TIDES,
SEICHES,
AND LONG WAVES

TIDES, SEICHES, AND LONG WAVES

by

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CONTENTS

Chapter 2. (continued)

PREFACE

This material is a revision of my lecture notes on tides originally issued in 1965 by the Chesapeake Bay Institute. There is only one excuse for preparing yet another screed on tides when there are already so many books on the subject. In my opinion, none of them present tides at the level necessary to provide the tyro oceanographer with a foundation on which he can build, in whatever the direction his evolving interests may lead him.

Darwin (1898) is, perhaps, the most profoundly insightful popular account of the subject. It is in the great tradition of the British scholars who, now and again, feel impelled to lay aside the paraphernalia and jargon of science and tell laymen, clearly, what they have found. Every intelligent person, no matter what his background and interests, should have read this book at least once.

Defant (1958) is another good popular account but, like Darwin (1898), too light on the "flute music" to satisfy an oceanographer.

Defant (1961) overwhelms with its coverage of observational material but, if that's what you want, Defant (1961) is the place to go.

Proudman (1954) is good--and out of print. My Chapter 2 depends heavily on this source.

The monographs in "The Sea, Vol. 1," Darbyshire, Groen and Groves, Hansen, Lafond and Cox, Munk, and Rossiter, give you a fine picture of the "state of the art" but only if you are already familiar to some extent with the topics they cover.

Dietrich (1963) gives a reasonable sort of discussion for oceanographers and Pillsbury (1956) treats tides from the point of view of engineers who must build structures in tideways.

Dronkers (1964) is the place to go if you need guidance on tidal computations but the title is a bit misleading. The tidal computations are there, based on the author's 20 years of work in Netherlands' waters, but he begins with a lengthy and excellent exposition of the theory of the tide.

The bibliography included with these notes is sketchy. There is no need for it to be otherwise when one has the first three references listed there. In addition, the Corps of Engineers, U.S. Army has, since 1954, been publishing an annotated "Bibliography on Tidal Hydraulics." it is called Report No. 2 and every so often another Supplement appears.

Aspirant oceanographers should add to their private collections the publications of the U.S.C.&G.S.: specifically, Marmer (1951), Schureman (1941, 1949), and U.S.C.&G.S. (1950, 1950, 1952). Harris is out of print. You should also own Doodson and Warburg (1941) and Dronkers (1964).

The first chapter of these notes describes the phenomenon, discusses the astronomical background, and goes into the data reduction. Its purpose is to enable you to understand, as opposed to use, the tide and current tables. It is, in a sense, practical rather than theoretical although it may be difficult for you to see this at first reading.

The second chapter is theoretical. It is devoted to the solution of continuity and motion under various boundary conditions and simplifying assumptions. One assumption always made in Chapter 2 is that friction is negligible. Because of the oversimplifications the solutions derived fit nothing in nature very well. Their value is that they show clearly the relations that must obtain among the various aspects of the tide and, thus, contribute to our understanding.

The third chapter recounts some efforts to include friction and its effects. It is, of necessity, very incomplete and unsatisfying.

Blair Kinsman

August 1978

... cal Analysis of the Tide.

1.01. _.. Bill of Fare and a Bit of History._

Schureman (1949) defines the _tide_ as "the periodic rising and falling of the water that results from the gravitational attraction of the moon and sun acting upon the rotating earth." This identification of the term "tide" with the _vertical_ movements of the water is conventional usage in the United States and England. There are also periodic horizontal movements of the water, called "tidal currents," caused by the same gravitational attractions. While it is useful to have separate terms for motion in the vertical and motion in the horizontal, the two, taken together, form the phenomenon we want to study.

If you go out and make a time record of water surface elevation at a point, you will have considerable difficulty seeing the tide. There are many forces which act to change the water level. Wind makes waves. Wind stress may tilt the water surface as a whole. Pressure differences from place to place distort the water surface. Your composite record may contain all of these and more in addition to the tide--and they don't come labeled. One of our problems will be to untangle the astronomical tide from surface fluctuations due to other causes.

The subject of tides has five aspects:

(1) The tide-generating forces: The tide-generating forces arise from the gravitational attraction of the celestial bodies of which only the sun, which is large enough, and the moon, which is near enough, are of practical importance. The net force acting at any moment on a particle of water is the result of the relative configuration of the earth, sun, and moon. The study of the relative motions of these bodies belongs to celestial mechanics and the problem has been solved for practical purposes--anything you may have heard about the "Three-Body Problem" not withstanding. Consequently, for practical purposes the tide generating forces have been well established.

(2) The tidal dynamics of the sea: The tidal dynamics of the sea concerns itself with the way in which the sea responds to the generating forces. In essence, this is a problem in fluid dynamics. The kinds of boundary value problems you meet are of the sorts you are used to seeing. In tides, however, the important driving forces are among those usually neglected in water tunnel experiments or in studies of flows in lakes and reservoirs. Their solutions will depend on the geometry of the ocean basins and on the viscosity of the water. As you might suspect, a general theory which would permit the prediction of tides in the oceans would be very complicated and no one has as yet managed it--or even come close. We do have a handful of limited solutions for such things as rectangular oceans in case you ever discover one.

(3) Observations and analysis of tide records: Lacking an adequate theoretical basis for the tide, we must obviously fall back on an empirical approach to keep our ships off the mud. Tide gages are set up at points of interest and careful records are made over many years. These give the rise and fall of the water. There are probably more of these measurements than there are of any other parameter in the field of oceanography. Observations of tidal currents are generally limited to inshore waters and are far fewer and less accurate than are the tide gage data. Once records are available, they are analysed into harmonics. (For "harmonics" read "sines" and "cosines" if it makes you more comfortable.) These analyses are a messy business but the procedure is pretty cut and dried.

(4) Prediction of tides: Basically, this is a matter of extrapolation from the observed tide. The extrapolation day by day for a year at a time is what is contained in the "Tide Tables." Since tide prediction depends on the maintenance of a large network of observing stations, and since the data reduction, usually done with special purpose computing machines, cf. Schureman (1941), pp. 126-152, all the major maritime nations have set up special organizations to carry out the work. The smaller countries depend on the larger ones for their tide information.

(5) Miscellaneous: This is a grab-bag of modifications of the tide and of problems associated with, or similar to, the tide. It includes:

. . ications of the tide produced by meteorological con. . ns: In Chesapeake Bay and many other shallow water coastal areas these a.e so pronounced that they become the problem of primary interest.

(b) Tidal waves: Tidal waves, usually called "tsunamis" by oceanographers because they are produced by seismic activity and not by the tide-producing forces, are included because they, like the tide, have very long wave lengths. Consequently, their mathematics is quite similar to the mathematics of tides.

(c) Seiches: These are surges in confined bodies of water. Such bodies have natural resonance periods. Seiches can be set going by tides, wind stress, or atmospheric pressure differences.

(d) Earth tides: The "solid" earth is a plastic, albeit a stiff one. It too responds to the tide-generating forces but to a much smaller degree than does the ocean which is highly mobile.

The Mediterranean, around which the ancient civilizations clustered, is virtually tideless. In a few places the tidal range reaches a maximum of 3 feet but such extremes are uncommon. The Greeks noticed the tide, as they noticed practically everything, but it was of no practical importance to them and we have no evidence that they gave it much thought. Among the first recorded ideas on the tide that have survived are those of Curtius Rufus, the biographer of Alexander the Great, and Pythias of Marseillia who make a voyage to England around 300 B.C.. There he encountered tides of size that made them hard to ignore. He connected them with the phases of the moon; as did the Romans who came to England a bit later. The Romans also distinguished between spring and neap tides. By the 13th century tide tables were being constructed. For example, there is one extant that relates the height of the water at London Bridge to the age of the moon. Until surprisingly recently the construction of such tables was a private enterprise of single families, the methods being jealously guarded and passed from father to son. Kepler and Gallileo first noticed that the tide could be described as a progressive wave and related its speed to the depth of the water.

Our modern approach to tides began with Newton and his law of gravitation. He discovered that the perturbations in the moon's motion are due to the sun and he developed a kinetic theory of tides, the "equilibrium" theory, the basic idea of which is that there will be a bulge in the direction of a distant attractive body. This bulge he supposed to travel around the earth always pointing exactly in the direction of the attractive body as though the water had no inertia--no mass, hence kinetic theory. The agreement between tides predicted on this basis and tides as actually observed is pretty poor but the idea is still a useful one. Most subsequent work on tides begins with the equilibrium tide as a zero-order approximation and brings it into closer agreement with the observed tide by adding corrections. Bernoulli was one who contributed to this.

Laplace attempted to replace the kinetic equilibrium theory with a dynamic theory. He also tried to derive the forces which produce the horizontal motions. He showed that tidal periods fall into three natural groups: semidiurnal, diurnal, and long-period. He also showed that a large number of small periodic perturbations could be treated separately and then combined. This is the basic idea of all harmonic (Fourier) analysis.

During the last century the English mathematicians and physicists followed out these leads. Kelvin did the most to bring the problem to its present unrewarding state. Other contributers were Airy, George Darwin (son of the evolution Darwin), Rayleigh, and Lamb.

In the United States Harris wrote a book on tides which was published over a seven-year period as an appendix to the superintendent's reports. A meteorologist named Ferrel helped to develop a special-purpose computer to carry out the analysis.

In the 20th century there has not been much creative work on tides although Dr. W. H. Munk seems to be reviving the interest of the problem by applying the methods of spectral analysis to it. The United States government is doing only routine predictions with little support for basic research. In Europe there is an active group at the Liverpool Tidal Institute which includes Proudman, Bowden, and Doodson. Liverpool atural r research on tides since the average tide range in the . sey is 20 feet while at springs the range increases to 26 feet. The docks at Liverpool have locks at the ends. The ships enter and leave at high tide and the lock gates are closed behind them to trap enough water to float the docked ships at low tide.

1.02. Terms and a Description of the Phenomenon.

If you observe the changes in sea level with a tide gage fixed to a pier in a harbor and plot the elevation against time, you have the tidal curve for that harbor and that period of time. It might look like Fig. 1.02-1.

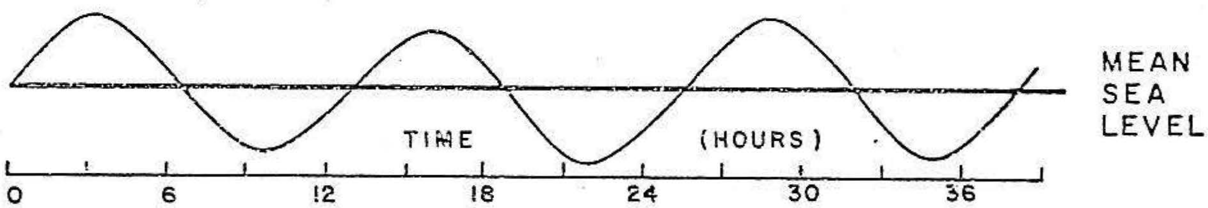


Fig. 1.02-1. Schematic Tidal Curve.

The curve, as sketched, looks rather sinusoidal and shows the particulars of the tide. The water rises for about 6 hours, a period called rising tide, flooding tide, rise, or flood, until it reaches a maximum, called high tide or high water. The time at which high water occurs is called the hour of the tide. For a little while the water elevation shows no perceptible change, high water slack, after which it begins to fall and continues to do so for about 6 hours, falling tide, ebbing tide, fall, or ebb, until it reaches a minimum, low tide, or low water. Again, for a little while there is no change in the elevation, low water slack, after which the tide again floods. The rhythm of this repetition is generally two high waters (HW) and two low waters (LW) every 24 hours and 50 minutes, the period of the lunar day. The expression tidal range refers to any of the values secured by taking the absolute difference of a high (low) water and the next succeeding low (high) water.

The flood current is the current which sets in after a low water and which produces the rise of the water level. The ebb current begins after high water and produces the fall of water level. It is important to distinguish carefully between the terms "ebb" and "flood" applied to tidal currents and the same terms applied to tides since the change in the current direction does not necessarily coincide with the times of high or low water but may lag them by as much as 3 hours. Near land the change in current direction usually does occur at the times of low and high water but in the open ocean it more commonly occurs at half tide.

Since we have been talking about an oscillation in water height we are, necessarily, thinking of it as taking place about some fixed level. There are a number of these reference levels in common use. There is mean sea level which is, practically, the average height of the water as it would be determined from a very long record. The half-tide level is the average of any pair of successive extrema. The mean daily level averages the heights over a day, the mean monthly level over a month, and the mean yearly level over a year. The heights of these levels all differ from each other but their differences are usually small. Mean daily levels may differ among themselves by 1.0 to 1.3 feet; mostly because of meteorological differences. The same is true for mean monthly levels. The mean yearly levels show a variation of the order of a few inches and are usually quite close to mean sea level. Marmer (1951) is devoted to the methods of determining fixed reference levels for tide measurements.

The study of mean yearly levels for a place can be quite interesting. They often show long-term general variations in the sea level which can be related to climatological changes. They may also show tectonic movements of the earth's crust, i.e., rising or sinking of the land to which the gage is attached.

The tidal curve with which we began, Fig. 1.02-1, is not representative of conditions everywhere. A particular place may have a semi-diurnal tide with two high waters and two low waters in a bit more than ve a diurnal tide with only one high water and one lo... ın a loose way the harbors of the world can be arranged in four c . sses:

(1) Harbors with regular semidiurnal tides: These harbors have two high waters and two low waters each day. The two highs and the two lows are about of the same heights and they are evenly spaced. New York, Brest, indeed, nearly all the harbors along the east coast of the United States and in Europe belong to this class.

(2) Harbors with diurnal inequalities: These harbors have semidiurnal tides but the heights of the two high waters may be markedly unequal. So may the two low waters. Further, the spacing of the highs and lows may be quite uneven. Many of the harbors of the Indian and Pacific Oceans, including specifically Saigon, belong to this class. To distinguish the highs and lows of any one day from each other we use the terms "higher" and "lower" as shown in Fig. 1.02-2.

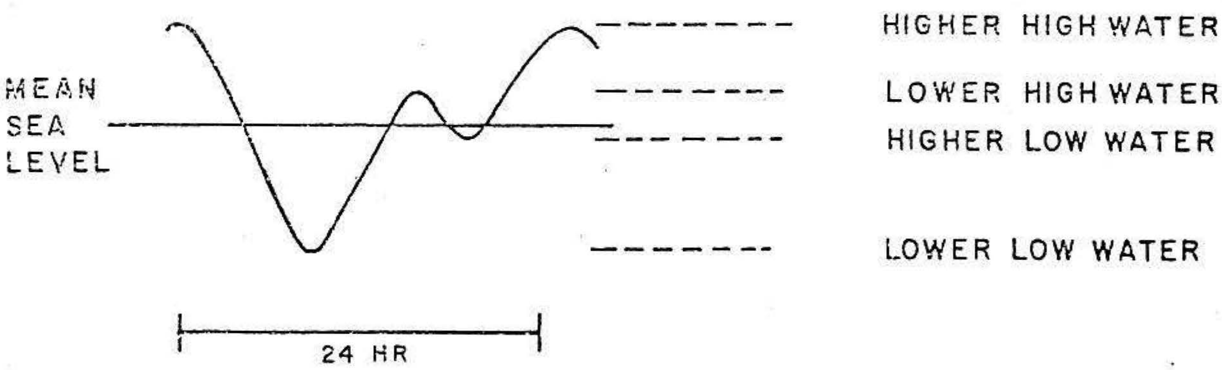


Fig. 1.02-2. Tidal Curve for a Harbor with a Diurnal Inequality.

The order of succession of high and low waters is called the sequence of the tide. It may have any order but for a particular harbor, whatever it is, it is always the same. Figure 1.02-3, page 8, shows some possible sequences.

(3) Harbors with mixed tides: In harbors with mixed tides one observes successively in the course of a fortnight two high waters and two low waters a day and then a period with a single high water and a single low water in a day. Tides of this class are very frequent in the Asiatic

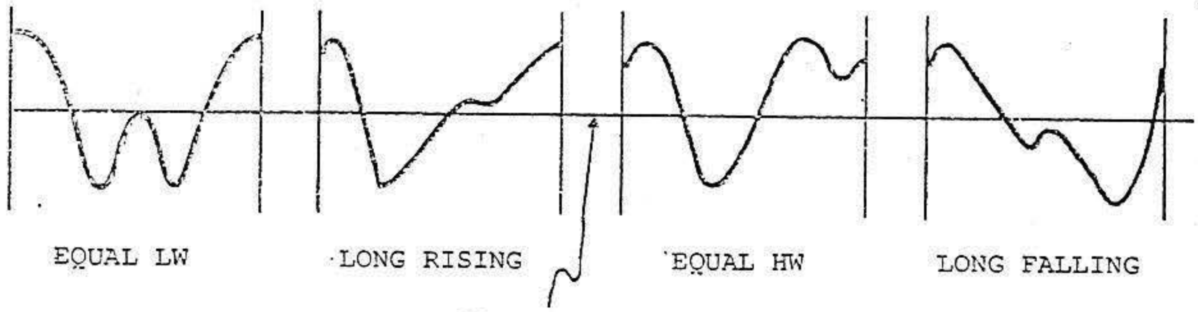


Fig. 1.02-3. Possible Sequences of the Tide.

archipelago, in Indochina, near the coasts of Siberia, and in Alaska. They are also found occasionally in the Atlantic as in Fort de France, Martinique.

(4) Harbors with diurnal tides: Harbors of this class have only one high and one low water a day and are rather rare. Examples include Tonkin, Dason in Indochina, Bangkok in Siam, St. Michael, Alaska, and Copenhagen.

This classification is not comprehensive. For instance, in Tahiti the tide occurs at just about the same time every day and is called a solar tide. At LeHavre and Southhampton there is a double high water at each tide while at Portland and the Hook of Holland there is a double low water.

In any one harbor the tidal range varies systematically increasing, priming, during 7 days to a maximum, spring tide or springs, then decreasing, lagging, during the next 7 days until it reaches a minimum, neap tide or neaps. The spring tides are associated with the conjunction or opposition of the sun and moon, syzygy, full and new moon, while the neap tides correspond to quadrature when the sun and moon are at 90° to each other, half moon. While the tide is lagging strands of sea weed are left in rows on the beach by the high tides as each high water fails to reach as far up the beach as its predecessor. These lines are erased again as the tide primes. The point reached by the highest high tide is the high water mark. In harbors with semidiurnal tides springs usually follow syzygy by one or two days called the age of the semidiurnal tide. Neaps lag quadrature by the same amount. The strongest spring tides of the year take place near the equinoctial syzygies, spring and autumn, when the sun and moon are most nearly in line. They need not occur exactly at these lag by as much as two months.

In harbors with semidiurnal tides it is customary to refer tidal ranges to the mean equinoctial spring tide range. The variation can be quite remarkable. For example, at Brest we have:

Range	Per Cent of Mean Equinoctial Spring Tide Range
Extraordinary equinoctial spring tides	120
Mean equinocital spring tides	100
Mean spring tides	94
Mean tides	70
Mean neap tides	45
Extraordinary neap tides	20

The French tide tables give similar coefficients for Brest for each day of the year. These can be used all along the French coast without noticible error.

It may be of some interest to see what the extreme tidal ranges are. The fifteen greatest are listed in order in the following table:

Country	Place	Mean Spring Range (ft)
Canada	Bassin des Mines (Fundy)	44.6
Canada	Frobisher Bay	44.6
England	Severn River	43.0
France	Mont-St. Michael	41.3
Patagonia	Magellan Strait (east)	39.0
Siberia	--	37.7
Australia	Collier Bay	36.1
China	Rambler Isle	34.1
Canada	Kotsoak River	33.1
Alaska	Sunrise	33.1
Mexico	Rio Colorado	31.5
Brazil	Maraca Isle	29.9
Corsica	Masamplio	29.9
Australia	Mangrove Isle	29.9
Indies	Banhagar	28.5

These extreme tidal ranges generally occur at the heads of bays or large estuaries where the tide is amplified by resonance; much as a sound wave is resonated in an organ pipe of suitable dimensions and shape. You will notice that Africa with its smooth coastline is absent from this list. On isolated oceanic islands the tide seldom exceeds 1.6 to 2.0 feet. In closed seas the tide is hardly felt.

On pages 11 through 14 extracts from "Oceanographic Atlas of the North Atlantic Ocean. Section I. Tides and Currents." (NAVOCEANO Pub. No. 700, 1965) are shown. These are characteristic tide curves for the East and Gulf coasts of the United States and for the Caribbean. Each curve covers 17 days and is keyed to the phases of the moon. They are worth considerable study.

The curve for Charleston, #6, is typical for the entire Atlantic coast and the Bahamas. I find a tendency among students (and others!) to feel that whatever the tide was where they grew up is the way the tide is everywhere. These curves should help you get that out of your system. Key West, #7, alternates between type (1), days 1-5, and type (2), days 7-14. Pensacola, #8, is type (4). The curve for Reykjavik, #19, was included here because it shows springs and neaps so clearly. You should try to find examples of the phenomena described by the terms we have introduced in these curves.

1.03. Gravitation and Gravity.

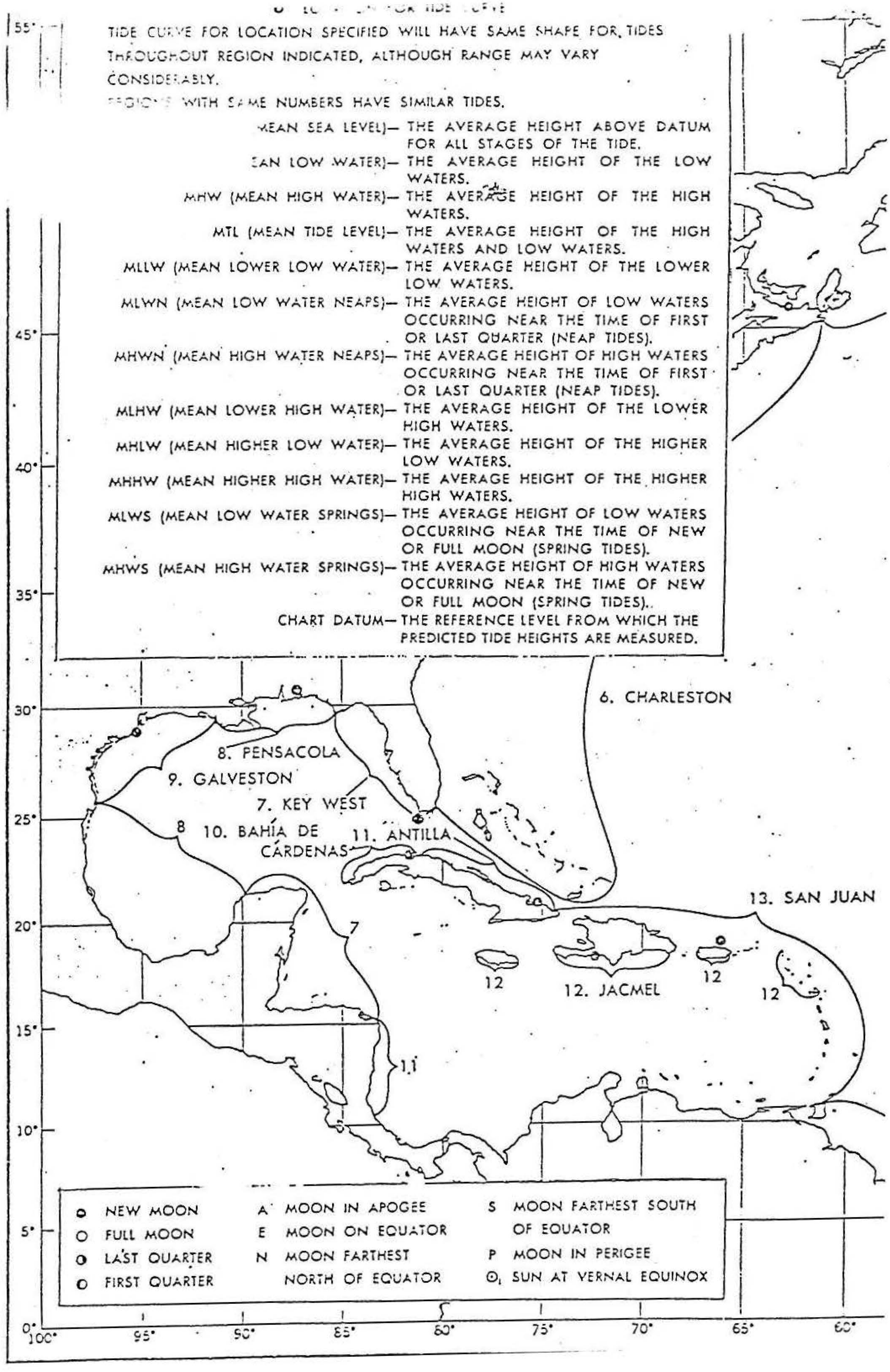
Newton's law of gravitation says that two masses, m_1 and m_2, separated by a distance, r, attract each other with a force, F, which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. In other words,

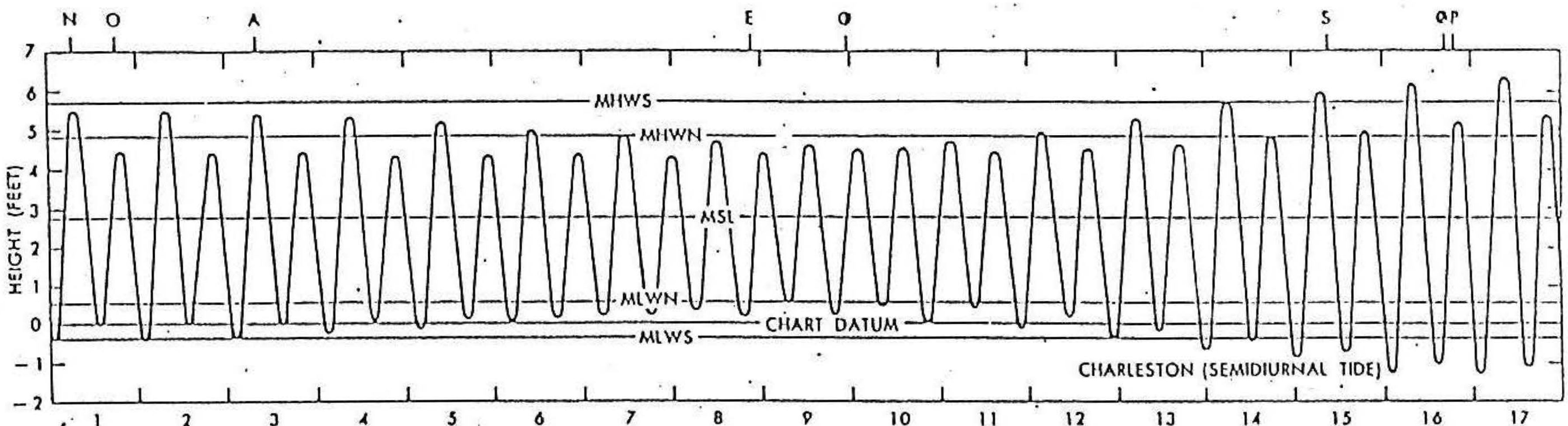
O LOC... FOR TIDE CURVE

TIDE CURVE FOR LOCATION SPECIFIED WILL HAVE SAME SHAPE FOR TIDES THROUGHOUT REGION INDICATED, ALTHOUGH RANGE MAY VARY CONSIDERABLY.

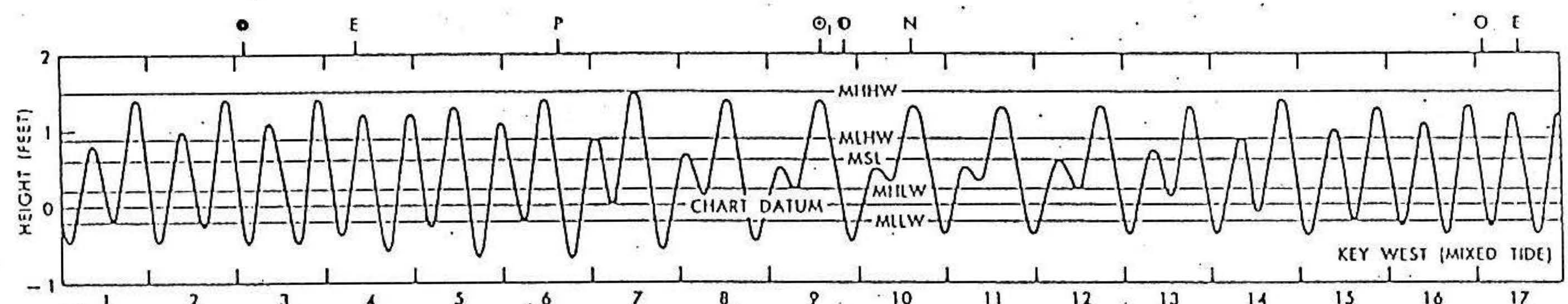
REGIONS WITH SAME NUMBERS HAVE SIMILAR TIDES.

MEAN SEA LEVEL)—	THE AVERAGE HEIGHT ABOVE DATUM FOR ALL STAGES OF THE TIDE.
EAN LOW WATER)—	THE AVERAGE HEIGHT OF THE LOW WATERS.
MHW (MEAN HIGH WATER)—	THE AVERAGE HEIGHT OF THE HIGH WATERS.
MTL (MEAN TIDE LEVEL)—	THE AVERAGE HEIGHT OF THE HIGH WATERS AND LOW WATERS.
MLLW (MEAN LOWER LOW WATER)—	THE AVERAGE HEIGHT OF THE LOWER LOW WATERS.
MLWN (MEAN LOW WATER NEAPS)—	THE AVERAGE HEIGHT OF LOW WATERS OCCURRING NEAR THE TIME OF FIRST OR LAST QUARTER (NEAP TIDES).
MHWN (MEAN HIGH WATER NEAPS)—	THE AVERAGE HEIGHT OF HIGH WATERS OCCURRING NEAR THE TIME OF FIRST OR LAST QUARTER (NEAP TIDES).
MLHW (MEAN LOWER HIGH WATER)—	THE AVERAGE HEIGHT OF THE LOWER HIGH WATERS.
MHLW (MEAN HIGHER LOW WATER)—	THE AVERAGE HEIGHT OF THE HIGHER LOW WATERS.
MHHW (MEAN HIGHER HIGH WATER)—	THE AVERAGE HEIGHT OF THE HIGHER HIGH WATERS.
MLWS (MEAN LOW WATER SPRINGS)—	THE AVERAGE HEIGHT OF LOW WATERS OCCURRING NEAR THE TIME OF NEW OR FULL MOON (SPRING TIDES).
MHWS (MEAN HIGH WATER SPRINGS)—	THE AVERAGE HEIGHT OF HIGH WATERS OCCURRING NEAR THE TIME OF NEW OR FULL MOON (SPRING TIDES).
CHART DATUM—	THE REFERENCE LEVEL FROM WHICH THE PREDICTED TIDE HEIGHTS ARE MEASURED.

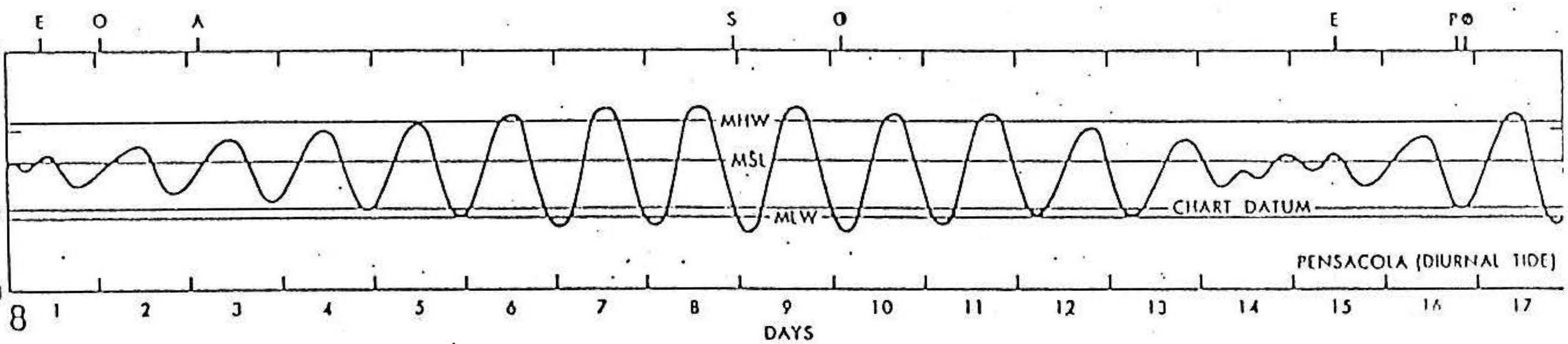



N O A E O S OP
MHWS
MHWN
MSL
MLWN
CHART DATUM
MLWS
HEIGHT (FEET)
CHARLESTON (SEMIDIURNAL TIDE)
DAYS

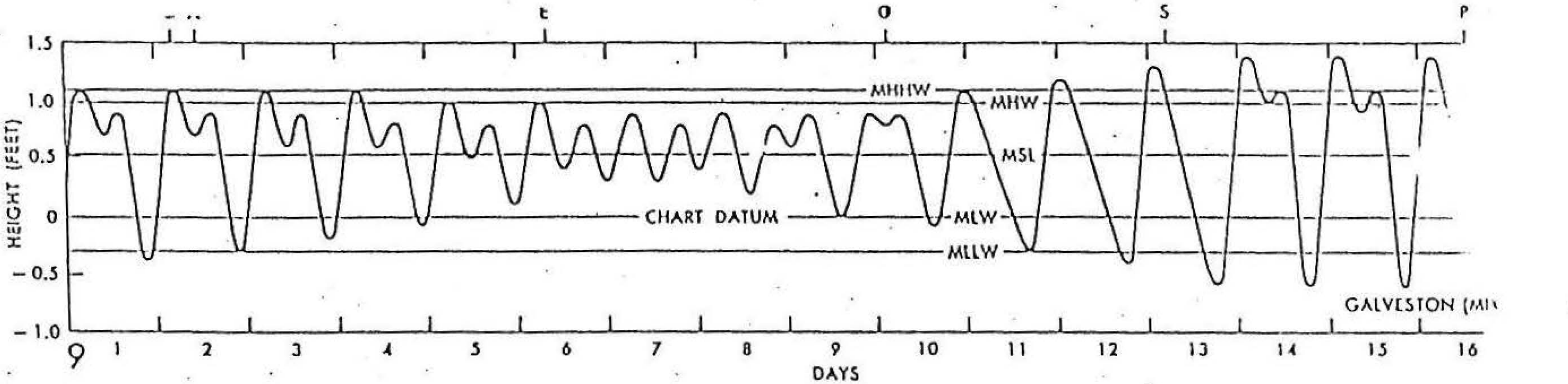
6


E P N O E
MHHW
MLHW
MSL
MHLW
CHART DATUM
MLLW
HEIGHT (FEET)
KEY WEST (MIXED TIDE)
DAYS

7

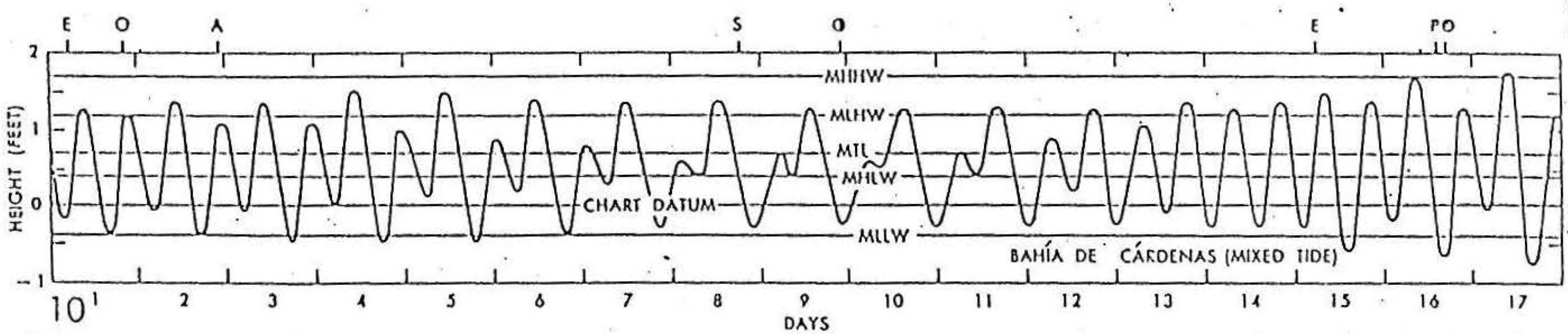

E O A S O E P
MHW
MSL
CHART DATUM
MLW
HEIGHT (FEET)
PENSACOLA (DIURNAL TIDE)
DAYS

8



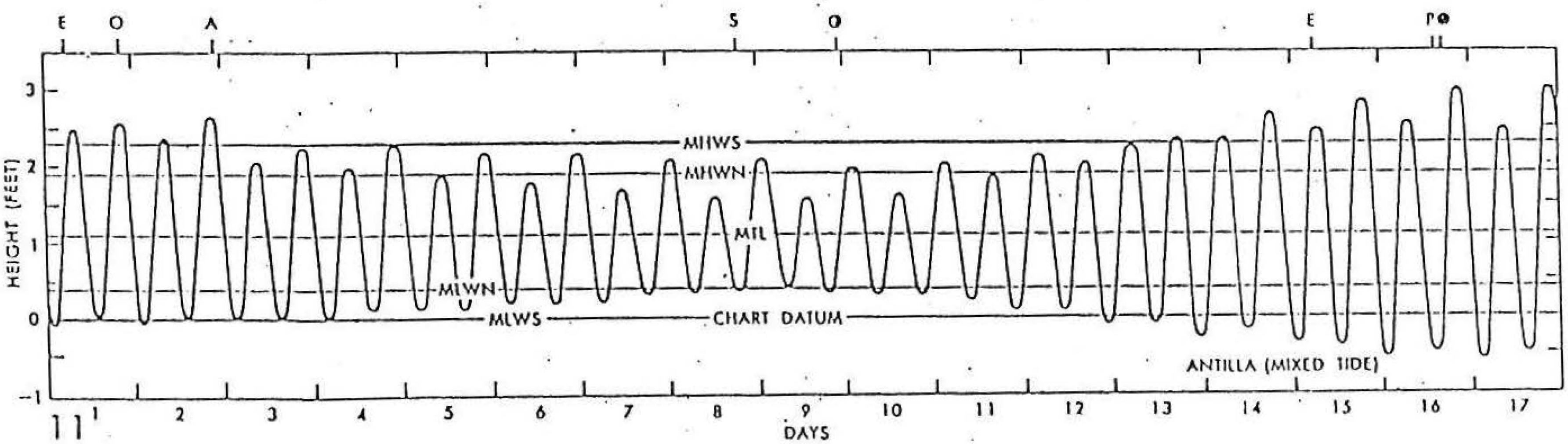
HEIGHT (FEET)
DAYS
MHHW
MHW
MSL
CHART DATUM
MLW
MLLW
GALVESTON (MI

9



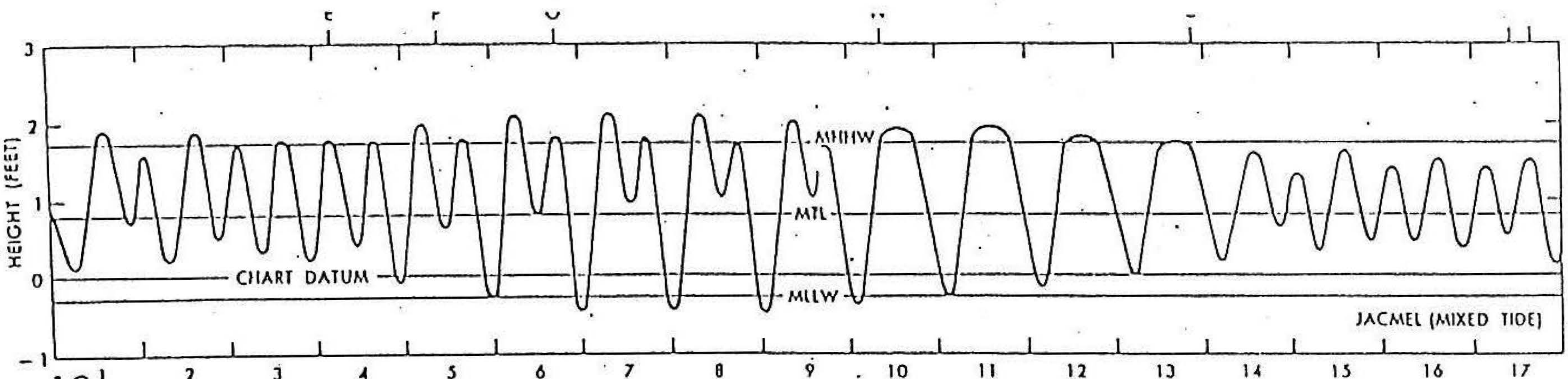
HEIGHT (FEET)
DAYS
MHHW
MLHW
MTL
MHLW
CHART DATUM
MLLW
BAHÍA DE CÁRDENAS (MIXED TIDE)

10

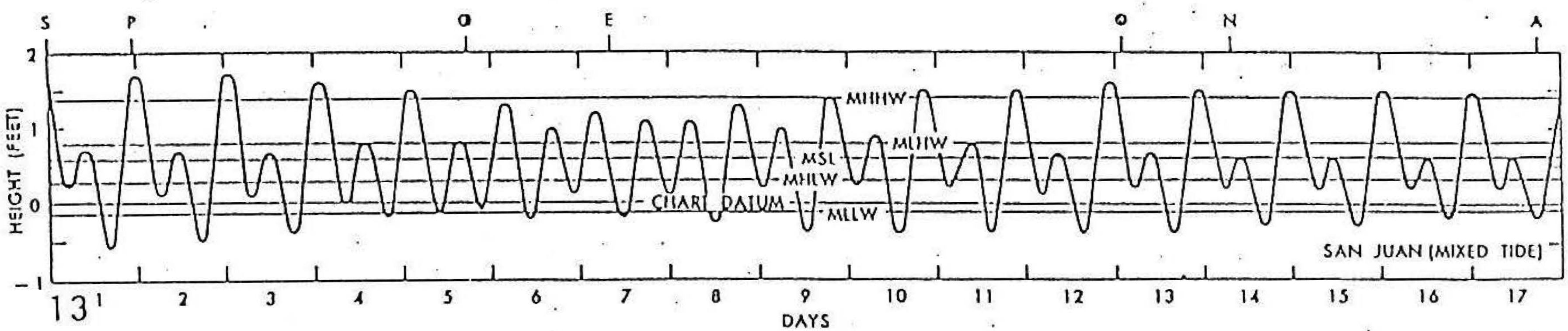


HEIGHT (FEET)
DAYS
MHWS
MHWN
MTL
MLWN
MLWS
CHART DATUM
ANTILLA (MIXED TIDE)

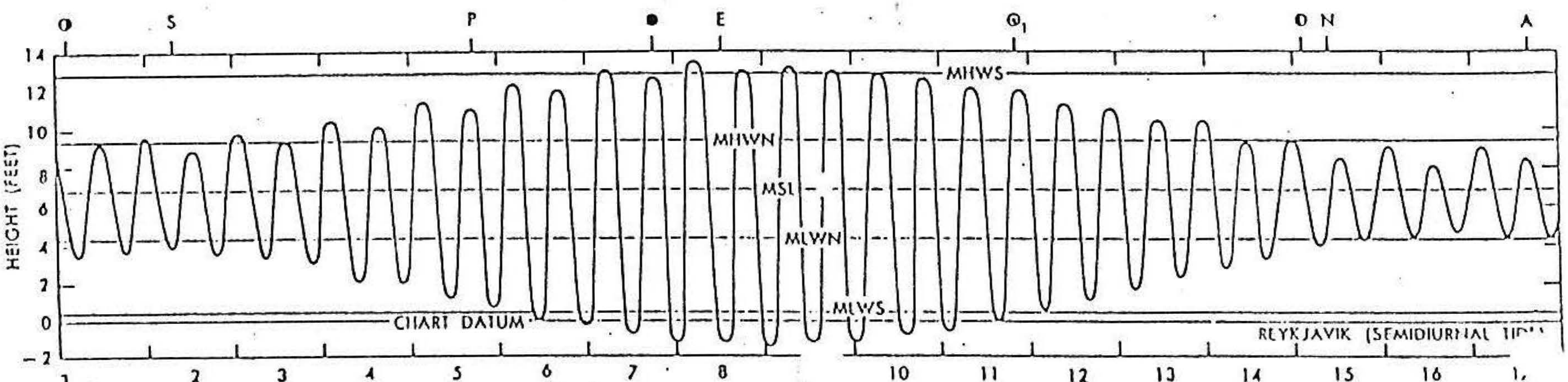
11


HEIGHT (FEET)
MHHW
MTL
CHART DATUM
MLLW
JACMEL (MIXED TIDE)
DAYS
3
2
1
0
-1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

12


S P O E O N A
HEIGHT (FEET)
MHHW
MHW
MSL
MHLW
CHART DATUM
MLLW
SAN JUAN (MIXED TIDE)
DAYS
2
1
0
-1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

13


O S P E O N A
HEIGHT (FEET)
MHWS
MHWN
MSL
MLWN
MLWS
CHART DATUM
14
12
10
8
6
4
2
0
-2
1 2 3 4 5 6 7 8 10 11 12 13 14 15 16

1

$$\gamma(m_1 m_2/r^2)$$

$$\gamma = 6.67\times10^{-1}\ g^{-1}cm^3sec^{-2}$$

is the universal gravitational constant when m_1 and m_2 are measured in grams and r is measured in centimeters.

If m_1 is unit mass and we use $m_2 \equiv m$, then the gravitational force per unit mass is

$$F = \gamma(m/r^2) \quad .$$

At the earth's surface the gravitational acceleration, g_o, defined as the force exerted by the earth's mass on a particle of unit mass is

$$g_o = \gamma\sum_i(m_i/r_i^{\,2}) = \gamma\ell^3\sum_i(\rho_i/r_i^{\,2})$$

where ℓ is unit length and ρ is density. With ρ in g cm^{-3} the unit of ℓ is cm. The summation is to be extended throughout the earth's volume. If the earth were a perfect sphere of mass M_e and radius a, we could write

$$\sum_i(m_i/r_i^{\,2}) \equiv M_e/a^2 \quad .$$

This amounts to acting as though the entire mass of the earth were concentrated at its center from which our unit mass is separated by a distance a equal to the earth's radius.

Sadly enough, the earth is not a sphere. Due to its rotation it bulges at the equator and a better approximation to its shape is an ellipsoid of revolution with the major axis in the equatorial plane. If a is the equatorial radius (semi-major axis) and b the polar radius (semi-minor axis), then the <u>ellipticity</u>, e, of the earth is

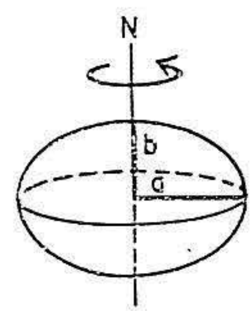


Fig. 1.03-1

$$e \equiv (a - b)/a \quad .$$

Naturally, measurements of e made in different places give different values. The figure of earth really isn't exactly an ellipsoid of revolution.

In practice, e is determined by very carefully measuring the length of an arc along a meridian and the angles made by the verticals at the ends of the arc. From these data, on the assumption that the arc along a meridian is a segment of an ellipse, a and b can be computed and from them the eccentricity. As one might expect, measurements in different places give different estimates of e. Each country tends to use the value that best fits its own geodesy. The International Spheroid of Revolution uses e = 1/297. The Clark Spheroid (United States) uses e = 1/293. In general, the estimates range from 1/300 to 1/290.

For an elliptical section the gravitational "constant," g_o, isn't constant but a function of latitude. Toward the poles a unit mass on the earth's surface is closer to the earth's center than it is at the equator and g_o is a function of latitude.

If the earth were at rest, this would be the whole story. It isn't. Since the earth rotates we must further modify g_o to include the centrifugal force which acts against gravitation. The centrifugal force is given by

$$F_c = \omega^2 R$$

where

$\omega \equiv$ the rate of angular rotation,

$R \propto \cos\phi$,

and

$\phi \equiv$ the latitude.

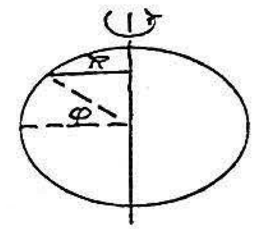


Fig. 1.03-2

F_c is greatest at the equator where $R = R_{max} = a$ and $\phi = 0$ so that $\cos\phi = 1$. It decreases to zero at the poles where $R = 0$ and $\phi = 90^o$ which makes $\cos\phi = 0$ as well.

When the gravitation, the ellipticity, and the centrifugal effects are combined we get the apparent gravitation which, by definition, is the <u>gravity</u>, g. Observationally, the components of gravity are inseparable. On the equator g = 978.05 cm sec^{-2}. At the poles g = 988.07 cm sec^{-2}.

WARNING: Gravitation and gravity are not the same thing. I have seen altogether too many students foul up their oral examinations by failing to make the distinction between them. A word to the wise ... .

ravity ction of latitude, a number of empirical equations ha... n set up to describe the variation. The most commonly used is Clairaut's formula:

$$g = 978.05(1 + 0.0053\sin^2\phi)$$

where ϕ is the latitude. Obviously, Clairaut's formula uses the value of g at the equator and adds a correction term. An alternative formula by Helmert does the same sort of thing but uses the value of g at latitude 45° as the base.

In spite of all this, one needs to go only a little distance away from the surface of the earth before earth behaves substantially like a point mass. Consider two unit masses, one on the equator but not rotating with the earth and the other at the pole. Now move them away from the earth radially. If you plot g_o as a function of distance from the earth's center you get something that looks like Fig. 1.03-3. The two curves merge very rapidly after which the earth is indistinguishable from a point mass, i.e., you can't tell from a knowledge of the curve whether you are moving out on a line with the pole or on a line in the equatorial plane. We treat the earth as a point mass in connection with the forces produced by the sun and moon. These are the principal extraterrestrial forces but we will have to make an argument for this later.

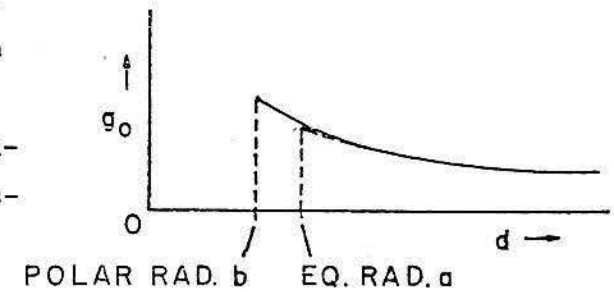


Fig. 1.03-3

There are other terrestrial forces on a particle on the earth's surface. The earth is not a true ellipsoid of revolution and mountain ranges over a few miles high and unusually dense slugs in the earth's crust, e.g., lodes of ore, produce local anomalies in g. These result in a permanent deformation of the sea surface. However, since they are permanent deformations, they need not be taken into account in studying a time dependent process such as the tide.

From here on out we will consider the earth to be a sphere. Let's take a look at the geometry. Consider any point P on the surface of the earth, Fig. 1.03-4.

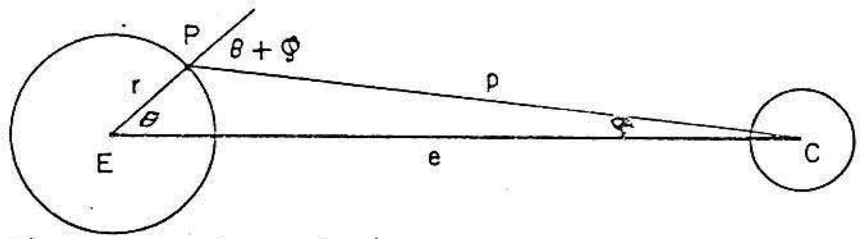


Fig. 1.03-4

Let E be the mass of the earth concentrated at its center,

C be the mass of any celestial body concentrated at its center,

e be the distance between centers,

r be the radius of the earth,

p be the distance from the center of the celestial body to P,

and ϕ and θ the angles indicated in Fig. 1.03-4.

The attracting force of C on E is directed along the line of centers, CE, which is of length e. The attracting force of C on P is directed along the line CP with length p. The attraction of C on a unit mass located at E, A_E, is

$$A_E = \gamma(C/e^2)$$

where γ is the universal gravitational constant. Similarly, the attraction of C on a unit mass located at P, A_P, is

$$A_P = \gamma(C/p^2) \quad .$$

We may consider that r, e, θ, and ϕ are known from geodesy and celestial mechanics. Then, using the law of cosines with

$$p^2 = e^2 + r^2 - 2er[\cos\theta] \quad ,$$

$$A_P = (\gamma C)/(e^2 + r^2 - 2er[\cos\theta]) \quad .$$

This is directed along the line PC. The component parallel to the line of centers, CE, is obviously

$$\{(\gamma C)/(e^2 + r^2 - 2er[\cos\theta])\}\cos\phi \quad .$$

T. [illegible] :s viewed from a celestial body will subtend a maxim[illegible] [illegible]le, ϕ_{max}, and for any point of earth we have $0 \leq |\phi| \leq \phi_{max}$, Fig. 1.03-5. The moon is our nearest celestial neighbor and so has the largest

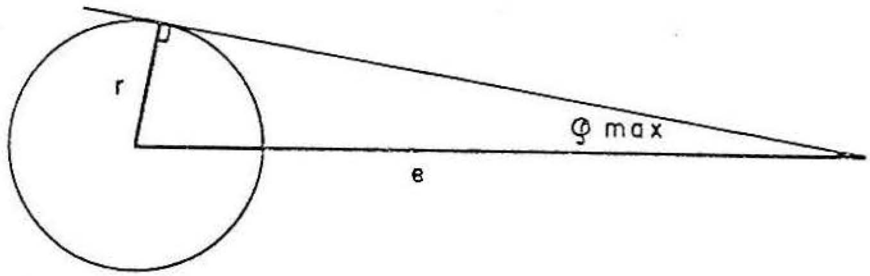


Fig. 1.03-5

ϕ_{max}, $\phi_{max} \simeq 1^{\circ}$. Consequently, for the moon $1 \geq \cos\phi \geq 0.99985$ and the component of A_p parallel to the line of centers, CE, differs from the full force along AP by at most 0.02%. For all other celestial bodies the difference is even smaller. Thus, the discrepancy is always so small that we frequently don't bother to distinguish between the two. However, in very precise work you may want to retain the distinction and the component of A_p, as we have expressed it, has an explicit dependence on both ϕ and θ. This is not really necessary since ϕ and θ are functionally dependent. The component can be expressed entirely in terms of θ by solving for $\phi = f(\theta)$ and substituting to get an expression in θ alone.

It is often useful to measure the length of the line of centers in units of the earth's radius, e/r. For instance, the center of the moon is approximately 60 earth radii from the center of the earth. However, in the course of our analyses we will often find that it is the reciprocal of this distance, r/e, which appears in our equations. The parameter r/e has been named the <u>parallax</u>. The parallax is a sort of upside down measure of distance to a celestial body in units of the earth's radius. The moon's parallax is $r/e \simeq 1/60$. The sun's parallax is $r/e \simeq (4\times10^3)/(92\times10^6) = 4/92{,}000 = 1/23{,}000$.

With the forces acting one might expect the two bodies to approach each other and they would were it not for the counterbalance of the centrifugal force produced by the motion of the moon about the earth in its orbit:

$$R\,\omega^2 = \gamma C/R^2 \quad .$$

Actually, the centrifugal force is higher on the far side but negligibly so. Our main problem is to determine the differences of the forces acting on points of the earth's surface. We have

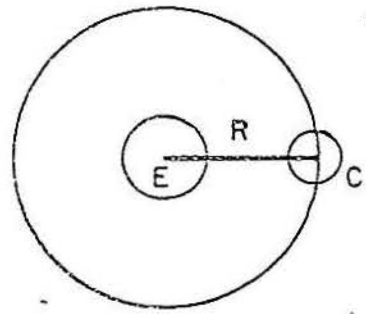


Fig. 1.03-6

$$A_P - A_E = (\gamma C)/(e^2 + r^2 - 2er[\cos\theta]) - (\gamma C)/(e^2) \quad .$$

Notice that the first term on the right-hand side is the approximate form since the factor $\cos\phi$ has been omitted. This may be rewritten as

$$A_P - A_E = \frac{\gamma C}{e^2}\left(\frac{1}{1 - 2(r/e)\cos\theta + (r/e)^2} - 1\right)$$

$$= \frac{\gamma C}{e^2}\left(\frac{2(r/e)\cos\theta - (r/e)^2}{1 - 2(r/e)\cos\theta + (r/e)^2}\right) \quad .$$

Notice the appearance of the parallax as a parameter.

We want to find the values of the parameters that can give appreciable differences between A_P and A_E. For the moon $r/e \simeq 1/60$ so that $r/e \leq 1/60$ for any celestial body. The extreme value of θ gives $\cos\theta = 1$. With these values

$$A_P - A_E \simeq \frac{\gamma C}{e^2}\,\frac{2(r/e)}{1} = 2\gamma r(C/e^3) \quad .$$

For any celestial body, $2\gamma r$ is a constant. Only C and e are different for different celestial bodies so that C/e^3 is the critical factor. It provides us with a means of comparing the effects to be expected from different celestial bodies. Take the obvious four for a check: the moon, the sun, Venus (the nearest planet), and Jupiter (the largest planet). Use the moon as a unit of measure. The numbers for comparison are shown in the table on page 21. While the sun has a mass 27 million times as great as the moon, the mass enters only linearly while the distance enters in cube so that the force exerted by the sun is less than half that exerted by the moon. The nearest fixed stars have masses comparable to that of the sun

Body	Mass (C)	Minimum Distance (e)	Ratio C/e^3 $\sim A_P - A_E$
Moon	1	1	1
Sun	27.1×10^6	389	4.6×10^{-1}
Venus	66	108	5×10^{-5}
Jupiter	26×10^3	1630	6×10^{-6}

but the distances, 4.3 light years for the nearest as against 7 light minutes for the sun, are so enormously greater that their effect is negligible. Even the planets, e.g., Venus and Jupiter, while closer, have effects many orders of magnitude smaller than the moon's. These calculations support the assertion made earlier that as tide producing bodies only the moon and the sun need be considered.

1.04. On Astronomy and Time.

1.04.1. The Sun and the Earth.

Rotation: The rotation of a body is a motion about an internal exis, e.g., the earth's daily rotation on its axis.

Revolution: The revolution of a body is a motion about an external axis, e.g., the earth's yearly motion around the sun.

Eccentricity: The eccentricity of an ellipse is one-half the focal distance divided by the semi-major axis,

$$\varepsilon \equiv c/a = \sqrt{a^2 - b^2}/a \quad .$$

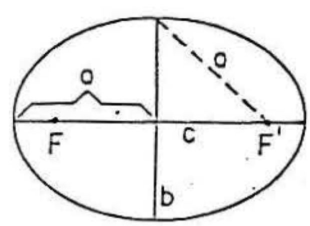


Fig. 1.04.1-1

Ellipticity: The ellipticity of an ellipse is the difference between the semi-major and semi-minor axes divided by the semi-major axis, $e \equiv (a - b)/a$.

The orbits of the planets are nearly elliptical with the sun located at one focus. All the orbits lie very nearly in one plane. The eccentricity of the earth's orbit is $0.017 \approx 1/60$.

The terms perihelion, aphelion, and line of apsides can best be understoon from Fig. 1.04.1-2 which is greatly exagerated. The difference between the distances of the earth to the sun at perihelion and aphelion is only about 3.4%.

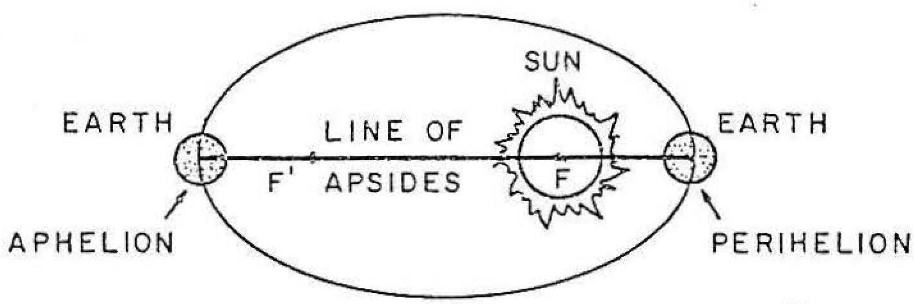


Fig. 1.04.1-2

The earth's mean distance to the sun is, by definition, one-half the line of apsides, i.e., it is the semi-major axis. A more refined distance measure might be secured by integrating around the orbit but it wouldn't turn out to be much different.

Viewed from outside the system, the earth rotates from west to east on its axis while revolving from west to east about the sun, the two motions being like a set of gears, Fig. 1.04.1-3

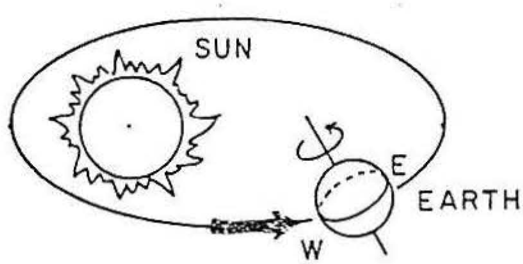


Fig. 1.04.1-3

The earth does not travel uniformly on its orbit. It moves faster when it is nearer the sun and more slowly when it is farther away in accordance with Kepler's Law which, incidently, is derivable from Newton's Laws. Keppler's Law says that a planet moves always in such a way that the

:n sweeps out equal areas in equal times. In Fig. ... arcs $\widehat{A_1A_2}$, $\widehat{B_1B_2}$, $\widehat{C_1C_2}$ are traver... in equal time intervals and have quite different lengths. The shaded areas, however, are all equal. This conforms to the requirement that gravitational attraction be balanced by centrifugal force:

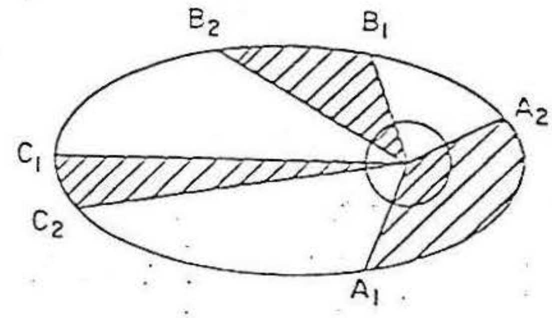


Fig. 1.04.1-4

$$R\omega^2 = \gamma C/R^2 \quad .$$

As R increases, the right member (gravitation) decreases; and in square. The left member (centrifugal force), on the contrary, grows with R; and linearly. The equality (balance) would be destroyed if the angular velocity ω did not decrease sufficiently to restore the equality.

If a line is drawn through the center of the earth perpendicular to the plane of the earth's orbit, the earth's axis makes an angle of $23^{o}27'$ with it. Except for a <u>very slow</u> wobble, this angle is maintained at all times as is the orientation of the axis.

The description of the motions of the earth and sun has so far been tacitly heliocentric; a view of things that you may have been taught somewhere or other was true with a capital "T." If so, you can forget it. Truth doesn't reside in a set of coordinates. For tides, life becomes simpler if we revert to Ptolemaic astronomy. From here on in the earth is a small sphere situated at the center of the universe. That universe consists of a celestial sphere on which the fixed stars are hung. The earth is at the center of the celestial sphere and the axes of the two coincide so that the North Celestial Pole is directly above the North Terrestrial Pole. The celestial equator lies directly above the terrestrial equator. the earth is stationary but the celestial sphere rotates evenly on its axis from east to west completing one rotation in something around 24 hours. The planets, which include the sun and the moon, share this movement of the celestial sphere but, unlike the stars, which remain permanently fixed in position on the celestial sphere, wander about with motions of their own. During the course of a year, for example, the sun is seen against different parts of the celestial sphere at different seasons. The path which it follows on the celestial sphere is a great circle called the ecliptic and is marked on the celestial sphere by the signs of the zodiac. The ecliptic intersects the equator in two points called the equinoctial points. The points where it departs most widely from the equator are called the solstices. The angle between the ecliptic and the equator is $23^{\circ}27'$. The way it looks is shown in Fig. 1.04.1-5.

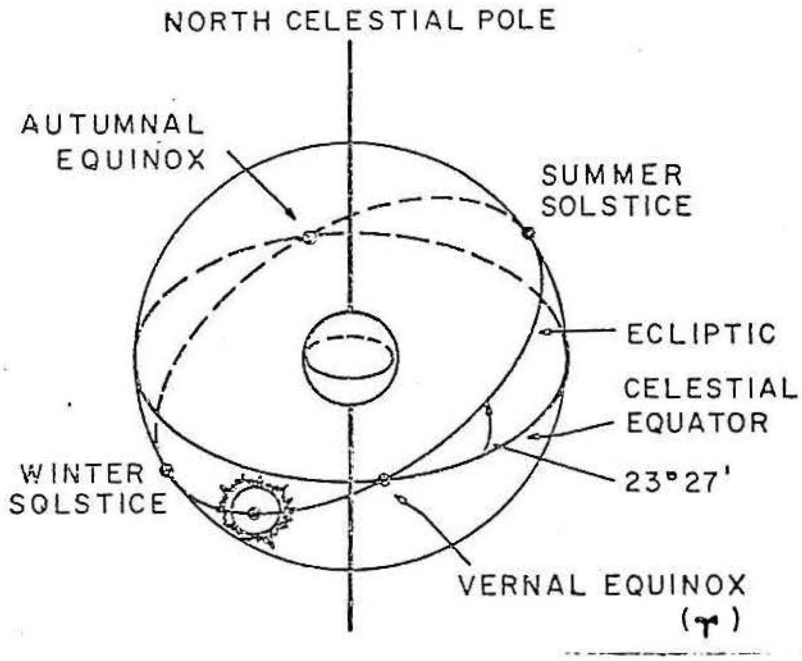


Fig. 1.04.1-5

The celestial sphere goes around once a day (more or less) carrying the sun with it. The sun meanwhile drifts slowly backward along the ecliptic making one circuit in a year. When the sun crosses the equator from south to north we have the vernal equinox. When it reaches its maximum elevation in the northern sky, the summer solstice. When it recrosses the equator from north to south, the autumnal equinox. And when it declines to its lowest point in the sky, the winter solstice. It is to these phenomena that the first day of each of the seasons, as noted on the calender, refer.

The equinoxes are used as reference points in one set of celestial coordinates, Fig. 1.04.1-6, page 25. This is the coordinate set

.tion. Consider a star Let a great circle through X and the north pole intersect the equator at A. This circle through X is called an hour circle. I have indicated the vernal equinox by a stylized ram's head which is the old symbol for the constellation Aries, ♈. In Ptolemy's time the vernal equinox was located at the first point of Aries but not now. Of this more later. The distance ♈A is the right ascension of the star at X and the distance AX is its declination. These two numbers specify its position relative to the equinoctial point.

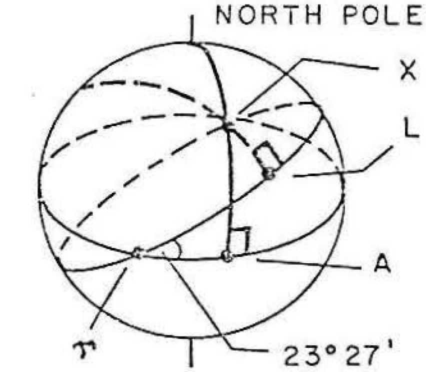


Fig. 1.04.1-6

Another system of coordinates passes the great circle through X perpendicular to the ecliptic instead of through the pole. In this system specification of ♈L, the celestial longitude, and XL, the celestial latitude, locates X. Since the angle between the celestial equator and the ecliptic is a constant $23^{\circ}27'$ one of the systems easily converts into the other. There are other systems in use which are relative to the observer but we need not discuss them here.

It was previously stated that the earth's axis maintains its orientation except for a slow wobble. Figure 1.04.1-7 shows the heliocentric picture while Fig. 1.04.1-8, page 26, shows the geocentric version.

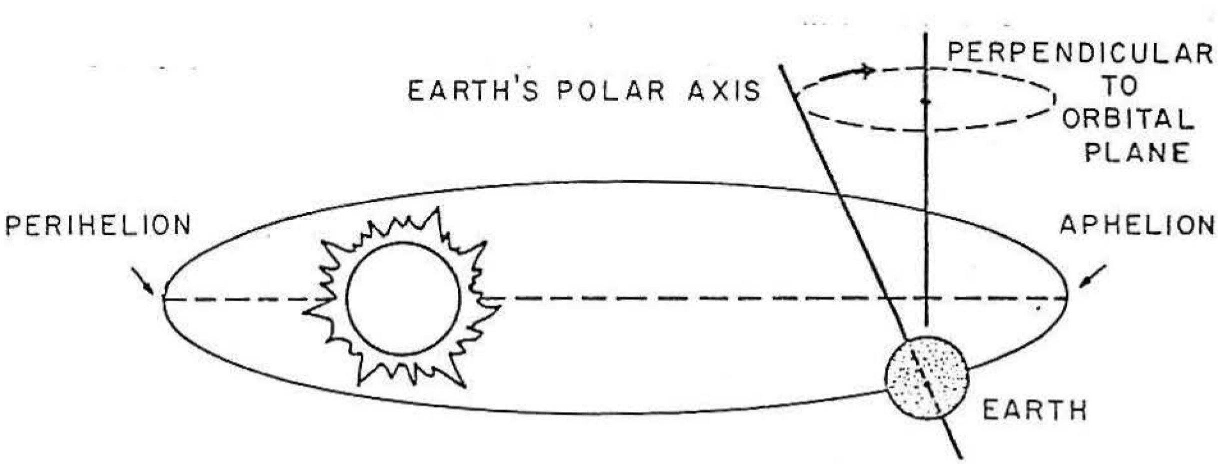


Fig. 1.04.1-7

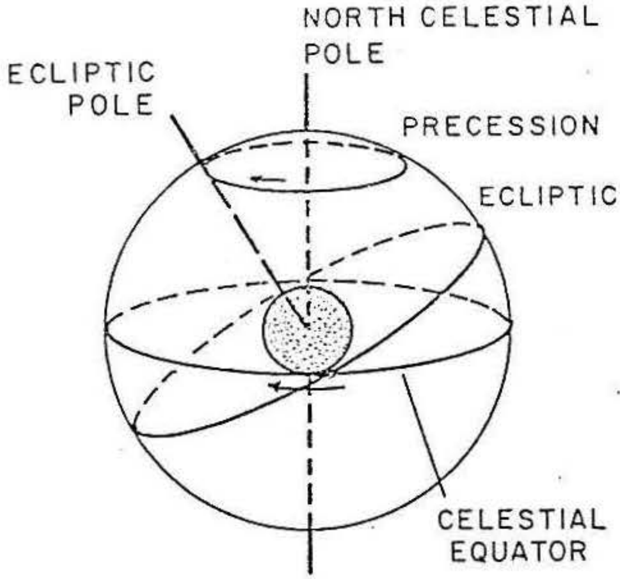


Fig. 1.04.1-8

The effect of this motion is to make the equinoxes drift slowly westward, i.e., the ecliptic, while maintaining its angle with the equator, moves relative to it. It is to this phenomenon that the term, precession of the equinoxes, refers. The period of the precession is very long; about 26,000 years. This corresponds to a shift in the equinoctial point of about 50" of arc per year. When Ptolemy first worked out his astronomy (c. 100 A.D.) the vernal equinox was located at the first point of Aries.

I would like to digress a bit here on the zodiac. The zodiac is a band on the celestial sphere 16° wide and centered on the ecliptic. It marks the region of the sky in which the sun, moon, and planets are always found. Hundreds of years before Christ the Babylonian astronomers grouped the stars along this band into 12 constellations each occupying 30° of arc and gave them animal names (more or less)--hence the "zo." Each sign of the zodiac was divided into thirty points. In other words, the ecliptic on the celestial sphere was divided into what we now call degrees. The signs of the zodiac reading from Aries (which marked the spring equinox in Ptolemy's time) in the order in which the sun moves through them are Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Saggitarius, Capricorn, Aquarius, and Pisces.

) years since Ptolemy, at 50" per year (0.01389°), the [illegible] quinox has moved about 26° and is now at the fourth point of Pisces This fact is well known to astronomers but seems to have escaped the astrologers who make horoscopes entirely. They still work from the first point of Aries.

Another slow motion is that of the line of apsides. It rotates eastward at about 11" per year. The relative motion of precession and the line of apsides is about 1'01" per year so that the period of the double motion is about 21,000 years.

These slow motions are not very important for tides but they must be taken into account in any long range "geological scale" study, e.g., a long range study of the fluctuations in solar radiation received by the earth. At present, perihelion occurs in December within 9 days of the winter solstice. This means that the northern hemisphere is now enjoying optimal conditions of solar radiation. The extremes of insolation produced by the height of the sun in the sky are counterbalanced by the distance to the sun. The worst conditions for the northern hemisphere will occur when aphelion coincides with the winter solstice.

1.04.2. Time.

One of the most important features of the celestial movements so far as tides are concerned it that they determine the periods of the components which make up the tide. The tide is a forced wave and the celestial bodies provide the forcing function. The three most important periods are the year, associated with the earth's revolution about the sun, the lunar month, associated with the moon's revolution about the earth, and the day, associated with the earth's rotation on its axis.

To measure anything you need a reference point. Relative to you, as an observer standing on a meridian of the earth, a celestial body is in upper transit when it is directly above your meridian. Upper transits are particularly important for tides. For example, crude tide tables can be constructed reckoned from the last upper transit of the moon. When the celestial body is directly opposed to upper transit it is said to be in lower transit.

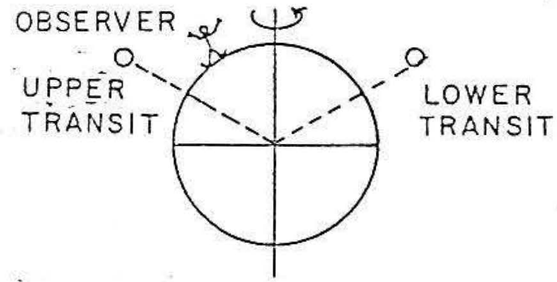


Fig. 1.04.2-1

The day: The day comes in three fundamental sizes depending on the reference point:

Reference Point	Definition	Name
Vernal equinox	The time at a local meridian from upper transit to upper transit of the vernal equinox	sidereal day
The sun	The time from meridian transit to meridian transit of the sun	solar day
The moon	The time from meridian transit to meridian transit of the moon	lunar day

There is a slight foul up in applying the term "sidereal" to a day measured from the vernal equinox. In its derivation the word "sidereal" means "constellation," i.e., it refers to the fixed stars. The vernal equinox, as we have seen, has a slow motion of its own relative to the fixed stars so that you don't get quite the same value from it as you would using transits of Sirius, for instance. Such a day would be a true sidereal day. However, it isn't used and differs from the sidereal day by only about 0.01 sec/day. In terms of civil time, the sidereal day comes out to be 23h 56m 04.1s. Whenever the angular speed of the earth is required in Coriolis acceleration calculations, and such like, it is this value that is intended.

The length of the true solar day is another kettle of fish. As we have seen, the speed of the earth along its orbit varies with its distance from the sun. This means that the amount by which the sun lags the return of the celestial sphere differs at different times of the year and, in consequence, its time from transit to transit varies. In the course of a year this variation amounts to as much as 48 sec. The range of the solar day is 23h 59m 41s to 24h 00m 29s. The differences are cumulative so that the average occurance may get out of whack with the mean solar day by as much as 16 minutes. The relation between mean solar time and apparent solar time is known as the equation of time. It is frequently graphed on terrestrial globes where it appears as an elongated "figure 8" called the analemma. The mean solar day is the average solar day and is the basis of civil time.

N.B.: When hours and days are used without specification civil hours and mean solar days are intended.

Even if the sun were to move evenly on the ecliptic, the solar day would still vary since the arc through which it moves is carried by the celestial sphere. This means that the arc must be projected on the equator. For example, the situations near equinox and near solstice are shown in Fig. 1.04.2-2.

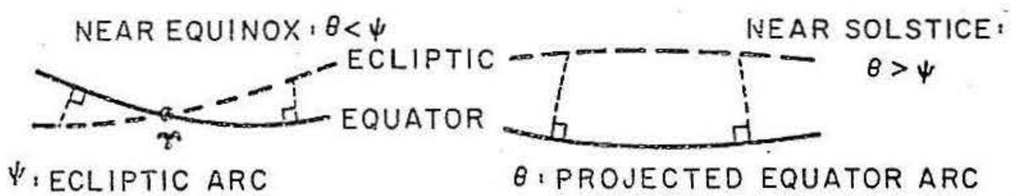


Fig. 1.04.2-2

A similar situation exists for the lunar day but it is much more erratic than the solar day. It may vary as much as 15 minutes from the mean lunar day. The lunar period is of the utmost importance for tides. The average lunar day is 24.84 hours as compared with a mean solar day of 24 hours. We will have more to say about the moon later.

The year: The year also comes in various sizes. The tropical year is 365.2422 mean solar days in length. It is the time taken for the sun to make one round of the ecliptic from vernal equinox to vernal equinox. In other words, it represents one complete cycle of the seasons and is sometimes called the year of the seasons. It is the year the common calender tries to keep in order.

The Julian calender introduced in 46 B.C. by Julius Caesar used a year of 365 days divided into months as we do today. Every fourth year was a leap year. This gives a year that is about 11 minutes too long on the average. By 1582 A.D. the vernal equinox had drifted 10 days and Pope Gregory XIII introduced the Gregorian calender now in general use. He dropped ten days to rectify the vernal equinox making October 15 come the day after October 5 and arranged that of the years divisible by 100 only those also divisible by 400 should be leap years. The Julian calender was otherwise unaltered. Great Britain and her colonies did not adopt the Gregorian calender until 1752.

The number of sidereal days in a tropical year is exactly one more than the number of solar days, 366.2422 sidereal days, because, in the course of a year, the sun slips back along the ecliptic one full turn which must be made up by the celestial sphere.

The year with respect to a fixed star is the sidereal year. It is 365.2564 days. The difference between the tropical and the sidereal years matches the 50" per year of precession.

WARNING: Here comes the confusion!

The sidereal year is taken with respect to the fixed stars.

The sidereal day is taken with respect to the vernal equinox.

The tropical year is taken with respect to the vernal equinox.

Watch your step!!!

Another year, the anomalistic year, taken with respect to the line of apsides from perihelion to perihelion is 365.2596 days in length.

1.04.3. The Moon.

We've ducked the moon as long as possible but I'm afraid that the lunacy must now begin. The moon's orbit about the earth is analogous to the orbit of the earth about the sun. It is an ellipse (mainly) with the earth at one focus but, because of the gravitational effect of the sun on the moon, it is much more deformed, i.e., it has many more anomalies. The eccentricity of the moon's orbit is 0.055. The earth's is 0.017. The motion of the moon in its orbit is, like the earth's about the sun, from west to east, Fig. 1.04.3-1.

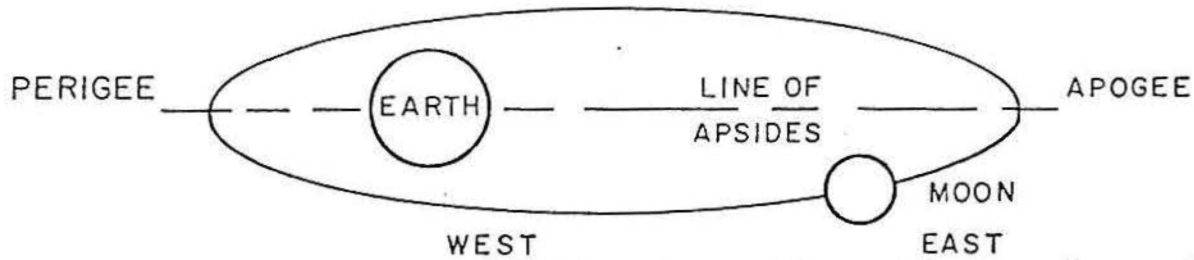


Fig. 1.04.3-1

The plane of the moon's orbit is inclined to the plane of the earth's orbit at a constant 5°. The heliocentric picture is shown in Fig. 1.04.3-2.

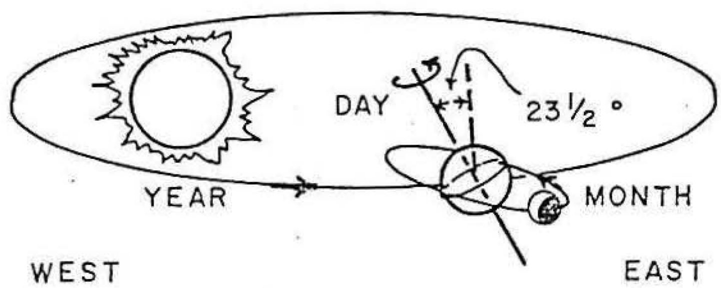


Fig. 1.04.3-2

The geocentric version of the same thing is shown in Fig. 1.04.3-3, page 32. The distance from the vernal equinox to the node is a slowly changing value with a period of about 19 years. The moon covers its orbit in a month (more or less) just as the sun covers the ecliptic in a year (more or less). An attempt at precise definition of the month results in the same sort of mess as the definition of the year. Different lengths result from different reference points.

Three commonly used reference points for definition of the month are: (1) conjunction with the sun, (2) conjunction with the vernal equinox, and (3) conjunction with a fixed star. Two celestial bodies are said to be

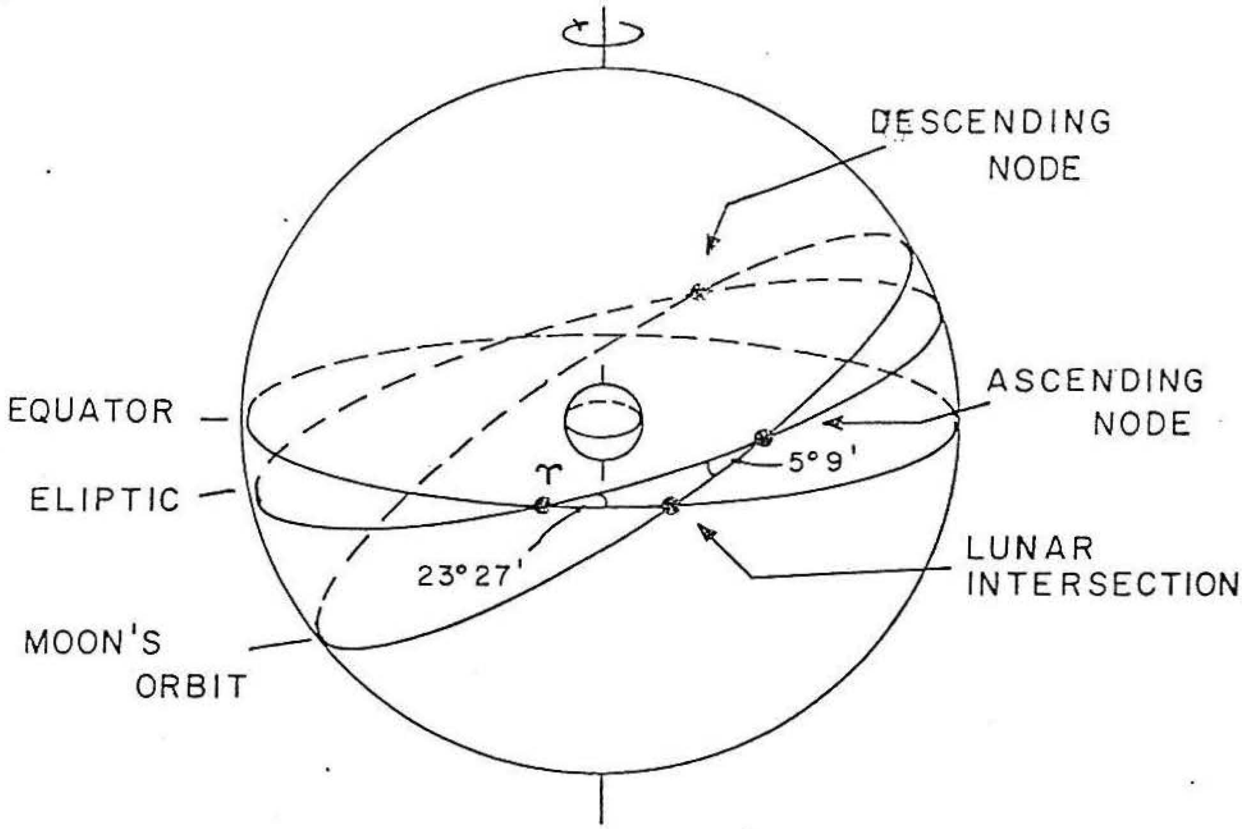


Fig. 1.04.3-3

in conjunction when they have the same right ascension, i.e., when they are located on the same hour circle. The first reference point gives you the synodical month of 29.5306 ± 0.5 days, the second the tropical month of 27.3216 ± 0.3 days, and the third the sidereal month of 27.3217 days. These values are averages taken over a year.

Other kinds of months arise from the motion of the nodes and the line of apsides. The movement of the nodes relative to the ecliptic is called the regression of the nodes and is analogous to the precession of the equinoxes. It causes a westward displacement of the nodes with a cycle of 18.61 years. This amounts to about 3' per day. The motion changes the declination of the moon but in tidal work we usually take the moon's declination as constant during a year and then jump it to the next value. This is rather hairy since the change comes to 16° per year. The saving feature is that the terms in which it enters our calculations are not very important.

The fourth kind of month, then, is the nodical month of 27.2122 days which represents a complete revolution with respect to the celestial latitude of the node.

During the 19-year cycle or the regression of the node the moon's maximum declination varies from $23^{\circ}27' + 5^{\circ}09' = 28^{\circ}36'$ to $23^{\circ}27' - 5^{\circ}09' = 18^{\circ}18'$ which has quite an effect on the tides.

Another long-period motion which produces yet another kind of month is the eastward motion of the line of apsides with a period of 8.85 years. This gives us the anomalistic month of 27.5546 days measured with respect to the moon's return to perigee.

Every last bleeding one of these months is important for tides. For example, take the anomalistic month. During this month the moon goes through one complete cycle of distances from the earth. Since the ellipticity of its orbit is 0.055 this means that the variation experienced is 11% of the mean distance and, since the distance enters the tide producing forces in cube, a variation in force of the order of 30% is to be expected with a period of one anomalistic month.

The speed of the moon along its orbit follows Kepler's Law and, since its eccentricity is greater, so also are the variations. In addition, the orbit itself changes shape because of the gravitational attraction of the sun. The lunar day varies from the average by as much as 15 minutes. The last time I noticed (which was a long time ago), astronomers had piled up 51 anomalies trying to keep track of the irregularities in the moon's motion. Because of these irregularities tide tables can not be extrapolated over long periods without serious error.

The phases of the moon--new, full, and half--are optical properties and, as such, of no interest in tides. However, they do mark relative positions of the sun and the moon, Fig. 1.04.3-4, page 34, and the positions are important. The moon is in opposition to the sun at full moon and in conjunction with the sun at new moon. At half moon it is in quadrature.

At conjunction and opposition the highest or spring tides occur. At quadrature the tidal range is at its lowest and neaps occur.

This is the period of the synodical month.

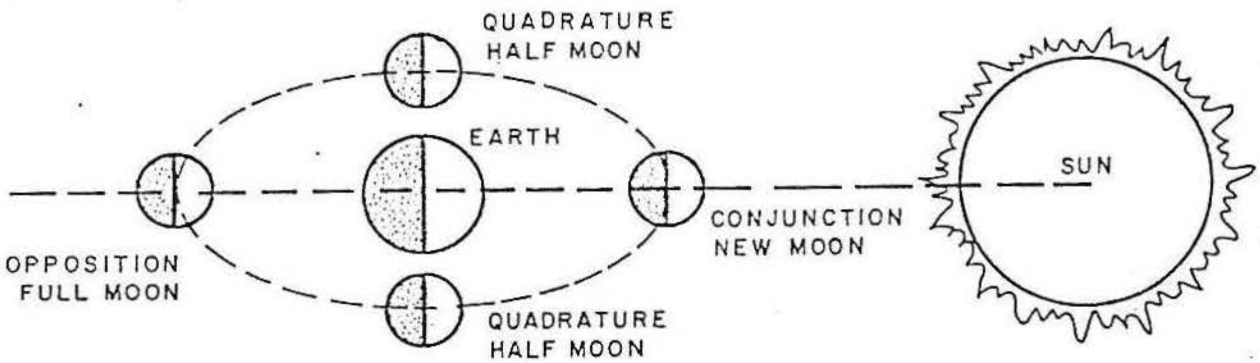


Fig. 1.04.3-4

The moon also rotates on its own axis, the period of the rotation being just equal to the period of revolution so that the same face is always turned toward the earth.

This account barely begins to scratch the surface of the motions of the sun and moon but I think it will be enough for our uses.

1.05. The Tide-Generating Forces.

We have been talking about the tide-generating forces in a very loose way. Our next job is to be more precise about them so that we can write their equations in terms of the astronomical parameters we have just discussed. The mathematical equations for the tide-generating forces due to the sun and moon express the combined effect of forces arising from:

(1) the rotation of the earth,

(2) the revolution of the moon about the earth in an orbit inclined to the earth's equator, and

(3) the motion of the earth about the sun which is also inclined to the equator.

The first step is to account for the "double bulge," Fig. 1.05-1.

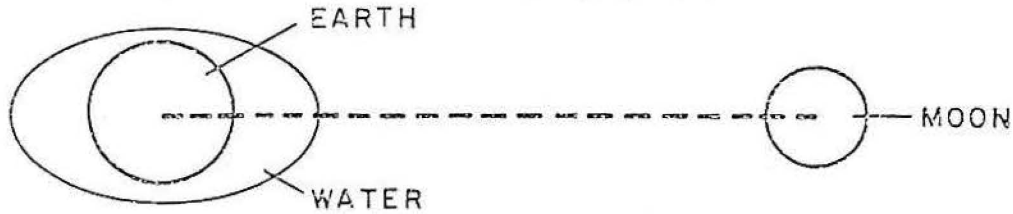


Fig. 1.05-1

We have noted that it seems reasonable that there should be a bulge in the hydrosphere in the direction of a distant attractive force. What seems unreasonable at first blush is that there should be a second bulge away from the direction of the attractive force. Unreasonable as it may seem, the double bulge is there. The "unreason" comes from taking too parochial a view of the earth-moon system.

We have said that the moon revolves about the earth in its orbit and that this revolution developes a centrifugal force which balances the gravitational attraction so that the earth and moon maintain their distance instead of falling into each other. The statement is not strictly accurate. The moon does not orbit the earth as a center. Both earth and moon, considered as a system, rotate about their common center of gravity.

Consider the earth-moon system alone for a start. Since the mass of the earth is roughly 80 times that of the moon, the center of gravity of the system lies within the earth and the period of rotation about this center is one lunar month. Simplifying considerably, the distance between centers is 237,000 miles and the position of the center of gravity can be calculated from the Law of Levers, Fig. 1.05-2.

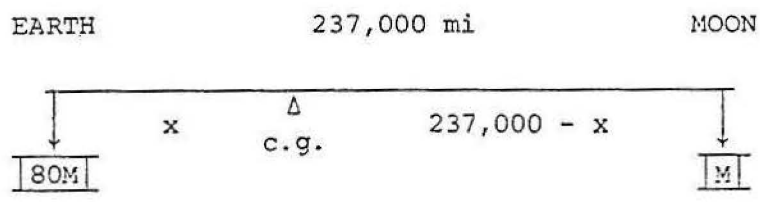


Fig. 1.05-2

$$80Mx = M(237{,}000 - x)$$

$$79x = 237{,}000$$

Thus, the distance of the common center of gravity from the earth's center is

$$x = 3000 \text{ mi} \quad .$$

The radius of the earth is approximately 3850 miles.

Simplify the situation further by stopping the diurnal rotation of the earth. This means stop all rotation of the earth. The motion under consideration is not that of a rigid rotating dumbbell. Consider Fig. 1.05-3.

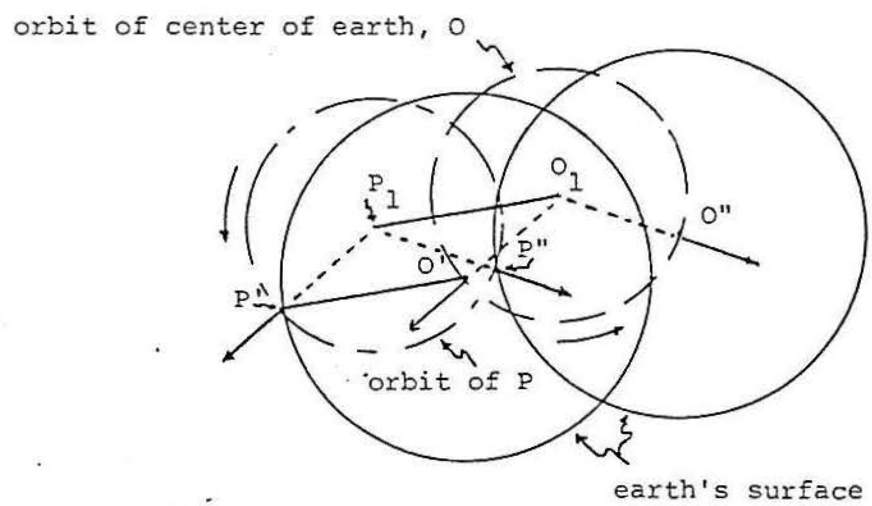


Fig. 1.05-3

Let O_1 be the common center of gravity and O' and O" be the positions of the earth's center O at times t' and t" during a rotation of the system. Let P be a point on the earth's surface and P' and P" its positions at times t' and t". Then P' and P" lie on a circle with center at P_1 which is the orbit described by P. It will have the same radius as the orbit described by the center of the earth, O. Thus, the radius vector of P will be parallel to the radius vector of O.

Conclusion: P_1P' is parallel to O_1O' and P_1P'' is parallel to O_1O''.

Warning: Valid only for no diurnal rotation.

Consequence: The centrifugal force vectors at P and O will be equal and their magnitude will be

$$4\pi^2 r_1/T^2 \quad \text{time}^{-1} \text{ mass}^{-1}$$

where r_1 ≡ the radius of the orbit of the earth's center O

and T ≡ the period of the motion of the system.

Clearly, the total centrifugal force acting on the earth must be balanced by the total attractional force exerted by the moon. However, they need not be, and are not, everywhere in balance in detail. There are points of earth where the attraction is stronger than the mean value of the attractive force so that the attractive force is greater than the centrifugal force. At points where the attraction is weaker than the mean attractive force the reverse holds.

We define the tide-generating force at any point P of earth as the difference between the attractive force at P, A_P, and that at the center of the earth, A_E, where attractive force and centrifugal force are exactly in balance. Note that this exact balance is implicit in our calculation for the system c.g.. If it were not true, it would make no sense to treat the earth an moon as point masses located at their centers.

As we know from Sir Isaac, the attractive force between two mass particles varies directly as the product of their masses and inversely as the square of the distance between them. The force is exerted along their line of centers. In Fig. 1.05-4 let O and C be the centers of the earth and moon.

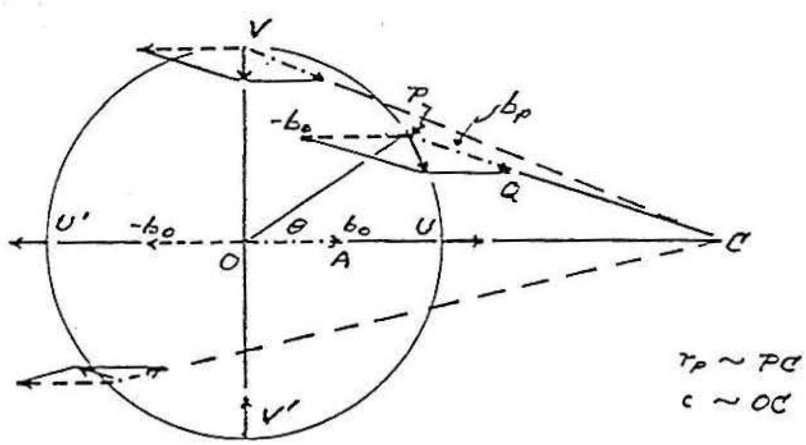


Fig. 1.05-4

Remark: Use mean positions of the earth and moon. Or simplify in any way you want to make the motions circular and uniform.

Let UVU'V' ≡ the great circle cut by a plane through OC.

a ≡ the radius of the earth.

c ≡ the distance from the earth to the moon. $c \simeq 60.26a$.

r_P ≡ the distance of a point of earth P from the moon, PC.

M ≡ the mass of the moon.

E ≡ the mass of the earth. $E = 81.53M$.

g_o ≡ the mean gravitational acceleration. $g_o = 980.6$ cm/sec^2 at 45^o latitude.

There is the usual hassel here over gravity and gravitation but for our purposes

(1.05:1) $$\gamma = g_o(a^2/E) \simeq 6.67\times10^{-8} \text{ dynes cm}^2/\text{grams}^2$$

is satisfactory. Thus, the attractive force of the moon acting on a particle of <u>unit mass located at P</u> is

(1.05:2.1) $$A_P = \gamma(M/r_P{}^2) = g_o(M/E)(a^2/r_P{}^2)$$

directed along PC and for <u>unit mass located at O</u> is

(1.05:2.2) $$A_E = \gamma(M/c^2) = g_o(M/E)(a^2/c^2)$$

directed along OC.

Remark: This is something of a replay of our discussion in section 1.03.

By definition, the tide-generating force is the difference between vector PQ with magnitude given by (1.05:2.1) and vector OA with magnitude from (1.05:2.2). Simple vector subtraction gives the tide-generating force. The subtraction is particularly easy when P is at U or U' where the vectors are in line. At U,

$$v_P/a = v_U/a = 59.26$$

and at U',

$$v_P/a = v_{U'}/a = 61.26 \quad .$$

If you slug reasonable values into M/E and a/c, the tide-generating force at U works out to

$$1.5\times10^{-7}g_o \text{ grams } \underline{\text{toward}} \text{ the moon}$$

and at U' to

$$1.10\times10^{-7}g_o \text{ grams } \underline{\text{away from}} \text{ the moon.}$$

Remark: Grams force--obviously.

You can get an approximation for the tide-generating force at V by multiplying A_E by a/c, i.e., effectively, by the cotangent of angle OVC which assumes that the component of the attractive force at V parallel to A_E equals A_E. One gets at V and V' 0.56 $10^{-7}g_o$ grams.

For U $\leqq$ P $\leqq$ V the size of the tide-producing force is intermediate to the values at U and V. Similarly for the other arcs. At U and V the force acts vertically and there is no horizontal component. In between the force will always have nonzero components in the directions of the local vertical and horizontal. It is the horizontal component which produces the lateral motion which makes the tide. The vertical component is just one more modification of earth's gravitation; at most of the order of one ten-millionth of g_o ($10^{-7}g_o$) which is of no practical interest.

The horizontal component of the tide-producing force has been named the tractive force by Doodson. It is zero at U, U', V, and V'. On the hemisphere VUV' it lies in the earth's surface and points toward the moon. The tractive forces at all points on the earth's surface lying in any plane perpendicular to OC are equal in magnitude though not in direction, Fig. 1.05-5.

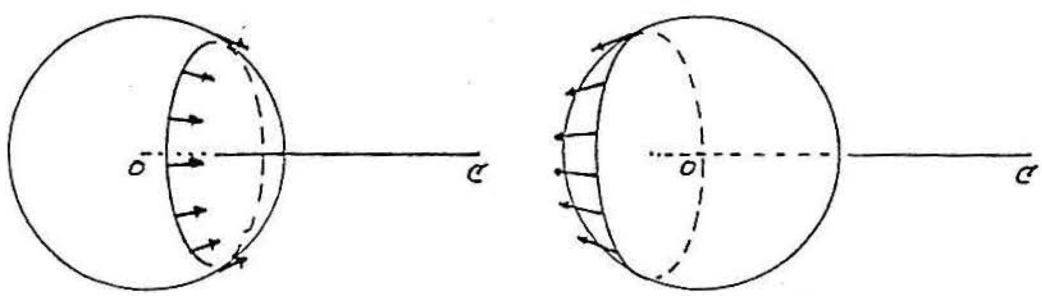


Fig. 1.05-5

Since the tractive force is zero at U, V, and V' it must have some intermediate maximum. The same is true for the other hemisphere except that the tractive force is away from the moon. Figure 1.05-6, page 40, attempts to suggest the distribution of tractive forces on the sphere rather than in section as in Fig. 1.05-4.

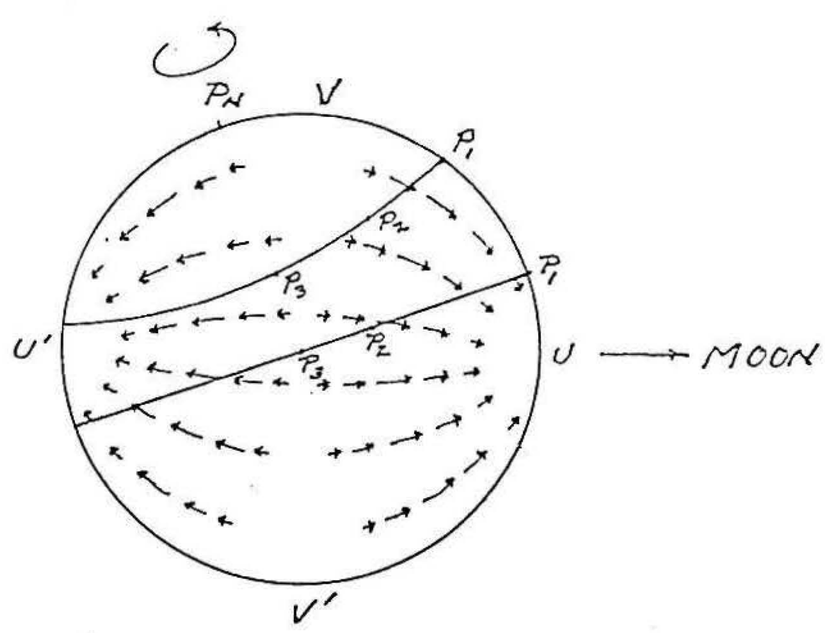


Fig. 1.05-6

So far we have not permitted our earth its diurnal rotation. If now we permit rotation, the tractive forces at any point P will generally change continuously with time. Let P_N be the north pole. Notice that it doesn't coincide with V. Let P_1, P_2, P_3 be a parallel of latitude. The earth's rotation will move any point through the tractive field along a parallel of latitude. This gives you the variation in tractive force experienced by any position. In Fig. 1.05-6 it increases from P_1 to maximum at P_2 and then decreases to P_3. For a point at P_3 the moon is setting. After moonset the force reverses its direction until moonrise.

This whole argument goes through for the earth-sun considered as an isolated system. The chief difference is that the common center of gravity is interior to the sun rather than to the earth.

Having disposed of the "double bulge" we can get along to expressing the tractive forces in terms of astronomical parameters. We will want to resolve our forces into local horizontal and vertical components, Fig. 1.05-7. From section 1.03 we have

(1.05:2.1) $$A_P = (\gamma C)/(e^2 + r^2 - 2er[\cos\theta])$$

and

(1.05:2.2) $$A_E = (\gamma C)/e^2$$

where A_P is in the approximate form secured by deleting the factor $\cos\phi$, see page 19. Their vertical and horizontal components are:

(1.05:3.1, 2)

$$A_P(V) = \frac{\gamma C\cos[\theta + \phi]}{e^2 + r^2 - 2er[\cos\theta]} \quad ; \quad A_P(H) = \frac{\gamma C\sin[\theta + \phi]}{e^2 + r^2 - 2er[\cos\theta]}$$

(1.05:3.3, 4)

$$A_E(V) = \frac{\gamma C\cos[\theta]}{e^2} \quad ; \quad A_E(H) = \frac{\gamma C\sin[\theta]}{e^2} \quad .$$

Their difference is the tide-generating force and we will use the notation

$$F_V \equiv A_P(V) - A_E(V)$$

for the vertical component and

$$F_H \equiv A_P(H) - A_E(H)$$

for the horizontal component or tractive force. From equations (1.05:3.i)

(1.05:4)
$$F_V = \frac{\gamma C}{e^2}\left(\frac{\cos[\theta + \phi]}{1 - 2(r/e)\cos[\theta] + (r/e)^2} - \cos[\theta]\right)$$

and

(1.05:5)
$$F_H = \frac{\gamma C}{e^2}\left(\frac{\sin[\theta + \phi]}{1 - 2(r/e)\cos[\theta] + (r/e)^2} - \sin[\theta]\right) \quad .$$

It will be handy if we eliminate ϕ from equations (1.05:4) and (1.05:5) and express F_V and F_H as functions of r, e, and θ. By the Law of Sines we have

$$\frac{\sin[\phi]}{r} = \frac{\sin[\theta]}{p}$$

and by the Law of Cosines

$$p = \sqrt{e^2 + r^2 - 2er[\cos\theta]}$$

so that

$$\sin[\phi] = \frac{r\sin[\theta]}{\{e^2 + r^2 - 2er[\cos\theta]\}^{\frac{1}{2}}}$$

and

$$\cos[\phi] = \{1 - \sin^2\phi\}^{\frac{1}{2}}$$

or

$$\cos[\phi] = \left(1 - (r/e)^2\frac{\sin^2\theta}{1 - 2(r/e)\cos[\theta] + (r/e)^2}\right)$$

or

$$\cos[\phi] = \frac{1 - (r/e)\cos[\theta]}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{\frac{1}{2}}} \quad .$$

Using the angle sum formulae from trigonometry

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta + \phi) = \frac{\cos\theta[1 - (r/e)\cos\theta]}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{\frac{1}{2}}} - \frac{\sin\theta[(r/e)\sin\theta]}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{\frac{1}{2}}}$$

$$= \frac{\cos\theta - (r/e)}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{\frac{1}{2}}}$$

and

$$\sin(\theta + \phi) = \frac{\sin\theta}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{\frac{1}{2}}}$$

Feeding all this back to equations (1.05:4) and (1.05:5) we get

$$(1.05:6) \qquad F_V = \frac{\gamma C}{e^2}\left(\frac{\cos\theta - (r/e)}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{3/2}} - \cos[\theta]\right)$$

$$(1.05:7) \qquad F_H = \frac{\gamma C}{e^2}\left(\frac{\sin\theta}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{3/2}} - \sin[\theta]\right)$$

No approximations have been made so far from equations (1.05:2.i). In (1.05:6) and (1.05:7) things are pretty messy. To get a useful result we have to begin throwing stuff away. It is the denominator that gives the trouble. Consider

$$1/\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{3/2} \equiv 1/(1 - x)^{3/2} \quad .$$

This can be expanded in a Maclaurin series which converges in the region $x^2 < 1$.

$$(1 - x)^{-3/2} = 1 + (3/2)x + (15/8)x^2 + (105/48)x^3 + \ldots \quad .$$

We have shown that the maximum value of r/e is 1/60. Thus, the maximum value of

$$x \equiv 2(r/e)\cos[\theta] - (r/e)^2$$

must be not greater than

$$2(1/60)(1) + (1/60)^2 \quad .$$

x^2 is certainly less than 1 and the expansion converges. Explicitly, in terms of the harmonic functions when the higher order terms have been suppressed

$$(1.05{:}8)\qquad 1/\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{3/2}$$
$$= 1 + 3(r/e)\cos\theta + (3/2)(r/e)^2(5\cos\theta - 1)$$
$$+ (5/2)(r/e)^3(7\cos^3\theta - 3\cos\theta) \quad .$$

The neglected terms introduce an error which is, at worst, less than 2×10^{-6}.

Substituting (1.05:8) in (1.05:6) and (1.05:7) and neglecting terms containing $(r/e)^4$ and higher powers, we have

$$(1.05{:}9)\qquad F_V = \frac{\gamma C}{e^2}\left((r/e)(3\cos^2\theta - 1) + (3/2)(r/e)^2(5\cos^3\theta - 3\cos\theta) + (1/2)(r/e)^3(35\cos^4\theta - 30\cos^2\theta + 3)\right)$$

and

$$(1.05{:}10)\qquad F_H = \frac{\gamma C}{e^2}\left((r/e)(3\sin\theta\cos\theta) + (3/2)(r/e)^2(\sin\theta)(5\cos^2\theta - 1) + (5/2)(r/e)^3(\sin\theta)(7\cos^3\theta - 3\cos\theta)\right) \quad .$$

If the term in $(r/e)^3$ were neglected, the maximum error that could be introduced in F_V would be less than 6×10^{-4} and in F_H less than 10^{-3} which differ only by about one order of magnitude. These are small errors so we will throw out the terms in $(r/e)^3$. Thus, after factoring out an (r/e), (1.05:9) and (1.05:10) take the forms

$$(1.05{:}11)\qquad F_V = \frac{\gamma Cr}{e^2}\left((3\cos^2\theta - 1) + (3/2)(r/e)(5\cos^3\theta - 3\cos\theta)\right)$$

and

$$(1.05{:}12)\qquad F_H = \frac{\gamma Cr}{e^2}\left(3\sin\theta\cos\theta + (3/2)(r/e)(\sin\theta)(5\cos^2\theta - 1)\right)$$

Some books on tides toss out the $(r/e)^2$-terms as well. However, for the moon it is sometimes interesting to retain the $(r/e)^2$-terms as we have done in equations (1.05:11) and (1.05:12). For the moon

$$r/e = 1/60 = 1.\dot{6}\times10^{-2} \quad .$$

For the sun

$$r/e = 4.26\times10^{-5} \quad .$$

so that we will never retain the $(r/e)^2$-terms in dealing with the sun.

So far we have been drawing our picture in two dimensions. We must now extend it to three, Fig. 1.05-8, page 44. F_V lies along the line EP extended. F_H lies in the intersection of the plane PEC with the plane

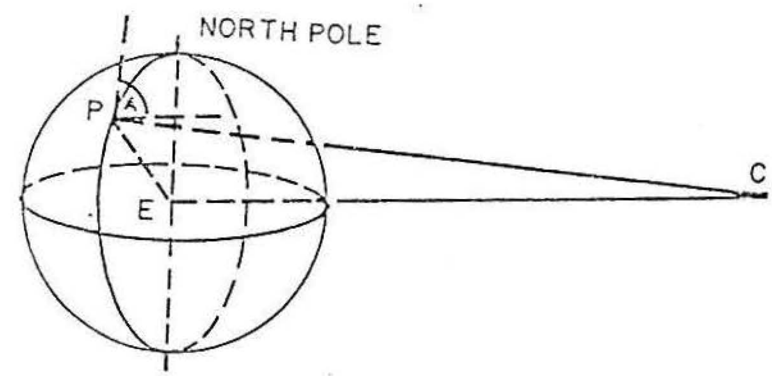


Fig. 1.05-8

tangent to the earth at P. To specify this direction we use the azimuth A which is the angle between the line of intersection of the meridian plane and the plane PEC with the plane tangent at P. A is independent of the height at which C appears in the sky. By convention, A is measured from the north in a clockwise direction. This conforms to British naval usage. To louse things up the U.S.C.&G.S. and the U.S. Navy Oceanographic Office view the angle from the south.

To resolve F_H into an east component, F_E, and a north component, F_N all that is required is to multiply F_H by a sine or cosine of A. Thus, equations (1.05:13) and (1.05:14) replace equation (1.05:12):

(1.05:13) $$F_E = F_H \sin A$$

(1.05:14) $$F_N = F_H \cos A \quad .$$

For a celestial body which appear due east $A = \frac{1}{2}\pi$ and $F_H = F_E$ while $F_N = 0$.

The variables θ and A are functions of time and of position on the earth. The distance between centers, e, is a function of time alone and the radius of earth, r, is a function of position. So far as r is concerned

$$\frac{r_E - r_P}{\bar{r}} = 0.0034$$

so that the difference between the equatorial and polar radii is about one-third of one percent of the mean radius. Consequently, for tidal work we use r constant equal to $\bar{r}$. This is not absolutely necessary. We could carry a variable r but nobody ever seems to do it.

For comparison, we can have

$$\frac{e_{max} - e_{min}}{\bar{e}} = 1/32 = 0.031 \simeq 3\% \quad .$$

We have considered the motions of the sun and moon in a qualitative way and stated some quantitative results. The exact formulation and derivation of the quantitative results belongs to celestial mechanics, a subject into which we can't go in this course. We will simply assume that the astronomers know their business, accept their results, and try to convert them to forms useful for tides.

We must first express θ and A in both terrestrial and celestial coordinate frames. We will use the symbols

L $\equiv$ the latitude of a point P on the earth.

D $\equiv$ the declination of the celestial body.

h $\equiv$ the hour angle of the celestial body from P used without regard to direction. (See Fig. 1.05-9.)

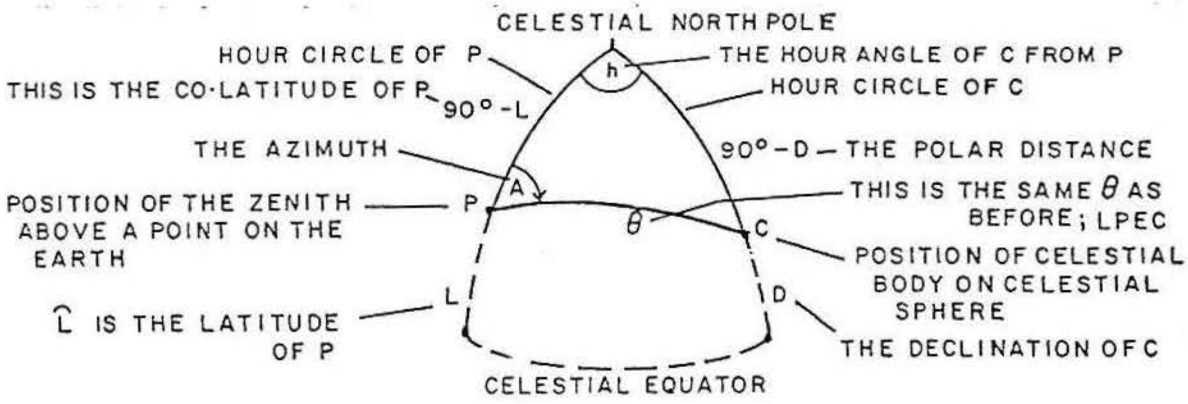


Fig. 1.05-9

One has, clearly, a spherical triangle which has 3 of its sides given by 90° - L, 90° - D, and θ and 2 of its angles by h and A. These parts are related by spherical trigonometry. In general we have

$$\cos x = \tfrac{1}{2}(1 + \cos[2x])$$

$$\sin x = \tfrac{1}{2}(1 - \cos[2x]) \qquad \{\text{multiple angle formulae}\}$$

$$\sin[x]\cos[x] = \tfrac{1}{2}\sin[2x]$$

and

$$\cos x = \tfrac{1}{4}(\cos[3x] - 3\cos[x]) \quad .$$

We want to get θ in terms of L, D, and h, i.e., we want the distance between the zenith of a point of observation and a given celestial body, θ, in terms of the latitude of the point, L, the declination of the celestial body, D, and the angle between their hour circles, h. We assume that the latitude of any point on the earth is known and that the declinations of the celestial bodies have been determined by celestial mechanics and published in tables.

From spherical trigonometry we have

(1.05:15.1) $$\cos\theta = (\sin[L]\sin[D]) + (\cos[L]\cos[D])\cos[h]$$

and

(1.05:15.2) $$\sin[D] = \sin[L]\cos[\theta] + \cos[L]\sin[\theta]\cos[A] \quad .$$

The first of these equations gives the required relation for θ. It will also be useful to have similar expressions for $\cos^2\theta$ and $\cos^3\theta$. Solving (1.05:15.2) for $\sin[\theta]\cos[A]$,

$$\sin[\theta]\cos[A] = \frac{\sin[D] - \sin[L]\cos[\theta]}{\cos[L]}$$

and substituting for $\cos[\theta]$ from (1.05:15.1)

$$\sin[\theta]\cos[A] = \frac{\sin[D] - \sin^2[L]\sin[D] - \sin[L]\cos[L]\cos[D]\cos[h]}{\cos[L]}$$
$$= \frac{\cancel{\sin[D]} - \cancel{\sin[D]} + \cos^2[L]\sin[D] - \sin[L]\cos[L]\cos[D]\cos[h]}{\cos[L]}$$

so that

(1.05:16) $$\sin[\theta]\cos[A] = \cos[L]\sin[D] - \sin[L]\cos[D]\cos[h] \quad .$$

Again, from spherical trigonometry

$$\frac{\sin[A]}{\sin[90 - D]} = \frac{\sin[h]}{\sin[\theta]}$$

or, since $\sin[90 - D] = \cos[D]$,

$$\frac{\sin[A]}{\cos[D]} = \frac{\sin[h]}{\sin[\theta]} \quad .$$

Therefore,

(1.05:17) $$\sin[\theta]\sin[A] = \cos[D]\sin[h] \quad .$$

Squaring (1.05:15.1) we get

$$\cos^2[\theta] = \sin^2[L]\sin^2[D] + 2\sin[L]\cos[L]\sin[D]\cos[D]\cos[h] + \cos^2[L]\cos^2[D]\cos^2[h]$$

and using the double angle relation for $\cos^2[h]$ this can be rearranged as

$$\cos^2[\theta] = \sin^2[L]\sin^2[D] + \cos^2[L]\cos^2[D]\{\tfrac{1}{2}(1 + \cos[2h])\}$$
$$+ 2\sin[L]\cos[L]\sin[D]\cos[D]\cos[h]$$
$$= \{\sin^2[L]\sin^2[D] + \tfrac{1}{2}\cos^2[L]\cos^2[D]\}$$
$$+ 2\{\sin[L]\cos[L]\}\{\sin[D]\cos[D]\}\cos[h]$$
$$+ \tfrac{1}{2}\cos^2[L]\cos^2[D]\cos[2h] \quad .$$

Therefore,

(1.05:18) $$\cos^2[\theta] = \{\sin^2[L]\sin^2[D] + \tfrac{1}{2}\cos^2[L]\cos^2[D]\}$$
$$+ \tfrac{1}{2}\{\sin[2L]\sin[2D]\}\cos[h]$$
$$+ \tfrac{1}{2}\{\cos^2[L]\cos^2[D]\}\cos[2h] \quad .$$

This is the expression for $\cos^2[\theta]$.

To get the expression for $\cos^3[\theta]$ begin with equation (1.05:15.1) and cube it.

$$\cos^3[\theta] = (\sin[L]\sin[D])^3 + 3(\sin[L]\sin[D])^2(\cos[L]\cos[D])\cos[h]$$
$$+ 3(\sin[L]\sin[D])(\cos[L]\cos[D])^2\cos^2[h]$$
$$+ (\cos[L]\cos[D])^3\cos^3[h] \quad .$$

For convenience let

$(\sin[L]\sin[D])^3 \equiv A$; $(\sin[L]\sin[D])^2(\cos[L]\cos[D]) \equiv B$

$(\sin[L]\sin[D])(\cos[L]\cos[D])^2 \equiv C$; $(\cos[L]\cos[D])^3 \equiv E$

From trigonometry

$$\cos\ [h] = \tfrac{1}{2}(1 + \cos[2h])$$

and

$$\cos\ [h] = \tfrac{1}{4}(3\cos[h] + \cos[3h]) \quad .$$

Therefore,

$$\cos^3[\theta] = A + 3B\cos[h] + (3/2)C(1 + \cos[2h])$$
$$+ \tfrac{1}{2}E(3\cos[h] + \cos[3h])$$
$$= \{A + (3/2)C\} + \{3B + (3/4)E\}\cos[h] + \{(3/2)C\}\cos[2h]$$
$$+\{\tfrac{1}{4}E\}\cos[3h]$$

where

$$A + (3/2)C = \tfrac{1}{2}\{2\sin^3[L]\sin^3[D] + 3\sin[L]\sin[D]\cos^2[L]\cos^2[D]$$
$$3B + (3/4)E = \tfrac{1}{4}\{12\sin^2[L]\sin^2[D]\cos[L]\cos[D] + 3\cos^3[L]\cos^3[D]\}$$
$$(3/2)C = (3/2)\{\sin[L]\sin[D]\cos^2[L]\cos^2[D]\}$$

and

$$\tfrac{1}{4}E = \tfrac{1}{4}\{\cos^3[L]\cos^3[D]\} \quad .$$

Therefore,

(1.05:19) $$\cos^3[\theta] = \tfrac{1}{4}\{2\sin^3[L]\sin^3[D] + 3\sin[L]\sin[D]\cos^2[L]\cos^2[D]\}$$
$$+ \tfrac{1}{4}\{12\sin^2[L]\sin^2[D]\cos[L]\cos[D] + 3\cos^3[L]\cos^3[D]\}\cos[h]$$
$$+ (3/2)\{\sin[L]\sin[D]\cos^2[L]\cos^2[D]\}\cos[2h]$$
$$+ \tfrac{1}{4}\{\cos^3[L]\cos^3[D]\}\cos[3h] \quad .$$

Equations (1.05:15), (1.05:18), and (1.05:19) give us expressions for powers of the cosines of θ in terms of L, D, and h. For any given point of earth, P, its latitude, L, is constant. The declination of the sun or moon is a slowly varying function of time which can be treated as constant for periods of useful length. The hour angle, h, is "the variable" and appears in multiples, i.e., $\cos[\theta] = f_1(h)$, $\cos^2[\theta] = f_2(h, 2h)$, and $\cos^3[\theta] = f_3(h, 2h, 3h)$. Since h makes a complete cycle in one day, 2h in half a day, ..., the probable advantage of this form of expression for $\cos^n[\theta]$ should be readily appreciated.

In equation (1.05:11) we have an expression for the vertical tide-generating force, F_V, in terms of the powers of $\cos[\theta]$. Substituting (1.05:15), (1.05:18), and (1.05:19) in (1.05:11) gives

(1.05:20)

$$F_V = \frac{3\gamma Cr}{2e^3}\Big(\{2\sin^2[L]\sin^2[D] + \cos^2[L]\cos^2[D] - (2/3)\}$$
$$+ \{\sin[2L]\sin[2D]\}\cos[h] + \{\cos^2[L]\cos^2[D]\}\cos[2h]\Big)$$
$$+ \frac{3\gamma Cr}{2e^3}\frac{r}{e}\Big(\{5(\sin[L]\sin[D])(\sin^2[L]\sin^2[D] + (3/2)\cos^2[L]\cos^2[D])$$
$$- 3(\sin[L]\sin[D])\}$$
$$+ \{15(\cos[L]\cos[D])(\sin^2[L]\sin^2[D] + \tfrac{1}{4}\cos^2[L]\cos^2[D])$$
$$- 3(\cos[L]\cos[D])\}\cos[h]$$
$$+ \{(15/2)(\sin[L]\sin[D])(\cos^2[L]\cos^2[D])\}\cos[2h]$$
$$+ \{(5/4)(\cos[L]\cos[D])(\cos^2[L]\cos^2[D])\}\cos[3h]\Big) \quad .$$

A similar expression holds for F_H.

A bit of contemplation shows that the second term, lines 3 to 8 above, is small. The quantity enclosed in the large parentheses is always rather small and, further, it is multiplied by r/e which is, at most, 1/60. The entire second term may be neglected in comparison with the first. We are impelled to this step by the messiness of the expression.

The easiest place to suppress the powers of r/e that you want to neglect is in equations (1.05:11), (1.05:13) and (1.05:14). If you do, only the first term remains on the right-hand side of equation (1.05:20). For the sun $r/e = 4.26\times10^{-5}$ and the error introduced is really negligible. In the case of the moon we had better have a closer look. What are the maximum values that can result in equation (1.05:11)? For the first term in (1.05:11)

$$(3\cos^2[\theta] - 1)_{max} = 2$$

and for the second

$$(3/2)(r/e)(5\cos^3[\theta] - 3\cos[\theta])_{extreme} = \pm 0.05 \quad .$$

Thus, neglect of the (r/e)-term could amount to

$$\pm 0.05/2 = \pm 5/200 = \pm 2.5\% \text{ of the retained term.}$$

For a 10-foot tide the error introduced is of the order of a quarter of a foot which is hardly serious for practical work.

To get an estimate for (1.05:13) and (1.05:14) we have to consider (1.05:12). The first term of (1.05:12) will exhibit extrema for

$$d(2\sin[\theta]\cos[\theta])/d\theta = 0 \quad .$$

The maxima occur at $\theta = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4,\ \ldots$ and they have the value 1.5. In the same way we compute

$$\frac{d\{(3/2)(r/e)\sin[\theta](5\cos\ [\theta] - 1)\}}{d\theta} = 0 \quad .$$

The resulting extrema from this term are $\pm$ 0.0344 which introduces an error on neglect of r/e of

$$\pm 0.0334/1.5 = \pm 0.0688/3 = \pm 0.023 = \pm 2.3\% \quad .$$

Thus, errors introduced in the horizontal force by the neglect of r/e are of the same order as those introduced in the vertical force and, even in the case of the moon, are of no great size.

Our approximation for the tide-producing force is now:

(1.05:21) $$F_V = \frac{3\gamma Cr}{2e^3}\Big(\{2\sin^2[L]\sin^2[D] + \cos^2[L]\cos^2[D] - (2/3)\} + \{\sin[2L]\sin[2D]\}\cos[h] + \{\cos^2[L]\cos^2[D]\}\cos[2h]\Big)$$

(1.05:22) $$F_E = \frac{3\gamma Cr}{2e^3}\Big(\{\sin[L]\sin[2D]\}\sin[h] + \{\cos[L]\cos^2[D]\}\sin[2h]\Big)$$

(1.05:23) $$F_N = \frac{3\gamma Cr}{2e^3}\Big(\{\sin[2L](\sin^2[D] - \tfrac{1}{2}\cos^2[D])\} + \{\cos[2L]\sin[2D]\}\cos[h] - \{\tfrac{1}{2}\sin[2L]\cos^2[D]\}\cos[2h]\Big)$$

As suggested before, the grouping of these forces into terms according to multiples of h is very useful for describing tides. Consider the changes in L, D, and h. For any particular point of earth the latitude, L, is constant. For the sun the declination, D, changes about 15.5' per day. For the moon the declination, D, exhibits changes ranging from 4°12' per day to 2°41' per day. These are relatively slow changes compared with h, the hour angle, which goes through 360° in a day and 2h which goes through 720° in a day. These changes in h and 2h are not slow. Thus, the terms not involving h vary slowly and contribute the long period components to the force. For the sun, D runs through one cycle in a year while, for the moon, D runs through one cycle in a month. The terms involving only L and D are called long period terms. The cos[h] term runs through one cycle in a day and is called the diurnal term. A term containing cos[2h] completes two cycles per day and is called the semidiurnal term. Semidiurnal terms are prominent in the tides of the Atlantic which show two high waters per day.

Should any further indication of the unimportance of the (r/e)-terms which we have neglected be needed, the fact that no place in the world has three high waters per day would offer confirmation. Should such a triple tide ever be found it would correspond to the cos[3h] term which has been suppressed by the approximation.

Let us consider now the coefficient common to all three force components called, naturally enough, the common coefficient:

$$\frac{3\gamma Cr}{2e^3}$$

Unfortunately, e is a variable and must be retained as such. The use of $\bar{e}$ or any other single representative value is inadequate. For the moon, e cycles in a month and, for the sun, in a year.

For practical work we use the common coefficient in another, approximate form. The apparent gravity is given by

$$g = \gamma E/r^2$$

so that

$$\gamma = gr^2/E$$

and we may write

$$(3\gamma Cr)/(2e^3) = (3/2)(C/E)(g)(r/e)^3$$

Consider, for example, the vertical tide-generating force as given by (1.05:21).

$$F_V/g = (3/2)(C/E)(r/e)^3\{\text{terms in L, D, and h}\} \quad .$$

Both g and r are variable. How big is the error introduced by taking g and r constant? At the equator g = 978 gal. At the pole g = 983 gal. Working from a mean value for g, $g = \overline{g}(1 \pm 0.0026)$. If we take r constant in a similar way, $r^3 = \overline{r}^3(1 \pm 0.0051)$. Further, r and g have countervailing effects at pole and equator. Thus, the error introduced by using constant g and r is about 0.25%. It would be possible to retain the variation in g but this never seems to be done.

Let d ≡ the mean distance of a celestial body.

e ≡ the true, variable distance of the celestial body.

Then we can write the common coefficient as

$$\{(3/2)(g)(C/E)(r/d)^3\}(d/e)^3 \quad .$$

In this expression only d/e remains variable.

In tidal work it is customary to define a constant factor by

$$U \equiv (C/E)(r/d)^3$$

which has the numerical values

$$\text{Sun:} \quad U_S = 2.569\times10^{-8}$$
$$\text{Moon:} \quad U_M = 5.582\times10^{-8} \quad .$$
$$U_S/U_M = 0.4602$$

is the ratio of the solar to the lunar tide-generating forces.

We have, then, as final approximate forms for F_V, F_E, and F_N equations (1.05:24), (1.05:25), and (1.05:26) shown on page 52.

The only remaining step is to introduce numerical values for the parameters U, d/e, g, L, D, and h. The ultimate objective is to express all the variables as simple functions of mean time in such a way that the rates of change with time will be constant. If this could be done, the components could be fixed once for all.

From a theoretical point of view, Laplace was the first to suggest that the tides could be represented as the sum of small perturbations linearly combined. That is, he pointed out that, if the tide-generating

		common coefficient	geodetic coefficient	celestial factors	
(1.05:24)	$F_V =$	$(3/2)gU$	$(\{(3\sin^2[L] - 1)\}$	$(d/e)^3\{(2/3) - \cos^2[D]\}$	long-period component
			$+ \{\sin[2L]\}$	$(d/e)^3(\sin[2D])\cos[h]$	diurnal component
			$+ \{\cos^2[L]\}$	$(d/e)^3(\cos^2[D])\cos[2h])$	semidiurnal component
(1.05:25)	$F_E =$	$(3/2)gU$	$(\ \{\sin[L]\}$	$(d/e)^3(\sin[2D])\sin[h]$	diurnal component
			$+ \{\cos[L]\}$	$(d/e)^3(\cos^2[D])\sin[2h])$	semidiurnal component
(1.05:26)	$F_N =$	$(3/2)gU$	$(\ \{(3/2)\sin[2L]\}$	$(d/e)^3\{(2/3) - \cos^2[D]\}$	long-period component
			$+ \{\cos[2L]\}$	$(d/e)^3(\sin[2D])\cos[h]$	diurnal component
			$- \{\tfrac{1}{2}\sin[2L]\}$	$(d/e)^3(\cos^2[D])\cos[2h])$	semidiurnal component

forces could be expressed as a sum of cosines then they could be expressed as $\sum A_i \cos[\theta_i + k_i]$. Lord Kelvin was the first to do the practical work. The problem of expressing (1.05:24), (1.05:25), and (1.05:26) as sums of cosine terms is so difficult, tedious, and unproductive of insight that we won't go into it in detail. We will only discuss some aspects of the problem in a general way.

To begin with, let us take the astronomical motions and express them as functions of time. The method is to introduce mean longitudes for the motions of sun and moon and then pile anomalies on them as needed. Perigee and perihelion are used as reference points, Fig. 1.05-10. The rate of motion on the orbits is not uniform but for the sun to a good approximation we can use

$$d\theta/dt = \text{a constant} \quad .$$

The moon is more complicated since its motion is influenced by the sun as well as by the earth. The position of the moon's nodes moves around the ecliptic once in 18.6 years. This position has an important effect on the tide. In practice, in any one year we assign it to its mean position and then jump it ahead for the next year. The movement of the vernal equinox due to precession is even slower and, a fortiori, can be taken constant over a year's time.

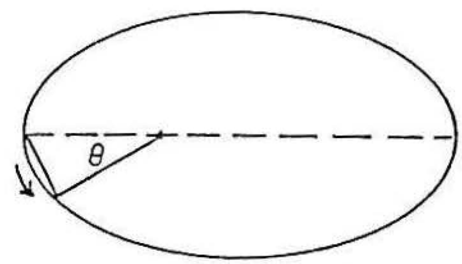


Fig. 1.05-10

In the development of harmonic terms the British and Americans pursue different routes. The U.S.C.&G.S. uses an obsolete method due to Charles Darwin which refers the motion of each celestial body to its own orbit. The British Admiralty method refers all motions to a reference point on the ecliptic. The differences in the results secured by the two methods are negligible even though the methods of attack are radically different. Schureman (1941) presents the U.S.C.&G.S. method in gory detail.

We may sketch the mode of attack as follows:

Let

$X \equiv$ the true longitude measured on the ecliptic from the vernal equinox, ♈.

$\overline{X} \equiv$ the mean longitude.

$d\overline{X}/dt \equiv$ a constant.

$P \equiv$ the longitude of perigee.

$N \equiv$ the longitude of the node.

$R_a \equiv$ the right ascension.

$R_{aG} \equiv$ the right ascension of the meridian of Greenwich.

$L_o \equiv$ the longitude of a point of earth.

As an example, for the moon

(1.05:27) $$(d/e)_M = 1 + 0.055\cos[\overline{X}_M - P_M] + \underbrace{0.010\cos[\overline{X}_M - 2\overline{X}_S + P_M]}_{\text{variational term}} + \underbrace{0.008\cos[2(\overline{X}_M - \overline{X}_S)]}_{\text{advectional term}} + 0.003\cos[2(\overline{X}_M - P_M)] \quad .$$

The advectional term covers the effect of the sun on the moon's motion. For $(d/e)_S$ there is no corresponding term. The numerical coefficients come from the characteristics of the moon's orbit. To our degree of approximation, all the variables change uniformly with time. (d/e) is a long-period term.

The moon's true longitude is given by

$$X_M = \overline{X}_M + 0.110\sin[\overline{X}_M - P_M] + 0.022\sin[\overline{X}_M - 2\overline{X}_S + P_M] + 0.011\sin[2(\overline{X}_M - \overline{X}_S)] + 0.004\sin[2(\overline{X}_M - P_M)] \quad .$$

The sine of the moon's declination is a function of its right ascension,

$$\sin[D_M] = f(R_{aM}) \quad ,$$

and the moon's hour angle is given by

$$h_M = R_{aM} - (R_{aG} - L_o) \quad .$$

We have mentioned that for the sun the advectional term drops out. In addition, any expression for the moon which involves the longitude of the node, N, will have no counterpart in the comparable expression for the sun.

Harmonic expansions of this sort always force us into a certain amount of approximation to keep them in hand. For example, when $(d/e)^3$ is computed from $(d/e)_M$, 35 cosine terms result. When you begin to feed this and the other required expansions back to get F_V, F_E, and F_N the expressions can run to hundreds of terms without half trying. The number of terms you elect to keep depends strongly on the particular point of earth for which you want to predict. In the Chesapeake Bay you can get by with relatively few. In the Bay of Fundy you have to retain many more.

In approximating, the small angle relationships,

$$\sin[\theta] \approx \theta \qquad \text{and} \qquad \cos[\theta] \approx 1$$

are frequently used.

For no discernable reason it has become conventional in tidal work to use only cosine terms. Whenever a $\sin[\theta]$ appears it is replaced with

$$\sin[\theta] = \cos[\theta - 90] \quad .$$

Any time $\cos^n[\theta]$ appears it can be reduced to expressions in multiple angles by means of the trigonometric identities.

With the procedure of breaking the forces into harmonics has emerged the notion of "astres fictifs." The idea is that you junk the celestial bodies which actually generate the forces and replace them with a whole swarm of imaginary celestial bodies of the proper mass which move uniformly around the earth; one for each harmonic component. The idea is fun in a macabre sort of way but fruitless. It should be a strong warning to you not to go around naively assigning physical interpretations to the components of an harmonic analysis.

For long-period components there is (almost) no change in the hour angle per hour.

For the diurnal constituent the hour angle of the mean sun changes 15° per hour.

For the semidiurnal constituent, the hour angle of the mean sun changes 30° per hour.

For the mean longitude, $\overline{X}$, the longitude of perigee (or perihelion), and the longitude of the node, N, we have the following rates of change:

$$\overline{X}_S \sim (d\overline{X}/dt)_S = 0.041069\ °/\text{hr} \quad ; \quad \overline{X}_M \sim (d\overline{X}/dt)_M = 0.549017\ °/\text{hr}$$

$$P_S \sim (dP/dt)_S = 0.000002\ °/\text{hr} \quad ; \quad P_M \sim (dP/dt)_M = 0.004642\ °/\text{hr}$$

$$N \sim dN/dt = -\ 0.002206\ °/\text{hr}$$

1.06. The Equilibrium Tide.

Up to now we have been primarily concerned with the development of the tide-generating forces. The time has come to look at the response of the water to those forces. The simplest approach is due to Isaac Newton. It is based on a number of quite unrealistic assumptions that reduce the problem from one in dynamics to on in statics. In spite of the unrealistic assumptions the approach still gives useful results. The theory is called the equilibrium theory of tides or, alternatively, the equilibrium tide, the astronomical tide, or the gravitational tide.

The two primary assumptions made in the equilibrium theory are:

(A) The entire earth is covered by water to a uniform depth.

(B) Friction and inertia are negligible.

The first assumption means that any perturbation in phase with a force can be propagated entirely around the earth without interference from land masses. In effect, it throws out the boundary conditions.

Assumption (B) is a little trickier. The neglect of friction (viscosity) is a common one in many approximate solutions to fluid problems. It feels comfortable. We're used to it and it seems reasonable here. But what about the neglect of inertia? Inertia is a property of matter, i.e., mass, as is the very gravitational attraction we assign as the cause of the tide. It seems a bit inconsistent to neglect inertia while retaining gravitational attraction. Careful inspection of the equilibrium theory seems to indicate that the function of this assumption is to provide that there shall be no lag in the response of the water to the tide-generating forces. At every instant the water is in equilibrium. We ask no questions about how it passes from equilibrium at one instant to equilibrium at another instant. In effect, the water is to have no memory at all about its past.

Despite the highly artificial assumptions just made, the equilibrium theory provides a valuable adjunct to tidal theory. For one thing, it offers a reference for actual tidal measurements. For example, at a particular place the measured tide might come to high water one hour after the computed equilibrium tide. It also gives a visualization of the forces of attraction that are at work.

The force components we have been studying--diurnal, semidiurnal, and long-period--are all present in the equilibrium tide; as they are in the actual tide. But the actual tide also includes such things as the effects of changing pressure gradient and other large scale phenomena, wind stress, which can be fairly steady in the Trades, and Coriolis force. However, our initial assumptions are so crude that it is not worth our while to introduce these refinements into the analysis. Our ocean will be homogeneous as well as ubiquitous. Clearly, assumption (B) will rule out Coriolis force.

To assumptions (A) and (B) we now add:

(C) The earth is a sphere.

(D) The tide-generating forces are such that the centripetal acceleration is negligible.

This means that we can write gravity in the simplified form

$$g = \gamma(E/r_E^{\,2})$$

where

$r_E \equiv$ the mean radius of the earth.

We will begin by using potential theory. By its very name the equilibrium theory suggests that the sea surface will be everywhere perpendicular to the resultant of the acting forces. Now an equilibrium surface is also an equipotential surface. The potential of a force is defined by the amount of work required to move a unit mass from the surface on which it is located to a position where the force is zero. Further, the amount of work done in moving the unit mass must be independent of the path along which it is moved.

So the first question before the house is: Where are the positions where the forces are zero? Consider gravity. If the earth were a point mass, then we would have

$$r_E \to 0 \quad , \quad g \to \infty \quad ; \quad r_E \to \infty \quad , \quad g \to 0 \quad .$$

Actually, the earth is <u>not</u> a point mass so that, appealing to Poisson's equation, we have

$$r_E \to 0 \quad , \quad g \to 0$$

instead of the first relation above.

The tide-generating forces as expressed in equations (1.05:6) and (1.05:7) were

$$F_V = \frac{\gamma C}{e^2}\left(\frac{\cos[\theta] - (r/e)}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{3/2}} - \cos[\theta]\right)$$

$$F_H = \frac{C}{e}\left(\frac{\sin[\theta]}{\{1 - 2(r/e)\cos[\theta] + (r/e)^2\}^{3/2}} - \sin[\theta]\right)$$

C is the mass of the celestial body and e is the length of the line of centers. We will take them as constants. F_V and F_H go to zero as r goes to zero. This is the place to which we must move our unit mass to get the potentials.

Let

$\Omega_g \equiv$ the potential of gravity

and

$\Omega \equiv$ the potential of the tide-producing forces .

By definition we have

(1.06:1) $$\Omega_g = \int_r^\infty -g \, dr \quad .$$

Substituting for g

$$\Omega_g = \gamma E \int_r^\infty - \frac{dr}{r^2} = - \gamma E \frac{1}{r} \Big|_r^\infty = \frac{\gamma E}{r} \quad .$$

Therefore,

(1.06:2) $$\Omega_g = \frac{\gamma E}{r} \quad .$$

From equation (1.06:1), clearly,

$$g = - \partial\Omega_g/\partial r \quad .$$

The proper expression for Ω is somewhat less obvious. Ω can be written as a function of r and θ,

$$\Omega = \Omega(r, \theta) \quad ,$$

so that, formally,

$$d\Omega = \frac{\partial\Omega}{\partial r} dr + \frac{\partial\Omega}{\partial\theta} d\theta \quad .$$

Whence

$$\Omega = \int_0^r \frac{\partial\Omega}{\partial r} dr + \int_{\theta_1}^{\theta_2} \frac{\partial\Omega}{\partial\theta} d\theta \quad .$$

At the center of the earth r = 0, Ω = 0 and the angle is immaterial. We can, if we choose, set $\theta_1 = \theta_2$ so that

$$\int_{\theta_1}^{\theta_2} \frac{\partial\Omega}{\partial\theta}\, d\theta = 0 \quad .$$

Then

$$\Omega = \int_0^r \frac{\partial\Omega}{\partial r}\, dr \quad .$$

This says that the effect of moving a unit mass from the surface to the center of the earth along any path is the same. That is, it is independent of the angle θ. For simplicity, take a radial path,

$$\theta = \text{a constant} \quad .$$

The work is done against F_V so that

(1.06:5) $$\Omega = \int_0^r F_V\, dr \quad ; \qquad \theta \text{ constant} \quad .$$

This tells us that

(1.06:6) $$F_V = \frac{\partial\Omega}{\partial r}$$

so that we have

$$\Omega = \frac{\gamma C}{e^3}\left((3\cos^2[\theta] - 1)\int_0^r r\,dr + \frac{3}{2}\,\frac{\gamma C}{e^4}\,(5\cos^3[\theta] - 3\cos[\theta])\int_0^r r^2 dr\right.$$

and, integrating,

(1.06:7) $$\Omega = \frac{3}{2}\,\frac{\gamma C r^2}{e^3}\left((\cos^2[\theta] - \frac{1}{3}) + \frac{1}{3}\,(\frac{r}{e})\,(5\cos^3[\theta] - 3\cos[\theta]\right)$$

gives Ω in terms of the geocentric angle θ. For use this expression would have to be hacked over to right ascension and declination as we did in section 1.05.

We can now write

(1.06:8) $$F_H = \frac{1}{r}\,\frac{\partial\Omega}{\partial\theta} \quad .$$

The condition that the sea surface be an equipotential surface may be written

$$\Omega_g + \Omega = \text{a constant} \quad .$$

To determine the constant consider equation (1.06:7). If it can be shown that Ω becomes zero at any point, then the constant must be the mean value of Ω_g. With this in mind we set equation (1.06:7) equal to zero and solve for θ either graphically or by inspection. Looking at Fig. 1.06-1, page 60,

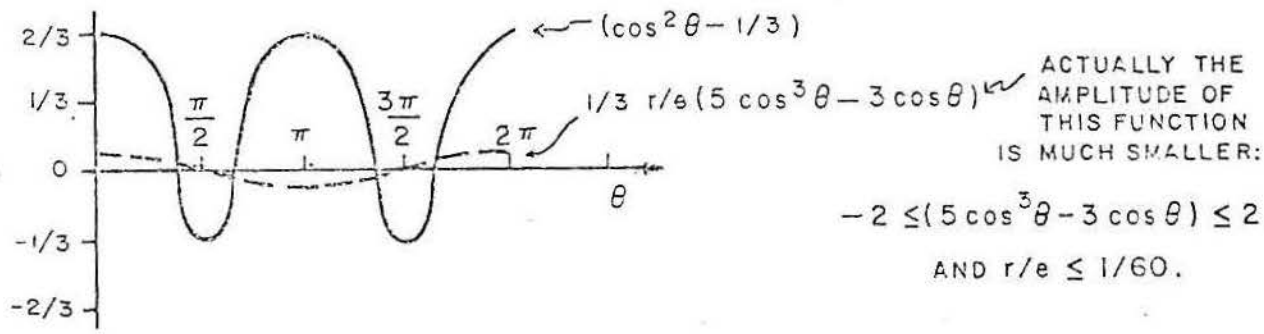


Fig. 1.06-1

it is obvious that there is at least one θ for which $\Omega = 0$. For this θ the condition becomes Ω_g = a constant or, since we are at the earth's surface where $r = r_E$, the constant is $\gamma E/r_E$.

Remark: You might find it profitable to check this through in the light of the Gauss mean value theorem.

Now, in general, $\Omega_g \doteq \gamma E/r$ so that $\Omega_g + \Omega$ = a constant becomes

$$(1.06:9) \qquad \frac{\gamma E}{r} + \frac{3}{2}\frac{\gamma C r^2}{e^3}\left((\cos^2[\theta] - \frac{1}{3}) + \frac{1}{3}(\frac{r}{e})(5\cos^3[\theta] - 3\cos[\theta])\right) = \frac{\gamma E}{r_E}\ . \ .$$

Remark: This same basic development is to be found in Schureman, U.S.C.&G.S. Sp. Pub. No. 98.

Define the displacement of the sea surface from mean sea level, η, by $r = r_E + \eta$. Our problem now is to express η as a function of θ and the constants. Begin by dividing equation (1.06:9) through by $\gamma E/r$ to get

$$(1.06:10) \qquad 1 + \frac{3}{2}\frac{C}{E}(\frac{r}{e})^3\left((\cos^2[\theta] - \frac{1}{3}) + \frac{1}{3}(\frac{r}{e})(5\cos^3[\theta] - 3\cos[\theta])\right) = \frac{r}{r_E}$$

and, substituting for r,

(1.06:11)
$$1 + \frac{3}{2}\frac{C}{E}\left[\frac{r_E + \eta}{e}\right]^3\left((\cos^2[\theta] - \frac{1}{3}) + \frac{1}{3}\left(\frac{r_E + \eta}{e}\right)(5\cos^3[\theta] - 3\cos[\theta])\right) = \frac{r_E + \eta}{r_E}\ .$$

This can be rewritten as

(1.06:12)
$$\frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^3(1 + \frac{\eta}{r_E})^3(\cos^2[\theta] - \frac{1}{3}) + \frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^4(1 + \frac{\eta}{r_E})^4(\frac{5}{3}\cos^3[\theta] - \cos[\theta]) = \frac{\eta}{r_E}\ .$$

Consider the ratio η/r_E. $r_E = 6.37\times10^6$ m and η is certainly much much smaller than this so that $\eta/r_E << 1$. Expanding the powers of $(1 + \eta/r_E)$ and neglecting all terms containing η/r_E to the second power and higher we have

(1.06:13)

$$\frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^3 \left(1 + 3\frac{\eta}{r_E}\right)\left(\cos^2[\theta] - \frac{1}{3}\right) + \frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^4 \left(1 + 4\frac{\eta}{r_E}\right)\left(\frac{5}{3}\cos^3[\theta] - \cos[\theta]\right) = \frac{r_E + \eta}{r_E} \ .$$

Solving (1.06:13) for η/r_E gives

(1.06:14)

$$\frac{\eta}{r_E} = \frac{\frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^3\left(\cos^2[\theta] - \frac{1}{3}\right) + \frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^4\left(\frac{5}{3}\cos^3[\theta] - \cos[\theta]\right)}{1 - 3\left[\frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^3\left(\cos^2[\theta] - \frac{1}{3}\right)\right] - 4\left[\frac{3}{2}\frac{C}{E}\left(\frac{r_E}{e}\right)^4\left(\frac{5}{3}\cos^3[\theta] - \cos[\theta]\right)\right]}$$

In working out the tide-generating forces we used

$$r_E/e = (r_E/d)(d/e)$$

and defined a numerical constant

$$U \equiv (C/E)(r_E/d)^3$$

where d was the mean distance from the earth to the celestial body. For the moon

$$U = U_M = 5.6\times10^{-8} \ .$$

The denominator can thus be written as

$$1 - 3\left[\frac{3}{2}U\left(\frac{d}{e}\right)^3\left(\cos^2[\theta] - \frac{1}{3}\right)\right] - 4\left[\frac{3}{2}U\left(\frac{r_E}{d}\right)\left(\frac{d}{e}\right)^4\left(\frac{5}{3}\cos^3[\theta] - \cos[\theta]\right)\right] \ .$$

d/e is of order 1 and $r_E/d \leq 1/60$. Consequently, the second and third terms are negligible in comparison with 1. With this in mind, equation (1.06:14) is approximately

$$(1.06{:}15) \qquad \eta = \frac{3}{2} r_E U\left(\frac{d}{e}\right)^3\left(\left(\cos^2[\theta] - \frac{1}{3}\right) + \left(\frac{r_E}{e}\right)\left(\frac{5}{3}\cos^3[\theta] - \cos[\theta]\right)\right) \ .$$

Equation (1.06:15) is used to get the magnitudes for the equilibrium tides shown on page 62. The values are remarkably small. The tide, as observed, is generally much greated. However, tides in small enclosed seas are close to these values and no data are available from the open ocean where assumption (A) is best represented. The numbers given are for the moon's mean distance. Variations up to 24% result from its closer approaches

	Lunar tide		Solar tide	
	(cm)	(ft)	(cm)	(ft)
Greatest rise	36	1.2	17	0.6
Greatest fall	18	0.6	8	0.3

Putting the equilibrium tide, (1.06:15), in terms of latitude, L, declination, D, and hour angle, h, we have, after suppressing the r_E/e term,

(1.06:16)

$$\eta = \underbrace{\frac{3}{2} U\left(\frac{d}{e}\right)^3 r_E}_{\text{common coefficient}} \left\{ \begin{array}{ll} \frac{3}{4}\left(\frac{1}{3} - \sin^2[L]\right)\left(\frac{2}{3} - 2\sin^2[D]\right) & \text{: long-period species} \\ + \left(\tfrac{1}{2}\sin[2L]\sin[2D]\right)\cos[h] & \text{: diurnal species} \\ + \left(\tfrac{1}{2}\cos^2[L]\cos^2[D]\right)\cos[2h] & \text{: semidiurnal species} \end{array} \right.$$

Thus, the equilibrium tide exhibits the same kinds of constituents as the tide producing forces.

Let me reiterate: The equilibrium tide is a construct of the mind. There is no reason to suppose that an observed tide behaves--or should behave--like an equilibrium tide. Its chief uses are two. First, it ties the astronomical forces in with the tides as they appear. Second, it is useful as a reference for observed tides. For this second use it is indeed fortunate that the characteristic variations of the equilibrium tide exhibit the characteristic variations of the observed tide to a greater or lesser degree.

The variations in the equilibrium tide and the way the various tidal species enter may be visualized as follows. Under the equilibrium theory, the moon would tend to draw the figure of earth into a prolate spheroid with the long axis directed toward the moon, Fig. 1.06-2, page 63. The picture, grossly exaggerated, is shown with the moon on the equator and with the moon a a declination of 15°N.

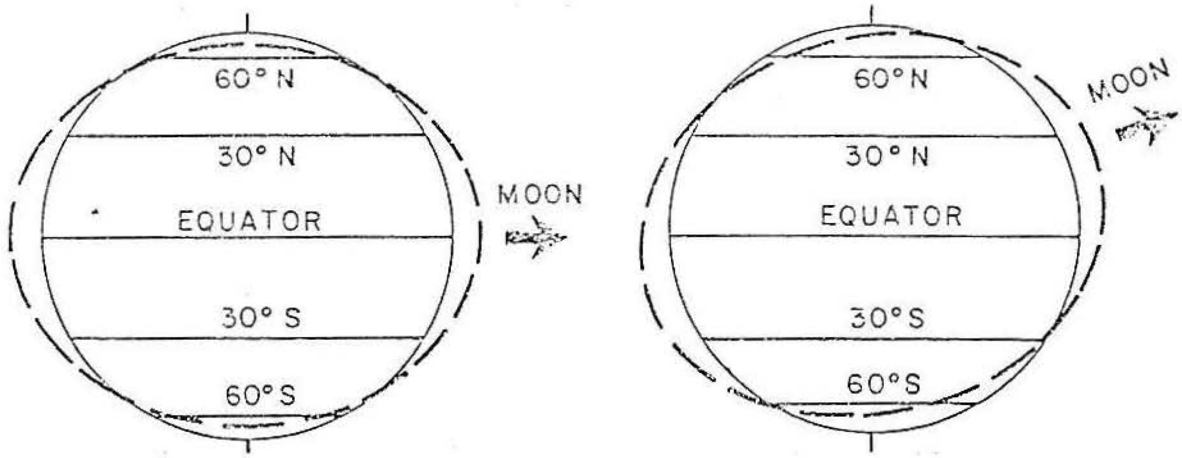


Fig. 1.06-2

The moon produces one high water immediately beneath it and another diametrically opposed. Low water extends like a belt entirely around the earth halfway between. Since we have assumed that the water has no inertia, the bulge will track the motion of the moon exactly always remaining directly beneath it.

When the moon is above the equator, declination 0°, the maximum tide range occurs on the equator and diminishes to zero at the poles. Whatever the range at any particular point, the highs and lows will be of the same size.

For declinations other than 0° a declinational inequality is introduced, the two high waters being of unequal size everywhere except at the equator. This inequality increases with latitude until, near the poles, there is only one high water per day. The variations in the lunar equilibrium tide with declination are illustrated in Fig. 1.06-3 for D_M = 15° N and latitudes of 0°, 30°, and 60°.

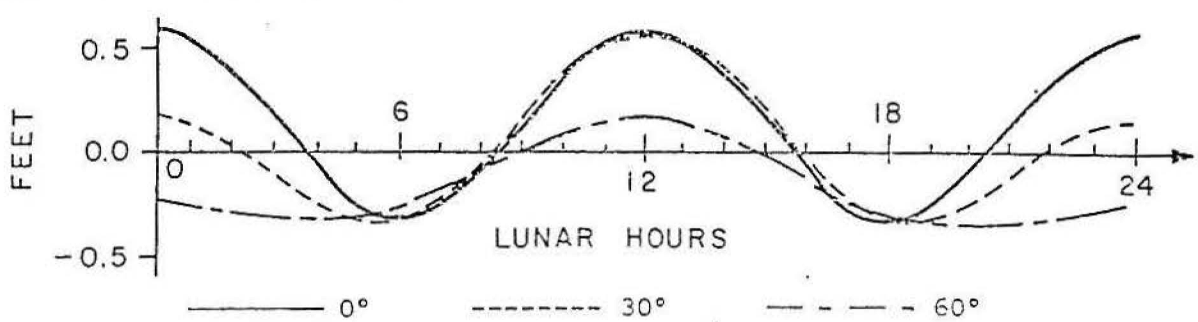


Fig. 1.06-3

From these considerations a number of deductions can be made about the equilibrium tide:

(1) The equilibrium tide is composed of "constant" terms, diurnal oscillations, and semidiurnal oscillations.

(2) The diurnal part increases with declination and vanishes with zero declination

(3) The semidiurnal part decreases with declination.

(4) The tide varies with the lunar distance approximately as the cube of the parallax.

Similar conclusions can be reached about the solar equlibrium tide.

One important point to note is that the effects of declination and parallax tend to appear as common factors to all tides of the same species, e.g., the diurnal forces and the diurnal equilibrium tide have factors depending on parallax and declination which are quite independent of position on the earth's surface. If declination or parallax reduces the diurnal tide at one point of earth, it will reduce it everywhere else in the same ratio. This was shown when we wrote η in equation (1.06:16) as a sum of terms in h, and 2h and separated each coefficient into a product of factors each of which contained only one parameter. While it does not follow that the actual tide will be governed exactly by this rule, there is a strong presumption that the rule will be followed approximately.

The cube of the parallax, $(r_E/e)^3$, or its equivalent, $(d/e)^3$, in equation (1.06:16), is a factor common to all species so that we expect changes in tides due to changes in the parallax to affect all species in the same ratio.

The diurnal constituents have a common factor which depends on the declination, sin[2D], so we expect all diurnal constituents to be affected to the same relative degree by changes in D. Similarly for the semidiurnal constituents affected by $\cos^2[D]$. However, since

$$\frac{2}{3} - 2\sin\,[D] \sim \text{long-period species} \quad ,$$

$$\sin[2D] \sim \text{diurnal species} \quad , \quad \text{and}$$

$$\cos^2[D] \sim \text{semidiurnal species} \quad ,$$

the different species are affected in different ways by a change in D.

Therefore, even if you could neglect the long-period species, it would be absolutely necessary to work up each species separately and then combine the four essential contributions: (1) lunar semidiurnal, (2) solar semi-diurnal, (3) lunar diurnal, and (4) solar diurnal.

For the lunar semidiurnal equilibrium tide high water occurs when $\cos[2h] = 1$, i.e., when the hour angle is $h = 0°$ or $180°$. This corresponds to the times of upper and lower lunar transit. In nature it doesn't happen that way. However, Laplace suggested that, if the forces vary with a known periodicity, the tides must exhibit the same periodicity. This would justify the notion that high water must lag transit by some fixed amount which may be different at different places. A similar statement can be made for the lunar diurnal constituent but the constant lags appropriate to the two species need not be the same. This whole notion of Laplace's is basically independent of any theory of tidal motion. However, it can only be exactly true if the periodicity of the forces is strictly constant. As you know, they are not.

People working with tides found that these relations were approximately true for observed tides but they had the devil's own time of it when they tried to combine the four basic constituents to give the whole tide. Further, they found that the variations in the separate constituents due to changes in parallax and declination didn't always follow the simple relations indicated by the forces. The reason is not too difficult to indicate. Any oscillation, left to itself, will ultimately die out because some frictional damping is always present. To maintain the oscillation, force must be applied <u>in phase</u> with the oscillation. If the force is seriously out of phase, it won't help maintain the motion. If it is nearly in phase, it will prolong the oscillation. The tidal forces have periods of roughly 12 and 24 hours. The natural period of oscillation of a body of water depends on the depth and the surface dimensions. For a simple rectangular basin of depth h and length ℓ the natural period is $2\ell/\sqrt{gh}$. Such a basin would have to be large and quite shallow to have a natural period of 24 hours. There are few oceans that come even close to matching this. But earth's geometry often comes close to a 12-hour natural period.

Because earth favors the 12-hour period, semidiurnal tides tend to be greater than diurnal tides. In general, local configuration modifies each species. Instead of having the semidiurnal real tide always related to the semidiurnal equilibrium tide by a constant factor and a constant time lag, it is found that both the factor and the lag vary with the length of the lunar day.

1.07. Harmonic Constituents.

In equation (1.06:16) we have an expression for the equilibrium tide which organizes it into a sum of terms, each of a different species. Each term is composed of factors dependent on a single parameter; e, L, D, or h. These terms are quite complex. What we want to do is to resolve the expression into "simple harmonics" in which the angles change uniformly with time and the amplitudes remain substantially constant over reasonably long period of time.

In general, a simple harmonic term can be written in the form

(1.07:1) $$A\cos[\sigma t - \omega)]$$

where

$A \equiv$ the amplitude (a constant)

$t \equiv$ the time

$\sigma \equiv$ the speed (increment of angle/unit time, constant)

$\omega \equiv$ the phase lag (constant)

$\sigma t - \omega \equiv$ the argument

$2\pi/\sigma \equiv$ the period .

To avoid continual use of the jawbreaker "the numerical value of the speed in degrees per mean solar hour" we will use the briefer, but less explicit, term speed number.

A compound harmonic is one in which at least one of the quantities A, σ, ω are variable. It is usually possible to break such a compound harmonic into simple harmonics. For example, suppose that in (1.07:1) A is variable and, specifically, that it is composed of a constant and a simple harmonic part. Then instead of (1.07:1) we would write

(1.07:2) $$(A_o + A_1\cos[\sigma_1 t - \omega_1])\cos[\sigma t - \omega]$$

where

$$A_o,\ A_1,\ \sigma,\ \sigma_1,\ \omega,\ \omega_1 \text{ are constant.}$$

Expanding,

$$A_o\cos[\sigma t - \omega] + A_1\cos[\sigma_1 t - \omega_1]\cos[\sigma t - \omega]$$

and using the cosine product relation from trigonometry we have

(1.07:3) $$A_o\cos[\sigma t - \omega] + \tfrac{1}{2}A_1\cos[(\sigma+\sigma_1)t - (\omega+\omega_1)] + \tfrac{1}{2}A_1\cos[(\sigma-\sigma_1)t - (\omega-\omega_1]$$

each term of which is simple harmonic. Thus, if the amplitude of any compound harmonic term with speed σ and phase lag ω includes a variable term which can be written as a simple harmonic with speed σ_1 and phase lag ω_1, then the expression can be represented by harmonic terms with speeds $\sigma + \sigma_1$ and $\sigma - \sigma_1$ and corresponding phase lags $\omega + \omega_1$ and $\omega - \omega_1$. A similar result holds for sines. Clearly, this will have its uses in equation (1.06:16).

In the example just offered only the amplitude was allowed a variation. However, consider the first two terms of (1.07:3). For convenience take the phase lags as zero. Then

(1.07:4) $$A_o\cos[\sigma t] + \tfrac{1}{2}A_1\cos[(\sigma + \sigma_1)t] \quad .$$

For $t = 0$ both cosines are 1 and, if this represents a tide, then high water comes to $A_o + \tfrac{1}{2}A_1$ and occurs at $t = 0$. If A_1 is small in comparison with A_o, near high water the compound wave is behaving like neither $\cos[\sigma t]$ nor like $\cos[(\sigma + \sigma_1)t]$ but like an oscillation with some intermediate speed. Although only the amplitude varied, at high water the speed is somewhat greater than σ ($\sigma_1 > 0$) and the amplitude is a bit greater than A. (If $\sigma_1 < 0$, the speed is less than σ near $t = 0$.)

From equation (1.06:16) the equilibrium semidiurnal tide, ignoring the geodetic factor, varies as

$$\left(\frac{d}{e}\right)^3 \cos^2[D]\cos[2h] \quad . \qquad (1.07:5)$$

Suppose the moon in the plane of the equator, $D = 0°$, and moving always at its mean distance, $e = d \sim (d/e)^3 = 1$. Then the semidiurnal tide is proportional to $\cos[2h]$ and, if the moon moves uniformly on its orbit, then $2h$ increases uniformly. Under these conditions (1.07:5) is a simple harmonic oscillation as it stands. Its period is half a lunar day, 12.42 mean solar hours. Its speed number is 360/12.42. This mean tide is denoted by the symbol M_2.

M_2 ≡ the principal lunar semidiurnal constituent
speed = 28.984° per mean solar hour.

Now, suppose e is not constant but, as in actuality, varies with a period of 27.555 days (661.3 hours). The speed number of this variation is 360/661.3 = 0.544. Looking at (1.07:5) we see $(d/e)^3$ present as a variable amplitude and the previous results can be applied. In addition to M_2 we get two other simple harmonic constituents with speed numbers

$$28.984 + 0.544 = 29.528$$

and

$$28.984 - 0.544 = 28.440 \quad .$$

These constituents would have equal amplitudes if only the amplitude of the main term were modified; see (1.07:3). However, the change in the distance e also changes the speed on the orbit. Bodies move more slowly at greater distances according to Kepler's Law. This means that the hour angle, h, can no longer be treated as increasing uniformly. Since the moon is revolving with the earth's rotation,

larger speed $\sim e < d \sim$ a slower rate for the increase in h $\sim$ a greater tide.

Conversely,

slower speed $\sim e > d \sim$ a faster rate for the increase in h $\sim$ a lower tide.

From this we associate the larger of the two speed numbers with a smaller tide and conversely. Thus, the two additional constituents acting with M_2 are:

N_2 ≡ the larger lunar elliptic semidiurnal constituent
speed = 28.440° per mean solar hour.

L_2 ≡ the smaller lunar elliptic semidiurnal constituent
speed = 29.528° per mean solar hour.

In this simple minded explanation we have tacitly assumed that the variation in d/e could be expressed by a single simple harmonic term. A better expression in terms of longitude was presented in (1.05:27). It was

$$(d/e)_M = 1 + 0.055\cos[\overline{X}_M - P_M] + 0.010\cos[\overline{X}_M - 2\overline{X}_S + P_M] + 0.008\cos[2(\overline{X}_M - \overline{X}_S)] + 0.003\cos[2(\overline{X}_M - P_M)]$$

where

$\overline{X}_M \equiv$ the moon's mean longitude,

$\overline{X}_S \equiv$ the sun's mean longitude,

and

$P_M \equiv$ the longitude of lunar perigee .

The terms have the following speed numbers:

$$\overline{X}_M - P_M \sim 0.544$$
$$\overline{X}_M - 2\overline{X}_S + P_M \sim 0.471$$
$$\overline{X}_M - \overline{X}_S \sim 0.507 \quad .$$

According to the development of equation (1.07:3), each simple harmonic term in this equation will produce two constituents whose speed numbers are sums and differences and, by analogy with the argument for N_2 and L_2, we have the following: (Among others!)

ν_2 : speed 28.513° per mean solar hour
λ_2 : speed 29.455° per mean solar hour
μ_2 : speed 27.968° per mean solar hour
S_2 : speed 30.000° per mean solar hour
$2N_2$: speed 27.895° per mean solar hour .

WARNING: The S_2 constituent listed here is not a solar constituent as the symbol might lead you to suppose.

The relative importance of each of these constituents could be established by checking through their amplitudes.

We still have the declinational factor to worry about. In the semidiurnal tide the declinational factor is $\cos^2[D]$. Suppose the moon were to move exactly on the ecliptic. Of course, it doesn't but the attack is the standard one: a constituent for a restricted case plus more stuff to allow for the staggers. Then a full cycle of declinations is run through in one revolution, 27.3216 mean solar days or 655.7 mean solar hours.

Since $\cos^2[D] = \frac{1}{2}(1 + \cos[2D])$, the declinational term has a period of 327.85 hours and, consequently, a speed number of 1.098. This variable factor in the lunar amplitude will give rise to two harmonic terms in association with the principal term M_2--and a pair for every other bloody constituent we've picked up so far as well. Their speed numbers will be the sum and difference of the M_2 speed number and 1.098, i.e.,

$$28.984 + 1.098 = 30.082$$

and

$$28.984 - 1.098 = 27.886 \quad .$$

We have mentioned that motion at the equinoxes is different from motion at the solstices. If this were not so, these two terms would be equal. When the moon is at ♈, D = 0. However, the change in right ascension is then less than average. Consequently, h is changing more rapidly than usual and the speed of the complex harmonic representing the tide is greater than the average speed. Thus, we find from the second situation initially discussed that the more important of the two constituents is the one with the greater speed. Therefore, add to the roster of constituents:

K_2 ≡ the lunar declinational semidiurnal constituent

speed = 30.082° per mean solar hour .

We will discuss the changes in declination due to the 18.61-year cycle a bit later.

You can see that there are many constituents of the lunar semidiurnal tide that we haven't mentioned. Fortunately, they are quite small. The ones listed are the ones usually used; and often not all of them.

The harmonic constituents of the solar semidiurnal tide come from an expression of exactly the same form as the corresponding lunar ones,

$$(d/e)^3\cos^2[D]\cos[2h] \quad ,$$

but solar instead of lunar values are used for the parameters.

The parallax factor has a period of 365.24 days so that the corresponding harmonic constituent has a speed number of 0.041. The cycle of declination has the same period but the declinational factor involves cos[2D] so that its speed number is 0.082, twice that of the parallax factor. Repeating the rock hockey all over again, if the sun moved on the equator uniformly at a constant distance, the tide would vary as cos(2h) and a steady rate of 30° per mean solar hour would result. This gives

S_2 ≡ the principal solar semidiurnal constituent

speed = 30.000° per mean solar hour.

The parallax factor applied to S_2 produces two constituents with speeds 30.041 and 29.959. By an argument analogous to that for L_2 and N_2 we find the smaller the more important and list

T_2 ≡ the principal solar elliptic semidiurnal constituent

speed = 29.959° per mean solar hour.

The declinational factor applied to S_2, as the similar one was to M_2, gives constituents with speed numbers 30.082 and 29.918 of which the greater is the more important. Thus,

K_2 ≡ the luni-solar declinational semidiurnal constituent

speed = 30.082° per mean solar hour.

The name "luni-solar" applies because the speed is exactly the same as the lunar declinational tide, K_2.

Of the constituents called "luni-solar" the principal ones are K_1 and K_2. It means that each constituent is made up of two components, one from the moon and the other from the sun, each having the same speed number. They could be symbolized, for example, by

$$K_2 = K_{2_S} + K_{2_M} \quad .$$

The coefficient of the constituent is very nearly, but not quite, the sum of the coefficients of the components. If we plot amplitude against time for the components, Fig. 1.07-1, we get curves which are a bit displaced. The phase difference is mainly due to the presence of a u in the argument of the lunar component and not in the solar component. We will get into this a bit later. Consequently, for a compound constituent like K_2 we have the maximum amplitude of the tidal constituent not equal to the sum of the maximum amplitudes of the components.

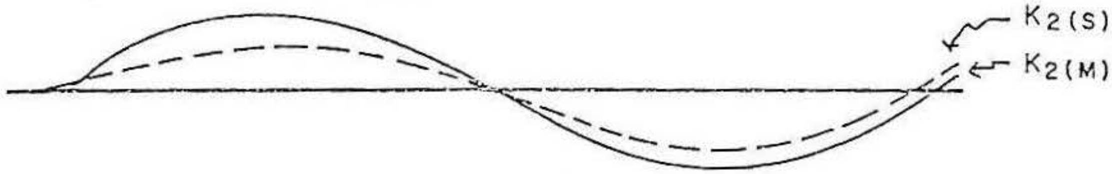


Fig. 1.07-1

We turn now to the harmonic constituents of the equilibrium diurnal tide. The variation for either the sun or the moon goes as

$$(d/e)^3 \sin[2D]\cos[h] \quad .$$

If we attacked this as before with the celestial body on the equator we would have D = 0 and nothing would result. Instead, hit the declination first. It is harmonic with mean period 655.7 hours and mean speed number 0.549 for the moon. This is the same as we had in the semidiurnal case. Since the speed number of the hour angle factor is 14.492, the result is a pair of terms with speed numbers

$$14.492 + 0.549 = 15.041$$
$$14.492 - 0.549 = 13.943 \quad .$$

These are

K_1 ≡ a lunar declinational diurnal constituent
speed = 15.041° per mean solar hour

and O_1 ≡ a lunar declinational diurnal constituent
speed = 13.943° per mean solar hour.

They have equal amplitudes and neutralize each other to give zero when D = 0. There are minor refinements on all this that spew up additional constituents but we won't bother.

Now stir in the changes in parallax. This variation is exactly the same as for the semidiurnal tides. The terms proliferate on the same arguments as before.. Of them we retain

Q_1 ≡ a lunar diurnal constituent
speed = 13.399° per mean solar hour.

M_1 ≡ a lunar diurnal constituent
speed = 14.492° per mean solar hour.

J_1 ≡ a lunar diurnal constituent
speed = 15.585° per mean solar hour.

Actually, there should be four constituents on this list but two of them have speed numbers 14.497 and 14.487 so that, practically, they are nearly inextricable. For them we use the average for M_1.

A serious problem arises when two constituents have very nearly the same speed number, say a difference of 0.01°/msh or less. A short observation of the tide can not possibly separate such constituents. They will, however, gradually get out of phase. For 0.01°/msh they will completely separate in 360/0.01 ≈ 4 years. At least 4 years of continuous observation would be necessary to untangle such a pair.

There are a number of rough ways to handle such a situation. Very often the amplitude of one of the components has a much greater magnitude. In this case the smaller can be thought of as a perturbation on the dominant component. Examples are the M_1 and L_2 constituents. Two assumptions are made:

(1) The ratio of the observed amplitudes is the same as the ratio of the equilibrium coefficients of the components multiplied by the corresponding node factor (if applicable).

(2) The local epochs of the two constituents, i.e., their phase shifts, are the same.

These assumptions are reasonably good when the constituents in question are of the same species and have very nearly the same speed numbers.

Consider two constituents

$$A_1\cos[V_1 + u_1] \quad \text{and} \quad A_2\cos[V_2 + u_2] \quad .$$

No phase shift has been included in the arguments because of assumption (1). Let $A_1 >> A_2$ so that the component subscripted with "1" is dominant. We can write

$$\begin{aligned} A_1\cos[V_1 + u_1] &= A_1\cos[n_1t + a_1 + u_1] \\ A_2\cos[V_2 + u_2] &= A_2\cos[n_2t + a_2 + u_2] \\ &= A_2\cos[n_1t + \Delta nt + a_1 + \Delta a + u_1 + \Delta u] \end{aligned}$$

where

$$a_1,\ a_2 = 0,\ \pm 90°,\ \pm 180°$$

and

$$\begin{aligned} \Delta n &= n_2 - n_1 \\ \Delta a &= a_2 - a_1 \\ \Delta u &= u_2 - u_1 \quad . \end{aligned}$$

The combination resembles the A_1-curve with A_2 adding a small perturbation. If we rewrite the argument as $\theta + \Delta\theta$, the combined tide is

$$A = A_1\cos[\theta] + A_2\cos[\theta + \Delta\theta]$$
$$= A_1\cos[\theta] + A_2\cos[\theta]\cos[\Delta\theta] - A_2\sin[\theta]\sin[\Delta\theta]$$

or

$$A = (A_1 + A_2\cos[\Delta\theta])\cos[\theta] - (A_2\sin[\Delta\theta])\sin[\theta] \quad .$$

Define

$$R\cos[r] \equiv A_1 + A_2\cos[\Delta\theta]$$
$$R\sin[r] \equiv A_2\sin[\Delta\theta] \quad .$$

Then

$$R^2 = (A_1 + A_2\cos[\Delta\theta])^2 + (A_2\sin[\Delta\theta])^2$$
$$= A_1{}^2 + 2A_1A_2\cos[\Delta\theta] + A_2{}^2\cos^2[\Delta\theta] + A_2{}^2\sin [\Delta\theta]$$
$$= A_1{}^2 + A_2{}^2 + 2A_1A_2\cos[\Delta\theta] \quad .$$

But

$$A = R\cos[r]\cos[\theta] - R\sin[r]\sin[\theta] = R\cos[\theta + r] \quad .$$

Therefore,

$$A = (A_1{}^2 + A_2{}^2 + 2A_1A_2\cos[\Delta\theta])^{\frac{1}{2}}\cos[\theta + r] \quad .$$

Also,

$$\tan[r] = \frac{R\sin[r]}{R\cos[r]} = \frac{A_2\sin[\Delta\theta]}{A_1 + A_2\cos[\Delta\theta]} \quad .$$

Therefore,

$$r = \tan^{-1}\{A_2\sin[\Delta\theta])/(A_1 + A_2\cos[\Delta\theta])\}$$

and

$$A = (A_1{}^2 + A_2{}^2 + 2A_1A_2\cos[\Delta\theta])^{\frac{1}{2}}\cos(\theta + \tan^{-1}\left(\frac{A_2\sin[\Delta\theta]}{A_1 + A_2\cos[\Delta\theta]}\right))$$

If $\Delta\theta = 0$, i.e., if there were no differences between the two arguments, this would reduce to

$$A = (A_1 + A_2)\cos[\theta]$$

as is only to be expected. Therefore

$$A \rightarrow (A_1 \pm A_2)\cos[\theta] \quad \text{as} \quad \Delta\theta \rightarrow 0,\ 180°,\ \ldots \quad .$$

Define

$$R_1 \equiv R/A_1 = \{1 + (A_2/A_1)^2 + 2(A_2/A_1)\cos[\Delta\theta]\}^{\frac{1}{2}} \quad .$$

Also

$$R = A_1R_1 \quad .$$

Then

$$A = R_1A_1\cos[\theta + r]$$

or, if you trace back up the symbolism,

$$A = R_1 A_1 \cos[V_1 + u_1 + r]$$

which can be compared with the dominant component,

$$A_1 \cos[V_1 + u_1] \quad .$$

The same kind of hassel with the sun leaves us with:

K_1 ≡ the luni-solar declinational diurnal constituent

speed = 15.041° per mean solar hour.

P_1 ≡ the solar declinational diurnal constituent

speed = 14.959° per mean solar hour.

The harmonic constituents of the equilibrium long-period tide vary as

$$(d/e)^3(\frac{1}{3} - \sin^2[D]) = (d/e)^3(\tfrac{1}{2}\cos[2D] - \frac{1}{6}) \quad .$$

By this time you should be able to whomp this up yourself. The results are:

M_f ≡ the lunar fortnightly constituent

speed = 1.098° per mean solar hour.

M_m ≡ the lunar monthly constituent

speed = 0.544° per mean solar hour.

S_a ≡ the solar annual constituent

speed = 0.041° per mean solar hour.

S_{sa} ≡ the solar semi-annual constituent

speed = 0.082° per mean solar hour.

As we have pointed out, the line of apsides of the moon goes through a cycle in 18.61 years. This moves the moon's nodes around the ecliptic and introduces a variation of the same period in the declination and all terms depending on it. Instead of introducing more harmonic constituents on the pattern used up to now, we allow for the variation by applying a factor, f, and an increment in the phase, u. The f,u-values are not the same for all lunar constituents. We will only point out here that all constituents can be written in the form

$$fH\cos[V + u]$$

where

f ≡ a factor varying with a period of 18.61 years

$u \equiv$ an angle varying with a period of 18.61 year

$V \equiv$ an angle changing uniformly at the mean speed of the constituent

$H \equiv$ the amplitude of the constituent.

This applies only to lunar constituents.

For solar constituents, $f \equiv 1$ and $u \equiv 0$ since the sun has no nodes.

We have repeatedly referred to the equilibrium tide as a reference for the actual tide. Using Laplace's constant lag idea, the actual tidal constituent corresponding to the equilibrium tidal constituent will be given by

$$fH\cos[V + u - \kappa]$$

where f, V, u, and H are as before and

κ = the lag of the phase of the tidal constituent behind the phase of the corresponding equilibrium constituent. It is called the epoch.

H and κ are called the harmonic constants of the tidal constituent. For further details about the harmonic constituents of the tide--should any vestige of interest or curiosity still remain with you--consult

(1) The Admiralty Manual, Ch. VII, pp. 50-61

and

(2) Schureman, U.S.C.&G.S. Sp. Pub. No. 98.

I might mention, in passing, that the U.S.C.&G.S. uses $F \equiv 1/f$ so that they have

$$FH_{observed} \equiv F_{equilibrium}$$

Tables of the epoch, κ, are based on the Greenwich meridian. To get the local epoch this must be corrected to your local meridian and time.

Even when only $(r/d)^3$ and $(r/d)^4$ are retained in the approximate equations some 124 constituent tides result. This includes both solar and lunar tides. Actually, very few have amplitudes that amount to much. The greatest number usually used in tide work is about 30. In Chesapeake Bay where the tide is about 2 feet the lesser constituents with amplitudes of the order of 0.01 inch aren't worth bothering about. The first 20 constituents include the most important ones.

The symbolism that we have been using for constituents is composed of a letter and a number. The long-term constituents use a different system omitting the subscript number.

The equilibrium tide can be written, as we have seen,

$$fRGA\cos[V + u]$$

where

$f \equiv$ the node factor

$R \equiv$ the general coefficient

$G \equiv$ the geodetic coefficient

$A \equiv$ the astronomical coefficient.

G depends only on latitude and is different for different species but the same for any one species. A depends on astronomical parameters and is different for every constituent, even those of the same species. Tide Tables give only A which they call "the" coefficient or, sometimes, the relative coefficient. To compare the sizes of constituents, for example, the size of the lunar semidiurnal M_2 with the size of the solar declinational diurnal P_1, we begin by checking up on their coefficients--the astronomical ones--in some place like Schureman, Table 2. They turn out to be:

$$M_2 \sim 0.9085 \qquad \text{(astronomical)}$$
$$P_1 \sim 0.1755 \quad .$$

The common coefficients from Schureman's Table 1 are:

0.5582×10^{-7} (for the moon)

and 0.2569×10^{-7} (for the sun) .

Their ratio is 0.4602 .

All solar astronomical coefficients have been adjusted by this ratio so that, as listed, they give direct comparisons in lunar terms. Also, differences arising from approximations using the third and fourth powers of the parallax have been incorporated.

WARNING: You have to watch your step with different tables on this sort of thing. We compared F_V/g from the tide generating forces with η/r in the equilibrium tide. They aren't equal. For terms containing $(r/d)^3$ there is a factor of ½ and for terms containing $(r/d)^4$ a factor of 1/3. Some tables use F_V/g as Schureman does; others use η/r.

Even with all these adjustments made for you, you still have to worry about the geodetic and nodal factors. For example, at 45°N in 1958 we have:

$M_2 \sim 0.500$	$M_2 \sim 1.033$
$P_1 \sim 1.000$	$P_1 \sim$ -----
(Schureman, Table 3)	(Schureman, Table 14)

Therefore,

$$\frac{M_2}{P_1} = \frac{0.9085 \times 0.500 \times 1.033}{0.1755 \times 1.000} = \frac{1.3872795}{0.3510} = 2.67 \quad .$$

Thus, the M_2 tide has an amplitude about 2.5 times as great as does the P_1 tide.

1.08. A Few Remarks on Shallow-Water Tides.

Before going on to the practical analysis of tide records a few remarks on shallow-water tides are in order. If coastal waters were deep, we wouldn't have to bother with this but, actually, estuaries and bays are quite shallow and the shallow-water effects are quite prominent. Mathematically, not much is known about the distortions of standing waves in shallow water. Progressive waves are in a bit better shape and, working empirically, we can do something about them. Viscosity, bottom friction, and interference from reflected waves are the kinds of things which act to distort the wave profiles.

Suppose we have a simple harmonic wave, curve (a) Fig. 108-1, page 79, entering shallow water. The time interval from LW to HW is the same as that from HW to LW. the wave is nicely regular and symmetrical. What happens? In gravity wave theory we use as an approximation for the speed of a wave in shallow water, $c = \sqrt{gh}$. This won't do for tides. It

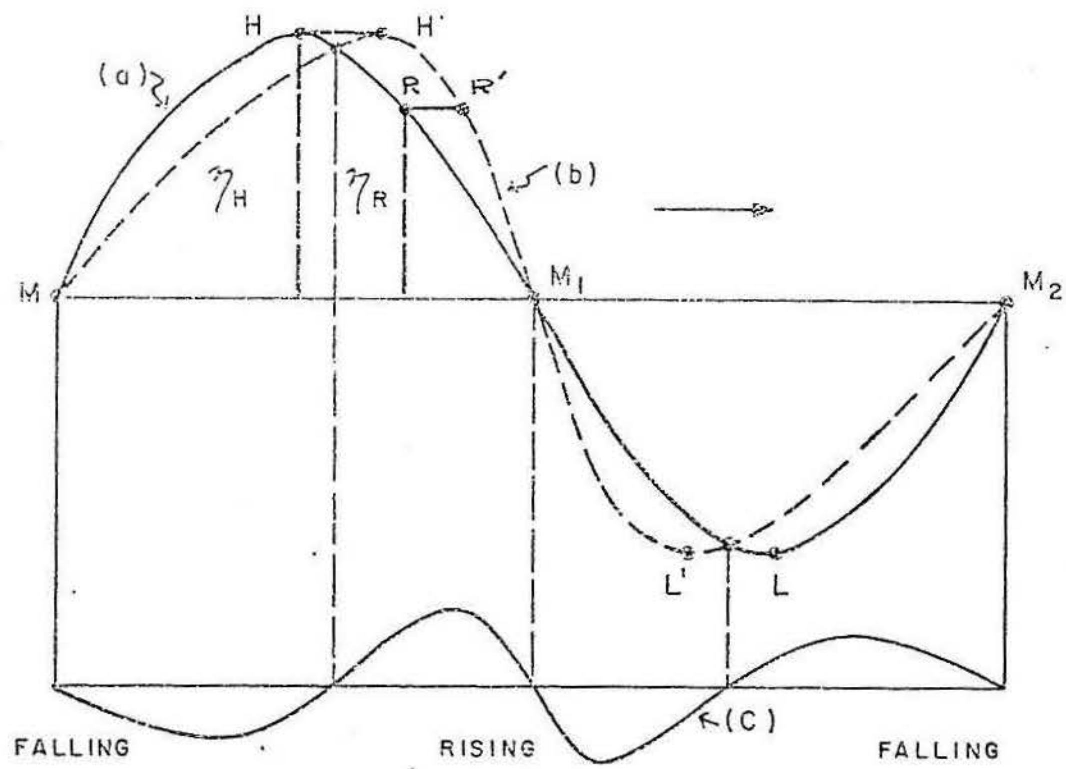


Fig. 1.08-1

can be shown that different parts of the wave profile travel at different speeds so that

(1.08:1) $$c = (1 + \frac{3}{2}\frac{\eta}{h})\sqrt{gh}$$

where

$c \equiv$ speed of a point on the wave profile,

$g \equiv$ gravity,

$h \equiv$ the mean depth of the water, and

$\eta \equiv$ the displacement of the point on the wave profile from mean water level.

By equation (1.08:1) points on the profile with different η's will travel at different speeds so that, as time passes, the wave profile will distort. Let the wave travel, and distort, during a time τ. Then freeze the profile and overlay it on the original wave for comparison. Make the points at mean level M coincide. This is the dotted curve (b) in Fig. 1.08-1. We find that high water H as pulled ahead to H' while low water L has fallen back to L'. In time τ each point on the profile will have moved a distance : which is a function of elevation through equation (1.08:1). The point M moves $c\tau = \tau\sqrt{gh}$ since $\eta = 0$. Relative to M all the other points move

$$(1.08{:}2) \qquad \tau\left(1 + \frac{3}{2}\frac{\eta}{h}\right)\sqrt{gh} - \tau\sqrt{gh} = \frac{3\eta}{2h}\sqrt{gh} \quad .$$

Therefore, the distance RR' traveled by the point R relative to M is proportional to its elevation η_R. Similarly for H so that

$$\frac{RR'}{HH'} = \frac{\eta_R}{\eta_H} \quad .$$

Figure 1.08-1 has been drawn on the assumption that τ was picked long enough to let H' get 30° out of phase with the position H would have occuied had there been no distortion.

Now, if the differences of (a) and (b) are plotted you get the wave (c), Fig. 1.08-1. Clearly, the distorted wave can be thought of as being made up of the undistorted wave, (a), plus the wave, (c). Wave (c) has two complete oscillations where the original wave (a) has only one.

We have been looking at the wave profile in space. We could, of course, sit at a point and get the same picture as a function of time. Then, if (a) had a period of, say, 12 hours, (c) would have a "period" of 6 hours.

Suppose, for the moment, that (c) were a pure harmonic. Then what we have just done would mean that the actual tide (b) could be represented by a pure harmonic, (a), plus a pure harmonic, (c), with (c) having a period half that of (a). The additional tide (c) is called a <u>shallow-water tide</u>.

Even a glance reveals that (c) is <u>not</u> a pure harmonic so that we have to analyze this curve further if we want to work only with pure harmonics. Pick up (c) and draw Fig. 1.08-2, page 81. Superimpose on (c) a simple harmonic wave (d) with an amplitude equal to the average amplitude of (c) and with a period that is strictly half the period of (a). Then, taking the differences exactly as before we can construct a curve, (e), which exhibits a "period" (?!?!?!) one-third of that shown by (a).

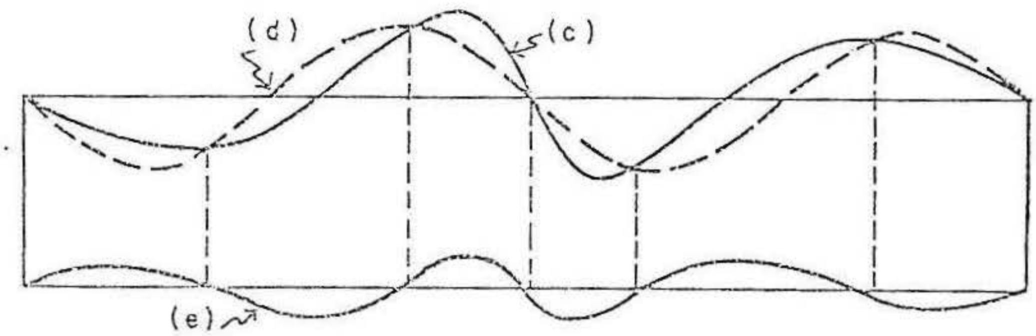


Fig. 1.08-2

For example, if (a) is semidiurnal, (e) is one-sixth diurnal. At this point the actual tide is represented by

$$(b) = (a) + (d) + (e)$$

where

(a) is pure harmonic; period 12 hours

(d) is pure harmonic; period 6 hours

(e) is compound; average period 4 hours.

Obviously, this process can be repeated until the actual tide is represented by a sum of pure harmonic terms of decreasing period plus a negligible compound harmonic term.

Thus, we expect any tide observed on earth to contain terrestrially generated components. If the primary is semidiurnal then the <u>overtides</u> will be quarter-diurnal, sixth-diurnal, ... while if the primary is diurnal, the overtides will be semidiurnal, third-diurnal, ...

Checking back to Fig. 1.08-1, one sees that the primary, (a), and the overtide, (c), are zero together. The overtide, (c), of course, has some extra zeros. Further, when (a) is at rising half-tide (c) is rising. However, when (a) is at falling half-tide (c) is rising. In the former case (a) and (c) reinforce each other to produce the rapid rise shown by (b). In the latter case they oppose each other to produce the slow fall shown by (b).

Suppose that

$$\text{wave (a)} \sim \cos[nt - k] \quad .$$

Then, if the secondary wave were a pure harmonic it would be

$$\text{wave (c)} \sim \cos[2nt - 2k - 90°]$$

This means, for example, that if the primary is an M_2 tide with a phase lag k, the shallow-water tide, in this case an M_4, would have a phase lag of 2k + 90°. Observation shows that for many places this relationship fits the facts pretty well. However, don't jump to conclusions. It isn't universally applicable. Results of this sort gained from theory are chiefly useful as a guide to what to look for.

For standing waves we know just enough about shallow-water distortion to indicate that the phase relationships are different. If the primary tide is cos[nt - k], then the secondary tide will be approximately either cos[2nt - 2k] or cos[2nt - 2k - 180°]. Hence, if k is the phase lag of the M_2 tide, then the phase lag of the M_4 tide due to a standing oscillation will be either 2k or 2k + 180°. This means that when the primary is at HW the secondary is at either HW or LW. In the first case the distorted high water will be more peaked while the low water will be flattened out, Fig. 1.08-3. In the second case the reverse will be true.

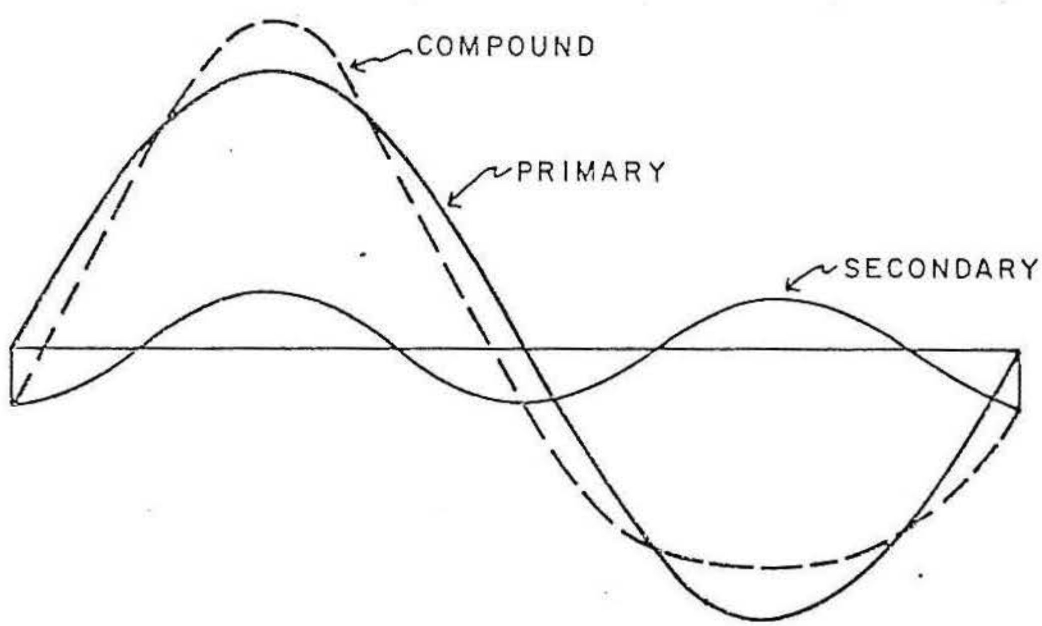


Fig. 1.08-3

The amplitude relations of these terrestrially generated tides to the amplitude of the primary is a matter of some importance. Suppose we take a section of Fig. 1.08-1 and compare it with the same thing for a primary of half the amplitude, Fig. 1.08-4. Since the distortion is proportional to the elevation it follows that

$$H_1H_1' = \tfrac{1}{2}HH' \quad \text{and } R_1R_1' = \tfrac{1}{2}RR' \quad .$$

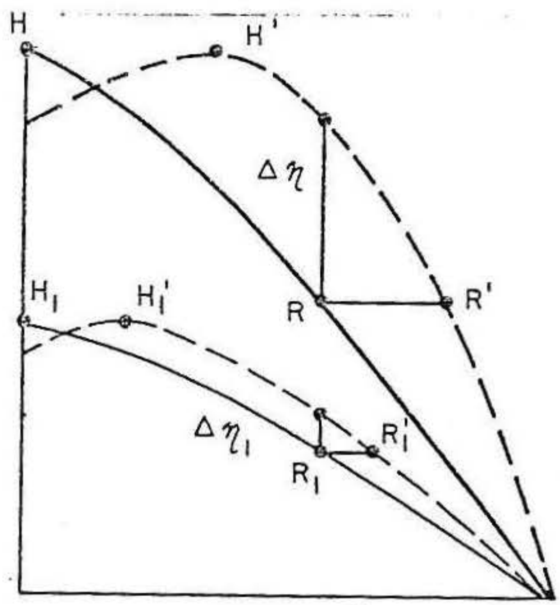


Fig. 1.08-4

On the other hand, when we come to consider the differences between the distorted and undistorted curves, we find that the ratio $\Delta\eta_1/\Delta\eta$ is about ¼. In other words, at any place the amplitude of the quarter-diurnal tide varies approximately as the square of the amplitude of the semidiurnal tide. According to this the quarter-diurnal at neaps will be less than those at springs in the ratio of the squares of neap and spring ranges. The is generally approximately true and can be used to estimate the relative importance of the shallow-water tides when the amplitude of the primary is known. A similar relation holds for the sixth-diurnal tide which goes as the cube, etc..

For Chesapeake Bay only the M_4 and, possibly, the M_6 shallow-water tides appear to be of any real interest. However, in many estuaries the distortional effects are so great that the number of shallow-water constituents becomes completely unmanageable. It was originally thought that the overtides dropped in importance very rapidly but recent work at Liverpool suggests that, in many cases, the convergence is much slower than was hoped. This indicates that the higher constituents, neglected up to now, may be worth looking into.

The attack we have outlined is based on a pure harmonic primary wave. Such a wave is seldom--if ever--met with in an estuary. The wave which enters the estuary might, conceivably, be pure harmonic but, in the estuary, it is damped by friction, refracted by variable depth, bounced back off barriers to form standing waves, and, in general, treated like the ball in a pinball machine. No one has made much progress with the problem. However, the explanation we have offered is actually pretty fair. One warning though is in order. Amplitudes and phase lags must, repeat MUST, be gotten from observation; not from theory.

Just to suggest to you that there are miseries as yet unmentioned, in addition to the overtides there are things known as compound tides which have speeds which are combinations of the primaries. For example, there is an MS_4 tide whose argument is $\arg\{M_2\} + \arg\{S_2\}$ and a long term tide whose argument is $\arg\{S_2\} - \arg\{M_2\}$.

The whole subject of shallow-water tides is quite analogous to the results Helmholtz got in studying sound. He found both overtones and compound tones in music. If you plan to do anything serious about tides, I suggest that you begin by becoming very familiar with sound.

1.09. The Harmonic Analysis of a Tide Record.

The time has come to go into the methods of analysis used by the U.S.C.&G.S. Suppose you have a continuous record of sea surface elevation made by a tide gage at some point. The problem is to determine from the observed record the amplitudes and epochs of the constituent tides at the point of observation. The theory of how to proceed is simple enough. Carrying out the necessary manipulations is extremely cumbersome. As I have mentioned before, the Admiralty and the U.S.C.&G.S. follow different procedures. My remarks will cover the U.S.C.&G.S. method. The Admiralty method is described in the Admiralty Manual.

Consider a tide having only two constituents and, for simplicity, no phase difference. Then the record will be described by

(1.09:1) $$\eta(t) = A\cos[at] + B\cos[bt] \quad ; \quad a \neq b \quad .$$

Assuming we know the species of constituent that enter, the fundamental periods are known so that at and bt are known. Since we have η from the record as a function of time, t, it would seem that any two values of η at two different times, $\eta(t_1)$ and $\eta(t_2)$, would be enough to set up a pair of simultaneous equations in A and B as unknowns. If so, they could be solved for A and B. For three constituents the system would need three equations, etc. for as many constituents as you wanted to use. This sounds simple, and it is, but it doesn't work in practice for a number of reasons:

First: The actual tide usually contains so many constituents that the system of simultaneous equations is very large. With hand-computing this is a real objection. With the advent of high-speed computers it really doesn't have too much force.

BUT Second: The trigonometric arguments must contain phase lags of assorted sizes. The constituents are certainly not all in phase and this doubles the number of unknowns. We have instead of (1.09:1)

(1.09:2) $$\eta(t) = A\cos[at + \alpha] + B\cos[bt + \beta] \quad ; \quad a \neq b \quad .$$

The variables are A, B, α, and β. Much worse than the doubling of the number of unknowns is the fact that some of them are outside the trigonometric functions and some inside. This means that you no longer have a simple system of linear equations and that the methods for such equation systems no longer apply.

AND

Third: The values of η read from the record are subject to errors of various sorts. There are the inevitable instrument errors. Wind-wave motion, although usually pretty well damped out by the tide gage, may not be entirely removed. Meteorological conditions may alter the sea level for days at at time. And on, and on. All these things will introduce spurious oscillations of various frequencies in the recorded tide. They force us to use averages in order to stabilize the measured values of $\eta(t)$.

The principle of the method of averaging is to isolate the effect of a single component. For example, in equation (1.09:2) suppose we want to take the average in such a way that

(1.09:3) $$\langle B\cos[bt + \beta]\rangle = 0 \quad ,$$

i.e.,

$$\eta(t) = A\cos[at + \alpha] + B\cos[bt + \beta]$$

$$\langle\eta(t)\rangle_\tau = \langle A\cos[at + \alpha] + B\cos[bt + \beta]\rangle_\tau$$

$$\langle\eta(t)\rangle_\tau = \langle A\cos[at + \alpha]\rangle_\tau + \langle B\cos[bt + \beta]\rangle_\tau$$

If we can choose τ adroitly enough, we can satisfy equation (1.09:3) so that, for that period of averaging, τ,

$$\langle\eta(t)\rangle_\tau = \langle A\cos[at + \alpha]\rangle_\tau \quad .$$

The ideal length of tidal record to pick for τ is some multiple of the synodic period of the constituents involved. The <u>synodic period</u> of two or more constituents is the time between successive conjunctions of like phases, Fig. 1.09-1, page 87.

The method of averaging to knock out a constituent is based on the fact that for a simple harmonic, say

$$\eta(t) = B\cos[\theta] \quad ,$$

averaging over a period, or any multiple of a period, gives zero.

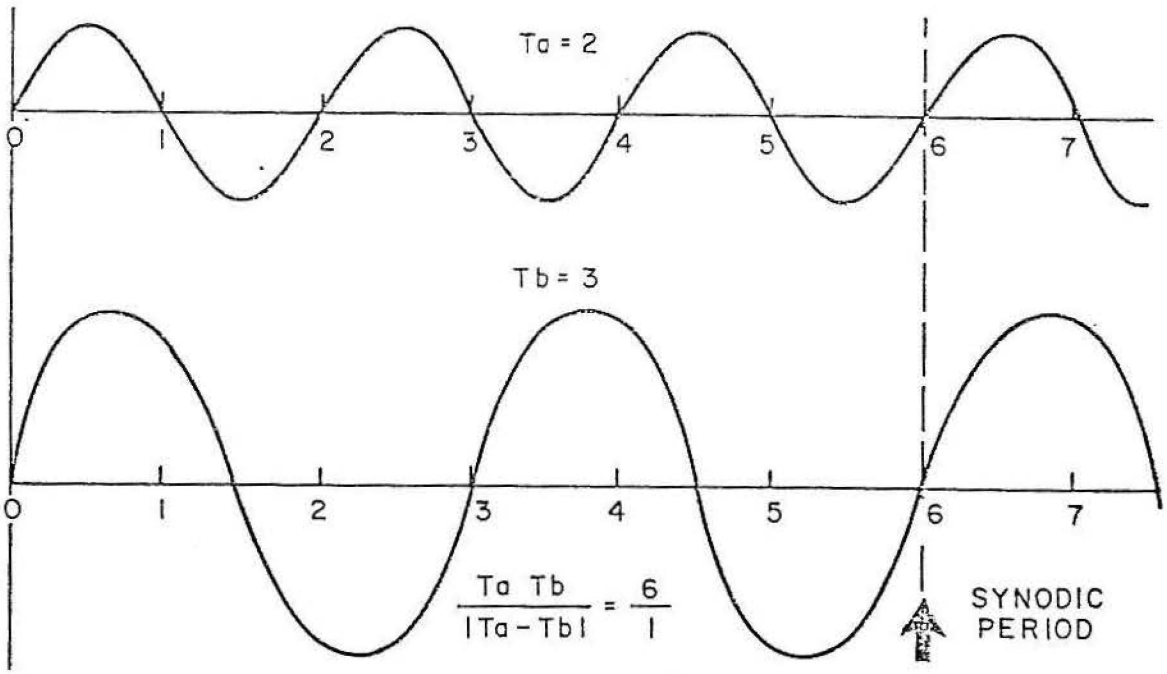


Fig. 1.09-1

$$\frac{1}{2k\pi}\int_{\theta=\kappa_1}^{\theta=\kappa_1+2k\pi} \eta d\theta = \frac{1}{2k\pi}\int_{\theta=\kappa_1}^{\theta=\kappa_1+2k\pi} B\cos[\theta]d\theta = 0 \quad .$$

This is also true if the record is read at a discrete set of points evenly spaced over an interval equal to a period or to some multiple of a period. In other words, if one complete cycle is divided into equal increments, m, then

$$\sum_{i=0}^{m-1} \eta_i = A\sum_{i=0}^{m-1} \cos\left(\frac{2\pi}{m}\, i\right) = 0 \quad .$$

One can start anywhere in the record and this is still true.

Suppose we have recorded our two-constituent tide, equation (1.09:2), over many cycles and we want to determine the value of A. We know that constituent $A\cos[at + \alpha]$ has a period $\tau_a = 360°/a$. Just to make the discussion definite, suppose $\tau_a = 24$ hours. Our record is then many days long. We read off the values of η at hourly intervals on the record starting anywhere and tabulate by hours, Fig. 1.09-2, page 88.

hour \ period	1	2	...	n
0	$\eta_{0,1}$	$\eta_{0,2}$	...	$\eta_{0,n}$
1	$\eta_{1,1}$	$\eta_{1,2}$	...	$\eta_{1,n}$
2	$\eta_{2,1}$	$\eta_{2,2}$	...	$\eta_{2,n}$
⋮	⋮	⋮		⋮
23	$\eta_{23,1}$	$\eta_{23,2}$	...	$\eta_{23,n}$

Fig. 1.09-2

Now, because we are using the period of $A\cos[at + \alpha]$, the contribution of this term to η at any particular hour is always the same, but because its period is different, the term $B\cos[bt + \beta]$ contributes different amounts to η at any one hour on different days. We can hope that for a long record, n large, the contributions from the second component will, on the whole, tend to cancel each other so that when we average we get

$$\frac{1}{n}\sum_{i=1}^{n}\eta_{0,i} = \frac{1}{n}\sum_{i=1}^{n}A\cos[at_{0,i} + \alpha] + \underbrace{\frac{1}{n}\sum_{i=1}^{n}B\cos[bt_{0,i} + \beta]}_{0}$$

$$\frac{1}{n}\sum_{i=1}^{n}\eta_{1,i} = \frac{1}{n}\sum_{i=1}^{n}A\cos[at_{1,i} + \alpha] + \underbrace{\frac{1}{n}\sum_{i=1}^{n}B\cos[bt_{1,i} + \beta]}_{0}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{1}{n}\sum_{i=1}^{n}\eta_{23,i} = \frac{1}{n}\sum_{i=1}^{n}A\cos[at_{23,i} + \alpha] + \underbrace{\frac{1}{n}\sum_{i=1}^{n}B\cos[bt_{23,i} + \beta]}_{0}$$

or

$$\langle\eta_0\rangle = \langle A\cos[at_0 + \alpha]\rangle$$

$$\langle\eta_1\rangle = \langle A\cos[at_1 + \alpha]\rangle$$

$$\vdots$$

$$\langle\eta_{23}\rangle = \langle A\cos[at_{23} + \alpha]\rangle \quad .$$

Thus, you have 24 estimates of the first constituent spaced equally over its period. Of course, they won't be pure but, if n is large, the disturbing residues will tend to be small. The points can be plotted and a curve representing $A\cos[at + \alpha]$ faired through them. From the plot the value of the amplitude A can be estimated. The value of the phase lag, α, can be secured by noting the time of occurance of the maximum and subtracting from it the hour of the equilibrium tide for the constituent.

This process can be repeated for each constituent in turn. Suppose the $B\cos[bt + \beta]$ constituent had a period of 25 hours. You could divide the 25 hours into 24 equal parts each $1\frac{1}{24}$ hours long, re-read the curve at these points, and again get 24 estimates. The only essential thing is that they be equally spaced over the period. So--in this case you could use the values already read but run from η_0 to η_{24} and get 25 estimates. That could save a lot of work.

The operation outlined above is simple minded enough but the mechanics can get fairly involved. Try to picture to yourself the amount of bonehead labor required to resolve a record from a major station which could well have ten to twenty constituents and be a year or more long. Again we sing the old refrain: It's a nice idea but it breaks down in practice.

Quite aside from that, there is another reservation that must be made about the method. Suppose you had 3 constituents with periods $\tau_a = 10$ hours, $\tau_b = 5$ hours, and $\tau_c = 4$ hours. The separation method depends on the contributions from one component being the same for a given time while those for the other components are randomly distributed if the record is long enough; the record being analysed over one of the given periods at equal time intervals. It is easy to see that the constituents with periods $\tau_b = 5$ and $\tau_c = 4$ fit the requirement. But what about $\tau_a = 10$ and $\tau_b = 5$? From Fig. 1.09-3, page 90, it can be seen that the wave produced by two pure harmonics, one with a period double the other, is an oscillation that is anything but pure harmonic--which is no surprise. More to the point is the fact that, if you take the 10-hour period, setting zero anywhere, and even if there is a phase difference, the two waves are in

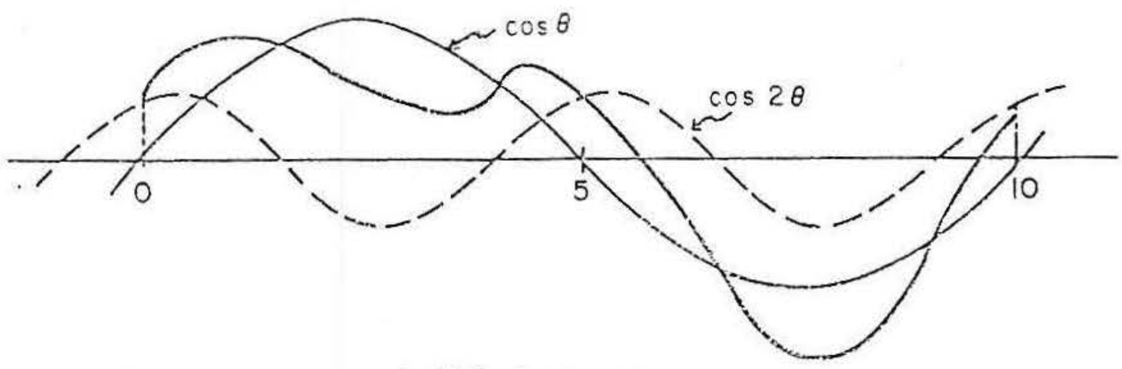


Fig. 1.09-3

exactly the same relation to each other at hour 10 as they were at hour 0. The heavy line simply repeats itself over and over. In other words, the sum of the contributions of the two constituents is always constant at a particular time. The method will not separate them.

In general, the method will not separate constituents whose periods are integral submultiples of a primary period. For example, suppose the primary tide had a period of 24 hours. The the method would fail to separate constituents with periods of 24, 12, 8, 6, 4, 3, 2, 1, ½, ..., hours. This is the Fourier series situation where you have

$$f(t) = a_0 + a_1\cos[\theta t] + a_2\cos[2\theta t] + a_3\cos[3\theta t] + \ldots \quad . \tag{1.09:4}$$

Advice: If you are not familiar with Fourier series perhaps you should poke about in the texts a bit. R. V. Churchill, "Fourier Series and Boundary Problems" is simple and I have found it useful.

The difficulty raised here is important in tidal work because some of the major constituents have such relationships. For example, $M_1 \sim 14.492°$/msh while $M_2 \sim 28.984°$/msh. The shallow-water tides can show the same property.

What the averaging method will do is to pull a record apart into constituents which are either pure cosines (if there are no important submultiples corresponding to them in the record) or deformed oscillations like the heavy line in Fig. 1.09-3.

To meet the difficulty we call upon some results from Fourier analysis. We begin by considering that a deformed "constituent" of the of the kind under discussion can be written as

(1.09:5) $$\eta_M = f_{\tau_{M_1}}(t) + f_{\tau_{M_2}}(t) + \dots + f_{\tau_{M_k}}(t) \quad .$$

Just to be specific, suppose I am talking here about the distorted curve you would get by averaging for the M_1 constituent when the submultiples M_2, M_3, ... were present and important.

Now Fourier analysis can always duplicate a curve exactly by taking an infinite number of such terms. The routine of Fourier analysis just goes ahead and generates the whole infinity of terms. The Fourier mechanism won't tell you ahead of time which components are present and important in your record. You have to make up your mind before you begin how many and which ones you think are there. Having made your decision, you carry out your computation and then check back to see how much residual wobble is unaccounted for.

Since each of the f-terms in equation (1.09:5) is actually a pure harmonic of the form $f = A\cos[at - \alpha]$, a vital relation for us is

(1.09:6) $$a\cos[at - \alpha] = C\cos[at] + S\sin[at]$$

where it is not necessary that C equal S and

(1.09:7) $$A = (C^2 + S^2)^{\frac{1}{2}}$$

while

(1.09:8) $$\alpha = \tan^{-1}[S/C] \quad .$$

Thus, the analysis for (1.09:5) can be put in the form

(1.09:9) $$\begin{aligned} \eta = H_0 &+ C_1\cos[at] + C_2\cos[2at] + \dots + C_k\cos[kat] \\ &+ S_1\sin[at] + S_2\sin[2at] + \dots + S_k\sin[kat] \\ &\quad \underbrace{}_{M_1\text{ constituent}} \quad \underbrace{}_{M_2\text{ constituent}} \quad \dots \quad \underbrace{}_{M_k\text{ constituent}} \end{aligned}$$

The constant, H_0, relates mean sea level to the height of the tide gage. If the relation between mean sea level and the height of the tide gage, which usually has an arbitrary zero level, is known, then we can remove this constant displacement from the record by forming $\eta - H_0$ but it hardly matters since H_0 is quite easy to determine from the record. Suppose the

M_1 tide has a period of 24 hours and that the record is divided into hourly intervals. Then

(1.09:10) $$H_0 = \frac{1}{24} \sum_{i=0}^{23} \eta_i \quad ,$$

i.e., H_0 is $\langle\eta\rangle$ over a complete cycle or over an integral multiple of a complete cycle of the primary wave.

It is well known from statistics that in sampling an oscillation you must sample at at least twice the frequency of the most rapid oscillation present. If you sample less often, a high frequency oscillation will look to you like a slower one. This is illustrated in Fig. 1.09-4. Suppose

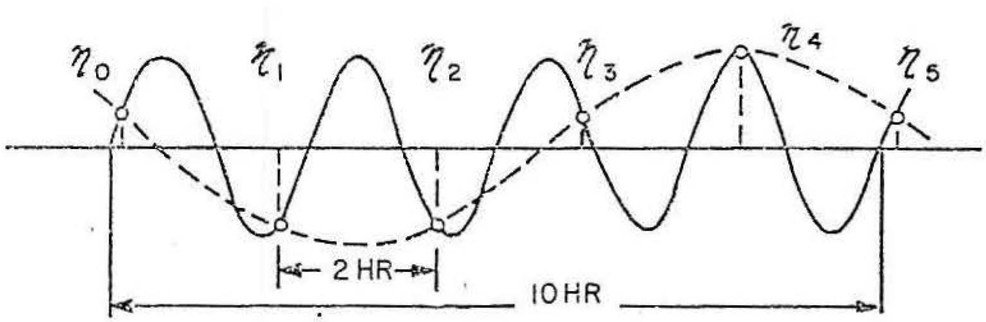


Fig. 1.09-4

you have a wave that oscillates 4 times in 10 hours and you want to explore it by sampling it at a finite number of points spaced equally over the 10 hours. Suppose you decide that anything smaller than 2-hour intervals would make too much work. So you read off the solid curve at 2-hour intervals. What you know about the curve is a set of values

$$(\eta_0, t_0), (\eta_1, t_0+\Delta t), (\eta_2, t_0+2\Delta t), \ldots, (\eta_5, t_0+5\Delta t) \quad .$$

If only this information is available to you, it looks like Fig. 1.09-5.

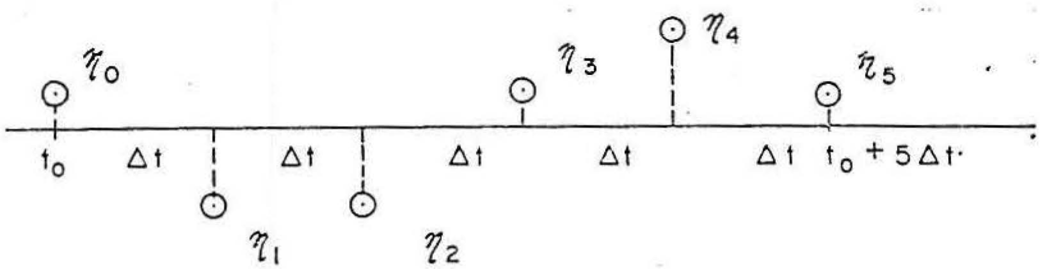


Fig. 1.09-5

I am willing to bet that, if any one of you were handed Fig. 1.09-5 and asked to fair a curve through the points, you would draw the dashed curve in Fig. 1.09-4 and not the solid curve that produced the given values of η_i. The sampling rate is too slow to even suggest the high frequency curve. To pick up the solid curve in Fig. 1.09-4 you must sample at least every 1.25 hours.. Thus, the number of intervals into which you divide your primary must be at least double the frequency of the highest constituent present, say k. Then, if m is the number of subdivisions of the period of the primary cycle, we require

(1.09:11) $$m \geq 2k \quad .$$

For any k, Fourier analysis gives us

(1.09:12) $$C_k = \frac{2}{m}\sum_{i=0}^{m-1} \eta_i \cos\left(k\frac{2\pi}{m}i\right)$$

and

(1.09:13) $$S_k = \frac{2}{m}\sum_{i=0}^{m-1} \eta_i \sin\left(k\frac{2\pi}{m}i\right)$$

where

k ∿ the harmonic

and

m ∿ the total number of increments used in the primary cycle.

From (1.09:12) and (1.09:13) you can compute A_k and α_k from equations (1.09:7) and (1.09:8).

Let me recapitulate the "ideal" method outlined so far:

Suppose we wish to analyse a tide in which we know or hope that the constituents $M_1, M_2, \ldots, M_\nu$ are important. The M is used in the generic sense to represent any constituent here. For these tides we know the corresponding periods $T_1, T_2, \ldots, T_\nu$. What we want to calculate are the amplitudes $A_1, A_2, \ldots, A_\nu$ and the epochs $\alpha_1, \alpha_2, \ldots, \alpha_\nu$.

Step 1: Determine the length of record to be used by computing the synodical period T_s and using a record length $R = \mu T_s$ where μ is any convenient integer.

Step 2: Divide the period T_1 into a sufficient number of equal intervals so that a good plot can be made. Carry out an average for each point within T_1, plot the averages, and fair a curve through them. Repeat Step 2 for each period T_i; $i = 1, 2, \ldots, \nu$.

From Step 2 two kinds of curves will result: pure harmonic and compound.

<u>For the pure harmonic curves</u>:

Step 3: From the graphs read off A_i and from the position of A_i on the time axis relative to the time of the corresponding equilibrium constituent determine α_i for each of the pure harmonics.

<u>For the compound curves</u>:

Since the compound curves are composed of pure harmonics whose frequencies are multiples of the frequency of some primary constituent further steps are necessary. For any such compound curve resulting from an analysis for a constituent with period T_λ, the frequency of the primary is $1/T_\lambda$.

Step 4: Determine, as well as you can, the numbers and frequencies of the constituents present in the compound curve. Suppose T_k to be the period of the shortest constituent present.

Step 5: Divide T_λ into at least 2k equal intervals m and read the corresponding η_i.

Step 6: Compute $H_0 = \langle\eta_i\rangle$.

Step 7: Replace η_i with $\eta_i - H_0 = \eta_i - \langle\eta_i\rangle$.

Step 8: With the new η_i compute C_k and S_k according to equations (1.09:12) and (1.09:13).

Step 9: Compute A_k according to equation (1.09:7) and α_k by equation (1.09:8).

Step 10: Test how closely the values secured in Step 9 agree with the initial compound curve. If the agreement is good, the job is done. If not, repeat Steps 4-10 using more harmonics and keep it up until the agreement is satisfactory.

The values found in Steps 3 and 9 are the values of $A_1, A_2, \ldots, A_\nu$ and $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ required.

It takes very little thought to see why this is still an "ideal" program rather than a practical one. Suppose that you must take into account a modest 20 constituents. Even one synodical period could easily require a 50-year record (Step 1). For each of the 20 periods the record would have to be redivided into equal increments (Step 2). If you think back to the speed numbers you have seen, you will appreciate how messy this could be. For instance, M_1 has a period of 24.84 hours requiring for a 24-point division an interval of 1.03 hours. N_2 has a period of 12.66 hours requiring an interval of 0.55 hours, etc.. After each re-division the values of η must be read off the curve all over again.

The U.S.C.&G.S. avoids the problem by using what is known as the "Standard System." Instead of adjusting the division for the period of each constituent, the values of η are read only at integral values of the mean solar hour. This gives you 24 values of η per mean solar day. These values are then used for every constituent regardless of period. Naturally, the values of η will be a bit wrong for every constituent that has a period not made up of an integral number of mean solar hours.

Remark: With a sampling rate of one per hour, according to what I have said earlier, we can work down to the 6th harmonic of a semidiurnal tide. The U.S.C.&G.S. seems to think that they can get down to the 12th but I don't see how they figure.

The practice with constituents which do not fit this Procustian bed is to assign the value of η read at an integral mean solar hour to the nearer constituent hour. What is going on is shown in Fig. 1.09-6, page 96. If the constituent day is longer than the mean solar day you will now and then pick up a double assignment of η's as at the 15th constituent hour in Fig. 1.09-6. If the constituent day is shorter than the mean solar day, then some constituent hours will have no η assigned to them. In the long run this evens out. Since you are taking averages it just means that there may be one number more or less in the average for some of the η's.

The assignment of η's is carried out at the U.S.C.&G.S. physically by means of stencils. The values of η are tabulated in a standard form, Fig. 1.09-7, page 97. Each constituent has an overlay with holes cut out in the appropriate places. The averages are formed from the numbers which appear when the overlay is in place. I won't bother to describe the stencil. You can find the story in sordid detail in Schureman (1941), pp. 104 et seq.. Actually, the stencil device is a good dodge and might lighten your own data processing chores on occasion. Look into it.

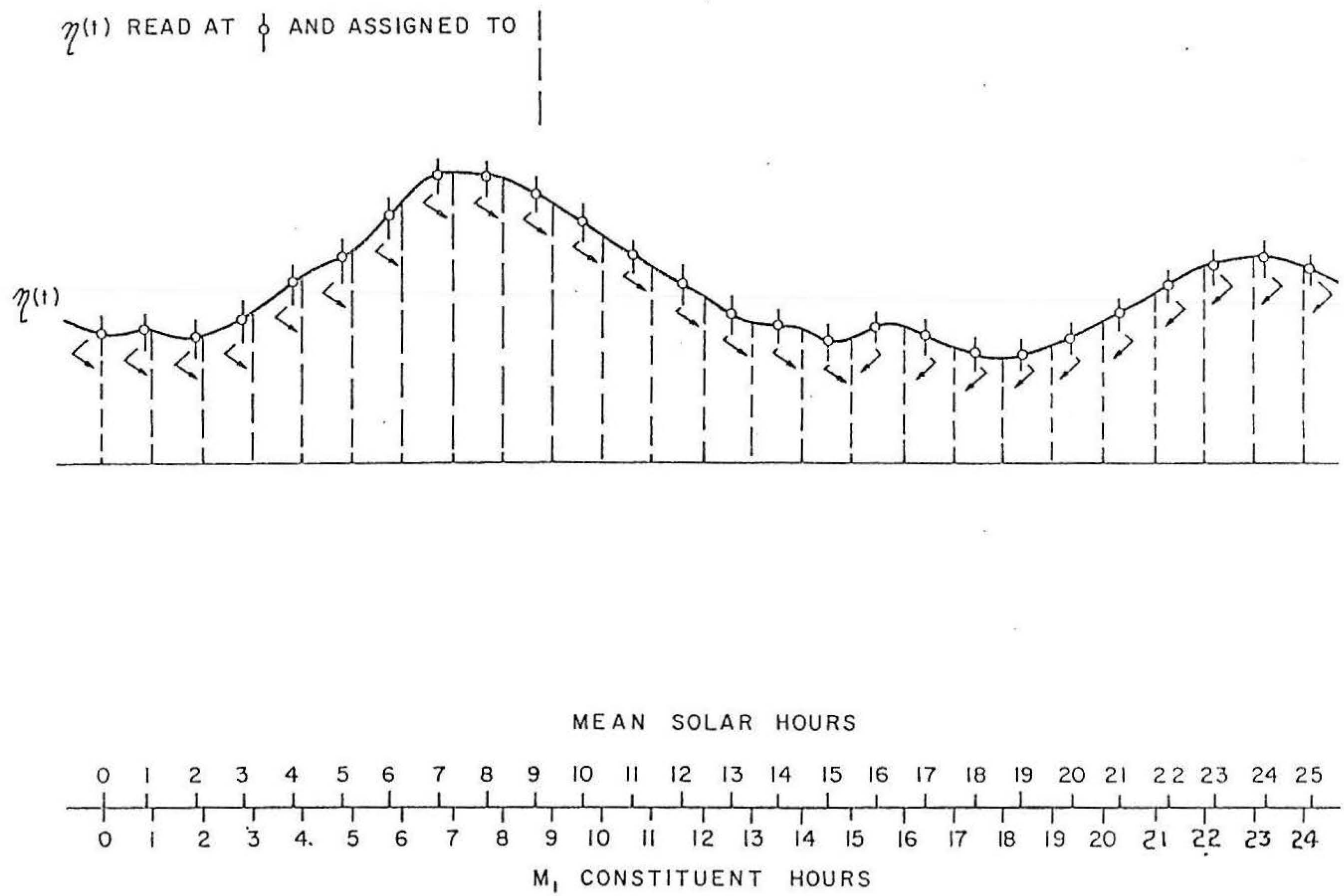

η(t) READ AT AND ASSIGNED TO
η(t)
MEAN SOLAR HOURS
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
M1 CONSTITUENT HOURS

Fig. 1.09-6

Days (mean solar)	1	2	3	4	5	6	7
Hours (mean solar)							
0							
1							
⋮			Measured η's				
23							

Fig. 1.09-7

In this life you don't get something for nothing. The principle of TANSTAFL is universal. While we have reduced reading labor considerably by using the "Standard System," we have introduced a certain amount of error into the averages. This arises because the values of η used to form them differ, more or less seriously, from the values we would have used had we read the record at the correct intervals. The U.S.C.&G.S. compensates for this be extending the constituent to a half-hour overhang at each end of the record. The necessity for the half-interval overhang can be shown as follows:

Consider $\eta = a \cos[at - \alpha]$ with t in solar hours. Let

$t_o \equiv$ the solar time of the exact constituent hour in question

and $\Delta t \equiv$ the solar time span of a constituent hour,

g., for M_1, $\Delta t = 1.035$ hours. The amount of the miss may range from $_o + \frac{\Delta t}{2}$ to $t_o - \frac{\Delta t}{2}$. The mean η over the interval is

$$\bar{\eta} = \frac{1}{\Delta t} \int_{t_o - \frac{1}{2}\Delta t}^{t_o + \frac{1}{2}\Delta t} A \cos[at - \alpha]dt$$

or, integrating and converting to radians,

$$\bar{\eta} = \frac{A}{a} \frac{1}{\Delta t} \frac{180}{\pi} \sin[at - \alpha] \Bigg|_{t_o - \frac{1}{2}\Delta t}^{t_o + \frac{1}{2}\Delta t}$$

or

$$\bar{\eta} = \frac{180A}{\pi a \Delta t} \{\sin[at_o - \alpha + \tfrac{1}{2}a\Delta t] - \sin[at_o - \alpha - \tfrac{1}{2}a\Delta t]\} \quad .$$

Using the trigonometric relation for the sum of sines this can be written

$$\overline{\eta} = \frac{180A}{\frac{1}{2}\pi a\Delta t} \cos[at_o - \alpha]\sin[\tfrac{1}{2}a\Delta t] \quad .$$

Suppose η_o is the value η should have had at t_o. Then

$$\eta_o = A\cos[at_o - \alpha] \quad .$$

Now, if the record with which we are working is very long, it seems reasonable to assume that the repeated misread values of η will be evenly distributed on the interval $(t_o - \frac{1}{2}\Delta t.\ t_o + \frac{1}{2}\Delta t)$, i.e., the average we compute will be a pretty damned good estimate of the value of $\overline{\eta}$ on the interval around t_o. With this in mind, form the ratio

$$\frac{\eta_o}{\overline{\eta}} = \frac{A\cos[at_o - \alpha]}{\frac{180}{\pi}\frac{A}{\frac{1}{2}a\Delta t}\cos[at_o - \alpha]\sin[\frac{1}{2}a\Delta t]}$$

or

(1.09:14) $$\frac{\eta_o}{\overline{\eta}} = \frac{\pi}{180}(\tfrac{1}{2}a\Delta t)(\sin[\tfrac{1}{2}a\Delta t])^{-1} \quad .$$

This ratio is called an <u>augmenting factor</u>. All the material for it you know. Using the mean solar hourly observations and stencils we get $\overline{\eta}$. Therefore

(1.09:15) $$\overline{\eta} \times (\text{the augmenting factor}) = \eta_o \quad .$$

The validity of relation (1.09:15) hinges entirely on the assumption that the record is long enough so that the values going into our estimate of $\overline{\eta}$ are evenly distributed over the interval about t_o.

For most work the U.S.C.&G.S. has tabulated a and Δt for each constituent. They can be found in Schureman (1941), page 228. They are independent of phase.

Another difficulty arises because the tide records with which you must work are truncated, i.e., they are of finite, not of infinite, duration. For one thing, the record has to be at least as long as one cycle of the slowest constituent present. Lopping it off too soon will louse you up and some of the long-period stuff is really long-period, e.g., 19 years. Another gimmick is that if you are trying to separate out constituents of very nearly the same period, you have to have a very very long record to let them appear in a truly representative set of phase relations. Some constituents always seem to foul up on this because they are so close that the records are never, repeat NEVER, long enough. For instance, S_2, the principal solar semi-diurnal constituent, and K_2 and T_2 which differ from it by less than 0.1°/msh. Also, there are K_1, the luni-solar diurnal constituent, and P_1 which have speeds of 15.0411°/msh and 14.9589°/msh and a speed difference of only 0.0822°/msh.

The values for S_2 and K_1 obtained by following the routines outlined above are only approximate, constituents with closely associated periods being, to some extent, still tangled up in them. We allow for this with yet another special correction formula. We take, for example,

(1.09:16) $$\frac{S_2 \text{ observed}}{K_2 \text{ observed}} = \frac{S_2 \text{ equilibrium}}{K_2 \text{ equilibrium}} \quad .$$

This seems a reasonable assumption; particularly since the speeds are about the same. The waves are pretty much alike and what happens to one will probably happen to the other. For example, if one constituent runs into shallow water the other constituent will, inevitably, be running into shallow water too and they can be expected to altered proportionally. The U.S.C.&G.S. calls these things <u>infering constants</u>. The British use a slightly different method for infering constants but they come out about in the same place as we do.

Another correction process called <u>elimination</u> is used because no record is ever long enough to cover enough multiples of every constituent period present in a tide to yield a good statistical estimate.

Translation: Some constituents will have too few numbers in the average to provide a stable estimate.

Suppose we consider

(1.09:17) $$\eta = A\cos[at - \alpha)] + \sum_i B_i\cos[b_i t - \beta_i]$$

where the first term on the right-hand side represents the constituent of interest and the summation all the other constituents present in the record. If the series is long enough--which it never is in practice--

$$\sum_i B_i\cos[b_i t - \beta_i] = 0 \quad .$$

The effect of the non-zero residue is that our estimate is

$$\eta = A'\cos[at - \alpha'] \quad ,$$

i.e., our estimates A' and α' are a bit off from the true A and α. Elimination gives a small correction factor,

$$A = A' \times (\text{a correction factor})$$

$$\alpha = \alpha' \times (\text{a correction factor}) \quad .$$

These elimination factors are given in Table 29 in Schureman (1941).

The methods by which such tables are constructed are always similar and always untidy. Roughly, a guess is made about the residuals B_i, etc.. This guess is fed in and worked out to get a first correction. This gives an ammended guess which is fed back in again. And round and round we go.

The methods discussed so far are for the semidiurnal and diurnal constituents. They are impractical for the long-period constituents which have periods ranging upward from 14 days. For these, hourly values of the record are discarded and mean daily values used instead. Some of the longer ones, e.g., the annual solar constituent, use mean monthly values from the tide record. With this change the methods developed carry through as before. However, if you are sometimes in a bind from short records on the fast stuff, you are always in the bind on the slow constituents.

If you have not already done so, be sure to read "Tidal Datum Planes," especially Chapters II, IV-VIII, and XI at this point.

1.10. Tidal Currents.

So far, we have centered our attention on the vertical component, the tide, to the exclusion of the horizontal components, the tidal current. We must now take steps to repair this omission and to relate the tide and the tidal current.

In Section 1.05 we developed an equation for the vertical tide producing force, (1.05:24), which was expressed in terms of a common coefficient, geodetic coefficients, and long- and short-period celestial factors. At the same time, almost as a by-product, we exhibited the east and north components of the tide-producing force in exactly the same terms, equations (1.05:25) and (1.05:26). It should be clear without further comment that tidal currents are, therefore, subject to exactly the same miserable harmonic analysis as tides and that such things as variation with apogee and perigee, declination, and parallax will appear in tidal currents just as they did in tides.

At the very beginning we mentioned that Gallileo noticed the wave-like appearance of the tide and, perhaps, the best way to get at the relation between the vertical and horizontal motion of the water will be through wave theory. Suppose we have a progressive wave of small amplitude whose profile is

(1.10:1) $$\eta = A\cos[\kappa x - \sigma t]$$

where $\kappa \equiv 2\pi/L$ is the wave number

and $\sigma \equiv 2\sigma/T$ is the frequency.

For a given position, which we may take equal to zero for convenience, what one sees passing the position is

(1.10:2) $$\eta = A\cos[\sigma t] \quad .$$

The horizontal component of the water motion associated with this wave, u, is

(1.10:3) $$u = A\sigma e^{\kappa z}\cos[\kappa x - \sigma t]$$

if the water is <u>deep</u>; $h/L > \frac{1}{2}$, while for <u>shallow</u> water; $h/L < 1/40$, it is

(1.10:4) $$u = \frac{A\sigma}{\kappa h}\cos[\kappa x - \sigma t]$$

or, with $x = 0$,

(1.10:3.1) $$u = A\sigma e^{\kappa z}\cos[\sigma t] \qquad \text{(deep water)}$$

and

(1.10:4.1) $$u = \frac{A\sigma}{\kappa h}\cos[\sigma t] \qquad \text{(shallow water)} \quad .$$

The parameter h is the depth of the water and z is measured along the vertical axis from an origin at the mean sea surface and negative downward.

The tide, considered as a wave, is always in shallow water. For example, consider a semidiurnal tide with $T = 12$ hours = 43,200 seconds. From wave theory we have

(1.10:5) $$L \simeq 5T^2 \quad ; \quad L \text{ in feet and } T \text{ in seconds} \quad .$$

Then the corresponding length is

$$L \simeq 5(43200)^2 = 5\times 432^2 \times 10^4 = 5\times 186624\times 10^4 \simeq 10^{10} \text{ feet} \quad .$$

The ocean is nowhere deeper than 10^5 feet. Therefore

$$h/L \leqq 10^5/10^{10} = 1/10^5$$

which is certainly less than 1/40.

The amplitude in equations (1.10:4) and (1.10:4.1) is

$$\frac{A\sigma}{\kappa h}$$

which, for a particular water depth and progressive wave with constant κ and σ, is constant from surface to bottom since nothing involving z enters.

Remark: In contrast, in equations (1.10:3) and (1.10:3.1) for deep water there is an exponential decay of the amplitude with depth z.

From equation (1.10:2) high tide occurs at $x = 0$ when $t = 0$, $2\pi/\sigma$, $4\pi/\sigma$, ..., $2n\pi/\sigma$, ... and, from equation (1.10:4.1), u also attains its largest positive values at the same times. For $t = \pi/\sigma$, $3\pi/\sigma$, ..., $(2n-1)\pi/\sigma$, ... low tide occurs and u attains its largest value in the reverse direction. Thus, for a progressive wave the strength of the current coincides with the high and low waters.

A standing wave may have the form

(1.10:6) $$\eta = A\cos[\kappa x]\sin[\sigma t]$$

and the horizontal velocity for shallow water is given by

(1.10:7) $$u = \frac{A\sigma}{\kappa h}\sin[\kappa x]\cos[\sigma t] \quad .$$ (See Lamb, Ch. IX.)

Again, the current is unmodified by depth and for $t = \pi/2\sigma$, $3\pi/2\sigma$, ..., $(2n-1)\pi/2\sigma$, ... and a given x position, η is either at high or low tide while $u = 0$. The tidal current, u, has its maximum values when η is at half-tide, $t = 0, \pi/\sigma, \ldots, n\pi/\sigma, \ldots$. Thus, for the progressive wave the strength of the tide corresponds to high and low water while, for the standing wave, the strength of the tide corresponds to half-tide. Tides occuring in nature are seldom purely progressive or purely standing so that the relations exhibited can not be expected to hold exactly.

The results for progressive and standing waves are intuitively appealing. For a progressive wave, Fig. 1.10-1, in progressing from a crest at position A to a crest at position A' the water must rise to a peak

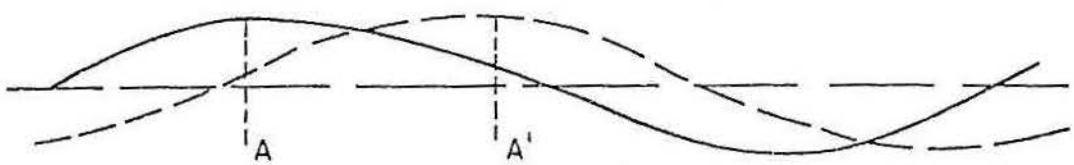


Fig. 1.10-1

at each successive point between A and A'. Under the crest there can be no vertical component of the water motion or the tide would continue to rise. The entire motion must be horizontal.

For a standing wave, Fig. 1.10-2, the water sloshes to one end and then back to the other. When we have high water at B it is clear that there can be no flow of water since such a flow would force a further change in the water level. At half-tide, when the water surface is level, the

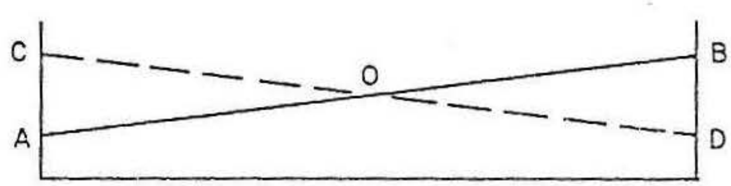


Fig. 1.10-2

the current is flowing most strongly toward AC to drain B down to a low at D and raise A to a high at C. When the high at C is reached the current must again be zero since, if it were not, there would be further change in the water level.

Tidal currents must be carefully distinguished from currents due to other causes such as gravity, density, wind, etc.. Tidal and nontidal currents almost always occur together; the actual current observed being the resultant. However, in the open ocean one or another of the kinds of currents may predominate. In the open ocean tidal currents are usually very weak. They are stronger near the coasts and in constrictions, e.g., entrances to bays and straits, where they may really roar. Let's begin by looking at the tidal currents in bays and rivers.

In the entrance to a bay or in a river and, in general, where the water is constricted, the tidal current is of the reversing rectilinear type. The flood current runs upstream for about six hours followed by the ebb current which runs downstream for about the same length of time. Figure 1.10-3 shows a current curve for the Narrows in New York Harbor. A curve like this is produced by measuring the current velocity every hour, plotting flood values above the zero velocity line and ebb values below, and fairing a curve through the points. The curve looks a lot like the

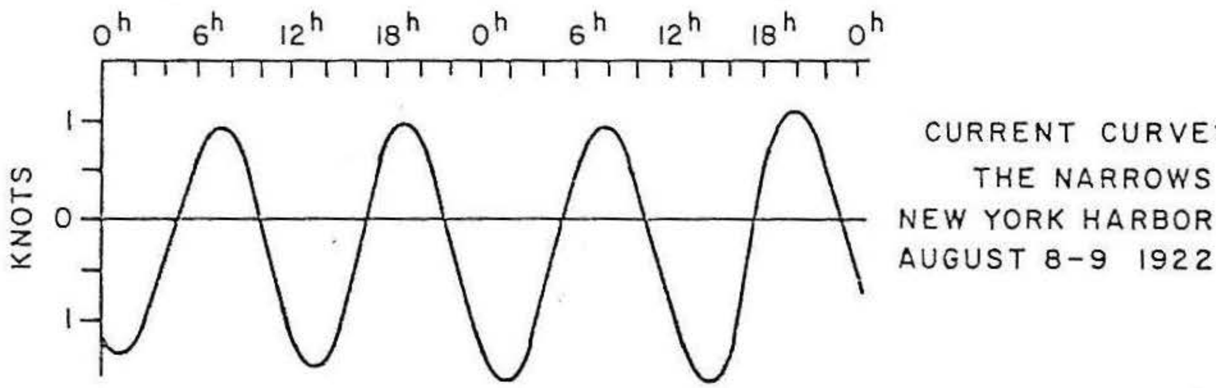


Fig. 1.10-3

tidal curve. Maximum velocity of the flood current is called the <u>strength of flood</u> and is analogous to high water on the tide curve although it doesn't necessarily occur at the same time. Maximum velocity of the ebb is called the <u>strength of ebb</u> and corresponds to low water. The current day, like the tidal day, is best expressed in lunar time.

Theoretically, the current in shallow water should extend unaltered from top to bottom, equations (1.10:4) and (1.10:7). Actually, for a real fluid, which has viscosity, this can not be the case. Tidal currents are the same over much of the depth but near the bottom they drop to about 2/3 of the surface value. There are many effects that can obscure this simple picture. Due to drag, the currents are generally slower near the sides of a channel than they are in the middle. As a rule of thumb, the average velocity of the tidal current across a section will be about 3/4 the central surface velocity. To show how badly this rule of thumb may serve you, consider Fig. 1.10-4, page 106. It shows a current profile taken off Bloody Point, Maryland by the Chesapeake Bay Institute. Here the surface current is ebbing at 0.26 kt. At 10 feet it is ebbing at 0.43 kt. From 35 to 45 feet it is practically slack while below that it is still flooding more strongly than the surface layer is ebbing. Current profiles like this are characteristic of two-layered estuaries like the Chesapeake Bay.

The effects of nontidal currents on tidal currents can be intuited in a general way. Consider Fig. 1.10-5. Referred to the zero velocity line, AB, a pure tidal current would be as shown. Strength of ebb

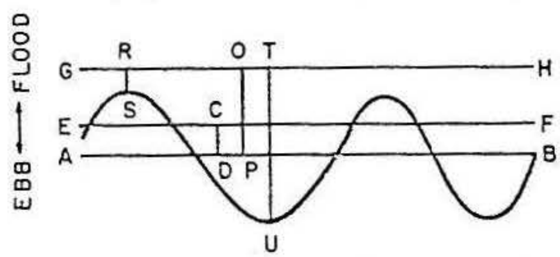


Fig. 1.10-5

and flood are equal and so are their durations. Now, suppose a nontidal current of velocity CD in the ebb direction is superimposed. Then the strength of ebb is increased by CD while the strength of flood is reduced by the same amount. The current picture can be had by moving the zero velocity line parallel to AB through a distance CD which puts it at EF. The tide obviously ebbs longer and stronger. If the velocity of the superimposed nontidal current is greater than the strength of flood, say CP, then the zero reference line moves to GH and there are no slack waters; the current is constantly ebbing. The composite current appears as a pulsating direct current with minimum speed RS and maximum speed TU. It should

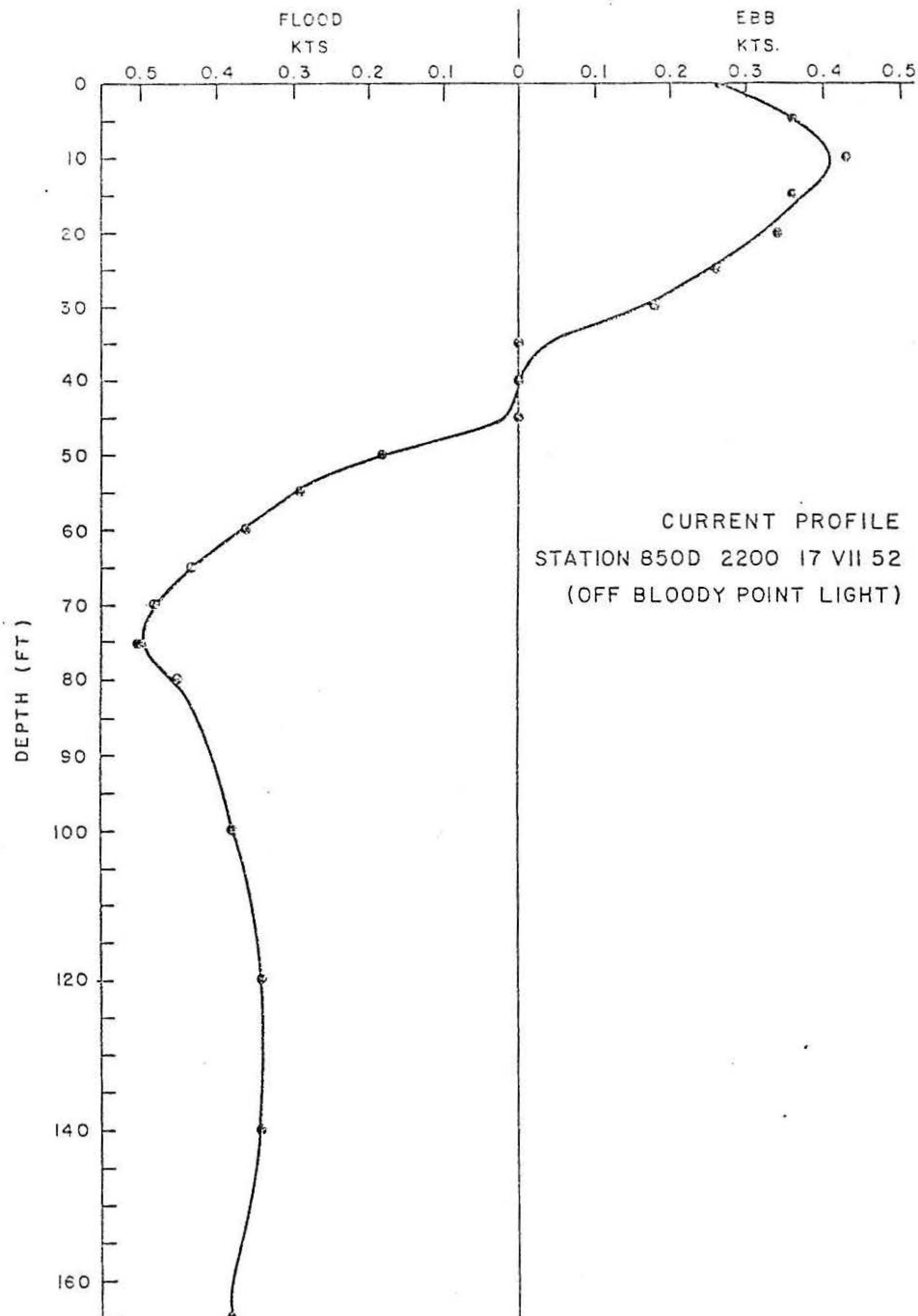

FLOOD
KTS
EBB
KTS.
0.5 0.4 0.3 0.2 0.1 0 0.1 0.2 0.3 0.4 0.5
0
10
20
30
40
50
60
70
80
90
100
120
140
160
168
DEPTH (FT)
CURRENT PROFILE
STATION 850D 2200 17 VII 52
(OFF BLOODY POINT LIGHT)
BOTTOM
BOTTOM

Fig. 1.10-4

hardly be necessary to mention that the direction of a composite current will be in the direction of the vector resultant of the component current vectors.

Tidal current curves come in the same assortment as tide curves. As examples, the Hudson has an equal semi-diurnal tidal current curve, Mobile Bay a diurnal, and Rich Passage, Puget Sound an unequal semidiurnal. Where there is an inequality the difference in the tidal current curve is usually less than the difference in the tide curve.

As we have pointed out, for a pure progressive wave the strength of the current occurs at the times of high and low water while for a standing wave the strength of the current occurs at half-tide. Since neither of these cases occur pure in nature, we expect a difference in the times of occurance, but, whatever relation strength of current and high water have at a particular place, the relation is maintained.

The distance traveled by a floating object during a tidal cycle can be determined from tidal current curves. The vertical distance traveled by such an object constrained at a fixed point is, of course, simply the range of the tide curve. The horizontal distance covered by a free floating object is not quite so simple. If the tidal current were a step function, you could multiply the velocity of flood--there would only be one--by the duration and similarly for the ebb: take the difference to get the distance. However, the tidal current is continually changing so that you must use the average velocities over the duration to get the distance. You can estimate this average in several ways. You can read off along the tidal current curve at more or less closely spaced intervals and use a desk calculator. You can planimeter the area between the curve and the zero line and then divide by the length of the zero line. Or, if you can kid yourself that the curve is pretty close to a cosine curve over a zero to zero loop, you can use

$$\frac{2}{\pi} v_{max} = 0.637 v_{max} = \overline{v}$$

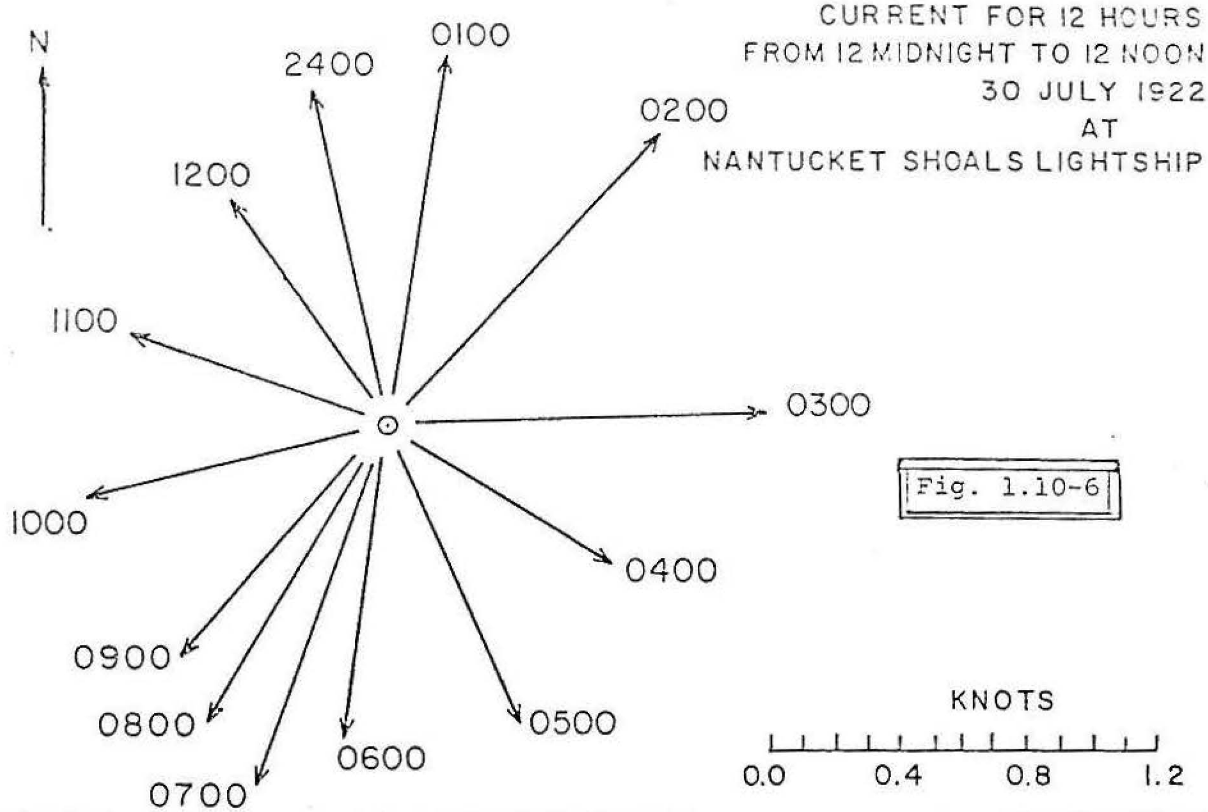
for the average velocity over the loop. The tidal excursion can be computed from the mean velocity by multiplying it by the duration. All this is very approximate. It is based on the assumption that the object in question moves exactly with the water which surrounds it. Where this is not the case you had better start piling on the corrections.

The duration of slack water needs brief comment. Mathematically, zero velocity occurs only at a point in time so that can't be what we mean by slack water. Because of the difficulty of measuring very slow currents it has become the custom to consider any period during which currents are less than 0.1 kt as slack water. Slack water thus becomes an interval about the zero velocity crossing. With a limit of 0.1 kt it is easy to compute the duration of slack water for a pure cosine curve.

Strength (kt)	Duration of Slack (min)	
	Semidiurnal	Diurnal
1	24	48
2	12	24
3	8	16
4	6	12
5	5	10
6	4	8
8	3	6

You should distinguish carefully between the velocity of the current which is an actual bodily movement of the water and the rate of advance (progression) of the tide which is the movement of the wave-form. The progression of the tide is usually many times faster than the velocity of the current.

In the open sea, where tidal currents are not restricted by banks, tidal currents may flow in any direction and are not usually of the reversing type. They change direction as well as magnitude continuously and are called rotary currents. Figure 1.10-6, page 109, shows the current for 12 hours from 12 midnight to 12 noon, 30 July 1922, at Nantucket Shoals Lightship. The current is seen to have rotated once clockwise in a little more than 12 hours. In a crude way, the tips of the vectors have traced out an ellipse. Since this is a single record accidental superimposed

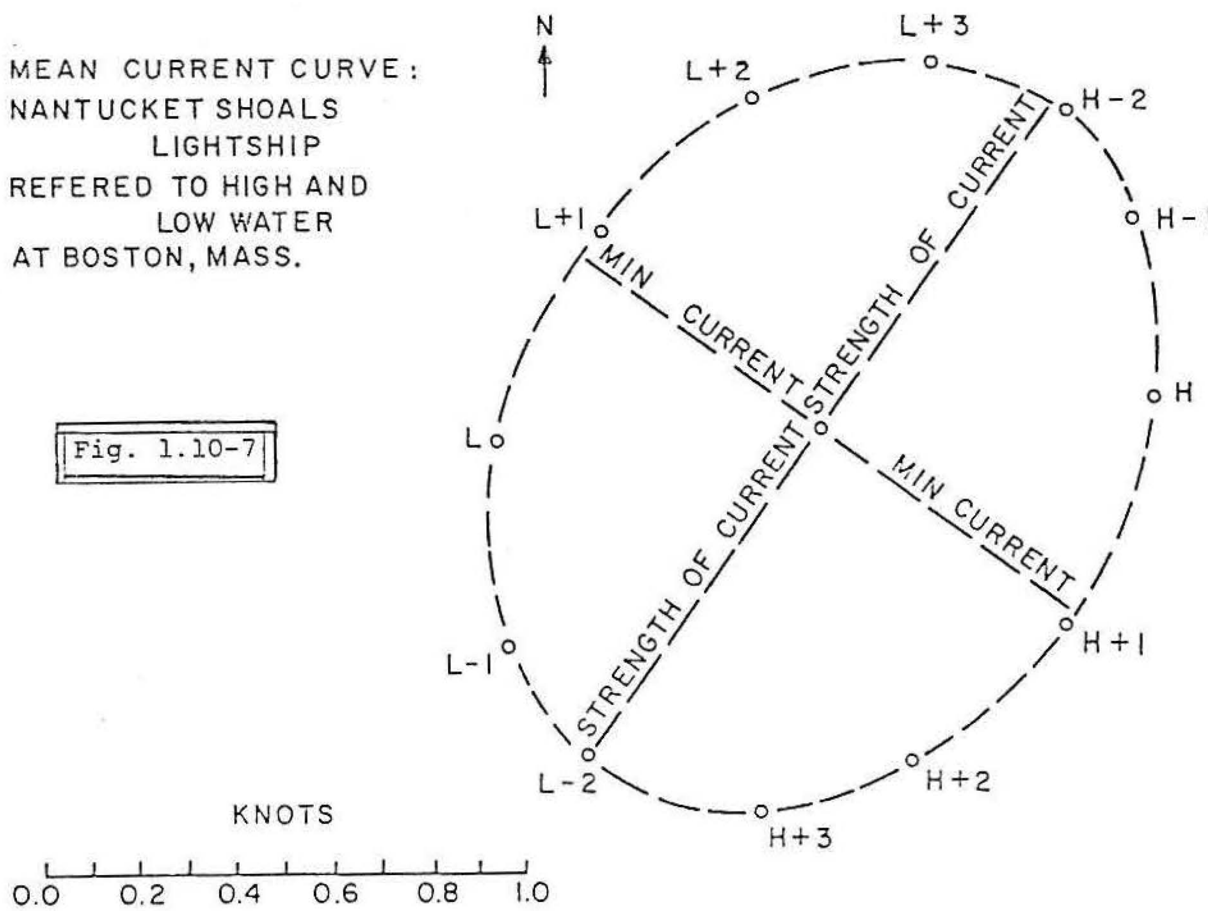


CURRENT FOR 12 HOURS
FROM 12 MIDNIGHT TO 12 NOON
30 JULY 1922
AT
NANTUCKET SHOALS LIGHTSHIP

Fig. 1.10-6

MEAN CURRENT CURVE:
NANTUCKET SHOALS
LIGHTSHIP
REFERED TO HIGH AND
LOW WATER
AT BOSTON, MASS.

Fig. 1.10-7



currents are present. If a long record were taken and the average considered it would be found that the ellipse was much better and the period about 12 hours 25 minutes, i.e., the current day for a rotary current, like the tidal day, is 24 hours 50 minutes.

Characteristically, a rotary current shows no slack water but there are maxima and minima of the speed corresponding to the semi-major and semi-minor axes of the ellipse. These are related to each other in the same way that high and low water are related to slack water in the reversing tide.

Since the current day corresponds to the tidal day, it is convenient in determining the average hourly velocity and direction of a rotary current to use times of high and low water at some nearby place as a reference. For example, in Fig. 1.10-7 the average hourly values of current velocity at Nantucket Shoals Lightship are referred to the times of high and low water at Boston, Massachusetts. In this figure H and L stand for high and low water at Boston and the numbers give the hours before or after.

The major features of a rotary tidal current at any place are specified by the major and minor axes of the ellipse which determine the ellipticity, the direction of rotation, and the direction of the major axis. In general, tidal currents rotate clockwise in the northern hemisphere and counterclockwise in the southern hemisphere although local hydrographic conditions may produce exceptions. Rotary tidal currents show the same periodic fluctuations as reversing tidal currents and can also be grouped into semidiurnal, diurnal, and mixed.

Nontidal currents can distort rotary currents in many ways, Fig. 1.10-8, page 111. A strong current can move the "center" entirely outside the "ellipse" as shown.

I suggest that you consult U.S.C.&G.S. sp. Pub. No. 215 and Sp. Pub. No. 230, page 16 et seq. for further information on measurement and reduction of tidal current data.

It remains only to mention briefly hydraulic currents. These currents are not really tidal currents. They are found in narrow straits

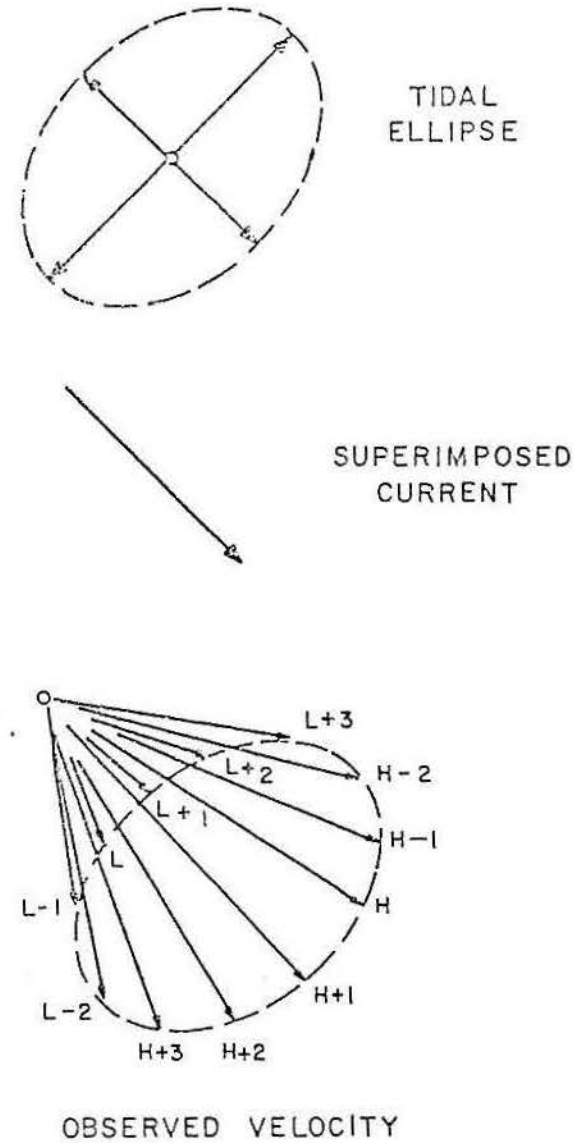


Fig. 1.10-8

connecting two bodies of water which have unequal tidal ranges or in which, although the tides have the same range, they are out of phase. The currents are of the reversing type but are not simple cosines. They are not directly caused by the tide producing forces but result from the difference in head at the ends of the channel--hence "hydraulic" currents--and, consequently, are more appropriately considered in a fluid mechanics course. For the velocity of an hydraulic current where $\eta_A(t)$ and $\eta_B(t)$ are the elevations at the two ends of the channel we have

$$v = C\sqrt{|\eta_A - \eta_B|} \quad .$$

The sign difficulty is met by using the absolute difference of the heights and then assigning the direction after the square root is taken. Although there is no natural ebb and flood direction, the terms are often used. They are assigned arbitrarily. The velocity would follow this formula if there were no inertia or friction. Actually, there is usually a time lag of 10 to 15 minutes in the response. The C includes gravity and other things that adjust the units. A theoretical value may be computed for C but in practice C varies enough from the theoretical to make it an empirical constant for each hydraulic current. You can find material on hydraulic currents under discussions of flow in open channels in most of the regular references: Rouse, Venard, Lamb, Milne-Thompson, etc.

For the oceanographer and the seaman hydraulic currents may be important in narrow straits and behind islands. Examples are: Messina, Woods Hole, Cape Cod Canal, and Hellgate.

This concludes our obligation to tides as a howling empiricism in the service of practical people. We are now free to move on to the never-never land of tidal dynamics.

Chapter 2. Tidal Dynamics.

2.01. Introduction.

Tides, as it now stands, is essentially an empirical science. The dynamics of tides, that is, the response of the sea to the tide generating forces, is embryonic. The problem is extremely difficult. Briefly it may be stated as: Given an ocean basin of a specified shape, filled with water of known properties, and a specific tide-producing force find the tides induced in the water. Obviously, to solve such a problem for any realistic set of conditions is a task to stagger the imagination.

We must content ourselves with setting up simplified mathematical models whose solutions are within our powers. From these we can hope for some insight into the necessary connections among things. In this chapter we will make a rapid Cook's Tour of some of the more interesting mathematical models. We will frequently discuss the possibility of applying these highly simplified mathematical models to the real ocean. This is a useful exercise--particularly if you are an oceanographer and care about what goes on in the ocean rather than a mathematician with a primary concern with interesting mathematical problems. That the mathematical models fail to describe the real ocean very well will be no surprise since they have been chosen for their mathematical tractability rather than for their realism.

2.02. Airy's Wave in a Canal.

There are two facts about tides which are pretty definitely established:

(1) The tide generating forces: These are well determined and are inaccurate only to the extent of the neglected terms in the series

expansion exhibited in Chapter 1. In setting up mathematical models it is customary to simplify the tide producing force by making it a simple cosine. Since the curve describing any force can be synthesized from simple harmonics this is a reasonable simplification and it does cut down on the work.

(2) The form of the sea surface: The tide has the form of a sum of small-amplitude, Airy waves,

$$\eta = \sum A_i \cos[\theta_i]$$

where it is assumed that $A_i/\lambda_i << 1$, Fig. 2.02-1

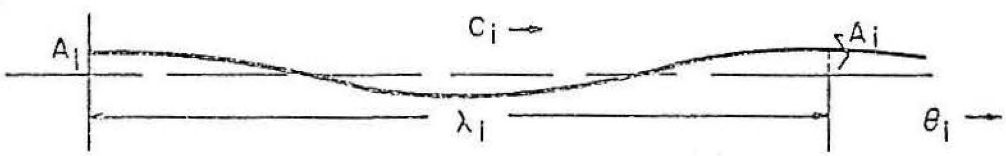


Fig. 2.02-1

This has been fairly well established for a record at any single point with time and also for the entire ocean at any given instant of time. For shallow water, of course, the component cosine waves become distorted. For the ocean the maximum known amplitude for the tide is about 10 meters. At a conservative estimate the corresponding wave length is roughly $\lambda = 10$ kilometers. Consequently, in the extreme case $A/\lambda = 10/10^4 = 10^{-3} << 1$ so that the small amplitude condition is always met and the Airy wave is a valid approximation.

In tidal dynamics it is essential to distinguish between a free oscillation and a forced oscillation. A free oscillation results when a stable equilibrium is disturbed by a force which is then removed. The system continues to oscillate about its equilibrium position with a period which depends in no way on the initial disturbing force. A forced oscillation is produced by the continuous application of a periodic force which determines the period of oscillation of the system to which it is applied.

Any system in stable equilibrium, if disturbed, has its own natural period of oscillation. For example, a simple pendulum that is initially displaced through a small angle has a natural period of

(2.02:1) $$T = 2\pi\sqrt{L/g}$$.

Basins of water also have natural periods.

For a progressive wave in shallow water $\lambda >> h$. Its phase speed is

(2.02:2) $$c = \sqrt{gh}$$.

Therefore, its period is

(2.02:3) $$T = \lambda/\sqrt{gh}$$.

This is the period of a progressive Airy wave in shallow water of depth h. It has sometimes been called a natural or free period but the usage seems to me a bit forced.

As our first model, consider an earth whose ocean consists of a narrow equatorial canal of depth h, Fig. 2.02-2. Suppose that there is a wave in this canal whose wave length is the earth's circumference, i.e.,

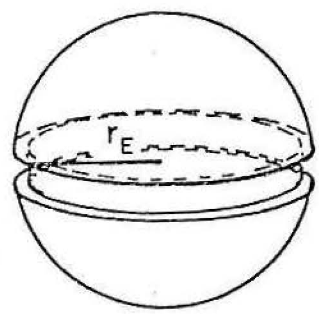


Fig. 2.02-2

$\lambda = 2\pi r_E$. Then the period is

(2.02:4) $$T = 2\pi r_E/\sqrt{gh}$$.

If $r_E \simeq 6\times10^3$ km ; $g \simeq 10^3$ cm/sec^2 ; $h = 1$ km ,

then $$T \simeq 105 \text{ hours}$$.

If h = 10 km, then

$$T \simeq 33.2 \text{ hours}$$.

In any case, if the wave is produced by a force with exactly the same period as the free oscillation you have resonance and the wave grows higher and higher as the energy feeds in.

Airy developed a model for a forced progressive wave in a canal like that of Fig. 2.02-2 but circling any parallel of latitude. Clearly, such a model has little hope of any application since the Antarctic is the only uninterrupted band of water at any latitude. Another feature of Airy's model is his neglect of Coriolis force so that it could apply only at the equator unless adequate restraining walls were present. Basically, what Airy did was to apply Bernoulli's theorem to a stream tube using some assumptions borrowed from Laplace. He neglected friction and he assumed that any such wave would have complete symmetry with the tide generating force, i.e., you have one or the other of the situations shown in Fig. 2.02-3.

Fig. 2.02-3

The wave is a forced wave moving at the same speed as the generating force. We will symbolize this forced speed by c_F. Don't confuse it with the free speed usually symbolized by c.

Remark: We get c for speed from our British cousins. It comes from "celerity" which also means speed.

We consider a canal in which the forced wave is traveling from east to west with a speed c_F. Using the artifice of steady motion, we take a set of axes moving from east to west at the speed c_F so that the wave remains in the same position relative to the moving axes at all times.

With an Airy wave in shallow water and with the dimensions we are considering, the vertical component of the water particle motion is negligible in comparison with the horizontal component, u. Under the crest of the wave the horizontal component of the velocity has the same direction as the wave profile motion while under the trough it is opposite.

Let Ω be the potential of the tide generating force. Then with

$u \equiv$ the horizontal component of the orbital velocity

$c_F \equiv$ the forced wave velocity

$g \equiv$ the acceleration due to gravity

$h \equiv$ the depth of the canal, assumed constant

$\eta \equiv$ the surface elevation of the wave form

$p \equiv$ the pressure

and $\rho \equiv$ the density of the water,

if the motion is irrotational, we may write Bernoulli's equation in the form

(2.02:5) $$\tfrac{1}{2}(u - c_F)^2 + g(h + \eta) + \frac{p}{\rho} + \Omega = \text{a constant} \quad .$$

Of the terms on the left-hand side the first represents the kinetic energy per unit mass, the second the potential energy per unit mass, the third is the pressure term, and the fourth the tide generating potential.

At the sea surface p is atmospheric pressure and is often taken as constant since the variations in atmospheric pressure are a negligible part of the total pressure only a short distance beneath the water surface. Let us also assume that g, ρ, and c_F are constants. The canal depth, h, has already been taken constant. Then collecting all the constant terms in (2.02:5) and transposing them to the right-hand side

(2.02:6) $$\tfrac{1}{2}u^2 - c_F u + g\eta + \Omega = \text{a constant} - \tfrac{1}{2}c_F^{\,2} - gh - \frac{p}{\rho} \quad .$$

The right-hand side of (2.02:6) is just another constant so that

(2.02:7) $$\tfrac{1}{2}u^2 - c_F u + g\eta + \Omega = \text{another constant} \quad .$$

The volume flow rate through any two sections across the canal must be equal or there will be a pile-up of water somewhere. (Remember that the canal is a ring.) Consider the mean section with depth h and width W. Then we have

$$c_F hW = (c_F - u)(h + \eta)W \quad ,$$

u being zero at the mean section. This is a version of the equation of continuity. Doing a bit of juggling we get

(2.02:8) $$\frac{u}{c_F} = \frac{\eta}{(h + \eta)} \quad .$$

If now we assume that the canal is quite deep so that $\eta \ll h$, equation (2.02:8) tells us that

(2.02:9) $$\frac{u}{c_F} \approx \frac{\eta}{h} \ll 1$$

so that

(2.02:10) $$u \ll c_F$$

and

(2.02:11) $$u^2 \ll c_F u \quad .$$

We can therefore under these conditions neglect $\frac{1}{2}u^2$ in comparison with $c_F u$ in equation (2.02:7) to get

(2.02:12) $$c_F u - g\eta + \Omega = \text{a constant} \quad .$$

We are now ready to pick up some results from our previous work on the tide producing forces. In those discussions it was shown that the horizontal component of the tide producing forces, F_H, could be written in terms of the potential as

(2.02:13) $$F_H = \frac{1}{r_E}\frac{\partial\Omega}{\partial\theta} \qquad (1.06:8)$$

or alternatively as

(2.02:14) $$F_H/g = -\frac{1}{r_E}\frac{\partial\bar{\eta}}{\partial\theta}$$

where

$\theta \equiv$ the geocentric angle

and $\bar{\eta} \equiv$ the equilibrium tide.

Eliminating F_H between (2.02:13) and (2.02:14)

(2.02:15) $$\frac{\partial\Omega}{\partial\theta} = -g\frac{\partial\bar{\eta}}{\partial\theta} \quad .$$

Integrating (2.02:15) we have

(2.02:16) $$\Omega = -g\bar{\eta} + \text{a constant} \quad .$$

As previously set up, when $\bar{\eta} = 0$, $\Omega = 0$ so that the constant is zero and

(2.02:17) $$\Omega = -g\bar{\eta} \quad .$$

Substituting (2.02:17) in the form of Bernoulli's equation given by (2.02:12) we have

(2.02:18) $$-c_F u + g\eta - g\bar{\eta} = \text{a constant}$$

which may be written

(2.02:19) $$g(\eta - \bar{\eta}) = c_F u + \text{a constant} \quad .$$

WARNING: $\bar{\eta}$ is the equilibrium tide. It is not the same thing as η nor is it the mean value of η. Watch your step!

Now η, $\bar{\eta}$, and u are all periodic functions of the geocentric angle, θ, and they are all zero together at half-tide for a progressive wave. Therefore, the constant is zero and

(2.02:20) $$g(\eta - \bar{\eta}) = c_F u \quad .$$

Now, by (2.02:9) we saw that $u/c_F \approx \eta/h$ so that

(2.02:21) $$u \approx \frac{c_F \eta}{h}$$

and, substituting in (2.02:20),

(2.02:22) $$g(\eta - \bar{\eta}) = \frac{c_F{}^2 \eta}{h} \quad .$$

Solving for η gives

(2.02:23) $$\eta = \frac{\bar{\eta}}{1 - \dfrac{c_F{}^2}{gh}} \quad .$$

Since the square of the free wave speed is

(2.02:24) $$c^2 = gh$$

this can be put in the quite attractive form

(2.02:25) $$\eta = \frac{\bar{\eta}}{1 - \left(\dfrac{c_F}{c}\right)^2} \quad .$$

There are three cases depending on the relative values of the free and forced speeds:

Case I:

$$c_F > c \quad ; \quad (c_F/c)^2 > 1 \quad \text{and} \quad \bar{\eta} > 0 \sim \eta < 0 \quad .$$

In other words, a crest in the equilibrium tide corresponds to a trough in the forced wave. This is called an <u>inverted tide</u>.

Case II:

$$c_F < c \quad ; \quad (c_F/c)^2 < 1 \quad \text{and} \quad \bar{\eta} > 0 \sim \eta > 0 \quad .$$

In other words, the configuration of the forced wave is the same as that for the equilibrium tide but the amplitude is greater.

Case III:

$$c_F = c \quad ; \quad (c_F/c)^2 = 1 \quad \text{and} \quad \bar{\eta} > 0 \sim \eta = \infty \quad .$$

Resonance occurs----------Duck!! Since $(c_F/c)^2 = (c_F^2)/(gh)$, as $h \to \infty$, $(c_F/c)^2 \to 0$ and $\eta \to \bar{\eta}$, i.e., the forced wave approaches the equilibrium tide.

The value of c_F is determined by the wave length λ, in this model the length of the canal, and the period of the tide generating force, T; $c_F = \lambda/T$. As our canal is moved to higher parallels of latitude its length decreases so that $c_F \to 0$ as $\phi \to \pm 90°$. This implies that $(c_F/c)^2 \to 0$ as well and, from equation (2.02:25), that $\eta \to \bar{\eta}$. Actually, this is pretty realistic behavior since polar-basin tides correspond very well to equilibrium tides.

Proudman has computed some values for 60°S where the Antarctic forms the only available case of a continuous band of water along a parallel of latitude. Since the resonance condition is $(c_F/c)^2 = 1$ and $c^2 = gh$, we can compute the depth of water for which resonance should occur for any c_F,

$$h = c_F^2/g \quad .$$

For a semidiurnal c_F and 60°S latitude, resonant h turns out to be 5440 meters. The average depth at 60°S latitude is nearer 4000 meters so that conditions aren't right for resonance.

We could go on to compute the c_F appropriate for resonance with h = 4000 meters and, in general, butch equation (2.02:25) hither and yon but the model is so totally unrealistic that you have to be a real masochist to do it.

2.03. A Free Wave in Airy's Canal on a Rotating Earth.

Lord Kelvin worked out a version of Airy's canal that is modified in two respects. First, he included Coriolis force which is a distinct improvement on Airy and, second, he used a free wave rather than a forced wave which is rather a set back for tides. The result is still unrealistic so far as available configurations on this planet go but the inclusion of the earth's rotation brings out a property we often see in the tide.

If we use:

$u \equiv$ the east-west velocity component,

$v \equiv$ the north-south velocity component,

$w \equiv$ the vertical velocity component, positive downward,

$L \equiv$ the latitude of the canal,

$\omega \equiv$ the angular velocity of the earth's rotation,

$p \equiv$ the pressure,

and $\rho \equiv$ the water density,

then the equations of motion, neglecting friction and Reynolds stresses and such, are

$$\frac{Du}{Dt} - 2\omega(\sin[L])v - 2\omega(\cos[L])w = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.03:1}$$

$$\frac{Dv}{Dt} + 2\omega(\sin[L])u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.03:2}$$

$$\frac{Dw}{Dt} + 2\omega(\cos[L])u = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g \tag{2.03:3}$$

where D/Dt is the material or Stokes derivative, formally

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad . \tag{2.03:4}$$

If we assume that the gradients of the velocity components are small in comparison with the velocities, i.e., that u, v, and w are slowly changing functions of position so that we may write for the material derivative in every dase $D/Dt \equiv \partial/\partial t$ and, if we assume that the vertical velocity w is negligibly small, $w \approx 0$, equations (2.03:1, 2, and 3) become

$$\frac{\partial u}{\partial t} - 2\omega(\sin[L])v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.03:5}$$

$$\frac{\partial v}{\partial t} + 2\omega(\sin[L])u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.03:6}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g \quad . \tag{2.03:7}$$

Equation (2.03:7) is the hydrostatic equation.

If now we further assume that the atmospheric pressure on the water surface is constant, from equation (2.03:7) we get

(2.03:8) $$p = p_{atm} + g\rho(\eta + z)$$

so that

(2.03:9) $$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

and

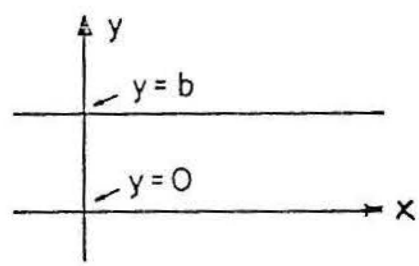
(2.03:10) $$-\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \eta}{\partial y}$$

and, substituting in (2.03:5) and (2.03:6),

(2.03:11) $$\frac{\partial u}{\partial t} - 2\omega(\sin[L])v = -g\frac{\partial \eta}{\partial x}$$

(2.03:12) $$\frac{\partial v}{\partial t} + 2\omega(\sin[L])u = -g\frac{\partial \eta}{\partial y} \quad .$$

To get specific, let's take our canal in the northern hemisphere and let x be along the canal with the positive sense in the direction in which the wave is traveling. Let the y-axis be across the canal with the origin at one side and let the canal's width be b. The Coriolis force will deflect any current to the right. On the boundaries



at y = 0 and y = b we must have the cross-channel velocity component v = 0. Kelvin made the assumption that the canal was so narrow that v = 0 on $0 \leq y \leq b$. With this assumption (2.03:11) and (2.03:12) reduce to

(2.03:13) $$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

and

(2.03:14) $$2\omega(\sin[L])u = -g\frac{\partial \eta}{\partial y} \quad .$$

The form of the equation of continuity which applies here is

(2.03:15) $$h\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t} \quad .$$

It relates the amount of water flowing into or out of a region and the local rate of change of elevation. A solution of (2.03:13) and (2.03:15) is

(2.03:16) $$u/c = \eta/h \qquad \text{where} \qquad c = \sqrt{gh} \quad .$$

Remark: This conforms to the results of classical wave theory:

$$u = u_o \cos[(2\pi/\lambda)x - (2\pi/\tau)t] \quad ; \quad \eta = \eta_o \cos[(2\pi/\lambda)x - (2\pi/\tau)t] \quad .$$

What we want from (2.03:13), (2.03:14), and (2.03:16) is a functional relation between η and y, i.e., we want to know how the elevation of the water surface changes across the channel.

Solving (2.03:16) for u and substituting in (2.03:14),

$$2\omega(\sin[L]) \frac{c\eta}{h} = - g \frac{\partial \eta}{\partial y}$$

or

$$\frac{\partial \eta}{\partial y} = - 2\omega(\sin[L]) \frac{c}{gh} \eta$$

or, since $c = \sqrt{gh}$,

(2.03:17) $$\frac{\partial \eta}{\partial y} = - \frac{2\omega(\sin[L])}{c} \eta \quad .$$

Because we are dealing with a narrow canal oriented east-west, we may take L = constant. Integrating (2.03:17) with respect to y gives

(2.03:18) $$\ln\{\eta\} = - \frac{2\omega(\sin[L])}{c} y + B'$$

or

(1.03:19) $$\eta = B \exp\{- \frac{2\omega(\sin[L])}{c} y\} \quad .$$

For y = 0: $\eta = B \equiv \eta_o \sim \eta \text{ at } y = 0$.

Therefore,

(2.03:20) $$\eta = \eta_o \exp\{- \frac{2\omega(\sin[L])}{c} y\} \quad .$$

This is something you see over and over again: exponential decay. The surface displacement η has its largest value in the cross-channel direction on the right-hand side looking in the direction of the current and it falls off exponentially toward the left-hand side. The fall off is governed by two things: the Coriolis force represented by $2\omega(\sin[L])$ and the wave speed c.

Exercise: Run through the effects of changing latitude and changing depth for yourself.

Looking at a cross-section in the direction of the wave, Fig. 2.03-1, page 124, the "tides" are less on the left. Naturally, this reverses in the

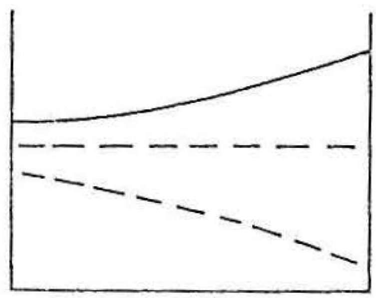


Fig. 2.03-1

Southern hemisphere.

Although the assumptions of the model are unrealistic, this cross-channel feature does seem to occur in nature. Martin Pollak found a tendency to greater tidal ranges along the Eastern Shore of Chesapeake Bay and similar observations have been obtained for Long Island Sound and for the Straits of Dover. They prove nothing. There are too many neglected features of the real world. Still, the correspondence is intriguing.

If one chugs through the rest of the solution, one gets, for Kelvin's version of Airy's canal,

$$\eta = \eta_o \exp\{-\frac{2\omega\sin[L]}{c} y\} \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \qquad (2.03:21)$$

and

$$u = \eta_o \frac{c}{h} \exp\{-\frac{2\omega\sin[L]}{c} y\} \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \qquad (2.03:22)$$

which give the "tidal" height and the current velocity at all points and times in the canal.

2.04. Horizontal Crested Waves and Rotating Currents.

In section 2.03 we saw that in a narrow canal where v was held zero in the cross channel direction, the surface of the water, η, responded by taking on an exponential variation with y. We got this result by solving the equations of motion and continuity. Other solutions of the equations under other assumptions will show quite different properties.

Suppose that our ocean is not restricted to a canal but is effectively unlimited and that our wave is long crested. In other words, the section for any y = a constant is the same as for any other. Whatever η is, it is certainly not a function of y. Further let the direction of travel be east along the positive x-axis, Fig. 2.04-1.

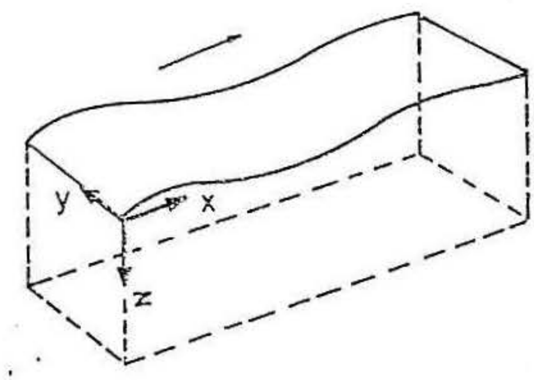


Fig. 2.04-1

WARNING: Note that the coordinate system here is left-handed.

Since we no longer have a narrow canal the assumption that v = 0 is not attractive. We would rather permit the velocity to have a y-component. We will retain the small-amplitude assumption so that our wave is still an Airy wave.

In section 2.03 the forms we arrived at for the equations of motion were

(2.04:1) $$\frac{\partial u}{\partial t} - 2\omega(\sin[L])v = -g\frac{\partial \eta}{\partial x}$$ (2.03:11)

(2.04:2) $$\frac{\partial v}{\partial t} + 2\omega(\sin[L])u = -g\frac{\partial \eta}{\partial y}$$ (2.03:12)

on the following assumptions:

(a) Neglect friction.

(b) Neglect Reynolds stress.

(c) The velocity gradients are small enough compared with the velocities to permit replacement of D/Dt by $\partial/\partial t$.

(d) The vertical velocity component w is negligibly small, $w \approx 0$.

(e) The atmospheric pressure is constant.

For long-crested waves the orbital velocity is the same in any y-section since all are identical. If we are to allow a transverse velocity component, v, as we wish to do, then that v will have to be constant for all y values or there will be a pile-up of water along some y-section and a consequent disparity among the profiles. There is zero accumulation of water in any vertical section so that we may again use the simple form of the equation of continuity,

(2.04:3) $$u/c = \eta/h \quad .$$

Since we have assumed a simple Airy wave, the x-velocity component, u, is given by

(2.04:4) $$u = u_o \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

where $\lambda \equiv$ the wave length

and $\tau \equiv$ the wave period.

Solving (2.04:3) for u and substituting in (2.04:4) we get for the surface

(2.04:5) $$\eta = \frac{u_o h}{c} \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \quad .$$

As we have said, η has no variation with y so that $\partial\eta/\partial y = 0$. Substituting in the equations of motion, (2.03:1) and (2.04:2), we get

(2.04:6) $$\frac{\partial u}{\partial t} - 2\omega(\sin[L])v = -g\frac{\partial \eta}{\partial x}$$

(2.04:7) $$\frac{\partial v}{\partial t} + 2\omega(\sin[L])u = 0 \quad .$$

We want to use (2.04:4)-(2.04:7) to determine v.

From equation (2.04:4)

(2.04:8) $$\frac{\partial u}{\partial t} = \frac{2\pi u_o}{\tau} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \quad .$$

From (2.04:5)

(2.04:9) $$\frac{\partial \eta}{\partial x} = -\frac{u_o h}{c}\frac{2\pi}{\lambda} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \quad .$$

Substituting (2.04:8) and (2.04:9) in (2.04:6)

$$\frac{2\pi u_o}{\tau} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] - 2\omega(\sin[L])v = \frac{2\pi u_o gh}{c\lambda} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

or, solving for v,

(2.04:10) $$v = \frac{\pi u_o}{\omega \sin[L]} \left(\frac{1}{\tau} - \frac{gh}{c\lambda}\right) \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \quad .$$

Solving equation (2.04:7) for $\partial v/\partial t$ and substituting for u from (2.04:4),

(2.04:11) $$\frac{\partial v}{\partial t} = -2\omega(\sin[L])u_o \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \quad .$$

Holding x constant and integrating with respect to t gives

$$v = - 2\omega(\sin[L])u_o(- \frac{\tau}{2\pi}) \int \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t](- \frac{2\pi}{\tau})dt + K$$

or

(2.04:12) $$v = \frac{\omega(\sin[L])u_o \tau}{\pi} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] + K \quad .$$

Equations (2.04:10) and (2.04:12) give us two different versions of the same thing, namely v. From (2.04:10) there exist some x and t for which v = 0 and for these same values, by (2.04:12), v = K. Therefore, K must be zero.

Setting K = 0 in (2.04:12) and equating it with (2.04:10),

$$\frac{\pi u_o}{\omega \sin[L]} \left(\frac{1}{\tau} - \frac{gh}{c\lambda}\right) \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] = \frac{\omega(\sin[L])u_o \tau}{\pi} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

or

(2.04:13) $$\frac{\pi}{\omega(\sin[L])\tau} \left(1 - \frac{gh}{c\lambda}\right) = \frac{\omega \sin[L]}{\pi} \tau \quad .$$

Since $c = \lambda/\tau$ or $\lambda = c\tau$, (2.04:13) can be written

(2.04:14) $$\frac{\pi}{\omega(\sin[L])\tau} \left(1 - \frac{gh}{c^2}\right) = \frac{\omega(\sin[L]))\tau}{\pi}$$

and, solving for c^2 we get

(2.04:15) $$c^2 = \frac{gh}{1 - \frac{\tau^2}{(\pi/\omega \sin[L])^2}} \quad .$$

The quantity $\pi/\omega \sin[L]$ is called the <u>half-pendulum day</u> and symbolized by τ_p.

(2.04:16) $$\tau_p \equiv \frac{\pi}{\omega \sin[L]} \quad .$$

It is a quantity that arises, almost inevitably, when Coriolis force enters a problem.

Remark: We could have introduced it in section 2.03 had we wanted to. Equation (2.03:20) was

$$\eta = \eta_o \exp\{-\frac{2\omega(\sin[L])}{c}y\}$$

which could have been written

$$\eta = \eta_o \exp\{-\frac{2\pi}{\frac{\pi c}{\omega \sin[L]}}y\} = \eta_o \exp\{-\frac{2\pi}{\tau_p c}y\} \quad .$$

Remark: The name "half-pendulum day" comes from the Foucault pendulum where it describes the period required for a Foucault pendulum to restore the orientation of its swing. A Foucault pendulum swinging above the North Pole will maintain the plane of its swing relative to the fixed stars. The earth turns under it. If it began swinging in the plane 0°-180° of longitude, then in 11 hours 59 minutes it would again be swinging in the same plane relative to the earth. At the North Pole the pendulum day, $2\tau_p$, is the sidereal day of 23 hours 58 minutes. As we move away from the pole, L decreases and sin[L] decreases so that the pendulum day increases. At the equator $2\tau_p$ is infinite. A Foucault pendulum at the equator does not change its orientation with respect to the earth.

Using τ_p, equation (2.04:15) assumes a particularly neat appearance.

(2.04:17) $$c^2 = \frac{gh}{1 - (\tau/\tau_p)^2} \quad .$$

At the equator $L = 0$, $\tau_p = \infty$, and $c^2 = gh$, i.e., we have an ordinary gravity wave. The minimum value of τ_p occurs for $L = 90°$ where τ_p = 11 hours 59 minutes.

If the waves we consider have periods $\tau < \tau_p$, equation (2.04:17) yields reasonable numbers for c^2 in that they are positive. For $\tau = \tau_p$ the wave speed is infinite and for $\tau > \tau_p$ we get imaginary values for c.

The model seems to have some significant features for semi-diurnal tides at all latitudes but for diurnal tides we must stay below 30°

where the half-pendulum day is longer than 24 hours.

Our analysis so far has been predicated on the assumption that the latitude L was constant. Hence, it is applicable only if the dominant component is oriented east-west. If there is a substantial north-south component, it doesn't apply and we would have to make some further modifications.

Consider the equation for v that arose from integrating equation (2.04:11), i.e., equation (2.04:12), having first introduced $1/\tau_p$ for $\omega \sin[L]/\pi$:

$$v = u_o\left(\frac{\tau}{\tau_p}\right) \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] \quad . \tag{2.04:18}$$

We want to compare the y-component, v, with the x-component, u. The comparison may be simplified by selecting a single convenient x value, say x = 0, and carrying out the comparison at the single x-position. Setting x = 0 in equations (2.04:4) and (2.04:18) we have

$$u = u_o \cos[\frac{2\pi}{\tau} t] \tag{2.04:19}$$

$$v = - u_o\left(\frac{\tau}{\tau_p}\right)\sin[\frac{2\pi}{\tau} t] \tag{2.04:20}$$

or, using the cofunction of the complement,

$$v = u_o\left(\frac{\tau}{\tau_p}\right)\cos[\frac{2\pi}{\tau} t + \frac{\pi}{2}] \quad . \tag{2.04:20.1}$$

Therefore, u and v are $\pi/2$ out of phase. Remembering the orientation of the x- and y-axes originally selected, v is positive when it is to the left of u.

Consider the current vector of which u, given by equation (2.04:19), and v, given either by (2.04:20) or (2.04:20.1), are the components. I'm sure that you will recognize the equations as the parametric equations of an ellipse. Certainly, when the ratio of the period to the half-pendulum day is one they are obviously the parametric equations of a circle. We need to check the direction of the rotation. This is most easily done for the special case where $\tau/\tau_p = 1$. In that case

t	0	τ/4	τ/2	3τ/4	τ	...
u	u_o	0	$-u_o$	0	u_o	...
v	0	$-u_o$	0	u_o	0	...

Plotting these we get Fig. 2.04-2. Clearly, the current vector rotates

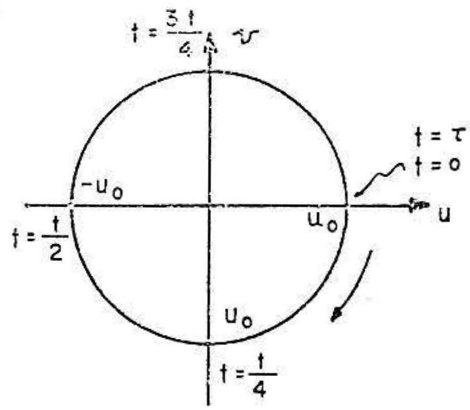


Fig. 2.04-2

clockwise completing a full circle in one period, .

Suppose $\tau/\tau_p = \alpha < 1$. Then

t	0	τ/4	τ/2	3τ/4	τ	...
u	u_o	0	$-u_o$	0	u_o	...
v	0	$-\alpha u_o$	0	αu_o	0	...

The plot is the ellipse of Fig. 2.04-3. The rotation is still clockwise at

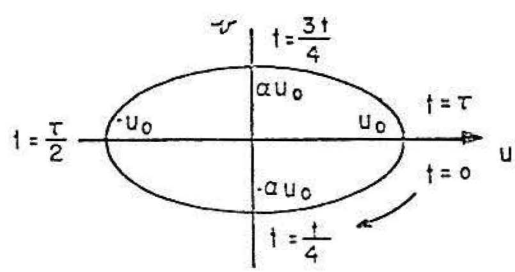


Fig. 2.04-3

one cycle per period τ but the magnitude now varies to trace out an ellipse with the major axis in the x-direction.

Should we have $\tau/\tau_p = 0$, then v vanishes and the motion reduces to a simple harmonic rectilinear reversing current of amplitude u_o.

We could consider the case where $\tau/\tau_p > 1$ formally but this would give imaginary values for the phase speed c as was shown in connection with equation (2.04:17). It hardly seems worth while to do so. In fact, even $\tau/\tau_p = 1$ is straining the analysis a bit since for that case equation (2.04:17) calls for $c = \infty$. The circular case is thus to be considered as an upper bound to the ellipses just as the rectilinear motion associated with $\tau/\tau_p = 0$ serves as a lower bound. The major axis is always lined up east-west since our analysis is no good if there is a substantial v.

The really important feature of this mathematical model is that it suggests one way that a rotary current can be produced. In the Southern hemisphere the rotation turns out to be counterclockwise. This comes out at once if we realize that we have been considering north latitude as positive in τ_p. South latitude will thus be negative and the sign of τ_p will be reversed. This reverses the sign of v in equations (2.04:20) and (2.04:20.1) but not the sign of u in equation (2.04:19). The result is a reversed sense of rotation.

As you know, observations of currents made at lightships do show rotation, Fig. 1.10-6, page 109, and in the same direction as that indicated by this model. However, a rotary current set up by a mechanism like that suggested by the model would be most likely to occur in a small oblong enclosed sea where a progressive wave was introduced by the tide at one end and where the width would discourage the development of large v. The lower Chesapeake Bay might be such a place although I don't know that anyone has looked into it. It might be good sport for somebody to pull the data from the CBI Blue Crab cruise where stations were made hourly for several weeks down there and do a few calculations to see whether rotary currents do exist and, if by any chance, they are of the right size according to this model.

An entirely different mechanism that can produce rotary currents is the refraction of waves around an island. Suppose we had a configuration like that shown in Fig. 2.04-4. Assume that a simple harmonic progressive wave is refracted around the island. On the back side the parts cross each

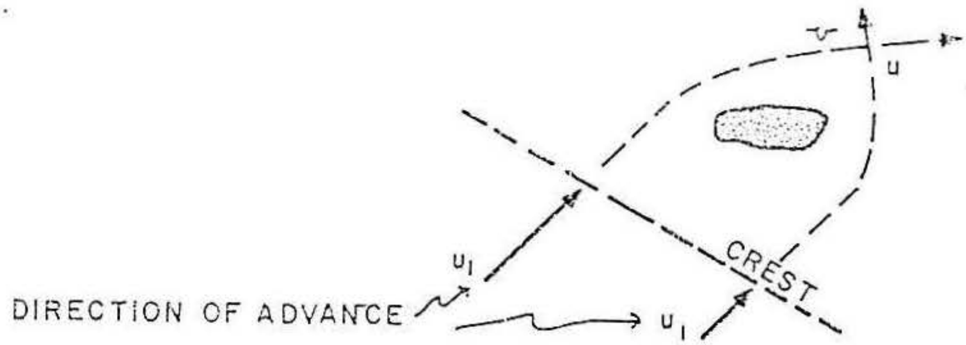


Fig. 2.04-4

other and are out of phase. The equations are similar to (2.04:19) and (2.04:20) or (2.04:20.1) but in this case the rotation may be in either direction depending on the amplitudes and the phase shift. The only case where you fail to get a rotary current occurs when the refracted parts are so bent that they meet in the same direction, either following or opposing. One seldom finds a good "ellipse" behind an island. The figure is usually wildly distorted.

Yet a third mechanism that can produce rotary currents is provided by the tide producing forces themselves. These forces have been shown to have both a north-south and an east-west component which, when combined, show the same characteristics as rotary currents. We can assume that the tidal streams in the ocean are set in motion by these horizontal forces and it seems intuitively appealing that, to a first approximation at least, they should reflect the rotary nature of the generating forces. There will, of course, be suitable modifications arising from various combinations of latitude, constituents, etc. so that any observed rotary current is likely to be pretty complex.

In deriving the properties of Kelvin's wave in a canal and waves of constant amplitude we have resorted to some quite unrealistic assumptions. In nature we can expect to encounter situations intermediate to them. For example, we may have constraining walls but they may be at a sufficient distance to permit the development of sizable cross-channel velocities as in lower Chesapeake Bay. There we might anticipate the $v \neq 0$ except near the boundaries and that $\eta = f(y)$ although, in all probability, the function won't be anything very wild. In these more complicated cases the simplified versions of the equations of motion and continuity will no longer serve us. We would need much more complete versions.

2.05. The Poincaré Wave.

The Poincaré wave is a generalization of the previous waves that includes them as special cases. It also includes a rather remarkable connection with inertial currents.

We start, as before, with the equations of continuity and motion. Instead of the simple form of the equation of continuity, however, we will use

(2.05:1) $$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \eta}{\partial t} = 0$$

This may be arrived at in the usual way by doing book keeping on the volume flow rate of water into a prism with section dxdy extending vertically from η to h.

The equations of motion will be picked up from (2.03:11) and (2.03:12) together with all the assumptions necessary to get them in that form. In the interests of neatness, however, we will use the half-pendulum day, τ_p, instead of the $\omega\sin[L]$ factors. The definition of τ_p is given by equation (2.04:16).

(2.05:2) $$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = - g \frac{\partial \eta}{\partial x}$$

(2.05:3) $$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = - g \frac{\partial \eta}{\partial y}$$

Equations (2.05:1)-(2.05:3) supply us with three equations in three unknowns, η, u, and v, which is nice.

If our solution is to cover both the Kelvin wave and the horizontal wave, we must allow for an exponential variation with y and also include some parameter that will remove the exponential variation so that for some cases we can get the horizontal crested wave. Further, we want the motion to be simple harmonic. To fit these stipulations it seems worth while to try for solutions of the form

(2.05:4) $$\eta = \eta_o \exp\{\frac{2\pi}{b} y\}\cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

(2.05:5) $$u = u_o \exp\{\frac{2\pi}{b} y\}\cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

(2.05:6) $$v = v_o \exp\{\frac{2\pi}{b} y\}\sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

where η_o, u_o, v_o are constants to be determined, b is a parameter which is assigned various constant values to produce the assorted cases, and λ, τ are constants specified ahead of time for any particular wave.

If (2.05:4)-(2.05:6) are to be solutions of (2.05:1)-(2.05:3), then we can use (2.05:4)-(2.05:6) to feed (2.05:1)-(2.05:3) and the equations must reduce to identities. We will need to compute

η	$\partial\eta/\partial t$	$\partial\eta/\partial x$	$\partial\eta/\partial y$
u	$\partial u/\partial t$	$\partial u/\partial x$	
v	$\partial v/\partial t$		$\partial v/\partial y$.

From (2.05:4)

$$\frac{\partial\eta}{\partial t} = \frac{2\pi\eta_o}{\tau} \exp\{\frac{2\pi}{b} y\}\sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

$$\frac{\partial\eta}{\partial x} = -\frac{2\pi\eta_o}{\lambda} \exp\{\frac{2\pi}{b} y\}\sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

$$\frac{\partial\eta}{\partial y} = \frac{2\pi\eta_o}{b} \exp\{\frac{2\pi}{b} y\}\cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

From (2.05:5)

$$\frac{\partial u}{\partial t} = \frac{2\pi u_o}{\tau} \exp\{\frac{2\pi}{b} y\} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

$$\frac{\partial u}{\partial x} = - \frac{2\pi u_o}{\lambda} \exp\{\frac{2\pi}{b} y\} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

From (2.05:6)

$$\frac{\partial v}{\partial t} = - \frac{2\pi v_o}{\tau} \exp\{\frac{2\pi}{b} y\} \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

$$\frac{\partial v}{\partial y} = \frac{2\pi v_o}{b} \exp\{\frac{2\pi}{b} y\} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

Substituting in (2.05:1) gives

(2.05:7) $$\frac{u_o}{\lambda} - \frac{v_o}{b} = \frac{\eta_o}{h\tau} .$$

Substituting in (2.05:2) gives

(2.05:8) $$\frac{u_o}{\tau} - \frac{v_o}{\tau_p} = \frac{g\eta_o}{\lambda} .$$

Substituting in (2.05:3) gives

(2.05:9) $$\frac{u_o}{\tau_p} - \frac{v_o}{\tau} = - \frac{g\eta_o}{b} .$$

The values of u_o, v_o, and η_o must satisfy equations (2.05:7)-(2.05:9) simultaneously if (2.05:4)-(2.05:6) are to be solutions of (2.05:1)-(2.05:3). The constant b is to be set at various values and the characteristics of the solutions explored.

Equations (2.05:7)-(2.05:9) can be put in slightly more usable form. From (2.05:8) and (2.05:9) we can find expressions for the ratios $u_o/g\eta_o$ and $v_o/g\eta_o$. From (2.05:8)

$$\frac{\tau_p}{\tau} u_o - v_o = \frac{g\tau_p}{\lambda} \eta_o .$$

From (2.05:9)

$$\frac{\tau}{\tau_p} u_o - v_o = - \frac{g\tau}{b} \eta_o .$$

Subtracting and solving for $u_o/g\eta_o$

(2.05:10)
$$\frac{u_o}{g\eta_o} = \frac{\frac{\tau_p}{\lambda} + \frac{\tau}{b}}{\frac{\tau_p}{\tau} - \frac{\tau}{\tau_p}} .$$

Also,
$$u_o - \frac{\tau}{\tau_p} v_o = \frac{g\tau}{\lambda} \eta_o$$
$$u_o - \frac{\tau_p}{\tau} v_o = \frac{g\tau_p}{b} \eta_o .$$

Subtracting the first from the second and solving for $v_o/g\eta_o$

(2.05:11)
$$\frac{v_o}{g\eta_o} = \frac{\frac{\tau}{\lambda} + \frac{\tau_p}{b}}{\frac{\tau_p}{\tau} - \frac{\tau}{\tau_p}} .$$

Equation (2.05:7) may be rewritten
$$\frac{1}{\lambda} \frac{u_o}{g\eta_o} - \frac{1}{b} \frac{v_o}{g\eta_o} = \frac{1}{gh\tau} .$$

Substituting from (2.05:10) and (2.05:11)

(2.05:12)
$$\frac{\frac{1}{\tau 2} - \frac{1}{\tau_p 2}}{\frac{1}{\lambda 2} - \frac{1}{b2}} = gh .$$

This can be shown to argree with Kelvin's wave on the condition that the y-component of the velocity, v, is zero. Equations (2.03:21) and (2.03:22) gave the final results for Kelvin's wave in the form
$$\eta = \eta_o \exp\{- \frac{2\pi}{\tau_p c} y\} \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] .$$
$$u = \eta_o \frac{c}{h} \exp\{- \frac{2\pi}{\tau_p c} y\} \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] .$$

if we use the half-pendulum day. Since $v = 0$, $v_o = 0$ and solution (2.05:6) drops out. From (2.05:4) and (2.05:5) it is clear that we must relate the following parameters:

Kelvin wave:	η_o	$-\tau_p c$	$\eta_o \frac{c}{h}$
Poincaré wave:	η_o	b	u_o

Since $v_o = 0$, equation (2.05:11) gives

(2.05:13) $$\frac{\tau}{\lambda} + \frac{\tau_p}{b} = 0$$

and plugging this in (2.05:12),

(2.05:14) $$\frac{\lambda^2}{\tau^2} = gh \quad ,\text{i.e.,} \quad c^2 = gh \quad .$$

Plugging (2.05:13) into (2.05:10), i.e., $\tau_p = -\frac{b\tau}{\lambda}$, gives

(2.05:15) $$u_o = g\frac{\tau}{\lambda}\eta_o \quad , \text{i.e.,} \quad u_o = \frac{g\eta_o}{c} \quad .$$

The b is a free parameter. Let it be assigned the value $b = -\tau_p c$. Then from (2.05:13) $c = \lambda/\tau$, as it should, and through equation (2.05:14) $c = \lambda/\tau = \sqrt{gh}$ so that

$$u_o = \eta_o\sqrt{g/h} = \eta_o\frac{\sqrt{gh}}{h} = \eta_o\frac{c}{h} \quad .$$

This is precisely the correct value for the Kelvin wave so that, for this selection of the parameter b, the general solution becomes the Kelvin wave.

We can also show that these solutions produce the horizontal-crest wave as a special case when the parameter b is set $b = \infty$. If this is done, the exponential factor $\exp\{\frac{2\pi y}{b}\} = 1$. Equation (2.05:12) becomes

$$\frac{\frac{1}{\tau^2} - \frac{1}{\tau_p^2}}{\frac{1}{\lambda^2}} = gh$$

or

(2.05:16) $$\frac{\lambda^2}{\tau^2} = \frac{gh}{1 - (\tau/\tau_p)^2}$$

while (2.05:10) becomes

(2.05:16.1) $$\frac{u_o}{g\eta_o} = \frac{\tau_p/\lambda}{\tau_p/\tau - \tau/\tau_p} = \frac{\tau/\lambda}{1 - (\tau/\tau_p)^2}$$

while (2.05:11) becomes

(2.05:16.2) $$\frac{v_o}{g\eta_o} = \frac{\tau/\lambda}{\tau_p/\tau - \tau/\tau_p} = \frac{(\tau/\lambda)(\tau/\tau_p)}{1 - (\tau/\tau_p)^2} \quad .$$

Dividing (2.05:16.2) by (2.05:16.1) gives

(2.05:17) $$\frac{v_o}{u_o} = \frac{\tau}{\tau_p}$$

The ratio of η_o to u_o can be found from equation (2.05:16.1). It is

(2.05:18) $$\frac{\eta_o}{u_o} = \frac{1}{g}\ [1 - (\tau/\tau_p)^2]\ \frac{\lambda}{\tau} = \frac{\lambda\tau}{g}\ (\frac{1}{\tau^2} - \frac{1}{\tau_p^{\ 2}})$$.

This agrees with the results for horizontal-crested waves as given in section 2.04.

$$\eta = \frac{u_o h}{c} \cos[\frac{2\pi}{\lambda}\ x - \frac{2\pi}{\tau}\ t] \qquad (2.04:5)$$

$$u = u_o \cos[\frac{2\pi}{\lambda}\ x - \frac{2\pi}{\tau}\ t] \qquad (2.04:4)$$

$$v = u_o\ \frac{\tau}{\tau_p}\ \sin[\frac{2\pi}{\lambda}\ x - \frac{2\pi}{\tau}\ t] \qquad (2.04:18)$$

$$c^2 = \frac{gh}{1 - (\tau/\tau_p)^2} \qquad (2.04:17)$$

The last relation is already directly established by equation (2.05:16). The ratio of the amplitudes of (2.04:4) and (2.04:18) is correct by equation (2.05:17). It remains to show that the η_o/u_o ratio is correct. From section 2.04 it should be h/c. By equation (2.05:18) it is

$$(\lambda\tau/g)(1/\tau^2 - 1/\tau_p^{\ 2})$$.

We need to prove the identity

$$h/c \equiv (\lambda\tau/g)(1/\tau^2 - 1/\tau_p^{\ 2})$$.

$$h/c = (1/g)(\lambda/\tau)[1 - (\tau/\tau_p)^2]$$

$$gh/c = c[1 - (\tau/\tau_p)^2]$$,

$$c^2 = \frac{gh}{1 - (\tau/\tau_p)^2}$$.

which is known true from equation (2.05:16). Thus, for $b = \infty$, the relations among the constants in equations (2.05:4)-(2.05:6) are properly adjusted to give horizontal-crested waves.

All this has been a matter of tidying up and showing agreement with previous work. A far more interesting case results when the period of the wave coincides with the period of the half-pendulum day, i.e., when $\tau = \tau_p$. We will consider two subcases: Case I, $b \equiv +\lambda$ and Case II, $b \equiv -\lambda$.

Case I: With $b = +\lambda$ and $\tau = \tau_p$ equation (2.05:10) solved for η_o gives

$$\eta_o = \frac{u_o}{g} \frac{\tau_p/\tau - \tau/\tau_p}{\tau_p/\lambda + \tau/b} .$$

Substituting,

$$\eta_o = \frac{u_o}{g} \frac{\tau/\tau - \tau/\tau}{\tau/\lambda + \tau/\lambda} = \frac{u_o}{g} \frac{0}{2(\tau/\lambda)} = 0 \qquad (2.05:19)$$

This means that $\eta = 0$. We are confronting a "wave" of zero amplitude with a "wave length," λ, and a "period," τ_p!

Dividing (2.05:10) by (2.05:11) we get

$$\frac{u_o}{v_o} = \frac{\tau_p/\lambda + \tau/b}{\tau/\lambda + \tau_p/b}$$

and, plugging in the values of the parameters,

$$\frac{u_o}{v_o} = \frac{\tau/\lambda + \tau/\lambda}{\tau/\lambda + \tau/\lambda} = 1$$

so that

$$u_o = v_o \qquad (2.05:20)$$

This means that the solutions, (2.05:4)-(2.05:6), come out to be

$$\eta = 0 \qquad (2.05:21)$$

$$u = u_o \exp\{\frac{2\pi}{\lambda} y\} \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau_p} t] \qquad (2.05:22)$$

$$v = u_o \exp\{\frac{2\pi}{\lambda} y\} \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau_p} t] \qquad (2.05:23)$$

The unholy feature of these equations is that they are the equations for an inertial current in a perfectly general form since they are not necessarily uniform over level surfaces! Discussions of inertial currents can be found in the standard hydrodynamic places, among them Proudman, "Dynamic Oceanography," section 48. This was the kind of current Stommel was after when he was flinging buoys overboard off Bermuda. You might check up on some of that work.

Case II: With $b = -\lambda$ and $\tau = \tau_p$, working with equation (2.05:7),

$$\frac{u_o}{\lambda} + \frac{v_o}{\lambda} = \frac{\eta_o}{h\tau_p} \qquad \text{or} \qquad u_o + v_o = \frac{\lambda\eta_o}{\tau_p h}$$

and with (2.05:8),

$$\frac{u_o}{\tau_p} - \frac{v_o}{\tau_p} = \frac{g\eta_o}{\lambda} \qquad \text{or} \qquad u_o - v_o = \frac{\tau_p g\eta_o}{\lambda}$$

and with (2.05:9)

$$\frac{u_o}{\tau_p} - \frac{v_o}{\tau_p} = \frac{g\eta_o}{\lambda} \qquad \text{or} \qquad u_o - v_o = \frac{\tau_p g\eta_o}{\lambda} \quad .$$

The first two give us the ratios u_o/η_o and v_o/η_o. The last two are identical.

(2.05:24) $$\frac{u_o}{\eta_o} = \frac{1}{2}\left(\frac{\lambda}{\tau_p h} + \frac{\tau_p g}{\lambda}\right)$$

(2.05:25) $$\frac{v_o}{\eta_o} = \frac{1}{2}\left(\frac{\lambda}{\tau_p h} - \frac{\tau_p g}{\lambda}\right)$$

Equations (2.05:24) and (2.05:25) are servicable only in a limited region since with $b = -\lambda$ we face an exponential blow-up in the y-direction in equations (2.05:4)-(2.05:6). Only the Kelvin wave can tolerate a coast line parallel to the x-axis. In a general way, equation (2.05:12) gives b^2 as a function of λ^2, τ^2, and $\tau_p^{\,2}$. Should the b^2 defined by (2.05:12) turn out to be negative, define a new parameter k^2 such that $b^2 \equiv -k^2$. Then (2.05:12) becomes

(2.05:26) $$\frac{1/\tau^2 - 1/\tau_p^{\,2}}{1/\lambda^2 + 1/k^2} = gh \quad .$$

Since gh is positive and the denominator is positive this requires

$$1/\tau^2 - 1/\tau_p^{\,2} > 0$$

or
$$(\tau_p{}^2 - \tau^2)/(\tau^2\tau_p{}^2) > 0$$
or
$$\tau_p{}^2 - \tau^2 > 0$$
or
$$\tau_p{}^2 > \tau^2$$
or
$$\tau_p > \tau$$

if both are positive which, being periods, they had better be.

In this case, replace the solutions (2.05:4)-(2.05:6) suggested originally with

(2.05:27) $$\eta = \eta_o \cos[\frac{2\pi}{k} y] \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

(2.05:28) $$u = \eta_o \frac{g\tau\tau_p}{\tau_p{}^2 - \tau^2} \left(\frac{\tau_p}{\lambda} \cos[\frac{2\pi}{k} y] - \frac{\tau}{k} \sin[\frac{2\pi}{k} y]\right) \cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

(2.05:29) $$v = \eta_o \frac{g\tau\tau_p}{\tau_p{}^2 - \tau^2} \left(\frac{\tau}{\lambda} \cos[\frac{2\pi}{k} y] - \frac{\tau_p}{k} \sin[\frac{2\pi}{k} y]\right) \sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

That (2.05:27)-(2.05:29) are solutions of (2.05:1)-(2.05:3) may be verified directly by substitution. They have the great virtue of beating the exponential blow-up to which equations (2.05:4)-(2.05:6) are liable and they can tolerate a coast line along a line in any direction.

(2.05:30) $$\tan[\frac{2\pi}{k} y] = \frac{k}{\lambda} \frac{\tau}{\tau_p}$$

or

(2.05:31) $$y = \frac{k}{2\pi} \tan^{-1}\left(\frac{k}{\lambda} \frac{\tau}{\tau_p}\right) .$$

This is as far as we will go with mathematical models of this kind. The next item on the docket is seiches in lakes and tides in gulfs.

2.06. Seiches in Lakes and Tides in Gulfs.

2.06.1. Seiches in a Narrow Lake.

Our starting point, as usual, is the equations of continuity and motion. Our axes are taken, as usual, at the undisturbed water surface and, as usual, η is the surface elevation.

For homogeneous water, constant atmospheric pressure, and no friction where the half-pendulum day represents the Coriolis force, the w-equation is the hydrostatic equation and, if w is negligible, the equations of motion take the form

(2.06.1:1) $$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = - g \frac{\partial \eta}{\partial x}$$

(2.06.1:2) $$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = - g \frac{\partial \eta}{\partial y} \quad .$$

Since $\partial u/\partial z$ and $\partial v/\partial z$ do not appear in (2.06.1:1) and (2.06.1:2), we can conclude that u and v are not functions of z.

Consider an elongated lake where the currents are pretty much in the long direction. Let x be along the axis of the lake. Let A be the area of a vertical cross-section at some point along x and let b be the width of the lake at the surface. At any time, t, let η be the mean elevation across the section and u the x-component of velocity averaged over the whole cross-section. By book keeping over the segment of the lake between x and x + δx we have

$$Au - (A + \delta A)(u + \delta u) = b \; \delta x \frac{d\eta}{dt} \quad .$$

Expanding and dividing by δx

$$- \frac{\delta A u}{\delta x} - \frac{\delta A \delta u}{\delta x} = b \frac{d\eta}{dt} \quad .$$

As $x \to 0$, the second term $\to 0$ and

(2.06.1:3) $$\frac{\partial A u}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

is the form of the equation of continuity we want.

Since we have banks in the way and are using averages, v averaged over the y-direction must be zero. Thus, for the equations of motion, (2.06.1:1) and (2.06:2), we have

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{2.06.1:4}$$

$$\frac{2\pi}{\tau_p}u = -g\frac{\partial \eta}{\partial y} \quad . \tag{2.06.1:5}$$

The general equation of continuity for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \tag{2.06.1:6}$$

Since u and v are independent of z, this can be integrated with respect to z from surface to bottom.

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) z + w = -\frac{\partial \eta}{\partial t} \tag{2.06.1:6.1}$$

since, at z = 0, we take $w = -\partial\eta/\partial t$--with a small departure, we take the boundary condition on the mean rather than on the actual surface--and at z = h, w = 0 if the bottom is taken level, we have

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) h = -\frac{\partial \eta}{\partial t} \quad . \tag{2.06.1:7}$$

Solving for w between (2.06.1:7) and (2.06.1:6.1) gives

$$w = -\left(1 - \frac{z}{h}\right)\frac{\partial \eta}{\partial t} \quad . \tag{2.06.1:8}$$

This has been obtained by neglecting vertical acceleration and (2.06.1:8) says, in effect, that the vertical acceleration is small.

We need to manufacture some criterion for the neglect of vertical acceleration. The version of the equations of motion that we are using arose from setting $\partial w/\partial t$, the vertical acceleration, zero in the third component equation,

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g \quad , \tag{2.06.1:9}$$

so that we got the hydrostatic equation. From (2.06.1:8)

$$\frac{\partial w}{\partial t} = - (1 - \frac{z}{h}) \frac{\partial^2 \eta}{\partial t^2}$$

so that (2.06.1:9) reads

$$- (1 - \frac{z}{h}) \frac{\partial \eta^2}{\partial t^2} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + g$$

or

$$\frac{\partial p}{\partial z} = \rho g + \rho (1 - \frac{z}{h}) \frac{\partial^2 \eta}{\partial t^2} \quad .$$

Integrating this over z gives

$$p - p_a = \rho g(z + \eta) + \rho (z - \frac{1}{2} \frac{z^2}{h}) \frac{\partial^2 \eta}{\partial t^2} \quad .$$

At the bottom $z = h$ and

$$p - p_a = \rho g(h + \eta) + \frac{1}{2} \rho h \frac{\partial^2 \eta}{\partial t^2} \quad .$$

Putting this in nondimensional form

$$\text{(2.06.1:10)} \qquad \frac{p - p_a}{\rho g \eta} = \frac{h + \eta}{\eta} + \frac{1}{2} \frac{h \frac{\partial^2 \eta}{\partial t^2}}{g \eta} \quad .$$

The left-hand member and the first member on the righ-hand side constitute the hydrostatic equation where $\partial w/\partial t = 0$. The last term on the right-hand side is the departure from the hydrostatic equation and must be small. Therefore, the criterion we require is

$$\text{(2.06.1:11)} \qquad \frac{h \frac{\partial^2 \eta}{\partial t^2}}{g \eta} << 1 \quad .$$

Consider an elongated basin oriented east-west for simplicity. Assume a periodic motion possible with period τ_1. At time $t = 0$ suppose that the water is at rest but piled up at the west end and depressed at the east. By time $t = \tau_1/4$ the lake will be level, by $t = \tau_1/2$ the high and low areas will be reversed, at $t = 3\tau_1/4$, level again, and by $t = \tau_1$ back to the initial configuration. With uniform density, the pressure gradient at any time will be the same at all points in a vertical line anywhere in the lake. Therefore, the accelerations and the currents are also the same.

Suppose that the basin is rectangular, depth h, width b, both constant, and that the ends are at $x = 0$ and $x = \ell$, Fig. 6.01.1-1, page 145.

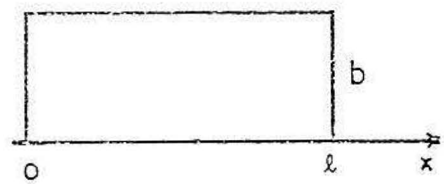


Fig. 2.06.1-1

Picking up continuity, equation (2.06.1:3), since the cross-section A is A = bh

$$\frac{\partial bhu}{\partial x} + b\,\frac{\partial \eta}{\partial t} = 0$$

or

$$bh\,\frac{\partial u}{\partial x} + b\,\frac{\partial \eta}{\partial t} = 0$$

or

(2.06.1:12) $$h\,\frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial t} = 0 \quad .$$

We want solutions for η and u as functions of x and t which are harmonic and which satisfy the boundary conditions, u = 0 at x = 0 and u = 0 at $x = \ell$, i.e., there is to be no velocity component in the x-direction at the ends of the basin.

Having in mind that we have just described the motion as a standing wave, it seems worth while to give a bloody go to

(2.06.1:13) $$u = C\sin[\frac{\pi}{\ell}\,x]\sin[\frac{2\pi}{\tau_1}\,t] \quad .$$

At least, this will fit the boundary conditions. From the first factor u = 0 for x = 0 and $x = \ell$. Further, by the second factor, for all x, u = 0 for t = 0 and $t = \tau_1$ and it also fits the initial and periodic conditions.

Stuff (2.06.1:13) back into (2.06.1:12).

$$\frac{\partial \eta}{\partial t} = -\,\frac{\pi h C}{\ell}\,\cos[\frac{\pi}{\ell}\,x]\sin[\frac{2\pi}{\tau_1}\,t] \quad .$$

Integrating with respect to t gives

$$\eta = -\,\frac{\pi h C}{\ell}\,\cos[\frac{\pi}{\ell}\,x]\,\int\sin[\frac{2\pi}{\tau_1}\,t]dt + \text{a constant} \quad .$$

The constant may be made zero by putting our origin of coordinates at the mean water level so that we get

(2.06.1:14) $$\eta = \frac{\tau_1 hC}{2\ell} \cos[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau_1} t] \quad .$$

Notice that this also satisfies the initial and boundary conditions as well as the periodic condition.

No claim is made for (2.06.1:13) and (2.06.1:14) as the only solution. There may be others but, at least, this one looks good. Feeding (2.06.1:13) and (2.06.1:14) into the equation of motion, (2.06.1:4),

$$\frac{\partial u}{\partial t} = \frac{2\pi C}{\tau_1} \sin[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau_1} t]$$

$$\frac{\partial \eta}{\partial x} = - \frac{\pi\tau_1 hC}{2\ell^2} \sin[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau_1} t]$$

so that

$$\frac{2\pi C}{\tau_1} \sin[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau_1} t] = \frac{\pi\tau_1 hCg}{2\ell^2} \sin[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau_1} t]$$

or

(2.06.1:15) $$\tau_1 = \frac{2\ell}{\sqrt{gh}} \quad .$$ <u>MERIAN'S FORMULA</u> (<u>VERY IMPORTANT</u>)

Merian's formula is very frequently used. It defines the natural period of a confined parallelopipedal body of water with a free surface. It depends only on the length and depth.

The coefficient of (2.06.1:14) is often written in the form

(2.06.1:16) $$H \equiv \frac{\tau_1 hC}{2\ell}$$

so that, with (2.06.1:15)

(2.06.1:17) $$\frac{H}{C} = \frac{\tau_1 h}{2\ell} = \frac{2\ell h}{2\ell\sqrt{gh}} = \sqrt{h/g} \quad .$$

Picking up our criterion for the neglect of vertical acceleration, (2.06.1:11)

$$\frac{h \frac{\partial^2 \eta}{\partial t^2}}{g\eta} = \frac{h\left(- \frac{2\pi^2 hC}{\ell\tau_1} \cos[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau_1} t]\right)}{g\left(\frac{\tau_1 hC}{2\ell} \cos[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau_1} t]\right)}$$

or

$$\frac{h \frac{\partial^2 \eta}{\partial t^2}}{g\eta} = - \frac{\pi^2 h^2}{\ell^2} .$$

Therefore, the requirement for the validity of the neglect of the vertical acceleration is that h/ℓ be small, i.e., the lake must be long compared with its depth.

Natural lakes aren't rectancular but Merian's formula is often used to provide an order of magnitude estimate of their natural periods in spite of the difficulty in deciding just what ℓ and h are to mean. Some examples are:

Lake	Length (km)	Depth (m)	Merian Natural Period (min)	Observed Period (min)	Comment
Loch Earn Scotland	10	60	14	14.5	
Lake George New S. Wales	30	5.5	136	131	Unusually shallow
Lake Geneva Switzerland	70	160	59	73.5	Seriously out of rectangular
Lake Baikal Siberia	665	680	270	278.4	

When the basin is not rectangular the fundamental equations are (2.06.1:3) and (2.06.1:4), i.e.,

$$\frac{\partial Au}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

$$\frac{\partial u}{\partial t} = - g \frac{\partial \eta}{\partial x}$$

where A and b are slowly varying functions of x. A possible solution here is

(2.06.1:18) $$\eta = Z(x) \cos[\frac{2\pi}{\tau_1} t]$$

(2.06.1:19) $$u = U(x) \sin[\frac{2\pi}{\tau_1} t] .$$

Z(x) and U(x) are functions of x only and we require A(x)U(x) = 0 at the ends of the lake since we must have u = 0 there.

$$\frac{\partial u}{\partial x} = \frac{dU}{dx} \sin[\frac{2\pi}{\tau_1} t]$$

$$\frac{\partial u}{\partial t} = \frac{2\pi}{\tau_1} U \cos[\frac{2\pi}{\tau_1} t]$$

$$\frac{\partial \eta}{\partial x} = \frac{dZ}{dx} \cos[\frac{2\pi}{\tau_1} t]$$

$$\frac{\partial \eta}{\partial t} = - \frac{2\pi}{\tau_1} Z \sin[\frac{2\pi}{\tau_1} t] \quad .$$

Substituting in (2.06.1:3) and (2.06.1:4), remembering that

$$\frac{\partial uA}{\partial x} = u \frac{\partial A}{\partial x} + A \frac{\partial u}{\partial x} \quad ,$$

$$U(x) \sin[\frac{2\pi}{\tau_1} t] \frac{dA}{dx} + A \frac{dU}{dx} \sin[\frac{2\pi}{\tau_1} t] - b \frac{2\pi}{\tau_1} Z \sin[\frac{2\pi}{\tau_1} t] = 0$$

and

$$\frac{2\pi}{\tau_1} U(x) \cos[\frac{2\pi}{\tau_1} t] = - g \frac{dZ}{dx} \cos[\frac{2\pi}{\tau_1} t]$$

or

$$U(x) \frac{dA}{dx} + A \frac{dU}{dx} = \frac{2\pi}{\tau_1} bZ(x)$$

and

$$\frac{2\pi}{\tau_1} U(x) = - g \frac{dZ(x)}{dx}$$

or

(2.06.1:20) $$\frac{d}{dx}\{AU(x)\} = \frac{2\pi}{\tau_1} bZ(x)$$

(2.06.1:21) $$\frac{2\pi}{\tau_1} U(x) = - g \frac{d}{dx}\{Z(x)\} \quad .$$

Solving (2.06.1:21) for U(x)

$$U(x) = - \frac{g\tau_1}{2\pi} \frac{d}{dx}\{Z(x)\}$$

and substituting in (2.06.1:20) gives

$$\frac{d}{dx}\{- \frac{g\tau_1 A}{2\pi} \frac{d}{dx}[Z(x)]\} = \frac{2\pi}{\tau_1} bZ(x)$$

or

$$\tau_1^{\,2} = \frac{-\,4\pi^2 bZ(x)}{g\,\frac{d}{dx}\{A(x)\,\frac{d}{dx}[Z(x)]\}} \qquad (2.06.1{:}22)$$

Equation (2.06.1:22) gives a means of making a more refined estimate of τ_1 in terms of the elevation Z, its gradient, the cross-sectional area, and the width of the lake. A finite difference equation could be used for the computation. Using (2.06.1:22), an estimate of τ_1 for Lake Geneva is 74.45 minutes which is in better accord with the observed 73.5 minutes than is the estimate of 59 minutes from Merian's formula.

There is one line across the lake on which there is no rise or fall of the water. Such a line is called a nodal line. The oscillations so far discussed have one such line and are called uninodal seiches. There may, however, be two or more nodal lines. In a simple-minded way, suppose you partitioned the lake into n sections. By adjusting the spacing of the barriers you could get smaller basins each with the same natural period which would, of course, be shorter than τ_1 since in Merian's formula ℓ would be reduced. If you started seiches in each compartment so that the elevation was the same on the two sides of each barrier, the elevations would always match, because the τ's are all equal, and the partitions might just as well not be there. Each compartment would have a node. If the motion has two nodes it is called binodal, if three, trinodal, etc..

For the uninodal seiche in a rectangular basin described by equation (2.06.1:14), the node occurs for that value of x between 0 and ℓ for which the $\cos[\pi x/\ell]$ factor is zero. For this value, clearly, no time varying factor can have any effect. There is just one value of x, i.e., $x = \ell/2$, for which $\cos[\pi x/\ell] = 0$.

In general, if the x-dependent cosine factor in (2.06.1:14) is replaced by $\cos[n\pi x/\ell]$, then between $x = 0$ and $x = \ell$ there are zeros at $x = (2\nu - 1)\ell/2n$; $\nu = 1, 2, \ldots, n$. If $n = 2$, nodes are located at $x = \ell/4$ and $x = 3\ell/4$. The period $\tau_2 = (1/2)\tau_1$. If $n = 3$, nodes occur at $x = \ell/6$, $\ell/2$ and $5\ell/6$ while the period is $\tau_3 = (1/3)\tau_1$, etc., Fig. 2.06.1-2, page 150. In Loch Earn the rectangular approximation for the binodal seiche gives $\tau_2 = 7$ minutes. One based on (2.06.1:22) gives $\tau_2 = 8.1$ minutes. The observed $\tau_2 = 8.1$ minutes.

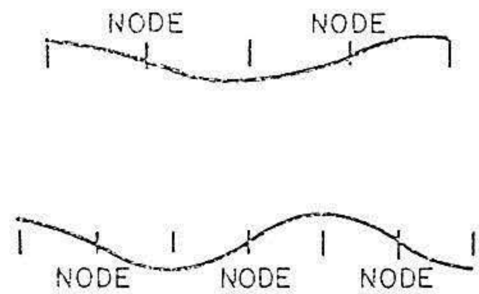


Fig. 2.06.1-2

We have been talking about longitudinal seiches but transverse seiches are also possible. Near Morges on Lake Geneva the lake is 13 kilometers wide and the mean depth is 170 meters. Using Merian's formula you get a uninodal transverse seiche period of $\tau_1 = 10.6$ minutes. The observed transverse seiche has a period of 10.3 minutes.

Since the seiche is a standing wave, if E is its total energy, P its potential energy, and K its kinetic energy, we have from wave theory

(2.06.1:23) $$P = E\ \cos^2 [\frac{2\pi}{\tau_1}\ t]$$

and

(2.06.1:24) $$K = E\ \sin^2 [\frac{2\pi}{\tau_1}\ t] \quad .$$

For an elongated basin of variable section the basic equations are (2.06.1:3) and (2.06.1:4). Multiplying the equation of motion by Au and integrating from end to end, x_1 to x_2, of the lake

$$\int_{x_1}^{x_2} Au\ \frac{\partial u}{\partial t}\ dx = -\ g \int_{x_1}^{x_2} Au\ \frac{\partial \eta}{\partial x}\ dx$$

or, interchanging operations on the left-hand side and integrating by parts on the right, remembering that $u(\partial u/\partial t) = \frac{1}{2}[\partial u^2/\partial t]$,

$$\frac{\partial}{\partial t}\{\tfrac{1}{2} \int_{x_1}^{x_2} Au^2 dx\} = -\ gAu\eta \Big|_{x_1}^{x_2} + g \int_{x_1}^{x_2} \frac{\partial}{\partial x}\ (Au)\ \eta\ dx \quad .$$

But $(Au)_{x_1} = (Au)_{x_2} = 0$ and, feeding in the value of $\partial(Au)/\partial x$ from the continuity equation,

$$\frac{\partial}{\partial t}\{\tfrac{1}{2}\int_{x_1}^{x_2} Au^2 dx\} = -g \int_{x_1}^{x_2} b\eta \frac{\partial \eta}{\partial t} dx\} \quad .$$

Again interchanging operations

$$\frac{\partial}{\partial t}\{\tfrac{1}{2}\int_{x_1}^{x_2} Au^2 dx = \frac{\partial}{\partial t}\{-\tfrac{1}{2}g \int_{x_1}^{x_2} b\eta^2 dx\} \quad .$$

Integrating with respect to t

(2.06.1:25) $$\tfrac{1}{2}\int_{x_1}^{x_2} Au^2 dx + \tfrac{1}{2}g \int_{x_1}^{x_2} b\eta^2 dx = \text{a constant} \quad .$$

But

$$K = \tfrac{1}{2}\rho \int_{x_1}^{x_2} Au^2 dx$$

and

$$P = \tfrac{1}{2}\rho g \int_{x_1}^{x_2} b\eta^2 dx$$

the only thing lacking in (2.06.1:25) being a factor of ρ which can be pulled out of the constant. Thus, (2.06.1:25) is a statement of the conservation of energy.

For the special case of a rectangular basin $x_1 = 0$, $x_2 = \ell$, u and η are to be had from (2.06.1:13) and (2.06.1:14) using (2.06.1:16) in (2.06.1:14). From (2.06.1:25)

$$K = \tfrac{1}{2}\rho b h \ell C^2 \sin^2[\frac{2\pi}{\tau_1} t]$$

$$P = \tfrac{1}{2}\rho g b \ell H^2 \cos^2[\frac{2\pi}{\tau_1} t] \quad .$$

By (2.06.1:17)

$$\frac{H^2}{C^2} = \frac{h}{g}$$

and the results, (2.06.1:23) and (2.06.1:24), are verified.

For Loch Earn with ρ = 1 grams/cm , ℓ = 10 km, b = 1 km, h = 60 m, and C = 4 cm/sec we find $E = 2.4\times10^{15}$ erg = 67 kw-hr.

2.06.2. Tidal Co-oscillations in a Narrow Gulf.

We are now going to knock out the eastern end of our long narrow lake and hitch it up to an ocean to make a gulf of it, Fig. 2.06.2-1. Take a cross-section at B. Between B and A we will continue to maintain a "lake like"

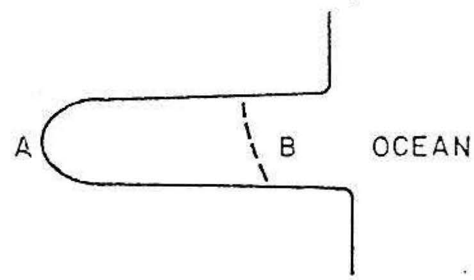


Fig. 2.06.2-1

behavior, i.e., we want exactly the same elevations and currents as we had before we knocked the end out. Clearly, there must be some sort of oscillation going on out in the ocean beyond B to maintain what we want. What goes on in the ocean is harmonic constituents of the tide so that's what we will take. We name the sloshing in the gulf the tidal co-oscillation. Since we are thinking about tidal constituents we can suppose that the period, τ, is specified. We will assume that the elevations along B are determined by the tide of the open sea and we will ask ourselves how the features of the co-oscillation depend on the dimensions of the gulf.

The first nodal line in the gulf may occur on either side of B, Fig. 2.06.2-2. If N is outside AB, then the tide at all points in the gulf

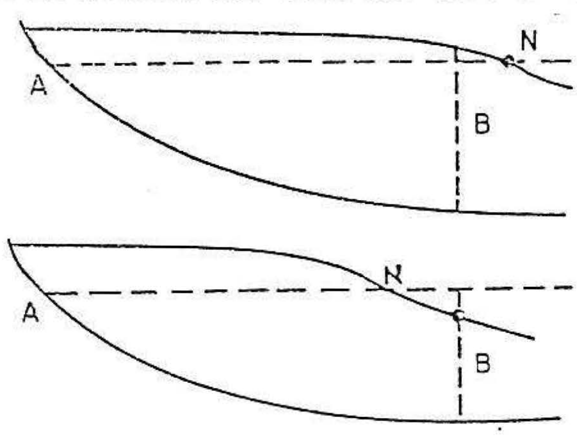


Fig. 2.06.2-2

will have the same phase. If N is between A and B, then N divides the gulf into two regions whose tides are 180° out of phase with each other. In either case the ratio of the tide range at A to the tide range at B increases as $N \to B$. When N is near B the tides at A are comparatively large. It is to this phenomenon that the term resonance refers.

Suppose a rectangular gulf of uniform depth h and let $AB \equiv L$. Then the pertinent equations are (2.06.1:13), (2.06.1:14), (2.06.1:16), and (2.06.1:17). For $x \leqq L$, from Merian's formula

(2.06.2:1) $$\ell = \tfrac{1}{2}\tau\sqrt{gh} \quad .$$

At B, x = L and the elevation at B comes from (2.06.1:14),

$$\eta_B = H\cos[\frac{\pi}{\ell} L]\cos[\frac{2\pi}{\tau} t]$$

or, if we define H_B as the amplitude at B, i.e.,

(2.06.2:2) $$H_B \equiv H\cos[\frac{\pi}{\ell} L] \quad ,$$

then $$\eta_B = H_B\cos[\frac{2\pi}{\tau} t] \quad .$$

The tide at any point within the gulf can then be expressed as

(2.06.2:3) $$\eta = H_B \frac{\cos[\frac{\pi}{\ell} x]}{\cos[\frac{\pi}{\ell} L]} \cos[\frac{2\pi}{\tau} t] \quad .$$

For a nodal line in the gulf between A and B we require $L > \frac{1}{2}\ell$ or, using (6.02.2:1),

(2.06.2:4) $$L > \tfrac{1}{4}\tau\sqrt{gh} \quad .$$

Clearly, the condition for resonance is

(2.06.2:5) $$L = \tfrac{1}{4}\tau\sqrt{gh} \quad .$$

Semidiurnals in the Bay of Fundy approach resonance of this kind. At the head of the bay are the largest tides in the world. Naturally, Fundy isn't rectangular but, taking as rough approximations, $\tau = 12.4$ hr and h = 75 m equation (2.06.2:1) gets us ℓ = 600 km. The length, L, of the Bay of Fundy is about 270 km. Therefore, $L < \frac{1}{2}\ell$ and the tides will be simultaneous with the strength 3 hours before high and low waters. This is roughly what is observed.

$$\cos[\frac{\pi}{\ell} L] = \cos[\frac{270}{600}\pi] \approx \cos[81°] = 0.156 \quad .$$

Therefore, from (2.06.2:2),

$$H_B = 0.156H \quad \text{or} \quad H = 6.4H_B \quad .$$

When the gulf is not rectangular we do better with equations (2.06.1:18)-(2.06.1:22) with the condition that AU(x) = 0 at the head of the gulf. We can get b and A from a chart of the gulf and, unlike the seiche case, we now know τ. This means that we can have at equations (2.06.1:20) and (2.06.1:21) by practical numerical methods of integration. Starting at the head of the bay with any handy value of Z(x), successive values of Z(x) and U(x) can be calculated down the bay. Then Z(x) and U(x) can be adjusted to make them agree with the observed values for some x and comparisons made between the adjusted computed values and the observed values. For this operation we can replace the differential equations, 2.06.1:20) and (2.06.1:21), with their finite difference analogs:

(2.06.2:6) $$AU(x_{\nu+2}) - AU(x_\nu) = \frac{2\pi}{\tau} bZ(x_{\nu+1})\{x_{\nu+2} - x_\nu\}$$

(2.06.2:7) $$Z(x_{\nu+2}) - Z(x_\nu) = -\frac{2\pi}{g\tau} U(x_{\nu+1})\{x_{\nu+2} - x_\nu\} \quad .$$

Proudman (pages 233-236) gives a calculation of this sort for the Red Sea. He divides it into 40 sections and works out the values section by section for the M_2 constituent. You would do well to check this over. The final upshot of his calculation is:

Section	Station	Co-oscillation			
		Calculated		Observed	
		H (cm)	γ (°)	H (cm)	γ (°)
1 *	Shadwan	25.1	117	25.1	117
3	Koseir	24.4	117	21.9	112
17 **	Jidda	4.4	117	7.4	124
19	Port Sudan	0.2	117	0.9	204
29	Massawa	25.4	297	34.4	327
33	Kamaran	30.3	297	32.8	303

* Point of forced agreement. ** There is a node somewhere around in here.

The agreement could be a lot worse.

For the Bay of Fundy the factor, 6.4, that we got is of the right order of magnitude but it is about twice as large as it should be. The measured tides at the head of the bay are about 40 feet and those at the mouth 12 feet which comes out to 40/12 = 3.3. If you use Proudman's values to compute u at the mouth you get about 11 knots which is off by an order of magnitude. All this isn't too surprising since no natural body of water is a rectangular gulf. What to take for its length is a question. We have the same difficulty, only more so, with so branched a structure as the Chesapeake Bay. There is really no good answer.

The same sort of problem arises in picking a number to represent the depth. Bays shoal toward their heads and this has an effect on the speed of propagation. Martin Pollak once ciphered it out for the Chesapeake Bay using h = 36 feet and wound up with the tide moving 70% too fast. He then worked back from the known travel time for the Bay and got h = 12 feet which is pretty silly since it is less than the average depth of the Bay.

Another source of disagreement between observed tides in the Chesapeake Bay and those computed from Merian's formula arises from the geometry of the Bay. Where sharp constrictions exist, e.g., as at Kent Island, the tide usually runs higher in the vicinity of the constriction.

If one starts a progressive wave moving up the bay and computes the potential and kinetic energies for a small section of the wave length, $m\lambda \ll \lambda$ across the breadth, b, and over the depth, h, of the bay, one can replace η by $\overline{\eta}$ and u by $\overline{u}$ over the small section. For the potential energy, P, we get

$$P = \tfrac{1}{2}\rho g b \overline{\eta}^2 \lambda m$$

and for the kinetic energy, K, of the same section assuming frictionless flow and u uniform from top to bottom,

$$K = \tfrac{1}{2}\rho b(h + \overline{\eta}) u^2 \lambda m \quad .$$

If we follow this element up the bay, i.e., ride it up, the P will remain constant and, in general, to first order $P = K$. If η is small, we can neglect $\overline{\eta}$ in comparison with h in the kinetic energy and write

$$K = \tfrac{1}{2}\rho bhu^2\lambda m \quad .$$

The wave length, λ, can be removed from these equations by

$$\lambda = c\tau = \sqrt{gh}\ \tau \quad .$$

The the potential energy is

$$P = (\tfrac{1}{2}\rho g^{3/2}\tau m)h^{1/2}b\overline{\eta^2} = \text{a constant} \quad .$$

Therefore,

$$\eta^2 bh^{\frac{1}{2}} = \text{another constant}$$

or

$$\eta \sim b^{-\frac{1}{2}}h^{-\frac{1}{4}} \quad .$$

Doing the same thing with the kinetic energy leads to

$$u \sim b^{-1/2}h^{-3/4} \quad .$$

This indicates that the dependence of the amplitude and the orbital velocity on the breadth is the same but that the orbital velocity is more strongly affected by the depth than is the amplitude.

The same sort of thing holds for a reflected wave in a bay. At the head of a bay the u's for the incident and reflected waves neutralize each other but the amplitudes add. In the Chesapeake Bay the tides show a slight increase north of Annapolis.

One of the big problems is how to separate the effects of resonance from the effects of a constriction. I know of no way to do this very well. In any case, these mathematical models are much too simple. To name just one important neglected feature: How about friction? Bottom friction in a bay would eventually damp out the tide if the bay were long enough.

Proudman's more elaborate method which he used for the Red Sea could be applied to places like the Chesapeake Bay or Long Island Sound--if anyone felt like it and had the energy to waste.

2.06.3. Standing Oscillations in a Channel.

We can make a quick profit by extending seiches in a narrow lake and tidal co-oscillations in a narrow gulf to standing oscillations in a channel. Just as in the case of the gulf where we had to specify τ at the open end, we must now specify τ at both ends of the channel. As before, lines of zero range are nodes. Lines of maximum range are known as loops, Fig. 2.06.3-1.

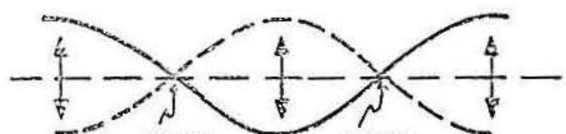


Fig. 2.06.3-1

With our usual channel with parallel vertical banks and uniform depth equations (2.06.1:4) and (2.06.1:13)-(2.06.1:17) apply. Four times the distance from a node to the nearest loop is called a "wave length" and is usually denoted by λ so that $\lambda = 2\ell$.

From (2.06.1:14) this gives

(2.06.3:1) $$\eta = H \cos[\frac{2\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

since $$H = \frac{\tau h C}{2\ell}$$

and from (2.06.1:13)

(2.06.3:2) $$u = C \sin[\frac{2\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

with $$\frac{H}{C} = (\frac{h}{g})^{\frac{1}{2}} \qquad \text{and} \qquad \frac{\lambda}{\tau} = \sqrt{gh} \quad .$$

This is all analogous to the results from the study of sound in closed and open pipes.

2.06.4. Geostrophic Effects.

We will now consider briefly, and qualitatively, the geostrophic effects on a tidal co-oscillation in a narrow gulf. Consider the situation covered by equations (2.06.2:1)-(2.06.2:7) and measure time t from the instant of high water at the head of the gulf. This, of course, will also be the time of high water as far as the first nodal line.

At t = 0 the currents will be zero. As we have seen, at $t = \tau/4$ the currents reach their maximum speed toward the sea.

Let us restrict our attention to the northern hemisphere. Owing to the geostrophic effects the surface at $t = \tau/4$ will not be level. It will be high to the right and low to the left of the current. Between is a line, nn', which will be at mean sea level. Without the geostrophic effect, the whole area would be at mean sea level at $t = \tau/4$.

At $t = \tau/2$ the first picture is reversed. At $t = 3\tau/4$ the second picture is reversed. Finally, at $t = \tau$ the whole cycle starts over again.

Continuity suggests that to get this progression of high and low water we will need currents at time t = 0 which may be as shown in Fig. 2.06.4-1. A similar picture holds for time $t = \tau/2$.

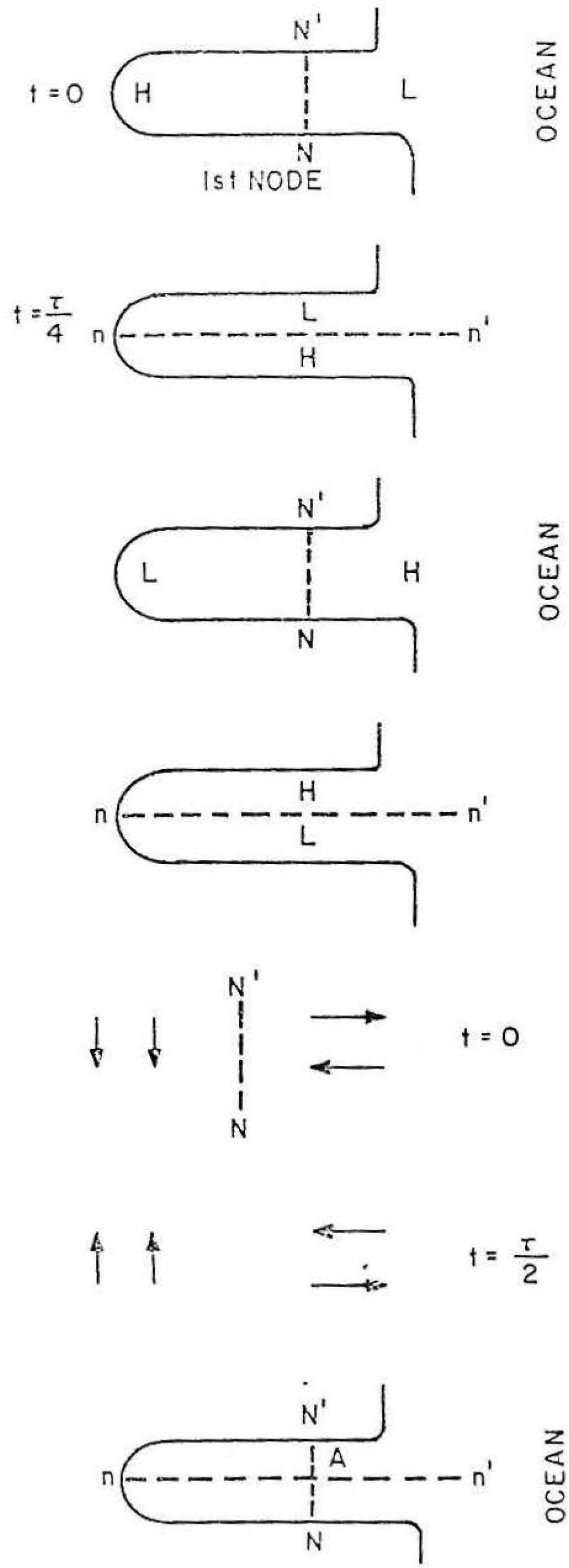


Fig. 2.06.4-1

The geostrophic effects of these currents will modify the elevations at $t = 0$ and $t = \tau/2$ but, if the gulf is narrow, the transverse components of the currents are small.

At the intersection, A, of NN' and nn' there will be no change in the water height.

At $t = 0$ high water occurs along An.

At $t = \tau/4$ high water occurs along AN.

At $t = \tau/2$ high water occurs along An'.

At $t = 3\tau/4$ high water occurs along AN'.

At any one time all the points of high water lie on a line through A. This line of high water is called a cotidal line and A is called an amphidromic point. The region around A is called an amphidromic region. The cotidal line rotates about the amphidromic point; in the northern hemisphere, counterclockwise.

An amphidromic region of this sort exists in the Adriatic. The mean depth of the inner 250 kilometers of the Adriatic is about 50 meters. For the M_2 constituent, by (2.06.2:5) for the resonance condition, we get

$$\tfrac{1}{2}\ell = \tfrac{1}{2}\times 12.4\text{hr}(9.81\text{m/sec}^2\times 50\text{m})^{\frac{1}{2}} = 247 \text{ km} \quad .$$

By observation, the amphidromic is located about 240 kilometers from Venice. High water at Zara on the east coast preceeds high water at Venice by about 2 hours 53 minutes while high water at Viesti on the west coast lags it by about 4 hours 37 minutes.

The same situation occurs in channels where an amphidromic region is associated with each nodal line. An example occurs in the southern part of the North Sea.

Cotidal lines and amphidromic points have interested investigators for quite a while. Whole oceans, for example the Atlantic, have been analysed into cotidal lines and amphidromic regions. If it could be done, it would be nice. But no two investigators seem to come up with the same picture. There is really very little data except along the coasts and everyone seems to be playing it by ear.

Harris worked out a theory in which he divided up the ocean into basins with small overlaps at the edges. Each of these basins would then have its own natural period which would single out constituents of the tide that would be resonant, and therefore dominant, for that basin. At least, he managed to get pretty consistent results, but his theory, as a picture of reality has grave defects. The sections are arbitrary, no flow can take place between sections, and Coriolis force was neglected.

Proudman and Doodson worked out cotidal lines and amphidromic points for the North Sea and the English Channel. In this area tide measurments are much more numerous, both for shore stations and for the open sea from lightships and oceanographic expeditions. They went through a very elaborate reconciliation of orbital velocities and tidal heights and their result is a very realistic picture of the region.

We can get mathematical expressions for an amphidromic region in a channel by using the Kelvin wave. Suppose we have a combination of two harmonic Kelvin waves traveling in opposite directions. Take the origin of co-ordinates at a point where their elevations are always equal and opposite. Then

(2.06.4:1) $$\eta = H \exp\{\frac{2\pi}{c} y\}\cos[\frac{2\pi}{\lambda} x + \frac{2\pi}{\tau} t] - H \exp\{-\frac{2\pi}{c} y\}\cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

and

(2.06.4:2) $$u = - C \exp\{\frac{2\pi}{c} y\}\cos[\frac{2\pi}{\lambda} x + \frac{2\pi}{\tau} t] - C \exp\{-\frac{2\pi}{c} y\}\cos[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t]$$

where H and C are constants and the channel is sufficiently restricted in latitude so that τ_p is approximately constant.

At high water $\partial\eta/\partial t = 0$ so that, from (2.06.4:1),

(2.06.4:3) $$\exp\{\frac{2\pi}{c} y\}\sin[\frac{2\pi}{\lambda} x + \frac{2\pi}{\tau} t] + \exp\{-\frac{2\pi}{c} y\}\sin[\frac{2\pi}{\lambda} x - \frac{2\pi}{\tau} t] = 0 \quad .$$

Equation (2.06.4:3) is the equation of a cotidal line at the time t. It can be rolled over into a neater form as follows:

Using the angle sum formulae from trigonometry and then grouping and factoring (2.06.4:33) takes the form

(2.06.4:4) $$\sin[\frac{2\pi}{\lambda} x]\cos[\frac{2\pi}{\tau} t](\exp\{\frac{2\pi}{c} y\} + \exp\{-\frac{2\pi}{c} y\}) + \cos[\frac{2\pi}{\lambda} x]\sin[\frac{2\pi}{\tau} t](\exp\{\frac{2\pi}{c} y\} - \exp\{-\frac{2\pi}{c} y\}) = 0$$

or

$$\frac{\exp\{\frac{2\pi}{c}y\} - \exp\{-\frac{2\pi}{c}y\}}{\exp\{\frac{2\pi}{c}y\} + \exp\{-\frac{2\pi}{c}y\}} = -\tan[\frac{2\pi}{\lambda}x]\cot[\frac{2\pi}{\tau}t] = \tan[\frac{2\pi}{\lambda}x]\tan[\frac{2\pi}{\tau}t + \frac{\pi}{2}]$$

Invoking the definitions of the hyperbolic functions, (2.06.4:4) is

$$\frac{\tanh[\frac{2\pi}{c}y]}{\tan[\frac{2\pi}{\lambda}x]} = \tan[\frac{2\pi}{\tau}(t + \frac{\tau}{4})] \quad . \qquad (2.06.4:5)$$

For $t = 0, \tau/4, \tau/2, 3\tau/4, \ldots$ the cotidal lines are the axes. Near the origin x and y are small so that the arguments of "tanh" and "tan" on the left side of (2.06.4:5) are small and the functions may be approximated by their arguments. This leads to

$$\frac{y}{x} = \frac{c}{\lambda}\tan[\frac{2\pi}{\tau}(t + \frac{\tau}{4})]$$

in the vicinity of the origin. But $c/\lambda = \tau_p/\tau$ so that, near the origin,

$$\frac{y}{x} \doteq \frac{\tau_p}{\tau}\tan[\frac{2\pi}{\tau}(t + \frac{\tau}{4})] \quad . \qquad (2.06.4:6)$$

This reproduces the features of the surface described at the beginning of this section fairly well. However, the velocities, being those of a pair of directly opposed Kelvin waves, produce a rectilinear current rather than a rotating current.

Proudman discusses the cotidal lines and amphidromic regions for the M_2 constituent in the English Channel on pages 256-262 of "Dynamic Oceanography." You should check this over.

2.06.5. Compound Seiches.

As a final move, let us consider some cases in which we remove the narrowness restriction. If a transverse seiche and a longitudinal seiche exist together, considerable variety is possible depending on the phase differences and amplitude differences of the two oscillations. Consider the case where one seiche is at mean level when the other is at its extreme displacement, Fig. 2.06.5-1. The high water travels around the lake and

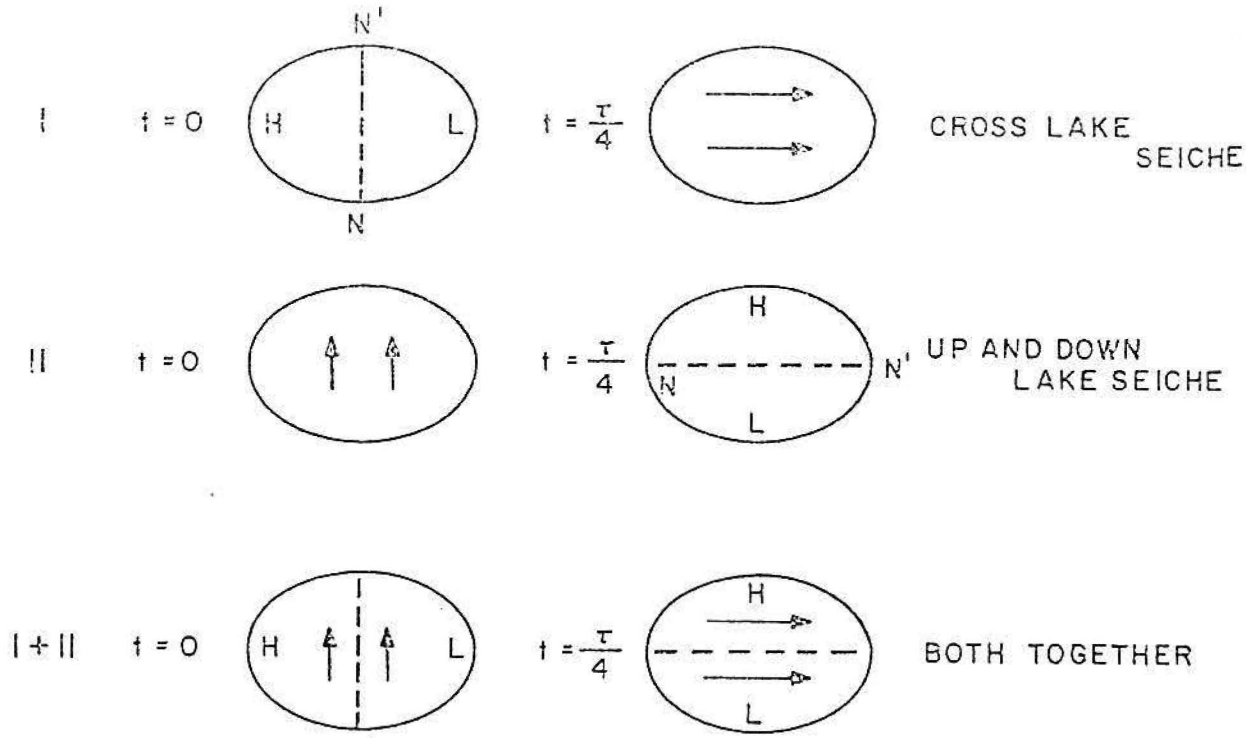


Fig. 2.06.5-1

and so does the mean water level line. Where the NN' intersect there will be an amphidromic point.

Suppose we had a square of uniform depth h and sides at x = 0, x = ℓ, y = 0, y = ℓ, Fig. 2.06.5-2. Then the two simple seiches can be

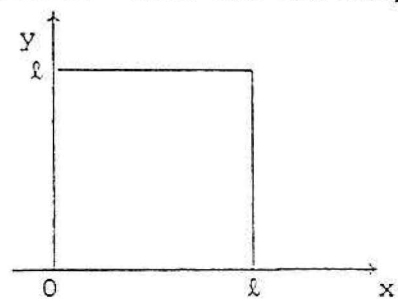


Fig. 2.06.5-2

written as

(2.06.5:1) $$\eta_1 = H_1 \cos[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau} t]$$

(2.06.5:2) $$u_1 = C_1 \sin[\frac{\pi}{\ell} x] \sin[\frac{2\pi}{\tau} t]$$

(2.06.5:3) $$\eta_2 = - H_2 \cos[\frac{\pi}{\ell} y] \sin[\frac{2\pi}{\tau} t]$$

(2.06.5:4) $$v_2 = C_2 \sin[\frac{\pi}{\ell} y] \cos[\frac{2\pi}{\tau} t]$$

where

(2.06.5:5) $$\frac{H_1}{C_1} = \frac{H_2}{C_2} = \left(\frac{h}{g}\right)^{1/2}$$

and

(2.06.5:6) $$\tau = \frac{2\ell}{\sqrt{gh}} \quad .$$

The compound seiche is the sum of the two.

Suppose, for instance, that $H_1 = H_2 = H$ while $C_1 = C_2 = C$. Then for the compound seiche

(2.06.5:7) $$\eta = H\{\cos[\frac{\pi}{\ell} x] \cos[\frac{2\pi}{\tau} t] - \cos[\frac{\pi}{\ell} y] \sin[\frac{2\pi}{\tau} t]\}$$

(2.06.5:8) $$u = C \sin[\frac{\pi}{\ell} x] \sin[\frac{2\pi}{\tau} t]$$

(2.06.5:9) $$v = C \sin[\frac{\pi}{\ell} y] \cos[\frac{2\pi}{\tau} t] \quad .$$

For high water $\partial\eta/\partial t = 0$ so that, from (2.06.5:7),

$$H\{- \frac{2\pi}{\tau} \cos[\frac{\pi}{\ell} x] \sin[\frac{2\pi}{\tau} t] - \frac{2\pi}{\tau} \cos[\frac{\pi}{\ell} y] \cos[\frac{2\pi}{\tau} t]\} = 0$$

or

(2.06.5:10) $$\frac{\cos[\frac{\pi}{\ell} y]}{\cos[\frac{\pi}{\ell} x]} = - \tan[\frac{2\pi}{\tau} t] \quad .$$

This connects up the where and when of high water.

2.06.6. Seiches in a Broad Lake.

We can generalize this a bit to a rectangular basin. Neglecting geostrophic effects, the equations of continuity and motion are

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \eta}{\partial t} = 0$$

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \quad .$$

Let the sides of the rectangle be x = 0, x = a, y = 0, y = b and try for a solution

(2.06.6:1) $$u = A\sin[\frac{m\pi}{a}x]\cos[\frac{n\pi}{b}y]\sin[\frac{2\pi}{\tau}t]$$

; A, B constants; m, n integers .

(2.06.6:2) $$v = B\cos[\frac{m\pi}{a}x]\sin[\frac{n\pi}{b}y]\sin[\frac{2\pi}{\tau}t]$$

The boundary conditions at the coasts are

$$u = 0 \text{ at } x = 0 \text{ and } x = a$$

$$v = 0 \text{ at } y = 0 \text{ and } y = b \quad .$$

Clearly, whatever else they do or don't do, (2.06.6:1) and (2.06.6:2) meet these conditions. Substituting them in the equations of motion gets you

$$\frac{\partial \eta}{\partial x} = -\frac{2\pi}{g\tau}A\sin[\frac{m\pi}{a}x]\cos[\frac{n\pi}{b}y]\cos[\frac{2\pi}{\tau}t]$$

$$\frac{\partial \eta}{\partial y} = -\frac{2\pi}{g\tau}B\cos[\frac{m\pi}{a}x]\sin[\frac{n\pi}{b}y]\cos[\frac{2\pi}{\tau}t]$$

and integrating,

$$\eta = \frac{2a}{mg\tau}A\cos[\frac{m\pi}{a}x]\cos[\frac{n\pi}{b}y]\cos[\frac{2\pi}{\tau}t]$$

$$= \frac{2b}{ng\tau}B\cos[\frac{m\pi}{a}x]\cos[\frac{n\pi}{b}y]\cos[\frac{2\pi}{\tau}t] \quad .$$

These are identical in form, viz.,

(2.06.6:3) $$\eta = H\cos[\frac{m\pi}{a}x]\cos[\frac{n\pi}{b}y]\cos[\frac{2\pi}{\tau}t]$$

with

(2.06.6:4) $$aA/m = bB/n = \tfrac{1}{2}g\tau H \quad .$$

Feeding (2.06.6:1)-(2.06.6:3) back to continuity we get

(2.06.6:5) $$\tau = \frac{2}{\sqrt{gh}}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-\frac{1}{2}} .$$

Equation (2.06.6:5) includes Merian's formula as a special case when $m = 1$ and $n/b << m/a$.

2.06.7. Seiches Across a Uniform Channel.

The last model for consideration in section 2.06 treats seiches across a uniform channel when geostrophic effects are important. Free waves are taken and we will suppose all cross-sections are equal and that conditions on all of them are the same. Take the latitude constant. Take axes Ox cross channel and Oy with the channel. For continuity we have

(2.06.7:1) $$\frac{\partial}{\partial x}(hu) + \frac{\partial \eta}{\partial t} = 0 .$$

For motion,

(2.06.7:2) $$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = - g \frac{\partial \eta}{\partial x}$$

(2.06.7:3) $$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = 0 .$$

At the sides of the channel

(2.06.7:4) $$hu = 0 .$$

Back a stretch we hit these equations without the geostrophic terms--equations (2.06.1:18), (2.06.1:19), ... for the non-rectangular basin--and got forms

$$\eta = Z(x)\cos[\frac{2\pi}{\tau_s} t] \quad ; \quad u = U(x)\sin[\frac{2\pi}{\tau_s} t]$$

where τ_s is the period of the ordinary transverse seiche and Z(x) and U(x) are functions determined up to an arbitrary factor which satisfy

(2.06.7:5) $$\frac{d}{dx}[hU(x)] = \frac{2\pi}{\tau_s} Z(x)$$

[See (2.06.1:20) and (2.06.1:21).]

(2.06.7:6) $$\frac{2\pi}{\tau_s} U(x) = - g \frac{d}{dx}[Z(x)]$$

across the channel and hU(x) = 0 at the boundaries.

Working on this pattern, let's try

(2.06.7:7) $$u = U(x)\sin[\frac{2\pi}{\tau} t]$$

keeping the requirements on U(x) expressed by (2.06.7:5) and (2.06.7:6) but letting τ be something other than τ_s. The boundary conditions, (2.06.7:4) are automatically satisfied.

Feeding (2.06.7:7) into (2.06.7:1) and (2.06.7:3) we get

(2.06.7:8) $$\frac{\partial \eta}{\partial t} = - \frac{d}{dx}[hU(x)]\sin[\frac{2\pi}{\tau} t]$$

(2.06.7:9) $$\frac{\partial v}{\partial t} = - \frac{2\pi}{\tau_p} U(x) \sin[\frac{2\pi}{\tau} t]$$

Using ((2.06.7:5) on (2.06.7:8) it can be written

(2.06.7:10) $$\frac{\partial \eta}{\partial t} = - \frac{2\pi}{\tau_s} Z(x) \sin[\frac{2\pi}{\tau} t] \quad .$$

Integrating (2.06.7:9) and (2.06.7:10) gets

(2.06.7:11) $$\eta = \frac{\tau}{\tau_s} Z(x) \cos[\frac{2\pi}{\tau} t]$$

(2.06.7:12) $$v = \frac{\tau}{\tau_p} U(x) \cos[\frac{2\pi}{\tau} t] \quad .$$

There are no constants of integration here since the motion is harmonic.

Feeding (2.06.7:7) (2.06.7:11), and (2.06.7:12) into (2.06.7:2),

(2.06.7:13) $$\frac{2\pi}{\tau} U(x) - 2\pi \frac{\tau}{\tau_p^2} U(x) = - g \frac{\tau}{\tau_s} \frac{d}{dx}[Z(x)]$$

and then working on (2.06.7:13) with (2.06.7:6) gets you

(2.06.7:14) $$1/\tau^2 = 1/\tau_s^2 + 1/\tau_p^2 \quad .$$

From (2.06.7:14) it can be seen that, independent of the dimensions of the channel, we must have

$$\tau < [\tau_s, \tau_p]_{min} \quad .$$

Also from (2.06.7:14), if either $\tau_s << \tau_p$ or $\tau_p << \tau_s$, then $\tau \approx \tau_s$ or $\tau \approx \tau_p$.

For a very wide channel τ_s is very large and equations (2.06.7:5) and (2.06.7:6) tell us that U(x) and Z(x) are practically independent of x.

2.07. Response to the Tide-Generating Forces.

2.07.1. Introduction.

We have already discussed the equilibrium tide in section 1.06 and, in section 2.02, developed the forced wave in a canal along a parallel of latitude. The problem of the response of the sea to the tide-generating forces is a most difficult one and we will restrict ourselves to those cases in which transverse currents may be neglected. We will consider a forced tidal wave in a narrow meridional ocean, the tides in a closed basin, shallow-water constituents, and, because of the mathematical similarity, changes in water elevation due to changing atmospheric pressure.

The fundamental equations are, as usual, continuity and motion. If η is not negligible in comparison with h, then continuity takes the form

$$\frac{\partial}{\partial x}[(h + \eta)u] + \frac{\partial}{\partial y}[(h + \eta)v] + \frac{\partial \eta}{\partial t} = 0 \quad . \qquad (2.07.1:1)$$

For an elongated basin continuity, as before, is

$$\frac{\partial}{\partial x}(Au) + b\,\frac{\partial \eta}{\partial t} = 0 \quad . \qquad (2.07.1:2)$$

In the equations of motion we must now include a body force per unit mass, $\bar{F}$, with components F_x, F_y, and F_z. The vertical component may be neglected since its only contribution is to make a slight modification of the value of gravity. Including these forces, the horizontal components of the equation of motion are

(2.07.1:3) $$\frac{Du}{Dt} - \frac{2\pi}{\tau_p} v = - \frac{1}{\rho} \frac{\partial p}{\partial x} + F_x$$

(2.07.1:4) $$\frac{Dv}{Dt} + \frac{2\pi}{\tau_p} u = - \frac{1}{\rho} \frac{\partial p}{\partial y} + F_y \quad .$$

Neglecting F_z and using p_a for the atmospheric pressure the vertical component of the equation of motion becomes the hydrostatic equation,

$$p = p_a + \rho g(\eta + z)$$

from which

$$\frac{\partial p}{\partial x} = \frac{\partial p_a}{\partial x} + \rho g \frac{\partial \eta}{\partial x}$$

and

$$\frac{\partial p}{\partial y} = \frac{\partial p_a}{\partial y} + \rho g \frac{\partial \eta}{\partial y} \quad .$$

Therefore, (2.07.1:3) and (2.07.1:4) can be written

(2.07.1:5) $$\frac{Du}{Dt} - \frac{2\pi}{\tau_p} v = - g \frac{\partial \eta}{\partial x} + F_x - \frac{1}{\rho} \frac{\partial p_a}{\partial x}$$

(2.07.1:6) $$\frac{Dv}{Dt} + \frac{2\pi}{\tau_p} u = - g \frac{\partial \eta}{\partial y} + F_y - \frac{1}{\rho} \frac{\partial p_a}{\partial y} \quad .$$

2.07.2. A Narrow Ocean Bounded by Two Meridians.

Consider an ocean of uniform depth, h, bounded by two meridians so close together that we can make all the approximations usual to the neglect of transverse currents. Let a be the radius of earth and ϕ the latitude; $\phi > 0$ for north latitude. Put the x-axis along the ocean with its origin at the equator so that $x \equiv a\phi$. Let b_o be the breadth of the ocean at the equator. Then b at latitude ϕ will be $b = b_o \cos(\phi)$ and the cross-sectional area, A, will be $A = b_o h \cos(\phi)$. If u is the northerly current, then continuity, by equation (2.07.1:2), is

(2.07.2:1) $$\frac{h}{a}\frac{\partial}{\partial\phi}\{u \cos(\phi)\} + \cos(\phi)\frac{\partial\eta}{\partial t} = 0 \quad .$$

It will be convenient, as in the case of the forced wave along a parallel of latitude, to express our results as a modification of the equilibrium tide. The equilibrium tide is one in which the accelerations relative to the earth are zero. Using $\overline{\eta}$ for the equilibrium tide the horizontal components of the equation of motion are

(2.07.2:2) $$g\frac{\partial\overline{\eta}}{\partial x} = F_x - \frac{1}{\rho}\frac{\partial p_a}{\partial x}$$

(2.07.2:3) $$g\frac{\partial\overline{\eta}}{\partial y} = F_y - \frac{1}{\rho}\frac{\partial p_a}{\partial y}$$

so that (2.07.1:5) and (2.07.1:6) can be written

(2.07.2:4) $$\frac{Du}{Dt} - \frac{2\pi}{\tau_p} v = - g\{\frac{\partial}{\partial x}(\eta - \overline{\eta})\}$$

(2.07.2:5) $$\frac{Dv}{Dt} + \frac{2\pi}{\tau_p} u = - g\{\frac{\partial}{\partial y}(\eta - \overline{\eta})\}$$

Since $v = 0$ and we may neglect geostrophic effects while $\frac{Du}{Dt} \approx \frac{\partial u}{\partial t}$, equation (2.07.2:4) becomes

(2.07.2:6) $$\frac{\partial u}{\partial t} = - \frac{g}{a}\{\frac{\partial}{\partial\phi}(\eta - \overline{\eta})\} \quad .$$

The equilibrium tide for a single semidiurnal constituent may be written

(2.07.2:7) $$\overline{\eta} = \overline{H}\cos^2(\phi)\cos(\frac{2\pi}{\tau}t)$$

where $\overline{H}$ = a constant.

What we want is a solution of (2.07.2:1) and (2.07.2:6) of the form

(2.07.2:8) $$\eta = H\{\cos^2(\phi) - \frac{2}{3}\}\cos(\frac{2\pi}{\tau}t) \quad .$$

The total volume of water in the ocean basin must remain constant and equation (2.07.2:8) takes care of this all right. With the limits of integration at $\phi = -\frac{1}{2}\pi$ for the south pole and $\phi = +\frac{1}{2}\pi$ at the north pole, we have

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \eta b_o \cos[\phi]d\phi = Hb_o\cos[\frac{2\pi}{\tau}t]\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}(\cos^2[\phi] - \frac{2}{3})\cos[\phi]d\phi$$

$$= Hb_o\cos[\frac{2\pi}{\tau}t]\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos^3[\phi]d\phi$$

$$- \frac{2Hb_o\cos[\frac{2\pi}{\tau}t]}{3}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos[\phi]d\phi \quad .$$

Now,

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos^3[\phi]d\phi = \frac{1}{3}\sin[\phi](\cos^2[\phi] + 2)\Bigg|_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} = \frac{4}{3}$$

and

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos[\phi]d\phi = \sin[\phi]\Bigg|_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} = 2$$

so that

$$\frac{4}{3}Hb_o\cos[\frac{2\pi}{\tau}t] - 2\,\frac{2Hb_o\cos[\frac{2\pi}{\tau}t]}{3} = 0 \quad .$$

Consequently, we at least conserve the water.

Feeding (2.07.2:7) and (2.07.2:8) back to (2.07.2:6)

$$\frac{\partial u}{\partial t} = -\frac{g}{a}\left(\frac{\partial}{\partial\phi}\{H(\cos^2[\phi] - \frac{2}{3}) - \overline{H}\cos^2[\phi]\}\cos[\frac{2\pi}{\tau}t]\right)$$

$$= -\frac{g}{a}\cos[\frac{2\pi}{\tau}t]\{(H - \overline{H})\frac{\partial}{\partial\phi}(\cos^2[\phi])\}$$

or

(2.07.2:9) $$\frac{\partial u}{\partial t} = \frac{2g}{a}(H - \overline{H})\cos[\phi]\sin[\phi]\cos[\frac{2\pi}{\tau}t] \quad .$$

Integrating with respect to t,

(2.07.2:10) $$u = \frac{g\tau}{\pi a}(H - \overline{H})\cos[\phi]\sin[\phi]\sin[\frac{2\pi}{\tau}t]$$

No additive constant is required since the motion is harmonic in t.

Feeding (2.07.2:8) and (2.07.2:10) to (2.07.2:1),

$$\frac{h}{a}\frac{\partial}{\partial\phi}\left(\frac{g\tau}{\pi a}(H - \overline{H})\cos^2[\phi]\sin[\phi]\sin[\frac{2\pi}{\tau}t]\right) - \cos[\phi]\,H\,\frac{2\pi}{\tau}(\cos^2[\phi] - \frac{2}{3})\sin[\frac{2\pi}{\tau}t]$$

$$= 0$$

or

$$\frac{gh\tau}{\pi a^2}(H - \overline{H})\{-2\cos[\phi]\sin^2[\phi] + \cos^3[\phi]\}\sin[\frac{2\pi}{\tau}t]$$

$$- \frac{2\pi H}{\tau}\{\cos^3[\phi] - \frac{2}{3}\cos[\phi]\}\sin[\frac{2\pi}{\tau}t] = 0 \quad .$$

Therefore,

$$H = (H - \overline{H}) \frac{\frac{gh\tau}{\pi a^2}(-2\cos[\phi]\sin^2[\phi] + \cos^3[\phi])\sin[\frac{2\pi}{\tau}t]}{\frac{2\pi}{\tau}(\cos^3[\phi] - \frac{2}{3}\cos[\phi])\sin[\frac{2\pi}{\tau}t]}$$

$$H = (H - \overline{H}) \frac{gh}{a^2}\frac{\tau^2}{2\pi^2}\frac{(-2\sin^2[\phi] + \cos^2[\phi])}{(\cos^2[\phi] - \frac{2}{3})}$$

$$H = (H - \overline{H}) \frac{3gh\tau^2}{2\pi^2 a^2}\frac{-2\sin^2[\phi] + \cos^2[\phi]}{3\cos^2[\phi] - 2}$$

$$H = (H - \overline{H}) \frac{3gh\tau^2}{2\pi^2 a^2}\frac{-2 + 2\cos^2[\phi] + \cos^2[\phi]}{3\cos^2[\phi] - 2}$$

$$H = (H - \overline{H}) \frac{3gh\tau^2}{2\pi^2 a^2}$$

and

(2.07.2:11) $$H = \overline{H}\frac{1}{1 - \frac{2\pi^2 a^2}{3gh\tau^2}} \quad .$$

Clearly, the resonance condition is

$$\frac{2\pi^2 a^2}{3gh\tau^2} = 1$$

or

(2.07.2:12) $$\tau = \left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{\pi a}{\sqrt{gh}}$$

where πa is the distance from pole to pole and $\sqrt{gh}$ is the free-wave speed. Therefore, $\pi a/\sqrt{gh}$ is the time taken for a free wave to travel from pole to pole. On the other hand, $\pi a/\tau$ is the speed of the forced wave which we will call c_F to distinguish it from $c = \sqrt{gh}$. Then in a manner similar to that used for Airy's canal we get

(2.07.2:13) $$\eta = \overline{\eta}\frac{1}{1 - \frac{2}{3}\left(\frac{c_F}{c}\right)^2} \quad .$$

We have no oceans which extend from pole to pole between closely spaced meridians. The only sea that even remotely matches this model is the Red Sea which is, at least, narrow and lies substantially north-south. Proudman has used the model and numerical methods to calculate the M_2 tides in the Red Sea. You will find his discussion on pages 282-285 of "Dynamic Oceanography." The results are

Section	Station	M_2 Tide			
		Calculated		Observed	
		H (cm)	γ (°)	H (cm)	γ (°)
1	Shadwan	25.1	117	25.1	117
3	Koseir	24.2	118	21.9	112
17	Jidda	4.0	149	7.4	124
19	Port Sudan	2.0	232	0.9	204
29	Massawa	24.8	298	33.4	327
33	Kamaran	28.4	304	32.8	303

You should compare these results with the results shown on page 154 that Proudman got by treating the tides in the Red Sea as a tidal co-oscillation. The order of agreement isn't particularly different.

2.07.3. Tides in a Closed Basin.

In a real tide problem the basin and forces are given and one must calculate the water motion. It is much easier, however, to specify the water motion and then work back to the forces necessary to produce the specified oscillation.

Suppose that we have a motion just like the seiches already discussed except that its period is τ rather than the natural period τ_1 given by Merian's formula. From the varying water level the accompanying accelerations can be calculated. In the case of a seiche oscillating at τ_1

the accelerations are due entirely to the pressure gradients resulting from surface gradients. If the seiche is oscillating at some other period, the resulting accelerations will not be due entirely to the pressure gradients and the generating forces must make up the difference. When the water surface is level the pressure gradients are zero so that there is no difference to be made up by the generating forces. When the surface elevations reach their extrema the difference to be made up reaches its greatest value and, therefore, so also must the generating forces.

If $\tau < \tau_1$, the accelerations are smaller than those of the natural seiche so that the generating forces will have to oppose the pressure gradients. When $\tau = \infty$ the accelerations are zero and the generating forces exactly balance the pressure gradients. This is the equilibrium case. For $\tau_1 < \tau < \infty$, the generating forces are less than the pressure gradients and for $\tau = \tau_1$ they vanish. For $\tau < \tau_1$ the period is less than the natural period and the generating forces must help the pressure gradients to speed up the accelerations. As $\tau \to 0$ the generating forces become infinite.

We may restate all this on the basis of generating forces of constant amplitude and take up the variation on the displacement of the water surface which will now vary with τ. When $\tau = \infty$ the elevation has the equilibrium form. When $\tau_1 < \tau < \infty$ the oscillation is in phase with the equilibrium form although its amplitude is modified. As τ decreases the amplitude increases and for $\tau = \tau_1$ it hits resonance and becomes, theoretically infinite. When $\tau < \tau_1$ the oscillation is 180° out of phase with the equilibrium form and, as $\tau \to 0$, the amplitude also goes to 0, Fig. 2.07.3-1, page 174.

Let's get down to cases again with our usual rectangular basin, length ℓ, depth h, small amplitude, no transverse currents, etc.. Continuity and motion are

(2.07.3:1) $$h \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial t} = 0$$

(2.07.3:2) $$\frac{\partial u}{\partial t} = - g \left[\frac{\partial}{\partial x}(\eta - \overline{\eta})\right]$$

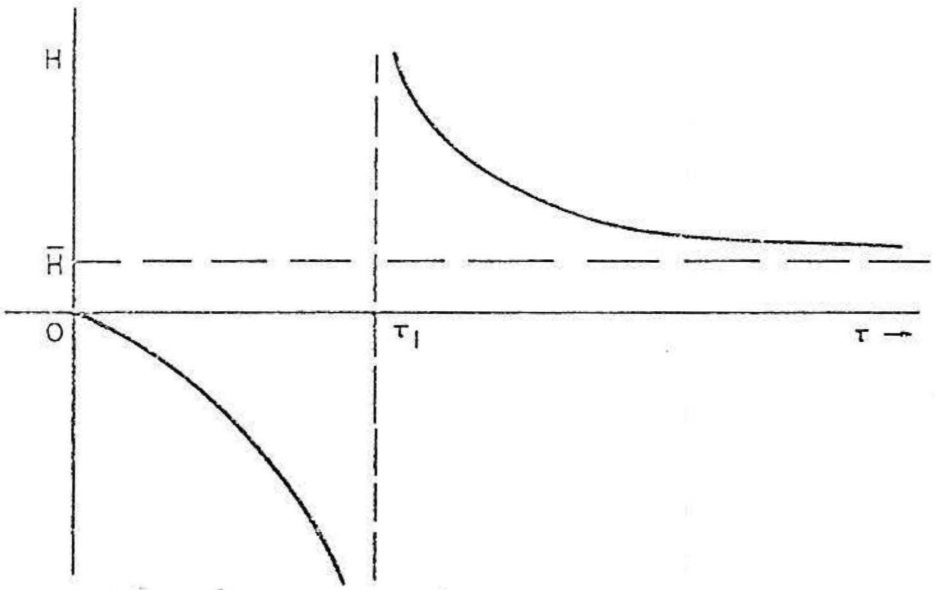


Fig. 2.07.3-1

We want a solution,

(2.07.3:3) $$u \doteq C \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] \quad ,$$

which has the form, but not the period, of the natural seiche and which kills off the boundary conditions, $u = 0$ at $x = 0$ and $x = \ell$ without further ado.

Firing (2.07.3:3) into (2.07.3:1) and solving for $\partial\eta/\partial t$,

$$\frac{\partial \eta}{\partial t} = -\frac{hC\pi}{\ell} \cos[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] \quad .$$

Integrating,

$$\eta = \frac{h\tau}{2\ell} C \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] \quad .$$

No additive constant is needed since η is harmonic in t.

Feeding (2.07.3:3) and (2.07.3:4) into (2.07.3:2) solved for $\partial\bar{\eta}/\partial x$,

$$\frac{\partial \bar{\eta}}{\partial x} = (\frac{\partial \eta}{\partial x} + \frac{1}{g}\frac{\partial u}{\partial t})$$

$$= -\frac{\pi h\tau}{2\ell^2} C \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] + \frac{2\pi}{g\tau} C \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

$$= (\frac{2\pi}{g\tau} - \frac{\pi h\tau}{2\ell^2}) C \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

which, on integration, gives

$$\bar{\eta} = -\frac{\ell}{\pi}\left(\frac{2\pi}{g\tau} - \frac{\pi h\tau}{2\ell^2}\right) C \cos\left[\frac{\pi}{\ell} x\right] \cos\left[\frac{2\pi}{\tau} t\right]$$

where the additive constant is zero if

$$\int_0^\ell \bar{\eta}\, dx = 0 \quad .$$

Thus,

$$\bar{\eta} = \left(\frac{h\tau}{2\ell} - \frac{2\ell}{g\tau}\right) C \cos\left[\frac{\pi}{\ell} x\right] \cos\left[\frac{2\pi}{\tau} t\right] \quad . \tag{2.07.3:5}$$

If we gather up the amplitude factors and write

$$\bar{\eta} = \bar{H} \cos\left[\frac{\pi}{\ell} x\right] \cos\left[\frac{2\pi}{\tau} t\right] \tag{2.07.3:6}$$

$$\eta = H \cos\left[\frac{\pi}{\ell} x\right] \cos\left[\frac{2\pi}{\tau} t\right] \tag{2.07.3:7}$$

then

$$\frac{\bar{H}}{C} = \frac{h\tau}{2\ell} - \frac{2\ell}{g\tau} \tag{2.07.3:8}$$

and

$$\frac{H}{C} = \frac{h\tau}{2\ell} \quad , \tag{2.07.3:9}$$

then the ratio of the amplitudes is

$$\frac{\bar{H}}{H} = \frac{h\tau/2\ell - 2\ell/g\tau}{h\tau/2\ell} = 1 - \frac{4\ell^2}{gh\tau^2} \quad .$$

Using Merian's formula,

$$\frac{\bar{H}}{H} = 1 - \left(\frac{\tau_1}{\tau}\right)^2 = \frac{\tau^2 - \tau_1^2}{\tau^2} \tag{2.07.3:10}$$

or

$$\frac{H}{\bar{H}} = \frac{\tau^2}{\tau^2 - \tau_1^2} \quad . \tag{2.07.3:11}$$

From (2.07.3:8)

$$\frac{C}{\bar{H}} = \frac{2\ell g\tau}{gh\tau^2 - 4\ell^2} = \frac{\tau}{\tau^2 - \frac{4\ell^2}{gh}} \frac{2\ell}{h}$$

or

$$\frac{C}{\bar{H}} = \frac{\tau}{\tau^2 - \tau_1^2} \frac{2\ell}{h} \quad . \tag{2.07.3:12}$$

An inspection of equation (2.07.3:11) will confirm the qualitative relations described at the beginning of our discussion. In particular, direct tides occur for $\tau > \tau_1$, resonance for $\tau = \tau_1$, and inverted tides for $\tau < \tau_1$.

Our solution can be extended from a uninodal forced tide to a polynodal forced tide in the same way we extended the uninodal seiche by using

(2.07.3:13) $$\bar{\eta} = \bar{H} \cos[\frac{n\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] \quad ,$$

n an integer, with solutions of (2.07.3:1) and (2.07.3:2),

(2.07.3:14) $$\eta = H_n \cos[\frac{n\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

(2.07.3:15) $$u = C_n \sin[\frac{n\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

where the relations between the H_n and C_n must be determined.

Remark: You might try this for the exercise.

If instead of (2.07.3:5) we have the more general

(2.07.3:16) $$\bar{\eta} = f(x) \cos[\frac{2\pi}{\tau} t] \quad ,$$

then, clearly, since x is defined over a finite interval, 0 to ℓ, f(x) can be fanned out in a Fourier series and the solution of (2.07.3:1) and (2.07.3:2) will be a sum of terms of the form (2.07.3:14) and (2.07.3:15).

A particularly interesting case arises when the tide generating forces are uniform over the entire basin at any one instant. This corresponds, roughly, to the state of affairs for, say, the M_2 constituent acting on a not so very large lake.

Ignoring the ends of the basin for the moment, the uniform forces oscillating in a definite period correspond to uniform accelerations with the same period and so to uniform currents over the whole lake with the same period. Now we have to fix up the boundary conditions. Suppose we extend the length of the basin and, in the enlarged basin, produce a seiche whose accelerations are just right to neutralize the currents at the places where the ends of our original basin used to be located. The elevation of the water surface will be produced only by the seiche motion. If the tide-generating forces attain their maximum toward the east at t = 0, then the currents generated by them will reach their maximum toward the east at $t = \frac{1}{4}\tau$. Then, to counteract this at the ends of the old basin the seiche currents will have to pick up maximum velocities to the west at time $t = \frac{1}{4}\tau$.

Let a denote the length of the enlarged basin, depth h, and τ_1 the period of the uninodal seiche. Then, with $\ell < a$, the tidal elevation will be in phase with the equilibrium tide. As $a \to \ell$ the amplitude of the seiche will have to become large to produce current enough to counteract the tide producing forces and, again, resonance sets in for $a = \ell$.

Take

(2.07.3:17) $$g\,\frac{\partial\bar{\eta}}{\partial x} = X_o \cos[\frac{2\pi}{\tau}t] \quad ; \quad X_o \text{ constant.}$$

Ignoring the boundary conditions, a solution of (2.07.3:1) and (2.07.3:2) is

$$\eta = 0 \quad ; \quad \frac{\partial u}{\partial t} = X_o \cos[\frac{2\pi}{\tau}t]$$

or, integrating

$$\eta = 0 \quad ; \quad u = \frac{\tau}{2\pi} X_o \sin[\frac{2\pi}{\tau}t] \quad .$$

Now put the origin at the center of the basin so that the boundary condition $u = 0$ applies at $x = -\frac{1}{2}\ell$ and $x = +\frac{1}{2}\ell$. Solutions for (2.07.3:1) and (2.07.3:2) with $\partial\bar{\eta}/\partial x = 0$, i.e., seiche motion, are

(2.07.3:19) $$\eta = H' \sin[\frac{\pi}{a}x]\cos[\frac{2\pi}{\tau}t]$$

(2.07.3:20) $$u = -C' \cos[\frac{\pi}{a}x]\sin[\frac{2\pi}{\tau}t]$$

with

(2.07.3:21) $$\tau = 2a/\sqrt{gh} \quad ; \quad H'/C' = h\tau/2a$$

as in section 2.06.1.

To make u given by (2.07.3:20) neutralize u given by (2.07.3:18) at $x = \pm\frac{1}{2}\ell$ we require

$$\frac{\tau}{2\pi} X_o \sin[\frac{2\pi}{\tau}t] + \{-C' \cos[\frac{\pi(\pm\frac{1}{2}\ell)}{a}]\sin[\frac{2\pi}{\tau}t]\} = 0$$

or

(2.07.2:22) $$C' \cos[\frac{\pi\ell}{2a}] = \frac{\tau X_o}{2\pi}$$

so that the elevation of the resultant motion from (2.07.3:18) and (2.07.3:19) using $H' = C'(h\tau/2a)$ from (2.07.3:21) is

(2.07.3:23) $$\eta = \frac{h\tau^2}{4\pi a} X_o \frac{\sin[\frac{\pi}{a} x]}{\cos[\frac{\pi \ell}{2a}]} \cos[\frac{2\pi}{\tau} t]$$

while u will be given by

$$u = \frac{\tau}{2\pi} X_o \sin[\frac{2\pi}{\tau} t] - \frac{\tau}{2\pi} X_o \frac{\cos[\frac{\pi}{a} x]}{\cos[\frac{\pi \ell}{2a}]} \sin[\frac{2\pi}{\tau} t]$$

or

(2.07.3:24) $$u = \frac{\tau}{2\pi} X_o \left(1 - \frac{\cos[\frac{\pi}{a} x]}{\cos[\frac{\pi \ell}{2a}]}\right) \sin[\frac{2\pi}{\tau} t] \quad .$$

The amplitude of η at $x = \pm \frac{1}{2}\ell$ is, from (2.07.3:23),

(2.07.3:25) $$H = \frac{h\tau^2}{4\pi a} X_o \tan[\frac{\pi \ell}{2a}] \quad .$$

The amplitude of $\overline{\eta}$ at $x = \pm \frac{1}{2}\ell$ from (2.07.3:17) is

$$\int \frac{\partial \overline{\eta}}{\partial x} dx = \frac{X_o}{g} \int dx \cos[\frac{2\pi}{\tau} t]$$

$$\overline{\eta} = \frac{X_o}{g} x \cos[\frac{2\pi}{\tau} t]$$

$$\overline{\eta} = \pm \frac{\ell}{2g} X_o \cos[\frac{2\pi}{\tau} t]$$

so that

(2.07.3:26) $$\overline{H} = \frac{\ell}{2g} X_o \quad .$$

Consequently, from (2.07.3:25) and (2.07.3:26)

$$\frac{H}{\overline{H}} = \frac{2gh\tau^2}{4\pi \ell a} \tan[\frac{\pi \ell}{2a}] = \frac{2}{\pi} \frac{\tau}{2\ell/\sqrt{gh}} \frac{\tau}{2a/\sqrt{gh}} \tan[\frac{\pi \ell}{2a}]$$

or, using Merian's formula,

$$\tau_1 = 2\ell/\sqrt{gh} \qquad \text{and} \qquad \tau = 2a/\sqrt{gh} \quad ,$$

(2.07.3:27) $$\frac{H}{\overline{H}} = \frac{2}{\pi} \frac{\tau}{\tau_1} \tan[\frac{\pi \ell}{2a}] \quad .$$

Resonance occurs for $\tan[\frac{\pi \ell}{2a}] = \infty$, i.e., for $\ell = (2\nu - 1)a$ with $\nu = 1, 2, \ldots$.

Proudman on pages 290-291 of "Dynamic Oceanography" gives calculations based on this model for the M_2 constituent in Lake Baikal. For Pestschannaja Bay his result is

$$0.82 \cos[360° \frac{t}{\tau} + 164°] \text{ cm}$$

as against an observed value of

$$0.48 \cos[360° \frac{t}{\tau} + 173°] \text{ cm} \quad .$$

He suggests that the discrepancy may arise because the model does not take into account the elastic yield of the "solid" earth to the tide-generating forces.

2.07.4. Shallow-Water Constituents.

We will consider the rectangular basin of section 2.07.3 and show how allowance may be made for the development of overtides by taking the higher order terms in the equations into account.

From equation (2.07.1:1), after deleting the cross-channel components and gradients, we have for continuity

(2.07.4:1) $$\frac{\partial}{\partial x} [(h + \eta)u] + \frac{\partial \eta}{\partial t} = 0 \quad .$$

Similarly, from equation (2.07.1:5) after deleting, in addition, the geostrophic term and replacing $F_x - \frac{1}{\rho} (\partial p_a/\partial x)$ with $g(\partial\overline{\eta}/\partial x)$ from the equilibrium tide, we have for motion

(2.07.4:2) $$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - g \frac{\partial}{\partial x}(\eta - \overline{\eta}) \quad .$$

Suppose the solution expanded in powers of some small ordering parameter.

(2.07.4:3) $$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots$$

(2.07.4:4) $$u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots \quad .$$

Substituting (2.07.4:3) and (2.07.4:4) in (2.07.4:1) and (2.07.4:2) and retaining only the terms to order ε gives

(2.07.4:5) $$h \frac{\partial u_1}{\partial x} + \frac{\partial \eta_1}{\partial t} = 0$$ (2.07.3:1)

(2.07.4:6) $$\frac{\partial u_1}{\partial t} + g \frac{\partial \eta_1}{\partial x} = g \frac{\partial \overline{\eta}}{\partial x}$$ (2.07.3:2)

These are the equations solved in section 2.07.3. They gave

(2.07.4:7) $$u_1 = C_1 \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$ (2.07.3:3)

(2.07.4:8) $$\eta_1 = H_1 \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$ (2.07.3:7)

(2.07.4:9) $$\overline{\eta} = \overline{H} \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$ (2.07.3:6)

where

(2.07.4:10) $$C_1 = H_1 \frac{2\ell}{h\tau} \quad .$$ (2.07.3:9)

Substituting (2.07.4:3) and (2.07.4:4) in (2.07.4:1) and (2.07.4:2) and equating terms of order ε^2 gives the equations for the second-order correction.

(2.07.4:11) $$h \frac{\partial u_2}{\partial x} + \frac{\partial \eta_2}{\partial t} = - \frac{\partial}{\partial x}(\eta_1 u_1)$$

(2.07.4:12) $$\frac{\partial u_2}{\partial t} + g \frac{\partial \eta_2}{\partial x} = - u_1 \frac{\partial u_1}{\partial x} \quad .$$

Multiplying (2.07.4:12) by h and taking the partial derivative with respect to x, then taking the partial with respect to t of (2.07.4:11), and forming the difference we get

(2.07.4:13) $$\frac{\partial^2 \eta_2}{\partial t^2} - gh \frac{\partial^2 \eta_2}{\partial x^2} = - \frac{\partial^2}{\partial x \partial t}(\eta_1 u_1) + h \frac{\partial}{\partial x}(u_1 \frac{\partial u_1}{\partial x}) \quad .$$

Substituting in the right-hand side of (2.07.4:13) from (2.07.4:7) and (2.07.4:8) gives, with the use of (2.07.4:10)

$$\frac{\partial^2 \eta_2}{\partial t^2} - gh \frac{\partial^2 \eta_2}{\partial x^2} = - H_1{}^2 \frac{1}{h}\left(\frac{2\pi}{\tau}\right)^2 \cos[2\frac{\pi}{\ell} x]\{\cos[2\frac{2\pi}{\tau} t] - \sin^2[\frac{2\pi}{\tau} t]$$

or

(2.07.4:14) $$\frac{\partial^2 \eta_2}{\partial t^2} - gh \frac{\partial^2 \eta_2}{\partial x^2} = H_1{}^2 \frac{1}{2h}\left(\frac{2\pi}{\tau}\right)^2 \cos[2\frac{\pi}{\ell} x]\{1 - 3\cos[2\frac{2\pi}{\tau} t] \quad .$$

Since the driver contains doubles of the arguments of the η_1-solution it seems reasonable to try a particular solution of the form

(2.07.4:15) $$\eta_2 = D_2 \cos[2\frac{\pi}{\ell} x] + E_2 \cos[2\frac{\pi}{\ell} x]\cos[2\frac{2\pi}{\tau} t] \quad .$$

Feeding (2.07.4:15) to (2.07.4:14) the left-hand side becomes

(2.07.4:16)

$$D_2(4gh\left(\frac{\pi}{\ell}\right)^2)\cos[2\frac{\pi}{\ell}\ x] + E_2\{4(gh\left(\frac{\pi}{\ell}\right)^2 - \left(\frac{2\pi}{\tau}\right)^2)\}\cos[2\frac{\pi}{\ell}\ x]\cos[2\frac{2\pi}{\tau}\ t]$$

Clearly, we require

$$D_2(4gh\left(\frac{\pi}{\ell}\right)^2) = H_1^{\ 2}\ \frac{1}{2h}\left(\frac{2\pi}{\tau}\right)^2$$

and

$$E_2\{4(gh\left(\frac{\pi}{\ell}\right)^2 - \left(\frac{2\pi}{\tau}\right)^2)\} = -\ 3H_1^{\ 2}\ \frac{1}{2h}\left(\frac{2\pi}{\tau}\right)^2$$

or

(2.07.4:17) $$D_2 = H_1^{\ 2}\ \frac{1}{2h}\ \frac{1}{gh}\left(\frac{\ell}{\tau}\right)^2$$

and

(2.07.4:18) $$E_2 = H_1^{\ 2}\ \frac{3}{2h}\ \frac{1}{4 - gh(\tau/\ell)^2}\ .$$

Using Merian's formula for the natural period of the basin $\tau_1 = 2\ell/\sqrt{gh}$ (2.07.4:17) and (2.07.4:18) become

(2.07.4:19) $$D_2 = \frac{H_1^{\ 2}}{gh}\left(\frac{\tau_1}{\tau}\right)^2$$

and

(2.07.4:20) $$E_2 = -\ \frac{3H_1^{\ 2}}{gh}\ \frac{\tau_1^{\ 2}}{\tau^2 - \tau_1^{\ 2}}\ .$$

Thus, the solution in terms of the natural period is

(2.07.4:21) $$\eta_2 = \frac{H_1^{\ 2}}{gh}\ \{\left(\frac{\tau_1}{\tau}\right)^2\cos[2\frac{\pi}{\ell}\ x] - 3\left(\frac{\tau_1^{\ 2}}{\tau^2 - \tau_1^{\ 2}}\right)\cos[2\frac{\pi}{\ell}\ x]\cos[2\frac{2\pi}{\tau}\ t]\}\ .$$

Expressed in terms of the equilibrium tide, since by (2.07.3:11)

$$H_1 = \frac{\tau^2}{\tau^2 - \tau_1^{\ 2}}\ \overline{H}\quad ,$$

we have

$$\overline{\eta} = \overline{H}\ \cos[\frac{\pi}{\ell}\ x]\cos[\frac{2\pi}{\tau}\ t]$$

$$\eta_1 = \overline{H}\left(\frac{\tau^2}{\tau^2 - \tau_1^{\ 2}}\right)\cos[\frac{\pi}{\ell}\ x]\cos[\frac{2\pi}{\tau}\ t]$$

(2.07.4:23) $$\eta_2 = \overline{H}\ \frac{\overline{H}}{gh}\ \left(\frac{\tau_1\tau}{\tau^2 - \tau_1^{\ 2}}\right)^2\{(1 - \left(\frac{\tau_1}{\tau}\right)^2)\cos[2\frac{\pi}{\ell}\ x] - 3\cos[2\frac{\pi}{\ell}\ x]\cos[2\frac{2\pi}{\tau}\ t]\}\ .$$

Equation (2.07.4:23) is the first overtide expressed in terms of the amplitude of the equilibrium tide, $\overline{H}$, the length, depth, and natural period, ℓ, h, τ_1, of the rectangular basin, and the period, τ, of the tide producing force.

The corresponding expression for the current, u_2, can be had by feeding η_2 as given by (2.07.4:23), along with (2.07.4:7) and (2.07.4:8) into (2.07.4:11) and (2.07.4:12).

The next overtide is secured by extending the ordering series, (2.07.4:3) and (2.07.4:4), to ε^3-terms and equating terms of order ε^3 after substitution in (2.07.4:1) and (2.07.4:2). Successive overtides will contain the factors

$$(\overline{H}/h),\ (\overline{H}/h)^2,\ \ldots,\ (\overline{H}/h)^{\nu},\ \ldots$$

where $\nu = 1, 2, \ldots$. In other words, the amplitude dependence of the overtides on the basin depth obeys a reciprocal power law.

2.07.5. A Traveling Atmospheric Disturbance.

The equations of continuity and motion are

(2.07.5:1) $$h\,\frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial t} = 0$$

(2.07.5:2) $$\frac{\partial u}{\partial t} = -\,g\,\frac{\partial}{\partial x}(\eta - \overline{\eta}) \quad .$$

Let the traveling atmospheric pressure disturbance be

(2.07.5:3) $$\overline{\eta} = F(x - Ut)$$

where U is a constant velocity and F is any physically possible function. We want a solution of (2.07.5:1) and (2.07.5:2) such that

(2.07.5:4) $$\eta = M\,F(x - Ut) \quad ; \quad M \text{ constant} \quad .$$

Feeding (2.07.5:4) to (2.07.5:1)

$$\frac{\partial u}{\partial x} = -\,\frac{U}{h}\,M\,\frac{\partial}{\partial t}[F(x - Ut)] \quad .$$

Now, since the argument of F is (x - Ut)--N.B.: not (x, Ut)--integration and differentiation with respect to either x or t bring about the same changes in the function. Therefore, if we integrate with respect to x, we will bet back F, i.e.,

$$u = -\frac{U}{h} M\, F(x - Ut) + \text{a constant} \quad .$$

But, if we suppose that things are at rest when there is no pressure disturbance, in other words, that u = 0 when $\bar{\eta} = 0$, the constant of integration is zero so that

(2.07.5:5) $$u = -\frac{U}{h} M\, F(x - Ut) \quad .$$

Feeding (2.07.5:3, 4, & 5) into (2.07.5:2)

$$\frac{U^2}{h} M\, F'(x - Ut) = -g[M\, F'(x - Ut) - F'(x - Ut)]$$

or

(2.07.5:6) $$-\frac{U^2}{h} = g(1 - M) \quad .$$

Consequently, solving for M, we get

$$M = \left(1 - \frac{U^2}{gh}\right)^{-1} \quad .$$

Therefore, from (2.07.5:3 & 4),

(2.07.5:7) $$\eta = \bar{\eta}\left(1 - \frac{U^2}{gh}\right)^{-1} \quad .$$

From (2.07.5:7) the elevation of the surface is a constant multiple of the equilibrium elevation. When U = 0, i.e., when the atmospheric disturbance just sits there without going anywhere, the elevation is the equilibrium elevation. When the atmospheric disturbance travels slower than the free wave speed, $U < \sqrt{gh}$, the phase is the same as the equilibrium form but the amplitude is greater. When $U = \sqrt{gh}$ we have resonance. When $U > \sqrt{gh}$ the wave is inverted. When $U >> \sqrt{gh}$ the amplitude is small.

If $\bar{\eta}$ contains the geostrophic factor $\exp\{-\frac{2\pi}{c} y\}$, where $c = gh\tau_p/U$, complete allowance can be made by inserting the same factor in η and u.

So far we have been using an infinitely long canal. Now, suppose a barrier at x = 0 where u = 0 for all t. For convenience, shift (2.07.5:3) to

(2.07.5:8) $\quad \overline{\eta} = f(t - \frac{x}{U})$

and try for solutions of (2.07.5:1) and (2.07.5:2) that will make u = 0 when x = 0. As usual, try

(2.07.5:9) $\quad \eta = M\, f(t - \frac{x}{U})$.

Substituting in (2.07.5:1)

$$\frac{\partial u}{\partial x} = - \frac{M}{h}\, f'(t - \frac{x}{U})$$

and integrating with respect to x

(2.07.5:10) $\quad u = \frac{MU}{h}\, f(t - \frac{x}{U})$

where the constant of integration is zero for the same reason as before.

For x = 0 this says that

(2.07.5:11) $\quad u = \frac{MU}{h}\, f(t) = 0$.

This can not be true for all t unless $f(t) \equiv 0$ for all t. If that were the case, $\overline{\eta}$ would show no variation with time and the baby would go out with the bath.

To get away with it without actually cheating we must superimpose another motion with the following characteristics:

(a) It must correspond to $\overline{\eta} = f(t - \frac{x}{U}) = 0$.

(b) At x = 0 it must take on just the right value to offset the value of u given by (2.07.5:11).

Let's try a free wave for the additional motion, say one with u given by

(2.07.5:12) $\quad u = - \frac{MU}{h}\, f(t - \frac{x}{U_c})$

where $U_c^2 = gh$ is the square of the free wave speed. Equation (2.07.5:12) will certainly do our business at x = 0 and the motion can exist in the presence of $\overline{\eta} = 0$.

That (2.07.5:12) is a solution of (2.07.5:1) and (2.07.5:2) may be verified. From (2.07.5:1)

$$\frac{\partial \eta}{\partial t} = - h\left(- \frac{MU}{h}\right)\left(- \frac{1}{U_c}\right) f'(t - \frac{x}{U_c})$$

or

$$\frac{\partial \eta}{\partial t} = - M\left(\frac{U}{U_c}\right) f'(t - \frac{x}{U_c})$$

or, integrating with respect to t

(2.07.5:13) $$\eta = - M \frac{U}{U_c} f(t - \frac{x}{U_c})$$.

Substituting from (2.07.5:12) and (2.07.5:13) in (2.07.5:2)

$$- \frac{MU}{h} f'(t - \frac{x}{U_c}) = - g \; [M \frac{U}{U_c^2} f'(t - \frac{x}{U_c}) - 0]$$

$$- \frac{MU}{h} = - g \frac{MU}{U_c^2}$$

or

$$U_c^2 = gh$$

as it should.

The complete solution, i.e., the traveling disturbance plus the patch to make it behave at the barrier, is

(2.07.5:14) $$\eta = M[f(t - \frac{x}{U}) - \frac{U}{U_c} f(t - \frac{x}{U_c})]$$

(2.07.5:15) $$u = \frac{MU}{h}[f(t - \frac{x}{U}) - f(t - \frac{x}{U_c})]$$.

Inspection of (2.07.5:14) shows that we have two waves traveling along the canal. One rides with the pressure disturbance at the speed U while the other travels away from the barrier at the free wave speed $U_c = \sqrt{gh}$.

Remark: Equation (2.07.5:14) is sometimes referred to as "Lagrange's formula." I wish they'd leave off this kind of thing. Any famous scientist or mathematician has so many different results named for him that you have one hell of a time finding out which one is being referred to.

When the atmospheric disturbance travels away from the barrier at a speed $U < U_c$ the result looks like the first two sketches in Fig. 2.07.5-1 when $M > 1$, page 186. Since $M > 1$, there is a magnified wave keeping pace with the disturbance at speed U and, since $U < U_c$, there is an inverted wave with speed U_c running out ahead of it. If $U > U_c$ and $M < 0$, with the disturbance traveling away from the barrier, you get the response shown in the third picture in Fig. 2.07.5-1. It is an inverted wave keeping pace with the disturbance and a direct wave trailing behind it.

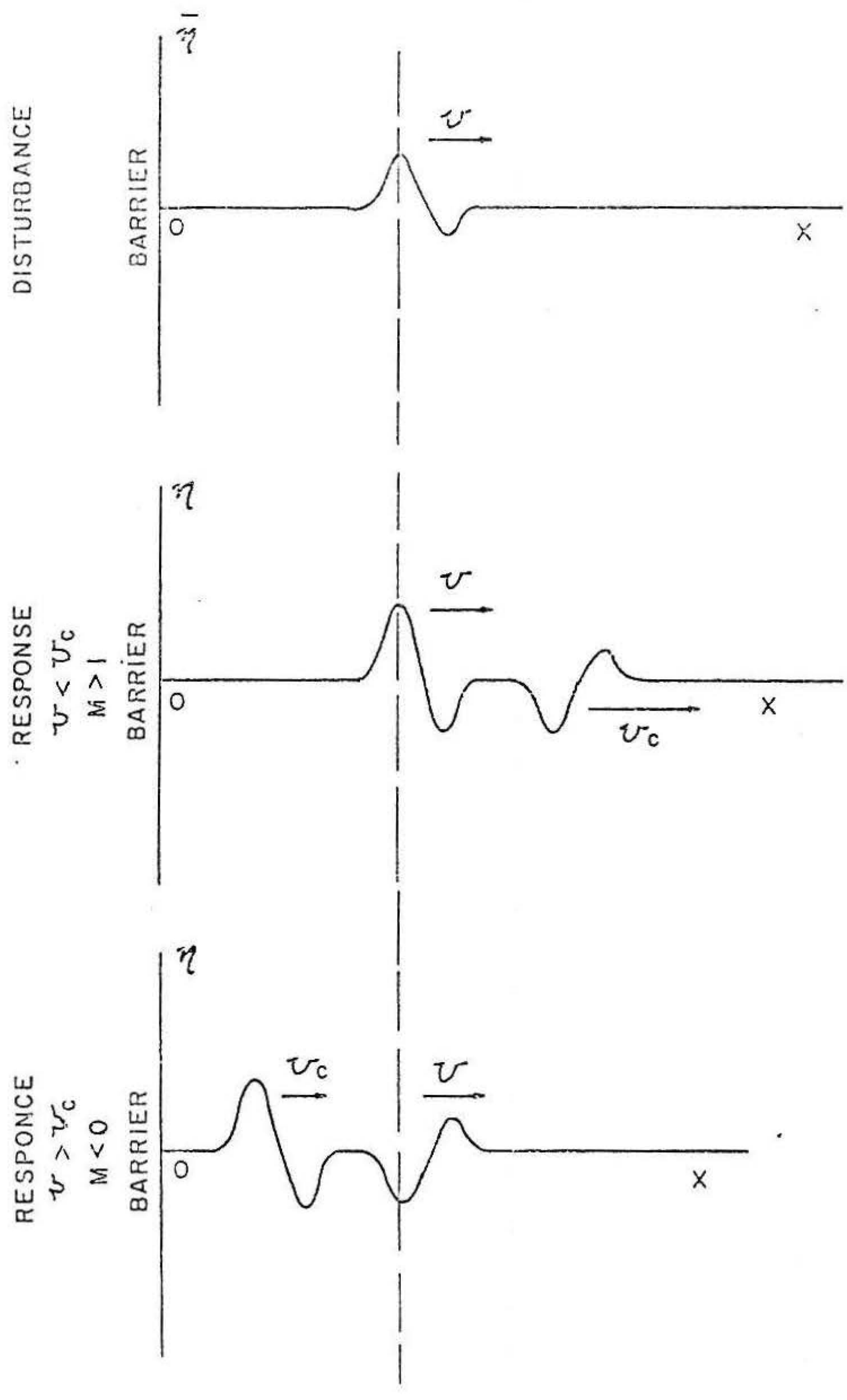

DISTURBANCE
BARRIER
O
X
RESPONSE
$v < v_c$
$M > 1$
BARRIER
O
X
v_c
RESPONCE
$v > v_c$
$M < 0$
BARRIER
O
X
v_c

Fig. 2.07.5-1

This analysis can be adapted for a moving pressure jump such as you find in a frontal passage. The equilibrium response is an adjustment from one level ahead of the jump to a lower level behind it if the jump is an increase. In the region of the front there is a traveling column of water whose volume is equal to the volume of water displaced behind it, Fig. 2.07.5-2.

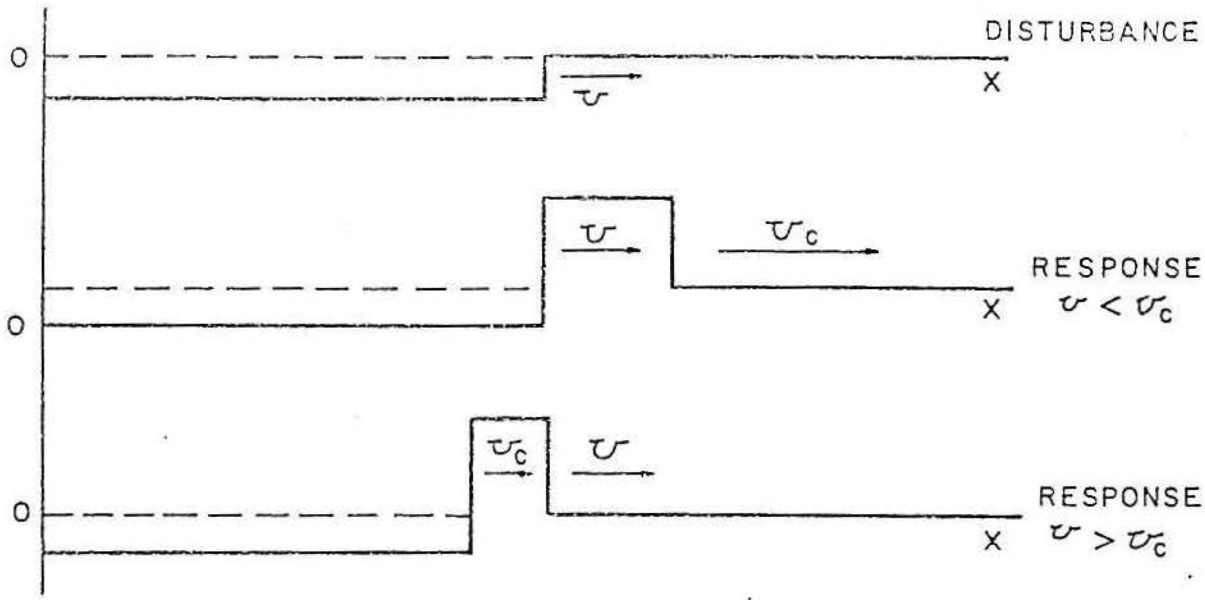


Fig. 2.07.5-2

Let $\overline{H}$ be the decrease in the equilibrium height $\overline{\eta}$. Then the height of the wave which travels with the speed U is

$$- M\overline{H} = - \frac{U_c^{\,2}\overline{H}}{U_c^{\,2} - U^2}$$

from (2.07.5:9) and (2.07.5:7) and the height of the wave which travels with the speed U_c is

$$\frac{U}{U_c} M\overline{H} = \frac{UU_c\overline{H}}{U_c^{\,2} - U^2}$$

from (2.07.5:13) and (2.07.5:7). The superposition of the two gives

$$- \frac{U_c\overline{H}}{U_c + U} .$$

If b is the width of the canal perpendicular to the direction of travel and s is the distance from the barrier traveled by the wave of speed U, then the distance traveled by the wave of speed U_c is $(U_c s/U)$. If $U_c > U$, then the traveling column is $[(U_c/U) - 1]$s long and the water volume is

$$\frac{UU_c\overline{H}}{U_c^2 - U^2}\left(\frac{U_c}{U} - 1\right)sb = \frac{U_c}{U_c + U}\, sb\overline{H} \quad .$$

This is equal to the volume of water displaced behind the column. If $U_c < U$, the length of the traveling column is

$$\frac{U_c^2\,\overline{H}}{U^2 - U_c^2}\left(1 - \frac{U_c}{U}\right)sb = \frac{U_c^2}{U(U_c + U)}\, sb\overline{H}$$

and is also equal to the volume of the displaced water behind the column.

Surges caused by frontal passages are not uncommon. Proudman gives an example of one that hit Sussex, "Dynamic Oceanography," page 300. A number of them that have been rather disasterous to Chicago have been analysed in the literature. Try the Transactions of the American Geophysical Union.

2.08. Internal Seiches, Tides, and Waves.

2.08.1. Introduction.

So far we have been discussing situations where the maximum vertical variation in the water motion occured at the water surface and where the currents were substantially the same from top to bottom. Another situation is possible. In it the maximum vertical motion occurs somewhere between surface and bottom and along any one vertical there are large changes of current in both size and direction. This case can occur only in the presence of large vertical density gradients such as those indicated by sharp thermoclines or haloclines. When oscillations of this sort are present there will be a wave-like undulation of the thermocline and halocline. What this can do to your ideas about the structure of the ocean deduced from classical Nansen bottle measurements taken at different times and places I leave to your imagination!

In our detailed work we will simplify things by considering a two layered system having homogeneous water of different densities in each layer so that there will be a surface of discontinuity of density rather than a layer of rapid change. Friction will be neglected and, whenever geostrophic effects are considered, latitude will be taken uniform.

The direct effect of the tide-generating forces is still ordinary tides on the surface but we will see that in some cases they may produce internal tides as well.

2.08.2. The Equations for a Two-Layered System.

For our two-layered system suppose an upper layer with density, ρ, depth, h, lying on top of a lower layer of density, ρ', and depth, h'. The depth of the water from surface to bottom is then $h + h'$. When a disturbance from the level position occurs, use η as the surface displacement and η' as the displacement of the surface of discontinuity. Take the Oxy-plane in the mean surface with Oz vertical and let z denote depth below the mean surface while z' denotes depth below the mean discontinuity surface. u, v will be the current components in the upper layer and u', v' the current components in the lower layer. Ω is the tide-generating potential and p_a the atmospheric pressure.

Continuity for the upper layer will be the same as usual except that the displacement must be taken relative to the discontinuity surface, i.e., instead of η we use $\eta - \eta'$.

(2.08.2:1U) $$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) + \frac{\partial}{\partial t}(\eta - \eta') = 0 \quad .$$

For the lower layer there is no modification so that

(2.08.2:1L) $$\frac{\partial}{\partial x}(h'u') + \frac{\partial}{\partial y}(h'v') + \frac{\partial \eta'}{\partial t} = 0 \quad .$$

The equations of motion for the upper layer are

(2.08.2:2U) $$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial \Omega}{\partial x}$$

(2.08.2:3U) $$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial \Omega}{\partial y}$$

and for the lower layer

(2.08.2:2L) $$\frac{\partial u'}{\partial t} - \frac{2\pi}{\tau_p} v' = -\frac{1}{\rho'}\frac{\partial p'}{\partial x} - \frac{\partial \Omega}{\partial x}$$

(2.08.2:3L) $$\frac{\partial v'}{\partial t} + \frac{2\pi}{\tau_p} u' = -\frac{1}{\rho'}\frac{\partial p'}{\partial y} - \frac{\partial \Omega}{\partial y}$$

where p and p' are pressures in the upper and lower layers.

On the neglect of the vertical velocities and accelerations, the third member of the equation of motion trio produces the hydrostatic equation so that we have

(2.08.2:4U) $$p = p_a + \rho g(\eta + z)$$

(2.08.2:4L) $$p' = p_a + \rho g(\eta + h - \eta') + \rho' g(\eta' + z') \quad .$$

Substitution of (2.08.2:4U&L) in (2.08.2:2U)-(2.08.3L) gives

(2.08.2:5U) $$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = -g\frac{\partial \eta}{\partial x} - \frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p_a}{\partial x}$$

(2.08.2:6U) $$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = -g\frac{\partial \eta}{\partial y} - \frac{\partial \Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p_a}{\partial y}$$

(2.08.2:5L) $$\frac{\partial u'}{\partial t} - \frac{2\pi}{\tau_p} v' = -g\frac{\rho}{\rho'}\frac{\partial \eta}{\partial x} - g\left(1 - \frac{\rho}{\rho'}\right)\frac{\partial \eta'}{\partial x} - \frac{\partial \Omega}{\partial x} - \frac{1}{\rho'}\frac{\partial p_a}{\partial x}$$

(2.08.2:6L) $$\frac{\partial v'}{\partial t} + \frac{2\pi}{\tau_p} u' = -g\frac{\rho}{\rho'}\frac{\partial \eta}{\partial y} - g\left(1 - \frac{\rho}{\rho'}\right)\frac{\partial \eta'}{\partial y} - \frac{\partial \Omega}{\partial y} - \frac{1}{\rho'}\frac{\partial p_a}{\partial y}$$

or, if we take p_a uniform and invoke the equilibrium form through $\Omega = -g\overline{\eta}$,

(2.08.2:7U) $$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = -g\frac{\partial}{\partial x}(\eta - \overline{\eta})$$

(2.08.2:8U) $$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = -g\frac{\partial}{\partial y}(\eta - \overline{\eta})$$

(2.08.2:7L) $$\frac{\partial u'}{\partial t} - \frac{2\pi}{\tau_p} v' = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g\left(1 - \frac{\rho}{\rho'}\right)\frac{\partial \eta'}{\partial x} + g \frac{\partial \overline{\eta}}{\partial x}$$

(2.08.2:8L) $$\frac{\partial v'}{\partial t} + \frac{2\pi}{\tau_p} u' = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial y} - g\left(1 - \frac{\rho}{\rho'}\right)\frac{\partial \eta'}{\partial y} + g \frac{\partial \overline{\eta}}{\partial y} \quad .$$

If, instead, in (2.08.2:5U)-(2.08.2:6L) we neglect the tide-generating forces and use

$$p_a = \text{a constant} - \rho g \overline{\eta} \quad ,$$

equations (2.08.2:5U) and (2.08.2:6U) again pass over into (2.08.2:7U) and (2.08.2:8U) but (2.08.2:5L) and (2.08.2:6L) become

(2.08.2:9L) $$\frac{\partial u'}{\partial t} - \frac{2\pi}{\tau_p} v' = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g\left(1 - \frac{\rho}{\rho'}\right)\frac{\partial \eta'}{\partial x} + g \frac{\rho}{\rho'} \frac{\partial \overline{\eta}}{\partial x}$$

(2.08.2:10L) $$\frac{\partial v'}{\partial t} + \frac{2\pi}{\tau_p} u' = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial y} - g\left(1 - \frac{\rho}{\rho'}\right)\frac{\partial \eta'}{\partial y} + g \frac{\rho}{\rho'} \frac{\partial \overline{\eta}}{\partial y} \quad .$$

The final bit of business is to get a criterion for the neglect of the vertical acceleration. For the upper layer the criterion is the same as that given by equation (2.06.1:11),

(2.08.2:11U) $$\frac{h \dfrac{\partial^2 \eta}{\partial t^2}}{g\eta} << 1 \quad .$$

The lower layer requires a bit of modification. For it the hydrostatic equation arose from the neglect of $\partial w'/\partial t$ in

$$\frac{\partial w'}{\partial t} = - \frac{1}{\rho'} \frac{\partial p'}{\partial z'} + g$$

so that, as before,

$$w' = - \left(1 - \frac{z'}{h'}\right)\frac{\partial \eta'}{\partial t}$$

and

$$\frac{\partial p'}{\partial z'} = \rho' g + \rho'\left(1 - \frac{z'}{h'}\right)\frac{\partial^2 \eta'}{\partial t^2} \quad .$$

or integrating

$$p' = p_a + \rho g(\eta + h - \eta') + \rho' g(\eta' + z') + \rho'\left(z' - \frac{1}{2}\frac{z'}{h'}\right)^2 \frac{\partial^2 \eta'}{\partial t^2} \ .$$

Comparing this with (2.08.2:4L) we see that, as before, the last term is the departure from the hydrostatic case so that what we require at $z' = h'$ is that

$$\tfrac{1}{2}\rho' h' (\partial^2 \eta'/\partial t^2)$$

be small in the lower layer in comparison with the hydrostatic terms

$$- \rho g \eta' + \rho' g \eta'$$

so that our criterion for the lower layer will be

$$\frac{\rho'}{\rho' - \rho} \frac{h' \frac{\partial^2 \eta'}{\partial t^2}}{g \eta'} << 1 \qquad (2.08.2:11L)$$

2.08.3. Internal Seiches in a Narrow Lake.

Consider a long narrow lake with two layers of water. If the upper layer were drawn off, an ordinary seiche would be possible in the lower layer and it is useful to use it as a reference. Its period will be the τ_1 of Merian's formula.

Now suppose that a free uninodal oscillation is possible with both layers in place; one of such a nature that the oscillation of the free surface is very small compared with that of the discontinuity surface. It will have some period τ_i which is not necessarily the same as τ_1. Continuity in the lower layer will be as before with τ_i replacing τ_1. To compensate for the movement of water in the lower layer without much change in the surface elevation, the volume of water moved in the upper layer will have to be about equal to the volume moved in the lower but it must move in the opposite direction. One can expect that the currents in the two layers will be inversely proportional to the depths of the layers, more or less, and oppositely directed, Fig. 2.08.3-1, page 193. Since the currents are always opposite in the two layers so also are the accelerations and, consequently, the horizontal pressure gradients as well. It follows that the elevations of the two surfaces are opposite in sign.

In an ordinary seiche water of density ρ' displaces air. In an internal seiche it displaces water of density ρ. At any point in the lower layer the changes in pressure due to the two layers will be opposite in sign.

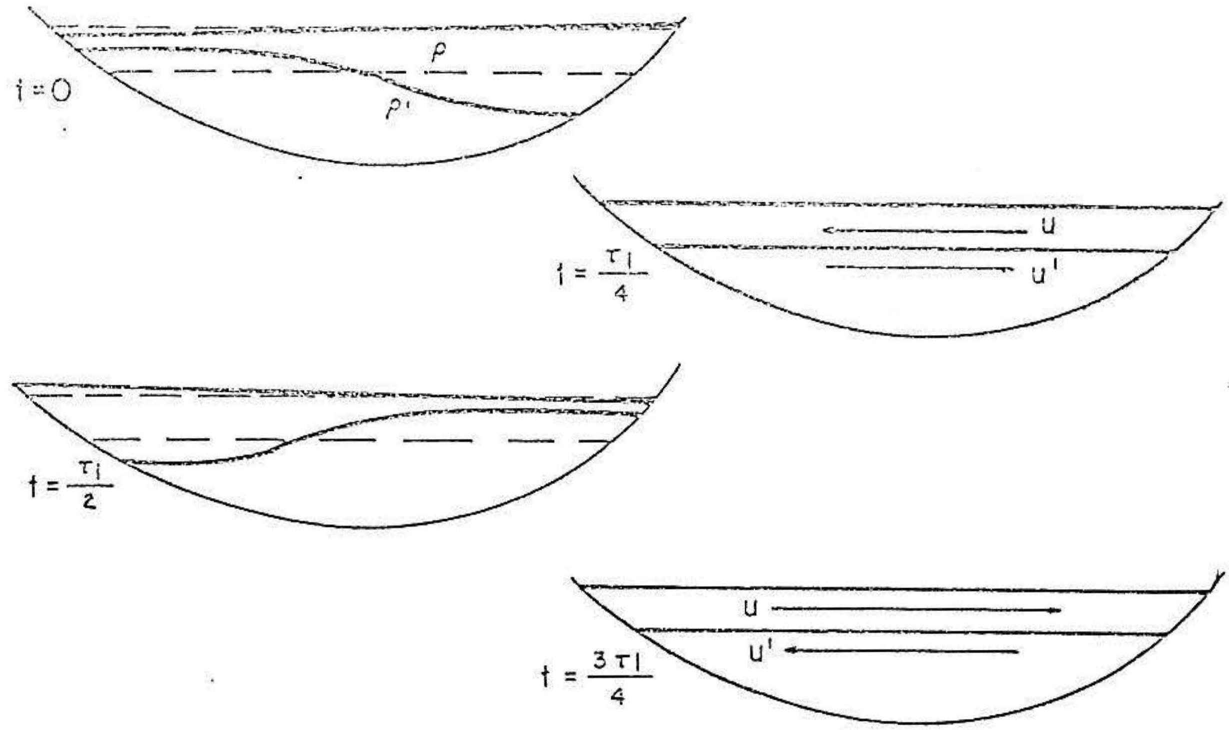


Fig. 2.08.3-1

The pressure gradients in the lower layer have the same direction as those of the ordinary seiche so that changes due to the motion of the discontinuity surface must preponderate over those due to the free surface. This confirms our assertion that the motion of the free surface must be relatively small. It also follows that the horizontal pressure gradients in the lower layer are reduced from those in the ordinary seiche. This means that the accelerations are smaller and that τ_i must be much greater than τ_1.

Consider the usual rectangular basin of length, ℓ, constant depth h and h', basin ends at $x = 0$ and $x = \ell$ where we have the boundary conditions $u = 0$ and $u' = 0$; $\overline{\eta} = 0$; $v = 0$, $v' = 0$.

Continuity from (2.08.2:1U&L) is

(2.08.3:1U) $$h \frac{\partial u}{\partial x} + \frac{\partial}{\partial t}(\eta - \eta')$$

(2.08.3:1L) $$h' \frac{\partial u'}{\partial x} + \frac{\partial \eta'}{\partial t} = 0$$

From (2.08.2:5U&L) the equations of motion, after neglecting body and geostrophic forces, are

(2.08.3:2U) $$\frac{\partial u}{\partial t} = - g \frac{\partial \eta}{\partial x}$$

(2.08.3:2L) $$\frac{\partial u'}{\partial t} = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g\left(1 - \frac{\rho}{\rho'}\right)\frac{\partial \eta'}{\partial x} \quad .$$

As usual, we try for a solution

(2.08.3:3) $$u = C \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

where C is a constant speed. Equation (2.08.3:3) certainly takes care of the boundary conditions at $x = 0$ and $x = \ell$.

Feeding (2.08.3:3) into (2.08.3:2U)

$$\frac{\partial \eta}{\partial x} = - \frac{2\pi}{g\tau} C \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

which, on integration, gives

(2.08.3:4) $$\eta = \frac{2\ell}{g\tau} C \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] \quad .$$

No additive constant is necessary.

Substituting (2.08.3:3) in (2.08.3:1U) gives

$$\frac{\partial}{\partial t}(\eta - \eta') = - \frac{\pi h}{\ell} C \cos[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

which, on integration, gives

$$\eta - \eta' = \frac{h\tau}{2\ell} C \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] \quad .$$

Substituting from (2.08.3:4) and solving for η' gives

(2.08.3:5) $$\eta' = \left(\frac{2\ell}{g\tau} - \frac{h\tau}{2\ell}\right) C \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

which is the elevation of the interface corresponding to the free surface elevation, (2.08.3:4).

To complete our analysis we need to find u' which can be done be feeding (2.08.3:5) into (2.08.3:1L) to get

$$\frac{\partial u'}{\partial x} = \frac{\pi}{h'}\left(\frac{4\ell}{g\tau^2} - \frac{h}{\ell}\right) C \cos[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] \quad .$$

Integration with respect to x then gives

(2.08.3:6) $$u' = \frac{1}{h'}\left(\frac{4\ell^2}{g\tau^2} - h\right) C \sin[\frac{\pi}{\lambda} x]\sin[\frac{2\pi}{\tau} t] \quad .$$

If we define

(2.08.3:7) $$\beta \equiv \frac{4\ell^2}{gh\tau^2} = \frac{2\ell}{g\tau}\,\frac{2\ell}{h\tau} = \frac{H}{C}\,\frac{2\ell}{h\tau}$$

or

$$C = \frac{1}{\beta}\,\frac{2\ell}{h\tau}\,H$$

where

(2.08.3:8) $$H \equiv \frac{2\ell}{g\tau}\,C \qquad --$$

H is thus a constant length--we can gather up our solutions in the form

(2.08.3:9) $$\eta = H\cos[\frac{\pi}{\ell}\,x]\cos[\frac{2\pi}{\tau}\,t]$$

(2.08.3:10) $$u = \frac{1}{\beta}\,\frac{2\ell}{h\tau}\,H\sin[\frac{\pi}{\ell}\,x]\sin[\frac{2\pi}{\tau}\,t]$$

(2.08.3:11) $$\eta' = (1 - \frac{1}{\beta})\,H\cos[\frac{\pi}{\ell}\,x]\cos[\frac{2\pi}{\tau}\,t]$$

since

$$\left(\frac{2\ell}{g\tau} - \frac{h\tau}{2\ell}\right)C = \left(1 - \frac{gh\tau^2}{4\ell^2}\right)\frac{2\ell}{g\tau}\,C = (1 - \frac{1}{\beta})\,H$$

and

(2.08.3:12) $$u' = (1 - \frac{1}{\beta})\frac{2\ell}{h'\tau}\,H\sin[\frac{\pi}{\ell}\,x]\sin[\frac{2\pi}{\tau}\,t]$$

since

$$\begin{aligned}\frac{1}{h'}\left(\frac{4\ell^2}{g\tau^2} - h\right)C &= \left(1 - \frac{gh\tau^2}{4\ell^2}\right)\frac{4\ell^2}{g\tau^2}\,\frac{1}{h'}\,C \\ &= (1 - \frac{1}{\beta})\frac{4\ell^2}{g\tau^2}\,\frac{1}{h'}\,\frac{1}{\beta}\,\frac{2\ell}{h\tau}\,H \\ &= (1 - \frac{1}{\beta})\frac{4\ell^2}{gh\tau^2}\,\frac{1}{\beta}\,\frac{2\ell}{h'\tau}\,H \\ &= (1 - \frac{1}{\beta})\frac{2\ell}{h'\tau}\,H \quad .\end{aligned}$$

β may be expressed as a function of h, h', ρ, and ρ' by substituting (2.08.3:9), (2.08.3:11), and (2.08.3:12) in the only fundamental equation as yet unused, (2.08.3:2L). The makings are

$$\frac{\partial u'}{\partial t} = (1 - \frac{1}{\beta})\frac{4\pi\ell}{h'\tau^2}\,H\sin[\frac{\pi}{\ell}\,x]\cos[\frac{2\pi}{\tau}\,t]$$

$$\frac{\partial\eta}{\partial x} = -\,\frac{\pi}{\ell}\,H\sin[\frac{\pi}{\ell}\,x]\cos[\frac{2\pi}{\tau}\,t]$$

$$\frac{\partial\eta'}{\partial x} = -\,(1 - \frac{1}{\beta})\frac{\pi}{\ell}\,H\sin[\frac{\pi}{\ell}\,x]\cos[\frac{2\pi}{\tau}\,t] \quad .$$

Clearly, the factor $\pi H\sin[\frac{\pi}{\ell}\,x]\,\cos[\frac{2\pi}{\tau}\,t]$ common to all terms will divide out so that

$$(1 - \frac{1}{\beta})\frac{4\ell}{h'\tau^2} = - g\,\frac{\rho}{\rho'}(-\frac{1}{\ell}) - g(1 - \frac{\rho}{\rho'})[-(1 - \frac{1}{\beta})\frac{1}{\ell}]$$

$$\frac{4\ell}{h'\tau^2} - \frac{4}{h'\tau^2}\,\frac{1}{\beta} = \frac{g}{\ell}\frac{\rho}{\rho'} + \frac{g}{\ell}(1 - \frac{\rho}{\rho'} - \frac{1}{\beta} + \frac{1}{\beta}\,\frac{\rho}{\rho'})$$

$$\frac{4\ell^2}{gh'\tau^2}\,\beta - \frac{4\ell^2}{gh'\tau^2} = \beta - 1 + \frac{\rho}{\rho'} \quad .$$

But by (2.08.3:7)

$$\frac{4\ell^2}{gh'\tau^2}\,\beta = \frac{4\ell^2}{gh\tau^2}\,\frac{h}{h'}\,\beta = \frac{h}{h'}\,\beta^2$$

so that

$$\frac{h}{h'}\,\beta^2 - \frac{h}{h'}\,\beta = \beta - 1 + \frac{\rho}{\rho'}$$

or

(2.08.3:13) $$\frac{h}{h'}\,\beta^2 - (1 + \frac{h}{h'})\,\beta + (1 - \frac{\rho}{\rho'}) = 0 \quad .$$

Equation (2.08.2:13) is sometimes called Stokes' Equation.

Solving for β we get the roots

$$\beta = \frac{(1 + \frac{h}{h'}) \pm \sqrt{(1 + \frac{h}{h'})^2 - 4\,\frac{h}{h'}(1 - \frac{\rho}{\rho'})}}{2\,\frac{h}{h'}}$$

or

(2.08.3:14) $$\beta = \tfrac{1}{2}(1 + \frac{h'}{h}) \pm [\tfrac{1}{4}(1 + \frac{h'}{h})^2 - \frac{h'}{h}(1 - \frac{\rho}{\rho'})]^{\frac{1}{2}}$$

Call the root with the positive sign β_o and the one with the negative sign β_i,

$$\beta_o = \tfrac{1}{2}(1 + \frac{h'}{h}) + [\tfrac{1}{4}(1 + \frac{h'}{h})^2 - \frac{h'}{h}(1 - \frac{\rho}{\rho'})]^{\frac{1}{2}}$$

$$\beta_i = \tfrac{1}{2}(1 + \frac{h'}{h}) - [\tfrac{1}{4}(1 + \frac{h'}{h})^2 - \frac{h'}{h}(1 - \frac{\rho}{\rho'})]^{\frac{1}{2}} \quad .$$

In actual cases, ρ and ρ' are nearly equal so that $1 - \frac{\rho}{\rho'}$ is very small. We will approximate β_o by neglecting the $(1 - \frac{\rho}{\rho'})$-factor entirely and β_i by retaining only linear terms in $(1 - \frac{\rho}{\rho'})$. We get

(2.08.3:15) $$\beta_o = 1 + \frac{h'}{h} \quad .$$

Remark: You can see that if we used the same approximation for β_i that we are using for β_o, we would get the rather uninteresting result $\beta_i = 0$.

Expanding the bracket by the binomial theorem

$$[\tfrac{1}{4}(1 + \frac{h'}{h})^2 - \frac{h'}{h}(1 - \frac{\rho}{\rho'})]^{\frac{1}{2}} = [\tfrac{1}{4}(1 + \frac{h'}{h})^2]^{\frac{1}{2}} - \tfrac{1}{2}[\tfrac{1}{4}(1 + \frac{h'}{h})^2]^{-\frac{1}{2}}[\frac{h'}{h}(1 - \frac{\rho}{\rho'})] \quad ,$$

the following terms being dropped since they are quadratic or higher in $(1 - \frac{\rho}{\rho'})$. Simplifying and substituting

(2.08.3:16) $$\beta_i = (1 - \frac{\rho}{\rho'}) \frac{h'}{h + h'} \quad .$$

From (2.08.3:15), and to the same order of approximation, it follows from (2.08.3:7) and (2.08.3:9)-(2.08.3:12) that

(2.08.3:17) $$\tau = \frac{2\ell}{\sqrt{g(h + h')}}$$

(2.08.3:18) $$\frac{\eta'}{\eta} = \frac{h'}{h + h'}$$

(2.08.3:19) $$\frac{u'}{u} = 1 \quad .$$

Equations (2.08.3:17)-(2.08.3:19) are the relations for an ordinary seiche. Apparently, the presence of a density stratification does not prevent, or seriously alter, the possibility of the development of an ordinary seiche.

Corresponding to (2.08.3:16), and to the same order of approximation, (2.08.3:7) and (2.08.3:9)-(2.08.3:12) give

(2.08.3:20) $$\tau_i = 2\ell[\frac{\rho'}{\rho' - \rho} \frac{1}{g}(\frac{1}{h} + \frac{1}{h'})]^{\frac{1}{2}} \quad .$$

$\eta'/\eta = 1 - 1/\beta = (\beta - 1)/\beta$. Therefore $\eta/\eta' = \beta/(\beta - 1) = f(\beta)$ and, using Maclaurin's expansion,

$$f(\beta) = \beta/(\beta - 1) \quad ; \quad f(0) = 0 \quad ; \quad f'(\beta) = -1/(\beta - 1)^2 \; ; \; f'(0) = -1 \; .$$

Higher order terms will yield $(1 - \rho/\rho')^2$ etc.. Therefore $\eta/\eta' = -\beta_i$ so that

(2.08.3:21) $$\frac{\eta'}{\eta} = -\frac{1}{\beta_i} = -\frac{\rho'}{\rho' - \rho}(1 + \frac{h}{h'}) \quad .$$

$u'/u = \beta(h/h')(1 - 1/\beta) = (h/h')(\beta - 1) = g(\beta)$, and using Maclaurin's expansion,

$$g(\beta) = (h/h')(\beta - 1) \quad ; \quad g(0) = -h/h' \quad ; \quad g'(\beta) = (h/h')(1 - 1) = 0 \quad .$$

Therefore,

$$\frac{u'}{u} = -\frac{h}{h'} \quad . \qquad (2.08.3:22)$$

Equations (2.08.3:17)-(2.08.3:22) given the features of the internal seiche.

We can rework our criterion, (2.08.2:14L), for the neglect of vertical acceleration by using (2.08.3:11) and (2.08.3:16).

$$\frac{\partial \eta'}{\partial t} = -\frac{2\pi}{\tau}\left(1 - \frac{1}{\beta}\right) H \cos\left[\frac{\pi}{\ell} x\right] \sin\left[\frac{2\pi}{\tau} t\right]$$

$$\frac{\partial^2 \eta'}{\partial t^2} = -\frac{4\pi^2}{\tau^2}\left(1 - \frac{1}{\beta}\right) H \cos\left[\frac{\pi}{\ell} x\right] \cos\left[\frac{2\pi}{\tau} t\right]$$

$$\frac{\rho'}{\rho' - \rho} \frac{h'}{g} \frac{\partial^2 \eta'/\partial t^2}{\eta'} = \frac{\rho'}{\rho' - \rho} \frac{h'}{g}\left(-\frac{4\pi^2}{\tau^2}\right)$$

$$= -\frac{\rho'}{\rho' - \rho} \frac{hh'\pi^2}{\ell^2} \frac{4\ell^2}{gh\tau^2}$$

$$= -\frac{\rho'}{\rho' - \rho} \frac{hh'\pi^2}{\ell^2} \beta$$

and, using β_i to approximate for β,

$$= -\frac{\rho'}{\rho' - \rho} \frac{hh'\pi^2}{\ell^2} \frac{\rho' - \rho}{\rho'} \frac{h'}{h + h'}$$

$$= -\frac{h'}{h + h'} \frac{\pi^2 hh'}{\ell^2} \quad .$$

In the rectangular case this is the quantity that must be made small. Equation (2.08.3:20) plays the same role for an internal seiche that Merian's formula plays for an ordinary seiche. Proudman applies it to Loch Earn and gets τ_i = 18 hours as against an observed value of 15.2 hours.

2.08.4. Internal Seiches Across a Uniform Channel.

In section 2.06.7 we discussed free oscillations across a channel with geostrophic effects. This discussion will now be extended to a two-layer system. Take the depth as uniform and suppose all conditions uniform along the channel.

For continuity

$$h \frac{\partial u}{\partial x} + \frac{\partial}{\partial t}(\eta - \eta') = 0 \qquad (2.08.4{:}1U)$$

$$h' \frac{\partial u'}{\partial x} + \frac{\partial \eta'}{\partial t} = 0 \qquad (2.08.4{:}1L)$$

For motion

$$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = - g \frac{\partial \eta}{\partial x} \qquad (2.08.4{:}2U)$$

$$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = 0 \qquad (2.08.4{:}3U)$$

$$\frac{\partial u'}{\partial t} - \frac{2\pi}{\tau_p} v' = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g(1 - \frac{\rho}{\rho'})\frac{\partial \eta'}{\partial x} \qquad (2.08.4{:}2L)$$

$$\frac{\partial v'}{\partial t} + \frac{2\pi}{\tau_p} u' = 0 \qquad (2.08.4{:}3L)$$

Try for solutions

$$u = C \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] \qquad (2.08.4{:}4U)$$

$$u' = C' \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] \qquad (2.08.4{:}4L)$$

where C and C' are constant speeds. The boundary conditions are $u = 0$ and $u' = 0$ at $x = 0$ and $x = \ell$. From (2.08.4:3U) and (2.08.4:3L)

$$\frac{\partial v}{\partial t} = - \frac{2\pi}{\tau_p} C \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

$$\frac{\partial v'}{\partial t} = - \frac{2\pi}{\tau_p} C' \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

or

$$v = \frac{\tau}{\tau_p} C \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] \qquad (2.08.4{:}5U)$$

(2.08.4:5L) $$v' = \frac{\tau}{\tau_p} C' \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

No additive constant required.

With (2.08.4:4L) in (2.08.4:1L)

$$\frac{\partial \eta'}{\partial t} = - h' \frac{\pi}{\ell} C' \cos[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] \quad .$$

Integrating with respect to t

(2.08.4:6L) $$\eta' = \frac{h'\tau}{2\ell} C' \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] \quad .$$

No additive constant required.

With (2.08.4:4U) and (2.08.4:6L) in (2.08.4:1U)

$$\frac{\partial \eta}{\partial t} = - h' \frac{\pi}{\ell} C' \cos[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] - h \frac{\pi}{\ell} C \cos[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$
$$= - \frac{\pi}{\ell}(h'C' + hC)\cos[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t] \quad .$$

Integrating with respect to t

(2.08.4:6U) $$\eta = \frac{\tau}{2\ell}(hC + h'C')\cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

or, if we define

(2.08.4:7U) $$H \equiv \frac{\tau}{2\ell}(hC + h'C')$$

(2.08.4:7L) $$H' \equiv \frac{\tau}{2\ell} h'C' \quad ,$$

then

(2.08.4:8U) $$\eta = H \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

(2.08.4:8L) $$\eta' = H' \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

Using (2.08.4:4), (2.08.4:5), and (2.08.4:8) in (2.08.4:2)

$$\frac{2\pi}{\tau} C \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] - \frac{2\pi\tau}{\tau_p^2} C \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] = g \frac{\pi}{\ell} H \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

$$\frac{2\pi}{\tau} C' \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] - \frac{2\pi\tau}{\tau_p^2} C' \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$
$$= g \frac{\rho}{\rho'} \frac{\pi}{\ell} H \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] + g(1 - \frac{\rho}{\rho'})\frac{\pi}{\ell} H' \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

or
$$\frac{2C}{\tau} - \frac{2\tau C}{\tau_p^2} = \frac{gH}{\ell}$$

and
$$\frac{2C'}{\tau} - \frac{2\tau C'}{\tau_p^2} = \frac{g}{\ell} \frac{\rho}{\rho'} H + \frac{g}{\ell}(1 - \frac{\rho}{\rho'}) H' \quad .$$

Solving for C and C' in (2.08.4:7)

$$C' = \frac{2\ell}{\tau h'} H'$$

$$C = \frac{2\ell}{\tau h} H - \frac{h'}{h} C' = \frac{2\ell}{\tau h} H - \frac{2\ell}{\tau h} H' = \frac{2\ell}{\tau h}(H - H')$$

and substituting

$$\frac{2}{\tau} \frac{2\ell}{\tau h}(H - H') - \frac{2\tau}{\tau_p^2} \frac{2\ell}{\tau h}(H - H') = \frac{g}{\ell} H$$

$$\frac{4\ell}{h} \frac{1}{\tau^2} (H - H') - \frac{4\ell}{h} \frac{1}{\tau_p^2}(H - H') = \frac{g}{\ell} H$$

$$\frac{4\ell^2}{gh}(\frac{1}{\tau^2} - \frac{1}{\tau_p^2})(H - H') = H$$

for the first and

$$\frac{2}{\tau} \frac{2\ell}{\tau h'} H' - \frac{2\tau}{\tau_p^2} \frac{2\ell}{\tau h'} H' = \frac{g}{\ell}[\frac{\rho}{\rho'} H + (1 - \frac{\rho}{\rho'}) H']$$

$$\frac{4\ell^2}{gh'}(\frac{1}{\tau^2} - \frac{1}{\tau_p^2}) H' = [\frac{\rho}{\rho'} H + (1 - \frac{\rho}{\rho'}) H']$$

for the second.

Defining

(2.08.4:9) $$\beta \equiv \frac{4\ell^2}{gh}(\frac{1}{\tau^2} - \frac{1}{\tau_p^2})$$

gives

$$\beta(H - H') = H$$

and

$$\beta \frac{h}{h'} H' = \frac{\rho}{\rho'} H + (1 - \frac{\rho}{\rho'}) H$$

or

(2.08.4:10) $$(1 - \beta)H + \beta H' = 0$$

and

(2.08.4:11) $$\frac{\rho}{\rho'} H + (1 - \frac{\rho}{\rho'} - \beta \frac{h}{h'})H' = 0 \quad .$$

Consequently, we have from (2.08.4:10)

(2.08.4:12) $$H' = \frac{\beta - 1}{\beta} H = (1 - \frac{1}{\beta})H$$

together with

(2.08.4:13) $$C = \frac{2\ell}{\tau h}[H - (1 - \frac{1}{\beta})H] = \frac{2\ell}{\tau h} \frac{1}{\beta} H$$

and

(2.08.4:14) $$C' = \frac{2\ell}{\tau h'}(1 - \frac{1}{\beta})H$$

so that our solutions may be written with directly comparable amplitudes.

(2.08.4:15U)
$$\eta = H \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

(2.08.4:16U)
$$u = \frac{1}{\beta}\frac{2\ell}{h\tau} H \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

(2.08.4:17U)
$$v = \frac{1}{\beta}\frac{2\ell}{h\tau_p} H \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

(2.08.4:15L)
$$\eta' = (1 - \frac{1}{\beta})H \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

(2.08.4:16L)
$$u' = (1 - \frac{1}{\beta})\frac{2\ell}{h'\tau} H \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

(2.08.4:17L)
$$v' = (1 - \frac{1}{\beta})\frac{2\ell}{h'\tau_p} H \sin[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

Compared with (2.08.3:9)-(2.08.3:12) η, η', u, and u' are seen to be formally identical. However, the β used here differs from the β in section 2.08.3 by a term in τ_p.

From (2.08.4:9)
$$\frac{H}{H'} = - \frac{\beta}{-\beta}$$

while from (2.08.4:10)
$$\frac{H}{H'} = - (1 - \frac{\rho}{\rho'} - \frac{h}{h'})/(\frac{\rho}{\rho'})$$

so that
$$\beta \frac{\rho}{\rho'} = (1 - \beta)(1 - \frac{\rho}{\rho'} - \frac{h}{h'})$$

or

(2.08.4:18) $$\frac{h}{h'}\beta^2 - (1 + \frac{h}{h'})\beta + (1 - \frac{\rho}{\rho'}) = 0$$

which is Stokes' Equation again.

Let τ_s be the period of the transverse seiche, ordinary or internal, without geostrophic effects as in section 2.08.3. Then from that section
$$\beta = \frac{4\ell^2}{gh}\frac{1}{\tau_s^2} .$$

Since the two possible values of β are the same in this section as in 2.08.3 being roots of the same equation through (2.08.4:9) we have
$$1/\tau_s^2 = 1/\tau^2 - 1/\tau_p^2$$

or

(2.08.4:19) $$1/\tau^2 = 1/\tau_s^2 + 1/\tau_p^2 .$$

When, as often happens, $\tau_s >> \tau_p$ we have $\tau \approx \tau_p$. In this case geostrophic effects dominate the period of internal seiches.

Proudman cites observations of an internal seiche in the Kattegat with an amplitude of 31.5 cm and a period of 14.3 hours. τ_p for this lattitude is 14.2 Hours.

2.08.5. Internal Waves with Horizontal Crests.

The analysis of section 2.08.4 also applies to standing waves of length, λ, with horizontal crests. From (2.08.4:9), if $\lambda = 2\ell$,

(2.08.5:1) $$\lambda^2 = gh\left(\frac{1}{\tau^2} - \frac{1}{\tau_p^{\,2}}\right)^{-1}\beta$$

and, for internal waves with $\beta = \beta_i$,

(2.08.5:2) $$\lambda_i^{\,2} = \left(1 - \frac{\rho}{\rho'}\right)\left(\frac{1}{\tau^2} - \frac{1}{\tau_p^{\,2}}\right)^{-1}\frac{ghh'}{h + h'} \quad .$$

Therefore, for real values of λ_i we require $\tau < \tau_p$. If the waves are of tidal period, (2.08.5:2) tells us that the wave length, λ_i, is small compared with ordinary tidal oscillations.

Now consider progressive waves. The fundamental equations are still (2.08.4:1)-(2.08.4:3) but the solutions we try for, instead of (2.08.4:4), will be

(2.08.5:3U) $$u = C\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right]$$

(2.08.5:3L) $$u' = C'\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right] \quad .$$

The net result of precisely the same attack will be

(2.08.5:4U) $$\eta = H\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right]$$

(2.08.5:4L) $$\eta' = H'\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right]$$

(2.08.5:5U) $$v = \frac{\tau}{\tau_p} C\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right]$$

(2.08.5:5L) $$v' = \frac{\tau}{\tau_p} C'\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right] \quad .$$

The connections between H, H', C, and C' are just the same with λ replacing 2ℓ throughout. Since $U = \lambda/\tau$, (2.08.4:19) tells us that

$$U^2 = \frac{\lambda^2}{\tau^2} = \frac{\lambda^2}{\tau_s^{\ 2}} + \frac{\lambda^2}{\tau_p^{\ 2}}$$

or

$$U^2 = U_s^{\ 2} + \frac{\lambda^2}{\tau_p^{\ 2}} \qquad (2.08.5:6)$$

where U_s is the speed of propagation of waves, either ordinary or internal, where there are no geostrophic effects. From (2.08.5:6) it is clear that $U > \lambda/\tau_p$ always.

It seems that geostrophic effects may dominate the speed of propagation of internal waves. From (2.08.5:1)

$$\frac{\lambda^2}{\tau^2} = U^2 = gh(1 - \frac{\tau^2}{\tau_p^{\ 2}})^{-1}\beta$$

or

$$U^2 = \frac{gh}{1 - (\tau^2/\tau_p^{\ 2})}\ \beta \quad .$$

When

$$\beta = \beta_o = 1 + \frac{h'}{h}$$

$$U_o^{\ 2} = \frac{g(h + h')}{1 - (\tau^2/\tau_p^{\ 2})}$$

which is the result we got before for surface waves.

When

$$\beta = \beta_i = (1 - \frac{\rho}{\rho'})\frac{h'}{h + h'}$$

$$U_i^{\ 2} = \frac{1}{1 - (\tau^2/\tau_p^{\ 2})}(1 - \frac{\rho}{\rho'})\frac{ghh'}{h + h'} \quad .$$

If h/h' is small, i.e., if the surface layer is shallow, then

$$U_i = \frac{[1 - (\rho/\rho')]}{[1 - (\tau^2/\tau_p^{\ 2})]}\ gh \quad .$$

U_i is generally much smaller than U_o but, if $\tau \approx \tau_p$, $U_i \to \sqrt{gh}$.

2.08.6. Internal Kelvin Waves.

If we drop the transverse currents and retain the geostrophic effects, we can produce the Kelvin wave. The equations of continuity are

(2.08.6:1U) $$h \frac{\partial u}{\partial x} + \frac{\partial}{\partial t}(\eta + \eta')$$

(2.08.6:1L) $$h' \frac{\partial u'}{\partial x} + \frac{\partial \eta'}{\partial t} = 0 \quad .$$

The equations of motion are

(2.08.6:2U) $$\frac{\partial u}{\partial t} = - g \frac{\partial \eta}{\partial x}$$

(2.08.6:3U) $$\frac{2\pi}{\tau_p} u = - g \frac{\partial \eta}{\partial y}$$

(2.08.6:2L) $$\frac{\partial u'}{\partial t} = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g(1 - \frac{\rho}{\rho'})\frac{\partial \eta'}{\partial x}$$

(2.08.6:3L) $$\frac{2\pi}{\tau_p} u' = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial y} - g(1 - \frac{\rho}{\rho'})\frac{\partial \eta'}{\partial y} \quad .$$

We start out in the plane $y = 0$ looking for a solution

(2.08.6:3) $$\eta = F(x - Ut)$$

where U is a constant speed whose value is to be determined. By the usual rock hockey, and by defining

(2.08.6:5) $$\beta \equiv \frac{U^2}{gh} \quad ,$$

we again wind up with Stokes' Equation controlling β and

(2.08.6:6U) $$\eta = \exp\{- \frac{2\pi}{c} y\}F(x - Ut)$$

(2.08.6:6L) $$\eta' = (1 - \frac{1}{\beta})\exp\{- \frac{2\pi}{c} y\}F(x - Ut)$$

(2.08.6:7U) $$u = \frac{1}{\beta} \frac{U}{h} \exp\{- \frac{2\pi}{c} y\}F(x - Ut)$$

$$u' = (1 - \frac{1}{\beta})\frac{U}{h'} \exp\{- \frac{2\pi}{c} y\}F(x - Ut)$$

and, corresponding to the two solutions of Stokes' Equation, β_o and β_i,

(2.08.6:8) $$U_o^{\,2} = g(h + h')$$

(2.08.6:9) $$U_i^{\,2} = (1 - \frac{\rho}{\rho'})\frac{ghh'}{h + h'} \quad .$$

U_o corresponds to the ordinary surface wave. U_i corresponds to the internal wave and, clearly, from a comparison of equations (2.08.6:8) and (2.08.6:9), is much smaller than U_o. Since $c = U\tau_p$ this means that c is much smaller for U_i than it is for U_o. Consequently, the cross-channel change for an internal Kelvin wave is much more marked than it is for an ordinary surface Kelvin wave.

2.08.7. A Submarine Barrier.

Consider a two-layer system with a thin barrier at x = 0 extending from the bottom to the interface. This is a simplified model of a sill. Take the usual rectangular channel kind of thing with v and geostrophic effects included. The equations are still (2.08.4:1)-(2.08.4:3) but the barrier adds the boundary condition u' = 0 at x = 0.

From the results of sections 2.08.4 and 2.08.5 the ordinary wave corresponding to β_o with

$$\beta \equiv \frac{\lambda^2}{gh}\left(\frac{1}{\tau^2} + \frac{1}{\tau_p^{\,2}}\right)$$

gives

(2.08.7:1U)

$$\eta = H_o \cos[2\pi(\frac{x}{\lambda_o} - \frac{t}{\tau})]$$

(2.08.7:2U)

$$u = \frac{1}{\beta_o}\frac{\lambda_o}{h\tau} H_o \cos[2\pi(\frac{x}{\lambda_o} - \frac{t}{\tau})]$$

(2.08.7:3U)

$$v = \frac{1}{\beta_o}\frac{\lambda_o}{h\tau_p} H_o \sin[2\pi(\frac{x}{\lambda_o} - \frac{t}{\tau})]$$

(2.08.7:1L)

$$\eta' = (1 - \frac{1}{\beta_o}) H_o \cos[2\pi(\frac{x}{\lambda_o} - \frac{t}{\tau})]$$

(2.08.7:2L)

$$u' = (1 - \frac{1}{\beta_o})\frac{\lambda_o}{h'\tau} H_o \cos[2\pi(\frac{x}{\lambda_o} - \frac{t}{\tau})]$$

(2.08.7:3L)

$$v' = (1 - \frac{1}{\beta_o})\frac{\lambda_o}{h'\tau_p} H_o \sin[2\pi(\frac{x}{\lambda_o} - \frac{t}{\tau})]$$

The internal wave corresponding to β_i is

(2.08.7:4U)

$$\eta = H_i \cos[2\pi(\frac{x}{\lambda_i} - \frac{t}{\tau})]$$

(2.08.7:4L)

$$\eta' = (1 - \frac{1}{\beta_i}) H_i \cos[2\pi(\frac{x}{\lambda_i} - \frac{t}{\tau})]$$

(2.08.7:5U)

$$u = \frac{1}{\beta_i} \frac{\lambda_i}{h\tau} H_i \cos[2\pi(\frac{x}{\lambda_i} - \frac{t}{\tau})]$$

(2.08.7:5L)

$$u' = (1 - \frac{1}{\beta_i}) \frac{\lambda_i}{h'\tau} H_i \cos[2\pi(\frac{x}{\lambda_i} - \frac{t}{\tau})]$$

(2.08.7:6U)

$$v = \frac{1}{\beta_i} \frac{\lambda_i}{h\tau_p} H_i \sin[2\pi(\frac{x}{\lambda_i} - \frac{t}{\tau})]$$

(2.08.7:6L)

$$v' = (1 - \frac{1}{\beta_i}) \frac{\lambda_i}{h'\tau_p} H_i \sin[2\pi(\frac{x}{\lambda_i} - \frac{t}{\tau})]$$

We have the values of H_o and H_i at our disposal. Let H_o/H_i be such that

(2.08.7:7)
$$(1 - \frac{1}{\beta_o})\lambda_o H_o + (1 - \frac{1}{\beta_i})\lambda_i H_i = 0$$

and superpose the wave, (2.08.7:1)-(2.08.7:3) on the wave (2.08.7:4)-(2.08.7:6) with the same τ in each case. On the plane $x = 0$ the combination gives

(2.08.7:8U)

$$\eta = (H_o + H_i)\cos[\frac{2\pi}{\tau} t]$$

(2.08.7:8L)

$$\eta' = [(1 - \frac{1}{\beta_o})H_o + (1 - \frac{1}{\beta_i})H_i]\cos[\frac{2\pi}{\tau} t]$$

(2.08.7:9U)

$$u = \frac{1}{h\tau}\left(\frac{\lambda_o H_o}{\beta_o} + \frac{\lambda_i H_i}{\beta_i}\right)\cos[\frac{2\pi}{\tau} t]$$

(2.08.7:9L)

$$u' = 0$$

(2.08.7:10U)

$$v = -\frac{1}{h\tau_p}\left(\frac{\lambda_o H_o}{\beta_o} + \frac{\lambda_i H_i}{\beta_i}\right)\sin[\frac{2\pi}{\tau} t]$$

(2.08.7:10L)

$$v' = 0$$

This suggests that an ordinary tide wave passing over a sill of the right height may give rise to an internal tide wave since the condition at the sill can not be satisfied without both. The two waves could have been made to progress in opposite directions.

2.08.8. A Forced Internal Tide Wave with Horizontal Crests.

Consider a forced tide wave with conditions uniform transverse to the propagation direction.

Continuity is

(2.08.8:1U) $$h \frac{\partial u}{\partial x} + \frac{\partial}{\partial t}(\eta - \eta') = 0$$

(2.08.8:1L) $$h' \frac{\partial u'}{\partial x} + \frac{\partial \eta'}{\partial t} = 0 .$$

Motion is

(2.08.8:2U) $$\frac{\partial u}{\partial t} - \frac{2\pi}{\tau_p} v = - g \frac{\partial}{\partial x}(\eta - \overline{\eta})$$

(2.08.8:3U) $$\frac{\partial v}{\partial t} + \frac{2\pi}{\tau_p} u = 0$$

(2.08.8:2L) $$\frac{\partial u'}{\partial t} - \frac{2\pi}{\tau_p} v' = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g(1 - \frac{\rho}{\rho'})\frac{\partial \eta'}{\partial x} + g \frac{\partial \overline{\eta}}{\partial x}$$

(2.08.8:3L) $$\frac{\partial v'}{\partial t} + \frac{2\pi}{\tau_p} u' = 0 .$$

Let

(2.08.8:4) $$\overline{\eta} = \overline{H} \cos[2\pi(\frac{x}{\lambda} - \frac{t}{\tau})] \quad ; \quad \overline{H},\ \lambda,\ \text{and}\ \tau\ \text{given.}$$

The form of equations (2.08.8:1)-(2.08.8:3) suggests that we try for

(2.08.8:5U) $$\eta = H \cos[2\pi(\frac{x}{\lambda} - \frac{t}{\tau})]$$

(2.08.8:6U) $$u = C \cos[2\pi(\frac{x}{\lambda} - \frac{t}{\tau})]$$

(2.08.8:7U) $$v = \frac{\tau}{\tau_p} C \sin[2\pi(\frac{x}{\lambda} - \frac{t}{\tau})]$$

(2.08.8:5L) $$\eta' = H' \cos[2\pi(\frac{x}{\lambda} - \frac{t}{\tau})]$$

(2.08.8:6L) $$u' = C' \cos[2\pi(\frac{x}{\lambda} - \frac{t}{\tau})]$$

(2.08.8:7L) $$v' = \frac{\tau}{\tau_p} C' \sin[2\pi(\frac{x}{\lambda} - \frac{t}{\tau})],$$

where H, H', C, and C' are constants.

Substituting (2.08.8:5)-(2.08.8:7) into (2.08.8:1)-(2.08.8:4) you ultimately wind up with

(2.08.8:8) $$\frac{h}{\lambda} C = \frac{1}{\tau}(H - H')$$

(2.08.8:9) $$\frac{h'}{\lambda} C' = \frac{1}{\tau} H'$$

(2.08.8:10) $$\left(\frac{1}{\tau} - \frac{\tau}{\tau_p^2}\right)C = \frac{g}{\lambda}(H - \overline{H})$$

(2.08.8:11) $$\left(\frac{1}{\tau} - \frac{\tau}{\tau_p^2}\right)C' = \frac{g}{\lambda}[\frac{\rho}{\rho'} H + (1 - \frac{\rho}{\rho'})H' - \overline{H}] \quad .$$

Substituting (2.08.8:8) and (2.08.8:9) into (2.08.8:10) and (2.08.8:11) and using

$$\beta \equiv \frac{\lambda^2}{gh}\left(\frac{1}{\tau^2} - \frac{1}{\tau_p^2}\right)$$

one gets

(2.08.8:12) $$(1 - \beta)H + \beta H' = \overline{H}$$

(2.08.8:13) $$\frac{\rho}{\rho'} H + (1 - \frac{\rho}{\rho'} - \beta \frac{h}{h'})H' = \overline{H} \quad .$$

Solving (2.08.8:12) and (2.08.8:13) for the ratios $H/\overline{H}$ and $H'/\overline{H}$

(2.08.8:14) $$\frac{H}{\overline{H}} = \frac{1}{\Delta}[1 - \frac{\rho}{\rho'} - \beta(1 + \frac{h}{h'})]$$

(2.08.8:15) $$\frac{H'}{\overline{H}} = \frac{1}{\Delta}(1 - \frac{\rho}{\rho'} - \beta)$$

where

(2.08.8:16) $$\Delta \equiv \frac{h}{h'}(\beta - \beta_o)(\beta - \beta_i)$$

and β_o and β_i are the roots of Stokes' Equation.

Equations (2.08.8:14)-(2.08.8:16) show that two kinds of resonance are possible:

(a) $\beta \to \beta_o$ $\sim$ ordinary surface waves.

(b) $\beta \to \beta_i$ $\sim$ internal waves.

The ratios of the elevations, η'/η, and the velocities, u'/u, are interesting. From (2.08.8:5, 6, 8, 9 & 14-16) we get

(2.08.8:17) $$\frac{\eta'}{\eta} = \frac{1 - \frac{\rho}{\rho'} - \beta}{1 - \frac{\rho}{\rho'} - \beta(1 + \frac{h}{h'})} \quad .$$

(2.08.8:18) $$\frac{u'}{u} = 1 - \frac{1}{\beta}(1 - \frac{\rho}{\rho'}) \quad .$$

Compare (2.08.8:17 & 18) with the same relations derivable from equations (2.08.4:15-17). For $\beta = \beta_o$ or $\beta = \beta_i$ the two sets agree.

When β is not small

$$\frac{\eta'}{\eta} \simeq \frac{h'}{h + h'} \qquad ; \qquad \frac{u'}{u} \simeq 1$$

and an ordinary forced wave is produced.

When $\tau \simeq \tau_p$ then β is small. If it is nearly equal to β_i, then η'/η will be large.

This indicates the possibility of direct generation of forced internal waves with a period of nearly the half-pendulum day. In section 2.08.5 we saw that the propagation speed of internal waves may be nearly λ/τ_p. This is often the magnitude of the speed of the tide-generating potential.

2.08.9. Forced Internal Tides in a Closed Basin.

Consider a narrow rectangular closed basin bounded at $x = 0$ and $x = \ell$. Neglect v and the geostrophic effect.

Continuity is

(2.08.9:1U) $$h \frac{\partial u}{\partial x} + \frac{\partial}{\partial t}(\eta - \eta') = 0$$

(2.08.9:1L) $$h' \frac{\partial u'}{\partial x} + \frac{\partial \eta'}{\partial t} = 0 \quad .$$

Motion is

(2.08.9:2U) $$\frac{\partial u}{\partial t} = - g \frac{\partial}{\partial x}(\eta - \overline{\eta})$$

(2.08.9:2L) $$\frac{\partial u'}{\partial t} = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g(1 - \frac{\rho}{\rho'})\frac{\partial \eta'}{\partial x} + g \frac{\partial \overline{\eta}}{\partial x} \quad .$$

Try for a solution in the form

(2.08.9:3U) $$\eta = H \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

(2.08.9:3L) $$\eta' = H' \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t]$$

(2.08.9:4U)

$$u = C \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

(2.08.9:4L)

$$u' = C' \sin[\frac{\pi}{\ell} x]\sin[\frac{2\pi}{\tau} t]$$

where H, H', C, and C' are constants.

With (2.08.9:1-4) we need for the equilibrium elevation

(2.08.9:5) $$\overline{\eta} = \overline{H} \cos[\frac{\pi}{\ell} x]\cos[\frac{2\pi}{\tau} t] \quad .$$

Suppose that $\overline{H}$ and τ are given. The usual feedback ultimately produces

(2.08.9:6) $$\frac{4\ell^2}{gh\tau^2}(H - H') = H - \overline{H}$$

(2.08.9:7) $$\frac{4\ell^2}{gh'\tau^2} H' = \frac{\rho}{\rho'} H + (1 - \frac{\rho}{\rho'})H' - \overline{H} \quad .$$

Defining

(2.08.9:8) $$\beta \equiv \frac{4\ell^2}{gh\tau}$$

leads to

(2.08.9:9) $$\frac{H}{\overline{H}} = \frac{1}{\Delta}[1 - \frac{\rho}{\rho'} - \beta(1 + \frac{h}{h'})]$$

(2.08.9:10) $$\frac{H'}{\overline{H}} = \frac{1}{\Delta}(1 - \frac{\rho}{\rho'} - \beta)$$

where

(2.08.9:11) $$\Delta \equiv \frac{h}{h'}(\beta - \beta_o)(\beta - \beta_i) \quad .$$

Formally, (2.08.9:9-11) are the same as (2.08.8:12-14). The β used there can be identified with the β used here if $\lambda \equiv 2\ell$ and $\tau_p \to \infty$. The formalism continues into

(2.08.9:12) $$\frac{\eta'}{\eta} = \frac{1 - \frac{\rho}{\rho'} - \beta}{1 - \frac{\rho}{\rho'} - \beta(1 + \frac{h}{h'})}$$ (2.08.8:17)

(2.08.9:13) $$\frac{u'}{u} = 1 - \frac{1}{\beta}(1 - \frac{\rho}{\rho'}) \quad .$$ (2.08.8:18)

Again, resonance is possible either for ordinary surface waves or for internal oscillations. As before, when β is not small

$$\frac{\eta'}{\eta} \approx \frac{h'}{h + h'} \quad ; \quad \frac{u'}{u} \approx 1 \quad .$$

For β near β_i, η'/η will be large. This case is of frequent occurance in small basins so that one may expect forced internal tides of some importance in such places. Once having been started, they can be propagated out into the sea as free internal waves.

2.08.10. Internal Response to a Traveling Atmospheric Pressure Disturbance.

Neglect geostrophic effects and v.

Continuity is

(2.08.10:1U) $$h \frac{\partial u}{\partial x} + \frac{\partial}{\partial t}(\eta - \eta') = 0$$

(2.08.10:1L) $$h' \frac{\partial u'}{\partial x} + \frac{\partial \eta'}{\partial t} = 0 \quad .$$

Motion is

(2.08.10:2U) $$\frac{\partial u}{\partial t} = - g \frac{\partial}{\partial x}(\eta - \overline{\eta})$$

(2.08.10:2L) $$\frac{\partial u'}{\partial t} = - g \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} - g(1 - \frac{\rho}{\rho'})\frac{\partial \eta'}{\partial x} + g \frac{\rho}{\rho'} \frac{\partial \overline{\eta}}{\partial x} \quad .$$

Suppose the equilibrium form given by

(2.08.10:3) $$\overline{\eta} = F(x - Ut)$$

where F is any physically possible function while U is the speed of propagation of the pressure disturbance.

Try for the solution

(2.08.10:4U) $$\eta = M\, F(x - Ut)$$

(2.08.10:4L) $$\eta' = M'\, F(x - Ut)$$

where M and M' are constants to be determined.

Feeding (2.08.10:4) to (2.08.10:1)

$$\frac{\partial u}{\partial x} = \frac{MU}{h} F'(x - Ut) - \frac{M'U}{h} F'(x - Ut)$$

or

$$\frac{\partial u}{\partial x} = (M - M')\frac{U}{h} F'(x - Ut)$$

and

$$\frac{\partial u'}{\partial x} = M' \frac{U}{h'} F'(x - Ut) \quad .$$

Therefore,

(2.08.10:5U) $$u = (M - M')\frac{U}{h} F(x - Ut)$$

(2.08.10:5L) $$u' = M' \frac{U}{h'} F(x - Ut)$$

with no additive constants if we take $u = u' = 0$ when $\overline{\eta} = 0$, i.e., at rest.

Feeding (2.08.10:3-5) to (2.08.10:2) and reducing gives

$$-(M - M')\frac{U^2}{h} = -gM + g$$

$$-M' \frac{U^2}{h'} = -g\frac{\rho}{\rho'} M - g(1 - \frac{\rho}{\rho'})M' + g\frac{\rho}{\rho'}$$

which, on defining

(2.08.10:6) $$\beta \equiv \frac{U^2}{gh} \quad ,$$

boils down to

(2.08.10:7) $$(1 - \beta)M + \beta M' = 1$$

(2.08.10:8) $$\frac{\rho}{\rho'} M + (1 - \frac{\rho}{\rho'} - \beta\frac{h}{h'})M' = \frac{\rho}{\rho'} \quad .$$

Equations (2.08.10: 7 & 8) are similar to the equations for $H/\overline{H}$ and $H'/\overline{H}$ in sections 2.08.8 & 9. Their solution is

(2.08.10:9) $$M = \frac{1}{\Delta}[1 - \frac{\rho}{\rho'} - \beta(\frac{\rho}{\rho'} + \frac{h}{h'})]$$

(2.08.10:10) $$M' = \frac{1}{\Delta}\beta\frac{\rho}{\rho'}$$

where

(2.08.10:11) $$\Delta \equiv \frac{h}{h'}(\beta - \beta_o)(\beta - \beta_i) \quad .$$

Again, the two kinds of resonance are possible. The ratios in the two layers are

(2.08.10:12) $$\frac{\eta}{\eta'} = 1 + \frac{\rho'}{\rho}\frac{h}{h'} + \frac{1}{\beta}(1 - \frac{\rho'}{\rho})$$

(2.08.10:13) $$\frac{u}{u'} = \frac{\rho'}{\rho} + \frac{1}{\beta}(1 - \frac{\rho'}{\rho})\frac{h'}{h} \quad .$$

When β is not small

$$\frac{\eta}{\eta'} \simeq 1 + \frac{h}{h'} \quad ; \quad \frac{u}{u'} \simeq 1$$

as in unstratified water. (See section 2.07.5.)

Suppose an internal wave is generated for which $\eta/\eta' = 0$. In other words, suppose that all the disturbance is internal. Then from (2.08.10:12 & 13) it follows that

$$\beta = \frac{\frac{\rho'}{\rho} - 1}{1 + \frac{h}{h'}} \quad ; \quad \frac{u}{u'} = -\frac{h'}{h} \quad .$$

This value of β is very approximately equal to β_i. Thus, it seems that the passage of a front could set off an internal wave in stratified water even though no disturbance appeared on the surface.

Proudman, "Dynamic Oceanography," page 359 discusses a possible case observed by the METEOR. Pages 364-367 give a discussion of observations of internal tides. You should take a look at it.

Chapter 3. The Effect of Friction on the Tide.

3.01. Introduction.

Up to now we have consistently neglected the effects of friction on our oscillations. It is pleasant to neglect friction since it makes such a mess of the equations. However, shallow water will require some consideration of friction if we are to derive realistic models.

We will assume water of uniform density and constant atmospheric pressure. To make things simpler, let all the motions be parallel to one vertical plane and let there be no wind so that what friction there is is entirely with the bottom. In reality, most of the geostrophic effects will be balanced out by transverse surface gradients but we will ignore them.

Let ρ be the water density and let the x-axis be taken in the direction of the current, u. F_z will represent the force of the internal friction per unit area.

The appropriate form of the equation of motion may be deduced on the following argument. Take an element of length δx and width b between depths z and z + δz, Fig. 3.01-1. Doing our book keeping on this element

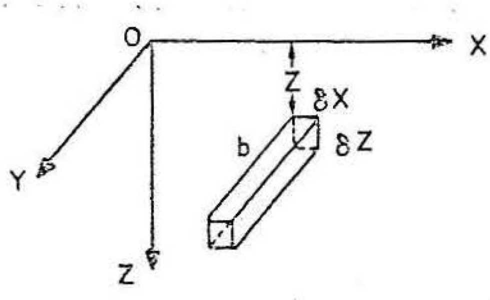


Fig. 3.01-1

the force components parallel to the x-axis are

the pressure force:	$- b\ \delta z\ \rho g\ \delta\eta$	
the tide-generating force:	$- b\ \delta x\ \delta z\ \rho g\ \frac{\partial \bar{\eta}}{\partial x}$	
the friction force:	$- b\ \delta x\ F_z$	(upper face)
	$- b\ \delta x (F_z + \delta F_z)$	(lower face) .

The mass-acceleration of the water in the element parallel to the x-axis, to a linear approximation, is

$$b\ \delta x \delta z \rho \frac{\partial u}{\partial t} \quad .$$

Therefore

$$b\delta x\delta z\rho \frac{\partial u}{\partial t} = - b\delta z\rho g\delta\eta + b\delta x\delta z\rho g \frac{\partial \overline{\eta}}{\partial x} + b\delta x F_z - b\delta x(F_z + \delta F_z)$$

or

$$\delta x\delta z\rho \frac{\partial u}{\partial t} = - \delta z\rho g\delta\eta + \delta x\delta z\rho g \frac{\partial \overline{\eta}}{\partial x} - \delta x\delta F_z$$

or

$$\frac{\partial u}{\partial t} = - g \frac{\delta\eta}{\delta x} + g \frac{\partial \overline{\eta}}{\partial x} - \frac{1}{\rho}\frac{\delta F_z}{\delta z}$$

or

(3.01:1)
$$\frac{\partial u}{\partial t} = - g \frac{\partial}{\partial x}(\eta - \overline{\eta}) - \frac{1}{\rho}\frac{\partial F_z}{\partial z} \quad .$$

Let [u] be the mean value of u from the surface to the depth z,

$$[u] \equiv \frac{1}{z}\int_0^z u\ dz \quad ,$$

and take the mean of (3.01:1) from the surface, 0, to depth z.

$$\frac{1}{z}\int_0^z \frac{\partial u}{\partial t} dz = \frac{1}{z}\int_0^z \{- g \frac{\partial}{\partial x}(\eta - \overline{\eta})\}dz + \frac{1}{z}\int_0^z \{- \frac{1}{\rho}\frac{\partial F_z}{\partial z}\}dz$$

$$\frac{\partial}{\partial t}\{\frac{1}{z}\int_0^z u\ dz\} = \{- g \frac{\partial}{\partial x}(\eta - \overline{\eta})\}\frac{1}{z}\int_0^z dz - \frac{1}{\rho}\frac{1}{z}\int_0^z dF_z$$

(3.01:2)
$$\frac{\partial}{\partial t}[u] = - g \frac{\partial}{\partial x}(\eta - \overline{\eta}) - \frac{F_z}{\rho z} \quad .$$

When the upper limit of integration is z = h, then F_z becomes F_b, the bottom friction, and we will use $[u] \equiv \overline{u}$ for this particular mean.

It has been observed that, to a good approximation, bottom friction is proportional to the square of the bottom current and directed opposite to it. We can express this by writing

$$F_b \sim u_b|u_b| \quad .$$

Remark: Note this trick with the absolute value sign which, in effect, squares u_b but does not destroy its sign.

We can make an equation of this in the usual way by including a constant of proportionality and writing

(3.01:3) $$F_b = k\rho u_b|u_b|$$.

If there is no spatial gradient in the tide-producing force, $\partial\bar{\eta}/\partial x = 0$ and no mean mass-acceleration from top to bottom, $\partial\bar{u}/\partial t = 0$, then we can write (3.01:2) as

$$0 = -g\frac{\partial\eta}{\partial x} - \frac{F_b}{\rho h}$$

or, using (3.01:3) and rearranging,

(3.01:4) $$\frac{\partial\eta}{\partial x} = -k\frac{|u_b|u_b}{gh}$$.

Before developing and applying the more general forms, equations (3.01:1) and (3.01:2), we will go into a use of (3.01:4) in the Irish Sea. The initial work was done by G. I. Taylor with later contributions by Proudman, Doodson, and Corkan.

Remark: If you would like to have the pleasure of reading a really well written scientific paper for a change, you should take the time to go through "Tidal friction in the Irish Sea" by G. I. Taylor.

3.02. G. I. Taylor's Analysis of the Tide in the Irish Sea.

Taylor was started on this study when he found some previous work based on a mathematical model using laminar flow and which suggested quite low energy dissipation rates. His own practical experience as a yachtsman, as well as considerations based on Reynolds number, made him feel that the flow must be turbulent and that any model based on laminar flow must be inadequate.

As a first step it was considered that the mechanism for the dissipation of energy from a tidal current must be analogous to that in the flow of a river over its bed or to that in the flow of air over the ground. By this analogy we write at once

(3.02:1) $$F = K\rho v^2$$

where

$F \equiv$ the skin friction per square centimeter.

$\rho \equiv$ the density of the fluid.

$v \equiv$ the velocity of the fluid.

$K \equiv$ a constant whose value depends on the nature of the surface over which the fluid flows.

This is all very well but it won't do us much good unless we can find some numbers for K.

If we turn to flow in rivers we find Bazin's formula pretty well established. It says

(3.02:2) $$rs = \frac{1}{7569}\left(1 - \frac{\gamma}{\sqrt{r}}\right)^2 v^2$$

where

$r \equiv$ the hydraulic radius of the channel, i.e., the ratio $\frac{\text{cross-sectional area}}{\text{wetted part of perimeter}}$.

$s \equiv$ the slope of the river bed.

$\gamma \equiv$ a constant depending on the nature of the bottom.

To connect (3.02:1) with (3.02:2) we must equate the resistance acting upstream to the component of the fluid weight acting downstream which gives

$$F \times (\text{wetted part of perimeter}) = s\rho g \times (\text{cross-sectional area}) \quad \text{, i.e.,}$$

$$\frac{K\rho v^2}{\rho g} = rs$$

so that

$$K = \frac{g}{7569}(1 - \gamma/\sqrt{r})^2 = 0.0013(1 - \gamma/\sqrt{r})^2$$

where

$$g = 9.81 \text{ m/sec}^2 \quad .$$

In the Irish Sea the depth is about 80 meters. In a very broad stream the depth is, effectively, the hydraulic radius. Therefore, use

$$\sqrt{r} = \sqrt{80} \approx 9 \quad .$$

The value of γ depends on the bottom roughness. Bazin's values are $\gamma = 0.85$ for clean stoney or smooth earth bottoms and $\gamma = 1.7$ for uneven or weedy bottoms. These give, respectively,

(3.02:3.1) $$K = 0.0016$$ and (3.02:3.2) $$K = 0.0018 \quad .$$

You see that large changes in bottom roughness make very slight changes in the amount of friction on the bottom. From equation (3.02:2) it is obvious that this results from the great depth of the sea. In order for bottom roughness to have much of an effect in slowing up a stream, the stream has to be shallow. It seems that the size of the projections which make up the bottom roughness have to be some appreciable fraction of r if they are to be felt by the stream as a whole. For our purposes this is important since it permits us to make a reasonable stab at the friction in the Irish Sea without knowing in any detail what the bottom is like. Also, our estimate will be, if anything, an underestimate since the friction will be larger in shoal areas over banks and shelves where the depth is small enough for roughness to be important.

We've made some milage out of the analogy with flow in rivers. Now let's see what can be had from the analogy with air flow over the ground. According to the principle of dynamic similarity the flow patterns in sea and air will be the same if the scales of roughness are the same and if

(3.02:4) $$\frac{v_w}{v_a} = \frac{\mu_w \rho_a}{\mu_a \rho_w}$$

where v, μ, and ρ are speed, viscosity, and density and the subscripts a and w identify air and water. From the tables for viscosity and density equation (3.02:4) gives

$$v_w/v_a = 1/11 \quad .$$

Measurements over grass on Salisbury Plain with wind speeds ranging from 6 to 30 mph fit

$$F = 0.002\rho_a v_a^2 \quad .$$

By dynamic similarity we expect the same relation to hold for water moving at speeds of (1/11)×6 to (1/11)×30 mph, i.e. from about 0.5 knots to 3 knots; which is the kind of speed one has for the tide in the Irish Sea.

Of course, for this to be valid the roughness of the bottom in the Irish Sea has to look to the water about the way grass looks to the wind. But this needn't worry us. We've already shown that the friction is not very sensitive to the roughness. So--let's take

(3.02:5) $F = 0.002\rho v^2$

for (3.02:1) and be done with it.

Bowden and Fairbairn (1952) measured turbulence associated with tidal currents at 75 cm above the bottom of an estuary. They computed Reynolds stresses and found them consistent with a K between 0.0020 and 0.0025. It certainly looks as though G. I. Taylor, in 1919, were in the right ball park.

Remark: Incidently, in my opinion, one of the stigmata of a real geophysicist or oceanographer is that, in most cases, he can intuit his way into the right ball park before any decent data are available. Sverdrup had this ability to a marked degree and so has Pritchard.

The rate of dissipation of energy by friction is the friction multiplied by the relative velocity of the surfaces between which the friction acts. If we use (3.02:5) for the friction on the bottom, then the amount of energy dissipated per square centimeter per second will be given by

(3.02:6) $w = 0.002\rho v^2 |v|$.

Since the currents in the Irish Sea vary in both time and space what we need here is an estimate of the average over the whole sea and over a tidal cycle. Let V be the maximum tidal velocity and, as an approximation, take v sinusoidal.

$$v = V \cos[\frac{2\pi}{T} t] = V \cos[\sigma t]$$

where T = 12 hours 25 minutes is the semidiurnal tidal period. This gives

(3.02:7) $w = 0.002\rho V^3 |\cos^3[\sigma t]|$.

The mean of $|\cos^3[\sigma t]|$ over T is $\frac{4}{3\pi}$. If we can figure out what would be reasonable for V, the maximum tidal current averaged over the whole Irish Sea, we'd be in. It could be gotten from tidal current measurements but G. I. Taylor, being just as human as the rest of us, notes that somebody has already published a mean for V^2 at spring tides over the Irish Sea, viz., $\overline{V^2} = 5\ kt^2$. This gives $(\overline{V^2})^{\frac{1}{2}} \simeq 2\frac{1}{4}$ kt. If we use this for the mean V--which it isn't--we probably won't be too far out of line and, in any case, we can expect to underestimate rather than overestimate the frictional effect.

With ρ = 1.03 g cm^{-3}, V = 2.25 kt = 114 cm sec^{-1}, and $\overline{|\cos^3[\sigma t]|} = \frac{4}{3\pi}$ equation (3.02:7) gives us

(3.02:8.1) $$\overline{w} \approx 1300 \text{ ergs cm}^{-2}\text{sec}^{-1} \quad .$$

If instead of K = 0.002 we use the smaller value K = 0.0016 we still get

(3.02:8.2) $$\overline{w} \approx 1040 \text{ ergs cm}^{-2}\text{sec}^{-1} \quad .$$

This is a hair raiser. You will have noticed that all along G. I. Taylor has been careful to keep this an underestimate. The best previous estimate, the one that got Taylor thinking about this, was Street's at

$$\overline{w} \approx 7 \text{ ergs cm}^{-2}\text{sec}^{-1} \quad .$$

Our smaller estimate is 150 times bigger.

All this is pretty unsettling and it would be nice if we could find another, independent, way of estimating the energy dissipation. If we can, and if it confirms the general size of our first estimate, then we can feel a little better about suggesting that everybody accept our estimate in preference to others. This is exactly what Taylor did.

His program is simple enough. Taylor says:

> "Instead of trying to measure the rate of dissipation at every point of the Irish Sea, I have calculated the rate at which energy enters the Irish Sea through the North and South Channels. To this must be added the rate at which work is done by lunar attraction on the waters of the Irish Sea. The sum of these will give the rate at which the energy of that sea is increasing plus the rate of dissipation of energy. When the average values of these expressions are taken during a complete tidal period it is evident that, since the energy of the Irish Sea does not increase or decrease continually, the average rate of dissipation by tidal currents can be found."

By the usual sort of book keeping Taylor finds for the amount of energy crossing a vertical cylindrical control surface, S, in time, dt, (3.02:9) $\rho g dt \int h\eta v \sin[\theta] ds + \int \tfrac{1}{2}\rho v \sin[\theta]\, dt\, (2g\eta^2 + hv^2 + \eta v^2) ds$,

the integrals to be taken around s,

where

$s \equiv$ the intersection of the cylinder, S, with the water surface.

$v \equiv$ the speed of the current at any point on s.

$\theta \equiv$ the angle between any element, ds, and the current direction.

$\eta \equiv$ the height of the tide above mean sea level.

and ρ and g are as usual.

Assuming η small compared with h, as it certainly is in the Irish Sea where the mean spring rise is about 6 feet as against a mean h of about 40 fathoms, and noting that v must be roughly the size of $c\eta/h$ where c is the speed of the tide wave in water of depth h, $(c = \sqrt{gh})$ we see that the second integral is much smaller than the first and we take

(3.02:10) $\rho g dt \int h\eta v \sin[\theta] ds$

as a sufficient approximation.

For energy to be conserved (3.02:10) must be equal to the sum of the increase in kinetic energy of the sea enclosed by s, the energy dissipated by friction, and the work done by the moon; all during the time dt.

Taking the mean over a tidal period the first of these must be zero and we can write

(3.02:11) $W - W_m = \langle \rho g \int h\eta v \sin[\theta] ds \rangle_T$

where

$W \equiv$ the mean rate at which energy is dissipated by the tidal friction within s.

$W_m \equiv$ the average rate at which work is done by the moon's attraction on the region enclosed by s.

To apply (3.02:11) to the Irish Sea we need a few numbers. Taylor got them from published Admiralty sources. The principal feature of the tide is a lunar semidiurnal constituent with a period T = 12 hr 25 min. To a sufficient approximation we can represent the height of the tide above mean sea level by a sinusoid,

(3.02:12) $\eta = A \cos[\sigma(t - t_1)]$

where

$H = 2A$ is the tidal range.

$T \equiv$ the tidal period: 12 hr 25 min $\sim \sigma = \dfrac{2\pi}{12\frac{25}{60}}$.

$t \equiv$ the time measured from the time of the moon's passage of Greenwich so that at full and change t is Greenwich Mean Time.

$t_1 \equiv$ the time of high water at full and change, i.e., it is the "establishment" of the place under discussion.

To evaluate (3.02:11) we must know the heights of the tide across a section. Fortunately, since the Irish Sea, Fig. 3.02-1, page 224 is open only at the two ends, integration around δ is simply integration across the ends.

That still leaves us up against it as by far the greater part of the tide records have been made along the coasts.

Remark: By the way, if making bricks without straw and silk purses out of sows' ears isn't to your taste, perhaps you'd better go into something other than geophysics.

To make the cheese more binding, the tidal ranges are not the same on the two sides of the channel, Table 3.02-1.

Table 3.02-1. Tidal Ranges in the Irish Sea.

Welsh Side		Irish Side	
Place	Spring Range (ft)	Place	Spring Range (ft)
Beardsey Island	15	Arklow	4
St. Tudwell Road	14	Courtown	3.75
Port Dynllayn	12.25	Arklow Bank	4.25
Llanddwyn Island	14.5	Kilmichael Point	4.34
Holyhead	16		

The question is: What is the shape of the water surface across the South Channel from A to B? To get an answer we ring in the geostrophic force. The current through the South Channel is substantially rectilinear

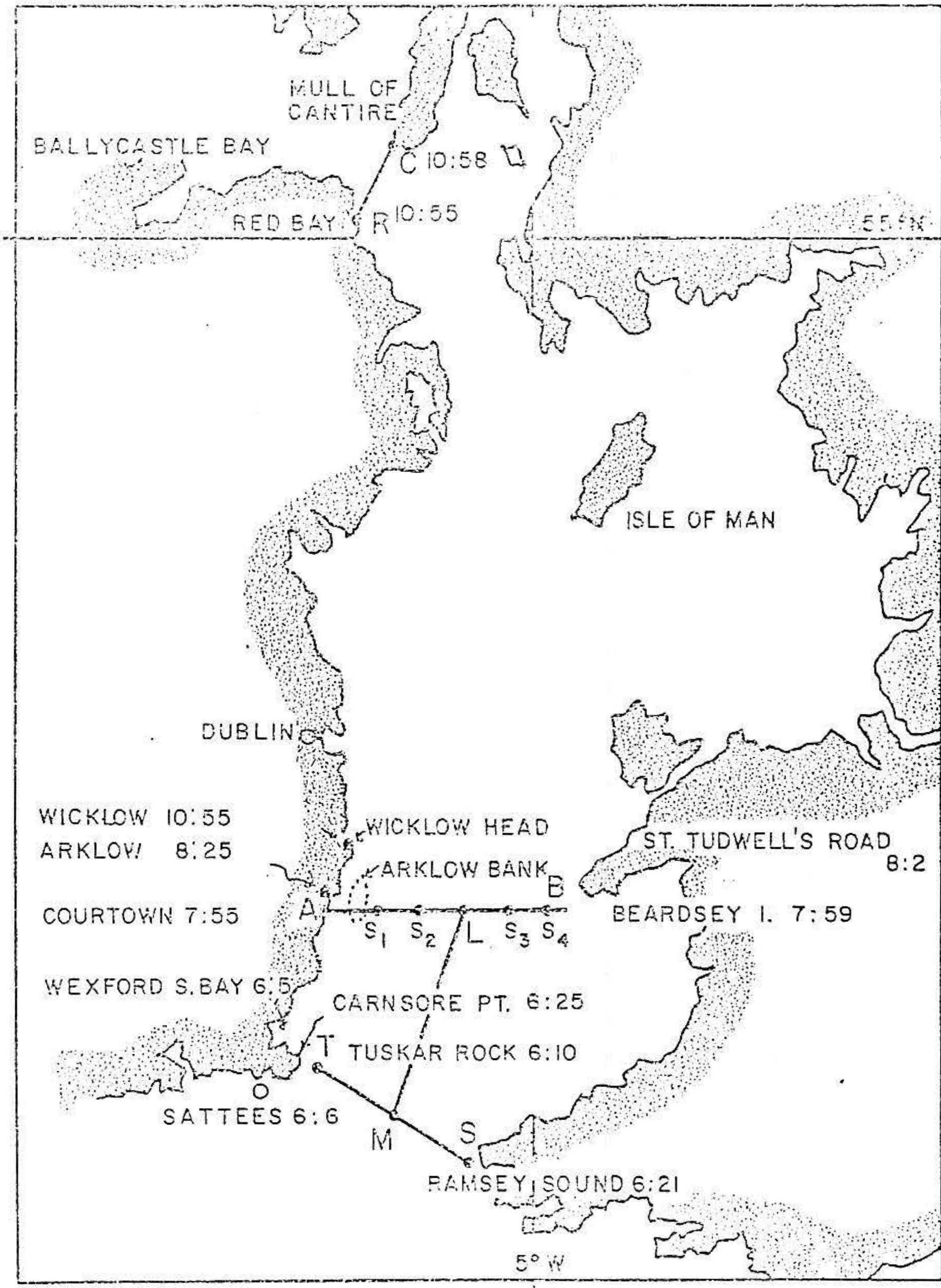


Fig. 3.02-1. The Irish Sea. $S_1S_2 \simeq S_3S_4 \simeq 10$ km. The numbers are the times of high water at full and change of the moon. (Remember the British habit of writing 8:02 as 8:2.)

reversing. Any current left to itself on this earth flows in a curved path. To keep the current flowing straight we invoke a pressure gradient just sufficient to balance the geostrophic force; one created by a linear tilt of the sea surface. This implies nearly uniform current across South Channel. From this, knowing the tide ranges at the ends of the section, we can get the tide range at mid-channel.

The geostrophic force is $2\omega\rho v \sin[\phi]$ and the cross-stream gradient just necessary to balance it is

$$\frac{2\omega\rho v \sin[\phi]}{\rho g} = \frac{2\omega v \sin[\phi]}{g} .$$

The measured maximum speed at springs across section AB, both ebb and flood, is 3.2 kt ≈ 162 cm sec^{-1}. With $\omega = 0.000073$, $\phi = 52°$, and $g = 981$ cm sec^{-2} the slope comes out to be 1.9×10^{-5} radians. The distance from A to B is 48 NM ≈ 288,000 ft. Consequently, the difference in level between Beardsey Island and Arklow at the time of maximum current should be about 5.7 feet.

The maximum current in these parts occurs substantially at high water and low water so that the difference in range, on this argument, sould be about 2×5.7 = 11.4 ft. From Table 3.02-1, page 223 you can see that the measured value is 11 feet which is not bad. Perhaps we can get away with a linear slope of the sea surface between A and B.

Remark: Taylor says, rather wistfully, that it would be nice if someone would actually measure this someday.

The next order of business is the speed of the tidal currents. If you have been wondering why G. I. Taylor chose the line AB, the answer is easy: There are a few measurements there at the positions S_1, S_2, S_3, and S_4, Table 3.02-2.

Table 3.02-2. Results of Measurements along AB.

Station	Maximum Flood (kt)	Direction (°T)	Maximum Ebb (kt)	Direction (°T)
S_1	3.6	032	3.6	206
S_2	3.2	035	3.3	212
S_3	3.2	025	3.0	208
S_4	3.0	016	2.3	196

The mean speed is 3.2 kt, the mean flood directions is 027°T and the mean ebb direction is 206°T. The directions are very nearly opposed and, for convenience, we will assume that they are exactly so.

There are no measurements available at the stations, S_i, for intermediate stages of the tide. The points nearest to the line AB at which such measurements are to be had are the South Arklow, North Arklow, and Carnarvon Bay light-ships, Table 3.02-3 and Fig. 3.02-2, page 227.

Table 3.02-3. Direction and Speed of the Tidal Streams at Three Light-Ships at Various Stages of the Tide.

Hours from HW at Dover	North Arklow Light-Ship		South Arklow Light-Ship		Carnarvon Bay Light-Ship	
	Direction (°T)	Speed (kt)	Direction (°T)	Speed (kt)	Direction (°T)	Speed (kt)
-5	043	1.7	043	1.0	021	---
-4	043	3.5	043	2.5	021	1.2
-3	043	3.7	043	3.2	021	2.0
-2	043	3.2	043	3.0	021	2.2
-1	043	2.2	043	2.0	021	2.0
0	043	0.7	054	1.0	021	1.0
1	223	1.2	112	1.5	slack	---
2	223	3.2	223	2.7	201	1.2
3	223	4.0	223	3.5	201	2.0
4	223	3.5	223	3.2	201	2.0
5	223	2.0	223	2.5	201	1.7
6	223	0.7	223	1.2	201	1.0

You can see that

(3.02:13) $$v = V \cos[\sigma(t + t_o)]$$

won't be too bad if we use V = 3.2 kt from the AB section and t_o = 8 hr 20 min to adjust the time to the Dover reference point.

We still have η to go. To begin with, AB is practically a co-tidal line, Table 3.02-4, page 227. Therefore, take the time of high

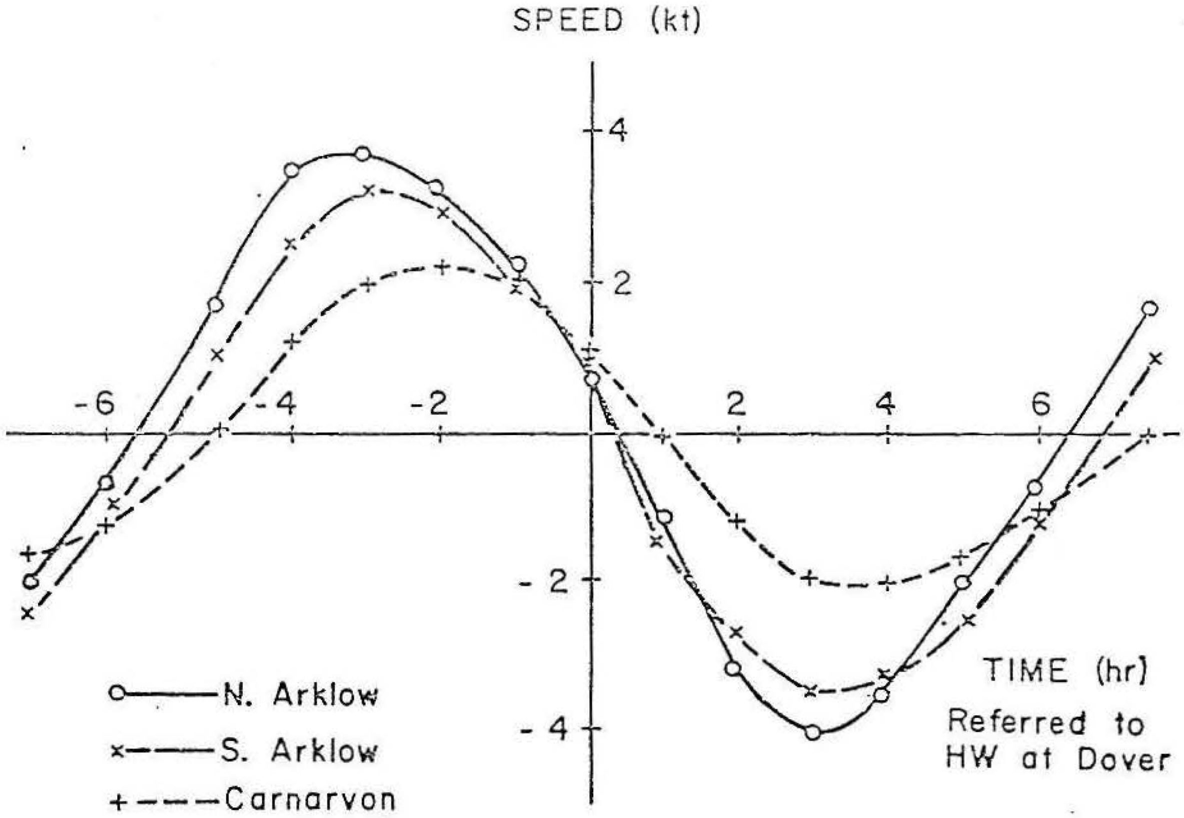


Fig. 3.02-2. Speed of the Tidal Stream at Three Light-Ships in the South Channel of the Irish Sea.

Table 3.02-4. Times of High Water on the Welsh and Irish Sides of the South Channel of the Irish Sea.

	Place	Time of HW	
Irish Side	Arklow Bank	8h 24m	
	Arklow	8h 25m	
	Kilmichael Point	8h 25m	Where AB comes ashore
	Courtown	7h 55m	4 mi south of Kilmichael Pt.
Welsh Side	St. Tudwell's Rd.	8h 02m	
	Beardsey Island	7h 55m	

water everywhere along AB as 8 hours 10 minutes after high water at Dover. This is the mean of the ends; more or less. Picking up (3.02:12), since the water surface slopes across the channel, A must vary. Let the cross-channel axis be y and, invoking the gradient already derived and the assumption of constant slope, we can write

$$A = A_1 - \frac{2\omega v \sin[\phi]}{g}\, y \qquad (3.02{:}14)$$

where A_1 is half the tide range at mid-channel; explicitly here,

$$A_1 = \frac{1}{2}\left(\frac{15 + 4}{2}\right) = 4.75 \text{ feet} \quad .$$

Further, if s is the distance measured from the mid-channel point L of line AB,

$$y = s \sin[\theta] \quad . \qquad (3.02{:}15)$$

All of this gets us the average rate at which energy enters the Irish Sea across the section AB,

$$\dot{W}_{AB} = \left\langle \rho g \int_A^B h \eta v \sin[\theta]\, ds \right\rangle_T \quad . \qquad (3.02{:}16)$$

Substituting (3.02:13, 14, & 15) in (3.02:16)

$$W_{AB} = \left\langle \rho g \int_A^B h\left(A_1 - \frac{2\omega v \sin[\phi]}{g}\, s \sin[\theta]\right)\cos[\sigma(t + t_1)]\; V \cos[\sigma(t + t_o)]\sin[\theta]\, ds \right\rangle_T .$$

Only the two cosine terms contain t so that, averaging them first over T we get

$$\tfrac{1}{2} \cos[\sigma(t_1 - t_o)] \quad .$$

Thus, after taking everything which is substantially constant on the section AB outside the integral sign, we come down to

$$W_{AB} = \tfrac{1}{2}\rho g V \sin[\theta]\cos[\sigma(t_1 - t_o)] \int_A^B h\left(A_1 - \frac{2\omega v \sin[\phi]}{g}\, s \sin[\theta]\right) ds \quad .$$

If we measure the depth across the section, the integral can be evaluated. Actually, it isn't necessary, the depth being constant at 37 fathoms along AB to a sufficient degree. The result is

$$W_{AB} = \tfrac{1}{2}\rho g V \sin[\theta]\cos[\sigma(t_1 - t_o)]\, h A_1 \ell \qquad (3.02{:}17)$$

where ℓ is the length of AB.

The roster of numerical values to use is:

$g = 981 \text{ cm sec}^{-2}$

$\rho = 1.03 \text{ g cm}^{-3}$

$V = 3.2 \text{ kt} = 163 \text{ cm sec}^{-1}$

$\theta = 60°$ AB lies 086°T. The current sets 026°T.

$T = 12.40$ hr

$t_1 = 8$ hr 10 min $\quad \sim \; (t_1 - t_o) = 10$ min

$t_o = 8$ hr 20 min

$h = 37$ fathoms $= 6800$ cm

$A_1 = 4.75$ ft $= 145$ cm

$\ell = 50$ NM $= 9.1\times10^6$ cm .

Consequently,

$$W_{AB} = 6.4\times10^{17} \text{ ergs sec}^{-1} \quad . \tag{3.02:18}$$

The same razzle-dazzle can be used for the energy coming into the Irish Sea through the North Channel across a section RC from Red Bay to the Mull of Cantire, Fig. 3.02-1, page 224. Fortunately, it isn't necessary since almost no energy comes through.

Remark: G. I. Taylor says it's obvious. Maybe it is--to him!

There is a strong tidal current through the North Channel. It runs in from 5 to 11 hours and out from 11 to 5 hours. The neck between the Mull of Cantire and the Irish coast is a loop in a stationary oscillation. The times of high water are: at the Mull of Cantire, 10 hr 58 min, and at Red Bay, 10 hr 55 min, i.e., RC is a co-tidal line. It was chosen for just that reason. The tidal streams change direction at the time of high water, Dover. That puts them 87° out of phase with the local high water and low water. They lack only 12 minutes of being exactly 90° out of phase. The maximum current speed is about 4 kt and there is quite a small tide range. Red Bay and the Mull of Cantire both show about 4 feet. The geostrophic tilt doesn't show up because, when the current is running full strength, which is the time the maximum tilt should occur, the water is at its mean level. At high water and low water the current is slack which rules out a geostrophic effect.

For the energy across RC we have

$$W_{RC} = \rho g \frac{hVA}{2} \cos[\theta]\cos[\sigma(t_o - t_1)] \times RC$$

The values to use are:

$$A = \tfrac{1}{2}(4 \text{ ft}) = 61 \text{ cm}$$
$$\sigma(t_o - t_1) = 87°$$
$$V = 4 \text{ kt} = 200 \text{ cm sec}^{-1}$$
$$RC \times \cos[\theta] = 11 \text{ NM} = 2 \times 10^6 \text{ cm}$$
$$h = 65 \text{ fathoms} = 10^4 \text{ cm} \quad , \quad \text{the mean depth.}$$

Consequently, we have

(3.02:19) $$W_{RC} = 6.2 \times 10^{15} \text{ ergs sec}^{-1} \quad .$$

From (3.02:18 & 19)

$$\frac{W_{RC}}{W_{AB}} = \frac{6.2 \times 10^{15}}{6.4 \times 10^{17}} < 1\% \quad .$$

Therefore, we neglect W_{RC} in comparison with W_{AB}.

The remaining bit of business is to assess the amount of work done by the moon's attraction on the Irish Sea between AB and RC. The moon's attraction can be expressed as a potential function, Ω. If an element of volume fixed with respect to the earth contains water at all times during a lunar day, then no net work will be done since potentials are conservative. However, if the volume element contains water only part of the time, net work is possible.

Take a column of water 1 centimeter square extending from surface to bottom. Then the work done on it is

$$m = \int_{\substack{\text{lunar} \\ \text{day}}} \rho \eta d\Omega \quad .$$

The energy communicated by the moon in a lunar day is

$$E_M = \iint_{\substack{\text{Irish Sea} \\ \text{AB-RC}}} m \, d\sigma$$

where $d\sigma$ is an area element.

In Chapter 1 we worked out Ω and expressed it in terms of latitude and hour angle. Taking the simplest sort of approximation for a dominant semidiurnal tide, and taking $(d/e)^3\cos^2[D] = 1$, we get

$$m = -\frac{3\pi}{2} U\rho g A r \cos^2[\phi]\sin^2[\psi_o]$$

where ψ_o is the phase of the tide at the time of the moon's upper transit.

The numerical values to use are:

$$U = 5.582\times10^{-8} \quad \text{for the moon}$$
$$\rho = 1.03 \text{ g cm}^{-3}$$
$$g = 981 \text{ cm sec}^{-2}$$
$$r = 6.4\times10^{8} \text{ cm} \quad .$$

These give

$$m = -6.6\times10^4 A \sin^2[\psi_o] \text{ ergs.}$$

Thus, the mean rate at which work is done by the moon's attraction per square centimeter of the Irish Sea is

$$W_M = \frac{-6.6\times10^4}{\begin{bmatrix}\text{length of the}\\ \text{lunar day (sec)}\end{bmatrix}} \times \langle A \sin^2[\psi_o]\rangle_{\substack{\text{over the Irish Sea}\\ \text{from AB to RC}}}$$

The mean value of H for the Irish Sea is about 14 feet or 420 centimeters. Take A = 210 cm. The average time of high water is about 1.5 hours before the moon's meridian transit. Take $\psi_o = 22.5°$. This gives a rough approximation of 150 cm for $\langle A \sin^2[\psi_o]\rangle$ and

(3.02:20) $$W_M = -110 \text{ ergs cm}^{-2} \text{ sec}^{-1} \quad .$$

Remark: Don't be panicked, as I was the first time I saw it, by the minus sign. Since high water in the Irish Sea comes shortly before meridian transit of the moon, the tides in the Irish Sea do work on the moon rather than the moon doing work on the tide; hence the minus sign.

The area of the Irish Sea included between the sections AB and RC is about 11,600 NM or 3.9×10^{14} cm . Using the estimate from (3.02:18) we have

$$(6.4\times10^{17})/(3.9\times10^{14}) = 1640 \text{ ergs cm}^{-2} \text{ sec}^{-1} \quad .$$

This, combined with (3.02:20) gives the independent estimate we wanted, (3.02:21) $$1640 - 110 = 1530 \text{ ergs cm}^{-2} \text{ sec}^{-1}$$.

This independent estimate agrees quite well with our previous estimates, 1040 and 1300 ergs cm^{-2} sec^{-1}, and not at all well with Street's estimate of 7 ergs cm^{-2} sec^{-1}.

So great! The big estimate for frictional dissipation is the one to use. What can we get out of it? Well, for one thing, the large amounts of tidal energy sopped up in the Irish Sea could make us wonder how much of the tide wave is absorbed and how much is reflected back out again. The very fact that you can find regions where high water and the strength of the current are substantially in phase suggests that not much energy is reflected out again. In other words, the observed facts match the progressive wave. If lots of the wave were reflected, one would expect a standing wave with the strength of the current $\pi/2$ out of phase with high and low water. Of course, any observed tidal curve will be a tangle of the incoming and the reflected waves. When you get them untangled what looks like a very complex tidal situation in the South Channel turns out to be remarkably simple.

In a general way, the Irish Sea acts like a resonator with two open ends which are "loops" with small tide range and maximum currents. In the middle, near the Isle of Man, Fig. 3.02-1, page 224 the ranges are large and the currents are small. If the analogy with a loop in a stationary oscillation is any good at all, it will be worthwhile to analyse the motion into two opposed waves.

Consider the motion in the South Channel to be entirely rectilinear reversing and work along the axis in the center of the channel. Suppose we assume two waves and write

(3.02:22) $$\eta = \underbrace{a \cos[\sigma t - \kappa x]}_{\text{entering wave}} - \underbrace{b \cos[\sigma t + \kappa x]}_{\text{reflected wave}}$$;

x positive into the Irish Sea. Let c be the speed of a long wave in shallow water of depth h, i.e., $c = \sqrt{gh}$. The problem is to determine a and b so that they make η track the observed tide and then use (3.02:22) to explain the various characteristic features of the tide in South Channel.

The tidal current corresponding to (3.02:22) is

(3.02:23) $$v = a\sqrt{g/h}\cos[\sigma t - \kappa x] + b\sqrt{g/h}\cos[\sigma t - \kappa x] \quad .$$

The maximum of v occurs at x = 0 when t = 0 and is

(3.02:24) $$V = (a + b)\sqrt{g/h} \quad .$$

Also, at x = 0 the phases of the current and the height of the water surface are the same.

At x = 0, t = 0 $\eta(0, 0) = a - b$. Thus the tide range is 2(a - b). At point L on the AB-section, taken as x = 0, we know the half-range, the depth, h, and the maximum velocity, V, so that we also know a and b.

$$a - b = 145 \text{ cm}$$

$$a + b = V\sqrt{h/g} = 163\sqrt{6800/981} = 430 \text{ cm} \quad .$$

Solving,

$$a = 287 \text{ cm}$$

$$b = 143 \text{ cm}$$

$$a/b = 2.0 \quad .$$

Conclusion: At springs the tide wave is cut almost in half during its passage into and out of the Irish Sea and the energy is, therefore, reduced to one quarter.

This conclusion seems to have a sound basis in theory as exhibited but, at the time G. I. Taylor made this argument, the generally accepted idea was that friction had very little effect on tidal regimes. To combat this entrenched idea Taylor again tried for some confirmation of his result. One argument is based on the movement of co-tidal lines in the South Channel.

Since for progressive waves co-tidal lines drawn for successive instants of time mark successive position of the crest and represent wave velocity, there is a tendency to interpret co-tidal lines in this way for all waves. You can't do it. In the case of two superposed progressive waves moving in opposite directions, for example, the co-tidal line moves in the direction of the wave of greater amplitude but at a different speed.

Suppose we write the tide given by equation (3.02:22) as

$$\eta = A \cos[\sigma(t - t_x)] \tag{3.02:25}$$

where

$$A \equiv \sqrt{a^2 + b^2 - 2ab \cos[2\kappa x]} \tag{3.02:26}$$

and

$$\cot[\sigma t_x] = \frac{a - b}{a + b} \cot[\kappa x] \quad . \tag{3.02:27}$$

The symbol t_x represents the time it takes a co-tidal line to move a distance x from the place where the phases of current and tide are the same.

Equation (3.02:27) relates t_x and x. The velocity of the co-tidal line, V_c, from (3.02:27), is

$$V_c = \frac{dx}{dt_x} = c \, \frac{\left(\frac{a - b}{a + b}\right)^2 \cot^2[\kappa x] - 1}{\cot^2[\kappa x] - 1} \left(\frac{a + b}{a - b}\right) \quad . \tag{3.02:28}$$

For x = 0 where the amplitudes are opposed and the tidal currents concur, we have

$$V_c = \frac{a - b}{a + b} c$$

which is smaller than the wave speed, c.

It would be nice if there were enough data to permit us to follow the co-tidal line in the vicinity of the Arklow-Beardsey section but there aren't. Krümmel's co-tidal lines show a crowding near AB which is what you would expect from equation (3.02:28); at least since V_c is a minimum near x = 0. However, there are two sections of the channel where we can nail down single co-tidal lines pretty firmly. Between them we can, at least, get a mean speed for the co-tidal line and compare it with our theory.

On AB you can argue yourself into a co-tidal line at 8 hr 10 min without much effort. We will take this as x = 0. The other section runs from the vicinity of Tuskar Rock to Ramsey Sound, TS on Fig. 3.02-1, page 224. There the co-tidal line is present at 6 hr 15 min. We can use equation (3.02:28) and these data to estimate the ratio of a to b again.

The distance between midpoints of the co-tidal lines, ML, is about 43 NM so that x = − 43 NM. The mean depth between M and L is around 45 fathoms so that c will be about 56 kt. With T the period of the semidiurnal tide

$$\cot[\kappa x] = \cot[-\ 22.3°] = -\ 2.44$$

and

$$t_x = 6 \text{ hr } 15 \text{ min} - 8 \text{ hr } 10 \text{ min} = -\ 1.92 \text{ hr}$$

so that

$$\cot[\sigma t_x] = \cot[-\ 56°] = -\ 0.67$$

and, stuffing (3.02:28),

$$\frac{a - b}{a + b} = \frac{\cot[\kappa x]}{\cot[\sigma t_x]} = \frac{2.44}{0.67}$$

or, getting the ratio of a to b

$$\frac{a}{b} = \frac{2.44 + 0.67}{2.44 - 0.67} = 1.8$$

which is amazingly close to the value of 2.0 based on quite different data. Note that this second method involves only measurements of depth and time. No data on currents or tidal ranges are used.

From here on Taylor, in his paper, goes on to clear out a few more things like the change in direction of the co-tidal line between AB and TS and the effect that the shape of the coast has on the times of high water along the coast. We won't follow him further. Perhaps you should give yourself the pleasure of reading Taylor's paper.

3.03. Munk and Macdonald's Oceanic Tidal Dissipation.

Day after day the moon and the sun do work on the earth's oceans yet the tides neither grow nor diminish. All this energy must go somewhere. But where? That question needs an answer. One possibility is that the energy of the tide which must be dissipated acts to slow the earth's rotation and increase the length of the day.

The material in this section is taken directly from Munk and Macdonald's "The Rotation of the Earth." This is a book that you, as geophysicists, can not afford to neglect; principally because it is such an excellent example of what the practice of geophysics, as distinct from physics, is.

The data bothering Munk and Macdonald are the irregularities in the rotation of the earth which come in two varieties:

(1) the wobble, and

(2) the changes in the rate of rotation, i.e., in the length of the day, abbreviated l.o.d..

The astronomers consider these irregularities a big pain and have simply legislated variations in the l.o.d. out of astronomy by redefining time in terms of the length of the year rather than in terms of the length of the day. This is fine for astronomers but it has become clearer and clearer that these rotational irregularities have a close connection with events on earth and, thus, offer a means of studying large scale geophysical phenomena.

However, like any truly geophysical problem, the diversity of fields of knowledge that must be brought into play is hair raising. It touches every branch of geophysics. One needs to know about winds and air masses, about atmospheric, oceanic, and earth tides, and about motion in the earth's fluid core. In each case, what one needs to know are integrated, i.e., averaged, quantities over the entire globe. This is the weakness of this method of attack; and also its strength. "In principle" we get the integrals by summing over data taken at properly spaced observation stations at sufficiently rapid rates and over long enough times. Actually, of course, there aren't enough stations. They aren't properly spaced. And, the data are not taken at the proper rates or for sufficient lengths of time. This is true now. VERY PROBABLY IT WILL <u>ALWAYS</u> BE TRUE.

Remark: Gentlemen, this is the field in which you aspire to work so bear this fact in mind. The men who have gone before you were well aware of the inadequacy of the data and they didn't rectify the situation to any great extent. You aren't going to either. Your real problem is learning how to live with inadequate data and still get something useful, and perhaps interesting, done in the next 40 years.

Since Munk and Macdonald need so many tools they have to give the basic information. This makes their book an excellent source for a lot of things not necessarily limited to the rotation of the earth. They give you chapters and appendices on:

(1) The dynamic equations in a form sufficiently general to impose no restrictions on deformation.

(2) The stress-strain relations.

(3) The technique of reducing stress-strain relations to dimensionless terms by the use of Love numbers.

(4) A lucid exposition of perturbation methods.

(5) A discussion of power spectra with particular attention to handling the spectrum of both wobble and rotation

Remark: You won't find this anywhere else in the literature on spectra.

(6) A development of the "ocean function" which expresses the location and boundaries of the world's oceans as an expansion in spherical harmonics. They also give the "continentality function."

$$C(\text{oceans}) = C(\theta, \lambda) = \begin{cases} 1 & \text{where there is water} \\ 0 & \text{where there is land} \end{cases} .$$

$$C(\text{continents}) = C(\theta, \lambda) = \begin{cases} 0 & \text{where there is water} \\ 1 & \text{where there is land} \end{cases} .$$

where θ = co-latitude and λ = east longitude.

These parts of "The Rotation of the Earth" are worth reading for themselves.

The irregularities of the earth's rotation can conveniently be separated by period, Fig. 3.03-1, page 238. For those with time scales of a year or less the evidence on the wobble comes largely from the observations of the International Latitude Service and for the variability of the l.o.d. from comparisons of clock time with astronomic time. The annual wobble is largely due to seasonal shifts in the air mass, that in the l.o.d. to winds. The shorter period terms in the l.o.d. are due to short period terms in earth tides. The Chandler wobble is a 14-month variation

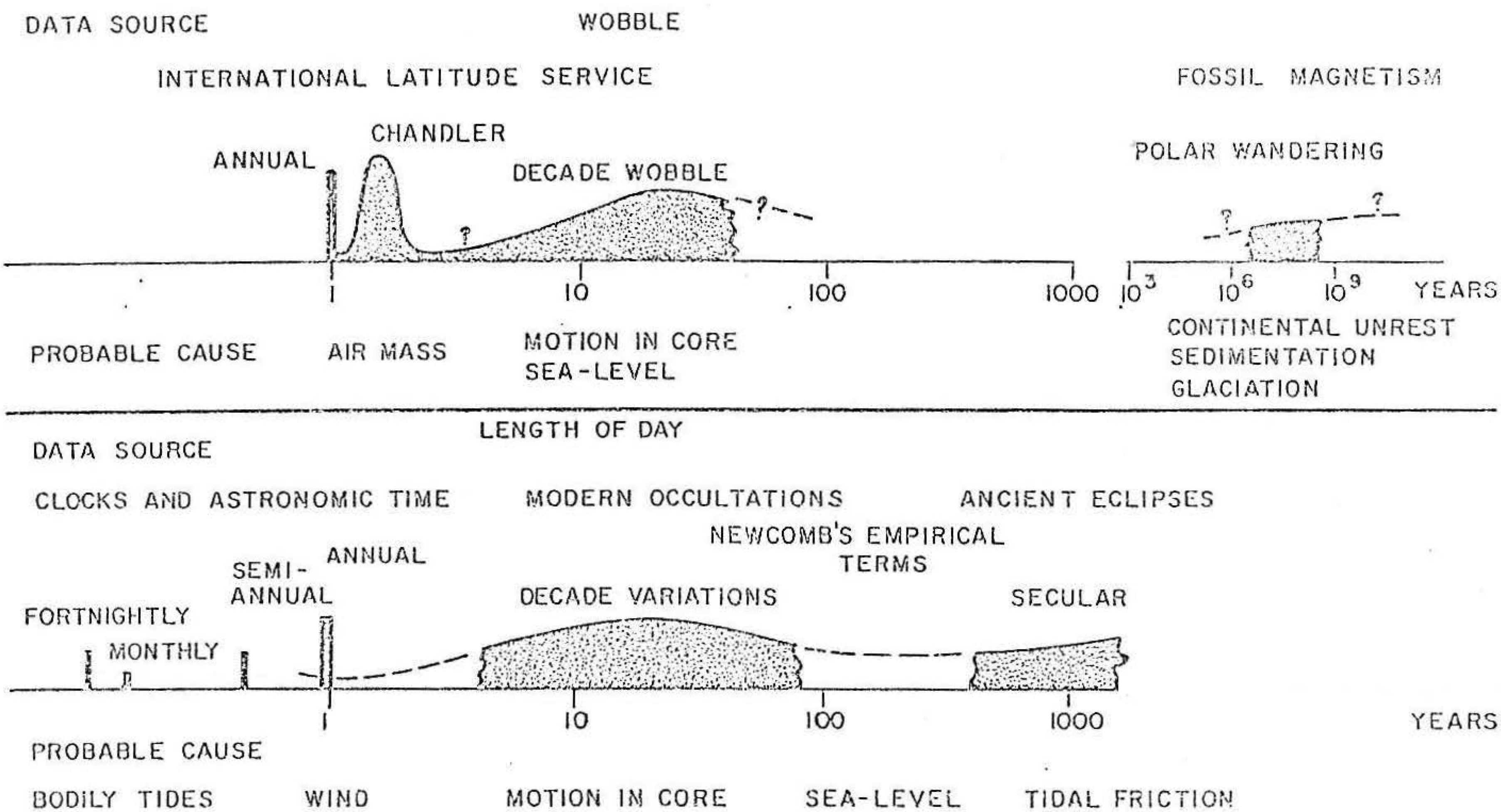


Fig. 3.03-1. The Spectrum of Rotation. The "wobble components (top) and the "length of the day" components (bottom) are schematically arranged according to their time scales in years. Vertical lines indicate discrete frequencies. Shaded portions indicate a continuous or noisy spectrum. Principle sources of the observations are shown above and presumable geophysical causes beneath each sketch. [from Munk and Macdonald (1960)]

governed by the ellipticity and rigidity of the earth. Says Munk, "... the wobble is generated by random impulses of unknown origin and damped by some unknown imperfections from elasticity, or by some other means." An assertion to conjure with!

For the longer periods, evidence comes mainly from modern observations of occultations and from records of ancient eclipses. Very large irregular variations in the l.o.d. with a time scale around a decade may be due to electromagnetic coupling of the earth's mantle to a turbulent fluid core. The century scale stuff may be due to changes in the earth's moment of inertia. And so on, and on. Fig. 3.03-1, page 238, which was lifted from Munk and Macdonald, summarizes the whole situation as they see it.

Munk and Madonald say, "Changes over the last few thousand years are predominantly the result of tidal friction, but here again changes in inertia (presumably associated with a variable sea-level) must play an important part." You see that they have to worry about tidal dissipation. It's all very well to estimate the value for the Irish Sea--if all you are thinking about is the Irish Sea. But, if you are considering the Earth-Moon-Sun as a closed system, energy blotted up in one part of the system has got to show up as a change in some other part of it. And as a great deal of energy does get dumped into the tides, and as it does get dissipated, and to assess the effect of this process on the earth's rotation, the when, where, and how of this dissipation becomes very important. Let's follow Munk and Macdonald through their analysis of oceanic tides.

The flux of tidal energy in the oceans can be represented quite simply. Suppose R_M and R_S represent the mean rates per unit surface area at which the moon's and the sun's gravitational attractions do work on the water. The usual assumption is that within the volume of the oceans there is little or no dissipation; substantially all of it taking place on the boundaries, i.e., on the bottom. But, over most of the ocean the bottom currents are very weak and there is negligible dissipation. Consequently, the dissipation must be concentrated in the few shallow seas.

If we let R_a be the flux across a unit vertical surface in the entrances to the shallow seas and R_b be the work done by tidal currents on the sea bottom (units: ergs cm^{-2} sec^{-1}), then schematically the flow is shown in Fig. 3.03-2.

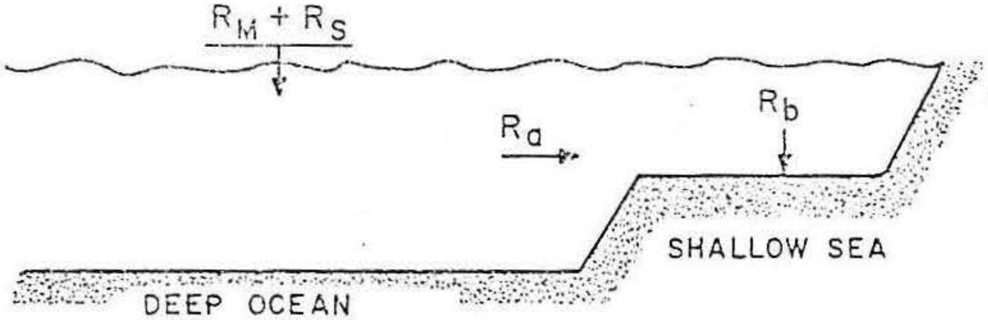


Fig. 3.03-2

You have three ways to arrive at the energy dissipation:

(3.03:1.1) $$-\frac{dE}{dt} = \int_{\substack{\text{Total Sea}\\ \text{Surface}}} (R_M + R_S)\,d\Sigma$$

(3.03:1.2) $$-\frac{dE}{dt} = \int_{\substack{\text{Entrances to}\\ \text{Shallow Seas}}} R_a\,d\Sigma$$

(3.03:1.3) $$-\frac{dE}{dt} = \int_{\substack{\text{Area of}\\ \text{Shallow Seas}}} R_b\,d\Sigma$$

All three have been used.

Let the observed variation of the M_2-tide about mean sea-level be approximated by the sinusoid,

$$\eta_M = A_M \cos[2(\sigma_M t - \phi_M)] \quad .$$

Then, with Ω_M the potential function for the moon,

(3.03:2.1) $$R_M = \langle \rho\Omega_M \frac{d\eta_M}{dt} \rangle = \tfrac{1}{2}\rho g K_M b_M \sigma_M A_M \sin^2[\theta]\sin[2\phi_M] \quad .$$

Similarly for the S_2-tide

(3.03:2.2) $$R_S = \tfrac{1}{2}\rho g K_M b_S \sigma_S A_S \sin^2[\theta]\sin[2\phi_S]$$

where the b's are amplitude factors,

$$\frac{2\pi}{\sigma_M} = \frac{2\pi}{12.42} \quad , \quad b_M = 0.908 \quad , \quad \frac{2\pi}{\sigma_S} = \frac{2\pi}{12} \quad , \quad b_S = 0.423,$$

and K_M is the general lunar coefficient,

$$\frac{3}{2}\,\frac{C_M}{E}\left(\frac{r_E}{d_M}\right)^3 r_E = 53.7 \text{ cm} \quad .$$

For R_b you can use equation (3.02:6) from the Taylor discussion.

(3.03.3) $$R_b = \gamma\rho\langle u|u^2|\rangle = \frac{4}{3\pi}\,\gamma\rho u_o^{\,3}$$

where

$$u = u_o\cos[\sigma t] \qquad \text{and} \qquad \langle|\cos^3[\sigma t]|\rangle = \frac{4}{3\pi}$$

and you can use $\gamma = 0.002$ and be pretty safe, G. I. Taylor (1919); Bowden and Fairbairn (1952).

If we can now produce an expression for the flux of energy by tidal currents, R_a, we will have all the makings for equations (3.03:1.i).

With u(t) and $\Delta p(t)$ for the tidal velocity and pressure departure, the flux of energy across a vertical face of unit area normal to u is Δpu. For a progressive wave

(3.03:4) $$\langle\Delta pu\rangle h = \rho g\langle\eta^2\rangle\upsilon = \tfrac{1}{2}\rho g A^2\upsilon = E\upsilon \quad .$$

A is the surface amplitude and E is the wave energy per unit area. For shallow water, i.e., tide waves, υ is the group velocity so all we've said here is that the energy flux is the energy times the group velocity.

If all the wave energy crossing the entrance to a shallow sea were absorbed, then this would be it. Unfortunately, some of it is reflected. If all were reflected, there would be a standing wave and no dissipation. The case lies somewhere between "progressive" and "standing" but is usually found to be closer to standing.

To treat partial reflection set

$$\Delta p = \rho g A \cos[2\ t] \quad \text{and} \quad u = \frac{\upsilon A}{h}\sin[2(\sigma t + \phi)] \quad .$$

In shallow water the phase speed, $c = \sqrt{gh}$, is the group velocity so that we have

$$\langle \Delta pu \rangle h = \tfrac{1}{2}\rho g A^2 c \sin[2\phi] = Ec \sin[2\phi] \quad .$$

We can consider that the incident wave carries carries Ec ergs sec^{-1} cm^{-1} into the bay and that the reflected wave carries r^2Ec out where $0 \leq r^2 \leq 1$. The net gain to the bay is $Ec(1 - r^2)$ and we can form the equation $\sin[2\phi] = 1 - r^2$. Perfect reflection corresponds to $r = 1$ and $2\phi = 0$ while perfect absorption corresponds to $r = 0$ and $2\phi = \tfrac{1}{2}\pi$.

Let S be the surface area of the bay. Then the average dissipation is

$$R_a = \frac{1}{S}(\tfrac{1}{2}\rho g c) \int_{\substack{\text{across the}\\ \text{entrances}}} A^2 \sin[2\phi]\,dx \quad .$$

In most cases you can use the shallow-water approximation, $c = \sqrt{gh}$, so that all you need to know to make this work is the tidal amplitude and the relative phase of height and current across the entrances.

Remark: Again, revert to section 3.02 and G. I. Taylor (1919).

If, because of friction and the earth's rotation $c = \sqrt{gh}$ isn't good enough, you can still get by with a knowledge of the magnitudes and phases of <u>both</u> height and current.

We have gone along with the assumption that dissipation on the deep-sea bottom was negligible but we had better make some argument for it. The phase speeds we are concerned with are roughly $c = O(\sigma a) = 5\times10^4$ cm sec^{-1} and the amplitudes like $A = 25$ cm. Therefore, $u = cA/h = 3$ cm sec^{-1} at most. Swallow (1955) using his neutrally buoyant float finds tidal components of the order of 1 cm sec^{-1}. This would make R_D around 0.002 ergs cm^{-2} sec^{-1} and $-\frac{dE}{dt} \approx 10^{16}$ ergs sec^{-1} which is completely negligible in comparison with the "observed" dissipation--meaning "the-energy-income-you-have-to-get-rid-of-if-you-aren't-going-to-accumulate-energy;" that amounts to about 3×10^{19} ergs sec^{-1}.

With the makings in hand let's get to it. Continental shelves with ocean depths less than 200 meters cover about 5.5% of the earth's surface, about 2.8×10^{17} cm . If we dissipated all the tidal energy on the shelves, it would come to an average of 114 ergs cm^{-2} sec^{-1}. If we equate this with $0.002\rho<|u^3|>$, we come down to something like 38 cm sec^{-1} or about 0.75 kt for the average current required over the shelves. The observed currents are roughly of the right order. But don't jump yet.

We've been through Taylor (1919) and found that his results look pretty good. A second look shows that the numerical value derived by Taylor for the Irish Sea alone accounts for about 2% of the total required dissipation. That's pretty big. Perhaps the shallow seas play a special role.

Once G. I. Taylor had broken the ice it didn't take any longer than usual for everybody else to get into the swim. Jeffreys (1920) extended the estimate to all oceans producing 1.1×10^{19} ergs sec^{-1} which was 80% of what he needed to get according to his interpretation of the astronomical requirements.

Remark: But only 34% of what he needed according to Munk and Macdonald.

Heiskanen (1920) also did the same thing based on substantially the same data and, as revised by Lambert (1928), got 1.9×10^{19} ergs sec^{-1}. Both of these estimates depend heavily on the bottom friction argument, equation (3.03:1.3). Heiskanen (and Lambert) also had a go at the work done by the moon and sun, equation (3.03:1.1). There was a slight foul up since Heiskanen added where shouldn't have according to Lambert. But, aside from that, the problem was considered closed.

Remark: Munk and Macdonald have taken a can opener to it.

Table 3.03-1, page 244 gives a summary of Jeffreys' and Heiskanen's results after two adjustments have been made:

(1) Multiplication by $\frac{4}{3\pi}$ to get mean dissipation over a tidal cycle in line with equation (3.03:3).

(2) A reduction by a factor of 0.51 [Jeffreys (1952), page 230] to allow for the fact that spring tides have maximum velocities. Heiskanen overlooked this correction so that his published values are about double what they should be.

Table 3.03-1. Energy Dissipation in Units of 10^{19} ergs sec^{-1}.

Region	Jeffreys	Heiskanen
European Waters		
Irish Sea	0.06	0.04
English Channel	0.11	0.23
North Sea	0.07	----
Other Seas	----	0.16
	0.24	0.43
Asiatic Waters		
Yellow Sea	0.11	----
Malacca Strait	0.11	0.18
Other seas	0.01	0.73
	0.23	0.91
North American Waters		
Northwest Passage	0.16	----
Bay of Fundy *	0.04	0.04
Other Seas	----	0.30
	0.20	0.34
South American Waters	----	0.40
Australian Waters	----	0.34
African Waters	----	0.08
Arctic Waters	----	0.13
Bering Sea	1.50	0.96
Total:Spring Tide	2.17	3.60
Total × 0.51	1.1	1.9

* The largest known tides are in the Bay of Fundy but the dissipation there is relatively small. A new calculation by McLellan (1958) gives 0.027×10^{19} ergs sec^{-1} by the flux method, 0.029×10^{19} ergs sec^{-1} by the bottom friction method (using $0.002\rho < |u^3| >$), as compared to Jeffreys' 0.04 and Heiskanens' 0.05. [from Munk and Macdonald (1960)

Now--how does this stack up? One-third to one-half of Heiskanen's total dissipation occurs along the open coastlines of continents. Jeffreys doesn't even consider this as a possibility. Munk and Macdonald think that Heiskanen is generally too large. As an example they cite the region from the mouth of the Gulf of California to Vancouver Island where Heiskanen assumed a tide current of 1.5 kt. Measurements of the total current [Shepard, Revelle, and Dietz (1939)] are rarely higher than 0.5 kt. Systematic measurement off Los Angeles [Stevenson, Tibby, and Gorsline (1956); San Diego Geologic Diving Consultants (1956)] showed maxima of 0.1 and 0.25 kt during springs in 100 feet of water. In general, you can rule out velocities above 0.5 kt on the open California coast.

As an example of a place where Heiskanen seems to be on, consider Patagonia. The shelf is about 500 km wide and about 2000 km long. The tidal amplitudes decrease from 12 feet at 50° South to 1 foot at 37° South. Formula (3.03:4) suggests that a maximum velocity of 10^2 cm sec^{-1} over the shelf might not be too far out. Measurements taken by the Argentine Hydrographic Service show velocities varying from 2.5 kt in the Falkland Passage to 1 kt off the Rio de la Plata which is in good agreement. The corresponding dissipation is about 0.2×10^{19} ergs sec^{-1} which matches Heiskanen.

Another estimate by the flux method supplied to Munk and Macdonald by Redfield considers the tide as a wave that enters from the Antarctic between Falkland and Staten Islands and which is attenuated as it rolls north over the shelf. With $A = 1.5$ m, $h = 50$ m, and an entrance of 500 km one gets $\frac{1}{2}\rho g A^2\sqrt{gh} \times 500 \approx 10^{18}$ ergs sec^{-1} which is of the same order of magnitude.

What about the Bering Sea? It has by far the largest estimate entered in Table 3.03-1. In the eastern Bering there is a large shoal area roughly 1000 km × 1000 km with depths generally less than 60 meters. Both Heiskanen and Jeffreys agree that this shelf is of paramount importance as a sink for tidal energy. In fact, the whole problem gets real local along about here. Three-quarters of the global dissipation occurs in this one relatively small region according to Jeffreys.

Actually, the whole thing hinges on what you elect to use for u_o on the shelf. Says Jeffreys, "... it is stated that the maximum rate of water, where clear of the passes between the Aleutian Islands, is usually about 2½ knots when the depth is less than 100 fathoms." Both Jeffreys and Heiskanen accept reported speeds of 2½ knots at St. Mathew Island and the Pribilof Islands. Jeffreys takes $u_o = 2\frac{1}{2}$ kt and Heiskanen uses $u_o = 2$ knots.

The U.S.C.&G.S. Coast Pilot for Alaska and "Current Tables, Pacific Ocean" aren't very useful here but at least nothing they picture requires a great sloshing about of water on the Bering shelf. Forty years are <u>vorbei</u> and still there aren't any decent measurements. The icebreaker NORTHWIND [U.S.H.O. (1958)] drifted around up there and found no strong tidal currents over the northern part of the shelf east of St. Lawrence Island. In the Bering Strait there is a northward current of about 1 knot but the superposed tidal oscillation is only about ½ knot. To the south there have been numerous oceanographic anchor stations but no current measurements.

Remark: !

However, there was never any evidence of the kind of strain you get on an anchor line if a 2-knot current is running. The NAUTILUS during its journey to the north pole encountered none of the discrepancies in navigation that would be expected if a 2-knot current were present. In short, even though there are no measurements, 2 knots looks excessive and Munk and Macdonald think 1 knot, as an absolute blithering upper limit, is the most that should be allowed.

Since the dependence on u_o is in cube, the reduction in u_o from 2 to 1 brings the dissipation down by an order of magnitude and the Bering Sea isn't anything special at all.

The "data" for this argument aren't exactly the high quality kind of thing one likes to have. Fortunately, one can get at the dissipation by the advection method, equation (3.03:1.2), and this also indicates that Heiskanen's and Jeffreys' estimates are high by an order of magnitude.

Item: Along the portion of the Aleutian chain which forms the southernmost boundary of the Bering shelf, high water and maximum northward current occur at about the same time which corresponds to a progressive wave traveling north onto the shelf. Deduction: where it must be just about all absorbed.

Item: Semidiurnal amplitudes are about 25 cm at the Aleutians and decrease northward, Bristol Bay excepted.

Item: There is no possibility of a large standing wave with nodes at the southern boundary of the shelf.

Numbers: Set $c = \sqrt{gh}$, $h = 60\text{m}$, $A = 25$ cm.

This gives us an $R_a S$ of

$$-\frac{dE}{dt} = R_a S = \tfrac{1}{2}\rho g c \int_0^{1000\text{ km}} A^2 dx = 0.8\times10^{17} \text{ ergs sec}^{-1}$$

or only about 1% of the Munk and Macdonald reduced estimate.

The formula used for c implies that $u_o = cA/h = 20 \text{ cm sec}^{-1} = 0.4$ kt; way low from the u_o used by Heiskanen or Jeffreys. Even if you assume that the whole Bering Sea rises synchronously, the required inflow, neglecting island obstructions, can hardly be more than $u_o = (\sigma_M AL^2)/(hL)$ where L is the surface area and h the depth just north of the Aleutians. With L = 1000 km this gives $u_o = 60 \text{ cm sec}^{-1}$ and $-dE/dt = 2.4\times10^{17}$ ergs sec^{-1}.

We have one more shot in the locker. We can estimate $-dE/dt$ from the work done by the moon and sun. Grab yourself some co-tidal charts and read off the amplitudes, $A(\lambda, \theta)$, and phases, $\phi(\lambda, \theta)$. λ is the east longitude and θ is the co-latitude. Get R as a function of λ and θ from equations (3.03:2.i) and get the dissipation by integrating over all the oceans according to (3.03:1.1).

This program has its difficulties. There are large variabilities in phase associated with amphidromic points and positive and negative contributions to $-dE/dt$ are not very widely separated. Even worse, the data on which the co-tidal charts are based are pretty vague over wide expanses of the open sea. Heiskanen had a crack at it with co-tidal charts published by von Sterneck in 1920. Groves and Munk (1958) did a couple more laps using Dietrich's (1944) compilation based on about three times as many tide stations as von Sterneck had. There wasn't much improvement, really. The big hole is in the Southern Ocean and Dietrich didn't have stations from there either.

Remark: Maybe the IGY data and the Indian Ocean data will bring in some worthwhile information if anyone ever gets around to doing something with it.

The results are shown in Table 3.03-2. All Munk and Macdonald will say

Table 3.03-2. Work Done by the Tidal Forces of the Moon and Sun on the Oceans. (units: 10^{19} ergs sec^{-1})

Ocean	Heiskanen	Lunar			Solar			Total
	Lunar Semi	Semi	Diur	Total	Semi	Diur	Total	
Pacific	3.8	2.8	0.6	3.4	0.7	0.9	0.9	4.3
-	-2.5	-1.9	-0.0	-1.9	-0.5	-0.0	-0.5	-2.4
Total	1.3	0.9	0.6	1.5	0.2	0.2	0.4	1.9
Atlantic	1.8	2.1	0.1	2.2	0.5	0.0	0.5	2.7
-	-1.2	-1.1	-0.1	-1.2	-0.2	-0.0	-0.2	-1.2
Total	0.6	1.0	0.0	1.0	0.3	0.0	0.3	1.3
Indian	1.8	1.5	0.2	1.7	0.4	0.1	0.5	2.2
-	-1.6	-0.9	-0.1	-1.0	-0.2	-0.0	-0.2	-1.2
Total	0.2	0.6	0.1	0.7	0.2	0.1	0.3	1.0
Total	2.1	2.5	0.7	3.2	0.7	0.3	1.0	4.2

[from Munk and Macdonald (1960)]

is "... that the totals may be far smaller than we obtained, or twice as large."

Remark: By this time it is a matter for congratulation that the totals for each ocean are, at least, positive, i.e., no ocean is actually adding energy to the tide.

For the lunar terms (Astronomical Input)

$$-\frac{dE}{dt} = 3.2\times10^{19} \text{ ergs sec}^{-1} \quad : \quad 2.7\times10^{19} \text{ ergs sec}^{-1}$$

For the solar terms

$$-\frac{dE}{dt} = 1.0\times10^{19} \text{ ergs sec}^{-1} \quad : \quad \begin{array}{c}\text{(Astronomical Input)}\\ 0.6\times10^{19} \text{ ergs sec}^{-1}\end{array}$$

Munk and Macdonald say, "The agreement is better than we had any right to expect."

They next calculate the Q^{-1} of the oceans. Q^{-1} is the specific dissipation function defined by

$$Q^{-1} \equiv \frac{1}{2\pi E} \oint \frac{dE}{dt}\, dt \quad .$$

It is the dimensionless measure of the rate at which energy is dissipated in a vibrating system.

$$\oint \frac{dE}{dt}\, dt$$

is the energy dissipated over a complete cycle and E is the peak energy stored in the system during a cycle. You'll find Q^{-1} particularly useful in discussions of observational data since it does not depend on the details of the mechanism which causes the dissipation.

Aside from local resonances, the total energy contained in the ocean tide at any moment can't be different from the total energy contained in the equilibrium tide by more than Q^{-1}. The energy of the lunar equilibrium tide is

$$\rho g \int \left(\frac{\Omega_M}{g}\right)^2 d\Sigma = \rho g K_M{}^2 b_M{}^2 \, \frac{1}{2} \, 2\pi a^2 \int_0^{\pi} (\tfrac{1}{2}\sin^2[\theta])^2 \sin[\theta]\, d\theta = 8\times10^{23} \text{ ergs} \; .$$

Since the earth isn't entirely covered by water, use 5×10^{23} ergs.
From astronomical data one has

$$\frac{2\pi}{Q} = 2\pi \, \frac{dE/dt}{2\sigma_M E} = 2\pi \, \frac{2.7\times10^{19}}{(1.4\times10^{-4})(5\times10^{23})} = \frac{2\pi}{2.6} = 2.4 \quad .$$

for the relative dissipation per cycle. What this says is that once every $24/2.4 = 10$ hours all of the tidal energy is dissipated!

If we take Jeffreys' estimate for the dissipation on the Bering Sea shelf, $0.51\times1.5\times10^{19} = 0.77\times10^{19}$ ergs sec^{-1}, then once every 18 hours all of the global tidal energy must find its way into the Bering Sea. The rate of energy dissipation derived by Jeffreys for the Bering Sea is remarkable enough by itself. The concentration of so much energy in one area is even more remarkable to the point of being entirely incredible. Jeffreys, himself, didn't really feel comfortable about it and said as much in a 1958 paper.

Suppose the ocean tide approximates the equilibrium configuration; which, of course, it doesn't. Then the maximum dissipation would occur with $2\phi = \frac{1}{2}\pi$ in equation (3.03:1.2). One gets 11.2×10^{19} ergs sec^{-1} which is a bit more than the 2.7×10^{19} ergs sec^{-1} required by astronomical considerations. If you stick with the equilibrium configuration but put a phase lag of $2\phi = 14°$ on it, then you produce exactly the required dissipation. It begins to look as though we should not have assumed that the dissipation within the bulk of the ocean is negligible.

A little independent evidence would be useful. A related problem is that of tsunamis. Tsunamis, like that following the Kamtschatka earthquake of 4 November 1952, release energy into the Pacific of the same order as the tide. The waves are long and their bottom currents should be comparable to those for the tides. As it turns out, most of the energy of a tsunami is dissipated in one day although the activity can remain above background for a week or so. As far as this goes, it doesn't conflict with the rate of tidal dissipation.

Well--where do we stand? Astronomical observations call for a tidal dissipation of 3×10^{19} ergs sec^{-1}. Table 3.03-2 shows that there is no sweat in getting this much energy into the ocean. So far as the how and where of the dissipation goes, we are staring a dilemma right in the eyeball. If Munk and Macdonald are right about the Bering Sea,

Remark: I, personally, think they are.

then the dissipation in the shallow seas is, at most, 10^{19} ergs sec^{-1}. And how do we get rid of the other 2×10^{19} ergs sec^{-1}? Of course, Munk and Macdonald may have overlooked regions of concentrated dissipation but, on principle, concentrated dissipation has a sort of sky blue pink color to it when one thinks about the ocean. What we really need is some process that produces dissipation more evenly. Munk and Macdonald suggest, as the only notion that has hit them, that perhaps the bulk of the dissipation is associated with internal (baroclinic) tides. BT measurements, where ever taken, reveal fluctuations in the isotherms which have tidal frequencies. Typical M_2-amplitudes are around 10 meters which is one or two orders of magnitude larger than the equilibrium tide. Working the other way is the density difference of around 10^{-3} times that at the surface so that the work done by the moon and the sun on internal tides is pretty small.

We have no satisfactory theory of internal tides. For free internal waves, neglecting the earth's rotation, you have phase speeds of roughly 20 km hr^{-1} in the open sea. For comparison, the speed of the equilibrium tidal bulge is around 1500 km hr^{-1}. It's a pretty impressive mismatch. Defant (1950) suggested that it could be reconciled by allowing for the effect of rotation on the phase velocity and that the resulting coupling could lead to a large internal response to the tide-generating forces. Recent observations by Ried (1956) make this look unlikely as a way out. Ried measured simultaneously at two points 100 kilometers apart and found no obvious phase relations. If the internal tides were due to the tide-generating forces, then the phases at the two stations should have been virtually identical.

Chip Cox (in a personal or "beer-hall" communication to Munk) suggests that in regions of variable depth the internal (baroclinic) and external (barotropic) modes are not independent and that a flux of energy must take place from each mode to all the others. More specifically, the degree of coupling depends on the extent to which a spectrum of the sea-bottom topography contains "power" at the local wave length of the tides. The results of Ried (1956) are consistent with the hypothesis that internal tides are generated all along the coast line with the degree of conversion depending on local bottom topography. Cox estimates that the conversion from surface to internal modes may amount to 5 ergs cm^{-2} sec^{-1} for the North Atlantic deep waters. There may also be appreciable reconversion to surface modes so that perhaps 5 ergs cm^{-2} sec^{-1} is some sort of an upper limit. At this rate you would have a global conversion around 1.5×10^{19} ergs sec^{-1}.

At least none of this is inconsistent with the notion that the energy of surface tides is converted to internal wave motion and then dissipated within the volume of the ocean. If this is so, then our appeal to tsunamis is not exactly pertinent as it stands. Perhaps one of you ought to try to do something about tides in your own research. The dissertation that Dr. E. J. Katz wrote at The Hopkins dealing with the spectra of bottom topography and his (1962) and (1963) papers should have some relevance.

3.04. Redfield's Reflected Wave with Damping.

Redfield (1950) created a model along the same lines as the incident wave/reflected wave of both Taylor and Munk and Macdonald. It differs in subjecting the waves to frictional damping and looking for the details of the motion. He begins by pointing out that, in most natural cases, the principal disagreements with the standing wave tidal co-oscillation model are:

(1) High water does not occur simultaneously so that the wave can not be strictly standing.

(2) The nodal "line" is usually present only as a region of relatively small tidal range.

These discrepancies can be taken into account by considering the tide as a sum of an incident and a reflected wave if the wave is subject to frictional damping.

The problem, then, is to determine numerically the properties of the primary and reflected waves in such a way as to force agreement with the observed displacements and velocities and to relate these numerical values to the geometry of the bay.

Redfield says of his paper:

"By expressing the relationship of the several aspects of a reflected wave in a form in which the wave period is taken as the unit of time and distance is given in terms of the related pahse changes, it is possible to eliminate the purely geographical dimensions and to obtain a wholly general description of the tide which may be used to indicate how any give channel distorts the behavior of the wave as it advances.

"In the case of irregular channels, in order to justify the applications of relations deduced for uniform channels, in which the change in phase of the primary and reflected waves and their damping is proportional to the distance traveled and in which the velocity of the waves is constant, it is necessary to make the following assumptions:

1. That the effect of irregularities in cross section is to alter the velocity of the primary and reflected waves, i.e., to distort the geographical distribution of phase differences.

2. That damping is proportional to the phase change in the waves rather than to the distance traveled.

3. That the damping coefficient, as defined is constant along the length of the channel."

Let's follow Redfield through and see what he's up to and how it comes out. We begin by taking a narrow bay which rules out v and we also ignore geostrophic terms. Our origin of coordinates is taken at the head of the bay at that point which may be considered the reflecting point. The water motion will be rectilinear reversing.

If no damping were present, we could write

$$\eta_1 = A \cos[\sigma t - \kappa x]$$

for the primary wave and

$$\eta_2 = A \cos[\sigma t + \kappa x]$$

for the reflected wave where $\sigma \equiv 2\pi/\tau$ and $\kappa \equiv 2\pi/\lambda$. Their sum is a standing wave if the channel is uniform. However, if we assume exponential damping we must write instead

(3.04:1) $\eta_1 = A \exp\{-\mu x\} \cos[\sigma t - \kappa x]$

for the primary and

(3.04:2) $\eta_2 = A \exp\{\mu x\} \cos[\sigma t + \kappa x]$

for the reflected wave where

- $A \equiv$ the amplitude of each wave at the reflecting barrier, $x = 0$
- $\sigma \equiv$ the change in phase per unit time
- $t \equiv$ the time measured from high water at the barrier when $t = 0$
- $\kappa \equiv$ the change in phase per unit distance with x measured from the barrier
- $\mu \equiv$ the damping coefficient.

To get generality, use the wave period 2π as a unit of time so that σt then expresses the phase change during the lapse of time measured from high water at the barrier. Denote the true angle for high water at any point along the channel by σt_H. Similarly, σt_S will be the local time angle of slack water relative to high water at the barrier.

For nonuniform channels, the velocity changes and κ can not be a constant. However, we can use κx as the phase difference relative to the barrier due to position. Thus, x represents the part of a cycle completed by the wave in passing between any point and the barrier.

3.04.1. The Time of High Water at Any Point.

The elevation at any time and place is given by $\eta = \eta_1 + \eta_2$ or

(3.04.1:1) $\eta = A\left(\exp\{-\mu x\} \cos[\sigma t - \kappa x] + \exp\{\mu x\} \cos[\sigma t + \kappa x]\right)$.

High water means $\partial\eta/\partial t = 0$,

$$A\left(-\exp\{-\mu x\}\, \sigma \sin[\sigma t - \kappa x] - \exp\{\mu x\}\, \sigma \sin[\sigma t + \kappa x]\right) = 0 ,$$

or

(3.04.1:2.1) $\exp\{-\mu x\} \sin[\sigma t - \kappa x] + \exp\{\mu x\} \sin[\sigma t + \kappa x]$.

Using angle sums,

$$\exp\{-\mu x\}\left(\sin[\sigma t]\cos[\kappa x] - \cos[\sigma t]\sin[\kappa x]\right) + \exp\{\mu x\}\left(\sin[\sigma t]\cos[\kappa x] + \cos[\sigma t]\sin[\kappa x]\right) = 0$$

or

(3.04.1:2.2) $$\left(\exp\{\mu x\} + \exp\{-\mu x\}\right)\sin[\sigma t]\cos[\kappa x] + \left(\exp\{\mu x\} - \exp\{-\mu x\}\right)\cos[\sigma t]\sin[\kappa x] = 0$$

or

$$\frac{\sin[\sigma t]}{\cos[\sigma t]} = -\frac{\exp\{\mu x\} - \exp\{-\mu x\}}{\exp\{\mu x\} + \exp\{-\mu x\}}\frac{\sin[\kappa x]}{\cos[\kappa x]}$$

or

$$\tan[\sigma t] = -\tanh[\mu x]\tan[\kappa x] \quad .$$

Therefore,

(3.04.1:3) $$\sigma t_H = \tan^{-1}\left(-\tanh[\mu x]\tan[\kappa x]\right) \quad .$$

Equation (3.04.1:3) gives the time of high water at any point along the bay, x, and for any damping coefficient, μ, Fig. 3.04.1-1.

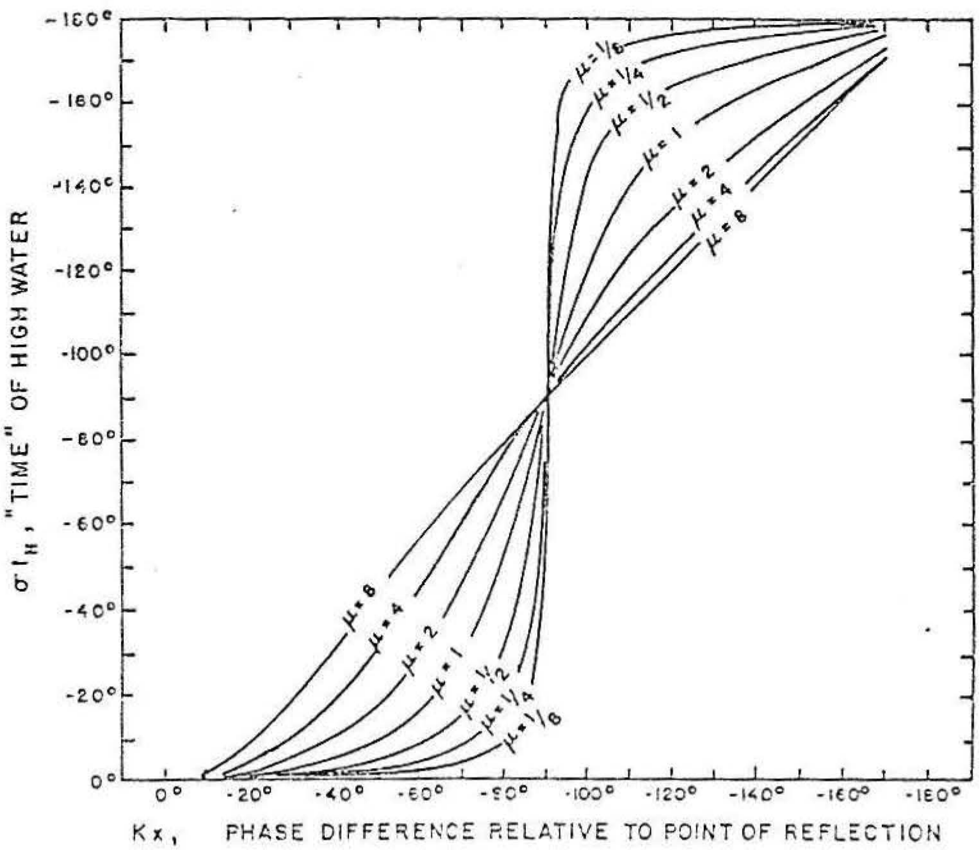


Fig. 3.04.1-1. Time of High Water as a Function of x and μ. [from Redfield (1950)]

3.04.2. To Find the Relative Heights of High Water at Any Point.

Use the height of high water at the barrier as a reference. By the same manipulation used to secure (3.04.1:2.2) from (3.04.1:2.1) equation (3.04.1:1) can be put in the form

$$\eta = 2A\left(\cos[\sigma t]\cos[\kappa x]\cosh[\mu x] - \sin[\sigma t]\sin[\kappa x]\sinh[\mu x]\right) \quad (3.04.2:1)$$

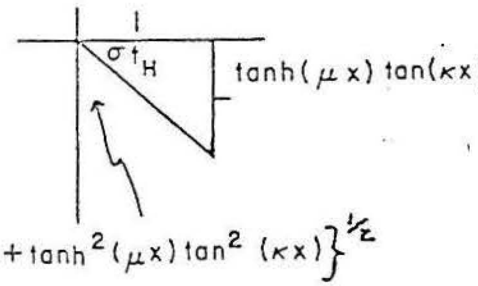
From (3.04.1:3) we have

$$\tan[\sigma t_H] = -\tanh[\mu x]\tan[\kappa x] \quad .$$

Therefore,

$$\sin[\sigma t_H] = -\frac{\tanh[\mu x]\tan[\kappa x]}{(1 + \tanh^2[\mu x]\tan^2[\kappa x])^{\frac{1}{2}}}$$

$$\cos[\sigma t_H] = \frac{1}{(1 + \tanh^2[\mu x]\tan^2[\kappa x])^{\frac{1}{2}}} \quad .$$



Therefore, substituting in (2.04.2:1) gives

$$\eta = 2A\,\frac{\cos[\kappa x]\cosh[\mu x]}{(1 + \tanh^2[\mu x]\tan^2[\kappa x])^{\frac{1}{2}}} + \frac{\tanh[\mu x]\tan[\kappa x]\sin[\kappa x]\sinh[\mu x]}{(1 + \tanh^2[\mu x]\tan^2[\kappa x])^{\frac{1}{2}}}$$

which reduces to

$$\eta = 2A\left(\tfrac{1}{2}(\cos[2\kappa x] - \cosh[2\mu x])\right)^{\frac{1}{2}} \quad .$$

At the barrier where $x = 0$, $\eta_o = 2A$. Therefore,

$$\frac{\eta}{\eta_o} = \left(\tfrac{1}{2}(\cos[2\kappa x] + \cosh[2\mu x])\right)^{\frac{1}{2}} \quad (3.04.2:2)$$

gives the ratio of the height of high water at any place, x, for any damping coefficient, μ, to the height of high water at the reflecting barrier, Fig. 3.04.2-1, page 257.

3.04.3. To Find the Time of Slack Water or Maximum Current at Any Point.

Appealing to wave theory we have for the current

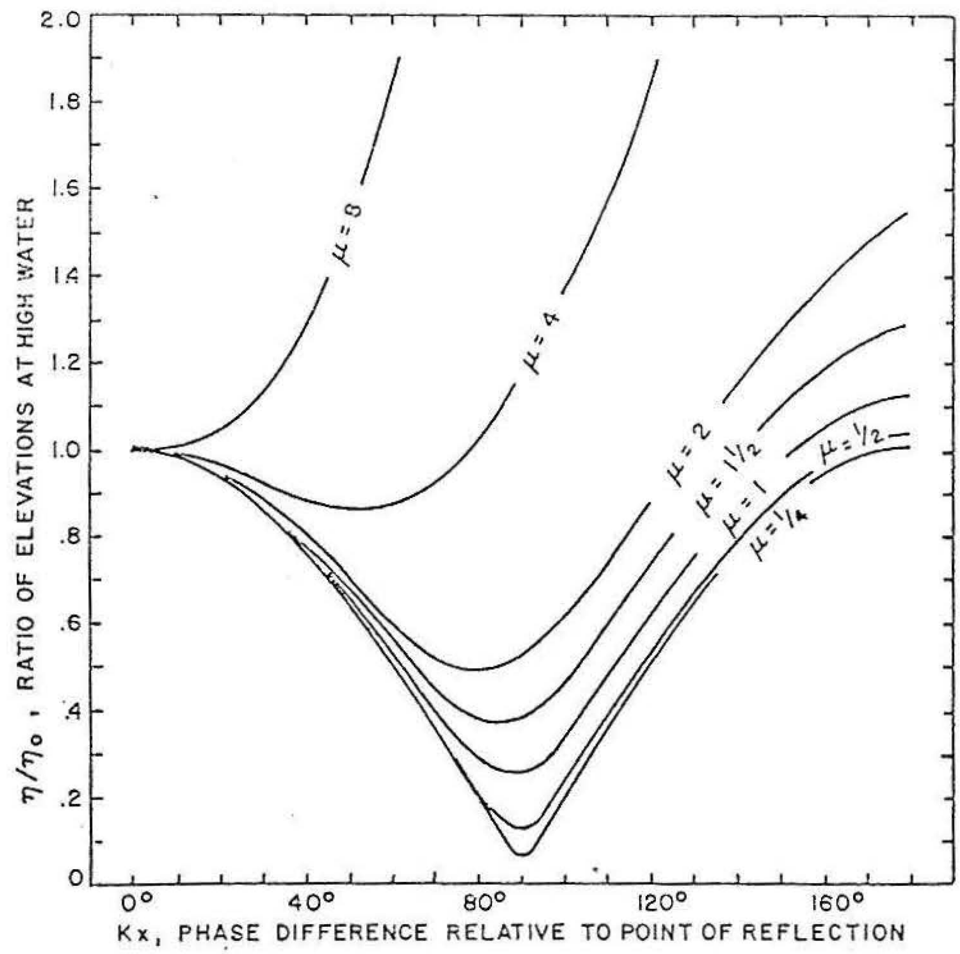


Fig. 3.04.2-1. Height of High Water as a Function of x and μ. [from Redfield (1950)]

(3.04.3:1) $$u_1 = A\,\frac{\sigma}{h}\,(\mu^2 + \kappa^2)^{-\frac{1}{2}}\,e^{-\mu x}\cos[\sigma t - \kappa x + \phi]$$

(3.04.2:2) $$u_2 = -\,A\,\frac{\sigma}{h}\,(\mu^2 + \kappa^2)^{-\frac{1}{2}}\,e^{\mu x}\cos[\sigma t + \kappa x + \phi]$$

where

$h \equiv$ the water depth

$\phi \equiv \tan^{-1}(\mu/\kappa)$ is the phase difference between the time of high water and the maximum current for a progressive wave with damping.

The resultant velocity is $u = u_1 + u_2$ and slack water occurs for $u = 0$. For $x = 0$, $u = 0$ for all t and it is always "slack water" as it should be at the reflecting barrier. For any other point, x, slack water occurs at

$$e^{-\mu x} \cos[\sigma t - \kappa x + \phi] - e^{\mu x} \cos[\sigma t + \kappa x + \phi] = 0$$

which reduces to

(3.04.3:3) $$\tan[\sigma t + \phi] = \frac{\tanh[\mu x]}{\tan[\kappa x]} .$$

If σt_S denotes the time of slack water,

(3.04.3:4) $$\sigma t_S = \tan^{-1}\left(\frac{\tanh[\mu x]}{\tan[\kappa x]}\right) - \phi .$$

Equation (3.04.3:4) relates the time of slack water to the phase difference due to position along the channel for any damping coefficient, Fig. 3.04.3-1. The maximum current will preceed or follow slack water by one-quarter of the period, 90° or $\pi/4$.

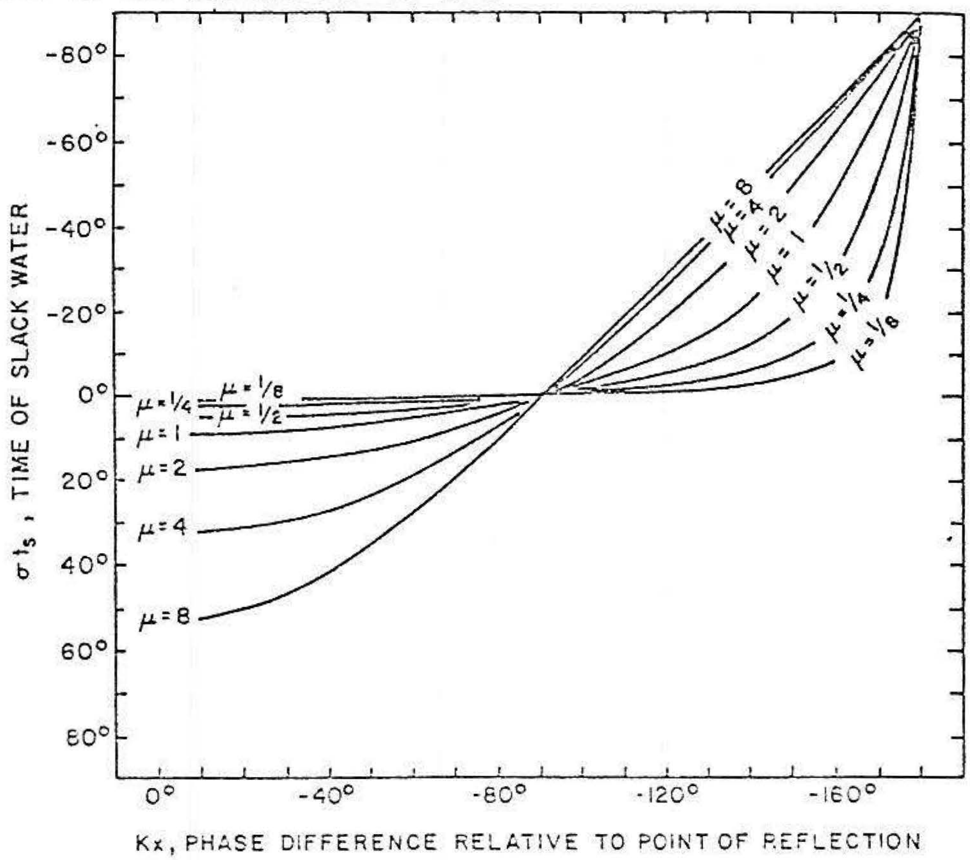


Fig. 3.04.3-1. Time of Slack Water. [from Redfield (1950)]

3.04.4. Application.

To apply this model we generally have at our disposal records of the times of high water, the elevation of high water, the time of slack water, and the time of maximum current at various points around the bay. To figure out what any particular bay will do in the line of modifying the primary and reflected waves we also need information on the distribution of phase differences and on the damping. Fig. 3.04.4-1 cross breeds equations (3.04.1:3) and (3.04.2:2), i.e., Figs. 3.04.1-1 and 3.04.2-1.

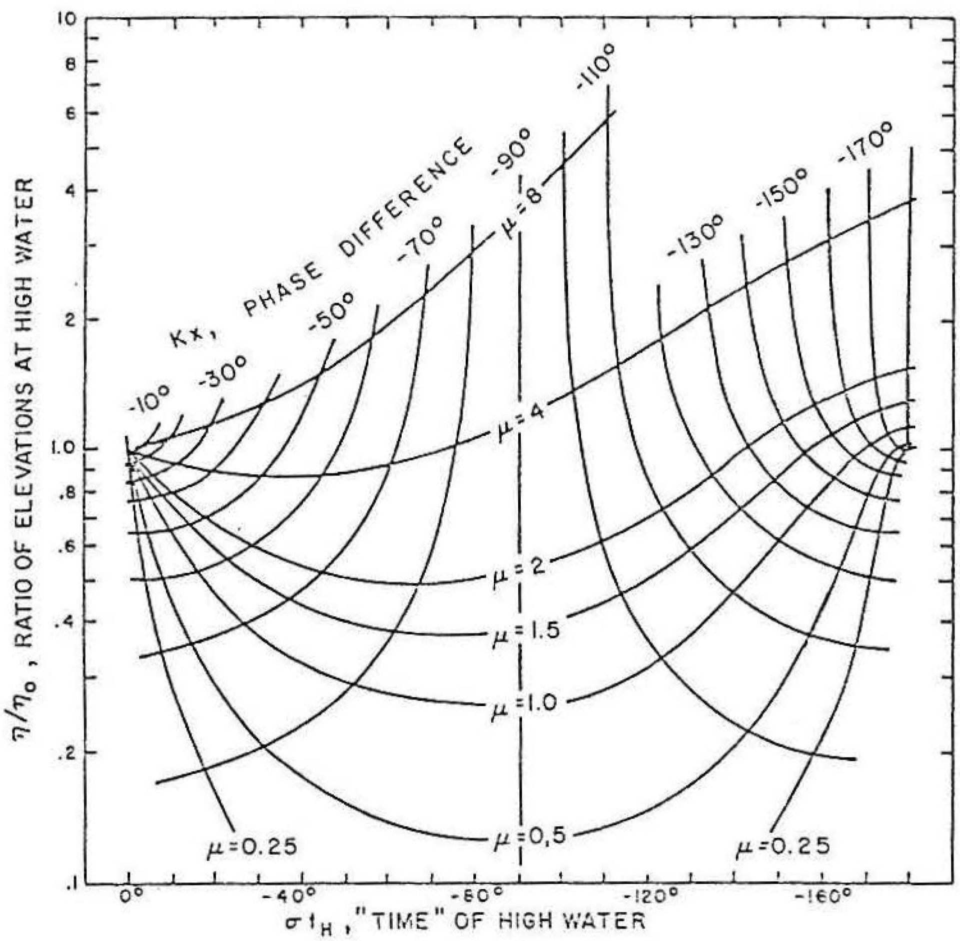


Fig. 3.04.4-1. [from Redfield (1950)]

Any clutch of data giving elevations and times of high water can be plotted on semi-log paper and overlaid on Fig. 3.04.4-1 to see whether a fit is possible. If one is possible, then the plot can be fanned over to geographical distances to determine the phase differences of the primary wave in the channel. By changing the sign you pick up the reflected wave.

In the same way, equations (3.04.1:3) and (3.04.3:4), i.e., Figs. 3.04.1-1 and 3.04.3-1, can be crossbred to give Fig. 3.04.4-2.

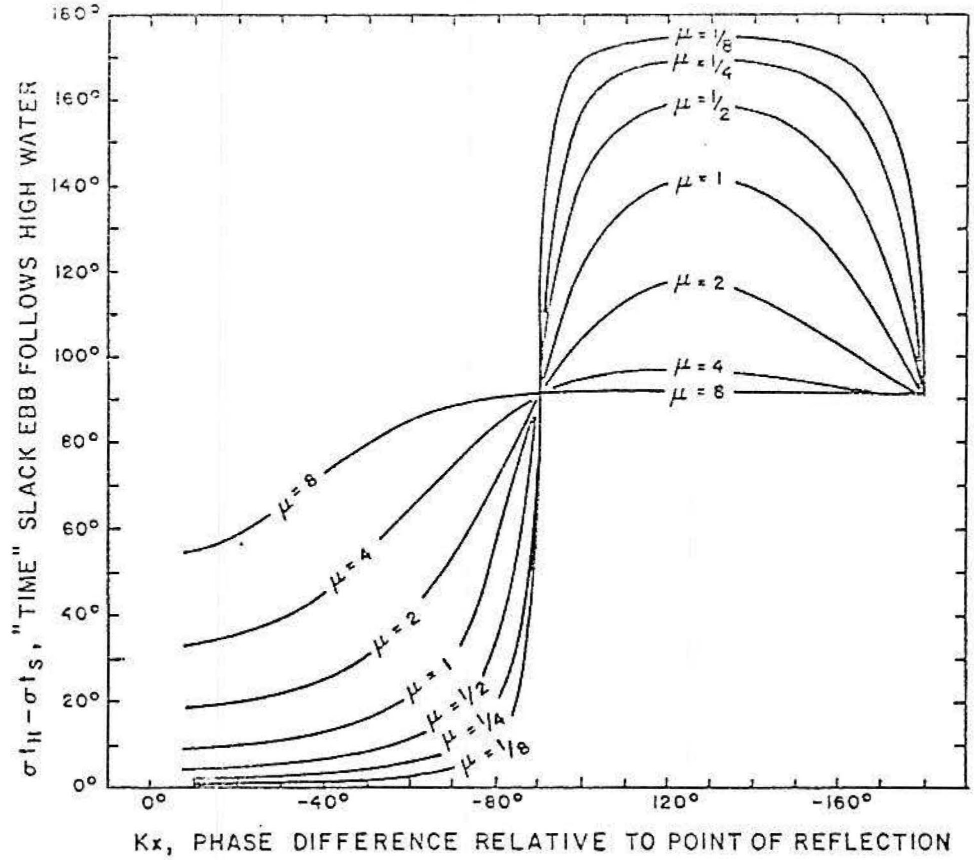


Fig. 3.04.4-2. [from Redfield (1950)]

Redfield has used this model on Long Island Sound and on the Bay of Fundy with good success. The damping coefficient, μ, turned out to be about 1. He also gave a whirl to the Juan de Fuca-Georgia Straits system with equal success but found in this case that μ had to be somewhat greater than 1.

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