Modeling Financial Durations in Ultra-High-Frequency Data: A Markov-Switching Multifractal Approach

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Abstract of the Dissertation

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This thesis focus on the modeling and forecasting financial durations by the Markov-Switching Multifractal Duration (MSMD) model with using ultra-high-frequency (UHF) data. We first review the literatures of both of autoregressive conditional duration (ACD) models and MSMD models, and point out the limitations of applying these models in the analysis of UHF data. We then study the facts of inter-trade, price and volume durations from the NASDAQ TotalView-ITCH database for US stocks, which consists of high precision transaction time stamp. We reveal a discovery of spike and diurnal periodic pattern in intra-day durations and introduce a new procedure for the adjustment of periodicity. We discuss the properties of MSMD model and show that the model is capable of generating high persistence and long memory series. We assume that innovations follow exponential, gamma, Weibull, Burr or generalized gamma distributions and propose an improvement for the Gam-MSMD process. The new process captures dynamics of both of the shape and scale parameters in gamma distribution. We perform Maximum-Likelihood Estimation for the MSMD models. The empirical evidence we present here is in conformance with the theoretical properties of the model, and both gamma and Weibull are shown to better fit the data than the exponential in certain aspects. The estimation and simulation results show that higher number of the Markov components does not necessarily bring better performance. Finally, we compare the out-of-sample forecasting performance of the ACD and MSMD models based on inter-trade, price and volume durations of four major equities (MSFT, INTC, FB and QCOM) traded on NASDAQ, and conclude that the MSMD models outperform ACD models. The Gamma and Weibull innovations are superior to the exponential in all types of durations. The complex distributions, Burr and generalized gamma, could not provide any improvements on fitting or forecasting. We also show that the modified Gam-MSMD process performs better in terms of capturing the variance of sequence.

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1 Introduction

1.1 Algorithmic Trading and Limit Order Book

A limit order market is an order-driven market that automatically collects orders, and matches the buyers and sellers in a centralized limit order book (LOB) based on some priority rules. As frequencies of trading have become extremely high in recent years, a great number of order events can be generated for a single stock in a very short time period. Correspondingly, the time interval between two order events has reached the nanosecond-scale (i.e., 10^{-9} second), and tends to be even finer with the rapid development of information technology. Such high frequency brings unparalleled challenge to researchers on modeling and analyzing trading events. As an example, although the NASDAQ stock market was initially a quote-driven market in which only market makers facilitate transactions, nowadays it has become a hybrid market where customer limit orders are allowed in addition to on-exchange market making, by using the electronic communication networks (ECN). In terms of total market activities, the share of using ECN has increased dramatically in the past years, and recently ECN capture more than 40% volume in NASDAQ securities (Fink et al. [2006], Hendershott [2003]). Given the high frequency feature of electronic limit orders in the market, the providers of market liquidity have transferred from the orders given by traditional market makers to the LOBs. Therefore, understanding the dynamics of the LOB has become extremely important for both market participants and academic researchers, as it allows them to gain more insight into the market mechanism and merits of orders in different situations (Harris and Hasbrouck [1996]) and optimize order execution strategies (Obizhaeva and Wang [2005]).

1.2 Motivation and Goal

In this paper, we focus on empirical features of financial durations in the LOB market. Three types of financial durations will be investigated, which include inter-trade duration, price duration and volume duration. Inter-trade duration is the time elapsed between transactions. It is highly related with trading behaviors and price formulation processes. Price duration is the time required to observe a change in the mid-price greater than a given threshold. It measures the price movement and is closely related to volatility. Volume duration is defined as the time until a certain amount of volume is traded on the market. It reflects the intensity of liquidity demand.

Recent studies have demonstrated the following important properties of financial durations. First, the transaction-time trade arrivals are intimately related to the calendar time volatility in price (Clark [1973], O'Hara [1997], Engle and Russell [1998], Engle [2000], Bauwens and Veredas [2004]). Specifically, the serial correlation in transaction-time trade arrivals drives the serial correlation of calendar-time trade counts, which further drives the serial correlation in calendar-time volatility. This feature is crucially important in terms of risk management, portfolio allocation and asset pricing. Second, returns interact with trade duration. For instance, short duration moves price more than long duration across stocks and across time (Manganelli [2005], Furfine [2007]). Third, transaction-time trade arrival intensity is ultimately driven by serial correlation in the information flow that drives trading (Diamond and Verrecchia [1987], Dufour and Engle [2000a], Simonsen [2006]). The time elapsed between transactions is believed to contain some message on the information flow and can be passed to market participants. Relevant information may be related to the valuation of the stock.

1.3 Literature Review

We note that some stylized facts of financial durations have been studied intensively over the past decades, for example, the long-range dependence (i.e., duration tends to be persistent), the heavy tailedness (i.e., extremely short or long duration can often be observed), and the clustering (i.e., short duration follows short duration and long duration follows long duration); see discussions in Jasiak [1999], Bauwens and Giot [2000], Dufour and Engle [2000b] and Chen et al. [2013]. Among these efforts, one breakthrough in modeling financial market durations is the autoregressive conditional duration (ACD) model of Engle and Russell [1998], which expresses the conditional expectation of duration as a linear function of past duration and past conditional expectation. And it is an analog of Engle [1982]'s autoregressive conditional heteroskedasticity model in duration analysis. Bauwens and Veredas [2004] extended the discussion and proposed a stochastic conditional duration (SCD) model, which is similar to the stochastic volatility model in Ghysels et al. [2004] and allows the conditional mean duration to depend on some latent information [4]. These models have been extended to discuss different features of the trade data in the past years; see brief reviews in Bauwens and Giot [2000], Feng [2004], Simonsen [2006], Zhang et al. [2001], and Hautsch [2011]. For example, Bauwens and Giot [2000] extended the ACD model to the logarithmic ACD model. Feng [2004] proposed a linear non-Gaussian state-space version of the SCD model to capture the leverage effect of the expected durations. Simonsen [2006] extended the ACD model to examine the dependence between durations. One of the most flexible parametric ACD specifications was proposed by Zhang et al. [2001], which has a regime switching structure that allows different regimes to have different persistence, conditional means, and error distributions. Besides the multiplicative frameworks discussed above, another way of studying the empirical features of trading is to directly model hazard rates or intensities (Russell [1999], Bauwens and Hautsch [2009]). Note that, base on the point process theory, the duration between two events follows an exponential distribution conditional on the hazard rate (or trade arrival intensity), hence the dynamics of the trade durations may be modeled directly. Recently, Chen et al. [2013] proposed a Markov-switching multifractal inter-trade duration (MSMD) model, which was inspired by the success of the Markov-Switching Multifractal (MSM) stochastic volatility model in forecasting persistent volatility of financial returns (Calvet and Fisher [2004]). The MSMD model uses an elegant structure and parsimonious parameters to generate rich dynamics of inter-trade durations, and the inter-trade durations are composed of multiple fractals, with each of them following a distinct hidden Markov process. The model was applied to the 1993 NYSE stock data and showed to capture the long memory property of inter-trade duration very well. However, we notice that the trading frequency in the 1993 NYSE stock market is not very high, comparing to the recent ultra-high frequency data in the LOB market. Specifically, the LOB data we collected from the NASDAQ market has a large dispersion in the inter-trade duration, as it ranges from 10^{-5} seconds to 10^2 seconds, while the inter-trade duration in the 1993 NYSE stock data ranges from 1 second to several hundreds of seconds. Due to these issues, the MSMD model in Chen et al. [2013] could not fit the recent NASDAQ LOB data well.

1.4 Our Contributions

We study the features of ultra-high-frequency (UHF) data in nanosecond and propose a new method for removing the seasonality. We extend the MSMD model by modifying their assumption for error distributions. Specifically, we keep the multifractal feature of the MSMD model, but relax the assumption of a single exponential distribution for error distribution by mixtures of exponentials (i.e., gamma, Weibull, Burr or generalized gamma distributions). We modify the structure of the MSMD model and introduce a new process to balance the shape and scale parameter of gamma distribution. We show that the extended model fits current UHF data better than the original one and demonstrate the empirical features of UHF data that can be captured by the modified model. We report the empirical studies of the MSMD models and also the implementation in Flash Crash.

1.5 Structure of the Thesis

The rest of the paper is organized as follows. In Chapter 2, we use the NASDAQ LOB data as an example, and demonstrate some important empirical facts on financial durations in the UHF data. We also introduce a new approach for the removal of observed seasonalities. In Chapter 3, we present the details of our models. In Chapter 4, we introduce the estimation method and the simulation results. We also investigate the finite sample properties of our models. In Chapter 5, we apply the modified MSMD models to the 2015 NASDAQ LOB stock data and compare their performance with the traditional MSMD model and ACD model. We also investigate the Flash Crash implementation. In Chapter 6, we draw our conclusion of this dissertation and a brief summary of the problems we solve.

2 Properties of Ultra-High-Frequency Financial Durations

2.1 Characteristics of Transaction Data

2.1.1 Data description and data handling

Our data is downloaded from LOBSTER (https://lobsterdata.com/), which is an online LOB data tool with the goal of providing high-quality LOB data of the entire universe of NASDAQ stocks from June 2007. The LOB data reconstructed by LOBSTER are based on NASDAQ's Historical TotalView-ITCH data (i.e., the historic record of what NASDAQ calls). NASDAQ's TotalView-ITCH data feed contains event message data, in which the information on the limit order event that changes the order book is recorded.

Tables 1 and 2 show a sample of LOB event message and order book, which are LOB events of Microsoft Corporation recorded in a time period of May 13, 2015. In Table 1, the Event Type 1, 2, and 3 mean submission, cancellation and deletion of a limit order, respectively. Event Type 4 represents execution of a visible limit order. In Table 2, we only display the level of one bid and ask (best bid and ask) price and volume here. Thus, from the event message and order book files provided in the LOB, one can easily construct the durations series for a particular stock in a certain time period.

Figure 1 shows two examples from the LOB database, the stocks of Microsoft (MSFT) and Google (GOOG). The red and blue lines are the bid and ask prices. We can see the bid and ask price changes of MSFT are much less frequently compare to the GOOG. Moreover, the bid-ask spread of MSFT remains in one tick at most of time, while the bid-ask spread of GOOG vary over time. The green circles are the executed market orders, and the black circles are the executed hidden orders. We find that the green circles are exactly at the bid-ask lines, but the black circles could be anywhere within the spread. That is because the hidden orders have the priority to be executed when market orders appear. In all, the microstructures are quite different between stocks, and it would be interesting to explore the features of LOB data.

Table 1: Message file of LOB event

Time (sec)	Event Type	Order ID	Size	Price	Direction
43357.245326775	1	121043331	100	479800	1
43357.245347603	1	121043332	100	479900	-1
43357.252946609	3	120818656	10	479800	1
43357.252994607	3	121043331	100	479800	1
43357.253010209	3	121043332	100	479900	-1

Table 2: Order book file of LOB event

Ask Price	Ask Size	Bid Price	Bid Size
479900	8850	479800	3256
479900	8950	479800	3256
479900	8950	479800	3246
479900	8950	479800	3146
479900	8850	479800	3146

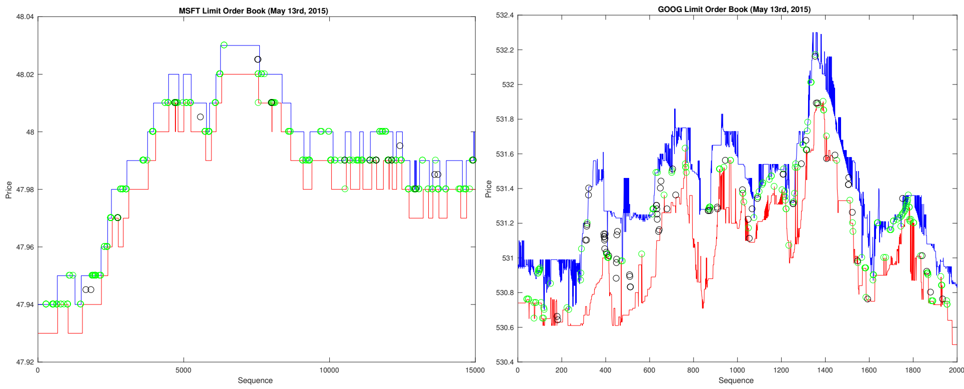


Figure 1: Limit order book sequence

2.1.2 Definition of Financial Durations

In this section, we discuss the definitions of inter-trade, price and volume durations.

Inter-trade duration In the trade executions process, a inter-trade duration X_i^{Trade} is defined as the time elapsed between two consecutive successive trades executed at t_i and t_{i-1}, such that

$$X_i^{\text{Trade}} = \begin{cases} t_i - t_{i-1} & i = 2, \ldots, n, \\ t_i & i = 1, \end{cases}$$

with $t_0 = 0$.

In order to reduce noise from transaction data, we aggregate inter-trade duration under the following criterions: (i) Orders executed simultaneously. (ii) Trading volume smaller than five, which filter test trades. (iii) Price does not change and inter-arrival time smaller than ten seconds.

Price duration To minimized biases caused by bid-ask bounce, we implement midquote duration instead of price duration. Midquote mq is defined as the midpoint of bid-ask spread, so price duration is the time required to observe a change in midquote greater than a given threshold ζ. We have

$$mq_i = \frac{bid_i + ask_i}{2}$$

$$X_i^{\text{Price}} = t_i - t_{i'}, \quad \text{if } |mq_i - mq_{i'}| \geq \zeta,\, i' < i$$

We focus on the large tick stocks that the bid-ask spread is almost always at one tick, and we deploy tick time sampling which is sampling whenever midquotes change by a tick.

Volume duration Volume duration is defined as the time until the traded volume aggregation reach certain amount ϑ. It can be showed as

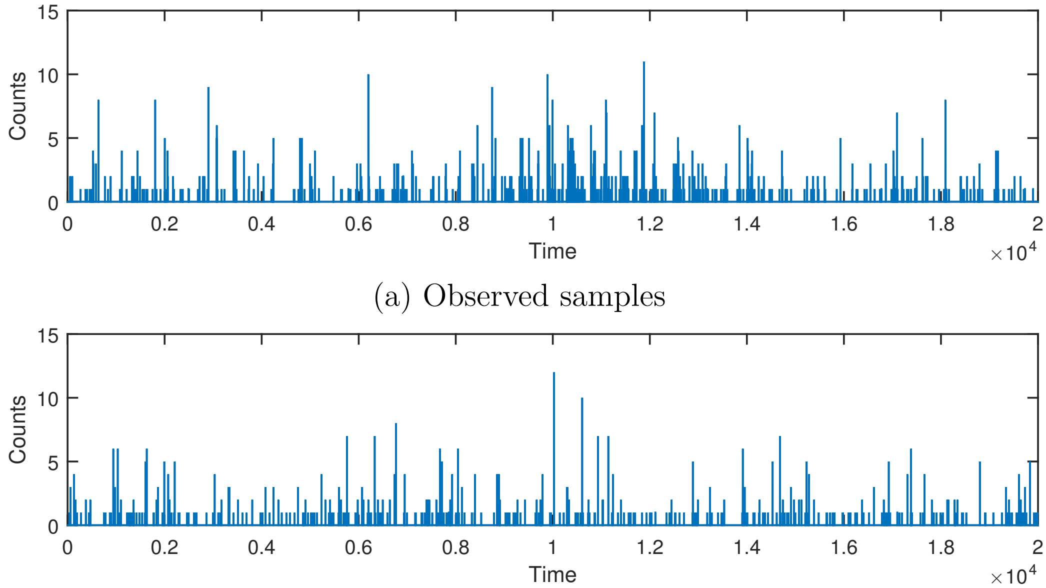
$$X_i^{\text{Volume}} = t_i - t_{i'}, \quad \text{if } \sum_{j=i'+1}^{i} v_j \geq \vartheta,\, i' < i, i > 1$$

where v_i the volume at trade i. We set ϑ to be equal to ten times of the average trading volume of the day.

2.2 Features of Ultra-High-Frequency Durations

Clustering

Figure 2 illustrates trading counts per one-tenth second. In subfigure (a), it shows a trend that large counts tend to be followed by large counts, of either sign, small counts tend to be followed by small counts. To check this, we permute the observed time series randomly, however, similar pattern does not occur in the new series. That is to say, the trades are clustering. Large durations come with large durations, small durations come with small durations.



(a) Observed samples

(b) Random permuted sequence

Figure 2: Time series of trading counts. The figure shows the number of trades executed in 0.1 second.

Small duration / zero duration

Another feature is zero or extremely small duration, the instances of multiple trades that occur simultaneously. The zero duration trades are discarded by most of the researchers who argued that these trades are executed simultaneously and they may arise from split transactions. However, we should not ignore the fact that most of the previous studies of HF financial duration are based on the data with timestamps in seconds. As we show in 3, by studying the UHF data in nanoseconds, there are about seventy percent of raw trade durations below one second after removing the 'real' trades with zero durations. Simply ignoring these small durations would cause a huge information loss without a doubt. During our UHF study, we keep meaningful small durations which come with price changes.

Large variation

From Tables 3, 4 and 5, the standard deviations are generally very large. As we observe, some durations are extremely short (to the extend of 10^{-7} seconds), while some are very long (to the extend of $10^2 \sim 10^3$ seconds). In the histogram of logarithm of MSFT durations, which is shown as Figure 3, the logarithmic trade duration is showed to be bimodal distributed with two peaks at e^{-9} seconds and e^1 seconds, while price and volume durations exhibit unimodal at around $e^{2.5}$ seconds with long and thick left tails. The fact that these durations span such a large range is probably related to the high frequency trading and has never been discussed before.

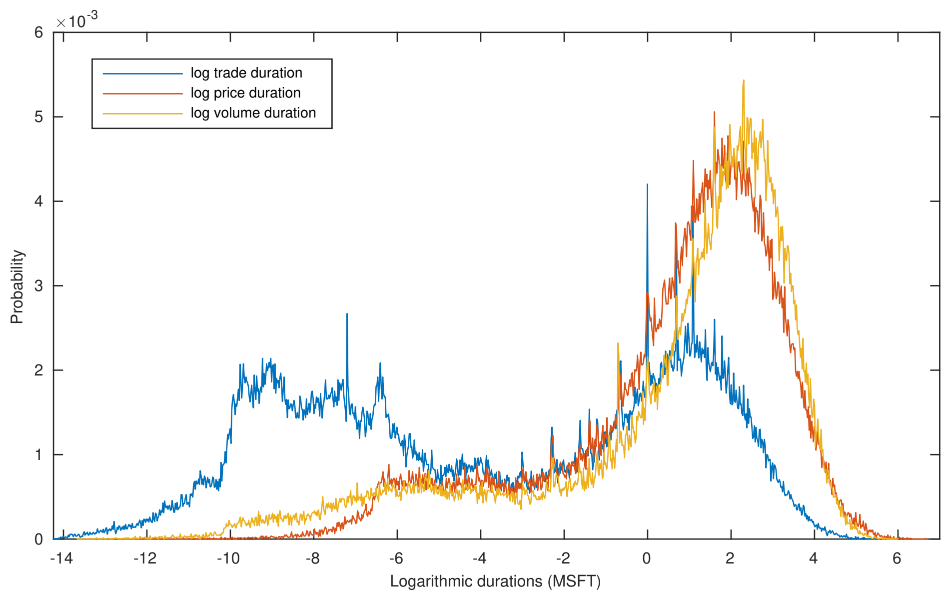


Figure 3: Histogram of financial durations of MSFT during May 2015

Overdispersion and heavily tail

This refers to the standard deviation exceeding the mean to a huge extent. Standard homogeneous Poisson process suggests that the duration should be an independent and identical exponential distribution. However, in the exponential $Q-Q$ plot for this duration series shown in Figure 4, we find all types of financial durations are non-exponential distributed with extraordinarily heavy tails. From the previous studies, trade and price durations exhibit overdispersion, while volume duration shows underdispersion. Our observations confirm that trade and price durations are highly overdispersion (in Table 3 and Table 4). The dispersion of volume duration is lower than trade and price durations, however some of the samples also show overdispersion as we can see in Table 5. The features of UHF data are quite different compare to lower frequency data. The Hill estimator of the tail index (Embrechts et al. [1997]) is reported in the Tables.

Daily Periodic or Diurnal Pattern Effects

Figure 5 represents moving average of the financial durations of MSFT during May 2015. In the figure, the x-axis is divided into 19 portions, each of the portion represents a trading day of the month. We note the following features: In most of days the durations tend to have a U-shaped transaction intensity. It implies that the trading frequencies are higher at opening and closing hours, while the activity is relatively lower at lunch time. This is somehow reflecting the behavior of traders: At the market opening, the high frequency transactions may be effected by the overnight information, and the traders tend to trade around the closing time due to lower market impacts. We can extend the research to all the stocks to observe the similar

Table 3: Descriptive statistics of inter-trade duration in May 2015

	MSFT		FB		INTC		QCOM	
	raw	adj	raw	adj	raw	adj	raw	adj
Mean	2.098	0.894	2.386	1.206	3.843	0.940	3.648	1.097
Median	0.018	0.504	0.235	0.716	0.103	0.475	0.211	0.523
Mode	2.4e-6	1.2e-6	4.1e-6	1.9e-4	8.5e-5	4.5e-7	3.3e-5	1.2e-6
Minimum	8.3e-7	1.2e-6	8.9e-7	3.0e-6	8.5e-7	4.5e-7	9.7e-7	1.2e-6
Maximum	194.4	22.06	86.34	20.90	322.3	30.71	111.8	35.95
S.D.	5.991	1.286	5.020	1.571	11.03	1.447	8.330	1.738
Dispersion	2.856	1.438	2.104	1.303	2.871	1.539	2.284	1.585
Skewness	7.177	4.675	4.407	3.230	7.013	5.257	4.423	4.820
Kurtosis	93.47	41.26	33.11	21.97	85.86	55.36	30.04	48.11
$\leq 1s$	0.711	0.727	0.627	0.600	0.646	0.722	0.603	0.677
$\geq 10s$	0.056	0.003	0.063	0.004	0.101	0.003	0.108	0.005
Tail	3.457	0.636	2.874	0.912	2.772	0.730	3.825	1.079

phenomenon.

We also find that the diurnal pattern of price duration is not showed as clear as the trade and volume durations. That is because, at the end of the trading day, the price fluctuation slows down when the volume of limit orders increased. The pattern of trade duration has longer period as rich amount of transactions are gathered during the day.

Table 4: Descriptive statistics of price duration in May 2015

	MSFT		FB		INTC		QCOM	
	raw	adj	raw	adj	raw	adj	raw	adj
Mean	9.837	0.841	12.15	0.965	16.91	0.853	21.51	0.845
Median	3.475	0.408	5.483	0.581	5.518	0.409	10.29	0.496
Mode	9.1e-6	5.3e-4	4.2e-5	0.129	7.1e-6	1.7e-4	1.1e-5	1.0e-6
Minimum	9.1e-6	3.0e-6	4.2e-5	1.4e-5	7.1e-6	1.7e-6	1.1e-5	1.0e-6
Maximum	428.31	25.08	295.8	15.58	891.8	25.88	602.8	15.15
S.D.	18.906	1.325	19.10	1.192	34.16	1.389	32.86	1.119
Dispersion	1.922	1.576	1.571	1.235	2.020	1.630	1.527	1.324
Skewness	6.310	4.853	3.881	2.920	7.022	5.563	4.270	3.555
Kurtosis	79.589	47.78	27.38	17.74	99.28	61.15	36.39	23.60
$\leq 1s$	0.288	0.737	0.207	0.667	0.245	0.734	0.129	0.723
$\geq 10s$	0.267	0.002	0.346	0.001	0.374	0.003	0.507	0.001
Tail	1.701	1.169	1.107	0.716	1.830	1.179	0.928	0.699

Spike Periodic Pattern

Figure 8 shows the average durations over a trading day by aggregating two years' samples of stock list in NASDAQ-100 Index. The durations are calculated by taking the average of each 30 seconds windows for each day, across all days and all stocks. Other than the U-shape daily pattern, we also find 5, 10, 30 minutes patterns and a 2:00 pm pattern. The durations drop sharply around the beginning of each 5, 10, 30 minutes intervals. This may be caused by the traders tend to set up electronic orders at the round time, and algorith-

Table 5: Descriptive statistics of volume duration in May 2015

	MSFT		FB		INTC		QCOM	
	raw	adj	raw	adj	raw	adj	raw	adj
Mean	10.21	1.021	30.02	1.040	16.92	1.054	53.65	1.195
Median	4.907	0.540	21.44	0.836	7.472	0.541	40.87	0.836
Mode	2.7e-6	2.8e-7	1.1e-5	8.3e-7	3.0e-6	1.8e-6	1.2e-5	6.4e-7
Minimum	2.7e-6	2.8e-7	1.1e-5	8.3e-7	3.0e-6	1.4e-7	1.2e-5	6.4e-7
Maximum	209.1	53.22	233.3	12.31	342.3	34.59	441.3	44.42
S.D.	15.10	1.760	28.65	0.924	26.03	1.715	49.11	1.641
Dispersion	1.480	1.723	0.954	0.888	1.539	1.627	0.916	1.374
Skewness	3.169	8.288	1.622	2.229	3.538	5.678	1.498	7.330
Kurtosis	18.91	150.3	6.314	13.46	23.27	61.22	6.088	106.4
$\leq 1s$	0.274	0.676	0.061	0.586	0.233	0.668	0.043	0.575
$\geq 10s$	0.321	0.005	0.726	0.0002	0.436	0.006	0.826	0.006
Tail	3.619	1.596	0.399	0.225	1.974	1.315	0.348	0.404

mic trading tries to catch the high volume during that time. Another spike pattern happens on Wednesday at 2:00PM as we observed, which appears a significant dip for all types of durations. It implies that the market may have important announcements released at this time.

ACF, long memory / high persistence

We present the autocorrelation of the observed financial durations of MSFT, INTC, QCOM and FB during May 2015 in Figure 9. Two features are showed in these figures: Firstly, all the sample autocorrelations at lag 1 are larger than 0.1, which are substantially larger than the 95% confidence bound. The autocorrelations gradually decrease to zero. Secondly, the autocorrelations display two seasonal patterns. A clearly daily periodic pattern repeats around every certain lags, which confirms what we discussed. We also see the long-term autocorrelations dropping from positive to negative, this suggests that the observation sequence may contain trend.

2.3 Stationary Financial Durations

We mainly observe two periodic patterns of financial durations in the previous section, which are the daily pattern and spike pattern. In this section, we show a brand new procedure to deseasonalize these patterns. Then, we summarize the results of the method and examine the adjusted sequence. The long-term trend that we presented can be removed by standardizing the series. As we focus on modeling the high frequency changes of duration, it is not necessary to consider this seasonal variations in this paper.

2.3.1 Adjustment for the Diurnal and Spike Pattern Effects

We estimate the spline functions for each weekdays. The spike pattern is defined as a linear spline function with using two years' NASDAQ-100 stocks data, while the diurnal pattern is defined as a cubic spline function which generated by three months' stock data. Since durations drop off quickly at the round-off five, ten or thirty minutes, so the spike pattern effects are generated by setting the nodes at every half minutes of the day and calculating the average duration across stocks. The trend of spike pattern is removed by implementing the Savitzky–Golay filter. The diurnal pattern effects are calculated by taking the average of thirty minutes sub-periods of duration, and they are set as the start points of intervals. The nodes are 9:30, 10:00, 10:30,..., 15:30, 16:00. The diurnal and spike pattern effects can be removed from the durations by implementing two steps: Firstly, the spike pattern is adjusted by dividing the linear spline $S_j^S(t_i)$ from original duration X_i

$$\widehat{X_i^S} = \frac{X_i}{S_j^S(t_i)}.$$

Secondly, we estimate the diurnal pattern spline $S_k^D(t_i)$ with using the spike adjusted duration $\widehat{X_i^S}$ and remove it from durations

$$\widehat{X_i} = \frac{\widehat{X_i^S}}{S_k^D(t_i)}.$$

Finally, we obtain the deseasonalized duration $\widehat{X_i}$. t_i is the start time of duration X_i. The spline functions are given by

$$\begin{aligned}
\widehat{S_j^{\mathrm{S}}(t)} &= u_j + \frac{(u_{j+1} - u_j)(t - t_j)}{t_{j+1} - t_j}, \quad t \in [t_j, t_{j+1}],\ j = 1, 2, \ldots, n \\
\widehat{S_k^{\mathrm{D}}(t)} &= v_k + \omega_{k,1}(t - t_k) + \omega_{k,2}(t - t_k)^2 + \omega_{k,3}(t - t_k)^3, \\
& \qquad t \in [t_k, t_{k+1}],\ k = 1, 2, \ldots, m
\end{aligned}$$

where u_j is the spike pattern effects in $[t_j, t_{j+1})$ after removing trend by Savitzky–Golay filter. v_k is the diurnal pattern effects in $\left[t_k - \frac{I}{2}, t_k + \frac{I}{2}\right)$, while I is the time interval between effects. v_1, v_m is assigned the intervals $\left[t_0, t_0 + \frac{I}{2}\right)$ and $\left[t_m - \frac{I}{2}, t_m\right]$.

The results of estimated spike and diurnal pattern effects are shown in Figures 10, 11, 12 and 13. In Figure 10, all weekday plots have similar features except Wednesday which drops substantially at two o'clock in the afternoon. We also report a huge fluctuation during the market closing time. Diurnal pattern effects calculated after removed spike pattern from durations (Figures 11, 12 and 13). We can see that all types of financial duration effects are U-shaped, however, the price duration effects are moving less than the other durations in the last hour. This might cause by the high volume of market orders and the market price can be hardly changed.

2.3.2 Analysis of Adjusted Durations

Figure 14 reports adjusted duration series of MSFT in May 2015. The seasonalities which described in previous section do not exist any more. Figure 15 compares the ACFs of raw and adjusted financial durations for MSFT during May 2015. It presents the autocorrelation of adjusted durations at lag 1 is approximately 0.15, which is slightly lower than the original autocorrelation except for inter-trade duration. The value of the autocorrelations decreases very slowly with increasing order of the lag. In fact the autocorrelation remains significantly larger than the confidence bound even at lag 2000. Therefore, it confirms the diurnal pattern is successfully removed. Inter-trade and price durations decay at similar speed, while volume duration decays faster before lag 200.

2.4 Fitting Distributions

In Figure 16, we report the sample fit with exponential, gamma and Weibull distributions on all types of financial durations. The performance of gamma and Weibull distributions are hard to distinct, and both of them successfully capture the peak of small size durations. The exponential distribution performs worse in terms of the tails fitting. In Q-Q plots, the adjusted durations versus exponential shows flatter than the line $y = x$ which indicates that the sample plotted on the horizontal axis is more dispersed than the theoretical exponential distribution. The Weibull distribution fits straightly on the $y = x$ of inter-trade duration, but it displays steeper than the line $y = x$ of price and volume durations which describes that it is more disperse than our samples. The gamma Q-Q plot presents good linear relation until five which covers the major quantiles of the samples, comparing to the exponential Q-Q plot starts to disperse at around two.

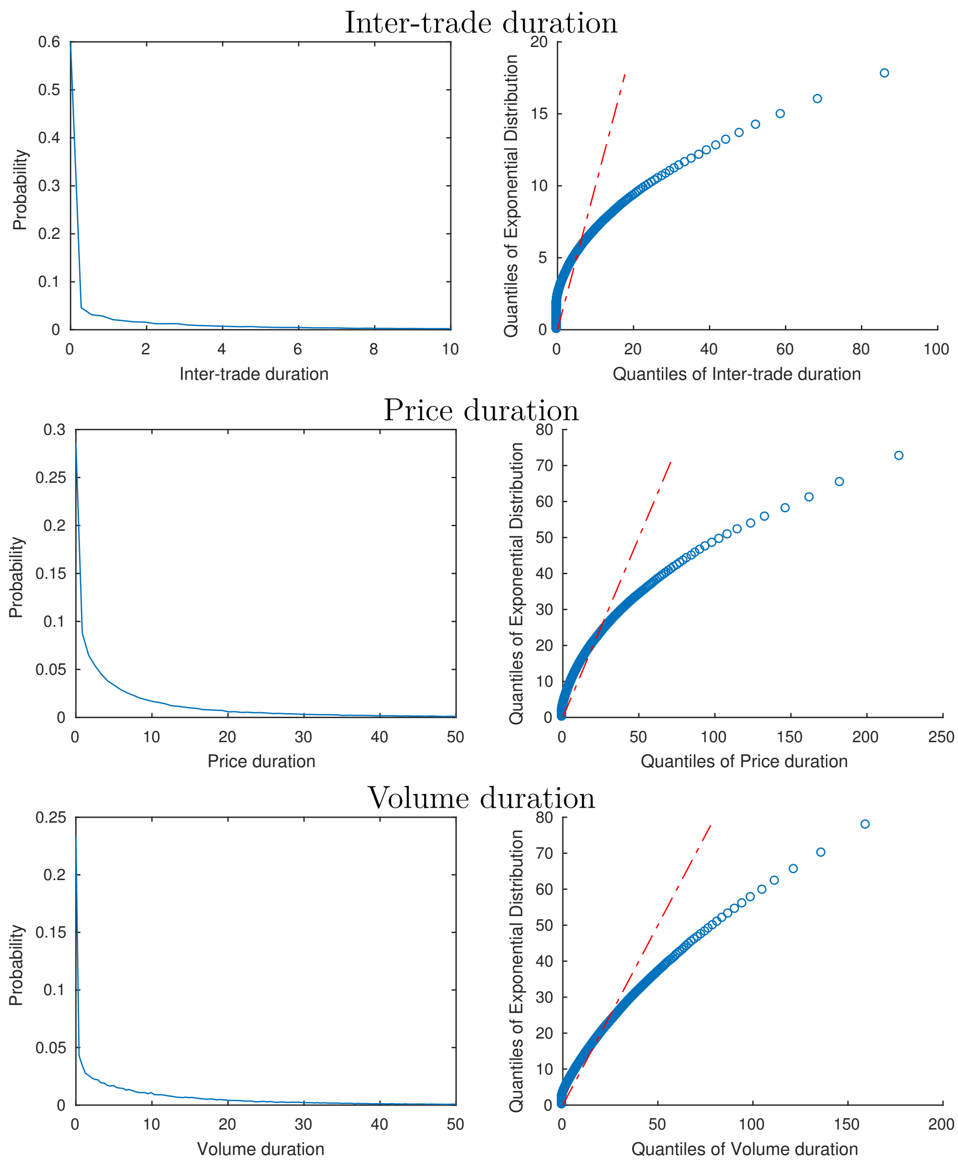


Figure 4: Density and Q-Q plot of the raw financial durations of MSFT during May 2015 based on exponential distribution.

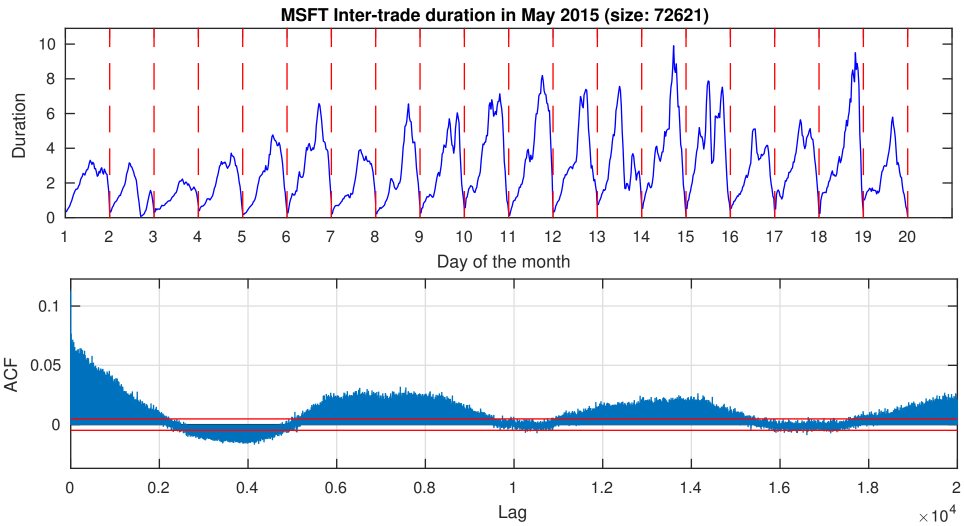


Figure 5: Diurnal pattern of inter-trade durations of MSFT. The x-axis denotes the trading days of May 2015, while y-axis is average duration size. The sequence is smoothed and set the start point of day to zero.

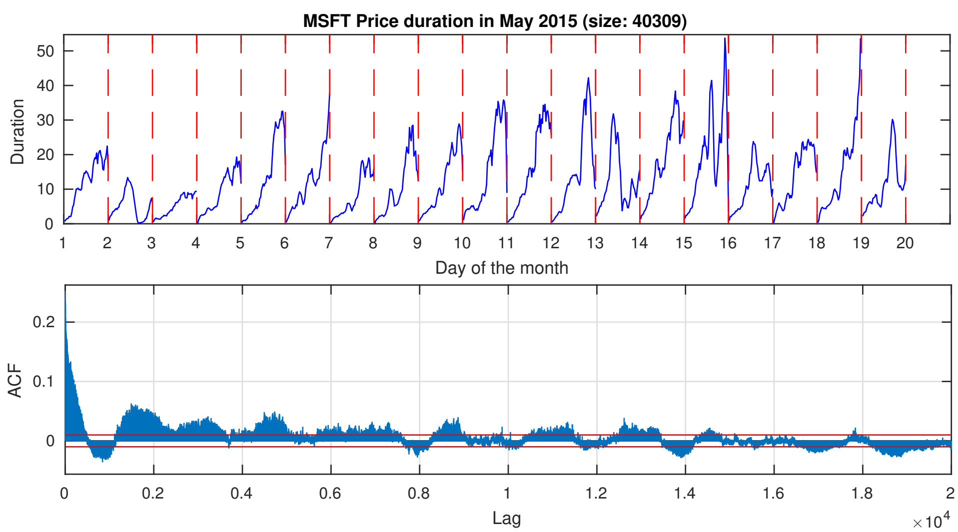


Figure 6: Diurnal pattern of price durations of MSFT. The x-axis denotes the trading days of May 2015, while y-axis is average duration size. The sequence is smoothed and set the start point of day to zero.

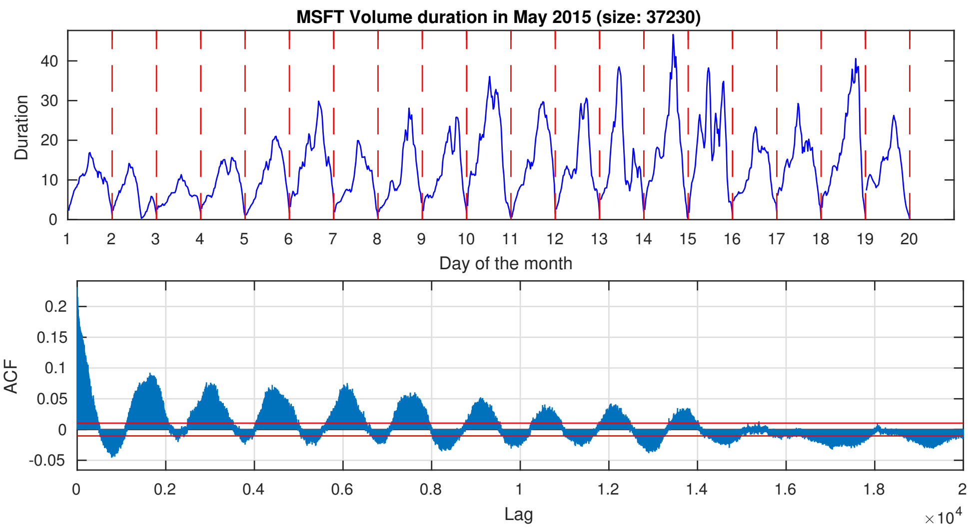


Figure 7: Diurnal pattern of volume durations of MSFT. The x-axis denotes the trading days of May 2015, while y-axis is average duration size. The sequence is smoothed and the start point of day is set to zero.

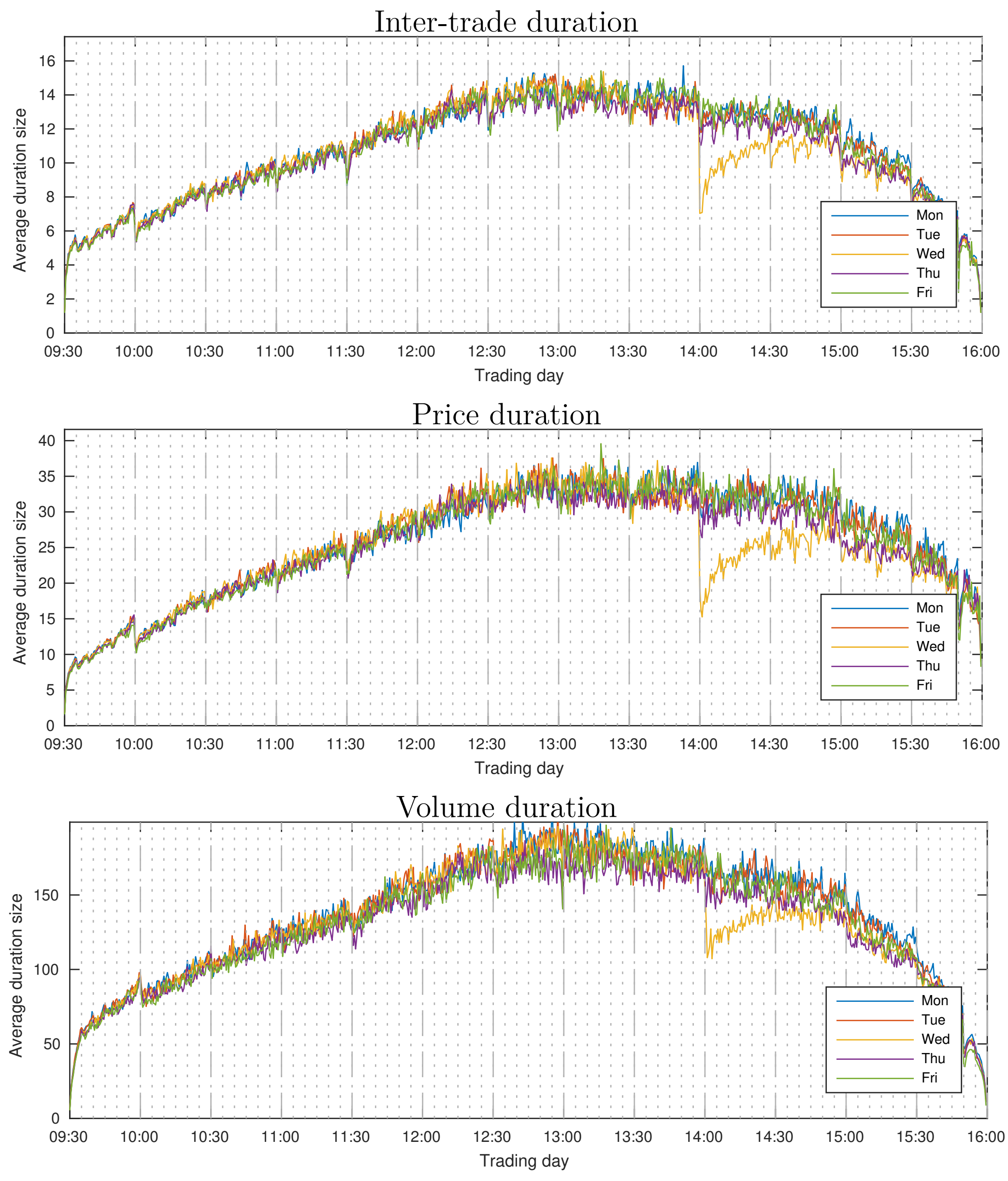


Figure 8: Spike periodic pattern of financial durations by using the NASDAQ-100 index data (20140101-20151231)

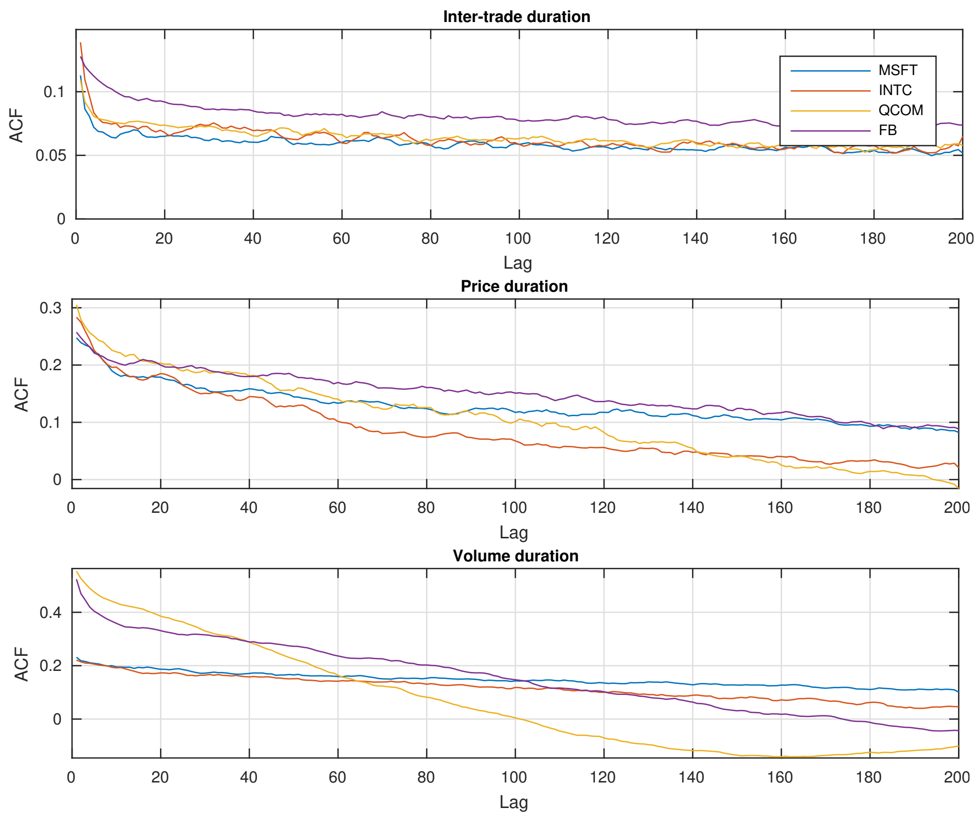


Figure 9: Autocorrelation function of raw financial durations

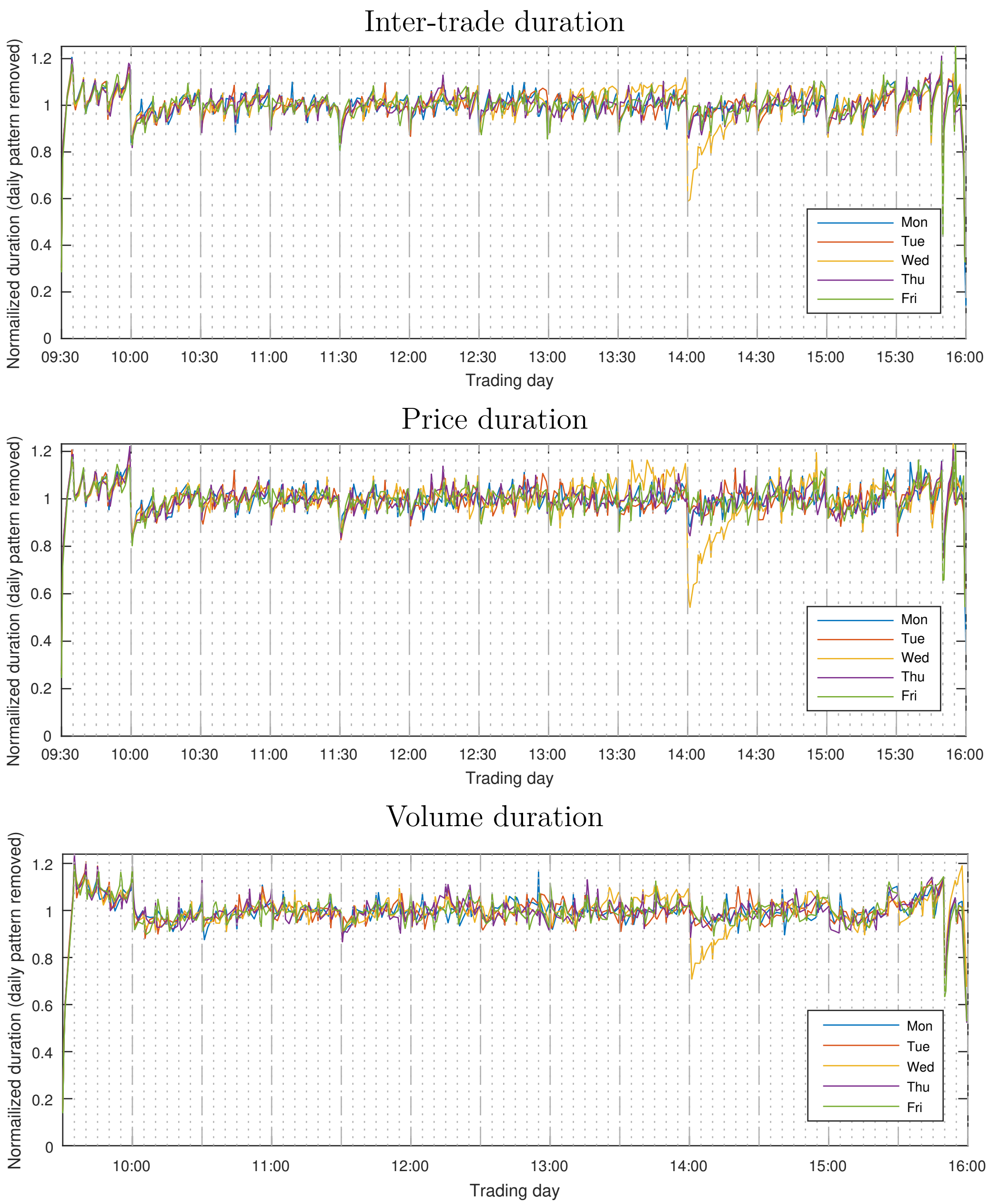


Figure 10: Spike pattern effects of financial duration by using NASDAQ-100 index data (20140101-20151231)

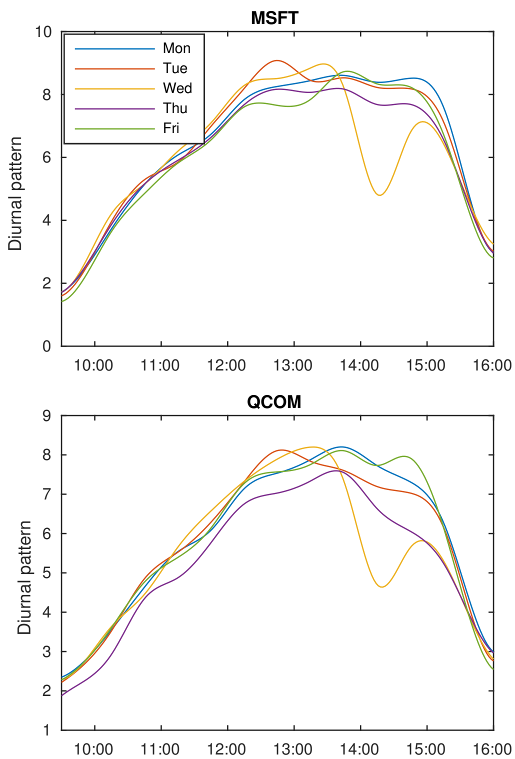


Figure 11: Weekday diurnal pattern of inter-trade duration (20150401-20150630)

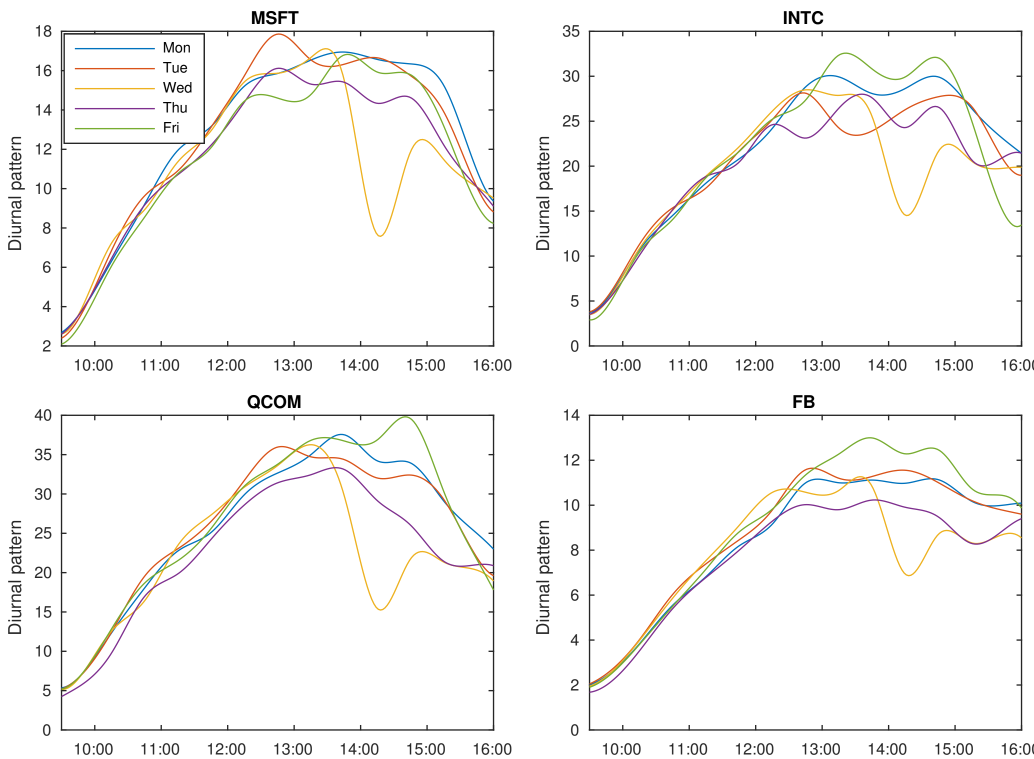


Figure 12: Weekday diurnal pattern of price duration (20150401-20150630)

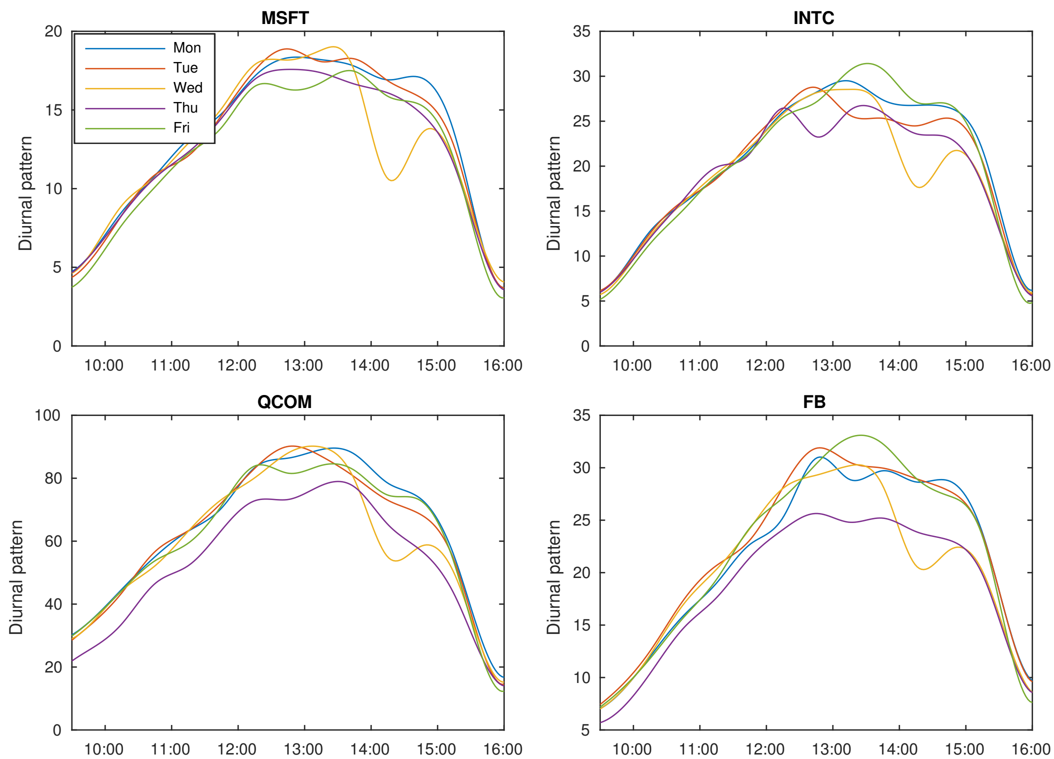


Figure 13: Weekday diurnal pattern of volume duration (20150401-20150630)

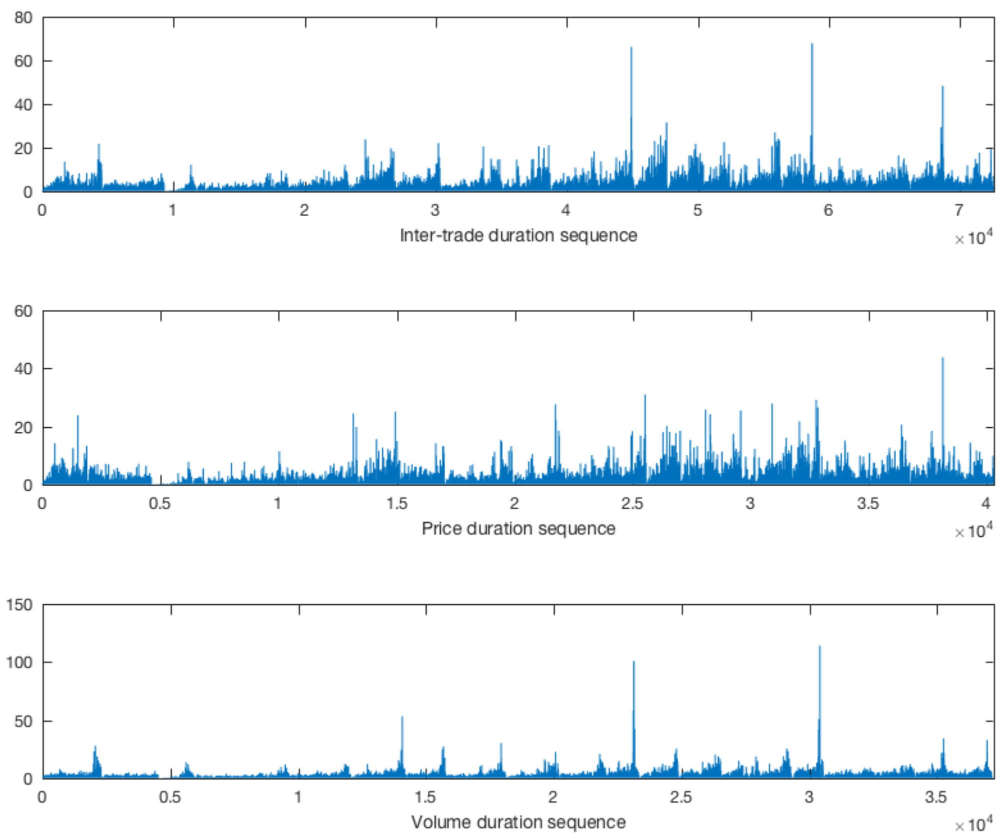


Figure 14: Adjusted duration series of MSFT in May 2015.

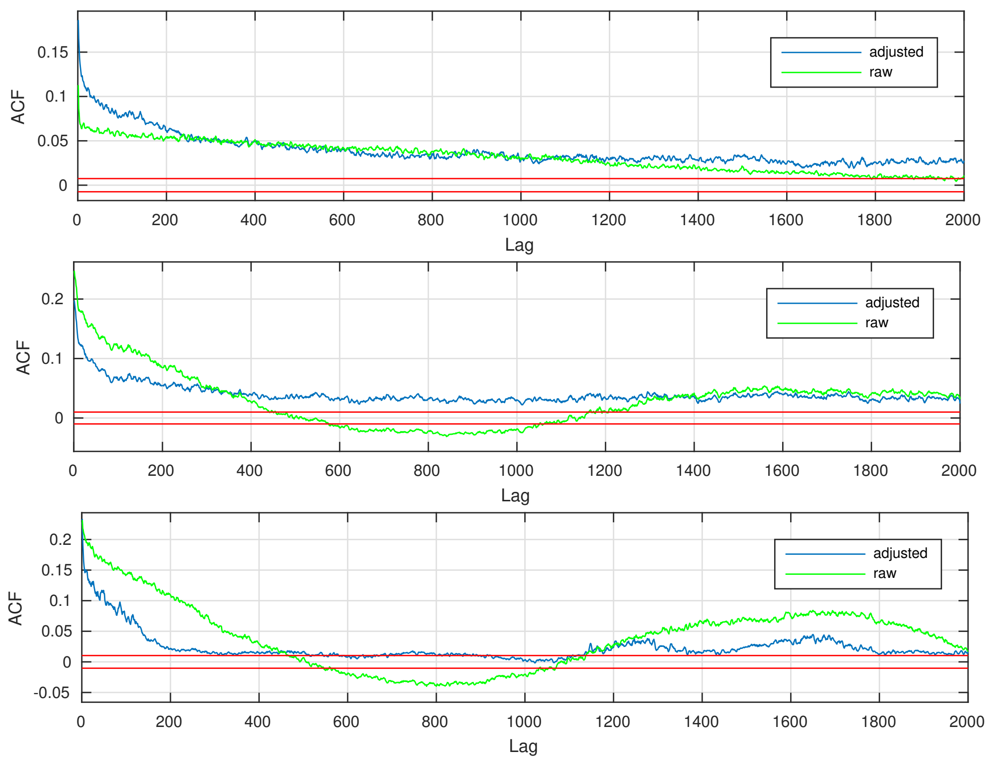


Figure 15: Comparison of the ACFs of raw and adjusted financial durations of MSFT during May 2015.

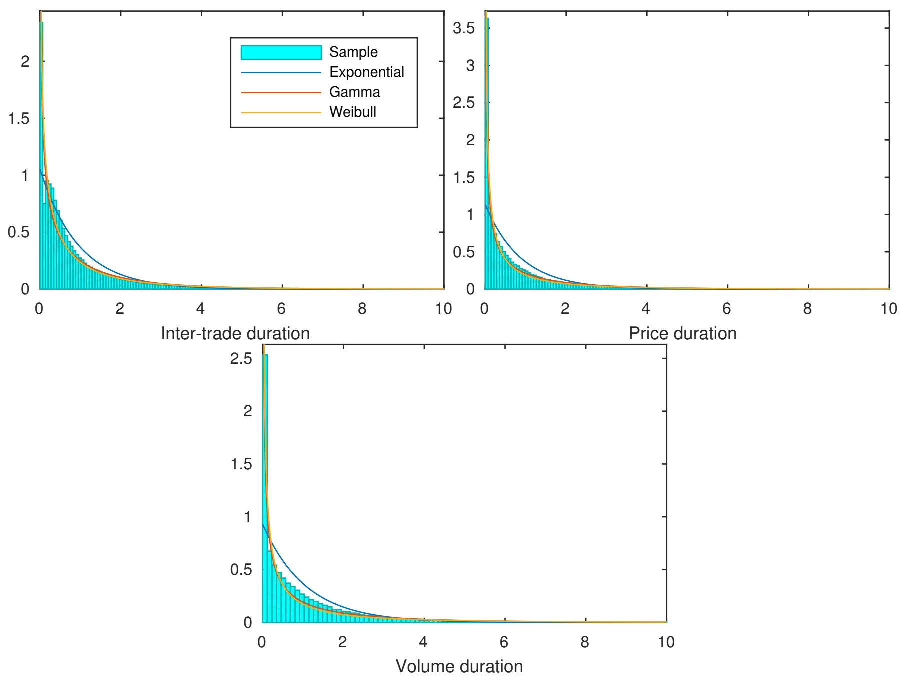


Figure 16: Fitting adjusted financial durations with distributions of exponential, gamma and Weibull using MSFT data in May 2015.

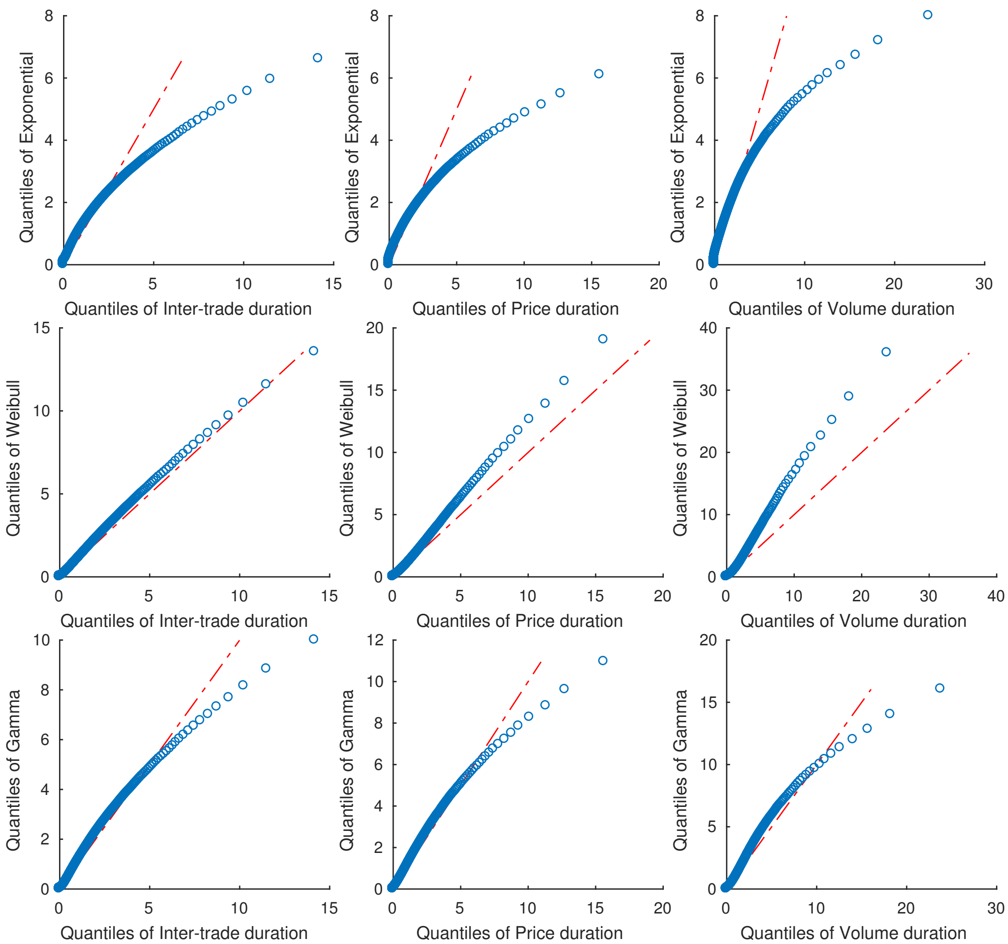


Figure 17: Q-Q plot of adjusted financial durations versus theoretical distributions using MSFT during May 2015. The columns from left to right are: Inter-trade, price and volume durations. The rows from top to down are: Exponential, Weibull and gamma distributions

3 The Markov-Switching Multifractal Duration Model

The Markov-Switching Multifractal (MSM) model is proposed by Calvet and Fisher [2004], and it has been proved that the MSM model can be successfully applying on forecasting persistent volatility of financial returns. Recently, Chen et al. [2013], Žikeš et al. [2014] adapted the MSM model to financial durations data, named the new model the Markov-Switching Multifractal Duration (MSMD). While Chen et al. [2013] introduced a MSMD model with mixture-of-exponentials representations, Žikeš et al. [2014] extended the results to the binomial and log-normal MSMD model with exponential or Weibull distributed innovations. In this section, we start by defining a binomial MSMD model, and then present the application of multiple distributed innovations. We also propose an innovative model, a modified gamma MSMD model.

3.1 Model Specification

Let X_i denotes the financial durations variable, $i = 1, 2, \ldots, T$. The Markov-Switching Multifractal Duration (MSMD) model is defined by

$$X_i = \psi_i \epsilon_i, \quad i \in \mathbb{Z}, \tag{1}$$

where ψ_i is the conditional expected duration at time i, $\{\epsilon_i\}$ are independent and identically distributed innovations with unit mean.

We now consider the Markov-switching multifractal model by Calvet and Fisher [2004]. The observed durations can be specified as a mixing process. The conditional expected duration function ψ_i follows the Markov-switching multifractal process as explained below.

$$\psi_i = \bar{\psi} \prod_{k=1}^{\bar{k}} M_{k,i}, \tag{2}$$

where $\bar{\psi}$ is a positive constant. In Equation (2), the system is assumed to be driven by a first-order Markov state vector with $\bar{k}$ components, the Markov state vector at time i is then written as

$$M_i = \left(M_{1,i}, M_{2,i}, \ldots, M_{\bar{k},i}\right) \in \mathbb{R}_+^{\bar{k}},$$

where each of component of M_i is independent and shares the same non-negative marginal distribution with unit-mean, its frequency heterogeneously decays at $\gamma_1, \gamma_2, \ldots, \gamma_{\bar{k}}$ which follows the range from low to high $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{\bar{k}} < 1$. Each stochastic multiplier can be treated as one type of effect factors for the economic system, and there is no correlation between different types of factors. The Markov-switching multifractal process is then shown as

$$\begin{array}{ccccccc}
\psi_1 & = & \bar{\psi} & M_{1,1} & M_{2,1} & \cdots & M_{\bar{k},1} \\
\psi_2 & = & \bar{\psi} & M_{1,2} & M_{2,2} & \cdots & M_{\bar{k},2} \\
\vdots & \vdots & & & \vdots & & \\
\psi_T & = & \bar{\psi} & M_{1,T} & M_{2,T} & \cdots & M_{\bar{k},T} \\
 & & & \downarrow & \downarrow & & \downarrow \\
 & & & type1 & type2 & \cdots & type\bar{k}
\end{array}.$$

By inserting Equation (2) to (1), X_i is then defined as

$$X_i = \bar{\psi}\epsilon_i \prod_{k=1}^{\bar{k}} M_{k,i}, \quad i = 1, \ldots, T, \tag{3}$$

Markov process. As we have defined, the $M_{k,i}$'s are statistically independent at time i. The sequence of $M_{k,i}$ follows Markov renewal processes, which means that at time i, $M_{k,i}$ either takes a new value drawn from a fixed distribution M with probability γ_k, or remains unchanged with probability $1-\gamma_k$. We can summarize $M_{k,i}$ as

$$M_{k,i} = \begin{cases} M & \text{drawn from distribution } f_M \text{with probability } \gamma_k \\ M_{k,i-1} & \text{with probability } 1-\gamma_k. \end{cases} \tag{4}$$

The new draws from f_M are assumed to be independent across k and i, with $M > 0$ and $\mathrm{E}(M) = 1$. Thus, the multipliers $\{M_{k,i}\}$ share identical marginal distribution f_M, the difference between them is their transition probabilities γ_k. A large γ_k gets a low persistence M, and a small γ_k gets high persistence M, M has little chance to change if γ_k is close to 0.

Calvet and Fisher [2004] has proposed the binomial and lognormal distributions for M. In this paper, we assume that the distribution of the components f_M is binomial, which draws values m_0 and $2-m_0$ with equal probability.

$$M = \begin{cases} m_0 & \text{with probability } 1/2 \\ 2-m_0 & \text{with probability } 1/2, \end{cases} \tag{5}$$

where $m_0 \in (1,2)$, ensuring that M is nonnegative and its mean is equal to one as defined previously. Thus, $M_{k,i}$ is a two-state Markov process with states $\pi_1 = m_0$ and $\pi_2 = 2-m_0$. From Equations (4) and (5), we could obtain the transition probabilities as

$$\begin{aligned}
P\left[S_{k,i}=\pi_1|S_{k,i-1}=\pi_1\right] &= p_{11} = & 1-\frac{1}{2}\gamma_k \\
P\left[S_{k,i}=\pi_2|S_{k,i-1}=\pi_1\right] &= p_{12} = & \frac{1}{2}\gamma_k \\
P\left[S_{k,i}=\pi_1|S_{k,i-1}=\pi_2\right] &= p_{21} = & \frac{1}{2}\gamma_k \\
P\left[S_{k,i}=\pi_2|S_{k,i-1}=\pi_2\right] &= p_{22} = & 1-\frac{1}{2}\gamma_k,
\end{aligned}$$

Then the transition matrix of $M_{k,i}$ could be summarized as

$$\boldsymbol{P}_k = \begin{bmatrix} p_{k,11} & p_{k,21} \\ p_{k,12} & p_{k,22} \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{2}\gamma_k & \frac{1}{2}\gamma_k \\ \frac{1}{2}\gamma_k & 1-\frac{1}{2}\gamma_k \end{bmatrix}. \tag{6}$$

Here the Markov process has an equilibrium distribution as the largest eigenvalue of $\boldsymbol{P}_k$ is equal to one. Since the components $\{M_{k,i}\}$ are independent, the transition matrix of M_i is given by

$$\boldsymbol{P} = \boldsymbol{P}_1 \otimes \boldsymbol{P}_2 \otimes \cdots \otimes \boldsymbol{P}_{\bar{k}}, \tag{7}$$

where $\otimes$ denotes the Kronecker product. The dimension of $\boldsymbol{P}$ is $2^k \times 2^k$.

Transition probabilities. The transition probabilities $(\gamma_1, \gamma_2, \ldots, \gamma_{\bar{k}})$ are parsimoniously parametrized by

$$\gamma_k = 1 - (1-\gamma_{\bar{k}})^{b^{k-\bar{k}}}, \tag{8}$$

where $\gamma_{\bar{k}} \in (0,1)$ and $b \in (1,\infty)$. Equation (8) can be alternatively shown as

$$\begin{aligned} 1-\gamma_k &= (1-\gamma_{\bar{k}})^{b^{k-\bar{k}}} \\ \log(1-\gamma_k) &= b^{k-\bar{k}} \log(1-\gamma_{\bar{k}}), \end{aligned}$$

which infers that the logarithms of unchanged probabilities $1-\gamma_k$ are exponentially increasing by k. The value of $1-\gamma_k$ is large when the value k is small, which implies that the component M_k has high persistence and low renewal probability. Hence, with this method, we could generate a series of γ_k ranging from low frequencies to high frequencies, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{\bar{k}} < 1$ as required. By combining Equations (8), (4) and (6), we have

$$M_{k,t} = \begin{cases} M & \text{w.p. } 1-(1-\gamma_{\bar{k}})^{b^{k-\bar{k}}} \\ M_{k,t-1} & \text{w.p. } (1-\gamma_{\bar{k}})^{b^{k-\bar{k}}} \end{cases}, \tag{9}$$

$$\boldsymbol{P}_k = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(1-\gamma_{\bar{k}})^{b^{k-\bar{k}}} & \frac{1}{2}-\frac{1}{2}(1-\gamma_{\bar{k}})^{b^{k-\bar{k}}} \\ \frac{1}{2}-\frac{1}{2}(1-\gamma_{\bar{k}})^{b^{k-\bar{k}}} & \frac{1}{2}+\frac{1}{2}(1-\gamma_{\bar{k}})^{b^{k-\bar{k}}} \end{bmatrix} \tag{10}$$

where $\gamma_{\bar{k}} \in (0,1)$ and $b \in (1,\infty)$. To summarize, the general MSMD model is defined by Equations (3), (5) and (9). Conditional on $\bar{k}$, the parameter vector is then

$$\boldsymbol{\zeta}_{\bar{k}} \equiv \left(\bar{\psi}, b, \gamma_{\bar{k}}, m_0\right) \in \mathbb{R}^4_+$$

where $\bar{\psi}$ governs the unconditional mean intensity of the system, m_0 characterizes the distribution of the components, and b and $\gamma_{\bar{k}}$ define the set of switching probabilities.

3.2 Alternative Markov-Switching Multifractal models

In this section, we discuss multiple distributional assumptions for the innovation ϵ_i. We explore the standard exponential distribution, and extend it to two parameter distributions Weibull and gamma. We also check out the generalized version of Burr and gamma.

The exponential distributed MSMD (Exp-MSMD)

For the Exp-MSMD, we assume the innovations, $\{\epsilon_i\}$, to be a exponential distribution with unit mean

$$f_E(\epsilon_i) = \exp\left(-\epsilon_i\right), \quad \epsilon_i \geq 0 \tag{11}$$

So we can get the density of the durations $\{x_i\}$ as

$$f_E(x_i;\psi_i) = \frac{\exp\left(-x_i/\psi_i\right)}{\psi_i}, \quad x_i \geq 0 \tag{12}$$

The Weibull distributed MSMD (Wbl-MSMD)

The Weibull distribution with $\epsilon_i \geq 0$ can be defined as

$$f_W\left(\epsilon_i;\kappa,\theta\right) = \frac{\kappa}{\theta}\left(\frac{\epsilon_i}{\theta}\right)^{\kappa-1}\exp\left\{-\left(\frac{\epsilon_i}{\theta}\right)^{\kappa}\right\}$$

where κ and θ are the shape and scale parameters. We normalize the distribution so that $\mathbb{E}\left(\epsilon_i\right) = \theta\Gamma\left(1+1/\kappa\right) = 1$ and the corresponding probability density function is given by

$$f_W\left(\epsilon_i;\kappa\right) = \kappa\left[\Gamma\left(1+\frac{1}{\kappa}\right)\right]^{\kappa}\epsilon_i^{\kappa-1}\exp\left\{-\left[\Gamma\left(1+\frac{1}{\kappa}\right)\epsilon_i\right]^{\kappa}\right\}, \quad \epsilon_i \geq 0 \tag{13}$$

the density of x_i is then

$$f_W\left(x_i;\psi_i,\kappa\right)=\frac{\kappa}{x_i}\left[\Gamma\left(1+\frac{1}{\kappa}\right)\frac{x_i}{\psi_i}\right]^{\kappa}\exp\left\{-\left[\Gamma\left(1+\frac{1}{\kappa}\right)\frac{x_i}{\psi_i}\right]^{\kappa}\right\},\quad x_i\geq 0 \tag{14}$$

The gamma distributed MSMD (Gam-MSMD)

The gamma distribution has scale and shape parameters, θ and κ, which gives it more flexibility compared to the exponential. Similar to exponential, gamma distribution also exhibits infinitely divisible. Following the definition of Exp-MSMD, we define innovations $\{\epsilon_i\}$ as Gamma distributed

$$f_G\left(\epsilon_i;\kappa\right)=\frac{\epsilon_i^{\kappa-1}}{\kappa^{-\kappa}\Gamma\left(\kappa\right)}\exp\left(-\epsilon_i\kappa\right),\quad \epsilon_i>0,\kappa>0 \tag{15}$$

where ϵ_i is normalized as $\mathbb{E}\left(\epsilon_i\right)=\kappa\theta=1$. The corresponding density of $\{x_i\}$ could be showed as

$$f_G\left(x_i;\psi_i,\kappa\right)=\frac{\left(\kappa x_i/\psi_i\right)^{\kappa}}{x_i\Gamma\left(\kappa\right)}\exp\left(\frac{-\kappa x_i}{\psi_i}\right),\quad x_i>0,\kappa>0 \tag{16}$$

The Burr distribution (Bur-MSMD)

We assume that the innovations $\{\epsilon_i\}$ are Burr distributed and normalize the distribution so that $\mathbb{E}\left(\epsilon_i\right)=1$, the corresponding probability density function is given by Grammig and Maurer [2000]

$$f_B\left(\epsilon_i;\kappa,\theta\right)=\omega\kappa\epsilon_i^{\kappa-1}\left(1+\theta^2\omega\epsilon_i^{\kappa}\right)^{-\left(\frac{1}{\theta^2}+1\right)},\quad \epsilon_i\geq 0 \tag{17}$$

where

$$\omega=\theta^{2\left(\frac{1}{\kappa}+1\right)}\frac{\Gamma\left(\frac{1}{\theta^2}+1\right)}{\Gamma\left(\frac{1}{\kappa}+1\right)\Gamma\left(\frac{1}{\theta^2}-\frac{1}{\kappa}\right)}\quad \text{and}\quad 0<\theta^2<\kappa$$

The distribution will become Weibull when $\theta^2 \to 0$. The density of x_i can be denoted as

$$f_B\left(x_i; \psi_i, \kappa, \theta^2\right) = \frac{\kappa\omega x_i^{\kappa-1}}{\psi_i^{\kappa}\left[1+\theta^2\omega\left(\frac{x_i}{\psi_i}\right)^{\kappa}\right]^{\left(\frac{1}{\theta^2}+1\right)}}, \quad x_i \geq 0 \tag{18}$$

The generalized gamma distributed MSMD (GG-MSMD)

Assuming the innovations $\{\epsilon_i\}$ to be generalized gamma distributed, we normalize the distribution so that $E\left[\epsilon_i\right] = 1$ and the corresponding probability density function is given by Lunde [1999]

$$f_{GG}\left(\epsilon_i; \kappa, \theta\right) = \frac{\theta\epsilon_i^{\kappa\theta-1}}{\omega^{\kappa\theta}\Gamma\left(\kappa\right)}\exp\left\{-\left(\frac{\epsilon_i}{\omega}\right)^{\theta}\right\}, \quad \epsilon_i \geq 0 \tag{19}$$

where

$$\omega = \frac{\Gamma\left(\kappa\right)}{\Gamma\left(\kappa+\frac{1}{\theta}\right)} \quad \text{and} \quad \kappa > 0, \theta > 0.$$

κ and θ are the shape parameters, while ω is the scale parameter. The density of x_i is then

$$f_{GG}\left(x_i; \psi_i, \kappa, \theta\right) = \frac{\theta\epsilon_i^{\kappa\theta-1}}{\left(\omega\psi_i\right)^{\kappa\theta}\Gamma\left(\kappa\right)}\exp\left\{-\left(\frac{\epsilon_i}{\omega\psi_i}\right)^{\theta}\right\}, \quad x_i \geq 0 \tag{20}$$

3.3 The Modified Gamma MSMD Model

In standard MSMD model, ψ_i governs the dynamic mean of the system, the parameters of the innovation distribution are fixed. Here we consider implementing the multifractal process for the innovation parameters. However, there are multiple challenges we have to face: Firstly, we need to decide whether the multifractal processes are independent or correlated. Secondly, as it is unlikely that the two processes are independent, the dramatically increased state space of the system is inevitable. Last but not least, we will have to deal with the high computation demanding. In this section, we will introduce a modified multifractal process so that the system gains more flexibility on controlling the distribution parameters without adding too much states.

We define the model based on the assumption that, conditional on the shape parameter κ_i and the scale parameter θ_i, the duration variables X_i at time i is gamma distributed

$$X_i \overset{d}{=} \text{Gamma}\left(\kappa_i, \theta_i\right), \quad \kappa_i > 0,\ \theta_i > 0,\ i \in \mathbb{Z}$$

then the probability density function is

$$f_{X_i}(x_i; \kappa_i, \theta_i) = \frac{1}{\Gamma\left(\kappa_i\right)\theta_i^{\kappa_i}} x_i^{\kappa_i - 1} \exp\left(-\frac{x_i}{\theta_i}\right). \tag{21}$$

We modify the MSMD model by defining the dynamic process to control both of the distribution parameters

$$\kappa_i = \psi_i^c \kappa, \tag{22}$$

$$\theta_i = \psi_i^{1-c}\theta, \tag{23}$$

where $c \in \mathbb{R}$, $\kappa > 0$ and $\theta > 0$. So the expectation of the system is $\psi_i \kappa \theta$, remaining the same as in the standard Gam-MSMD model. To combine the Equations (21), (22) and (23), we have

$$f_{G2}\left(x_i; \psi_i, \kappa, \theta, c\right) = \frac{x_i^{\psi_i^c \kappa - 1}}{\left(\psi_i^{1-c}\theta\right)^{\psi_i^c \kappa} \Gamma\left(\psi_i^c \kappa\right)} \exp\left(-\frac{x_i}{\psi_i^{1-c}\theta}\right).$$

To be consistent with the definition of the MSMD model, we let $\kappa\theta = 1$, then the pdf of x_i can be denoted as

$$f_{G2}\left(x_i;\psi_i,\kappa,c\right)=\frac{\left(\kappa x_i\psi_i^{c-1}\right)^{\psi_i^c\kappa}}{x_i\Gamma\left(\psi_i^c\kappa\right)}\exp\left(-\kappa x_i\psi_i^{c-1}\right),\quad x_i\geq 0,\kappa>0,\theta>0,c\in\mathbb{R}. \tag{24}$$

Conditional on $\bar{k}$, the parameter vector of the system is then

$$\boldsymbol{\zeta}_{\bar{k}}\equiv\left(\bar{\psi},b,\gamma_{\bar{k}},m_0,\kappa,c\right)\in\mathbb{R}_+^6$$

3.4 Statistical Properties

Stationarity and ergodicity. It is easy to find that the transition probability matrix (6) of two-state Markov processes $M_{k,i}$ is strictly stationary and ergodic with the conditions $b>1$ and $0<\gamma_k<1$. Thus, all the elements in P_k are positive, and it is straightforward to show that the elements of the transition matrix of M_i in Equation (7) are also strictly positive. It is possible to go from every state to every state. We say $\boldsymbol{P}$ is irreducible and strictly stationary and ergodic.

Moments, autocovariance and density. From equation (1), the innovation ϵ_i distribution has finite moments, the lower bound of the intensity ψ_i is always positive and it is also bounded by a upper limit. Hence, we can conclude that the moments of the events X_i are finite. Since all the components

$M_{k,i}$ and ϵ_i are mutually independent, the first two moments are given by

$$\begin{aligned} \mathbb{E}\left(X_i\right) &= \mathbb{E}\left(\psi_i\right)\mathbb{E}\left(\epsilon_i\right) = \bar{\psi}\mathbb{E}\left(M\right)^{\bar{k}} = \bar{\psi}, &(25)\\ \mathrm{Var}\left(X_i\right) &= \mathbb{E}\left(\psi_i^2\right)\mathbb{E}\left(\epsilon_i^2\right) - \mathbb{E}\left(\psi_i\right)^2\mathbb{E}\left(\epsilon_i\right)^2 \\ &= \bar{\psi}^2\mathbb{E}\left(M^2\right)^{\bar{k}}\mathbb{E}\left(\epsilon_i^2\right) - \bar{\psi}^2\mathbb{E}\left(M\right)^{\bar{k}} \\ &= \bar{\psi}^2\left[\mathbb{E}\left(M^2\right)^{\bar{k}}\mathbb{E}\left(\epsilon_i^2\right) - 1\right]. &(26) \end{aligned}$$

It implies that the MSMD models are over-dispersion only when $\mathbb{E}\left(M^2\right)^{\bar{k}}\mathbb{E}\left(\epsilon_i^2\right) > 2$, otherwise it can be both under- or over-dispersion. For the Exp-MSMD model, as the innovations ϵ_i are exponential distributed with intensity one, we have $\mathrm{E}\left(\epsilon_i^2\right) = 2$. By Jensen's inequality and $\bar{k} \geq 1$, we can show

$$\begin{aligned} \mathrm{Var}\left(X_i^{exp}\right) &= \bar{\psi}^2\left[2\mathbb{E}\left(M^2\right)^{\bar{k}} - 1\right] \\ &\geq \bar{\psi}^2\left[2\mathbb{E}\left(M\right)^{2\bar{k}} - 1\right] \\ &= \bar{\psi}^2, \end{aligned}$$

alternatively this can be written as

$$\sqrt{\mathrm{Var}\left(X_i^{exp}\right)} \geq \mathbb{E}\left(X_i^{exp}\right). \tag{27}$$

The inequality implies that the MSM model brings over-dispersion property to the Exp-MSMD model.

Long memory. The long memory property of the MSMD process can be showed by hyperbolically decay of the duration autocorrelation function

$$\begin{aligned}\rho(h) &= \operatorname{Corr}(X_i, X_{i-h}) \\ &= \frac{\operatorname{Cov}(X_i, X_{i-h})}{\sqrt{\operatorname{Var}(X_i)\operatorname{Var}(X_{i-h})}} \\ &= \frac{\prod_{k=1}^{\bar{k}}\left[1+\operatorname{Var}(M)(1-\gamma_k)^h\right]-1}{\mathbb{E}(M^2)^{\bar{k}}\,\mathbb{E}(\epsilon_i^2)-1}\end{aligned}$$

where the $\operatorname{Cov}(X_i, X_{i-h})$ can be derived as $\bar{\psi}^2\left(\prod_{k=1}^{\bar{k}}\left[1+\operatorname{Var}(M)(1-\gamma_k)^h\right]-1\right)$.

Follow the Proposition 1 in Calvet and Fisher [2004], we set two arbitrary numbers $\alpha_1 < \alpha_2$ in the interval $(0,1)$, and denote the set of integers $I_{\bar{k}} = \left\{h \,:\, \alpha_1 \log_b\left(b^{\bar{k}}\right) \le \log_b h \le \alpha_2 \log_b\left(b^{\bar{k}}\right)\right\}$ containing a broad range of lags. As $\bar{k} \to \infty$, we have

$$\sup_{h\in I_{\bar{k}}}\left|\frac{\log\rho(h)}{\log h^{-\delta}}-1\right| \to 0,$$

where $\delta = \log_b\left(E(M^2)/[E(M)]^2\right)$ and $\tilde{M}$ has a binomial distribution,

$$\tilde{M} = \begin{cases} \frac{2m_0^{-1}}{\left(m_0^{-1}+(2-m_0)^{-1}\right)} & \text{w.p. } 1/2 \\ \frac{2(2-m_0)^{-1}}{\left(m_0^{-1}+(2-m_0)^{-1}\right)} & \text{w.p. } 1/2 \end{cases}$$

4 Estimation, Inference and Simulation

4.1 Likelihood Inference of the MSMD Model

The likelihood is given by the joint density of the durations $X_1, X_2, \ldots, X_T$

$$\begin{aligned} p(X_1, X_2, \ldots, X_T; \boldsymbol{\theta}) &= \prod_{i=1}^{T} p(X_i | X_{i-1}, \ldots, X_1; \boldsymbol{\theta}) \\ &= \prod_{i=1}^{T} \left[\sum_{j=1}^{N} p\left(X_i, M_i = m^j | X_{i-1}, \ldots, X_1; \boldsymbol{\theta} \right) \right] \\ &= \prod_{i=1}^{T} \left\{ \sum_{j=1}^{N} \left[P\left[M_i = m^j | X_{i-1}, \ldots, X_1; \boldsymbol{\theta} \right] \times \right. \right. \\ & \qquad \left. \left. p\left(X_i | M_i = m^j, X_{i-1}, \ldots, X_1; \boldsymbol{\theta} \right) \right] \right\}, \end{aligned}$$

where $\boldsymbol{\theta}$ is a vector of the parameters. (see Hamilton [1989, 1994])

We know that each of the volatility components has 2 states, therefore the Markov state vector $M_i = (M_{1,i}, \ldots, M_{\bar{k},i})$ takes $2^{\bar{k}}$ possible values. Let $N = 2^{\bar{k}}$, then we assume $m^1, \ldots, m^N \in \mathbb{R}_+^{\bar{k}}$ are the values of state vector M_i. Then m^j, $j = 1, \ldots, N$, is defined as

$$m^j = m_0^A (2 - m_0)^B, \quad A + B = \bar{k}.$$

As we showed before, the transition matrix $\boldsymbol{P}$ is $(N \times N)$ and it is equal to $\boldsymbol{P_1} \otimes \cdots \otimes \boldsymbol{P_{\bar{k}}}$. The row j, column h element of $\boldsymbol{P}$ is the transition probability p_{hj}, then we have $\boldsymbol{P} = (p_{hj})_{1 \le h, j \le N}$. p_{hj} gives the probability that state

h will be followed by state j

$$
\begin{aligned}
p_{hj} &= P\left\{M_{i+1} = m^j | M_i = m^h\right\} \\
&= \prod_{k=1}^{\bar{k}} \left\{ \left[\frac{1}{2} + \frac{1}{2}\left(1-\gamma_{\bar{k}}\right)^{b^{k-\bar{k}}}\right] \cdot 1_{\left\{m_k^h = m_k^j\right\}} + \left[\frac{1}{2} - \frac{1}{2}\left(1-\gamma_{\bar{k}}\right)^{b^{k-\bar{k}}}\right] \cdot 1_{\left\{m_k^h \neq m_k^j\right\}} \right\},
\end{aligned}
$$

where $1_{\{m_k^h = m_k^j\}}$ is the dummy variable equal to one if $m_k^h = m_k^j$, and zero otherwise. The same case in $1_{\{m_k^h \neq m_k^j\}}$. Conditional on the state vector M_i, X_i is exponential, gamma, Weibull, Burr, generalized gamma or modified gamma distributed which the densities are defined in the Equations (12), (16), (14), (18), (20) and (24). If we denote the conditional density of X_i to be β_i^j, then we have

$$
\beta_i^j = p\left(X_i | M_i = m^j\right),
$$

$$
\boldsymbol{\beta}_i = \left(\beta_i^1, \ldots, \beta_i^N\right)' = \begin{bmatrix} p\left(X_i | M_i = m^1\right) \\ p\left(X_i | M_i = m^2\right) \\ \vdots \\ p\left(X_i | M_i = m^N\right) \end{bmatrix}
$$

M_i is unobservable, however, we can compute the probabilities of M_i conditional on the observed durations $X_1, \ldots, X_i$

$$
\delta_{i|i}^j = P\left\{M_i = m^j | X_1, \ldots, X_i\right\}
$$

over the unobserved states $m^1, \ldots, m^N$. We stack these probabilities in the vector

$$
\boldsymbol{\delta}_{i|i} = \left(\delta_{i|i}^1, \ldots, \delta_{i|i}^N\right)' \in \mathbb{R}_+^N.
$$

Let $\mathbf{1} = (1, \dots 1)' \in \mathbb{R}^N$, we have $\mathbf{1}'\boldsymbol{\delta}_{i|i} = 1$. By Bayes' rule

$$
\begin{aligned}
P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\
P(A|B \cap C) &= \frac{P(B|A \cap C)P(A|C)}{\sum_s P(B|A_s \cap C)P(A_s|C)},
\end{aligned}
$$

$$
\begin{aligned}
P\left\{M_i = m^j | X_1, \dots X_i\right\} &= P\left\{M_i = m^j | X_i \cap X_1, \dots X_{i-1}\right\} \\
&= \frac{P\{X_i|M_i = m^j\} P\{M_i = m^j|X_1, \dots X_{i-1}\}}{\sum_s P\{X_i|M_i = m^s\} P\{M_i = m^s|X_1, \dots X_{i-1}\}} \\
&= \frac{\beta_i^j \cdot \delta_{i|i-1}^j}{\sum_s \left(\beta_i^s \cdot \delta_{i|i-1}^s\right)}
\end{aligned}
$$

$\boldsymbol{\delta}_{i|i}$ can be expressed as a function of the probabilities vector of previous time $\boldsymbol{\delta}_{i-1|i-1}$ and the innovation X_i

$$
\boldsymbol{\delta}_{i|i} = \frac{\boldsymbol{\delta}_{i|i-1} \odot \boldsymbol{\beta}_i}{\mathbf{1}'\left(\boldsymbol{\delta}_{i|i-1} \odot \boldsymbol{\beta}_i\right)} = \frac{\left(\boldsymbol{P}\boldsymbol{\delta}_{i-1|i-1}\right) \odot \boldsymbol{\beta}_i}{\mathbf{1}'\left[\left(\boldsymbol{P}\boldsymbol{\delta}_{i-1|i-1}\right) \odot \boldsymbol{\beta}_i\right]},
$$

where "$\odot$" is the Hadamard product, $\boldsymbol{a} \odot \boldsymbol{b} = (a_1 b_1, \dots, a_N b_N)'$. Thus, we have the log-likelihood function with written in vector

$$
\log \mathcal{L}\left(X_1, X_2, \dots, X_T; \boldsymbol{\theta}\right) = \sum_{i=1}^{T} \log\left[\mathbf{1}'\left(\boldsymbol{\delta}_{i|i-1} \odot \boldsymbol{\beta}_i\right)\right]
$$

MLE algorithm:

Step 1: Calculate the predicted probabilities of M_i

$$
\boldsymbol{\delta}_{i|i-1} = \boldsymbol{P}\boldsymbol{\delta}_{i-1|i-1}
$$

Step 2: Calculate the joint conditional density distribution of X_i and M_i

$$\boldsymbol{\delta}_{i|i-1} \odot \boldsymbol{\beta}_i$$

Step 3: Calculate the conditional density function of X_i

$$\mathbf{1}' \left(\boldsymbol{\delta}_{i|i-1} \odot \boldsymbol{\beta}_i \right)$$

Step 4:

$$\boldsymbol{\delta}_{i|i} = \frac{\boldsymbol{\delta}_{i|i-1} \odot \boldsymbol{\beta}_i}{\mathbf{1}' \left(\boldsymbol{\delta}_{i|i-1} \odot \boldsymbol{\beta}_i \right)}$$

Starting the algorithm

Set $\boldsymbol{\delta}_{1|0}$ equals to the vector of unconditional probabilities $\boldsymbol{\pi}$. The vector $\boldsymbol{\pi} \equiv (\pi_1, \pi_2, \ldots, \pi_N)'$ is described as the unconditional expectation of δ_i

$$\boldsymbol{\pi} = \mathrm{E}\left(\boldsymbol{\delta}_i\right).$$

Since

$$\mathrm{E}\left(\boldsymbol{\delta}_{i+1}\right) = \boldsymbol{P}\mathrm{E}\left(\boldsymbol{\delta}_i\right),$$

assuming the process is stationary, we have

$$\boldsymbol{\pi} = \boldsymbol{P}\boldsymbol{\pi}.$$

This implies an ergodic N-state process, vector $\boldsymbol{\pi}$ is the eigenvector of $\boldsymbol{P}$ associated with the unit eigenvalues. We also have

$$\mathbf{1}'\boldsymbol{\pi} = 1.$$

By the properties of eigenvalue and eigenvector, we seek $\boldsymbol{\pi}$ satisfying

$$\boldsymbol{A}\boldsymbol{\pi} = \boldsymbol{e}_{N+1},$$

where $\boldsymbol{e}_{N+1}$ is the $(N+1)$th column of the identity matrix $\boldsymbol{I}_{N+1}$, and

$$\boldsymbol{A}_{(N+1)\times N} = \begin{bmatrix} \boldsymbol{I}_N - \boldsymbol{P} \\ \mathbf{1}' \end{bmatrix}.$$

Thus, we have

$$\boldsymbol{\pi} = \left(\boldsymbol{A}'\boldsymbol{A}\right)^{-1} \boldsymbol{A}'\boldsymbol{e}_{N+1},$$

which implies that $\boldsymbol{\pi}$ is the $(N+1)$th column of the matrix $\left(\boldsymbol{A}'\boldsymbol{A}\right)^{-1} \boldsymbol{A}'$.

4.2 Simulation Results

The properties of maximum likelihood estimator are necessary to test by Monte Carlo simulations before analyzing the real data. In this section, we simulate the MSMD models and then compare the results with the original samples.

We use the calibrated results for inter-trade duration of Microsoft (MSFT) in the later section as the parameters of the MSMD process. For the number of components $\bar{k} = 3, \ldots, 8$, we compute the renewal probabilities $\gamma_k/2$ in Table 6. The result shows that the first two components $(k = 1, 2)$ renew in very low frequencies. $\gamma_k = 10^{-4}$ implies that the component only renews several times in a month. As we are interested in the high frequency microstructure of the market, the components with extremely low renewal probabilities are reasonably to be treated as constant values.

To illustrate the renewal process of the model, we present the simulations of multifractal process $M_{k,i}$, ψ_i and duration sequence X_i in the Exp-MSMD and Gam2-MSMD models with using six components (shown in Figures 19 and 20). For the Exp-MSMD model, the parameters are set as $m_0 = 1.314$, $\bar{\psi} = 0.656$, $\gamma_{\bar{k}} = 0.118$ and $b = 3.482$, while the Gam2-MSMD model parameters are $m_0 = 1.359$, $\bar{\psi} = 1.103$, $\gamma_{\bar{k}} = 0.999$, $b = 7.273$, $\kappa = 1.287$ and $c = 0.694$. The simulation size is 10,000. For both cases, we could see that renewal frequencies increase from component one to component six. The renewal frequencies increase very slowly before component four for the Exp-MSMD model and component three for the Gam2-MSMD model. There are barely any changes in M_1 and M_2 for the Exp-MSMD model. The frequencies start to increase rapidly after that, and it is hard to find any unswitching moments in higher frequency components such as M_5 and M_6.

From the Q-Q plot of the MSMD simulated series versus sample durations (shown in Figures 21), we can see that the Exp-MSMD process is successful to generate over-dispersion series and the other MSMD processes are capable of creating both under- or over-dispersion sequences. For inter-trade duration, all the simulated series are nearly linearly related to the samples and located within the confident bounds except the GG-MSMD model. The magnitude of the Exp-MSMD simulations is slightly smaller than the samples', while the Wbl-MSMD simulations are a little bit greater. The Gam- and Burr-MSMD simulation quantiles are able to match the sample quantiles perfectly.

For the price duration, the Exp-, Gam2 and GG simulations appear to be linear fit with the sample. The Gam-MSMD simulations are out of the confident bounds. The tail of the Wbl- and Burr-MSMD simulations is fatter than the sample and their quantile-quantile lines tend to be a quadratic fit.

For the volume duration, most of the MSMD models fail to capture the magnitude of the sample except the Gam-MSMD model. The tail of the Gam-MSMD is lighter than the sample. The QQ plot of Gam2-MSMD displays linearly, however the quantiles are much smaller than the sample quantiles.

In Figures 22, it confirms the long memory property of the process, the simulated series are all persistent. For the inter-trade and price durations, all the simulation ACFs decay similarly with the sample ACF. The Gam-MSMD ACFs are shown stronger than the sample ACFs. For the volume duration, the sample ACF tends to decay faster than the simulation ACFs. The Wbl-, Gam2- and Burr MSMD ACFs are weaker than the sample ACFs.

4.3 Finite-Sample Properties

We next investigate the finite-sample properties of the ML estimators by Monte Carlo experiments. For each MSMD models, we use a fixed number of intensity components $\bar{k}$ equal to six which is proved to reasonable computation and good performance. The parameters across all MSMD models are the binomial value m_0, the unconditional intensity $\bar{\psi}$, the high-frequency switching probability $\gamma_{\bar{k}}$ and the frequency growth rate b. The Wbl and Gam-MSMD models have an extra shape parameter κ, while the two additional parameters of the Gam2-, Burr and GG-MSMD model are κ and c/θ. The parameter values are set to be consistent with the empirical results showed in later section: $m_0 = 1.3$, $\bar{\psi} = 1$, $\gamma_{\bar{k}} = 0.5$, $b = 5$, $\kappa = 0.9$, $c = 0.1$, $\theta_{\text{Burr}} = 0.1$ and $\theta_{\text{GG}} = 4$. We consider the sample length of 5,000 and 10,000.

We simulate 100 sample paths for different cases with the parameters above. We then estimate the parameters for each paths, the average estimated parameter values, the standard error (SE) and the root-mean-square error (RMSE) are calculated. The results are reported in Table 7. The biases of all estimated parameters are reasonably small and decline when sample length is increased. We can also find that the estimations of m_0, κ and c are more consistent compared to the other parameters.

Table 6: Renewal probabilities of the MSMD components. The table shows the renewal probabilities $\gamma_k/2$ associate with the components M_k, $k = 1, 2, \ldots, 8$. The parameter $\gamma_{\bar{k}}$ is the estimation of inter-trade duration of MSFT.

	$\bar{k}$	$\gamma_k/2$							
		$k=1$	2	3	4	5	6	7	8
Exp	8	0.00012	0.00035	0.00101	0.00286	0.00810	0.02271	0.06198	0.15698
	7	0.00010	0.00035	0.00114	0.00376	0.01232	0.03958	0.11928	
	6	0.00012	0.00043	0.00149	0.00516	0.01775	0.05914		
	5	0.00012	0.00059	0.00301	0.01523	0.07320			
	4	0.00022	0.00150	0.00996	0.06304				
	3	0.00046	0.00465	0.04560					
Wbl	8	0.00169	0.00170	0.00170	0.00170	0.00170	0.00170	0.00170	0.00171
	7	0.00183	0.00184	0.00185	0.00186	0.00187	0.00187	0.00188	
	6	0.00127	0.00152	0.00183	0.00219	0.00263	0.00315		
	5	0.00158	0.00193	0.00237	0.00290	0.00354			
	4	0.00050	0.00129	0.00331	0.00844				
	3	0.00057	0.00265	0.01213					
Gam	8	0.00163	0.00163	0.00163	0.00163	0.00163	0.00163	0.00163	0.00163
	7	0.00167	0.00173	0.00179	0.00185	0.00191	0.00198	0.00205	
	6	0.00119	0.00146	0.00179	0.00219	0.00268	0.00329		
	5	0.00183	0.00209	0.00238	0.00270	0.00307			
	4	0.00045	0.00122	0.00329	0.00886				
	3	0.00055	0.00255	0.01175					
Gam2	8	0.00050	0.00195	0.00758	0.02892	0.10373	0.29819	0.48550	0.50000
	7	0.00045	0.00224	0.01107	0.05280	0.21331	0.46873	0.50000	
	6	0.00034	0.00246	0.01764	0.11494	0.42519	0.50000		
	5	0.00103	0.00924	0.07765	0.39140	0.50000			
	4	0.00165	0.02588	0.28776	0.50000				
	3	0.00052	0.00295	0.01648					
Burr	8	0.00167	0.00167	0.00168	0.00168	0.00168	0.00169	0.00169	0.00170
	7	0.00182	0.00183	0.00185	0.00186	0.00187	0.00188	0.00189	
	6	0.00128	0.00153	0.00182	0.00217	0.00259	0.00309		
	5	0.00153	0.00190	0.00235	0.00292	0.00363			
	4	0.00051	0.00130	0.00330	0.00836				
	3	0.00059	0.00267	0.01194					
GG	8	0.00007	0.00028	0.00114	0.00469	0.01904	0.07390	0.24122	0.46681
	7	0.00006	0.00030	0.00150	0.00762	0.03767	0.16467	0.43483	
	6	0.00021	0.00109	0.00571	0.02922	0.13537	0.40445		
	5	0.00016	0.00114	0.00786	0.05213	0.26745			
	4	0.00025	0.00181	0.01316	0.08902				
	3	0.00052	0.00448	0.03767					

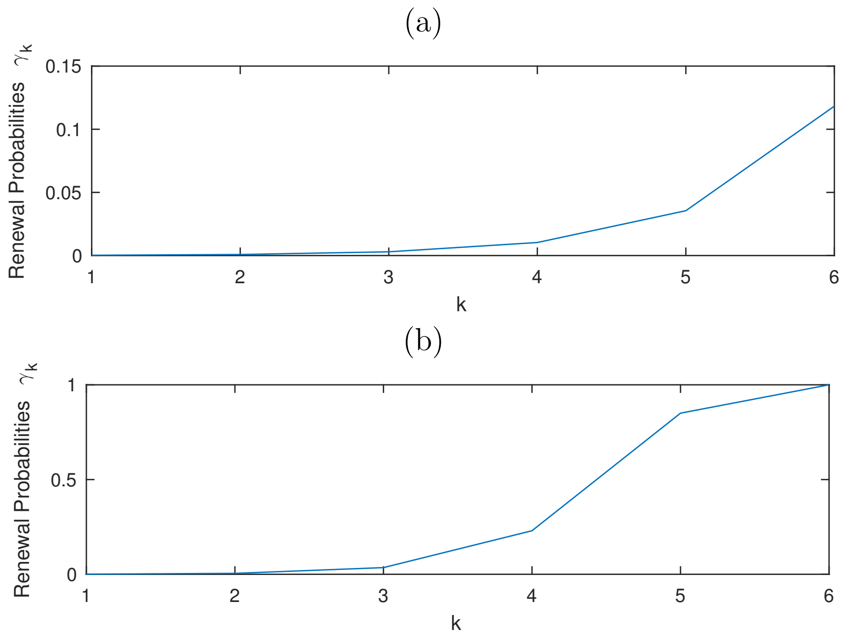


Figure 18: Renewal probabilities of (a) Exp-MSMD and (b) Gam2-MSMD components. The figure shows the renewal probabilities γ_k. The calibrated parameters are (a) $\gamma_{\bar{k}} = 0.327$, $b = 7.300$ and (b) $\gamma_{\bar{k}} = 0.999$, $b = 6.210$ with $\bar{k} = 6$.

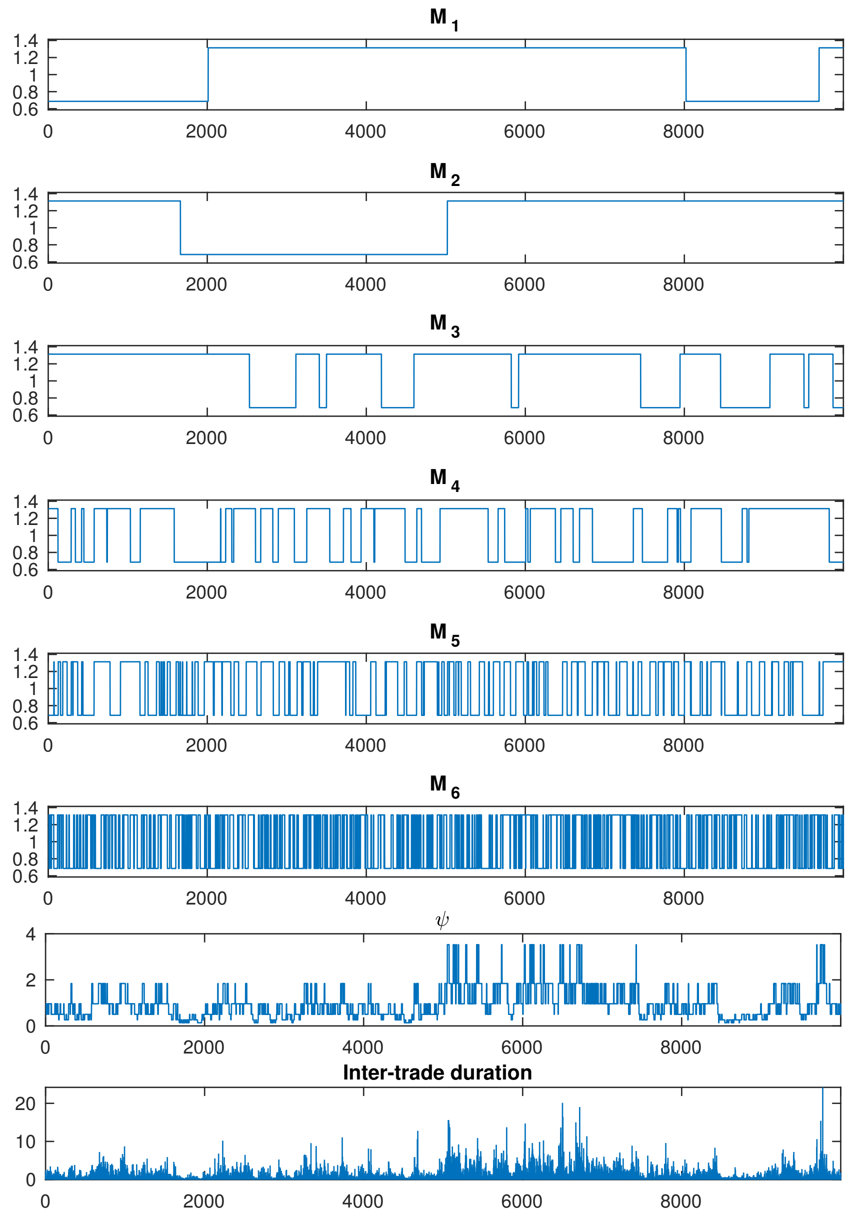


Figure 19: Simulated Exp-MSMD process with six stochastic components $M_{1,i}, \ldots, M_{6,i}$, the mean ψ_i and the durations X_i. Sample size is 10,000. Model parameters are $m_0 = 1.314$, $\bar{\psi} = 0.656$, $\gamma_{\bar{k}} = 0.118$ and $b = 3.482$.

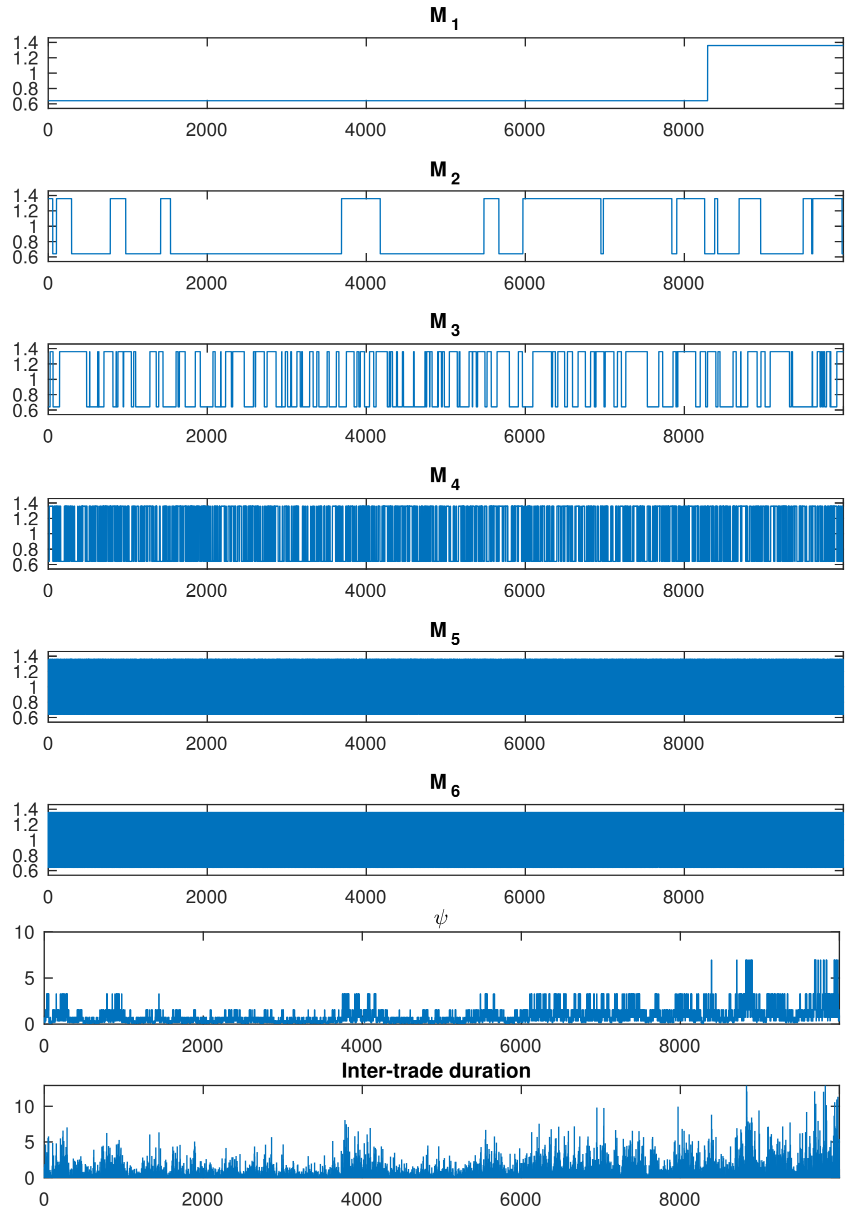


Figure 20: Simulated Gam2-MSMD process with six stochastic components $M_{1,i}, \ldots, M_{6,i}$, the mean ψ_i and the durations X_i. Sample size is 10,000. Model parameters are $m_0 = 1.359$, $\bar{\psi} = 1.103$, $\gamma_{\bar{k}} = 0.999$, $b = 7.273$, $\kappa = 1.287$ and $c = 0.694$.

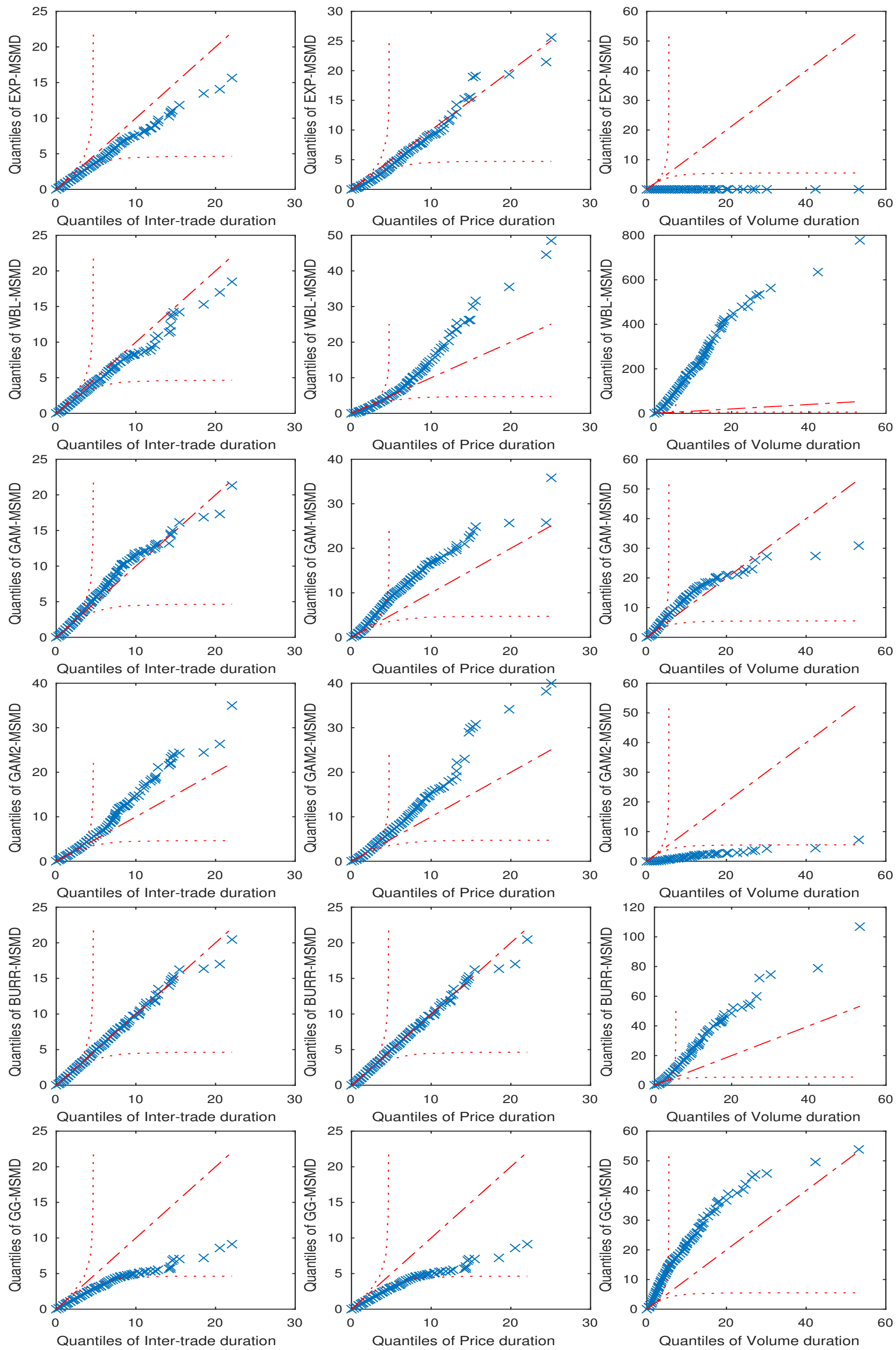


Figure 21: Q-Q plot of the sample durations versus the MSMD simulated series (left). The first to the sixth rows are the Exp-, Wbl-, Gam-, Gam2-, Burr- and GG-MSMD simulations relatively. The first column is inter-trade duration, the second column is price duration and the third column is volume duration.

/Users/kinsley/Google Drive/GitHub/MCD/stats/dissertation/MSFT_T_

/Users/kinsley/Google Drive/GitHub/MCD/stats/dissertation/MSFT_P_

/Users/kinsley/Google Drive/GitHub/MCD/stats/dissertation/MSFT_V_

Figure 22: ACF of sample compare to simulated series

Table 7: Monte Carlo MLE results

	Exp-MSMD		Wbl-MSMD		Gam-MSMD		Gam2-MSMD		Bur-MSMD		GG-MSMD	
	5,000	10,000	5,000	10,000	5,000	10,000	5,000	10,000	5,000	10,000	5,000	10,000
$\bar{m}_{sim}$	1.309	1.304	1.312	1.308	1.310	1.306	1.315	1.309	1.313	1.309	1.316	1.311
SE	(0.051)	(0.038)	(0.079)	(0.047)	(0.054)	(0.030)	(0.083)	(0.069)	(0.077)	(0.061)	(0.093)	(0.073)
RMSE	(0.057)	(0.044)	(0.093)	(0.062)	(0.072)	(0.037)	(0.107)	(0.088)	(0.082)	(0.063)	(0.099)	(0.081)
$\bar{\psi}_{sim}$	1.079	1.035	1.112	1.053	1.120	1.064	1.129	1.061	1.118	1.051	1.131	1.111
SE	(0.152)	(0.088)	(0.214)	(0.166)	(0.222)	(0.132)	(0.221)	(0.156)	(0.211)	(0.135)	(0.223)	(0.166)
RMSE	(0.166)	(0.095)	(0.218)	(0.169)	(0.232)	(0.139)	(0.234)	(0.161)	(0.214)	(0.142)	(0.232)	(0.167)
$\bar{\gamma}_{sim}$	0.529	0.513	0.530	0.524	0.522	0.515	0.520	0.511	0.522	0.513	0.523	0.512
SE	(0.195)	(0.102)	(0.128)	(0.092)	(0.158)	(0.106)	(0.173)	(0.112)	(0.176)	(0.120)	(0.183)	(0.131)
RMSE	(0.195)	(0.106)	(0.143)	(0.093)	(0.157)	(0.109)	(0.156)	(0.120)	(0.159)	(0.133)	(0.177)	(0.129)
$\bar{b}_{sim}$	5.221	5.134	5.115	5.085	5.214	5.125	5.109	5.088	5.116	5.088	5.109	5.088
SE	(1.021)	(0.057)	(1.718)	(1.003)	(1.234)	(0.074)	(1.300)	(0.090)	(1.450)	(0.123)	(1.421)	(0.083)
RMSE	(1.270)	(0.068)	(1.779)	(1.010)	(1.310)	(0.080)	(1.398)	(0.088)	(1.461)	(0.118)	(1.439)	(0.094)
$\bar{\kappa}_{sim}$			0.896	0.902	0.906	0.903	0.910	0.908	0.909	0.907	0.913	0.912
SE			(0.017)	(0.008)	(0.019)	(0.010)	(0.118)	(0.066)	(0.078)	(0.057)	(0.104)	(0.075)
RMSE			(0.017)	(0.009)	(0.019)	(0.010)	(0.116)	(0.065)	(0.083)	(0.069)	(0.109)	(0.082)
$\bar{c}_{sim}$							0.114	0.109				
SE							(0.071)	(0.044)				
RMSE							(0.069)	(0.043)				
$\bar{\theta}_{sim}$									0.094	0.097	4.117	4.113
SE									(0.049)	(0.037)	(0.093)	(0.063)
RMSE									(0.051)	(0.041)	(0.098)	(0.071)

5 Empirical Studies

5.1 Adjusted Financial Durations

In this section, we investigate the estimation and forecast performance of the MSMD models. To do this, four US equities (MSFT, INTC, QCOM and FB) are selected from NASDAQ-100. The LOB data are extracted from LOBSTER database which is based on the NASDAQ TotalView-ITCH. The database provides transaction data up to nanosecond. The sample period covers trading days in April and May 2015, 9:30 - 16:00. We also use data in 2014 and 2015 for the adjustments of seasonality. All financial duration samples are cleaned and adjusted as already described in previous sections. We focus on discussing one equity result from each types of financial durations, results of other equities are also showed individually.

5.2 Model Selection Criteria

We briefly describe the model selection criteria would be used in the following sections, which is included the Akaike information criterion, the Bayesian information criterion and the Vuong test.

5.2.1 The Akaike Information Criterion (AIC)

The AIC is introduced by Akaike [1974], which is defined as

$$\text{AIC} = 2k - 2\log\mathcal{L}\left(\boldsymbol{X};\hat{\boldsymbol{\zeta}}\right)$$

where k is the number of independently adjusted parameters in the model, $\log\mathcal{L}\left(\boldsymbol{X};\hat{\boldsymbol{\zeta}}\right)$ is the log-likelihood function of durations $\boldsymbol{X}$ with estimated parameters $\hat{\boldsymbol{\zeta}}$.

5.2.2 The Bayesian Information Criterion (BIC)

The BIC is proposed by Schwarz [1978], which is defined as

$$\mathrm{BIC} = k \log(T) - 2 \log \mathcal{L}\left(\boldsymbol{X}_T; \hat{\boldsymbol{\zeta}}\right)$$

where T is the sample size and k is the number of independently adjusted parameters in the model. $\log \mathcal{L}\left(\boldsymbol{X}_T; \hat{\boldsymbol{\zeta}}\right)$ is the log-likelihood function of durations $\boldsymbol{X}_T$ with estimated parameters $\hat{\boldsymbol{\zeta}}$.

5.2.3 Vuong Test

We implement a likelihood ratio test for the model selection between different MSMD($\bar{k}$) processes, and it is introduced by Vuong [1989]. By setting the target model to be the MSMD process with highest $\bar{k}$ which denote as MSMD($\bar{k}^*$), the approach uses the Kullback-Leibler information criterion (KLIC) to examine the closeness of the model MSMD($\bar{k}$) to MSMD($\bar{k}^*$), where $\bar{k} < \bar{k}^*$. Consider the conditional densities of the two competing models are $f\left(X_i \,|\boldsymbol{X}_{i-1}; \boldsymbol{\zeta}\right)$ and $g\left(X_i \,|\boldsymbol{X}_{i-1}; \boldsymbol{\eta}\right)$. The null hypothesis of the test is

$$H_0: \quad \mathrm{E}_0\left[\log \frac{f\left(X_i \,|\boldsymbol{X}_{i-1}; \boldsymbol{\zeta}\right)}{g\left(X_i \,|\boldsymbol{X}_{i-1}; \boldsymbol{\eta}\right)}\right] = 0$$

Vuong [1989] proved that the null hypothesis expectation can be estimated by mean of the likelihood ratio statistic. As the KLIC and Central Limit Theorem, by assuming the observations X_i are independent and identically distributed (i.i.d.), the likelihood ratio statistic is asymptotically normally distributed, then we have

$$\text{under } H_0 : \quad \frac{LR_T\left(\hat{\boldsymbol{\zeta}}_T, \hat{\boldsymbol{\eta}}_T\right)}{\hat{\sigma}_T\sqrt{T}} \xrightarrow{D} \mathcal{N}\left(0, 1\right),$$

where $LR_T\left(\hat{\boldsymbol{\zeta}}_T, \hat{\boldsymbol{\eta}}_T\right)$ can be calculated as

$$LR_T\left(\hat{\boldsymbol{\zeta}}_T, \hat{\boldsymbol{\eta}}_T\right) = \mathcal{L}_T^f\left(\hat{\boldsymbol{\zeta}}_T\right) - \mathcal{L}_T^g\left(\hat{\boldsymbol{\eta}}_T\right) = \sum_{i=1}^{T} \log \frac{f\left(X_i \left| \boldsymbol{X}_{i-1}; \hat{\boldsymbol{\zeta}}_T\right.\right)}{g\left(X_i \left| \boldsymbol{X}_{i-1}; \hat{\boldsymbol{\eta}}_T\right.\right)},$$

and $\hat{\sigma}_T$ is the estimated sample standard deviation of the addends $\hat{a}_i = \log\left[f\left(X_i \left| \boldsymbol{X}_{i-1}; \hat{\boldsymbol{\zeta}}_T\right.\right) / g\left(X_i \left| \boldsymbol{X}_{i-1}; \hat{\boldsymbol{\eta}}_T\right.\right)\right]$, which is given by

$$\hat{\sigma}_T = \frac{1}{T}\sum_{i=1}^{T}\hat{a}_i^2 - \left(\frac{1}{T}\sum_{i=1}^{T}\hat{a}_i\right)^2$$

5.2.4 Heteroscedasticity and Autocorrelation Consistent (HAC) Adjusted Vuong Test

Now we consider the observations X_i are non-i.i.d., then the addends $\hat{a}_i$ are also non-i.i.d.. We estimate the standard deviation σ_T by implementing the method from Newey and West [1987]

$$\hat{\sigma}_T = \hat{\mathit{\Omega}}_0 + 2\sum_{j=1}^{m} w\left(j, m\right)\hat{\mathit{\Omega}}_j, \quad w\left(j, m\right) = 1 - \frac{j}{m+1},$$

where m is the bound of the number of sample autocovariances $\hat{\mathit{\Omega}}_j$, which is chosen to be a function of sample size $m\left(T\right)$ (White [2014]). $w\left(j, m\right)$ is the modified Bartlett weight which is used to smooth the sample autocovariance function (Anderson [1994]). $\hat{\mathit{\Omega}}_j$ denotes the sample covariance of the addends

a_i, which is estimated by

$$\hat{\Omega}_j = \frac{1}{T} \sum_{i=j+1}^{T} \hat{a}_i \hat{a}_{i-j}$$

5.3 Estimation results

We estimate the MSMD models with the 10,000 observations before May 13th, 2015, 11:00am, and compare the results of different number of multipliers $\bar{k}$. The estimation results are displayed in $\bar{k} = 3, \ldots, 8$ for all the models. Since $\bar{k} = 1, 2$ usually lead to bad estimation results, these values are not considered in this paper. To compute maximum likelihood estimation (MLE), we minimize the negative likelihood function by using the NLopt nonlinear optimization library for constrained minimization. There is no upper-bound for the parameters $\bar{\psi}$ and b from the assumption of the model. However, from the experiments, we find that the expected value of $\bar{\psi}$ should be around the mean of the estimated sample, and b is smaller than 30 in order to estimate reasonable γ_k's. Therefore, we set the parameter space as $m_0 \in [1.001, 1.999]$, $\lambda \in [0.001, 10]$, $\gamma_{\bar{k}} \in [0.001, 0.999]$ and $b \in [1.001, 50]$. For the shape parameter κ, in order to generate an L-shape density, it should be a small value $\kappa \in [0.001, 5]$. The extra parameters in modified gamma, Burr and generalized gamma are set as $c \in [-5, 5]$, $\theta \in [0.001, 10]$. The boundaries settings will benefit the optimal routines searching in MLE.

Here we will focus on discussing the estimation results of MSFT. We will also present the results of all types of financial durations for the other equities (INTC, FB and QCOM). The full estimation results of inter-trade durations are showed in Tables 10, 11, 12 and 13. The results for price durations are in Tables 14, 15, 16 and 17.. Tables 18, 19, 20 and 21 present the volume durations results.

For the Exp-MSMD model, the log-likelihood $\ln \mathcal{L}$ is improved when the number of multipliers $\bar{k}$ increased from three to eight, and it is maximized at $\bar{k} = 8$. However, the improvement tends to be slower after five, and it is hard to tell the difference from $\bar{k} = 7$ and 8. We also find that when $\bar{k}$ increases, the renewal probability $\gamma_{\bar{k}}$ increases and the rate parameter b decreases. All estimated parameters have reasonable asymptotic standard errors which are showed in parentheses. m_0 is estimated around 1.3 on inter-trade and price durations, but it reaches the upper-bound on volume duration. $\bar{\psi}$ is reasonably closed to the sample mean except on volume duration. $\gamma_{\bar{k}}$ is found to be more fluctuating and harder to estimate compared to the other parameters, that is because its closely related to the particular sample sequence. b generally ranges from three to twenty depending on $\gamma_{\bar{k}}$ and $\bar{k}$. Overall, the Exp-MSMD model performs well on the estimations of inter-trade and price durations, however it has limitation in estimating the volume duration.

For the Wbl-MSMD and Gam-MSMD models, the likelihood does not exhibit significant improvement after $\bar{k} = 4$. It indicates that these models are possible to implement lower dimension transition states. The estimators m_0 and $\bar{\psi}$ are consistent with the Exp-MSMD model. The renewal probability $\gamma_{\bar{k}}$ is generally below 0.1, which means the system rarely changes states and the frequency components do not have many variations. We can also observe that b is below 5 in most cases. κ is less than one in all types of financial durations, which indicates an L-shape density. By implementing the two parameter distributions Weibull and gamma for the innovations, the estimation results are more stable and accurate especially for the volume duration.

For the Bur-MSMD model, after increasing $\bar{k}$ to certain level, the likelihood does not improvement but fluctuates within a small range. θ^2 is closed to zero, which means the Burr here is a Weibull distribution. We actually find that some results of the Bur-MSMD and Wbl-MSMD are very closed to each other. Another observation is that the estimations of Bur-MSMD are unstable based on some cases. For the GG-MSMD model, the likelihood keeps increasing until eight and the estimations of the basic parameters are consistent with the Wbl- and Gam-MSMD models.

Finally, the estimation results of the Gam2-MSMD are similar to the Gam-MSMD in some cases, especially on the estimators m_0, $\bar{\psi}$ and κ. The value of $\gamma_{\bar{k}}$ changes significantly using different samples. The $\gamma_{\bar{k}}$ reaches upper-bound on inter-trade duration, and becomes very small on price duration. On volume duration, the estimation is about 0.5. $\bar{\psi}$ is around the sample mean, while κ is below one which indicates an 'L' shape. The constant c which balancing the multifractal processes is estimated within $[0, 1]$ in general.

Based on the cross-model tests, we can observe that the Gam2- and GG-MSMD models have greater likelihood and lower AIC, BIC compare to the other models. The alternative MSMD models outperform the classical Exp-MSMD model in almost all cases.

Tables 22, 23 and 24 show the t-ratios and one-sided p-values of classical and HAC-adjusted Vuong test for the estimations of 10,000 duration samples in different types.

On inter-trade duration (Table 22), we can see that the significant level of each models are generally lower or equal to 1% level in both tests when $\bar{k} = 3$ or 4. For $\bar{k} = 5$, most of the models reach 5% level in HAC-adjusted test except the GG-MSMD is rejected in both tests. For $\bar{k} = 6$ or above, all the MSMD models significantly outperform the lower $\bar{k}$ cases except we only see improvement from the GG-MSMD model after using up to seven components. In some cases, the level start to decrease at $\bar{k} = 7$. The test results indicate that the estimation performance increases when the number of multifractal components less than six. After $\bar{k} = 6$, the performance diverge between different models. Higher $\bar{k}$ performs even worse in certain cases, the reason for this could be the models get harder to be correctly estimated when the state dimension increase.

In the results of price duration (Table 23), we can see that the performance of the Exp-MSMD model is different from the other models. For the Exp-MSMD model, the null hypothesis is rejected at $\bar{k} = 3$, and the level increases with $\bar{k}$ increases. For other models, larger $\bar{k}$ does not improve the performance of the models, and some models start at a high level of p-values at $\bar{k} = 3$. For consistency in analysis, we will set MSMD($\bar{k} = 7$) when generating results of price duration in the following sections.

On volume duration (Table 24), the null of most of the cases are rejected when $\bar{k} \leq 5$. We can also observe that almost all cases work better when $\bar{k}$ is larger. The models with less parameters are easier to reach their best performance by adding more components. For the complex model, the Gam2-, Burr and GG-MSMD models, the 1% significant level is rejected until $\bar{k} = 7$. From the analysis of the results of all types of financial durations, we can conclude that the optimal selection of multifractal components can be related to the changing frequencies of the data, higher frequency data need less components for estimation. More multifractal components do not necessary create better performance, the results become even worse in some cases.

5.4 Out-of-Sample Forecasts

In this section, we compare the forecasting performance of the MSMD models with the benchmark autoregressive conditional duration (ACD) model. We are interested in examining how well a model captures the dynamics of financial durations around the mean, so the one, five, ten and twenty steps ahead forecasts will be evaluated for all types of financial durations. To test the forecasting performance, we implement the Mincer-Zarnowitz ordinary least squares (OLS) regressions

$$X_{i+h} = \omega_0 + \omega_1 \mathbb{E}_i X_{i+h} + u_i,$$

where h denotes the number of steps ahead forecast, u_i is the unforecastable error term at time i. For an unbiased forecast, the test hypothesis is then $\omega_0 = 0$ and $\omega_1 = 1$.

On inter-trade duration, we focus on discussing the forecasting results of QCOM which are presented in Figure 23. Results for all equities are reported in Tables 25, 26, 27 and 28. On one-step ahead forecast, both of the ACD and MSMD models perform well. The estimated intercept $\hat{\omega}_0$ is positive and the slope $\hat{\omega}_1$ is less than one for the ACD models and the Exp-MSMD model, while $\hat{\omega}_0$ is negative and $\hat{\omega}_1$ is greater than one for the Gam-, Gam2-, Burr and GG-MSMD models. The Wbl-MSMD model has negative $\hat{\omega}_0$ and $\hat{\omega}_1 < 1$. The results suggest that the forecasts of ACD, Exp-MSMD and Wbl-MSMD models are too variable and the other MSMD forecasts are not enough variable. The root mean squared errors of the MSMD forecasts are generally less than the ACD forecasts. On larger step forecasts, all the ACD models perform poorly. As we show in Figure 23, the Gam-MSMD(6) and Gam2-MSMD(8) model outperform the other models especially on larger step forecasts, while the performance of the Gam2-MSMD model is slightly better than the Gam-MSMD model in terms of capturing the variance. The classical Exp-MSMD(8) model starts off similarly with other MSMD models at one-step forecast, but the adjusted R^2 drops faster when step size increases.

On price duration, we select to show the results of INTC in Figure 24. Tables 29, 30, 31 and 32 present all the price duration forecasts. Similar to the results on inter-trade duration, the ACD models only perform well on one-step ahead forecast, changing the innovation distribution does not improve the performance. For the MSMD models, the Gam2-MSMD(6) forcast, again, outperforms the other MSMD models in terms of capturing the variance of the sequence. The Wbl-MSMD(6) model is capable to predict the peak, however, the trade-off is greater errors on the small duration forecasts. The Exp-MSMD(6) forecast is generally worse than other MSMD forecasts in this case.

On volume duration, the results of FB is presented in Figure 25. We report the full results of volume duration forecasts in Tables 33, 34, 35 and 36. The features of volume duration are different from inter-trade and price durations. The variance is smaller and the sequence appears to have less fluctuation at most time. However, it also displays some spikes in the sequence, which rapidly increase and then decrease in a short period. In this case, the process need to react quickly and be more flexible to change states. So we can find that using more components $\bar{k}$ are preferred in all MSMD models. Consider the computation issue, we use $\bar{k} = 7$ here. The results of Exp-, Wbl- and Gam-MSMD models are preferred in this case. The Gam2-MSMD model fails on capturing the magnitude.

Based on the empirical studies above, we conclude that the Weibull and gamma distributed innovations enhance the forecast performance of the MSMD model. The flexible distributions (Burr and generalized gamma) offer little improvement. To get better performance of fitting UHF financial durations, the MSMD model need to balance the choices between the innovation distributions and the multifractal processes. The modified gamma MSMD model is capable of capturing the high volatility of the UHF data. The Exp-MSMD model performs worse in high dispersion durations and fails in certain point.

5.5 Flash Crash

The May 6, 2010, Flash Crash started at 2:32 pm and last until 3:08 pm. Stocks and indexes collapsed and rebounded within a short period of time (Kirilenko et al. [2017]). The Dow Jones Industrial Average went down about 9% which is recorded as one of the biggest intraday drop in the history. As the conclusion from a CFTC report, high-frequency traders "did not cause the Flash Crash, but contributed to it by demanding immediacy ahead of other market participants." Therefore, the failure of high-frequency trading algorithms in such situation is definitely a huge potential risk which should not be ignored.

In this section, we are interested in implementing the MSMD models on predicting inter-trade durations during Flash Crash. We are using the last 10,000 data points before 11:00 am on May 6, 2010 to feed in our models for parameter estimation. In Table 8, we report the estimation results of the MSMD models with using six multifractal components. Here m_0 is estimated around 1.3 to 1.4 in most of the models. For the unconditional mean intensity $\bar{\psi}$, the smallest estimation is from the Gam-MSMD model which is actually quite reasonable for predicting Flash Crash, while the largest estimation is from the Exp-MSMD model which is approximately the average of durations before Flash Crash happened. The estimated renew frequencies are low in most cases except the Exp- and GG-MSMD models, and the innovation parameters are in reasonable values. When comparing the likelihood, AIC and BIC, we find that the GG-MSMD model performs the best, and the Gam- and Gam2 MSMD models are also well.

We report the forecasts results in Table 9 and Figure 26. As we can see, the gamma family MSMD models outperform other models in one-step ahead forecast. The Exp- and Wbl-MSMD forecasts have greater variance than the samples as $\omega_1 < 1$. We also observe that the Gam- and Gam2-MSMD models perform the best in twenty-step ahead forecast. Overall, the MSMD models succeed to predict the huge drop during Flash Crash, and the Gam- or Gam2-MSMD models are preferred in this case.

Table 8: Estimation comparison of Flash Crash with using six multifractal components

	Exp	Wbl	Gam	Gam2	Burr	GG
k	6	6	6	6	6	6
$\hat{m}_0$	1.486	1.356	1.315	1.31	1.312	1.383
	(0.010)	(0.012)	(0.009)	(0.012)	(0.013)	(0.009)
$\hat{\psi}$	0.852	0.462	0.362	0.651	0.371	0.483
	(0.037)	(0.018)	(0.012)	(0.026)	(0.018)	(0.018)
$\hat{\gamma}_{\bar{k}}$	0.994	0.011	0.009	0.006	0.006	0.652
	(0.007)	(0.004)	(0.003)	(0.003)	(0.001)	(0.082)
$\hat{b}$	9.556	1.964	1.754	1.326	1.286	7.678
	(1.271)	(0.365)	(0.282)	(0.302)	(0.029)	(0.895)
$\hat{\kappa}$		0.696	0.563	0.562	0.696	0.182
		(0.006)	(0.007)	(0.007)	(0.006)	(0.015)
$\hat{\theta}$				-0.017	0.006	2.764
				(0.021)	(0.000)	(0.207)
$\ln \mathcal{L}$	-8283.9	-7588.1	-7369.9	-7369.7	-7594.8	-7255.8
AIC	16576	15186	14750	14751	15202	14524
BIC	16605	15222	14786	14795	15245	14567

Table 9: Forecasting performance of Flash Crash (MSFT)

	Steps	ω_0	ω_1	RMSE	R^2_{adj}
Exp	1	-0.010	0.905	0.041	0.981
	20	-0.024	0.908	0.086	0.909
Wbl	1	-0.008	0.944	0.054	0.969
	20	0.011	0.810	0.075	0.918
Gam	1	-0.005	0.990	0.058	0.962
	20	0.001	0.936	0.082	0.913
Gam2	1	-0.021	1.011	0.058	0.962
	20	-0.021	0.948	0.086	0.909
Burr	1	-0.022	1.779	0.058	0.961
	20	-0.024	1.683	0.086	0.908
GG	1	-0.010	0.990	0.041	0.981
	20	-0.001	0.901	0.085	0.911

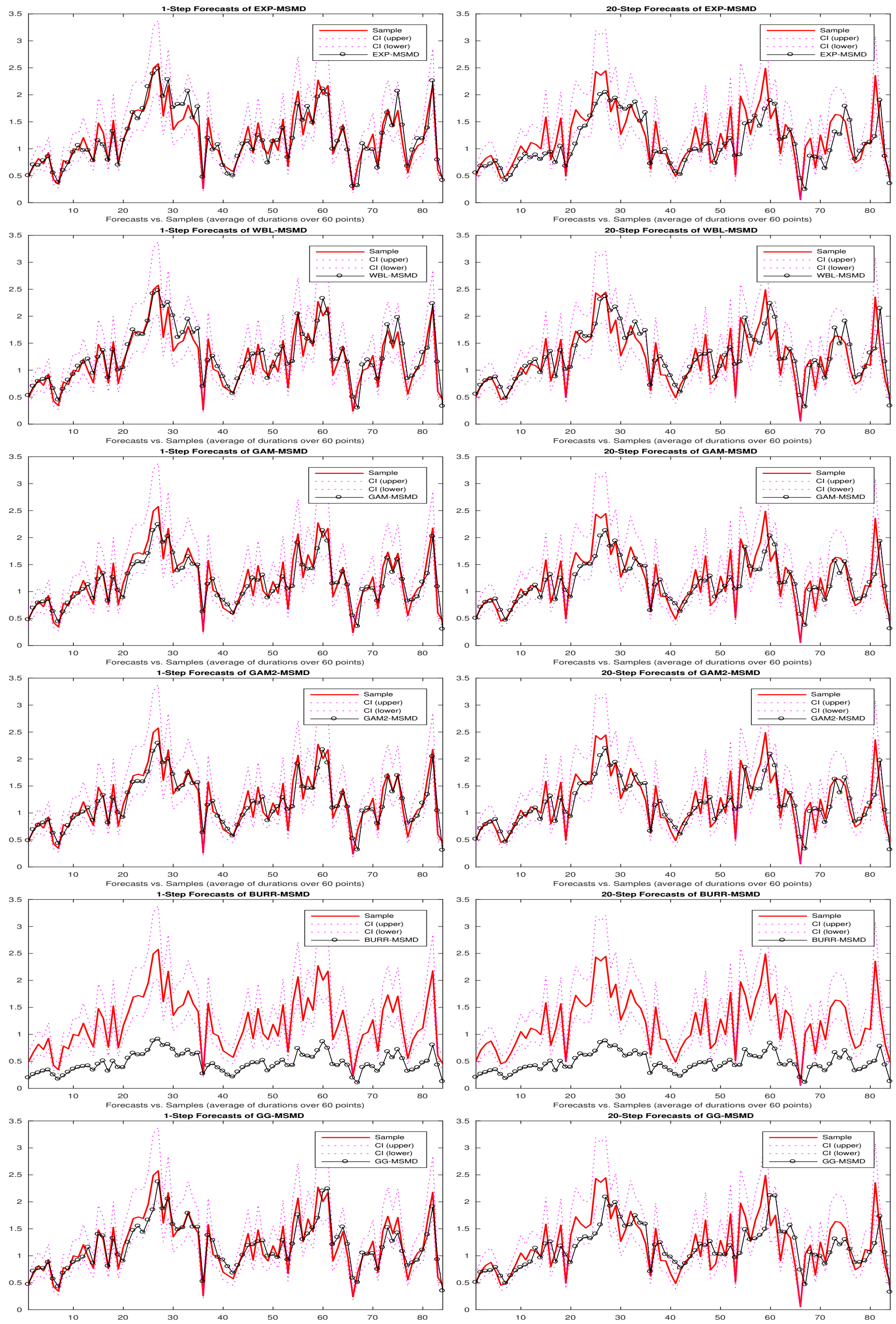


Figure 23: Multi-step forecasts of inter-trade duration (QCOM, starts at 11:00am on May 13, 2015). From top to bottom are the Exp-, Wbl-, Gam-, Gam2-, Burr- and GG-MSMD models. Left column is 1-step forecasts, and right column is 20-step forecasts.

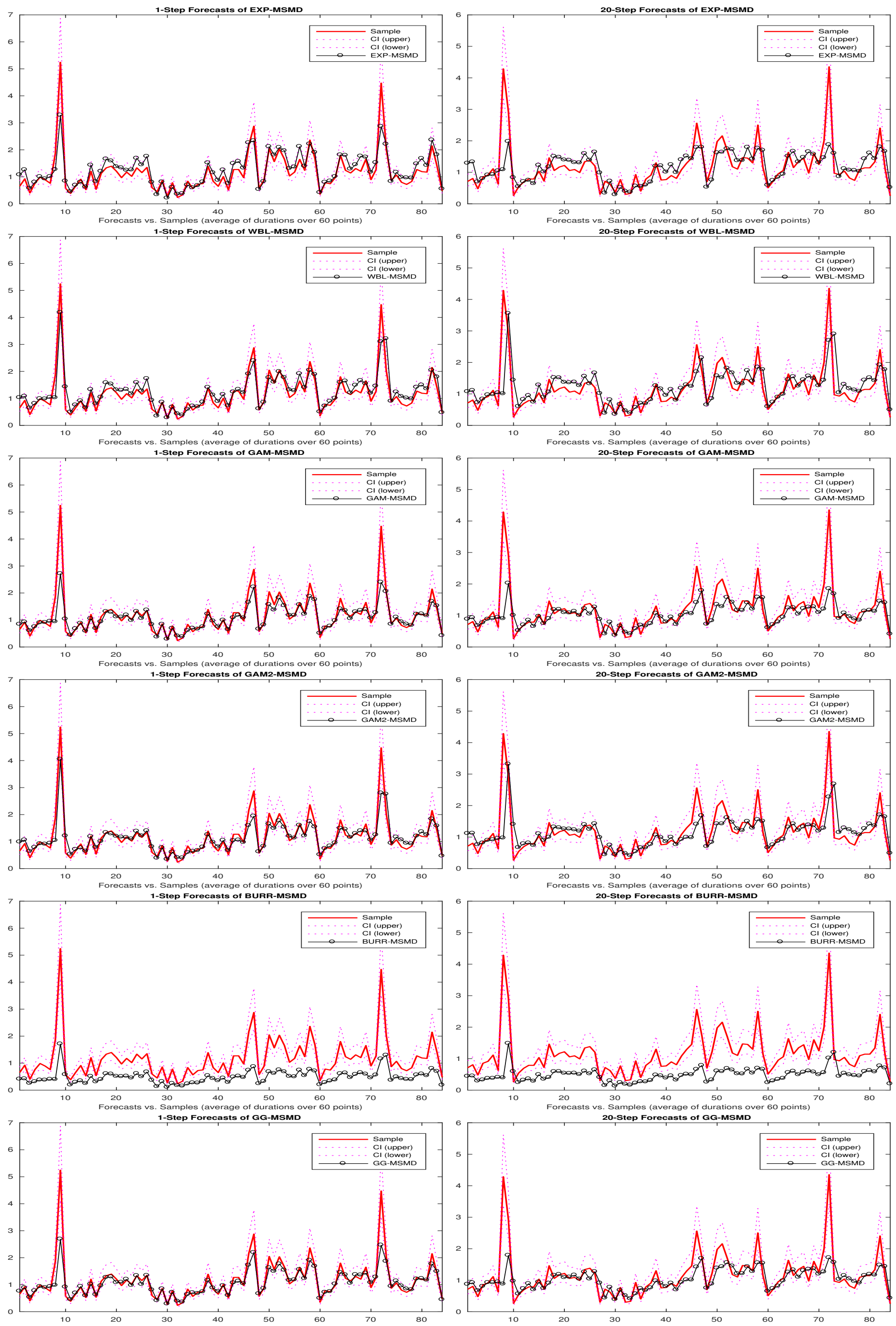


Figure 24: Multi-step forecasts of price duration (INTC, starts at 11:00am on May 13, 2015). From top to bottom are the Exp-, Wbl-, Gam-, Gam2-, Burr- and GG-MSMD models. Left column is 1-step forecasts, and right column is 20-step forecasts.

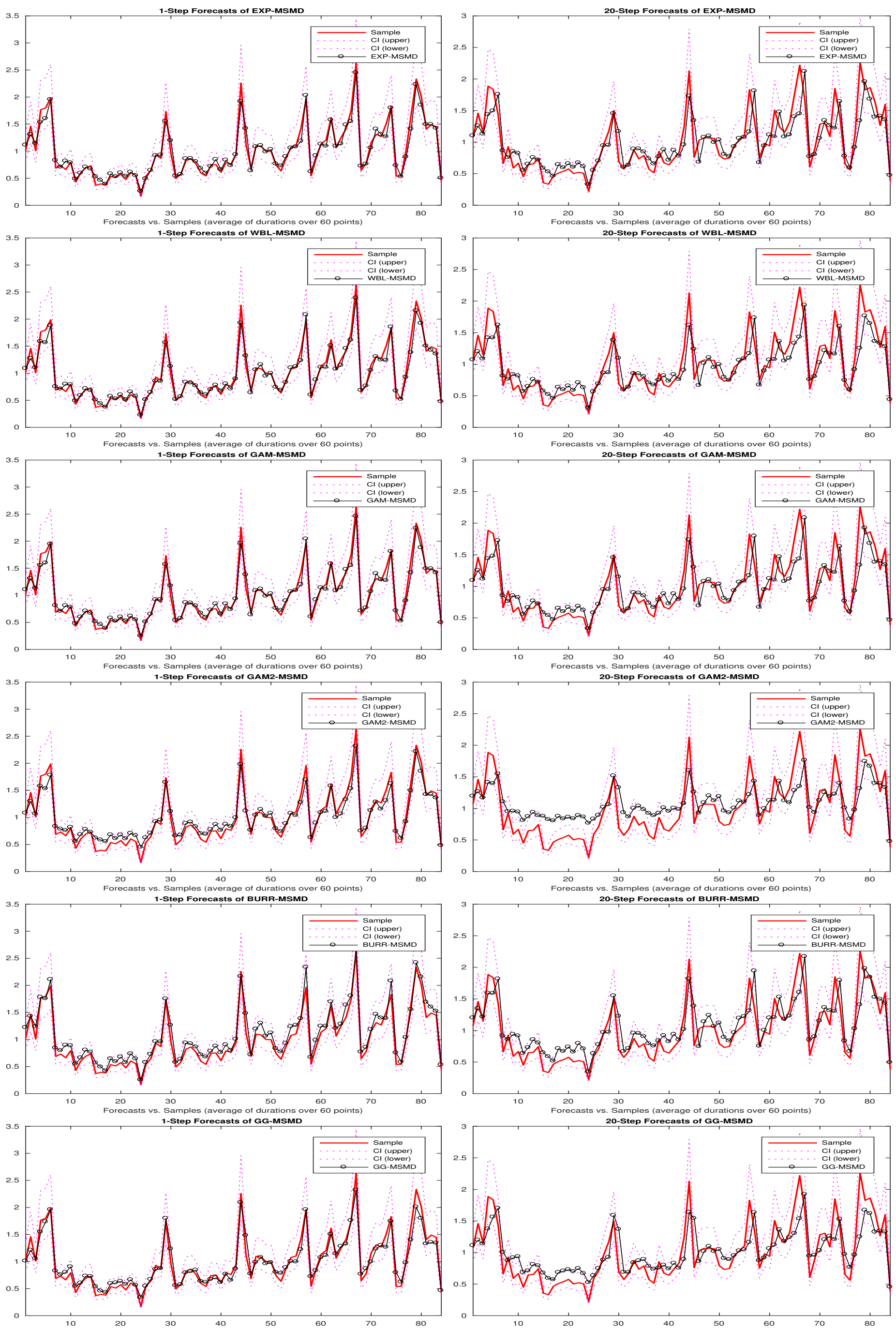


Figure 25: Multi-step forecasts of volume duration (FB, starts at 11:00am on May 13, 2015). From top to bottom are the Exp-, Wbl-, Gam-, Gam2-, Burr- and GG-MSMD models. Left column is 1-step forecasts, and right column is 20-step forecasts.

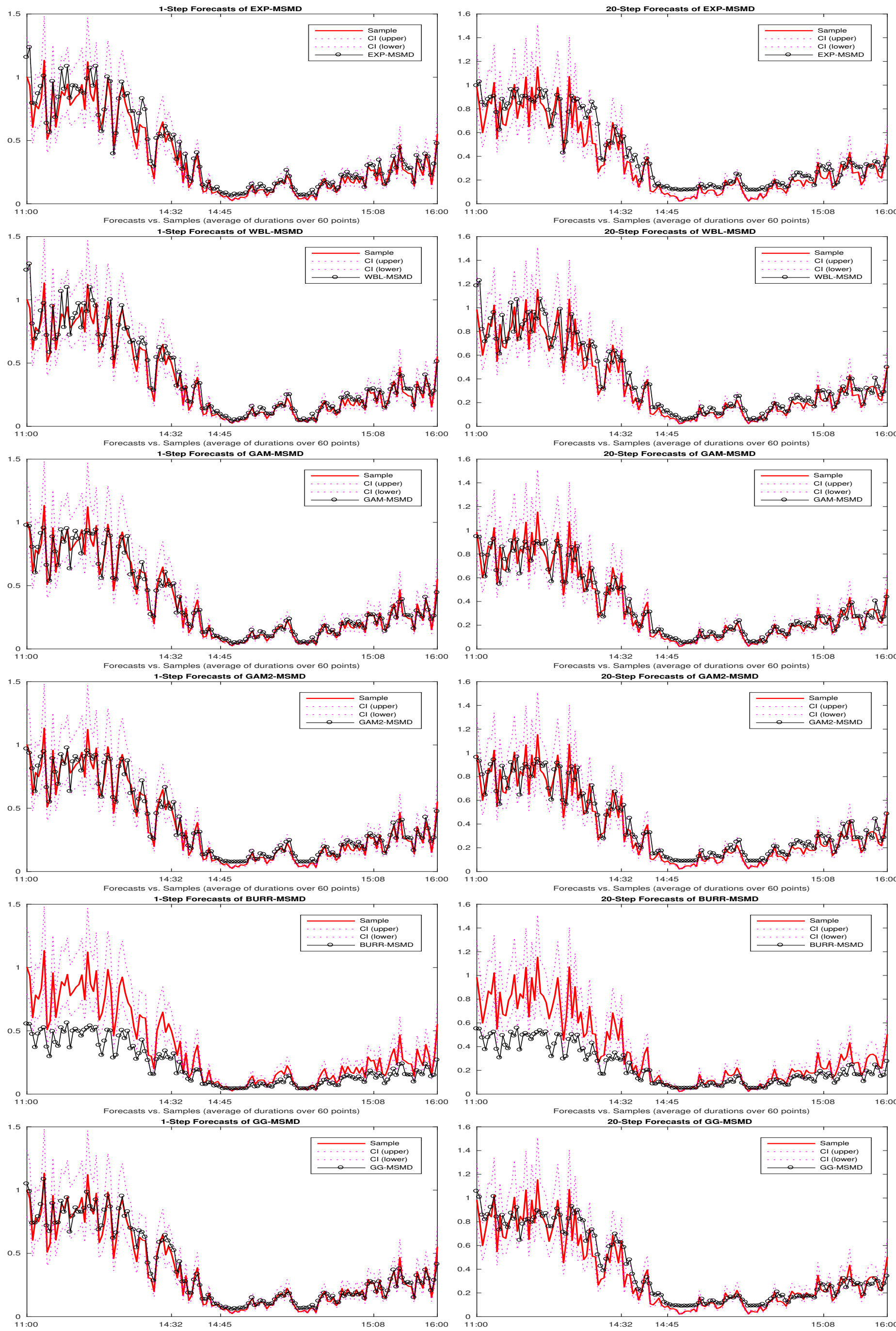


Figure 26: Multi-step forecasts of inter-trade duration for Flash Crash (MSFT, start at 11:00am on May 6, 2010). From top to bottom are the Exp-, Wbl-, Gam-, Gam2-, Burr- and GG-MSMD models with using six multifractal components. Left column is 1-step forecasts, and right column is 20-step forecasts.

Table 10: MLE results of inter-trade duration (MSFT, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.423	1.380	1.342	1.314	1.282	1.257	1.362	1.332	1.296	1.279	1.277	1.266
	(0.010)	(0.013)	(0.009)	(0.010)	(0.009)	(0.008)	(0.015)	(0.012)	(0.011)	(0.010)	(0.010)	(0.012)
$\hat{\psi}$	0.692	0.618	0.560	0.686	0.659	0.623	0.693	0.582	0.750	0.606	0.830	0.656
	(0.020)	(0.034)	(0.023)	(0.039)	(0.042)	(0.043)	(0.032)	(0.022)	(0.028)	(0.018)	(0.029)	(0.020)
$\hat{\gamma}_{\bar{k}}$	0.091	0.126	0.146	0.118	0.239	0.314	0.024	0.017	0.007	0.006	0.004	0.003
	(0.014)	(0.028)	(0.034)	(0.034)	(0.107)	(0.117)	(0.006)	(0.005)	(0.004)	(0.003)	(0.001)	(0.001)
$\hat{b}$	10.225	6.697	5.117	3.482	3.305	2.847	4.628	2.565	1.225	1.201	1.004	1.001
	(2.168)	(1.199)	(0.821)	(0.554)	(0.557)	(0.348)	(1.343)	(0.640)	(0.445)	(0.284)	(0.119)	(0.098)
$\hat{\kappa}$							0.801	0.802	0.802	0.803	0.804	0.804
							(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)
$\ln \mathcal{L}$	-7711.6	-7677.6	-7659.9	-7653.2	-7650.6	-7649.9	-7366.2	-7341.1	-7325.7	-7316.1	-7314.0	-7314.3
AIC	15431.1	15363.2	15327.8	15314.4	15309.3	15307.7	14742.4	14692.2	14661.3	14642.3	14638.1	14638.5
BIC	15460.0	15392.0	15356.6	15343.3	15338.1	15336.5	14778.5	14728.3	14697.4	14678.3	14674.1	14674.6
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.351	1.331	1.293	1.280	1.277	1.276	1.360	1.400	1.140	1.359	1.329	1.304
	(0.014)	(0.011)	(0.010)	(0.009)	(0.010)	(0.010)	(0.014)	(0.012)	(0.009)	(0.011)	(0.012)	(0.010)
$\hat{\psi}$	0.718	0.581	0.751	0.600	0.824	1.126	0.720	0.926	0.924	1.103	1.014	0.931
	(0.034)	(0.019)	(0.025)	(0.017)	(0.028)	(0.050)	(0.031)	(0.038)	(0.022)	(0.062)	(0.094)	(0.058)
$\hat{\gamma}_{\bar{k}}$	0.024	0.018	0.006	0.007	0.004	0.003	0.033	1.000	1.000	1.000	1.000	1.000
	(0.005)	(0.005)	(0.004)	(0.003)	(0.001)	(0.001)	(0.007)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\hat{b}$	4.656	2.706	1.138	1.225	1.035	1.000	5.661	16.123	9.048	7.273	4.984	3.902
	(1.398)	(0.662)	(0.388)	(0.283)	(0.134)	(0.092)	(1.603)	(0.087)	(0.211)	(0.137)	(0.197)	(0.133)
$\hat{\kappa}$	0.675	0.677	0.677	0.678	0.679	0.679	0.709	1.206	0.589	1.287	1.285	1.272
	(0.009)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.012)	(0.048)	(0.031)	(0.066)	(0.086)	(0.074)
$\hat{c}$							0.153	0.664	-2.863	0.694	0.693	0.690
							(0.030)	(0.023)	(0.170)	(0.026)	(0.032)	(0.029)
$\ln \mathcal{L}$	-7186.3	-7162.1	-7143.8	-7135.3	-7133.1	-7134.1	-7172.7	-7162.2	-7224.0	-7127.4	-7121.8	-7118.8
AIC	14382.6	14334.3	14297.7	14280.6	14276.2	14278.2	14357.3	14336.3	14459.9	14266.9	14255.7	14249.6
BIC	14418.7	14370.3	14333.7	14316.6	14312.3	14314.2	14400.6	14379.6	14503.2	14310.1	14298.9	14292.9
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.453	1.866	1.383	1.995	1.373	1.358	1.377	1.379	1.365	1.332	1.307	1.291
	(0.018)	(0.014)	(0.022)	(0.000)	(0.019)	(0.016)	(0.012)	(0.011)	(0.011)	(0.009)	(0.011)	(0.008)
$\hat{\psi}$	0.275	10.000	0.252	5.519	0.297	0.240	1.161	0.897	0.681	0.651	0.676	0.665
	(0.016)	(1.917)	(0.022)	(0.213)	(0.020)	(0.027)	(0.039)	(0.026)	(0.023)	(0.036)	(0.064)	(0.032)
$\hat{\gamma}_{\bar{k}}$	0.031	0.013	0.020	0.076	0.011	0.011	0.140	0.156	0.171	0.427	0.319	0.621
	(0.005)	(0.002)	(0.004)	(0.008)	(0.002)	(0.002)	(0.023)	(0.024)	(0.030)	(0.087)	(0.069)	(0.107)
$\hat{b}$	8.274	16.084	2.881	49.999	1.730	1.621	9.178	6.868	6.030	5.291	3.782	3.676
	(0.020)	(1.932)	(0.024)	(0.222)	(0.025)	(0.028)	(1.824)	(1.078)	(0.844)	(0.630)	(0.514)	(0.358)
$\hat{\kappa}$	0.663	0.657	0.663	0.725	0.663	0.665	0.175	0.167	0.164	0.129	0.138	0.098
	(0.006)	(0.006)	(0.006)	(0.008)	(0.006)	(0.006)	(0.009)	(0.009)	(0.010)	(0.011)	(0.012)	(0.011)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	2.547	2.707	2.764	3.501	3.270	4.589
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.124)	(0.137)	(0.151)	(0.278)	(0.279)	(0.509)
$\ln \mathcal{L}$	-9476.2	-9481.3	-9438.4	-9420.1	-9428.8	-9424.4	-8796.0	-8749.5	-8727.3	-8720.1	-8712.0	-8700.6
AIC	18964.3	18974.5	18888.8	18852.3	18869.5	18860.8	17604.0	17511.0	17466.6	17452.2	17436.0	17413.3
BIC	19007.6	19017.8	18932.1	18895.6	18912.8	18904.1	17647.3	17554.3	17509.9	17495.5	17479.2	1.362

Table 11: MLE results of inter-trade duration (INTC, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.469	1.456	1.422	1.370	1.337	1.310	1.340	1.305	1.284	1.271	1.272	1.353
	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.009)	(0.013)	(0.016)	(0.013)	(0.013)	(0.013)	(0.013)
$\hat{\psi}$	0.734	0.565	0.783	0.689	0.667	0.662	0.923	0.785	0.646	0.862	1.168	8.087
	(0.034)	(0.026)	(0.034)	(0.032)	(0.043)	(0.069)	(0.032)	(0.037)	(0.024)	(0.032)	(0.052)	(0.886)
$\hat{\gamma}_{\bar{k}}$	0.574	0.635	0.706	0.939	0.986	0.993	0.019	0.008	0.006	0.004	0.004	0.035
	(0.056)	(0.058)	(0.064)	(0.063)	(0.020)	(0.012)	(0.005)	(0.005)	(0.003)	(0.002)	(0.001)	(0.008)
$\hat{b}$	15.353	12.096	9.327	6.908	5.105	3.979	3.846	1.459	1.182	1.001	1.014	50.000
	(2.085)	(1.620)	(1.081)	(0.995)	(0.609)	(0.395)	(1.287)	(0.634)	(0.365)	(0.168)	(0.143)	(0.018)
$\hat{\kappa}$							0.736	0.735	0.737	0.738	0.738	0.737
							(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
$\ln \mathcal{L}$	-8139.8	-8097.9	-8070.9	-8062.4	-8058.9	-8057.2	-7550.3	-7529.9	-7523.1	-7516.7	-7518.3	-7577.5
AIC	16287.6	16203.8	16149.9	16132.8	16125.7	16122.4	15110.7	15069.7	15056.1	15043.5	15046.6	15165.0
BIC	16316.5	16232.6	16178.7	16161.6	16154.6	16151.2	15146.7	15105.8	15092.2	15079.5	15082.6	15201.1
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.345	1.319	1.293	1.267	1.256	1.256	1.337	1.272	1.299	1.232	1.228	1.241
	(0.013)	(0.017)	(0.015)	(0.015)	(0.012)	(0.012)	(0.016)	(0.016)	(0.010)	(0.019)	(0.020)	(0.022)
$\hat{\psi}$	0.930	0.737	1.001	0.835	0.687	0.912	0.817	0.764	0.674	0.795	0.670	0.547
	(0.027)	(0.030)	(0.040)	(0.035)	(0.026)	(0.033)	(0.031)	(0.032)	(0.020)	(0.038)	(0.038)	(0.032)
$\hat{\gamma}_{\bar{k}}$	0.021	0.014	0.010	0.005	0.004	0.004	0.307	0.020	0.011	0.005	0.004	0.011
	(0.006)	(0.005)	(0.005)	(0.002)	(0.002)	(0.001)	(0.072)	(0.007)	(0.005)	(0.003)	(0.003)	(0.004)
$\hat{b}$	4.388	2.191	1.574	1.000	1.000	1.000	8.410	2.446	1.641	1.000	1.000	1.562
	(1.479)	(0.721)	(0.543)	(0.207)	(0.159)	(0.116)	(1.295)	(0.676)	(0.471)	(0.289)	(0.259)	(0.261)
$\hat{\kappa}$	0.604	0.606	0.606	0.607	0.608	0.608	0.711	0.613	0.614	0.605	0.606	0.606
	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.023)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)
$\hat{c}$							0.534	0.087	0.063	0.004	0.000	0.000
							(0.052)	(0.038)	(0.024)	(0.035)	(0.034)	(0.036)
$\ln \mathcal{L}$	-7336.1	-7319.1	-7312.3	-7304.9	-7303.7	-7302.9	-8335.0	-8382.6	-7800.6	-8390.9	-8392.3	-8392.5
AIC	14682.2	14648.2	14634.6	14619.8	14617.4	14615.7	16681.9	16777.2	15613.3	16793.7	16796.6	16797.0
BIC	14718.3	14684.3	14670.7	14655.8	14653.5	14651.8	16725.2	16820.4	15656.5	16837.0	16839.8	16840.3
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.342	1.310	1.285	1.272	1.262	1.261	1.430	1.388	1.362	1.353	1.327	1.297
	(0.013)	(0.016)	(0.013)	(0.013)	(0.013)	(0.013)	(0.014)	(0.012)	(0.009)	(0.009)	(0.010)	(0.007)
$\hat{\psi}$	0.577	0.488	0.408	0.546	0.444	0.594	1.119	1.014	0.984	0.759	0.742	0.781
	(0.022)	(0.026)	(0.016)	(0.021)	(0.018)	(0.023)	(0.047)	(0.035)	(0.032)	(0.027)	(0.038)	(0.046)
$\hat{\gamma}_{\bar{k}}$	0.018	0.009	0.006	0.004	0.004	0.003	0.120	0.405	0.916	0.905	0.999	0.999
	(0.003)	(0.002)	(0.001)	(0.001)	(0.001)	(0.001)	(0.024)	(0.086)	(0.075)	(0.072)	(0.000)	(0.000)
$\hat{b}$	3.782	1.587	1.182	1.002	1.004	1.001	15.324	11.040	9.080	7.679	6.107	4.093
	(0.031)	(0.025)	(0.019)	(0.027)	(0.021)	(0.029)	(3.814)	(1.732)	(1.447)	(1.167)	(0.387)	(0.218)
$\hat{\kappa}$	0.737	0.737	0.739	0.740	0.741	0.741	0.338	0.276	0.209	0.208	0.167	0.149
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.016)	(0.017)	(0.015)	(0.015)	(0.015)	(0.016)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	1.619	1.980	2.602	2.627	3.272	3.651
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.066)	(0.115)	(0.179)	(0.185)	(0.278)	(0.390)
$\ln \mathcal{L}$	-7555.5	-7535.1	-7528.1	-7522.0	-7520.7	-7520.0	-7250.9	-7205.8	-7181.4	-7172.2	-7160.4	-7155.9
AIC	15123.1	15082.1	15068.3	15055.9	15053.5	15052.1	14513.8	14423.7	14374.9	14356.4	14332.8	14323.9
BIC	15166.3	15125.4	15111.5	15099.2	15096.8	15095.4	14557.1	14467.0	14418.1	14399.7	14376.1	14367.2

Table 12: MLE results of inter-trade duration (FB, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
	Exp-MSMD						Wbl-MSMD					
$\hat{m}_0$	1.965	1.601	1.991	1.512	1.930	1.466	1.453	1.401	1.381	1.373	1.374	1.357
	(0.002)	(0.011)	(0.000)	(0.010)	(0.002)	(0.011)	(0.018)	(0.019)	(0.021)	(0.018)	(0.019)	(0.016)
$\hat{\psi}$	10.000	0.415	10.000	0.315	10.000	0.364	0.587	0.689	0.536	0.419	0.631	0.511
	(0.457)	(0.018)	(0.324)	(0.018)	(0.622)	(0.028)	(0.032)	(0.037)	(0.045)	(0.053)	(0.041)	(0.053)
$\hat{\gamma}_{\bar{k}}$	0.346	0.631	0.320	0.799	0.304	0.824	0.031	0.019	0.020	0.019	0.011	0.012
	(0.012)	(0.043)	(0.009)	(0.060)	(0.014)	(0.055)	(0.006)	(0.005)	(0.006)	(0.006)	(0.004)	(0.004)
$\hat{b}$	50.000	11.479	50.000	6.864	20.131	4.934	8.224	3.234	2.885	2.498	1.722	1.622
	(11.761)	(1.261)	(6.064)	(0.629)	(3.295)	(0.443)	(2.496)	(0.820)	(0.597)	(0.439)	(0.319)	(0.225)
$\hat{\kappa}$							0.661	0.660	0.662	0.663	0.662	0.663
							(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
$\ln \mathcal{L}$	-10004.6	-10533.2	-9740.3	-10499.4	-10228.3	-10490.2	-9467.4	-9443.7	-9429.6	-9422.1	-9419.9	-9415.6
AIC	20017.1	21074.4	19488.7	21006.9	20464.6	20988.4	18944.9	18897.5	18869.2	18854.1	18849.9	18841.1
BIC	20046.0	21103.2	19517.5	21035.7	20493.5	21017.2	18980.9	18933.5	18905.3	18890.2	18886.0	18877.2
	Gam-MSMD						Gam2-MSMD					
$\hat{m}_0$	1.369	1.361	1.343	1.332	1.322	1.312	1.389	1.359	1.237	1.352	1.354	1.319
	(0.015)	(0.016)	(0.015)	(0.020)	(0.030)	(0.028)	(0.017)	(0.013)	(0.009)	(0.012)	(0.012)	(0.011)
$\hat{\psi}$	0.853	0.652	0.522	0.439	0.355	0.521	1.426	1.335	1.268	1.730	1.297	2.333
	(0.030)	(0.029)	(0.039)	(0.050)	(0.044)	(0.048)	(0.053)	(0.059)	(0.027)	(0.104)	(0.073)	(0.109)
$\hat{\gamma}_{\bar{k}}$	0.020	0.020	0.020	0.019	0.017	0.011	0.650	0.998	0.176	1.000	1.000	1.000
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.004)	(0.056)	(0.000)	(0.013)	(0.000)	(0.000)	(0.000)
$\hat{b}$	4.439	3.710	3.061	2.481	2.128	1.604	11.629	10.021	1.000	10.569	11.307	6.308
	(1.385)	(0.976)	(0.635)	(0.428)	(0.304)	(0.222)	(1.713)	(1.048)	(0.000)	(0.122)	(0.092)	(0.132)
$\hat{\kappa}$	0.517	0.520	0.521	0.523	0.524	0.523	0.678	0.759	0.562	0.778	0.787	0.792
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.023)	(0.027)	(0.015)	(0.029)	(0.030)	(0.031)
$\hat{c}$							0.986	0.958	0.885	0.944	0.944	0.941
							(0.038)	(0.031)	(0.047)	(0.030)	(0.029)	(0.029)
$\ln \mathcal{L}$	-9073.1	-9047.3	-9034.4	-9029.0	-9026.5	-9023.0	-8790.5	-8773.6	-9665.2	-8755.0	-8755.3	-8752.0
AIC	18156.2	18104.6	18078.8	18068.0	18063.1	18055.9	17592.9	17559.2	19342.5	17522.1	17522.7	17515.9
BIC	18192.3	18140.7	18114.9	18104.1	18099.1	18092.0	17636.2	17602.5	19385.7	17565.4	17565.9	17559.2
	Burr-MSMD						GG-MSMD					
$\hat{m}_0$	1.453	1.866	1.383	1.995	1.373	1.358	1.377	1.379	1.365	1.332	1.307	1.291
	(0.018)	(0.014)	(0.022)	(0.000)	(0.019)	(0.016)	(0.012)	(0.011)	(0.011)	(0.009)	(0.011)	(0.008)
$\hat{\psi}$	0.275	10.000	0.252	5.519	0.297	0.240	1.161	0.897	0.681	0.651	0.676	0.665
	(0.016)	(1.917)	(0.022)	(0.213)	(0.020)	(0.027)	(0.039)	(0.026)	(0.023)	(0.036)	(0.064)	(0.032)
$\hat{\gamma}_{\bar{k}}$	0.031	0.013	0.020	0.076	0.011	0.011	0.140	0.156	0.171	0.427	0.319	0.621
	(0.005)	(0.002)	(0.004)	(0.008)	(0.002)	(0.002)	(0.023)	(0.024)	(0.030)	(0.087)	(0.069)	(0.107)
$\hat{b}$	8.274	16.084	2.881	49.999	1.730	1.621	9.178	6.868	6.030	5.291	3.782	3.676
	(0.020)	(1.932)	(0.024)	(0.222)	(0.025)	(0.028)	(1.824)	(1.078)	(0.844)	(0.630)	(0.514)	(0.358)
$\hat{\kappa}$	0.663	0.657	0.663	0.725	0.663	0.665	0.175	0.167	0.164	0.129	0.138	0.098
	(0.006)	(0.006)	(0.006)	(0.008)	(0.006)	(0.006)	(0.009)	(0.009)	(0.010)	(0.011)	(0.012)	(0.011)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	2.547	2.707	2.764	3.501	3.270	4.589
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.124)	(0.137)	(0.151)	(0.278)	(0.279)	(0.509)
$\ln \mathcal{L}$	-9476.2	-9481.3	-9438.4	-9420.1	-9428.8	-9424.4	-8796.0	-8749.5	-8727.3	-8720.1	-8712.0	-8700.6
AIC	18964.3	18974.5	18888.8	18852.3	18869.5	18860.8	17604.0	17511.0	17466.6	17452.2	17436.0	17413.3
BIC	19007.6	19017.8	18932.1	18895.6	18912.8	18904.1	17647.3	17554.3	17509.9	17495.5	17479.2	17456.5

Table 13: MLE results of inter-trade duration (QCOM, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
	Exp-MSMD						Wbl-MSMD					
$\hat{m}_0$	1.584	1.548	1.511	1.488	1.469	1.434	1.360	1.361	1.304	1.282	1.318	1.242
	(0.010)	(0.012)	(0.013)	(0.011)	(0.011)	(0.012)	(0.014)	(0.015)	(0.019)	(0.018)	(0.017)	(0.014)
$\hat{\psi}$	0.934	0.938	0.865	0.689	0.584	0.625	1.103	1.717	1.280	1.008	2.907	1.153
	(0.053)	(0.045)	(0.058)	(0.000)	(0.000)	(0.000)	(0.041)	(0.078)	(0.081)	(0.043)	(0.256)	(0.056)
$\hat{\gamma}_{\bar{k}}$	0.827	0.998	0.999	0.999	0.999	0.999	0.012	0.013	0.008	0.007	0.011	0.006
	(0.043)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.004)	(0.003)	(0.004)	(0.003)	(0.003)	(0.003)
$\hat{b}$	13.726	18.357	8.777	7.171	5.638	4.119	2.892	3.714	1.645	1.438	2.610	1.263
	(1.748)	(2.161)	(0.805)	(0.589)	(0.375)	(0.232)	(1.108)	(1.223)	(0.650)	(0.401)	(0.773)	(0.267)
$\hat{\kappa}$							0.614	0.614	0.614	0.615	0.614	0.615
							(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
$\ln \mathcal{L}$	-9085.8	-9014.5	-8984.3	-8972.2	-8971.1	-8970.4	-7427.1	-7428.8	-7416.0	-7412.2	-7418.1	-7412.2
AIC	18179.6	18037.0	17976.7	17952.4	17950.1	17948.9	14864.3	14867.5	14842.0	14834.4	14846.1	14834.3
BIC	18208.4	18065.9	18005.5	17981.3	17979.0	17977.7	14900.3	14903.6	14878.1	14870.5	14882.2	14870.4
	Gam-MSMD						Gam2-MSMD					
$\hat{m}_0$	1.357	1.359	1.285	1.247	1.360	1.237	1.359	1.317	1.287	1.247	1.400	1.233
	(0.013)	(0.013)	(0.014)	(0.012)	(0.013)	(0.017)	(0.013)	(0.015)	(0.014)	(0.012)	(0.012)	(0.015)
$\hat{\psi}$	1.017	1.576	1.095	0.970	5.988	1.038	1.017	0.848	1.089	0.973	3.220	1.051
	(0.035)	(0.065)	(0.050)	(0.039)	(0.541)	(0.045)	(0.034)	(0.039)	(0.055)	(0.043)	(0.188)	(0.048)
$\hat{\gamma}_{\bar{k}}$	0.013	0.014	0.010	0.008	0.015	0.007	0.015	0.012	0.012	0.009	1.000	0.008
	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.006)	(0.005)	(0.000)	(0.004)
$\hat{b}$	3.275	4.276	1.669	1.424	4.730	1.295	3.584	2.486	1.810	1.497	10.825	1.330
	(1.325)	(1.666)	(0.537)	(0.373)	(1.803)	(0.285)	(1.355)	(0.919)	(0.590)	(0.387)	(0.229)	(0.283)
$\hat{\kappa}$	0.475	0.475	0.476	0.475	0.475	0.475	0.477	0.477	0.478	0.478	0.757	0.478
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.032)	(0.006)
$\hat{c}$							0.057	0.052	0.058	0.056	0.694	0.058
							(0.026)	(0.025)	(0.025)	(0.025)	(0.025)	(0.024)
$\ln \mathcal{L}$	-6946.5	-6947.9	-6934.2	-6930.7	-6950.1	-6930.5	-6943.9	-6934.4	-6931.4	-6928.1	-6958.6	-6927.6
AIC	13902.9	13905.7	13878.5	13871.5	13910.2	13871.0	13899.8	13880.8	13874.8	13868.2	13929.2	13867.1
BIC	13939.0	13941.8	13914.5	13907.5	13946.2	13907.0	13943.1	13924.1	13918.1	13911.5	13972.4	13910.4
	Burr-MSMD						GG-MSMD					
$\hat{m}_0$	1.360	1.361	1.304	1.304	1.280	1.242	1.399	1.334	1.334	1.310	1.296	1.296
	(0.014)	(0.014)	(0.018)	(0.014)	(0.017)	(0.014)	(0.011)	(0.010)	(0.009)	(0.008)	(0.008)	(0.008)
$\hat{\psi}$	0.400	0.622	0.463	0.206	0.504	0.421	1.075	1.253	1.131	1.058	0.929	1.322
	(0.018)	(0.034)	(0.029)	(0.012)	(0.028)	(0.023)	(0.055)	(0.050)	(0.055)	(0.049)	(0.033)	(0.049)
$\hat{\gamma}_{\bar{k}}$	0.012	0.013	0.008	0.011	0.007	0.006	0.058	0.083	0.283	0.729	0.872	0.870
	(0.002)	(0.003)	(0.002)	(0.002)	(0.002)	(0.001)	(0.019)	(0.018)	(0.089)	(0.123)	(0.090)	(0.090)
$\hat{b}$	2.813	3.699	1.620	2.369	1.470	1.277	8.026	5.343	6.596	6.336	5.146	5.126
	(0.027)	(0.046)	(0.041)	(0.016)	(0.040)	(0.030)	(2.055)	(1.047)	(1.273)	(0.838)	(0.630)	(0.620)
$\hat{\kappa}$	0.615	0.615	0.616	0.616	0.616	0.617	0.212	0.192	0.153	0.119	0.104	0.104
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.012)	(0.010)	(0.012)	(0.011)	(0.010)	(0.010)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	1.987	2.177	2.742	3.511	4.025	4.050
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.103)	(0.100)	(0.208)	(0.323)	(0.383)	(0.376)
$\ln \mathcal{L}$	-7436.5	-7438.2	-7425.4	-7424.9	-7422.1	-7421.6	-6728.5	-6695.8	-6672.1	-6662.1	-6655.0	-6655.6
AIC	14885.0	14888.3	14862.8	14861.8	14856.2	14855.3	13469.0	13403.5	13356.2	13336.1	13322.0	13323.1
BIC	14928.3	14931.6	14906.0	14905.1	14899.5	14898.5	13512.3	13446.8	13399.4	13379.4	13365.3	13366.4

Table 14: MLE results of price duration (MSFT, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.584	1.534	1.483	1.466	1.425	1.424	1.403	1.381	1.368	1.370	1.362	1.400
	(0.010)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.015)	(0.014)	(0.015)	(0.016)	(0.022)	(0.021)
$\hat{\psi}$	0.678	0.586	0.597	0.452	0.484	0.348	0.670	0.525	0.792	0.586	0.913	8.274
	(0.031)	(0.028)	(0.053)	(0.027)	(0.000)	(0.024)	(0.031)	(0.025)	(0.038)	(0.030)	(0.049)	(1.769)
$\hat{\gamma}_{\bar{k}}$	0.959	0.962	0.999	0.999	0.999	0.999	0.021	0.019	0.008	0.015	0.005	0.025
	(0.026)	(0.021)	(0.000)	(0.000)	(0.000)	(0.000)	(0.006)	(0.005)	(0.006)	(0.006)	(0.002)	(0.006)
$\hat{b}$	21.719	13.130	7.848	6.859	4.576	4.474	2.640	2.238	1.149	1.778	1.001	3.514
	(3.730)	(1.525)	(0.482)	(0.406)	(0.212)	(0.217)	(0.822)	(0.481)	(0.445)	(0.416)	(0.150)	(0.927)
$\hat{\kappa}$							0.635	0.637	0.637	0.637	0.637	0.636
							(0.006)	(0.006)	(0.005)	(0.006)	(0.006)	(0.006)
$\ln \mathcal{L}$	-6393.0	-6337.1	-6318.3	-6310.4	-6308.1	-6307.1	-5043.8	-5037.1	-5033.9	-5036.2	-5035.6	-5049.4
AIC	12794.0	12682.1	12644.7	12628.8	12624.2	12622.1	10097.5	10084.2	10077.8	10082.4	10081.1	10108.8
BIC	12822.8	12711.0	12673.5	12657.7	12653.1	12651.0	10133.6	10120.3	10113.9	10118.5	10117.2	10144.9
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.395	1.370	1.355	1.305	1.360	1.282	1.411	1.340	1.343	1.295	1.291	1.292
	(0.016)	(0.015)	(0.018)	(0.013)	(0.017)	(0.023)	(0.015)	(0.015)	(0.018)	(0.015)	(0.018)	(0.018)
$\hat{\psi}$	0.636	0.513	0.758	0.670	1.825	0.749	0.872	0.654	0.740	0.680	0.897	0.723
	(0.033)	(0.030)	(0.044)	(0.036)	(0.173)	(0.046)	(0.030)	(0.055)	(0.049)	(0.047)	(0.067)	(0.054)
$\hat{\gamma}_{\bar{k}}$	0.028	0.025	0.016	0.020	0.026	0.008	0.942	0.060	0.026	0.036	0.025	0.036
	(0.007)	(0.006)	(0.009)	(0.010)	(0.008)	(0.007)	(0.039)	(0.015)	(0.012)	(0.017)	(0.017)	(0.017)
$\hat{b}$	3.518	2.686	1.568	1.449	2.410	1.023	26.050	3.277	1.906	1.773	1.456	1.822
	(1.054)	(0.586)	(0.571)	(0.357)	(0.468)	(0.271)	(5.528)	(0.683)	(0.588)	(0.378)	(0.415)	(0.432)
$\hat{\kappa}$	0.499	0.501	0.501	0.502	0.501	0.502	0.702	0.528	0.521	0.522	0.521	0.522
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.028)	(0.009)	(0.008)	(0.009)	(0.009)	(0.009)
$\hat{c}$							0.509	0.118	0.097	0.096	0.091	0.094
							(0.037)	(0.027)	(0.024)	(0.026)	(0.025)	(0.025)
$\ln \mathcal{L}$	-4667.1	-4659.7	-4657.6	-4655.7	-4660.8	-4657.2	-4725.6	-4651.6	-4648.6	-4648.0	-4648.4	-4649.5
AIC	9344.1	9329.4	9325.1	9321.5	9331.5	9324.5	9463.2	9315.2	9309.2	9308.1	9308.8	9311.1
BIC	9380.2	9365.5	9361.2	9357.5	9367.6	9360.5	9506.5	9358.4	9352.4	9351.3	9352.0	9354.3
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.404	1.381	1.420	1.368	1.363	1.421	1.400	1.388	1.352	1.326	1.309	1.286
	(0.015)	(0.014)	(0.016)	(0.015)	(0.020)	(0.013)	(0.011)	(0.011)	(0.010)	(0.010)	(0.010)	(0.013)
$\hat{\psi}$	0.276	0.218	0.786	0.506	0.378	0.673	0.883	0.707	0.686	0.718	0.854	0.767
	(0.014)	(0.012)	(0.050)	(0.033)	(0.022)	(0.033)	(0.035)	(0.032)	(0.041)	(0.046)	(0.055)	(0.068)
$\hat{\gamma}_{\bar{k}}$	0.020	0.019	0.045	0.009	0.005	0.045	0.138	0.152	0.284	0.323	0.440	0.535
	(0.003)	(0.003)	(0.008)	(0.002)	(0.001)	(0.008)	(0.020)	(0.023)	(0.059)	(0.076)	(0.091)	(0.124)
$\hat{b}$	2.608	2.221	50.000	1.242	1.001	50.000	6.440	4.969	3.869	2.955	2.802	2.445
	(0.022)	(0.018)	(0.067)	(0.045)	(0.034)	(0.045)	(1.095)	(0.795)	(0.481)	(0.303)	(0.317)	(0.250)
$\hat{\kappa}$	0.637	0.639	0.638	0.638	0.639	0.638	0.206	0.197	0.167	0.146	0.127	0.113
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.005)	(0.010)	(0.011)	(0.012)	(0.013)	(0.014)	(0.016)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	2.135	2.244	2.635	3.007	3.437	3.872
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.092)	(0.109)	(0.171)	(0.253)	(0.378)	(0.545)
$\ln \mathcal{L}$	-5051.9	-5045.2	-5090.4	-5044.2	-5043.7	-5090.1	-4470.5	-4444.5	-4433.0	-4423.7	-4418.7	-4413.7
AIC	10115.8	10102.5	10192.7	10100.4	10099.3	10192.3	8952.9	8900.9	8878.1	8859.5	8849.4	8839.4
BIC	10159.0	10145.8	10236.0	10143.7	10142.6	10235.6	8996.2	8944.2	8921.4	8902.7	8892.7	8882.7

Table 15: MLE results of price duration (INTC, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.645	1.620	1.598	1.528	1.598	1.578	1.393	1.377	1.395	1.377	1.376	1.377
	(0.009)	(0.010)	(0.010)	(0.011)	(0.010)	(0.010)	(0.017)	(0.018)	(0.017)	(0.020)	(0.021)	(0.020)
$\hat{\psi}$	0.583	1.263	0.802	0.713	4.972	10.000	0.806	0.605	2.208	0.700	1.120	0.815
	(0.019)	(0.064)	(0.033)	(0.109)	(0.393)	(0.883)	(0.037)	(0.030)	(0.191)	(0.035)	(0.078)	(0.047)
$\hat{\gamma}_{\bar{k}}$	0.999	0.999	0.999	0.999	0.999	0.997	0.031	0.031	0.032	0.032	0.031	0.032
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)	(0.006)	(0.006)	(0.006)	(0.006)	(0.007)	(0.006)
$\hat{b}$	22.241	18.969	14.176	5.827	14.200	12.654	4.311	3.538	5.174	3.720	3.662	3.765
	(2.036)	(2.000)	(1.205)	(0.360)	(1.203)	(1.290)	(1.224)	(0.904)	(1.207)	(0.964)	(0.981)	(0.958)
$\hat{\kappa}$							0.621	0.622	0.622	0.622	0.622	0.622
							(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
$\ln \mathcal{L}$	-6543.5	-6496.9	-6471.3	-6464.9	-6472.6	-6475.3	-5199.5	-5198.6	-5201.8	-5199.4	-5199.5	-5199.7
AIC	13095.0	13001.9	12950.6	12937.8	12953.2	12958.7	10408.9	10407.1	10413.7	10408.8	10409.1	10409.4
BIC	13123.8	13030.7	12979.5	12966.6	12982.1	12987.5	10445.0	10443.2	10449.7	10444.8	10445.1	10445.4
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.389	1.322	1.306	1.264	1.367	1.368	1.406	1.351	1.354	1.355	1.359	1.359
	(0.015)	(0.015)	(0.035)	(0.020)	(0.018)	(0.018)	(0.017)	(0.016)	(0.016)	(0.016)	(0.012)	(0.012)
$\hat{\psi}$	0.769	0.807	0.767	0.765	4.475	7.141	0.873	0.808	1.246	1.926	1.333	0.982
	(0.031)	(0.077)	(0.078)	(0.100)	(0.650)	(1.254)	(0.046)	(0.050)	(0.089)	(0.160)	(0.084)	(0.059)
$\hat{\gamma}_{\bar{k}}$	0.039	0.060	0.042	0.059	0.039	0.040	0.073	0.183	0.186	0.187	1.000	1.000
	(0.007)	(0.015)	(0.016)	(0.018)	(0.007)	(0.007)	(0.018)	(0.047)	(0.047)	(0.047)	(0.000)	(0.000)
$\hat{b}$	5.032	3.258	2.179	2.012	4.522	4.584	7.250	5.770	6.067	6.135	6.582	6.606
	(1.334)	(0.751)	(0.606)	(0.393)	(0.979)	(1.001)	(2.232)	(1.218)	(1.178)	(1.159)	(0.121)	(0.096)
$\hat{\kappa}$	0.486	0.488	0.488	0.488	0.486	0.487	0.541	0.564	0.565	0.565	0.843	0.842
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.012)	(0.014)	(0.015)	(0.014)	(0.036)	(0.036)
$\hat{c}$							0.252	0.306	0.308	0.309	0.617	0.616
							(0.035)	(0.039)	(0.039)	(0.038)	(0.026)	(0.026)
$\ln \mathcal{L}$	-4849.0	-4848.3	-4846.3	-4846.4	-4852.1	-4852.8	-4811.4	-4802.6	-4803.7	-4804.5	-4786.5	-4786.6
AIC	9707.9	9706.6	9702.6	9702.8	9714.2	9715.6	9634.8	9617.2	9619.4	9621.0	9585.0	9585.3
BIC	9744.0	9742.7	9738.7	9738.9	9750.3	9751.7	9678.1	9660.5	9662.7	9664.2	9628.2	9628.6
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.393	1.394	1.397	1.395	1.376	1.375	1.465	1.393	1.361	1.343	1.343	1.289
	(0.017)	(0.017)	(0.017)	(0.017)	(0.018)	(0.018)	(0.011)	(0.014)	(0.010)	(0.008)	(0.007)	(0.007)
$\hat{\psi}$	0.307	0.505	0.844	1.382	0.427	0.680	0.972	0.909	0.899	0.843	1.283	0.819
	(0.017)	(0.035)	(0.076)	(0.152)	(0.031)	(0.064)	(0.029)	(0.051)	(0.044)	(0.032)	(0.056)	(0.035)
$\hat{\gamma}_{\bar{k}}$	0.031	0.032	0.032	0.032	0.031	0.031	0.171	0.216	0.442	0.788	0.811	0.881
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.022)	(0.041)	(0.077)	(0.094)	(0.089)	(0.079)
$\hat{b}$	4.270	4.946	5.398	5.168	3.628	3.627	49.979	5.423	4.413	3.964	4.159	2.645
	(0.030)	(0.050)	(0.092)	(0.168)	(0.047)	(0.080)	(0.000)	(0.964)	(0.559)	(0.438)	(0.431)	(0.211)
$\hat{\kappa}$	0.623	0.623	0.623	0.623	0.624	0.624	0.234	0.205	0.160	0.113	0.113	0.090
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.011)	(0.012)	(0.013)	(0.011)	(0.011)	(0.012)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	1.869	2.125	2.700	3.845	3.852	4.815
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.080)	(0.114)	(0.199)	(0.378)	(0.370)	(0.619)
$\ln \mathcal{L}$	-5206.9	-5208.4	-5209.4	-5210.0	-5207.0	-5207.3	-4737.6	-4686.2	-4669.7	-4661.3	-4663.0	-4652.9
AIC	10425.8	10428.9	10430.7	10432.1	10425.9	10426.6	9487.2	9384.5	9351.5	9334.5	9338.1	9317.9
BIC	10469.1	10472.2	10474.0	10475.3	10469.2	10469.9	9530.5	9427.7	9394.7	9377.8	9381.3	9361.1

Table 16: MLE results of price duration (FB, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.443	1.381	1.371	1.325	1.318	1.318	1.379	1.306	1.306	1.302	1.300	1.300
	(0.013)	(0.011)	(0.011)	(0.010)	(0.012)	(0.012)	(0.014)	(0.014)	(0.014)	(0.014)	(0.015)	(0.016)
$\hat{\psi}$	0.580	0.637	0.494	0.581	0.470	0.687	0.517	0.643	0.924	2.347	0.564	0.811
	(0.024)	(0.030)	(0.023)	(0.038)	(0.035)	(0.046)	(0.021)	(0.031)	(0.046)	(0.178)	(0.030)	(0.045)
$\hat{\gamma}_{\bar{k}}$	0.254	0.641	0.646	0.644	0.659	0.653	0.025	0.040	0.040	0.045	0.036	0.035
	(0.032)	(0.131)	(0.115)	(0.115)	(0.110)	(0.110)	(0.005)	(0.010)	(0.010)	(0.011)	(0.010)	(0.011)
$\hat{b}$	14.007	8.517	7.518	4.660	4.255	4.273	6.569	3.940	4.009	6.314	3.076	2.932
	(2.552)	(1.556)	(1.195)	(0.523)	(0.494)	(0.511)	(2.087)	(0.882)	(0.926)	(1.504)	(0.814)	(1.008)
$\hat{\kappa}$							0.776	0.781	0.781	0.778	0.781	0.782
							(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)
$\ln \mathcal{L}$	-8786.4	-8780.4	-8767.5	-8767.4	-8765.8	-8766.0	-8392.4	-8383.7	-8384.4	-8395.5	-8382.7	-8382.8
AIC	17580.8	17568.7	17543.0	17542.7	17539.5	17540.0	16794.8	16777.4	16778.8	16801.0	16775.4	16775.5
BIC	17609.6	17597.6	17571.9	17571.6	17568.4	17568.9	16830.9	16813.5	16814.9	16837.1	16811.5	16811.6
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.286	1.288	1.284	1.283	1.287	1.284	1.356	1.348	1.317	1.317	1.317	1.336
	(0.013)	(0.014)	(0.016)	(0.016)	(0.014)	(0.015)	(0.013)	(0.012)	(0.011)	(0.011)	(0.011)	(0.012)
$\hat{\psi}$	0.799	0.642	0.518	0.721	1.770	0.782	0.898	0.724	0.752	1.099	1.610	3.728
	(0.030)	(0.027)	(0.029)	(0.036)	(0.115)	(0.039)	(0.025)	(0.019)	(0.023)	(0.033)	(0.060)	(0.239)
$\hat{\gamma}_{\bar{k}}$	0.043	0.041	0.040	0.038	0.042	0.038	0.979	1.000	1.000	1.000	1.000	0.986
	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.025)	(0.000)	(0.000)	(0.000)	(0.000)	(0.017)
$\hat{b}$	5.715	4.424	3.539	3.394	4.642	3.499	17.487	22.532	9.299	9.419	9.437	14.910
	(1.561)	(1.240)	(0.746)	(0.918)	(1.183)	(0.943)	(3.595)	(0.049)	(0.058)	(0.070)	(0.097)	(2.413)
$\hat{\kappa}$	0.652	0.655	0.656	0.656	0.655	0.656	0.996	1.034	1.079	1.077	1.076	0.980
	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.032)	(0.032)	(0.038)	(0.037)	(0.038)	(0.031)
$\hat{c}$							0.809	0.791	0.809	0.808	0.808	0.784
							(0.031)	(0.029)	(0.028)	(0.028)	(0.028)	(0.031)
$\ln \mathcal{L}$	-8209.7	-8204.1	-8204.1	-8204.1	-8206.1	-8204.5	-8002.8	-7994.6	-7974.7	-7976.0	-7976.8	-7991.9
AIC	16429.3	16418.1	16418.1	16418.2	16422.2	16419.0	16017.6	16001.2	15961.4	15964.1	15965.6	15995.7
BIC	16465.4	16454.2	16454.2	16454.3	16458.2	16455.0	16060.9	16044.5	16004.7	16007.4	16008.9	16039.0
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.380	1.306	1.302	1.298	1.301	1.300	1.336	1.300	1.297	1.296	1.296	1.296
	(0.014)	(0.014)	(0.015)	(0.016)	(0.016)	(0.015)	(0.013)	(0.011)	(0.009)	(0.009)	(0.009)	(0.009)
$\hat{\psi}$	0.364	0.459	0.369	0.527	0.407	1.071	0.745	0.831	0.820	0.644	0.914	0.706
	(0.016)	(0.023)	(0.021)	(0.031)	(0.023)	(0.083)	(0.030)	(0.027)	(0.025)	(0.021)	(0.030)	(0.023)
$\hat{\gamma}_{\bar{k}}$	0.025	0.039	0.037	0.030	0.035	0.034	0.175	0.443	0.952	0.953	0.955	0.955
	(0.005)	(0.007)	(0.007)	(0.006)	(0.007)	(0.006)	(0.033)	(0.116)	(0.052)	(0.048)	(0.047)	(0.046)
$\hat{b}$	6.518	3.856	3.197	2.487	2.993	2.834	11.199	7.493	7.016	6.541	6.583	6.568
	(0.019)	(0.031)	(0.027)	(0.041)	(0.031)	(0.095)	(2.310)	(1.373)	(0.978)	(0.885)	(0.872)	(0.881)
$\hat{\kappa}$	0.779	0.783	0.785	0.784	0.784	0.784	0.302	0.254	0.180	0.177	0.177	0.177
	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.015)	(0.018)	(0.016)	(0.016)	(0.016)	(0.016)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	1.874	2.205	3.101	3.157	3.164	3.160
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.077)	(0.141)	(0.268)	(0.267)	(0.271)	(0.270)
$\ln \mathcal{L}$	-8398.3	-8389.8	-8388.4	-8388.4	-8388.7	-8389.6	-8057.9	-8040.1	-8025.8	-8021.6	-8022.2	-8022.3
AIC	16808.5	16791.6	16788.9	16788.7	16789.5	16791.2	16127.9	16092.2	16063.6	16055.2	16056.4	16056.5
BIC	16851.8	16834.8	16832.1	16832.0	16832.7	16834.5	16171.1	16135.5	16106.9	16098.5	16099.6	16099.8

Table 17: MLE results of price duration (QCOM, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.400	1.337	1.298	1.283	1.252	1.296	1.293	1.262	1.253	1.257	1.258	1.259
	(0.011)	(0.012)	(0.009)	(0.017)	(0.011)	(0.010)	(0.013)	(0.012)	(0.013)	(0.014)	(0.014)	(0.013)
$\hat{\psi}$	0.810	0.894	0.856	0.743	0.794	0.742	0.916	1.103	0.895	0.721	0.967	0.770
	(0.025)	(0.038)	(0.037)	(0.075)	(0.082)	(0.038)	(0.036)	(0.074)	(0.038)	(0.029)	(0.047)	(0.034)
$\hat{\gamma}_{\bar{k}}$	0.228	0.439	0.647	0.668	0.756	0.655	0.016	0.015	0.009	0.016	0.016	0.017
	(0.032)	(0.145)	(0.107)	(0.113)	(0.131)	(0.103)	(0.007)	(0.006)	(0.007)	(0.006)	(0.006)	(0.006)
$\hat{b}$	15.246	8.249	5.588	4.404	3.296	5.331	2.328	2.033	1.219	2.014	2.087	2.223
	(2.950)	(1.685)	(0.727)	(0.726)	(0.381)	(0.821)	(1.201)	(0.783)	(0.586)	(0.539)	(0.621)	(0.517)
$\hat{\kappa}$							0.780	0.779	0.780	0.780	0.780	0.780
							(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)
$\ln \mathcal{L}$	-7363.3	-7354.9	-7349.6	-7349.1	-7349.0	-7349.8	-6947.0	-6946.8	-6944.4	-6945.9	-6946.3	-6946.6
AIC	14734.6	14717.7	14707.2	14706.2	14706.0	14707.6	13904.0	13903.6	13898.8	13901.7	13902.6	13903.1
BIC	14763.4	14746.6	14736.1	14735.0	14734.9	14736.4	13940.0	13939.7	13934.8	13937.8	13938.6	13939.2
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.293	1.294	1.264	1.254	1.257	1.256	1.410	1.379	1.347	1.326	1.287	1.326
	(0.013)	(0.014)	(0.012)	(0.014)	(0.013)	(0.013)	(0.011)	(0.012)	(0.011)	(0.011)	(0.009)	(0.011)
$\hat{\psi}$	0.903	0.707	1.521	0.705	1.601	0.755	1.126	0.908	0.905	1.179	0.995	1.321
	(0.034)	(0.023)	(0.097)	(0.024)	(0.106)	(0.028)	(0.036)	(0.022)	(0.032)	(0.050)	(0.053)	(0.061)
$\hat{\gamma}_{\bar{k}}$	0.017	0.020	0.018	0.017	0.018	0.017	1.000	1.000	1.000	1.000	1.000	1.000
	(0.006)	(0.005)	(0.005)	(0.006)	(0.005)	(0.005)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\hat{b}$	2.431	3.095	2.899	2.166	2.455	2.329	60.000	17.292	8.418	7.732	4.202	7.710
	(1.172)	(0.977)	(0.674)	(0.533)	(0.516)	(0.523)	(0.079)	(0.070)	(0.085)	(0.098)	(0.103)	(0.108)
$\hat{\kappa}$	0.652	0.653	0.651	0.652	0.652	0.651	1.206	1.242	1.329	1.295	1.318	1.291
	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.038)	(0.045)	(0.056)	(0.052)	(0.060)	(0.052)
$\hat{c}$							0.792	0.782	0.797	0.811	0.819	0.811
							(0.025)	(0.024)	(0.024)	(0.025)	(0.026)	(0.025)
$\ln \mathcal{L}$	-6741.0	-6742.3	-6743.6	-6740.9	-6742.3	-6741.5	-6586.5	-6540.8	-6524.5	-6521.2	-6519.6	-6521.8
AIC	13492.1	13494.7	13497.3	13491.7	13494.7	13493.0	13185.0	13093.5	13061.0	13054.3	13051.1	13055.7
BIC	13528.1	13530.7	13533.3	13527.8	13530.7	13529.0	13228.3	13136.8	13104.3	13097.6	13094.4	13099.0
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.292	1.349	1.253	1.260	1.999	1.260	1.376	1.325	1.304	1.276	1.259	1.250
	(0.012)	(0.012)	(0.013)	(0.012)	(0.000)	(0.012)	(0.011)	(0.010)	(0.009)	(0.007)	(0.006)	(0.017)
$\hat{\psi}$	0.650	1.262	0.638	0.865	8.855	0.933	0.788	0.934	0.905	0.868	0.833	0.723
	(0.026)	(0.067)	(0.028)	(0.049)	(0.092)	(0.056)	(0.022)	(0.028)	(0.028)	(0.028)	(0.033)	(0.090)
$\hat{\gamma}_{\bar{k}}$	0.016	0.024	0.008	0.017	0.010	0.017	0.224	0.390	0.914	0.999	0.999	0.999
	(0.003)	(0.005)	(0.002)	(0.004)	(0.002)	(0.004)	(0.037)	(0.092)	(0.069)	(0.000)	(0.000)	(0.000)
$\hat{b}$	2.250	49.983	1.174	2.180	50.000	2.281	13.261	8.726	7.573	6.093	4.210	3.578
	(0.037)	(0.079)	(0.036)	(0.058)	(0.095)	(0.065)	(2.527)	(1.631)	(1.097)	(0.404)	(0.287)	(0.600)
$\hat{\kappa}$	0.781	0.781	0.782	0.782	0.746	0.782	0.266	0.241	0.182	0.153	0.124	0.112
	(0.007)	(0.007)	(0.007)	(0.007)	(0.005)	(0.007)	(0.015)	(0.016)	(0.016)	(0.013)	(0.014)	(0.015)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	2.121	2.329	3.078	3.656	4.493	4.978
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.105)	(0.142)	(0.268)	(0.292)	(0.495)	(0.638)
$\ln \mathcal{L}$	-6953.4	-6977.2	-6950.7	-6952.9	-7273.7	-6953.2	-6578.0	-6549.8	-6533.2	-6524.2	-6518.6	-6519.4
AIC	13918.7	13966.5	13913.4	13917.8	14559.3	13918.4	13168.0	13111.5	13078.4	13060.3	13049.2	13050.8
BIC	13962.0	14009.7	13956.6	13961.0	14602.6	13961.6	13211.2	13154.8	13121.7	13103.6	13092.4	13094.0

Table 18: MLE results of volume duration (MSFT, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
	Exp-MSMD						Wbl-MSMD					
$\hat{m}_0$	1.974	1.988	1.994	1.997	1.998	1.999	1.391	1.378	1.398	1.385	1.271	1.195
	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.025)	(0.028)	(0.027)	(0.025)	(0.049)	(0.029)
$\hat{\psi}$	10.000	10.000	10.000	10.000	10.000	8.886	2.117	1.510	6.011	4.052	1.104	1.233
	(0.495)	(0.338)	(0.327)	(0.163)	(0.000)	(0.234)	(0.151)	(0.086)	(0.985)	(0.509)	(0.086)	(0.145)
$\hat{\gamma}_{\bar{k}}$	0.323	0.490	0.999	0.447	0.431	0.416	0.006	0.003	0.007	0.006	0.003	0.005
	(0.024)	(0.010)	(0.000)	(0.009)	(0.008)	(0.020)	(0.002)	(0.001)	(0.002)	(0.002)	(0.001)	(0.003)
$\hat{b}$	1.001	18.639	50.000	50.000	50.000	50.000	2.767	1.001	4.267	2.175	1.001	1.001
	(0.081)	(2.244)	(3.584)	(4.125)	(0.000)	(0.000)	(1.525)	(0.238)	(1.798)	(0.546)	(0.162)	(0.132)
$\hat{\kappa}$							0.520	0.522	0.520	0.522	0.523	0.523
							(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
$\ln \mathcal{L}$	-7035.5	-6679.9	-7644.0	-5789.2	-5505.7	-5407.6	-5356.1	-5346.2	-5358.3	-5350.2	-5347.7	-5348.0
AIC	14079.0	13367.7	15296.0	11586.4	11019.5	10823.1	10722.3	10702.5	10726.6	10710.4	10705.4	10706.1
BIC	14107.9	13396.6	15324.9	11615.3	11048.3	10852.0	10758.3	10738.5	10762.7	10746.4	10741.4	10742.1
	Gam-MSMD						Gam2-MSMD					
$\hat{m}_0$	1.327	1.300	1.292	1.284	1.276	1.272	1.532	1.487	1.686	1.520	1.435	1.692
	(0.016)	(0.023)	(0.020)	(0.019)	(0.026)	(0.029)	(0.025)	(0.016)	(0.014)	(0.018)	(0.012)	(0.014)
$\hat{\psi}$	1.133	1.507	1.193	0.941	1.260	0.998	1.204	0.782	0.564	1.068	1.654	0.118
	(0.041)	(0.115)	(0.070)	(0.044)	(0.114)	(0.073)	(0.049)	(0.014)	(0.011)	(0.044)	(0.064)	(0.002)
$\hat{\gamma}_{\bar{k}}$	0.004	0.003	0.003	0.003	0.003	0.002	0.502	0.325	0.569	0.498	0.302	0.572
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.000)	(0.022)	(0.017)	(0.018)	(0.021)	(0.016)	(0.019)
$\hat{b}$	1.000	1.001	1.000	1.000	1.002	1.011	21.576	17.602	24.789	21.529	16.114	24.928
	(0.327)	(0.234)	(0.150)	(0.116)	(0.096)	(0.087)	(1.649)	(1.475)	(1.685)	(1.608)	(1.425)	(2.216)
$\hat{\kappa}$	0.389	0.389	0.390	0.391	0.391	0.391	0.905	0.823	0.981	0.899	0.814	0.984
	(0.004)	(0.004)	(0.005)	(0.005)	(0.005)	(0.005)	(0.026)	(0.025)	(0.026)	(0.026)	(0.026)	(0.026)
$\hat{c}$							1.451	1.533	1.023	1.489	1.697	1.009
							(0.086)	(0.059)	(0.032)	(0.064)	(0.052)	(0.032)
$\ln \mathcal{L}$	-4447.3	-4444.0	-4433.7	-4431.8	-4432.5	-4432.5	-3158.4	-3466.1	-3153.0	-3157.2	-3465.3	-3154.4
AIC	8904.5	8898.0	8877.4	8873.7	8875.0	8875.1	7687.1	7770.2	7828.2	7673.6	7741.4	7837.4
BIC	8940.6	8934.1	8913.4	8909.7	8911.1	8911.1	7730.4	7813.4	7871.4	7716.8	7784.6	7880.6
	Burr-MSMD						GG-MSMD					
$\hat{m}_0$	1.995	1.379	1.353	1.999	1.280	1.194	1.404	1.351	1.317	1.307	1.305	1.261
	(0.000)	(0.026)	(0.038)	(0.000)	(0.031)	(0.026)	(0.009)	(0.008)	(0.016)	(0.007)	(0.007)	(0.007)
$\hat{\psi}$	9.999	0.244	0.178	10.000	0.182	0.202	1.251	1.602	2.181	1.706	1.312	1.446
	(0.834)	(0.017)	(0.012)	(0.521)	(0.014)	(0.024)	(0.036)	(0.044)	(0.095)	(0.057)	(0.038)	(0.056)
$\hat{\gamma}_{\bar{k}}$	0.034	0.003	0.003	0.398	0.003	0.006	0.013	0.021	0.028	0.021	0.006	0.018
	(0.007)	(0.001)	(0.001)	(0.009)	(0.001)	(0.002)	(0.004)	(0.005)	(0.007)	(0.006)	(0.003)	(0.008)
$\hat{b}$	49.987	1.001	1.001	50.000	1.001	1.001	2.266	2.424	2.819	1.858	1.010	1.383
	(0.844)	(0.040)	(0.043)	(4.368)	(0.038)	(0.040)	(0.900)	(0.520)	(0.459)	(0.303)	(0.165)	(0.218)
$\hat{\kappa}$	0.527	0.523	0.524	0.926	0.524	0.524	0.140	0.114	0.103	0.102	0.103	0.096
	(0.006)	(0.005)	(0.005)	(0.001)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
$\hat{\theta}$	0.006	0.006	0.006	0.456	0.006	0.006	2.476	3.046	3.325	3.392	3.367	3.593
	(0.000)	(0.000)	(0.000)	(0.019)	(0.000)	(0.000)	(0.081)	(0.120)	(0.155)	(0.164)	(0.152)	(0.193)
$\ln \mathcal{L}$	-5433.3	-5358.8	-5359.1	-4984.3	-5360.5	-5360.9	-3967.2	-3893.4	-3880.0	-3852.8	-3840.2	-3836.7
AIC	10878.6	10729.6	10730.1	9980.7	10733.0	10733.7	7946.4	7798.9	7772.1	7717.6	7692.3	7685.3
BIC	10921.8	10772.9	10773.4	10023.9	10776.3	10777.0	7989.7	7842.1	7815.3	7760.8	7735.6	7728.6

Table 19: MLE results of volume duration (INTC, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
	Exp-MSMD						Wbl-MSMD					
$\hat{m}_0$	1.974	1.818	1.994	1.913	1.999	1.678	1.437	1.364	1.354	1.348	1.292	1.339
	(0.001)	(0.000)	(0.000)	(0.003)	(0.000)	(0.008)	(0.023)	(0.017)	(0.017)	(0.020)	(0.021)	(0.027)
$\hat{\psi}$	10.000	10.000	10.000	10.000	10.000	3.247	0.946	1.086	0.838	1.243	1.243	1.383
	(0.497)	(0.000)	(0.267)	(0.616)	(0.189)	(0.249)	(0.043)	(0.052)	(0.050)	(0.078)	(0.129)	(0.121)
$\hat{\gamma}_{\bar{k}}$	0.469	0.497	0.415	0.483	0.339	0.999	0.006	0.006	0.005	0.005	0.006	0.004
	(0.011)	(0.018)	(0.009)	(0.015)	(0.016)	(0.000)	(0.003)	(0.002)	(0.002)	(0.002)	(0.002)	(0.001)
$\hat{b}$	50.000	11.852	38.920	6.621	50.000	1.700	1.003	1.001	1.001	1.001	1.001	1.005
	(0.000)	(0.189)	(7.084)	(0.668)	(0.000)	(0.032)	(0.458)	(0.234)	(0.206)	(0.139)	(0.119)	(0.103)
$\hat{\kappa}$							0.550	0.551	0.552	0.552	0.553	0.552
							(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
$\ln \mathcal{L}$	-7881.9	-8510.0	-7278.2	-8235.2	-8270.1	-8378.7	-6059.0	-6048.3	-6047.3	-6045.4	-6046.4	-6046.7
AIC	15771.9	17028.1	14564.5	16478.4	16548.3	16765.4	12127.9	12106.7	12104.7	12100.7	12102.7	12103.4
BIC	15800.7	17056.9	14593.3	16507.2	16577.1	16794.2	12164.0	12142.7	12140.7	12136.8	12138.8	12139.5
	Gam-MSMD						Gam2-MSMD					
$\hat{m}_0$	1.347	1.364	1.291	1.291	1.275	1.274	1.632	1.620	1.651	1.278	1.285	1.385
	(0.013)	(0.021)	(0.013)	(0.018)	(0.015)	(0.017)	(0.014)	(0.012)	(0.010)	(0.009)	(0.008)	(0.011)
$\hat{\psi}$	1.161	1.064	0.983	0.833	1.036	0.848	1.505	3.850	2.726	0.981	0.994	0.379
	(0.058)	(0.068)	(0.065)	(0.065)	(0.071)	(0.057)	(0.045)	(0.217)	(0.120)	(0.026)	(0.029)	(0.013)
$\hat{\gamma}_{\bar{k}}$	0.007	0.005	0.006	0.005	0.005	0.005	0.417	0.408	0.489	1.000	1.000	1.000
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.001)	(0.019)	(0.019)	(0.021)	(0.000)	(0.000)	(0.000)
$\hat{b}$	1.000	1.000	1.000	1.000	1.000	1.000	39.679	38.789	26.802	3.387	2.788	10.579
	(0.319)	(0.209)	(0.149)	(0.125)	(0.102)	(0.094)	(4.271)	(4.234)	(2.720)	(0.052)	(0.058)	(0.046)
$\hat{\kappa}$	0.411	0.413	0.414	0.415	0.415	0.415	0.826	0.810	0.738	0.822	0.866	0.851
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.022)	(0.021)	(0.021)	(0.030)	(0.035)	(0.030)
$\hat{c}$							1.000	1.000	0.848	1.107	1.000	1.000
							(0.034)	(0.032)	(0.021)	(0.032)	(0.026)	(0.028)
$\ln \mathcal{L}$	-5276.2	-5264.7	-5258.6	-5257.8	-5255.8	-5257.3	-4741.6	-4746.5	-5932.6	-4793.4	-4815.5	-4820.9
AIC	10562.4	10539.3	10527.1	10525.7	10521.6	10524.6	9495.1	9505.1	11877.2	9598.8	9643.0	9653.9
BIC	10598.5	10575.4	10563.2	10561.7	10557.7	10560.6	9538.4	9548.3	11920.5	9642.0	9686.2	9697.1
	Burr-MSMD						GG-MSMD					
$\hat{m}_0$	1.437	1.363	1.999	1.345	1.293	1.999	1.379	1.321	1.312	1.289	1.285	1.268
	(0.022)	(0.017)	(0.000)	(0.019)	(0.020)	(0.000)	(0.032)	(0.008)	(0.008)	(0.008)	(0.011)	(0.014)
$\hat{\psi}$	0.205	0.241	10.000	0.274	0.277	0.001	1.476	1.391	1.266	1.550	1.142	1.044
	(0.012)	(0.015)	(0.837)	(0.020)	(0.030)	(0.000)	(0.491)	(0.053)	(0.072)	(0.096)	(0.044)	(0.111)
$\hat{\gamma}_{\bar{k}}$	0.006	0.006	0.309	0.005	0.006	0.297	0.050	0.040	0.095	0.068	0.011	0.151
	(0.001)	(0.001)	(0.008)	(0.001)	(0.001)	(0.009)	(0.107)	(0.014)	(0.036)	(0.038)	(0.007)	(0.117)
$\hat{b}$	1.009	1.020	50.000	1.011	1.002	50.000	6.546	2.545	2.839	2.244	1.002	2.088
	(0.025)	(0.026)	(5.755)	(0.034)	(0.036)	(0.033)	(12.759)	(0.605)	(0.575)	(0.528)	(0.206)	(0.463)
$\hat{\kappa}$	0.551	0.553	0.862	0.553	0.553	0.813	0.133	0.122	0.100	0.102	0.114	0.083
	(0.005)	(0.005)	(0.002)	(0.005)	(0.005)	(0.012)	(0.122)	(0.007)	(0.009)	(0.011)	(0.007)	(0.018)
$\hat{\theta}$	0.006	0.006	0.559	0.006	0.006	0.376	2.775	2.970	3.618	3.532	3.190	4.371
	(0.000)	(0.000)	(0.020)	(0.000)	(0.000)	(0.016)	(2.340)	(0.165)	(0.315)	(0.364)	(0.170)	(0.932)
$\ln \mathcal{L}$	-6070.9	-6060.4	-6277.4	-6057.4	-6058.5	-6158.1	-4849.3	-4802.1	-4779.4	-4774.0	-4764.1	-4763.7
AIC	12153.8	12132.7	12566.9	12126.8	12128.9	12328.1	9710.5	9616.3	9570.8	9560.1	9540.3	9539.4
BIC	12197.0	12176.0	12610.1	12170.0	12172.2	12371.4	9753.8	9659.5	9614.1	9603.3	9583.5	9582.6

Table 20: MLE results of volume duration (FB, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.346	1.340	1.273	1.278	1.265	1.266	1.361	1.355	1.310	1.284	1.299	1.276
	(0.012)	(0.019)	(0.010)	(0.015)	(0.012)	(0.012)	(0.010)	(0.011)	(0.010)	(0.012)	(0.011)	(0.011)
$\hat{\psi}$	0.968	0.769	0.857	0.762	0.901	0.739	0.977	0.763	0.720	0.793	0.786	0.850
	(0.029)	(0.049)	(0.045)	(0.051)	(0.045)	(0.062)	(0.023)	(0.023)	(0.035)	(0.037)	(0.042)	(0.042)
$\hat{\gamma}_{\bar{k}}$	0.018	0.021	0.013	0.018	0.010	0.018	0.037	0.036	0.038	0.016	0.028	0.013
	(0.012)	(0.007)	(0.007)	(0.008)	(0.005)	(0.007)	(0.009)	(0.008)	(0.009)	(0.017)	(0.010)	(0.009)
$\hat{b}$	1.270	1.577	1.001	1.281	1.005	1.311	1.998	1.700	1.606	1.006	1.317	1.002
	(0.950)	(0.498)	(0.287)	(0.297)	(0.190)	(0.221)	(0.633)	(0.372)	(0.277)	(0.419)	(0.231)	(0.204)
$\hat{\kappa}$							1.201	1.218	1.223	1.223	1.229	1.227
							(0.011)	(0.011)	(0.011)	(0.011)	(0.012)	(0.011)
$\ln \mathcal{L}$	-9657.7	-9641.7	-9630.9	-9627.6	-9622.8	-9621.3	-9462.2	-9424.7	-9409.1	-9400.2	-9393.6	-9388.7
AIC	19323.3	19291.4	19269.7	19263.2	19253.5	19250.6	18934.3	18859.3	18828.1	18810.4	18797.3	18787.4
BIC	19352.2	19320.3	19298.6	19292.0	19282.3	19279.4	18970.4	18895.4	18864.2	18846.5	18833.3	18823.4
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.355	1.352	1.281	1.287	1.273	1.347	1.403	1.361	1.322	1.302	1.276	1.272
	(0.011)	(0.014)	(0.011)	(0.015)	(0.012)	(0.015)	(0.006)	(0.004)	(0.005)	(0.004)	(0.003)	(0.003)
$\hat{\psi}$	0.975	0.761	0.839	0.781	0.897	8.029	1.320	1.401	1.235	1.472	1.275	1.729
	(0.027)	(0.029)	(0.045)	(0.039)	(0.045)	(1.133)	(0.035)	(0.026)	(0.037)	(0.034)	(0.023)	(0.027)
$\hat{\gamma}_{\bar{k}}$	0.023	0.025	0.014	0.017	0.011	0.035	0.387	0.582	0.702	0.833	0.918	0.957
	(0.010)	(0.007)	(0.010)	(0.010)	(0.006)	(0.005)	(0.026)	(0.030)	(0.037)	(0.033)	(0.026)	(0.019)
$\hat{b}$	1.459	1.628	1.000	1.180	1.003	3.155	4.970	4.113	3.047	3.006	2.598	2.774
	(0.798)	(0.421)	(0.362)	(0.323)	(0.199)	(0.462)	(0.491)	(0.253)	(0.179)	(0.148)	(0.126)	(0.123)
$\hat{\kappa}$	1.093	1.108	1.106	1.113	1.111	1.092	2.599	3.199	3.496	3.992	4.390	4.614
	(0.014)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.074)	(0.093)	(0.112)	(0.136)	(0.147)	(0.153)
$\hat{c}$							1.371	1.486	1.499	1.637	1.673	1.753
							(0.029)	(0.027)	(0.028)	(0.030)	(0.028)	(0.030)
$\ln \mathcal{L}$	-9635.4	-9612.9	-9603.1	-9597.0	-9592.4	-9646.5	-8271.1	-8140.7	-8108.2	-8058.1	-8016.9	-7997.7
AIC	19280.9	19235.9	19216.2	19204.1	19194.9	19303.0	16554.3	16293.5	16228.3	16128.2	16045.7	16007.4
BIC	19316.9	19271.9	19252.2	19240.1	19230.9	19339.0	16597.5	16336.8	16271.6	16171.5	16089.0	16050.7
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.361	1.355	1.311	1.284	1.299	1.276	1.303	1.256	1.242	1.238	1.211	1.206
	(0.010)	(0.011)	(0.010)	(0.012)	(0.010)	(0.011)	(0.005)	(0.006)	(0.005)	(0.005)	(0.006)	(0.005)
$\hat{\psi}$	1.088	0.854	0.807	0.889	0.883	0.954	1.151	1.262	1.121	0.935	1.125	0.952
	(0.026)	(0.026)	(0.038)	(0.042)	(0.048)	(0.047)	(0.022)	(0.034)	(0.026)	(0.020)	(0.028)	(0.021)
$\hat{\gamma}_{\bar{k}}$	0.037	0.035	0.038	0.018	0.027	0.013	0.099	0.106	0.133	0.133	0.148	0.143
	(0.004)	(0.003)	(0.004)	(0.003)	(0.003)	(0.003)	(0.013)	(0.016)	(0.024)	(0.025)	(0.027)	(0.028)
$\hat{b}$	1.993	1.687	1.603	1.036	1.307	1.002	4.506	3.217	2.481	2.100	1.909	1.701
	(0.033)	(0.029)	(0.035)	(0.046)	(0.046)	(0.050)	(0.715)	(0.428)	(0.291)	(0.234)	(0.162)	(0.138)
$\hat{\kappa}$	1.204	1.220	1.226	1.225	1.232	1.229	0.203	0.175	0.157	0.153	0.134	0.130
	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	4.018	4.593	5.194	5.371	6.064	6.295
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.148)	(0.213)	(0.279)	(0.285)	(0.385)	(0.422)
$\ln \mathcal{L}$	-9471.4	-9433.7	-9418.3	-9409.6	-9402.9	-9398.0	-8658.4	-8592.6	-8537.5	-8516.1	-8500.0	-8486.6
AIC	18954.7	18879.4	18848.6	18831.1	18817.7	18808.1	17328.7	17197.3	17087.0	17044.1	17012.0	16985.1
BIC	18998.0	18922.7	18891.8	18874.4	18861.0	18851.3	17372.0	17240.6	17130.3	17087.4	17055.3	17028.4

Table 21: MLE results of volume duration (QCOM, $n = 10,000$).

$\bar{k} =$	3	4	5	6	7	8	3	4	5	6	7	8
			Exp-MSMD						Wbl-MSMD			
$\hat{m}_0$	1.504	1.393	1.379	1.337	1.335	1.331	1.517	1.413	1.391	1.345	1.342	1.305
	(0.010)	(0.010)	(0.008)	(0.008)	(0.008)	(0.010)	(0.007)	(0.009)	(0.006)	(0.007)	(0.006)	(0.006)
$\hat{\psi}$	1.054	1.446	1.074	1.401	1.060	1.552	1.048	1.493	1.163	1.394	1.059	1.276
	(0.019)	(0.045)	(0.032)	(0.054)	(0.034)	(0.080)	(0.015)	(0.048)	(0.026)	(0.052)	(0.032)	(0.042)
$\hat{\gamma}_{\bar{k}}$	0.011	0.011	0.010	0.010	0.009	0.008	0.023	0.023	0.017	0.015	0.014	0.014
	(0.004)	(0.004)	(0.004)	(0.003)	(0.003)	(0.003)	(0.009)	(0.009)	(0.011)	(0.006)	(0.005)	(0.004)
$\hat{b}$	1.001	1.001	1.001	1.001	1.001	1.002	1.354	1.358	1.001	1.013	1.001	1.001
	(0.350)	(0.257)	(0.172)	(0.129)	(0.103)	(0.095)	(0.634)	(0.403)	(0.320)	(0.169)	(0.109)	(0.088)
$\hat{\kappa}$							1.266	1.306	1.339	1.346	1.359	1.363
							(0.011)	(0.012)	(0.012)	(0.012)	(0.013)	(0.013)
$\ln\mathcal{L}$	-9784.0	-9691.8	-9664.5	-9645.4	-9639.7	-9640.1	-9453.3	-9283.0	-9196.2	-9156.2	-9129.9	-9122.5
AIC	19576.1	19391.6	19337.0	19298.8	19287.4	19288.2	18916.6	18576.0	18402.4	18322.5	18269.8	18254.9
BIC	19604.9	19420.5	19365.8	19327.6	19316.2	19317.0	18952.6	18612.1	18438.5	18358.5	18305.8	18291.0
			Gam-MSMD						Gam2-MSMD			
$\hat{m}_0$	1.512	1.404	1.387	1.345	1.341	1.330	1.508	1.486	1.468	1.432	1.360	1.368
	(0.008)	(0.009)	(0.007)	(0.008)	(0.007)	(0.019)	(0.007)	(0.004)	(0.004)	(0.005)	(0.005)	(0.004)
$\hat{\psi}$	1.058	1.479	1.123	1.427	1.076	1.511	1.507	2.553	4.542	3.056	1.874	5.000
	(0.018)	(0.047)	(0.036)	(0.056)	(0.035)	(0.181)	(0.039)	(0.045)	(0.095)	(0.061)	(0.056)	(0.231)
$\hat{\gamma}_{\bar{k}}$	0.015	0.013	0.013	0.012	0.011	0.010	0.205	0.311	0.332	0.393	0.532	0.525
	(0.006)	(0.007)	(0.005)	(0.004)	(0.003)	(0.004)	(0.016)	(0.018)	(0.018)	(0.023)	(0.037)	(0.032)
$\hat{b}$	1.000	1.006	1.000	1.004	1.000	1.003	4.913	4.674	4.553	3.434	2.559	2.970
	(0.439)	(0.347)	(0.188)	(0.136)	(0.100)	(0.097)	(0.518)	(0.327)	(0.281)	(0.185)	(0.114)	(0.121)
$\hat{\kappa}$	1.244	1.261	1.295	1.298	1.310	1.310	1.962	2.579	2.706	2.942	3.203	3.185
	(0.017)	(0.017)	(0.018)	(0.018)	(0.018)	(0.018)	(0.044)	(0.063)	(0.066)	(0.079)	(0.103)	(0.095)
$\hat{c}$							0.686	0.899	0.969	0.944	0.995	1.060
							(0.018)	(0.018)	(0.020)	(0.018)	(0.021)	(0.023)
$\ln\mathcal{L}$	-9658.6	-9549.0	-9492.9	-9469.6	-9453.0	-9454.4	-9131.7	-8837.7	-8798.1	-8717.2	-8707.8	-8696.8
AIC	19327.3	19107.9	18995.8	18949.2	18916.0	18918.9	18275.4	17687.5	17608.2	17446.4	17427.6	17405.5
BIC	19363.3	19144.0	19031.9	18985.2	18952.1	18954.9	18318.7	17730.8	17651.5	17489.7	17470.8	17448.8
			Burr-MSMD						GG-MSMD			
$\hat{m}_0$	1.516	1.412	1.391	1.345	1.342	1.305	1.442	1.423	1.357	1.301	1.298	1.296
	(0.007)	(0.009)	(0.006)	(0.006)	(0.006)	(0.006)	(0.005)	(0.006)	(0.005)	(0.004)	(0.003)	(0.003)
$\hat{\psi}$	1.187	1.710	1.343	1.608	1.228	1.479	2.153	1.580	1.954	1.973	1.583	1.258
	(0.018)	(0.052)	(0.030)	(0.057)	(0.035)	(0.043)	(0.052)	(0.030)	(0.059)	(0.044)	(0.036)	(0.029)
$\hat{\gamma}_{\bar{k}}$	0.023	0.024	0.017	0.015	0.014	0.015	0.026	0.046	0.049	0.048	0.035	0.018
	(0.002)	(0.002)	(0.002)	(0.003)	(0.002)	(0.003)	(0.006)	(0.008)	(0.008)	(0.008)	(0.010)	(0.011)
$\hat{b}$	1.358	1.355	1.010	1.001	1.001	1.020	1.893	2.155	1.929	1.660	1.299	1.003
	(0.027)	(0.058)	(0.036)	(0.061)	(0.040)	(0.046)	(0.506)	(0.356)	(0.236)	(0.152)	(0.147)	(0.176)
$\hat{\kappa}$	1.270	1.309	1.343	1.350	1.361	1.367	0.303	0.311	0.257	0.234	0.237	0.236
	(0.011)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.009)
$\hat{\theta}$	0.006	0.006	0.006	0.006	0.006	0.006	2.934	3.009	3.589	3.910	3.932	3.989
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.079)	(0.079)	(0.118)	(0.139)	(0.134)	(0.135)
$\ln\mathcal{L}$	-9457.1	-9290.1	-9203.1	-9164.1	-9137.6	-9130.6	-8948.1	-8765.1	-8623.9	-8560.1	-8486.1	-8451.9
AIC	18926.2	18592.2	18418.3	18340.2	18287.3	18273.2	17908.3	17542.2	17259.7	17132.2	16984.1	16915.9
BIC	18969.5	18635.5	18461.5	18383.4	18330.5	18316.5	17951.5	17585.4	17303.0	17175.5	17027.4	16959.1

Table 22: Multifractal model selection of inter-trade duration ($n = 10,000$)

MSMD	Vuong test					HAC-adjusted Vuong test				
	$\bar{k}=3$	4	5	6	7	3	4	5	6	7
					MSFT					
Exp	-4.728	-3.211	-1.550	-0.784	-0.588	-3.992	-2.953	-1.498	-0.720	-0.585
	(0.000)	(0.001)	(0.061)	(0.216)	(0.278)	(0.000)	(0.002)	(0.067)	(0.236)	(0.279)
Wbl	-4.569	-3.244	-2.058	-0.745	0.089	-3.303	-2.813	-1.529	-0.700	0.083
	(0.000)	(0.001)	(0.020)	(0.228)	(0.465)	(0.000)	(0.002)	(0.063)	(0.242)	(0.467)
Gam	-4.645	-2.969	-2.313	-0.316	0.628	-3.351	-2.646	-1.667	-0.302	0.575
	(0.000)	(0.001)	(0.010)	(0.376)	(0.265)	(0.000)	(0.004)	(0.048)	(0.381)	(0.283)
Gam2	-2.262	-3.444	-2.120	-2.126	-1.720	-2.154	-3.393	-1.666	-2.288	-1.862
	(0.012)	(0.000)	(0.017)	(0.017)	(0.043)	(0.016)	(0.000)	(0.048)	(0.011)	(0.031)
Bur	-4.578	-3.244	-2.065	-0.754	0.085	-3.307	-2.809	-1.531	-0.711	0.079
	(0.000)	(0.001)	(0.019)	(0.225)	(0.466)	(0.000)	(0.002)	(0.063)	(0.239)	(0.469)
GG	-7.238	-6.169	-3.873	-2.736	-2.593	-6.088	-5.558	-3.834	-2.632	-2.574
	(0.000)	(0.000)	(0.000)	(0.003)	(0.005)	(0.000)	(0.000)	(0.000)	(0.004)	(0.005)
					INTC					
Exp	-5.649	-3.525	-2.084	-1.302	-0.904	-4.458	-3.405	-2.209	-1.425	-0.948
	(0.000)	(0.000)	(0.019)	(0.097)	(0.183)	(0.000)	(0.000)	(0.014)	(0.077)	(0.172)
Wbl	4.046	4.810	4.747	5.325	5.023	3.081	3.436	3.102	3.655	3.550
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.001)	(0.000)	(0.000)
Gam	-3.292	-2.282	-1.711	-0.881	-0.466	-2.253	-1.961	-1.162	-0.570	-0.456
	(0.000)	(0.011)	(0.043)	(0.189)	(0.320)	(0.012)	(0.025)	(0.123)	(0.284)	(0.324)
Gam2	2.103	1.450	14.921	0.422	0.109	1.679	0.934	4.363	0.283	0.089
	(0.018)	(0.073)	(0.000)	(0.336)	(0.457)	(0.047)	(0.175)	(0.000)	(0.389)	(0.464)
Bur	-3.548	-2.401	-1.721	-0.814	-0.435	-2.336	-1.784	-1.692	-0.532	-0.409
	(0.000)	(0.008)	(0.043)	(0.208)	(0.332)	(0.010)	(0.037)	(0.045)	(0.297)	(0.341)
GG	-9.157	-6.390	-3.834	-2.725	-0.883	-7.440	-5.528	-3.198	-2.607	-0.836
	(0.000)	(0.000)	(0.000)	(0.003)	(0.189)	(0.000)	(0.000)	(0.001)	(0.005)	(0.202)
					FB					
Exp	4.880	-2.480	6.291	-0.933	3.046	4.360	-2.132	5.231	-0.964	2.778
	(0.000)	(0.007)	(0.000)	(0.175)	(0.001)	(0.000)	(0.016)	(0.000)	(0.168)	(0.003)
Wbl	-4.270	-3.070	-2.326	-2.082	-1.212	-2.446	-1.537	-1.211	-1.213	-0.879
	(0.000)	(0.001)	(0.010)	(0.019)	(0.113)	(0.007)	(0.062)	(0.113)	(0.113)	(0.190)
Gam	-4.183	-2.751	-1.851	-2.145	-1.930	-2.165	-1.627	-1.190	-1.400	-1.606
	(0.000)	(0.003)	(0.032)	(0.016)	(0.027)	(0.015)	(0.052)	(0.117)	(0.081)	(0.054)
Gam2	-4.599	-3.752	-43.106	-0.796	-0.756	-3.980	-2.800	-15.110	-0.850	-0.821
	(0.000)	(0.000)	(0.000)	(0.213)	(0.225)	(0.000)	(0.003)	(0.000)	(0.198)	(0.206)
Bur	-4.232	-1.633	-2.376	0.056	-1.239	-2.425	-1.492	-1.219	0.049	-0.872
	(0.000)	(0.051)	(0.009)	(0.478)	(0.108)	(0.008)	(0.068)	(0.111)	(0.480)	(0.192)
GG	-7.706	-5.199	-3.225	-2.504	-2.710	-4.290	-4.043	-3.194	-2.487	-2.638
	(0.000)	(0.000)	(0.001)	(0.006)	(0.003)	(0.000)	(0.000)	(0.001)	(0.006)	(0.004)
					QCOM					
Exp	-6.229	-3.243	-1.512	-0.336	-0.195	-4.761	-2.754	-1.321	-0.357	-0.222
	(0.000)	(0.001)	(0.065)	(0.368)	(0.423)	(0.000)	(0.003)	(0.093)	(0.361)	(0.412)
Wbl	-2.459	-2.889	-1.104	-0.014	-1.520	-1.710	-1.974	-0.811	-0.011	-1.157
	(0.007)	(0.002)	(0.135)	(0.494)	(0.064)	(0.044)	(0.024)	(0.209)	(0.496)	(0.124)
Gam	-2.325	-2.694	-0.945	-0.163	-2.981	-1.823	-2.068	-0.768	-0.114	-2.272
	(0.010)	(0.004)	(0.172)	(0.435)	(0.001)	(0.034)	(0.019)	(0.221)	(0.454)	(0.012)
Gam2	-2.407	-1.327	-1.047	-0.361	-1.133	-1.887	-1.151	-0.843	-0.250	-0.980
	(0.008)	(0.092)	(0.148)	(0.359)	(0.129)	(0.030)	(0.125)	(0.200)	(0.401)	(0.164)
Bur	-2.456	-2.888	-1.089	-0.782	-0.211	-1.704	-1.967	-0.797	-0.661	-0.179
	(0.007)	(0.002)	(0.138)	(0.217)	(0.416)	(0.044)	(0.025)	(0.213)	(0.254)	(0.429)
GG	-6.588	-5.516	-2.983	-1.597	0.208	-6.138	-4.567	-2.843	-1.539	0.248
	(0.000)	(0.000)	(0.001)	(0.055)	(0.418)	(0.000)	(0.000)	(0.002)	(0.062)	(0.402)

Table 23: Multifractal model selection of price duration ($n = 10,000$)

MSMD	Vuong test					HAC-adjusted Vuong test				
	$\bar{k} = 3$	4	5	6	7	3	4	5	6	7
					MSFT					
Exp	-5.150	-2.748	-1.336	-0.667	-0.304	-3.382	-1.863	-0.771	-0.464	-0.150
	(0.000)	(0.003)	(0.091)	(0.252)	(0.381)	(0.000)	(0.031)	(0.220)	(0.321)	(0.440)
Wbl	1.399	1.856	3.048	1.965	2.015	0.973	1.300	1.918	1.253	1.282
	(0.081)	(0.032)	(0.001)	(0.025)	(0.022)	(0.165)	(0.097)	(0.028)	(0.105)	(0.100)
Gam	-1.329	-0.459	-0.081	0.781	-0.767	-1.084	-0.407	-0.073	0.721	-0.653
	(0.092)	(0.323)	(0.468)	(0.218)	(0.222)	(0.139)	(0.342)	(0.471)	(0.235)	(0.257)
Gam2	-3.872	-0.454	0.305	0.927	0.582	-3.216	-0.370	0.264	0.765	0.519
	(0.000)	(0.325)	(0.380)	(0.177)	(0.280)	(0.001)	(0.356)	(0.396)	(0.222)	(0.302)
Bur	4.708	4.708	-0.254	4.451	4.045	3.128	2.951	-0.215	2.965	2.554
	(0.000)	(0.000)	(0.400)	(0.000)	(0.000)	(0.001)	(0.002)	(0.415)	(0.002)	(0.005)
GG	-5.310	-3.512	-2.744	-2.432	-1.778	-4.476	-3.690	-2.872	-2.311	-1.796
	(0.000)	(0.000)	(0.003)	(0.008)	(0.038)	(0.000)	(0.000)	(0.002)	(0.010)	(0.036)
					INTC					
Exp	-3.235	-1.926	0.856	1.399	0.635	-2.769	-1.627	0.707	1.374	0.556
	(0.001)	(0.027)	(0.196)	(0.081)	(0.263)	(0.003)	(0.052)	(0.240)	(0.085)	(0.289)
Wbl	0.049	0.520	-0.453	0.583	0.191	0.042	0.467	-0.363	0.612	0.160
	(0.481)	(0.302)	(0.325)	(0.280)	(0.424)	(0.483)	(0.320)	(0.358)	(0.270)	(0.436)
Gam	0.834	0.929	1.571	1.481	2.427	0.614	0.749	1.181	1.163	1.207
	(0.202)	(0.176)	(0.058)	(0.069)	(0.008)	(0.270)	(0.227)	(0.119)	(0.122)	(0.114)
Gam2	-1.964	-1.609	-1.752	-1.824	0.161	-1.621	-1.329	-1.407	-1.456	0.142
	(0.025)	(0.054)	(0.040)	(0.034)	(0.436)	(0.052)	(0.092)	(0.080)	(0.073)	(0.444)
Bur	0.092	-0.287	-0.509	-0.662	0.794	0.078	-0.238	-0.415	-0.524	0.510
	(0.463)	(0.387)	(0.305)	(0.254)	(0.213)	(0.469)	(0.406)	(0.339)	(0.300)	(0.305)
GG	-5.945	-5.298	-3.622	-1.642	-1.996	-5.645	-4.988	-3.438	-1.618	-1.917
	(0.000)	(0.000)	(0.000)	(0.050)	(0.023)	(0.000)	(0.000)	(0.000)	(0.053)	(0.028)
					FB					
Exp	-1.694	-1.572	-0.240	-0.425	0.257	-1.386	-1.018	-0.232	-0.250	0.224
	(0.045)	(0.058)	(0.405)	(0.335)	(0.399)	(0.083)	(0.154)	(0.408)	(0.401)	(0.411)
Wbl	-1.075	-0.192	-0.331	-1.356	0.091	-0.930	-0.129	-0.217	-0.788	0.076
	(0.141)	(0.424)	(0.370)	(0.088)	(0.464)	(0.176)	(0.449)	(0.414)	(0.215)	(0.470)
Gam	-0.681	0.113	0.327	0.423	-0.385	-0.433	0.088	0.309	0.463	-0.275
	(0.248)	(0.455)	(0.372)	(0.336)	(0.350)	(0.332)	(0.465)	(0.379)	(0.322)	(0.392)
Gam2	-1.399	-0.494	3.789	3.828	3.629	-1.044	-0.504	3.736	3.699	3.578
	(0.081)	(0.311)	(0.000)	(0.000)	(0.000)	(0.148)	(0.307)	(0.000)	(0.000)	(0.000)
Bur	-0.989	-0.050	0.541	1.192	0.338	-0.856	-0.033	0.440	0.770	0.248
	(0.161)	(0.480)	(0.294)	(0.117)	(0.368)	(0.196)	(0.487)	(0.330)	(0.221)	(0.402)
GG	-3.598	-2.458	-0.556	0.254	0.028	-3.232	-1.911	-0.425	0.253	0.027
	(0.000)	(0.007)	(0.289)	(0.400)	(0.489)	(0.001)	(0.028)	(0.335)	(0.400)	(0.489)
					QCOM					
Exp	-1.327	-0.701	0.030	0.168	0.148	-1.172	-0.535	0.023	0.134	0.119
	(0.092)	(0.242)	(0.488)	(0.433)	(0.441)	(0.121)	(0.296)	(0.491)	(0.447)	(0.453)
Wbl	-0.072	-0.047	0.705	0.771	0.238	-0.056	-0.036	0.602	0.661	0.196
	(0.471)	(0.481)	(0.240)	(0.220)	(0.406)	(0.477)	(0.486)	(0.273)	(0.254)	(0.422)
Gam	0.081	-0.205	-0.473	0.947	-0.363	0.068	-0.179	-0.374	0.801	-0.288
	(0.468)	(0.419)	(0.318)	(0.172)	(0.358)	(0.473)	(0.429)	(0.354)	(0.212)	(0.387)
Gam2	-4.376	-1.788	-0.305	0.279	0.367	-3.644	-1.752	-0.292	0.189	0.345
	(0.000)	(0.037)	(0.380)	(0.390)	(0.357)	(0.000)	(0.040)	(0.385)	(0.425)	(0.365)
Bur	-0.031	-2.719	1.062	0.235	-5.688	-0.025	-2.255	0.947	0.192	-5.159
	(0.487)	(0.003)	(0.144)	(0.407)	(0.000)	(0.490)	(0.012)	(0.172)	(0.424)	(0.000)
GG	-6.185	-4.556	-2.368	-0.948	0.266	-5.877	-4.561	-2.325	-0.892	0.250
	(0.000)	(0.000)	(0.009)	(0.172)	(0.395)	(0.000)	(0.000)	(0.010)	(0.186)	(0.401)

Table 24: Multifractal model selection of volume duration ($n = 10,000$)

MSMD	Vuong test					HAC-adjusted Vuong test				
	$\bar{k}=3$	4	5	6	7	3	4	5	6	7
					MSFT					
Exp	-17.481	-13.214	-25.645	-7.210	-3.824	-13.120	-10.651	-17.575	-6.491	-3.637
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Wbl	-1.739	0.441	-2.111	-0.505	0.129	-1.196	0.317	-1.474	-0.383	0.076
	(0.041)	(0.330)	(0.017)	(0.307)	(0.449)	(0.116)	(0.376)	(0.070)	(0.351)	(0.470)
Gam	-2.138	-2.232	-0.446	0.526	0.021	-1.389	-1.174	-0.255	0.451	0.013
	(0.016)	(0.013)	(0.328)	(0.299)	(0.492)	(0.082)	(0.120)	(0.399)	(0.326)	(0.495)
Gam2	2.899	1.194	2.396	3.177	1.543	4.331	1.627	2.536	4.686	1.998
	(0.002)	(0.116)	(0.008)	(0.001)	(0.061)	(0.000)	(0.052)	(0.006)	(0.000)	(0.023)
Bur	-1.200	0.493	0.410	3.886	0.121	-1.326	0.353	0.247	3.583	0.071
	(0.115)	(0.311)	(0.341)	(0.000)	(0.452)	(0.092)	(0.362)	(0.403)	(0.000)	(0.472)
GG	-4.682	-4.114	-5.093	-2.426	-0.494	-3.717	-3.313	-3.363	-2.137	-0.439
	(0.000)	(0.000)	(0.000)	(0.008)	(0.311)	(0.000)	(0.000)	(0.000)	(0.016)	(0.330)
					INTC					
Exp	6.127	-1.817	10.997	2.164	0.735	4.962	-1.338	8.441	1.845	0.536
	(0.000)	(0.035)	(0.000)	(0.015)	(0.231)	(0.000)	(0.090)	(0.000)	(0.033)	(0.296)
Wbl	-1.553	-0.339	-0.204	0.755	0.094	-1.187	-0.212	-0.172	0.527	0.067
	(0.060)	(0.367)	(0.419)	(0.225)	(0.462)	(0.118)	(0.416)	(0.432)	(0.299)	(0.473)
Gam	-1.946	-1.110	-0.252	-0.230	0.614	-1.307	-0.864	-0.179	-0.189	0.453
	(0.026)	(0.133)	(0.400)	(0.409)	(0.270)	(0.096)	(0.194)	(0.429)	(0.425)	(0.325)
Gam2	3.338	3.283	-22.272	2.307	0.577	3.208	3.160	-7.686	2.226	0.596
	(0.000)	(0.001)	(0.000)	(0.011)	(0.282)	(0.001)	(0.001)	(0.000)	(0.013)	(0.276)
Bur	0.943	1.051	-18.856	1.077	1.064	0.969	1.078	-13.992	1.103	1.090
	(0.173)	(0.147)	(0.000)	(0.141)	(0.144)	(0.166)	(0.141)	(0.000)	(0.135)	(0.138)
GG	-5.800	-3.217	-2.064	-1.473	-0.055	-4.530	-2.632	-1.721	-1.251	-0.052
	(0.000)	(0.001)	(0.019)	(0.070)	(0.478)	(0.000)	(0.004)	(0.043)	(0.105)	(0.479)
					FB					
Exp	-3.288	-2.600	-1.655	-1.630	-0.522	-1.563	-1.440	-0.689	-0.843	-0.228
	(0.001)	(0.005)	(0.049)	(0.052)	(0.301)	(0.059)	(0.075)	(0.245)	(0.200)	(0.410)
Wbl	-4.541	-3.544	-2.150	-2.657	-0.664	-2.446	-2.191	-1.446	-1.285	-0.546
	(0.000)	(0.000)	(0.016)	(0.004)	(0.253)	(0.007)	(0.014)	(0.074)	(0.099)	(0.292)
Gam	2.386	3.189	4.721	4.691	4.755	1.627	1.863	2.940	2.495	2.556
	(0.009)	(0.001)	(0.000)	(0.000)	(0.000)	(0.052)	(0.031)	(0.002)	(0.006)	(0.005)
Gam2	-8.852	-4.994	-3.793	-3.109	-0.843	-8.262	-5.260	-4.014	-3.430	-0.946
	(0.000)	(0.000)	(0.000)	(0.001)	(0.199)	(0.000)	(0.000)	(0.000)	(0.000)	(0.172)
Bur	-4.536	-3.518	-2.141	-2.623	-0.649	-2.437	-2.165	-1.439	-1.269	-0.533
	(0.000)	(0.000)	(0.016)	(0.004)	(0.258)	(0.007)	(0.015)	(0.075)	(0.102)	(0.297)
GG	-7.723	-6.835	-4.662	-3.363	-2.825	-6.034	-4.971	-3.565	-3.081	-1.908
	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.001)	(0.028)
					QCOM					
Exp	-7.062	-4.046	-2.760	-0.976	0.177	-4.428	-2.997	-2.093	-0.831	0.130
	(0.000)	(0.000)	(0.003)	(0.164)	(0.430)	(0.000)	(0.001)	(0.018)	(0.203)	(0.448)
Wbl	-8.889	-7.230	-4.034	-3.175	-0.726	-5.625	-5.368	-3.192	-2.622	-0.585
	(0.000)	(0.000)	(0.000)	(0.001)	(0.234)	(0.000)	(0.000)	(0.001)	(0.004)	(0.279)
Gam	-7.613	-5.436	-3.095	-2.091	0.347	-4.969	-4.028	-2.452	-1.682	0.294
	(0.000)	(0.000)	(0.001)	(0.018)	(0.364)	(0.000)	(0.000)	(0.007)	(0.046)	(0.384)
Gam2	-8.341	-3.964	-3.685	-0.838	-0.327	-5.833	-4.169	-3.218	-0.720	-0.400
	(0.000)	(0.000)	(0.000)	(0.201)	(0.372)	(0.000)	(0.000)	(0.001)	(0.236)	(0.345)
Bur	-8.905	-7.217	-4.016	-3.175	-0.691	-5.622	-5.345	-3.175	-2.616	-0.554
	(0.000)	(0.000)	(0.000)	(0.001)	(0.245)	(0.000)	(0.000)	(0.001)	(0.004)	(0.290)
GG	-13.109	-9.313	-7.634	-5.420	-3.666	-9.023	-7.262	-5.696	-3.664	-2.615
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.004)

Table 25: Prediction performance comparison of inter-trade duration (MSFT, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.093	0.931	0.324	0.949	1.003	0.084	0.677	-0.003	1.030	0.064	0.674	-0.007	1.060	0.047	0.667	-0.009
	(0.028)	(0.024)			(0.103)	(0.098)			(0.103)	(0.097)			(0.101)	(0.095)		
Wbl-ACD	0.089	0.959	0.324	0.951	1.033	0.088	0.690	-0.003	1.071	0.068	0.691	-0.007	1.124	0.051	0.691	-0.009
	(0.028)	(0.024)			(0.107)	(0.102)			(0.107)	(0.102)			(0.108)	(0.102)		
Bur-ACD	0.252	0.676	0.199	0.986	0.854	0.047	0.462	0.000	0.825	0.021	0.353	-0.004	0.792	0.006	0.205	-0.007
	(0.010)	(0.009)			(0.048)	(0.046)			(0.028)	(0.027)			(0.010)	(0.009)		
GG-ACD	0.098	0.919	0.328	0.945	0.994	0.082	0.673	-0.003	1.018	0.061	0.669	-0.007	1.041	0.045	0.660	-0.009
	(0.028)	(0.024)			(0.102)	(0.097)			(0.101)	(0.096)			(0.098)	(0.093)		
Exp-MSMD(7)	-0.124	1.155	0.108	0.950	-0.159	1.210	0.153	0.902	-0.149	1.215	0.179	0.859	-0.123	1.206	0.260	0.709
	(0.032)	(0.029)			(0.047)	(0.044)			(0.057)	(0.054)			(0.088)	(0.085)		
Wbl-MSMD(7)	-0.112	1.138	0.138	0.925	-0.103	1.134	0.153	0.907	-0.094	1.124	0.177	0.873	-0.005	1.023	0.243	0.734
	(0.040)	(0.035)			(0.045)	(0.040)			(0.052)	(0.047)			(0.074)	(0.067)		
Gam-MSMD(7)	-0.105	1.141	0.130	0.934	-0.090	1.129	0.145	0.915	-0.079	1.114	0.173	0.876	-0.001	1.026	0.236	0.748
	(0.037)	(0.033)			(0.042)	(0.038)			(0.051)	(0.046)			(0.072)	(0.065)		
Gam2-MSMD(7)	-0.206	1.265	0.147	0.908	-0.256	1.320	0.188	0.854	-0.298	1.372	0.204	0.831	-0.228	1.292	0.272	0.683
	(0.047)	(0.044)			(0.063)	(0.060)			(0.071)	(0.068)			(0.101)	(0.096)		
Burr-MSMD(7)	-0.113	1.516	0.139	0.925	-0.104	1.510	0.154	0.906	-0.095	1.498	0.177	0.872	-0.006	1.363	0.243	0.734
	(0.040)	(0.048)			(0.045)	(0.053)			(0.053)	(0.063)			(0.074)	(0.090)		
GG-MSMD(7)	-0.050	1.038	0.103	0.951	-0.073	1.073	0.145	0.905	-0.069	1.078	0.195	0.833	-0.029	1.049	0.261	0.706
	(0.030)	(0.026)			(0.043)	(0.038)			(0.059)	(0.053)			(0.082)	(0.074)		

Table 26: Prediction performance comparison of inter-trade duration (INTC, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.172	0.860	0.422	0.916	1.390	-0.192	0.775	0.004	1.391	-0.191	0.767	0.005	1.385	-0.181	0.751	0.005
	(0.040)	(0.029)			(0.166)	(0.165)			(0.160)	(0.159)			(0.153)	(0.151)		
Wbl-ACD	0.157	0.909	0.421	0.924	1.459	-0.205	0.800	0.004	1.478	-0.206	0.797	0.005	1.509	-0.203	0.792	0.005
	(0.040)	(0.029)			(0.176)	(0.176)			(0.173)	(0.171)			(0.170)	(0.168)		
Bur-ACD	0.416	1.043	0.265	0.990	2.840	-0.258	0.881	0.006	4.136	-0.281	0.929	0.005	7.226	-0.379	1.034	0.009
	(0.016)	(0.011)			(0.214)	(0.213)			(0.235)	(0.233)			(0.290)	(0.287)		
GG-ACD	0.179	0.851	0.427	0.910	1.382	-0.190	0.772	0.004	1.381	-0.189	0.764	0.005	1.372	-0.178	0.748	0.005
	(0.041)	(0.029)			(0.164)	(0.164)			(0.159)	(0.157)			(0.151)	(0.150)		
Exp-MSMD(8)	-0.276	1.280	0.195	0.892	-0.136	1.138	0.214	0.821	-0.110	1.123	0.261	0.726	-0.152	1.213	0.341	0.594
	(0.059)	(0.049)			(0.069)	(0.059)			(0.089)	(0.078)			(0.123)	(0.111)		
Wbl-MSMD(8)	-0.022	1.011	0.278	0.805	0.075	0.922	0.294	0.753	0.144	0.856	0.306	0.709	0.304	0.723	0.374	0.539
	(0.071)	(0.055)			(0.075)	(0.058)			(0.078)	(0.060)			(0.095)	(0.074)		
Gam-MSMD(8)	-0.078	1.066	0.195	0.908	-0.094	1.083	0.205	0.899	-0.053	1.041	0.272	0.815	0.130	0.866	0.301	0.696
	(0.049)	(0.037)			(0.052)	(0.040)			(0.071)	(0.054)			(0.081)	(0.063)		
Gam2-MSMD(8)	-0.186	1.192	0.221	0.852	-0.104	1.121	0.219	0.833	-0.044	1.078	0.272	0.737	0.046	1.020	0.329	0.605
	(0.071)	(0.054)			(0.071)	(0.055)			(0.090)	(0.071)			(0.113)	(0.091)		
Burr-MSMD(8)	-0.074	1.616	0.210	0.885	-0.098	1.658	0.222	0.874	-0.012	1.520	0.272	0.790	0.143	1.297	0.310	0.663
	(0.054)	(0.064)			(0.058)	(0.069)			(0.072)	(0.086)			(0.085)	(0.103)		
GG-MSMD(8)	-0.261	1.297	0.173	0.928	-0.133	1.160	0.212	0.849	-0.155	1.198	0.285	0.741	-0.178	1.258	0.326	0.662
	(0.048)	(0.040)			(0.064)	(0.054)			(0.090)	(0.078)			(0.111)	(0.099)		

Table 27: Prediction performance comparison of inter-trade duration (FB, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.071	0.995	0.369	0.952	1.542	-0.179	0.790	0.011	1.567	-0.175	0.794	0.009	1.648	-0.194	0.801	0.012
	(0.034)	(0.024)			(0.155)	(0.129)			(0.157)	(0.131)			(0.163)	(0.138)		
Wbl-ACD	0.024	1.143	0.370	0.963	1.816	-0.217	0.874	0.011	1.992	-0.231	0.916	0.009	2.442	-0.304	1.006	0.012
	(0.034)	(0.025)			(0.189)	(0.158)			(0.208)	(0.174)			(0.258)	(0.217)		
Bur-ACD	0.004	1.065	0.335	0.971	1.622	-0.194	0.829	0.010	1.713	-0.199	0.854	0.009	1.952	-0.245	0.904	0.011
	(0.028)	(0.020)			(0.171)	(0.143)			(0.181)	(0.152)			(0.208)	(0.175)		
GG-ACD	0.169	0.867	0.377	0.933	1.428	-0.154	0.730	0.011	1.419	-0.146	0.720	0.010	1.421	-0.149	0.699	0.012
	(0.036)	(0.025)			(0.132)	(0.110)			(0.129)	(0.108)			(0.124)	(0.105)		
Exp-MSMD(6)	-0.113	1.165	0.227	0.852	-0.146	1.282	0.323	0.704	-0.155	1.361	0.392	0.588	-0.124	1.418	0.478	0.434
	(0.067)	(0.053)			(0.104)	(0.091)			(0.134)	(0.125)			(0.177)	(0.177)		
Wbl-MSMD(7)	0.002	0.934	0.165	0.927	0.012	0.924	0.234	0.856	0.004	0.942	0.283	0.800	-0.003	0.960	0.356	0.706
	(0.043)	(0.029)			(0.061)	(0.042)			(0.075)	(0.052)			(0.097)	(0.068)		
Gam-MSMD(5)	-0.151	1.151	0.176	0.913	-0.149	1.158	0.243	0.837	-0.188	1.214	0.294	0.779	-0.196	1.247	0.384	0.648
	(0.051)	(0.039)			(0.072)	(0.056)			(0.090)	(0.071)			(0.124)	(0.101)		
Gam2-MSMD(7)	-0.400	1.357	0.176	0.918	-0.473	1.406	0.239	0.854	-0.521	1.449	0.280	0.807	-0.535	1.455	0.355	0.700
	(0.058)	(0.045)			(0.082)	(0.064)			(0.100)	(0.078)			(0.134)	(0.104)		
Burr-MSMD(6)	-2.487	4.619	0.467	0.518	-3.319	4.241	0.449	0.609	-3.944	3.900	0.444	0.651	-4.311	3.269	0.466	0.619
	(0.395)	(0.487)			(0.402)	(0.372)			(0.418)	(0.313)			(0.478)	(0.281)		
GG-MSMD(6)	-0.255	1.243	0.163	0.930	-0.307	1.296	0.253	0.833	-0.349	1.337	0.320	0.743	-0.353	1.355	0.416	0.588
	(0.049)	(0.037)			(0.082)	(0.064)			(0.109)	(0.086)			(0.154)	(0.124)		

Table 28: Prediction performance comparison of inter-trade duration (QCOM, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.207	0.839	0.390	0.893	1.183	0.035	0.676	-0.009	1.187	0.035	0.667	-0.008	1.197	0.034	0.648	-0.008
	(0.042)	(0.032)			(0.099)	(0.065)			(0.097)	(0.063)			(0.090)	(0.058)		
Wbl-ACD	0.207	0.946	0.411	0.896	1.352	0.040	0.731	-0.009	1.412	0.043	0.738	-0.008	1.537	0.046	0.751	-0.008
	(0.047)	(0.035)			(0.116)	(0.076)			(0.119)	(0.077)			(0.121)	(0.078)		
Bur-ACD	0.257	1.073	0.308	0.973	2.168	0.048	0.822	-0.009	3.158	0.061	0.910	-0.009	5.864	0.095	1.116	-0.008
	(0.026)	(0.020)			(0.147)	(0.096)			(0.180)	(0.117)			(0.267)	(0.173)		
GG-ACD	0.217	0.812	0.388	0.889	1.156	0.034	0.664	-0.009	1.153	0.034	0.651	-0.008	1.149	0.032	0.627	-0.008
	(0.042)	(0.031)			(0.096)	(0.062)			(0.092)	(0.060)			(0.084)	(0.055)		
Exp-MSMD(8)	0.071	0.939	0.195	0.867	0.133	0.906	0.255	0.762	0.165	0.900	0.279	0.706	0.231	0.871	0.348	0.555
	(0.053)	(0.040)			(0.071)	(0.056)			(0.079)	(0.064)			(0.102)	(0.085)		
Wbl-MSMD(6)	-0.019	0.965	0.226	0.820	0.018	0.929	0.255	0.765	0.056	0.900	0.271	0.722	0.134	0.842	0.330	0.588
	(0.068)	(0.050)			(0.077)	(0.056)			(0.083)	(0.061)			(0.104)	(0.077)		
Gam-MSMD(6)	-0.117	1.133	0.221	0.829	-0.089	1.105	0.249	0.777	-0.038	1.059	0.275	0.716	0.047	0.993	0.333	0.582
	(0.071)	(0.057)			(0.081)	(0.065)			(0.091)	(0.073)			(0.113)	(0.092)		
Gam2-MSMD(8)	-0.096	1.107	0.217	0.837	-0.063	1.073	0.240	0.792	-0.017	1.031	0.263	0.739	0.070	0.959	0.327	0.598
	(0.068)	(0.054)			(0.076)	(0.060)			(0.084)	(0.067)			(0.108)	(0.086)		
Burr-MSMD(8)	-0.021	2.629	0.233	0.809	0.009	2.545	0.258	0.760	0.052	2.447	0.273	0.717	0.133	2.271	0.332	0.583
	(0.071)	(0.140)			(0.079)	(0.156)			(0.084)	(0.168)			(0.105)	(0.210)		
GG-MSMD(8)	-0.119	1.125	0.211	0.838	-0.056	1.072	0.276	0.715	0.017	1.016	0.315	0.619	0.168	0.892	0.378	0.445
	(0.068)	(0.054)			(0.092)	(0.074)			(0.107)	(0.087)			(0.132)	(0.109)		

Table 29: Prediction performance comparison of price duration (MSFT, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.137	0.878	0.384	0.935	1.204	-0.097	0.739	-0.005	1.209	-0.100	0.712	-0.004	1.210	-0.099	0.660	-0.001
	(0.033)	(0.025)			(0.132)	(0.130)			(0.121)	(0.119)			(0.107)	(0.105)		
Wbl-ACD	0.133	0.972	0.400	0.937	1.381	-0.113	0.799	-0.005	1.470	-0.126	0.796	-0.003	1.639	-0.143	0.790	-0.001
	(0.035)	(0.028)			(0.155)	(0.152)			(0.152)	(0.148)			(0.154)	(0.151)		
Bur-ACD	-0.001	1.126	0.459	0.921	1.457	-0.140	0.899	-0.006	1.589	-0.170	0.941	-0.004	1.887	-0.234	1.033	-0.002
	(0.046)	(0.036)			(0.196)	(0.192)			(0.213)	(0.208)			(0.263)	(0.258)		
GG-ACD	0.141	0.862	0.385	0.932	1.182	-0.094	0.731	-0.005	1.178	-0.097	0.702	-0.004	1.165	-0.094	0.647	-0.002
	(0.033)	(0.026)			(0.129)	(0.127)			(0.118)	(0.115)			(0.103)	(0.101)		
Exp-MSMD(7)	-0.043	1.017	0.160	0.884	-0.061	1.072	0.212	0.785	-0.015	1.034	0.226	0.721	-0.060	1.150	0.303	0.570
	(0.047)	(0.041)			(0.067)	(0.063)			(0.077)	(0.075)			(0.111)	(0.116)		
Wbl-MSMD(7)	-0.032	0.926	0.186	0.906	-0.005	0.891	0.206	0.874	0.061	0.816	0.226	0.823	0.081	0.801	0.293	0.705
	(0.044)	(0.033)			(0.050)	(0.037)			(0.056)	(0.042)			(0.076)	(0.058)		
Gam-MSMD(5)	-0.046	1.030	0.155	0.923	-0.062	1.046	0.185	0.889	-0.104	1.102	0.232	0.836	0.022	0.966	0.290	0.681
	(0.039)	(0.033)			(0.048)	(0.041)			(0.063)	(0.054)			(0.083)	(0.074)		
Gam2-MSMD(5)	-0.097	1.098	0.163	0.917	-0.113	1.114	0.195	0.880	-0.158	1.175	0.233	0.837	-0.024	1.028	0.281	0.692
	(0.043)	(0.036)			(0.052)	(0.045)			(0.065)	(0.057)			(0.084)	(0.076)		
Burr-MSMD(5)	0.000	2.258	0.255	0.792	-0.018	2.332	0.300	0.742	0.045	2.170	0.300	0.712	0.265	1.571	0.327	0.548
	(0.067)	(0.127)			(0.078)	(0.151)			(0.078)	(0.152)			(0.084)	(0.165)		
GG-MSMD(7)	-0.139	1.147	0.155	0.932	-0.170	1.175	0.210	0.865	-0.208	1.221	0.266	0.787	-0.030	1.012	0.297	0.635
	(0.040)	(0.034)			(0.059)	(0.051)			(0.079)	(0.070)			(0.095)	(0.087)		

Table 30: Prediction performance comparison of price duration (INTC, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.219	0.779	0.382	0.944	0.847	0.306	0.731	0.047	0.928	0.189	0.684	0.019	1.025	0.039	0.595	-0.010
	(0.029)	(0.021)			(0.132)	(0.136)			(0.114)	(0.116)			(0.083)	(0.084)		
Wbl-ACD	0.207	0.885	0.397	0.949	0.964	0.369	0.801	0.047	1.118	0.246	0.778	0.020	1.364	0.060	0.727	-0.009
	(0.032)	(0.023)			(0.159)	(0.163)			(0.147)	(0.150)			(0.124)	(0.125)		
Bur-ACD	0.045	1.494	0.428	0.975	2.125	1.145	1.389	0.051	4.730	1.637	1.952	0.023	21.809	2.015	3.829	-0.008
	(0.037)	(0.026)			(0.476)	(0.489)			(0.926)	(0.946)			(3.435)	(3.475)		
GG-ACD	0.229	0.758	0.381	0.942	0.837	0.295	0.719	0.047	0.911	0.180	0.669	0.019	0.996	0.036	0.576	-0.010
	(0.029)	(0.021)			(0.128)	(0.131)			(0.109)	(0.111)			(0.078)	(0.079)		
Exp-MSMD(6)	0.026	0.841	0.164	0.906	0.029	0.852	0.242	0.782	0.101	0.791	0.275	0.678	0.173	0.745	0.315	0.544
	(0.042)	(0.030)			(0.068)	(0.050)			(0.081)	(0.062)			(0.100)	(0.079)		
Wbl-MSMD(6)	-0.064	0.947	0.182	0.915	-0.068	0.959	0.220	0.874	-0.089	0.985	0.258	0.828	0.131	0.773	0.289	0.677
	(0.045)	(0.032)			(0.056)	(0.040)			(0.067)	(0.049)			(0.080)	(0.060)		
Gam-MSMD(6)	-0.149	1.159	0.165	0.912	-0.166	1.192	0.203	0.861	-0.098	1.131	0.240	0.773	-0.039	1.087	0.290	0.626
	(0.047)	(0.040)			(0.061)	(0.053)			(0.076)	(0.068)			(0.102)	(0.095)		
Gam2-MSMD(6)	-0.179	1.159	0.175	0.930	-0.188	1.169	0.235	0.867	-0.158	1.138	0.274	0.805	0.082	0.888	0.310	0.637
	(0.044)	(0.035)			(0.062)	(0.050)			(0.076)	(0.062)			(0.091)	(0.075)		
Burr-MSMD(6)	-0.060	2.444	0.192	0.909	-0.065	2.472	0.233	0.866	-0.055	2.454	0.265	0.817	0.159	1.928	0.305	0.651
	(0.046)	(0.085)			(0.058)	(0.107)			(0.068)	(0.127)			(0.083)	(0.158)		
GG-MSMD(6)	-0.195	1.209	0.152	0.930	-0.207	1.227	0.212	0.843	-0.122	1.138	0.251	0.739	0.000	1.016	0.286	0.594
	(0.043)	(0.037)			(0.067)	(0.059)			(0.085)	(0.076)			(0.106)	(0.098)		

Table 31: Prediction performance comparison of price duration (FB, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.248	0.756	0.301	0.922	1.314	-0.427	0.518	0.215	1.315	-0.417	0.493	0.232	1.263	-0.334	0.455	0.219
	(0.024)	(0.024)			(0.080)	(0.088)			(0.074)	(0.081)			(0.062)	(0.068)		
Wbl-ACD	0.226	0.813	0.300	0.932	1.392	-0.465	0.540	0.215	1.417	-0.462	0.520	0.232	1.402	-0.386	0.491	0.216
	(0.024)	(0.024)			(0.087)	(0.096)			(0.082)	(0.091)			(0.072)	(0.079)		
Bur-ACD	1.313	0.037	0.040	0.990	1.364	0.000	0.003	0.214	1.364	0.000	0.000	0.223	1.364	0.000	0.000	0.494
	(0.000)	(0.000)			(0.000)	(0.000)			(0.000)	(0.000)			(0.000)	(0.000)		
GG-ACD	0.272	0.718	0.309	0.905	1.283	-0.409	0.507	0.215	1.280	-0.401	0.483	0.234	1.222	-0.323	0.445	0.224
	(0.026)	(0.026)			(0.077)	(0.084)			(0.071)	(0.078)			(0.059)	(0.065)		
Exp-MSMD(7)	-0.104	1.147	0.128	0.896	-0.114	1.169	0.171	0.804	-0.129	1.213	0.210	0.715	-0.119	1.217	0.247	0.595
	(0.040)	(0.043)			(0.058)	(0.063)			(0.075)	(0.084)			(0.096)	(0.110)		
Wbl-MSMD(7)	-0.060	1.056	0.142	0.867	-0.055	1.048	0.165	0.816	-0.064	1.073	0.195	0.751	-0.065	1.084	0.225	0.664
	(0.044)	(0.045)			(0.053)	(0.055)			(0.065)	(0.068)			(0.079)	(0.084)		
Gam-MSMD(7)	-0.107	1.131	0.149	0.852	-0.102	1.123	0.174	0.796	-0.102	1.134	0.207	0.720	-0.093	1.132	0.238	0.631
	(0.049)	(0.052)			(0.059)	(0.062)			(0.072)	(0.077)			(0.088)	(0.095)		
Gam2-MSMD(7)	-0.291	1.315	0.161	0.844	-0.279	1.287	0.198	0.750	-0.280	1.285	0.229	0.670	-0.245	1.222	0.253	0.573
	(0.059)	(0.062)			(0.078)	(0.081)			(0.094)	(0.099)			(0.111)	(0.116)		
Burr-MSMD(7)	-0.055	1.467	0.141	0.869	-0.049	1.453	0.164	0.817	-0.060	1.491	0.194	0.755	-0.060	1.502	0.225	0.667
	(0.044)	(0.063)			(0.052)	(0.075)			(0.064)	(0.093)			(0.078)	(0.116)		
GG-MSMD(7)	-0.181	1.207	0.142	0.872	-0.182	1.204	0.179	0.788	-0.164	1.194	0.221	0.685	-0.163	1.210	0.258	0.586
	(0.048)	(0.051)			(0.065)	(0.068)			(0.083)	(0.089)			(0.103)	(0.111)		

Table 32: Prediction performance comparison of price duration (QCOM, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.234	0.751	0.354	0.906	1.143	-0.030	0.618	-0.011	1.120	-0.029	0.590	-0.011	1.062	-0.005	0.538	-0.012
	(0.035)	(0.027)			(0.106)	(0.122)			(0.097)	(0.113)			(0.078)	(0.089)		
Wbl-ACD	0.227	0.786	0.354	0.913	1.187	-0.031	0.633	-0.011	1.175	-0.030	0.607	-0.011	1.133	-0.004	0.558	-0.012
	(0.035)	(0.027)			(0.111)	(0.128)			(0.103)	(0.120)			(0.084)	(0.096)		
Bur-ACD	0.171	1.043	0.433	0.892	1.553	-0.048	0.776	-0.011	1.676	-0.056	0.799	-0.011	1.896	-0.016	0.847	-0.012
	(0.052)	(0.040)			(0.167)	(0.193)			(0.179)	(0.207)			(0.193)	(0.222)		
GG-ACD	0.246	0.730	0.357	0.898	1.127	-0.029	0.610	-0.011	1.103	-0.029	0.582	-0.011	1.041	-0.005	0.531	-0.012
	(0.035)	(0.027)			(0.103)	(0.119)			(0.095)	(0.110)			(0.076)	(0.087)		
Exp-MSMD(6)	-0.093	1.132	0.158	0.877	-0.051	1.105	0.204	0.777	-0.011	1.077	0.233	0.695	0.043	1.052	0.277	0.567
	(0.054)	(0.047)			(0.074)	(0.065)			(0.088)	(0.079)			(0.111)	(0.103)		
Wbl-MSMD(6)	-0.035	1.051	0.200	0.835	0.037	0.976	0.217	0.782	0.105	0.912	0.236	0.717	0.204	0.829	0.273	0.589
	(0.063)	(0.051)			(0.070)	(0.057)			(0.077)	(0.063)			(0.092)	(0.077)		
Gam-MSMD(6)	-0.048	1.076	0.209	0.823	0.052	0.971	0.231	0.750	0.115	0.912	0.252	0.680	0.218	0.822	0.286	0.552
	(0.067)	(0.055)			(0.075)	(0.062)			(0.083)	(0.069)			(0.098)	(0.082)		
Gam2-MSMD(6)	-0.153	1.226	0.145	0.917	-0.126	1.185	0.179	0.861	-0.078	1.126	0.212	0.789	0.036	1.004	0.255	0.652
	(0.047)	(0.040)			(0.061)	(0.052)			(0.075)	(0.064)			(0.095)	(0.081)		
Burr-MSMD(6)	-0.039	1.435	0.201	0.838	0.034	1.327	0.218	0.784	0.104	1.230	0.237	0.718	0.203	1.108	0.275	0.589
	(0.063)	(0.069)			(0.070)	(0.077)			(0.077)	(0.085)			(0.093)	(0.102)		
GG-MSMD(6)	-0.174	1.176	0.188	0.826	-0.126	1.134	0.240	0.702	-0.054	1.066	0.269	0.600	0.013	1.028	0.317	0.461
	(0.071)	(0.059)			(0.096)	(0.082)			(0.112)	(0.097)			(0.140)	(0.123)		

Table 33: Prediction performance comparison of volume duration (MSFT, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.246	0.826	0.465	0.932	1.423	-0.043	0.906	-0.011	1.434	-0.052	0.897	-0.010	1.457	-0.069	0.880	-0.008
	(0.041)	(0.024)			(0.160)	(0.127)			(0.156)	(0.124)			(0.153)	(0.123)		
Wbl-ACD	0.346	0.976	0.487	0.941	1.860	-0.051	1.010	-0.011	2.039	-0.067	1.035	-0.010	2.425	-0.104	1.086	-0.008
	(0.045)	(0.027)			(0.199)	(0.158)			(0.208)	(0.166)			(0.233)	(0.187)		
Bur-ACD	0.897	0.553	0.211	0.993	1.921	-0.006	0.391	-0.011	1.998	-0.002	0.177	-0.011	2.017	0.000	0.037	-0.009
	(0.008)	(0.005)			(0.030)	(0.024)			(0.006)	(0.005)			(0.000)	(0.000)		
GG-ACD	1.010	0.002	0.035	0.791	1.013	0.000	0.050	-0.011	1.011	0.000	0.048	-0.011	1.011	0.000	0.043	-0.011
	(0.000)	(0.000)			(0.000)	(0.000)			(0.000)	(0.000)			(0.000)	(0.000)		
Exp-MSMD(8)	0.630	0.021	0.505	0.742	0.180	0.016	0.561	0.543	-0.268	0.017	0.582	0.514	-1.181	0.018	0.596	0.496
	(0.072)	(0.001)			(0.130)	(0.002)			(0.181)	(0.002)			(0.284)	(0.002)		
Wbl-MSMD(8)	-0.215	0.955	0.285	0.810	-0.227	0.965	0.257	0.836	-0.262	0.998	0.270	0.823	-0.233	0.995	0.392	0.658
	(0.086)	(0.051)			(0.079)	(0.047)			(0.085)	(0.051)			(0.129)	(0.079)		
Gam-MSMD(8)	-0.042	1.035	0.262	0.887	-0.030	1.033	0.257	0.888	-0.024	1.035	0.296	0.852	0.119	0.913	0.401	0.698
	(0.061)	(0.041)			(0.060)	(0.040)			(0.070)	(0.047)			(0.097)	(0.066)		
Gam2-MSMD(8)	-0.090	1.067	0.239	0.817	-0.152	0.987	0.275	0.740	-0.436	1.077	0.306	0.699	-0.655	1.074	0.380	0.544
	(0.075)	(0.055)			(0.097)	(0.064)			(0.127)	(0.077)			(0.198)	(0.107)		
Burr-MSMD(8)	-0.218	5.878	0.286	0.808	-0.230	5.941	0.259	0.834	-0.265	6.137	0.272	0.819	-0.237	6.126	0.393	0.656
	(0.087)	(0.317)			(0.080)	(0.294)			(0.086)	(0.319)			(0.130)	(0.490)		
GG-MSMD(8)	0.110	0.929	0.255	0.916	0.088	0.951	0.263	0.909	0.004	1.032	0.269	0.912	0.066	0.968	0.377	0.796
	(0.050)	(0.031)			(0.052)	(0.033)			(0.055)	(0.035)			(0.083)	(0.054)		

Table 34: Prediction performance comparison of volume duration (INTC, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.209	0.857	0.412	0.957	1.502	0.078	0.894	-0.009	1.499	0.071	0.874	-0.009	1.464	0.090	0.835	-0.007
	(0.037)	(0.020)			(0.175)	(0.158)			(0.166)	(0.149)			(0.157)	(0.143)		
Wbl-ACD	0.188	1.117	0.449	0.965	2.113	0.111	1.087	-0.009	2.433	0.121	1.153	-0.010	3.126	0.217	1.295	-0.007
	(0.044)	(0.023)			(0.259)	(0.233)			(0.290)	(0.260)			(0.379)	(0.343)		
Bur-ACD	1.338	0.824	0.339	0.979	5.632	0.032	0.642	-0.010	7.471	0.015	0.427	-0.010	8.649	0.005	0.190	-0.007
	(0.025)	(0.013)			(0.090)	(0.081)			(0.040)	(0.036)			(0.008)	(0.007)		
GG-ACD	0.266	0.741	0.411	0.945	1.347	0.069	0.820	-0.009	1.307	0.059	0.784	-0.009	1.215	0.066	0.718	-0.007
	(0.037)	(0.020)			(0.147)	(0.133)			(0.134)	(0.120)			(0.116)	(0.105)		
Exp-MSMD(8)	-0.340	0.418	0.388	0.586	-2.532	1.098	0.465	0.546	-8.308	2.890	0.478	0.746	-57.181	17.935	0.487	0.987
	(0.188)	(0.045)			(0.456)	(0.129)			(0.663)	(0.199)			(0.731)	(0.225)		
Wbl-MSMD(8)	-0.091	0.870	0.269	0.872	-0.068	0.852	0.303	0.835	0.163	0.699	0.300	0.778	0.054	0.779	0.467	0.614
	(0.074)	(0.037)			(0.084)	(0.042)			(0.085)	(0.042)			(0.136)	(0.069)		
Gam-MSMD(8)	-0.039	1.057	0.240	0.909	-0.052	1.066	0.311	0.851	-0.056	1.085	0.353	0.811	0.044	1.035	0.452	0.673
	(0.061)	(0.037)			(0.081)	(0.049)			(0.094)	(0.058)			(0.125)	(0.079)		
Gam2-MSMD(5)	-0.616	1.501	0.345	0.921	-0.710	1.446	0.468	0.853	-0.900	1.447	0.538	0.806	-1.190	1.443	0.669	0.714
	(0.087)	(0.048)			(0.126)	(0.066)			(0.156)	(0.078)			(0.219)	(0.100)		
Burr-MSMD(8)	0.238	5.981	0.620	0.127	0.088	9.369	0.633	0.103	0.128	10.155	0.607	0.114	-0.261	13.944	0.629	0.125
	(0.667)	(3.472)			(0.907)	(6.479)			(0.911)	(7.300)			(0.945)	(8.005)		
GG-MSMD(8)	-0.042	1.116	0.245	0.919	-0.030	1.112	0.312	0.850	0.014	1.088	0.339	0.792	-0.134	1.253	0.453	0.681
	(0.060)	(0.036)			(0.081)	(0.051)			(0.094)	(0.061)			(0.138)	(0.094)		

Table 35: Prediction performance comparison of volume duration (FB, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.135	0.893	0.231	0.986	0.954	0.123	0.625	0.000	1.001	0.103	0.569	0.001	1.081	0.062	0.471	-0.003
	(0.013)	(0.012)			(0.121)	(0.122)			(0.100)	(0.100)			(0.071)	(0.071)		
Wbl-ACD	0.150	0.859	0.230	0.985	0.923	0.114	0.604	0.000	0.948	0.092	0.538	0.001	0.994	0.051	0.429	-0.003
	(0.013)	(0.011)			(0.113)	(0.114)			(0.089)	(0.090)			(0.058)	(0.059)		
Bur-ACD	0.268	0.779	0.195	0.991	1.040	0.081	0.493	0.002	1.114	0.044	0.364	0.002	1.178	0.012	0.198	-0.002
	(0.009)	(0.008)			(0.075)	(0.076)			(0.041)	(0.041)			(0.012)	(0.013)		
GG-ACD	0.224	0.733	0.234	0.979	0.861	0.089	0.547	-0.001	0.854	0.068	0.475	-0.001	0.852	0.033	0.359	-0.004
	(0.014)	(0.012)			(0.093)	(0.093)			(0.070)	(0.070)			(0.041)	(0.041)		
Exp-MSMD(7)	-0.090	1.094	0.092	0.967	-0.085	1.081	0.127	0.932	-0.069	1.059	0.151	0.895	-0.069	1.055	0.218	0.767
	(0.024)	(0.022)			(0.035)	(0.032)			(0.043)	(0.040)			(0.068)	(0.064)		
Wbl-MSMD(7)	-0.098	1.128	0.073	0.980	-0.108	1.135	0.117	0.945	-0.114	1.142	0.144	0.908	-0.147	1.181	0.211	0.787
	(0.019)	(0.018)			(0.032)	(0.030)			(0.042)	(0.040)			(0.069)	(0.068)		
Gam-MSMD(7)	-0.089	1.092	0.085	0.972	-0.086	1.080	0.121	0.938	-0.080	1.069	0.146	0.902	-0.093	1.081	0.214	0.774
	(0.022)	(0.020)			(0.033)	(0.030)			(0.042)	(0.039)			(0.068)	(0.064)		
Gam2-MSMD(7)	-0.298	1.286	0.072	0.981	-0.519	1.470	0.136	0.928	-0.634	1.544	0.170	0.874	-0.706	1.545	0.245	0.700
	(0.022)	(0.020)			(0.049)	(0.045)			(0.071)	(0.064)			(0.124)	(0.111)		
Burr-MSMD(7)	-0.098	1.005	0.073	0.980	-0.108	1.011	0.117	0.945	-0.115	1.018	0.144	0.908	-0.148	1.053	0.211	0.786
	(0.019)	(0.016)			(0.032)	(0.027)			(0.042)	(0.036)			(0.069)	(0.060)		
GG-MSMD(7)	-0.144	1.162	0.100	0.963	-0.217	1.225	0.130	0.934	-0.249	1.242	0.160	0.890	-0.313	1.288	0.227	0.766
	(0.027)	(0.025)			(0.039)	(0.036)			(0.052)	(0.048)			(0.083)	(0.078)		

Table 36: Prediction performance comparison of volume duration (QCOM, $n = 10,000$)

	One-step				Five-step				Ten-step				Twenty-step			
	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}	ω_0	ω_1	RMSE	R^2_{adj}
Exp-ACD	0.093	0.940	0.288	0.993	1.463	-0.167	0.958	0.020	1.482	-0.160	0.920	0.024	1.510	-0.147	0.850	0.025
	(0.014)	(0.009)			(0.144)	(0.101)			(0.132)	(0.092)			(0.115)	(0.083)		
Wbl-ACD	0.111	0.899	0.286	0.993	1.381	-0.153	0.917	0.020	1.351	-0.139	0.857	0.024	1.298	-0.114	0.750	0.026
	(0.014)	(0.009)			(0.132)	(0.093)			(0.114)	(0.080)			(0.090)	(0.064)		
Bur-ACD	0.101	0.935	0.274	0.994	1.464	-0.160	0.941	0.020	1.484	-0.149	0.888	0.024	1.512	-0.127	0.792	0.025
	(0.013)	(0.008)			(0.139)	(0.098)			(0.123)	(0.086)			(0.100)	(0.072)		
GG-ACD	0.164	0.810	0.281	0.991	1.253	-0.128	0.837	0.021	1.173	-0.106	0.746	0.024	1.055	-0.072	0.592	0.026
	(0.014)	(0.008)			(0.110)	(0.077)			(0.087)	(0.060)			(0.056)	(0.040)		
Exp-MSMD(8)	-0.092	1.050	0.145	0.975	-0.115	1.056	0.194	0.953	-0.106	1.022	0.247	0.915	0.033	0.854	0.344	0.769
	(0.028)	(0.019)			(0.039)	(0.026)			(0.052)	(0.034)			(0.078)	(0.052)		
Wbl-MSMD(8)	-0.080	1.076	0.111	0.983	-0.123	1.098	0.160	0.962	-0.172	1.118	0.211	0.928	-0.111	1.019	0.310	0.787
	(0.022)	(0.015)			(0.034)	(0.024)			(0.048)	(0.034)			(0.080)	(0.059)		
Gam-MSMD(8)	-0.095	1.055	0.122	0.981	-0.137	1.074	0.173	0.960	-0.149	1.053	0.222	0.927	-0.053	0.919	0.318	0.794
	(0.024)	(0.016)			(0.036)	(0.024)			(0.048)	(0.032)			(0.077)	(0.052)		
Gam2-MSMD(8)	-0.219	1.160	0.089	0.992	-0.495	1.336	0.217	0.950	-0.580	1.350	0.290	0.901	-0.252	0.996	0.404	0.680
	(0.017)	(0.012)			(0.049)	(0.034)			(0.073)	(0.049)			(0.116)	(0.076)		
Burr-MSMD(8)	-0.081	0.929	0.111	0.983	-0.124	0.948	0.160	0.962	-0.173	0.965	0.210	0.929	-0.114	0.881	0.310	0.787
	(0.022)	(0.013)			(0.034)	(0.021)			(0.048)	(0.029)			(0.080)	(0.051)		
GG-MSMD(8)	-0.070	1.083	0.139	0.975	-0.095	1.081	0.181	0.950	-0.163	1.121	0.230	0.911	-0.138	1.045	0.317	0.759
	(0.027)	(0.019)			(0.038)	(0.027)			(0.053)	(0.039)			(0.087)	(0.066)		

6 Conclusion

In this thesis, we introduce the Markov-switching multifractal model in financial durations modeling and forecasting with ultra-high frequency data. The UHF data have brought much challenge to statistical analysis, we find that there exists both spike and diurnal patterns at UHF financial durations and propose a new procedure to adjust these patterns. To model the dynamics of financial durations in UHF data, we extend the classical MSMD model which was originally proposed for the lower frequency data (Chen et al. [2013], Žikeš et al. [2014]) and discuss multiple extensions of exponentials, Weibull, gamma, Burr and generalized gamma distributions for the noise. We also propose a new multifractal process for the gamma MSMD model which grants more flexibility on controlling the scale and shape. We compare the in-sample and out-of-sample performance of all six models using the 2015 NASDAQ LOB data. The Gam2-MSMD model is able to capture the heavy tail property of durations, and it enhances the model forecast in terms of generating the high variance in sequence. The empirical results show that our model outperform the traditional MSMD and ACD models in terms of fitting and forecasting financial data. Finally, we investigate the MSMD models in Flash Crash and prove that the models could survive in such extreme situation.

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