1) who receives a signal of $X_1 = x$ is:

$$\begin{aligned} e^{III}(x) &= \int_0^x E\Big[b(Y_2)|X_1 = x, Y_1 = y\Big] f_{Y_1}(y|x) dy \\ &= \int_0^x \left(\int_0^y b(z) f_{Y_2}(z|x, y) dz \right) f_{Y_1}(y|x) dy \\ &\geq \int_0^x \left(\int_0^y b(z) f_{Y_2}(z|y, y) dz \right) f_{Y_1}(y|x) dy \\ &= \int_0^x v(y, y) f_{Y_1}(y|x) dy = e^{II}(x) \end{aligned}$$

The inequality follows from Fact 2, and the second last equality follows from Theorem 1. This completes the proof. □

Chapter 2

Licensing of a Technology with Unknown Use[1]

[1]This is joint work with Bruno D. Badia

Abstract

Suppose an inventor holds the patent of a technology that could potentially reduce the costs of firms operating in a given industry. Also assume that inventor and licensed firms could each discover, with some probability, the cost reducing use of this technology. The inventor thus face the problem: should he first try to discover the use for the technology and then license it, or should he license the technology before a use has been discovered, leaving the discovery task to the licensees? We show that the answer to this question depends on how discovery by each agent is related to discovery by other agents. If discovery is independent across agents, then the inventor is better-off choosing the former alternative. If, on the other hand, discovery is fully correlated across agents, then the inventor should optimally choose the latter alternative, even when costs associated to a trial are absent. We also study the effect of these choices on the expected number of firms operating with a reduced cost, our measure of technology diffusion. We show that the inventor's choice is not necessarily the alternative leading to the highest diffusion of the technology.

Keywords: Process innovation; Patent licensing; Cournot competition
JEL classification: D43; D45; L13

2.1 Introduction

Empirical evidence suggests that the search for uses and improvements to patented technologies still occurs during the first few years of a patent's life. For instance, Pakes (1986) shows that in France and Germany the returns to holding a patent increases for some years after the patent has been granted. Furthermore, Boldrin and Levine (2013) argue that the increasing number of licenses issued each year in the United States is not being followed by a correspondent productivity growth in the country's economy.[2] Both phenomena are arguably associated with the fact that

[2]For precise figures on the number of patents issued yearly in the United States, see Lemley and Shapiro (2005) and Boldrin and Levine (2013), and the references thereof.

many technologies are patented before a mature stage of their development has been reached. As time passes, (alternative) uses and enhancements to these technologies may eventually be discovered.

Therefore, in principle, licensing of a new technology could take place in environments where uses for it are, to some extent, unknown. In this paper we consider the problem facing an inventor who holds the patent of a technology with unknown use.

In particular, suppose an outside inventor holds the patent of a technology that could potentially be used to reduce the costs of firms operating in a Cournot industry. Suppose further that inventor and licensed firms could discover, with some probability, the cost reducing use of this technology. Finally, assume that the inventor has decided to license the technology by means of an auction. Under these circumstances, we address the following question: should the inventor first try to discover the cost reducing use of his technology and only then license it, or should he license the technology as soon as he is granted the patent, leaving to licensees the task of discovery?

The answer to this question depends on how discovery by any agent is related to discovery by other agents. More specifically, we consider two possible scenarios. In the *independent discoveries scenario*, use discovery is independent across agents. That is, the probability that any agent discovers the use for the technology is not affected by the success or failure of other agents at this enterprise. In the (fully) *correlated discoveries scenario*, if one agent succeeds (fails) at discovering the use for the technology, then the probability that any other agent discovers it is one (zero).

These scenarios can be interpreted as representing two distinct industry structures. For instance, in the independent discoveries scenario, in spite of firms initially having the same marginal cost, firms' technologies are assumed to be heterogeneous. Thus, uses for the new technology are firm-specific, and discovery is likely to be independent across firms (and the inventor). Differently, in the correlated discoveries scenario, firms are viewed as being homogeneous, i.e. their common marginal cost is derived from the same technology. In this context, a firm discovering the use for the invention is likely to be certain about the other firms also discovering it.

We show that in the independent discoveries scenario the inventor should try to find the use before licensing, whereas in the correlated discoveries scenario the opposite holds. We notice that the latter is true even when there are no costs associated to a trial. Intuitively, in the independent discoveries scenario, a failure by the inventor does not alter the value firms attribute to the technology. Hence, the inventor can only gain by trying to discover the use before licensing: if he succeeds, he will license a stronger (and therefore more valuable) technology; if he fails, the terms of trade remain the same. The intuition for the correlated discoveries scenario is not as clear. If the inventor succeeds, he again licenses a more valuable technology. However, if he fails, the value firms attribute to the technology is updated to zero. The balance of these forces will determine the inventor's behavior.

We also study the effect of the inventor's choices on the expected number of firms operating with a reduced cost, our measure of technology diffusion. We show that, in the independent discoveries scenario, the inventor does not necessarily choose the alternative leading to the highest diffusion of the technology. That is, higher diffusion would be achieved if the inventor did not try to discover the use for his technology before licensing. In the correlated discoveries scenario, the inventor's choice is always the one associated with the largest expected number of firms producing with reduced costs.

The model we analyze is close in spirit to those in Kamien and Tauman (1986), Kamien et al. (1992), and Sen and Tauman (2007), among others, in that it takes the Cournot industry structure in which the potential licensees operate explicitly into account. Kamien (1992) provides a review of the basic model. Our model extends the previous literature in that it allows for licensing to take place in an environment where neither the inventor nor the firms are certain about the cost reducing use of the new technology.

Different from previous studies, we do not consider the problem of choosing among different licensing strategies. Instead, in order to focus on the question we pose above, we assume that the inventor has exogenously chosen to license his technology by means of an auction. This assumption is justified by the fact that usually auction licensing revenue-dominates other licensing mechanisms.[3] An interesting question,

[3]See Sen (2005) for an illuminating discussion on the comparison of revenues from different

that we do not address in this paper, is how the revenue from different licensing mechanisms relate in our setting.

It is worth noticing that in our model, even though the use of the patented technology is, to some degree, unknown, the patent does give the inventor complete rights over the technology. A recent literature on "probabilistic" (or "weak") patents considers situations where these rights are uncertain.[4]

The paper is organized as follows. In the next section we introduce the main elements of our model. In section 2.3 we analyze the game arising from the assumption that discoveries happen independently across agents. In section 2.4 we make the assumption that discoveries are fully correlated across agents and study the resulting game. In section 2.5 we present our concluding remarks.

2.2 The model

Consider an industry with $n \geq 2$ firms producing a homogeneous good and competing in quantities. To produce quantity q_i, firm i incurs cost $c_i(q_i) = c_H q_i$. The market inverse demand for the homogeneous good is given by $p(Q) = \max\{a - Q, 0\}$, where $Q = \sum_{j=1}^{n} q_j$.

An outside inventor holds the patent of a technology that could potentially reduce firms' marginal costs to $c_L < c_H$. Specifically, any agent (i.e. firms and inventor) with access to the patented technology succeeds at discovering its cost reducing use with unconditional probability $\alpha \in (0, 1)$, and fails with the remaining probability.[5] In our analysis we consider the licensing of this technology under two distinct scenarios.

In the *independent discoveries* (ID) scenario, the probability that an agent succeeds at discovering the use for the technology conditional on some other agent's

licensing strategies. See Sen and Tauman (2007) for a discussion on optimal licensing strategies.

[4]See, for instance, Lemley and Shapiro (2005), Farrell and Shapiro (2008), and Amir et al. (2013).

[5]We assume that, for any agent, trying to discover the use for the technology carries no cost. In section 2.5 we indicate how our results would change in the presence of a fixed cost associated to a trial.

outcome is given by alpha. That is, for any agents i and j we have

$$\Pr\left\{i \text{ succeeds} \mid \omega_j\right\} = \alpha, \tag{2.1}$$

where $\omega_j \in \{\text{success}, \text{failure}\}$ is the outcome of agent j's trial at trying to discover the use.

In the *correlated discoveries* (CD) scenario, the conditional probability (2.1) is either one or zero, corresponding to $\omega_j = \text{success}$ or $\omega_j = \text{failure}$, respectively.[6]

Suppose the inventor has decided to auction licenses to the firms in a first-price sealed-bid auction in which ties are randomly resolved with even probabilities. A licensing strategy to the inventor, therefore, constitutes of a number $k \in \{0, 1, \ldots, n\}$ of licenses to be auctioned and sold to the k highest bidders. Before the auction takes place, however, the inventor has a decision to make: either he tries to discover the use for his invention (alternative a) or he leaves this task to the licensed firms (alternative b). Each of these decisions gives rise to a distinct game, Γ^a or Γ^b, respectively. The game Γ^a, in turn, has two relevant subgames for our analysis. The game Γ^{a_s} follows a successful attempt by the inventor; the game Γ^{a_f} follows a failure. In our analysis we assume that firms observe whether the inventor has chosen alternative a or alternative b. We also assume that, following the choice of alternative a by the inventor, firms observe whether he is successful or not in his attempt to discover the use for the technology. The situation described above is illustrated in Figure 2.1.

Each of the games Γ^{a_s}, Γ^{a_f}, and Γ^b, has the inventor and the firms as players, and happens in three stages. In the first stage the inventor announces a number $k \in \{0, 1, \ldots, n\}$ of licenses to be auctioned. In the second stage, firms simultaneously offer bids. The k highest bidders win the licenses, paying the respective bids to the inventor. The set of firms then partitions into the sets of k licensees and $n - k$ nonlicensees and, in the third stage, Cournot competition takes place. The inventor's payoff is given by the revenue he obtains in the auction. The firms' payoffs are given by their Cournot profits net of bid expenses (if any).

Clearly, some aspects of these games may change as we change the scenario (ID or

[6] Similarly, $\Pr\left\{i \text{ fails} \mid \omega_j = \text{failure}\right\} = 1$.

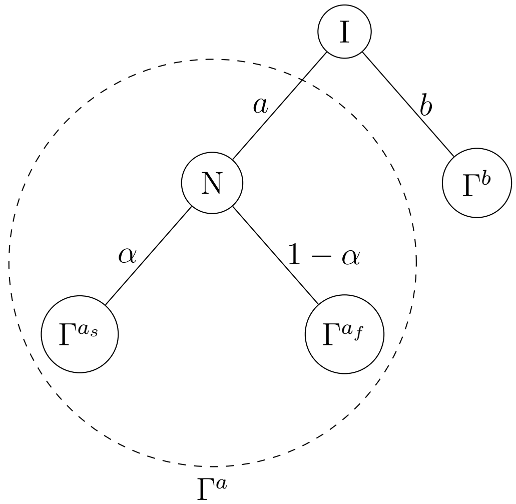

Figure 2.1: The game tree. “I” stands for “innovator”, “N” for “nature”.

CD) under consideration. These details are explained below, in the relevant sections.

To carry our analysis we adopt the subgame-perfect equilibrium solution concept. Thus, we study the above games using backward induction.

Before proceeding, we introduce two simple notations. For each $\alpha \in (0, 1]$, we define $\varepsilon_\alpha = \alpha(c_H - c_L)$ and $k_\alpha = (a - c_H)/\varepsilon_\alpha$.

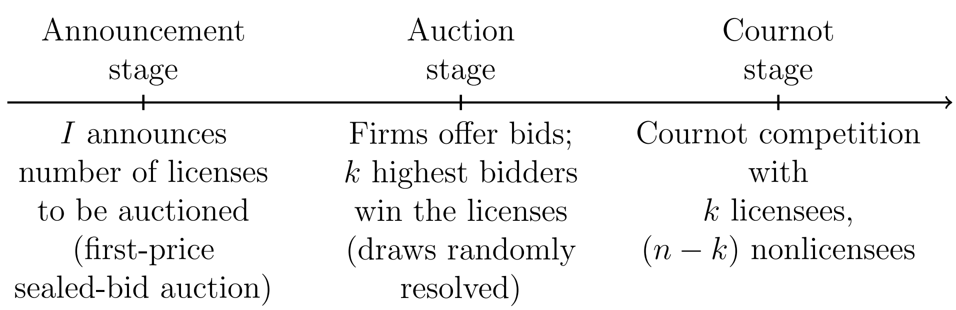

Figure 2.2: Timing in Γ^{a_s}, Γ^{a_f}, and Γ^b.

2.3 The independent discoveries scenario

2.3.1 The game Γ^a

As noted above, two relevant subgames, Γ^{a_s} and Γ^{a_f}, unfold from Γ^a. We analyze each one in turn.

The game Γ^{a_s}

Suppose the inventor has succeeded in discovering the use for his patented technology. The game following this event has been extensively analyzed in the literature.[7] After the auction takes place the set of firms is partitioned into the subsets of k licensees and $n-k$ nonlicensees. Cournot competition then happens with each licensee having marginal cost c_L and each nonlicensee having marginal cost c_H. Let $q^{a_s}(k)$ and $q_\ell^{a_s}(k)$ denote the Cournot equilibrium quantities of nonlicensees and licensees, respectively, when there are k licensees. One can show that

$$q^{a_s}(k) = \varepsilon_1 \cdot \begin{cases} \frac{k_1-k}{n+1}, \text{ if } k < k_1 \\ 0, \text{ if } k_1 \leq k, \end{cases}$$

and

$$q_\ell^{a_s}(k) = \varepsilon_1 \cdot \begin{cases} \frac{k_1-k}{n+1} + 1, \text{ if } k < k_1 \\ \frac{k_1+1}{k+1}, \text{ if } k_1 \leq k. \end{cases}$$

One can also show that, for each k, the Cournot equilibrium profits, $\pi^{a_s}(k)$, and $\pi_\ell^{a_s}(k)$, are given by the squares of these quantities.

Now, since firms are symmetric, in the auction stage of the game they will all submit the same bid. Because a licensee's payoff is given by its Cournot profit minus its bid and a nonlicensee's payoff is simply its Cournot profit, it follows that the

[7]We follow roughly the exposition in Kamien (1992).

equilibrium bid submitted by firms, $\beta^{a_s}(k)$, is given by[8]

$$\beta^{a_s}(k) = \begin{cases} \pi_\ell^{a_s}(k) - \pi^{a_s}(k), \text{ if } k < n \\ \pi_\ell^{a_s}(k) - \pi^{a_s}(k-1), \text{ if } k = n. \end{cases} \tag{2.2}$$

Given k, it is clear that a licensee would not bid more than $\beta^{a_s}(k)$, for by increasing its bid it would still get the license, however lowering its payoff. On the other hand, by bidding below $\beta^{a_s}(k)$ it would become a nonlicensee, not benefiting from a payoff increase. Similarly, nonlicensees have no incentives to deviate from $\beta^{a_s}(k)$.

From the above considerations, it follows that a subgame-perfect equilibrium strategy for the inventor must involve a choice of k solving

$$\begin{aligned} &\underset{k}{\text{maximize}} \quad k\beta^{a_s}(k) \equiv \rho^{a_s}(k) \\ &\text{s.t.} \qquad\quad\; k \in \{0, 1, \ldots, n\}, \end{aligned} \tag{2.3}$$

where

$$\beta^{a_s}(k) = \varepsilon_1^2 \cdot \begin{cases} \frac{2(k_1-k)}{n+1} + 1, \text{ if } 1 \leq k < k_1 \\ \left(\frac{k_1+1}{k+1}\right)^2, \text{ if } k_1 \leq k. \end{cases}$$

We denote by k^{a_s} the solution to the above problem.

The game Γ^{a_f}

In this subgame firms obtaining a license in the auction stage succeed to reduce costs independently, each with probability α. The information on whether a licensee has succeeded or not is kept private by the licensee. It then follows from (2.1) that in the Cournot competition stage, each firm believes that each licensee has marginal cost c_L with probability α and c_H with probability $1-\alpha$. Of course, each firm believes that each nonlicensee has marginal cost c_H.

[8]Whenever the inventor intends to auction $k = n$ licenses he should also require from firms the minimum bid reported in (2.2), otherwise no firm would place a positive bid: by bidding zero, any firm would be among the n-highest bidders. From now on, for simplicity, we focus on the case $k < n$. We observe that this does not change our results.

The situation just described defines a Bayesian game played by the firms. Firm i's type space consists of c_L and c_H if it is a licensee, and only c_H if it is a nonlicensee. Suppose there are k licensees. Given a profile of other firms' marginal costs having j entries equal to c_L, firm i, conditional on its own marginal cost, assigns probability $\alpha^j(1-\alpha)^{\tilde{k}-j}$ to it, where $\tilde{k}=k$ if i is a nonlicensee and $\tilde{k}=k-1$ if i is a licensee. In particular, given i's marginal cost, i's belief that exactly j licensees have succeeded is given by

$$\binom{\tilde{k}}{j}\alpha^j(1-\alpha)^{\tilde{k}-j}.$$

Strategies and payoffs are defined in an obvious manner and this structure is common knowledge.

We denote by $q^{a_f}(k;\alpha)$ the (Bayesian) equilibrium quantity produced by nonlicensees. Similarly, $q^{a_f}_{\ell,H}(k;\alpha)$ and $q^{a_f}_{\ell,L}(k;\alpha)$ denote the equilibrium quantities produced by the high and low cost types, respectively, of each licensee. The Cournot equilibrium in the present case is characterized in the following lemma.

Lemma 1. *Consider the independent discoveries scenario. The Cournot game played by the firms in Γ^{a_f} has a unique (Bayesian) equilibrium. Equilibrium quantities are given by*

$$q^{a_f}(k;\alpha)=\varepsilon_\alpha\cdot\begin{cases}\frac{k_\alpha-k}{n+1}, & \text{if } k<k_\alpha\\ 0, & \text{if } k_\alpha\le k,\end{cases}$$

$$q^{a_f}_{\ell,H}(k;\alpha)=\varepsilon_\alpha\cdot\begin{cases}\frac{k_\alpha-k}{n+1}+\frac{1}{2}, & \text{if } k<k_\alpha\\ \frac{2k_\alpha+1-k}{2(k+1)}, & \text{if } k_\alpha\le k<2k_\alpha+1\\ 0, & \text{if } 2k_\alpha+1\le k,\end{cases}$$

and

$$q^{a_f}_{\ell,L}(k;\alpha)=\varepsilon_\alpha\cdot\begin{cases}\frac{k_\alpha-k}{n+1}+\frac{1+\alpha}{2\alpha}, & \text{if } k<k_\alpha\\ \frac{2k_\alpha+1-k}{2(k+1)}+\frac{1}{2\alpha}, & \text{if } k_\alpha\le k<2k_\alpha+1\\ \frac{k_\alpha+1/\alpha}{2+\alpha(k-1)}, & \text{if } 2k_\alpha+1\le k.\end{cases}$$

Moreover, the Cournot equilibrium profits, $\pi^{a_f}(k;\alpha)$, $\pi^{a_f}_{\ell,H}(k;\alpha)$, and $\pi^{a_f}_{\ell,L}(k;\alpha)$, *are given by the square of the corresponding equilibrium quantities.*

Proof. See Appendix 2.A. □

From now on, for simplicity, we suppress from our notation the dependence of the quantities given in Lemma 1 on α. Hence, we write $q^{a_f}(k)$ instead of $q^{a_f}(k;\alpha)$, and so on, and do the same for corresponding profits.

As in the Γ^{a_s}, in the stage preceding the Cournot competition, the inventor announces a number k of licenses to be sold to the k highest bidders in an auction. Symmetry implies that, given the announcement k, firms in equilibrium will place the same bid $\beta^{a_f}(k;\alpha)$ given by

$$\beta^{a_f}(k;\alpha) = \begin{cases} \mathrm{E}_\alpha[\pi_\ell^{a_f}(k)] - \pi^{a_f}(k), \text{ if } k < n \\ \mathrm{E}_\alpha[\pi_\ell^{a_f}(k)] - \pi^{a_f}(k-1), \text{ if } k = n, \end{cases} \tag{2.4}$$

where, for each $k \in \{1,\ldots,n\}$,

$$\mathrm{E}_\alpha[\pi_\ell^{a_f}(k)] = \alpha(q_{\ell,L}^{a_f}(k))^2 + (1-\alpha)(q_{\ell,H}^{a_f}(k))^2.$$

As for the Cournot equilibrium quantities and profits, we write $\beta^{a_f}(k)$, suppressing the dependence of β^{a_f} on α. The proof that, for each k, firms place bid $\beta^{a_f}(k)$ in equilibrium in the auction stage, follows a line of argument similar to that given in subsection 2.3.1. For instance, a licensee would not bid more than $\beta^{a_f}(k)$, for its expected payoff would decrease, whereas by bidding less its gains would be unchanged. Similarly, none of these deviation would increase a licensee's expected payoff.[9]

As in subsection 2.3.1, the considerations thus far imply that the inventor's equilibrium choice of k should be a solution to the following problem

$$\begin{aligned} &\underset{k}{\text{maximize}} && k\beta^{a_f}(k) \equiv \rho^{a_f}(k) \\ &\text{s.t.} && k \in \{0,1,\ldots,n\}, \end{aligned} \tag{2.5}$$

[9]For the case $k = n$ see footnote 8.

where

$$\beta^{a_f}(k) = \varepsilon_\alpha^2 \cdot \begin{cases} \frac{2(k_\alpha - k)}{n+1} + \frac{1+3\alpha}{4\alpha}, & \text{if } k < k_\alpha \\ \left(\frac{k_\alpha+1}{k+1}\right)^2 + \frac{1-\alpha}{4\alpha}, & \text{if } k_\alpha \leq k < 2k_\alpha + 1 \\ \alpha\left(\frac{k_\alpha + 1/\alpha}{2+\alpha(k-1)}\right)^2, & \text{if } 2k_\alpha + 1 \leq k \leq n, \end{cases}$$

by (2.4), Lemma 1, and some algebra.

We denote by k^{a_f} the solution to the above problem.

2.3.2 The game Γ^b

It is easy to see that Γ^b is equivalent Γ^{a_f}. In particular, in the Cournot stage firms' beliefs are as described in subsection 2.3.1. This is so because, in the ID scenario, a failure by the inventor (in Γ^a) does not alter the perceived likelihood that each (licensed) firm succeeds at discovering the use for the patented invention, as stated in equation (2.1).

To keep our notation consistent, we write a b superscript for equilibrium values of the endogenous variables. Hence, $q^b(\cdot) = q^{a_f}(\cdot)$ stands for the Cournot equilibrium output produced by nonlicensees in the third stage of Γ^b. Similarly, $q^b_{\ell,H}(\cdot) = q^{a_f}_{\ell,H}(\cdot)$ and $q^b_{\ell,H}(\cdot) = q^{a_f}_{\ell,L}(\cdot)$ denote the equilibrium outputs of high and low cost licensees; $\beta^b(\cdot) = \beta^{a_f}(\cdot)$ denotes the equilibrium bid in the auction stage; and $k^b = k^{a_f}$ the solution to the inventor's problem.

The next result identifies the alternative (a or b) that should be chosen by the inventor in his first move.

Proposition 2. *In the independent discoveries scenario,*

$$\alpha\rho^{a_s}(k^{a_s}) + (1-\alpha)\rho^{a_f}(k^{a_f}) \geq \rho^b(k^b).$$

That is, the expected revenue to the inventor from alternative a is at least the revenue the inventor obtains from alternative b.

Proof. See appendix 2.A. □

The next result deals with the question of technological diffusion. In particular, we ask whether the inventor's choice identified above lead to a more efficient industry configuration. Since the use of the invention is unknown to begin with, we do not measure diffusion by the (expected) number of licensees. Instead we focus on the expected number of firms operating with low marginal cost technology.

We say a firm is *efficient* if it operates with the low marginal cost technology. For each game $\Gamma \in \left\{\Gamma^a, \Gamma^b\right\}$ we denote by $ENEF(\Gamma)$ the expected number of efficient firms in Γ. We then have

Proposition 3. *Consider the independent discoveries scenario.*

1. *If $k_\alpha \leq k^b$, then $ENEF(\Gamma^a) \leq ENEF(\Gamma^b)$.*
2. *If $k^b < k_\alpha$, then $ENEF(\Gamma^a) \geq ENEF(\Gamma^b)$.*

Proof. See appendix 2.A. □

Proposition 3 says that if the solution to the inventor's problem in Γ^b is relatively large in comparison to the minimum number of licensees required to drive nonlicensees out of the industry, then alternative b is the alternative leading to the highest expected number of efficient firms. On the other hand, if k^b is relatively small, then alternative a is the alternative that carries this distinction.

Hence, the alternative chosen by the inventor (alternative a by Proposition 2) is not necessarily the one associated with the highest diffusion of the technology.

Next, we turn to the analysis of the CD scenario.

2.4 The correlated discoveries scenario

Recall that in the CD scenario, for any players i and j, and outcomes $\omega_i, \omega_j \in \{\text{success}, \text{failure}\}$, it is common knowledge that

$$\Pr\left\{\omega_i \mid \omega_j\right\} = \begin{cases} 1, & \text{if } \omega_i = \omega_j \\ 0, & \text{if } \omega_i \neq \omega_j. \end{cases}$$

In particular, if the inventor tries to discover the use for his technology and fails, then firms attribute probability zero to the event that any of them, becoming a licensee, will discover the use.

2.4.1 The game Γ^a

The above observation implies that no licensing occurs in Γ^{a_f}. Thus, in this subgame the equilibrium payoff to the inventor is zero, whereas the equilibrium payoff to each firm is given by its (homogeneous) Cournot profit. As for Γ^{a_s}, it is easily seen that this subgame is the same as Γ^{a_s} in the ID scenario, analyzed in subsection 2.3.1. These observations conclude the analysis of Γ^a in the CD scenario.

2.4.2 The game Γ^b

The analysis here is similar to the one carried in subsection 2.3.1. However, the Cournot stage differs from that summarized in Lemma 1. In the present case, every firm is informed of nonlicensees' marginal costs, c_H. Furthermore, licensees are also informed of each others' costs, since the probability they attribute to the event that all others succeed (fail) conditional on own cost is either one (in case own cost is c_L) or zero (in case own cost is c_H). Nonlicensees, in turn, attribute probability α, respectively $1-\alpha$, to the event that all licensees have marginal cost c_L, respectively c_H. This structure is common knowledge among the firms. From the discussion in subsection 2.3.1 it is clear that this environment defines a Bayesian game between the firms. The following lemma characterizes equilibrium in the Cournot stage.

Lemma 4. *Consider the correlated discoveries scenario. The Cournot game played by the firms in Γ^b has a unique (Bayesian) equilibrium. Equilibrium quantities are given by*

$$q^b(k;\alpha) = \varepsilon_\alpha \cdot \begin{cases} \frac{k_\alpha - k}{n+1}, & \text{if } k < k_\alpha \\ 0, & \text{if } k_\alpha \leq k, \end{cases}$$

$$q^b_{\ell,H}(k;\alpha) = \varepsilon_\alpha \cdot \begin{cases} \frac{k_\alpha - k}{n+1} + \frac{k}{k+1}, & \textit{if } k < k_\alpha \\ \frac{\varepsilon_\alpha k_\alpha}{k+1}, & \textit{if } k_\alpha \leq k, \end{cases}$$

and

$$q^b_{\ell,L}(k;\alpha) = \varepsilon_\alpha \cdot \begin{cases} \frac{k_\alpha - k}{n+1} + \frac{k+1/\alpha}{k+1}, & \textit{if } k < k_\alpha \\ \frac{k_\alpha + 1/\alpha}{k+1}, & \textit{if } k_\alpha \leq k. \end{cases}$$

Moreover, the Cournot equilibrium profits, $\pi^b(k;\alpha)$, $\pi^b_{\ell,H}(k;\alpha)$, and $\pi^b_{\ell,L}(k;\alpha)$, are given by the square of the corresponding equilibrium quantities.

Proof. See Appendix 2.B. □

The equilibrium bid by the firms in the auction stage can be easily seen to be

$$\beta^b(k;\alpha) = \begin{cases} \mathrm{E}_\alpha[\pi^b_\ell(k)] - \pi^b(k), & \text{if } k < n \\ \mathrm{E}_\alpha[\pi^b_\ell(k)] - \pi^b(k-1), & \text{if } k = n, \end{cases} \tag{2.6}$$

where, as in section 2.3, we again suppressed the dependence of Cournot profits on α.

Using (2.6) and Lemma 4 we then obtain

$$\beta^b(k) = \varepsilon^2_\alpha \cdot \begin{cases} \frac{2(k_\alpha - k)}{n+1} + 1 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{k+1}\right)^2, & \text{if } k < k_\alpha \\ \left(\frac{k_\alpha+1}{k+1}\right)^2 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{k+1}\right)^2, & \text{if } k_\alpha \leq k. \end{cases}$$

Finally, we observe that the equilibrium number of licenses to be auctioned by the inventor, k^b, is, therefore, the solution to

$$\begin{aligned} &\underset{k}{\text{maximize}} && \rho^b(k) \\ &\text{s.t.} && k \in \{0, 1, \ldots, n\}. \end{aligned} \tag{2.7}$$

We then have the following result.

Proposition 5. *In the correlated discoveries scenario,*

$$\alpha\rho^{a_s}(k^{a_s}) + (1-\alpha)\rho^{a_f}(k^{a_f}) \leq \rho^b(k^b).$$

That is, the expected revenue to the inventor from alternative a is at most the revenue the inventor obtains from alternative b.

Proof. See appendix 2.B. □

Hence, propositions 2 and 5 together imply that the inventor's choice depends on the underlying scenario defining how discovery is related across players. Furthermore, we highlight that Proposition 5 establishes that, in the CD scenario, the inventor should not try to discover the use for his technology, even when there is no cost associated with such a trial.

The next result is the CD scenario counterpart of Proposition 3.

Proposition 6. *In the correlated discoveries scenario,*

$$ENEF(\Gamma^a) \leq ENEF(\Gamma^b).$$

That is, alternative b always leads to the highest expected number of efficient firms.

Proof. See appendix 2.B. □

Therefore, different from the result obtained for the ID scenario, the above proposition shows that in the CD scenario one can be sure that the inventor ultimately chooses the alternative associated to the highest diffusion of the technology.

2.5 Conclusion

In this paper we studied the problem facing an inventor who holds the patent of a technology which could be potentially used by firms in a given industry to reduce costs. The main question we addressed was whether the inventor should try or not to discover the use of the technology before licensing. We showed that the answer to this question depends on how discovery by one player is related to discovery by other players. Furthermore, it was showed that the inventor's ultimate decision has implications for technological diffusion in the industry.

We notice that our analysis can be adjusted to allow for a fixed cost, say F, associated to the effort of trying to discover the use for the technology. In this case, conditional on licensing taking place, firms would place bid equal to $\beta^x(k) - F$, for each $k \in \{1, \dots, n\}$ and $x \in \{a_s, a_f, b\}$, where β^x is as in the text. Our results, in particular Proposition 2, would then change. Specifically, threshold levels of F would be specified, below which the inequality in the referred proposition would hold.

We conclude by indicating some interesting questions for future investigation. A natural question is whether the above results extend to environments with more general demands. Also, one could investigate whether the availability of different licensing mechanisms changes the above findings, and if the decisions of an insider inventor are consistent to those of an outside inventor.

2.A Omitted Proofs: Independent discoveries scenario

Proof of Lemma 1. Suppose k firms were licensed in the auction stage. Each nonlicensee has marginal cost c_H and solves

$$\max_{\tilde{q} \geq 0} \quad \Big[a - \sum_{j=0}^{k} \binom{k}{j} \alpha^j (1-\alpha)^{k-j} \big(jq_L + (k-j)q_H\big) - (n-k-1)q - \tilde{q} - c_H\Big]\tilde{q}$$

where, for brevity, we adopt the simplified notation $q = q^{a_f}(k;\alpha)$, $q_H = q^{a_f}_{\ell,H}(k;\alpha)$, and $q_L = q^{a_f}_{\ell,L}(k;\alpha)$. Assuming interior solution, one can easily derive the first order condition

$$a - c_H - (n-k+1)q - kq_H = (q_L - q_H) \sum_{j=0}^{k} \binom{k}{j} \alpha^j (1-\alpha)^{k-j} j. \tag{2.8}$$

Type c_H of each licensee firm solves

$$\max_{\tilde{q} \geq 0} \quad \Big[a - \sum_{j=0}^{k-1} \binom{k-1}{j} \alpha^j (1-\alpha)^{k-1-j} \big(jq_L + (k-1-j)q_H\big) - (n-k)q - \tilde{q} - c_H\Big]\tilde{q},$$

Again assuming interior solution, the first order condition can be written as

$$a - c_H - (n-k)q - (k+1)q_H = (q_L - q_H)\sum_{j=0}^{k-1}\binom{k-1}{j}\alpha^j(1-\alpha)^{k-1-j}j. \quad (2.9)$$

Finally, type c_L of each licensee firm solves

$$\max_{\tilde{q}\geq 0} \quad \Big[a - \sum_{j=0}^{k-1}\binom{k-1}{j}\alpha^j(1-\alpha)^{k-1-j}\big(jq_L + (k-1-j)q_H\big) - (n-k)q - \tilde{q} - c_L\Big]\tilde{q},$$

leading to the (interior) first order condition

$$a - c_L - (n-k)q - (k-1)q_H - 2q_L = (q_L - q_H)\sum_{j=0}^{k-1}\binom{k-1}{j}\alpha^j(1-\alpha)^{k-1-j}j. \quad (2.10)$$

From (2.9) and (2.10) it easily follows that

$$q_L = q_H + \frac{\Delta c}{2}, \quad (2.11)$$

where $\Delta c = c_H - c_L$.

Substituting (2.11) into (2.9) and observing that

$$\sum_{j=0}^{k}\binom{k}{j}\alpha^j(1-\alpha)^{k-j}j = \alpha k \quad (2.12)$$

we obtain

$$q_H = \frac{a - c_H - (n-k)q - \alpha(k-1)\Delta c/2}{k+1}.$$

Recalling the notation adopted in the beginning of the proof, the above equality can then be substituted into (2.8) to give

$$q^{a_f}(k;\alpha) = \frac{a - c_H - k\alpha\Delta c}{n+1},$$

using again equality (2.12). Making $\varepsilon_\alpha = \alpha\Delta c$ and $k_\alpha = (a - c_H)/\varepsilon_\alpha$, $q^{a_f}(k;\alpha)$ can then be written as

$$q^{a_f}(k;\alpha) = \frac{\varepsilon_\alpha(k_\alpha - k)}{n+1},$$

for $k < k_\alpha$ and zero otherwise.

Substituting for $q = q^{a_f}(k;\alpha)$ in (2.9), we obtain

$$q^{a_f}_{\ell,H}(k;\alpha) = \frac{\varepsilon_\alpha(k_\alpha - k)}{n+1} + \frac{\varepsilon_\alpha}{2},$$

provided $0 < q^{a_f}(k;\alpha)$. Substituting for $q^{a_f}(k;\alpha) = 0$ in (2.9), we get

$$q^{a_f}_{\ell,H}(k;\alpha) = \frac{\varepsilon_\alpha(2k_\alpha + 1 - k)}{2(k+1)},$$

for $k_\alpha \leq k < 2k_\alpha + 1$ and zero otherwise.

Substituting for $q = q^{a_f}(k;\alpha)$ and $q_H = q^{a_f}_{\ell,H}(k;\alpha)$ in (2.11), we obtain

$$q^{a_f}_{\ell,L}(k;\alpha) = \frac{\varepsilon_\alpha(k_\alpha - k)}{n+1} + \frac{(1+\alpha)\varepsilon_\alpha}{2\alpha},$$

for $k < k_\alpha$,

$$q^{a_f}_{\ell,L}(k;\alpha) = \frac{\varepsilon_\alpha(2k_\alpha + 1 - k)}{2(k+1)} + \frac{\varepsilon_\alpha}{2\alpha},$$

for $k_\alpha \leq k < 2k_\alpha + 1$, and

$$q^{a_f}_{\ell,L}(k;\alpha) = \frac{\varepsilon_\alpha(k_\alpha + 1/\alpha)}{2 + \alpha(k-1)},$$

for $2k_\alpha + 1 \leq k$.

Clearly, equilibrium is unique. Profits being the square of quantities is a general property of the Cournot model with our demand specification and can be easily checked with some algebra. □

Proof of Proposition 2. Recall that $\beta^{a_f}(\cdot) = \beta^b(\cdot)$ and, therefore, $k^{a_f} = k^b$. Thus, it is sufficient to show that

$$\rho^{a_s}(k^{a_s}) \geq \rho^b(k^b).$$

We consider three cases.

Case 1 ($k^b < k_\alpha$). Since $k^b < k_\alpha$, we have $\alpha k^b < k_1$. Using the formulas for $\beta^{a_s}(\cdot)$ and $\beta^b(\cdot)$, in the appropriate intervals, gives

$$\begin{aligned}
\rho^{a_s}(k^{a_s}) &\geq \rho^{a_s}(\alpha k^b) \\
&= k^b \alpha \varepsilon_1^2 \left(\frac{2(k_1 - \alpha k^b)}{n+1} + 1 \right) \\
&\geq k^b \alpha \varepsilon_1^2 \left(\frac{2(\alpha k_\alpha - \alpha k^b)}{n+1} + \frac{1+3\alpha}{4} \right) \\
&= k^b \varepsilon_\alpha^2 \left(\frac{2(k_\alpha - k^b)}{n+1} + \frac{1+3\alpha}{4\alpha} \right) \\
&= \rho^b(k^b),
\end{aligned}$$

where the first inequality follows from the optimality of k^{a_s}.

Case 2 ($k_\alpha \leq k^b < 2k_\alpha + 1$). Since $k_\alpha \leq k^b$, we have $k_1 \leq \alpha k^b$. Since $k^b < 2k_\alpha + 1$, we have $1/4 < [(k_\alpha + 1)/(k^b + 1)]^2$. Furthermore, $[(k_\alpha + 1)/(k^b + 1)]^2 \leq [(k_\alpha + 1/\alpha)/(k^b + 1/\alpha)]^2$. Using the formulas for $\beta^{a_s}(\cdot)$ and $\beta^b(\cdot)$, in the appropriate intervals, we get

$$\begin{aligned}
\rho^{a_s}(k^{a_s}) &\geq \rho^{a_s}(\alpha k^b) \\
&= k^b \alpha \varepsilon_1^2 \left(\frac{k_1 + 1}{\alpha k^b + 1} \right)^2 \\
&= k^b \alpha \varepsilon_1^2 \left(\frac{k_\alpha + 1/\alpha}{k^b + 1/\alpha} \right)^2 \\
&\geq k^b \alpha \varepsilon_1^2 \left(\frac{k_\alpha + 1}{k^b + 1} \right)^2 \\
&\geq k^b \alpha \varepsilon_1^2 \left(\alpha \left(\frac{k_\alpha + 1}{k^b + 1} \right)^2 + \frac{1-\alpha}{4} \right) \\
&= k^b \varepsilon_\alpha^2 \left(\left(\frac{k_\alpha + 1}{k^b + 1} \right)^2 + \frac{1-\alpha}{4\alpha} \right) \\
&= \rho^b(k^b).
\end{aligned}$$

Case 3 ($2k_\alpha + 1 \leq k^b$). Clearly, $k_1 < \alpha k^b$. Therefore

$$\begin{aligned}
\rho^{a_s}(k^{a_s}) &\geq \rho^{a_s}(\alpha k^b) \\
&= k^b \alpha \varepsilon_1^2 \left(\frac{k_1 + 1}{\alpha k^b + 1} \right)^2 \\
&\geq k^b \alpha \varepsilon_1^2 \left(\frac{k_1 + 1}{2 + \alpha(k^b - 1)} \right)^2 \\
&= k^b \alpha \varepsilon_1^2 \left(\frac{\alpha k_\alpha + 1}{2 + \alpha(k^b - 1)} \right)^2 \\
&= k^b \alpha \varepsilon_\alpha^2 \left(\frac{k_\alpha + 1/\alpha}{2 + \alpha(k^b - 1)} \right)^2 \\
&= \rho^b(k^b).
\end{aligned}$$

Case 3 exhausts the possibilities and concludes the proof of the proposition. $\square$

Proof of Proposition 3. First observe that

$$ENEF(\Gamma^b) = \sum_{j=0}^{k^b} \binom{k^b}{j} \alpha^j (1 - \alpha)^{k^b - j} j = \alpha k^b,$$

since in Γ^b each licensee discovers with probability α, and independent from others, the use for the invention.

Next, recall that in the ID scenario Γ^{a_f} and Γ^b lead to exactly the same outcomes. Hence,

$$ENEF(\Gamma^a) = \alpha k^{a_s} + (1 - \alpha)(\alpha k^b).$$

But $k^{a_s} \leq k_1$, and $k_\alpha \leq k^b \Leftrightarrow k_1 \leq \alpha k^b$. This and the above observations then imply 1.

To prove 2, first observe that, since $k^b < k_\alpha$, we have

$$k^b = \min \left\{ k_\alpha, \frac{k_\alpha}{2} + \left(\frac{1 + 3\alpha}{4\alpha} \right) \frac{n+1}{4} \right\}.$$

But,

$$k^{a_s} = \min\left\{k_1, \frac{k_1}{2} + \frac{n+1}{4}\right\}.$$

Hence, $\alpha k^b \leq k^{a_s}$, concluding the proof. □

2.B Omitted Proofs: Correlated discoveries scenario

Proof of Lemma 4. The calculations carried in this proof are similar to those carried in the proof of Lemma 1. Suppose k firms were licensed in the auction stage. Each nonlicensee has marginal cost c_H and solves

$$\max_{\tilde{q}\geq 0} \quad \left[a - k(\alpha q_L + (1-\alpha)q_H) - (n-k-1)q - \tilde{q} - c_H\right]\tilde{q}$$

where we use the fact that discoveries are fully correlated, and, as in the proof of Lemma 1, for brevity, we adopt the simplified notation $q = q^b(k;\alpha)$, $q_H = q^b_{\ell,H}(k;\alpha)$, and $q_L = q^b_{\ell,L}(k;\alpha)$. The first order condition for interior solution can be easily seen to be

$$a - k(\alpha q_L + (1-\alpha)q_H) - (n-k+1)q - c_H = 0. \tag{2.13}$$

Let $t \in \{H, L\}$. Type t of each licensee then solves

$$\max_{\tilde{q}\geq 0} \quad \left[a - (k-1)q_t - (n-k)q - \tilde{q} - c_t\right]\tilde{q}.$$

The first order condition for an interior solution to the above problem is

$$a - (k+1)q_t - (n-k)q - c_t = 0. \tag{2.14}$$

These equations then imply

$$q_L = q_H + \frac{\Delta c}{k+1}. \tag{2.15}$$

Now, equations (2.13), (2.14) for $t = H$, and (2.15) give

$$q_H = q + \alpha \Delta c \frac{k}{k+1}. \tag{2.16}$$

Using these relations in (2.14), $t = L$, lead to $q = q^b(k;\alpha)$ (and hence $q_H = q^b_{\ell,H}(k;\alpha)$ and $q_L = q^b_{\ell,L}(k;\alpha)$) for the case $k < k_\alpha$.

For the case $k_\alpha \leq k$, we observe that, since nonlicensees are driven out of the industry, the Cournot competition is one of complete information among homogeneous firms. Hence, type t firms will produce $(a - c_t)/(k+1)$. Using the definitions of ε_α and k_α we obtain the expressions stated in the lemma.

To conclude the proof of the lemma, we again observe that profits being the square of quantities is a general property of the Cournot model with our demand specification and can be easily checked with some algebra. □

Proof of Proposition 5. Recall that in the CD scenario no licensing occurs after a failure by the inventor. Thus, $\rho^{a_f}(k^{a_f}) = 0$. Next, observe that $\rho^{a_s}(k^{a_s})$ is decreasing over $k_1 \leq k$. Hence, it must be $k^{a_s} \leq k_1 \Leftrightarrow k^{a_s}/\alpha \leq k_\alpha$. Therefore,

$$\begin{aligned}
\rho^b(k^b) &\geq \rho^b(k^{a_s}/\alpha) \\
&= (k^{a_s}/\alpha)\varepsilon_\alpha^2 \left(\frac{2(k_\alpha - k^{a_s}/\alpha)}{n+1} + 1 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{k^{a_s}/\alpha + 1}\right)^2 \right) \\
&\geq (k^{a_s}/\alpha)\varepsilon_\alpha^2 \left(\frac{2(k_\alpha - k^{a_s}/\alpha)}{n+1} + 1 \right) \\
&= k^{a_s}\alpha\varepsilon_1^2 \left(\frac{2(k_\alpha - k^{a_s}/\alpha)}{n+1} + 1 \right) \\
&= k^{a_s}\alpha\varepsilon_1^2 \left(\frac{2(k_1 - k^{a_s})/\alpha}{n+1} + 1 \right) \\
&\geq k^{a_s}\alpha\varepsilon_1^2 \left(\frac{2(k_1 - k^{a_s})}{n+1} + 1 \right) \\
&= \alpha\rho^{a_s}(k^{a_s}),
\end{aligned}$$

concluding the proof of the proposition. □

Proof of Proposition 6. Observe that, since no licensing takes place in Γ^{a_f}, $k^{a_f} = 0$. Thus, $ENEF(\Gamma^a) = \alpha k^{a_s}$. Now, since $ENEF(\Gamma^b) = \alpha k^b$, it is sufficient to show that $k^{a_s} \leq k^b$.

We consider two cases.

Case 1 ($k^{a_s} = k_1/2 + (n+1)/4$). For all $k \leq k^{a_s}$, we have

$$\begin{aligned}
\rho^b(k) &= k\varepsilon_\alpha^2 \left(\frac{2(k_\alpha - k)}{n+1} + 1 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{k+1}\right)^2 \right) \\
&= k\varepsilon_\alpha^2 \left(\frac{2(k_1 - k)}{n+1} + 1 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{2k_1}{n+1} + \left(\frac{1}{k+1}\right)^2\right) \right) \\
&= k\varepsilon_\alpha^2 \left(\frac{2(k_1 - k)}{n+1} + 1 \right) + k\varepsilon_\alpha^2 \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{2k_1}{n+1} + \left(\frac{1}{k+1}\right)^2\right) \\
&\leq k^{a_s}\varepsilon_\alpha^2 \left(\frac{2(k_1 - k^{a_s})}{n+1} + 1 \right) + k^{a_s}\varepsilon_\alpha^2 \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{2k_1}{n+1} + \left(\frac{1}{k^{a_s}+1}\right)^2\right) \\
&\leq \rho^b(k^b),
\end{aligned}$$

where the first inequality follows from the optimality of k^{a_s} and the fact that the second term in the sum is increasing over $k \leq k^{a_s}$. Hence, it must be $k^{a_s} \leq k^b$.

Case 2 ($k^{a_s} = k_1$). Suppose $k^b \leq k^{a_s} < k_\alpha$. Then, k^b must satisfy the first-order condition

$$\beta^b(k) = -k \cdot \frac{\mathrm{d}}{\mathrm{d}k}\beta^b(k).$$

That is, at $k = k^b$, we have

$$\frac{2(k_\alpha - k^b)}{n+1} + 1 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{k^b+1}\right)^2 = k^b \left(\frac{2}{n+1} + \left(\frac{1-\alpha}{\alpha}\right)\frac{2}{(k^b+1)^3} \right).$$

It then follows that

$$
\begin{aligned}
\rho^b(k^b) &= k^b\varepsilon_\alpha^2\left(\frac{2(k_\alpha - k^b)}{n+1} + 1 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{k^b+1}\right)^2\right) \\
&= k^b\varepsilon_\alpha^2\left(\frac{2k^b}{n+1} + \left(\frac{1-\alpha}{\alpha}\right)\frac{2k^b}{(k^b+1)^3}\right) \\
&\leq k^b\varepsilon_\alpha^2\left(1 + \frac{1-\alpha}{\alpha}\right) \\
&= (k^b/\alpha)\varepsilon_\alpha^2 \\
&\leq k_\alpha\varepsilon_\alpha^2 \\
&< k_\alpha\varepsilon_\alpha^2\left(1 + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{k_\alpha+1}\right)^2\right) \\
&= \rho^b(k_\alpha),
\end{aligned}
$$

where the first inequality follows from the facts that $k^{a_s} = k_1$, and thus $k_1 \leq (n+1)/2$, and $2k < (k+1)^3$. Therefore, the optimality of k^b is contradicted and we must have $k^{a_s} < k_\alpha \leq k^b$.

Case 2 concludes the proof of the proposition. $\square$

Chapter 3

A Note on Cournot Equilibrium with Positive Price[1]

[1] This is joint work with Bruno D. Badia and Yair Tauman, and it appeared in *Economics Bulletin (2014)*

Abstract

Consider an oligopoly in which n firms compete in quantity, the market inverse demand is strictly decreasing (on the set of quantities for which the price is positive), twice differentiable and log-concave, and each of the firms has nondecreasing, twice differentiable cost of production. It is observed that, under additional mild assumptions, Cournot equilibrium with positive price is unique. The result also holds if the costs are piecewise linear, nondecreasing, and convex. In addition, if at least one firm incurs positive cost to produce positive quantities, then the equilibrium is unique and the corresponding price is positive.

Keywords: Cournot game; Cournot equilibrium; Existence; Uniqueness
JEL classification: L13

3.1 Introduction

We study the existence and uniqueness of Cournot equilibrium. Our conditions are weaker than the ones appearing in the literature, on the expense of multiple equilibrium points. However, we show that among these equilibrium points only one has a positive price. If we add the requirement that at least one firm produces at a positive cost any positive quantity, then the equilibrium is unique and the equilibrium price is positive. Furthermore, existence and uniqueness of equilibrium with a positive price is preserved if costs are piecewise linear, nondecreasing, and convex.

Let the market inverse demand be given by $P(Q) = \max\{0, \widehat{P}(Q)\}$. To show the existence and uniqueness of Cournot equilibrium with positive price, we require that (i) $\widehat{P}(\cdot)$ is log-concave, strictly decreasing, and twice differentiable, (ii) the cost function of each firm is nondecreasing and twice differentiable, and (iii) $\widehat{P}'(Q) - c_j''(q_j) < 0$ for all industry aggregated output Q and all individual firm's output q_j, where $c_j(\cdot)$ is firm j's cost function. If in addition, there exists a firm j such that $c_j(q_j) > 0$ whenever $q_j > 0$, then Cournot equilibrium is unique and the market price is positive.

Several other papers have addressed the uniqueness of Cournot equilibrium. We weaken the assumptions that profits are concave and marginal costs are strictly positive (Szidarovszky and Yakowitz (1977), Gaudet and Salant (1991), and Van Long and Soubeyran (2000)). Furthermore, we do not require convex costs as in Szidarovszky and Yakowitz (1977) and Van Long and Soubeyran (2000). Kolstad and Mathiesen (1987) provide necessary and sufficient conditions for the existence of a unique Cournot equilibrium. Some of their assumptions, however, are not globally stated and they require certain properties to hold at all equilibrium points. Their regularity conditions require (*i*) the Jacobian of the marginal profits for the firms with positive output to be nonsingular at every equilibrium point, and (*ii*) all Cournot equilibria to be non degenerate (that is, firms producing zero at equilibrium have marginal costs greater than the equilibrium price).

3.2 Setup and Results

Let $N = \{1, \dots, n\}$ be the set of firms producing a homogeneous good in a market with inverse demand given by $P(Q) = \max\{0, \widehat{P}(Q)\}$. For each $j \in N$, let $c_j(\cdot)$ be the cost function of firm j. The profit of firm j when producing q_j units of the good is

$$\Pi_j(q_1, \dots, q_n) = q_j P(Q) - c_j(q_j). \tag{3.1}$$

where $Q = \sum_{i=1}^{n} q_i$. Firms are assumed to choose production levels simultaneously and independently. Given a profile $(q_i)_{i \in N}$ of quantities, each firm receives a payoff given by (3.1). We refer to this game as *Cournot game* and denote it by G. A *Cournot equilibrium* is a Nash equilibrium of G.

We make the following assumptions:

Assumption 1. $\widehat{P}(\cdot)$ *is a strictly decreasing, twice differentiable log-concave function and* $\lim_{Q \to \infty} P(Q) = 0$.

Assumption 2. *For each* $j \in N$, $c_j(\cdot)$ *is twice differentiable with* $c_j'(q) \geq 0$ *for all* $q \in \mathbf{R}_+$.

Assumption 3. *For each* $j \in N$ *and* $(q, Q) \in \mathbf{R}^2_+$, $\widehat{P}'(Q) - c''_j(q) < 0$.

Remark 1. Under Assumption 1 there exists a unique $0 < Q^0 \leq +\infty$ such that $\widehat{P}(Q^0) = 0$. As noted in Amir (1996) the log-concavity assumption relaxes the so called Novshek's condition, $P'(Q) - QP''(Q) \leq 0$ for all $Q \in [0, Q^0)$ (Novshek (1985)). Assumption 3 is standard in the literature and can be interpreted as relaxing the requirement that costs are convex.

Without loss of generality we normalize $c_j(0) = 0$ and assume $P(0) > c'_j(0)$ for all $j \in N$. For instance, if the last inequality does not hold for some firm, then by Assumption 3 this firm will optimally produce zero.

The following is a useful observation.

Lemma 1. *Suppose assumptions 1-3 hold. Then, for each* $j \in N$ *and each* $q_{-j} = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$*, the function given by* $q_j \widehat{P}(Q) - c_j(q_j)$ *is quasiconcave in* q_j.

Proof. See Appendix. □

The next proposition is the main result of this paper.

Proposition 1. *Suppose assumptions 1-3 hold. Then* G *has a unique Cournot equilibrium with positive price.*

Before turning to the proof of the above proposition, we illustrate the importance of Assumption 3 for that result.

Example 1. Suppose $N = \{1, 2\}$ and $P(Q) = \max\{0, 10 - Q\}$. Define the strictly decreasing function $f : \mathbf{R}_+ \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 + 4(1-x)^2 & \text{if } x \leq 1 \\ 1 - 4(x-1)^2 & \text{if } x > 1. \end{cases}$$

Each firm $j \in N$ produces q_j units of output at cost

$$c(q_j) = 10q_j - q_j^2 - \int_0^{q_j} f(x)\, dx.$$

It can be easily checked that all assumptions, except Assumption 3, hold. Clearly, $(q_1, q_2) = (1, 1)$, $(q_1, q_2) = (3/4, 5/4)$, and $(q_1, q_2) = (5/4, 3/4)$ are equilibrium points, each of them associated with a positive price.

Let $\widehat{G}$ be the *auxiliary Cournot game* having $\widehat{P}(\cdot)$ as inverse demand function. The proof of Proposition 1 relies on the following

Proposition 2. *The game $\widehat{G}$ has a unique equilibrium $q^* = (q_i^*)_{i \in N}$ and $\widehat{P}(Q^*) > 0$.*

Proof. See Appendix. □

We can now prove Proposition 1.

Proof of Proposition 1. Let q^* be the unique equilibrium of $\widehat{G}$. Let $q_j \geq 0$. Suppose first that $q_j + \sum_{i \neq j} q_i^* \leq Q^0$. Since $\widehat{P}(Q^*) > 0$, $\Pi_j(q^*) = \widehat{\Pi}_j(q^*) \geq \widehat{\Pi}_j(q_j, q_{-j}^*) = \Pi_j(q_j, q_{-j}^*)$. Suppose next that $q_j + \sum_{i \neq j} q_i^* > Q^0$. Since $Q^* < Q^0$, it follows that $q_j > q_j^*$ and, by Assumption 2, $c_j(q_j) \geq c_j(q_j^*)$. Thus, $\Pi_j(q^*) \geq -c_j(q_j) = \Pi_j(q_j, q_{-j}^*)$. That is, q^* is an equilibrium of the Cournot game.

Let us next prove uniqueness. Assume $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$ is also an equilibrium of the Cournot game and $P(\bar{Q}) > 0$. We will show that $\bar{q}$ is also an equilibrium of $\widehat{G}$. Let $q_j \geq 0$. Suppose first that $q_j + \sum_{i \neq j} \bar{q}_i \leq Q^0$. Since $P(\bar{Q}) > 0$, $\widehat{\Pi}_j(\bar{q}) = \Pi_j(\bar{q}) \geq \Pi_j(q_j, \bar{q}_{-j}) = \widehat{\Pi}_j(q_j, \bar{q}_{-j})$. Suppose next that $q_j + \sum_{i \neq j} \bar{q}_i > Q^0$. Then $q_j > \bar{q}_j$, and $c_j(q_j) \geq c_j(\bar{q}_j)$. Hence, $\widehat{\Pi}_j(\bar{q}) \geq \widehat{\Pi}_j(q_j, \bar{q}_{-j})$. That is, $\bar{q}$ is an equilibrium of $\widehat{G}$. By Proposition 2, $\bar{q} = q^*$, and the proof of Proposition 1 is complete. □

Remark 2. The result is intact even if Assumption 2 does not hold for some firms, provided that for those firms costs are piecewise linear, nondecreasing, and convex. (Of course, in this case, Assumption 3 does not hold either.) Let us outline the proof of this claim. Suppose that, for each $j \in N$, $c_j(\cdot)$ satisfies the latter requirements. Equilibrium existence follows by Novshek (1985). Suppose there are two equilibrium points with positive price. Starting with firm 1, and proceeding firm by firm, one at a time, "smooth" the firm's cost such that assumptions 2 and 3 hold and the firm's best response is unchanged at each equilibrium (see appendix 3.B for one such smoothing). Clearly, at each step the equilibrium points are preserved. However, after the process terminates, Proposition 1 applies, which leads to a contradiction. Szidarovszky and Yakowitz (1982) prove uniqueness of Cournot equilibrium allowing for non differentiable costs. However, the authors require concavity of inverse demand. Our result relaxes this condition.

As the next result shows, equilibrium uniqueness can be obtained with an additional assumption.

Proposition 3. *Suppose that assumptions 1-3 hold and there exists $j \in N$ such that $c_j(q_j) > 0$ for all $q_j > 0$. Then G has a unique equilibrium q^* and $P(Q^*) > 0$.*

Observe that uniqueness is obtained without the standard assumption that marginal costs are strictly positive. Also note that the additional assumption is essential. For instance, if $N = \{1, \ldots, 102\}$, $\widehat{P}(Q) = 100 - Q$, and $c_j(q_j) = 0$ for $q_j \in [0, 1]$ and $c_j(q_j) = q_j - 1$ for $q_j > 1$, then $(q_i^*)_{i \in N} = (1, \ldots, 1)$ is a Cournot equilibrium, but $P(Q^*) = 0$.

Proof of Proposition 3. Suppose to the contrary that $\bar{q}$, with $P(\bar{Q}) = 0$, is an equilibrium of G. Hence, it must be $\Pi_j(\bar{q}) \geq \Pi_j(0, \bar{q}_{-j})$ for each $j \in N$. However, this inequality implies that, for each $j \in N$, $c_j(\bar{q}_j) \leq 0$, a contradiction. □

3.A Omitted Proofs

Proof of Lemma 1. For each q_{-j}, we show that $\widehat{\Pi}_j(\cdot, q_{-j}) = q_j \widehat{P}(Q) - c_j(q_j)$ is single peaked. If $\widehat{\Pi}'_j(0, q_{-j}) \leq 0$ then there is nothing to show. Thus, suppose $\widehat{\Pi}'_j(0, q_{-j}) > 0$. We claim that there is a unique q_j such that $\widehat{\Pi}'_j(q_j, q_{-j}) = 0$. In fact, this last equality holds if, and only if,

$$f(q_j) \equiv \frac{c'_j(q_j) - q_j \widehat{P}'(Q)}{\widehat{P}(Q)} = 1. \tag{3.A.2}$$

By assumptions 1, 2, and 3

$$f'(q_j) = \frac{\widehat{P}(Q)\big(c''_j(q_j) - \widehat{P}'(Q)\big) + q_j\big((\widehat{P}'(Q))^2 - \widehat{P}(Q)\widehat{P}''(Q)\big) - \widehat{P}'(Q)c'(q_j)}{(\widehat{P}'(Q))^2} > 0,$$

so that $f(\cdot)$ is strictly increasing. Furthermore, $f(0) < 1$ and

$$\lim_{q_j \to Q^0 - Q_{-j}} f(q_j) = +\infty,$$

where $Q_{-j} = \sum_{i \neq j} q_i$. Therefore, there is a unique q_j satisfying (3.A.2), as claimed. $\square$

Next we present the proof of Proposition 2. Despite similar proofs for existence being available in the literature for the case of positive marginal costs (see Szidarovszky and Yakowitz (1977) and Friedman (1982)), we present it here both for completeness and to develop notation used in the uniqueness part of our proof.

Proof of Proposition 2. For each $j \in N$ and every $(q_j, Q) \in \mathbf{R}^2_+$, define

$$F_j(q_j, Q) = \widehat{P}(Q) + q_j \widehat{P}'(Q) - c_j'(q_j). \tag{3.A.3}$$

In what follows, we treat q_j and Q independently. That is, we do not assume $Q = \sum_i q_i$.

Claim 1. *For each $j \in N$ and $Q \in \mathbf{R}_+$, $F_j(q_j, Q) < 0$ for sufficiently large q_j.*

Proof. Follows by (3.A.3), and the assumptions $\widehat{P}'(Q) < 0$ for all $Q \in \mathbf{R}_+$ and $c_j'(q_j) \geq 0$ for all $q_j \in \mathbf{R}_+$. $\square$

Claim 2. *For each $j \in N$, there exists a unique $Q_j \in \mathbf{R}_+$ such that $\widehat{P}(Q_j) = c_j'(0)$.*

Proof. There are two cases to consider. If $c_j'(0) > 0$, then there exists $Q \in \mathbf{R}_+$ such that $\widehat{P}(Q) < c_j'(0) < \widehat{P}(0)$. Since $\widehat{P}(\cdot)$ is continuous, there exists Q_j such that $\widehat{P}(Q_j) = c_j'(0)$. Since $\widehat{P}'(Q) < 0$, for all $Q \in \mathbf{R}_+$, Q_j is uniquely defined. If $c_j'(0) = 0$, then $Q_j = Q^0$ is the unique solution to $\widehat{P}(Q) = c_j'(0)$. $\square$

For each $j \in N$, define

$$S_j = \big\{Q \in \mathbf{R}_+ : \exists q_j \geq 0 \text{ such that } F_j(q_j, Q) = 0\big\}.$$

Claim 3. *For each $j \in N$, $Q_j \in S_j$ and $F_j(0, Q_j) = 0$.*

Proof. Follows by (3.A.3) and Claim 2. □

Claim 4. *For each $j \in N$, if $Q_j < +\infty$ and $Q > Q_j$, then $Q \notin S_j$.*

Proof. Let $Q > Q_j$ and $q_j \geq 0$. By assumptions 1 and 3,

$$\widehat{P}(Q)- \leq c_j'(q_j) < \widehat{P}(Q_j) = c_j'(0).$$

Hence, for every $q_j \geq 0$,

$$F_j(q_j, Q) = \widehat{P}(Q) + q_j \widehat{P}'(Q) - c_j'(q_j) < 0,$$

and $Q \notin S_j$. □

Claim 5. *Let $Q \in \mathbf{R}_+$ such that $Q \leq Q_j$. Then $Q \in S_j$.*

Proof. If $Q = Q_j$, then $Q \in S_j$ by Claim 3. Thus, let $Q < Q_j \leq +\infty$. Since $\widehat{P}(\cdot)$ is strictly decreasing, it follows that

$$F_j(0, Q) = \widehat{P}(Q) - c_j'(0) > \widehat{P}(Q_j) - c_j'(0) = 0,$$

where the second equality follows by Claim 2. Hence, by Claim 1 and the continuity of $F_j(\cdot, Q)$, there exists $q_j \geq 0$ such that $F_j(q_j, Q) = 0$. That is, $Q \in S_j$. □

It follows by claims 4 and 5 that $S_j = [0, Q_j]$ whenever Q_j is finite and $S_j = \mathbf{R}_+ \cup \{+\infty\}$ otherwise.

Claim 6. *For each $j \in N$, $F_j(\cdot, Q)$ is strictly decreasing and the solution to $F_j(q_j, Q) = 0$ is unique for all $Q \in S_j$.*

Proof. Fix $Q \in \mathbf{R}_+$. By Assumption 3, for all $q_j \in \mathbf{R}_+$,

$$\frac{\partial F_j}{\partial q_j}(q_j, Q) = \widehat{P}'(Q) - c_j''(q_j) < 0.$$

In particular, the solution to $F_j(q_j, Q) = 0$ is unique. □

For every $Q \in S_j$, let $q_j(Q)$ denote the unique solution to $F_j(q_j, Q) = 0$. Clearly, for all $Q \in S_j$,

$$\widehat{P}(Q) + q_j(Q)\widehat{P}'(Q) - c_j'(q_j(Q)) = 0. \tag{3.A.4}$$

If $Q \notin S_j$, put $q_j(Q) = 0$. Note that, for each $j \in N$, claims 3 and 6 imply that $q_j(Q_j) = 0$, whenever Q_j is finite.

Claim 7. *For each $j \in N$, and every $Q \in S_j$,*

$$\widehat{P}'(Q) + q_j(Q)\widehat{P}''(Q) \leq 0. \tag{3.A.5}$$

Proof. Fix $Q \in S_j$. If $\widehat{P}''(Q) \leq 0$, then (3.A.5) follows trivially from the assumption that $\widehat{P}(\cdot)$ is strictly decreasing. Thus, suppose $\widehat{P}''(Q) > 0$. Since $\widehat{P}(\cdot)$ is log-concave,

$$(\widehat{P}'(Q))^2 \geq \widehat{P}''(Q)\widehat{P}(Q). \tag{3.A.6}$$

It follows that

$$\begin{aligned} \widehat{P}'(Q)\big(\widehat{P}'(Q) + q_j(Q)\widehat{P}''(Q)\big) &\geq \widehat{P}''(Q)\widehat{P}(Q) + q_j(Q)\widehat{P}'(Q)\widehat{P}''(Q) \\ &= \widehat{P}''(Q)\big(\widehat{P}(Q) + q_j(Q)\widehat{P}'(Q)\big) \\ &= \widehat{P}''(Q)c_j'(q_j(Q)) \\ &\geq 0, \end{aligned}$$

where the inequality follows from (3.A.6) and the last equality follows from (3.A.4). Since $\widehat{P}'(Q) < 0$, the above inequality implies $\widehat{P}'(Q) + q_j(Q)\widehat{P}''(Q) \leq 0$. □

It then follows from the Implicit Function Theorem and Claim 7 that $q_j(\cdot)$ is decreasing in S_j since

$$q_j'(Q) = -\frac{\widehat{P}'(Q) + q_j(Q)\widehat{P}''(Q)}{\widehat{P}'(Q) - c_j''(q_j(Q))}, \tag{3.A.7}$$

for all $Q \in S_j$. Hence, $q_j(\cdot)$ is continuous in S_j, and, since $q_j(Q) = 0$ for all $Q \geq Q_j$, $q_j(\cdot)$ is continuous in $\mathbf{R}_+$.

Now, since $q_j(\cdot)$ is decreasing, $0 \leq q_j(Q) \leq q_j(0)$, for each $j \in N$. Let $b = \sum_{i=1}^n q_i(0)$, and define $G : [0, \infty) \to [0, \infty)$ by $G(Q) = \sum_{i=1}^n q_i(Q) - Q$. Clearly, $G(\cdot)$ is continuous and strictly decreasing. In addition, $G(0) = b \geq 0$ and $G(b) \leq 0$. Hence, there exists $Q^* \in [0, b]$ such that $G(Q^*) = 0$. That is, $\sum_{i=1}^n q_i(Q^*) = Q^*$.

Claim 8. *For each $j \in N$, if $Q^* > Q_j$, then $F_j(q_j(Q^*), Q^*) < 0$.*

Proof. Since $Q^* > Q_j$, $q_j(Q^*) = 0$, and

$$
\begin{aligned}
F_j(q_j(Q^*), Q^*) &= F_j(0, Q^*) \\
&= \widehat{P}(Q^*) - c_j'(0) \\
&< \widehat{P}(Q_j) - c_j'(0) = 0,
\end{aligned}
$$

where the inequality follows by the assumption that $\widehat{P}(\cdot)$ is strictly decreasing and the last equality follows by Claim 2. □

For each $j \in N$, the definition of $q_j(\cdot)$ implies that $F_j(q_j(Q^*), Q^*) = 0$ whenever $Q^* \in S_j$. This observation and Claim 8 imply that, at $q_j(Q^*)$, first order condition for maximization holds for each $j \in N$. Using Lemma 1, we conclude that $q^* = (q_i(Q^*))_{i \in N}$ is an equilibrium of the auxiliary game.

Claim 9. *The equilibrium is unique.*

Proof. Suppose $\bar{q} = (\bar{q}_i)_{i \in N}$ is also an equilibrium and define the set $J = \{j \in N : \bar{q}_j > 0\}$. Since the first order condition holds for each $j \in J$, $F_j(\bar{q}_j, \bar{Q}) = 0$. Hence, $\bar{Q} \in S_j$ for each $j \in J$ and $q_j(\bar{Q}) = \bar{q}_j$. If $j \notin J$, $\bar{q}_j = 0$ and $F_j(0, \bar{Q}) \leq 0$. Since $F_j(\cdot, \bar{Q})$ is strictly decreasing, $F_j(q_j, \bar{Q}) < 0$, for all $q_j > 0$. Thus, $\bar{Q} \geq Q_j$ and $q_j(\bar{Q}) = 0 = \bar{q}_j$. It then follows that, for each $j \in N$, $q_j(\bar{Q}) = \bar{q}_j$ and $G(\bar{Q}) = 0 = G(Q^*)$. Since $G(\cdot)$ is strictly decreasing over $[0, \infty)$, it must be $\bar{Q} = Q^*$ and $\bar{q}_j = q_j(\bar{Q}) = q_j(Q^*) = q_j^*$, for each $j \in N$. □

Claim 10. $\widehat{P}(Q^*) > 0$.

Proof. Suppose $\widehat{P}(Q^*) < 0$. Since $P(0) > c_j'(0)$, there is $j \in N$ such that $q_j^* > 0$. Then, $\widehat{\Pi}_j(q^*) < -c_j(q_j^*) \leq -c_j(0) = \widehat{\Pi}_j(0, q_{-j}^*)$, a contradiction. Hence, it must be $\widehat{P}(Q^*) \geq 0$. However, if $\widehat{P}(Q^*) = 0$, then there is $j \in N$ such that $q_j^* > 0$. But,

$$\begin{aligned}
\widehat{\Pi}_j(q^*) &= -c_j(q_j^*) \\
&\leq -c_j(q_j^* - \varepsilon) \\
&< (q_j^* - \varepsilon)\widehat{P}(Q^* - \varepsilon) - c_j(q_j^* - \varepsilon) \\
&= \widehat{\Pi}_j(q_j^* - \varepsilon, q_{-j}^*)
\end{aligned}$$

for sufficiently small $\varepsilon > 0$. Again a contradiction. Therefore, $\widehat{P}(Q^*) > 0$. □

This concludes the proof of Proposition 2. □

3.B Cost Function Smoothing

The argument given in Remark 2 relied on "smooth" functions that substitute for the non differentiable costs. To be concrete, suppose $j \in N$ is a firm whose cost function is piecewise linear, nondecreasing, and convex. That is, for $m = 0, 1, 2, \ldots,$ $c_j : \mathbf{R}_+ \to \mathbf{R}_+$ is given by $c_j(q) = \alpha_m + \beta_m q$ for $q \in [a_m, a_{m+1})$, with $0 = a_0 < a_1 < a_2 < \cdots < a_m < \cdots$ such that $\alpha_{m+1} < \alpha_m$, and $\beta_m < \beta_{m+1}$ for all m.

Suppose, $\bar{q}$ is an equilibrium point of the Cournot game such that $\alpha_m + \beta_m \bar{q}_j = \alpha_{m+1} + \beta_{m+1}\bar{q}_j$ for some m. That is, $(\bar{q}_j, c_j(\bar{q}_j))$ is located at a kink of j's cost. We provide a function, $\tilde{c}_j$, that is smooth over $[a_m, a_{m+2}]$ and coincides with c_j at all points of this interval except at an arbitrarily small interval around $\bar{q}_j$ and such that $\bar{q}_j$ is best response to $\bar{q}_{-j}$ even when $\tilde{c}_j$ replaces c_j, j's original cost.

Let γ denote the marginal revenue of firm j at $\bar{q}$. For small $\varepsilon > 0$ and $q \in (\bar{q}_j - \varepsilon, \bar{q}_j]$ define the function g^1 by

$$g^1(q) = \beta_m + (\gamma - \beta_m)\left[3\left(\frac{q - \bar{q}_j + \varepsilon}{\varepsilon}\right)^2 - 2\left(\frac{q - \bar{q}_j + \varepsilon}{\varepsilon}\right)^3\right].$$

For $\delta = (\gamma - \beta_m)\varepsilon/(\beta_{m+1} - \gamma)$ and $q \in [\bar{q}_j, \bar{q}_j + \delta)$ define the function g^2 by

$$g^2(q) = \beta_{m+1} + (\beta_{m+1} - \gamma)\left[3\left(\frac{q - \bar{q}_j}{\delta}\right)^2 - 2\left(\frac{q - \bar{q}_j}{\delta}\right)^3\right].$$

Now, define

$$\tilde{c}_j'(q) = \begin{cases} \beta_m & \text{if } q \in [a_m, \bar{q}_j - \varepsilon] \\ g^1(q) & \text{if } q \in (\bar{q}_j - \varepsilon, \bar{q}_j] \\ g^2(q) & \text{if } q \in [\bar{q}_j, \bar{q}_j + \delta) \\ \beta_{m+1} & \text{if } q \in [\bar{q} + \delta, a_{m+2}] \end{cases}$$

Clearly, $\tilde{c}_j'$ is continuous and $\tilde{c}_j'(\bar{q}_j) = \gamma$. Hence, $\bar{q}_j$ is a best response to $\bar{q}_{-j}$ when firm j operates under cost $\tilde{c}_j = \int \tilde{c}_j'$. Finally, observe that δ is chosen such that, over $(\bar{q}_j - \varepsilon, \bar{q}_j + \delta)$, the areas below c_j' and $\tilde{c}_j'$ coincide. Therefore, $\tilde{c}_j$ is smooth over $[a_m, a_{m+2}]$, coincides with c_j over $[a_m, \bar{q}_j - \varepsilon]$ and $[\bar{q}_j + \delta, a_{m+2}]$, and approaches c_j as ε approaches zero. This concludes the argument.

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